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DEPARTMENT-CSE

SUBJECT- MA-101

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1) Let $q \geq 0$, $y_1 = q$ and $y_{n+1} = \frac{1}{4}(y_n^2+3)$, $\forall n \geq 1$.
For what values of q the sequence $\{y_n\}$ converges? Give a proper justification of your answer. Also find $\lim_{n\to\infty} y_n$ whenever it exists.

Solⁿ) Given — $y_1 = q$ and $q \geq 0$

$$y_{n+1} = \frac{1}{4}(y_n^2+3) \quad \forall n \geq 1$$

Let $\lim_{n\to\infty} y_n = l$ and $l \in R$

if l exists then

$$l = \frac{1}{4}(l^2+3)$$

$$4l = l^2+3$$

$$l^2 - 4l + 3 = 0$$

$$(l-3)(l-1) = 0$$ so l can be 1 or 3.

1.1) If we suppose that

$$y_2 > y_1$$

then

$$\frac{1}{4}(q^2+3) > q$$

$$(q-1)(q-3) > 0$$

so $q > 3$ or $q < 1$

If $q > 3$

then

if $y_n > y_{n-1}$

then $y_{n+1} - y_n$

$$\frac{1}{4}\left(y_n^2 - y_{n-1}^2\right) > 0$$

hence y_n is an montonically increasing sequence.

By MCT hence $\lim y_n = \sup(y_n)$

and $y_2 = \frac{1}{4}(q^2+3)$ for $q > 3$

$$y_2 > 3$$

but l can be 1 or 3.

so for $q > 3$ no limit of y_n is possible.

(iv) If $0 \le q < 1$

then

if $y_n > y_{n-1}$

then $y_{n+1} - y_n = \frac{1}{4}\left(y_n^2 - y_{n-1}^2\right) > 0$

hence y_n is again an monotonically increasing sequence by principle of mathematical induction.

By MCT so $l = \sup(y_n)$

and $y_2 = \frac{1}{4}(q^2+3)$

$$\frac{3}{4} < y_2 < 1$$

$$y_3 < 1$$

$$\vdots$$

$$y_n < 1$$

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so for $0 \leq q \leq 1$

$$y_n < 1$$

hence 1 is an upper bound of y_n
as l can be 1 or 3.

so $l = 1$ or $\lim_{n \to \infty} y_n = 1$

2.) If $q = 3$

then $y_n = 3$

hence y_n will be a constant sequence

hence $\lim_{n \to \infty} y_n = 3$.

3.) If $1 < q < 3$

then $y_2 < y_1$

if we suppose $y_n < y_{n-1}$

then $y_{n+1} - y_n = \frac{1}{4}(y_n^2 - y_{n-1}^2) < 0$.

hence y_n is an monotonically decreasing sequence
by principle of mathematical induction

By MCT so $\lim_{n \to \infty} y_n = \inf(y_n)$

$$y_2 = \frac{1}{4}(q^2 + 3)$$

Parthgp.

so, $1 < y_2 < 3$

and $1 < y_3 < 3$

hence $1 < y_n < 3$

so 1 is a lower bound of y_n
and l can be either 1 or 3
so $l = 1$ or $\lim_{n \to \infty} y_n = 1$.

Q.4 Let $f: X \to \mathbb{R}$ be a continuous function where $X \subset \mathbb{R}$. If (x_n) is a sequence in X with the property that for every $\varepsilon > 0$ there exists a positive integer N such that $|x_n - x_m| < \varepsilon$ for all $n, m \geq N$. Then prove or disprove that $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$.

Sol^n: Let $f(x) = \frac{1}{x}$ for $x \in (0,1)$ and $x_n = \frac{1}{n}$ for $n \in \mathbb{N}$

and ~~Then~~ f is continuous on $(0,1)$

The sequence (x_n) is a Cauchy sequence.

and let $y_n = f(x_n) = n$

$|y_n - y_m| \geq 1$ for every $m, n \in \mathbb{N}$ with $m \neq n$

so (y_n) is not Cauchy.

Hence proved.

Parth

3) Determine the value of (a, b, c) for which the series $\sum_{n=3}^{\infty} \frac{a^n}{n^b (\ln n)^c}$

1) Converges absolutely 2) Converges but not absolutely

3) diverges.

Sol^n) Let $x_n = \frac{a^n}{n^b (\ln n)^c}$

By root test to check absolute convergence.

$$\alpha = \lim_{n\to\infty} \left| \frac{x_{n+1}}{x_n} \right| = \left| \frac{a^{n+1}}{a^n} \times \left(\frac{n}{n+1}\right)^b \left(\frac{\ln n}{\ln n+1}\right)^c \right|$$

$$\alpha = |a| \quad \forall\ b, c \in R$$

Now for absolute convergence,

$$|a| < 1 \ \forall \ b, c \in R$$

and for divergence

$$|a| > 1$$

If $a = 1$

$$x_n = \frac{1}{n^b (\ln n)^c}$$

so $x_n \leq \frac{1}{n^b} \quad \forall c \geq 0$

$\sum \frac{1}{n^b}$ converges for $b > 1$

so $\sum x_n$ will converge absolutely for $b > 1$ $(a=1)$

By comparison Test

if $b = 1$,

$x_n \leq \frac{1}{(\ln n)^c}$

$\sum \frac{1}{(\ln n)^c}$ will converge only if $c > 1$

By comparison Test $\sum x_n$ will converge absolutely

for $b = 1$, $c > 1$ $(a = 1)$

also $x_n > \frac{1}{n(\ln n)^{c-1}}$

$\sum \frac{1}{n(\ln n)^{c-1}}$ will diverge for $c < 1$.

$\Rightarrow \sum x_n$ will diverge for

$b = 1$, $c \leq 1$ $(a = 1)$

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if $b < 1$

$$x_n = \frac{1}{n^b (\ln n)^c}$$

then $\sum x_n$ will diverge

$(a=1, b<1, c \in R)$

for $a = -1$

$$x_n = \frac{(-1)^n}{n^b (\ln n)^c}$$

$$|x_n| = \frac{1}{n^b (\ln n)^c}$$

similarly $\sum |x_n|$ will converge for $b > 1$

similarly $\Rightarrow \sum_{n=3}^{\infty} |x_n|$ will converge for $b=1$ if $c>1$

so, $\sum_{n=3}^{\infty} x_n$ will converge absolutely for

$b > 1$ $(a=-1)$

if $b=1$, $c>1$ $(a=-1)$

$\rightarrow \sum_{n=3}^{\infty} x_n$ will converge conditionally

for $b=1$, $c \leq 1$

and it will also converge conditionally for

$0 < b < 1$ $(c \in R, a = -1)$

if $b < 0$

$\sum_{n=3}^{\infty} x_n$ will diverge

so

1) Absolutely convergent if

a) $|a| < 1$, $b \in R$, $c \in R$

b) $a = 1$, $b > 1$, $c \in R$

c) $a = 1$, $b < 1$, $c > 1$

d) $a = 1$, $b = 1$, $c > 1$

e) $a = -1$, $b > 1$

f) $a = -1$, $b = 1$, $c > 1$

2) Conditionally convergent if

a) $a = -1$, $b = 1$, $c \leq 1$

b) $a = -1$, $0 < b < 1$, $c \in R$

3) Divergent for.

a) $|a| > 1$, $b \in R$, $c \in R$

b) $a = 1$, $b = 1$, $c \leq 1$

c) $a = 1$, $b = 1$, $c \in R$

d) $a = -1$, $b < 0$, $c \in R$

4.) Find the range of positive values of x for which the series $\sum_{n=1}^{\infty} \frac{1}{x^n + x^{-n}}$ is convergent.

Sol^n)

$$\sum_{n=1}^{\infty} \frac{1}{x^n + x^{-n}}$$

Let $y_n = \frac{1}{x^n + x^{-n}}$

By ratio test $\alpha = \lim_{n\to\infty} \frac{y_{n+1}}{y_n}$

$$\alpha = \lim_{n\to\infty} \frac{1}{x^{n+1} + x^{-n+1}} (x^n + x^{-n})$$

$$\alpha = \lim_{n\to\infty} \frac{(x^{2n}+1)}{(x^{2n+2}+1)} \frac{x^{n+1}}{\cancel{x^n}}$$

$$\alpha = \lim_{n\to\infty} x \frac{(x^{2n}+1)}{(x^{2n+2}+1)}$$

1.) if $x<1$

then

$\alpha = x$

and $\alpha < 1$

so $\sum y_n$ is convergent for $x<1$

2.) if $x=1$

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Then $\alpha = \lim_{x \to \infty} 1 \left(\frac{1+1}{1+1}\right)$

$\alpha = 1$

so

$$\sum y_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

if $S_n = \sum_{n=1} \frac{1}{2}$

if $\sum y_n$ is convergent then

$\lim_{n \to \infty} S_n$ should exist

but $\lim_{n \to \infty} S_n \longrightarrow \infty$

hence for $x = 1$ $\sum y_n$ is divergent

3. for $x > 1$

then $\alpha = \lim_{n \to \infty} x \left(\frac{x^{2n}+1}{x^{2n+2}+1}\right)$

$$= \lim_{n \to \infty} \frac{x^{2n+1}\left(1+\frac{1}{x^{2n}}\right)}{x^{2n+2}\left(1+\frac{1}{x^{2n+2}}\right)}$$

$$= \lim_{n \to \infty} \frac{1}{x}$$

so $\alpha = \frac{1}{x}$

and if $x > 1$ then $\frac{1}{x} < 1$

so $\alpha < 1$

so $\sum y_n$ is convergent for $x > 1$

Parth Singh

So the range of positive values of x

is $x \in (0,1) \cup (1,\infty)$

Q5. Find the radius and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(ax-b)^n}{na^n}$, where $a, b > 0$.

Sol^n: Series $= \sum_{n=1}^{\infty} \frac{(ax-b)^n}{na^n}$, where $a, b > 0$.

Using Ratio Test

$$x_n = \frac{(ax-b)^n}{na^n}$$

so $\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = \alpha$

$$\alpha = \lim_{n\to\infty} \left| \frac{(ax-b)^{n+1}}{(n+1)a^{n+1}} \times \frac{n \times a^n}{(ax-b)^n} \right|$$

$$\alpha = \lim_{n\to\infty} \left| \frac{(ax-b)}{a} \frac{n}{n+1} \right|$$

$$\alpha = \left| \frac{ax-b}{a} \right|$$

For series to be convergent $\alpha < 1$

so $\left|\frac{ax-b}{a}\right| < 1$

$$-1 < \frac{ax-b}{a} < 1$$

$$-a < ax - b < a$$

$$\frac{b-a}{a} < x < \frac{a+b}{a}$$

and The standard eq^n is

$$a - R < x < a + R$$

so $\frac{b}{a} - 1 < x < 1 + \frac{b}{a}$

so by comparison

$R = 1$

so Radius of convergence $= 1$

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a) $x = \frac{b-a}{a}$

$$x_n = \frac{(-a)^n}{n \times a^n} = \frac{(-1)^n}{n}$$

$\cancel{x_{n+1}} \geq \cancel{\frac{1}{n+1}}$

$x_n = \frac{(-1)^n}{n}$ is a alternating series

so $u_n = \frac{1}{n}$

and $u_n \geq u_{n+1}$

and as $\lim_{n\to\infty} \frac{1}{n} \to 0$ so by Leibnitz's Theorem

$\sum_{n=1}^{\infty} \left(\frac{-1}{n}\right)^n$ is convergent.

b.Y for $x = \frac{b+a}{a}$

$x_n = \frac{a^n}{na^n} = \frac{1}{n}$

and $\frac{1}{n}$ is divergent

So the interval of convergence

$= \frac{b-a}{a} \leq x < \frac{b+a}{a}$

and Radius of convergence is 1.

Ans

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6.) Use $\varepsilon - \delta$ definition of limit to show that

$$\lim_{x \to 1} \frac{x^2 - x + 1}{x+1} = \frac{1}{2}$$

Solⁿ) Let $\varepsilon > 0$ be any real number.

To find ∃ $\delta > 0$ such that

$$0 < |x-1| < \delta \Rightarrow \left| \frac{x^2 - x + 1}{x+1} - \frac{1}{2} \right| < \varepsilon$$

$$\left| \frac{2(x^2 - x + 1) - (x+1)}{2(x+1)} \right| < \varepsilon$$

$$\left| \frac{2x^2 - 2x + 2 - x - 1}{2(x+1)} \right| < \varepsilon$$

$$\left| \frac{2x^2 - 3x + 1}{2(x+1)} \right| < \varepsilon$$

$$\frac{|2x-1|\,|x-1|}{2|(x+1)|} < \varepsilon$$

let $|x-1| < \frac{1}{2}$

$$\frac{1}{2} < x < \frac{3}{2}$$

so

$$\left|\frac{x^2-x+1}{x+1}-\frac{1}{2}\right| \leq \frac{4}{3}|x-1|$$

and $\delta = \min\left\{\frac{3}{4}\varepsilon, \frac{1}{2}\right\}$

so $0 < |x-1| < \delta \Rightarrow \left|\frac{x^2-x+1}{x+1}-\frac{1}{2}\right| < \varepsilon$

Hence Proved.

Parth

7) Let $f:[0,1] \to R$ be a continuous function which satisfies $f(x) = f(x^{\frac{1}{3}})$ $\forall x \in [0,1]$. Show that f is a constant function.

Sol n. So $f(x) = f(x^{\frac{1}{3}})$

for $x > 0$

$f(x) = f(x^{\frac{1}{3}})$

$f(x^{\frac{1}{3}}) = f(x^{\frac{1}{9}})$

|

|

$f(x) = f(x^{\frac{1}{3^n}})$ for some n

and $\lim_{n \to \infty} x^{\frac{1}{3^n}} \to 1$

so $\lim_{n \to \infty} f(x^{\frac{1}{3^n}}) \to f(1)$

ie $f(x) = f(1)$

and at $x = 0$

as the function is continuous

so $\lim_{x \to 0} f(x) = f(0)$

so $f(0) = f(1)$

Hence $f(x) = f(1)$ for $x \in [0,1]$

hence $f(x)$ is a constant function

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8. Let $f: R \rightarrow (0,\infty)$ be a function which satisfies $f(x+y) = f(x)f(y)$ for all $x, y \in R$. If f is continuous at $x=0$, then prove that $f(x) = k^x$ for some real number $k \neq 0$.

Solⁿ: Range of $f(x)$ is $(0,\infty)$,

so $f(x) > 0 \ \forall x \in R$

$f(0) = f(0)^2$

so $f(0) = 1$ or 0

and $f(0) = 1$ as $f(x) > 0$

for a positive integer n

$$f(n) = f(1+1+1 --- +1) = f(1+(n-1)) = f(1)f(n-1)$$
$$= f(1)^2 \times f(n-2) = --- = f(1)^n$$

so $f(n) = f(1)^n$

if $n = -1$

$$f(-1) = f(-2+1) = f(1)f(-2) = f(1) \times f(-1)$$

so $f(-1) = f(-1)^2 f(1)$

$f(-1) = f(1)^{-1}$

for a negative integer m

$f(m) = f(-1+(-1) - - - -) = f(-1)^{m}$

$= \left(f(1)^{-1}\right)^{-m} = f(1)^{m}$

$f(m) = f(1)^{m}$

so $f(0) = 1 = f(1)^{0}$, $f(n) = f(1)^{n}$ holds for all integers.

Let q and m be positive integers

then $f(m) = f\left(\frac{1}{q} + \frac{1}{q} - - - + \frac{1}{q}\right) = f\left(\frac{1}{q}\right)^{qm}$

$f(1)^{m} = f\left(\frac{1}{q}\right)^{qm}$

so $f(1) = f\left(\frac{1}{q}\right)^{q}$

so $f\left(\frac{1}{q}\right) = f(1)^{\frac{1}{q}}$

and this will hold for negative integers q, and m,

Then $f\left(\frac{s}{t}\right) = f(1)^{\frac{s}{t}}$ for any rational number.

so $f(n) = f(1)^{n}$ for all real numbers.

since $f(1) > 0$ and is real valued,

so $\log f(1) = c$ exists

so $f(x) = f(1)^x = e^{cx} = b^x$

where $b = e^c$

Hence proved.