Laplacian in Cylindrical coordinate system

parton [*]

1 Laplacian

Cartesian coordinates are given as (X, Y, Z).
In this system, laplacian is defined as

$$\Delta f = \frac{\partial^2 f}{\partial X^2} + \frac{\partial^2 f}{\partial Y^2} + \frac{\partial^2 f}{\partial Z^2}$$

2 Cylindrical coordinate system

Cylindrical coordinates are given as (r, θ, z).

2.1 mapping

Cartesian coordinates and cylindrical coordinates are mapped as follows

$$X = r\cos\theta \tag{1}$$
$$Y = r\sin\theta \tag{2}$$
$$Z = z \tag{3}$$

By using this definition,

$$r = (X^2 + Y^2)^{\frac{1}{2}} \tag{4}$$
$$\theta = \tan^{-} 1\frac{Y}{X} \tag{5}$$
$$z = Z \tag{6}$$

2.2 Preparation

By (4),

$$\frac{\partial r}{\partial X} = \frac{\partial}{\partial X}(X^2 + Y^2)^{\frac{1}{2}} = X(X^2 + Y^2)^{-}\frac{1}{2} = \cos\theta \tag{7}$$

$$\frac{\partial r}{\partial X} = \frac{\partial}{\partial Y}(X^2 + Y^2)^{\frac{1}{2}} = Y(X^2 + Y^2)^{-}\frac{1}{2} = \sin\theta \tag{8}$$

$$\frac{\partial r}{\partial X} = \frac{\partial}{\partial Z}(X^2 + Y^2)^{\frac{1}{2}} = 0 \tag{9}$$

By (5)[*1],

$$\frac{\partial \theta}{\partial X} = \frac{\partial}{\partial X}\tan^{-1}\frac{Y}{X} = -\frac{Y}{X^2 + Y^2} = -\frac{1}{r}\sin\theta \tag{10}$$

$$\frac{\partial \theta}{\partial Y} = \frac{\partial}{\partial Y}\tan^{-1}\frac{Y}{X} = \frac{Y}{X^2 + Y^2} = \frac{1}{r}\cos\theta \tag{11}$$

$$\frac{\partial \theta}{\partial Z} = \frac{\partial}{\partial Z}\tan^{-1}\frac{Y}{X} = 0 \tag{12}$$

By (6),

$$\frac{\partial z}{\partial X} = 0 \tag{13}$$

$$\frac{\partial z}{\partial Y} = 0 \tag{14}$$

$$\frac{\partial z}{\partial Z} = 1 \tag{15}$$

3 function in cylindrical coordinate system

Let f be a function in cylindrical coordinate system.

$$f = (r, \theta, z)$$

Cylindrical coordinate system can be regarded as a function of orthogonal coordinates[*2]. Therefore, function f can be regarded as a composite function.

$$f(r(X, Y, Z), \theta(X, Y, Z), z(X, Y, Z)) \tag{16}$$

Partially differentiate equation (16) with X, Y, Z.

$$\frac{\partial f}{\partial X} = \frac{\partial r}{\partial X}\frac{\partial f}{\partial r} + \frac{\partial \theta}{\partial X}\frac{\partial f}{\partial \theta} + \frac{\partial z}{\partial X}\frac{\partial f}{\partial z} \tag{17}$$

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[*] aqua

[*1]

$$\frac{\partial}{\partial u}\tan^{-1} u = \frac{1}{1 + u^2}$$

[*2] (4),(5), and (6)

$$\frac{\partial f}{\partial Y} = \frac{\partial r}{\partial Y}\frac{\partial f}{\partial r} + \frac{\partial \theta}{\partial Y}\frac{\partial f}{\partial \theta} + \frac{\partial z}{\partial Y}\frac{\partial f}{\partial z} \tag{18}$$

$$\frac{\partial f}{\partial Z} = \frac{\partial r}{\partial Z}\frac{\partial f}{\partial r} + \frac{\partial \theta}{\partial Z}\frac{\partial f}{\partial \theta} + \frac{\partial z}{\partial Z}\frac{\partial f}{\partial z} \tag{19}$$

Then,

$$\frac{\partial^2 f}{\partial X^2} = \frac{\partial}{\partial X}\frac{\partial f}{\partial X}$$
$$= \frac{\partial r}{\partial X}\frac{\partial}{\partial r}\frac{\partial f}{\partial X} + \frac{\partial \theta}{\partial X}\frac{\partial}{\partial \theta}\frac{\partial f}{\partial X} + \frac{\partial z}{\partial X}\frac{\partial}{\partial z}\frac{\partial f}{\partial X}$$
$$= \frac{\partial r}{\partial X}\frac{\partial}{\partial r}(17) + \frac{\partial \theta}{\partial X}\frac{\partial}{\partial \theta}(17) + \frac{\partial z}{\partial X}\frac{\partial}{\partial z}(17)$$

$$\frac{\partial^2 f}{\partial Y^2} = \frac{\partial}{\partial Y}\frac{\partial f}{\partial Y}$$
$$= \frac{\partial r}{\partial Y}\frac{\partial}{\partial r}\frac{\partial f}{\partial Y} + \frac{\partial \theta}{\partial Y}\frac{\partial}{\partial \theta}\frac{\partial f}{\partial Y} + \frac{\partial z}{\partial Y}\frac{\partial}{\partial z}\frac{\partial f}{\partial Y}$$
$$= \frac{\partial r}{\partial Y}\frac{\partial}{\partial r}(18) + \frac{\partial \theta}{\partial Y}\frac{\partial}{\partial \theta}(18) + \frac{\partial z}{\partial Y}\frac{\partial}{\partial z}(18)$$

$$\frac{\partial^2 f}{\partial Z^2} = \frac{\partial}{\partial Z}\frac{\partial f}{\partial Z}$$
$$= \frac{\partial r}{\partial Z}\frac{\partial}{\partial r}\frac{\partial f}{\partial Z} + \frac{\partial \theta}{\partial Z}\frac{\partial}{\partial \theta}\frac{\partial f}{\partial Z} + \frac{\partial z}{\partial Z}\frac{\partial}{\partial z}\frac{\partial f}{\partial Z}$$
$$= \frac{\partial r}{\partial Z}\frac{\partial}{\partial r}(19) + \frac{\partial \theta}{\partial Z}\frac{\partial}{\partial \theta}(19) + \frac{\partial z}{\partial Z}\frac{\partial}{\partial z}(19)$$

Using these equations, laplacian is

$$\Delta f = \frac{\partial^2 f}{\partial Z^2}$$
$$= \frac{\partial r}{\partial X}\frac{\partial}{\partial r}(17) + \frac{\partial \theta}{\partial X}\frac{\partial}{\partial \theta}(17) + \frac{\partial z}{\partial X}\frac{\partial}{\partial z}(17)$$
$$+ \frac{\partial r}{\partial Y}\frac{\partial}{\partial r}(18) + \frac{\partial \theta}{\partial Y}\frac{\partial}{\partial \theta}(18) + \frac{\partial z}{\partial Y}\frac{\partial}{\partial z}(18)$$
$$+ \frac{\partial r}{\partial Z}\frac{\partial}{\partial r}(19) + \frac{\partial \theta}{\partial Z}\frac{\partial}{\partial \theta}(19) + \frac{\partial z}{\partial Z}\frac{\partial}{\partial z}(19)$$

Substitute (7)∼(15) to this equation and got

$$\Delta f = \cos\theta\frac{\partial}{\partial r}(\cos\theta\frac{\partial f}{\partial r} - \frac{1}{r}\sin\frac{\partial f}{\partial \theta})$$
$$- \frac{1}{r}\sin\theta\frac{\partial}{\partial \theta}(\cos\theta\frac{\partial f}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial f}{\partial \theta})$$
$$+ \sin\theta\frac{\partial}{\partial r}(\sin\theta\frac{\partial f}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial f}{\partial \theta})$$
$$+ \frac{1}{r}\cos\theta\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial f}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial f}{\partial \theta})$$
$$+ \frac{\partial}{\partial z}(\frac{\partial f}{\partial z})$$

Transform the equation

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial f}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \tag{20}$$

4 参考

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