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Determinant Formulas & Cofactors

→ We can split the determinant of matrices into smaller cofactors to find the determinnat.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$

$$= \cancel{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}} + \underbrace{\begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix}}_{\text{These contribute to determinant}} + \cancel{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}}$$

$$\boxed{ad - bc}$$

→ We notice a pattern when finding out the determinant of 2 X 2, 3 X 3 matrices.

2 X 2 → 2 terms

3 X 3 → 6 terms.

4 X 4 → 24 terms. [4! to be precise]

Big formula to find determinant of N X N matrix

$$\det A = \underset{n!\ \text{terms}}{\sum} \pm a_{1\alpha} a_{2\beta} a_{3\gamma} \cdots a_{n\omega}$$

$(\alpha, \beta, \gamma \ldots \omega) = $ Perm $(1, 2, 3 \ldots$

→ The formula tells to take 1 element from each row and then a permutation from every column and then find the correct combination.

→ Cofactors of 3X3

$$\det = a_{11}(a_{22}a_{33} - a_{23}a_{32})$$
$$+ a_{12}(a_{21}a_{33} - a_{23}a_{31})$$
$$a_{13}(\quad)$$

Cofactor of $a_{ij} = c_{ij}$

$\pm \det \left(\begin{matrix} n-1 \text{ matrix} \\ \text{with row } i \\ \text{erased col } j \end{matrix} \right)$

$i+j$ even (+)

$i+j$ odd (−)

Cofactor ⇒ A number by eliminating the row and column of particular element which is in form of square or rectangle + or − sign based on elements position.

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Cramers Rule Inverse matrix and volume.

→ There is a need to find inverse of a matrix and we can use →

$$A^{-1} = \frac{1}{\det A} C^T \rightarrow \text{product of } n-1 \text{ entries}$$

↘ Product of N entries

- C^T is the transpose of cofactor matrix
- We also call it Adjoint of A

→ We can just verify →»

$$AC^T = (\det A) I$$

$$Ax = b$$

$$x = A^{-1} b$$

$$\Rightarrow \boxed{\frac{1}{\det A} \times C^T \times b}$$

Gramers Rule

→ Gives out Solution (unique) solution to a system of equation

$$X_1 = \frac{\det B_1}{\det X}$$

$$X_2 = \frac{\det B_1}{\det X}$$

A with column 1 replaced by b

$$\Rightarrow B_1 = \begin{bmatrix} b \cdots & \begin{matrix} n-1 \\ \text{colums} \\ \text{of } A \end{matrix} \end{bmatrix}$$

B_j = A with column j replaced by B

→ Grammers rule is complicated and not much used, we use normal transformation

Determinant Tells you the Area (change)

Determinant of a matrix tells us how much change in area we see

$A = Q$ (orthogonal matrix diag entries are 1)

Q

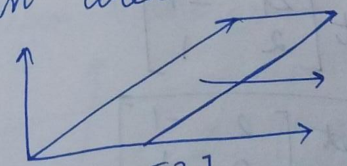
The determinant of an identity matrix will preserve itself (shape).

Scaling factor tells us how all other areas will change

→ Orthonormal are unit vector that are ⊥ and remain the same even after transforming often called rotation matrices.

→ The area of the ||gm gives us the change in area.

$A\begin{bmatrix} x \\ y \end{bmatrix}$



$\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

This is the signed area that can be -ve or +ve.

Signed Area = (det A) y

then $y = \frac{\text{signed area}}{\det(A)}$

→ we create a new matrix where our first column is same as our matrix but 2nd column is output vector.

$$\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$y = \frac{\text{Area}}{\det(A)} = \frac{\det \begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}}{\det \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}}$$

→ output vector

same 1st column

we can do the same for x

$$x = \frac{\text{Area}}{\det(A)} = \frac{\det \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}}{\det \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}}$$

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Eigen Values and Eigen Vectors

→ Eigenvalues - Eigen Vectors.

$$\det [A - \lambda I] = 0$$

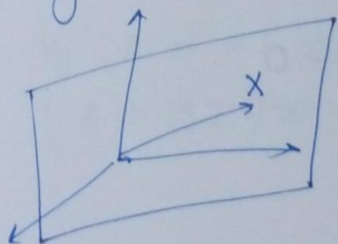
$$\text{TRACE} = \lambda_1 + \lambda_2 + \cdots \lambda_n$$

So we need to find Ax so why not find a vector parallell to x.

That will give us Eigenvector

$$\boxed{Ax = \lambda x}$$

If A is a singular Matrix $\lambda = 0$ is eigen Value.



What are x's & λ's for projection Matrix.

Any x in the plane

$$Px = x, \ \lambda = 1$$

Any X in plane $\rightarrow Px = X, \lambda = 1$

$\rightarrow$ Any X ~~in~~ $\perp$ to plane $Px = 0.X, \lambda = 0$

Example

$A \rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $Ax = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\lambda_1 = 1$

$x = \begin{bmatrix} -1 \\ +1 \end{bmatrix}$ $Ax = \begin{bmatrix} +1 \\ -1 \end{bmatrix}$ $\lambda_2 = -1$

. Trace is the sum of diagnol that also gives you the sum of eigen vectors.

Sum of λ's $= a_{11} + a_{22} \cdots + a_{nn}$

$\rightarrow$ How to solve $Ax = \lambda x$.

Rewrite $(A - \lambda I)x = 0$

$\det(A - \lambda I) = 0$

$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \rightarrow$ we can see $3+3 = 6$

~~we can just calculate the eigen v are 3 & 3 is degenerate.~~

$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1$

$\Rightarrow \lambda^2 - 6\lambda + 8$

This is the trace i.e. $(3+3)$

$\boxed{\lambda_1 = 4}$ & $\boxed{\lambda_2 = 2}$

$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\rightarrow$ The ~~to~~ eigen vector will always stay the same even if you perform addition / subtraction.

if $Ax = \lambda x$

$(A + 3I)x = \lambda x + 3x = (\lambda + 3)x$

stays the same.

ex. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ trace $= 0+0$
$= \lambda_1 + \lambda_2$

det $= 1 = \lambda_1 \lambda_2$

$\det \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = \lambda^2 + 1$ | $\lambda_1 = i$, $\lambda_2 = -i$

→ If our matrix is not symmetric then the eigenvalues will be imaginary.

→ eigen val/vectors for....

$\begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)(3-\lambda)$

$\lambda_1 = 3, \lambda_2 = 3$

$(A - \lambda I)[x] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$

$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x_2 =$ No 2nd eigen vector

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Diagnolisation and Powers of A.

→ Suppose n independant eigenvectors of A put them in columns of S

$AS = A[x_1, x_2, \dots x_n] = [\lambda_1 x_1 \dots\dots \lambda_2 x_2]$

$= [x_1 \dots\dots x_n] \begin{bmatrix} \lambda_1 & & 0 \\ 0 & \ddots & \vdots \\ 0 & & \lambda_n \end{bmatrix} = S\Lambda$

→ Λ is the diagnol eigenvalue Matrix

→ Now we will try to inverse S

$$AS = S\Lambda$$

$$S^{-1}AS = \Lambda$$

$$A = S\Lambda S^{-1}$$

if $Ax = \lambda x$

$A^2 x = \underbrace{\lambda Ax}_{\text{replace by } \lambda x} = \lambda^2 x$

$A^2 = S\Lambda S^{-1} S \Lambda S^{-1} = S \Lambda^2 S^{-1}$

Thus

$A^K = S \Lambda^K S^{-1}$

→ we can't get a diagnol eigen matrix if ~~A is~~ the eigenvectors are not independant.

• Theorem

when $A^K \to 0$ as $K \to \infty$

only if all $|\lambda_i| < 1$

all independent diagnol eigen values need to be less than 1.

• A is sure to have n independent eigenvectors (and be diagnolisable) if all the ~~matrix~~ λ are different.

• The diagnol values are the eigen vectors in a Δ matrix.

Repeated Eigenvalues.

Equation $u_{k+1} = A u_k$ $\boxed{k \text{ tells how many repetitio}}$

start with given vector u_0

$u_1 = A u_0$, $u_2 = A^2 u_0$

To solve - - :

$u_0 = c_1 x_1 + c_2 x_2 \text{ - - - } c_n x_n$

$A^{100} u_0 = c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + c_n \lambda_n x_n$

$= \Lambda^{100} S c$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow |A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$

$= \lambda^2 - \lambda - 1$

$\lambda_1 = \frac{1}{2}(1 + \sqrt{5})$

$\lambda_2 = \frac{1}{2}(1 - \sqrt{5})$

Using Fibonnaci Example

→ $0, 1, 1, 2, 3, 5, 8, 13, \quad F_{100} = ?$

$F_{k+2} = F_{k+1} + F_k$

$F_{k+1} = F_{k+1}$

$u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}, \quad u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k$

$F_{100} = c_1 \left(\frac{1+\sqrt{5}}{2} \right)^{100}$

$\boxed{\begin{matrix}\text{Linear Algebra} \\ 23 \\ \text{Guilbert Strang}\end{matrix}}$

Differential Equations and Exponents of A^T

Differential Equations $\frac{du}{dt} = Au$

$\frac{du_1}{dt} = -u_1 + 2u_2$

$\frac{du_2}{dt} = u_1 - 2u_2$

$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$

Lets assume a starting point

$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

① Find eigen value of the matrix A

$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \quad \lambda = 0, -3$

Singular matrix has 1 eigen value 0

$x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Soution.

$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$

Check $\Rightarrow \frac{du}{dt} = Au$ Plug in $e^{\lambda_1 t} x_1$

solving and getting:

$\lambda_1 \cancel{e^{\lambda_1 t}} x_1 = A \cancel{e^{\lambda_1 t}} x_1$

$\boxed{Ax_1 = \lambda_1 x_1}$ thus proved.

$\cong c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 : u_{k+1} = Au_k$

$\hookrightarrow c_1 \cdot 1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

- we know everything except c_1 & c_2

→ We can use $u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to find out

At $t = 0$, $c_1 \begin{bmatrix} +2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$c_1 = 1/3$ & $c_2 = 1/3$

So the steady state is when

$u(\infty) = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ will give out steady state.

How do we get stability

① Stability $u(t) \to 0$ / need $e^{\lambda t} \to 0$ / $\lambda < 0$

but what if λ is a complex number?

eg $\left| e^{(-3+6i)t} \right| = e^{-3t}$

since $|e^{6it}| = 1$

② Steady state

$\lambda_1 = 0$ & other Real $\lambda < 0$

③ It will blow up if any Real $\lambda > 0$

2x2 Stability with Real $\lambda_1 < 0$, Real $\lambda_2 < 0$.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; trace $a + d = \lambda_1 + \lambda_2 < 0$

thus both are negative

Trace may still blow up if $\lambda_1 + \lambda_2 < 0$

$\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow$ here $-2+1 < 0$

Exponential

$\frac{du}{dt} = Au$, $\frac{dv}{dt} = S^{-1} A S_v = \Lambda_v$

set $u = Sv$
↳ eigen vector matrix.

$v(t) = e^{\Lambda t} v(0)$

$u(t) = S e^{\Lambda t} S^{-1} u(0)$

$e^{At} = S e^{\Lambda t} S^{-1}$

$= e^{At} (u(0))$

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Markov Matrices
Fourier Series

→ Markov Matrices
Properties of the matrix (Markov)

① All entries ≥ 0
② All columns add to 1.

$\begin{bmatrix} 0.1 & 0.01 & 0.3 \\ 0.2 & 0.99 & 0.3 \\ 0.7 & 0 & 0.4 \end{bmatrix}$

for every Markov matrix
1. $\lambda = 1$ is an eigenvalue
2. All other $|\lambda_i|$ (eigenvalue) < 1

$u_k = A^k u_0 = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 + \ldots$

→ so we need to find the steady state from u_k... we already know that $\lambda_1 = 1$ & $(\lambda_2) < 1$

for steady state c_2 in the end will turn to 0

- λ_1 part of u_0 will give you the steady state

→

If we know that one eigen value is 1 so we can find the vectors....

$$A - 1I = \begin{bmatrix} -0.9 & .01 & .3 \\ .2 & -.01 & .3 \\ .7 & 0 & -.6 \end{bmatrix}$$

All columns add to zero → of $A - I$

$A - I$ is singular

- Eigenvalues of <u>A & A^T</u> are the <u>same</u>

$\boxed{\text{Linear Algebra 25 Guillent Strang}}$

Symmetric Matrices & Positive Definiteness

Start → Positive Definite Matrices.

$A = A^T$

(1) The eigen values are ~~positive~~ <u>REAL</u>

(2) The eigen vector are <u>PERPENDICULAR</u>

usual case $A = S \Lambda S^{-1}$

Symmetric case $A = Q \Lambda Q^{-1} = Q \Lambda Q^T$

$Q^{-1} = Q^T$ for orthonormal eigen vectors

$\boxed{Q \Lambda Q^T}$ → Also called the <u>spectral</u> or <u>axis</u>

<u>why real eigenvalues ?</u>

$A\bar{x} = \bar{\lambda}\bar{x} \Rightarrow \bar{x}^T A = \bar{x}^T \bar{\lambda}$

$\lambda \bar{x}^T x = \bar{\lambda} \bar{x}^T x$

$\lambda = \bar{\lambda}$ → This proves lamda is real.

Good Matrices

$$\boxed{A = A^T}$$

Real λ's
Perpendicular x's

$A = A^T$

$A = Q \Lambda Q^T$

$$\begin{bmatrix} q_1 & q_2 & \cdots \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \end{bmatrix} =$$

$$\lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T$$

Every symmetric matrix is a combination of perpendicular projection matrices.

→ Pivots and eigen values have a thing in common ie ___. They are products of the diagnol. So if a matrix with 50x50 then we can calculate the pivot

→ Signs of Pivots is same as signs of λ's

\# positive pivots = # positive λ's

Positive definite symmetric Matrix

all eigen values are positive

all pivots are positive

all subdeterminants are +ve.

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Complex Matrix
Fast Fourier Transformation

Fourier Matrix → Matrix we use to perform fourier transformation

Brings ~~out~~ down calculation matter from something like n^2 down to $n \log n$.

$$Z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \text{ length}$$

z_n is complex number in C^n

$z^T z$ is no good

→ we need to conjugate the transpose to get meaning out of vector in C^n

$$\bar{z}_1 z_1 = |z_1|^2$$

→ we do $\bar{z}^T$ $\quad [\bar{z}_1, \bar{z}_2, \dots \bar{z}_n] \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$

→ We use z^H more that says $(\bar{z})^T$

→ Also called Hermite and this manipulation is Hermitian matrices.

New inner product for matrices is not $y^T x$ its $y^H x$

- Symmetric means ~~$\bar{A}$~~ $A^T = A$ but no good when matrix is complex.
- we can use ~~symmetric~~ Hermitian matrices

$$\bar{A}^T = A = \begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix}$$

Hermitian matrix

→ $Q^H Q = I$ is unitary orthogonal matrix

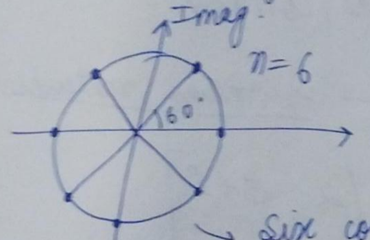
Fourier Matrices

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{n-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(n-1)} \\ \vdots & \vdots & \vdots & \cdots & \\ 1 & w^{n-1} & w^{2(n-1)} & \cdots & w^{?} \end{bmatrix} \quad i, j = 0 \ldots$$

$$(F_n)_{ij} = w^{ij}$$

$$w^n = 1, \quad w_n = e^{i 2\pi/n} = \cos\frac{2\pi}{n} + i\sin\ldots$$

→ 2π because of the full circle



$n = 6$

$w = e^{i 2\pi/n}$ is somewhere in the circle

→ Six complete parts.

$w, w^2, w^3 \cdots w^6$

$n = 4, \quad w^4 = 1$

$w = e^{2\pi i/4} = i$

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

These all columns are orthogonal

while orthonormal means ⇒ $F_4^H F_4 = I$

we can find relation b/w F_{64} & F_{32}

$$(W_{64})^2 = W_{32}$$

we can even further break down the matrices and form and bring it down to W_2

→ Fast fourier Transformation multiplied by $n \times n$ matrix but not in n^2 step but rather $\boxed{\frac{1}{2} n \log_2 n}$

$n = 1024 = 2^{10}$

$n^2 > 1000000$ → 1024×1024

→ This is n

$\frac{1}{n} \log_2 n = (1024)\frac{10}{2} = 5 \times 1024$

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Positive Definite Matrices & Minima

→ Positive definite matrices (Test)

→ Test for minimum ($X^T A x > 0$)

(I) Test for +ve definite Matrices. $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$

① $\lambda_1 > 0$, $\lambda_2 > 0$ eigen value is +ve

② $a > 0$, $ac - b^2 > 0$ | det > 0

③ Pivot $a > a$, $\frac{ac - b^2}{a} > 0$

④ $X^T A x > 0$

Example.

$\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$ Pivots → 2

positive semidefinite, $\lambda = 0, 20$

$$\underset{X^T}{[x_1 \ x_2]} \underset{A}{\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}} \underset{X}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

$X^T A X \Rightarrow 2x_1^2 + 12x_1x_2 + 18x_2^2$

$ax^2 + 2bxy + cy^2 > 0$?

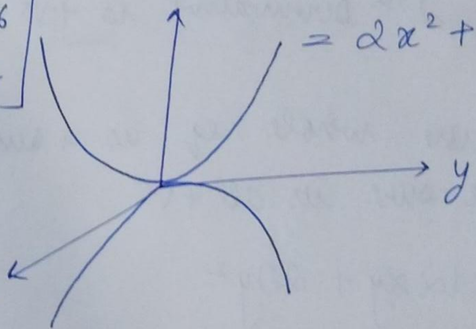
→ The equation for +ve definite would've totally failed if we had any number like 7 in place of 18 → $\begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$

Graphs for $f(x,y)$ $= \vec{X}^T A \vec{X}$

$= ax^2 + 2bxy + cy^2$

$= 2x^2 + 12xy + 7y^2$

$\begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$

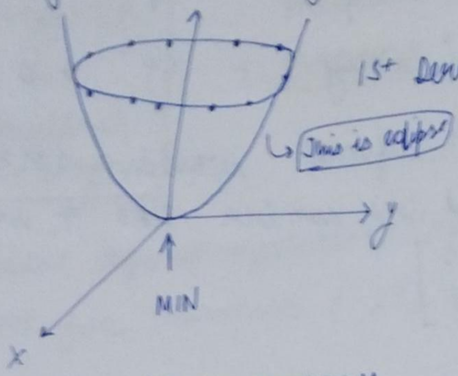


If $\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$ det = 4, Trace = 22

→ $\boxed{2x_1^2 + 12x_1x_2 + 20x_2^2}$

This will not fail & be +ve definite since we have 20 ie > 18.

$f(x,y) = 2x^2 + 12xy + 20y^2$



1st Derivative = 0

The 2nd derivative needs to +ve

It will be MINIMUM.

→ MIN ~ MATRIX OF

$f(x_1, x_2 \dots x_n)$; 2nd Derivative is +ve definite

• we can make the whole eq as a sum of square to make sure its all +ve

$f(x,y) = 2x^2 + 12xy + 20y^2$

$\Rightarrow \underset{\text{Pivot}}{\underbrace{2}}(x + \underset{\text{These are the multipliers}}{\underbrace{3}}y)^2 + \underset{\text{Pivot}}{\underbrace{2}}y^2 = 1$

Matrix of 2nd Der ↳ $\begin{bmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{bmatrix}$

3x3 example

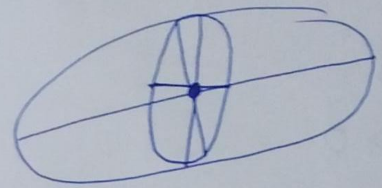
$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

determinant ⇒ 2, 3, 4

Pivot ⇒ $2, \frac{3}{2}, \frac{4}{3}$

eigenvalues ⇒ $2-\sqrt{2}, 2, 2+\sqrt{2}$

$x^T A x = 2x_1^2 + 2x_2 + 2x_3^2 - 2 > 0$



Three diff eigenvalues tells us a story of 3 sides

Thus $x^T A x > 0$ (except $x = 0$)

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Similar Matrices A Jordan Form

Similar Matrices A, B

$B = M^{-1} A M$

+ve definite means

- $x^T A x > 0$

(except for $x = 0$)

- If A, B are positive definite

$\boxed{x^T (A + B) x > 0}$ so it $A + B$

Now A m by n.

- $x^T A^T A x$

square, symmetric, pos definite

$x^T A^T A x = (Ax)^T (Ax) = \|Ax\|^2 \geq 0$

always +ve.

columns are independent so....

A m by n rank n.

Similarity.

$n \times n$ matrices.

A & B are similar means for some M.

$B = M^{-1} A M$

Example.

A is similar to Λ

$S^{-1} A S = \Lambda$

→ Similar matrices have same λ's !!

$Ax = \lambda x$ ($B = M^{-1} A M$)

$M^{-1} A M M^{-1} x = \lambda M^{-1} x$

$(M^{-1} A M) M^{-1} x = \lambda M^{-1} x$

$B M^{-1} x = \lambda M^{-1} x$

$[B = M^{-1} A M]$

Eigen vector of B is M^{-1} which is eigen vector A

Bad case

when $\lambda_1 = \lambda_2 = 4$

family has $M^{-1}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} M = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

Both are same even after $M^{-1}AM$

Big family has. $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$

This form is called Jordan Form

Upper triangular form is Jordan if.

(1) Diagnol entries are equal to its eigenvalues

(2) Super diagnol is 0 or 1

(3) All other entries are 0

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

$\lambda = 0, 0, 0, 0$

2 eigen vector

dim $N(A) = 2$

Null space dimension

got 2 similar matrices

$$\left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Jordan block / with 1 eigen Vect

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ 0 & & \lambda_i & \ddots \end{bmatrix}$$

So jordan block has 1 eigen vector that gets repeated

Every square A is similar to a Jordan Matrix J

$$J = \begin{bmatrix} \boxed{J_1} & & & \\ & \boxed{J_2} & & \\ & & \ddots & \\ & & & \boxed{J_n} \end{bmatrix}$$

If diagnol eigenval sit on the diagnol thus

blocks = eigenvectors

Linear Algebra
2019
Gilbert Strang

Singular Value Decomposition

→ We can factorise every $m \times n$ matrix into $A = \underbrace{U}_{\text{Rotate}} \underbrace{\Sigma}_{\text{Stretch}} \underbrace{V^T}_{\text{Rotate}}$

Stretch ↓ Tells us about strength.

$A = [u_1 \, u_2 \cdots u_n]$

↓ This is orthogonal.

$\begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \sigma_3 \end{bmatrix}$ → Diagnol weights

$V_T = \begin{bmatrix} V^T_1 \\ V^T_2 \\ \vdots \end{bmatrix}$

$A = U \Sigma V^T$ what are U & Σ an V

$A^T A = (V \Sigma^T U^T) U \Sigma V^T = V \underbrace{(\Sigma^T \Sigma)}_{\lambda \text{ for } A^TA \text{ \& } \sigma^2 \text{ for } A} V^T$

$A A^T = U \Sigma V^T V \Sigma^T U^T$

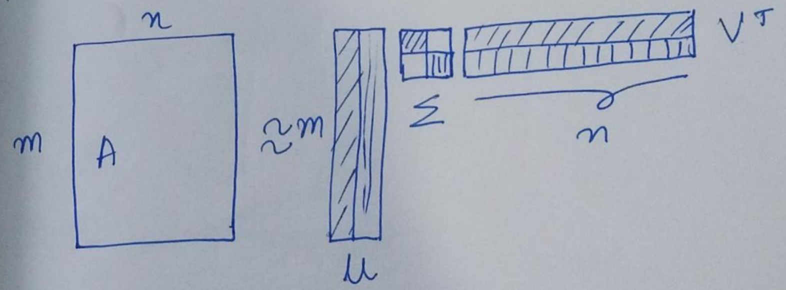
$A_{[m \times n]} = U_{[m \times r]} \Sigma_{[r \times r]} (V_{[n \times r]})^T$

A : Input Data Matrix
m → ~~docum~~ documents
n → n terms

<u>U: Left singular vectors</u>
m × r matrix (m docs r conceps)

<u>Σ : Singular values</u>
r × r → strength of each concept
diagnol matrix
r → rank of A

<u>V: Right singular vectors</u>



U & V are column orthonormal $U^TU = I$, $V^TV = I$

$\Sigma \rightarrow$ Diagnol

Entries are +ve (Singular Values)

$\boxed{\text{Linear Algebra 31 Gilbert Strang}}$

Change of Basis.

<u>Image compression</u>

ith Pixel
$0 \le X < 255$
8 bits

512 × 512

$X \in R^n$

$n = (512)^2$

JPEG

we need to compress the image

The most common basis is that of $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

but we need better bais

Standard Basis

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \dots \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

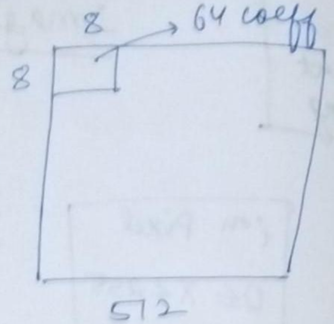
Better Basis

$\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ +1 \\ -1 \end{bmatrix}$

The better basis we might use is <u>fourier basis</u>.

we chose an 8 X 8 block for this,
first column/vector is all 1's.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ w_2 \\ w_3 \\ \vdots \\ w^{n-1} \end{bmatrix} \longrightarrow$$



each block is 8 X 8 with 64 coefficients
to create lossless basis.

Signal

[lossless] $\downarrow$ change basis

compression

$\downarrow$

left with fewer coeff $\leftarrow$ $\hat{c}$ (many zeros)

we reconstruct

$\hat{x} = \sum \hat{c}\, v_1$

$\downarrow$

lesser and more compressed.

Change of Basis.

columns of W = new Basis Vector

$x = Wc$

$[\,x\,]$ old Basis $\longrightarrow$ $[\,c\,]$ new Basis

T wrt $v_1, \dots v_8$ it has matrix A.

wrt to $w_1, \dots w_8$ it has Matrix B

Similar $B = M^{-1} A M$

Eigenvector Basis [Best Basis]

$\hookrightarrow T(v_i) = \lambda_i v_i$

what is A

$$A = \begin{bmatrix} \lambda_1 & & & \\ 0 & \lambda_2 & & \\ 0 & & \lambda_3 & \\ 0 & & & \lambda_4 \\ 0 & & & & \ddots \\ 0 \end{bmatrix}$$

first input is v_1
its output is $\lambda_1 v_1$
2^{nd} input is v_2 output $\lambda_2 v_2$