Guilbert Strang 4 LINEAR ALGEBRA

# Factorisation in $A = LU$

$A = L U$

our matrix

This is the inverse or the product of the inverses

This is the val

$A = LU$

If no rows exchanges.
Multipliers go directly into L

or just

$A \bar{x} = \vec{v}$

This is the starting pt.

this is the final vector.

$\vec{x} = A^{-1} \vec{v}$

→ when output of a transformation is a line meaning 1-D it means that transformation has rank 1

→ If all the vectors lie on 2-D plane after transformation that means rank 2

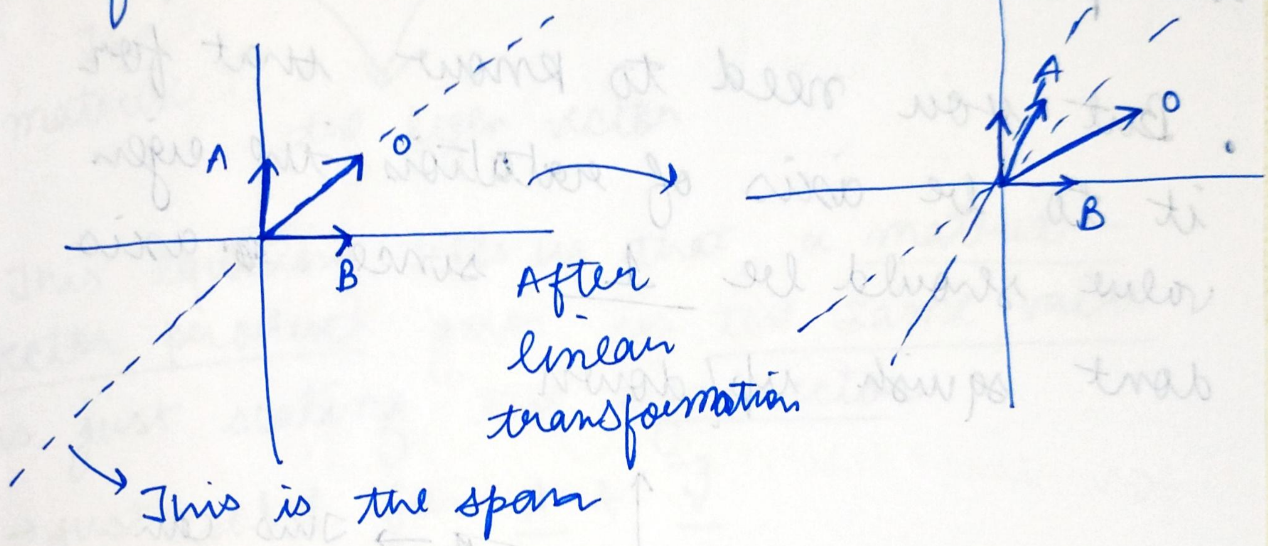
★ A set of all possible outputs of $\underline{A\vec{v}}$ is called coulumn space.

→ Rank is highest means all no. of coulumns its a full rank matrix.

3B1B
Linear
Algebra

# Eigen Vector & Eigen values

- Most vectors get knocked off the span during the transformation.

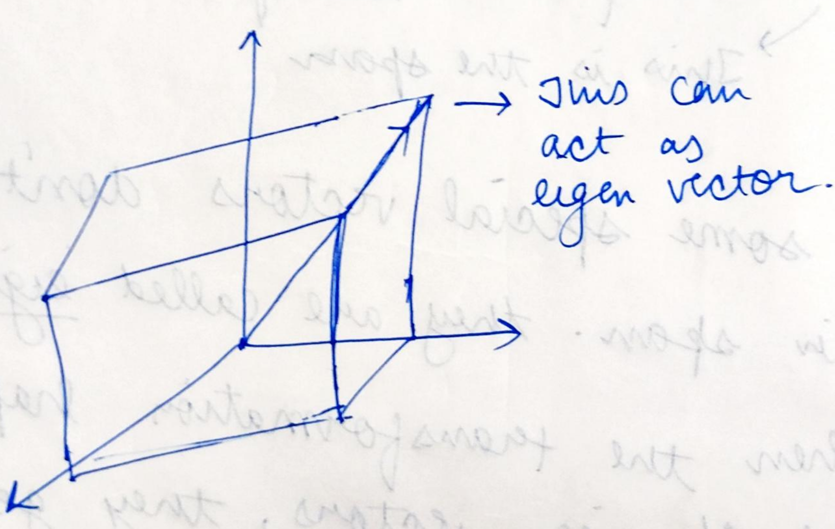



- But some special vectors don't shift from their span. they are called <u>eigen vectors</u>.

- when the transformation happens, in case of eigen vectors, they get stretched by certain <u>factor</u>, the factor is <u>eigen value</u>.

- It is useful because in a 3D/2D space, when you can an eigen vector for that span, you basically find the <u>axis of rotation</u> for the plane.

- But you need to know that for it to be axis of rotation the eigen value should be <u>1</u> since axis dont squish up/down




- If you want to be not dependent of coordinate system then we find the <u>eigen vector</u> and <u>eigen value</u>.

## IN A TRANSFORMATION

we have → the eigen values.

$$A \vec{v} = \lambda \vec{v}$$

The matrix

the eigen vector

- This equation tells us that a matrix-vector product gives us the same value as just scaling the eigen vector

→ just solve for $\lambda$ & $\vec{v}$

### SOLVING eigen vectors.

1. Scale $\lambda$ with an identity matrix

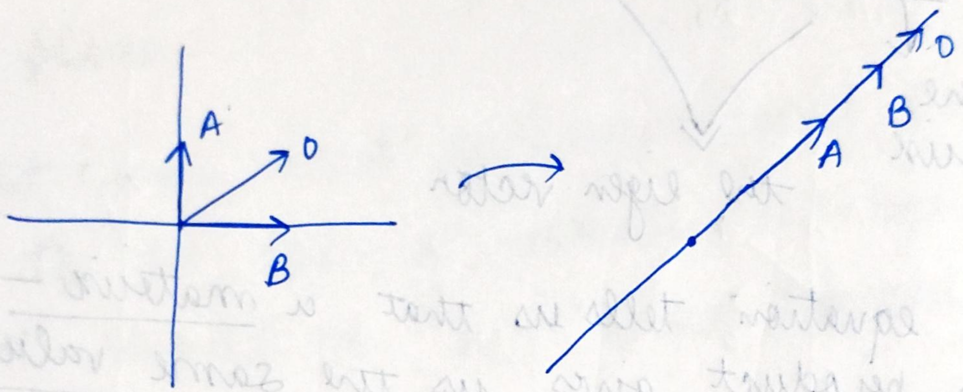
$$\boxed{A \vec{v} = (\lambda I) \vec{v}}$$

2. $\rightarrow A \vec{v} - (\lambda I) \vec{v} = 0$

$\rightarrow (A - \lambda I) \vec{v} = 0$

$\rightarrow \left( A - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right) \vec{v} = 0$

* The only way for product of matrix with non-zero vector to be zero if the transformation turns into lower dimension:




for this → determinant of $(A - \lambda I)$ needs to be 0

eg →
$$\begin{bmatrix} 3-\lambda & 5 & 1 \\ 7 & 2-\lambda & 3 \\ 9 & 1 & 1-\lambda \end{bmatrix} = 0$$

eg →
$$\det\left(\begin{bmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}\right)$$
$$= (3-\lambda)(2-\lambda) = 0$$

→ Note when eigen vectors value is imagina
- ry this means the plane is rotated,
eg → 90° rotation.

## Scaling eigen vectors n times

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \cdots \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

.... 100 times

↳ $\begin{bmatrix} 3^{100} & 0 \\ 0 & 2^{100} \end{bmatrix}$ it is just the basis vectors raised to $n$.

→ For function to be linear two conditions are to be followed.

Additivity : $L(\vec{v} + \vec{w}) = L(\vec{v}) + L(\vec{w})$

Scaling : $L(c\vec{v}) = cL(\vec{v})$

## working with polynomials

→ each polynomial has finitely many terms, but the full space is of arbitiary large degree.

→ we use the powers of $x$ as basis.

$$1x^2 + 3x + 5 \Rightarrow \underline{\text{Basis}} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ \vdots \end{bmatrix} \begin{matrix} \rightarrow b_0(x) \\ \rightarrow b_1(x) \\ \\ \\ \\ \end{matrix}$$

these basis are just like $\hat{i}, \hat{j}, \hat{k}$

- the derivative of a polynomial can be put in a matrix mostly full of zeros but some in diagnol.

$$\frac{d}{dx}(1x^3 + 5x^2 + 4x + 5) = 3x^2 + 10x + 4$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 5 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 \\ 2 \cdot 5 \\ 3 \cdot 5 \\ \vdots \end{bmatrix}$$

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TRANSPOSES
PERMUTATION
R^N

Permutations (P) are identity matrices with reordered rows.

How many possibilities?

↓

$n! = n(n-1) \ldots (3)(2)(1)$

that counts all the $n \times n$ permutations.

- They are invertible
- $P^{-1} = P^T$ or $P^T P = I$

Transpose →

$(A^T)_{ij} = A_{ji}$

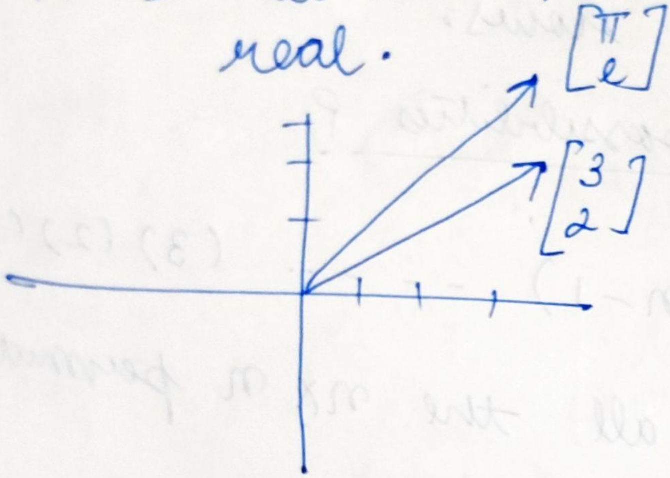
→ Symmetric matrices are unaffected by transposes.

$\boxed{A^T = A}$

• Span of 2 lines/vectors is the whole 2D shape.

## Vector Space

eg → $R^2$ = all 2-D vectors that are real.




This whole plane is $R^2$ or X-Y plane.

## Supspaces of $R^2$ (Real x-y Plane)

① all of $R^2$

② Any line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

③ zero vector only Z

### example

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$$

coulumns in $R^3$

all their combination form a subspace called the column space. $C(A)$

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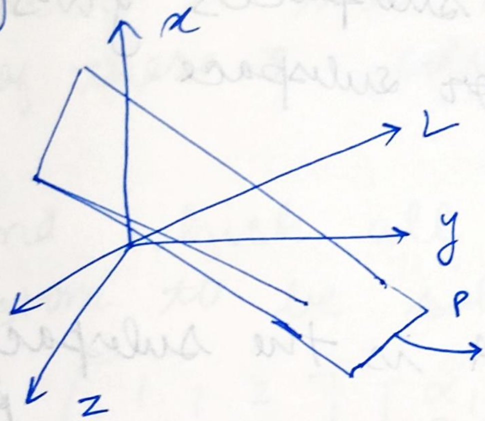
# COLUMN SPACE AND NULLSPACE

An $R^3$ vector space is $xyz$ region.




→ Subspace is a vector space inside a vector space.

This plane is a subspace (P)

→ The plane through $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a subspace

→ A line is also a subspace. (L)

Now the 2 subspaces : P and L.
where plane is P and L is a line.

• ~~Vector space requires v + w~~

we try to take $\cup$ of $P \cup L$ ~~and~~,
we won't be able to add ~~them~~
the elements of P or L outside $P \cup L$

→ But in case of P ∩ L
the value of both P ∩ L lie in the space, ie we can add them, so P ∩ L is a subspace.

→ zero vector by itself is a subspace.

→ Intersection of two subspaces gives you a smaller vector subspace

→

example

column space of A is the subspace of $R^4$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

$[\underbrace{4}_{\text{coulumn space}} \times 3]$

• we take the ~~subspace~~ linear combination of the of the column space to find out the vector space.

• Hence all the linear combination of our coulumns is the vector space

→ Just the coulumn of these matrices does not fill the whof x-D space.

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

there are going to be a lot of vector b that are not combination of these 3 - columns.

→ Find which all b's allow the system to be solved.

eg
$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

I can solve $\underline{Ax = b}$ exactly when $\underline{B}$ is in the <u>column space</u>.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

These two are pivot columns.

- Since these two contribute to the column space while the 3rd one ~~add~~ adds nothing new.
- 3rd one is in the same plane as $C_1$ & $C_2$ so nothing new added.

Thus this matrix is a two dimensional subspace of $R^4$

NULL SPACE

- It contains all the x-s that solve b
- $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is $n$ or $3^n$ big

in an $m \times n$ matrix $A$

Null $A$ is set of all solution to $Ax = 0$

If $Ax = 0$ & $Aw = 0$

then $A(x + w) = 0$

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# PIVOT VARIABLES, SPECIAL SOLUTION

solving $Ax = 0$ with pivoting.

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 1 & 8 & 10 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

→ we're going to perform elimination so we get the pivot position

→ while doing elimination we dont chang the nullspace or just say that the value of solution remains the same.

## Performing elimination

→ $\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ → This is echelon form or staircase.

we just found out 2 pivots of the matrix.

- Rank of a Matrix A is equal to no. of pivots

$$U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{cccc} \uparrow & 2 & \uparrow & 4 \end{array}$$

2 & 4 are free columns.

2 pivot coulumns.

- Pivot is the first non-zero number in any coulumn

so we need to solve the equations.

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$$

$$2x_3 + 4x_4 = 0$$

- we substitue $x_4 = 0$ then $x_3 = 0$ then $x_2 = 1$ then $x_1 = -2$.

hence the nullspace is

$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ → vector in the nullspace ← $\begin{bmatrix} -4 \\ 2 \\ 0 \\ 0 \end{bmatrix}$

same way ⇒ $\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ → another vector in the nullspace.

→ Now we have 2 special solution and take all the combination of that solution

$$x = c\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} \begin{matrix}+\\ \times\\ \div\end{matrix} d\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$$

- The nullspace contains exactly all the combinations of special solution

- How many special sol are there?

→ one for every free variable.

$$\boxed{\begin{aligned} r &= 2 \ (\text{pivot variables}) \\ m &= m \text{ columns.} \end{aligned}}$$

free - var ⇒ $m - r = 4 - 2 = 2$

→ That gives you complete algorithm to do all $Ax = 0$

REDUCED ROW ECHELON FORM

→ ~~the~~ clear the ~~rows~~ zeros above and below the pivot. = 1

change all the pivot numbers into 1