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Carothers' Analysis - Chapters 2 and 3

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1 Introduction

In this paper, I study N. L. Carothers' *Real Analysis* textbook (currently used for Math 446 and 447 at Texas A&M University). The topics covered are: the Cantor Set, Metric Spaces, Normed Vector Spaces, a study of more advanced inequalities, and Limits in Metric Spaces. For each of these, I give solutions for selected homework problems.

2 Chapter 2 - Countable and Uncountable Sets

2.1 The Cantor Set

Exercise 28: Let $f : \Delta \to [0, 1]$ be the Cantor function (as originally defined). Check that $f(x) = \sup\{f(y) : y \in \Delta, y \leq x\}$ for any $x \in \Delta$.

Proof. Fix $x \in \Delta$, and let $y \in \Delta$ be variable with $y < x$. Since f is increasing, this means:

$$f(y) \leq f(x) \tag{1}$$

Taking the supremum of both sides of (1) gives:

$$\sup\{f(y) : y \in \Delta, y < x\} \leq f(x)$$

And when we allow $y = x$,

$$\sup\{f(y) : y \in \Delta, y \leq x\} = f(x)$$

□

Exercise 32: Deduce from *Theorem 2.17* that a monotone function $f : \mathbb{R} \to \mathbb{R}$ has points of continuity in every open interval.

Proof. Let (a, b) be an open interval with $a < b$. Apply *Theorem 2.17* to obtain an enumeration, (x_n), of the discontinuities of f in (a, b). Then, since (a, b) is uncountable, and $\{x_n : n \in \mathbb{N}\}$ is countable, $(a, b) \setminus \{x_n : n \in \mathbb{N}\}$ is uncountable. Therefore, f has uncountably many points of continuity in (a, b). □

Exercise 35: Let $f : [a, b] \to \mathbb{R}$ be increasing, and let (x_n) be an enumeration of the discontinuities of $f(x)$. For each n, let $a_n = f(x_n) - f(x_n^-)$ and $b_n = f(x_n^+) - f(x_n)$. Define $a_n = 0$ if $x_n = a$ and $b_n = 0$ if $x_n = b$. Show that:

$$\sum_{n=1}^{\infty} a_n \leq f(b) - f(a) \text{ and } \sum_{n=1}^{\infty} b_n \leq f(b) - f(a)$$

Proof. We prove the inequality for a_n. The proof for b_n is similar, and is left to the reader. Suppose, by contradiction, that $\sum_{n=1}^{\infty} a_n > f(b) - f(a)$. Then, there exists N such that

$$\sum_{n=1}^{N-1} a_n \leq f(b) - f(a) < \sum_{n=1}^{N} a_n \implies f(b) < f(a) + \sum_{n=1}^{N} a_n$$

Thus, $f(x_N) \geq f(a) + \sum_{n=1}^{N} a_n > f(b)$, because $f(x)$ must at least attain the sum of the "left" jump discontinuities provided by $\{a_n\}$. Then, $f(x_N) > f(b)$, but $x_N < b$, a contradiction. □

3 Chapter 3 - Metrics and Norms

3.1 Metric Spaces

Exercise 1: Show that:

$$d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$$

defines a metric on $(0,\infty)$.

Proof. We verify all of the properties:
i. (non-negativity) This is trivial since d is enclosed by absolute values and $x \neq 0 \neq y$.
ii. (indiscernibility) Suppose $d(x,y) = 0$. Then:

$$\left|\frac{1}{x} - \frac{1}{y}\right| = 0 \implies \frac{1}{x} - \frac{1}{y} = 0 \implies \frac{1}{x} = \frac{1}{y} \implies x = y$$

iii. (symmetry) $d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{1}{y} - \frac{1}{x}\right| = d(y,x)$
iv. (triangle inequality) Compute:

$$\begin{aligned} d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right| &= \left|\frac{1}{x} - \frac{1}{z} + \frac{1}{z} - \frac{1}{y}\right| \\ &\leq \left|\frac{1}{x} - \frac{1}{z}\right| + \left|\frac{1}{z} - \frac{1}{y}\right| \\ &= d(x,z) + d(y,z) \end{aligned}$$

Which implies that $d(x,y) \leq d(x,z) + d(y,z)$. □

Exercise 2: If d is a metric on M, show that $|d(x,z) - d(y,z)| \leq d(x,y)$ for any $x, y, z \in M$.

Proof. By the triangle inequality:

$$\begin{aligned} & d(x,z) \leq d(x,y) + d(y,z) \\ \implies & d(x,z) - d(y,z) \leq d(x,y) \end{aligned} \tag{2}$$

and:

$$\begin{aligned} & d(y,z) \leq d(x,y) + d(x,z) \\ \implies & d(y,z) - d(x,z) \leq d(x,y) \end{aligned} \tag{3}$$

Together (2) and (3) imply that $|d(x,z) - d(y,z)| \leq d(x,y)$. □

Exercise 3: Suppose $d : M \times M \to \mathbb{R}$ satisfies $d(x,y) = 0 \iff x = y$ and $d(x,y) \leq d(x,z) + d(y,z)$ for all $x, y, z \in M$ (indiscernibility and the triangle inequality). Prove that d is a metric for M.

Proof. Let $x, y \in M$. We will show d satisfies the other metric properties. First, let's confirm that $0 \leq d(x, y)$. Compute:

$$\begin{aligned} 0 = d(x,x) \leq d(x,y) + d(x,y) = 2d(x,y) \\ \implies 0 \leq d(x,y) \end{aligned}$$

Now, we will show that $d(x, y) = d(y, x)$. By the triangle inequality:

$$d(x,y) \leq d(x,x) + d(y,x) = d(y,x) \tag{4}$$
$$d(y,x) \leq d(y,y) + d(x,y) = d(x,y) \tag{5}$$

Together, (4) and (5) imply that $d(x, y) = d(y, x)$. □

Exercise 5: Show that $\rho(a, b) = \sqrt{|a-b|}$, $\sigma(a, b) = \frac{|a-b|}{1+|a-b|}$, and $\tau(a, b) = \min\{|a-b|, 1\}$ each define metrics on $\mathbb{R}$.

Proof. Applying *Exercise 3*, we need only prove that the identity of indiscernibles and the triangle inequality hold.
$\underline{\rho(a, b)}$: Consider: $\sqrt{|a-b|} = 0 \iff |a-b| = 0 \iff a = b$, proving indiscernibility. Suppose $x, y \in \mathbb{R}$ with $x, y \geq 0$. Then, we have that;

$$\begin{aligned} x + y &\leq x + y + 2\sqrt{xy} \\ \implies (\sqrt{x+y})^2 &\leq (\sqrt{x} + \sqrt{y})^2 \\ \implies \sqrt{x+y} &\leq \sqrt{x} + \sqrt{y} \end{aligned} \tag{6}$$

We will use this to prove the modified triangle inequality for ρ. Let $a, b, c \in \mathbb{R}$. Compute:

$$\begin{aligned} \rho(a,b) = \sqrt{|a-b|} = \sqrt{|a-c+c-b|} &\leq \sqrt{|a-c| + |c-b|} \\ &\leq \sqrt{|a-c|} + \sqrt{|c-b|} \text{ from (6)} \\ &= \rho(a,c) + \rho(b,c) \end{aligned}$$

Thus, the proof for ρ is complete.
$\underline{\sigma(a, b)}$: Consider: $1 + |a-b| \geq 1$, so we multiply by the denominator to obtain $|a-b| = 0 \iff a = b$, proving indiscernibility. Next, we need to prove the modified triangle inequality. Let:

$$F(t) := \frac{t}{1+t} = \frac{1}{\frac{1}{t}+1} = (1 + \frac{1}{t})^{-1} \quad \text{for } t \neq 0$$

We claim that $F(t)$ is increasing. Suppose $x > y > 0$. Then, $\frac{1}{x} < \frac{1}{y} \implies \frac{1}{x} + 1 < \frac{1}{y} + 1 \implies (\frac{1}{x}+1)^{-1} > (\frac{1}{y}+1)^{-1} \implies F(x) > F(y)$. Then let $s, t \geq 0$ and compute:

$$F(s+t) = \frac{s+t}{1+s+t} = \frac{s}{1+s+t} + \frac{t}{1+s+t} \leq \frac{s}{1+s} + \frac{t}{1+t} = F(s) + F(t) \tag{7}$$

Then, let $a, b, c \in \mathbb{R}$. By the triangle inequality, we have $|a-b| \leq |a-c| + |c-b|$. Since F is increasing, we can write:

$$\begin{aligned} \sigma(a,b) = F(|a-b|) &\leq F(|a-c| + |c-b|) \\ &\leq F(|a-c|) + F(|c-b|) = \sigma(a,c) + \sigma(b,c) \end{aligned}$$

Thus, the proof for σ is complete.
$\underline{\tau(a,b)}$: Suppose $\tau(a,b) = 0$. Since $1 \neq 0$, we know that the min function selected $\overline{|a-b|}$, so $|a-b| = 0 \implies a = b$. The other direction is trivial. Now, we will prove the modified triangle inequality. Let:

$$G(t) := \min\{t, 1\} \text{ for } t \geq 0$$

G is very clearly increasing, as can be confirmed by a simple graph of the function. Now, we will prove that the modified triangle inequality holds for G. Let $s, t \geq 0$. We'll consider three cases. First, suppose $s, t \geq 1$. Then, $G(s+t) = 1 \leq 2 = G(s) + G(t)$. Second, suppose $s < 1 \leq t$. Then, $G(s+t) = 1 \leq 1 = G(t) \leq G(t) + s = G(t) + G(s)$. Third, and finally, suppose $s, t < 1$. Then, if $s + t \geq 1$, $G(s+t) = 1 \leq s+t = G(s)+G(t)$. If $s+t < 1$, then $G(s+t) = s+t = G(s)+G(t)$. We now apply the same technique from our proof for σ, completing the proof for the modified triangle inequality. □

Exercise 8: If d_1 and d_2 are both metrics on the same set M, does $d_1 + d_2$ yield a metric on M? If d is a metric, is d^2 a metric?
<u>Solution:</u> We apply *Problem 3*, and thus, need only show inseparability and the modified triangle inequality.
We claim that $d_1 + d_2$ is a metric.

Proof. Let $a, b \in M$. Compute:

$$d_1(a,b) + d_2(a,b) = 0 \iff d_1(a,b) = -d_2(a,b)$$

Since $d_1, d_2 \geq 0$, the only solution to the above equation is that $d_1(a,b) = 0 = d_2(a,b)$. Then, since both d_1 and d_2 are metrics, $a = b$. Now, we will prove the modified triangle inequality. Let $a, b, c \in M$. Compute:

$$\begin{aligned} d_1(a,b) + d_2(a,b) &\leq d_1(a,c) + d_1(b,c) + d_2(a,b) \\ &\leq d_1(a,c) + d_1(b,c) + d_2(a,c) + d_2(b,c) \\ &= (d_1 + d_2)(a,c) + (d_1 + d_2)(b,c) \end{aligned}$$

□

We claim that if d is a metric, then d^2 is NOT (necessarily) metric.

Proof. We give a counterexample. Let d be the Euclidean metric on $\mathbb{R}$. Then, let $x = 0$, $y = \frac{1}{2}$, and $z = 1$. Then, compute:

$$d^2(x,z) = 1 \not\leq \frac{1}{4} + \frac{1}{4} = d^2(x,y) + d^2(z,y)$$

□

Exercise 9: Recall that $2^{\mathbb{N}}$ denotes the set of sequences of 0s and 1s. Show that $d(a,b) = \sum_{n=1}^{\infty} 2^{-n}|a_n - b_n|$ defines a metric on $2^{\mathbb{N}}$.

Proof. We verify inseparability and the modified triangle inequality. Suppose $a = b$. Then $d(a,b) = 0$ clearly. Conversely, suppose $d(a,b) = 0$. Notice that each term in d is nonnegative. That is:

$$2^{-n}|a_n - b_n| \geq 0 \quad \forall n \in \mathbb{N}$$

Thus, $d(a,b) = 0$ implies each term in its sum is 0. That is:

$$\begin{aligned} 2^{-n}|a_n - b_n| = 0 \quad \forall n \in \mathbb{N} \\ \implies |a_n - b_n| = 0 \quad \forall n \in \mathbb{N} \\ \implies a_n = b_n \quad \forall n \in \mathbb{N} \\ \implies a = b \end{aligned}$$

Now, we will verify the triangle inequality. Let $a, b, c \in 2^{\mathbb{N}}$. Then, compute:

$$\begin{aligned} d(a,b) &= \sum_{n=1}^{\infty} 2^{-n}|a_n - b_n| \\ &\leq \sum_{n=1}^{\infty} 2^{-n}(|a_n - c_n| + |c_n - b_n|) \\ &= \sum_{n=1}^{\infty} 2^{-n}|a_n - c_n| + 2^{-n}|c_n - b_n| \\ &= \sum_{n=1}^{\infty} 2^{-n}|a_n - c_n| + \sum_{n=1}^{\infty} 2^{-n}|c_n - b_n| \\ &= d(a,c) + d(b,c) \end{aligned}$$

Thus, $d(a,b)$ is a metric for $2^{\mathbb{N}}$. $\square$

Exercise 14: We say that a subset A of a metric space M is *bounded* if there is some $x_0 \in M$, and some constant $C < \infty$ such that $d(a, x_0) \leq C$ for all $a \in A$. Show that a finite union of bounded sets is again bounded.

Proof. Let $A_1, A_2, \cdots, A_n \subseteq M$ be bounded sets. Since they are bounded, we can apply the definition and find $x_1, x_2, \cdots, x_n \in M$ and $C_1, C_2, \cdots, C_N < \infty$ such that $d(a, x_k) \leq C_k$ for all $a \in A_k$. Let $A := \cup_{k=1}^{n} A_k$. Then, let $C_1^* := \max_{1 \leq i < j \leq n}\{d(x_i, x_j)\}$. That is, C_1^* is the maximum distance between the x_k's. Then, let $C_2^* := \max\{C_1, \cdots, C_n\}$. Let $C := C_1^* + C_2^*$. Then let $x := x_n$. Let $a \in A$. Then, $a \in A_k$ for some $k = 1, 2, \cdots, n$. Let k take that value. Then, compute:

$$d(a,x) \leq d(a,x_k) + d(x,x_k) \leq C_k + C_1^* \leq C_2^* + C_1^* = C$$

Thus, A is bounded. $\square$

Exercise 15: We define the *diameter* of a nonempty subset A of M by $\operatorname{diam}(A) = \sup\{d(a,b) : a, b \in A\}$. Show that A is bounded if and only if $\operatorname{diam}(A)$ is finite.

Proof. Suppose A is bounded. Then, there exists $x_0 \in M$ and $C < \infty$ such that $d(a, x_0) \leq C$ for all $a \in A$. Suppose by contradiction that $\operatorname{diam}(A)$ is not finite. WLOG, suppose $\operatorname{diam}(A) = \infty$. Then, by the definition of sup, given $K > 0$, there exist $a, b \in A$ such that $d(a,b) \geq K$. Then, there exist $a, b \in A$ such that

$d(a,b) \geq 2C+1$. But, $d(a,b) \leq d(a,x_0)+d(b,x_0) \leq 2C$, a contradiction. Conversely, suppose $\text{diam}(A)$ is finite. Then, fix x_0 to be any point in A. Then, $x_0 \in M$. Let $a \in A$ be variable. Then, $d(a,x_0) \leq \text{diam}(A)$ by its definition, showing that A is bounded. $\square$

3.2 Normed Vector Spaces

Exercise 18: Show that $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$ for any $x \in \mathbb{R}^n$. Also check that $\|x\|_1 \leq n\|x\|_\infty$ and $\|x\|_1 \leq \sqrt{n}\|x\|_2$.

Proof. We'll start by proving $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$ for any $x \in \mathbb{R}^n$. We do this by induction on n.
Base Case ($n = 1$): This is trivial. Clearly $|x_1| = |x_1| = |x_1|$.
Inductive Step: Suppose the hypothesis is true up to n. We will show it holds for $n+1$. We first prove the left half of the inequality. By its definition, $\|x\|_\infty$ is either equal to $\sup_{k=1,\cdots,n}|x_k|$ or $|x_{n+1}|$. If it is the former, then we can apply the inductive hypothesis to obtain:

$$\begin{aligned}\|x\|_\infty = \sup_{k=1,\cdots,n}|x_k| &\leq \sqrt{|x_1|^2+|x_2|^2+\cdots+|x_n|^2}\\ &\leq \sqrt{|x_1|^2+|x_2|^2+\cdots+|x_n|^2+|x_{n+1}|^2} = \|x\|_2\end{aligned}$$

If it is the latter, the conclusion is trivial:

$$\|x\|_\infty = |x_{n+1}| = \sqrt{|x_{n+1}|^2} \leq \sqrt{|x_1|^2+|x_2|^2+\cdots+|x_n|^2+|x_{n+1}|^2} = \|x\|_2$$

We now prove the right half of the inequality. It is sufficient to show

$$\sqrt{|x_1|^2+\cdots+|x_{n+1}|^2} \leq |x_1|+\cdots+|x_{n+1}| \iff |x_1|^2+\cdots+|x_{n+1}|^2 \leq (|x_1|+\cdots+|x_{n+1}|)^2$$

Compute:

$$\begin{aligned}(|x_1|+\cdots+|x_{n+1}|)^2 &= (|x_1|+\cdots+|x_n|)^2+(|x_{n+1}|)^2+2(|x_1|+\cdots+|x_n|)(|x_{n+1}|)\\ &\geq (|x_1|+\cdots+|x_n|^2+|x_{n+1}|^2\end{aligned}$$

Then we apply the inductive hypothesis to obtain:

$$(|x_1|+\cdots+|x_n|)^2+|x_{n+1}|^2 \geq |x_1|^2+\cdots+|x_n|^2+|x_{n+1}|^2$$

Thus $(|x_1|+\cdots+|x_{n+1}|)^2 \geq |x_1|^2+\cdots+|x_{n+1}|^2 \iff \|x\|_2 \leq \|x\|_1$. Thus, combining both parts, $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$, completing the proof.

Now we will prove $\|x\|_1 \leq n\|x\|_\infty$. Since n is finite, $\|x\|_\infty = \sup_{k=1,\cdots,n}|x_k| = \max_{k=1,\cdots,n}|x_k|$. Then, pick i such that $|x_i| = \max_{k=1,\cdots,n}|x_k| = \|x\|_\infty$. Then, by definition, $|x_i| \geq |x_k|$ for all $k = 1,\cdots,n$. Then, compute

$$\begin{aligned}\|x\|_1 &= |x_1|+\cdots+|x_n|\\ &\leq |x_i|+\cdots+|x_i|\\ &= n|x_i| = n\|x\|_\infty\end{aligned}$$

Thus, the proof is complete.

Now we will check that $\|x\|_1 \leq \sqrt{n}\,\|x\|_2$. We want to show that $(|x_1|+\cdots+|x_n|)^2 \leq n(|x_1|^2+\cdots+|x_n|^2)$. It is sufficient to show that $(|x_1|+\cdots+|x_n|)^2+\sum_{1\leq i<j\leq n}(|x_i|-|x_j|)^2 = n(|x_1|^2+\cdots+|x_n|^2)$, because $\sum_{1\leq i<j\leq n}(|x_i|-|x_j|)^2$ is non-negative, so our desired conclusion follows. (For simplicity, we write $\sum_{1\leq i<j\leq n}(|x_i|-|x_j|)^2$ in place of $\sum_{j=1}^{n-1}\sum_{i=j}^{n}(|x_i|-|x_j|)^2$). Compute:

$$\begin{aligned}
(|x_1|+\cdots+|x_n|)^2 &+ \sum_{1\leq i<j\leq n}(|x_i|-|x_j|)^2 \\
&= \sum_{i=1}^{n}|x_i|^2 + \sum_{1\leq i<j\leq n}2|x_i||x_j| + \sum_{1\leq i<j\leq n}(|x_i|-|x_j|)^2 \\
&= \sum_{i=1}^{n}|x_i|^2 + \sum_{1\leq i<j\leq n}2|x_i||x_j| + \sum_{1\leq i<j\leq n}(|x_i|^2+|x_j|^2-2|x_i||x_j|) \\
&= \sum_{i=1}^{n}|x_i|^2 + \sum_{1\leq i<j\leq n}(|x_i|^2+|x_j|^2)
\end{aligned}$$

It is easy to verify that every $|x_k|$ appears $n-1$ times in the expanded version of the sum over $1\leq i<j\leq n$ for all $k=1,2,\cdots,n$. Thus, the equation simplifies to become:

$$\begin{aligned}
(|x_1|+\cdots+|x_n|)^2 + \sum_{1\leq i<j\leq n}(|x_i|-|x_j|)^2 &= \sum_{i=1}^{n}|x_i|^2 + \sum_{i=1}^{n}(n-1)|x_i|^2 \\
&= n\sum_{i=1}^{n}|x_i|^2
\end{aligned}$$

Thus, the proof is complete. $\square$

Exercise 20: Show that $\|A\| = \max_{1\leq i\leq n}\left(\sum_{j=1}^{m}|a_{i,j}|^2\right)^{1/2}$ is a norm on the vector space $\mathbb{R}^{n\times m}$ of all $n\times m$ real matrices $A=[a_{i,j}]$.

Proof. We will refer to $\|\cdot\|$ as "the norm".
i. The norm is clearly positive.
ii. If $A=0$, $\|A\|=0$ clearly. Suppose $\|A\|=0$. Then

$$\left(\sum_{j=1}^{m}|a_{k,j}|^2\right)^{1/2} = 0$$

for some k. Notice that the Euclidean norm of a vector in $\mathbb{R}^m$ takes the same form. We thus conclude that $a_{k,1}=a_{k,2}=\cdots=a_{k,m}=0$. Moreover, if any other row had a nonzero term, the max function would have selected that row. Thus, all rows are 0, so $A=0$.

iii. Compute:

$$\begin{aligned}
\|\alpha A\| &= \max_{1\leq i\leq n}\left(\sum_{j=1}^{m}|\alpha a_{i,j}|^2\right)^{1/2} \\
&= \max_{1\leq i\leq n}\left(\sum_{j=1}^{m}|\alpha|^2|a_{i,j}|^2\right)^{1/2} \\
&= \max_{1\leq i\leq n}\left(|\alpha|^2\sum_{j=1}^{m}|a_{i,j}|^2\right)^{1/2} \\
&= \max_{1\leq i\leq n}|\alpha|\left(\sum_{j=1}^{m}|a_{i,j}|^2\right)^{1/2}
\end{aligned}$$

And since multiplying by a constant affects all rows equally, we can pull $|\alpha|$ through the max function, concluding that $\|\alpha A\| = |\alpha|\,\|A\|$.
iv. Let $A, B \in \mathbb{R}^{n\times m}$. Then:

$$\|A+B\| = \max_{1\leq i\leq n}\left(\sum_{j=1}^{m}|a_{i,j}+b_{i,j}|^2\right)^{1/2}$$

By the properties of the Euclidean norm, we can simplify this:

$$\|A+B\| \leq \max_{1\leq i\leq n}\left(\left(\sum_{j=1}^{m}|a_{i,j}|^2\right)^{1/2} + \left(\sum_{j=1}^{m}|b_{i,j}|^2\right)^{1/2}\right)$$

Then, applying the max function to each term can only make them bigger or cause them to remain the same. It cannot make either term smaller. Thus, we can write:

$$\|A+B\| \leq \max_{1\leq i\leq n}\left(\sum_{j=1}^{m}|a_{i,j}|^2\right)^{1/2} + \max_{1\leq i\leq n}\left(\sum_{j=1}^{m}|b_{i,j}|^2\right)^{1/2} = \|A\| + \|B\|$$

Thus, we have verified all the properties of a norm, so $\|\cdot\|$ is indeed a norm on $\mathbb{R}^{n\times m}$. □

3.3 More Inequalities

Exercise 24: The conclusion of *Lemma 3.7* also holds in the case $p = 1$ and $q = \infty$. Why?

Proof. Let $x \in \ell_1$ and $y \in \ell_\infty$. Then, $\|y\|_\infty = \sup\{|y_n| : n \in \mathbb{N}\}$. That is,

$|y_i| \le \|y\|_\infty$ for all $i \in \mathbb{N}$. Then, compute:

$$\begin{aligned}
\sum_{i=1}^{\infty} |x_i y_i| &= \sum_{i=1}^{\infty} |x_i||y_i| \\
&\le \sum_{i=1}^{\infty} |x_i| \, \|y\|_\infty \\
&= \|y\|_\infty \sum_{i=1}^{\infty} |x_i| \\
&= \|y\|_\infty \, \|x\|_1
\end{aligned}$$

Thus, the proof is complete. □

3.4 Limits in Metric Spaces

Exercise 27: Show that $\operatorname{diam}(B_r(x)) \le 2r$.

Proof. Let $a, b \in B_r(x)$. Then, $d(a,b) \le d(a,x) + d(b,x) < r + r = 2r$, so $d(a,b) < 2r$. Taking the sup of both sides gives $\sup\{d(a,b) : a, b \in B_r(x)\} \le 2r$. Thus, $\operatorname{diam}(B_r(x)) \le 2r$. □

Exercise 31: Give an example where $\operatorname{diam}(A \cup B) > \operatorname{diam}(A) + \operatorname{diam}(B)$. If $A \cap B \ne \emptyset$ show that $\operatorname{diam}(A \cup B) \le \operatorname{diam}(A) + \operatorname{diam}(B)$.

Proof. Let $A = [-3, -2]$ and $B = [2, 3]$. Then, $\operatorname{diam}(A) = \operatorname{diam}(B) = 1$, but $\operatorname{diam}(A \cup B) = (3) - (-3) = 6$. Suppose $A \cap B \ne \emptyset$. □

Exercise 33: Prove that limits are unique. [*Hint:* $d(x,y) \le d(x,x_n) + d(y,x_n)$]

Proof. Suppose $x_n \to x$ and $x_n \to y$. We will show $x = y$. Let $\epsilon > 0$ be given. Then, there exists $N_1 \in \mathbb{N}$ such that $d(x, x_n) < \epsilon/2$ for all $n \ge N_1$. Similarly, there exists $N_2 \in \mathbb{N}$ such that $d(y, x_n) < \epsilon/2$ for all $n \ge N_2$. Let $N = \max\{N_1, N_2\}$. Then, compute:

$$\begin{aligned}
d(x,y) &\le d(x,x_n) + d(y,x_n) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

But, since ϵ is arbitrary, $d(x,y) = 0$, thus, $x = y$. □

Exercise 34: If $x_n \to x$ in (M, d), show that $d(x_n, y) \to d(x, y)$ for any $y \in M$. More generally, if $x_n \to x$ and $y_n \to y$, show that $d(x_n, y_n) \to d(x, y)$.

Proof. Let $\epsilon > 0$ be given. Since $x_n \to x$, there exists $N \in \mathbb{N}$ such that for all $n \ge N$, $d(x_n, x) < \epsilon$. Compute, (using the euclidean norm in $\mathbb{R}$):

$$\begin{aligned}
|d(x_n, y) - d(x, y)| &\le |d(x_n, x) + d(x, y) - d(x, y)| \\
&\le |d(x_n, x)| = d(x_n, x) < \epsilon
\end{aligned}$$

Thus, $d(x_n, y) - d(x, y) \to 0$, and since $d(x_n, y), d(x, y) \ge 0$, $d(x_n, y) \to d(x, y)$.

For the more general proposition, let $\epsilon > 0$ be given. Since $x_n \to x$ and $y_n \to y$, we can find $N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \implies d(x_n, x) < \epsilon/2$ and $n \geq N_2 \implies d(y_n, y) < \epsilon/2$. Then, let $N = \max\{N_1, N_2\}$. Suppose $n \geq N$. Then, compute:

$$\begin{aligned}|d(x_n, y_n) - d(x, y)| &\leq |d(x_n, x) + d(y_n, x) - d(x, y)| \\ &\leq |d(x_n, x) + d(y_n, y) + d(x, y) - d(x, y)| = |d(x_n, x) + d(y_n, y)| \\ &\leq |d(x_n, x)| + |d(y_n, y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

□

Exercise 35: If $x_n \to x$, then $x_{n_k} \to x$ for any subsequence (x_{n_k}) of (x_n).

Proof. Let $\epsilon > 0$ be given. Then, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq N$. Suppose $n \geq N$. Then, by definition $n_k \geq n \geq N$. Thus, n_k satisfies the limit definition, so $d(x_{n_k}, x) < \epsilon$, so $x_{n_k} \to x$ □

Exercise 40: Here is a positive result about ℓ_1 that may restore your faith in intuition. Given any (fixed) element $x \in \ell_1$, show that the sequence $x^{(k)} := (x_1, \cdots, x_k, 0, \cdots) \in \ell_1$ (i.e., the first k terms of x followed by all 0s) converges to x in ℓ_1-norm. Show that the same holds true in ℓ_2 but give an example showing that it fails (in general) in ℓ_∞.

Proof. Let $x \in \ell_1$. Then, compute:

$$\begin{aligned}\left\|x - x^{(k)}\right\| &= \sum_{i=1}^{\infty} |x_i - x_i^{(k)}| \\ &= \sum_{i=1}^{k} |x_i - x_i^{(k)}| + \sum_{i=k+1}^{\infty} |x_i - x_i^{(k)}| \\ &= 0 + \sum_{i=k+1}^{\infty} |x_i - 0| \\ &= |x_{k+1}| + |x_{k+2}| + \cdots = \|x\| - \sum_{i=1}^{k} |x_i| < \infty\end{aligned}$$

To verify that $x^{(k)} \to x$ as $k \to \infty$, it suffices to show that $\left\|x - x^{(k)}\right\| \to 0$ as $k \to \infty$. Note that $\|x\| < \infty$ and that $\sum_{i=1}^{k} |x_i|$ is a monotone increasing sequence. Further, it converges to $\|x\|$ by definition. Thus, as $k \to \infty$, $\left\|x - x^{(k)}\right\| = \|x\| - \sum_{i=1}^{k} |x_i| \to 0$.

Similarly, let $x \in \ell_2$. Again, it suffices to verify that $\left\|x - x^{(k)}\right\| \to 0$ as $k \to \infty$.

Compute:

$$\begin{aligned}
\left\|x - x^{(k)}\right\| &= \left(\sum_{i=1}^{\infty} |x_i - x_i^{(k)}|^2\right)^{1/2} \\
&= \left(\sum_{i=1}^{k} |x_i - x_i^{(k)}|^2 + \sum_{i=k+1}^{\infty} |x_i - x_i^{(k)}|^2\right)^{1/2} \\
&= \left(\sum_{i=k+1}^{\infty} |x_i - x_i^{(k)}|^2\right)^{1/2} \\
&= \left(\sum_{i=k+1}^{\infty} |x_i|^2\right)^{1/2} = \left(\|x\|^2 - \sum_{i=1}^{k} |x_i|^2\right)^{1/2}
\end{aligned}$$

Again, by definition, $\sum_{i=1}^{k} |x_i|^2 \to \|x\|^2$ as $k \to \infty$. Thus, $\left\|x - x^{(k)}\right\| \to 0$ in ℓ_2.

Finally, we give an example showing that this fails in ℓ_∞. Consider $x_n = 1 - \frac{1}{n}$. Then, x_n is increasing, so:

$$\left\|x_n - x_n^{(k)}\right\|_\infty = \sup\{|x_i - x_i^{(k)}| : i \in \mathbb{N}\} = \sup\{0, \cdots, 0, |x_{k+1}|, \cdots\} = |x_{k+1}|$$

But, as $k \to \infty$, $|x_{k+1}| \to 1$ since $x_n \to 1$. Thus, $\left\|x_n - x_n^{(k)}\right\|_\infty \not\to 0$. $\square$

Exercise 41: Given $x, y \in \ell_2$, recall that $\langle x, y\rangle := \sum_{i=1}^{\infty} x_i y_i$. Show that $\langle x^{(k)}, y^{(k)}\rangle \to \langle x, y\rangle$.

Proof. Suppose $x^{(k)} \to x$ and $y^{(k)} \to y$. Then, compute:

$$\begin{aligned}
\langle x^{(k)}, y^{(k)}\rangle &= \sum_{i=1}^{\infty} x_i^{(k)} y_i^{(k)} \\
&= \sum_{i=1}^{k} x_i^{(k)} y_i^{(k)} + \sum_{i=k+1}^{\infty} x_i^{(k)} y_i^{(k)} \\
&= \sum_{i=1}^{k} x_i y_i
\end{aligned}$$

Thus, as $k \to \infty$, $\langle x^{(k)}, y^{(k)}\rangle \to \sum_{i=1}^{\infty} x_i y_i = \langle x, y\rangle$, and the proof is complete, $\square$