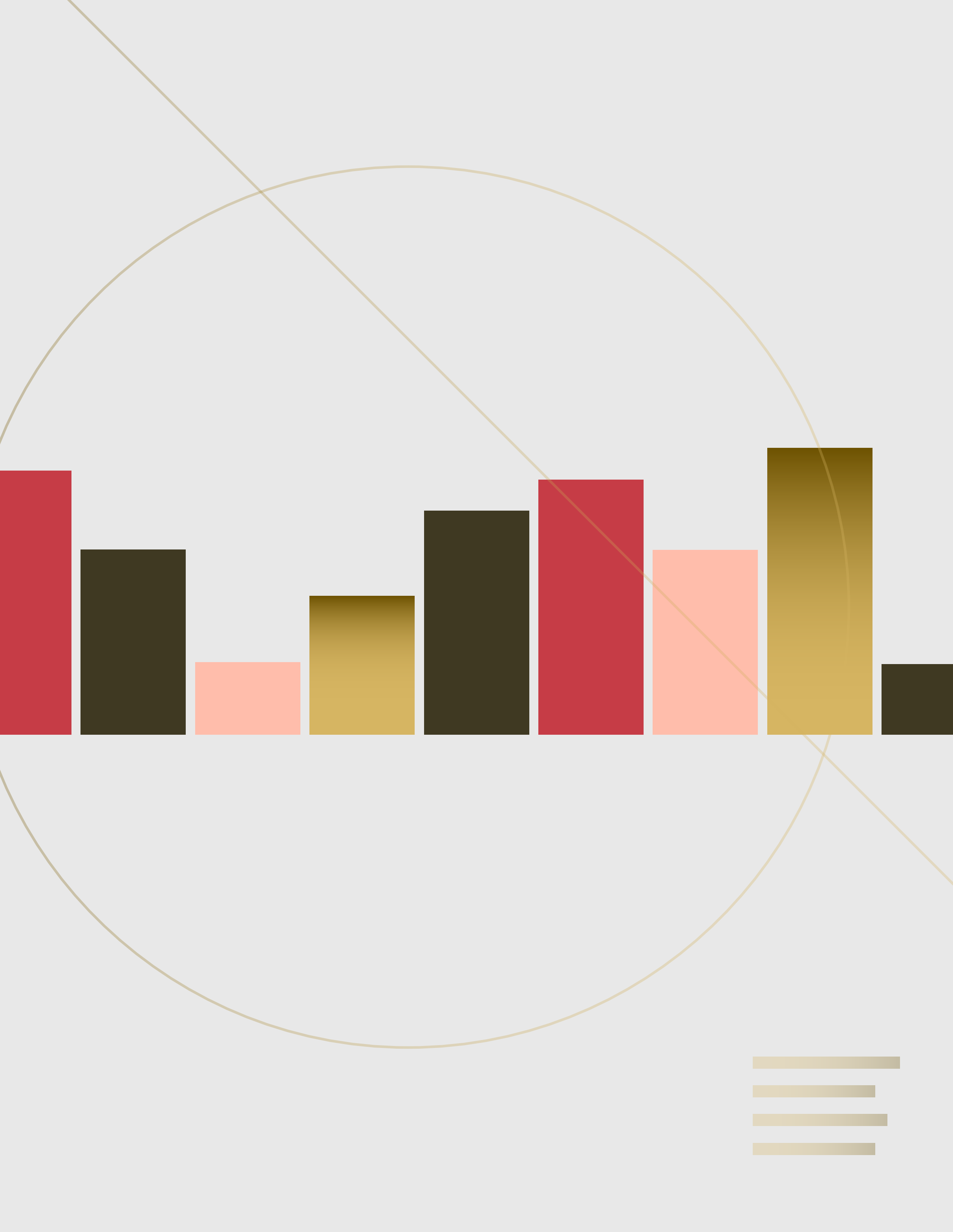

MCMC perspectives on simulated likelihood estimation
- Jeliazkov and Lee (2010)
Advances in econometrics

Basic Model

$$\underset{J\times 1}{z_i} = \underset{J\times k}{X_i} \underset{k\times 1}{\beta} + \underset{J\times 1}{\varepsilon_i} \qquad \varepsilon_i \sim N(0, \underset{J\times J}{\Omega})$$

$$y_{ij} = 1\{z_{ij} > 0\}$$

$$\Pr(y_i \mid \beta, \Omega) = \text{Prob}(z_i > 0 \mid \beta, \Omega)$$

$$= \int_{B_{iJ}}\int\cdots\int_{B_{i1}} f_N(z_i \mid X_i\beta, \Omega)\, dz_{i1}\, z_{i2} \cdots z_{iJ}$$

$$= \int 1\{z_i > 0\}\, f_N(z_i \mid X_i\beta, \Omega)\, dz_i$$

(It is not a proper truncated normal distribution as the normalizing constant is missing)

Can consider cdf of normal with non diagnal var-cov matrix

Research Problem

Estimate the probability of the observed response, given the model parameters.

Methods to calculate integration

1) **Accept - Reject Algorithm**

$$\Pr(y_i \mid \beta, \Omega) = G^{-1} \sum_{g=1}^{G} 1\{z_i^{(g)} \in B_i\}$$

2) Smooth Kernel Method

$$Pr^{\wedge}(y_i \mid \beta, \Omega) = \frac{1}{G} \sum_{i=1}^{G} K\left(\frac{z_i}{b}\right)$$

3) Stern Approach

Let $z_i = v_i + w_i$

$$v_i \sim N(X\beta, \Omega - \Lambda)$$
$$w_i \sim N(0, \Lambda) \quad , \Lambda = \text{diag}(\lambda)$$

$$Pr(y_{ij} = 1 \mid \beta, \Omega) = Pr(z_{ij} > 0)$$
$$= Pr(v_{ij} + w_{ij} > 0) \quad = \text{Joint distribution}$$
$$= \int Pr(v_{ij} + w_{ij} > 0 \mid v_{ij}) \, f(v_{ij}) \, dv_{ij}$$

$$= \int Pr_j(v_{ij} + w_{ij} > 0 \mid v_{ij}) \, f_N(v_{ij} \mid X\beta, \Omega - \Lambda) \, dv_{ij}$$
$$= \int Pr_j(w_{ij} > -v_{ij} \mid v_{ij}) \, f_N(v_{ij} \mid X\beta, \Omega - \Lambda) \, dv_{ij}$$
$$Pr(y_{ij} = 1 \mid \beta, \Omega) = \int Pr_j(\underbrace{w_{ij} < v_{ij}}_{\text{symmetric}} \mid v_{ij}) \, f_N(v_{ij} \mid X\beta, \Omega - \Lambda) \, dv_{ij}$$

$$Pr(y_{ij} = 0 \mid \beta, \Omega) = \int Pr_j(w_{ij} < -v_{ij} \mid v_{ij}) \, f_N(v_{ij} \mid X\beta, \Omega - \Lambda) \, dv_{ij}$$

$$Pr(y_i \mid \beta, \Omega) = \int \prod_{y_i=1} Pr_j(w_{ij} < v_{ij} \mid v_{ij}) \prod_{y_i=0} Pr_j(w_{ij} < -v_{ij} \mid v_{ij}) \, f_N(v_i \mid X\beta, \Omega - \Lambda) \, dv_i$$

$$Pr(y_i \mid \beta, \Omega) = \int \prod_{y_i=1} \Phi\left(\frac{v_{ij}}{\sqrt{\lambda}}\right) \prod_{y_i=0} \Phi\left(\frac{-v_{ij}}{\sqrt{\lambda}}\right) f_N(v_i \mid X\beta, \Omega - \Lambda) \, dv_i$$

The idea is to take care of the variance-covariance matrix. So divide the Normal into two parts, drawing from multinormal is not an issue but integrating is. So integrate the independent part (CDF)

$$\Pr(y_i \mid \beta, \Omega) = \int \prod_{j=1}^{J} \Phi\left(\frac{(-1)^{1-y_{ij}} v_{ij}}{\sqrt{\lambda}}\right) f_N(v_i \mid X\beta, \Omega - \Lambda)\, dv,$$

$$\hat{\Pr}(y_i \mid \beta, \Omega_i) = \frac{1}{G} \sum_{g=1}^{G} \left\{ \prod_{j=1}^{G} \Phi\left(\frac{(-1)^{1-y_{ij}} v_{ij}^{(g)}}{\sqrt{\lambda}}\right) \right\}$$

where $v_{ij}^{(g)} \sim N(v_i \mid X_i \beta, \Omega - \Lambda)$

4) GHK algorithm

$$\underset{J\times 1}{z_i} = \underset{J\times k}{X_i}\, \underset{k\times 1}{\beta} + \underset{J\times J}{\overset{\text{lower cholesky}}{L}}\, \underset{J\times 1}{\eta_i}, \quad \eta_i \sim N(0, I)$$

$$LL' = \Omega$$

$$h(z_i \mid y_i, \beta, \Omega) = h(z_{i1} \mid y_{i1}, \beta, \Omega)\, h(z_{i2} \mid z_{i1}, y_{i2}, \beta, \Omega) \cdots h(z_{iJ} \mid z_{i1}, z_{i2} \cdots z_{iJ-1}, y_{iJ}, \beta, \Omega)$$

$$= \prod_{j=1}^{J} h(z_{ij} \mid z_{il}\, [l < j], y_{ij}, \beta, \Omega)$$

why? Because of lower cholesky

$$\begin{bmatrix} z_{i1} \\ z_{i2} \\ \vdots \\ z_{iJ} \end{bmatrix} = \begin{bmatrix} X_{i11} & \cdots & X_{i1k} \\ \vdots & & \\ \vdots & & \\ X_{iJ1} & \cdots & X_{iJk} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \vdots \\ \beta_k \end{bmatrix} + \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{12} & l_{22} & 0 & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ l_{1J} & l_{2J} & & \cdots & l_{JJ} \end{bmatrix} \begin{bmatrix} \eta_{i1} \\ \eta_{i2} \\ \vdots \\ \eta_{iJ} \end{bmatrix} \quad -(1)$$

So z_{i1} is independent of z_{il} for $l > 1$

z_{i2} is dependent on only z_{i1}

z_{i3} is dependent on only z_{i1} and z_{i2} and so on.

Define the importance sampling density using ①

$$h(z_{i1} \mid y_{i1}, \beta, \Omega) = f_{TN_{B_{i1}}}(z_{i1} \mid x_{i1}'\beta, l_{11}^2)$$

$$= 1\{z_{i1} \in B_{i1}\} f_N(z_{i1} \mid x_{i1}'\beta, l_{11}^2) / c_{i1}$$

$$h(z_{i2} \mid z_{i1}, y_{i1}, \beta, \Omega) = f_{TN_{B_{i2}}}(z_{i2} \mid x_{i2}'\beta + l_{12}\eta_{i1}, l_{22}^2)$$

$$= 1\{z_{i2} \in B_{i2}\} f_N(z_{i2} \mid x_{i2}'\beta + l_{12}\eta_{i1}, l_{22}^2) / C_{i2}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$h(z_{ij} \mid \{z_{ik}\}_{k<j}, y_{ij}, \beta, \Omega) = f_{TN_{B_{ij}}}\left(z_{ij} \mid x_{ij}'\beta + \sum_{k=1}^{j-1} l_{jk} n_{ik}, l_{22}^2\right)$$

$$= 1\{z_{ik} \in B_{ij}\} f_N\left(z_{ij} \mid x_{ij}'\beta + \sum_{k=1}^{j-1} l_{jk} n_{ik}, l_{22}^2\right) / c_{ij}$$

normalizing constant $= c_{ij} = \Phi\left((-1)^{1-y_{ij}}\left(x_{ij}'\beta + \sum_{k=1}^{j-1} l_{jk}\eta_{ik}\right)/l_{jj}\right)$

of the truncated normal density

Deriving the estimator

$$h(z_i \mid y_i, \beta, \Omega) = \frac{\prod_{j=1}^{J} 1\{z_{ij} \in B_{ij}\} f_N\left(z_{ij} \mid x_{ij}'\beta + \sum_{k=1}^{j-1} l_{jk}\eta_{ik}, l_{jj}^2\right)}{\prod_{j=1}^{J} c_{ij}}$$

$$= \frac{1\{z_i \in B_i\} f_N(z_i \mid X_i\beta, \Omega)}{\prod_{j=1}^{J} c_{ij}}$$

$$\Pr(y_i \mid \beta, \Omega) = \int_{B_i} f_N(z_i \mid X_i\beta, \Omega)\, dz_i$$

multiplying dividing by $h(z_i \mid y_i, \beta, \Omega)$

$$= \int_{B_i} \frac{f_N(z_i \mid X_i\beta, \Omega)}{h(z_i \mid y_i, \beta, \Omega)} h(z_i \mid y_i, \beta, \Omega)\, dz_i$$

$$= \int_{B_i} \frac{f_N(z_i \mid X_i\beta, \Omega)}{f_N(z_i \mid X_i\beta, \Omega) / \prod_{j=1}^{J} c_{ij}} h(z_i \mid y_i, \beta, \Omega)\, dz_i$$

Can we draw from $h(z_i \mid y_i, \beta, \Omega)$ using MCMC?

just normalizing constant

$$= \int_{B_i} \prod_{j=1}^{J} c_{ij} \underbrace{h(z_i \mid y_i, \beta, \Omega)}_{\text{truncated normal distribution}}\, dz_i$$

(normalizing constant

$$\text{Estimator} = \widehat{\Pr}(y_i \mid \beta, \Omega) = \frac{1}{G} \sum_{g=1}^{G} \prod_{j=1}^{J} c_{ij}^{(g)}$$

$$z_{ij}^{(g)} \sim h(z_{ij} \mid z_{ik}\,(k<j), y_{ij}, \beta, \Omega)$$

CRB Method

$$\Pr(y_i \mid \beta, \Omega) = \int 1\{z_i \in B_i\} f_N(z_i \mid X_i \beta, \Omega)\, dz_i$$

According to Bayes formulae.

$$\text{Posterior} = \frac{\text{Likelihood} \times \text{Prior}}{\text{marginal}}$$

$$\underbrace{f_{TN}(z_i \mid X_i \beta, \Omega)}_{\text{posterior}} = \frac{\overbrace{1\{z_i \in B_i\}}^{\text{Likelihood}}\ \overbrace{f_N(z_i \mid X_i \beta, \Omega)}^{\text{Prior}}}{\underbrace{\int 1\{z_i \in B_i\} f_N(z_i \mid X_i \beta, \Omega)\, dz_i}_{\text{marginal}}}$$

$$1\{z_i \in B_i\} = \text{Prob}(y_i \mid z_i)$$

$$\text{marginal} = \text{Prob}(y_i)$$

$$\text{posterior} = \text{Prob}(z_i \mid y_i)$$

$$\text{Prob}(z_i \mid y_i) = \frac{\text{Prob}(y_i \mid z_i)\, \pi(z_i)}{\text{Prob}(y_i)}$$

$$\int 1\{z_i \in B_i\} f_N(z_i \mid X_i \beta, \Omega)\, dz_i = \frac{1\{z_i \in B_i\} f_N(z_i \mid X_i \beta, \Omega)}{f_{TN}(z_i \mid X_i \beta, \Omega)}$$

↳ normalizing constant of posterior

(valid for all z_i)

∴ The estimator is

$$\log \hat{\Pr}(y_i \mid \beta, \Omega) = \underbrace{\log(1 \mid z^* \in B_i)}_{0} + \log f_N(z_i^* \mid X_i \beta, \Omega) - \log \widehat{f_{TN}}(z_i^* \mid X_i \beta, \Omega)$$

where z_i^* is the mean from MCMC draws $z_i^{(g)} \sim TN_{B_i}(X_i\beta, \Omega)$. Thus $z_i^* \in B_i$ and $\log(1) = 0$.

given full conditionals $z_{ij} \sim f(z_{ij} \mid \{z_{ik}\}_{j \neq k}, y_{ij}, \beta, \Omega)$
$= TN_{B_{ij}}(\underbrace{\mu_{ij}, \sigma^2_{ij}})$ for $j = 1, \ldots J$

mean & variance of
z_{ij} conditional on z_{ik} $(k \neq j)$

$$= 1\{z_i \in B_i\} f_N(z_i \mid X_i\beta, \Omega)$$

$$= 1\{z_i \in B_i\} (2\pi)^{-1/2} |\Omega|^{-1/2} \exp\left\{-\frac{1}{2}(z_i - X_i\beta)\Omega^{-1}(z_i - X_i\beta)\right\}$$

$$= 1\left\{\begin{pmatrix} z_{i1} \\ z_{i2} \\ z_{i3} \end{pmatrix} \in \begin{pmatrix} (0,\infty) \\ (0,\infty) \\ (0,\infty) \end{pmatrix}\right\} (2\pi)^{-1/2} |\Omega)^{-1/2} \exp\left\{-\frac{1}{2}\left(\begin{bmatrix} z_{i1} \\ z_{i2} \\ z_{i3} \end{bmatrix} - \begin{bmatrix} 0 \\ 0.5 \\ 1 \end{bmatrix}\right)' \Omega^{-1} \ldots \right\}$$

$$\begin{bmatrix} z_{i1} & z_{i2} - 0.5 & z_{i3} - 1 \end{bmatrix} \frac{1}{p^2 - 1} \begin{bmatrix} -1 & p & 0 \\ p & -(p^2+1) & p \\ 0 & p & -1 \end{bmatrix} \underset{3 \times 1}{\begin{bmatrix} z_{i1} - 0 \\ z_{i2} - 0.5 \\ z_{i3} - 1 \end{bmatrix}}$$

$$\left[-z_{i1} + (z_{i2} - 0.5)p\right]\frac{1}{p^2-1}, \left[z_{i1}p - (z_{i2} - 0.5)(p^2+1) + (z_{i3} - 1)p\right]\frac{1}{p^2-1}, \left[p(z_{i2} - 0.5) - (z_{i3} - 1)\right]\frac{1}{p^2-1}$$

$$\begin{bmatrix} z_{i1} - 0 \\ z_{i2} - 0.5 \\ z_{i3} - 1 \end{bmatrix}$$

$$= \left[-z_{i1} + (z_{i2} - 0.5)p\right] z_{i1} / (p^2 - 1)$$
$$+ \left[z_{i1}p - (z_{i2} - 0.5)(p^2+1) + (z_{i3} - 1)p\right](z_{i2} - 0.5) / (p^2 - 1)$$
$$+ \left[p(z_{i2} - 0.5) - (z_{i3} - 1)\right](z_{i3} - 1) / (p^2 - 1)$$

$$= \left[- z_{i1} + (z_{i2} - 0.5)\rho \right] z_{i1} / (\rho^2 - 1)$$
$$+ \left[z_{i1}\rho - (z_{i2} - 0.5)(\rho^2+1) + (z_{i3}-1)\rho \right] (z_{i2} - 0.5) / (\rho^2 - 1)$$
$$+ \left[\rho(z_{i2} - 0.5) - (z_{i3} - 1) \right] (z_{i3} - 1) / (\rho^2 - 1)$$

$$z_{i1} \mid z_{i2}, z_{i3} \propto \left(\left[- z_{i1} + (z_{i2} - 0.5)\rho \right] z_{i1} + (z_{i2} - 0.5) z_{i1}\rho \right) \frac{1}{\rho^2 - 1}$$
$$\propto \left(\left[z_{i1} - (z_{i2} - 0.5)\rho \right] z_{i1} - (z_{i2} - 0.5) z_{i1}\rho \right) \frac{1}{1-\rho^2}$$
$$\propto = \frac{z_{i1}^2}{1-\rho^2} - z_{i1} \left[2(z_{i2} - 0.5) \frac{\rho}{1-\rho^2} \right.$$

$$\boxed{\begin{aligned} \sigma^2_{i1} &= 1-\rho^2 \\ \mu_{i1} &= (z_{i2} - 0.5)\rho \end{aligned}}$$

$$z_{i2} \mid z_{i1}, z_{i3} \propto \left[(z_{i2} - 0.5)\rho z_{i1} + \left(\left[z_{i1}\rho - (z_{i2} - 0.5)(\rho^2+1) + (z_{i3}-1)\rho \right] (z_{i2} - 0.5) \right) + \rho(z_{i2} - 0.5)(z_{i3} - 1) \right] \frac{1}{\rho^2 - 1}$$
$$\propto \left[z_{i2} \left[2\rho z_{i1} + 2(0.5(\rho^2+1)) + 2(z_{i3} - 1)\rho \right] \right] \frac{1}{\rho^2 - 1} - \frac{z_{i2}^2}{\rho^2 - 1}(\rho^2+1)$$
$$\propto \left[- z_{i2} \left[2(z_{i1}\rho + 0.5(\rho^2+1) + (z_{i3} - 1)\rho \right] \frac{1}{1-\rho^2} + \frac{z_{i2}^2}{1-\rho^2}(\rho^2+1) \right.$$

$$\boxed{\begin{aligned} \sigma^2_{i2} &= \frac{1-\rho^2}{1+\rho^2} \\ \mu_{i2} &= \left[z_{i1}\rho + 0.5(\rho^2+1) + (z_{i3} - 1)\rho \right] \frac{1}{1+\rho^2} \end{aligned}}$$

$$z_{i3} \propto (z_{i3}-1)p\,](z_{i2}-0.5)/(p^2-1)$$
$$+[\,p(z_{i2}-0.5)-(z_{i3}-1)](z_{i3}-1)/(p^2-1)$$

$$\propto z_{i3}[2(z_{i2}-0.5)p+2(1)]\frac{1}{p^2-1}-\frac{z_{i3}^2}{p^2-1}$$

$$\propto -z_{i3}\left[2[(z_{i2}-0.5)p+1]\frac{1}{1-p^2}\right]+\frac{z_{i3}^2}{1-p^2}$$

$$\boxed{\begin{aligned}\sigma^2_{i3} &= 1-p^2\\ \mu_{i3} &= (z_{i2}-0.5)p+1\end{aligned}}$$

General

$$= [-(z_{i1}-a_1)+(z_{i2}-a_2)p](z_{i1}-a_1)/(p^2-1)$$
$$+[(z_{i1}-a_1)p-(z_{i2}-a_2)(p^2+1)+(z_{i3}-a_3)p](z_{i2}-a_2)/(p^2-1)$$
$$+[\,p(z_{i2}-a_2)-(z_{i3}-a_3)](z_{i3}-a_3)/(p^2-1)$$

$$z_{i1} \propto \frac{1}{p^2-1}[-(z_{i1}-a_1)+(z_{i2}-a_2)p](z_{i1}-a_1)+(z_{i1}-a_1)p(z_{i2}-a_2)$$
$$\propto \frac{1}{p^2-1}[-z_{i1}^2+2z_{i1}a_1+z_{i2}pz_{i1}-a_2pz_{i1}+z_{i1}pz_{i2}-z_{i1}pa_2]$$
$$\propto \frac{1}{p^2-1}[-z_{i1}]^2+\frac{1}{p^2-1}z_{i1}[2a_1+2z_{i2}p-2a_2p]$$

$$\boxed{\begin{aligned}\mu_{i1} &= a_1+z_{i2}p-a_2p\\ \sigma^2_{i1} &= 1-p^2\end{aligned}}$$

$$\boxed{\begin{aligned} \mu_{i2} &= \left[(z_{i1} - a_1)\rho + a_2(\rho^2+1) + (z_{i3} - a_3)\rho\right]\frac{1}{1+\rho^2} \\ \sigma^2_{i2} &= \frac{1-\rho^2}{1+\rho^2} \end{aligned}}$$

$$\boxed{\begin{aligned} \mu_{i3} &= a_3 + z_{i2}\rho - a_2\rho \\ \sigma^2_{i3} &= 1-\rho^2 \end{aligned}}$$

There is no problem in sampling from truncated normal distribution. There is a problem of evaluating the integration of truncated distribution.

Now we have to estimate $f_{TN}(z_i^* \mid X_i\beta, \Omega)$

$$f_{TN}(z_i^* \mid X_i\beta, \Omega) = f_{TN}(z_{i1}^*, z_{i2}^* \cdots z_{iJ}^* \mid X_i\beta, \Omega)$$

$$= f_{TN}(z_{i1}^* \mid X_i\beta, \Omega)\, f_{TN}(z_{i2}^* \mid z_{i1}^*, X_i\beta, \Omega) \cdots \cdot$$
$$f_{TN}(z_{iJ}^* \mid z_{i1}^*, z_{i2}^* \cdots z_{iJ}^*, X_i\beta, \Omega)$$

$$= \prod_{j=1}^{J} f(z_{ij}^* \mid y_i, \{z_{ik}^*\}_{k<j}, \beta, \Omega)$$

Using Rao Blackwellization

$$\text{Prob}(x \mid y) = \int \text{Prob}(x \mid y, z)\, \text{Prob}(y \mid z)\, dy$$

Similarly,

$$f(z_{ij}^* \mid y_i, \{z_{ik}^*\}_{k<j}, \beta, \Omega) = \int \overbrace{f(z_{ij}^* \mid y_i, \{z_{ik}^*\}_{k<j}, \{z_{ik}\}_{k>j}, \beta, \Omega)}^{\text{full conditional}}$$
$$\times f(\{z_{ik}\}_{k>j} \mid y, \{z_{ik}^*\}_{k<j}, \beta, \Omega)\, d\{z_{ik}\}_{k>j}$$

$$\hat{f}(z_{ik}^* \mid y_i, \{z_{ik}^*\}_{k<j}, \beta, \Omega) = G^{-1} \sum_{g=1}^{G} \underbrace{f(z_{ij}^* \mid y_i, \{z_{ik}^*\}_{k<j}, \{z_{ik}^{(g)}\}_{k>j}, \beta, \Omega)}_{\text{we need to estimate } J \text{ ordinates}}$$

$$f(z_{i1}^* \mid y_{i1}, \beta, \Omega) = G^{-1} \sum_{g=1}^{G} f(z_{i1}^* \mid y_i, z_{i2}^{(g)} \cdots z_{iJ}^{(g)}, \beta, \Omega)$$

which will come from normal sampling

$$f(z_{i2}^* \mid z_{i1}^*, y_i, \beta, \Omega) = G^{-1} \sum_{g=1}^{G} f(z_{i2}^* \mid y_i, z_{i1}^*, z_{i3}^{(g)} \cdots z_{iJ}^{(g)}, \beta, \Omega)$$

new sampling required

⋮

$$f(z_{iJ-1}^* \mid z_{i1}^* \cdots z_{iJ-2}^*, z_{iJ}^{(g)}, \beta, \Omega) = G^{-1} \sum_{g=1}^{G} f(z_{iJ-1}^* \mid y_i, z_{i1}^* \cdots z_{iJ}^{(g)}, \beta, \Omega)$$

new sampling required

$f(z_{i,J}^* \mid z_{i1}^*, z_{i,2}^* \cdots z_{iJ-1}^*, \beta, \Omega)$ - already have full conditional so will calculate directly

$J-2$ extra reduced runs required.

CRT Method

Will use the same estimator of $P(y \mid \beta, \Omega)$ as in CRB

$$\log \hat{Pr}(y_i \mid \beta, \Omega) = \log f_N(z_i^* \mid X_i\beta, \Omega) - \log \widehat{f_{TN}}(z_i^* \mid X_i\beta, \Omega)$$

A better way to calculate $\widehat{f_{TN}}(z_i^* | X_i \beta, \Omega)$

$$f(z_{ij}^* | y_i, \{z_{ik}^*\}_{k<j}, \beta, \Omega) = \int f(z_{ij}^* | y_i, \{z_{ik}^*\}_{k<j}, \{z_{ik}\}_{k>j}, \beta, \Omega) \times f(\{z_{ik}\}_{k>j} | y, \{z_{ik}^*\}_{k<j}, \beta, \Omega) \, d\{z_{ik}\}_{k>j}$$

$$f(z_i^* | y_i \{z_{ik}^*\}_{k<j}, \beta, \Omega) = \int \prod_{j=1}^{J} f(z_{ij}^* | y_i, \{z_{ik}^*\}_{k<j}, \{z_{ik}\}_{k>j}, \beta, \Omega) \times f(\underbrace{z_{i1}, z_{i2} \ldots z_{iJ} | y_i, \beta,}_{\text{Truncated normal}} \Omega) \, dz_i$$

This was used in CRB. But the full conditi- -onals are actually Gibbs sampler

$$\underbrace{K(z_i, z_i^* | y_i, \beta, \Omega)}_{\text{gibbs sampler}} = \prod_{j=1}^{J} f(z_{ij}^* | y_i, \{z_{ik}^*\}_{k<j}, \{z_{ik}\}_{k>j}, \beta, \Omega)$$

Thus,

$$f_{TN}(z_i^* | X_i \beta, \Omega) = \int K(z_i, z_i^* | y, \beta, \Omega) f_{TN_{B_i}}(z_i | X_i \beta, \Omega) \, dz_i$$

$$\therefore \widehat{f_{TN}}(z_i^* | X_i \beta, \Omega) = G^{-1} \sum_{g=1}^{G} K(z_i^{(g)}, z_i^* | y_i, \beta, \Omega)$$

$z_i^{(g)}$ comes from main MCMC which we need to store.

ARK method

Will use the same estimator of $P(y | \beta, \Omega)$ as in CRB

$$\log \widehat{Pr}(y_i | \beta, \Omega) = \log f_N(z_i^* | X_i \beta, \Omega) - \log \widehat{f_{TN}}(z_i^* | X_i \beta, \Omega)$$

Another way to calculate $\widehat{f_{TN}}(z_i^* \mid X_i\beta, \Omega)$

$$f_{TN}(z_i^* \mid X_i\beta, \Omega) = \int K(z_i, z_i^* \mid y, \beta, \Omega)\, f_{TN_{B_i}}(z_i \mid X_i\beta, \Omega)\, dz_i$$

$$= \int K(z_i, z_i^* \mid y, \beta, \Omega)\, 1\{z_i \in B_i\}\, f_N(z_i \mid X_i\beta, \Omega)\, dz_i$$

Draw from $N(X_i\beta, \Omega)$.
Accept if $z_i \in B_i$, Reject otherwise.

$$\therefore \widehat{f_{TN}}(z_i^* \mid X_i\beta, \Omega) = G^{-1} \sum_{g=1}^{G} K(z_i^{(g)}, z_i^* \mid y_i, \beta, \Omega)$$

ASK method

Will use the same estimator of $P(y \mid \beta, \Omega)$ as in CRB

$$\log \widehat{Pr}(y_i \mid \beta, \Omega) = \log f_N(z_i^* \mid X_i\beta, \Omega) - \log \widehat{f_{TN}}(z_i^* \mid X_i\beta, \Omega)$$

Another way to calculate $\widehat{f_{TN}}(z_i^* \mid X_i\beta, \Omega)$

$$z_i = X_i\beta + L\eta_i \qquad , \eta_i \sim N(0, I)$$

$$\eta_i = L^{-1}(z_i - X_i\beta) \qquad [\text{ mahalanobis transformation }]$$

Gibbs Kernel :

$$K(\eta_i, \eta_i^* \mid y_i, \beta, \Omega) = \prod_{j=1}^{J} f(\eta_{ij}^* \mid y_i, \{\eta_{ik}^*\}_{k<j}, \{\eta_{ik}^{(g)}\}_{k>j}, \beta, \Omega)$$

So MCMC simulation $\eta_i \sim TN_{\varepsilon_i}(0, I)$ corresponds to $z_i \sim TN_{B_i}(x_i\beta, \Omega)$

Algorithm

1) Initalize $p_z \in (0,1)$ and $p_n = 1 - p_z$

2) Given z_i and corresponding η_i, Sample z_i with prob p_z or η_i with prob p_{n_i} and calculate the corresponding η_i or z_i value respectively.

3) Keep track of draws obtain from $K_n(\cdot)$ & $K_z(\cdot)$

4) Update p_z periodically.

$$p_n = \begin{cases} 1 & \text{if } r_z >> r_n \\ 0 & \text{if } r_n >> r_z \\ \dfrac{w' r_z}{w' r_z + w' r_n} & \text{otherwise} \end{cases}$$

$$\rho_{jl} = \text{correlation}\left(z_{ij}^{(g)}, z_{ij}^{(g-l)}\right) \quad - l^{th} \text{ autocorrelation.}$$

$$\text{Inefficiency factors} = 1 + \sum_{l=1}^{\infty} \rho_{jl}$$

$$r_z(j) = (1 - \rho_{j1})^{-1}$$

$$r_n(j) = (1 - \rho_{j1})^{-1}$$

$\therefore \hat{f}_{TN}(z_i^* \mid X_i\beta, \Omega) = G^{-1} \sum_{g=1}^{G} K(z_i^{(g)}, z_i^* \mid y_i, \beta, \Omega)$

where $z_i^{(g)}$ are drawn using algorithm.

Calculating full conditionals for η_i

$$z_i = X_i\beta + L\eta_i$$

$$\eta_i = L^{-1}(z_i - X_i\beta),$$

$$\begin{bmatrix} \eta_{i1} \\ \eta_{i2} \\ \eta_{i3} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{12} & l_{22} & 0 \\ l_{13} & l_{23} & l_{33} \end{bmatrix}^{-1} \begin{bmatrix} z_{i1} - 0 \\ z_{i2} - 0.5 \\ z_{i3} - 1 \end{bmatrix}$$

$$\eta_{i1} = z_{i1}/l_{11}$$

$$\eta_{i2} = (z_{i2} - 0.5)/l_{22} - (l_{12} z_{i1})/l_{11} \times l_{22}$$

$$\eta_{i3} = (z_{i3} - 1)/l_{33} - \left[l_{23}(z_{2i} - 0.5)/l_{22} l_{33} + z_{i1}(l_{12} l_{23} - l_{13} l_{22}) \right] / l_{11} l_{22} l_{33}$$

$$z_{i1} = l_{11}\eta_{i1}$$

$$z_{i2} = 0.5 + l_{12}\eta_{i1} + l_{22}\eta_{i2}$$

$$z_{i3} = 1 + l_{13}\eta_{i1} + l_{23}\eta_{i2} + l_{33}\eta_{i3}$$

$$z_{i1} = l_{11}\eta_{i1} > 0$$
$$\eta_{i1} > 0$$

$$z_{i2} = 0.5 + l_{12}\eta_{i1} + l_{22}\eta_{i2} > 0$$
$$l_{12}\eta_{i1} + l_{22}\eta_{i2} > -0.5$$
$$\eta_{i2} > (-0.5 - l_{12}\eta_{i1})/l_{22}$$

$$z_{i3} = 1 + l_{13}\eta_{i1} + l_{23}\eta_{i2} + l_{33}\eta_{13} > 0$$

$$\eta_{i3} > (-1 - l_{13}\eta_{i1} - l_{23}\eta_{i2})/l_{33}$$

General

$$z_{i1} = a_1 + l_{11}\eta_{i1} > 0$$
$$\eta_{i1} > -a_1/l_{11}$$

$$z_{i2} = a_2 + l_{12}\eta_{i1} + l_{22}\eta_{i2} > 0$$
$$l_{12}\eta_{i1} + l_{22}\eta_{i2} > -a_2$$
$$\eta_{i2} > (-a_2 - l_{12}\eta_{i1})/l_{22}$$

$$z_{i3} = a_3 + l_{13}\eta_{i1} + l_{23}\eta_{i2} + l_{33}\eta_{13} > 0$$

$$\eta_{i3} > (-a_3 - l_{13}\eta_{i1} - l_{23}\eta_{i2})/l_{33}$$

full conditional distribution of η_i

$$\begin{bmatrix} z_{i1} \\ z_{i2} \\ z_{i3} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \underbrace{\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}}_{L} \begin{bmatrix} \eta_{i1} \\ \eta_{i2} \\ \eta_{i3} \end{bmatrix}$$

Calculating region ε_i using B_i

$$\begin{cases} z_{i1} = a_1 + l_{11}\eta_{i1} > 0 \\ z_{i2} = a_2 + l_{22}\eta_{i2} + l_{21}\eta_{i1} > 0 \\ z_{i3} = a_3 + l_{31}\eta_{i1} + l_{32}\eta_{i2} + l_{33}\eta_{i3} > 0 \end{cases}$$

$$\eta_{i1} > -a_1/l_{11}$$

$$\eta_{i2} > (-a_2 - l_{21}\eta_{i1})/l_{22}$$

$$\eta_{i3} > (-a_3 - l_{31}\eta_{i1} - l_{32}\eta_{i2})/l_{33}$$

$$\eta_{i1} \mid y, \eta_{i2}, \eta_{i3} \sim 1\{z_{i1} > -a_1/l_{11}\}\, N(0, 1)$$

$$\eta_{i2} \mid y, \eta_{i1}, \eta_{i3} \sim 1\{z_{i2} > (-a_2 - l_{21}\eta_{i1})/l_{22}\}\, N(0, 1)$$

$$\eta_{i3} \mid y, \eta_{i1}, \eta_{i2} \sim 1\{z_{i3} > (-a_3 - l_{31}\eta_{i1} - l_{32}\eta_{i2})\}\, N(0, 1)$$

GHK $(J = 6)$

$$z_{i1} \propto \frac{1}{\rho^2-1}\left[2a_1 z_{i1} - z_{i1}^2 - 2a_2\rho z_{i1} + 2\rho z_{i1} z_{i2}\right]$$

$$\propto \frac{1}{1-\rho^2}\left[z_{i1}^2 - 2a_1 z_{i1} + 2a_2\rho z_{i1} - 2\rho z_{i1} z_{i2}\right]$$

$$\boxed{\begin{aligned} \mu_{i1} &= a_1 - a_2\rho + z_{i2}\rho \\ \sigma^2_{i1} &= 1-\rho^2 \end{aligned}}$$

$$z_{i2} \propto \frac{1}{\rho^2-1}\Big[-z_{i2}^2 - \rho^2 z_{i2}^2 + 2a_2 z_{i2} - 2a_1\rho z_{i2} - 2a_3\rho z_{i2}$$
$$+ 2\rho z_{i1} z_{i2} + 2\rho z_{i2} z_{i3} + 2a_2\rho^2 z_{i2}\Big]$$

$$\propto \frac{1}{1-\rho^2}\Big[(1+\rho^2) z_{i2}^2 - 2 z_{i2}\big[+a_2 - a_1\rho - a_3\rho + \rho z_{i1}$$
$$+ \rho z_{i3} + a_2\rho^2\big]\Big)$$

$$\boxed{\begin{aligned} \mu_{i2} &= \left[-(a_1+a_3)\rho + (z_{i1}+z_{i3})\rho + a_2(1+\rho^2)\right]/_{1+\rho^2} \\ \sigma^2_{i2} &= \frac{1-\rho^2}{1+\rho^2} \end{aligned}}$$

$$z_{i3} \propto \frac{1}{\rho^2-1}\Big[-z_{i3}^2 - \rho^2 z_{i3}^2 + 2a_3 z_{i3} - 2a_2\rho z_{i3} - 2a_4\rho z_{i3}$$
$$+ 2\rho z_{i2} z_{i3} + 2\rho z_{i3} z_{i4} + 2a_3\rho^2 z_{i3}\Big]$$

$$\propto \frac{1}{1-\rho^2}\Big[z_{i3}^2\left[1+\rho^2\right] - 2a_3 z_{i3} + 2a_2\rho z_{i3} + 2a_4\rho z_{i3}$$
$$- 2\rho z_{i2} z_{i3} - 2\rho z_{i3} z_{i4} - 2a_3\rho^2 z_{i3}\Big]$$

$$\propto \frac{1}{1-\rho^2}\Big[z_{i3}^2(1+\rho^2) - 2z_{i3}\big[a_3 - a_2\rho - a_4\rho + \rho z_{i2}$$
$$+ \rho z_{i4} + a_3\rho^2\big]\Big)$$

$$\propto \frac{1}{1-\rho^2}\left[z_{i2}^2(1+\rho^2) - 2 z_{i3}\left[a_3 - a_2\rho - a_4\rho + \rho z_{i2} + \rho z_{i4} + a_3\rho^2\right]\right)$$

$$\boxed{\begin{aligned} \mu_{i3} &= \left[-(a_2+a_4)\rho + (z_{i2}+z_{i4})\rho + a_3(1+\rho^2)\right] / 1+\rho^2 \\ \sigma^2_{i3} &= \frac{1-\rho^2}{1+\rho^2} \end{aligned}}$$

$$\boxed{\begin{aligned} \mu_{i4} &= \left[-(a_3+a_5)\rho + (z_{i3}+z_{i5})\rho + a_4(1+\rho^2)\right] / 1+\rho^2 \\ \sigma^2_{i4} &= \frac{1-\rho^2}{1+\rho^2} \end{aligned}}$$

$$\boxed{\begin{aligned} \mu_{i5} &= \left[-(a_4+a_6)\rho + (z_{i4}+z_{i6})\rho + a_5(1+\rho^2)\right] / 1+\rho^2 \\ \sigma^2_{i5} &= \frac{1-\rho^2}{1+\rho^2} \end{aligned}}$$

$$z_{i6} \propto \frac{1}{\rho^2-1}\left[-z_{i6}^2 + 2a_6 z_{i6} - 2a_5\rho z_{i6} + 2\rho z_{i5} z_{i6}\right]$$

$$\propto \frac{1}{1-\rho^2}\left[z_{i6}^2 - 2a_6 z_{i6} + 2a_5\rho z_{i6} - 2\rho z_{i5} z_{i6}\right]$$

$$\propto \frac{1}{1-\rho^2}\left[z_{i6}^2 - 2z_{i6}\left[a_6 - a_5\rho + \rho z_{i5}\right]\right)$$

$$\boxed{\begin{aligned} \mu_{i6} &= a_6 + z_{i5}\rho - a_5\rho \\ \sigma^2_{i6} &= 1-\rho^2 \end{aligned}}$$

$$z_i = X\beta + \varepsilon_i \qquad , \varepsilon_i \sim N(0, \Omega)$$

$$I(z_i \sim (0, \infty))\, N(X\beta, \Omega)$$

Or

$$z_i = X\beta + L\eta_i \qquad \eta_i \sim N(0, I)$$

$$\eta_i = L^{-1}(z_i - X\beta)$$

$$\exp\{(z_i - a)' L^{-1'} L^{-1} (z_i - a)\}$$

$$z_{i1} = a_1 + l_{11} n_{i1} > 0$$
$$z_{i2} = a_2 + l_{21} n_{i1} + l_{22} n_{i2} > 0$$
$$z_{i3} = a_3 + l_{31} \eta_{i1} + l_{32} \eta_{i2} + l_{33} \eta_{i3} > 0$$

1. $n_{i1} > -a_1 / l_{11}$
2. $n_{i1} > (-a_2 - l_{22} n_{i2}) / l_{21}$
3. $n_{i1} > (-a_3 - l_{32} n_{i2} - l_{33} n_{i3}) / l_{31}$
2. $n_{i2} > (-a_2 - l_{21} n_{i1}) / l_{22}$
3. $n_{i2} > (-a_3 - l_{31} n_{i1} - l_{33} n_{i3}) / l_{32}$
3. $n_{i3} > (-a_3 - l_{31} n_{i1} - l_{32} n_{i2}) / l_3$

$$n_{i1} > (-a_3 - l_{32}[-a_2 - l_{21}(\tfrac{-a_1}{l_{11}})]/l_{22} - l_{33}\eta_{i3})/l_{31}$$

$$n_{i2} > (-a_3 - l_{31}\eta_{i1} - l_{33}[-a_3 - l_{31}\eta_{i1} - l_{32}(-a_2 - l_{21}\eta_{i1})/l_{22})/l_{33})/l_{32}$$

$$n_{i3} > (-a_3 - l_{31} n_{i1} - l_{32} n_{i2})/l_{23}$$

$$z_i = X_i \beta + L \eta_i > 0$$

$$\eta_i > L^{-1}(-X\beta)$$

$$\begin{array}{l} \eta_{i1} = \\ \eta_{i2} = \\ \eta_{i3} = \end{array} \begin{bmatrix} 1/l_{11} & 0 & 0 \\ -l_{21}/l_{11} l_{22} & 1/l_{22} & 0 \\ \frac{(l_{21} l_{32} - l_{22} l_{31})}{l_{11} l_{22} l_{33}} & \frac{-l_{32}}{l_{22} l_{33}} & \frac{1}{l_{33}} \end{bmatrix} \begin{bmatrix} z_{i1} - u_{a1} \\ z_{i2} - u_{a2} \\ z_{i3} - u_{a3} \end{bmatrix}$$

2 $\quad \eta_{i2} > (-a_2 - l_{21} \eta_{i1}) / l_{22}$

3 $\quad \eta_{i2} > (-a_3 - l_{31} \eta_{i1} - l_{33} \eta_{i3}) / l_{32}$

$$y_i = 1\{z_{i1} > 0\}$$

$$= 1\left\{ \begin{bmatrix} u_{a1} + l_{11} n_1 \\ u_{a2} + l_{21} n_1 + l_{22} n_2 \\ u_{a3} + l_{31} n_1 + l_{32} n_2 + l_{33} n_3 \end{bmatrix} > \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Application : Women labor force participation

Data : 7 years (1975-1985) , 1545 married women (age - 17-66)

Model 1

$$y_{it} = 1\{\bar{x}_{it}'\delta + w_{it}'\beta_i + g(s_{it}) + \phi_1 y_{i,t-1} + \phi_2 y_{1,t-2} + \varepsilon_{it} > 0\}$$

$$\varepsilon_{it} \sim N(0,1)$$

$$z_{it} = \bar{x}_{it}'\delta + w_{it}'\underbrace{\beta_i}_{\text{varying}} + g(s_{it}) + \phi_1 y_{i,t-1} + \phi_2 y_{1,t-2} + \varepsilon_{it}$$

$$y_{it} = \overbrace{\text{Work}}^{\text{Dummy}}{}_{it}$$

$$x_{it}' = (\overbrace{\text{Race}}^{\text{Dummy}}{}_i, \overbrace{\text{Edu}}^{\text{ordinal}}{}_{it}, \log(\text{INC}_{it}))$$

$$s_{it} = \text{AGE}_{it} \in [17, 66] \in \mathbb{N}$$

$$w_{it}' = (1, \underbrace{\text{CH2}}_{\text{ordinal}}{}_{it}, \underbrace{\text{CH5}}_{\text{ordinal}}{}_{it})$$

$$\text{Work}_{it} = 1\Big\{ [\text{Race}_i, \text{Edu}_{it}, \log(\text{INC}_{it})] \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix}$$

$$+ [1, \text{CH2}_{it}, \text{CH5}_{it}] \begin{bmatrix} \beta_{i1} \\ \beta_{i2} \\ \beta_{i3} \end{bmatrix} + g(\text{AGE}_{it}) + \phi_1 \text{Work}_{i,t-1}$$

$$+ \phi_2 \text{Work}_{i,t-2} + \varepsilon_{it} > 0 \Big\}$$

Define $\beta_i = A_i \gamma + b_i \quad , b_i \sim N_3(0, D)$

$$\underset{3\times 1}{\begin{bmatrix} \beta_{i1} \\ \beta_{i2} \\ \beta_{i3} \end{bmatrix}} = \underset{3\times 7}{\begin{bmatrix} \bar{y}_{i0} & & & & & & \\ & 1 & \bar{y}_{i0} & \overline{\ln(INC_i)} & & & \\ & & & & 1 & \bar{y}_{i0} & \overline{\ln(INC_i)} \end{bmatrix}} \underset{7\times 1}{\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \vdots \\ \gamma_7 \end{bmatrix}} + \underset{3\times 1}{\begin{bmatrix} b_{i1} \\ b_{i2} \\ b_{i3} \end{bmatrix}}$$

Substituting β_i in $w_{oi}k_{it}$

$$y_{it} = 1\{ \bar{x}_{it}'\delta + w_{it}'(A_i\gamma + b_i) + g(s_{it}) + \phi_1 y_{i,t-1} + \phi_2 y_{i,t-2} + \varepsilon_{it} > 0\}$$

$$y_{it} = 1\{ \bar{x}_{it}'\delta + w_{it}'A_i\gamma + w_{it}'b_i + g(s_{it}) + \phi_1 y_{i,t-1} + \phi_2 y_{i,t-2} + \varepsilon_{it} > 0\}$$

$$y_{it} = 1\{ [x_{it}' \quad w_{it}'A_i \quad y_{i,t-1} \quad y_{i,t-2}] \begin{bmatrix} \delta \\ \gamma \\ \phi \\ \phi \end{bmatrix} + g_i + \varepsilon_i$$

$$\varepsilon_i \sim N(0, \Omega + W_i D W_i')$$

Stacking over t

$$\underset{5\times 1}{z_i} = \underset{5\times 3}{\bar{x}_i'} \underset{3\times 1}{\delta} + \underset{5\times 3}{w_i'} \underset{3\times 7}{A_i} \underset{7\times 1}{\gamma} + g(\underset{5\times 1}{s_i}) + \phi_1 \underset{5\times 1}{y_i} + \phi_2 \underset{6\times 1}{y_i} + \underset{5\times 1}{w_i' b_i} + \underset{5\times 1}{\varepsilon_i}$$

$$\underset{5\times 1}{z_i} = [\bar{x}_i' \quad w_i'A_i \quad y_i' \quad y_i''] \begin{bmatrix} \delta \\ \gamma \\ \phi_1 \\ \phi_2 \end{bmatrix} + \underset{5\times 1}{w_i'b_i} + \varepsilon_i$$

$$
z_{it} = \overbrace{\left[\underset{1\times 3}{[R \;\; Edu \;\; ln(I)]} : \underbrace{\underset{1\times 3}{[1 \;\; CH_1 \;\; CH_2]} \overbrace{\underset{3\times 3}{\begin{bmatrix} \bar{y} & & \\ & \cdots & \\ & & \cdots \end{bmatrix}}}^{A}}_{1\times 3} : \underset{1\times 2}{[y_i' \;\; y_i'']} \right]_{1\times 8}}^{X} \underset{8\times 1}{\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \phi_1 \\ \phi_2 \end{bmatrix}}
$$

Stack up over t

$$
\underset{5\times 1}{z_i} = \overbrace{\underset{5\times 8}{\begin{bmatrix} x_{i3}' & w_{i3}' A_i & y_{i2} & y_{i1} \\ x_{i4}' & w_{i4}' A_i & y_{i3} & y_{i2} \\ \vdots & \vdots & \vdots & \vdots \\ x_{i7}' & w_{i7}' A_i & y_{i6} & y_{i5} \end{bmatrix}}}^{F_i} \underset{8\times 1}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}} + \underset{5\times 1}{g_i} + \varepsilon_i
$$

Stack up over i

$$
\underset{(5\times 1545)\times 1}{Z} = \underset{(5\times 1545)\times 8}{\begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_{1545} \end{bmatrix}} \underset{8\times 1}{\beta} + \underset{(5\times 1545)\times 1}{\begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_{1545} \end{bmatrix}} + \underset{(5\times 1545)\times 1}{\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_{1545} \end{bmatrix}}
$$

$$
z = \underbrace{X\beta + g}_{a_1} + \varepsilon \qquad \varepsilon \sim N(0, \underbrace{\Omega + W'DW'}_{\text{Omega}})
$$

$$= 1\{z_i \in B_i\}\,(2\pi)^{-1/2}\,|\Omega|^{-1/2}\,\exp\left\{-\frac{1}{2}(z_i - x_i\beta)\Omega^{-1}(z_i - x_i\beta)\right\}$$

$$= 1\left\{\begin{pmatrix} z_{i1} \\ z_{i2} \\ \vdots \\ z_{in} \end{pmatrix} \in \begin{pmatrix} (0,\infty) \\ (0,\infty) \\ (0,\infty) \\ \vdots \end{pmatrix}\right\}(2\pi)^{-1/2}\,|\Omega)^{-1/2} \exp\left\{-\frac{1}{2}\left(\begin{bmatrix} z_{i1} \\ z_{i2} \\ z_{13} \\ \vdots \end{bmatrix} - \begin{bmatrix} 0 \\ 0.5 \\ 1 \\ \vdots \end{bmatrix}\right)' \Omega^{-1} \cdots \right\}$$

$$-\frac{1}{2}\left(\begin{bmatrix} z_{i1} \\ z_{i2} \\ z_{i3} \\ z_{i4} \\ z_{i5} \end{bmatrix} - \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}\right)' \Omega^{-1} \left(\begin{bmatrix} z_{i1} \\ z_{i2} \\ z_{i3} \\ z_{i4} \\ z_{i5} \end{bmatrix} - \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}\right)$$

$$-\frac{1}{2}\underbrace{\left(\begin{bmatrix} z_{i1} \\ z_{i2} \\ z_{i3} \\ z_{i4} \\ z_{i5} \end{bmatrix} - \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}\right)'}_{1\times 5} \underbrace{\begin{bmatrix} l_{11} & l_{12} & l_{13} & l_{14} & l_{15} \\ l_{21} & l_{22} & l_{23} & l_{24} & l_{25} \\ l_{31} & l_{32} & l_{33} & l_{34} & l_{35} \\ l_{41} & l_{42} & l_{43} & l_{44} & l_{45} \\ l_{51} & l_{52} & l_{53} & l_{54} & l_{55} \end{bmatrix}}_{5\times 5} \underbrace{\left(\begin{bmatrix} z_{i1} \\ z_{i2} \\ z_{i3} \\ z_{i4} \\ z_{i5} \end{bmatrix} - \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}\right)}_{5\times 1}$$

$$\underbrace{\begin{bmatrix} (z_{i1}-a_1)l_{11} + (z_{i2}-a_2)l_{21} + (z_{i3}-a_3)l_{31} + (z_{i4}-a_4)l_{41} + (z_{i5}-a_5)l_{55} \\ (z_{i1}-a_1)l_{12} + (z_{i2}-a_2)l_{22} + (z_{i3}-a_3)l_{32} + (z_{i4}-a_4)l_{42} + (z_{i5}-a_5)l_{52} \\ (z_{i1}-a_1)l_{13} + (z_{i2}-a_2)l_{23} + (z_{i3}-a_3)l_{33} + (z_{i4}-a_4)l_{43} + (z_{i5}-a_5)l_{53} \\ (z_{i1}-a_1)l_{14} + (z_{i2}-a_2)l_{24} + (z_{i3}-a_3)l_{34} + (z_{i4}-a_4)l_{44} + (z_{i5}-a_5)l_{54} \\ (z_{i1}-a_1)l_{15} + (z_{i2}-a_2)l_{25} + (z_{i3}-a_3)l_{35} + (z_{i4}-a_4)l_{45} + (z_{i5}-a_5)l_{55} \end{bmatrix}'}_{1\times 5} \underbrace{\begin{bmatrix} z_{i1}-a_1 \\ z_{i2}-a_2 \\ z_{i3}-a_3 \\ z_{i4}-a_4 \\ z_{i5}-a_5 \end{bmatrix}}_{5\times 1}$$

$$\begin{aligned}
&\left[(z_{i1}-a_1)l_{11} + (z_{i2}-a_2)l_{21} + (z_{i3}-a_3)l_{31} + (z_{i4}-a_4)l_{41} + (z_{i5}-a_5)l_{55}\right](z_{i1}-a_1) + \\
&\left[(z_{i1}-a_1)l_{12} + (z_{i2}-a_2)l_{22} + (z_{i3}-a_3)l_{32} + (z_{i4}-a_4)l_{42} + (z_{i5}-a_5)l_{52}\right](z_{i2}-a_2) + \\
&\left[(z_{i1}-a_1)l_{13} + (z_{i2}-a_2)l_{23} + (z_{i3}-a_3)l_{33} + (z_{i4}-a_4)l_{43} + (z_{i5}-a_5)l_{53}\right](z_{i3}-a_3) + \\
&\left[(z_{i1}-a_1)l_{14} + (z_{i2}-a_2)l_{24} + (z_{i3}-a_3)l_{34} + (z_{i4}-a_4)l_{44} + (z_{i5}-a_5)l_{54}\right](z_{i4}-a_4) + \\
&\left[(z_{i1}-a_1)l_{15} + (z_{i2}-a_2)l_{25} + (z_{i3}-a_3)l_{35} + (z_{i4}-a_4)l_{45} + (z_{i5}-a_5)l_{55}\right](z_{i5}-a_5) +
\end{aligned}$$

$z_{i1} \propto$

$$\left[(z_{i1}-a_1)\, l_{11} + (z_{i2}-a_2)\, l_{21} + (z_{i3}-a_3)\, l_{31} + (z_{i4}-a_4)\, l_{41} + (z_{i5}-a_5)\, l_{55}\right](z_{i1}-a_1)$$
$$\left[(z_{i1}-a_1)\, l_{12}\right](z_{i2}-a_2) + \left[(z_{i1}-a_1)\, l_{13}\right](z_{i3}-a_3) + \left[(z_{i1}-a_1)\, l_{14}\right](z_{i4}-a_4) +$$
$$\left[(z_{i1}-a_1)\, l_{15}\right](z_{i5}-a_5)$$

$$(z_{i1}-a_1)^2 l_{11} + (z_{i1}-a_1)\left[(z_{i2}-a_2)\, l_{21} + (z_{i3}-a_3)\, l_{31} + (z_{i4}-a_4)\, l_{41} + (z_{i5}-a_5)\, l_{55}\right]$$
$$+ \quad (z_{i1}-a_1)\left[(z_{i2}-a_2)\, l_{12} + (z_{i3}-a_3)\, l_{13} + (z_{i4}-a_4)\, l_{14} + (z_{i5}-a_5)\, l_{15}\right]$$

$$(z_{i1}-a_1)^2 l_{11} - 2(z_{i1})\left[(a_2-z_{i2})\, l_{21} + (a_3-z_{i3})\, l_{31} + (a_4-z_{i4})\, l_{41} + (a_5-z_{i5})\, l_{55}\right]$$
$$z_{i1}^2 l_{11} - 2(z_{i1})\left[a_1 l_{11} + (a_2-z_{i2})\, l_{21} + (a_3-z_{i3})\, l_{31} + (a_4-z_{i4})\, l_{41} + (a_5-z_{i5})\, l_{55}\right]$$

variance $= 1/l_{11}$

mean $= \big(a_1 l_{11} + (a_2-z_{i2})\, l_{21} + (a_3-z_{i3})\, l_{31} + (a_4-z_{i4})\, l_{41} + (a_5-z_{i5})\, l_{55}\big)\, l_{11}$

$$\left[-\frac{1}{2} \begin{bmatrix} z_1-a_1 & z_2-a_2 & \cdots & z_5-a_5 \end{bmatrix} \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{15} \\ L_{21} & L_{22} & \cdots & L_{25} \\ \vdots & & & \\ L_{51} & L_{52} & \cdots & L_{55} \end{bmatrix} \begin{bmatrix} z_1-a_1 \\ z_2-a_2 \\ \vdots \\ z_5-a_5 \end{bmatrix} \right]$$

$$\begin{bmatrix} (z_1-a_1)L_{11} + (z_2-a_2)L_{21} \cdots (z_5-a_5)L_{51} \\ (z_1-a_1)L_{12} + (z_2-a_2)L_{22} \cdots (z_5-a_5)L_{52} \\ \vdots \\ z_1-a_1)L_{15} + (z_2-a_2)L_{25} \cdots (z_5-a_5)L_{55} \end{bmatrix}_{1\times 5} \begin{bmatrix} z_1-a_1 \\ z_2-a_2 \\ \vdots \\ z_5-a_5 \end{bmatrix}_{5\times 1}$$

$$(z_1-a_1)\left[(z_1-a_1)L_{11} + (z_2-a_2)L_{21} \cdots (z_5-a_5)L_{51}\right] +$$
$$(z_2-a_2)\left[(z_1-a_1)L_{12} + (z_2-a_2)L_{22} \cdots (z_5-a_5)L_{52}\right] +$$
$$(z_3-a_3)\left[(z_1-a_1)L_{13} + (z_2-a_2)L_{23} \cdots (z_5-a_5)L_{53}\right] +$$
$$(z_4-a_4)\left[(z_1-a_1)L_{14} + (z_2-a_2)L_{24} \cdots (z_5-a_5)L_{54}\right] +$$
$$(z_5-a_5)\left[(z_1-a_1)L_{15} + (z_2-a_2)L_{25} \cdots (z_5-a_5)L_{55}\right]$$

$$z_1 \propto -1\,(z_1-a_1)\left[(z_1-a_1)L_{11} + (z_2-a_2)L_{21} \cdots (z_5-a_5)L_{51}\right]$$
$$+ (z_1-a_1)L_{12}(z_2-a_2) + (z_3-a_3)(z_1-a_1)L_{13}$$
$$+ (z_4-a_4)L_{14}(z_1-a_1) + (z_5-a_5)L_{15}(z_i-a_1)$$

$$\propto (z_1-a_1)^2 L_{11} + 2(z_1-a_1)\left[(z_2-a_2)L_{21} + (z_3-a_3)L_{13} \cdots (z_5-a_5)L_{51}\right]$$

$$z_1^2 L_{11} - 2z_1 L_{11} a_1 + 2z_1\left[\left[(z_2-a_2)L_{21} + (z_3-a_3)L_{13} \cdots (z_5-a_5)L_{51}\right]\right]$$
$$z_1^2 L_{11} - 2z_1\left[L_{11}a_1 - (z_2-a_2)L_{21} - (z_3-a_3)L_{13} \cdots (z_5-a_5)L_{15}\right]$$

sigma $= \frac{1}{L_{11}}$

mean $= \left[L_{11}a_1 - (z_2-a_2)L_{21} - \cdot\ (z_5-a_5)L_{15}\right]L_{11}$