CSC591: Foundations of Data Science
HW3: Regression Analysis

Released: 10/21/15
Due: **10/27/15 (23:55pm);** (One day late: -25%; -100% after that).

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Notes

- Submit single zip file containing: (1) all solutions as single pdf file (Filename: Lastname_StudentID.pdf); (2) separate R code file for Q4-Q5 with appropriately named readme files.
- You can also submit scanned hand written solution (should be legible, TA's interpretation is final).
- This h/w is worth 5% of total grade + **Bonus questions account for 3% of grade**
- You can discuss with your friends, but solution should be yours.
- Any kind of copying will result in 0 grade (minimum penalty), serious cases will be referred to appropriate authority.
- All submissions must be through Moodle (you can email to TA with cc to Instructor – only if these is a problem – if not received on time, then standard late submission rules apply)
- No makeups; for regarding policies, refer to syllabus and 1st day lecture slides.

Q#	Max Points	Your Score
1	55	
2	15	
3	1% of grade (Bonus)	
4	30 (R mini project)	
5	2% of grade (Bonus)	

Note:

(1) All questions and sub-parts (Q1-Q3) of this h/w require hand calculations using the formulas that you learned from the course materials. You can use any calculator.
(2) For all project/programming questions, you need implementations in R and submit codes and plots (as requested). Please provide comments (in the code), and separate readme.txt file describing steps to run the code.

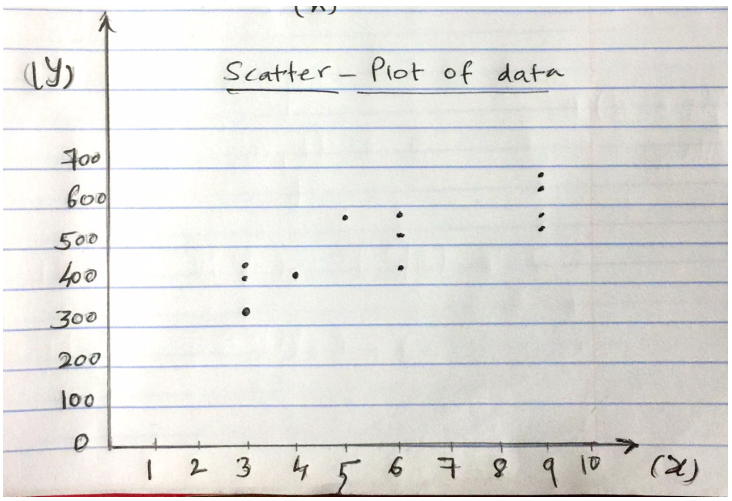
Q1. Simple Linear Regression (**55 points**)

Following table 1 show the data required to answer this question.

x	y
6	520
4	421
6	581
9	630
3	412
9	562
6	434
3	443
9	590
5	570
3	348
9	672

(a) Draw 2-d scatter plot (choose appropriate scaling for x and y axis). (5 points)

Answer



(b) Compute the slope and intercept of the simple linear regression equation (show all computations. **Hint**: use tabular format to compute intermediate quantities) (10 points)

Answer

The following table computes all the required statistics to find the slope and intercept of the simple linear regression. The last row in the table provides the summation of each column. The formula are as follows:

$y' = \beta_0 + \beta_1 x$

$\beta_0 = ((\sum y)(\sum x^2) - (\sum x)(\sum xy)) / (n(\sum x^2) - (\sum x)^2)$

$\beta_1 = (n(\sum xy) - (\sum x)(\sum y)) / (n(\sum x^2) - (\sum x)^2)$

x	y	xy	x2	y2
6	520	3120	36	270400
4	421	1684	16	177241
6	581	3486	36	337561
9	630	5670	81	396900
3	412	1236	9	169744
9	562	5058	81	315844
6	434	2604	36	188356
3	443	1329	9	196249
9	590	5310	81	348100
5	570	2850	25	324900
3	348	1044	9	121104
9	672	6048	81	451584
72	**6183**	**39439**	**500**	**3297983**

Thus using the above table we get

$\beta_0 = ((6183)(500) - (72)(39439)) / (12(500) - (72)^2)$

$\beta_0 = 308.691$

$\beta_1 = ((12)(39439) - (72)(6183)) / (12(500) - (72)^2)$

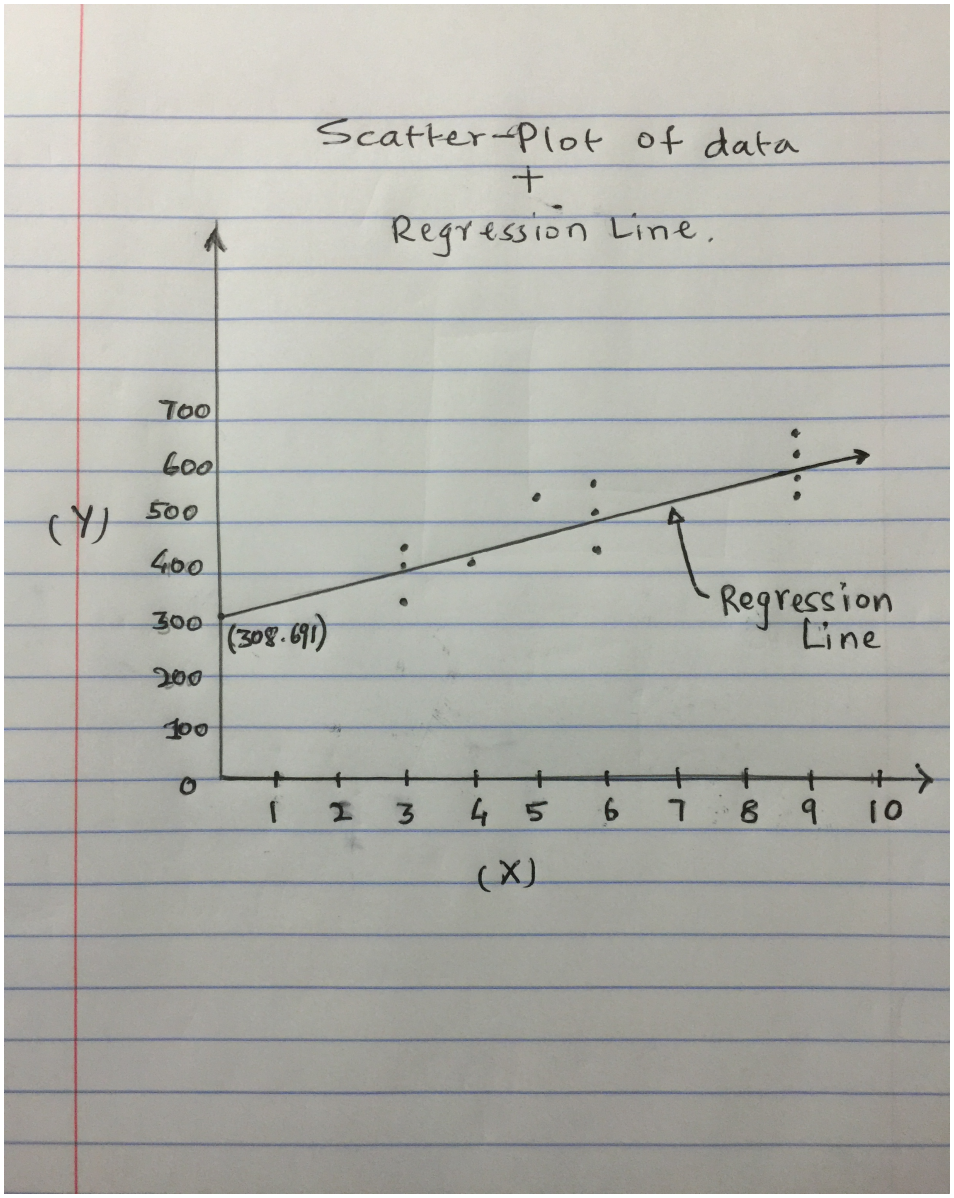
$\beta_1 = 34.4265$

Thus,

$y' = 308.691 + 34.4265x$

(c) Draw resulting regression line on the 2-d scatter plot (you can copy initial plot from (a)). (5 points)

Answer



(d) Compute the fitted values and residuals for each observation and verify that the residuals sum to zero (or approximately zero). (5 points)

Answer

In the following table as you can see the value of the residuals for each observation and sum is almost zero.

	x	y	y'	Residual = y - y'
	6	520	515.25	4.75

	4	421	446.397	-25.397
	6	581	515.25	65.75
	9	630	618.5295	11.4705
	3	412	411.9705	0.0295
	9	562	618.5295	-56.5295
	6	434	515.25	-81.25
	3	443	411.9705	31.0295
	9	590	618.5295	-28.5295
	5	570	480.8235	89.1765
	3	348	411.9705	-63.9705
	9	672	618.5295	53.4705
SUM				**7.38964E-13 ≈ 0**

(e) How much of variation in y is explained by x? (5 points)

Answer:

	x	**y**	**y'**	**y' - mean(y)**	**$(y' - mean(y))^2$**
	6	520	515.25	0	0
	4	421	446.397	-68.853	4740.735609
	6	581	515.25	0	0
	9	630	618.5295	103.2795	10666.65512
	3	412	411.9705	-103.2795	10666.65512
	9	562	618.5295	103.2795	10666.65512
	6	434	515.25	0	0
	3	443	411.9705	-103.2795	10666.65512
	9	590	618.5295	103.2795	10666.65512
	5	570	480.8235	-34.4265	1185.183902
	3	348	411.9705	-103.2795	10666.65512
	9	672	618.5295	103.2795	10666.65512
SUM	**72**	**6183**			**80592.50535**
MEAN	**6**	**515.25**			

Thus **80592.51** is the total variance explained by x.

(f) Compute the standard error of the estimate (5 points)

Answer:

	x	y	y'	y - y'	(y - y')2
	6	520	515.25	4.75	22.5625
	4	421	446.397	-25.397	645.007609
	6	581	515.25	65.75	4323.0625
	9	630	618.5295	11.4705	131.5723703
	3	412	411.9705	0.0295	0.00087025
	9	562	618.5295	-56.5295	3195.58437
	6	434	515.25	-81.25	6601.5625
	3	443	411.9705	31.0295	962.8298703
	9	590	618.5295	-28.5295	813.9323702
	5	570	480.8235	89.1765	7952.448152
	3	348	411.9705	-63.9705	4092.22487
	9	672	618.5295	53.4705	2859.09437
SUM					**31599.88235**

$$S_{est} = \frac{\sqrt{\sum(y-y')^2}}{\sqrt{n-2}}$$

$$S_{est} = \frac{\sqrt{31599.88235}}{\sqrt{12-2}} = \mathbf{56.214}$$

(g) What are the predicted values for x = 2, 4, 6, 7, 10. (5 points)

Answer

The following table shows the corresponding predicted values using the equation

y' = 308.691 + 34.4265x

x	y'
2	377.544
4	446.397
6	515.25

7	549.6765
10	652.956

(h) Test for significance of "r" (linear relationship) at $\alpha = 0.05$. (10 points)

Answer

The correlation coefficient 'r' can be calculated as

$$r = \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{[n(\sum x^2) - (\sum x)^2][n(\sum y^2) - (\sum y)^2]}}$$

The values of these parameters can be obtained from the table in answer to part(b). Thus replacing the values we get

$$r = \frac{12(39439) - (72)(6183)}{\sqrt{[12(500) - 72^2][n(3297983) - 6183^2]}}$$

r = 28092 / 33144.93 = **0.84755**

The obtained 'r' is a sample correlation coefficient and an estimate of the population correlation coefficient ρ. To verify the significance of this estimate we need to test for significance. The hypothesis testing procedure is as follows:

STEP 1: Determine Null hypothesis and the claim.

H_0: ρ = 0 and H_1: ρ ≠ 0

STEP 2: Calculate the critical value

Given α = 0.05 and conducting a two-tail t-test we get
Critical Value = 2.228

STEP 3: Calculate test Value

$$t = (r)\sqrt{\frac{n-2}{1-r^2}}$$

$$t = (0.84755)\sqrt{\frac{12-2}{1-0.84755^2}}$$

t = 0.84755 (5.958)
t = **5.05**

STEP 4: Compare values and make decision

The value 5.05 > 2.228 and hence we reject the null hypothesis.

STEP 5: Summary

Thus, there is enough evidence to support the claim that there is a strong between the variables.

(i) Compute the prediction interval for same α (5 points)

Answer

The prediction interval can be given by

$$y' \pm t_{\alpha/2} S_{est} \sqrt{1 + \frac{1}{n} + \frac{n(x - mean(x))^2}{n\sum x^2 - (\sum x)^2}}$$

The values of these variables can be found in table for part b and previous parts.

$$y' \pm 2.228 * 56.214 \sqrt{1 + \frac{1}{12} + \frac{12(x-6)^2}{12(500) - 72^2}}$$

$$y' \pm 125.24 \sqrt{1.083 + 0.015(x-6)^2}$$

Q2. (20 points)

(a) Sate the assumptions of Simple, Multiple, and Logistic regression (if you can't find answer in slides, you should look at any standard book and/or online; clearly cite the reference). (3x4 = 12 points).

Answer

Assumptions of Simple Linear Regression:

1. **Normality Assumption**: For a given value of x, the value of dependent variable y is normally distributed.
2. **Equal Variance Assumption**: For each value of the independent variable x, the variance for all the corresponding dependent variable y are the same. So the variance for all the y variables remains the same.
3. **Linearity Assumption**: The independent variable x, is linearly related to the dependent variable y.
4. **Independence Assumption**: The values of dependent variable y are independent of each other.
5. **Independence of Errors**: The errors are of the response variables are unrelated to each other.

Assumptions of Multiple Linear Regression:

All the assumptions of the simple linear regression hold good. Apart from these we have following additional assumptions for multiple Linear Regression:

1. **Non-multicollinearity Assumption**: Multiple linear regression has multiple independent variables on which the dependent variable y is related to. So all these independent variables should be unrelated to each other.

Assumptions of Logistic Linear Regression:

Following are the assumptions of Logistic Linear Regression:

1. **Non-multicollinearity Assumption**: In case of multiple independent variables, each independent variable should be unrelated to each other. This does not apply in case there exists only one independent variable.
2. **Independence Assumption:** The values of dependent variable y are independent of each other.
3. **Linearity Assumption with Log odds**: Even though logistic regression does not require the dependent variables to be linearly related to independent variables, it requires the independent variables and log odds to be linearly related.
4. **Independence of Error Term**: The errors are of the response variables are unrelated to each other.
5. **Asymptotically better:** The accuracy increases with increase in the number of samples since the theory is asymptotically based.

Note: All the assumptions are taken from Lecture Slides shared by professor and the references [1][2][3] as stated at the end.

(b) Pickup any one assumption from each group (i.e., simple, multiple, and logistic) and state what happens if that assumption is violated (3x2 = 6 points). (These three assumptions should be different from each other).

Answer

1. **Linearity Assumption**: In simple linear regression if this assumption is violated then we will not have any value of β_0 and β_1 that correctly fits y and it will be impossible to predict the values to y with a linear model. The accuracy will be bad and the errors will keep increasing. This also happens in case of multiple linear regression.
2. **Equal Variance Assumption:** In case of simple and multiple linear regression if the all the predicted values of y do not have the same variance, then for each predicted value variance will be different and it will not be possible to determine the confidence level of the prediction since the normal distribution of that predicted value will have unknown variance. Also since the sample values can also have difference variance, each sample will have different importance in building the model.
3. **Linearity Assumption with log odds:** This says that there should be a linear relationship between the independent variables and log odds. If not the case then the model underestimates the strength of the relationship and thus resulting in not rejecting Null hypothesis by considering the relationship not significant while it actually is significant and thus predicting wrong results.

4. **Errors are Independent:** There needs to be no correlation between errors. If there exists such a correlation among the error terms, then the model underestimates the value of standard error resulting in inaccurate confidence level and prediction level.

Note: All the assumptions are taken from Lecture Slides shared by professor and the references [1][2][3] as stated at the end.

Q3. Bonus Question (1% of grade)

In logistic regression analysis (Slide 8), we mentioned that

(a) Errors can't be normally distributed. With proper analysis (or sound arguments) show why this is the case? (5 points)

Answer

As stated in class discussion forum by professor, This question actually says that errors can't be normally distributed in case we use linear regression instead of logistic regression.

Logistic regression is used when the response variable is categorical and not continuous. Assuming we have a binary response variable then such that y = 1 or 0. This can also be represented as probability

$$P(Yi = 1) = \pi i$$
$$P(Yi = 0) = (1 - \pi i)$$

Thus,

$$E(Yi) = 1 * \pi i + 0 * (1 - \pi i) = \pi i$$

The binary response can thus be in general be represented as

$$\mathbf{E(Yi|Xi) = \beta o + \beta 1 Xi = \pi i}$$

Now if we try to fit the linear model to this, then we have a constraint on the response variable since the probability is always between 0 and 1. But additionally, binary nature of the response also creates difficulties in how the individual values are distributed around the mean. Since there are only two responses these cannot be normally distributed around the mean and thus violating the linear regression assumptions. Thus even the errors generated will not be normally distributed as there are only 2 error values possible.

Reference from [3].

(b) Error variance is not constant. Show why? (10 points)

Answer

As stated above if linear regression is applied on a binary response variable instead of logistic regression then the expected value in general can be represented as follows.

$$E(Yi|Xi) = \beta o + \beta 1 Xi = \pi i$$

The variance of the binary response function can be represented as

$$\mathbf{Var(Yi) = \pi i * (1 - \pi i)}$$

Now since 'πi' is itself a function of Xi the variance cannot be constant. Thus the again violating the assumption of linear regression.

Reference from [3].

<u>Mini R project (30 points; 10 points each for Q4-Q6)</u>

Note: You should provide answers (along with inline R code) in the pdf file. In addition submit R scripts.

Q4. Using "Advertising" data answer (a) and (b)

(a) Fit simple linear regression (separately) for each covariate. Provide scatter plots with fitted regression line. Which covariate provides best prediction?

Answer

R Code

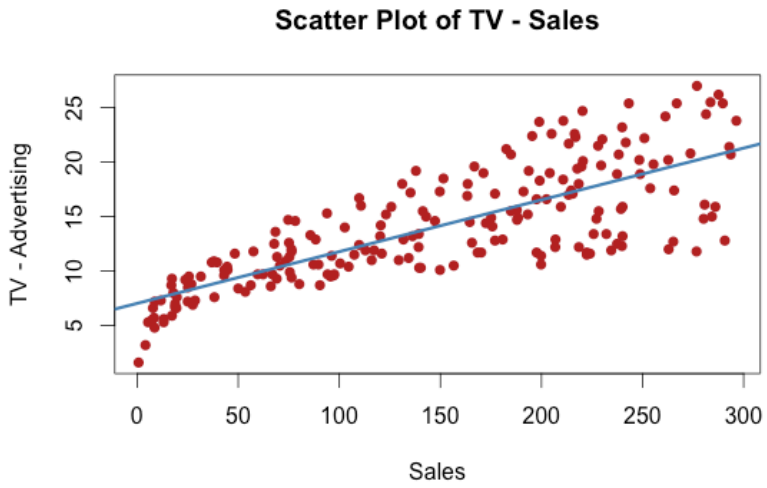
```
#Question 4(a)
# Clean the environment
rm(list = ls())
# Load data
adv_data = read.csv("Advertising.csv")
adv_data = adv_data[,-1]

#Simple linear Regression for TV covariate
plot(adv_data$TV, adv_data$Sales, xlab = "Sales", ylab = "TV - Advertising",
    main = "Scatter Plot of TV - Sales", col = "firebrick",
    type = "p", pch = 16)
model_TV = lm(adv_data$Sales ~ adv_data$TV)
abline(model_TV, col = "steelblue", lw = 3)
print(model_TV$coefficients)
print(summary(model_TV))

#Simple linear Regression for Radio covariate
plot(adv_data$Radio, adv_data$Sales, xlab = "Sales", ylab = "Radio - Advertising",
```

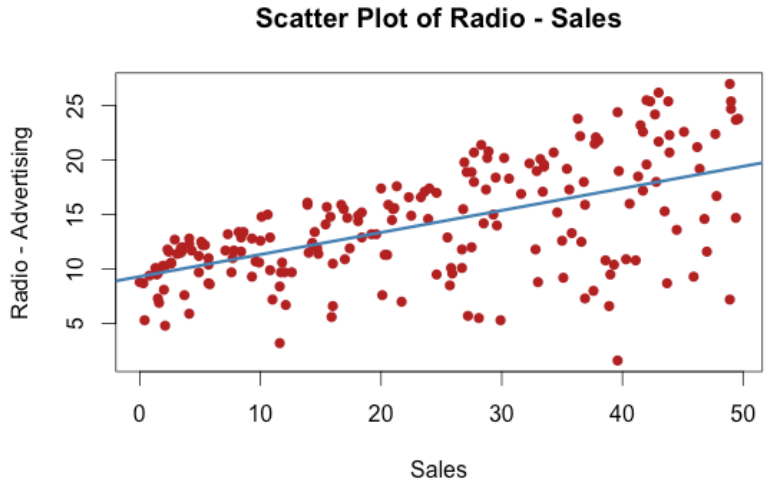
```
    main = "Scatter Plot of Radio - Sales", col = "firebrick",
    type = "p", pch = 16)
model_Radio = lm(adv_data$Sales ~ adv_data$Radio)
abline(model_Radio, col = "steelblue", lw = 3)
print(model_Radio$coefficients)
print(summary(model_Radio))

#Simple linear Regression for Newspaper covariate
plot(adv_data$Newspaper, adv_data$Sales, xlab = "Sales", ylab = "Newspaper -
Advertising",
    main = "Scatter Plot of Radio - Newspaper", col = "firebrick",
    type = "p", pch = 16)
model_NewsPaper = lm(adv_data$Sales ~ adv_data$Newspaper)
abline(model_NewsPaper, col = "steelblue", lw = 3)
print(model_NewsPaper$coefficients)
print(summary(model_NewsPaper))
```

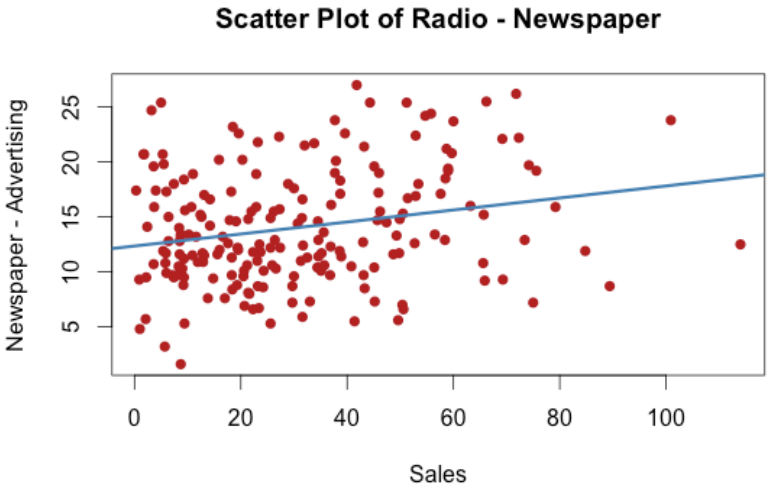
Beta – values for each covariate

TV:	(Intercept)	adv_data$TV
	7.03259355	0.04753664
Radio:	(Intercept)	adv_data$Radio
	9.3116381	0.2024958
Newspaper	(Intercept)	adv_data$Newspaper
	12.3514071	0.0546931

Scatter Plots







The '**TV**' covariate best predicts the model, since it's the most significant feature.

(b) Fit multiple linear regression model for the data. Show resulting equation. How do you compare the β's obtained with this model with corresponding β's found in (a).
Answer

R Code

```
# Clean the environment
rm(list = ls())
# Load data
adv_data = read.csv("Advertising.csv")
adv_data = adv_data[,-1]
#Question 4(b)
# Multiple Linear Regression and the Beta-coefficient
model_multivatiate = lm(adv_data$Sales ~ adv_data$TV +
```

```
                adv_data$Radio + adv_data$Newspaper)
print(model_multivatiate$coefficients)
```

Beta – Coefficients of multiple linear regression

(Intercept)	adv_data$TV	adv_data$Radio	adv_data$Newspaper
2.938889369	0.045764645	0.188530017	-0.001037493

Comparing the beta's from the simple linear regression for each covariate.

The beta's obtained from linear and multiple linear regression for TV and Radio covariates is almost the same. While there is a difference between the beta's for the covariate Newspaper. From linear regression it seems that there is a correlation between the Newspaper Covariate and the dependent variable sales. But from multiple linear regression we can see that beta value for Newspaper is slightly negative, which means that Newspaper advertisement does not affect the Sales. Since the value is only slightly negative we can say that its not adversely affecting the sales.

(c). Fit logistic regression model for the dataset (hw3-q4c.txt). Note that this dataset contain 3 covariates, therefore you should use multiple logistic regression which is straight forward generalization of simple logistic regression (simply replace $\beta_0 + \beta_1 X$ with $\beta_0 + \beta_1 X_1 + \ldots + \beta_n X_n$)

Answer

R Code

```
#Question 4(c)
rm(list = ls())
# Load the new data
log_data = read.table("hw3-q4c.txt", header = TRUE, sep = "\t")
log_model = glm(log_data$Y ~ log_data$X1 + log_data$X2 + log_data$X3,
          family = "binomial")
print(summary(log_model))
```

Model Summary:

```
Deviance Residuals:
    Min       1Q   Median       3Q      Max
-1.64148  -0.00008   0.00000   0.00135   1.41755

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -10.1535    10.8398  -0.937   0.3489
```

```
log_data$X1   0.3312     0.3007   1.101   0.2707
log_data$X2   0.1809     0.1069   1.692   0.0907 .
log_data$X3   5.0875     5.0820   1.001   0.3168
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 91.4954  on 65  degrees of freedom
Residual deviance:  5.8129  on 62  degrees of freedom
AIC: 13.813

Number of Fisher Scoring iterations: 12
```

Q5. Apply your data science skills to **improve** the model fitted in (4.c). In what sense your improved model is **better** than the model found in (4.c). [Note the term "improve"; that is you still have to use multiple logistic model only). Show your work. (Worth **2% of final grade points**)

Answer

R Code

```
rm(list =ls())
# Applying log transform
trans_data = read.table("hw3-q4c.txt", header = TRUE, sep = "\t")
trans_data$X3 = log(trans_data$X3)
trans_model = glm(trans_data$Y ~ trans_data$X1 + trans_data$X2 + trans_data$X3,
            family = "binomial")
print(summary(trans_model))
```

Model Summary

```
Deviance Residuals:
    Min       1Q   Median       3Q      Max
-1.57236  -0.00004   0.00000   0.00093   1.48872

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept)    -5.5860     7.3058  -0.765   0.4445
trans_data$X1   0.3493     0.3636   0.960   0.3368
trans_data$X2   0.1827     0.1084   1.685   0.0919 .
trans_data$X3   8.3599     8.9293   0.936   0.3492
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)
```

Null deviance: 91.4954 on 65 degrees of freedom
Residual deviance: 5.7925 on 62 degrees of freedom
AIC: 13.793

Number of Fisher Scoring iterations: 13

As seen in the above model summary we have some improvements on the in terms of the Residual deviance and AIC as compared to the original model.

The log transformation is applied to the X3 covariate, since applying log transforms the space into a more concise space. We can only apply log transform to X3 since other covariates have negative values and hence log cannot be applied.

References:

1) http://www.statisticssolutions.com/assumptions-of-logistic-regression/
2) https://en.wikipedia.org/wiki/Linear_regression
3) http://www.public.iastate.edu/~stat415/stephenson/stat415_chapter3.pdf