MA-101 END SEMESTER ANSWER SHEET

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SECTION-BA1

DISCIPLINE-COMPUTER SCIENCE AND ENGINEERING

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(1) A function f is defined on $[1,\infty)$ by.

$$F(x) = \frac{(-1)^{n-1}}{n}, \text{ for } n \le x < n+1 \; (n = 1,2,3 \ldots$$

Examine the convergence of the integrals

1.) $\int_1^\infty f(x)\,dx$

2.) $\int_1^\infty |f|(x)\,dx$

Sol^n.) $f(x) = \frac{(-1)^{n-1}}{n}$

Let $I = \int_1^\infty f(x)\,dx$

$I = \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx \; \text{------} \; \infty$

$I = \int_1^2 \frac{(-1)^{1-1}}{1}\,dx + \int_2^3 \frac{(-1)^{2-1}}{2}\,dx \; \text{------} \; \infty$

$I = \int_1^2 x + \int_2^3 \frac{-1}{2}(x) \; \text{------} \; \infty$

$I = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \; \text{------} \; \infty.$

This is an infinite alternating series

so for Alternating series convergence.

Leibnitz Theorem states that

A alternating series of the type $\Sigma (-1)^{n+1} U_n$ is convergent if

1) $U_n > 0 \; \forall n$.

2) $U_n \geq U_{n+1} \; \forall n \geq N$

3) $U_n \longrightarrow 0$ as $n \longrightarrow \infty$

for $I = 1 - \frac{1}{2} + \frac{1}{3} - - - - \infty$.

$I = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

so by comparing

we get $U_n = \frac{1}{n}$

so.

1) $\frac{1}{n} > 0 \; \forall n$.

2) $\frac{1}{n} \geq \frac{1}{n+1} \; \forall n \geq N \; (N=1)$.

3) $\frac{1}{n} \longrightarrow 0$ as $n \longrightarrow \infty$

Hence by Leibnitz Theorem

we can conclude that $I = \int_0^{\infty} f(x)\,dx = \int_0^{\infty} \frac{(-1)^{n-1}}{n}\,dx$

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converges

2) $\int_1^\infty |f|(x)\,dx$.

Let $Q = \int_1^\infty |f|(x)\,dx$

$Q = \int_1^2 |f|(x)\,dx + \int_2^3 |f|(x)\,dx \;---- \infty$

$Q = \left[|1.x|\right]_1^2 + \left[\left|\frac{1}{2}(x)\right|\right]_2^3 \;---- \infty$

$Q = 1 + \frac{1}{2} + \frac{1}{3} \;---- \infty$

Let $x_n = \frac{1}{n}$ be a sequence.

and S_k be the partial sum of x_n upto k terms

so $S_k = \sum_{n=1}^{k} x_n = \sum_{n=1}^{k} \frac{1}{n}$

so

$S_{2^k} = \sum_{n=1}^{2^k} \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + -- \left(\frac{1}{2^{k-1}+1}\right.$

$S_{2^k} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) ---- \left(\frac{1}{2^k} -- \frac{1}{2^k}\right)$

$S_{2^k} > 1 + \frac{k}{2}$

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and $1+\frac{k}{2} \longrightarrow \infty$ as $k \longrightarrow \infty$.

So S_{2^k} is unbounded.

and If the partial sum of an infinite series is unbounded then the infinite series is also divergent

Hence the series

$1+\frac{1}{2}$ - - - ∞ is divergent

Hence $Q = \int_1^\infty f(x)\, dx$ is divergent.

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2) Find the values of $\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dy\,dx$ and $\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dx\,dy$. Noting the values are different can we conclude that Fubini's Theorem is violated here?

Soln:

Let $\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dy\,dx \longrightarrow \mathrm{I}$

and $\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dx\,dy \longrightarrow \mathrm{II}$

Evaluating I

Let $\int_0^1 \frac{x-y}{(x+y)^3}\,dy = P$

so $P = \int_0^1 \frac{x-y}{(x+y)^3}\,dy = \int_0^1 \frac{x}{(x+y)^3} - \frac{y}{(x+y)^3}\,dy.$

Putting $y = x\tan^2\theta$ and solving we get

$$\frac{2}{x}\left[-\frac{\cos 4\theta}{4} - \frac{\sin 4\theta}{4}\right]$$

and $\tan\theta = \sqrt{\frac{y}{x}}$ so $\sin\theta = \sqrt{\frac{y}{x+y}}$.

and $\cos\theta = \sqrt{\frac{x}{x+y}}$

so $\frac{1}{2x}\left[\frac{-x^2}{(x+y)^2} - \frac{y^2}{(x+y)^2}\right]_{y=0}^{1} = P$

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$$P = \frac{1}{2x}\left[\frac{-x^2}{(x+1)^2} - \frac{1}{(x+1)^2} + 1\right]$$

$$I = \int_0^1 P\,dx = \frac{1}{2}\int_0^1 \left[\frac{-x}{(x+1)^2} - \frac{1}{x(x+1)^2} + \frac{1}{x}\right) dx.$$

$$= \frac{1}{2}\int_0^1 \left[\cancel{\frac{1}{x}} - \cancel{\frac{1}{(x+1)}} + \frac{1}{(x+1)^2} - \cancel{\frac{1}{x}} + \cancel{\frac{1}{(x+1)}} + \frac{1}{(x+1)^2}\right) dx$$

$$= \int_0^1 \frac{1}{(1+x)^2}\,dx = \left[\frac{-1}{(x+1)}\right]_{x=0}^{1} = \frac{1}{2} = I$$

And $II = \int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dx\,dy$

let $\int_0^1 \frac{x-y}{(x+y)^3}\,dx = Q.$

$$Q = \int_0^1 \frac{x-y}{(x+y)^3}\,dx$$

Putting $x = y\tan^2\theta$ and solving

we get

$$\frac{2}{y}\left[\frac{\sin 4\theta}{4} + \frac{\cos 4\theta}{4}\right] = \frac{1}{2y}\left[\frac{x^2}{(x+y)^2} + \frac{y^2}{(x+y)^2}\right]_{x=0}^{1}$$

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$$\frac{1}{2y}\left[\frac{1}{(1+y)^2}+\frac{y^2}{(1+y)^2}-0-1\right]=Q$$

$$II=\int_0^1 Q\,dy=\frac{1}{2}\int_0^1 \frac{1}{y(1+y)^2}+\frac{y}{(1+y)^2}-\frac{1}{y}\,dy$$

$$=\frac{1}{2}\int_0^1\left[\cancel{\frac{1}{y}}-\frac{1}{(1+y)}-\frac{1}{(1+y)^2}+\frac{1}{(1+y)}-\frac{1}{(1+y)^2}-\cancel{\frac{1}{y}}\right]$$

$$=\frac{1}{2}\int_0^1 \frac{-2}{(1+y)^2}=\left[\frac{1}{1+y}\right]_{y=0}^{1}=\frac{-1}{2}=II$$

So $I \neq II$

But Fubini's Theorem is not violated in this case as according to the definition of Fubini's Theorem it can be applied to only those function whose absolute integrals are convergent

but in this case.

$$\int_0^1\int_0^1\left|\frac{x-y}{(x+y)^3}\right|dy\,dx \longrightarrow \infty$$

Proof :-

Let $\int_0^1\int_0^1\left|\frac{x-y}{(x+y)^3}\right|dy\,dx = J$

$$J = \int_0^1 \left[\int_0^x \frac{x-y}{(x+y)^3}\, dy + \int_x^1 \frac{y-x}{(x+y)^3}\, dy \right] dx$$

$$J = \int_0^1 \left[\frac{y}{(x+y)^2}\right]_0^x - \left[\frac{y}{(x+y)^2}\right]_x^1 dx$$

$$= \int_0^1 \frac{x}{4x^2} + \frac{x}{4x^2} - \frac{1}{(x+1)^2}\, dx$$

$$= \int_0^1 \frac{1}{2x} - \frac{1}{(x+1)^2}\, dx$$

$$= \frac{1}{2}\int_0^1 \frac{dx}{x} - \frac{1}{2}$$

Clearly $\int_0^1 \frac{dx}{x}$ is divergent

Hence $J \longrightarrow \infty$.

or.

$$\int_0^1 \int_0^1 \left| \frac{x-y}{(x+y)^3} \right| dy\, dx \longrightarrow \infty$$

Therefore Fubini's Theoram is not violated as Fubini's Theoram was not applicable on this function

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3. Let $f : [0,1] \to (0,\infty)$ be a continuous function then show that

$$\lim_{n\to\infty} \left(f\left(\tfrac{1}{n}\right) f\left(\tfrac{2}{n}\right) \cdots f\left(\tfrac{n}{n}\right)\right)^{\frac{1}{n}} = e^{\int_0^1 \ln f(x)\,dx}$$

Sol. Let

$$L = \lim_{n\to\infty} \left[f\left(\tfrac{1}{n}\right) \cdots f\left(\tfrac{n}{n}\right)\right]^{\frac{1}{n}}$$

so taking ln on both side

$$\ln L = \lim_{n\to\infty} \frac{1}{n} \ln\left(f\left(\tfrac{1}{n}\right) f\left(\tfrac{2}{n}\right) \cdots f\left(\tfrac{n}{n}\right)\right)$$

$$\ln L = \lim_{n\to\infty} \frac{1}{n} \left[\ln f\left(\tfrac{1}{n}\right) + \ln f\left(\tfrac{2}{n}\right) \cdots \ln f\left(\tfrac{n}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \ln f\left(\tfrac{r}{n}\right)$$

As $\ln f(x)$ is continuous on $[0,1]$ as $f > 0$.

so $\lim_{n\to\infty} \sum_{r=1}^{n} f\left(\frac{r}{n}\right) \times \frac{1}{n}$ must be equal to $\int_0^1 f(x)\,dx$

Claim $\to$ ie $\lim_{n\to\infty} \sum_{r=1}^{n} f\left(\frac{r}{n}\right) \times \frac{1}{n} = \int_0^1 f(x)\,dx$

$\rightarrow$ Claim

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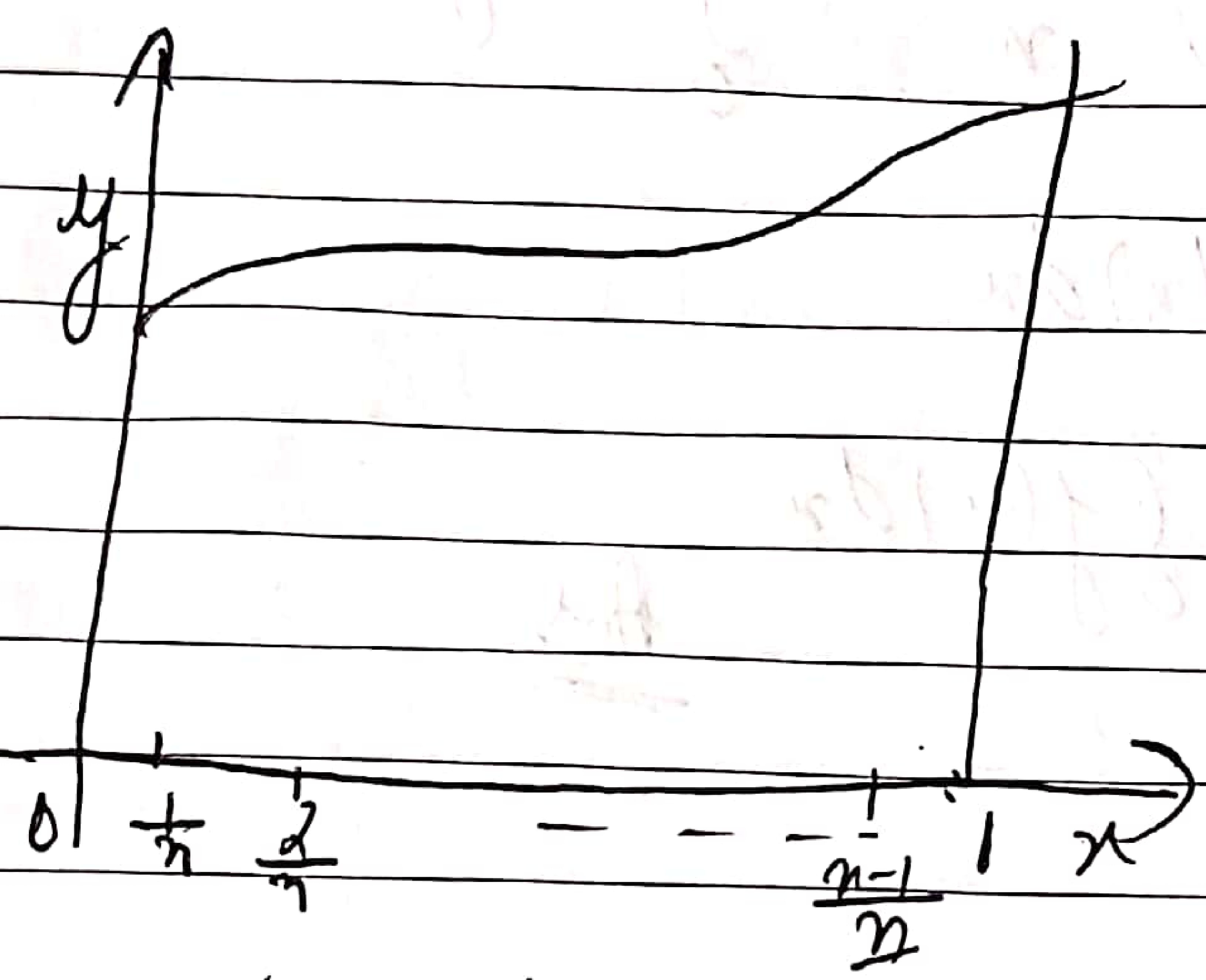
Proof of the claim

Let $h\left(\frac{r}{n}\right)$ or $h(x)$ is a continuous function on $x \in [0,1]$

So we need to prove that

$$\lim_{n\to\infty} \sum_{r=1}^{n} h\left(\frac{r}{n}\right) \times \frac{1}{n} = \int_0^1 h(x)\,dx$$



Consider the tagged partition

$$\tilde{P} = \left\{ \left(\left[0,\frac{1}{n}\right], \frac{1}{n}\right), \text{-----} \left(\left[\frac{n-1}{n}, 1\right], 1\right) \right\}$$

And the Riemann Sum corresponding to $\tilde{P}$

$$S(\tilde{P}, f) = \sum_{r=1}^{n} h\left(\frac{r}{n}\right) \times \frac{1}{n}$$

now as h is continuous
so h must be integrable on $[0,1]$

$$\text{so } \lim_{\|P\|\to 0} \sum_{r=1}^{n} h\left(\frac{r}{n}\right) \times \frac{1}{n} = \int_0^1 h(x)\,dx$$

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$$\boxed{\lim_{n\to\infty} \sum_{r=1}^{n} f\left(\frac{r}{n}\right) \times \frac{1}{n} = \int_0^1 f(x)\,dx}$$

Hence Proved

so we can say

$$\lim_{n\to\infty} \sum_{r=1}^{n} \ln f\left(\frac{r}{n}\right) \times \frac{1}{n} = \int_0^1 \ln f(x)\,dx$$

so $\ln L = \int_0^1 f(x)\,dx$

so

$$L = e^{\int_0^1 f(x)\,dx}$$ Ans

Hence Proved

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Q. Let f be a continuous function, then prove that

$$\int_0^y \left(\int_0^u f(x)\, dx \right) du = \int_0^y f(t)(y-t)\, dt$$

Sol^n:- Consider region R defined as $0 \le u \le y$ and $0 \le x \le u$

$\because$ f is a continuous function

$$\iint_R f(x)\, dA = \int_0^y \int_0^u f(x)\, dx\, du \longrightarrow (1)$$

Subsequently region R can also be defined as-

$0 \le x \le y$ and $\cancel{x} \le u \le y$.

$$\iint_R f(x)\, dA = \int_0^y \int_x^y f(x)\, du\, dx \longrightarrow (2)$$

$$\iint_R f(x)\, dA = \int_0^y \left(f(x) \int_x^y du \right) dx.$$

$$\iint_R f(x)\, dA = \int_0^y f(x)(y-x)\, dx \longrightarrow (3)$$

From (1) and (3)

$$\iint_R f(x)\, dA = \int_0^y f(x)(y-x)\, dx = \int_0^y \int_0^u (f(x)\, dx)\, du$$

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$$\int_0^y f(t)(y-t)\,dt = \int_0^y \int_0^u \left(f(x)\,dx\right) du$$

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(5) For what values of p and q the following integral is convergent

$$\int_0^1 \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}\,dx$$

Sol^n:- $\int_0^1 \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}\,dx$

$$= \underbrace{\int_0^1 \frac{x^{p-1}\,dx}{(1+x)^{p+q}}}_{I} + \underbrace{\int_0^1 \frac{x^{q-1}}{(1+x)^{p+q}}\,dx}_{J}$$

Checking convergence of I

Case I

$p \geq 1$

Here, $1 \geq x \geq 0$.

$1 \geq x^{p-1} \geq 0 \quad (p \geq 1)$

Also $2 \geq (1+x) \geq 1$.

$$\frac{1}{2} \leq \frac{1}{(1+x)} \leq 1$$

$$\left(\frac{1}{2}\right)^{p+q} \leq \frac{1}{(1+x)^{p+q}} \leq 1$$

Since both x^{p-1} and $\frac{1}{(1+x)^{p+q}}$ are bounded so

$\therefore \frac{x^{p-1}}{(1+x)^{p+q}}$ is bounded.

Hence, integral is convergent when $p \geq 1$

<u>Case 2.</u> $p < 1$

Here $1 \geq x \geq 0$
$1 \leq x^{p-1} < x \quad (p<1)$.

Since it tends to ∞

$\therefore \int_0^1 \frac{x^{p-1}}{(1+x)^{p+q}} dx$ is an improper integral

as it contains a vertical asymptote as $x \to 0$

Let $g(x) = \frac{1}{(x)^{1-p}}$ and $f(x) = \frac{1}{x^{1-p}(1+x)^{p+q}}$

By limit comparison test

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{\frac{1}{x^{1-p}(1+x)^{p+q}}}{\frac{1}{x^{1-p}}} = 1$$

So integral $f(x)$ and $g(x)$ behave similarly

so let $h(x) = \frac{1}{x^t}$ (checking the convergence of integral of $h(x)$ for comple

so let $A = \int_0^1 h(x) dx$.

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$A = \int_0^1 \frac{1}{x^t} dt = \lim_{a \to 0} \int_a^1 \frac{1}{x^t} dx = \begin{cases} \lim_{a\to 0^+} \left[\frac{x^{1-t}}{1-t}\right]_a^1 & \text{when } t \neq 1 \\ \lim_{a \to 0^+} \ln\frac{1}{a} & \text{when } t = 1. \end{cases}$

so $A = \int_0^1 \frac{1}{x^t} dx = \begin{cases} \lim_{a\to 0^+} \frac{1-a^{1-t}}{1-t}, & t \in (0,1) \\ \lim_{a\to 0^+} \ln\frac{1}{a} & t = 1 \\ \lim_{a \to 0^+} \frac{-1}{1-t} + \frac{1}{a^{t-1}(1-t)} & t > 1 \end{cases}$

Clearly A only converges for $0 < t < 1$

and $\int_0^1 f(x)dx$ behaves similarly as $\int_0^1 g(x)\,dx$.

$g(x) = \frac{1}{x^{1-p}} = \frac{1}{x^t}$

so $t = 1-p$.

$0 < 1-p < 1$.

so $0 < p < 1$

hence $I = \int_0^1 f(x)\,dx$ converges for

$0 < p\ 1$ and $p \geq 1 \rightarrow$ by case 1

so I converges for $p > 0$ and diverges for $q \leq 0$

and for J

$J = \int_0^1 \frac{x^{q-1} dx}{(1+x)^{p+q}}$

II is similar to I

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so, $\int_0^1 \frac{x^{q-1}}{(1+x)^{p+q}}$ converges for $q > 0$.

and diverges for $q \leq 0$

Hence as $\int_0^1 \frac{x^{p-1} + x^{q-1}}{(1+x)^{p+q}} = I + J$.

so $\int_0^1 \frac{x^{p-1} + x^{q-1}}{(1+x)^{p+q}}$ converges when both I and J converge

Hence $\int_0^1 \frac{x^{p-1} + x^{q-1}}{(1+x)^{p+q}}$ converges for $p > 0$ and q

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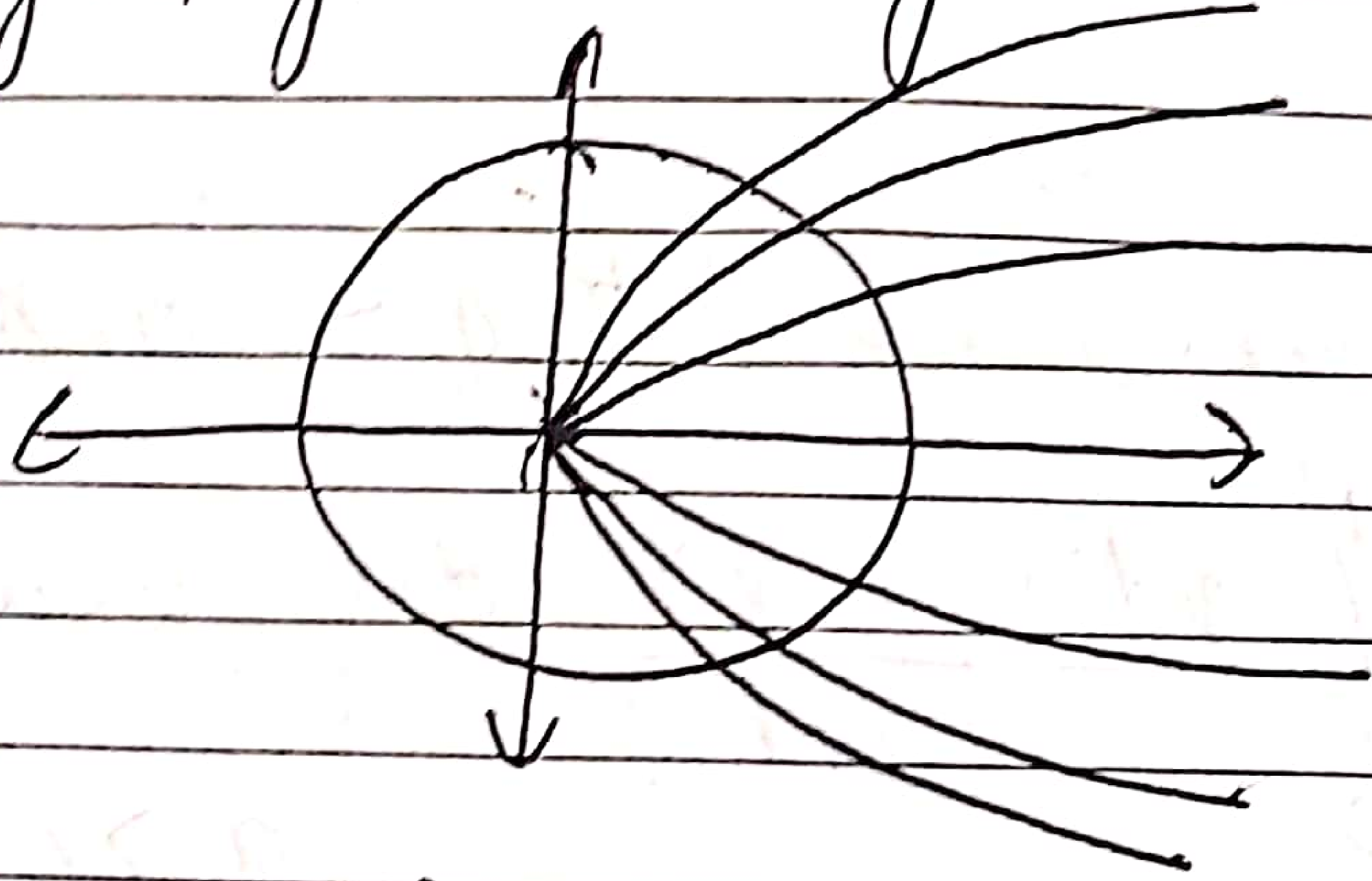
(6) Q. Describe the two regions (a, b) space for which the function $f_{a,b}(x,y) = ay^2 + bx$ restricted to the circle $x^2+y^2=1$ has exactly two and exactly four ~~tt~~ critical points.

Solⁿ. $f_{a,b}(x,y) = ay^2 + bx.$

and the circle is $x^2 + y^2 = 1$

let $g(x,y) = x^2 + y^2 - 1$



As $f(x,y)$ is a continuous function on a closed bounded set $\{g(x,y)=0\}$.

$\therefore$ it will also achieve its maximum and minimum value at $[g=0]$ (constraint).

Using Lagrange Multiplier

$$\boxed{\nabla f = \lambda \nabla g.}$$

$$\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} = \lambda\left(\frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j}\right)$$

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$b\hat{i} + 2ay\hat{j} = 2\lambda x\hat{i} + 2\lambda y\hat{j}$

so $\boxed{b = 2\lambda x}$ —— (1)

and $\cancel{2}ay = \cancel{2}\lambda y.$

$(a-\lambda)y = 0.$

$\boxed{\lambda = a}$ or $\boxed{y = 0}$ —— (2)

1. If $y = 0.$

Using $g(x,y) = 0 \Rightarrow x^2 + y^2 - 1 = 0.$

$x^2 = 1$ $\boxed{x = \pm 1}$

2. If $\lambda = a$

$\Rightarrow b = 2ax.$

$g(x,y) = 0.$ $\boxed{x = \frac{b}{2a}}$

$\left(\frac{b}{2a}\right)^2 + y^2 - 1 = 0.$

$y = \pm\sqrt{1 - \left(\frac{b}{2a}\right)^2}$

So possible critical points

a) $(1, 0)$

b) $(-1, 0)$

c) $\left(\frac{b}{2a}, \sqrt{1-\left(\frac{b}{2a}\right)^2}\right)$

d) $\left(\frac{b}{2a}, -\sqrt{1-\left(\frac{b}{2a}\right)^2}\right)$.

(1) For 4 critical point

$1-\left(\frac{b}{2a}\right)^2 > 0$ (i.e. points c and d are real)

$$\left(1-\frac{b}{2a}\right)\left(1+\frac{b}{2a}\right) > 0$$

$$-1 < \frac{b}{2a} < 1$$

$$\boxed{-2a < b < 2a.}$$

(2) For 2 critical points

A) $1-\left(\frac{b}{2a}\right)^2 < 0$ (i.e. points c and d are imaginary

$$\left(1-\frac{b}{2a}\right)\left(1+\frac{b}{2a}\right) < 0$$

~~so~~ $b \in (-\infty, -2a) \cup (2a, \infty)$

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B.P. If $1-\left(\frac{b}{2a}\right)^2 = 1$

$$1 = \left(\frac{b}{2a}\right)^2$$

so $\frac{b}{2a} = \pm 1$

so points c and d $= \frac{b}{2a}, \pm\sqrt{1-\left(\frac{b}{2a}\right)^2} = (1,0)$ or $(-1,0)$

i.e. we get the same points

So

1.) $-2a < b < 2a$ for exactly 4 critical points

2.) $b \in (-\infty, -2a] \cup [2a, \infty)$ for exactly 2 critical points.

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7.4. Let $a < b$ and

$$f(x) = \begin{cases} 0, & \text{if } x \in [a,b] \cap \mathbb{Q} \\ x, & \text{if } x \in [a,b] \text{ is irrational}, \end{cases}$$

where $\mathbb{Q}$ denotes the set of rational numbers. Find the upper and lower Riemann integrals of $f(x)$ over $[a,b]$ and conclude whether $f(x)$ is Riemann integrable.

Soln 7.4.

For finding the upper and lower integral ~~to~~ the partition that can be used is

$$P_n = \left\{a, a + \frac{b-a}{n}, a + \frac{2(b-a)}{n} ---- b\right\}$$

$$m_k(f) = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$$

$$M_k(f) = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$$

Case I : $b > a > 0$

$$L_P(f) = \sum_{k=1}^{n} m_k(f) \times \frac{(b-a)}{n}$$

and $f(x) = \begin{cases} 0, & x \in [a,b] \cap \mathbb{Q} \\ x, & x \in [a,b] \text{ irrational} \end{cases}$

so $m_k(f) = 0$.

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[$\because$, in an interval of real numbers both rational and irrational numbers are present]

$$\boxed{L_P(f) = \sum_{r=1}^{n} 0 \times \frac{(b-a)}{n} = 0}$$

$$\boxed{\text{Lower integral} = \lim_{n\to\infty} L_P(f) = 0} \left(\underline{\int_a^b} f\right)$$

and $M_r(f) = a + r\frac{(b-a)}{n}$

$$U_P(f) = \sum_{r=1}^{n} M_r(f) \times \Delta x.$$

$$U_P(f) = \sum_{r=1}^{n} \left[a + r\frac{(b-a)}{n}\right]\left(\frac{b-a}{n}\right)$$

$$U_P(f) = \sum_{r=1}^{n} \frac{a(b-a)}{n} + \frac{(b-a)^2}{n^2} \sum_{r=1}^{n} r$$

$$U_P(f) = \frac{a(b-a)\times \cancel{n}}{\cancel{n}} + \frac{(b-a)^2}{n^2} \times \frac{n(n+1)}{2}$$

$$U_P(f) = a(b-a) + \frac{(b-a)^2}{2}\left[1 + \frac{1}{n}\right]$$

Upper integral $= \lim_{n\to\infty} U_P(f)$

$$\left(\overline{\int_a^b} f\right) = a(b-a) + \frac{(b-a)^2}{2}$$

$$= \frac{2\cancel{ab} - 2a^2 + b^2 + a^2 - \cancel{2ab}}{2}$$

$$= \left(\frac{b^2 - a^2}{2}\right).$$

$0 \neq \left(\frac{b^2-a^2}{2}\right)$

Lower integral $\neq$ upper integral

So, it is not Riemann integrable in this case.

Case 2 $\quad a < b < 0$

The Partion can be, $P_n = \{a, a+\frac{(b-a)}{n}, a+2\frac{(b-a)}{n} \dots b\}$

so $m_k(f) = \inf\{f(x) ; x \in [x_{k-1}, x_k]\}$

$\boxed{m_k(f) = a+\frac{(k-1)(b-a)}{n}}$ (as $a, b < 0$; so all terms must be less than 0.

$$L_{P_n}(f) = \sum_{k=1}^{n}\left(a+\frac{k(b-a)}{n}-\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right)$$

$$= \left(\frac{b-a}{n}\right)\left(an - \frac{n(b-a)}{n} + \frac{n(n+1)}{2}\frac{(b-a)}{n}\right)$$

$$= \left(\frac{b-a}{n}\right)\left(a+\frac{b-a}{2}-\frac{b-a}{2n}\right).$$

Lower integral $= \underline{\int_a^b} f = \lim_{n\to\infty} L_{P_n}(f) = \lim_{n\to\infty}(b-a)\left(\frac{a+b}{2} \dots\right)$

$$= \left(\frac{b^2-a^2}{2}\right)$$

$M_k(f) = \sup\{f(x) ; x \in [x_{k-1}, x_k]\}$

$M_k(f) = 0$ (as all terms are $-ve$).

$U_{P_n}(f) = \sum_{k=1}^{n} M_k(f)\left(\frac{b-a}{n}\right) = 0$

$$0 \neq \left(\frac{b^2-a^2}{2}\right)$$

So, not Riemann integrable in this case.

Case 3 : $a < 0 < b$

Taking partition

$$P_n = \left\{a, \frac{(n-1)}{n}a, \dots, \frac{a}{n}, 0, \frac{b}{n}, \dots, b\right\}$$

$$m_k(f) = 0 \quad [\text{for } x \geq 0]$$

$$m_k(f) = \frac{ka}{n} \quad [x < 0]$$

$$L_P(f) = \sum_{k'=1}^{n} m_k(f) \times \frac{a}{n} + \sum_{k''=1}^{n} m_k(f) \times \frac{b}{n}$$

$$L_{P_n}(f) = \sum_{k'=1}^{n} \frac{k' a^2}{n^2} = \frac{n(n+1)}{2}\,\frac{a^2}{n^2}$$

$$\text{Lower Integral Sum} = \underline{\int_a^b} f = \lim_{n\to\infty} \frac{a^2}{2}\left(\frac{n(n+1)}{n^2}\right)$$

$$= \lim_{n\to\infty} \frac{a^2}{2}\left(1+\frac{1}{n}\right) = \frac{a^2}{2}$$

$$U_{P_n}(f) = \sum_{k''=1}^{n} M_k(f) \times \frac{b}{n} + \sum_{k'=1}^{n} M_k(f) \times \frac{a}{n}$$

$$M_k(f) = 0, \quad x < 0$$

Parth

$$M_k(f) = \frac{x''b}{n}, n \geq 0.$$

$$U_{P_n}(f) = \sum_{x''=1} \frac{b^2}{n^2} x''$$

$$= \frac{b^2}{2} \cdot \frac{n(n+1)}{n^2}$$

$$\text{Upper Integral} = \int_a^b f = \lim_{n\to\infty} U_{P_n}(f)$$

$$= \lim_{n\to\infty} \frac{b^2}{2} \frac{n(n+1)}{n^2} = \lim_{n\to\infty} \frac{b^2}{2}\left(1+\frac{1}{n}\right)$$

$$= \frac{b^2}{2}.$$

$$\frac{a^2}{2} \neq \frac{b^2}{2}.$$

Upper Integral $\neq$ Lower Integral

So, it is not Reimann Integrable in this case.

Hence, $f(x)$ is not Reimann integrable in all the cases.

Hence proved.

Parth

8.) Let $f:[0,1] \to R$ be Riemann integrable over $[x,1]$ for all x with $0 < x \le 1$. Prove or disprove that f is Riemann integrable over $[0,1]$.

Solⁿ) Given :- $f:[0,1] \to R$, be Riemann integrable over $[x,1]$ for all x with $0 < x \le 1$

→ We have to prove or disprove that f is Riemann integrable over $[0,1]$.

So let an example

if we take $f(x)$ to be:-

$$f(x) = \begin{cases} \frac{1}{x}, & x \in (0,1] \\ 1, & x = 0 \end{cases}$$

For every $x > 0$, function is Riemann integrable because f is ~~continuous~~ continuous on $[x,1]$

Let the tagged partition be

$$\dot{P} = \left\{ \left[\frac{k-1}{n}, \frac{k}{n} \right], \frac{k}{n} \right\}_{k=1}^{n}$$

Then, $S(\dot{P}, f) = \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\left(\frac{k}{n} - \frac{(k-1)}{n}\right)$

Riemann Sum for Partition

$$= \sum_{k=1}^{n} \frac{1}{\frac{k}{n}} \times \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$$

Now,

i) The function is integrable if

$\lim_{\|P\| \to 0} S(P, f)$ converges

ie $\lim_{n \to \infty} S(P, f)$ converges.

but here $\lim_{n\to\infty} (S(P, f)) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k}$.

is not convergent

as $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k}$ is divergent (harmonic series)

So, the function f is not integrable on [0,1] even though it satisfies the conditions given in the question ie Riemann integrable over $[x,1]$ $\forall\, x \in (0,1]$

The above case implies that f may or may not be integrable on [0,1]. The conclusion that can be drawn is

1) If f is bounded on [0,1], then it will be integrable.

2) But in case, if f is unbounded, there can be a lot of counter examples.

The example mentioned is able to disprove the statement

(29)

(i) Let $f: R^2 \longrightarrow R$ be a function defined by

$$f(x, y) = \begin{cases} \frac{\cos y \sin x}{x}, & \text{if } x \neq 0 \\ \cos y, & \text{if } x = 0. \end{cases}$$

Discuss the continuity of f

Sol:- Checking the continuity of f at the origin first

so $\lim_{(x,y) \to (0,0)} f(x, y) = f(0, 0)$ (for continuity)

$$\lim_{(x,y) \to (0,0)} \frac{\cos y \sin x}{x} = f(0,0)$$

and at $x=0$ $f(x,y) = \cos y$.

so at $f(0,0) = \cos 0 = 1$

so

$$\lim_{(x,y) \to (0,0)} \frac{\cos y \sin x}{x} = 1$$

and $\lim_{x \to 0} \frac{\sin x}{x} = 1$

so $\lim_{(x,y) \to (0,0)} \left(\frac{\sin x}{x}\right) \lim_{(x,y) \to (0,0)} (\cos y) = 1$

$1 = 1$

Hence $f(x, y)$ is continuous at the origin

And if we check the continuity for $x = 0$ and y any real value

so $\lim_{(x,y)\to(0,y)} f(x,y) = f(0,y)$ (for continuity

so $\lim_{(x,y)\to(0,y)} \left(\frac{\sin x}{x}\right) \lim_{(x,y)\to(0,y)} (\cos y)$

and $\lim_{x\to 0} \frac{\sin x}{x} = 1$

so $1 \times \cos y = \lim_{(x,y)\to(0,y)} f(x,y)$

$\lim_{(x,y)\to(0,y)} f(x,y) = \cos y.$

and $f(0,y) = \cos y$

Hence $\lim_{(x,y)\to(0,y)} = f(0,y)$

Hence the function will be continuous at every point in R^2.

As for $x \neq 0$, we can evaluate the limit directly and for $x = 0$ we have proved that continuity also holds for $x = 0$.

Parth