

Line Assignment

Manideep Parusha - FWC22004

September 17, 2022

Problem

Show that the diagonals of a square are equal and bisect each other at right angles

Solution

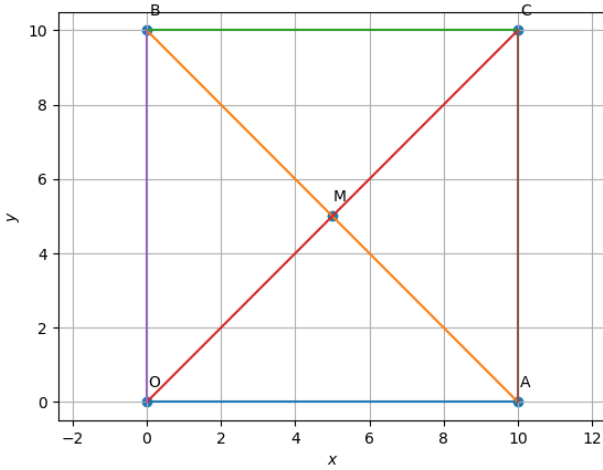


Figure 1: Diagram generated using python

Let OABC is a Square. Length of all sides are equal for a square and all interior angles equal to 90°

O at the origin and vectors A, B & C represent other vertices of the square.

$$OA = AB = BC = CO \quad (1)$$

$$\angle A = \angle B = \angle C = \angle O = 90^\circ \quad (2)$$

Here, D_1 and D_2 are the diagonals of the square and we can compute D_1 and D_2 as

$$D_1 = (A + B) \quad (3)$$

$$D_2 = (A - B) \quad (4)$$

To prove that the diagonals of the square are equal, we can find the length of the two diagonals and compare. Hence,

$$\|D_1\| = \|A + B\| \quad (5)$$

$$\|D_2\| = \|A - B\| \quad (6)$$

For finding length of D_1 , we can write from eqn. 5,

$$\|A + B\| = \sqrt{\|A\|^2 + \|B\|^2 + 2A^T B} \quad (7)$$

But, for a square we know that length of all sides are equal.

$$\|A\| = \|B\| \quad (8)$$

and, the angle between two adjacent sides is 90° . The dot product of two vectors which are separated by 90° angle is always '0'.

$$A^T B = 0 \quad (9)$$

So the eqn. 7 becomes

$$\|A + B\| = \sqrt{2} \|A\| \quad (10)$$

$$\|D_1\| = \sqrt{2} \|A\| \quad (11)$$

Similarly, for finding the length of D_2

$$\|A - B\| = \sqrt{\|A\|^2 + \|B\|^2 - 2A^T B} \quad (12)$$

But, from (8) and (9)

$$\|A - B\| = \sqrt{2} \|A\| \quad (13)$$

$$\|D_2\| = \sqrt{2} \|A\| \quad (14)$$

So, from the equations (11) and (14), we can say that the lengths of diagonals D_1 and D_2 are equal

$$\|D_1\| = \|D_2\| \quad (15)$$

We know that, if the dot product of two vectors is zero then the vectors are perpendicular to each

other.

So, by taking the dot product of \mathbf{D}_1 and \mathbf{D}_2

$$\mathbf{D}_1 \cdot \mathbf{D}_2 = \mathbf{D}_1^T \mathbf{D}_2 \quad (16)$$

$$\mathbf{D}_1 \cdot \mathbf{D}_2 = (\mathbf{A} + \mathbf{B})^T (\mathbf{A} - \mathbf{B}) \quad (17)$$

$$\mathbf{D}_1 \cdot \mathbf{D}_2 = \|\mathbf{A}\|^2 - \|\mathbf{B}\|^2 \quad (18)$$

From the equation (8),

$$\mathbf{D}_1 \cdot \mathbf{D}_2 = \|\mathbf{A}\|^2 - \|\mathbf{A}\|^2 \quad (19)$$

$$\mathbf{D}_1 \cdot \mathbf{D}_2 = 0 \quad (20)$$

as the dot product of the diagonals is equal to 0, we can say that both diagonals are perpendicular to each other.

Let diagonals \mathbf{D}_1 and \mathbf{D}_2 intersect at a point \mathbf{M} . We have to prove that \mathbf{M} is the mid point of \mathbf{D}_1 and \mathbf{D}_2 , in order to say that both diagonals bisect each other.

$$\mathbf{OM} = x\mathbf{D}_1 \quad (21)$$

$$\mathbf{MA} = y\mathbf{D}_2 \quad (22)$$

From the equations (3) and (4), the above equations can be written as

$$\mathbf{OM} = x\mathbf{A} + \mathbf{B} \quad (23)$$

$$\mathbf{MA} = y\mathbf{A} - \mathbf{B} \quad (24)$$

Now, if we consider

$$\mathbf{OA} = \mathbf{OM} + \mathbf{MA} \quad (25)$$

$$\mathbf{A} = x(\mathbf{A} + \mathbf{B}) + y(\mathbf{A} - \mathbf{B}) \quad (26)$$

$$\mathbf{A} = x\mathbf{A} + x\mathbf{B} + y\mathbf{A} - y\mathbf{B} \quad (27)$$

$$\mathbf{A} = (x + y)\mathbf{A} + (x - y)\mathbf{B} \quad (28)$$

Equating the co-efficient of \mathbf{A} and \mathbf{B} , we get

$$x + y = 1, x - y = 0 \quad (29)$$

$$2x = 1 \quad (30)$$

$$x = \frac{1}{2} \quad (31)$$

$$y = \frac{1}{2} \quad (32)$$

now we can say that

$$\mathbf{OM} = \frac{1}{2}\mathbf{D}_1 \quad (33)$$

$$\mathbf{MA} = \frac{1}{2}\mathbf{D}_2 \quad (34)$$

Hence, M is the mid point of diagonals \mathbf{D}_1 and \mathbf{D}_2 and we can say that both diagonals bisect each other.

we have proved that diagonals of a square are equal in length and bisect each other at right angles.