CSC 503 Homework Assignment 4

Due September 17, 2014

September 10, 2014

In using the Fitch macros to typeset proofs in first order logic, one introduces a dummy variable x by means of the command `\open[x]`.

Unless directed otherwise, follow the convention of the text and assume that a, b, c, d, e are constant symbols, f, g, h are function symbols, and u, v, x, y, z are variable symbols.

1. Let c and d be constants, f a function symbol with one argument, g a function symbol with two arguments, h a function symbol with three arguments, and P and Q predicate symbols with three arguments. Indicate, for each of the following strings, which strings are formulas in predicate logic, and specify a reason for failure for strings which are not.

   (a) [5 points] $\forall x P(f(d), h(g(c, x), d, y), x)$
   Answer
   This formula is a **valid** formula in predicate logic

   (b) [5 points] $\forall x P(f(d), h(P(x, y, d), d, y), x)$
   Answer
   This formula is **invalid** as $P(x, y, d)$ is not a term and all the paramters to the function must be a term.

   (c) [5 points] $\forall x(Q(z, z, z) \to P(z))$
   Answer
   This formula is **invalid** formula in predicate logic since the predicate P is incorrectly used with only one argument.

   (d) [5 points] $\forall x \forall y(g(x, y) \to P(x, y, x))$
   Answer
   This formula is **invalid** formula as $g(x, y)$ is a term and is not a formula. Hence $\phi_! \to \phi_2$ is a formula only when ϕ_1 and ϕ_2 are formulas.

   (e) [5 points] $Q(c, d, c)$
   Answer
   This formula is a **valid** formula in predicate logic

   (f) [5 points] $\forall x \forall y P(x, x, x)$
   Answer
   This formula is a **valid** formula in predicate logic

2. Let P be a predicate symbol with arity 2, and let ϕ be the formula

$$\exists x(P(y, z) \land (\forall y(\neg P(y, x) \lor P(y, z))))$$

   (a) [5 points] Indicate, for each occurrence of each variable in ϕ, whether that occurrence is free or bound.
   Answer
   The variables in ϕ include x, y and z. The highlighted variables in ϕ are free.

$$\exists x(P(\mathbf{y}, \mathbf{z}) \land (\forall y(\neg P(y, x) \lor P(y, \mathbf{z}))))$$

(b) [5 points] List all variables which occur free and bound in ϕ.

Answer

The variables in ϕ include x, y and z. The highlighted variables in ϕ are free and the rest of the occurences are bounded with x bounded by $\exists x$ and y on the right hand side by $\forall y$

$$\exists x(P(\mathbf{y}, \mathbf{z}) \wedge (\forall y(\neg P(y, x) \vee P(y, \mathbf{z}))))$$

(c) [10 points] Compute $\phi[t/x]$ for $t = g(f(g(y, y)), y)$. Is t free for x in ϕ?

Answer

t is not free for x in ϕ as there is no free x in ϕ to be replaced by t. Thus $\phi[t/x]$ will remain ϕ.

(d) [10 points] Compute $\phi[t/y]$ for $t = g(f(g(y, y)), y)$ Is t free for y in ϕ?

Answer

t is free for y in ϕ as there a free instance of y which on replacement with t does not bound it. Thus we have,

$$\exists x(P(\mathbf{g(f(g(y,y)),y)}, z) \wedge (\forall y(\neg P(y, x) \vee P(y, z))))$$

(e) [10 points] Compute $\phi[t/z]$ for $t = g(f(g(y, y)), y)$ Is t free for z in ϕ?

Answer

t is not free for z in ϕ as when we replace t for free instances of z in ϕ we add additional bounding condition to the variable y in t.

3. [30 points] Find a proof for $\forall x(P(x) \wedge Q(x)) \vdash \forall x(P(x) \rightarrow Q(x))$.

Answer

1	$\forall x(P(x) \wedge Q(x))$	premise
2	x_0 $P(x_0) \wedge Q(x_0)$	$\forall$e, 1
3	$P(x_0)$	assumption
4	$Q(x_0)$	$\wedge$e$_2$, 2
5	$P(x_0) \rightarrow Q(x_0)$	$\rightarrow$i, 3–4
6	$\forall x(P(x) \rightarrow Q(x))$	$\forall$i, 2–5