Landscape analysis for Quadratic feasibility problems

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***Abstract*—Instructions are given for the preparation and submission of papers for the *2020 International Symposium on Information Theory*.**

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I. Introduction

Finding solutions to systems of quadratic equations is an important problem with a wide range of applications. This arises in a diverse set of applications such as power system state estimation [1], phase retrieval [2]–[5], x-ray crystallography [6], the turnpike problem [7], and unlabeled distance geometry problems [8], [9] among others. At their heart, these problems maybe cast as ones of finding a *feasible* vector $\mathbf{x}$ that conforms to a set of quadratic observations with respect to matrices A_i of the form $\{\langle A_i\mathbf{x}, \mathbf{x}\rangle\}_{i=1}^m$ – hence this is also referred to as the *quadratic feasibility problem*.

The quadratic feasibility problem is an instance of quadratically constrained quadratic programs (QCQPs) [10], which has enjoyed a long and rich research history dating back to 1941 [11]. Given their broad applicability to critical problems, research in QCQPs continues to be of active interest [10], [12]–[14]. Unfortunately, it is known that solving QCQPs is an NP-hard problem [15]. This combined with the lack of tractable duality properties [16] has made it hard to establish a sound theoretical framework for understanding the solutions and computing them. However, an extremely productive line of research has instead considered subclasses of QCQPs that are both practically relevant and can in fact be analyzed

But what if we try to separate out conditions that make the quadratic feasibility problems NP-hard and analyse the rest? Even before that, is it possible to identify a subset of tractable problems that have interesting properties. With that aim, in this paper we analyse quadratic feasibility problems, a subset of QCQPs, and prove salient features which makes the feasibility setup quite nice to tackle. With the results in this paper, we work towards getting better understanding of general QCQPs.

We start by looking at the properties of quadratic functions $\{\langle A_i\mathbf{x}, \mathbf{x}\rangle\}_{i=1}^m$, where $\mathbf{x} \in \mathbb{C}^n$ and $A_i \in \mathbb{C}^{n\times n}$ are hermitian. We focus on their ability to generate injective maps which can only be unique upto a phase factor, since the quadratic function $\{\langle A_i\mathbf{x}, \mathbf{x}\rangle\}_{i=1}^m$, by itself, are invariant to phase shifts. We then show that, in real world scenarios, it is not difficult for a set of quadratic measurements $\{\langle A_i\mathbf{x}, \mathbf{x}\rangle\}_{i=1}^m$ to possess isometry by establishing RIP-like conditions when the measuring matrices $\{A_i\}_{i=1}^m$ are gaussian random matrices.

A natural way to deal with feasibility problems is to cast them as any ℓ_p loss functions and run optimization algorithms to search for their minimizerss. Though for the problems in consideration, its ℓ_2 loss function turns out to be nonconvex which renders gradient based algorithms useless. In recent years, there has been a lot of progress in solving nonconvex objective functions, like phase retrieval and matrix factorization [17], using gradient only methods either through good initialization within the basin of convergence of global minima, or establishing good properties about the function itself which might prove beneficial to many gradient algorithms like all local minima are globally optimum. For our problem we delve into the properties of the function landscape and uncover that the function has surprisingly salient features which, restricted to RIP-like conditions being satisfied, can make all gradient based algorithms with random initialization to recover the globally optimal solution with probability 1.

II. Related works

Quadratically constrained quadratic programs(QCQPs) has an enormous number of applications and hence has been an active research area for experimentalists and theoreticians alike for about a century. Early research on algorithmic advances on QCQPs halted cause of limitation of its duality properties [18]. Despite this, due to the wide applicability of QCQPs, a lot of experimental methods have been proposed [19]–[21]. Recently, the authors in [22] gives an asymptotic convergence statement based on the duality properties of a subclass of generic QCQPs. With the results in this paper we supplement the current insights by analysing a subset of QCQPs, the quadratic feasibility problems.

The quadratic feasibility problem (P1) covers a lot of applications like phase retrieval problem [2] and power system state estimation [1]. Phase retrieval itself has applications in numerous fields such as imaging, optics, quantum tomography, audio signal processing with a wide literature base including [2], [4], [5], [23]. Authors in [3] first established approximate ℓ_1 isometry for the phase retrieval problem, but these bounds are loose and cannot be used to establish RIP-like conditions. In this paper, we improve on these bounds to establish isometry results for a larger class of problems and establish RIP-like bounds for when $\{A_i\}_{i=1}^m$ are complex gaussian random matrices.

One of the more straightforward ways of dealing with feasibility problem is to cast it as a loss function and find its minimizer. Even with nonconvex objectives, using gradient based methods have been proved to work for phase retrieval [2], [5], [23], matrix factorization [24], [25] and robust linear regression [17]. The work in [26] establish landscape properties for the phase retrieval problem. Authors in [27] show that many of the nonconvex loss functions have salient landscape properties, which makes gradient based algorithm recover globally optimal solution without any additional information. For the quadratic feasibility problem, we cannot borrow the analysis cause of the stark difference in the behaviour between the problem in the real and complex domain. For e.g., the quadratic feasibility problem considered when considered in the real domain consists of two isolated local minima, but in the case of complex domain has a continuum of minima. Hence we delve deeper into the quadratic feasibility problem in the complex domain to discover that salient features does exists for our problem as well.

The quadratic feasibility problem was first considered in [28] which analysed the problem from a existential perspective and gave lower bounds on the minimum number of independent measurements required for successful recovery. Recently, [29] showed that, with high probability, the quadratic feasibility problem (P1) can be tackled by gradient descent provided a good initialization is used. We go on a parallel track compared to [29] and provide analysis that analyse the ℓ_2 loss function landscape. To this end, we analyse the ℓ_2 loss function corresponding to the quadratic feasibility problem and, restricted on the RIP conditions being satisfied, prove that all local minima are global minima and saddle points are strict saddle points. Thus our reuslts enables a large spectrum of gradient based algorithms with random initilization to recover the solution for the quadratic feasibility problem

We introduce our problem setup in section III. In section V-A we define injectivity property and establish its connection with retrievability. Section V-B analyses and establishes isometric properties of the mapping $\{A_i\mathbf{x}, \mathbf{x}\}$. Section V-C shows that the isometry property of the mapping $\{A_i\mathbf{x}, \mathbf{x}\}$ holds for complex gaussian random matrices. Section VI focuses on casting the problem into a loss framework suitable to solve it algorithmically and establishes properties of this loss function.

III. Problem structure

In this paper we analyze the quadratic feasibility problem which may be expressed as follows,

$$\begin{aligned} &\text{find} \quad \mathbf{x} \\ &\text{s.t.} \quad \langle A_i\mathbf{x}, \mathbf{x}\rangle = c_i, \quad \forall j = 1, 2, \dots m. \end{aligned} \tag{P1}$$

where $\mathbf{x} \in \mathbb{C}^n$, $A_i \in \mathbb{C}^{n\times n}$ are $n \times n$ hermitian matrices, and $c_i \in \mathbb{C}\ \forall i \in [m]$ are scalar constants.

A. Notation

Let $\mathbb{C}^n$ correspond to the n-dimensional complex vector space and $\mathbb{R}^{2n}$ denote the $2n$-dimensional real vector space. In this work, unless mentioned, bold letters $\mathbf{x}$ indicate vectors in $\mathbb{C}^n$, $\mathbf{x}_\mathbb{R}$ denotes the real part of the vector $\mathbf{x}$ and $\mathbf{x}_\mathbb{C}$ denotes the imaginary part of the vector $\mathbf{x}$. Big letters X denote matrices in $\mathbb{C}^{n\times n}$ where n indicates the dimension. Operators $\langle\cdot,\cdot\rangle$ and $[[\cdot,\cdot]]$ denotes the inner product and symmetric outer product respectively in the complex space. The use of i in the equation indicates the complex square root of -1. Let $\mathcal{S}^{a,b}(\mathbb{R}^{n\times n})$ denote the set of all elements $X \in \mathbb{R}^{n\times n}$ such that X has a non-negative eigenvalues and b non-positive eigenvalues, $a+b = n$. Let $\mathbf{H}_n(\mathbb{C})$ denote the $n\times n$ hermitian matrices over $\mathbb{C}$-domain. Let A^H denote the hermitian of a matrix A and A^T denote the transpose A. For any integer r, $[r]$ represents the set $\{1, 2, \dots, r\}$

IV. Main Results

Define the equivalence relation $\sim$ on $\mathbb{C}^n$ such that $\mathbf{x} \sim \mathbf{y}$ implies $\mathbf{x} = c\mathbf{y}$ such that $c \in \mathbb{C}, |c| = 1$. Define the quotient space $\hat{\mathbb{C}}^n := \mathbb{C}^n/\sim$.

Given a set of matrices $\mathcal{A} = \{A_i\}_{i=1}^m \subset \mathbf{H}_n^m(\mathbb{C})$, define the mapping $\mathcal{M}_\mathcal{A} : \hat{\mathbb{C}}^n \to \mathbb{C}^m$,

$$\mathcal{M}_\mathcal{A}(\mathbf{x}) = (\langle A_1\mathbf{x}, \mathbf{x}\rangle, \langle A_2\mathbf{x}, \mathbf{x}\rangle, \dots, \langle A_m\mathbf{x}, \mathbf{x}\rangle) \tag{1}$$

In Theorem 1, we establish an isometry result for the mapping $\mathcal{M}_\mathcal{A}$ and work out specific RIP-like bounds for the case when $\{A_i\}_{i=1}^m$ are complex gaussian matrices. Formally written,

Theorem 1. *Let $\mathcal{A} = \{A_i\}_{i=1}^m$ be the set of complex Gaussian random matrices. Let the number of measurements $m > Cn$ for a sufficiently large constant $C > 0$ Then with probability $1 - d_0 exp^{-c_0 m}$ the following biLipschitz condition holds,*

$$b_0 d(\mathbf{x}, \mathbf{y}) \geq \|\mathcal{M}_\mathcal{A}(\mathbf{x}) - \mathcal{M}_\mathcal{A}(\mathbf{y})\|_2 \geq a_0 d(\mathbf{x}, \mathbf{y})$$

where a_0, b_0 are given by,

$$a_0 := \frac{\big((1-2\delta)^2 - \gamma\big)}{(1+2\delta)^2}, b_0 := \frac{\big((1+2\delta)^2 + \gamma\big)}{(1-2\delta)^2}$$

For the quadratic feasibility problem (P1), the ℓ_2 loss function can be expressed as:

$$\min_\mathbf{x} f(\mathbf{x}) = \sum_{i=1}^m |\langle A_i\mathbf{x}, \mathbf{x}\rangle - c_i|^2$$

We prove the following result for the ℓ_2 loss function,

Theorem 2. *Let $\mathcal{A} = \{A_i\}$ be a set of complex gaussian random matrices. Then with high probability, if a point $\mathbf{x} \in \mathbb{C}^n$ satisfies $\|\nabla f(\mathbf{x})\| \leq \epsilon$ for some small $\epsilon > 0$, then either of the following holds*

- $$d(\mathbf{x}, \mathbf{x}^*) \leq \delta$$

  for some small $\delta > 0$

- $\exists \mathbf{z} \in \mathbb{C}^n$ *such that*

$$\mathbf{z}^H \nabla^2 f(\mathbf{x}) \mathbf{z} \leq -c$$

for some $c > 0$

Theorem 2 implies that in case the stopping criteria of gradient algorithms is reached ($\|\nabla f(\mathbf{x})\| \leq \epsilon$ for some $\epsilon > 0$), then the argument $\mathbf{x}$ is either near a solution of the quadratic feasibility problem (P1) or it is near a strict saddle point.

Corollary 1. *Consider $\|\boldsymbol{x}_0\| < C$ for some constant $C > 0$. Then gradient decent with x_0 as starting point converges to the global minima of the loss function of the quadratic feasibility* (7)

The above corollary can be easily seen as a consequence of Theorem 2 and [30], [31].

V. Properties of quadratic mapping

We start by analysing the properties of the mapping $\{\langle A_i \mathbf{x}, \mathbf{x} \rangle\}_{i=1}^m$ and work towards proving Theorem 1.

A. Injectivity

Definition 1. *Let $\mathcal{A} = \{A_i\}_{i=1}^m \subset \mathbf{H}_n^m(\mathbb{C})$. We say that the set of matrices $\mathcal{A}$ has phase retrieval property if the mapping $\mathcal{M}_\mathcal{A}$ is injective.*

The connection between phase retrievability and injectivity for the mapping $\mathcal{M}_\mathcal{A}$ was first considered in [28].

It is difficult to build up an intuitive understanding of the behavior of the mapping $\mathcal{M}_\mathcal{A}$ in the complex domain. We thus map the whole system from $\mathbb{C}^n$ to $\mathbb{R}^{2n}$ to look at the function properties in the real domain. We build up this and the mathematics surrounding this in Appendix **??**

We prove the following theorem to provide insights into the properties of the mapping $\mathcal{M}_\mathcal{A}$.

Theorem 3. *The following statements are equivalent:*

1) *The nonlinear map $\mathcal{M}_\mathcal{A} : \hat{\mathbb{C}}^n \to \mathbb{C}^m$ is injective*
2) *There exist constants $a_0, b_0 > 0$ such that $\forall \mathbf{u}, \mathbf{v} \in \mathbb{C}^n$, we have that,*

$$b_0 \|[[\mathbf{u}, \mathbf{v}]]\|_1^2 \geq \sum_{i=1}^m |\langle A_k, [[\mathbf{u}, \mathbf{v}]] \rangle|^2 \geq a_0 \|[[\mathbf{u}, \mathbf{v}]]\|_1^2 \tag{2}$$

We refer the reader to Appendix VII-A for a detailed proof. The statement in Theorem 1 shows the connection between the mapping $\mathcal{M}_\mathcal{A}$ being injective and the mapping $\mathcal{N}_\mathcal{A}$ satisfying isometry condition (2), where $\mathcal{N}_\mathcal{A}$ is defined as,

$$\mathcal{N}_\mathcal{A}(\mathbf{u}, \mathbf{v}) = (\langle A_1 \mathbf{u}, \mathbf{v} \rangle, \langle A_2 \mathbf{u}, \mathbf{v} \rangle, \ldots, \langle A_m \mathbf{u}, \mathbf{v} \rangle) \tag{3}$$

Next we focus on establishing a relation between the mapping $\mathcal{N}_\mathcal{A}$ and $\mathcal{M}_\mathcal{A}$.

B. Stability

The authors in [28] proved that the set of matrices $\mathcal{A} = \{A_i\}_{i=1}^m$ having phase retrieval property is an open set. This indicates that the quadratic feasibility system should work well with small perturbations in the system parameters. We would like to venture further in this direction by obtaining some form of quantification as to how robust the mapping $\mathcal{M}_\mathcal{A}$ is in terms of retaining its injectivity. This form of quantification can also serve as a measure to compare two different maps $\mathcal{M}_{\mathcal{A}'}$ and $\mathcal{M}_{\mathcal{A}''}$.

We would like to denote this form of quantization of robustness of mapping $\mathcal{M}_\mathcal{A}$ by the term "stability". Such a notion of stability is hardly new and sufficient researchers have analyzed this in different problem setups [4], [32]. Intuitively, stability indicates the separation power of the mapping $\mathcal{M}_\mathcal{A}$. Mathematically, $\mathbf{x}, \mathbf{y} \in \hat{\mathbb{C}}^n$ such that dist$(\mathbf{x}, \mathbf{y}) \geq \epsilon$ for some $\epsilon > 0$ should imply $\|\mathcal{M}_\mathcal{A}(\mathbf{x}) - \mathcal{M}_\mathcal{A}(\mathbf{y})\| \geq \delta$ should hold, for some $\delta > 0$.

For our quadratic feasibility problem, we define a λ-stable mapping $\mathcal{M}_\mathcal{A}$ if it satisfies the following statement for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$:

Definition 2 (λ-stability)**.** *The set of matrices $\mathcal{A} = \{A_i\}_{i=1}^m$ forms a λ-stable mapping $\mathcal{M}_\mathcal{A}$ if it satisfies the following inequality for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$,*

$$\begin{aligned} &\|\mathcal{M}_\mathcal{A}(\mathbf{x}) - \mathcal{M}_\mathcal{A}(\mathbf{y})\|_2 \\ &\geq \lambda \min_{\theta \in [0, 2\pi)} \|\mathbf{x} - e^{i\theta}\mathbf{y}\|_2 \max_{\theta \in [0, 2\pi)} \|\mathbf{x} + e^{i\theta}\mathbf{y}\|_2 \end{aligned} \tag{4}$$

The stability constant λ can be considered to be a measure of the robustness of the system. Work towards quantifying the stability constant, in the context of phase retreival, for different sample distributions was considered in [3], [4], [32], [33]. The stability property of mapping $\mathcal{M}_\mathcal{A}$ forms a central pillar to many high probability bounds and convergence performance of various algorithms, such as [32], [34].

In this paper, we consider an even stronger notion than the stability. We define biLipschitz property for a function f as follows:

Definition 3. *A function $f : (\mathcal{X}, d_x) \to (\mathcal{Y}, d_y)$ is (α, β)-biLipschitz if it satisfies the following for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$*

$$\alpha d_x(\mathbf{x}_1, \mathbf{x}_2) \leq d_y(f(\mathbf{x}_1), f(\mathbf{x}_2)) \leq \beta d_x(\mathbf{x}_1, \mathbf{x}_2) \tag{5}$$

where $0 < \alpha \leq \beta$.

The notion of distance metric in the above definition is quite crucial and has an impact on determining the constants α, β as well. This is a stronger notion than stability, since (α, β)-biLipschitz function always implies α-stability.

For the problem (P1), we want the distance metric in $\hat{\mathbb{C}}^n$ space. This requires the metric to be invariant to phase shifts.

In this paper, we take the distance metric to be,

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x}\mathbf{x}^H - \mathbf{y}\mathbf{y}^H\|_1 \tag{6}$$

Notice that this distance metric satisfies our requirements of being invariant under phase ambiguity. Such a notion of distance metric is not new and has been used in [3], [34], [35] to prove crucial robustness results for the phase retrieval setup in $\mathbb{R}^n$.

Using this distance metric, we prove the following biLipschitz property of mapping $\mathcal{M}_{\mathcal{A}}$,

Lemma 1. *Let $\mathcal{M}_{\mathcal{A}}$ be an injective mapping defined as in* (1) *and $d(.,.)$ be defined as in* (6)*, then $\exists 0 < \alpha \leq \beta$ such that the following holds,*

$$\alpha d(\mathbf{x},\mathbf{y}) \leq \|\mathcal{M}_{\mathcal{A}}(\mathbf{x}) - \mathcal{M}_{\mathcal{A}}(\mathbf{y})\|_2 \leq \beta d(\mathbf{x},\mathbf{y})$$

Proof. We want to examine the properties of the ratio,

$$V(\mathbf{x},\mathbf{y}) \ = \frac{\|\mathcal{M}_{\mathcal{A}}(\mathbf{x}) - \mathcal{M}_{\mathcal{A}}(\mathbf{y})\|_2}{d(\mathbf{x},\mathbf{y})}$$

This can be easily worked out to be,

$$V^2(\mathbf{x},\mathbf{y}) = \frac{\sum_{i=1}^m |\langle A_i, \mathbf{x}\mathbf{x}^H - \mathbf{y}\mathbf{y}^H\rangle|^2}{\|\mathbf{x}\mathbf{x}^H - \mathbf{y}\mathbf{y}^H\|_1^2}$$

Hence the biLipschitz nature of the mapping $\mathcal{M}_{\mathcal{A}}$ directly follows from Theorem 3 and lemma 6. $\square$

C. High probability bounds

We proved the biLipschitz property when the mapping $\mathcal{M}_{\mathcal{A}}$ is injective. In this subsection we show that the properties of injectivity are not rare by using gaussian random matrices. We focus on quantifying the biLipschitz parameters a_0, b_0 from Theorem 1 when the set of hermitian matrices $\mathcal{A} = \{A_i\}_{i=1}^m$ were sampled from a complex gaussian distribution.

Let $\phi \in [0, 2\pi]$. Note that,

$$\begin{aligned} &\|\mathcal{M}_{\mathcal{A}}(\mathbf{x}) - \mathcal{M}_{\mathcal{A}}(\mathbf{y})\|_2 \\ &= \|(\ldots, \mathbf{x}^H A_i \mathbf{x} - \mathbf{y}^H A_i \mathbf{y}, \ldots)\|_2 \\ &= \|(\ldots, (\mathbf{x} - e^{i\phi}\mathbf{y})^H A_i(\mathbf{x} + e^{i\phi}\mathbf{y}), \ldots)\|_2 \end{aligned}$$

Using lemma 6 we can find $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ such that the following is satisfied,

$$\mathbf{x}\mathbf{x}^H - \mathbf{y}\mathbf{y}^H = \mathbf{u}\mathbf{v}^H + \mathbf{v}\mathbf{u}^H = [[\mathbf{u},\mathbf{v}]]$$

Using lemma 8, we can further claim that $\forall \mathbf{x},\mathbf{y} \in \mathbb{C}^n$ such that $\|\mathbf{x}\mathbf{x}^H - \mathbf{y}\mathbf{y}^H\| \neq 0$, i.e. $\mathbf{x} \nsim \mathbf{y}$, $\exists$ unique $\mathbf{u}_0, \mathbf{v}_0 \in \mathbb{C}^n$ such that $\|\mathbf{u}_0\|_2 < \|\mathbf{v}_0\|_2$, $\|[[\mathbf{u}_0,\mathbf{v}_0]]\|_1 = \|\mathbf{x}\mathbf{x}^H - \mathbf{y}\mathbf{y}^H\|_1$ and $|\langle \mathbf{u}_0, \mathbf{v}_0\rangle| = \|\mathbf{u}_0\|\|\mathbf{v}_0\|$.

Thus the relation in equation (2) can be rewritten as,

$$b_0\|\mathbf{u}_0\|_2\|\mathbf{v}_0\|_2 \geq \sum_{i=1}^n |\langle A_i\mathbf{u}_0, \mathbf{v}_0\rangle|^2 \geq a_0\|\mathbf{u}_0\|_2\|\mathbf{v}_0\|_2$$

Further from corollary 3 we can state that $\mathbf{v}_0 = c\mathbf{u}_0$ where $c \in \mathbb{R}, |c| > 1$. Hence finally we are looking at the relation,

$$b_0\|\mathbf{u}_0\|_2^4 \geq \sum_{i=1}^n |\langle A_i\mathbf{u}_0, \mathbf{u}_0\rangle|^2 \geq a_0\|\mathbf{u}_0\|_2^4$$

Further, in order to obtain the values of a_0, b_0, we would need the following few intermediate results,

Lemma 2. *Let $\mathcal{A} = \{A_i\}_{i=1}^m$ be a set of complex hermitian gaussian random matrices for the measurement model given by* (P1)*. Then we have that,*

$$\mathbb{P}\left(\left|\sum_{i=1}^n \frac{1}{m}|\mathbf{x}^H A_i \mathbf{x}|^2 - \|\mathbf{x}\|_2^4\right| \geq t\right) \leq de^{-cm}$$

where c, d are constants depending on t.

Proof. Please refer to appendix VIII-A $\square$

It's common knowledge that [29],

$$\|A\| = \sup_{\mathbf{x}\in S^{n-1}} \langle A\mathbf{x}, \mathbf{x}\rangle$$

Following this, we prove a covering argument as follows,

Lemma 3. *Let $\mathcal{N}_\delta \subset S^{n-1}$ represents collection of δ radius balls, such that $\mathcal{N}_\delta$ covers S^{n-1} and $A \in \mathbb{C}^{n\times n}$ be any matrix. Then the following holds,*

$$(1+2\delta)\|A\| \geq \sup_{\mathbf{y}\in\mathcal{N}_\delta} |\langle A\mathbf{y}, \mathbf{y}\rangle| \geq (1-2\delta)\|A\|$$

where,

$$\|A\| = \sup_{\mathbf{x}\in S^{n-1}} |\langle A\mathbf{x}, \mathbf{x}\rangle|$$

This covering argument is quite standard and is not new [29]. We given the proof in Appendix IX-D for the sake of completeness.

As noted in [36], an upper bound on the size of the covering sets can be given by $|\mathcal{N}_\delta| \leq \left(\frac{12}{\delta}\right)^n$.

Finally we can quantify the constant a_0, b_0 as follows,

Theorem 4. *Let $\mathcal{A} = \{A_i\}$ be the set of complex Gaussian random matrices. Take the number of measurements $m > Cn$ for a sufficiently large constant $C > 0$ Then with probability $1 - d_0 exp^{-c_0 m}$ the following biLipschitz condition holds,*

$$b_0 d(\mathbf{x},\mathbf{y}) \geq \|\mathcal{M}_{\mathcal{A}}(\mathbf{x}) - \mathcal{M}_{\mathcal{A}}(\mathbf{y})\|_2 \geq a_0 d(\mathbf{x},\mathbf{y})$$

where a_0, b_0 are given by,

$$a_0 := \frac{\left((1-2\delta)^2 - \gamma\right)}{(1+2\delta)^2}, b_0 := \frac{\left((1+2\delta)^2 + \gamma\right)}{(1-2\delta)^2}$$

Proof. Using union bound and lemma 2, we can say that,

$$\begin{aligned} &\mathbb{P}\left(\sup_{\mathbf{x}\in\mathcal{N}_\delta}\left|\frac{1}{m}\sum_{i=1}^n |\langle A_i\mathbf{x}, \mathbf{x}\rangle|^2 - \mathbb{E}[|\langle A\mathbf{x}, \mathbf{x}\rangle|^2]\right| \geq \epsilon\right) \\ &\leq d_1 e^{-c_1 m}\left(\frac{12}{\delta}\right)^n \end{aligned}$$

where $c_1 := c_1(\epsilon)$ is the constant stated in lemma 2 depending on ϵ.

Thus,

$$\begin{aligned} &\mathbb{P}\left(\sup_{\mathbf{x}\in\mathcal{N}_\delta}\left|\frac{1}{m}\sum_{i=1}^n |\langle A_i\mathbf{x}, \mathbf{x}\rangle|^2 - \mathbb{E}[|\langle A\mathbf{x}, \mathbf{x}\rangle|^2]\right| \leq \epsilon\right) \\ &\geq 1 - d_1 e^{-c_1 m}\left(\frac{12}{\delta}\right)^n \end{aligned}$$

Note that,

$$\sup_{\mathbf{x}\in\mathcal{N}_\delta}\left|\frac{1}{m}\sum_{i=1}^{n}|\langle A_i\mathbf{x},\mathbf{x}\rangle|^2-\mathbb{E}[|\langle A\mathbf{x},\mathbf{x}\rangle|^2]\right|$$
$$\geq \sup_{\mathbf{x}\in\mathcal{N}_\delta}\left|\frac{1}{m}\sum_{i=1}^{n}|\langle A_i\mathbf{x},\mathbf{x}\rangle|^2\right|-\sup_{\mathbf{x}\in\mathcal{N}_\delta}|\mathbb{E}[|\langle A,\mathbf{x},\mathbf{x}\rangle|^2]|$$

and

$$\sup_{\mathbf{x}\in\mathcal{N}_\delta}\left|\frac{1}{m}\sum_{i=1}^{n}|\langle A_i\mathbf{x},\mathbf{x}\rangle|^2-\mathbb{E}[|\langle A\mathbf{x},\mathbf{x}\rangle|^2]\right|$$
$$\leq \sup_{\mathbf{x}\in\mathcal{N}_\delta}|\mathbb{E}[|\langle A\mathbf{x},\mathbf{x}\rangle|^2]|-\inf_{\mathbf{x}\in\mathcal{N}_\delta}\left|\frac{1}{m}\sum_{i=1}^{n}|\langle A_i\mathbf{x},\mathbf{x}\rangle|^2\right|$$

Therefore,

$$\mathbb{P}\left(\sup_{\mathbf{x}\in\mathcal{N}_\delta}\left|\frac{1}{m}\sum_{i=1}^{m}|\langle A_i\mathbf{x},\mathbf{x}\rangle|^2\right|-\sup_{\mathbf{x}\in\mathcal{N}_\delta}|\mathbb{E}[|\langle A\mathbf{x},\mathbf{x}\rangle|^2]|\leq\epsilon\right)$$
$$\geq 1-d_1e^{-c_1m}\left(\frac{12}{\delta}\right)^n$$
$$\mathbb{P}\left(\sup_{\mathbf{x}\in\mathcal{N}_\delta}\frac{1}{m}\sum_{i=1}^{m}|\langle A_i\mathbf{x},\mathbf{x}\rangle|^2-\sup_{\mathbf{x}\in\mathcal{N}_\delta}|\mathbb{E}[|\langle A\mathbf{x},\mathbf{x}\rangle|^2]|\leq\epsilon\right)$$
$$\geq 1-d_1e^{-c_1m}\left(\frac{12}{\delta}\right)^n$$

From covering argument in lemma 3, note that,

$$\sup_{\mathbf{x}\in\mathcal{N}_\delta}\frac{1}{m}\sum_{i=1}^{m}|\langle A_i\mathbf{x},\mathbf{x}\rangle|^2\geq\sup_{\mathbf{x}\in S^{n-1}}\frac{(1-2\delta)^2}{m}\sum_{i=1}^{m}|\langle A_i\mathbf{x},\mathbf{x}\rangle|^2$$

and,

$$\sup_{\mathbf{x}\in\mathcal{N}_\delta}|\mathbb{E}[|\langle A\mathbf{x},\mathbf{x}\rangle|^2]|=\sup_{\mathbf{x}\in\mathcal{N}_\delta}\|\mathbf{x}\|_2^4\geq(1-2\delta)\|\mathbf{x}\|_2^4$$

Thus we can conclude that,

$$\mathbb{P}\left(\sup_{\mathbf{x}\in S^{n-1}}\frac{1}{m}\sum_{i=1}^{m}|\langle A_i\mathbf{x},\mathbf{x}\rangle|^2\leq\frac{\left((1+2\delta)^2\|\mathbf{x}\|^4+\epsilon\right)}{(1-2\delta)^2}\right)$$
$$\geq 1-d_1e^{-c_1m}\left(\frac{12}{\delta}\right)^n$$

Similarly, we can prove,

$$\mathbb{P}\left(\inf_{\mathbf{x}\in S^{n-1}}\left|\frac{1}{m}\sum_{i=1}^{n}|\langle A_i\mathbf{x},\mathbf{x}\rangle|^2\right|\geq\frac{\left((1-2\delta)^2\|\mathbf{x}\|^4-\epsilon\right)}{(1+2\delta)^2}\right)$$
$$\geq 1-d_1e^{-c_1m}\left(\frac{12}{\delta}\right)^n$$

Hence taking $m\geq Cn$, such that $C>\frac{(log12-log\delta)}{c_1}$, we can finally state, for some constants d_0,c_0, with probability $\geq 1-d_0e^{-c_0m}$ that,

$$b_0d(\mathbf{x},\mathbf{y})\geq\|\mathcal{M}_{\mathcal{A}}(\mathbf{x})-\mathcal{M}_{\mathcal{A}}(\mathbf{y})\|_2\geq a_0d(\mathbf{x},\mathbf{y})$$

where a_0,b_0 are given by,

$$a_0:=\frac{\left((1-2\delta)^2\|\mathbf{x}\|^4-\epsilon\right)}{(1+2\delta)^2},\quad b_0:=\frac{\left((1+2\delta)^2\|\mathbf{x}\|^4+\epsilon\right)}{(1-2\delta)^2}$$

□

Corollary 2. *Given the power to choose the probability bounds ϵ,δ and the number of measurements m, we can guarantee that*

$$\frac{b_0}{a_0}=\frac{(1-2\delta)^2\left((1-2\delta)^2\|\mathbf{x}\|^4+\epsilon\right)}{(1+2\delta)^2\left((1+2\delta)^2\|\mathbf{x}\|^4-\epsilon\right)}=c>1$$

for $\forall c\in\mathbb{R}$

VI. Non-convex loss reformulation

Progress in nonconvex optimization has proved to be an important milestone in cracking problems like matrix completion [25], phase retrieval [2] and tools such as neural nets [37]. Classically, nonconvex optimization brings with it the issue of NP-hardness, which would lead force the use of its convex relaxations. From an algorithmic viewpoint, a major advantage of using convexified problem is that the gradient vanishing is sufficient condition to guarantee convergence to global minima. This crucial benefit is lost as soon as we shift to nonconvex problems. In order to prove convergence to global minima, use of second order information is necessary, since the gradient vanishing may imply local minima or saddle point as well. Requirement of this additional information can prove computationally challenging in many real world problems.

However, methods like SGD and other gradient based methods have been astonishingly successful in converging to global minimas of many nonconvex problems [38]–[40]. This begs the question, what factor is aiding this success?

Further study into this reveals that many nonconvex problems share surprisingly good landscape properties which proves crucial in the empirical success of gradient based methods [26], [41], [42]. The work in [26] proves that the ℓ_2 loss function for the phase retrieval problem enjoys properties like local minima being global minima and saddle points have a strict negative curvature. For the quadratic feasibility problem (P1), the ℓ_2 loss function can be expressed as:

$$\min_{\mathbf{x}}f(\mathbf{x})=\sum_{i=1}^{m}|\langle A_i\mathbf{x},\mathbf{x}\rangle-c_i|^2$$

Note that any direct analysis on the ℓ_2 loss function (7) is not possible as the function is not complex differentiable. We hence use concepts from wirtinger calculus [2] to work out the results. The basic principle of wirtinger calculus is to deal with the function $f:\mathbb{C}^n\to\mathbb{R}$ as a function of two variables $h:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$.

We prove the following result for the ℓ_2 loss function,

Theorem 5. *Let $\mathcal{A}=\{A_i\}$ be a set of complex gaussian random matrices. Then with high probability, if a point $\mathbf{x}\in\mathbb{C}^n$ satisfies $\|\nabla f(\mathbf{x})\|\leq\epsilon$ for some small $\epsilon>0$, then either of the following holds*

- $$d(\mathbf{x},\mathbf{x}^*)\leq\delta$$
  for some small $\delta>0$

- $\exists \mathbf{z} \in \mathbb{C}^n$ *such that*

$$\mathbf{z}^H \nabla^2 f(\mathbf{x}) \mathbf{z} \leq -c$$

for some $c > 0$

This theorem, hence, concludes that any first order method with any random initiation can work its way to find the global optimum of the above method.

The general approach we follow is quite standard and has been used in a lot of work before, for e.g. [27], [43]. Though most of these analysis is done in the real domain. And as far as we know this is the first work using this technique in the complex domain.

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VII. Appendix

A. Proof of Theorem 3

Proof. Prove 1⇒2.

The following result from [28] is quite crucial,

Theorem 6 (Theorem 2.1, [28]). *Let $\mathcal{A} = \{A_i\}_{i=1}^m \subset \mathbf{H}_n^m(\mathbb{C})$. The following statements are equivalent:*

1) *For a given $\mathcal{A} = \{A_i\}_{i=1}^m$, the mapping $\mathcal{M}_\mathcal{A}$ has phase retrieval property.*
2) *There exists no nonzero vector $\mathbf{v}, \mathbf{u} \in \mathbb{C}^n$ with $\mathbf{u} \neq ic\mathbf{v}$, $c \in \mathbb{R}$, such that $\langle A_j \mathbf{u}, \mathbf{v}\rangle = 0$ for all $1 \leq j \leq m$.*

For the mapping $\mathcal{M}_\mathcal{A}$ to be injective, we need the following to be true,

$$\mathcal{M}_\mathcal{A}(\mathbf{x}) = \mathcal{M}_\mathcal{A}(\mathbf{y}) \quad \text{iff} \quad \mathbf{x} \sim \mathbf{y} \tag{7}$$

Thus for $\mathbf{x} \nsim \mathbf{y}$, we can guarantee that $\mathcal{M}_\mathcal{A}(\mathbf{x}) \neq \mathcal{M}_\mathcal{A}(\mathbf{y})$. Then we can also conclude that,

$$\begin{aligned} &\|\mathcal{M}_\mathcal{A}(\mathbf{x}) - \mathcal{M}_\mathcal{A}(\mathbf{y})\|_2^2 \\ &= \sum_{i=1}^m \left|(\mathbf{x}^H A_i \mathbf{x}) - (\mathbf{y}^H A_i \mathbf{y})\right|^2 > 0 \end{aligned}$$

From lemma 6, for every $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ $\exists$ $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ satisfying

$$\left(\mathbf{x}\mathbf{x}^H - \mathbf{y}\mathbf{y}^H\right) = \left(\mathbf{u}\mathbf{v}^H + \mathbf{v}\mathbf{u}^H\right) = [[\mathbf{u}, \mathbf{v}]] \tag{8}$$

Thus,

$$\|\mathcal{M}_\mathcal{A}(\mathbf{x}) - \mathcal{M}_\mathcal{A}(\mathbf{y})\|_2^2 = \sum_{i=1}^m |\langle A_i, [[\mathbf{u}, \mathbf{v}]]\rangle|^2 \tag{9}$$

In order to find upper and lower bound required in (2), we can define the lower bound a_0 and upper bound b_0 as below,

$$\begin{aligned} a_0 &:= \min_{T \in S^{1,1}, \|T\|_1 = 1} \sum_{i=1}^m |\langle A_i, T\rangle|^2 \\ b_0 &:= \max_{T \in S^{1,1}, \|T\|_1 = 1} \sum_{i=1}^m |\langle A_i, T\rangle|^2 \end{aligned}$$

Since the set $T \in S^{1,1}, \|T\|_1 = 1$ is compact, the constants a_0, b_0 exists.

Prove 1⇐2

We argue the negation that 1⇏2.

Suppose the mapping $\mathcal{M}_\mathcal{A}$ is not injective, then $\exists \mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ such that,

$$\mathbf{x} \nsim \mathbf{y}, \quad \mathcal{M}_\mathcal{A}(\mathbf{y}) = \mathcal{M}_\mathcal{A}(\mathbf{x}) \tag{10}$$

Hence we can argue that $\|\mathbf{x}\mathbf{x}^H - \mathbf{y}\mathbf{y}^H\|_1 \neq 0$, but $\|\mathcal{M}_\mathcal{A}(\mathbf{y}) - \mathcal{M}_\mathcal{A}(\mathbf{x})\|_2 = 0$. Hence we have proved the negation 1⇏2. □

VIII. High probability bounds

A. Proof of Lemma 2

Proof. Let $A \in \mathbb{C}^{n \times n}$ be a complex Hermitian Gaussian random matrix, i.e. each $a_{ij} \sim \mathcal{N}(0, \frac{\sigma^2}{2}) + i\mathcal{N}(0, \frac{\sigma^2}{2})$. Define the random variable Y as,

$$\begin{aligned} Y &= |\langle A\mathbf{x}, \mathbf{x}\rangle|^2 = (\langle A\mathbf{x}, \mathbf{x}\rangle)^H \langle A\mathbf{x}, \mathbf{x}\rangle \\ &= \overline{\left(\sum_{ij} a_{ij} x_i \bar{x}_j\right)} \left(\sum_{ij} a_{ij} x_i \bar{x}_j\right) \end{aligned}$$

Next we analyze $E[Y]$

$$\mathbb{E}[Y] = \mathbb{E}\left(\overline{\left(\sum_{ij} a_{ij} x_i \bar{x}_j\right)} \left(\sum_{kl} a_{kl} x_k \bar{x}_l\right)\right)$$

Let us split the entire summation $(i, j, k, l) \in [1, n]^4$ into the following 4 sets such that:

1) $A := \{(i, j, k, l) | i = j = k = l\}$
2) $B := \{(i, j, k, l) | i = k, j = l\} \cap A^C$
3) $C := \{(i, j, k, l) | i = l, j = k\} \cap A^C$
4) $D := \{(i, j, k, l)\} \cap A^C \cap B^C \cap C^C$

Calculating the expectation of the sum of the elements in each individual sets:

1) For set A,

$$\begin{aligned} \mathbb{E}\left(\sum_{(i,j,k,l) \in A} \bar{a}_{ij} a_{kl} x_j \bar{x}_i x_k \bar{x}_l\right) &= \mathbb{E}\left(\sum_{i=1}^n |a_{ii}|^2 |x_i|^4\right) \\ &= \sigma^2 \sum_{i=1}^n |x_i|^4 \end{aligned} \tag{11}$$

2) For set B,

$$\begin{aligned} &\mathbb{E}\left(\sum_{(i,j,k,l) \in B} \bar{a}_{ij} a_{kl} x_j \bar{x}_i x_k \bar{x}_l\right) \\ &= \mathbb{E}\left(\sum_{i,j=1, i \neq j}^n |a_{ij}|^2 |x_i|^2 |x_j|^2\right) \\ &= \sigma^2 \sum_{i,j=1, i \neq j}^n |x_i|^2 |x_j|^2 \end{aligned} \tag{12}$$

3) For set C, since the matrix A is hermitian $a_{ij} = \bar{a}_{ji}$

$$\begin{aligned} &\mathbb{E}\left(\sum_{(i,j,k,l) \in C} \bar{a}_{ij} a_{kl} x_j \bar{x}_i x_k \bar{x}_l\right) \\ &= \mathbb{E}\left(\sum_{ij} (a_{ij})^2 (x_i)^2 (\bar{x}_j)^2\right) \\ &= 0 \end{aligned} \tag{13}$$

Notice that $\forall i, j$

$$(a_{ij})^2 (x_i)^2 (\bar{x}_j)^2 = ((a_{ij}^r)^2 - (a_{ij}^i)^2 + i a_{ij}^r a_{ij}^i)(x_i)^2 (\bar{x}_j)^2 \tag{14}$$

Thus,

$$\begin{aligned}&\mathbb{E}\left[(a_{ij})^2(x_i)^2(\bar{x}_j)^2\right]\\&=\mathbb{E}\left[(a_{ij})^2\right](x_i)^2(\bar{x}_j)^2\\&=\mathbb{E}\left[((a^r_{ij})^2-(a^i_{ij})^2+ia^r_{ij}a^i_{ij})\right](x_i)^2(\bar{x}_j)^2\end{aligned} \tag{15}$$

Since both the real and imaginary parts are independent and from the same distribution, we have that $\mathbb{E}\left[((a^r_{ij})^2-(a^i_{ij})^2+ia^r_{ij}a^i_{ij})\right]=0$

4) For set D, as all the elements $(i,j,k,l)\in D$ make a_{ij},a_{kl} independent of each other, we have,

$$\mathbb{E}\left(\sum_{(i,j,k,l)\in D}\bar{a}_{ij}a_{kl}x_j\bar{x}_ix_k\bar{x}_l\right)=0 \tag{16}$$

Hence we can conclude,

$$\mathbb{E}[Y]=\sigma^2\|\mathbf{x}\|_2^4 \tag{17}$$

We focus our attention on obtaining concentration bounds. Evaluating the behaviour in the individual sets we have,

1) For elements in set A,

$$\begin{aligned}&\sum_{(i,j,k,l)\in A}\bar{a}_{ij}a_{kl}x_j\bar{x}_ix_k\bar{x}_l\\&-\mathbb{E}\left(\sum_{(i,j,k,l)\in A}\bar{a}_{ij}a_{kl}x_j\bar{x}_ix_k\bar{x}_l\right)\\&=\sum_{i=1}^{n}\left(|a_{ii}|^2-\sigma^2\right)|x_i|^4\end{aligned} \tag{18}$$

We can easily see that $\forall i,|a_{ii}|^2-\sigma^2$ is a centered subexponential random variable. Hence using Bernstein type inequality [36], we can say that,

$$\begin{aligned}&Pr\left(\left|\sum_{i=1}^{n}\left(|a_{ii}|^2-\sigma^2\right)|x_i|^4\right|\geq t\right)\\&\leq 2\text{exp}\left(-c\min\left\{\frac{t^2}{K_1^2\|\mathbf{x}\|_2^8},\frac{t}{K_1\|\mathbf{x}\|_\infty^4}\right\}\right)\end{aligned} \tag{19}$$

where $K_1:=\max_i\|a_{ii}\|$ is the subexponential norm.

2) For elements in set B,

$$\begin{aligned}&\sum_{(i,j,k,l)\in B}\bar{a}_{ij}a_{kl}x_j\bar{x}_ix_k\bar{x}_l\\&-\mathbb{E}\left(\sum_{(i,j,k,l)\in B}\bar{a}_{ij}a_{kl}x_j\bar{x}_ix_k\bar{x}_l\right)\\&=\sum_{i,j=1,i\neq j}^{n}(|a_{ij}|^2-\sigma^2)|x_i|^2|x_j|^2\end{aligned} \tag{20}$$

We can easily see that $\forall i,j\in[1,n]^2,i\neq j,|a_{ij}|^2-\sigma^2$ is a centered subexponential random variable. Hence using Bernstein type inequality [36], we can say that,

$$\begin{aligned}&Pr\left(\left|\sum_{i=1}^{n}\left((|a_{ij}|^2-\sigma^2\right)|x_i|^2|x_j|^2\right|\geq t\right)\\&\leq 2\text{exp}\left(-c\min\left\{\frac{t^2}{K_2^2\|\mathbf{x}\|_2^8},\frac{t}{K_2\|\mathbf{x}\|_\infty^4}\right\}\right)\end{aligned} \tag{21}$$

where $K_2:=\max_{i,j}\|a_{ij}\|$ is the subexponential norm.

3) For elements in set C,

$$\begin{aligned}&\sum_{(i,j,k,l)\in C}\bar{a}_{ij}a_{kl}x_j\bar{x}_ix_k\bar{x}_l\\&-\mathbb{E}\left(\sum_{(i,j,k,l)\in C}\bar{a}_{ij}a_{kl}x_j\bar{x}_ix_k\bar{x}_l\right)\\&=\sum_{ij}(a_{ij})^2(x_i)^2(\bar{x}_j)^2\end{aligned} \tag{22}$$

Splitting up $a^2_{ij}=(a^r_{ij})^2-(a^i_{ij})^2+ia^r_{ij}a^i_{ij}$ makes it easier to argue that $\forall i,j\in[1,n]^2,i\neq j,(a_{ij})^2$ is a centered subexponential random variable. Hence using Bernstein type inequality [36], we can say that,

$$\begin{aligned}&Pr\left(\left|\sum_{ij}(a_{ij})^2(x_i)^2(\bar{x}_j)^2\right|\geq t\right)\\&\leq 4\text{exp}\left(-c\min\left\{\frac{t^2}{K_3^2\|\mathbf{x}\|_2^8},\frac{t}{K_3\|\mathbf{x}\|_\infty^4}\right\}\right)\end{aligned} \tag{23}$$

where $K_3:=\max_{i,j}\{\|a^r_{ij}\|,\|a^i_{ij}\|\}$ is the subexponential norm.

4) For elements in set D,

$$\begin{aligned}&\sum_{(i,j,k,l)\in D}\bar{a}_{ij}a_{kl}x_j\bar{x}_ix_k\bar{x}_l\\&-\mathbb{E}\left(\sum_{(i,j,k,l)\in D}\bar{a}_{ij}a_{kl}x_j\bar{x}_ix_k\bar{x}_l\right)\\&=\sum_{i,j,k,l\in D}\bar{a}_{ij}a_{kl}x_ix_k(\bar{x}_l)(\bar{x}_j)\end{aligned} \tag{24}$$

Similar to the case with the elements of set C we can see that the above is a centered subexponential random variable. Hence using Bernstein type inequality [36], we can say that,

$$\begin{aligned}&Pr\left(\left|\sum_{i,j,k,l\in D}\bar{a}_{ij}a_{kl}x_ix_k(\bar{x}_l)(\bar{x}_j)\right|\geq t\right)\\&\leq 4\text{exp}\left(-c\min\left\{\frac{t^2}{K_4^2\|\mathbf{x}\|_2^8},\frac{t}{K_4\|\mathbf{x}\|_\infty^4}\right\}\right)\end{aligned} \tag{25}$$

where $K_3:=\max_{i,j}\{\|a^r_{ij}\|,\|a^i_{ij}\|,\|a^r_{kl}\|,\|a^i_{kl}\|\}$ is the subexponential norm.

Taking $\sigma^2 = 1$, we have the Bernstein type inequality as,

$$\mathbb{P}\left(\left|\sum_{i=1}^{n} \frac{1}{m}|\mathbf{x}^H A_i \mathbf{x}|^2 - \|\mathbf{x}\|_2^4\right| \geq t\right) \leq de^{-mc}$$

For some constants $c, d > 0$

□

Lemma 4. *Let $\{A_i\}_{i=1}^n$ be Gaussian random matrices. Then with probability $1 - e^{-c\epsilon^2}$ we can say that,*

$$|\langle A, \Delta\bar{\Delta}^T\rangle\langle A, \mathbf{x}\bar{\mathbf{x}}^T\rangle - \langle A, \Delta\bar{\mathbf{x}}^T\rangle\langle A, \mathbf{x}\bar{\Delta}^T\rangle| \leq 2\epsilon^2\|\mathbf{x}\|^2\|\Delta\|^2$$

Proof. Evaluating the expression, we can see that,

$$\begin{aligned}
&\langle A, \Delta\bar{\Delta}^T\rangle\langle A, \mathbf{x}\bar{\mathbf{x}}^T\rangle - \langle A, \Delta\bar{\mathbf{x}}^T\rangle\langle A, \mathbf{x}\bar{\Delta}^T\rangle \\
&= \left(\sum_{ij} a_{ij}\Delta_i\bar{\Delta}_j\right)\left(\sum_{kl} a_{kl}\mathbf{x}_k\bar{\mathbf{x}}_l\right) \\
&- \left(\sum_{ij} a_{ij}\mathbf{x}_i\bar{\Delta}_j\right)\left(\sum_{kl} a_{kl}\Delta_k\bar{\mathbf{x}}_l\right) \\
&= \underbrace{\sum_{A}\left(a_{ij}^2\Delta_i\bar{\Delta}_j\mathbf{x}_i\bar{\mathbf{x}}_j - a_{ij}^2\Delta_i\bar{\Delta}_j\mathbf{x}_i\bar{\mathbf{x}}_j\right)}_{=0} \\
&+ \sum_{B}\left(a_{ij}a_{kl}\Delta_i\bar{\Delta}_j\mathbf{x}_k\bar{\mathbf{x}}_l - a_{ij}a_{kl}\Delta_k\bar{\mathbf{x}}_l\mathbf{x}_i\bar{\Delta}_j\right) \\
&= \sum_{B} a_{ij}a_{kl}\left(\Delta_i\bar{\Delta}_j\mathbf{x}_k\bar{\mathbf{x}}_l - \Delta_k\bar{\mathbf{x}}_l\mathbf{x}_i\bar{\Delta}_j\right) \\
&= \sum_{B} a_{ij}a_{kl}\bar{\Delta}_j\bar{\mathbf{x}}_l\left(\Delta_i\mathbf{x}_k - \Delta_k\mathbf{x}_i\right)
\end{aligned} \tag{26}$$

Let us split the entire summation $(i,j,k,l) \in [1,n]^4$ into the following 4 sets such that:

1) $A := \{(i,j,k,l) | i = j = k = l\}$
2) $B := \{(i,j,k,l) | i = k, j = l\} \cap A^C$
3) $C := \{(i,j,k,l) | i = l, j = k\} \cap A^C$
4) $D := \{(i,j,k,l)\} \cap A^C \cap B^C \cap C^C$

Calculating the expectation of the sum of the elements in each individual sets:

1) For set A,

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{ij} a_{ij}\Delta_i\bar{\Delta}_j\right)\left(\sum_{kl} a_{kl}\mathbf{x}_k\bar{\mathbf{x}}_l\right)\right. \\
&\left. - \left(\sum_{ij} a_{ij}\mathbf{x}_i\bar{\Delta}_j\right)\left(\sum_{kl} a_{kl}\Delta_k\bar{\mathbf{x}}_l\right)\right] \\
&= \mathbb{E}\left[a_{ii}^2\Delta_i\bar{\Delta}_i\mathbf{x}_i\bar{\mathbf{x}}_j - a_{ii}^2\Delta_i\bar{\Delta}_i\mathbf{x}_i\bar{\mathbf{x}}_i\right] \\
&= \mathbb{E}[0] = 0
\end{aligned} \tag{27}$$

2) For set B,

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{ij} a_{ij}\Delta_i\bar{\Delta}_j\right)\left(\sum_{kl} a_{kl}\mathbf{x}_k\bar{\mathbf{x}}_l\right)\right. \\
&\left. - \left(\sum_{ij} a_{ij}\mathbf{x}_i\bar{\Delta}_j\right)\left(\sum_{kl} a_{kl}\Delta_k\bar{\mathbf{x}}_l\right)\right] \\
&= \mathbb{E}\left[a_{ij}^2\Delta_i\bar{\Delta}_j\mathbf{x}_i\bar{\mathbf{x}}_j - a_{ij}^2\Delta_i\bar{\Delta}_j\mathbf{x}_i\bar{\mathbf{x}}_j\right] \\
&= \mathbb{E}[0] = 0
\end{aligned} \tag{28}$$

3) For set C, since the matrix A is hermitian $a_{ij} = \bar{a}_{ji}$

$$\begin{aligned}
&\left[\left(\sum_{ij} a_{ij}\Delta_i\bar{\Delta}_j\right)\left(\sum_{kl} a_{kl}\mathbf{x}_k\bar{\mathbf{x}}_l\right)\right. \\
&\left. - \left(\sum_{ij} a_{ij}\mathbf{x}_i\bar{\Delta}_j\right)\left(\sum_{kl} a_{kl}\Delta_k\bar{\mathbf{x}}_l\right)\right] \\
&= \left[|a_{ij}|^2\Delta_i\bar{\Delta}_j\mathbf{x}_j\bar{\mathbf{x}}_i - |a_{ij}|^2\Delta_j\bar{\Delta}_j\mathbf{x}_i\bar{\mathbf{x}}_i\right]
\end{aligned} \tag{29}$$

Notice that $\forall i, j$

$$\begin{aligned}
&= |a_{ji}|^2\Delta_j\bar{\Delta}_i\mathbf{x}_i\bar{\mathbf{x}}_j + |a_{ij}|^2\Delta_i\bar{\Delta}_j\mathbf{x}_j\bar{\mathbf{x}}_i \\
&-|a_{ji}|^2\Delta_i\bar{\Delta}_i\mathbf{x}_j\bar{\mathbf{x}}_j - |a_{ij}|^2\Delta_j\bar{\Delta}_j\mathbf{x}_i\bar{\mathbf{x}}_i
\end{aligned} \tag{30}$$

Since $|a_{ij}|^2 = |a_{ji}|^2$. Thus,

$$\begin{aligned}
&= |a_{ji}|^2\left[\Delta_j\bar{\Delta}_i\mathbf{x}_i\bar{\mathbf{x}}_j + \Delta_i\bar{\Delta}_j\mathbf{x}_j\bar{\mathbf{x}}_i\right. \\
&\left. -\Delta_i\bar{\Delta}_i\mathbf{x}_j\bar{\mathbf{x}}_j - \Delta_j\bar{\Delta}_j\mathbf{x}_i\bar{\mathbf{x}}_i\right] \\
&= |a_{ji}|^2\left[\Delta_j\bar{\Delta}_i\mathbf{x}_i\bar{\mathbf{x}}_j + \Delta_i\bar{\Delta}_j\mathbf{x}_j\bar{\mathbf{x}}_i\right. \\
&\left. -\|\Delta_i\|_2^2\|\mathbf{x}_j\|_2^2 - \|\Delta_j\|_2^2\|\mathbf{x}_i\|_2^2\right] \\
&\leq 0
\end{aligned} \tag{31}$$

4) For set D, as all the elements $(i,j,k,l) \in D$ make a_{ij}, a_{kl} independent of each other, we have,

$$\mathbb{E}\left(\sum a_{ij}a_{kl}\bar{\Delta}_j\bar{\mathbf{x}}_l\left(\Delta_i\mathbf{x}_k - \Delta_k\mathbf{x}_i\right)\right) = 0 \tag{32}$$

Hence we can conclude,

$$\mathbb{E}\left[\langle A, \Delta\bar{\Delta}^T\rangle\langle A, \mathbf{x}\bar{\mathbf{x}}^T\rangle - \langle A, \Delta\bar{\mathbf{x}}^T\rangle\langle A, \mathbf{x}\bar{\Delta}^T\rangle\right] = 0 \tag{33}$$

We focus our attention on obtaining concentration bounds. Evaluating the behaviour in the individual sets we have,

1) For elements in set D,

$$\begin{aligned}
&\sum_{(i,j,k,l)\in D} a_{ij}a_{kl}\bar{\Delta}_j\bar{\mathbf{x}}_l\left(\Delta_i\mathbf{x}_k - \Delta_k\mathbf{x}_i\right) \\
&- \mathbb{E}\left(\sum_{(i,j,k,l)\in D} a_{ij}a_{kl}\bar{\Delta}_j\bar{\mathbf{x}}_l\left(\Delta_i\mathbf{x}_k - \Delta_k\mathbf{x}_i\right)\right) \\
&= \sum_{(i,j,k,l)\in D} a_{ij}a_{kl}\bar{\Delta}_j\bar{\mathbf{x}}_l\left(\Delta_i\mathbf{x}_k - \Delta_k\mathbf{x}_i\right)
\end{aligned} \tag{34}$$

Similar to the case with the elements of set D we can see that the above is a centered subexponential random variable. Hence using Bernstein type inequality [36], we can say that,

$$Pr\left(\left|\sum_{(i,j,k,l)\in D} a_{ij}a_{kl}\bar{\Delta}_j\bar{\mathbf{x}}_l\left(\Delta_i\mathbf{x}_k-\Delta_k\mathbf{x}_i\right)\right|\geq t\right)$$
$$\leq 4\exp\left(-c\min\left\{\frac{t^2}{K_4^2\|\Delta\|_2^4\|\mathbf{x}\|_2^4},\frac{t}{K_4\|\Delta\|_\infty^2\|\mathbf{x}\|_\infty^2}\right\}\right) \tag{35}$$

where $K_4 := \max_{i,j}\{\|a_{ij}^r\|,\|a_{ij}^i\|,\|a_{kl}^r\|,\|a_{kl}^i\|\}$ is the subexponential norm.

Thus we can say Bernstein type inequality,

$$Pr\left(|\sum_{i=1}^m\langle A,\Delta\bar{\Delta}^T\rangle\langle A,\mathbf{x}\bar{\mathbf{x}}^T\rangle-\langle A,\Delta\bar{\mathbf{x}}^T\rangle\langle A,\mathbf{x}\bar{\Delta}^T\rangle|\geq t\right)$$
$$\leq f_0e^{-mf_1t} \tag{36}$$

for some constants $f_0,f_1>0$. □

IX. Non-convex lanscape

A. Supporting Lemmas

Taking inspiration from [27], we can prove the following:

Lemma 5. *Let $\mathbf{x},\mathbf{x}^*\in\mathbb{C}^n$. Then,*

$$\|(\mathbf{x}-e^{i\phi}\mathbf{x}^*)(\mathbf{x}-e^{i\phi}\mathbf{x}^*)^H\|_F^2\leq 2\|\mathbf{x}\mathbf{x}^H-\mathbf{x}^*(\mathbf{x}^*)^H\| \tag{37}$$

Proof. We first note that,

$$\begin{aligned}&\arg_\theta\min\|\mathbf{x}-e^{i\theta}\mathbf{x}^*\|^2\\&=\arg_\theta\min\left(\mathbf{x}-e^{i\theta}\mathbf{x}^*\right)^H\left(\mathbf{x}-e^{i\theta}\mathbf{x}^*\right)\\&=\arg_\theta\min\|\mathbf{x}\|^2+\|\mathbf{x}^*\|-e^{-i\theta}(\mathbf{x}^*)^H\mathbf{x}-e^{i\theta}\mathbf{x}^H\mathbf{x}^*\\&=\arg_\theta\min\|\mathbf{x}\|^2+\|\mathbf{x}^*\|-2Re(\langle\mathbf{x},e^{i\theta}\mathbf{x}^*\rangle)\end{aligned} \tag{38}$$

The minimum can only be achieved at a point where $\mathbf{x}^H(e^{i\phi}\mathbf{x}^*)\geq 0$. We already know that,

$$\begin{aligned}&\|\mathbf{x}\mathbf{x}^H-\mathbf{x}^*(\mathbf{x}^*)^H\|\\&=\|\mathbf{x}\Delta^H+\Delta\mathbf{x}^H-\Delta\Delta^H\|_F^2\\&=Tr\left((\mathbf{x}\Delta^H+\Delta\mathbf{x}^H-\Delta\Delta^H)^H(\mathbf{x}\Delta^H+\Delta\mathbf{x}^H-\Delta\Delta^H)\right)\\&=\big(\|\mathbf{x}\Delta^H\|_F^2+(\langle\mathbf{x},\Delta\rangle)^2+(\langle\Delta,\mathbf{x}\rangle)^2+\|\Delta\mathbf{x}^H\|_F^2\\&-2\langle\mathbf{x},\Delta\rangle\|\Delta\|_F^2-2\langle\Delta,\mathbf{x}\rangle\|\Delta\|_F^2+\|\Delta\Delta^H\|_F^2\big)\\&=\big(2\|\langle\mathbf{x},\Delta\rangle\|_F^2+2Re((\langle\mathbf{x},\Delta\rangle)^2)\\&-4Re(\langle\mathbf{x},\Delta\rangle)\|\Delta\|_F^2+\|\Delta\Delta^H\|_F^2\\&=\big(2\mathbf{x}^H\mathbf{x}\Delta^H\Delta+2Re((\langle\mathbf{x},\Delta\rangle)^2)\\&-4Re(\mathbf{x}^H\Delta\Delta^H\Delta)+\|\Delta\Delta^H\|_F^2\end{aligned}$$

Since $\mathbf{x}^H\mathbf{x}\Delta^H\Delta=\|\langle\mathbf{x},\Delta\rangle\|_F^2$, its a real quantity.

$$\begin{aligned}&\|\mathbf{x}\mathbf{x}^H-\mathbf{x}^*(\mathbf{x}^*)^H\|\\&=2\mathbf{x}^H\left(\mathbf{x}-\Delta\right)\Delta^H\Delta+2Re((\langle\mathbf{x},\Delta\rangle)^2)\\&-2Re(\mathbf{x}^H\Delta\Delta^H\Delta)+\|\Delta\Delta^H\|_F^2\\&=2\mathbf{x}^H\left(\mathbf{x}-\Delta\right)\Delta^H\Delta+(\langle\mathbf{x},\Delta\rangle)^2+(\langle\Delta,\mathbf{x}\rangle)^2\\&-\langle\mathbf{x},\Delta\rangle\|\Delta\|_F^2-\langle\Delta,\mathbf{x}\rangle\|\Delta\|_F^2+\|\Delta\Delta^H\|_F^2\\&=2\mathbf{x}^H\left(\mathbf{x}-\Delta\right)\Delta^H\Delta+\left(\langle\mathbf{x},\Delta\rangle-\frac{1}{2}\langle\Delta,\Delta\rangle\right)^2\\&+\left((\langle\Delta,\mathbf{x}\rangle-\frac{1}{2}\langle\Delta,\Delta\rangle\right)^2+\frac{1}{2}\|\Delta\Delta^H\|_F^2\\&=2\mathbf{x}^H\left(\mathbf{x}-\Delta\right)\Delta^H\Delta+2Re\left((\langle\Delta,\mathbf{x}-\frac{1}{2}\Delta\rangle\right)^2+\frac{1}{2}\|\Delta\Delta^H\|_F^2\\&=2\mathbf{x}^He^{i\phi}\mathbf{x}^*\Delta^H\Delta+\frac{1}{2}Re\left((\langle\Delta,\mathbf{x}+e^{i\phi}\mathbf{x}^*\rangle\right)^2+\frac{1}{2}\|\Delta\Delta^H\|_F^2\end{aligned}$$

It can be seen that $Im\left(\langle\Delta,\mathbf{x}+e^{i\phi}\mathbf{x}^*\rangle\right)=0$. We cannot say this statement for general θ. We also know that $\mathbf{x}^H(e^{i\theta}\mathbf{x}^*)\geq 0$. Hence we can say,

$$\|\mathbf{x}\mathbf{x}^H-\mathbf{x}^*(\mathbf{x}^*)^H\|\geq\frac{1}{2}\|\Delta^H\Delta\|_F^2 \tag{39}$$

□

Lemma 6. *[44] if the mapping is injective and hence $ker(\mathcal{M}_{\mathcal{A}})\cap(S^{1,0}-S^{1,0})=\{0\}$. And from [44], we also know that $S^{1,0}-S^{1,0}=S^{1,1}=S^{1,0}+S^{0,1}$. Hence we can say that if $T\in S^{1,1}$, $\exists\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v}\in\mathbb{C}^n$ such that,*

$$T=\frac{1}{2}\left(\mathbf{x}\mathbf{x}^H-\mathbf{y}\mathbf{y}^H\right)=\frac{1}{2}\left(\mathbf{u}\mathbf{v}^H+\mathbf{v}\mathbf{u}^H\right)=[[\mathbf{u},\mathbf{v}]] \tag{40}$$

Lemma 7.

$$\|\mathbf{x}\mathbf{x}^H-\mathbf{x}^*(\mathbf{x}^*)^H\|=\min\|\mathbf{x}^*-e^{i\theta}\mathbf{x}\|\max\|\mathbf{x}^*-e^{i\theta}\mathbf{x}\| \tag{41}$$

and the right hand sign is a distance metric (show it satisfies all the three laws).

Proof. We know from [44], that $\mathcal{S}^{1,1}=\mathcal{S}^{1,0}+\mathcal{S}^{0,1}=\mathcal{S}^{1,0}-\mathcal{S}^{1,0}$. So for $\mathbf{x}\mathbf{x}^H-(\mathbf{x}^*)(\mathbf{x}^*)^H$ we need to find appropriate vectors in $S^{1,0}+S^{0,1}$ i.e. of the form $\mathbf{v}\mathbf{u}^H+\mathbf{v}\mathbf{u}^H$, and we can see that the vectors $\mathbf{x}+\mathbf{x}^*,\mathbf{x}-\mathbf{x}^*$ can be one such pair as shown below :

$$\begin{aligned}&2(X-X^*)\\&=2\left(\mathbf{x}\mathbf{x}^H-\mathbf{x}^*(\mathbf{x}^*)^H\right)\\&=\left(\mathbf{x}\mathbf{x}^H\pm\mathbf{x}^*\mathbf{x}^H-\mathbf{x}^*(\mathbf{x}^*)^H\right)\\&+\left(\mathbf{x}\mathbf{x}^H\pm\mathbf{x}(\mathbf{x}^*)^H-\mathbf{x}^*(\mathbf{x}^*)^H\right)\\&=\left(\mathbf{x}-\mathbf{x}^*\right)\mathbf{x}^H+\mathbf{x}^*\left(\mathbf{x}-\mathbf{x}^*\right)^H\\&+\mathbf{x}\left(\mathbf{x}-\mathbf{x}^*\right)^H+\left(\mathbf{x}-\mathbf{x}^*\right)(\mathbf{x}^*)^H\\&=\left(\mathbf{x}-\mathbf{x}^*\right)\left(\mathbf{x}+\mathbf{x}^*\right)^H+\left(\mathbf{x}+\mathbf{x}^*\right)\left(\mathbf{x}-\mathbf{x}^*\right)^H\end{aligned} \tag{42}$$

Note that in the above derivation, we could might as well use $\mathbf{x}+e^{i\theta}\mathbf{x}^*$ and $\mathbf{x}-e^{i\theta}\mathbf{x}^*$, $\forall\theta\in[0,2\pi]$, instead of $\mathbf{x}+\mathbf{x}^*$ and $\mathbf{x}-\mathbf{x}^*$ respectively.

So the next thing is to see how to find the norm of $\|X - X^*\|_1$. On this we know from [44] again that,

$$\|X - X^*\|_1^2 = \|\mathbf{u}\|_2^2\|\mathbf{v}\|_2^2 - \langle i\mathbf{u}, \mathbf{v}\rangle_{\mathbb{R}} \tag{43}$$

where we have $\mathbf{u}, \mathbf{v}$ as mentioned before. We focus on $\langle i\mathbf{u}, \mathbf{v}\rangle$.

Next we try to argue that $\exists\theta$ such that $\langle i(\mathbf{x} - e^{i\theta}\mathbf{x}^*), (\mathbf{x} + e^{i\theta}\mathbf{x}^*)\rangle = 0$

The angle can be explicitly calculated as follows

$$\begin{aligned}
&\langle i\mathbf{u}, \mathbf{v}\rangle \\
&= (e^{i\frac{\pi}{2}}\mathbf{u})^T\bar{\mathbf{v}} \\
&= \left(e^{i\frac{\pi}{2}}(e^{i\phi}\mathbf{x} + \mathbf{x}^*)\right)^T\overline{(e^{i\phi}\mathbf{x} - \mathbf{x}^*)} \\
&= \left(e^{i(\phi+\frac{\pi}{2})}\mathbf{x} + e^{i\frac{\pi}{2}}\mathbf{x}^*\right)^T\left(e^{-i\phi}\bar{\mathbf{x}} - \bar{\mathbf{x}^*}\right) \\
&= e^{i\frac{\pi}{2}}\langle\mathbf{x}, \mathbf{x}\rangle - e^{i(\phi+\frac{\pi}{2})}e^{i\omega}\|\mathbf{x}\|\|\mathbf{x}^*\| \\
&+ e^{i(\frac{\pi}{2}-\phi)}e^{-i\omega}\|\mathbf{x}\|\|\mathbf{x}^*\| - e^{i\frac{\pi}{2}}\langle\mathbf{x}^*, \mathbf{x}^*\rangle \\
&= e^{i\frac{\pi}{2}}\left(\langle\mathbf{x}, \mathbf{x}\rangle - \langle\mathbf{x}^*, \mathbf{x}^*\rangle\right) \\
&+ \left(e^{i\pi}e^{i(\phi+\frac{\pi}{2})}e^{i\omega} + e^{i(\frac{\pi}{2}-\phi)}e^{-i\omega}\right)\|\mathbf{x}\|\|\mathbf{x}^*\| \\
&= e^{i\frac{\pi}{2}}\left(\langle\mathbf{x}, \mathbf{x}\rangle - \langle\mathbf{x}^*, \mathbf{x}^*\rangle\right) \\
&+ \left(e^{i(\phi+\omega+\frac{3\pi}{2})} + e^{i(\frac{\pi}{2}-\phi-\omega)}\right)\|\mathbf{x}\|\|\mathbf{x}^*\|
\end{aligned} \tag{44}$$

where ω is the angle between the two vectors. Now we are using the real part of this inner product above, which is 0 if and only if,

$$\begin{aligned}
&cos(\phi + \omega + \frac{3\pi}{2}) = -cos(\frac{\pi}{2} - \omega - \phi) \\
&cos(\phi + \omega + \frac{3\pi}{2}) + cos(\frac{\pi}{2} - \omega - \phi) = 0 \\
&2cos(\frac{(\phi + \omega + 2\pi - \omega - \phi)}{2})cos(\frac{(\phi + \omega + \pi + \omega + \phi)}{2}) = 0 \\
&cos\left(\phi + \omega + \frac{\pi}{2}\right) = 0 \\
&sin\,(\phi + \omega) = 0 \\
&\phi + \omega = n\pi
\end{aligned} \tag{45}$$

The parameter in our control is ϕ, hence we need This proves that we need -ω or $-\omega \pm \pi$ □

Corollary 3. *The vectors* $\min\|\mathbf{x}^* - e^{i\theta}\mathbf{x}\|\max\|\mathbf{x}^* - e^{i\theta}\mathbf{x}\|$ *are aligned*

Lemma 8. *in the case when* $\|\mathbf{x}\mathbf{x}^H - \mathbf{y}\mathbf{y}^H\| \neq 0$*, i.e.* $\mathbf{x} \nsim \mathbf{y}$*,* $\exists$ *unique* $\mathbf{u}_0, \mathbf{v}_0 \in \mathbb{C}^n$ *such that* $\|\mathbf{u}_0\|_2 < \|\mathbf{v}_0\|_2$*,* $\|[[\mathbf{u}_0, \mathbf{v}_0]]\|_1 = \|\mathbf{x}\mathbf{x}^H - \mathbf{y}\mathbf{y}^H\|_1$ *and* $|\langle\mathbf{u}_0, \mathbf{v}_0\rangle| = \|\mathbf{u}_0\|\|\mathbf{v}_0\|$

Proof. □

Lemma 9. *For any rank one matrix,*

$$\|\mathbf{x}\mathbf{x}^H\|_F^2 = \|\mathbf{x}\|_2^4 \tag{46}$$

Proof.

$$\begin{aligned}
\|\mathbf{x}\mathbf{x}^H\|_F^2 &= \sum_{i,j=1}^n |x_i\bar{x}_j|^2 = \sum_{i,j=1}^n (x_i\bar{x}_j)^H(x_i\bar{x}_j) = \sum_{i,j=1}^n x_j\bar{x}_i x_i\bar{x}_j \\
&= \sum_{i,j=1}^n |x_j|^2|x_i|^2 = (\sum_{i=1}^n |x_i|^2)^2 = \|\mathbf{x}\|_2^4
\end{aligned} \tag{47}$$

□

B. Wirtinger Calculus

Going with standard arguments from wirtinger calculus [28], note that,

$$\begin{aligned}
f(\mathbf{x}) = g(\mathbf{x}, \bar{\mathbf{x}}) &= \sum_{i=1}^m g_i(\mathbf{x}, \bar{\mathbf{x}}) \\
&= \frac{1}{2}\sum_{i=1}^n (\bar{\mathbf{x}}^T A_i\mathbf{x} - b_i)^2
\end{aligned}$$

For the gradient we have,

$$\nabla g(\mathbf{x}, \bar{\mathbf{x}}) = \sum_{i=1}^n \begin{bmatrix}(\bar{\mathbf{x}}^T A_i\mathbf{x} - b_i)A_i\mathbf{x} \\ (\mathbf{x}^T A_i\bar{\mathbf{x}} - b_i)A_i\bar{\mathbf{x}}\end{bmatrix} \tag{48}$$

For the hessian, we have

$$\nabla^2 g(\mathbf{x}, \bar{\mathbf{x}}) = \sum_{i=1}^m \begin{bmatrix}(2\bar{\mathbf{x}}^T A_i\mathbf{x} - b_i)A_i & (A_i\mathbf{x})(A_i\mathbf{x})^T \\ (A_i\bar{\mathbf{x}})(A_i\bar{\mathbf{x}})^T & (2\bar{\mathbf{x}}^T A_i\mathbf{x} - b_i)A_i\end{bmatrix} \tag{49}$$

For the rest of the write-up, define $\Delta = \mathbf{x} - e^{i\phi}\mathbf{x}^*$ such that $\phi = \min_{\theta\in[0,2\pi]}\|\mathbf{x} - e^{i\theta}\mathbf{x}^*\|$. Notice that the following relation holds,

$$\mathbf{x}\mathbf{x}^H - \mathbf{x}^*(\mathbf{x}^*)^H + \Delta\Delta^H = \mathbf{x}\Delta^H + \Delta\mathbf{x}^H \tag{50}$$

The following can be verified easily,

$$\begin{aligned}
&\langle\nabla g(\mathbf{x}), \begin{bmatrix}\Delta \\ \bar{\Delta}\end{bmatrix}\rangle \\
&= \langle A_i, \mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}^*})^T\rangle\langle A_i, \mathbf{x}\bar{\Delta}^T + \Delta\bar{\mathbf{x}}^T\rangle \\
&= \langle A_i, \mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}^*})^T\rangle\langle A_i, \mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}}^*)^T + \Delta\bar{\Delta}^T\rangle
\end{aligned} \tag{51}$$

$$\begin{aligned}
\begin{bmatrix}\Delta \\ \bar{\Delta}\end{bmatrix}^* \nabla^2 g_i(\mathbf{x}, \bar{\mathbf{x}})\begin{bmatrix}\Delta \\ \bar{\Delta}\end{bmatrix} &= (2\mathbf{x}^T A_i\bar{\mathbf{x}} - b_i)(\bar{\Delta}^T A_i\Delta + \Delta^T A_i\bar{\Delta}) \\
&+ \left((\Delta^T A_i\bar{\mathbf{x}})^2 + (\bar{\Delta}A_i\mathbf{x})^2\right)
\end{aligned} \tag{52}$$

C. Proof of Theorem 5

Proof. Notice that,

$$\begin{aligned}
&(\Delta A\bar{\mathbf{x}})^2 + (\bar{\Delta}^T A\mathbf{x})^2 \\
&= \left(\langle A, \mathbf{x}\bar{\Delta}^T + \Delta\bar{\mathbf{x}}^T\rangle\right)^2 - 2(\Delta^T A\bar{\mathbf{x}})(\bar{\Delta}^T A\mathbf{x}) \\
&= \left(\langle A, \mathbf{x}\mathbf{x}^T - \mathbf{x}^*(\mathbf{x}^*)^T + \Delta\Delta^T\rangle\right)^2 \\
&- 2(\langle A, \Delta\bar{\mathbf{x}}^T\rangle)(\langle A, \mathbf{x}\bar{\Delta}^T\rangle)
\end{aligned} \tag{53}$$

Thus we have,

$$\begin{bmatrix}\Delta\\ \bar{\Delta}\end{bmatrix}^* \nabla^2 g_i(\mathbf{x},\bar{\mathbf{x}}) \begin{bmatrix}\Delta\\ \bar{\Delta}\end{bmatrix}$$
$$= \langle A_i, 2\mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}}^*)^T\rangle\langle A_i, 2\Delta\bar{\Delta}^T\rangle$$
$$+ \left(\langle A, \mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}}^*)^T + \Delta\bar{\Delta}^T\rangle\right)^2$$
$$- 2(\langle A, \Delta\bar{\mathbf{x}}^T\rangle)(\langle A, \mathbf{x}\bar{\Delta}^T\rangle)$$
$$= 2\left(\langle A_i, 2\mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}}^*)^T\rangle\langle A_i, \Delta\bar{\Delta}^T\rangle\right)$$
$$+ \langle A_i, \Delta\bar{\Delta}^T\rangle\langle A_i, \Delta\bar{\Delta}^T + \mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}}^*)^T\rangle$$
$$+ \langle A_i, \mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}}^*)^T\rangle\langle A_i, \Delta\bar{\Delta}^T + \mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}}^*)^T\rangle$$
$$- 2(\langle A, \Delta\bar{\mathbf{x}}^T\rangle)(\langle A, \mathbf{x}\bar{\Delta}^T\rangle)$$
$$= 2\langle A_i, \Delta\bar{\Delta}^T\rangle\langle A_i, \mathbf{x}\bar{\mathbf{x}}^T\rangle$$
$$+ 2\left(\langle A_i, \mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}}^*)^T\rangle\langle A_i, \Delta\bar{\Delta}^T\rangle\right)$$
$$+ \langle A_i, \Delta\bar{\Delta}^T\rangle\langle A_i, \mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}}^*)^T\rangle$$
$$+ \langle A_i, \Delta\bar{\Delta}^T\rangle\langle A_i, \Delta\bar{\Delta}^T\rangle$$
$$+ \langle A_i, \mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}}^*)^T\rangle\langle A_i, \Delta\bar{\Delta}^T + \mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}}^*)^T\rangle$$
$$- 2(\langle A, \Delta\bar{\mathbf{x}}^T\rangle)(\langle A, \mathbf{x}\bar{\Delta}^T\rangle)$$
$$= 2\langle A_i, \Delta\bar{\Delta}^T\rangle\langle A_i, \mathbf{x}\bar{\mathbf{x}}^T\rangle$$
$$+ 2\left(\langle A_i, \mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}}^*)^T\rangle\langle A_i, \Delta\bar{\Delta}^T + \mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}}^*)^T\rangle\right)$$
$$- 2\langle A_i, \mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}}^*)^T\rangle\langle A_i, \mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}}^*)^T\rangle$$
$$+ \langle A_i, \Delta\bar{\Delta}^T\rangle\langle A_i, \mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}}^*)^T\rangle$$
$$+ \langle A_i, \Delta\bar{\Delta}^T\rangle\langle A_i, \Delta\bar{\Delta}^T\rangle$$
$$+ \langle A_i, \mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}}^*)^T\rangle\langle A_i, \Delta\bar{\Delta}^T + \mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}}^*)^T\rangle$$
$$- 2(\langle A, \Delta\bar{\mathbf{x}}^T\rangle)(\langle A, \mathbf{x}\bar{\Delta}^T\rangle)$$
$$= 2\langle A_i, \Delta\bar{\Delta}^T\rangle\langle A_i, \mathbf{x}\bar{\mathbf{x}}^T\rangle - 2(\langle A, \Delta\bar{\mathbf{x}}^T\rangle)(\langle A, \mathbf{x}\bar{\Delta}^T\rangle)$$
$$- 3\langle A_i, \mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}}^*)^T\rangle\langle A_i, \mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}}^*)^T\rangle$$
$$+ \langle A_i, \Delta\bar{\Delta}^T\rangle\langle A_i, \Delta\bar{\Delta}^T\rangle$$
$$+ 4\langle A_i, \mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}}^*)^T\rangle\langle A_i, \Delta\bar{\Delta}^T + \mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}}^*)^T\rangle$$
$$= 2\langle A_i, \Delta\bar{\Delta}^T\rangle\langle A_i, \mathbf{x}\bar{\mathbf{x}}^T\rangle - 2(\langle A, \Delta\bar{\mathbf{x}}^T\rangle)(\langle A, \mathbf{x}\bar{\Delta}^T\rangle)$$
$$- 3\langle A_i, \mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}}^*)^T\rangle\langle A_i, \mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}}^*)^T\rangle$$
$$+ \langle A_i, \Delta\bar{\Delta}^T\rangle\langle A_i, \Delta\bar{\Delta}^T\rangle$$
$$+ 4\langle \nabla g(\mathbf{x},\bar{\mathbf{x}}), \begin{bmatrix}\Delta\\ \bar{\Delta}\end{bmatrix}\rangle \tag{54}$$

Thus we can say overall that,

$$\begin{bmatrix}\Delta\\ \bar{\Delta}\end{bmatrix}^* \nabla^2 g(\mathbf{x},\bar{\mathbf{x}}) \begin{bmatrix}\Delta\\ \bar{\Delta}\end{bmatrix} = \sum_{i=1}^m \begin{bmatrix}\Delta\\ \bar{\Delta}\end{bmatrix}^* \nabla^2 g_i(\mathbf{x},\bar{\mathbf{x}}) \begin{bmatrix}\Delta\\ \bar{\Delta}\end{bmatrix}$$
$$= \sum_{i=1}^m \left(2\langle A_i, \Delta\bar{\Delta}^T\rangle\langle A_i, \mathbf{x}\bar{\mathbf{x}}^T\rangle - 2(\langle A, \Delta\bar{\mathbf{x}}^T\rangle)(\langle A, \mathbf{x}\bar{\Delta}^T\rangle)\right.$$
$$-3\langle A_i, \mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}}^*)^T\rangle\langle A_i, \mathbf{x}\bar{\mathbf{x}}^T - \mathbf{x}^*(\bar{\mathbf{x}}^*)^T\rangle$$
$$\left.+ \langle A_i, \Delta\bar{\Delta}^T\rangle\langle A_i, \Delta\bar{\Delta}^T\rangle + 4\langle \nabla g(\mathbf{x},\bar{\mathbf{x}}), \begin{bmatrix}\Delta\\ \bar{\Delta}\end{bmatrix}\rangle\right)$$
$$\leq 2\epsilon^2(\|\mathbf{x}\|_2^2\|\Delta\|_2^2) + 4\delta\|\Delta\|_2 + b_0\|\Delta\Delta^H\|_F^2$$
$$- 3a_0\|\mathbf{x}\mathbf{x}^H - \mathbf{x}^*(\mathbf{x}^*)^H\|$$
$$\leq 4\epsilon^2\|\mathbf{x}\|_2^2\|\mathbf{x}\mathbf{x}^H - \mathbf{x}^*(\mathbf{x}^*)^H\|_F + 4\delta\|\Delta\|_2$$
$$+ 2b_0\|\mathbf{x}\mathbf{x}^H - \mathbf{x}^*(\mathbf{x}^*)^H\|_F - 3a_0\|\mathbf{x}\mathbf{x}^H - \mathbf{x}^*(\mathbf{x}^*)^H\|$$
$$\leq \left(4\epsilon^2\|\mathbf{x}\|_2^2 + 2b_0 - 3a_0\right)\|\mathbf{x}\mathbf{x}^H - \mathbf{x}^*(\mathbf{x}^*)^H\|_F + 4\delta\|\Delta\|_2$$

Since ϵ, δ can be taken to be arbitrarily close to 0, and $a_0 \to b_0 \neq 0$, thus we can see that the strict saddle point condition holds. □

D. Proof Lemma 3

Proof. We aim to connect the supremum over $\mathbf{x}, \mathbf{y} \in S^{n-1}$ to the supremum over $\mathbf{x}, \mathbf{y} \in \mathcal{N}_\delta$.

Since $\mathcal{N}_\delta$ covers S^{n-1}. We can say that $\forall \mathbf{x} \in S^{n-1}$, $\exists \mathbf{u} \in \mathcal{N}_\delta$ such that $\|\mathbf{x} - \mathbf{u}\| \leq \delta$.

Hence $\forall \mathbf{x} \in S^{n-1}$, $\exists \mathbf{y} \in \mathcal{N}_\delta$ such that,

$$\begin{aligned}&|\langle A\mathbf{x}, \mathbf{x}\rangle - \langle A\mathbf{y}, \mathbf{y}\rangle| \\ &= |\langle A\mathbf{x}, \mathbf{x}\rangle - \langle A\mathbf{x}, \mathbf{y}\rangle + \langle A\mathbf{x}, \mathbf{y}\rangle - \langle A\mathbf{y}, \mathbf{y}\rangle| \\ &= |\langle A\mathbf{x}, \mathbf{x} - \mathbf{y}\rangle + \langle A\mathbf{y}, \mathbf{x} - \mathbf{y}\rangle| \\ &\leq \|A\|\|\mathbf{x} - \mathbf{y}\| + \|A\|\|\mathbf{x} - \mathbf{y}\| \\ &\leq 2\delta\|A\|\end{aligned}$$

We can thus conclude,

$$\begin{aligned}|\langle A\mathbf{x}, \mathbf{x}\rangle| - |\langle A\mathbf{y}, \mathbf{y}\rangle| &\leq 2\delta\|A\| \\ |\langle A\mathbf{y}, \mathbf{y}\rangle| &\geq |\langle A\mathbf{x}, \mathbf{x}\rangle| - 2\delta\|A\|\end{aligned}$$

And,

$$\begin{aligned}|\langle A\mathbf{y}, \mathbf{y}\rangle| - |\langle A\mathbf{x}, \mathbf{x}\rangle| &\leq 2\delta\|A\| \\ |\langle A\mathbf{y}, \mathbf{y}\rangle| &\leq |\langle A\mathbf{x}, \mathbf{x}\rangle| + 2\delta\|A\|\end{aligned}$$

Taking supremum,

$$\begin{aligned}\sup_{\mathbf{x}\in\mathcal{N}_\delta} |\langle A\mathbf{x}, \mathbf{x}\rangle| &\geq \sup_{\mathbf{x}\in S^{n-1}} |\langle A\mathbf{x}, \mathbf{x}\rangle| - 2\delta\|A\| \\ &= (1 - 2\delta)\|A\| \\ &= (1 - 2\delta)\sup_{\mathbf{x}\in S^{n-1}} |\langle A\mathbf{x}, \mathbf{x}\rangle| \\ \sup_{\mathbf{x}\in\mathcal{N}_\delta} |\langle A\mathbf{x}, \mathbf{x}\rangle| &\leq \sup_{\mathbf{x}\in S^{n-1}} |\langle A\mathbf{x}, \mathbf{x}\rangle| + 2\delta\|A\| \\ &= (1 + 2\delta)\|A\| \\ &= (1 + 2\delta)\sup_{\mathbf{x}\in S^{n-1}} |\langle A\mathbf{x}, \mathbf{x}\rangle|\end{aligned}$$

□

X. Applications

A. Power system state estimation problem

Apart from being a broader class of problems encompassing phase retrieval, the problem setup (P1) also has applications in power system engineering. Given a network of buses and transmission lines, the goal is to estimate complex voltages across all buses from a subset of noisy power and voltage magnitude measurements. In the AC power model, these measurements are quadratically dependent on the voltage values to be determined. Let $\{c_i\}_{i=1}^m$ be the set of measurements and

$\{A_i\}_{i=1}^m$ be the corresponding bus admittance value matrices. Then the problem boils down to an estimation problem

$$\begin{aligned} &\text{find} \quad \mathbf{x} \\ &\text{s.t.} \quad c_i = \mathbf{x}^H A_i \mathbf{x} + \nu_i \quad \forall i = 1, 2, \ldots, m. \end{aligned}$$

where $\nu_i \sim \mathcal{N}(0, \sigma_i^2)$ is gaussian noise associated with the readings. [20]. For details on the problem setup, please refer [1].

B. Fusion Phase retrieval

Let $\{W_i\}_{i=1}^m$ be a set of subspace of $\mathbb{R}^n/\mathbb{C}^n$. Fusion phase retrieval deals with the problem of recovering $\mathbf{x}$ upto a phase ambiguity from the measurements of the form $\{\|P_i\mathbf{x}\|\}_{i=1}^m$, where $P_i : \mathbb{C}^n/\mathbb{R}^n \rightarrow W_i$ are projection operators onto the subspaces. [45] had the initial results on this problem with regards to the conditions on the subspaces and minimum number of such subspaces required for successful recovery of $\mathbf{x}$ under phase ambiguity.

C. X-ray crystallography

In X-ray crystallography, especially in crystal twinning [6], the measurements are obtained with orthogonal matrices $Q_i^2 = Q_i$ which again would be solved by out setup.

In the worst case, a feasibility quadratic feasibility problem can be NP-hard, which makes the setup (**??**) we address all the more interesting as we can highlight properties about a subgroup of quadratic feasibility problems and take a shot at providing provably converging algorithm for the same. This question resonates quite closely with many applications of quadratic feasibility as discussed above. In this write-up we have only considered the noiseless system, which later can be extended to noisy system analysis.