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Number format:

$$N_B \quad = \quad N_X + N_Y \quad = \quad [d_{S_B-1}d_{S_B-2} \dots d_3 d_2 d_1 d_0] + 0.[p_1 p_2 p_3 p_4 \dots p_{T_B-2} p_{T_B-1} p_{T_B}]$$

$$N_B \quad = \quad [d_{S_B-1}d_{S_B-2} \dots d_3 d_2 d_1 d_0 . p_1 p_2 p_3 p_4 \dots p_{T_B-2} p_{T_B-1} p_{T_B}]$$

In algebraic form, the number N is:

$$N_X = [d_{S_B-1}d_{S_B-2} \dots d_3 d_2 d_1 d_0]$$

$$N_X = d_{S_B-1}(B)^{S_B-1} + d_{S_B-2}(B)^{S_B-2} + \cdots + d_3(B)^3 + d_2(B)^2 + d_1(B)^1 + d_0(B)^0$$

$$N_Y = 0.[p_1 p_2 p_3 p_4 \dots p_{T_B-2} p_{T_B-1} p_{T_B}]$$

$$N_Y = p_1(B^{-1})^1 + p_2(B^{-1})^2 + p_3(B^{-1})^3 + \cdots + p_{T_B-2}(B^{-1})^{T_B-2} + p_{T_B-1}(B^{-1})^{T_B-1} + p_{T_B}(B^{-1})^{T_B}$$

More details about the meaning of the sequences of symbols that were used:

Note: Any blue text is a computation done in decimal.

$x = 0, 1, 2, 3, \dots, (S_B - 2), (S_B - 1)$	Pick any single value for x in the list of numbers $0, 1, 2, 3, \dots, (S_B - 2), (S_B - 1)$.
$y = 1, 2, 3, \dots, (T_B - 2), (T_B - 1), T_B$	Pick any single value for y in the list of numbers $1, 2, 3, \dots, (T_B - 2), (T_B - 1), T_B$. In general, if you see an equal sign and commas used in this way, the meaning is choose one number in the list of numbers separated by commas and then assign that number as the value of the variable on the left side of the equation.
$[d_{S_B-1}d_{S_B-2} \dots d_3 d_2 d_1 d_0]$	Denotes the concatenation of the digits $d_{S_B-1}, d_{S_B-2}, \dots, d_3, d_2, d_1, d_0$ in the order specified by that sequence enclosed by the brackets. (In general, a bracket used in that manner denotes the concatenation of the sequence of symbols or digits inside it)
B	Base used in expressing N_B. A counting number or whole number.
$N_X = [d_{S_B-1}d_{S_B-2} \dots d_3 d_2 d_1 d_0]$	Part of N_B that is a whole number in which these are true: either $N_X = 0$ or $N_X \geq 1$. The "whole part" of the number N_B.
d_x	A digit in N_B that is: - Used in N_X - At a position $x = 0, 1, 2, 3, \dots, (S_B - 2), (S_B - 1)$ to the left, away from the decimal point of N_B - A whole number in which these are true: $0 \leq d_x \leq (B - 1)$
$N_Y = [0. p_1 p_2 p_3 p_4 \dots p_{T_B-2} p_{T_B-1} p_{T_B}]$	Part of N_B that is a rational number in which these are true: $0 < N_Y < 1$. The "fractional part" of the number N_B.
p_y	A digit in N_B that is: - Used in N_Y - At a position $y = 1, 2, 3, \dots, (T_B - 2), (T_B - 1), T_B$ to the right, away from the decimal point of N_B - A whole number in which these are true: $0 \leq p_y \leq (B - 1)$

Example usage:

$$343.1415_{10}$$

$$d_2 = 3 \quad d_1 = 4 \quad d_0 = 3 \quad p_1 = 1 \quad p_2 = 4 \quad p_3 = 1 \quad p_4 = 5 \quad B = 10$$

$$B_I = initial\ base \qquad B_F = final\ base$$

CONVERTING A NUMBER BETWEEN <u>A BASE THAT IS A POWER OF 2</u> AND <u>BASE 10</u>

$\{2, 8, 16\} \rightarrow 10$: Convert N from $B_I = 2, 8, 16$ to $B_F = 10$

Number N in the initial base or N_{B_I}:

$$N_{B_I} \quad = \quad [d_{S_{B-1}} d_{S_{B-2}} \dots d_3 d_2 d_1 d_0 . p_1 p_2 p_3 p_4 \dots p_{T_{B-2}} p_{T_{B-1}} p_{T_B}]$$

If $B_I = 2$, then:	If $B_I = 8$, then:	If $B_I = 16$, then:
$d_x = 0, 1$ $p_y = 0, 1$	$d_x = 0, 1, 2, 3, \dots, 7$ $p_y = 0, 1, 2, 3, \dots, 7$	$d_x = 0, 1, 2, 3, \dots, a, b, c, d, e, f$ $p_y = 0, 1, 2, 3, \dots, a, b, c, d, e, f$ $a = 10 \quad b = 11 \quad c = 12 \quad d = 13 \quad e = 14 \quad f = 15$

Number N in the final base or N_{B_F}:

$$N_X = d_{S_{B-1}}(B)^{S_{B-1}} + d_{S_{B-2}}(B)^{S_{B-2}} + \dots + d_3(B)^3 + d_2(B)^2 + d_1(B)^1 + d_0(B)^0$$

$$N_Y = p_1(B^{-1})^1 + p_2(B^{-1})^2 + p_3(B^{-1})^3 + \dots + p_{T_{B-2}}(B^{-1})^{T_{B-2}} + p_{T_{B-1}}(B^{-1})^{T_{B-1}} + p_{T_B}(B^{-1})^{T_B}$$

$$N_{B_F} = N_X + N_Y$$

In general, this works for any number with a whole number base that you want to convert to base 10. Just change B_I into any whole number that you want. (What if you are an alien who is only familiar with the base-8 system or you were only taught to compute in base-8?)

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$10 \rightarrow \{2, 8, 16\}$: Convert N from $B_I = 10$ to $B_F = 2, 8, 16$ (This method actually applies for any B_I, B_F)

Number N in the initial base or N_{B_I}:

$$N_{10} = \left[d_{S_B-1} d_{S_B-2} \dots d_3 d_2 d_1 d_0 . p_1 p_2 p_3 p_4 \dots p_{T_B-2} p_{T_B-1} p_{T_B}\right]$$

$$d_x = 0, 1, 2, 3, \dots, 9 \qquad p_y = 0, 1, 2, 3, \dots, 9$$

Number N in the final base or N_{B_F}:

Note: If you have $A = a\ sequence\ of\ symbols$, and A contains an asterisk (*), then you have a set of texts similar to A in which in place of that asterisk in A is a symbol that varies at a regular pattern for each of the texts in the set of texts similar to A.

Note: Extending the definition in page 1, this means that $N_{X,L0}$ is the whole part of $(N_X)(B_F)^{-1}$, while $N_{Y,L0}$ is the fractional part of $(N_X)(B_F)^{-1}$. In N_{L0}, the L indicates that it is a digit to the left of the decimal point. In N_{R0}, the R indicates that it is a digit to the right of the decimal point. See how this pattern is applied in the texts below:

Whole part of N_{B_F} is a sequence of digits N_{L*} (see explanation in page 9)

$(N_X)(B_F)^{-1} = N_{X,L0} + N_{Y,L0}$ $(N_{X,L0})(B_F)^{-1} = N_{X,L1} + N_{Y,L1}$ $(N_{X,L1})(B_F)^{-1} = N_{X,L2} + N_{Y,L2}$ $(N_{X,L2})(B_F)^{-1} = N_{X,L3} + N_{Y,L3}$ $\dots = \cdots$ Stop if you get $N_{X,L*} = 0$	$N_{L0} = (N_{Y,L0})(B_F)$ $N_{L1} = (N_{Y,L1})(B_F)$ $N_{L2} = (N_{Y,L2})(B_F)$ $N_{L3} = (N_{Y,L3})(B_F)$ $\dots = \cdots$ Remember to convert the computed decimal numbers here into their corresponding equivalent digit. For example, if $B_F = 16$ and you got a digit $N_{L*} = 13$, then in the sequence $[\dots N_{L3} N_{L2} N_{L1} N_{L0}]$, put $N_{L*} = d$. In the text below, N_{L*} became black because we converted that decimal number from our computation into a corresponding digit used in base B_F.
Whole number part of N_{B_F}	$[\dots N_{L3} N_{L2} N_{L1} N_{L0}]$

Fractional part of N_{B_F} is a sequence of digits N_{R*} (see explanation in page 9)

$(N_Y)(B_F) = N_{X,R1} + N_{Y,R1}$ $(N_{Y,R1})(B_F) = N_{X,R2} + N_{Y,R2}$ $(N_{Y,R2})(B_F) = N_{X,R3} + N_{Y,R3}$ $(N_{Y,R3})(B_F) = N_{X,R4} + N_{Y,R4}$ $\dots = \cdots$ Stop if you get $N_{Y,R*} = 0$	$N_{R1} = N_{X,R1}$ $N_{R2} = N_{X,R2}$ $N_{R3} = N_{X,R3}$ $N_{R4} = N_{X,R4}$ $\dots = \cdots$ Remember to convert the computed decimal numbers here into their corresponding equivalent digit. For example, if $B_F = 16$ and you got a digit $N_{R*} = 12$, then in the sequence $[0. N_{R1} N_{R2} N_{R3} N_{R4} \dots]$, put $N_{R*} = c$. In the text below, N_{R*} became black because we converted that decimal number from our computation into a corresponding digit used in base B_F.
Fractional number part of N_{B_F}	$[0. N_{R1} N_{R2} N_{R3} N_{R4} \dots]$

$$N_{B_F} = [\dots N_{L3} N_{L2} N_{L1} N_{L0} . N_{R1} N_{R2} N_{R3} N_{R4} \dots]$$

CONVERT N FROM $B_I = 2^m$ TO $B_F = 2^n$

Where m, n are whole numbers selected in the set $1, 2, 3, \dots$

CASE 1	CASE 2	CASE 3	CASE 4
$m < n$ $m \neq 2 \quad n \neq 2$	$m > n$ $m \neq 2 \quad n \neq 2$	$m \neq n, \quad n = 2$	$m \neq n, \quad m = 2$

Number N in the initial base or N_{B_I}:

$$N_{B_I} = \left[d_{S_B-1} d_{S_B-2} \dots d_3 d_2 d_1 d_0 . p_1 p_2 p_3 p_4 \dots p_{T_B-2} p_{T_B-1} p_{T_B}\right]$$

$$d_x = 0, 1, \dots, (2^m - 3), (2^m - 2), (2^m - 1) \qquad p_y = 0, 1, \dots, (2^m - 3), (2^m - 2), (2^m - 1)$$

Note: In the previous line, the symbols are black because those are just decimal numbers that represent a digit used in the base 2^m.

In N_{B_I}, there are S_B digits used in its whole part and T_B digits used in its fractional part. The subscript I was omitted because these symbols are just recycled, no further editing done so that the symbols are familiar throughout this e-note.

$$N_{B_I} = N_X + N_Y$$

$$N_X = \left[d_{S_B-1} d_{S_B-2} \dots d_3 d_2 d_1 d_0\right]$$

$$N_X = d_{S_B-1}(2^m)^{S_B-1} + d_{S_B-2}(2^m)^{S_B-2} + \dots + d_3(2^m)^3 + d_2(2^m)^2 + d_1(2^m)^1 + d_0(2^m)^0$$

$$N_Y = 0.\left[p_1 p_2 p_3 p_4 \dots p_{T_B-2} p_{T_B-1} p_{T_B}\right]$$

$$N_Y = p_1(2^{-m})^1 + p_2(2^{-m})^2 + p_3(2^{-m})^3 + \dots + p_{T_B-2}(2^{-m})^{T_B-2} + p_{T_B-1}(2^{-m})^{T_B-1} + p_{T_B}(2^{-m})^{T_B}$$

$$N_{B_I} = \sum_{x=0}^{S_B-1} d_x (2^m)^x + \sum_{y=1}^{T_B} p_y (2^{-m})^y$$

Since d_x and p_y can be expressed in base-2 as sequences both with length m bits, we can transform each of those digits into their corresponding polynomial equivalent. The polynomial D_x corresponds to the digit d_x. The polynomial P_y corresponds to the digit p_y

Both D_x, P_y are polynomials evaluated at 2^m with m monomial terms. For both D_x and P_y, the coefficient for each of their terms is either 1 or 0.

$d_x \equiv \left[d_{x,m-1} d_{x,m-2} d_{x,m-3} \dots d_{x,2} d_{x,1} d_{x,0}\right] = \langle d_x \rangle$	$d_{x,*}$ is a bit. A bit is either 1 or 0. $\langle d_x \rangle$ is a subsequence made of m bits. $\langle d_x \rangle$ is the number d_x expressed in base 2.
$p_y \equiv \left[p_{y,m-1} p_{y,m-2} p_{y,m-3} \dots p_{y,2} p_{y,1} p_{y,0}\right] = \langle p_y \rangle$	$p_{y,*}$ is a bit. A bit is either 1 or 0. $\langle p_y \rangle$ is a subsequence made of m bits. $\langle p_y \rangle$ is the number p_y expressed in base 2.

$$D_x = d_{x,m-1}(2)^{m-1} + d_{x,m-2}(2)^{m-2} + d_{x,m-3}(2)^{m-3} + \dots + d_{x,2}(2)^2 + d_{x,1}(2)^1 + d_{x,0}(2)^0$$

$$P_y = p_{y,m-1}(2)^{m-1} + p_{y,m-2}(2)^{m-2} + p_{y,m-3}(2)^{m-3} + \dots + p_{y,2}(2)^2 + p_{y,1}(2)^1 + p_{y,0}(2)^0$$

$$\langle\langle N_{B_I}\rangle\rangle = \sum_{x=0}^{S_B-1} D_x(2^m)^x + \sum_{y=1}^{T_B} P_y(2^{-m})^y$$

$$D_x(2^m)^x$$
$$= d_{x,m-1}(2)^{mx+m-1} + d_{x,m-2}(2)^{mx+m-2} + d_{x,m-3}(2)^{mx+m-3} + \cdots + d_{x,2}(2)^{mx+2} + d_{x,1}(2)^{mx+1} + d_{x,0}(2)^{mx+0}$$

$$P_y(2^{-m})^y$$
$$= p_{y,m-1}(2)^{-ym+m-1} + p_{y,m-2}(2)^{-ym+m-2} + p_{y,m-3}(2)^{-ym+m-3} + \cdots + p_{y,2}(2)^{-ym+2} + p_{y,1}(2)^{-ym+1} + p_{y,0}(2)^{-ym+0}$$

Observations about $\langle\langle N_{B_I}\rangle\rangle$:

Notice that $\langle\langle N_{B_I}\rangle\rangle$ is in the form of a polynomial evaluated at 2. There are $m(S_B + T_B)$ monomial terms in $\langle\langle N_{B_I}\rangle\rangle$.

The powers of 2 in $\langle\langle N_{B_I}\rangle\rangle$ is in the interval of the integers $[-mT_B, mS_B]$.

For every whole number θ in which $-mT_B \le \theta \le mS_B$ is true, there is one corresponding monomial term in $\langle\langle N_{B_I}\rangle\rangle$ that has a factor of 2^θ.

For every monomial term in $\langle\langle N_{B_I}\rangle\rangle$, the factor 2^θ is always multiplied to either 1 or 0. This means that $\langle\langle N_{B_I}\rangle\rangle$ is the algebraic form of the number N_{B_I} in base 2.

Define $\langle N_{B_I}\rangle$:

Observe that $\langle N_{B_I}\rangle$ is a sequence of $m(S_B + T_B)$ bits (either 1 or 0) or a concatenation of $(S_B + T_B)$ subsequences of m bits. Specifically

$$\langle N_{B_I}\rangle$$
$$= \left[\langle d_{S_B-1}\rangle\langle d_{S_B-2}\rangle\langle d_{S_B-3}\rangle \ldots \langle d_2\rangle\langle d_1\rangle\langle d_0\rangle || \langle p_1\rangle\langle p_2\rangle\langle p_3\rangle \ldots \langle p_{T_B-2}\rangle\langle p_{T_B-1}\rangle\langle p_{T_B}\rangle\right]$$
$$= \left[d_{S_B-1,m-1}d_{S_B-1,m-2}d_{S_B-1,m-3} \ldots d_{0,2}d_{0,1}d_{0,0} || p_{1,m-1}p_{1,m-2}p_{1,m-3} \ldots p_{T_B,2}p_{T_B,1}p_{T_B,0}\right]$$

The symbol || was used to separate the whole part and fractional part of $\langle N_{B_I}\rangle$.

In $\langle N_{B_I}\rangle$ above, we locate each bit using a subscript or coordinate.

If the bit is in the left part, (x, β_1) is its coordinate, and we can see this in the subscript of d_{x,β_1}. See that the bit d_{x,β_1} is inside $\langle d_x\rangle$.

The whole number x is chosen in the interval $0 \le x \le (S_B - 1)$.

The whole number β_1 is chosen in the interval $0 \le \beta_1 \le (m-1)$.

If the bit is in the right part, (y, β_0) is its coordinate, and we can see this in the subscript of p_{y,β_0}. See that the bit p_{y,β_0} is inside $\langle p_y\rangle$.

The whole number y is chosen in the interval $1 \le y \le T_B$.

The whole number β_0 is chosen in the interval $0 \le \beta_0 \le (m-1)$.

Remember that each of those bits with coordinate (x, β_1) or (y, β_0) mentioned above correspond to:

- A single number θ in the interval of integers $-mT_B \le \theta \le mS_B$

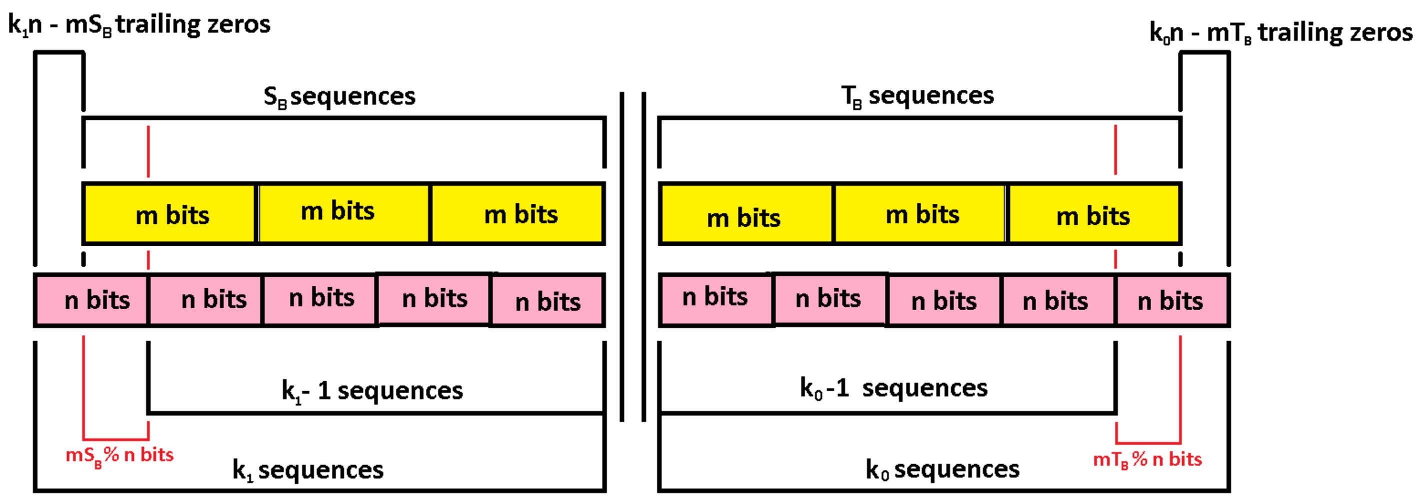
- A single monomial term in $\langle\langle N_{B_I}\rangle\rangle$.

We can then think in terms of these correspondences so that we avoid having to think with the individual monomial terms of $\langle N_{B_I}\rangle$. In the following steps, imagine how $\langle\langle N_{B_I}\rangle\rangle$ is affected by re-arranging bits in $\langle N_{B_I}\rangle$.

Solving these equations below, we can count how many sequences of n bits can be used to express $\langle N_{B_I}\rangle$

The symbol % denotes the modulo operation

In $\langle N_{B_I}\rangle$, let us say that mS_B bits to the left of || can be composed from a concatenation of k_1 sequences of n bits in which the single left-most sequence of n bits at the end have $(k_1 n - mS_B)$ trailing zeros. Those zeros are added because sometimes, there are not enough bits in the sequence of mS_B bits to make the left-most sequence of n bits. This is visualized by the following diagram:



If $(mS_B \;\%\; n) \neq 0$, then:	If $(mT_B \;\%\; n) \neq 0$, then:
For bits to the left of ||:	For bits to the right of ||:
$mS_B = (k_1 - 1)n + (mS_B \;\%\; n)$ $k_1 = \frac{mS_B - (mS_B \;\%\; n)}{n} + 1$	$mT_B = (k_0 - 1)n + (mT_B \;\%\; n)$ $k_0 = \frac{mT_B - (mT_B \;\%\; n)}{n} + 1$

If $(mS_B \;\%\; n) = 0$, then:	If $(mT_B \;\%\; n) = 0$, then:
For bits to the left of ||:	For bits to the right of ||:
$mS_B = k_1 n$ $k_1 = \frac{mS_B}{n}$	$mT_B = k_0 n$ $k_0 = \frac{mT_B}{n}$

$[Z_L]$	A sequence of $(k_1 n - mS_B)$ trailing zeros
$[Z_R]$	A sequence of $(k_0 n - mT_B)$ trailing zeros

Below, the sequence $\langle N_{B_I}\rangle$ was transformed into a concatenation of $n(k_1 + k_0)$ bits.

$$\langle N_{B_I}\rangle$$
$$= [Z_L d_{S_B-1,m-1} d_{S_B-1,m-2} d_{S_B-1,m-3} \dots d_{0,2} d_{0,1} d_{0,0} || p_{1,m-1} p_{1,m-2} p_{1,m-3} \dots p_{T_B,2} p_{T_B,1} p_{T_B,0} Z_R]$$

Re-indexed $\langle N_{B_I}\rangle$ is defined below as $[\![N_{B_I}]\!]$:

$$[\![N_{B_I}]\!] = [\langle H_{k_1-1}\rangle\langle H_{k_1-2}\rangle\langle H_{k_1-3}\rangle \ldots \langle H_2\rangle\langle H_1\rangle\langle H_0\rangle||\langle G_1\rangle\langle G_2\rangle\langle G_3\rangle \ldots \langle G_{k_0-2}\rangle\langle G_{k_0-1}\rangle\langle G_{k_0}\rangle]$$

H_{x_R,α_1} is a bit at position α_1 inside the subsequence $\langle H_{x_R}\rangle = [H_{x_R,n-1}H_{x_R,n-2}H_{x_R,n-3} \ldots H_{x_R,2}H_{x_R,1}H_{x_R,0}]$ Select α_1 in $0 \le \alpha_1 \le (n-1)$ Select x_R in $0 \le x_R \le (k_1-1)$	G_{y_R,α_0} is a bit at position α_0 inside the subsequence $\langle G_{y_R}\rangle = [G_{y_R,0}G_{x_R,1}G_{y_R,2} \ldots G_{y_R,n-3}G_{y_R,n-2}G_{y_R,n-1}]$ Select α_0 in $0 \le \alpha_0 \le (n-1)$ Select $y_R$in $1 \le y_R \le k_0$
To re-index digits to the left of $\langle N_{B_I}\rangle$ we use these equations If $[Z_L]$ exists and $(mS_B \,\%\, n) \ne 0$: $H_{x_R,\alpha_1} = d_{x,\beta_1}$ if we have x_R and α_1 in which: ➢ If $0 \le x_R \le (k_1-2)$, then $0 \le \alpha_1 \le (n-1)$ ➢ If $x_R = (k_1-1)$, then $0 \le \alpha_1 \le \big((mS_B \,\%\, n) - 1\big)$ $H_{x_R,\alpha_1} = 0$ if we have x_R and α_1 in which: ➢ If $x_R = (k_1-1)$, then $(mS_B \,\%\, n) \le \alpha_1 \le (n-1)$ If $[Z_L]$ does not exist and $(mS_B \,\%\, n) = 0$: $H_{x_R,\alpha_1} = d_{x,\beta_1}$ if we have x_R and α_1 in which: ➢ $0 \le x_R \le (k_1-1)$ and $0 \le \alpha_1 \le (n-1)$	To re-index digits to the right of $\langle N_{B_I}\rangle$ we use these equations If $[Z_R]$ exists and $(mT_B \,\%\, n) \ne 0$: $G_{y_R,\alpha_0} = p_{y,\beta_0}$ if we have y_R and α_0 in which: ➢ If $1 \le y_R \le (k_0-1)$, then $0 \le \alpha_0 \le (n-1)$ ➢ If $y_R = k_0$, then $0 \le \alpha_0 \le \big((mT_B \,\%\, n) - 1\big)$ $G_{x_R,\alpha_0} = 0$ if we have y_R and α_0 in which: ➢ If $y_R = k_0$, then $(mT_B \,\%\, n) \le \alpha_0 \le (n-1)$ If $[Z_R]$ does not exist and $(mT_B \,\%\, n) = 0$: $G_{y_R,\alpha_0} = p_{y,\beta_0}$ if we have y_R and α_0 in which: ➢ $1 \le y_R \le k_0$ and $0 \le \alpha_0 \le (n-1)$

Finding the coordinates (x_R, α_1) and (y_R, α_0) that correspond to the coordinates (x, β_1) and (y, β_0)

First convert the coordinates or subscripts as absolute distances from the separator ||:

- For (x, β_1) with $0 \le x \le (S_B - 1)$ and $0 \le \beta_1 \le (m-1)$: $|x, \beta_1| = 1 + \beta_1 + mx$
- For (y, β_0) with $1 \le y \le T_B$ and $0 \le \beta_0 \le (m-1)$: $|y, \beta_0| = 1 + \beta_0 + m(y-1)$

These formulas convert the coordinates of the original bits from $\langle N_{B_I}\rangle$ to coordinates in $[\![N_{B_I}]\!]$

Table U:

If $(\lvert x, \beta_1\rvert \,\%\, n) \ne 0$ or	If $(\lvert y, \beta_0\rvert \,\%\, n) \ne 0$:
$\alpha_1 = (\lvert x, \beta_1\rvert \,\%\, n) - 1$ $x_R = K_L - 1$ $\lvert x, \beta_1\rvert = (K_L - 1)n + (\alpha_1 + 1)$ $K_L = \dfrac{\lvert x, \beta_1\rvert - (\alpha_1 + 1)}{n} + 1$	$\alpha_0 = (\lvert y, \beta_0\rvert \,\%\, n) - 1$ $\lvert y, \beta_0\rvert = (y_R - 1)n + (\alpha_0 + 1)$ $y_R = \dfrac{\lvert y, \beta_0\rvert - (\alpha_0 + 1)}{n} + 1$

Table U+1:

If $(\lvert x, \beta_1\rvert \,\%\, n) = 0$:	If $(\lvert y, \beta_0\rvert \,\%\, n) = 0$:
$x_R = K_L - 1$ $\lvert x, \beta_1\rvert = K_L n$ $K_L = \dfrac{\lvert x, \beta_1\rvert}{n}$ $\alpha_1 = n - 1$	$\lvert y, \beta_0\rvert = y_R n$ $y_R = \dfrac{\lvert y, \beta_0\rvert}{n}$ $\alpha_0 = n - 1$

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For any value of $(\|x,\beta_1\| \,\%\, n) = 0,1,2,3,\ldots(n-1)$ and $(\|y,\beta_0\| \,\%\, n) = 0,1,2,3,\ldots(n-1)$, we can also use these equations. The function used here, $mod(*)$ is inspired from desmos	
$\alpha_1 = mod\big((mod(\|x,\beta_1\|,n)-1),n\big)$	$\alpha_0 = mod\big((mod(\|y,\beta_0\|,n)-1),n\big)$

How I got those formulas? Take for example about finding the corresponding α_1 for $|y,\beta_0|$:

$S_1\big(mod(\|x,\beta_1\|,n)\big) = mod(\|x,\beta_1\|,n)$	Graph that identity function S_1 from $mod(\|x,\beta_1\|,n)$ to $S_1\big(mod(\|x,\beta_1\|,n)\big)$ The domain and range of S_1 are both $[0,n-1]$. Horizontal axis is $mod(\|x,\beta_1\|,n)$ Vertical axis is $S_1\big(mod(\|x,\beta_1\|,n)\big)$
$S_0\big(mod(\|x,\beta_1\|,n)\big) = S_1\big(mod(\|x,\beta_1\|,n)\big)-1$ $= mod(\|x,\beta_1\|,n)-1$	Graph that function S_0 from $mod(\|x,\beta_1\|,n)$ to $S_0\big(mod(\|x,\beta_1\|,n)\big)$ The domain of S_0 is $[0,n-1]$ The range of S_0 is $[-1,n-2]$ Horizontal axis is $mod(\|x,\beta_1\|,n)$ Vertical axis is $S_0\big(mod(\|x,\beta_1\|,n)\big)$ What happened here is that the graph of S_1 vertically shifted down by 1 unit.
$S\big(mod(\|x,\beta_1\|,n)\big) = S_1\Big(S_0\big(mod(\|x,\beta_1\|,n)\big)\Big)$ $= mod\big((mod(\|x,\beta_1\|,n)-1),n\big)$ $\alpha_1 = mod\big((mod(\|x,\beta_1\|,n)-1),n\big)$	Graph that function S_1 from $mod(\|x,\beta_1\|,n)$ to $S\big(mod(\|x,\beta_1\|,n)\big)$ The domain and range of S are both $[0,n-1]$. Horizontal axis is $mod(\|x,\beta_1\|,n)$ Vertical axis is $S\big(mod(\|x,\beta_1\|,n)\big)$ What happened here is that the graph of S_1 vertically shifted down by 1 unit but at 0 in the domain, the corresponds to $n-1$ in the range. The points in the graph are $(0,n-1),(1,0),(2,1),(3,2),\ldots,(n-3,n-4),(n-2,n-3),(n-1,n-2)$

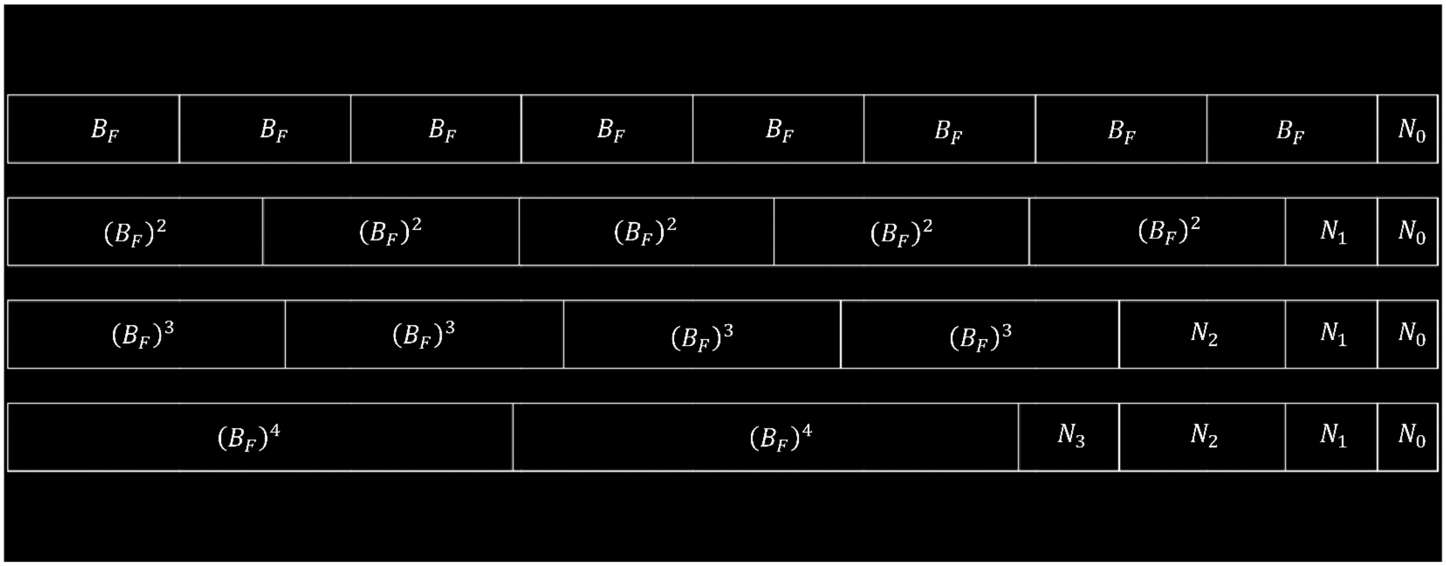
Observe that the graph of the function S exactly visualizes the conditions specified in Table U and U+1 to get α_1

In $[\![N_{B_I}]\!]$, converting every subsequence $\langle H_{x_R}\rangle$ and $\langle G_{y_R}\rangle$ into their equivalent digit in base 2^n, $[\![N_{B_I}]\!]$ becomes N_{B_F}. Explanation of why that is true is at page 14.

$$[\![N_{B_I}]\!] = [\langle H_{k_1-1}\rangle\langle H_{k_1-2}\rangle\langle H_{k_1-3}\rangle \ldots \langle H_2\rangle\langle H_1\rangle\langle H_0\rangle||\langle G_1\rangle\langle G_2\rangle\langle G_3\rangle \ldots \langle G_{k_0-2}\rangle\langle G_{k_0-1}\rangle\langle G_{k_0}\rangle]$$

Explanation behind the processes in page 3

For the whole part of N_{B_F}, consider this diagram below. They are bars with length N_X



The bars on the image represent the same number N_X, but cut in different ways. Each bar corresponds to the following equations:

In N_X, count how many $(B_F)^0$ are in it by removing all $(B_F)^1$. There are N_{L0} pieces of $(B_F)^0$.

$$N_X = N_{X,L0}B_F + N_0 \qquad N_{L0} = mod(N_X, B_F) \qquad N_0 = N_{L0}$$

$$N_{Y,L0} = (B_F)^{-1}mod(N_X, (B_F)^1)$$

$$N_{L0}(B_F)^{-1} = N_{Y,L0} \qquad (N_X)(B_F)^{-1} = N_{X,L0} + N_0(B_F)^{-1} \qquad (N_X)(B_F)^{-1} = N_{X,L0} + N_{Y,L0}$$

In E_1, count how many $(B_F)^1$ are in E_1 by removing all $(B_F)^2$. There are N_{L1} pieces of $(B_F)^1$.

$$E_1 = N_X - N_{L0} = N_{X,L1}(B_F)^2 + N_1 \qquad N_1 = N_{L1}B_F = mod(E_1, (B_F)^2)$$

$$N_{L1} = (B_F)^{-1}mod(E_1, (B_F)^2) \qquad N_{Y,L1} = (B_F)^{-2}mod(E_1, (B_F)^2)$$

Scale down the number E_1 by $(B_F)^{-1}$

$$E_1(B_F)^{-1} = N_{X,L0} \qquad E_1(B_F)^{-1} = N_{X,L1}(B_F) + N_{L1} \qquad N_{X,L0} = N_{X,L1}(B_F) + N_{L1}$$

$$N_{X,L0}(B_F)^{-1} = N_{X,L1} + N_{L1}(B_F)^{-1} \qquad N_{L1}(B_F)^{-1} = N_{Y,L1} \qquad N_{X,L0}(B_F)^{-1} = N_{X,L1} + N_{Y,L1}$$

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In E_2, count how many $(B_F)^2$ are in E_2 by removing all $(B_F)^3$. There are N_{L2} pieces of $(B_F)^2$.

$$E_2 = N_X - N_{L0} - B_F N_{L1} = N_{X,L2}(B_F)^3 + N_2 \qquad N_2 = N_{L2}(B_F)^2 = mod(E_2,(B_F)^3)$$

$$N_{L2} = (B_F)^{-2} mod(E_2,(B_F)^3) \qquad N_{Y,L2} = (B_F)^{-3} mod(E_2,(B_F)^3)$$

Scale down the number E_2 by $(B_F)^{-2}$

$$E_2(B_F)^{-2} = N_{X,L1} \qquad E_2(B_F)^{-2} = N_{X,L2}(B_F) + N_{L2} \qquad N_{X,L1} = N_{X,L2}(B_F) + N_{L2}$$

$$N_{X,L1}(B_F)^{-1} = N_{X,L2} + N_{L2}(B_F)^{-1} \qquad N_{L2}(B_F)^{-1} = N_{Y,L2} \qquad N_{X,L1}(B_F)^{-1} = N_{X,L2} + N_{Y,L2}$$

In E_3, count how many $(B_F)^3$ are in E_3 by removing all $(B_F)^4$. There are N_{L3} pieces of $(B_F)^3$.

$$E_3 = N_X - N_{L0} - B_F N_{L1} - (B_F)^2 N_{L2} = N_{X,L3}(B_F)^4 + N_3 \qquad N_3 = N_{L3}(B_F)^3 = mod(E_3,(B_F)^4)$$

$$N_{L3} = (B_F)^{-3} mod(E_3,(B_F)^4) \qquad N_{Y,L3} = (B_F)^{-4} mod(E_3,(B_F)^4)$$

Scale down the number E_3 by $(B_F)^{-3}$

$$E_3(B_F)^{-3} = N_{X,L2} \qquad E_3(B_F)^{-3} = N_{X,L3}(B_F) + N_{L3} \qquad N_{X,L2} = N_{X,L3}(B_F) + N_{L3}$$

$$N_{X,L2}(B_F)^{-1} = N_{X,L3} + N_{L3}(B_F)^{-1} \qquad N_{L3}(B_F)^{-1} = N_{Y,L3} \qquad N_{X,L2}(B_F)^{-1} = N_{X,L3} + N_{Y,L3}$$

and so on…

For the fractional part of N_{B_F}, consider this reasoning below:

If we compare what we do in the computations versus the polynomial form of the number N_Y in base B_F, you see that we are essentially isolating the digits in base B_F one by one, from the digit multiplied to $(B_F)^{-1}$ to the digit multiplied to $(B_F)^{-\cdots}$. Remember that in each of the equations below, those variables with subscript X represent the whole part of the number or expression on the left side of the equation, while those variables with subscript Y represent the fractional part of the number or expression on the left side of the equation.

$$(N_Y)(B_F) = N_{X,R1} + N_{Y,R1}$$

$$(N_{Y,R1})(B_F) = N_{X,R2} + N_{Y,R2}$$

$$(N_{Y,R2})(B_F) = N_{X,R3} + N_{Y,R3}$$

$$(N_{Y,R3})(B_F) = N_{X,R4} + N_{Y,R4}$$

$$\ldots = \cdots$$

$P_1 = N_Y B_F$ $N_{Y,R1} = (B_F)^0 mod(P_1, 1)$ $P_1 - N_{Y,R1} = N_{X,R1}$	$P_2 = N_{Y,R1} B_F$ $N_{Y,R2} = (B_F)^1 mod(P_1,(B_F)^{-1})$ $P_2 - N_{Y,R2} = N_{X,R2}$	$P_3 = N_{Y,R2} B_F$ $N_{Y,R3} = (B_F)^2 mod(P_1,(B_F)^{-2})$ $P_3 - N_{Y,R3} = N_{X,R3}$
$P_4 = N_{Y,R3} B_F$ $N_{Y,R4} = (B_F)^3 mod(P_1,(B_F)^{-3})$ $P_4 - N_{Y,R4} = N_{X,R4}$	$P_5 = N_{Y,R4} B_F$ $N_{Y,R5} = (B_F)^4 mod(P_1,(B_F)^{-4})$ $P_5 - N_{Y,R5} = N_{X,R5}$	And so on…

What if we remove the dependence on P_1? What if we start by doing the modulo operation from $(B_F)^{-1}$ instead of getting the fractional part of P_1 first?

$P_1 = N_Y B_F$ $N_{Y,R1} = (B_F)^1 mod(N_Y, (B_F)^{-1})$ $P_1 - N_{Y,R1} = N_{X,R1}$	$P_2 = N_{Y,R1} B_F$ $N_{Y,R2} = (B_F)^2 mod(N_Y, (B_F)^{-2})$ $P_2 - N_{Y,R2} = N_{X,R2}$	$P_3 = N_{Y,R2} B_F$ $N_{Y,R3} = (B_F)^3 mod(N_Y, (B_F)^{-3})$ $P_3 - N_{Y,R3} = N_{X,R3}$
$P_4 = N_{Y,R3} B_F$ $N_{Y,R4} = (B_F)^4 mod(N_Y, (B_F)^{-4})$ $P_4 - N_{Y,R4} = N_{X,R4}$	$P_5 = N_{Y,R4} B_F$ $N_{Y,R5} = (B_F)^5 mod(N_Y, (B_F)^{-5})$ $P_5 - N_{Y,R5} = N_{X,R5}$	And so on…

In words, we can describe what is happening at the upper-right corner as follows. It uses the same intuition in reasoning about the whole part of N_{B_I} , but done “backwards”. Notice that we just reversed some signs on the powers used in the modulo equations. We have here some re-arrangement of the equations to count how many of each negative powers are present.

The intuition behind all this is “scaling”. This is illustrated in the following pages.

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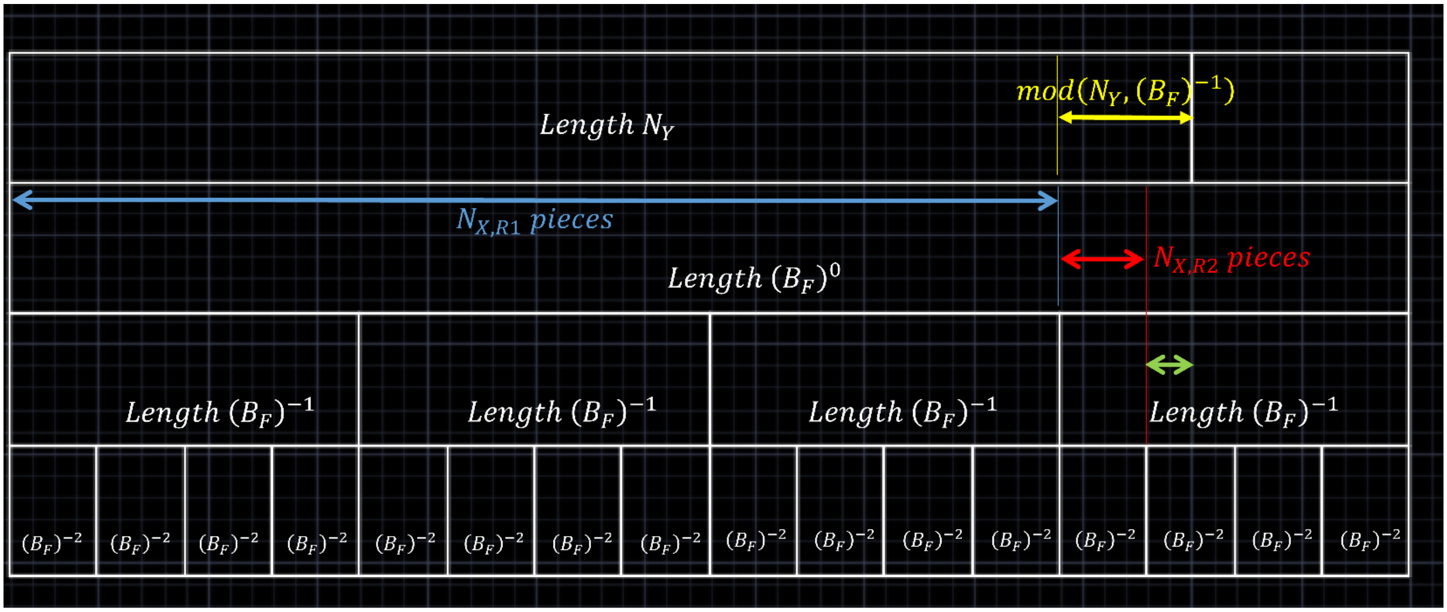

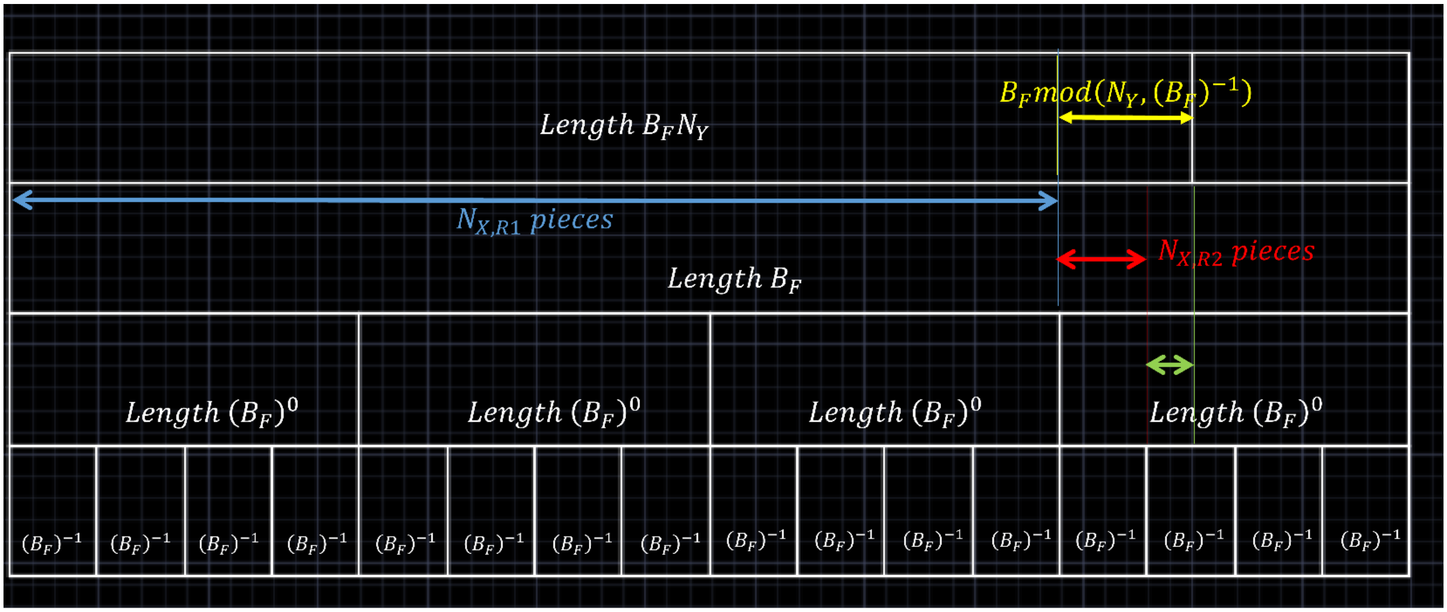

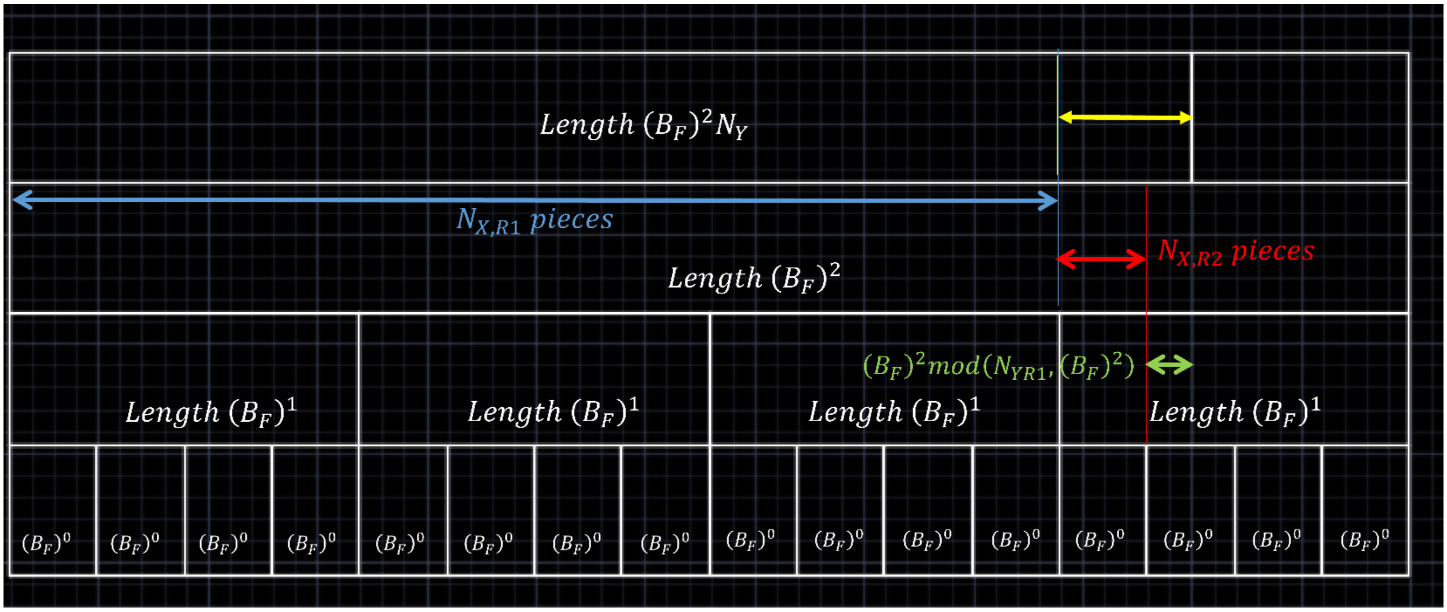

$$N_Y = N_{X,R1}(B_F)^{-1} + mod(N_Y,(B_F)^{-1})$$
$$(B_F)(N_Y) = N_{X,R1} + N_{Y,R1}$$

$$N_{YR1} = N_{X,R2}(B_F)^{-2} + mod(N_{YR1},(B_F)^{-2})$$
$$N_{YR1}B_F B_F = N_{X,R2} + (B_F)^2 mod(N_{YR1},(B_F)^{-2})$$
$$\left(N_{Y,R1}\right)(B_F) = N_{X,R2} + (B_F)^2 mod(N_{YR1},(B_F)^{-2})$$
$$\left(N_{Y,R1}\right)(B_F) = N_{X,R2} + N_{Y,R2}$$

$$N_{YR2} = N_{X,R3}(B_F)^{-3} + mod(N_{YR2},(B_F)^{-3})$$
$$N_{YR2}B_F B_F B_F = N_{X,R3} + (B_F)^3 mod(N_{YR2},(B_F)^{-3})$$
$$\left(N_{Y,R2}\right)(B_F) = N_{X,R3} + (B_F)^3 mod(N_{YR2},(B_F)^{-3})$$
$$\left(N_{Y,R2}\right)(B_F) = N_{X,R3} + N_{Y,R3}$$

$$N_{YR3} = N_{X,R4}(B_F)^{-4} + mod(N_{YR3},(B_F)^{-4})$$
$$N_{YR3}B_F B_F B_F B_F = N_{X,R4} + (B_F)^4 mod(N_{YR3},(B_F)^{-4})$$
$$\left(N_{Y,R3}\right)(B_F) = N_{X,R4} + (B_F)^4 mod(N_{YR3},(B_F)^{-4})$$
$$\left(N_{Y,R3}\right)(B_F) = N_{X,R4} + N_{Y,R4}$$
$$\ldots = \cdots$$

And so on…

$$N_{YR1} = mod(N_Y,(B_F)^{-1}) = mod(B_F N_Y,(B_F)^0)$$
$$N_{Y,R1} = B_F N_{YR1}$$

$$N_{YR2} = mod(N_{YR1},(B_F)^{-2})$$
$$N_{Y,R2} = (B_F)^2 mod(N_{YR1},(B_F)^{-2})$$

$$N_{YR3} = mod(N_{YR2},(B_F)^{-3})$$
$$N_{Y,R3} = (B_F)^3 mod(N_{YR2},(B_F)^{-3})$$

$$N_{YR4} = mod(N_{YR3},(B_F)^{-4})$$
$$N_{Y,R4} = (B_F)^4 mod(N_{YR3},(B_F)^{-4})$$

And so on…

Key concept: Scaling does not change the counts of pieces

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Explanation behind why you can convert every subsequence in $\llbracket N_{B_I} \rrbracket$ into a number in base B_F and so $\llbracket N_{B_I} \rrbracket$ becomes N_{B_F}:

Let's say you started from the binary sequence $\langle N_{B_I} \rangle$, and you have $B_F = 2^3$

In the polynomial form of $\langle N_{B_I} \rangle$: let's say that V_{num} stands for a bit in the position num in which $num \geq 0$ if the bit is located to the left of the separator and $num < 0$ if the bit is located to the right of the separator. V_{num} is multiplied to a number 2^{num} :

In the polynomial form of the whole part of $\langle N_{B_I} \rangle$, you have this sequence of powers that have a bit multiplied to them.

$$(V_0)(2)^0 + (V_1)(2)^1 + (V_2)(2)^2 + (V_3)(2)^3 + (V_4)(2)^4 + (V_5)(2)^5 + (V_6)(2)^6 + (V_7)(2)^7 + \cdots$$

That number can then be re-arranged as:

$$\left((V_0)(2)^0 + (V_1)(2)^1 + (V_2)(2)^2\right)(2^3)^0 + \left((V_3)(2)^0 + (V_4)(2)^1 + (V_5)(2)^2\right)(2^3)^1 + \left((V_6)(2)^0 + (V_7)(2)^1 + (V_8)(2)^2\right)(2^3)^2 + \cdots$$

In the polynomial form of the fractional part of $\langle N_{B_I} \rangle$, you have this sequence of powers that have a bit multiplied to them.

$$(V_{-1})(2)^{-1} + (V_{-2})(2)^{-2} + (V_{-3})(2)^{-3} + (V_{-4})(2)^{-4} + (V_{-5})(2)^{-5} + (V_{-6})(2)^{-6} + (V_{-7})(2)^{-7} + (V_{-8})(2)^{-8} + \cdots$$

That number can then be re-arranged as:

$$\left((V_{-1})(2)^2 + (V_{-2})(2)^1 + (V_{-3})(2)^0\right)(2^3)^{-1} + \left((V_{-4})(2)^2 + (V_{-5})(2)^1 + (V_{-6})(2)^0\right)(2^3)^{-2} + \left((V_{-7})(2)^2 + (V_{-8})(2)^1 + (V_{-9})(2)^0\right)(2^3)^{-3} + \cdots$$

Those numbers in orange color, corresponding to a certain subsequence in $\llbracket N_{B_I} \rrbracket$, can then be converted into a number in base B_F because of the re-arrangements we did for powers of 2. Observe that those numbers in orange do not exceed B_F.