ISL 2017 and IMO 2018 Collection

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Contents

Introduction

Problem 0.0.1 (APMO). A positive integer is called *fancy* if it can be expressed in the form

$$2^{a_1} + 2^{a_2} + \cdots + 2^{a_{100}},$$

where $a_1, a_2, \ldots, a_{100}$ are non-negative integers that are not necessarily distinct. Find the smallest positive integer n such that no multiple of n is a fancy number.

Link

Problem 0.0.2 (Argentina Intercollegiate Olympiad First Level). Find all positive integers a, b, c, and d, all less than or equal to 6, such that

$$\frac{a}{b} = \frac{c}{d} + 2.$$

Problem 0.0.3 (Argentina Intercollegiate Olympiad Third Level). Find a number with the following conditions:

1. it is a perfect square,
2. when 100 is added to the number, it equals a perfect square plus 1, and
3. when 100 is again added to the number, the result is a perfect square.

Problem 0.0.4 (Austria Regional Competition). Determine all positive integers k and n satisfying the equation

$$k^2 - 2016 = 3^n.$$

Problem 0.0.5 (Azerbaijan TST). The set A consists of positive integers which can be expressed as $2x^2 + 3y^2$, where x and y are integers (not both zero at the same time).

1. Prove that there is no perfect square in A.
2. Prove that the product of an odd number of elements of A cannot be a perfect square.

Link

Problem 0.0.6 (Baltic Way). For which integers $n = 1, 2, \ldots, 6$ does the equation

$$a^n + b^n = c^n + n$$

have a solution in integers? Link

Problem 0.0.7 (Bay Area Olympiad). Find a positive integer N and $a_1, a_2, \ldots, a_N$, where $a_k \in \{1, -1\}$ for each $k = 1, 2, \ldots, N$, such that

$$a_1 \cdot 1^3 + a_2 \cdot 2^3 + a_3 \cdot 3^3 + \cdots + a_N \cdot N^3 = 20162016,$$

or show that this is impossible. Link

Problem 0.0.8 (Belgium National Olympiad Final Round). Solve the equation

$$2^{2m+1} + 9 \cdot 2^m + 5 = n^2$$

for integers m and n.

Problem 0.0.9 (Bosnia and Herzegovina TST). Determine the largest positive integer n which cannot be written as the sum of three numbers bigger than 1 which are pairwise coprime. Link

Problem 0.0.10 (Bulgaria National Olympiad). Determine whether there exists a positive integer $n < 10^9$ such that n can be expressed as a sum of three squares of positive integers in more than 1000 distinct ways.

Preface

We will compile all the solutionds to IMO Shortlist 2017 and IMO 2018 problems, as well as ideas, lemmas, and useful facts used in solving those problems. All the contents is taken from AoPS forums, and hence is written by AoPS users. This content is free for public and should not be sold independently. The project websites:

`https://parvardi.com/downloads/ISL2017/`

Some more words...
Enjoy problem solving!

Amir Hossein Parvardi,
Pitchayut Saengrungkongka,
July 2018.

Problems

0.1 IMO 2018

0.1.1 Notes

This is an intro.

0.1.2 Problems of IMO 2018

Problem 0.1.1 (Austria Federal Competition for Advanced Students Final Round). Determine all composite positive integers n with the following property: If $1 = d_1 < d_2 < \ldots < d_k = n$ are all the positive divisors of n, then

$$(d_2 - d_1) : (d_3 - d_2) : \cdots : (d_k - d_{k-1}) = 1 : 2 \cdots : (k-1).$$

Problem 0.1.2 (Austria Beginners' Competition). Determine all nonnegative integers n having two distinct positive divisors with the same distance from $n/3$.

Problem 0.1.3 (Azerbaijan Balkan Math Olympiad First TST). Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(f(n)) = n + 2015,$$

for all $n \in \mathbb{N}$. Link

Problem 0.1.4 (Benelux). Let n be a positive integer. Suppose that its positive divisors can be partitioned into pairs (i.e. can be split in groups of two) in such a way that the sum of each pair is a prime number. Prove that these prime numbers are distinct and that none of these are a divisor of n. Link

Problem 0.1.5 (CCA Math Bonanza). Compute

$$\sum_{k=1}^{420} \gcd(k, 420).$$

Link

Problem 0.1.6 (China South East Mathematical Olympiad). Let n be a positive integer and let D_n be the set of all positive divisors of n. Define

$$f(n) = \sum_{d \in D_n} \frac{1}{1+d}.$$

Prove that for any positive integer m,

$$\sum_{i=1}^{m} f(i) < m.$$

Link

Problem 0.1.7 (China TST). Set positive integer $m = 2^k \cdot t$, where k is a non-negative integer, t is an odd number, and let $f(m) = t^{1-k}$. Prove that for any positive integer n and for any positive odd number $a \leq n$, $\prod_{m=1}^{n} f(m)$ is a multiple of a. Link

Problem 0.1.8 (China TST). For any two positive integers x and $d > 1$, denote by $S_d(x)$ the sum of digits of x taken in base d. Let a, b, b', c, m, and q be positive integers, where $m > 1, q > 1$, and $|b - b'| \geq a$. It is given that there exists a positive integer M such that

$$S_q(an + b) \equiv S_q(an + b') + c \pmod{m}$$

holds for all integers $n \geq M$. Prove that the above equation is true for all positive integers n. Link

Problem 0.1.9 (Estonia National Olympiad Eleventh Grade). Let n be a positive integer. Let $\delta(n)$ be the number of positive divisors of n and let $\sigma(n)$ be their sum. Prove that

$$\sigma(n) > \frac{(\delta(n))^2}{2}.$$

Problem 0.1.10 (European Mathematical Cup Juniors). Let $d(n)$ denote the number of positive divisors of n. For a positive integer n we define $f(n)$ as

$$f(n) = d(k_1) + d(k_2) + \cdots + d(k_m),$$

where $1 = k_1 < k_2 < \cdots < k_m = n$ are all divisors of the number n. We call an integer $n > 1$ *almost perfect* if $f(n) = n$. Find all almos perfect numbers.

0.2 ISL 2017

0.2.1 Notes

This is an intro.

0.3 Problems of ISL 2017

Problem 0.3.1 (Austria Federal Competition for Advanced Students Final Round). Determine all composite positive integers n with the following property: If $1 = d_1 < d_2 < \ldots < d_k = n$ are all the positive divisors of n, then

$$(d_2 - d_1) : (d_3 - d_2) : \cdots : (d_k - d_{k-1}) = 1 : 2 \cdots : (k-1).$$

Problem 0.3.2 (Austria Beginners' Competition). Determine all nonnegative integers n having two distinct positive divisors with the same distance from $n/3$.

Problem 0.3.3 (Azerbaijan Balkan Math Olympiad First TST). Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(f(n)) = n + 2015,$$

for all $n \in \mathbb{N}$. Link

Problem 0.3.4 (Benelux). Let n be a positive integer. Suppose that its positive divisors can be partitioned into pairs (i.e. can be split in groups of two) in such a way that the sum of each pair is a prime number. Prove that these prime numbers are distinct and that none of these are a divisor of n. Link

Problem 0.3.5 (CCA Math Bonanza). Compute

$$\sum_{k=1}^{420} \gcd(k, 420).$$

Link

Problem 0.3.6 (China South East Mathematical Olympiad). Let n be a positive integer and let D_n be the set of all positive divisors of n. Define

$$f(n) = \sum_{d \in D_n} \frac{1}{1+d}.$$

Prove that for any positive integer m,

$$\sum_{i=1}^{m} f(i) < m.$$

Link

Problem 0.3.7 (China TST). Set positive integer $m = 2^k \cdot t$, where k is a non-negative integer, t is an odd number, and let $f(m) = t^{1-k}$. Prove that for any positive integer n and for any positive odd number $a \leq n$, $\prod_{m=1}^{n} f(m)$ is a multiple of a. Link

Problem 0.3.8 (China TST). For any two positive integers x and $d > 1$, denote by $S_d(x)$ the sum of digits of x taken in base d. Let $a, b, b', c, m,$ and q be positive integers, where $m > 1, q > 1$, and $|b - b'| \geq a$. It is given that there exists a positive integer M such that

$$S_q(an + b) \equiv S_q(an + b') + c \pmod{m}$$

holds for all integers $n \geq M$. Prove that the above equation is true for all positive integers n. Link

Problem 0.3.9 (Estonia National Olympiad Eleventh Grade). Let n be a positive integer. Let $\delta(n)$ be the number of positive divisors of n and let $\sigma(n)$ be their sum. Prove that

$$\sigma(n) > \frac{(\delta(n))^2}{2}.$$

Problem 0.3.10 (European Mathematical Cup Juniors). Let $d(n)$ denote the number of positive divisors of n. For a positive integer n we define $f(n)$ as

$$f(n) = d(k_1) + d(k_2) + \cdots + d(k_m),$$

where $1 = k_1 < k_2 < \cdots < k_m = n$ are all divisors of the number n. We call an integer $n > 1$ *almost perfect* if $f(n) = n$. Find all almos perfect numbers.

Solutions

0.4 IMO 2018

0.4.1 Solutions of IMO 2018

Problem 0.4.1 (Argentina Intercollegiate Olympiad Second Level). Find all positive integers x and y which satisfy the following conditions:

1. x is a $4-$digit palindromic number, and

2. $y = x + 312$ is a $5-$digit palindromic number.

Note. A palindromic number is a number that remains the same when its digits are reversed. For example, 16461 is a palindromic number.

Problem 0.4.2 (Bundeswettbewerb Mathematik). A number with 2016 zeros that is written as $101010\ldots0101$ is given, in which the zeros and ones alternate. Prove that this number is not prime. Link

Problem 0.4.3 (Caltech Harvey Mudd Math Competition (CHMMC) Fall). We say that the string $d_k d_{k-1} \cdots d_1 d_0$ represents a number n in base -2 if each d_i is either 0 or 1, and $n = d_k(-2)^k + d_{k-1}(-2)^{k-1} + \cdots + d_1(-2) + d_0$. For example, 110_{-2} represents the number 2. What string represents 2016 in base -2? Link

Problem 0.4.4 (CentroAmerican). Find all positive integers n that have 4 digits, all of them perfect squares, and such that n is divisible by $2, 3, 5$, and 7. Link

Problem 0.4.5 (Croatia IMO TST, Bulgaria TST). Let $p > 10^9$ be a prime number such that $4p+1$ is also a prime. Prove that the decimal expansion of $\frac{1}{4p+1}$ contains all the digits $0, 1, \ldots, 9$. Link

Problem 0.4.6 (Germany National Olympiad Second Round Eleventh/Twelfth Grade). The sequence $x_1, x_2, x_3, \ldots$ is defined as $x_1 = 1$ and

$$x_{k+1} = x_k + y_k \quad \text{for } k = 1, 2, 3, \ldots$$

where y_k is the last digit of decimal representation of x_k. Prove that the sequence $x_1, x_2, x_3, \ldots$ contains all powers of 4. That is, for every positive integer n, there exists some natural k for which $x_k = 4^n$.

Problem 0.4.7 (Germany National Olympiad Fourth Round Tenth Grade[1]). A sequence of positive integers $a_1, a_2, a_3, \ldots$ is defined as follows: a_1 is a 3 digit number and a_{k+1} (for $k \geq 1$) is obtained by

$$a_{k+1} = a_k + 2 \cdot Q(a_k),$$

[1]Thanks to Arian Saffarzadeh for translating the problem.

where $Q(a_k)$ is the sum of digits of a_k when represented in decimal system. For instance, if one takes $a_1 = 358$ as the initial term, the sequence would be

$$\begin{aligned} a_1 &= 358, \\ a_2 &= 358 + 2 \cdot 16 = 390, \\ a_3 &= 390 + 2 \cdot 12 = 414, \\ a_4 &= 414 + 2 \cdot 9 = 432, \\ &\vdots \end{aligned}$$

Prove that no matter what we choose as the starting number of the sequence,

(a) the sequence will not contain 2015.

(b) the sequence will not contain 2016.

Problem 0.4.8 (IberoAmerican). Let k be a positive integer and suppose that we are given $a_1, a_2, \ldots, a_k$, where $0 \leq a_i \leq 9$ for $i = 1, 2, \ldots, k$. Prove that there exists a positive integer n such that the last $2k$ digits of 2^n are, in the following order, $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k$, for some digits $b_1, b_2, \ldots, b_k$. Link

Problem 0.4.9 (India IMO Training Camp). Given that n is a natural number such that the leftmost digits in the decimal representations of 2^n and 3^n are the same, find all possible values of the leftmost digit. Link

Problem 0.4.10 (Middle European Mathematical Olympiad). A positive integer n is called *Mozart* if the decimal representation of the sequence $1, 2, \ldots, n$ contains each digit an even number of times. Prove that:

1. All Mozart numbers are even.

2. There are infinitely many Mozart numbers.

Link

0.5 Solutions of ISL 2017

Problem 0.5.1 (Argentina Intercollegiate Olympiad Second Level). Find all positive integers x and y which satisfy the following conditions:

1. x is a 4−digit palindromic number, and

2. $y = x + 312$ is a 5−digit palindromic number.

Note. A palindromic number is a number that remains the same when its digits are reversed. For example, 16461 is a palindromic number.

Problem 0.5.2 (Bundeswettbewerb Mathematik). A number with 2016 zeros that is written as $101010\ldots0101$ is given, in which the zeros and ones alternate. Prove that this number is not prime. Link

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$$x_{k+1} = x_k + y_k \quad \text{for } k = 1, 2, 3, \ldots$$

where y_k is the last digit of decimal representation of x_k. Prove that the sequence $x_1, x_2, x_3, \ldots$ contains all powers of 4. That is, for every positive integer n, there exists some natural k for which $x_k = 4^n$.

Problem 0.5.7 (Germany National Olympiad Fourth Round Tenth Grade[2]). A sequence of positive integers $a_1, a_2, a_3, \ldots$ is defined as follows: a_1 is a 3 digit number and a_{k+1} (for $k \geq 1$) is obtained by

$$a_{k+1} = a_k + 2 \cdot Q(a_k),$$

where $Q(a_k)$ is the sum of digits of a_k when represented in decimal system. For instance, if one takes $a_1 = 358$ as the initial term, the sequence would be

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Problem 0.5.10 (Middle European Mathematical Olympiad). A positive integer n is called *Mozart* if the decimal representation of the sequence $1, 2, \dots, n$ contains each digit an even number of times. Prove that:

1. All Mozart numbers are even.
2. There are infinitely many Mozart numbers.

Link

Lemmas, Ideas, and Theorems

Problem 0.5.11 (Austria National Competition Final Round). Let a, b, and c be integers such that

$$\frac{ab}{c} + \frac{ac}{b} + \frac{bc}{a}$$

is an integer. Prove that each of the numbers

$$\frac{ab}{c}, \frac{ac}{b}, \text{ and } \frac{bc}{a}$$

is an integer.

Problem 0.5.12 (Azerbaijan Balkan Math Olympiad Third TST). Find all natural numbers n for which there exist primes p and q such that the following conditions are satisfied:

1. $p + 2 = q$, and
2. $2^n + p$ and $2^n + q$ are both primes.

Link

Problem 0.5.13 (Azerbaijan Junior Mathematical Olympiad). Given

$$34! = 295232799039a041408476186096435b0000000,$$

in decimal representation, find the numbers a and b. Link

Problem 0.5.14 (Azerbaijan Junior Mathematical Olympiad). A quadruple (p, a, b, c) of positive integers is called a *good quadruple* if

(a) p is an odd prime,

(b) a, b, and c are distinct,

(c) $ab + 1, bc + 1$, and $ca + 1$ are divisible by p.

Prove that for all good quadruples (p, a, b, c),

$$p + 2 \le \frac{a + b + c}{3},$$

and show the equality case.

Link

Problem 0.5.15 (Balkan). Find all monic polynomials f with integer coefficients satisfying the following condition: there exists a positive integer N such that p divides $2(f(p)!) + 1$ for every prime $p > N$ for which $f(p)$ is a positive integer.
Note. A monic polynomial has a leading coefficient equal to 1. Link

Problem 0.5.16 (Baltic Way). Let n be a positive integer and let a, b, c, d be integers such that $n | a+b+c+d$ and $n | a^2+b^2+c^2+d^2$. Show that

$$n | a^4+b^4+c^4+d^4+4abcd.$$

Link

Problem 0.5.17 (Bay Area Olympiad). The distinct prime factors of an integer are its prime factors listed without repetition. For example, the distinct prime factors of 40 are 2 and 5. Let $A = 2^k - 2$ and $B = 2^k \cdot A$, where k is an integer ($k \geq 2$). Show that for every integer k greater than or equal to 2,

1. A and B have the same set of distinct prime factors, and
2. $A+1$ and $B+1$ have the same set of distinct prime factors.

Link

Problem 0.5.18 (Belgium Flanders Math Olympiad Final Round). Find the smallest positive integer n which does not divide 2016!.

Problem 0.5.19 (Benelux). Find the greatest positive integer N with the following property: there exist integers $x_1, x_2, \ldots, x_N$ such that $x_i^2 - x_i x_j$ is not divisible by 1111 for any $i \neq j$. Link

Problem 0.5.20 (Bulgaria National Olympiad). Find all positive integers m and n such that

$$\left(2^{2^m}+1\right)\left(2^{2^n}+1\right)$$

is divisible by mn.

0.6 Lemmas

Problem 0.6.1 (Austria National Competition Final Round). Let a, b, and c be integers such that

$$\frac{ab}{c} + \frac{ac}{b} + \frac{bc}{a}$$

is an integer. Prove that each of the numbers

$$\frac{ab}{c}, \frac{ac}{b}, \text{ and } \frac{bc}{a}$$

is an integer.

Problem 0.6.2 (Azerbaijan Balkan Math Olympiad Third TST). Find all natural numbers n for which there exist primes p and q such that the following conditions are satisfied:

1. $p+2=q$, and
2. 2^n+p and 2^n+q are both primes.

Link

Problem 0.6.3 (Azerbaijan Junior Mathematical Olympiad). Given

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in decimal representation, find the numbers a and b. Link

Problem 0.6.4 (Azerbaijan Junior Mathematical Olympiad). A quadruple (p, a, b, c) of positive integers is called a *good quadruple* if

(a) p is an odd prime,

(b) a, b, and c are distinct,

(c) $ab+1, bc+1$, and $ca+1$ are divisible by p.

Prove that for all good quadruples (p, a, b, c),

$$p+2 \leq \frac{a+b+c}{3},$$

and show the equality case.

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Problem 0.6.5 (Balkan). Find all monic polynomials f with integer coefficients satisfying the following condition: there exists a positive integer N such that p divides $2(f(p)!)+1$ for every prime $p > N$ for which $f(p)$ is a positive integer.
Note. A monic polynomial has a leading coefficient equal to 1. Link

Problem 0.6.6 (Baltic Way). Let n be a positive integer and let a, b, c, d be integers such that $n|a+b+c+d$ and $n|a^2+b^2+c^2+d^2$. Show that

$$n|a^4+b^4+c^4+d^4+4abcd.$$

Link

Problem 0.6.7 (Bay Area Olympiad). The distinct prime factors of an integer are its prime factors listed without repetition. For example, the distinct prime factors of 40 are 2 and 5. Let $A = 2^k - 2$ and $B = 2^k \cdot A$, where k is an integer ($k \geq 2$). Show that for every integer k greater than or equal to 2,

1. A and B have the same set of distinct prime factors, and

2. $A+1$ and $B+1$ have the same set of distinct prime factors.

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Problem 0.6.9 (Benelux). Find the greatest positive integer N with the following property: there exist integers $x_1, x_2, \ldots, x_N$ such that $x_i^2 - x_i x_j$ is not divisible by 1111 for any $i \neq j$. Link

Problem 0.6.10 (Bulgaria National Olympiad). Find all positive integers m and n such that

$$\left(2^{2^m}+1\right)\left(2^{2^n}+1\right)$$

is divisible by mn.