[Gilbert Strang]
[LEC 1 MIT 18.06]

LINEAR ALGEBRA

- n linear equation, n unknowns.
  Row picture
- * Column picture
  Matrix form.

$$2x - y = 0$$
$$-x + 2y = 3$$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

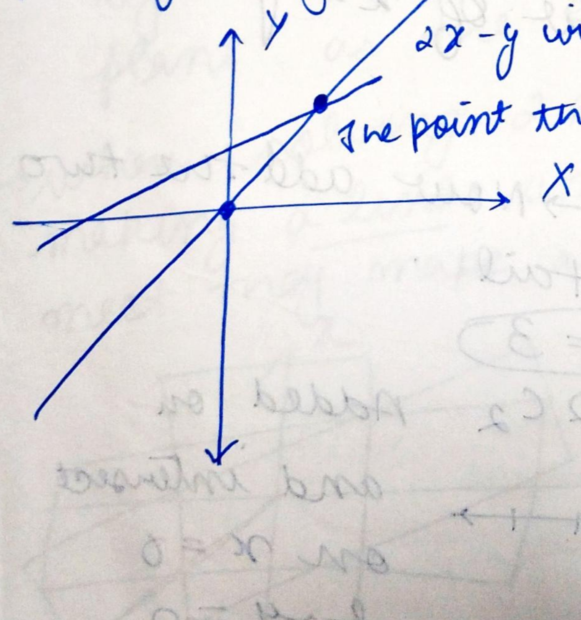
Matrix. vector

$$A \quad X = B$$

Plotting in graph. → $\boxed{\text{This is the Row picture}}$

2x-y will satisfy this line.

The point that lies on both lines and satisfies it.



$$\boxed{\begin{matrix} x = 1 \\ y = 2 \end{matrix}}$$

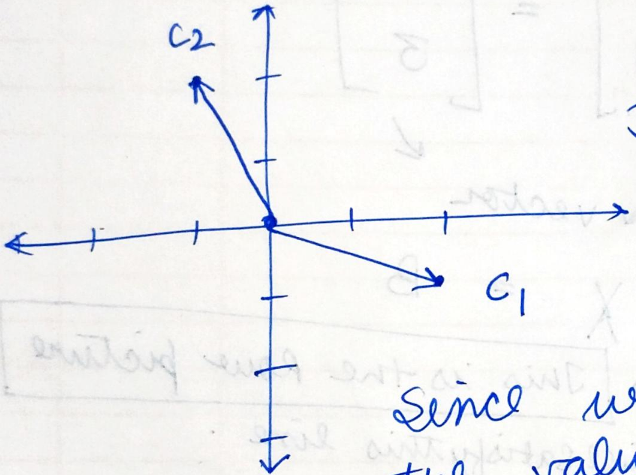
The coulumn picture

$$x\begin{bmatrix}2\\-1\end{bmatrix}+y\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}0\\3\end{bmatrix}$$

- The equation somehow wants us to combine the vector attached to vector 1 and 2 and give us $\begin{bmatrix}0\\3\end{bmatrix}$
- Find out the right linear combination of the coulumns

$\begin{bmatrix}2\\-1\end{bmatrix}$ $\begin{bmatrix}-1\\2\end{bmatrix}$

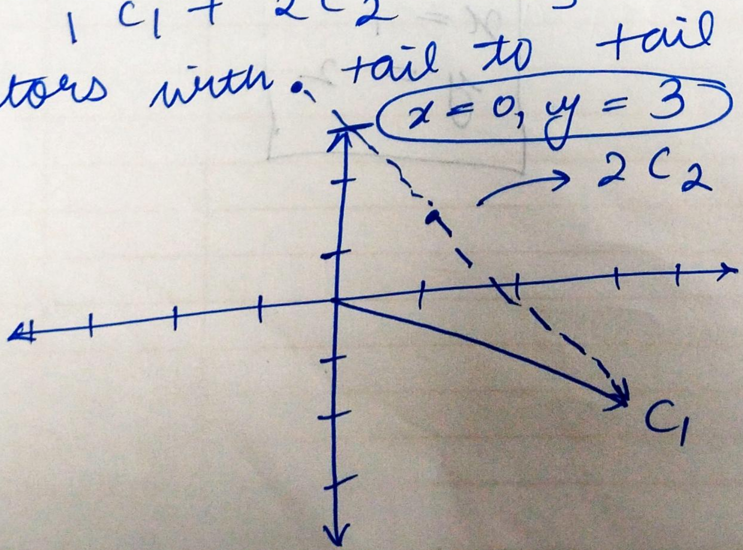
C_1 C_2



The combination that will produce $\begin{bmatrix}0\\3\end{bmatrix}$

Since we know that the value of x & y are 1 & 2.

$1C_1 + 2C_2 = C_3$ → Now add the two vectors with tail to tail.



Added on and intersect on $x=0$ & $y=0$

when we try all the possible combinations, it will fill the whole plane.

ANOTHER EXAMPLE

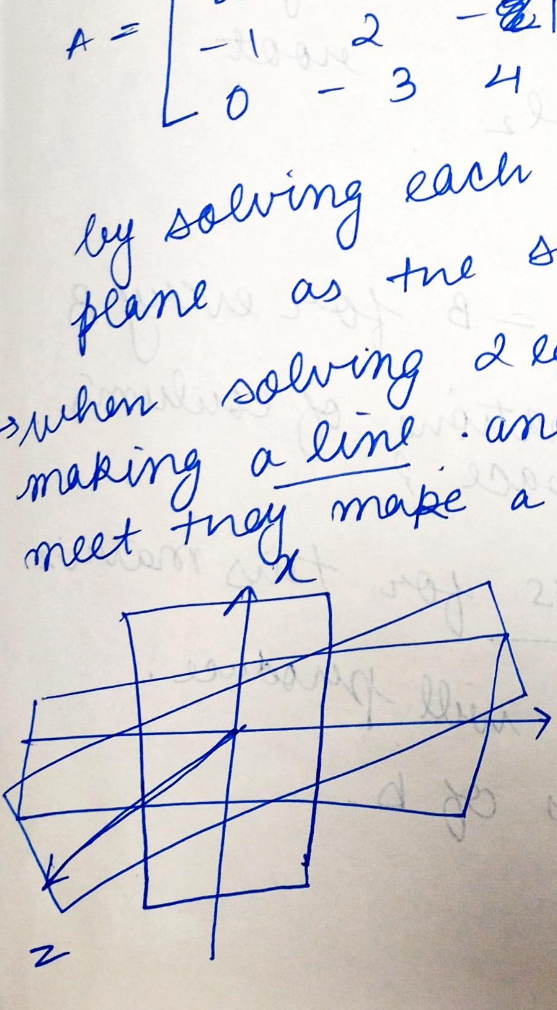
$$2x - y = 0$$
$$-x + 2y - z = -1$$
$$-3y + 4z = 4$$

form a matrix →

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

by solving each row we will get a plane as the solution.

→ when solving 2 eq we get 2 planes making a line. and when 3 of them meet they make a point.



This makes the solution very complex thus we move to coulumn picture.

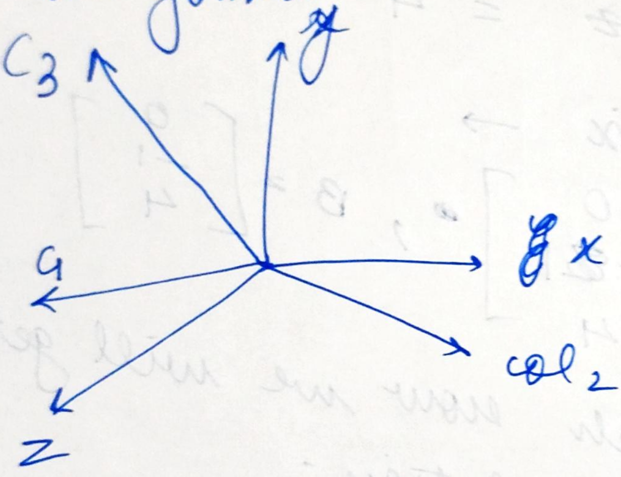
Coulumn picture

$$x\begin{bmatrix}2\\-1\\0\end{bmatrix}+y\begin{bmatrix}-1\\2\\-3\end{bmatrix}+z\begin{bmatrix}0\\-1\\4\end{bmatrix}=\begin{bmatrix}0\\-1\\4\end{bmatrix}$$

C_1 C_2 C_3 C_4 B

we're to solve a linear combination.

we are gonna



Just put
$x = 0, y = 0$
$z = 1$
to get 1
root

- Can we solve all $Ax = B$ for every B
- Do the linear combinations of coulums fill up the 3-D space?

↳ The answer is <u>YES</u> for this matrix

- Non singular matrix will produce all possible iterations of b.

when the three coulumns lie on the same plane, then their comb will lie on the same plane

If C_1 lies on same coulumn of C_2 & C_3 then we wont get any new combinations.

In this case singular matrix
the matrix would not be invertible

No solution for every B.

Non singular and invertible matrices can be solved.

If we create a 9-Dimensional matrix
→ we would get 9 coulumns / vectors
→ We plot them and solve it if its a good matrix.
→ But if 8th coulumn happens to be = 9th coulumm then it would be a 8-D plane in a 9-D space

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$A \; x = B$ (But how to solve these?)

A ↳ Matrix, x ↳ Vector

Q. $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$

→ This is linear comb of coulums.

Invertible Matrix

↳ $AB = BA = I_n$

$B = A^{-1}$

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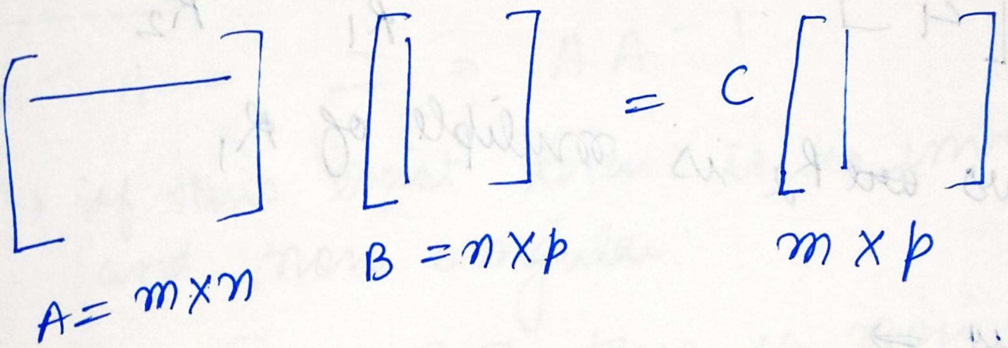
LEC 3 - -

MULTIPLICATION ~~ELIMINATION~~ WITH MATRICES

For multiplication to happen A has to be $\underline{m \times n}$ & B has to be $\underline{n \times b}$

The shape of the output will be $\underline{m \times p}$



- Matrix times the first coulumn will give out the first coulumn of C matrix
- Matrix times 2nd coulun will give out the ~~first~~ 2nd coulumn of C

These coulumns of C are combination of <u>coulunns of A</u>.

The rows of C are combination <u>of rows of B</u>

what happens when we multiply

coulumn of A $\times$ row of B

$$\underset{m \times 1}{\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}} \underset{1 \times p}{\begin{bmatrix} 1 & 6 \end{bmatrix}} = \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix}$$

R_1 R_2

Rows ~~and~~ R_2 is muliple of R_1

4th way $\rightarrow$

AB = sum of (cols of A) $\times$ (rows of B)

$$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} +$$

$$\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix}$$

The row space is a line where [1.6] lies and

coulumn space is also a line

⑤ Using blocks to multiply matrices.

$$\underset{A}{\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]} \underset{B}{\left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right]} = \left[\begin{array}{c|c} A_1B_1 & A_2B_2 \\ \hline & \end{array}\right]$$

INVERSE.

$$A^{-1} A = I = A A^{-1}$$

↳ if this exist then it is invertible and non singular

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$ this is non invertible since both of them.

→ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ & $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ lie on the same line; every combination is going to be that one same line

This matrix will have no inverse because we can always find a vector x with $Ax = 0$

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x \neq 0$ and still produces 0

Gauss - Jordan (solve 2 eq at once

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}}_{A} \left| \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I} \right]$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\downarrow$$

$$\left[\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I} \middle| \underbrace{\begin{matrix} 7 & -3 \\ -2 & 1 \end{matrix}}_{A^{-1}} \right] \leftarrow \left[\begin{matrix} 1 & 3 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} 1 & 0 \\ -2 & 1 \end{matrix} \right]$$

↓ Start with this double length matrix

3B1B
CH-3
ALGEBRA

Linear Transformation

Linear transformation
vector function

→ Happens for a vector ; goes through the function and spits out another vector.

Linear transformation needs to have 2 properties :-

- All lines after getting through the function should still remain lines.
- Origin must remain in place.

Just define where your $\hat{i}$ & $\hat{j}$ will land, it will give out the desired output vector.

$$\hat{i} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \hat{j} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow x\begin{bmatrix} 1 \\ -2 \end{bmatrix} + y\begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Linear transformation is just dependent on 2 points, the place where $\hat{i}$ lands and where $\hat{j}$ lands.

$2 \times 2 \Rightarrow$

$$\begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix} \quad \& \quad \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$\hookrightarrow 5\begin{bmatrix} 3 \\ -2 \end{bmatrix} + 7\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

A generic example

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

a & c is where first basis vector lands.

b & d is where 2nd basis vector lands.

when you apply these ~~vectors~~ linear transformation to some vector $\begin{bmatrix} x \\ y \end{bmatrix}$ the result is $\rightarrow$

$$x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix}$$

$$= \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

this is also called matrix vector multiplication

- A matrix is denoted by $[a_{ij}]_{m \times n}$, where i and j represent the position of element in matrix, row wise and coulumn wise.
- $\underline{m}$ is number of rows and $\underline{n}$ is numeer of coulumn

$$\begin{array}{l} \text{Row 1} \rightarrow \\ \text{row 2} \rightarrow \end{array} \begin{bmatrix} 2 & 3 & 0 \\ -6 & 8 & 2 \end{bmatrix}$$

2×3

rows ↓ ↓ coulumns

$$\begin{bmatrix} 1 & -8 \\ 7 & 2 \\ 8 & 9 \end{bmatrix} \begin{array}{l} \rightarrow \text{row 1} \\ \rightarrow \text{row 2} \\ \rightarrow \text{row 3} \end{array}$$

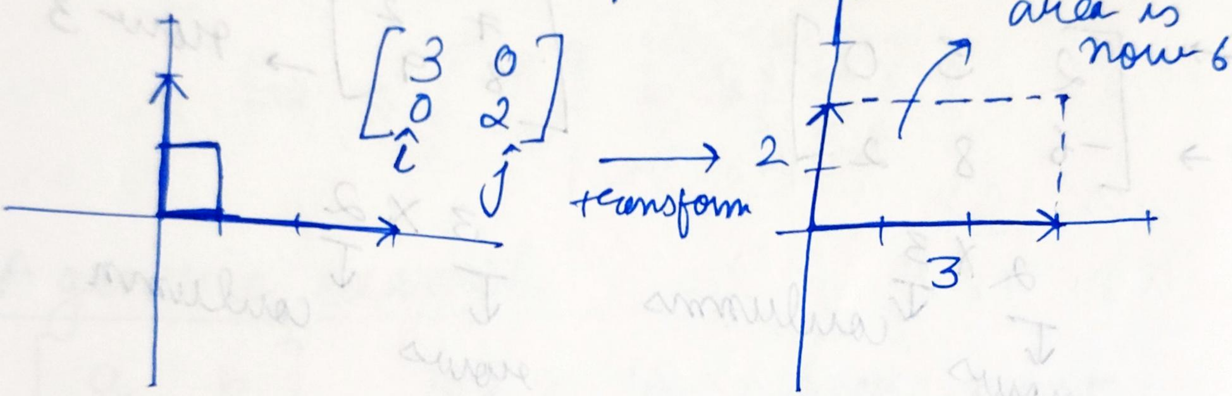
3×2

rows ↓ ↓ coulumn

3B1B
CH-6
LINEAR
ALGEBRA

THE DETERMINANT:

→ when doing linear transformation, the area also scales up and scales down



Linear transformation scaled area by a factor of 6.

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ means that $\hat{i}$ stays in place ~~by~~ but $\hat{j}$ moves by (1,1)

- Once we find in what way $\hat{i}$ and $\hat{j}$ area initially changes, we can find it for any other area.

- Determinant is the ~~sc~~ scaling factor of the initial area.

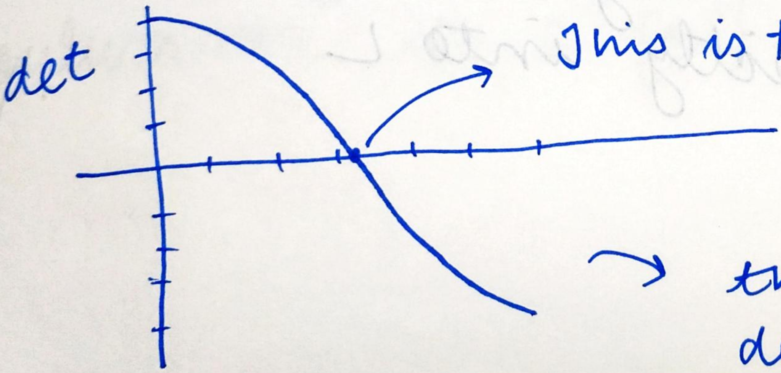
- Determinant of a 2-D space is zero, if it squishes down all the space into a line, and area of any space will become 0.

whenever the orientation of space changes, or you can say inverted, the determinant is negative.

$$\det\left(\begin{bmatrix} 2 & 1 \\ -1 & -3 \end{bmatrix}\right) = -5.0$$

This tells inversion

This is area scaling



This is the point where $\hat{i} = \hat{j}$

this is -ve determinant

In 3-D matrix the determinant scales by volume.