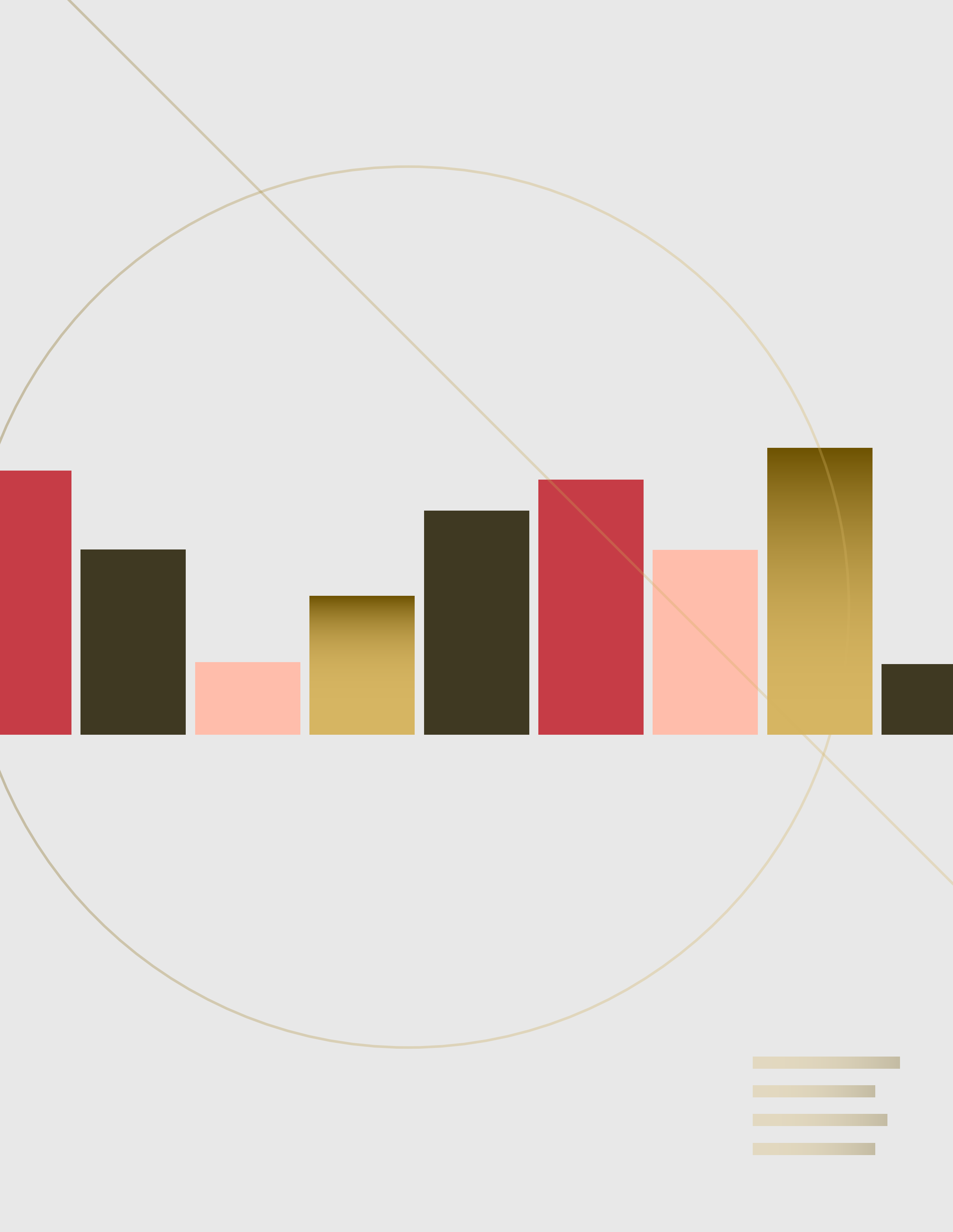

Efficient simulation and integrated likelihood estimation in state space models

- Chan and Jeliazkov (2009)

Mathematical modelling and Numerical optimization

Model Set up

Linear gaussian state space model

For any given t

$$\underset{n\times 1}{\begin{bmatrix} y_{1t} \\ y_{2t} \\ \vdots \\ y_{nt} \end{bmatrix}} = \underset{n\times k}{\begin{bmatrix} X_{11t} & \cdots & X_{1kt} \\ X_{21t} & & X_{2kt} \\ \vdots & & \vdots \\ X_{n1t} & & X_{nkt} \end{bmatrix}} \underset{k\times 1}{\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}} + \underset{n\times q}{\begin{bmatrix} G_{11t} & \cdots & G_{1qt} \\ G_{21t} & \cdots & G_{2qt} \\ \vdots & & \\ G_{n1t} & \cdots & G_{nqt} \end{bmatrix}} \underset{q\times 1}{\begin{bmatrix} \eta_{1t} \\ \vdots \\ \eta_{qt} \end{bmatrix}} + \underset{n\times 1}{\begin{bmatrix} \varepsilon_{1t} \\ \vdots \\ \varepsilon_{nt} \end{bmatrix}}$$

$$\underset{q\times 1}{\begin{bmatrix} \eta_{1t} \\ \vdots \\ \eta_{qt} \end{bmatrix}} = \underset{q\times l}{\begin{bmatrix} Z_{11t} & \cdots & Z_{1lt} \\ Z_{21t} & \cdots & Z_{2lt} \\ \vdots & & \\ Z_{q1t} & \cdots & Z_{qlt} \end{bmatrix}} \underset{l\times 1}{\begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_l \end{bmatrix}} + \underset{q\times q}{\begin{bmatrix} F_{11t} & \cdots & F_{1qt} \\ F_{21t} & \cdots & F_{2qt} \\ \vdots & & \\ F_{q1t} & \cdots & F_{qqt} \end{bmatrix}} \underset{q\times 1}{\begin{bmatrix} \eta_{1,t-1} \\ \vdots \\ \eta_{q,t-1} \end{bmatrix}} + \underset{q\times 1}{\begin{bmatrix} \nu_{1t} \\ \vdots \\ \nu_{qt} \end{bmatrix}}$$

Or

$$y_t = X_t\beta + G_t\eta_t + \varepsilon_t$$ - measurement equation

$$\eta_t = Z_t\gamma + F_t\eta_{t-1} + \nu_t$$ - transition equation

$$\text{parameters} = \theta = \{\beta, \gamma, \{G_t\}, \{F_t\}\}, \underbrace{\Omega_{11}, \Omega_{12}, D}_{\text{defined in next page}}\}$$

$$\begin{pmatrix}\varepsilon_t \\ \nu_t\end{pmatrix} \sim N\left(0, \begin{pmatrix}\overset{n\times n}{\Omega_{11}} & 0 \\ 0 & \underset{q\times q}{\Omega_{22}}\end{pmatrix}\right)$$

we initialize by $\underset{q\times 1}{\eta_1} \sim N(\underset{q\times l}{Z_1}\underset{l\times 1}{\gamma}, \underset{q\times q}{D})$

MCMC algorithms suffer due to

$\Rightarrow$ high dimentionality in n, q and T.

$\Rightarrow$ slow convergence & poor mixing

Methodological framework

Stacking the measurement equation over t we get

$$\underset{nT\times 1}{y} = \underset{nT\times k}{X}\,\underset{k\times 1}{\beta} + \underset{nT\times qT}{G}\,\underset{qT\times 1}{\eta} + \underset{nT\times 1}{\varepsilon}$$

$$X = \begin{bmatrix} X_1 \\ \vdots \\ X_T\end{bmatrix}, \quad G = \begin{bmatrix} G_1 & & 0 \\ & \ddots & \\ 0 & & G_T\end{bmatrix}, \quad \varepsilon = \begin{bmatrix}\varepsilon_1 \\ \vdots \\ \varepsilon_T\end{bmatrix}$$

$$\eta = \begin{bmatrix}\eta_1 \\ \vdots \\ \eta_T\end{bmatrix}$$

$$\varepsilon \sim N(0, \underset{nT\times nT}{I_T \otimes \Omega_{11}})$$

$$\begin{aligned} f(y\mid\theta,\eta) &\sim N(\overbrace{X\beta + G\eta}^{nT\times 1}, \overbrace{I_T \otimes \Omega_{11}}^{nT\times nT}) \\ &= f_N(y \mid X\beta + G\eta, I_T \otimes \Omega_{11}) \\ &= f_N(y \mid \theta, \eta)\end{aligned}$$

Now lets look at transition equations

$$\eta_t = Z_t\gamma + F_t\eta_{t-1} + \nu_t$$

$$\underbrace{\underset{q\times 1}{\eta_t} - \underset{q\times q}{F_t}\,\underset{q\times 1}{\eta_{t-1}}}_{\Downarrow} = \underset{q\times \ell}{Z_t}\,\underset{\ell\times 1}{\gamma} + \underset{q\times 1}{\nu_t}$$

$$\underbrace{\begin{bmatrix} \overset{q\times q}{I_q} & & & & \\ -F_2 & I_q & & 0 & \\ 0 & -F_3 & I_q & & \\ & & \ddots & \ddots & \\ & 0 & & -F_T & I_q\end{bmatrix}}_{H}\begin{bmatrix}\eta_1\\ \eta_2\\ \vdots\\ \eta_T\end{bmatrix} = \begin{bmatrix} Z_{11t} & \cdots & Z_{1\ell t}\\ Z_{21t} & \cdots & Z_{2\ell t}\\ \vdots & & \\ Z_{q1t} & \cdots & Z_{q\ell t}\end{bmatrix}\begin{bmatrix}\gamma_1\\ \vdots\\ \gamma_\ell\end{bmatrix} + \begin{bmatrix}\nu_1\\ \vdots\\ \nu_q\end{bmatrix}$$

Stacking over T

$$\underset{qT\times qT}{H}\,\underset{qT\times 1}{\eta} = \underset{qT\times \ell}{Z}\,\underset{\ell\times 1}{\gamma} + \underset{qT\times 1}{\nu}$$

A simple change in variable from ν to η

$$\underset{qT\times 1}{\eta} = \underset{qT\times qT}{H^{-1}}\,\underset{qT\times \ell}{Z}\,\underset{\ell\times 1}{\gamma} + \underset{qT\times qT}{H^{-1}}\,\underset{qT\times 1}{\nu}$$

So, $\eta_1 \sim N(Z_1\gamma, D)$

$\eta_2 \sim N(Z_2\gamma + F_2\eta_1, \Omega_{22})$

$\vdots$

$\eta_T \sim N(Z_T\gamma + F_T\eta_{T-1}, \Omega_{22})$

$$\therefore\ \eta \sim N\left(\underbrace{H^{-1}Z\gamma}_{qT\times 1},\ \underbrace{H^{-1}\underbrace{\begin{pmatrix} \overset{q\times q}{D} & & \\ \vdots & \overset{q\times q}{\Omega_{22}} & 0 \\ \vdots & & \ddots \\ \underset{T\text{ times}}{0} & & \Omega_{22}\end{pmatrix}}_{S_{qT\times qT}} H^{-1}}_{K^{-1}_{qT\times qT}}\right) = f_N(\eta\,|\,\theta)$$

$$\overset{\text{precision matrix}}{\overbrace{\underset{qT \times qT}{K}}} = H'S^{-1}H = \underbrace{\begin{bmatrix} F_2'\Omega_{22}^{-1}F_2 + D^{-1} & -F_2'\Omega_{22}^{-1} & & 0 \\ -\Omega_{22}^{-1}F_2 & F_3'\Omega_{22}^{-1}F_3 + \Omega_{22}^{-1} & F_3'\Omega_{22}^{-1} & \\ & \ddots & \ddots & \ddots \\ 0 & & -\Omega_{22}^{-1}F_T & \Omega_{22}^{-1} \end{bmatrix}}_{\text{Block banded sparse matrix}}$$

Now we have $f(y \mid \theta, \eta)$ and $f(\eta \mid \theta)$

Using Bayes Theorem

$$f(\eta \mid y, \theta) \propto f(y \mid \eta, \theta)\, f(\eta \mid \theta)$$

$$\propto f_N(y \mid X\beta + G\eta,\ I_T \otimes \Omega_{11}) * f_N(H^{-1}Z\gamma,\ K^{-1})$$

$$\propto \exp\{(y - X\beta - G\eta)'(I_T \otimes \Omega_{11})^{-1}(y - X\beta - G\eta)\}$$
$$\exp\{(\eta - H^{-1}Z\gamma)' K (\eta - H^{-1}Z\gamma)\}$$

$$\propto \exp\{-(y - X\beta)'(I_T \otimes \Omega_{11})^{-1}G\eta - \eta'G'(I_T \otimes \Omega_{11})^{-1}(y - X\beta) + \eta'G'(I_T \otimes \Omega_{11})^{-1}G\eta\}$$
$$\exp\{\eta'K\eta - \eta'KH^{-1}Z\gamma - \gamma'Z'H^{-1}K\eta\}$$

$\rightarrow$ precision $= \underset{qT \times qT}{P} = \underset{qT \times qT}{K} + \underset{qT \times nt}{G'}\underset{nt \times nt}{(I_T \otimes \Omega_{11}^{-1})}\underset{nt \times qT}{G}$

$\rightarrow$ mean $= \underset{qT \times 1}{\hat{\eta}} = \underset{qT \times qT}{P^{-1}}(\underset{qT \times qT}{K}\underset{qT \times qT}{H^{-1}}\underset{qT \times l}{Z}\underset{l \times 1}{\gamma} + \overset{qT \times nT}{G'}\overset{nT \times nT}{(I_T \otimes \Omega_{11}^{-1})}\overset{nT \times 1}{(y - X\beta)})$

Since $G'(I_T \otimes \Omega_{11}^{-1})G$ is banded, P is also banded.

Considering the posterior mean equation

$$\underset{qT\times 1}{\hat{\eta}} = P^{-1}(K\bar{\eta} + G'(I_T \otimes \Omega_{11}^{-1})(y - X\beta))$$

$$P\hat{\eta} = (K\tilde{\eta} + G'(I_T \otimes \Omega_{11}^{-1})(y - X\beta))$$

Using Cholesky decomposition $P = C'C$

$$\underset{qT\times qT}{C'}\,\underset{qT\times 1}{C\hat{\eta}} = K\tilde{\eta} + G'(I_T \otimes \Omega_{11}^{-1})(y - X\beta)$$

$$\underbrace{\begin{bmatrix} c_{11} & & 0 \\ \vdots & \ddots & \\ c_{qT} & \cdots & c_{qTqT} \end{bmatrix}}_{\text{lower triangular}} \underbrace{\begin{bmatrix} c_{11} & \cdots & c_{qT} \\ & \ddots & \\ 0 & & c_{qTqT} \end{bmatrix}}_{\text{upper triangular}}' \begin{bmatrix} \eta_{11} \\ \eta_{12} \\ \vdots \\ \eta_{1T} \\ \eta_{21} \\ \vdots \\ \eta_{2T} \\ \eta_{qT} \end{bmatrix}_{qT\times 1} = K\tilde{\eta} + G'(I_T \otimes \Omega_{11}^{-1})(y - X\beta)$$

Forward substitution

Let $C\hat{\eta} = \eta^*$

Solve $C\eta^* = K\tilde{\eta} + G'(I_T \otimes \Omega_{11}^{-1})(y - X\beta)$ for η^* using lower triangular matrix

Backward substitution

Then solve $C\hat{\eta}$ for $\hat{\eta}$ using upper triangular matrix

(What if we have solved directly using P? Will it increase efficiency using Cholesky decomposotion?)

we get $\hat{\eta}$ which is posterior mean.
we need to draw from $(\eta | y, Q)$. So, we will follow the following procedure

1) Sample $\mu = N(0, I)$
2) Solve $Cx = u$ for x by backward substitution as C is upper triangular matrix.

$$x \sim N(0, P^{-1})$$

3) Construct $\eta = \hat{\eta} + x$ and then $\eta \sim N(\hat{\eta}, P^{-1}) = f_N(\eta | y, Q)$

Applications

Time series data : GDP growth
unemployment rate
interest rate
inflation

Time period : (1948:Q1 to 2005 Q1)
229 observations

Time varying parameter model

$$y_t = \mu_t + \Gamma_t y_{t-1} + \varepsilon_t , \qquad \varepsilon_t \sim N(0, \Omega_{11})$$

$$\begin{bmatrix} Out_t \\ Unemp_t \\ Inf_t \\ Int_t \end{bmatrix} = \begin{bmatrix} \mu_{1t} \\ \mu_{2t} \\ \mu_{3t} \\ \mu_{4t} \end{bmatrix} + \begin{bmatrix} \beta_{1t} & \beta_{2t} & \beta_{3t} & \beta_{4t} \\ \beta_{5t} & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \beta_{16t} \end{bmatrix} \begin{bmatrix} Out_{t-1} \\ Unemp_{t-1} \\ Inf_{t-1} \\ Int_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \end{bmatrix}$$

$$\varepsilon_t \sim N(0, \underset{4\times 4}{\Omega_{11}})$$

Writing the model in Seemingly Unrelated Regression form

$$y_t = X_t \beta_t + \varepsilon_t$$

$$\begin{bmatrix} Out_t \\ Unemp_t \\ Inf_t \\ Int_t \end{bmatrix} = \begin{bmatrix} 1 & out_t & Unemp_t & Inf_t & Int_t & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & out & \\ & & & & & & \ddots & \\ & 0 & & & & & & Int_t \end{bmatrix} \begin{bmatrix} \mu_{1t} \\ \beta_{1t} \\ \beta_{2t} \\ \beta_{3t} \\ \beta_{4t} \\ \mu_{2t} \\ \vdots \\ \beta_{16t} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \end{bmatrix}$$

Or

$$y_t = \begin{bmatrix} (1, y_{t-1}) & & & \\ & (1, y_{t-1}) & & \\ & & (1, y_{t-1}) & \\ & & & (1, y_{t-1}) \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \mu_{1t} \\ \gamma_{1t} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \mu_{4t} \\ \gamma_{4t} \end{bmatrix} \end{bmatrix} + \varepsilon_t$$

$$\underset{20\times1}{\beta_t} = \underset{20\times1}{\beta_{t-1}} + \underset{20\times1}{\nu_t}$$

$$\beta_1 \sim N(0, \underset{20\times20}{D}) \quad , \quad \nu_t \sim N(0, \underset{20\times20}{\Omega_{22}})$$

$$\Omega_{22} = \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_{20}^2 \end{bmatrix}$$

Stacking the observations

$$y = X\beta + \varepsilon, \quad \varepsilon \sim N(0, \underbrace{I_T \otimes \Omega_{11}}_{\text{Blocked matrix}})$$

$$y = \underset{(4\times 228)\times 1}{\begin{bmatrix} \underset{4\times1}{y_1} \\ \vdots \\ y_T \end{bmatrix}}, \quad X = \underset{(4\times228)\times(4\times5\times228)}{\begin{bmatrix} \underset{4\times(4\times5)}{X_1} & & & \\ & X_2 & & \\ & & \ddots & \\ & & & X_T \end{bmatrix}}, \quad \beta = \underset{(4\times5\times228)\times1}{\begin{bmatrix} \underset{(4\times5)\times1}{\beta_1'} \\ \vdots \\ \beta_T' \end{bmatrix}}$$

rom AR1 representation of β

$$\beta_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{20 \times 1} \sim N(0, D)$$

$$\beta_2 = \beta_1 + \nu_2 \quad , \quad \beta_2 \sim N(0, \underset{20\times 20}{\Omega_{22}})$$

$$\beta_2 - \beta_1 = \nu_2$$

$$\beta_3 - \beta_2 = \nu_3$$

$$\vdots \qquad\qquad\qquad \vdots \quad \vdots \quad \vdots$$

$$\underset{20\times 1}{\beta_{228}} - \underset{20\times 1}{\beta_{227}} = \underset{20\times 1}{\nu_{228}} \qquad\qquad \beta_{228} \sim N(0, \underset{20\times 20}{\Omega_{22}})$$

$$\beta_t - \beta_{t-1} = \nu_t$$

$$\underbrace{\begin{bmatrix} \overset{20\times 20}{I} & 0 & \cdots & \cdots & 0 \\ -I & I & 0 & \cdots & 0 \\ 0 & -I & I & 0 \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \; -I & I \end{bmatrix}}_{\substack{H \\ (20\times 228)\times(228\times 20)}} \underbrace{\begin{bmatrix} \underset{20\times 1}{\beta_1} \\ \underset{20\times 1}{\beta_2} \\ \vdots \\ \beta_t \\ \vdots \\ \beta_T \end{bmatrix}}_{\substack{\beta \\ (20\times 228)\times 1}} = \underbrace{\begin{bmatrix} \nu_1 \\ \vdots \\ \vdots \\ \nu_T \end{bmatrix}}_{\substack{\nu \\ (228\times 20)\times 1}}$$

$$\beta = H^{-1} \nu$$

$$\beta \sim N(0, K^{-1})$$

$K^{-1} =$

$\downarrow$

$(20\times 228)\ (20\times 228)$

$(qT)(qT)$

$$K^{-1} = \underset{(20\times 228)\times(228\times 20)}{H^{-1}} \underbrace{\begin{pmatrix} \underset{20\times 20}{D} & & & \\ & \underset{20\times 20}{\Omega_{22}} & & \\ & & \ddots & \\ & & & \underset{20\times 20}{\Omega_{22}} \end{pmatrix}}_{S} \underset{(228\times 20)\times(228\times 20)}{H^{-1}}$$

$(20\times 228)\ (20\times 228)$

$$K^{-1} = H^{-1} S H^{-1}$$

$$K = H' S^{-1} H$$

Bayesian estimation

priors
$$\underset{4\times 4}{\Omega_{11}} \sim IW(v_1^{\circ}, S_1^{\circ})$$
$$\underset{1\times 1}{\sigma_i^2} \sim IG(v_{i2}^{\circ}/2,\ S_{i2}^{\circ}/2) \rightarrow \text{for } \Omega_{22}$$
$$\beta \sim N(0, \underset{(20\times 228)\times(20\times 228)}{K^{-1}})$$

$$\underset{4\times 1}{y_t} \sim N(\underset{4\times 20}{X_t}\, \underset{20\times 1}{\beta_t},\ \underset{4\times 4}{\Omega_{11}})$$

$$\pi(\Omega_{11}, \beta, \Omega_{22} \mid y) = f(y \mid \beta, \Omega_{11}, \Omega_{22})\, \pi(\beta \mid \Omega_{22})\, \pi(\Omega_{22})\, \pi(\Omega_{11})$$

$$= (2\pi)^{-T/2} (\Omega_{11})^{-T/2} \exp\left\{ \frac{1}{2} \sum_{t=1}^{T} (y_t - X_t\beta_t)\Omega_{11}^{-1}(y_t - X_t\beta_t) \right\}$$

$$* \ (2\pi)^{-1/2} (K^{-1})^{-1/2} \exp\left\{ -\frac{1}{2} \beta K \beta' \right\}$$

$$* \ \frac{|S_1^{\circ}|^{v_1^{\circ}/2}}{2^{v_1^{\circ} p/2}\, \Gamma_p\left(\frac{v_1^{\circ}}{2}\right)} |\Omega_{11}|^{-(v_i^{\circ} + p + 1)} e^{-1/2\, tr(S_i^{\circ} \Omega_{11}^{-1})}$$

$$* \ \prod_{i=1}^{20} \frac{\left(S_{i2}^{\circ}/2\right)^{\frac{v_{i2}}{2}}}{\Gamma\left(\frac{v_{i2}}{2}\right)} (\sigma_i^2)^{-\frac{v_{i2}}{2} - 1} e^{-\frac{S_{i2}^{\circ}}{2\sigma_i^2}}$$

$$= \cancel{(2\pi)^{-T/2}} (\Omega_{11})^{-T/2} \exp\left\{ \frac{-1}{2} \sum_{t=1}^{T} (y_t - X_t \beta_t) \Omega_{11}^{-1} (y_t - X_t \beta_t) \right\}$$

$$* \cancel{(2\pi)^{-1/2}} (K^{-1})^{-1/2} \exp\left\{ \frac{-1}{2} \beta K \beta' \right\}$$

$$* \frac{\cancel{|S_1^{\circ}|^{v_1^{\circ}/2}}}{2^{v_1^{\circ} p/2} \cancel{\Gamma_p\left(\frac{v_1}{2}\right)}} |\Omega_{11}|^{-(v_i^{\circ}+p+1)/2} e^{-1/2\, tr(S_i^{\circ} \Omega_{11}^{-1})}$$

$$* \prod_{i=1}^{20} \frac{\cancel{(s_{i2}^{\circ}/2)^{\frac{\nu_{i2}}{2}}}}{\cancel{\Gamma\left(\frac{\nu_{i2}}{2}\right)}} (\sigma_i^2)^{-\frac{\nu_{i2}}{2}-1} e^{-\frac{s_{i2}^{\circ}}{2\sigma_i^2}}$$

$$= (\Omega_{11})^{-T/2} \exp\left\{ \frac{1}{2} \sum_{t=1}^{T} (y_t - X_t \beta_t) \Omega_{11}^{-1} (y_t - X_t \beta_t) \right\} (K^{-1})^{-1/2}$$

$$* \exp\left\{ \frac{-1}{2} \beta K \beta' \right\} |\Omega_{11}|^{-(v_i^{\circ}+p+1)} e^{-1/2\, tr(S_i^{\circ} \Omega_{11}^{-1})}$$

$$* \prod_{i=1}^{20} (\sigma_i^2)^{-\frac{\nu_{i2}}{2}-1} e^{-\frac{s_{i2}^{\circ}}{2\sigma_i^2}}$$

$$(\beta | \Omega_{11}, \Omega_{22}, y) \propto \cancel{(\Omega_{11})^{-T/2}} \exp\left\{ \frac{-1}{2} \sum_{t=1}^{T} (y_t - X_t \beta_t) \Omega_{11}^{-1} (y_t - X_t \beta_t) \right\}$$

$$* \cancel{(K^{-1})^{-1/2}} \exp\left\{ \frac{-1}{2} \beta K \beta' \right\} \cancel{|\Omega_{11}|^{-(S_1^{\circ}+5)}}$$

$$* \cancel{e^{-1/2\, tr(v_1^{\circ} \Omega_{11}^{-1})}} \cancel{\prod_{i=1}^{20} (\sigma_i^2)^{-\frac{\nu_{i2}}{2}-1} e^{-\frac{s_{i2}^{\circ}}{2\sigma_i^2}}}$$

$$(\beta | \Omega_{11}, \Omega_{22}, y) \propto \exp\left\{ (y - X\beta)' [I \otimes \Omega_{11}^{-1}] (y - X\beta) + \beta K \beta' \right\}$$

$$\propto \exp\left\{ \frac{-1}{2} \left[\cancel{y' [I \otimes \Omega_{11}^{-1}] y} - y' [I \otimes \Omega_{11}^{-1}] X\beta - \beta' X' [I \otimes \Omega_{11}]^{-1} y + \beta' X' [I \otimes \Omega_{11}^{-1}] X\beta + \beta K \beta' \right] \right\}$$

$$[I \otimes \Omega_{11}]^{-1}$$

$$\propto \exp \left\{ \frac{-1}{2} \left[y'[I \otimes \Omega_{11}^{-1}] X\beta - \beta' X'[I \otimes \Omega_{11}]^{-1} y + \beta' X' X \beta + \beta' K \beta \right] \right\}$$

$$\propto \exp \left\{ \frac{-1}{2} \left[\beta'[X'[I \otimes \Omega_{11}]^{-1} X + K] \beta + \beta'[X'[I \otimes \Omega_{11}]^{-1} y] - y'[I \otimes \Omega_{11}^{-1}] X\beta \right] \right]$$

$$P = X'[I \otimes \Omega_{11}]^{-1} X + K = \text{precision}$$

$$\boxed{\beta \mid y, \Omega_{11}, \Omega_{22} \sim N\left(P^{-1}[X'[I \otimes \Omega_{11}]^{-1} y], P^{-1}\right)}$$

$$(\Omega_{11} \mid \beta, \Omega_{22}, y) \propto |\Omega_{11}|^{-T/2} \exp \left\{ \frac{-1}{2} \sum_{t=1}^{T} (y_t - X_t\beta_t)' \Omega_{11}^{-1} (y_t - X_t\beta_t) \right\}$$

$$* \cancel{(K^{-1})^{-1/2}} \cancel{\exp \left\{ -\frac{1}{2} \beta K \beta' \right\}} |\Omega_{11}|^{-(v_1^0 + 5)/2}$$

$$* e^{-1/2 \, tr(S_1^0 \Omega_{11}^{-1})} \cancel{\prod_{i=1}^{20} (\sigma_i^2)^{-\frac{v_{i0}}{2} - 1} e^{\frac{S_{i2}^0}{2\sigma_i^2}}}$$

$$= |\Omega_{11}|^{-T/2 - (v_1^0 + 5)/2} \exp \left\{ \frac{-1}{2} \sum_{t=1}^{T} (y_t - X_t\beta_t)' \Omega_{11}^{-1} (y_t - X_t\beta_t) - \frac{1}{2} tr(S_1^0 \Omega_{11}^{-1}) \right\}$$

↳ determinant

$$= |\Omega_{11}|^{-\left(\frac{T + v_1^0}{2}\right) - \frac{4}{2} - 1/2} \exp \left\{ -\frac{1}{2} \_\_\_\_ \Omega_{11}^{-1} \right\}$$

$$\Omega_{11} \sim IW\left(v_1^0 + T, \sum_{t=1}^{T} (y_t - X_t\beta_t)(y_t - X_t\beta_t + S_1^0\right)$$

$$(\Omega_{22}|\beta, \Omega_{11}, y) \propto \cancel{|\Omega_{11}|^{-T/2}} \exp \cancel{\left\{ -\frac{1}{2}\sum_{t=1}^{T} (y_t - X_t\beta_t)' \Omega_{11}^{-1} (y_t - X_t\beta_t) \right\}}$$

$$* \ (K^{-1})^{-1/2} \exp\left\{-\frac{1}{2}\beta K \beta'\right\} \cancel{|\Omega_{11}|^{-(v_1^0+5)/2}}$$

$$* \ \cancel{e^{-1/2 \, tr(S_1^0 \Omega_{11}^{-1})}} \prod_{i=1}^{20} (\sigma_i^2)^{-\frac{v_{i2}}{2}-1} e^{-\frac{S_{i2}^0}{2\sigma_i^2}}$$

$$(\Omega_{22}|\beta, \Omega_{11}, y) \propto (K^{-1})^{-1/2} \exp\left\{-\frac{1}{2}\beta K \beta'\right\} \prod_{i=1}^{20} (\sigma_i^2)^{-\frac{v_{i2}}{2}-1} e^{-\frac{S_{i2}^0}{2\sigma_i^2}}$$

$$\propto (S^{-1})^{-1/2} \exp\left\{-\frac{1}{2}\beta K \beta'\right\} \prod_{i=1}^{20} (\sigma_i^2)^{-\frac{v_{i2}}{2}-1} e^{-\frac{S_{i2}^0}{2\sigma_i^2}}$$

$$[\sigma^2_i | y, \beta] \sim IG\left[\frac{v_{i2}^0 + T - 1}{2}, \frac{S_{i2}^0 + \sum_{t=0}^{T} (\beta_{t,i} - \beta_{t-1,t})^2}{2}\right]$$

Dynamic Factor Model

$$\underset{4\times1}{y_t} = \underset{4\times1}{\mu} + \underset{4\times5}{\Gamma}\, \underset{5\times1}{y_{t-1}} + \underset{4\times1}{A}\, \underset{1\times1}{f_t} + \underset{4\times1}{\varepsilon_t} \qquad -(1)$$

$$\begin{bmatrix} GDP_t \\ unemp_t \\ inflat_t \\ inter_t \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix} + \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \beta_5 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \beta_{16} \end{bmatrix} \begin{bmatrix} GDP_{t-1} \\ unemp_{t-1} \\ inflat_{t-1} \\ inter_{t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} [f_t] + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \end{bmatrix}$$

$$\varepsilon_t \sim N(0, \underset{4\times4}{\Omega}) \quad \text{where} \quad \Omega = \begin{bmatrix} \omega_{11} & 0 & 0 & 0 \\ 0 & \omega_{22} & 0 & 0 \\ 0 & 0 & \omega_{33} & 0 \\ 0 & 0 & 0 & \omega_{44} \end{bmatrix}$$

$$\underset{1\times 1}{f_t} = \underset{1\times 1}{\gamma}\, \underset{1\times 1}{f_{t-1}} + \underset{1\times 1}{\nu_t} \qquad -\ (2)$$

$$\nu_t \sim N(0, \sigma^2)$$
$$t = 1, 2 \ldots \quad T$$

for f_1, the steady state solution is given as

$$f_1 \sim N\left(0, \frac{\sigma^2}{1-\gamma^2}\right)$$

Stacking up (2) for t

$$\begin{bmatrix} f_2 \\ \vdots \\ \\ f_T \end{bmatrix} = \gamma \begin{bmatrix} f_1 \\ \vdots \\ \\ f_{T-1} \end{bmatrix} + \begin{bmatrix} \nu_1 \\ \vdots \\ \\ \nu_T \end{bmatrix}$$

$$\begin{bmatrix} f_2 \\ \vdots \\ \\ f_T \end{bmatrix} - \gamma \begin{bmatrix} f_1 \\ \vdots \\ \\ f_{T-1} \end{bmatrix} = \begin{bmatrix} \nu_1 \\ \vdots \\ \\ \nu_T \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \sqrt{(1-\gamma^2)} & 0 & \cdots & \cdots & 0 \\ -\gamma & 1 & 0 & \cdots & 0 \\ 0 & -\gamma & 1 & 0 \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & -\gamma \ 1 \end{bmatrix}_{T\times T}}_{H} \begin{bmatrix} f_1 \\ \vdots \\ \\ f_T \end{bmatrix}_{T\times 1} = \begin{bmatrix} \nu_1 \\ \vdots \\ \\ \nu_T \end{bmatrix}_{228\times 1}$$

$$
\begin{bmatrix} f_1 \\ \vdots \\ f_T \end{bmatrix} = \underbrace{\begin{bmatrix} \sqrt{(1-\gamma^2)} & 0 & \cdots & & 0 \\ -\gamma & 1 & 0 & \cdots & 0 \\ 0 & -\gamma & 1 & 0 \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -\gamma & 1 \end{bmatrix}}_{H}{}^{-1} \begin{bmatrix} v_1 \\ \vdots \\ v_T \end{bmatrix}
$$

$$
\underset{228\times228}{H'}\ \underset{228\times228}{H} = \underset{\substack{228\times228 \\ T\times T}}{F_0} = \underset{228\times228}{\begin{bmatrix} 1 & -\gamma & & & \\ -\gamma & 1+\gamma^2 & -\gamma & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 1+\gamma^2 & -\gamma \\ & & & -\gamma & 1 \end{bmatrix}}
$$

$$
f \mid \gamma, \sigma^2 \sim N(0, \sigma^2 F_0^{-1}) = N(0, \sigma^2 (H'H)^{-1})
$$

Rewriting the model in SUR form as in TVP-VAR.

$$
\underset{4\times1}{y_t} = \underset{4\times(4\times5)}{X_t}\ \underset{(4\times5)\times1}{\beta} + \underset{4\times1}{A}\ \underset{1\times1}{f_t} + \underset{4\times1}{\varepsilon_t}
$$

$$
\underset{4\times1}{\begin{bmatrix} GDP_t \\ Unemp_t \\ Int_t \\ Infl_t \end{bmatrix}} = \underset{4\times(20)}{\begin{bmatrix} GDP_{t-1} & Unemp_{t-1} & Int_{t-1} & Inf_{t-1} & 0 & \cdots & & 0 \\ 0 & 0 & 0 & 0 & GDP_{t-1} & \cdots & & 0 \\ 0 & 0 & 0 & 0 & \cdots & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & & Int_{t-1} & Inf_{t-1} \end{bmatrix}} \underset{20\times1}{\begin{bmatrix} \mu_1 \\ \beta_1 \\ \vdots \\ \beta_5 \\ \mu_2 \\ \beta_6 \\ \vdots \\ \beta_{16} \end{bmatrix}}
$$

$$
+ \underset{4\times1}{\begin{bmatrix} 1 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}} \underset{1\times1}{f_t} + \underset{4\times1}{\begin{bmatrix} v_{1t} \\ v_{2t} \\ v_{3t} \\ v_{4t} \end{bmatrix}}
$$

Stacking equation over t

$$\underset{(4\times228)\times1}{y} = \underset{(4\times228)\times20}{X}\ \underset{20\times1}{\beta} + \underbrace{(\underset{228\times228}{I_T} \otimes \underset{4\times1}{A})}_{(228\times4)\ (228\times1)}\ \underset{(228\times1)}{f} + \underset{(228\times1)}{\varepsilon}$$

$$\underset{(4\times228)\times1}{\begin{bmatrix} y'_{2\,(4\times1)} \\ y'_{3\,(4\times1)} \\ \vdots \\ \vdots \\ y'_{229\,(4\times1)} \end{bmatrix}} = \underset{(228\times4)\times20}{\begin{bmatrix} GDP_1 \cdots Int_1 & & & & 0 \\ & GDP_1 \cdots Int_1 & & & \\ & & GDP_1 \cdots Int_1 & & \\ & & & GDP_1 \cdots Int_1 & \\ & & & \ddots & \ddots \\ 0 & & & & y_{228} \end{bmatrix}} \underset{20\times1}{\begin{bmatrix} \mu_1 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{16} \end{bmatrix}}$$

$$+ \underset{(4\times228)(228)}{\begin{bmatrix} \begin{matrix} a'_1 \\ a_2 \\ a_3 \end{matrix} & & & 0 \\ & \begin{matrix} a'_1 \\ a_2 \\ a_3 \end{matrix} & & \\ & & \ddots & \\ 0 & & & \begin{matrix} a'_1 \\ a_2 \\ a_3 \end{matrix} \end{bmatrix}} \underset{228\times1}{\begin{bmatrix} f_2 \\ f_3 \\ \vdots \\ f_{229} \end{bmatrix}} + \begin{bmatrix} \varepsilon_2 \\ \vdots \\ \vdots \\ \varepsilon_{229} \end{bmatrix}$$

Distributions

1) $[f \mid \gamma, \sigma^2] \sim N(0, \sigma^2 \underset{228\times228}{F_0^{-1}})$ → contains γ
2) $\beta \sim N(\beta_0, B_0)$
3) $a \sim N(a_0, A_0)$
4) $w_{ii} \sim IG(v_{i0}/2, \delta_{i0}/2)$ for $i = 1, 2, 3, 4$)

$y_t = \mu + \Gamma y_{t-1} + A f_t + \varepsilon_t,\ \varepsilon_t \sim N(0, \Omega)$ where $\Omega = \mathrm{diag}\{\omega_{11}, \ldots \omega_{44}\}$

5) $\sigma^2 \sim IG(s_0/2, \delta_0/2)$
6) $\gamma \sim TN_{(-1,1)}(\gamma_0, G_0)$ → truncated normal

$$\pi(\beta, a, f, \Omega, \gamma, \sigma^2 | y) \propto f(y | \beta, a, f, \Omega, \gamma, \sigma^2) * \pi_N(f)$$
$$* \pi_N(\beta) * \pi_N(a) * \prod_{i=1}^{4} \pi_{IG}(\omega_{ii})$$
$$* \pi_{IG}(\sigma^2) * \pi_{TN(-1,1)}(\gamma)$$

Scheme to draw from posterior

1) $[\beta | y, A, \Omega, \gamma, \sigma^2]$ - does not depend on f
2) $[a, f | y, \beta, \Omega, \gamma, \sigma^2]$
   a) $[a | y, \beta, \Omega, \gamma, \sigma^2]$ - does not depend on f
   b) $[f | y, \beta, A, \Omega, \gamma, \sigma^2]$
3) $[\Omega | y, \beta, A, f]$
4) $[\gamma | f, \sigma^2]$
5) $[\sigma^2 | f, \gamma]$

For step 1 of scheme, we have to integrate out f from.

$$y = X\beta + \underbrace{(I_t \otimes A)f + \varepsilon}_{\nu} \quad \text{using} \quad f \sim N(0, \sigma^2 F_0^{-1})$$

$$y = X\beta + \nu$$

$$\nu \sim N(0, \Sigma)$$

$$\underset{(228\times 4)(228\times 4)}{\Sigma} = \underbrace{(I_T \otimes A)}_{(228\times 4)\times(228)} \sigma^2 \underset{(228\times 228)}{F_0^{-1}} \underset{(228)\times(228\times 4)}{(I_T \otimes A)'} + \underbrace{I_T \otimes \Omega}_{(4\times 228)(4\times 228)}$$

posterior $[\beta | y, A, \Omega, \gamma, \sigma^2] \sim N(\hat{\beta}, B)$

$$B = (B_0^{-1} X' \Sigma^{-1} X)^{-1} \qquad , \quad \hat{\beta} = B(B_0^{-1}\beta_0 + X'\Sigma^{-1} y)$$

This will require brute force for inverting Σ which will be computationally costly.

So the new way to calculate is given by

$$\underset{20\times20}{B} = \Big[\underset{20\times20}{B_0^{-1}} + \sum_{t=1}^{T} \underbrace{\underset{20\times4}{X_t'}\,\underset{4\times4}{\Omega^{-1}}\,\underset{4\times20}{X_t}}_{20\times20} - \underset{20\times228}{\tilde{X}'} \Big[\underset{228\times228}{\sigma^{-2}F_0} + \underbrace{\underset{228\times228}{I_T}(\underset{1\times4}{A'}\,\underset{4\times4}{\Omega^{-1}}\,\underset{4\times1}{A})}_{228\times228} \Big]^{-1} \underset{228\times20}{\tilde{X}} \Big]$$

$$\underset{20\times20}{B} = \Big[B_0^{-1} + \sum_{t=1}^{T} X_t'\Omega^{-1}X_t - \tilde{X}' P^{-1} \tilde{X} \Big]^{-1}$$

$$\tilde{X} = \begin{array}{c} 1\times20 \\ \begin{bmatrix} A'\Omega^{-1}X_1 \\ \vdots \\ A'\Omega^{-1}X_{228} \end{bmatrix} \end{array} \; 228\times20 \qquad\qquad P = \left[\sigma^{-2}F_0 + I_T(A'\Omega^{-1}A) \right]$$

$$\hat{\beta} = \underset{20\times20}{B} \Big[\underset{20\times20}{B_0^{-1}}\,\underset{20\times1}{\beta_0} + \sum_{t=1}^{T} \underset{20\times4}{X_t'}\,\underset{4\times4}{\Omega^{-1}}\,\underset{4\times1}{y_t} - \underset{20\times228}{\tilde{X}'}\,\underset{228\times228}{P^{-1}}\,\underset{228\times1}{\tilde{y}} \Big]$$

$$\tilde{y} = \begin{array}{c} 1\times4 \;\; 4\times4 \;\; 4\times1 \\ \begin{bmatrix} A'\Omega^{-1}y_2 \\ \vdots \\ A'\Omega^{-1}y_{229} \end{bmatrix} \end{array}$$

$$\underbrace{\underset{20\times(228\times4)}{X'}\;\underset{(228\times4)(228\times4)}{(\overset{228\times228}{I_T}\otimes\overset{4\times4}{\Omega^{-1}})}\;\underset{(228\times4)\times20}{X}}_{20\times20}
+\underbrace{\underset{20\times(228\times4)}{X'}\;\underset{(228\times4)\times228}{(\overset{228\times228}{I_T}\otimes\overset{4\times1}{\overbrace{\overset{4\times4}{\Omega^{-1}}\,\overset{4\times1}{A}}})}}_{\tilde{X}'}\;\underset{228\times228}{P^{-1}}\;\underbrace{\underset{228\times(228\times4)}{(I_T\otimes\overset{1\times4}{A'}\,\overset{4\times4}{\Omega^{-1}})}\;\underset{(228\times4)\times20}{X}}_{\tilde{X}}$$

$$\underset{228\times(228\times4)}{\begin{bmatrix} \overset{1\times4}{A'\Omega^{-1}} & \cdots\cdots & 228\text{ times} \\ & \ddots & \\ & & A'\Omega^{-1}\end{bmatrix}}\;\underset{(4\times228)\times20}{\begin{bmatrix} \overset{4\times20}{X_1} \\ X_2 \\ \vdots \\ X_{228}\end{bmatrix}} = \underset{228\times20}{\begin{bmatrix} \overset{1\times4}{A'}\,\overset{4\times4}{\Omega^{-1}}\,\overset{4\times20}{X_1} \\ A'\Omega^{-1}X_2 \\ \vdots \\ A'\Omega^{-1}X_{228}\end{bmatrix}} = \tilde{X}$$

$$\underbrace{\underset{20\times(228\times4)}{X'}\;\underset{(228\times4)(228\times4)}{(I_T\otimes\Omega^{-1})}\;\underset{(228\times4)\times1}{y}}_{20\times1}
+\underbrace{\underset{20\times(228\times4)}{X'}\;\underset{(228\times4)\times228}{(\overset{228\times228}{I_T}\otimes\overset{4\times1}{\overbrace{\overset{4\times4}{\Omega^{-1}}\,\overset{4\times1}{A}}})}}_{\tilde{y}'}\;\underset{228\times228}{P^{-1}}\;\underbrace{\underset{228\times(228\times4)}{(I_T\otimes A'\Omega^{-1})}\;\underset{(228\times4)\times1}{y}}_{\tilde{y}}$$

$$\underset{228\times(228\times4)}{\begin{bmatrix} \overset{1\times4}{A'\Omega^{-1}} & \cdots\cdots & 228\text{ times} \\ & \ddots & \\ & & A'\Omega^{-1}\end{bmatrix}}\;\underset{(4\times228)\times1}{\begin{bmatrix} \overset{4\times1}{y_2} \\ y_3 \\ \vdots \\ y_{229}\end{bmatrix}} = \underset{228\times1}{\begin{bmatrix} \overset{1\times4}{A'}\,\overset{4\times4}{\Omega^{-1}}\,\overset{4\times1}{y_2} \\ A'\Omega^{-1}y_3 \\ \vdots \\ A'\Omega^{-1}y_{229}\end{bmatrix}} = \tilde{y}$$

For step 2 of the Scheme, a has to be sampled marginally of f similar to

$$f(y \mid \theta) = \frac{f(y \mid \theta, \eta)\, \pi(\eta \mid \theta)}{\pi(\eta \mid y, \theta)}$$ - marginal likelihood

$$\pi(a \mid \theta, A) = \frac{\pi(a \mid \theta, f)}{\pi(f \mid \theta, A)} \underbrace{\pi(f \mid \theta)}_{\text{can be ignored due to proportionality}}$$

A and a represent same element or group of elements

$$\pi(a \mid y, \beta, A, \Omega, \gamma, \sigma^2) \propto \frac{\pi(a \mid y, \beta, f, \Omega, \gamma, \sigma^2)}{\pi(f \mid y, \beta, A, \Omega, \gamma, \sigma^2)}$$

Calculating full conditionals from posterior distribution

$$\pi(\beta, a, f, \Omega, \gamma, \sigma^2 \mid y) \propto f(y \mid \beta, a, f, \Omega, \gamma, \sigma^2) * \pi_N(f) * \pi_N(\beta) * \pi_N(a) * \prod_{i=1}^{4} \pi_{IG}(w_{ii}) * \pi_{IG}(\sigma^2) * \pi_{TN(-1,1)}(\gamma)$$

$$\propto (2\pi)^{-1/2} |\Sigma|^{-1/2} \exp\left\{-\tfrac{1}{2}(y - X\beta)'\Sigma^{-1}(y - X\beta)\right\} * (2\pi)^{-1/2} (\sigma^2 F_0^{-1})^{-1/2}$$
$$\exp\left\{-\tfrac{1}{2}\left(f'(\sigma^2 F_0^{-1})^{-1} f\right)\right\} (2\pi)^{-1/2} (B_0)^{-1/2} \exp\left\{-\tfrac{1}{2}(\beta - \beta_0)' B_0^{-1} (\beta - \beta_0)\right\}$$
$$* (2\pi)^{-1/2} A_0^{-1/2} \exp\left\{-\tfrac{1}{2}(a - a_0)' A_0^{-1} (a - a_0)\right\}$$
$$* \prod_{i=1}^{4} \frac{(\gamma_{i0}/2)^{v_{i0}/2}}{\Gamma(v_{i0}/2)} (w_{ii})^{-\frac{v_{i0}}{2} - 1} \exp\left\{-\frac{\gamma_{i0}}{2 w_{ii}}\right\} * \frac{(\delta_0/2)^{s_0/2}}{\Gamma(s_0/2)} (\sigma^2)^{\frac{s_0}{2} - 1}$$
$$\exp\left\{-\frac{\delta_0}{2\sigma^2}\right\} * I(\gamma \in (-1,1)) (2\pi)^{-1/2} (G_0)^{-1/2} \exp\left\{-\tfrac{1}{2}(\gamma - \gamma_0)' G_0^{-1} (\gamma - \gamma_0)\right\}$$

$$\propto \cancel{(2\pi)^{-1/2}} (\Sigma)^{-1/2} \exp\left\{-\tfrac{1}{2}(y - X\beta)'\Sigma^{-1}(y - X\beta)\right\} * \cancel{(2\pi)^{-1/2}} (\sigma^2 F_0^{-1})^{-1/2}$$

$$\exp\left\{-\tfrac{1}{2}\left(f'(\sigma^2 F_0^{-1})^{-1} f\right)\right\} \cancel{(2\pi)^{-1/2}} \cancel{(B_0)^{-1/2}} \exp\left\{-\tfrac{1}{2}(\beta - \beta_0)' B_0^{-1}(\beta - \beta_0)\right\}$$

$$* \cancel{(2\pi)^{-1/2}} \cancel{A_0^{-1/2}} \exp\left\{-\tfrac{1}{2}(a - a_0)' A_0^{-1}(a - a_0)\right\}$$

$$* \prod_{i=1}^{4} \cancel{\frac{(r_{i0}/2)^{v_{i0}/2}}{\Gamma(v_{i0}/2)}} (w_{ii})^{-\frac{v_{i0}}{2} - 1} \exp\left\{-\frac{r_{i0}}{2 w_{ii}}\right\} * \cancel{\frac{(\delta_0/2)^{s_0/2}}{\Gamma(s_0/2)}} (\sigma^2)^{\frac{s_0}{2} - 1}$$

$$\exp\left\{-\frac{\delta_0}{2\sigma^2}\right\} * I(\gamma \in (-1, 1)) \cancel{(2\pi)^{-1/2}} \cancel{(G_0)^{-1/2}} \exp\left\{-\tfrac{1}{2}(\gamma - \gamma_0)' G_0^{-1}(\gamma - \gamma_0)\right\}$$

$$\propto (\Sigma)^{-1/2} \exp\left\{-\tfrac{1}{2}(y - X\beta)'\Sigma^{-1}(y - X\beta)\right\} * (\sigma^2 F_0^{-1}\ {}^{-1/2})$$

$$\exp\left\{-\tfrac{1}{2}\left(f'(\sigma^2 F_0^{-1})^{-1} f\right)\right\} \exp\left\{-\tfrac{1}{2}(\beta - \beta_0) B_0^{-1}(\beta - \beta_0')\right\}^{-1}$$

$$* \exp\left\{-\tfrac{1}{2}(a - a_0)' A_0^{-1}(a - a_0)\right\} * \prod_{i=1}^{4} (w_{ii})^{-\frac{v_{i0}}{2} - 1} \exp\left\{-\frac{r_{i0}}{2 w_{ii}}\right\} *$$

$$(\sigma^2)^{\frac{s_0}{2} - 1} \exp\left\{-\frac{\delta_0}{2\sigma^2}\right\} * I(\gamma \in (-1, 1)) \exp\left\{-\tfrac{1}{2}(\gamma - \gamma_0)' G_0^{-1}(\gamma - \gamma_0)\right\}$$

$$\beta \mid \theta, y, A \propto \exp\left\{-\tfrac{1}{2}(y - X\beta)'\Sigma^{-1}(y - X\beta)\right\} \exp\left\{-\tfrac{1}{2}(\beta - \beta_0) B_0^{-1}(\beta - \beta_0)'\right\}^{-1}$$

$$\propto \exp\left\{-\tfrac{1}{2}\left[\cancel{y'\Sigma^{-1}y} - y'\Sigma^{-1}X\beta - \beta'X'\Sigma^{-1}y + \beta'X'\Sigma^{-1}X\beta + \beta' B_0^{-1}\beta\right.\right.$$

$$\left.\left. - \beta' B_0^{-1} \beta_0 - \beta_0' B_0^{-1} \beta + \cancel{\beta_0' B_0^{-1} \beta_0}\right\}\right.$$

$$\propto \exp\left\{-\tfrac{1}{2}\left[\beta'[X'\Sigma^{-1}X + B_0^{-1}] - \beta'[X'\Sigma^{-1}y + B_0^{-1}\beta]\right.\right.$$

$$\left.\left. - [y'\Sigma^{-1}X - \beta_0' B_0^{-1}]\beta\right]\right.$$

$$\therefore \quad \beta \mid y, A, \Omega, \gamma, \sigma^2 \sim N(\hat{\beta}, B)$$

where $$B = (B_0^{-1} + X'\Sigma^{-1}X)^{-1}$$

$$\beta = B(B_0^{-1}\beta_0 + X'\Sigma^{-1}y)$$

$$a \mid \alpha, A \propto |\Sigma|^{-1/2} \exp\left\{-\frac{1}{2}\underbrace{(y - X\beta)'}_{1\times(228\times4)}\underbrace{\Sigma^{-1}}_{(228\times4)\times(228\times4)}\underbrace{(y - X\beta)}_{(228\times4)\times1}\right\} * \left[\exp\left\{-\frac{1}{2}\underset{1\times3}{(a-a_0)'}\underset{3\times3}{A_0^{-1}}\underset{3\times1}{(a-a_0)}\right\}\right]$$

Algorithm

1) Sample $[\beta \mid y, A, \Omega, \gamma, \sigma^2] \sim N(\hat{\beta}, B)$

$$\underset{20\times1}{\hat{\beta}} = B\left[B_0^{-1}\beta_0 + \sum_{t=1}^{T} X_t'\Omega^{-1}y_t - \tilde{X}'P^{-1}\tilde{y}\right]$$

$$\underset{20\times20}{B} = \left[B_0^{-1} + \sum_{t=1}^{T} X_t'\Omega^{-1}X_t - \tilde{X}'P^{-1}\tilde{X}\right]^{-1}$$

2) Sample in one block

a) Sample $a^+ \sim q(\underset{3\times1}{\hat{a}}, \underset{3\times3}{V})$ which is student's T distribution with low degree of freedom and $\hat{a}$ is mode & V is inverse of negative Hessian at the mode of $\pi(a \mid y, \beta, \Omega, \gamma, \sigma^2)$
Accept a^+ with probability

$$\alpha_{MH}(a, a^+) = \min\left\{1, \frac{\pi(a^+ \mid y, \beta, A, \Omega, \gamma, \sigma^2)\, q(a \mid \hat{a}, V)}{\pi(a \mid y, \beta, A, \Omega, \gamma, \sigma^2)\, q(a^+ \mid \hat{a}, V)}\right\}$$

b) Sample $[f \mid y, \beta, A, \Omega, \gamma, \sigma^2] \sim N(\hat{f}, F)$

$$F = \left(\overset{228\times228}{F_0/\sigma^2} + \overset{228\times228}{}\overset{228\times(228\times4)}{(I_T \otimes A)'}\overset{(228\times4)(228\times4)}{(I_T \otimes \Omega^{-1})}\overset{(228\times4)(228)}{(I_T \otimes A)}\right)^{-1}$$

$$\hat{f} = \underset{228\times228}{F}\underbrace{(I_T \otimes A)'}_{228\times(228\times4)}\underset{(228\times4)(228\times4)}{(I_T \otimes \Omega^{-1})}\underset{(228\times4)\times1}{(y - X\beta)}$$

using algorithm 1.

3) Sample $[\underset{4\times4}{\Omega} | y, \beta, A, f]$ by $w_{ii} \sim IG\left[\overset{1\times1}{\left(\frac{v_{i0}+T}{2}\right)}, \overset{1\times1}{\left(\frac{\gamma_{i0}+e_i'e_i}{2}\right)}\right]$ (4 times)

e_i is T vector of residuals.

4) Sample $[\gamma | f, \sigma^2]$ by MH proposal $\gamma^+ \sim N(\hat{\gamma}, G)$

$$G = (\overset{1\times1}{G_0^{-1}} + \overset{1\times227}{f_{1:T-1}'}\ \overset{227\times1}{f_{1:T-1}} / \sigma^2)^{-1}$$

$$\hat{\gamma} = G(\underset{1\times1}{G_0^{-1}}\ \underset{1\times1}{\gamma_0} + \underset{1\times227}{f_{1:T-1}'}\ \underset{227\times1}{f_{2:T}} / \sigma^2)$$

γ^+ is accepted with probability

$$\alpha_{MH}(\gamma, \gamma^+) = \min\left\{1, \frac{f_N(f_1 | 0, \sigma^2/(1-\gamma^+)^2)}{f_N(f_1 | 0, \sigma^2/(1-\gamma)^2)}\right\}$$

5) Sample $[\sigma^2 | f, \gamma] \sim IG\left(\overset{1\times1}{\frac{s_0+T}{2}}, \overset{1\times1}{\frac{\delta_0 + \overbrace{(f^*-\hat{f})'}^{1\times228}\overbrace{(f^*-\hat{f})}^{228\times1}}{2}}\right)$

$$f^* = \underset{228\times1}{(f_1\sqrt{1-\gamma^2}, f_2 \cdots f_T)'}$$

$$\hat{f} = (0, \gamma f_2 \cdots \gamma f_{T-1})$$

Distribution $(a | y, \beta, A, \Omega, \gamma, \sigma^2)$

$$(a | \cdots) \propto \cancel{(2\pi)^{-1/2}}\ \cancel{|I_T \otimes \Omega|^{-1/2}} \exp\{(y - X\beta - (I_T \otimes A)f)'(I_T \otimes \Omega)^{-1}$$
$$(y - X\beta - (I_T \otimes A))\}$$
$$* \cancel{(2\pi)^{-1/2}}\ \cancel{|A_0|^{-1/2}} \exp\{(a-a_0)' A_0^{-1} (a-a_0)\}$$

Professor's Mail

full conditionals for f.

$$y - X\beta = (I_t \otimes A) f + \varepsilon$$

$$\text{posterior} = (2\pi)^{-n/2} |I_T \otimes \Omega|^{-1/2} \exp\{ (y - X\beta - (I_t \otimes A) f)' (I_t \otimes \Omega)^{-1} (y - X\beta - (I_t \otimes A) f) \}$$

$$(2\pi)^{-1/2} |\sigma^2 F_0^{-1}|^{-1/2} \exp\{ f' (\sigma^{-2} F_0) f \}$$

$$= \exp\{ (y - X\beta)' (I_t \otimes \Omega)^{-1} (y - X\beta) - (y - X\beta)' (I_t \otimes \Omega)^{-1} (I_t \otimes A) f - f' [I_t \otimes A]' (I_t \otimes \Omega)^{-1} (y - X\beta) + f' (I_t \otimes A)' (I_t \otimes \Omega)^{-1} (I_t \otimes A) f + f' (\sigma^{-2} F_0) f \}$$

$$f \sim N(\hat{f}, F)$$

$$F = [(I_t \otimes A)' (I_t \otimes \Omega)^{-1} (I_t \otimes A) + \sigma^{-2} F_0]^{-1}$$

$$\hat{f} = F ((I_t \otimes A)' (I_t \otimes \Omega)^{-1} (y - X\beta))$$

full conditionals for A

full conditional for A

$$y_t = X_t \beta + \underset{4\times1}{A}\, \underset{1\times1}{f_t} + \varepsilon_t$$

$$y_t = X_t \beta + \underset{4\times4}{[I_n} \times \underset{1\times1}{f_t]} \underset{4\times1}{A}$$

stack up t

$$y = X\beta + \underset{(218\times n)\times 4}{I_f} \underset{(4\times1)}{A}$$

$$y - X\beta = I_f A + \varepsilon$$

$$\text{postrior} = (2\pi)^{-n/2} |I_T \otimes \Omega|^{-1/2} \exp\{(y - X\beta - I_f A)'(I_t \otimes \Omega)^{-1} (y - X\beta - I_f A)\}$$
$$(2\pi)^{-1/2} |A_0|^{-1/2} \exp\{(A - a_0) A_0^{-1} (A - a_0)\}$$

$$\cdot \exp\{ \cancel{(y - X\beta)'(I_t \otimes \Omega)^{-1}} (y - X\beta) - (y - X\beta)'(I_t \otimes \Omega)^{-1} I_f . A$$
$$- A' I_f' (I_t \otimes \Omega)^{-1} (y - X\beta) + A' I_f' (I_t \otimes \Omega)^{-1} I_f A$$
$$+ A' A_0^{-1} A - A' A_0^{-1} a_0 - a_0' A_0^{-1} A + \cancel{a_0' A_0^{-1} a_0} \}$$

$$\exp\{-\tfrac{1}{2} [A' [A_0^{-1} + I_f'(I_t \otimes \Omega)^{-1} I_f] A$$
$$+ A' [A_0^{-1} a_0 + I_f'(I_t \otimes \Omega)^{-1} (y - X\beta)]$$

$$\hat{A} = [A_0^{-1} + I_f'(I_t \otimes \Omega)^{-1} I_f]^{-1}$$
$$\hat{a} = \hat{A} [A_0^{-1} a_0 + I_f'(I_t \otimes \Omega)^{-1} (y - X\beta)]$$