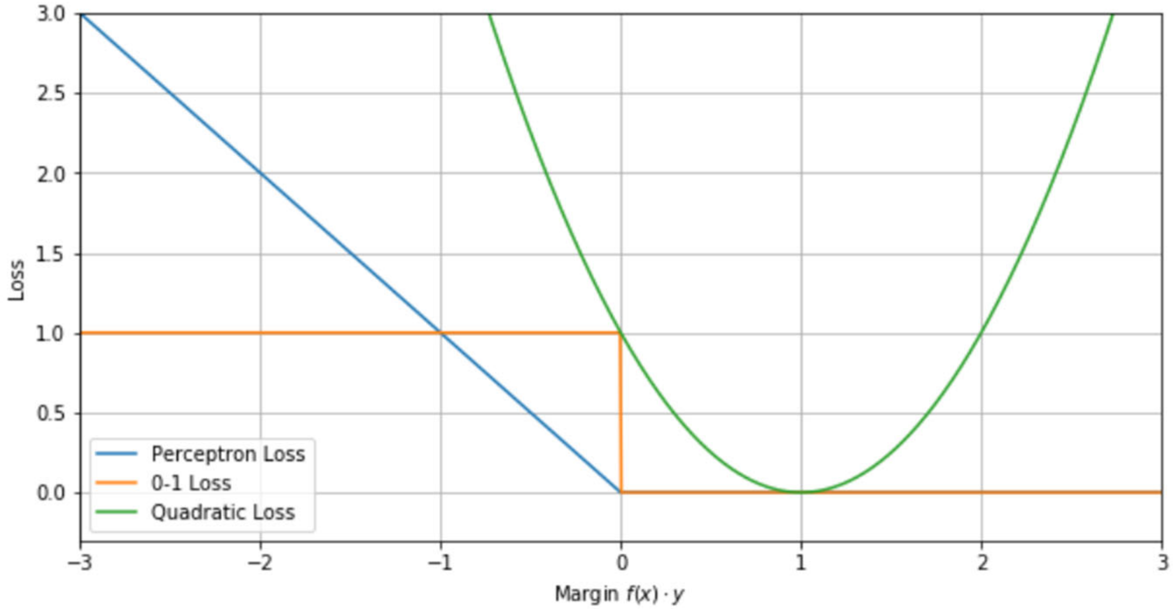
Questions

ADALINE Algorithm

- Questions on Piazza?
  - Please provide your feedback
- Question for You!

Could square Loss be used for Classification?



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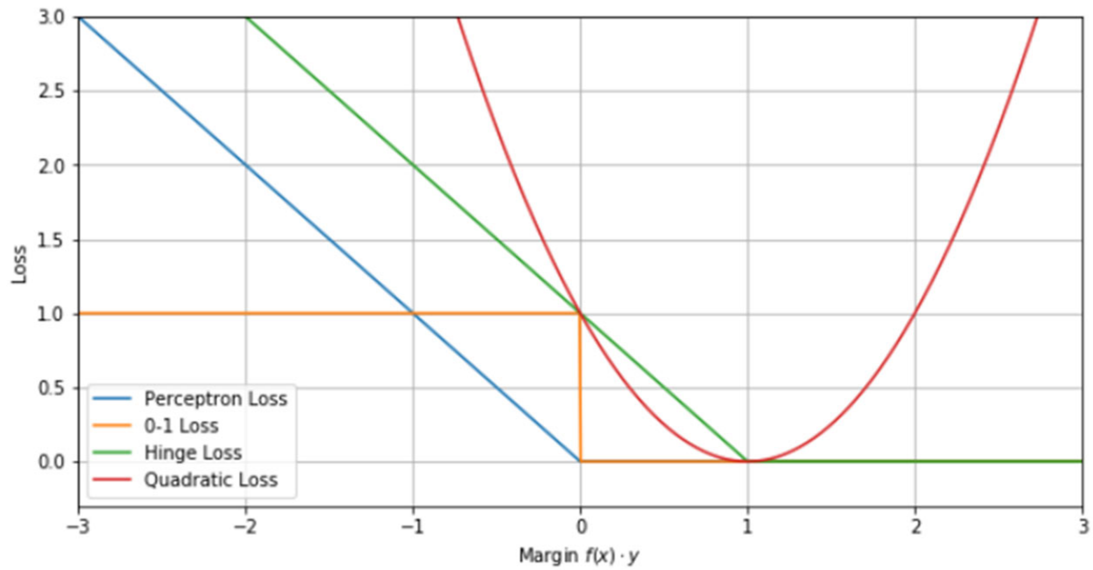


Center for Data Science
NYU

DS-GA 1003
Machine Learning

Week 5: Lecture 5

Support Vector Machines – Margin Based Classifiers

NYU Center for Data Science

DS-GA 1003
Machine Learning

How should we incorporate regularization into perceptron?

Week 5: Lecture 5

Support Vector Machines – Margin Based Classifiers

Adapted from Rosenberg, Miolane, Sontag, Rudin

Announcements

- Please check Week 5 agenda on NYU Classes
  - Homework 3
  - Midterm
  - Recordings
- Remember to post to Piazza

Check Calendar linked to NYU Classes for important dates

Review

- General minimization problems with constraints take the form

$$\text{minimize } f(x)$$
$$\text{subject to } g(x) \le 0$$

where $x \in \mathbb{R}^n$

- Suppose that the minimizer x occurs at the boundary of the constraint set
  - Here g(x) = 0 is an active constraint

- If we can find a vector *u* such that

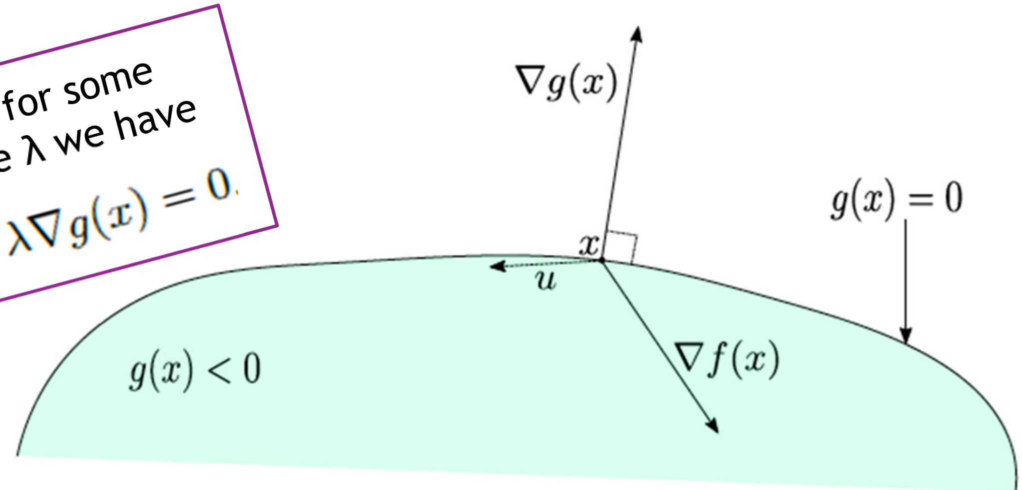
$$\langle u, \nabla g(x) \rangle < 0 \quad \text{and} \quad \langle u, \nabla f(x) \rangle < 0$$

then we can decrease the value of both *g* and *f* for some small number δ > 0

$$g(x + \delta u) \simeq g(x) + \delta \langle u, \nabla g(x) \rangle \le 0.$$
$$f(x + \delta u) \simeq f(x) + \delta \langle u, \nabla f(x) \rangle < f(x)$$

Therefore for some nonnegative λ we have

$$\nabla f(x) + \lambda \nabla g(x) = 0.$$



Review

- Suppose we want to solve

$$\text{minimize } x^2$$
$$\text{subject to } x \geq b$$

where $x \in \mathbb{R}$

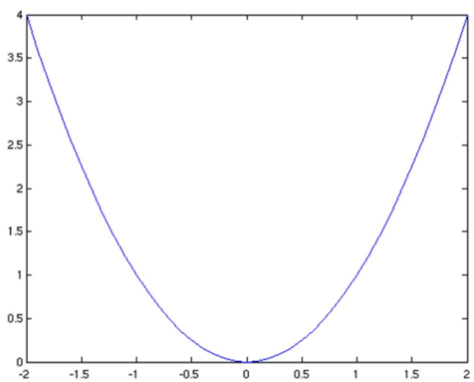
- We want to switch from constraint to penalization by studying the Lagrangian

$$L(x, \alpha) = x^2 - \alpha(x - b)$$

- We add a constraint on the additional variable to solve

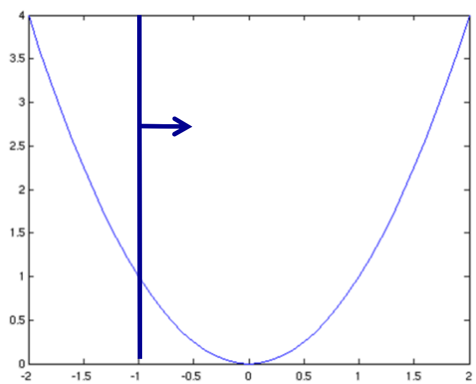
$$\min_x \max_\alpha \; L(x, \alpha)$$
$$\text{s.t.} \quad \alpha \geq 0$$

No Constraint



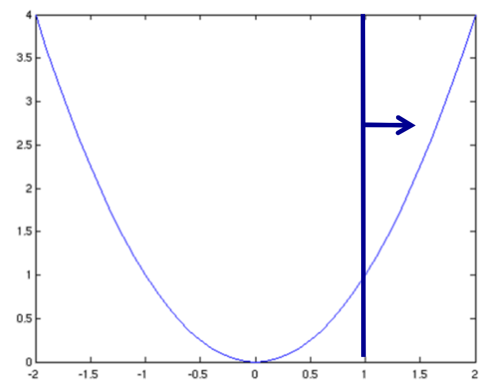
$x^*=0$

$x \geq -1$



$x^*=0$

$x \geq 1$



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$$\min_x \max_\alpha \; L(x, \alpha)$$
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Having *min* outside forces *max* to give us the constraints

$x<b \rightarrow (x-b)<0 \rightarrow \max_\alpha -\alpha(x-b) = \infty$

$x=b \rightarrow \alpha$ can be anything

$x>b,\ \alpha \geq 0 \rightarrow (x-b)>0 \rightarrow \max_\alpha -\alpha(x-b) = 0,\ \alpha^*=0$

Review

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- Switching the order we can solve

$$\max_\alpha \min_x \; L(x, \alpha)$$
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x<b → (x-b)<0 → $\max_\alpha -\alpha(x-b) = \infty$

x=b → α can be anything

x>b, α≥0 → (x-b)>0 → $\max_\alpha -\alpha(x-b) = 0$, α*=0

Review

- Suppose we want to solve

$$\text{minimize } x^2$$
$$\text{subject to } x \geq b$$

where $x \in \mathbb{R}$

$$\boxed{\frac{\partial}{\partial x} L(x, \alpha) = 2x - \alpha \Rightarrow x = \frac{\alpha}{2}}$$

- We want to switch from constraint to penalization by studying the Lagrangian

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- Suppose we want to solve

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where $x \in \mathbb{R}$

$$\frac{\partial}{\partial x} L(x, \alpha) = 2x - \alpha \Rightarrow x = \frac{\alpha}{2}$$

$$\max_\alpha \min_x L(x, \alpha) = \max_\alpha b\alpha - \frac{\alpha^2}{4}$$

- We want to switch from constraint to penalization by studying the Lagrangian

$$L(x, \alpha) = x^2 - \alpha(x - b)$$

- We add a constraint on the additional variable to solve

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- Switching the order we can solve

$$\max_\alpha \min_x \; L(x, \alpha)$$
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- We obtain the expected solution

$$x = \frac{2b}{2} = b$$

Agenda

- Convexity
  - Sets, Functions
- Duality
  - Min-Max Inequality
  - Complementary Slackness
- Support Vector Machines
  - Hard Margin, Soft Margin
- Understanding Support Vector Machines through Duality

References

- D. Rosenberg, Lecture Notes (link)
- Optional
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Convexity

- Convex applies to both sets and functions
- Set C is convex if for any x_1 and x_2 in C we have

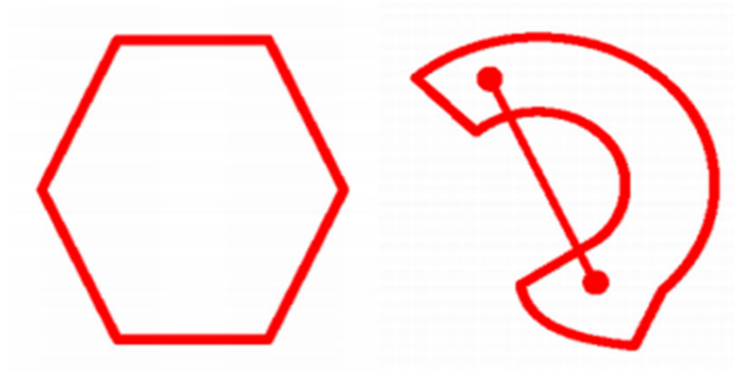
$$\theta x_1 + (1-\theta)x_2 \in C$$

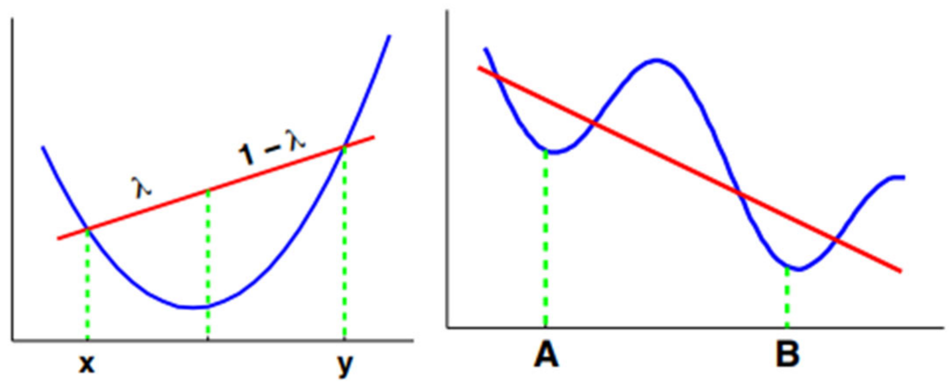
for any $0 \leqslant \theta \leqslant 1$

- Function $f : \mathbf{R}^n \to \mathbf{R}$

is convex if for any x,y and $0 \leqslant \theta \leqslant 1$ we have

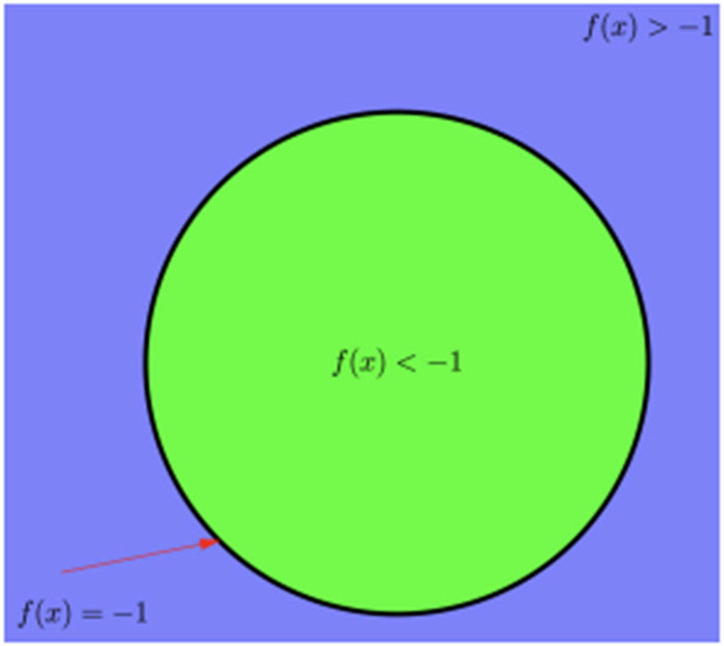
$$f(\theta x + (1-\theta)y) \leqslant \theta f(x) + (1-\theta)f(y).$$





Convexity

- A connection between convex sets and convex functions comes from looking at level sets
- Recall that a level set (aka contour line) for the value c is a points x such that f(x) = c.
- A sublevel set for value c is the set of points x such that f(x) ≤ c
- Sublevel sets of convex functions are convex.



Convexity

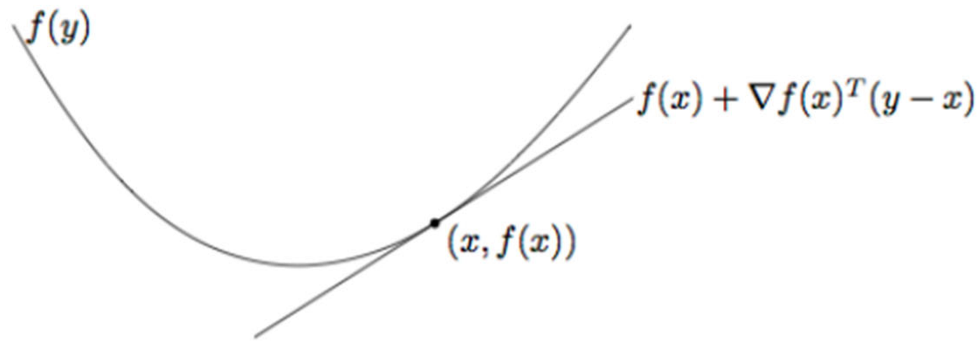
- Suppose $f : \mathbf{R}^d \to \mathbf{R}$ is differentiable. We can compute the gradient through each of the d partial derivatives. Can we predict f(y) from f(x) and $\nabla f(x)$?
- While convex functions are not linear functions, they behave like linear functions in a one-sided sense. The Taylor expansion near x is

$$f(y) \approx f(x) + \nabla f(x)^T (y - x)$$

- By convexity we have

$$f(y) \geqslant f(x) + \nabla f(x)^T (y - x)$$

- Therefore the linear approximation near x determine by the gradient is a global under-estimator of f



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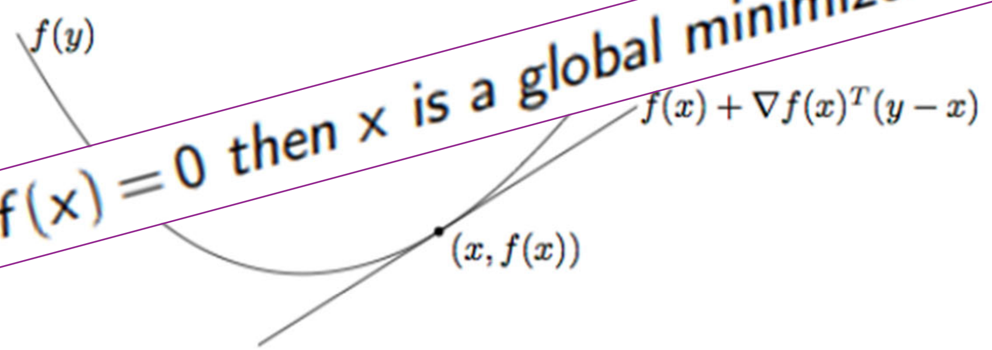


If $\nabla f(x) = 0$ then x is a global minimizer of f.

Convexity

- A function is strictly convex when the line segment connecting two points on the graph (aka secant line) lies strictly above the graph
- So when a function is convex, if we have a local minimum, then we know it's a global minimum
- Moreover with strict convexity the global minimum is unique

Examples of Convex Functions

$x \mapsto ax + b$ is both convex and concave on $\mathbf{R}$ for all $a, b \in \mathbf{R}$

$x \mapsto |x|^p$ for $p \geqslant 1$ is convex on $\mathbf{R}$

$x \mapsto e^{ax}$ is convex on $\mathbf{R}$ for all $a \in \mathbf{R}$

Every norm on $\mathbf{R}^n$ is convex (e.g. $\|x\|_1$ and $\|x\|_2$)

Max: $(x_1, \ldots, x_n) \mapsto \max\{x_1 \ldots, x_n\}$ is convex on $\mathbf{R}^n$

Optimization Problem

- We can study a more general minimization problem by incorporating equality constraints
- Note that equality constraints are not actually that different because

$$h(x) = 0$$

if and only if

$$h(x) \geqslant 0 \text{ AND } h(x) \leqslant 0$$

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leqslant 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots p, \end{array}$$

$$\boxed{\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & h(x) = 0 \end{array}}$$

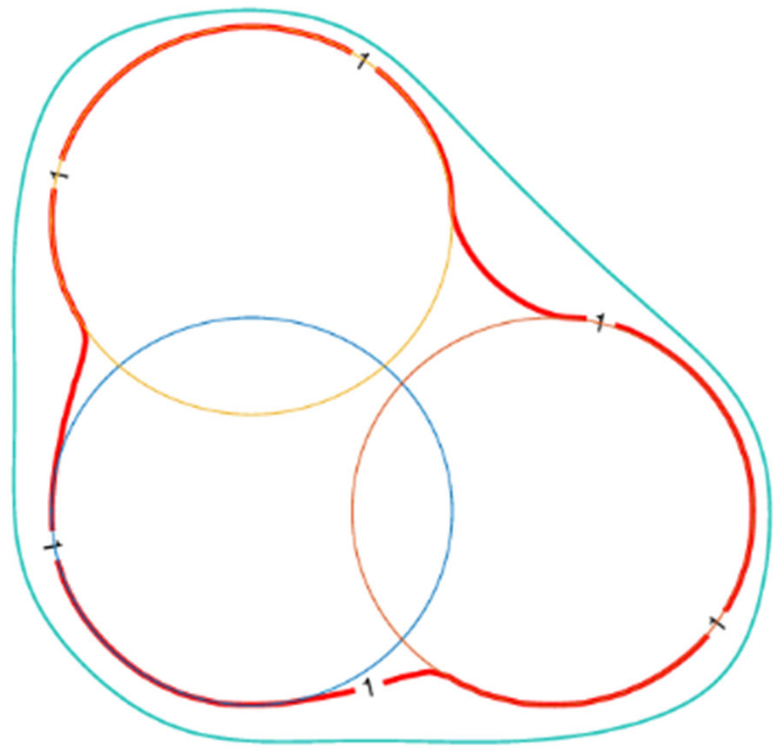
$$\boxed{\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & h(x) \leqslant 0 \\ & -h(x) \leqslant 0 \end{array}}$$

Optimization Problem

- Set of points satisfying the constraints is called the feasible set.
- Note that the intersection of convex sets is convex. So convex constraint functions give us convex feasible set.
- Optimal value p^* is

$$p^* = \inf\{f_0(x) \mid x \text{ satisfies all constraints}\}$$

- Optimal point x^* is a feasible point such that $f_0(x^*) = p^*$



Duality between Optimization Problems

- We can switch from the constraint form to the penalization form by forming the Lagrangian.

$$L(x,\lambda) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x)$$

- Note that the additional variables are called dual variables (aka Lagrange multipliers)
- The primal form of the optimization problem is

$$p^* = \inf_x \sup_{\lambda \succeq 0} L(x,\lambda)$$

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- The dual form of the optimization problem is

$$d^* = \sup_{\lambda \succeq 0} \inf_x L(x, \lambda)$$

- Note that we may not have equality between the primal form and the dual form with conditions called constraint qualifications.
- However, we have weak duality

$$p^* \geqslant d^*$$

even for problems without convexity assumptions

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$$p^* = \inf_x \sup_{\lambda \succeq 0} \left[f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) \right]$$
$$\geqslant \sup_{\lambda \succeq 0, \nu} \inf_x \left[f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) \right] = d^*$$

- The equality between primal and dual problems is called strong duality. Otherwise we have a duality gap $p^* - d^*$

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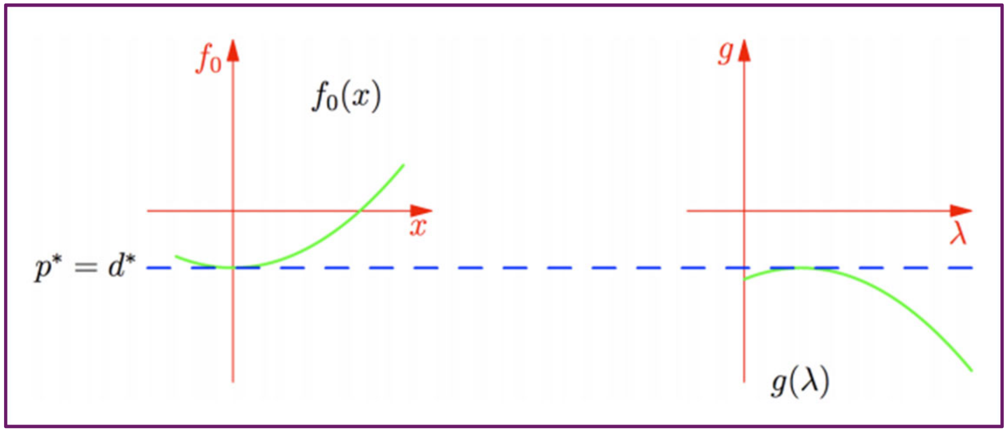

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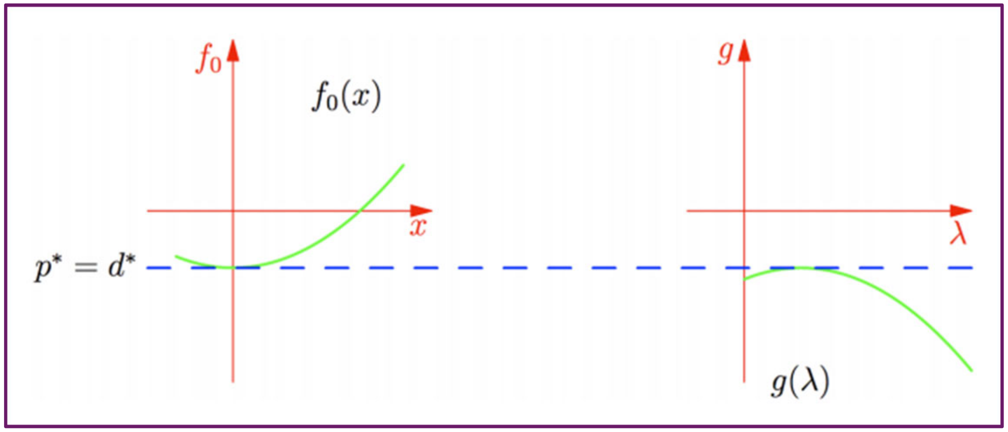

- The equality between primal and dual problems is called strong duality. Otherwise we have a duality gap $p^* - d^*$
- The dual function is always concave

$$g(\lambda) = \inf_x L(x, \lambda) = \inf_x \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) \right)$$

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- However, we have weak duality

$$p^* \geqslant d^*$$

even for problems without convexity assumptions

- For convex optimization problems where
  - f_0 is a convex function
  - f_i is a convex function

we have Slater's sufficient conditions for strong duality

$$f_i(x) < 0 \text{ for } i = 1, \ldots, m$$

This must hold for some x.

- Note that for affine functions f_i the inequality doe not have to be strict.

$$f_i(x) \leqslant 0$$

Complementary Slackness

- Weak duality implies the dual form allows us to search for the best lower bound for the primal problem.
- We need to constrain the dual variables to be positive

$$\lambda \succeq 0$$

Otherwise the dual variables are not dual feasible.

- The dual form can be more simple or more informative yielding insights into dual problem

$$\boxed{\begin{array}{ll} \text{maximize} & g(\lambda) \\ \text{subject to} & \lambda \succeq 0 \end{array}}$$

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$$\boxed{\begin{array}{ll} \text{maximize} & g(\lambda) \\ \text{subject to} & \lambda \succeq 0 \end{array}}$$

- Assuming we have strong duality $p^* = d^*$, we can connect the primal variables and the dual variables

$$\begin{aligned} f_0(x^*) &= g(\lambda^*) = \inf_x L(x, \lambda^*) \\ &\leqslant L(x^*, \lambda^*) \\ &= f_0(x^*) + \sum_{i=1}^{m} \underbrace{\lambda_i^* f_i(x^*)}_{\leqslant 0} \\ &\leqslant f_0(x^*). \end{aligned}$$

Complementary Slackness

- Here x^* is the primal optimal and λ^* is the dual optimal
- Each term in the sum

$$\sum_{i=1} \lambda_i^* f_i(x^*)$$

must be 0 because of the primal and dual constraints dictating the sign of the terms

- Therefore

$$\boxed{\lambda_i^* f_i(x^*) = 0, \quad i = 1, \ldots, m}$$

- We call the relationship complementary slackness

- Note that $L(x^*, \lambda^*) = \inf_x L(x, \lambda^*)$ implies
$$\nabla_x L(x^*, \lambda^*) = 0$$

for $x \mapsto L(x, \lambda^*)$ differentiable.

- Assuming we have strong duality $p^* = d^*$, we can connect the primal variables and the dual variables

$$\begin{aligned} f_0(x^*) &= g(\lambda^*) = \inf_x L(x, \lambda^*) \\ &\leqslant L(x^*, \lambda^*) \\ &= f_0(x^*) + \sum_{i=1}^{m} \underbrace{\lambda_i^* f_i(x^*)}_{\leqslant 0} \\ &\leqslant f_0(x^*). \end{aligned}$$

Kuhn Tucker Conditions

- Assume we have a convex optimization problem where
  - f_0 is a convex function
  - f_i is a convex function
- For example suppose we have checked Slater's sufficient conditions for $p^* = d^*$

$$f_i(x) < 0 \text{ for } i = 1, \dots, m$$

- Assume that $x \mapsto L(x, \lambda^*)$ is differentiable
- We want to determine optimal primal variables and optimal dual variables by checking conditions

- First Order Condition

$$\nabla_x L(x^*, \lambda^*) = 0$$

- Dual Feasible

$$\lambda^* \succeq 0$$

- Primal Feasible

$$f_i(x^*) \leqslant 0$$

- Complementary Slackness

$$\boxed{\lambda_i^* f_i(x^*) = 0, \quad i = 1, \dots, m}$$

Exercise

- Suppose that graders and students compete to minimize or maximize points on assignments.
- Assume that each assignment has five questions.
- The graders decide to score only one question from each assignment. Graders will set the number of points lost to $+\infty$ for any omitted problem.
- The students must decide on allocation of time for each problem to avoid losing points.

Each row is a student strategy

$$A = \begin{bmatrix} 5 & 5 & 5 & 5 & 5 \\ 8 & 8 & 1 & 8 & 8 \\ +\infty & +\infty & +\infty & 0 & +\infty \end{bmatrix}$$

Each column is a grader strategy

Exercise

- Assuming that
  - Student is forced to submit the homework without knowing the graders' choice of problem
  - Graders are allowed to decide which problem to grade after having seen the student's submission
- The number of points lost will be

$$p^* = \min_i \max_j a_{ij}$$

- Here students pick strategy first and graders pick strategy second.

$$A = \begin{bmatrix} 5 & 5 & 5 & 5 & 5 \\ 8 & 8 & 1 & 8 & 8 \\ +\infty & +\infty & +\infty & 0 & +\infty \end{bmatrix}$$

Exercise

- Assuming that
  - Graders announce the problem in advance
  - Students get to decide on allocation of time for that problem
- The number of points lost will be

$$d^* = \max_j \min_i a_{ij}$$

- Here graders pick strategy first and students pick strategy second.

$$A = \begin{bmatrix} 5 & 5 & 5 & 5 & 5 \\ 8 & 8 & 1 & 8 & 8 \\ +\infty & +\infty & +\infty & 0 & +\infty \end{bmatrix}$$

Exercise

- Show that for any matrix

$$A = (a_{ij}) \in \mathbb{R}^{m \times n}$$

we always have

$$\boxed{\max_j \min_i a_{ij} = d^* \leq p^* = \min_i \max_j a_{ij}}$$

- Check the inequality through showing

$$d^* = a_{i_d j_d} \leq a_{i_p j_d} \leq a_{i_p j_p} = p^*$$

Calculate p*

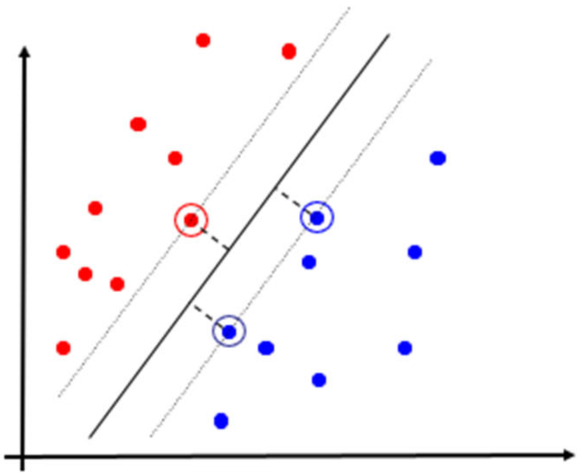
$$A = \begin{bmatrix} 5 & 5 & 5 & 5 & 5 \\ 8 & 8 & 1 & 8 & 8 \\ +\infty & +\infty & +\infty & 0 & +\infty \end{bmatrix}$$

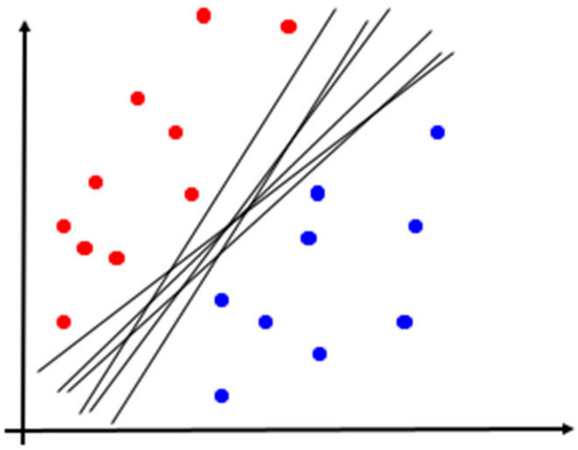
Calculate d*

Perceptron

- Remember some of the properties of perceptron algorithm
- Advantages
  - Error Bound
  - Online Algorithm
- Disadvantages
  - Many Decision Boundaries
  - Separable Data

Distance from the decision boundary should reflect confidence





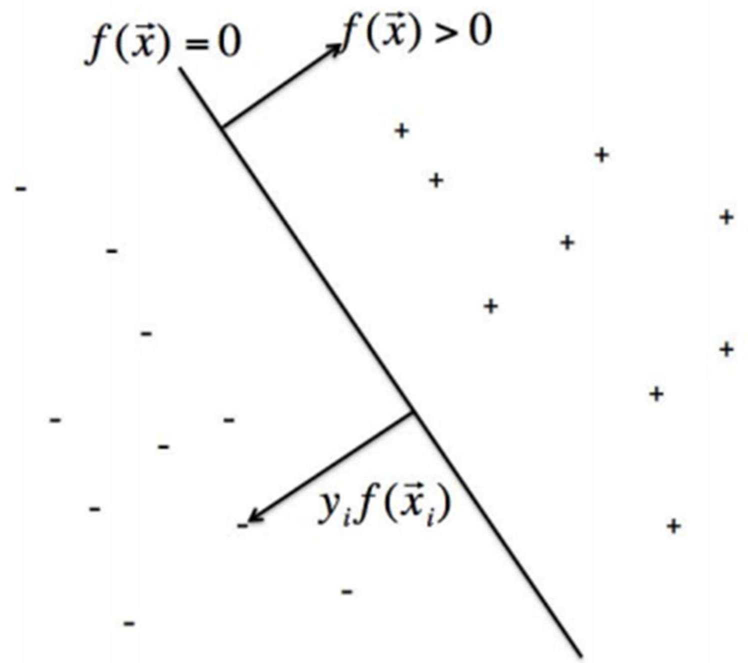
We want to make the distance from the decision boundary large

Margin

- Recall that margin intuitively means signed distance from decision boundary
- For hypothesis f, we could define the functional margin as

$$y\, f(x)$$

- The decision boundary was the level set of f for value 0.
- The decision boundary splits into two half-spaces depending on the sign of the margin
  - Positive
  - Negative



Margin

Focus on linear hypotheses

$$f(\mathbf{x}) = \sum_{j=1}^{n} w^{(j)} x^{(j)} + w_0$$

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- For hypothesis f, we could define the functional margin as

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  - Positive
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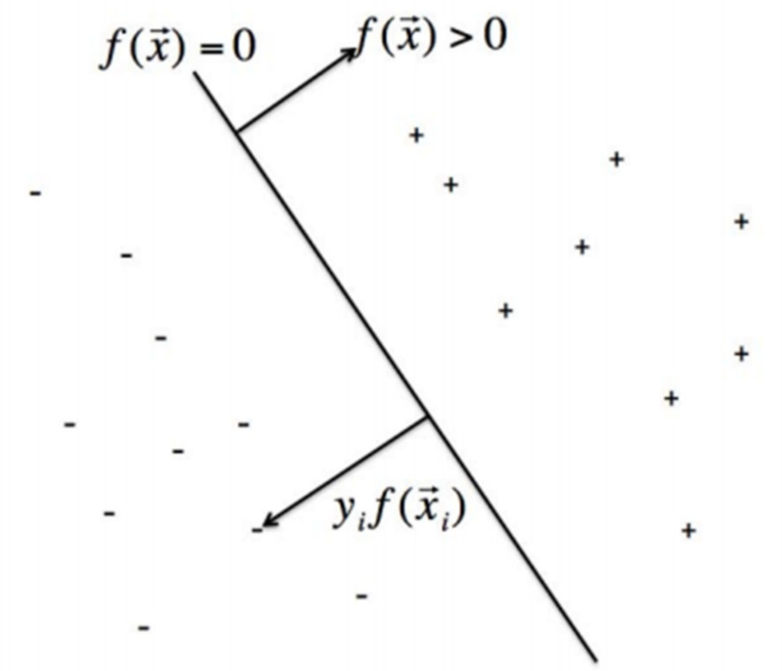
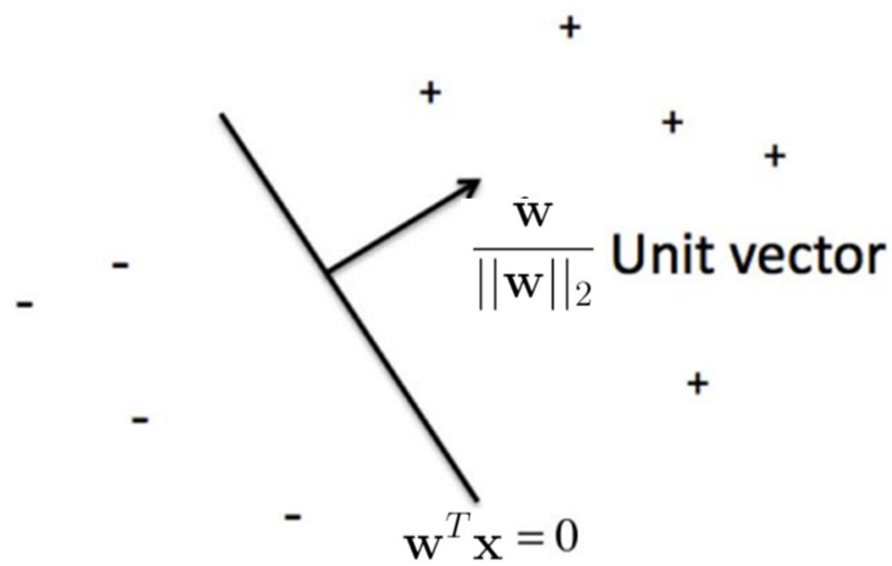


Support Vector Machine

- Note that scale the hypothesis f by a large number would increase margin for correct classifications
- We must recognize the scaling issue in maximizing the minimum distance of the training points from the decision boundary
- Remember that the weights determine a plane through the origin. The offset w_0 shifts the plane away from the origin



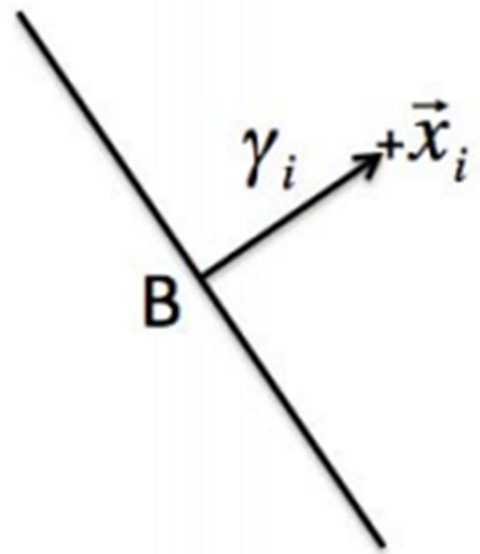
Support Vector Machine

- Recall projection of a point x_i onto the plane determine by w shifted by w_0
- Suppose the nearest point is B. Denote the signed distance by γ_i
- Note that B is

$$\mathbf{B} = \mathbf{x}_i - \gamma_i \frac{\mathbf{w}}{||\mathbf{w}||_2}$$

- Since B lies on the decision boundary, we have

$$w^T B + w_0 = 0$$



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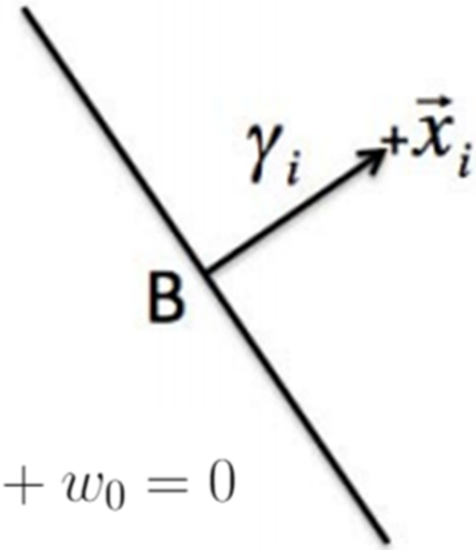

$$\mathrm{w}^T \left(\mathbf{x}_i - \gamma_i \frac{\mathbf{w}}{||\mathbf{w}||_2} \right) + w_0 = 0$$

$$\mathrm{w}^T \mathbf{x}_i - \gamma_i \frac{||\mathbf{w}||_2^2}{||\mathbf{w}||_2} + w_0 = 0$$

$$\gamma_i = \frac{\mathbf{w}^T \mathbf{x}_i + w_0}{||\mathbf{w}||_2}$$

Support Vector Machine

- Therefore to choose the hypothesis so the training points are far away from the decision boundary, we need to study maximize the minimum geometric margin

$$\gamma_i = \frac{\mathbf{w}^T \mathbf{x}_i + w_0}{||\mathbf{w}||_2}$$

$$\max_f \max_\gamma \gamma \text{ subject to } y_i f(x_i) \geq \gamma \text{ for } i = 1, \ldots, m$$

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$$\max_{\mathbf{w},w_0} \frac{1}{||\mathbf{w}||_2} \text{ subject to } y_i \left(\mathbf{w}^T\mathbf{x}_i + w_0\right) \geq 1 \text{ for } i = 1, \ldots, m$$

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$$\min_{\mathbf{w}, w_0} \frac{1}{2} ||\mathbf{w}||_2^2 \text{ subject to } 0 \geq 1 - y_i \left(\mathbf{w}^T \mathbf{x}_i + w_0\right) \text{ for } i = 1, \ldots, m$$

Support Vector Machine

- We can form the Lagrangian
- Note that we have a convex optimization problem with affine constraints
- Strict feasibility requires that the training data can be separated with a linear decision boundary

$$L\left(\mathbf{w}, w_0, \alpha\right) = \frac{1}{2}\sum_{j=1}^{n} w^{(j)2} + \sum_{i=1}^{m} \alpha_i \left(1 - y_i \left(\mathbf{w}^T \mathbf{x}_i + w_0\right)\right)$$

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$$\nabla_{\mathbf{w}} L\left(\mathbf{w}, w_0, \alpha\right) = \mathbf{w} - \sum_{i=1}^{m}\alpha_i y_i \mathbf{x}_i = 0 \Rightarrow \mathbf{w} = \sum_{i=1}^{m}\alpha_i y_i \mathbf{x}_i$$

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$$\alpha_i \left(-y_i \left(\mathbf{w}^T \mathbf{x}_i + w_0\right) + 1\right) = 0 \text{ for } i = 1, \ldots, m$$

Support Vector Machine

- Substituting the expressions in the Kuhn Tucker conditions arising from the derivative of the Lagrangian, we can simplify the Lagrangian.
- We obtain a quadratic programming problem in the dual variables α

$$L(\mathbf{w}, w_0, \alpha) = \underbrace{\frac{1}{2}\sum_{j=1}^{n} w^{(j)2}}_{\mathbf{w} = \sum_{i=1}^{m} \alpha_i y_i \mathbf{x}_i} + \underbrace{\sum_{i=1}^{m} \alpha_i \left(1 - y_i\left(\mathbf{w}^T\mathbf{x}_i + w_0\right)\right)}_{\sum_{i=1}^{m} \alpha_i y_i = 0}$$

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$$= \frac{1}{2}||\mathbf{w}||_2^2 - ||\mathbf{w}||_2^2 - w_0 \sum_{i=1}^{m}(y_i w_0) + \sum_{i=1}^{m} \alpha_i$$

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- Substituting the expressions in the Kuhn Tucker conditions arising from the derivative of the Lagrangian, we can simplify the Lagrangian.
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$$-\frac{1}{2}\sum_{j=1}^{n} w^{(j)2} = -\frac{1}{2}\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\alpha_i y_i x_i^{(j)}\right)^2$$

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\begin{aligned}
-\frac{1}{2}\sum_{j=1}^{n} w^{(j)2} &= -\frac{1}{2}\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\alpha_i y_i x_i^{(j)}\right)^2 \\
&= -\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m}\alpha_i\alpha_k y_i y_k x_i^{(j)} x_k^{(j)}
\end{aligned}
$$

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Support Vector Machine

- We can solve the quadratic programming problem with a package like CVXOPT.
- Another approach called Sequential Minimal Optimization applies coordinate descent to pairs of dual variables.

Dual of Hard Margin SVM

$$\max_{\boldsymbol{\alpha}} \mathcal{L}(\boldsymbol{\alpha})$$

where

$$\mathcal{L}(\boldsymbol{\alpha}) = \sum_{i=1}^{m} \alpha_i - \frac{1}{2} \sum_{i,k} \alpha_i \alpha_k y_i y_k \mathbf{x}_i^T \mathbf{x}_k$$

subject to constrains

$$\begin{cases} \alpha_i \geq 0 \quad i = 1 \ldots m \\ \sum_{i=1}^{m} \alpha_i y_i = 0 \end{cases}$$

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Dual of Hard Margin SVM

Note that we have not used the other two Kuhn Tucker conditions…and we have not computed w_0!

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Support Vectors

- Consider complementary slackness
- Note that the second and fourth situations are not possible by primal feasibility and dual feasibility.

$$\alpha_i^* \left(1 - y_i \left(\mathbf{w}^{*T}\mathbf{x}_i + w_0^*\right)\right) = 0$$

$$\downarrow$$

$$\begin{cases} \alpha_i^* > 0 \Rightarrow y_i \left(\mathbf{w}^{*T}\mathbf{x}_i + w_0^*\right) = 1 \\ \alpha_i^* < 0 \\ \alpha_i^* = 0 \Rightarrow 1 - y_i \left(\mathbf{w}^{*T}\mathbf{x}_i + w_0^*\right) < 0 \\ \alpha_i^* = 0 \Rightarrow 1 - y_i \left(\mathbf{w}^{*T}\mathbf{x}_i + w_0^*\right) > 0 \end{cases}$$

Support Vectors

- Consider complementary slackness
- Note that the second and fourth situations are not possible by primal feasibility and dual feasibility.
- So for the hypothesis with optimal (scaled) weights

$$f^*(x) = \mathbf{w}^{*T}\mathbf{x} + w_0^*$$

$$\begin{cases} \alpha_i^* > 0 & \Rightarrow \; y_i f^*(\mathbf{x}_i) = \text{scaled margin}_i = 1 \\ 1 < y_i f^*(\mathbf{x}_i) & \Rightarrow \; \alpha_i^* = 0 \end{cases}$$

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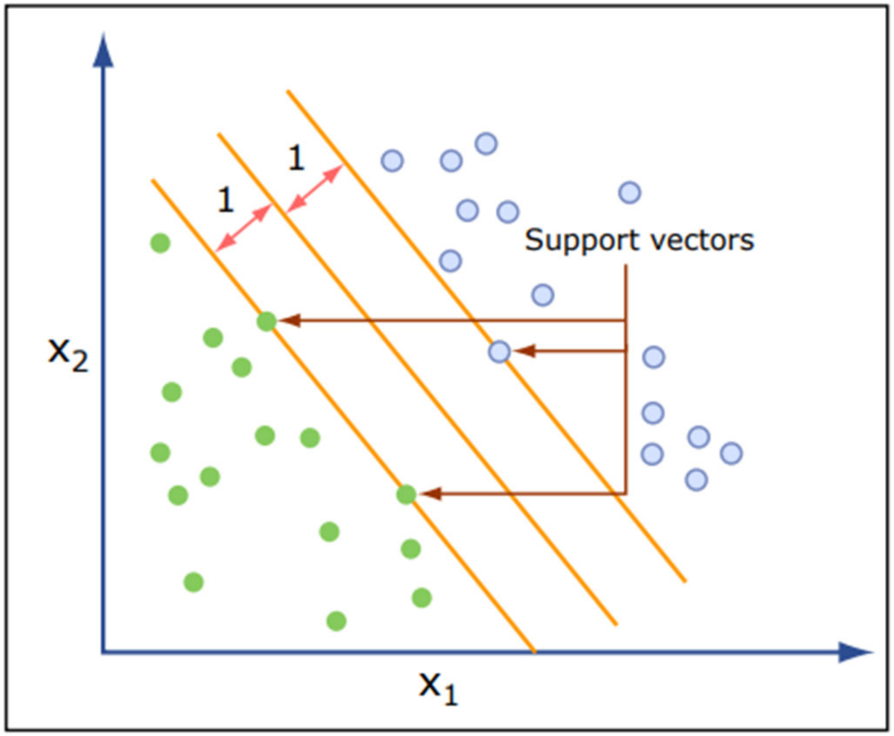


Offset Term

- For a support vector we have

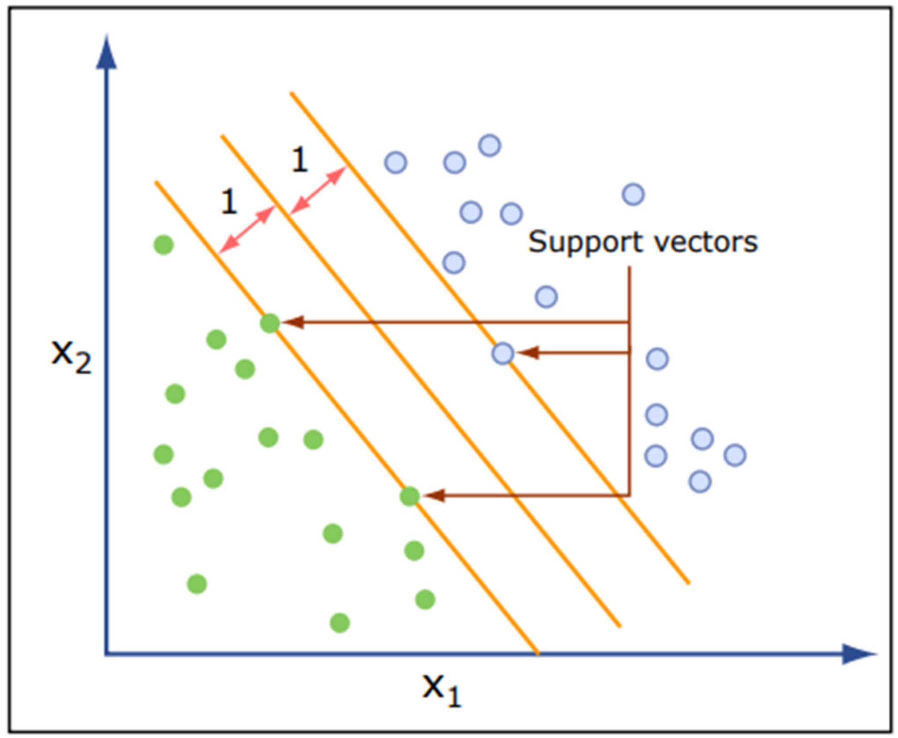
$$y_i\left(\mathbf{w}^{*T}\mathbf{x}_i + w_0^*\right) = 1$$

- For a $y_i = 1$ we obtain

$$w_0^* = 1 - \mathbf{w}^{*T}\mathbf{x}_i$$

- So we would compute the dual variables α to get w before computing w_0

$$\begin{cases} \alpha_i^* > 0 & \Rightarrow \quad y_i f^*(\mathbf{x}_i) = \text{scaled margin}_i = 1 \\ 1 < y_i f^*(\mathbf{x}_i) & \Rightarrow \quad \alpha_i^* = 0 \end{cases}$$

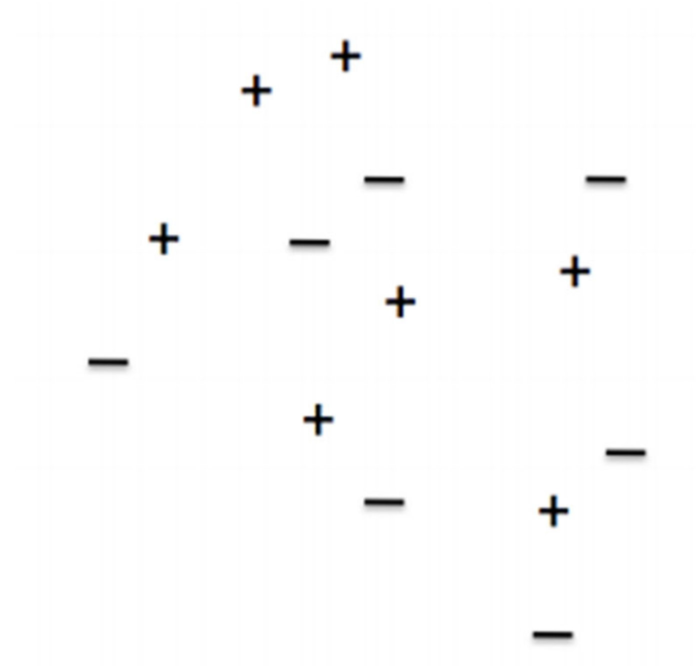


Relaxing the Constraint

- We cannot separate some training sets with a linear decision boundary
- We can relax the constraint by adding a slack variable that captures the violation of the margin constraint.

$$\min_{\mathbf{w},w_0} \frac{1}{2}||\mathbf{w}||_2^2 + C\sum_{i=1}^{m} \xi_i$$

$$\text{subject to } \begin{cases} y_i\left(w^T\mathbf{x}_i + w_0\right) \geq 1 - \xi_i \\ \xi_i \geq 0 \end{cases}$$



Relaxing the Constraint

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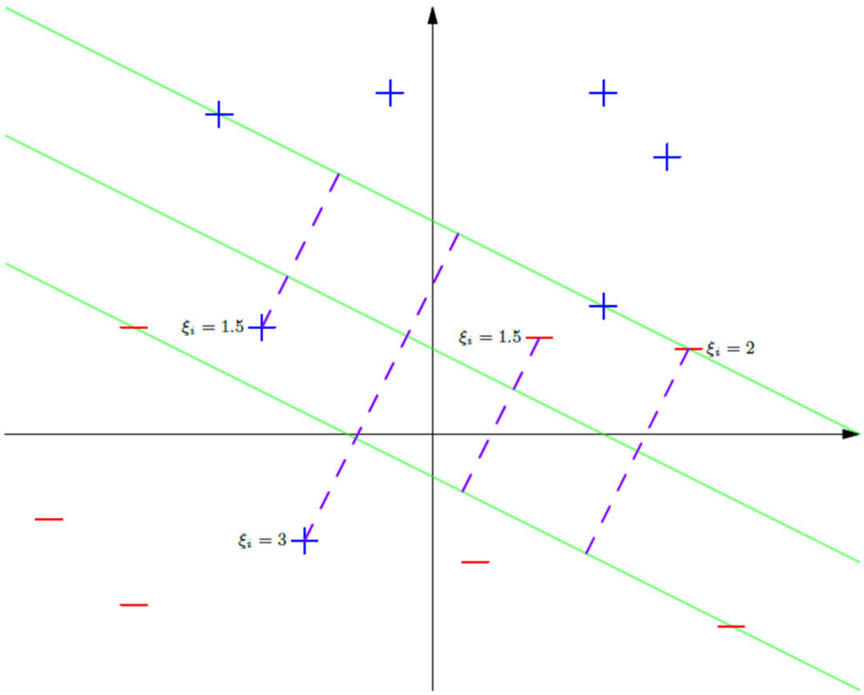


Relaxing the Constraint

- Note that we can combine the two constraints because

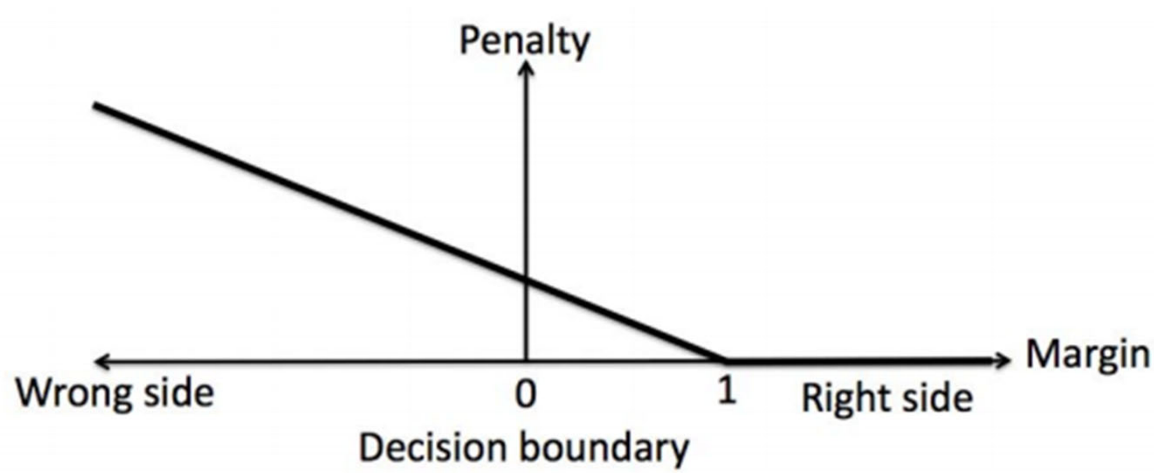
$$\begin{cases} y_i \left(w^T \mathbf{x}_i + w_0 \right) \geq 1 - \xi_i \\ \xi_i \geq 0 \end{cases}$$

if and only if

$$\xi_i \geq \max \left\{ 0, 1 - y_i \left(w^T \mathbf{x}_i + w_0 \right) \right\}$$

- Therefore we substitute into the objective function

$$\min_{\mathbf{w}, w_0} \frac{1}{2} ||\mathbf{w}||_2^2 + C \sum_{i=1}^{m} \max \left\{ 0, 1 - y_i \left(w^T \mathbf{x}_i + w_0 \right) \right\}$$



Support Vector Machine

- We can solve the quadratic programming problem with a package like CVXOPT.
- Another approach called Sequential Minimal Optimization applies coordinate descent to pairs of dual variables.

Dual of Soft Margin SVM

$$\max_{\boldsymbol{\alpha}} \mathcal{L}(\boldsymbol{\alpha})$$

where

$$\mathcal{L}(\boldsymbol{\alpha}) = \sum_{i=1}^{m} \alpha_i - \frac{1}{2} \sum_{i,k} \alpha_i \alpha_k y_i y_k \mathbf{x}_i^T \mathbf{x}_k$$

subject to constrains

$$\begin{cases} 0 \leq \alpha_i \leq C & i = 1 \ldots m \\ \sum_{i=1}^{m} \alpha_i y_i = 0 \end{cases}$$

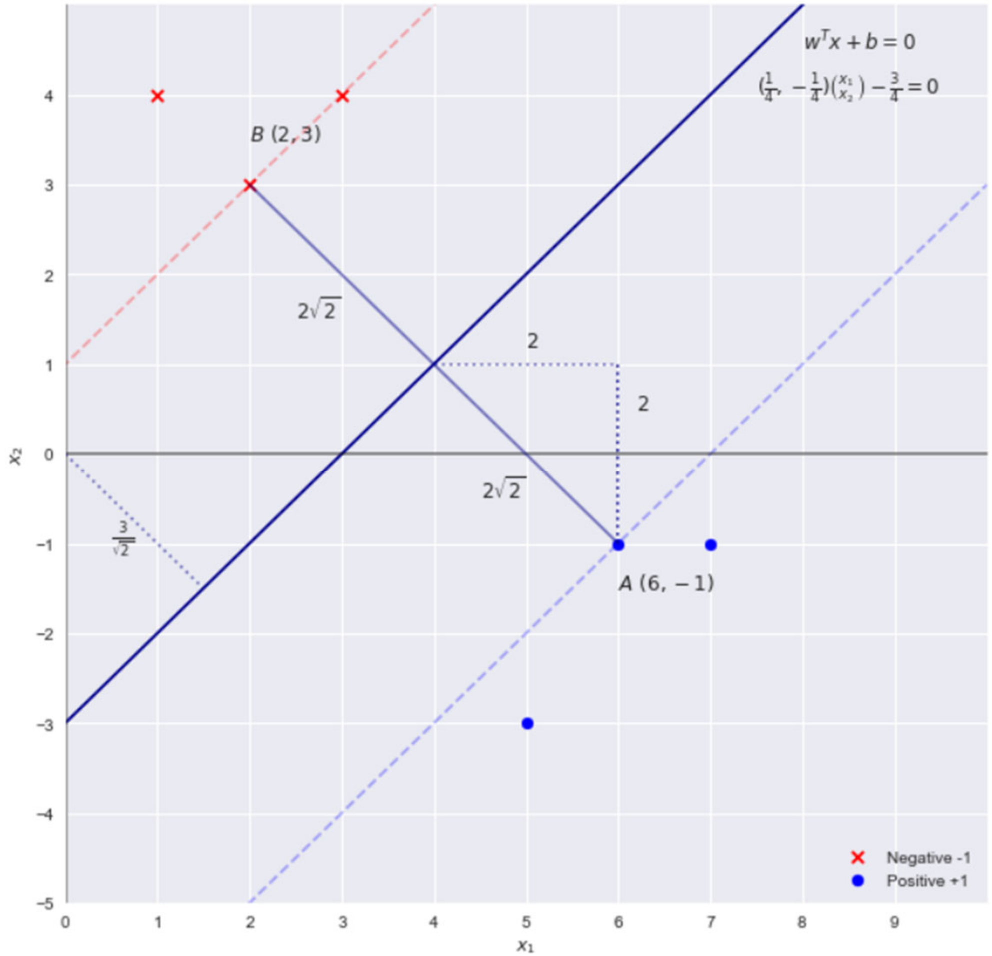
Exercise

```
import numpy as np
from sklearn.svm import SVC

X = np.array([[3,4],[1,4],[2,3],[6,-1],[7,-1],[5,-3]] )
y = np.array([-1,-1, -1, 1, 1 , 1 ])

clf = SVC(C = 1e5, kernel = 'linear')
clf.fit(X, y)
```

```
SVC(C=100000.0, cache_size=200, class_weight=None, coef0=0.0,
  decision_function_shape='ovr', degree=3, gamma='auto_deprecated',
  kernel='linear', max_iter=-1, probability=False, random_state=None,
  shrinking=True, tol=0.001, verbose=False)
```

```
clf.support_vectors_
```

```
array([[ 2.,  3.],
       [ 6., -1.]])
```



Summary

- Convexity
  - Sets, Functions
- Duality
  - Min-Max Inequality
  - Complementary Slackness
- Support Vector Machines
  - Hard Margin, Soft Margin
- Understanding Support Vector Machines through Duality

- Goals
  - How does the hinge loss increase margins? Why would large margins help us?
  - What are some advantages of the dual formulation of a minimization problem? What insights can we gain from the SVM dual problem?
  - What is a support vector? Why is SVM sparse in the data?

Questions

Kernel Methods

- Questions on Piazza?
  - Please provide your feedback
- Question for You!

The objective depends on dot products. Could we replace with other products?

$$\sup_{\alpha} \quad \sum_{i=1}^{n} \alpha_i - \frac{1}{2} \sum_{i,j=1}^{n} \alpha_i \alpha_j y_i y_j x_j^T x_i$$

$$\text{s.t.} \quad \sum_{i=1}^{n} \alpha_i y_i = 0$$

$$\alpha_i \in \left[0, \frac{c}{n}\right] \quad i = 1, \ldots, n.$$