Numerical Methods to solve a System of Linear Eqn

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots\cdots a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots\cdots a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad\qquad\qquad \vdots \qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots\cdots a_{nn}x_n = b_n$$

$$AX = B \Rightarrow X = A^{-1}B.$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} ; \quad B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix} ; \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}$$

JACOBI METHOD

Assumption: ① Unique Solution

② the Coefficient matrix 'A' has no zeros on its main diagonal

$$a_{11}, a_{22}, a_{33}, \cdots\cdots a_{nn} \neq 0$$

To begin with, solve the 1st equation for x_1, the second eq/n. for x_2, and so on,

$$\begin{cases} x_1 = \dfrac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n\right] \\ x_2 = \dfrac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n\right] \\ \vdots \\ x_n = \dfrac{1}{a_{nn}}\left[b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_n\right] \end{cases}$$

Initial guess

$$\{x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \ldots x_n^{(0)}\}$$

$$\rightarrow \{x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \ldots x_n^{(1)}\}$$

$$\rightarrow \{x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \ldots x_n^{(2)}\}$$

$$x^{(K)} = \{x_1^{(K)}, x_2^{(K)}, x_3^{(K)}, \ldots x_n^{(K)}\}$$

$K = 1, 2, 3, \ldots n$

For each $K \geq 1$, we generate the components $x_i^{(K)}$ for $x^{(K-1)}$

$$x_i^{(K)} = \frac{1}{a_{ii}}\left[b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}\, x_j^{(K-1)}\right]$$

$\forall i = 1, 2, \ldots n$

Numerical Algorithm :-

$$AX = B$$

$$A = [a_{ij}] \quad , \quad B = b_i \qquad X = x_i^{(k)}$$

Tolerance = TOL

Max. # of iterations = N

Step-1 $\quad x_i^{NEW} \qquad x_i^{OLD}$

Set $k = 1$

STEP2 $\rightarrow$ While $K \le N$, Do the following step

For $i = 1, 2, \ldots n$

STEP-3 $\rightarrow$ $$x_i^{NEW} = \frac{1}{a_{ii}} \left[b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} \, x_j^{OLD} \right]$$

STEP-4 $\rightarrow$ I $\quad |x^{NEW} - x^{OLD}| < TOL$, Print x_i
STOP.

$\downarrow$

STEP-5 $\rightarrow$ Set $K = K+1$ (Go back to step-2)
$x^{OLD} = x^{NEW}$

finally $\rightarrow$ OUTPUT $(x_1, x_2, x_3, \ldots x_n)$.

Ex-1

$$\left.\begin{aligned} 6x_1 - 2x_2 + x_3 &= 11 \\ 2x_1 - 7x_2 - 2x_3 &= -5 \\ x_1 + 2x_2 - 5x_3 &= -1 \end{aligned}\right\} \begin{aligned} x_1 &= 2 \\ x_2 &= 1 \\ x_3 &= 1 \end{aligned}$$

Soln:

$$\left.\begin{aligned} x_1 &= \frac{1}{6}\left(11 + 2x_2 - x_3\right) \\ x_2 &= \frac{1}{7}\left(5 + 2x_1 - 2x_3\right) \\ x_3 &= \frac{1}{5}\left(1 + x_1 + 2x_3\right) \end{aligned}\right\}$$

Initial gues $\quad x_1^{(0)} = 0, \quad x_2^{(0)} = 0, \quad x_3^{(0)} = 0$

$$x_1^{(1)} = 1.833, \quad x_2^{(1)} = 0.714, \quad x_3^{(1)} = 0.200$$

$$x_1^{(2)} = 2.038, \quad x_2^{(2)} = 1.181, \quad x_3^{(2)} = 0.852$$

$k \to$	0th	1	2	3	4	5	6	7	8
x_1	0	1.833	2.038	2.085					2.000
x_2	0	0.714	1.181	1.053	...	...	...		1.000
x_3	0	0.200	0.852	1.080					1.00

Gauss - Seidel Method :-

With the Jacobi method, the values of x_i obtained in the n-th iteration remain unchanged until the entire $(n+1)$th itsation is completed.

With Gauss-seidel method, you use the new values of each x_i as soon as they are known.

In other words, once you have determined x_1 (say) from the 1st eqn, its value is then used in the 2nd eqn to obtain the new x_2. Similarly the new x_1, & x_2 are used in the 3rd eqn to obtain the new x_3 and so on.

In general,

Once we have computed $x_1^{(k+1)}$ from the first eqn, its value is then used in the 2nd eqn to obtain the new $x_2^{(k+1)}$ and so on.

Formulate

For each $k \geq 1$, generate the component $x_i^{(k)}$ of $x^{(k)}$ from $x^{(k-1)}$, by using the following eqn

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[b_i - \underline{\sum_{j=1}^{i-1} a_{ij}\, x_j^{(k)}} - \sum_{j=i+1}^{n} a_{ij}\, x_j^{(k-1)}\right]$$

$\forall\ i = 1, 2, 3, \ldots n$

Initial guess $x_1^{(0)}, x_2^{(0)}, x_3^{(0)} \ldots x_n^{(0)}$

For $k=1$,

$$x_1^{(1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)} - \cdots a_{1n}x_n^{(0)}\right]$$

$$x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(1)} - a_{22}x_2^{(0)} \ldots\right]$$

$$x_3^{(1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)} - a_{33}x_3^{(0)} - \cdots\right]$$

Gauss - Seidel Method

$$6x_1 - 2x_2 + x_3 = 1$$
$$2x_1 - 7x_2 - 2x_3 = -5$$
$$x_1 + 2x_2 - 5x_3 = -1$$

Initial guess

$$x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$$

$$x_1 = \frac{1}{6}(11 + 2x_2 - x_3)$$
$$x_2 = \frac{1}{7}(5 + 2x_1 - 2x_3)$$
$$x_3 = \frac{1}{5}(1 + x_1 + 2x_3)$$

$x_1^{(1)} = \frac{11}{6} = 1.833$, $x_2^{(1)} = 1.238$, $x_3^{(1)} = 1.062$

$x_2^{(1)} = 2.069$, $x_2^{(2)} = 1.002$, $x_3^{(2)} = 1.015$

	0th	1	2	3	4	5
x_1	0	1.833	2.069	1.998	1.999	2.000
x_2	0	1.238	1.002	0.995	1.000	1.000
x_3	0	1.062	1.015	0.998	1.000	1.000

Example of Divergence

$$x_1 - 5x_2 = -4$$
$$7x_1 - x_2 = 6$$

$$x_1^{(0)} = 0, \qquad x_2^{(0)} = 0$$

$$x_1^{(1)} = -4, \qquad x_2^{(1)} = -6$$

	0	1	2	3	4	5
x_1	0	-4	-34	-174	-1244	-6124
x_2	0	-6	-34	-244	-1244	-8574

STRICTLY DIAGONALLY DOMINANT matrix

$$AX = B.$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

An $(n \times n)$ matrix A is strictly diagonally dominant if the absolute value of each entry on the main diagonal is greater than the sum of the absolute values of the other entries in the same row

$$|a_{11}| > |a_{12}| + |a_{13}| + \cdots + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}| + \cdots + |a_{2n}|$$

$$\vdots$$

$$|a_{nn}| > |a_{n1}| + |a_{n2}| + \cdots + |a_{n,n-1}|$$

$$3x_1 - x_2 = -4$$
$$2x_1 - 5x_2 = 2$$

$$A \begin{pmatrix} 3 & -1 \\ 2 & -5 \end{pmatrix}$$

$$|3| > |-1|$$
$$|-5| > |2|$$

$$\left.\begin{aligned} 4x_1 + 2x_2 - x_3 &= -1 \\ x_1 \qquad + 2x_3 &= -4 \\ 3x_1 - 5x_2 + x_3 &= 3 \end{aligned}\right\} \quad A = \begin{pmatrix} 4 & 2 & -1 \\ 3 & -5 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

If $\underset{\sim}{A}$ is strictly diagonally dominant, then the System of linear eq/ns $AX = B$ has a unique solution to which the Jocobi and Gauss-Seidel method will Converge for any initial approximation

$$\begin{cases} x_1 - 5x_2 = -4 \\ 7x_1 - x_2 = 6 \end{cases}$$

$AX = B$, $\quad A = \begin{pmatrix} 1 & -5 \\ 7 & -1 \end{pmatrix}$

$$7x_1 - x_2 = 6$$
$$x_1 - 5x_2 = 4$$

$AX = B$, $\quad A = \begin{pmatrix} 7 & -1 \\ 1 & -5 \end{pmatrix}$

$\downarrow$

Strictly diagonally Dominant

Jacobi-method

$$x_1 = \frac{1}{7}(6 + x_2)$$
$$x_2 = \frac{1}{5}(4 + x_2)$$

$$x_1^{(0)} = 0, \quad x_2^{(0)} = 0$$

$$x_1^{(1)} = 0.8571, \quad x_2^{(1)} = 0.9714$$

	0	1	2	3	4	5
x_1	0	0.8571	0.9959	0.9999	1.000	1.000
x_2	0	0.9714	0.9992	1.000	1.000	1.000