Conditional Covariance of Risky and Risk-Free Assets

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Abstract

This paper examines how the covariance between stock and bond returns changes with respect to market volatility. First, we establish a set of stylized facts that describe the qualitative nature of this relationship, and the existence of a *Tail Risk* in the conditional distribution. We construct an asset pricing model that replicates and explains this relationship in the context of a CRRA utility maximization model. The empirical strategy used is a non-parametric approach to the task of conditional covariance measurement.

We observe that covariances between stock and bond returns are decreasing with respect to market volatility up to a certain point, after which they begin increasing. This non-monotonic relationship is a consequence of how this covariance is driven by two opposing forces that we derive in the theoretical model: macroeconomic risk, which drives covariances up, and asset reallocation, which drives covariances down.

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The code used for this project is available upon request.

1 Introduction

1.1 Puzzle of Time Varying Covariances

Figure 1 below shows a graph with the sample rolling correlation between SP500 log returns and changes in the 10Y Treasury Yield. This correlation is to be interpreted as the negative of the relationship between stock and bond returns (given the monotonic decreasing relationship between changes in yield and changes in price of bonds). In other words, if the data shows a negative correlation, it implies that the correlation between stock and bond returns is positive, and vice versa. Note that this fact is relevant for all data analysis in this paper, as this paper uses yield data to examine bond returns. Figure 1 also includes the VIX index, representing the market price of volatility (in orange), scaled for the purposes of visualization.

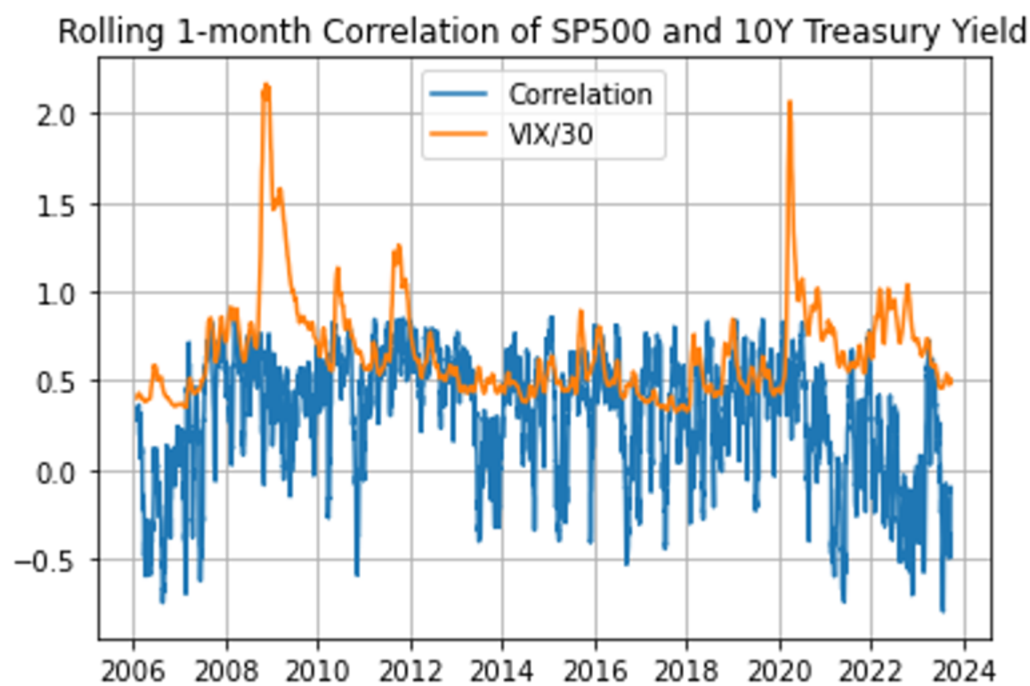

Figure 1

We observe three specific phenomena from this data that constitute different components of a puzzle with regards to the covariance and correlation structure of risky and risk-free assets.

First, we observe that correlation between stocks and bonds, represented by the blue line in Figure 1, is not stable over time. It takes values anywhere in the broad range of 0.5 to -0.5. The fact that the correlation coefficient between the asset returns does not appear to be stable around either a positive or a negative value implies that the covariance matrix of the asset returns is not stable. This is of great importance to the field of portfolio construction; the class of portfolio construction models that use Mean Variance optimization techniques, such as the Black-Litterman Model (Black and Litterman; 1992) typically depend on a stable covariance matrix as an input. If, as this graph suggests, covariance matrices do not have a static or stable value, then these models may not provide optimal portfolio constructions.

Second, we observe that there are several periods where the blue line falls below 0; in other words the covariance between stock returns and changes in bond yield is negative, which implies that the covariance between stock returns and bond returns is positive during certain periods. Why does this occur? One would expect that the covariance between stocks and bonds would be weakly negative, as the relative movement of the returns between both assets is driven by decisions of investors to reallocate funds between these assets. Any such reallocation of funds should cause a negative covariance between these asset returns. However, we observe the opposite relationship for several sustained periods.

Third, we observe that these periods of positive covariance and correlation between asset returns (visible in the graph as negative correlation between stock returns and changes in bond yield) tend to occur during periods when the VIX index, representing market volatility, has risen significantly above its usual levels. We observe this phenomenon with respect to the years immediately following the 2008 financial crisis. We also observe this more recently during the 2020-24 period, when markets became more volatile during the Covid-19 pandemic and the persistent inflation that followed it.

These three empirical facts, not well explained by existing literature, constitute the puzzle of time varying covariances that this paper seeks to tackle.

1.2 Research Question

In light of these puzzles, this paper attempts to answer two questions. First, does the correlation and covariance between stock and bond returns increase with an increase in market volatility? Second, does there exist a *Tail Risk* in this distribution, i.e., does the correlation and covariance sharply increase when market volatility crosses a certain threshold?

In order to answer these questions, this paper first develops a simple asset pricing model that can simulate and explain the stylized facts that we have observed. Second, the paper uses a non-parametric estimation method to provide empirical results that show that the answer to both of the questions above is 'yes'.

1.3 Existing Literature

First, it is well established that the correlation matrices of financial assets are unstable over time. This has been documented by Kaplanis (1988), specifically for the case of international equities. Empirical evidence from bond arbitrage desks in the 1990s, such as the infamous Long Term Capital Management, also provides evidence of cases where covariances of previously unrelated assets rise unexpectedly (Lowenstein; 2011). Furthermore, changes in covariance matrices across international equities have been linked to increases in volatility (King and Wadhwani; 1989), but not to macroeconomic changes (King et al.; 1994). In short, covariances do indeed change dynamically, but we have no conclusive answer as to what drives these changes.

The literature of changing covariance matrices of financial assets has primarily been done through the lens of GARCH (Generalized Auto-Regressive Conditional Heteroskedastic) models, and extensions such as BEKK models. These models provide for a framework to simulate the clustering nature of volatility and explain the remaining volatility through autoregressions. De Goeij (2004) observes significant and asymmetric relationships between covariances of stock and bond returns.

However, criticisms of the applications and limitations of GARCH (and BEKK) models are widespread in the econometric literature. (De Goeij; 2004, p. 7) acknowledges how "GARCH specification does not follow from any economic theory, it provides a good approximation to the heteroskedasticity typically found in financial time-series data". This paper attempts to explore the covariance matrices of asset returns in a method more grounded in economic theory, but without the same aim of simulating the phenomenon of volatility clustering.

2 Model

Our theoretical model is a three period asset pricing model. It is extremely similar to the standard two-period asset pricing model in several ways. The representative consumer solves the utility maximization problem below:

$$\max_{c_1,b_1,b_2} u(c_1) + \beta u(c_2) \tag{2.0.1}$$

$$s.t.\ c_1 + b_1 + b_2 \leq e_1 \tag{2.0.2}$$

$$c_2 \leq R^f b_1 + R^i b_2 + e_2 \tag{2.0.3}$$

where:

- c_1, c_2 denote consumption in periods 1 and 2 respectively
- b_1, b_2 respectively denote the quantities of the Risk-Free and Risky Asset purchased by the representative consumer
- e_1, e_2 denote the endowment of the representative consumer in periods 1 and 2 respectively
- $u(.)$, the utility function is a CRRA utility function; i.e., the function is of the form:

  $$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \tag{2.0.4}$$

  for some value of γ

2.1 Model Assumptions and Timeline

The most important assumption of our model is that we assume that $log\left(\frac{c_2}{c_1}\right) \sim N(0, \sigma^2)$, i.e., consumption growth follows a lognormal distribution. Intuitively, this assumption can be understood as the fact that consumption growth between both periods is completely driven by the endowments e_1, e_2, and the savings and investment allocation choices made by the representative consumer only have a marginal impact on the consumption growth. Our model extends this notion by also considering a stochastic value for σ^2. The list below describes this and other dynamics of our model:

- At t_0:
  - σ^2 is a random variable which follows a known distribution
  - R^f is a random variable whose distribution depends only on σ^2 (This fact is proved in the next section)
  - R^i, e_2 are random variables whose distributions are parameterized on σ^2

- At t_1:
  - σ^2 and R^f are both realized
  - R^i, e_2 are random variables that have the realized values of σ^2 as a parameter
  - The representative consumer makes their choice of c_1, b_1, b_2
- At t_2:
  - R^i, e_2 are realized

2.2 Solving the Model

Constructing the Lagrangian for this constrained maximization problem and taking the First Order Conditions leads to the standard Euler Equations with both the Risk-Free Rate, R^f, and the Rate of Return on the Risky Asset, R^i:

$$1 = \beta \mathbb{E}_1 \left[\left(\frac{c_2}{c_1} \right)^{-\gamma} R^f \right] \tag{2.2.1}$$

$$1 = \beta \mathbb{E}_1 \left[\left(\frac{c_2}{c_1} \right)^{-\gamma} R^i \right] \tag{2.2.2}$$

Note that $\mathbb{E}_i[.], \mathbb{V}_i[.], \mathbb{C}_i[.]$ is notation used in this paper to denote Expectation, Variance, and Covariance with respect to the set of information known at period t_i, where $i \in \{0, 1, 2\}$.

After defining the stochastic discount factor as $m = \beta \left(\frac{c_2}{c_1} \right)^{-\gamma}$, we can rewrite the equations above as:

$$1 = \mathbb{E}_1[mR^f] \tag{2.2.3}$$

$$\Rightarrow 1 = R^f \mathbb{E}_1[m] \tag{2.2.4}$$

$$1 = \mathbb{E}_1[mR^i] \tag{2.2.5}$$

Where equation (2.2.4) is a result of the fact that R^f is a constant and known to the representative consumer at t_1.

Using a known proof (Cochrane; 2005, p. 10-12) (which depends on our earlier assumption of lognormal consumption growth), we arrive at the following formula that determines the exact value of R^f at t_1 as a function of the model parameters:

$$log(R^f) = e^{-\beta} + \gamma \mathbb{E}_1 \left[log\left(\frac{c_2}{c_1}\right) \right] - \frac{\gamma^2}{2}\sigma^2 \tag{2.2.6}$$

We observe that in period t_1, when σ^2 is a constant, R^f is also a constant. However, in period t_0, when σ^2 is a random variable, this formula indicates that R^f must also be a random variable whose distribution depends on that of σ^2. This implies that:

$$\mathbb{V}_1[log(R^f)] = 0 \tag{2.2.7}$$

$$\mathbb{V}_0[log(R^f)] = \frac{\gamma^4}{4}\mathbb{V}_0[\sigma^2] \tag{2.2.8}$$

Hence, we obtain the result that at t_0, the variance of $log(R^f)$, and consequently, of R^f, is increasing in $\mathbb{V}_0[\sigma^2]$.

Setting this result aside, we proceed with equation (2.2.4) and (2.2.5) from earlier, and we obtain the standard result that describes the relationship between the expected return of the Risky Asset, the Risk-Free Rate, and the Risk Premium:

$$\begin{aligned} 1 &= \mathbb{E}_1[mR^i] && (2.2.9)\\ &= \mathbb{C}_1[m, R^i] + \mathbb{E}_1[m]\mathbb{E}_1[R^i] && (2.2.10)\\ \Rightarrow \mathbb{E}_1[R^i] &= \frac{\mathbb{E}_1[mR^i]}{\mathbb{E}_1[m]} - \frac{\mathbb{C}_1[m, R^i]}{\mathbb{E}_1[m]} && (2.2.11)\\ \mathbb{E}_1[R^i] &= R^f - \frac{\mathbb{C}_1[m, R^i]}{\mathbb{E}_1[m]} && (2.2.12)\\ \mathbb{E}_1[R^i] &= R^f + \left(\frac{\mathbb{C}_1[m, R^i]}{\mathbb{V}_1[m]}\right)\left(\frac{-\mathbb{V}_1[m]}{\mathbb{E}_1[m]}\right) && (2.2.13)\\ \mathbb{E}_1[R^i] &= R^f + \beta_{i,m}\lambda_m && (2.2.14) \end{aligned}$$

Where equation (2.2.12) follows from equation (2.2.9) combined with equation (2.2.4) from earlier. This result, well known in the asset pricing literature, expresses the expected return of the Risky Asset as being the rate of return of Risk-Free Asset and a Risk Premium, expressed by $\beta_{i,m}\lambda_m$.

Denote the variable $\epsilon = R^i - R^f + \beta_{i,m}\lambda_m$. Intuitively, ϵ can be understood to represent random noise that is independent of σ^2. By the equation (2.2.14) above, we have

$$\mathbb{E}_1[\epsilon] = 0 \tag{2.2.15}$$
$$\Rightarrow \mathbb{E}_0[\epsilon|\sigma^2] = 0 \tag{2.2.16}$$
$$\Rightarrow \mathbb{E}_0[\epsilon] = 0 \tag{2.2.17}$$

In a similar fashion, we expand equation (2.2.14) to our period t_0. Since the only change between t_0 and t_1 is the realization of σ^2, we can rewrite equation (2.2.14) as:

$$\mathbb{E}_0[R^i|\sigma^2] = \mathbb{E}_0[R^f|\sigma^2] + \mathbb{E}_0[\beta_{i,m}\lambda_m|\sigma^2] \tag{2.2.18}$$
$$\Rightarrow \mathbb{E}_0[R^i] = \mathbb{E}_0[R^f] + \mathbb{E}_0[\beta_{i,m}\lambda_m] \tag{2.2.19}$$

2.3 Deriving the Covariance between R^i and R^f

Using these results, we can now evaluate the term $\mathbb{C}_0[R^i, R^f]$, the covariance between the returns of the risky asset and the risk free asset.

$$\mathbb{C}_0[R^i, R^f] = \mathbb{C}_0[R^f, R^f] + \mathbb{C}_0[R^f, \beta_{i,m}\lambda_m] + \mathbb{C}_0[R^f, \epsilon] \tag{2.3.1}$$
$$\mathbb{C}_0[R^i, R^f] = \mathbb{V}_0[R^f] + \mathbb{C}_0[R^f, \beta_{i,m}\lambda_m] \tag{2.3.2}$$

This step requires the assumption that $\mathbb{C}_0[R^f, \epsilon] = 0$. Since the only stochastic component of R^f is σ^2, this is equivalent to assuming that $\epsilon \perp \sigma^2$. This assumption claims that σ^2 affects R^i, the Rate of return on the Risky Asset, solely through its effect on the Risk Free Rate (R^f) and the Risk Premium ($\beta_{i,m}\lambda_m$).

We examine the two terms on the right hand side of equation (2.3.2). $\mathbb{V}_0[R^f]$ is known to be increasing in $\mathbb{V}_0[\sigma^2]$. This is implied by equation (2.2.8) earlier. The second term, $\mathbb{C}_0[R^f, \beta_{i,m}\lambda_m]$ is decreasing in $\mathbb{V}_0[\sigma^2]$, under the assumption that the correlation coefficient between m and R^i is strictly negative. We provide the proof for this below:

2.4 Proof that $\mathbb{C}_0[R^f, \beta_{i,m}\lambda_m]$ is decreasing in $\mathbb{V}_0[\sigma^2]$

Assumption: We assume that the correlation coefficient between m and R^i is strictly negative. This assumption effectively states that R^i is positively correlated with consumption growth. Since m is related to the expected Marginal Rate of Substitution between consumption in t_2 and consumption in t_1, m is larger when consumption growth is smaller and vice versa. Therefore, a negative correlation between R^i and m represents a positive correlation between R^i and consumption growth. Intuitively, R^i represents the Rate of Return on an equity that is positively correlated to macroeconomic growth.

Step I: We first evaluate the term $\mathbb{E}_1[m]$:

$$\mathbb{E}_1[m] = \beta\mathbb{E}_1\left[\left(\frac{c_2}{c_1}\right)^{-\gamma}\right] \tag{2.4.1}$$

$$= \beta exp\left(\mathbb{E}_1[-\gamma log\left(\frac{c_2}{c_1}\right)] + \frac{\gamma^2\sigma^2}{2}\right) \tag{2.4.2}$$

$$= \beta exp\left(\frac{\gamma^2\sigma^2}{2}\right) \tag{2.4.3}$$

Step II:

$$m = \beta\left(\frac{c_2}{c_1}\right)^{-\gamma} \tag{2.4.4}$$

$$log(m) = log(\beta) - \gamma log\left(\frac{c_2}{c_1}\right) \tag{2.4.5}$$

$$\mathbb{V}_1[log(m)] = \gamma^2\sigma^2 \tag{2.4.6}$$

Furthermore, from the first order Taylor expansion of $\mathbb{V}_1[log(m)]$:

$$\mathbb{V}_1[log(m)] \approx \frac{\mathbb{V}_1[m]}{\mathbb{E}_1[m]^2} \tag{2.4.7}$$

$$\mathbb{V}_1[m] \approx \mathbb{V}_1[log(m)]\mathbb{E}_1[m]^2 \tag{2.4.8}$$

Step III:

$$\beta_{i,m}\lambda_m = -\frac{\mathbb{C}_1[m, R^i]}{\mathbb{E}_1[m]} \tag{2.4.9}$$

$$= -\frac{\rho\mathbb{V}_1[R^i]\mathbb{V}_1[m]}{\mathbb{E}_1[m]} \tag{2.4.10}$$

$$\approx -\rho\mathbb{V}_1[R^i]\mathbb{V}_1[log(m)]\mathbb{E}_1[m] \tag{2.4.11}$$

$\mathbb{V}_1[R^i], \mathbb{V}_1[log(m)], \mathbb{E}_1[m]$ are all increasing in σ^2, and we assume $\rho < 0$.
Therefore, $\beta_{i,m}\lambda_m$ must be increasing in σ^2. Consequently, $\mathbb{V}_0[\beta_{i,m}\lambda_m]$ is increasing in $\mathbb{V}_0[\sigma^2]$.

Step IV:

$$log(R^f) = e^{-\beta} + \gamma \mathbb{E}_1 \left[log \left(\frac{c_2}{c_1} \right) \right] - \frac{\gamma^2}{2}\sigma^2 \tag{2.4.12}$$

From this equation, we know that the only stochastic component of R^f is σ^2, and R^f is decreasing in σ^2. Since, $\beta_{i,m}\lambda_m$ is increasing in σ^2, this implies that the correlation coefficient between $\beta_{i,m}\lambda_m$ and R^f, denoted ρ_2, must be negative.

Step V:

$$\mathbb{C}_0[R^f, \beta_{i,m}\lambda_m] = \rho_2 \mathbb{V}_0[R^f]\mathbb{V}_0[\beta_{im}\lambda_m] \tag{2.4.13}$$

- $\rho_2 < 0$
- $\mathbb{V}_0[R^f]$ is increasing in $\mathbb{V}_0[\sigma^2]$
- $\mathbb{V}_0[\beta_{im}\lambda_m]$ is increasing in $\mathbb{V}_0[\sigma^2]$

This implies that $\mathbb{C}_0[R^f, \beta_{i,m}\lambda_m]$ is decreasing in $\mathbb{V}_0[\sigma^2]$. Hence Proved.

2.5 Interpretation of Equation

In summary, we have the result:

$$\mathbb{C}_0[R^i, R^f] = \mathbb{V}_0[R^f] + \mathbb{C}_0[R^f, \beta_{i,m}\lambda_m] \tag{2.5.1}$$

where the first right hand side term is increasing in $\mathbb{V}_0[\sigma^2]$, and the second right hand side term $\mathbb{C}_0[R^f, \beta_{i,m}\lambda_m]$ is decreasing in $\mathbb{V}_0[\sigma^2]$. Both of these phenomena can be interpreted in the context of the representative consumer's behavior in response to changing conditions of volatility:

- The fact that $\mathbb{V}_0[R^f]$ is increasing in $\mathbb{V}_0[\sigma^2]$ is a representation of how *variance of volatility* in consumption growth causes volatility in R^f and $\mathbb{E}[R^i] = R^f + \beta_{i,m}\lambda_m$. This phenomenon is essentially macroeconomic risk; it is the variance in σ^2 that adds uncertainty to both R^f and R^i, causing stocks and bonds to move in the same direction.

- That $\mathbb{C}_0[R^f, \beta_{i,m}\lambda_m]$ is decreasing in $\mathbb{V}_0[\sigma^2]$ is the phenomenon of consumers reallocating their portfolio, and therefore substituting between Risky and Risk Free goods. This action causes a negative covariance of returns.

In short, there are two phenomena that drive the covariance of risky and risk-free asset returns with respect to volatility. First, there is a macroeconomic risk that jointly affects the returns of both assets, driving the covariance between the assets up. Second, there is the effect of reallocation; a change in market conditions causes the representative consumer to reallocate their portfolio by buying bonds and selling stocks or vice versa, and in doing so, they drive down the covariance between the assets.

3 Econometric Model

3.1 Motivation for a Non-Parametric Estimator

Equation (2.5.1) is provides an elegant and intuitive decomposition of the covariance between the rate of return of Risky and Risk-Free Assets. However, it does not provide a closed form solution for the equation as a function of $\mathbb{V}_0[\sigma^2]$. Considering the fact that this equation suggests that the relationship between the covariance and volatility (as we shall term $\mathbb{V}_0[\sigma^2]$) consists of these two opposing forces, it is likely that the relationship is non-monotonic. Indeed, we shall observe from our empirical results that the *Tail Risk* phenomena described earlier can be understood in the context of one force dominating the other beyond a certain threshold of volatility, leading to a reversal in the direction of the relationship. Suffice it to say, the structure of this non-monotonic relationship is best suited to be estimated using a non-parametric method that relies only on the weak assumption of continuity of the function.

3.2 Deriving the Estimator

In the setup of our non-parametric estimator, we define the following values:

- $X = (X_1, X_2)^T \in \mathbb{R}^2$ is the random vector of asset returns (risk-free and risky)
- $U \in \mathbb{R}$ is the random scalar representing our independent variable, volatility
- X has a mean $m(u) = [m_1(u), m_2(u)]$, where $m(u)$ is a continuous function
- X has a covariance matrix $\Sigma(u) = \sigma_{ij}(u) \in \mathbb{R}^{2\times 2}$, where $\sigma_{ij}(u)$ is the continuous function that represents the covariance between X_i, X_j conditional on $U = u$

We use an Nadaraya-Watson estimator for both $m(.)$ and $\Sigma(.)$. This model is identical to the one derived in Yin, Geng, Li, and Wang (2010).

We seek to maximize the likelihood of the model:

$$\mathcal{L}(m, \Sigma) = (2\pi)^{-\frac{np}{2}} det(\Sigma(u))^{-\frac{n}{2}} \prod_{i=1}^{n} exp\left[-\frac{1}{2}(X_i - m(u))^T \Sigma^{-1}(u)(X_i - m(u))\right] \tag{3.2.1}$$

Where $m(u)$ and $\Sigma(u)$ are the mean and covariance of X conditional on u. Converting this to log-likelihood and aiming to find the log-likelihood using the kernel method leads to us to minimize:

$$\sum_{i=1}^{n} \left[[X_i - m(u)]^T \Sigma^{-1}(u) [X_i - m(u)] - log\left(det\left(\Sigma^{-1}(u)\right)\right)\right] K\left(\frac{U_i - u}{h}\right) \tag{3.2.2}$$

By minimizing this objective function, we obtain the following Nadaraya-Watson Kernel Estimators:

$$\hat{m}(u) = \frac{\sum_{i=1}^{n} K\left(\frac{U_i - u}{h}\right) X_i}{\sum_{i=1}^{n} K\left(\frac{U_i - u}{h}\right)} \tag{3.2.3}$$

$$\hat{\Sigma}(u) = \frac{\sum_{i=1}^{n} K\left(\frac{U_i - u}{h}\right) [X_i - \hat{m}(u)][X_i - \hat{m}(u)]^T}{\sum_{i=1}^{n} K\left(\frac{U_i - u}{h}\right)} \tag{3.2.4}$$

Note that h denotes the bandwidth in our model. We choose h by minimizing the Leave One Out Cross Validation Score.

4 Data

4.1 Equity Data

We use data from Yahoo Finance regarding equity returns of various indices, primarily the SP500, but also the NASDAQ, the Dow Jones Index, and the Russell 3000 index, for our results ahead. We calculate the log returns of the data at monthly intervals. Figure 2 below describes the distribution of the data for the SP500 along with the trends in the data over time. As one would expect, the distribution is qualitatively similar for other equity indices.

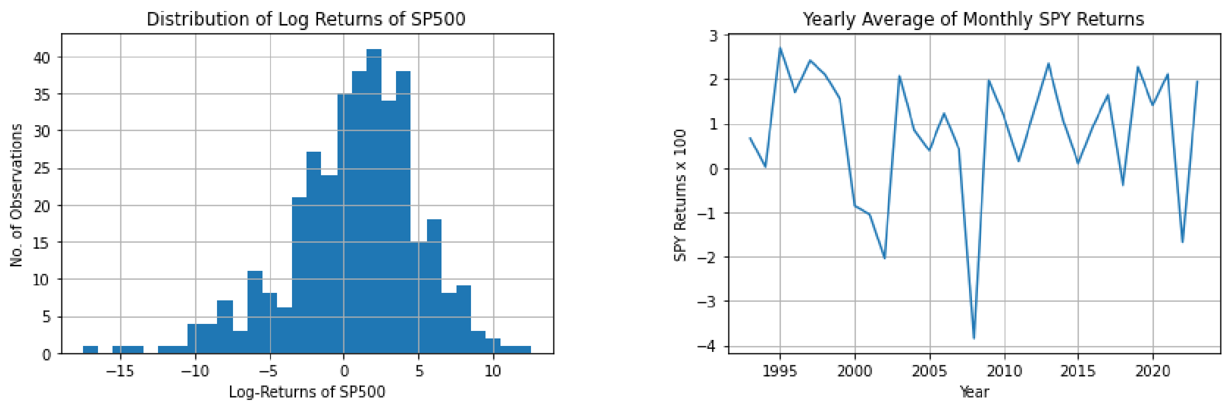

Figure 2: Summary Distribution Figures for Equity Data

4.2 Bond Data

We use data from Yahoo Finance regarding changes in yields of various tenors of USD bonds, primarily the 10Y, but also the 5Y and the 30Y for our results ahead. We take the data at a monthly frequency. Figure 3 below describes the distribution of the data for the 10Y bond along with the trends in the data over time. The distribution is qualitatively similar for the 5Y and the 30Y bond data. Note that an increase (or decrease) in the yield of a bond reflects a reduction (or an increase) in the price of the bond, and therefore a negative (or positive) return on the bond over that period. By using yield differences as our data source, we are effectively using the negative of the bond returns. As a result of this, when we observe a positive (negative) covariance in the data, it reflects a negative (or positive) covariance between stock and bond returns.

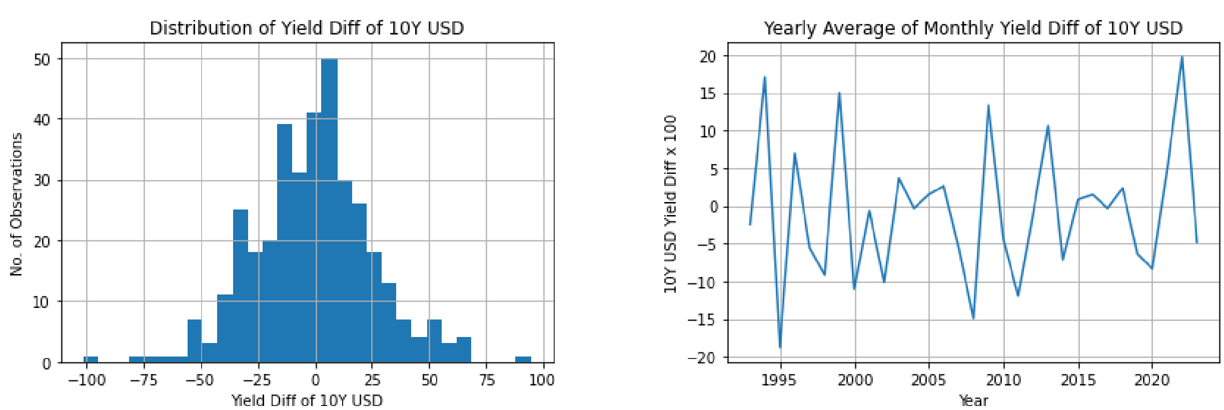

Figure 3: Summary Distribution Figures for Bond Data

4.3 Volatility Data

In order to measure volatility, we use the VIX volatility index (data from Yahoo Finance). Similar to the data sources described above, we take the monthly frequency of VIX. For our calculations, we use the log-levels of VIX. Figure 4 below describes the distribution of the data as well as its trends over time.

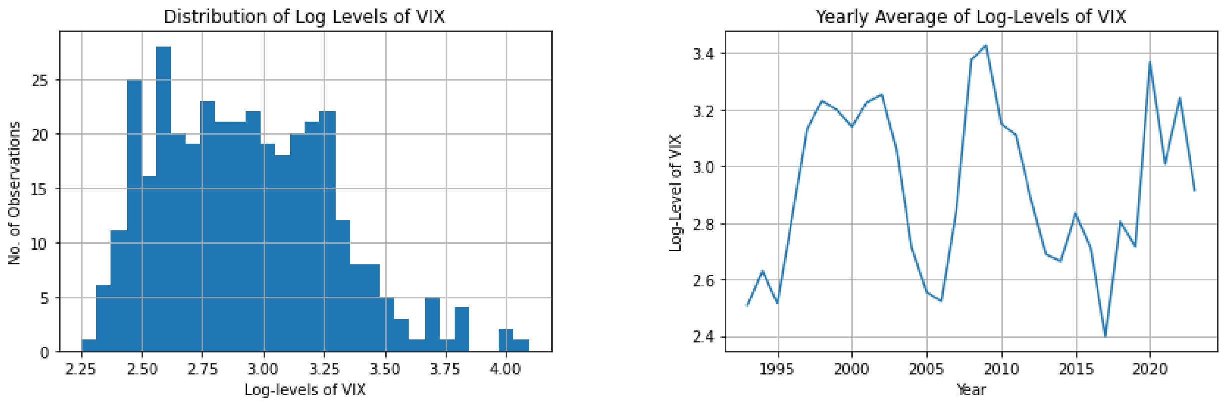

Figure 4: Summary Distribution Figures for Volatility Data

4.4 Summary Statistics

Table 1 below includes summary statistics for all the equity data mentioned. Note that the equity data in the summary statistics has been scaled by a factor of 100, to provide clear visualization of the distribution and trends in the data.

Similarly, Table 2 below includes summary statistics for the bond and volatility data mentioned. The changes in bond yield data is measured in basis points. The volatility data is measured in log-levels of VIX.

5 Results

5.1 Graphs of Covariances and Correlations

The figures below describe the results from calculating the conditional covariance and correlation between the monthly returns of the relevant equity index and monthly changes in yield of the relevant tenor of USD bond, with respect to log-levels of VIX. The graphs show these results with the true values of VIX, rather than their log-levels for the purpose of an easier

Table 1: Summary Statistics for Equity Returns Data

	SP500	Dow Jones	NASDAQ	Russell 3000
count	363.00	363.00	363.00	363.00
mean	0.79	0.65	0.84	0.64
std	4.41	4.31	6.37	4.47
min	-17.48	-16.41	-26.01	-19.58
25%	-1.63	-1.66	-2.02	-1.94
50%	1.41	1.02	1.50	1.28
75%	3.64	3.33	4.46	3.43
max	12.54	13.06	19.87	12.32

Table 2: Summary Statistics for Bond and Volatility Data

	10Y USD	5Y USD	30Y USD	VIX
count	363.00	363.00	363.00	363.00
mean	-0.66	-0.40	-0.84	2.93
std	25.78	27.31	22.83	0.35
min	-101.30	-87.70	-88.20	2.25
25%	-15.70	-15.15	-15.00	2.63
50%	0.50	-0.10	-1.00	2.90
75%	14.45	15.25	14.10	3.18
max	94.60	86.80	91.20	4.09

understanding of the relationship between market volatility, covariance, and correlation.

These figures also display the 95% confidence intervals in the shaded region. These confidence intervals were calculated using bootstrapping methods. We observe that for large values of VIX, the confidence intervals become significantly broader. This is a natural consequence of how there are fewer datapoints for larger values of VIX.

5.2 Observations on Empirical Results

With regards to every single equity-bond pairing that we observe in Figures 5 through 10, there are two main qualitative similarities, each of which yields an intuitive explanation that fits in with the theoretical framework given Section 2.

- *Low and Decreasing values of Covariance and Correlation of returns for VIX less than $\approx$ 30:* In all of the graphs, we observe that for low values of VIX, the covariance and correlation of the data are close to zero, and increasing in VIX until VIX reaches

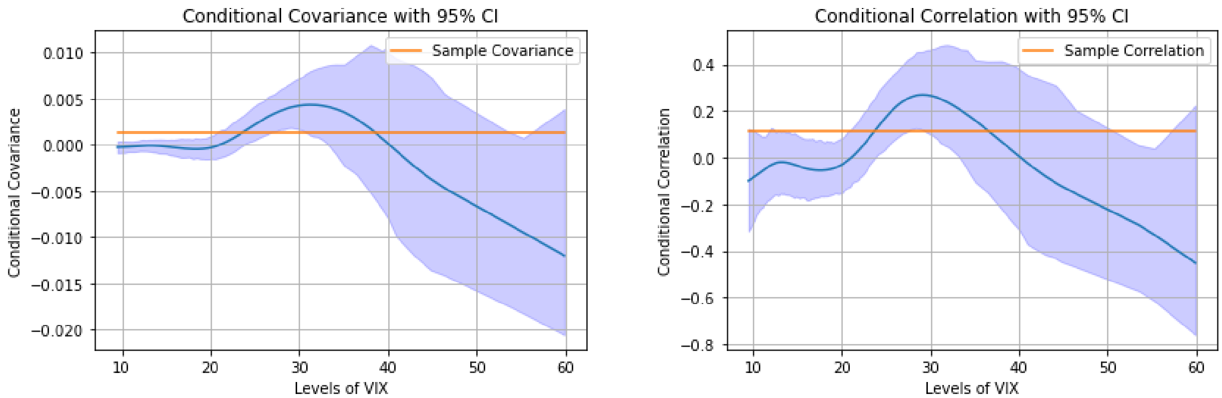

Figure 5: Conditional Covariance (Left) and Correlation (Right) of SP500 and 10Y USD

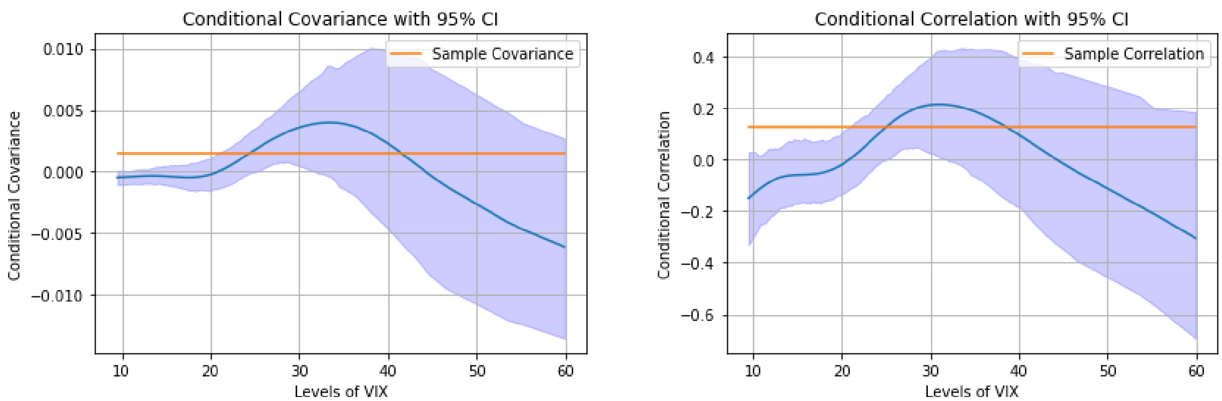

Figure 6: Conditional Covariance (Left) and Correlation (Right) of SP500 and 5Y USD

approximately 30. That is the point in most graphs where the blue line, representing covariances or correlations, reaches a peak. Since our data uses changes in bond yields, this observation reflects the phenomenon of covariances and correlations between stock and bond returns decreasing (and taking on negative values) for values of VIX less than approximately 30.

- *Increasing Covariance and Correlation of returns after VIX exceeds $\approx$ 30:* After the covariances and correlations in the data reach their peak at a point that corresponds to a value of VIX around 30, we observe a reversal of the trend. The covariances and correlations in the data decrease rapidly with increases in VIX and take on large negative values. This observation reflects how covariances and correlations between stock and bond returns increases (and takes on large positive values) as VIX increases beyond approximately 30.

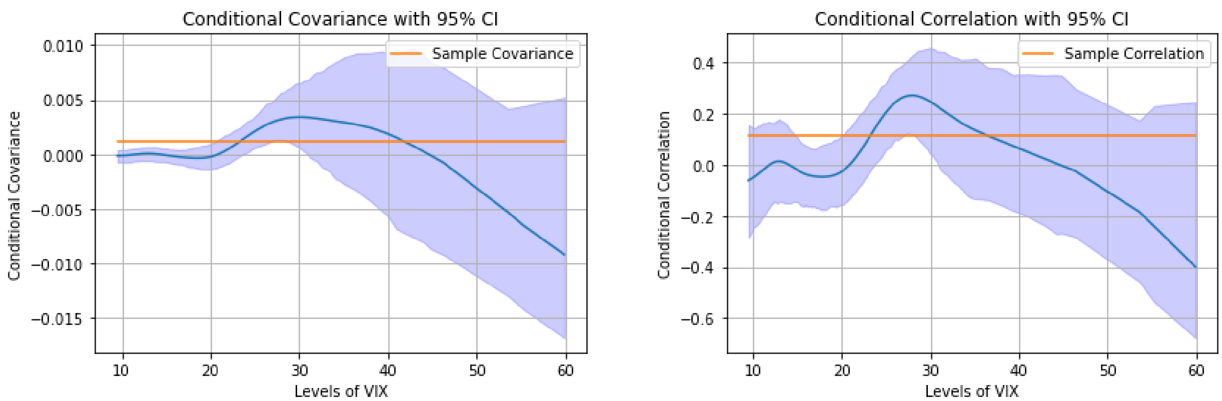

Figure 7: Conditional Covariance (Left) and Correlation (Right) of SP500 and 30Y USD

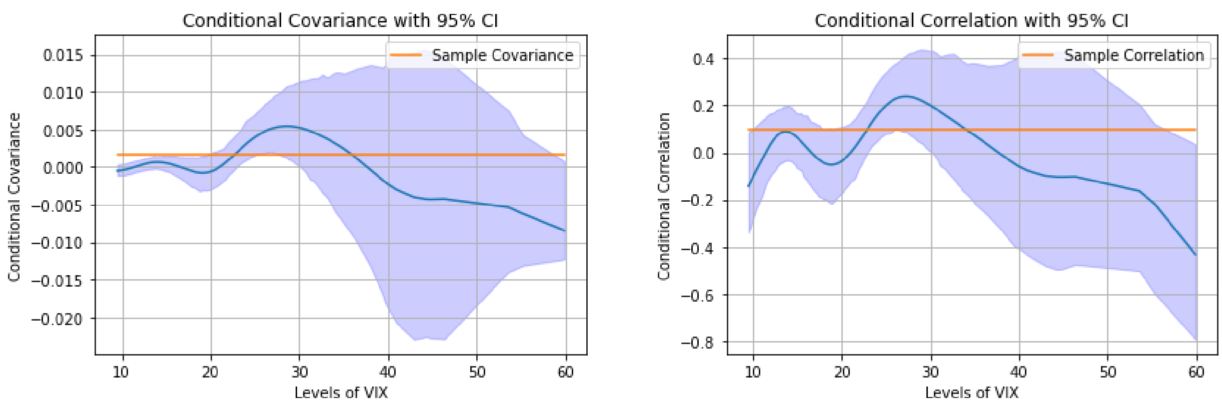

Figure 8: Conditional Covariance (Left) and Correlation (Right) of NASDAQ and 10Y USD

5.3 Interpretation of Results

We can interpret these results using our decomposition of $\mathbb{C}[R^i, R^f]$ described earlier. The covariance between the stock and bond returns is driven upwards by the force of macroeconomic risk, and the downwards by the force of portfolio reallocation. While VIX lies in a 'normal' range, i.e., below 30, we observe the latter force dominate the former. An increase in VIX prompts the representative consumer to optimally reallocate their portfolio between stocks and bonds, which leads a negative covariance and correlation between the assets.

However, as VIX exceeds this value, and there is increased uncertainty in the distribution of R^f, R^i, and future period consumption, the opposite occurs. The former force, of macroeconomic risk dominates, and increases in market volatility correspond with an increase in the covariance and correlation between stock and bond returns.

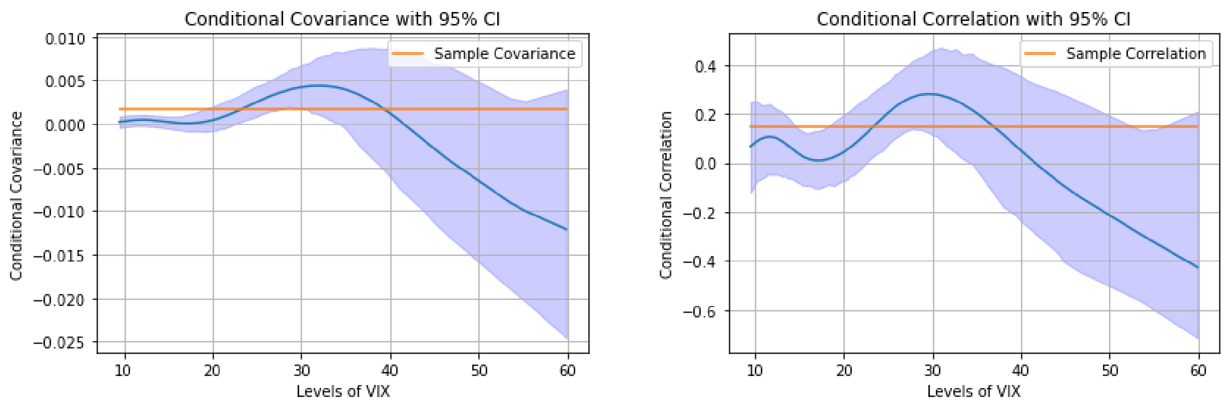

Figure 9: Conditional Covariance (Left) and Correlation (Right) of DJIA and 10Y USD

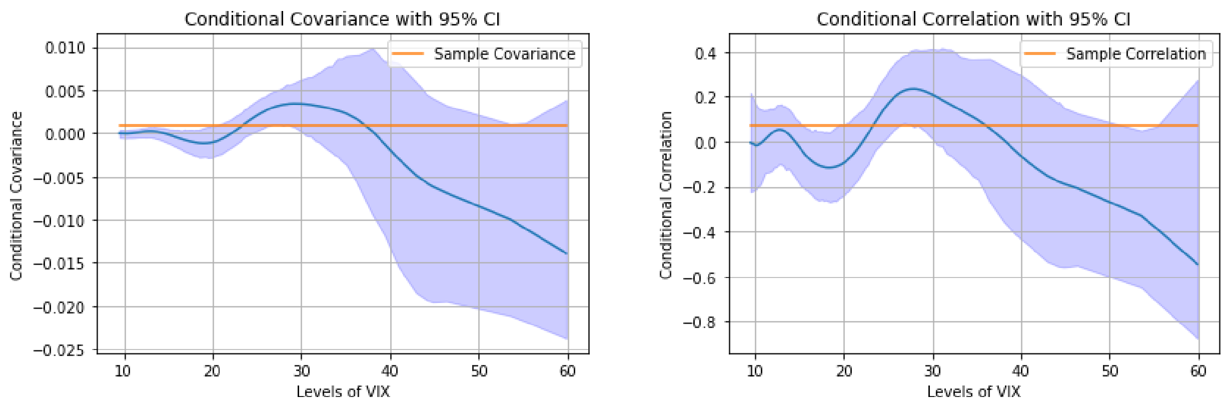

Figure 10: Conditional Covariance (Left) and Correlation (Right) of Russell 3000 and 10Y USD

5.4 Simulation vs Empirical

We attempt to connect further the link between the theoretical results and the empirical results we have observed here. We find that with some additional assumptions, we can approximate our Covariance expression as follows:

$$\mathbb{C}_0[R^f, R^i] \approx \alpha \mathbb{V}_0[\sigma^2](1 - f(\mathbb{V}_0[\sigma^2])) \tag{5.4.1}$$

where $f(.)$ is a convex quadratic. When we input suitable coefficients to construct the function and the other parameters of the expression above, we can replicate qualitatively the results we observe in our empirical data. Figure 11 below compares a theoretical simulation of $\mathbb{C}_0[-R^f, R^i]$ with the empirical result from Figure 5. Note that we use $-R^f$ rather than R^f so that our theoretical simulation can show the same directional relationship as our empirical results, which use yields rather than returns. The derivation and inputs of this theoretical simulation are described in further detail in the Appendix.

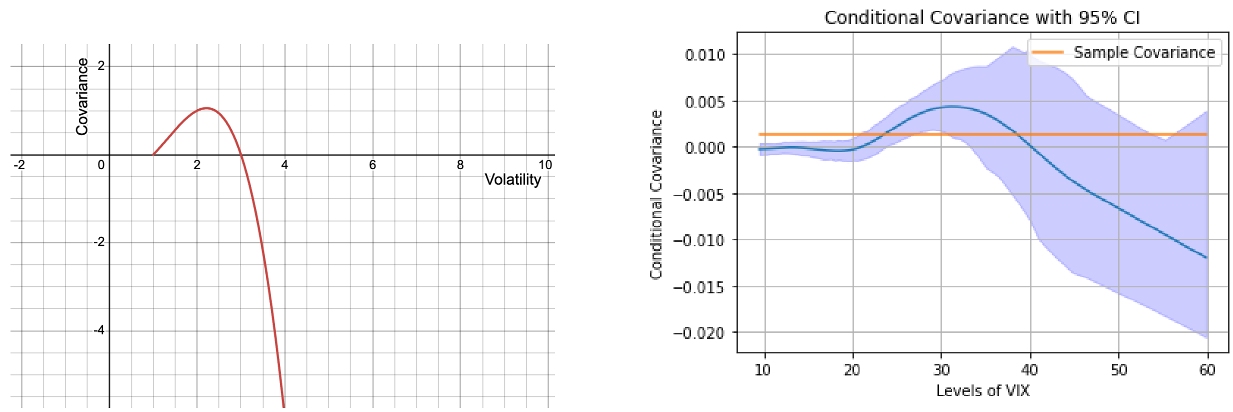

Figure 11: Theoretical vs Empirical Conditional Covariances between Stock Returns and Changes in Bond Yields

We observe that the theoretical form of our decomposition of $\mathbb{C}_0[-R^f, R^i]$ can produce results that are qualitatively similar to the empirical results. We observe a similar pattern of covariances rising with volatility until a peak, followed by a sharp fall in the covariances for all further increases in volatility. Quantitatively, the simulation does not quite match the empirical results; this is likely a combination of several factors:

- First, the relationship between changes in yields and returns is not linear; this might cause distortions between our theoretical result which uses returns (negative of bond returns in the graph), and our empirical results, which use changes in bond yields
- Second, although we treat VIX as the empirical analogue to volatility, VIX represents the market implied volatility of a specific quantity of equities, specifically of 1 unit of the SP500 index. $\mathbb{V}_0[\sigma^2]$ may be measured using VIX, but it may be that there is a significant scale difference between VIX and our theoretical $\mathbb{V}_0[\sigma^2]$.
- Third, our construction of a functional form for this covariance expression is dependent on a combination of several strong assumptions, which may distort the data, as well as inputted parameters, which may play a role in scaling and distorting the relationship to some extent

However, despite all of these reasons that could cause a disconnect between any theoretically driven simulation and our empirical results, we still observe important qualitative features of our empirical results replicated in the theoretical simulation. This is highly reassuring as it provides a confirmation of the intuition that we derive from our theoretical approach.

6 Conclusion

6.1 Returning to the Research Question

Our research question, described in Section 1.2, had two components, both of which have been explored in this paper. The first component of the research question was regarding the relationship between the covariance of stock and bond returns and market volatility. Our theoretical work in Section 2 provides an asset pricing framework that arrives at an expression for this conditional covariance, as well as intuition for what economic forces cause the covariance to increase or decrease with market volatility.

The second component of the research question was regarding the phenomenon of Tail Risk. What happens to covariances and correlations when market volatility reaches extremely high values? Our empirical results show us that once VIX exceeds values of around 30, i.e., once it reaches relatively high levels, the force of macroeconomic risk drives the covariance between assets upwards. This force only becomes stronger as markets become more volatile, and this leads to large and rapidly increasing values of covariance and correlation of stock and bond returns. In this way, we gain a stronger understanding of this 'Tail Risk' phenomena, and how such circumstances can lead to significantly different covariance and correlations than would occur normally.

6.2 Implications of Research

The results of this paper have important implications for the field of portfolio construction. Our model suggests that if an investor is able to optimally reallocate their portfolio according to changes in $\mathbb{V}_0[\sigma^2]$, then their portfolio can always be optimally adjusted in response to changes in market volatility. However, that may not be a given for all investors. Investigating these cases can produce further interesting research.

An investor who is unable to reallocate their portfolio according to changes in market volatility, perhaps as a result of transaction costs, regulatory conditions, a lack of knowledge of changes in market volatility, and other financial frictions, might find their portfolio vulnerable to having significantly more variance in response to changes in volatility. Constructing a model of an investor constrained in this manner could potentially yield important insights.

The observations regarding high values of VIX sheds light on the phenomenon of 'Tail Risk' and how financial crises occur. Understanding the reasons that drive this 'Tail Risk' phenomena, as well as the effects it has on financial markets is an important field that we hope this paper has provided valuable insight into.

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7 Appendix

7.1 Derivation and Construction of Simulation

Simplifying our equation on the covariance between Risky and Risk-Free Assets:

$$\mathbb{C}_0[R^f, R^i] = \frac{\gamma^4}{4}\mathbb{V}_0[\sigma^2](1 + \rho_2\mathbb{V}_0[\beta_{i,m}\lambda_m])$$

$\mathbb{V}_0[\beta_{i,m}\lambda_m]$ can be approximated as a convex quadratic function of $\mathbb{V}_0[\sigma^2]$. With suitable parameter inputs, we obtain a graph that is similar in terms of directional relationships to the empirical results:
We assume that $\mathbb{V}_1[R^i]$ is independent of $\mathbb{V}_0[\sigma^2]$. We also continue with the finding that $\rho_2 < 0$ based on the proof in Section 2.4.

$$\beta_{im}\lambda_m \approx \rho\mathbb{V}_1[R^i]\mathbb{V}_1[ln(m)]\mathbb{E}_1[m] \tag{7.1.1}$$

$$\mathbb{V}_1[ln(m)] = \gamma^2\sigma^2 \tag{7.1.2}$$

$$\mathbb{E}_1[exp(ln(m))] \approx exp(\mathbb{E}_1[ln(m)]) + \frac{exp(\mathbb{E}_1[ln(m)])}{2}\mathbb{V}_1[ln(m)] \tag{7.1.3}$$

$$= ln(\beta) + \frac{ln(\beta)}{2}\gamma^2\sigma^2 \tag{7.1.4}$$

$$\Rightarrow \beta_{im}\lambda_m \approx \rho\mathbb{V}_1[R^i](\gamma^2\sigma^2)(ln(\beta) + \frac{ln(\beta)}{2}\gamma^2\sigma^2) \tag{7.1.5}$$

$$= \rho\mathbb{V}_1[R^i]ln(\beta)[\frac{\gamma^4\sigma^4}{2} + \gamma^2\sigma^2] \tag{7.1.6}$$

$$\approx f(\mathbb{V}[\sigma^2]) \tag{7.1.7}$$

where $f(.)$ is a convex quadratic function.

7.2 Leave One Out Cross Validation

Figure 12 below depicts the methodology by which the bandwidth, h, for the non-parametric estimator is chosen. This figure is taken from the analysis of the covariance of SP500 and the 10Y USD. The LOOCV graphs with respect to bandwidth are very similar for all other stock-bond pairings analyzed in this paper. We calculate the Leave One Out Cross Validation Score with the same methodology as Yin et al. (2010).

Figure 12: LOOCV Score (y-axis) against bandwidth h (x-axis)