The Olympiad Algebra Book

VOLUME I:

1220 Polynomials and Trigonometry Problems

SUMMER KAYWAÑAN ALGEBRA COMPETITIONS

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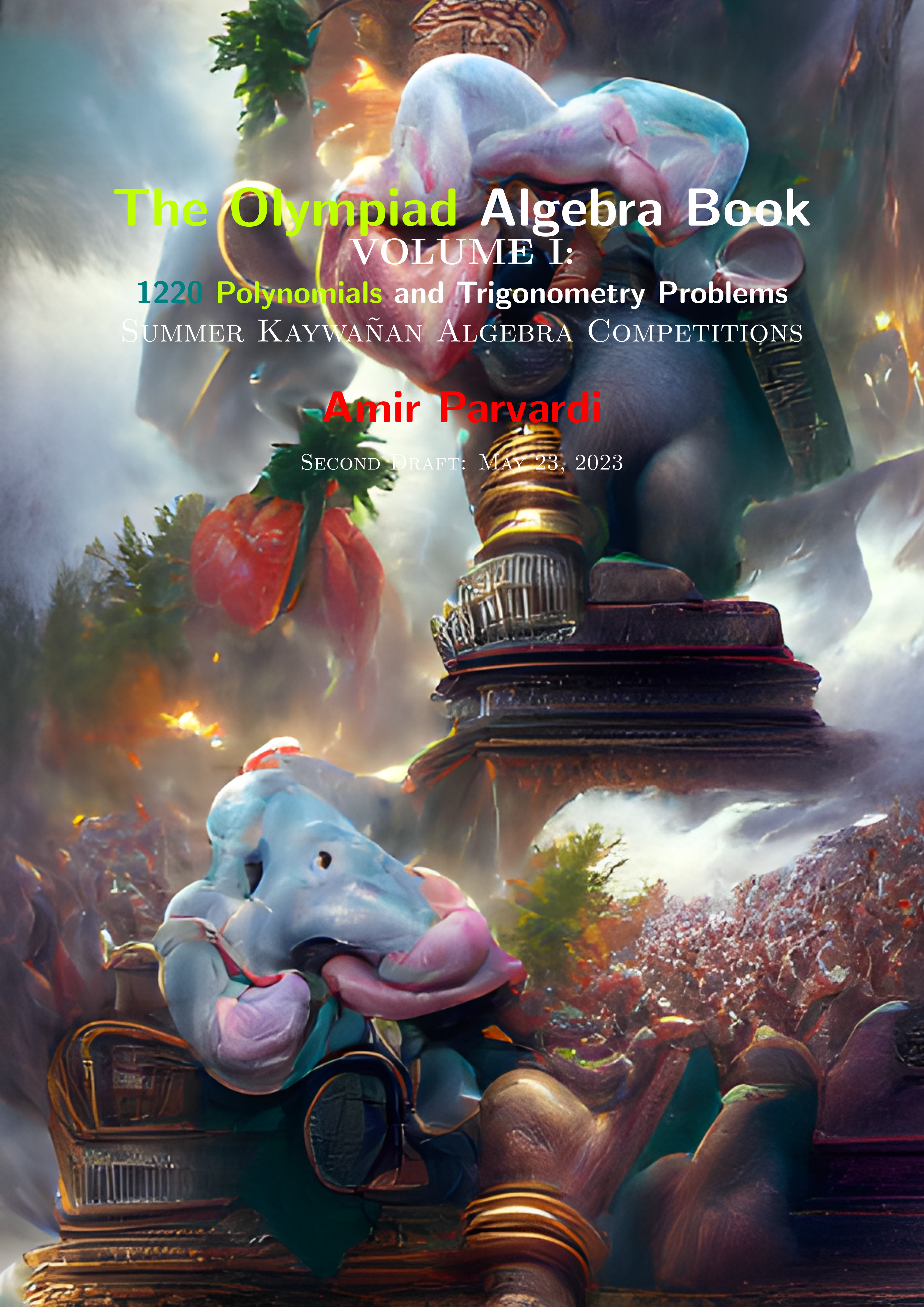
SECOND DRAFT: MAY 23, 2023



Preface

Foreword

Azermalohg speaks to **Rima**:

Why are you so afraid of the IllLyrans?
What has your fear had you achieved?
You have made yourself weary for lack of sleep,
You only fill your flesh with grief,
You only bring the distant days closer.
Humankind's fame is cut down
like reeds in a reed-bed.
A fine young man, a fine young girl...
at grip of Death.
You have seen Death,
You have touched the face of Death,
You hear the voice of Death lamenting in your ears,
Savage Death just cuts humankind down.
Sometimes we have hope,
sometimes we make a wish,
but then our airplanes are shot in the air.
Sometimes there is hostility in the land,
but in the end, only the most benevolent will remain.
The ruthless IllLyrans bring Death with themselves;
but the merciful Lyrans will always prevail.
Remember, the night is darkest just before dawn.

Synopsis

The Olympiad Algebra Book comes in two volumes. The first volume, dedicated to Polynomials and Trigonometry, is a collection of lesson plans containing 1220 beautiful problems, around two-thirds of which are polynomial problems and one-third are trigonometry problems. The second volume of The Olympiad Algebra Book contains 1220 Problems on Functional Equations and Inequalities, and I hope to finish it before the end of Summer 2023. I hope I can finish collecting the FE and INEQ problems by June 29^{th}, as a reminder of the 1220 Number Theory Problems published as the first 1220 set of J29 Project. The current volumes has 843 Polynomial problems and 377 Trigonometry questions, the last 63 of which are bizarre spherical geometry problems!

This book is supposed to be a problem bank for Algebra, and it forms the resource for the first series of the KAYWAÑAN Algebra Contest. I suggest you start with Polynomials, and before you get bored or exhausted, also start solving Trigonometry problems. If you find these problems easy and not challenging enough, the Spherical Trigonometry lessons and problems are definitely going to be a must try!

The Summer and Winter Kaywañan Algebra Contest are called KACY–I and KACY–II, respectively, in The Olympiad Algebra Book. This large, nearly 15,000–line TeX file, `'KACY-VOL-I.tex'`, contains the questions of the Summer Kaywañan Algebra Competitions, and in the very first version of the file, I compose nine Olympiad Pre–Algebra Contests. The numbers referred here are the question number out of the 1220 questions labeled from 1 to 1220. The competitions are titled "Kaywañan Olympiad Pre–Algebra Summer Contest 001–009," referred to below by labels `KACY--I001` to `KACY--I009`, respectively, for short.

“Kaywañan Olympiad Pre–Algebra Summer Contests”

a) `KACY--I001`: KACY–I $\{2, 37, 38, 58, 74, 75, 86, 103, 113\}$.
b) `KACY--I002`: KACY–I $\{3, 39, 40, 59, 76, 77, 87, 104, 114\}$.
c) `KACY--I003`: KACY–I $\{4, 41, 42, 60, 61, 78, 88, 105, 115\}$.
d) `KACY--I004`: KACY–I $\{5, 43, 44, 62, 79, 89, 90, 106, 116\}$.
e) `KACY--I005`: KACY–I $\{6, 45, 46, 63, 80, 91, 92, 107, 117\}$.
f) `KACY--I006`: KACY–I $\{13, 47, 64, 81, 93, 95, 108, 109, 118\}$.
g) `KACY--I007`: KACY–I $\{14, 48, 65, 66, 82, 96, 97, 98, 110, 119\}$.
h) `KACY--I008`: KACY–I $\{32, 49, 67, 68, 83, 84, 99, 100, 111, 120\}$.
i) `KACY--I009`: KACY–I $\{33, 55, 57, 72, 73, 85, 101, 102, 112, 121\}$.

Dates of `KACY--I001` to `KACY--I009`:

1. `KACY--I001`: June 3, 2023.
2. `KACY--I002`: June 10, 2023.
3. `KACY--I003`: June 17, 2023.
4. `KACY--I004`: June 24, 2023.
5. `KACY--I005`: July 1, 2023.
6. `KACY--I006`: July 8. 2023.
7. `KACY--I007`: July 15, 2023.
8. `KACY--I008`: July 22, 2023.
9. `KACY--I009`: July 29, 2023.

Amir Parvardi,
Vancouver, British Columbia,
May 23, 2023

Contents

List of Figures

Part I

KAYWAÑAN Algebra Contest
Summer KACY
Polynomials & Trigonometry

Chapter 1

Olympiad Algebra 101: Polynomials (KACY–I)

> "Let No One Ignorant of Algebra Enter!"
>
> Kaywan

The rules of the KACY Competitions are simple:

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a) All problems whose titles contain **KACY–I** are questions of the Summer KACY Series, and all problems with a title containing **KACY–II** are questions of the Winter KACY Series.

b) This is the first volume of KACY, and it contains the SUMMER KACY questions. For the SUMMER KACY 2023 held weekly in Summer and Fall of 2023, only questions with title containing "**KACY–I**" are to be used in the actual KAYWAÑAN competitions.

This is because all the questions whose source does not contain **KACY–I** are either from a legit mathematical competition such as IMO, IMO Shortlist/Longlist, MAA Series (AMC, AIME, USAMO, USATST, USATSTST, USAMTS, etc.), National or Regional Olympiads (USA, APMC, Canada, etc.), or maybe from a book/paper I found and referenced in the question's title.

This assures that no famous problems are used in KACY, and that we actually identify and solve the non–KACY problems as exercises and examples in our journey of learning algebra during KAYWAÑAN Algebra Contest.

1.1 Olympiad Algebra 001: Corollaries of Pre–Algebra

We begin by reminding ourselves of why polynomials are the first topic we need to study in Pre–Algebra. The study of the relationships between the roots of polynomials, a field whose master is undoubtedly Évariste Galois, is the hidden root of the stout tree of Algebra. We may begin stating the definitions of polynomials and how their roots are related to each other, involving the Fundamental Theorem of Algebra, which states that each polynomial $P(x)$ with complex coefficients has exactly n roots in the complex plane. This ferocious fact about the roots of any polynomial is, indeed, the Most Fundamental Theorem in Algebra.

1.1.1 Introduction to Olympiad Algebra

In olympiad Algebra, starting from polynomials, we seek to find special examples of equations that have solutions that seem interesting in some way. For instance, regarding the Fundamental Theorem of Algebra just stated, we my ask special questions, for instance, *Casus irreducibilis* (Latin for "the irrational case"): can we solve all third-degree polynomials with real radicals? And the answer is no. For example,

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Example 1. Prove that the cubic equation $2x^3 - 9x^2 - 6x + 3 = 0$ has three real roots. You can check this by finding the discriminant Δ, which is given by

$$\Delta := \big((x_1 - x_2)(x_1 - x_3)(x_2 - x_3)\big)^2 = 18abcd - 4ac^3 - 27a^2d^2 + b^2c^2 - 4b^3d,$$

where a, b, c, d must be replaced with the coefficients of our polynomial. Prove that if $\Delta > 0$, then x_1, x_2, x_3 would be three real roots, but, in the case of the three roots of our polynomial $2x^3 - 9x^2 - 6x + 3 = 0$, they are not presentable in any real radical form and we require imaginary radicals to solve this specific equation [from Wikipedia] "are given by:

$$t_k = \frac{3 - \omega_k \sqrt[3]{39 - 26i} - \omega_k^2 \sqrt[3]{39 + 26i}}{2},$$

for $k = 1, 2, 3$. The solutions are in radicals and involve the cube roots of complex conjugate numbers."

Definition of Polynomial and Roots

We state the definition of a "polynomial" in the broadest form:

Definition. A function $P(x)$ defined over complex numbers by

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \qquad \text{for} \quad n \geq 0,$$

where $a_0, a_1, \ldots, a_n$ are complex numbers, is called a **polynomial of degree** n with complex **coefficients** $a_0, a_1, \ldots, a_n$. We also write $\deg P := \deg(P(x)) = n$.

However, in most cases, we are interested in polynomials with real, rational, integer, or natural coefficients. For the sake of completeness and self-containment of Kaywañan, we need to discuss the definition of complex numbers in details, though, and we will mention the most important definitions, theorems, and identities for complex numbers. Start by studying the different representations of a complex number in the complex plane, once assuming the plane is Cartesian, and once in the Polar Plane. The most important result, then, would be the **De Moivre's Formula**, which will be mentioned just enough not to spoil the fun for later trigonometry lessons in Olympiad Algebra 401.

Equivalent Polynomials, Monic Polynomials

Definition. Two polynomials $P(x)$ and $Q(x)$ are **equivalent** if and only if

a) They have equal degrees, i.e., $\deg(P(x)) = \deg(Q(x))$; and

b) All their corresponding coefficients are the same.

In other words, assuming

$$P(x) = a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0 \quad \text{and} \quad Q(x) = b_mx^m+b_{m-1}x^{m-1}\cdots+b_1x+b_0,$$

we have $P(x) \equiv Q(x)$ if and only if $m = n$ and $a_i = b_i$ for all $i = 1, 2, \ldots, n$.

Definition. Let $P(x)$ be a polynomial of degree n, defined by

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0,$$

Then we say $P(x)$ is a **monic polynomial** if and only if $a_n = 1$.

Definition. We may introduce the derivative of polynomial $P(x)$ usually denoted $P'(x)$, where

$$\begin{aligned} P(x) &= a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0, \\ P'(x) &= na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2x + a_1. \end{aligned}$$

Theorem 1 (Fundamental Theorem of Algebra)**.** Any polynomial $P(x)$ with complex coefficients has precisely $\deg P$ complex roots.

Corollary 1. Let $P(x)$ be a polynomial of degree n with complex coefficients, defined by

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0.$$

If the equation $P(x) = 0$ has $n+1$ solutions in $\mathbb{C}$, then $P(x) \equiv 0$.

1.1.2 Essential Polynomial Theorems

Here are some theorems that you really need to prove on your own.

Theorem 2. For two polynomials

$$A(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \quad \text{and} \quad B(x) = b_mx^m + b_{m-1}x^{m-1} \cdots + b_1x + b_0,$$

assuming $n \geq m$, we have

a) The polynomial $S(x) = A(x) + B(x)$ is a polynomial of degree at most n, whose coefficients are the sum of the corresponding coefficients of $A(x)$ and $B(x)$.

b) The polynomial $\Pi(x) = A(x) \cdot B(x)$ is a polynomial of degree $m+n$ whose coefficients $\pi_0, \pi_1, \ldots, \pi_{m+n}$, where

$$\Pi(x) = \pi_{m+n}x^{m+n} + \cdots + \pi_1 x + \pi_0,$$

are calculated by

$$\begin{aligned}
\pi_0 &= a_0 b_0, \\
\pi_1 &= a_0 b_1 + a_1 b_0, \\
\pi_2 &= a_0 b_2 + a_1 b_1 + a_2 b_0, \\
\vdots \quad & \quad \vdots \\
\pi_{m+n-1} &= a_{n-1} b_m + a_n b_{m-1}, \\
\pi_{m+n} &= a_n b_m.
\end{aligned}$$

Theorem 3 (Polynomial Division Theorem)**.** For two polynomials $A(x)$ and $B(x)$ with

$$A(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \quad \text{and} \quad B(x) = b_m x^m + b_{m-1}x^{m-1} \cdots + b_1 x + b_0,$$

we can define the **quotient polynomial** $Q(x)$ and the **remainder polynomial** $R(x)$, assuming $n \geq m$, by

$$A(x) = B(x) \cdot Q(x) + R(x), \qquad \text{and} \quad \deg R < \deg B.$$

In the special case when the remainder is the zero polynomial, $R(x) \equiv 0$, we say $A(x)$ is divisible by $B(x)$.

Theorem 4 (Bézout's Theorem for Polynomials AKA Factor Theorem)**.** As a special case of the Polynomial Division Theorem, in the polynomial division $A(x) = B(x) \cdot Q(x) + R(x)$, let $B(x) = x - x_0$, where $x_0 \in \mathbb{R}$. Then, $A(x_0) = R(x_0)$ and we may write:

$$A(x) = (x - x_0) \cdot Q(x) + A(x_0).$$

The factor theorem says $x - x_0$ is a factor of $A(x)$ if and only if $A(x_0) = 0$.

Theorem 5 (Unique Factorization Theorem)**.** According to the Fundamental Theorem of Algebra, all polynomials $P(x)$ with complex coefficients in the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad \text{with} \quad a_n \neq 0.$$

Let $x_0, x_1, \ldots, x_n$ be the n complex roots of $P(x) = 0$. Then,

$$P(x) = a_n(x - x_1)(x - x_2) \cdots (x - x_n).$$

1.1.3 In Search of Rational Roots

Rational Root Theorem

Theorem 6. Let $P(x)$ be a polynomial with integer coefficients written as

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad \text{with} \quad a_n \neq 0.$$

. Show that $P(x)$ has a rational root $r = p/q$, where p and q are relatively prime positive integers, then p is a divisor of a_0 and q is a divisor of a_n.

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KACY–I 2. We call a polynomial **monic** if the coefficient of the highest exponent in the polynomial equals 1. Consider the monic polynomial $P(x)$ with integer coefficients:

$$P(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0.$$

1. Prove that the equation $P(x) = 0$ does not have any roots in the form $x = p/q$ where p and q are coprime integers.
2. If the equation $P(x) = 0$ has a rational root, then this root is an integer and it divides a_0.
3. Let $x = \alpha$ be an integer root of the equation $P(x) = 0$. Prove that α divides $a_1 + \frac{a_0}{\alpha}$.
4. Let $x = \alpha$ be an integer root of the equation $P(x) = 0$. Prove that the numbers $\alpha, \alpha^2, \ldots, \alpha^n$ divide the following numbers, respectively:

$$a_0, \quad a_0 + \alpha a_1, \quad a_0 + \alpha a_1 + \alpha^2 a_2, \quad \ldots, \quad a_0 + \alpha a_1 + \cdots + \alpha^{n-1} a_{n-1}.$$

KACY Summer League

KACY–I 3. Solve for x:

$$x^4 + 5x^3 - 2x^2 - 9x + 5 = 0.$$

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KACY–I 4. Solve for x:

$$2x^4 + 3x^3 - 10x^2 - 2x + 3 = 0.$$

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KACY–I 5. Solve for x:

$$x^4 - 5x^3 + 2x^2 + 20x - 24 = 0.$$

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KACY–I 6. Solve for x:

$$x^4 - 3x^3 - 8x^2 + 12x + 16 = 0.$$

KACY Summer League

2002 Croatia 7. Solve the equation

$$\left(x^2 + 3x - 4\right)^3 + \left(2x^2 - 5x + 3\right)^3 = \left(3x^2 - 2x - 1\right)^3.$$

KACY Summer League

2019 Greece 8. Solve in $\mathbb{R}$ the following equation

$$108(x-2)^4 + (4-x^2)^3 = 0.$$

KACY Summer League

2018 Romanian District 9. Show that the number

$$\sqrt[n]{\sqrt{2019} + \sqrt{2018}} + \sqrt[n]{\sqrt{2019} - \sqrt{2018}}$$

is irrational for any $n \geq 2$.

KACY Summer League

2017 Thailand 10. Let p be a prime. Show that $\sqrt[3]{p} + \sqrt[3]{p^5}$ is irrational.

KACY Summer League

2006 All–Russian 11. The sum and the product of two purely periodic decimal fractions a and b are purely periodic decimal fractions of period length T. Show that the lengths of the periods of the fractions a and b are not greater than T.
Note. A purely periodic decimal fraction is a periodic decimal fraction without a non-periodic starting part.

KACY Summer League

2006 Pan–African 12. Let a, b, c be three non-zero integers. It is known that the sums

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \qquad \text{and} \qquad \frac{b}{a} + \frac{c}{b} + \frac{a}{c},$$

are integers. Find these sums.

KACY Summer League

KACY–I 13. Let p be a prime number. Prove that the polynomial

$$P(x) = x^{p-1} + 2x^{p-2} + \ldots + (p-1)x + p,$$

is irreducible over $\mathbb{Z}[x]$.

KACY Summer League

KACY–I 14. If a and b are real numbers such that

$$\sqrt[3]{a} - \sqrt[3]{b} = 12 \quad \text{and} \quad ab = \left(\frac{a+b+8}{6}\right)^3,$$

find the value of $a - b$.

KACY Summer League

2014 Poland 15. Let x, y be positive integers such that

$$\frac{x^2}{y} + \frac{y^2}{x},$$

is an integer. Prove that $y \mid x^2$.

KACY Summer League

2022 Thailand 16. Determine all possible values of a_1 for which there exists a sequence $a_1, a_2, \ldots$ of rational numbers satisfying

$$a_{n+1}^2 - a_{n+1} = a_n,$$

for all positive integers n.

KACY Summer League

2017 Romania TST 17. Determine all integers $n \geq 2$ such that $a + \sqrt{2}$ and $a^n + \sqrt{2}$ are both rational for some real number a depending on n.

KACY Summer League

2006 Romania TST 18. Let p a prime number, $p \geq 5$. Find the number of polynomials of the form

$$x^p + px^k + px^l + 1, \quad k > l, \quad k, l \in \{1, 2, \cdots, p-1\},$$

which are irreducible in $\mathbb{Z}[X]$.

KACY Summer League

2021 Saudi Arabia TST 19. For a non-empty set T denote by $p(T)$ the product of all elements of T. Does there exist a set T of 2021 elements such that for any $a \in T$ one has that $P(T) - a$ is an odd integer? Consider two cases:

a) All elements of T are irrational numbers.

b) At least one element of T is a rational number.

KACY Summer League

2012 IMO Shortlist 20. Let f and g be two nonzero polynomials with integer coefficients and $\deg f > \deg g$. Suppose that for infinitely many primes p the polynomial $pf + g$ has a rational root. Prove that f has a rational root.

KACY Summer League

2003 Spain 21. Let x be a real number such that $x^3 + 2x^2 + 10x = 20$. Demonstrate that both x and x^2 are irrational.

KACY Summer League

1993 Italy 22. Find all pairs (p, q) of positive primes such that the equation $3x^2 - px + q = 0$ has two distinct rational roots.

KACY Summer League

2017 Romania 23. Define

$$P(x) = x^2 + \frac{x}{2} + b \quad \text{and} \quad Q(x) = x^2 + cx + d,$$

be two polynomials with real coefficients such that $P(x)Q(x) = Q(P(x))$ for all real x. Find all real roots of $P(Q(x)) = 0$.

KACY Summer League

2008 Iran Third Round 24. Let (b_0, b_1, b_2, b_3) be a permutation of the set $\{54, 72, 36, 108\}$. Prove that

$$x^5 + b_3x^3 + b_2x^2 + b_1x + b_0,$$

is irreducible in $\mathbb{Z}[x]$.

1.1.4 Viète's Formulas

Vieta's Formulas

Theorem 7 (François Viète's Formulas AKA Vieta's Formulas)**.** For any polynomial $P(x)$ with complex coefficients, written as

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0,$$

imagine the n roots are $x_1, x_2, \ldots, x_n$. Prove that

$$\begin{cases} \displaystyle\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n & = -\dfrac{a_{n-1}}{a_n}, \\ \displaystyle\sum_{i \neq j} x_ix_j = x_1x_2 + \cdots + x_{n-1}x_n & = +\dfrac{a_{n-2}}{a_n}, \\ \quad\vdots & \quad\vdots \\ \displaystyle\prod_{i=1}^{n} x_i = x_1x_2\cdots x_{n-1}x_n & = (-1)^n\dfrac{a_0}{a_n}. \end{cases}$$

Binomial Theorem

Theorem 8 (Binomial Theorem)**.** Prove that for all $x, y \in \mathbb{C}$, and positive integer n,

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{1}xy^{n-1} + y^n.$$

KACY Summer League

1993 AIME 25. Let $P_0(x) = x^3 + 313x^2 - 77x - 8$. For integers $n \geq 1$, define $P_n(x) = P_{n-1}(x-n)$. What is the coefficient of x in $P_{20}(x)$?

KACY Summer League

1996 AIME 26. Suppose that the roots of $x^3 + 3x^2 + 4x - 11 = 0$ are a, b, and c, and that the roots of $x^3 + rx^2 + sx + t = 0$ are $a+b, b+c$, and $c+a$. Find t.

KACY Summer League

2001 AIME 27. Find the sum of the roots, real and non-real, of the equation

$$x^{2001} + \left(\frac{1}{2} - x\right)^{2001} = 0,$$

given that there are no multiple roots.

KACY Summer League

2005 AIME 28. The equation

$$2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$$

has three real roots. Given that their sum is m/n where m and n are relatively prime positive integers, find $m+n$.

KACY Summer League

2008 AIME 29. Let r, s, and t be the three roots of the equation

$$8x^3 + 1001x + 2008 = 0.$$

Find $(r+s)^3 + (s+t)^3 + (t+r)^3$.

KACY Summer League

2014 AIME 30. Real numbers r and s are roots of $p(x) = x^3 + ax + b$, and $r + 4$ and $s - 3$ are roots of $q(x) = x^3 + ax + b + 240$. Find the sum of all possible values of $|b|$.

KACY Summer League

2015 AIME 31. Steve says to Jon, "I am thinking of a polynomial whose roots are all positive integers. The polynomial has the form

$$P(x) = 2x^3 - 2ax^2 + (a^2 - 81)x - c,$$

for some positive integers a and c. Can you tell me the values of a and c?" After some calculations, Jon says, "There is more than one such polynomial." Steve says, "You're right. Here is the value of a." He writes down a positive integer and asks, "Can you tell me the value of c?" Jon says, "There are still two possible values of c." Find the sum of the two possible values of c.

KACY Summer League

KACY–I 32. Define four real numbers A, B, C, D by

$$\begin{cases} A &= +\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4}, \\ B &= -\sqrt{1} + \sqrt{2} + \sqrt{3} - \sqrt{4}, \\ C &= +\sqrt{1} - \sqrt{2} + \sqrt{3} + \sqrt{4}, \\ D &= +\sqrt{1} + \sqrt{2} - \sqrt{3} + \sqrt{4}. \end{cases}$$

Prove that the product $ABCD$ of these four reals equals 8.

KACY Summer League

KACY–I 33. How many numbers in the 100^{th} row of the Pascal triangle (the one starting with $1, 100, \ldots$) are not divisible by 3?

KACY Summer League

2012 Serbia TST 34. Let $P(x)$ be a polynomial of degree 2012 with real coefficients satisfying the condition

$$P(a)^3 + P(b)^3 + P(c)^3 \geq 3P(a)P(b)P(c),$$

for all real numbers a, b, c such that $a + b + c = 0$. Is it possible for $P(x)$ to have exactly 2012 distinct real roots?

KACY Summer League

2014 USA TST 35. Let n be a positive even integer, and let $c_1, c_2, \ldots, c_{n-1}$ be real numbers satisfying

$$\sum_{i=1}^{n-1} |c_i - 1| < 1.$$

Prove that

$$2x^n - c_{n-1}x^{n-1} + c_{n-2}x^{n-2} - \cdots - c_1x^1 + 2$$

has no real roots.

KACY Summer League

2014 India Regional 36. The roots of the equation

$$x^3 - 3ax^2 + bx + 18c = 0,$$

form a non-constant arithmetic progression and the roots of the equation

$$x^3 + bx^2 + x - c^3 = 0,$$

form a non-constant geometric progression. Given that a, b, c are real numbers, find all positive integral values a and b.

1.2 Factorization

1.2.1 One–Variable Identities

KACY Summer League

KACY–I 37. Factorize $2x^4 + x^3 + 4x^2 + x + 2$.

KACY Summer League

KACY–I 38. Factorize $x^5 + x^4 + x^3 + x^2 + x + 1$.

KACY Summer League

KACY–I 39. Factorize $x^4 - x^2 + 7$.

KACY Summer League

KACY–I 40. Factorize $x^4 + 4x - 1$.

KACY Summer League

KACY–I 41. Factorize $(1 + x + x^2 + \cdots + x^n)^2 - x^n$.

KACY Summer League

KACY–I 42. Factorize $2x^4 + x^3 + 3x^2 + x + 2$.

KACY Summer League

KACY–I 43. Factorize $x^6 + 2x^5 + 3x^4 + 24x^3 + 23x^2 + 22x + 21$.

KACY Summer League

KACY–I 44. Factorize $x^4 + 6x^2 + 18$.

KACY Summer League

KACY–I 45. If n is a positive integer, factorize $a^{5n} + a^n + 1$.

KACY Summer League

KACY–I 46. Factorize $(x + 1)(x + 3)(x + 5)(x + 7) + 15$.

KACY Summer League

KACY–I 47. Factorize $x^3 + 5x^2 + 3x - 9$.

KACY Summer League

KACY–I 48. Factorize $x^3 + 9x^2 + 11x - 21$.

KACY Summer League

KACY–I 49. Factorize $x^3(x^2 - 7)^2 - 36x$.

1.2.2 Two–Variable Identities

KACY Summer League

(Positive Double–Variable Identity) 50. Factorize $x^2 + 2xy + y^2$.

KACY Summer League

(Negative Double–Variable Identity) 51. Factorize $x^2 - 2xy + y^2$.

KACY Summer League

(Difference of Squares Identity) 52. Factorize $x^2 - y^2$.

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(n^{th} Positive Double–Variable Identity) 53. Factorize $x^n + y^n$ for odd n.

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(n^{th} Negative Double–Variable Identity) 54. Factorize $x^n - y^n$ for all n.

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(2^{kth} Negative Double–Variable Identity) 55. Factorize $x^{2^k} - y^{2^k}$ for all k.

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(Sophie Germain Identity) 56. Factorize $x^4 + 4y^4$.

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(Sophie Parker Identity) 57. Factorize $x^4 + x^2y^2 + y^4$.

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KACY–I 58. Factorize $x^4 + y^4 + (x + y)^4$.

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KACY–I 59. Factorize $x^4 + y^4 + (x - y)^4$.

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KACY–I 60. Factorize $(x + y)^3 - x^3 - y^3$.

KACY Summer League

KACY–I 61. Factorize $(x + y)^5 - x^5 - y^5$.

KACY Summer League

KACY–I 62. Factorize $(x + y)^7 - x^7 - y^7$.

KACY Summer League

KACY–I 63. Show that $(x + y)^n - x^n - y^n$ always has a factor of

$$nxy(x + y)(x^2 + xy + y^2)^2,$$

if $n = 6k + 1$ for some integer $k \geq 1$.

KACY Summer League

KACY–I 64. Factorize $4(x^2 + xy + y^2)^3 - 27x^2y^2(x + y)^2$.

1.2.3 Three–Variable Identities

KACY Summer League

KACY–I 65. Factorize $x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$.

KACY Summer League

KACY–I 66. Factorize $(xy^3 + yz^3 + zx^3) - (x^3y + y^3z + z^3x)$.

KACY Summer League

KACY–I 67. Factorize $(x^2y^3 + y^2z^3 + z^2x^3) - (x^3y^2 + y^3z^2 + z^3x^2)$.

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KACY–I 68. Factorize $x^3 + y^3 + z^3 + (x + y)^3 + (y + z)^3 + (z + x)^3$.

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KACY–I 69. Factorize $(x + y)(y + z)(z + x) + xyz$.

KACY Summer League

KACY–I 70. Factorize $xy(x + y) + yz(y - z) - xz(x + z)$.

KACY Summer League

KACY–I 71. Factorize $2a^2b + 4ab^2 - a^2c + ac^2 - 4b^2c + 2bc^2 - 4abc$.

KACY Summer League

KACY–I 72. Factorize $a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2$.

KACY Summer League

KACY–I 73. Factorize $x^2y^2z^2 + (x^2 + yz)(y^2 + zx)(z^2 + xy)$.

KACY Summer League

KACY–I 74. Factorize $y(x - 2z)^2 + 8xyz + x(y - 2z)^2 - 2z(x + y)^2$.

KACY Summer League

KACY–I 75. Factorize $(a + b + c)^3 - a^3 - b^3 - c^3$.

KACY Summer League

KACY–I 76. Factorize $(ab + bc + ca)(a + b + c) - abc$.

KACY Summer League

KACY–I 77. Factorize $(xy^2 + yz^2 + zx^2) - (x^2y + y^2z + z^2x)$.

KACY Summer League

KACY–I 78. Factorize $(x - y)^3 + (y - z)^3 + (z - x)^3$.

KACY Summer League

KACY–I 79. Define $g(x, y, z) = x^2 + y^2 + z^2 - xy - yz - zx$. Show that

$$(x - y)^2 + (y - z)^2 + (z - x)^2,$$

is divisible by $g(x, y, z)$ and that

$$(x - y)^4 + (y - z)^4 + (z - x)^4,$$

is divisible by $(g(x, y, z))^2$.

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KACY–I 80. Factorize $(x-y)^5+(y-z)^5+(z-x)^5$.

KACY Summer League

KACY–I 81. Factorize $(x-y)^7+(y-z)^7+(z-x)^7$.

KACY Summer League

KACY–I 82. Factorize $(x^2+y^2+z^2)^2-2(x^4+y^4+z^4)$.

KACY Summer League

KACY–I 83. Factorize $x^3+y^3+z^3-3xyz$.

KACY Summer League

KACY–I 84. Factorize

$$(a^2-bc)^3+(b^2-ac)^3+(c^2-ab)^3-3(a^2-bc)(b^2-ac)(c^2-ab).$$

KACY Summer League

KACY–I 85. Factorize $(x+y+z)^5-x^5-y^5-z^5$.

KACY Summer League

KACY–I 86. Factorize $a^3(x-y)+x^3(a-y)+y^3(a-x)$.

KACY Summer League

KACY–I 87. Factorize $a^2b^2(b-a)+b^2c^2(c-b)+c^2a^2(a-c)$.

KACY Summer League

KACY–I 88. Factorize $8x^3(y+z) - y^3(z+2x) - z^3(2x-y)$.

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KACY–I 89. Factorize $x^2y + xy^2 + x^2z + xz^2 + y^2z + yz^2 + 2xyz$.

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KACY–I 90. Factorize $x^2y + xy^2 + x^2z + xz^2 + y^2z + yz^2 + 3xyz$.

KACY Summer League

KACY–I 91. Factorize $(x^2+y^2)^3 + (z^2-x^2)^3 - (y^2+z^2)^3$.

KACY Summer League

KACY–I 92. Factorize $x^3(y-z) + y^3(z-x) + z^3(x-y)$.

KACY Summer League

KACY–I 93. Factorize $x^3(z-y^2) + y^3(x-z^2) + z^3(y-x^2) + xyz(xyz-1)$.

KACY Summer League

2006 Korea 94. Find the number of positive integer triples (a, b, c) such that

$$\frac{a^2+b^2-c^2}{ab} + \frac{b^2+c^2-a^2}{bc} + \frac{c^2+a^2-b^2}{ca} = 2 + \frac{15}{abc}.$$

KACY Summer League

KACY–I 95. Factorize

$$[(x^2+y^2)(a^2+b^2) + 4abxy]^2 - 4[xy(a^2+b^2) + ab(x^2+y^2)]^2.$$

1.3 Exercises in Expressions

Here, we study algebraic expressions in the general form of Functional Expressions first, and then discuss problems of the special case of Polynomial Expressions.

1.3.1 Functional Expressions

Remember that a function $f : A \to B$ takes elements of the set A as input and assigns to them elements of the set B as outputs. We study simple functional expressions here and reserve the more advanced functional equations for a later chapter.

KACY Summer League

KACY–I 96. If $f(x) = x^2 - 2x$, find $f(2x + 1)$.

KACY Summer League

KACY–I 97. If

$$f(x) = x + \frac{1}{x},$$

find $f(f(x))$.

KACY Summer League

KACY–I 98. If

$$f(x) = \frac{x-1}{x+1},$$

find $f(f(f(x))) \cdot f(x)$.

KACY Summer League

KACY–I 99. If

$$f\left(\frac{x}{x+1}\right) = x^2,$$

find $f(x)$.

KACY Summer League

KACY–I 100. Write $x^3 - 3x + 4$ as a sum in terms of exponents of $(x + 2)$.

KACY Summer League

KACY–I 101. If $f(x) = ax^2 + bx + c$, what is the value of the following expression?

$$g(x) = f(x+3) - 3f(x+2) + 3f(x+1) - f(x).$$

KACY Summer League

KACY–I 102. Find a function in the form of $f(x) = a + bc^x$ such that

$$f(0) = 15, f(2) = 30, f(4) = 90.$$

KACY Summer League

KACY–I 103.

(a) Consider a linear function $f(x) = ax + b$. If the inputs $x = x_n$ (for $n = 1, 2, 3, \dots$) of the function form an arithmetic progression, then what kind of progression do the outputs $y_n = f(x_n)$ form?

(b) For $a > 0$, consider an exponential function $f(x) = a^x$. If the inputs $x = x_n$ (for $n = 1, 2, 3, \dots$) of the function form an arithmetic progression, then what kind of progression do the outputs $y_n = f(x_n)$ form?

KACY Summer League

KACY–I 104. For $a > 0$, define

$$f(x) = \frac{1}{2}\left(a^x + a^{-x}\right).$$

Find an alternative form for $f(x) + f(y)$ as a product.

KACY Summer League

KACY–I 105. If $f(x) + f(y) = f(z)$, find z in terms of x and y so that:

(a) $f(x) = ax$;

(b) $f(x) = \frac{1}{x}$;

(c) $f(x) = \arctan x$, where $|x| < 1$;

(d) $f(x) = \log \frac{1+x}{1-x}$.

KACY Summer League

KACY–I 106. Assuming

$$f(x) = \frac{1}{1-x},$$

Find $f(f(x))$ and $f(f(f(x)))$.

KACY Summer League

KACY–I 107. Assuming

$$f(x+1) = x^2 - 3x + 2,$$

Find $f(x)$.

KACY Summer League

KACY–I 108. Assuming

$$f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2},$$

Find $f(x)$ for $|x| \geq 2$.

KACY Summer League

KACY–I 109. If for $x > 0$,

$$f\left(\frac{1}{x}\right) = x + \sqrt{1+x^2},$$

Find $f(x)$.

KACY Summer League

KACY–I 110. If

$$f\left(\frac{2x-1}{x+2}\right) = \frac{3x^2-3x+7}{(x+2)^2},$$

Find $f(x)$.

KACY Summer League

KACY–I 111. If we define

$$f_n(x) = \underbrace{f(f(f(\dots(f}_{n \text{ times}}(x))\dots))),$$

find $f_n(x)$ given that

$$f(x) = \frac{x}{\sqrt{1+x^2}}.$$

KACY Summer League

KACY–I 112. The function $f(x)$ is defined for $x > 1$ as

$$f(x) = \log(x + \sqrt{x^2-1}).$$

Find $f(2x^2-1)$ and $f(4x^3-3x)$ in terms of $f(x)$.

KACY Summer League

KACY–I 113. If we know that

$$f\left(\frac{x+2}{x-2}\right) = \frac{x^2+4x+4}{8x},$$

Find $f(x)$.

KACY Summer League

KACY–I 114. If we know that

$$f(x) = \frac{4-x}{2x-4}, \text{ and}$$
$$f(\alpha+x)\cdot f(\alpha-x) = \text{constant},$$

Find α and the constant.

KACY Summer League

KACY–I 115. Consider the function

$$f(x) = \frac{a(x-b)(x-c)}{(a-b)(a-c)} + \frac{b(x-c)(x-a)}{(b-c)(b-a)} + \frac{c(x-a)(x-b)}{(c-a)(c-b)}.$$

Find the roots of $f(x) - x = 0$ and conclude that $f(x) = x$ for all x.

KACY Summer League

KACY–I 116. If n is an odd integer, $a^2 \neq 1$, and $f(x)$ is defined for all x by

$$af(x^n) + f(-x^n) = bx,$$

Find $f(x)$.

KACY Summer League

KACY–I 117.

(a) Find two roots for the following equation:

$$f(x) = f\left(\frac{x+8}{x-1}\right).$$

(b) If $f(x) = x^2 - 12x + 3$, find all the roots of the equation given in (a).

KACY Summer League

KACY–I 118. Find $f(x, y)$ given that

$$f\left(x+y, \frac{y}{x}\right) = x^2 - y^2.$$

KACY Summer League

KACY–I 119. For a real x and positive integer n, the function $F_n(x)$ is recursively defined by $F_1(x) = \cos x$ and

$$F_{n+1}(x) + F_n(x+1) = F_n(x).$$

Find $F_n(x)$ for different values of n modulo 4.

KACY Summer League

KACY–I 120. If for all $-\frac{1}{2} < x < \frac{1}{2}$, we have

$$f\left(\frac{x}{x^2+1}\right) = \frac{x^4+1}{x^2},$$

find $f(x)$.

1.3.2 Polynomial Expressions

We are now going to study special types of functional expressions called **polynomial expressions**, which are so important that the whole of Chapter 1 is dedicated to them.

KACY Summer League

KACY–I 121. What is the sum of coefficients of the following polynomial after expansion?

$$p(x) = (12x^3 - 54x^2 + 19x + 22)^{71}.$$

KACY Summer League

KACY–I 122. Given a polynomial

$$p(x) = (x + a)(x + a^2) \cdots (x + a^n),$$

find an alternative factorization for $a^n(x + 1)p(x)$.

KACY Summer League

KACY–I 123. Given

$$p(x) = (x - a)(x - b)(x - c),$$

Find alternative expressions for $p(a+b) \cdot p(b+c) \cdot p(c+a)$ and $p(-a) \cdot p(-b) \cdot p(-c)$.

KACY Summer League

KACY–I 124. (a) Write the given polynomial $p(x)$ as a sum of descending exponents of $(x - 1)$:

$$p(x) = 6x^4 + 19x^3 - 17x^2 - 72x - 36.$$

(b) Solve $f(x) = 0$.

KACY Summer League

KACY–I 125. If $a_0, a_1, \ldots, a_{50}$ are the coefficients of the polynomial

$$p(x) = (1 + x + x^2)^{50},$$

determine whether the sum $a_0 + a_2 + \cdots + a_{50}$ is odd or even.

KACY Summer League

KACY–I 126. Let $p(x) = x^2 + ax + b$ be a quadratic polynomial in which a and b are integers. Find all integers n for which there exists an integer m such that $p(n)p(n+1) = p(m)$.

KACY Summer League

KACY–I 127. Let $p(x) = x^3 + ax^2 + bx + c$ and $q(x) = x^3 + bx^2 + cx + a$ be polynomials with integer coefficients and $c \neq 0$. If we know that $p(1) = 0$ and that the roots of $q(x)$ are squares of roots of $p(x)$, find $a^{2023} + b^{2023} + c^{2023}$.

KACY Summer League

KACY–I 128. Let $p_k(x) = x^k + 1/x^k$. If x is a non-zero real number such that both $p_4(x)$ and $p_5(x)$ are rational numbers, prove that $p_1(x)$ is also rational.

KACY Summer League

KACY–I 129. Let $p(x) = x^2 + ax + b$ and $q(x) = x^2 + cx + d$ be quadratic polynomials with integer coefficients such that $a \neq c$ and there exist integers $m \neq n$ for which $p(m) = q(n)$ and $p(n) = q(m)$. What is the parity of $a - c$?

KACY Summer League

KACY–I 130. Given three real numbers x, y, z such that $x + y + z = 0$ and $xy + yz + zx = -3$, find the value of $x^3y + y^3z + z^3x$.

1.4 Polynomial Division

1.4.1 Remainder of Polynomial Division

KACY Summer League

KACY–I 131. Find a and b such that $x^4 - 3x^3 + ax + b$ is divisible by $x^2 - 2x + 4$.

KACY Summer League

KACY–I 132. What is the quotient of division of $nx^{n+1} - (n+1)x^n + 1$ by $(x-1)^2$?

KACY Summer League

KACY–I 133. Find m such that $x^4 + ma^2x^2 + a^4$ is divisible by $x^2 - ax + a^2$, and find the quotient of the division.

KACY Summer League

KACY–I 134. Find a and b such that $a(x-2)^n + b(x-1)^n - 1$ is divisible by $x^2 - 3x + 2$, and find the quotient of the division.

KACY Summer League

KACY–I 135.

(a) If $p(1) = 1$ and $p(3) = -4$, what is the remainder of the division of $p(x)$ by $(x-1)(x-3)$?

(b) If $p(a) = A$ and $p(b) = B$, find the remainder of the division of $p(x)$ by $(x-a)(x-b)$.

KACY Summer League

KACY–I 136.

(a) If $p(-1) = 1$ and $p(2) = -3$, and $p(-2) = 2$, what is the remainder of the division of $p(x)$ by $(x+1)(x^2-4)$?

(b) If $p(a) = A$ and $p(b) = B$, and $p(c) = C$, find the remainder of the division of $p(x)$ by $(x-a)(x-b)(x-c)$.

KACY Summer League

KACY–I 137.

(a) Find the polynomial $p(x)$ of degree 4 such that it is divisible by $x+2$, the sum of its coefficients is 15, and it has a remainder of 5, -13, and 92 upon division by $x+1$, $x+3$, and $x-2$, respectively.

(b) Find the polynomial $q(x)$ of degree 3 such that it is divisible by $x+1$, the sum of its coefficients is 2, and it has a remainder of $1-x$ upon division by x^2+1.

KACY Summer League

KACY–I 138.

(a) Find the remainder of the division of $p(x) = x^{4a} + x^{4b+1} + x^{4c+2} + x^{4d+3}$ by $x^3 + x^2 + x + 1$, where a, b, c, d, are positive integers.

(b) Find the remainder of the division of $q(x) = x^{na_1} + x^{na_2+1} + x^{na_3+2} + \cdots + x^{na_n+(n-1)}$ by $x^{n-1} + x^{n-2} + x + 1$, where $a_1, a_2, \ldots, a_n$, are positive integers.

KACY Summer League

KACY–I 139. If $p(x) = x^4 + px^2 + qx + a^2$ is divisible by $x^2 - 1$, find the remainder of the division of $p(x)$ by $x^2 - a^2$.

KACY Summer League

KACY–I 140. If $p(x) = x^3 + px + q$ has a remainder of β and α upon division by $x - \alpha$ and $x - \beta$, respectively,

(a) find $(\alpha + \beta)(\alpha\beta + 1)$ and $\alpha^2 + \beta^2 + \alpha\beta$ in terms of p and q.

(b) find α and β if $p = -22$ and $q = -19$.

KACY Summer League

KACY–I 141.

(a) For each positive integer n, define $p_n(x, y, z) = (x + y + z)^n - x^n - y^n - z^n$. Find all positive integers m such that $p_m(x, y, z)$ is divisible by $p_3(x, y, z)$.

(b) For each positive integer n, define $q_n(x, y) = x^n - y^n$. For what values of a and b, is $q_a(x, y)$ divisible by $q_b(x, y)$?

KACY Summer League

KACY–I 142. Define

$$\begin{aligned} p_n(x) &= x^{2n-2} + x^{2n-4} + \cdots + x^4 + x^2 + 1, \\ q_n(x) &= x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1. \end{aligned}$$

Find all n for which $p_n(x)$ is divisible by $q_n(x)$.

KACY Summer League

KACY–I 143. If $p(x, y)$ is a polynomial divisible by $x - y$ such that $p(x, y) = p(y, x)$, then find the remainder of division of $p(x, y)$ by $(x - y)^2$.

KACY Summer League

KACY–I 144.

(a) For which positive integers m is $x^{2m} + x^m + 1$ divisible by $x^2 + x + 1$?

(b) Find positive integers m and n such that $x^m + x^n + 1$ divisible by $x^2 + x + 1$.

KACY Summer League

KACY–I 145. Let a, b, x, y be integers. If $p(x, y) = a^n b^n (x^{2n} + y^{2n})$ is divisible by $q(x, y) = xy(a^2 + b^2) - ab(x^2 + y^2)$, then find the remainder of division of $s(x, y) = x^n y^n (a^{2n} + b^{2n})$ by $q(x, y)$.

KACY Summer League

KACY–I 146. Find the polynomial $p(x)$ of degree 4 such that $p(x + 1)$ is divisible by $(x - 1)^2$ and $p(x - 1)$ is divisible by $(x + 1)^2$, and also $p(1) = 1$.

KACY Summer League

KACY–I 147. Find all polynomials $p(x)$ of degree 3 such that $p(x) + 2$ is divisible by $(x - 1)^2$ and $p(x) - 2$ is divisible by $(x + 1)^2$.

KACY Summer League

KACY–I 148. Find all polynomials $p(x)$ of degree 3 such that $p(x) + 2$ is divisible by $(x - 1)^2$ and $p(x) - 2$ is divisible by $(x + 1)^2$.

KACY Summer League

KACY–I 149. If p, q, r are the roots of the cubic equation $x^3 - 3px^2 + 3q^2 x - r^3 = 0$, then what is $p + q - 2r$?

KACY Summer League

KACY–I 150. Let $P_1(x) = ax^2 - bx - c$, $P_2(x) = bx^2 - cx - a$, and $P_3(x) = cx^2 - ax - b$ be three quadratic polynomials where a, b, c are non-zero real numbers. Suppose there exists a real number α such that $P_1(\alpha) = P_2(\alpha) = P_3(\alpha)$. What is $a + b - 2c$?

KACY Summer League

KACY–I 151. Let $P(x) = x^2 + ax + b$ be a quadratic polynomials with real coefficients. Suppose there exist real numbers α and β such that $P(\alpha) = \beta$ and $P(\beta) = \alpha$. Find the remainder of the division of $x^2 + ax + b - \alpha\beta$ by $x - b + \alpha\beta$.

KACY Summer League

KACY–I 152. Define a sequence $\langle f_0(x), f_1(x), f_2(x), \dots \rangle$ of functions by $f_0(x) = 1$, $f_1(x) = x$, and for $n \geq 1$,

$$(f_n(x))^2 - 1 = f_{n+1}(x) f_{n-1}(x).$$

Show that $f_n(x)$ is a polynomial with integer coefficients for all n.

KACY Summer League

KACY–I 153. Find all real values of a for which the equation $x^4 - 2ax^2 + x + a^2 - a = 0$ has all its roots real.

KACY Summer League

KACY–I 154. Find all real values of a for which the equation $x^2 + (a - 5)x + 1 = 3|x|$ has exactly three distinct real solutions in x.

KACY Summer League

KACY–I 155. For positive reals $a, b, c,$ which one of the following statements necessarily implies $a = b = c$? Justify your answer.

(I) $a(b^3 + c^3) = b(c^3 + a^3) = c(a^3 + b^3),$

(II) $a(a^3 + b^3) = b(b^3 + c^3) = c(c^3 + a^3),$

KACY Summer League

KACY–I 156. If a, b, c are non-zero real numbers such that

$$(ab + bc + ca)^3 = abc(a + b + c)^3,$$

prove that a, b, c are terms of a geometric progression.

KACY Summer League

2017 Ecuador 157. If we know that $x^2 - x - 1$ is a factor of the polynomial $ax^7 + bx^6 + 1$, where a and b are integers, find the value of $a - b$.

1.4.2 Greatest Common Factor of Polynomials

KACY Summer League

KACY–I 158. Find the greatest common factor of

a) $x^{91} + 1$ and $x^{65} + 1$;

b) $x^5 - 3x^3 + x^2 + 2x - 1$ and $x^6 - 2x^5 + x^4 - x^2 + 2x - 1$.

KACY Summer League

KACY–I 159. Write the following fraction in its simplest form:

$$\frac{x^6 + 3x^5 + x^4 + 4x^3 - 5x^2 - x + 1}{x^7 + 6x^5 + 7x^4 - 8x^3 - 5x^2 + 2x + 1}.$$

KACY Summer League

KACY–I 160. Solve the following equation for real x:

$$(x^2+x-2)^3+(2x^2-x-1)^3=27(x^2-1)^3.$$

KACY Summer League

KACY–I 161. Let $P(x)=x^3+ax^2+b$ and $Q(x)=x^3+bx+a$, where a and b are non-zero real numbers. Suppose that if x is a root of $P(x)$, then $1/x$ is a root of $Q(x)$.

a) Find a and b.

b) For a positive integer n, if the greatest common factor of $P(n)$ and $Q(n)$ is the same as the greatest common factor of 3 and $R(n)$, find $R(n)$.

KACY Summer League

2018 Romanian Masters in Mathematics 162. Determine whether there exist non-constant polynomials $P(x)$ and $Q(x)$ with real coefficients satisfying

$$P(x)^{10}+P(x)^9=Q(x)^{21}+Q(x)^{20}.$$

KACY Summer League

2020 Balkan TST 163. Let $P(x), Q(x)$ be distinct polynomials of degree 2020 with non-zero coefficients. Suppose that they have r common real roots counting multiplicity and s common coefficients. Determine the maximum possible value of $r+s$.

KACY Summer League

2013 IMO Shortlist 164. Let $m\neq 0$ be an integer. Find all polynomials $P(x)$ with real coefficients such that

$$(x^3-mx^2+1)P(x+1)+(x^3+mx^2+1)P(x-1)=2(x^3-mx+1)P(x)$$

for all real number x.

KACY Summer League

2022 ELMO 165. Find all monic non–constant polynomials P with integer coefficients for which there exist positive integers a and m such that for all positive integers $n \equiv a \pmod{m}$, $P(n)$ is non–zero and

$$2022 \cdot \frac{(n+1)^{n+1} - n^n}{P(n)},$$

is an integer.

KACY Summer League

2004 Russia 166. The polynomials $P(x)$ and $Q(x)$ are given. It is known that for a certain polynomial $R(x, y)$ the following identity holds for all x, y:

$$P(x) - P(y) = R(x, y)(Q(x) - Q(y)).$$

Prove that there is a polynomial $S(x)$ so that $P(x) = S(Q(x)) \quad \forall x$.

KACY Summer League

2016 USA TST 167. Let p be a prime number. Let $\mathbb{F}_p$ denote the integers modulo p, and let $\mathbb{F}_p[x]$ be the set of polynomials with coefficients in $\mathbb{F}_p$. Define $\Psi : \mathbb{F}_p[x] \to \mathbb{F}_p[x]$ by

$$\Psi\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} a_i x^{p^i}.$$

Prove that for nonzero polynomials $F, G \in \mathbb{F}_p[x]$,

$$\Psi(\gcd(F, G)) = \gcd(\Psi(F), \Psi(G)).$$

Here, a polynomial Q divides P if there exists $R \in \mathbb{F}_p[x]$ such that $P(x) - Q(x)R(x)$ is the polynomial with all coefficients 0 (with all addition and multiplication in the coefficients taken modulo p), and the gcd of two polynomials is the highest degree polynomial with leading coefficient 1 which divides both of them. A non-zero polynomial is a polynomial with not all coefficients 0. As an example of multiplication, $(x + 1)(x + 2)(x + 3) = x^3 + x^2 + x + 1$ in $\mathbb{F}_5[x]$.

KACY Summer League

2016 USA TSTST 168. Let $A = A(x, y)$ and $B = B(x, y)$ be two-variable polynomials with real coefficients. Suppose that $A(x, y)/B(x, y)$ is a polynomial in x for infinitely many values of y, and a polynomial in y for infinitely many values of x. Prove that B divides A, meaning there exists a third polynomial C with real coefficients such that $A = B \cdot C$.

1.5 Advanced Polynomial Problems & Theorems

1.5.1 Symmetric Sums

Symmetries of Things

Definition (Symmetric Sums σ_k)**.** Define σ_k, known as the k^{th} **symmetric sum** of the n numbers $x_1, x_2, \dots, x_n$ by

$$\sigma_k(x_1, x_2, \dots, x_n) = \sum_{1 \le n_1 < n_2 < \cdots < n_k \le n} x_{n_1} \cdot x_{n_2} \cdots x_{n_k}.$$

Then Vieta's Formulas for a polynomial $P(x) = a_n x^n + \cdots + a_0$ reduce to:

$$\sigma_k(x_1, x_2, \dots, x_n) = (-1)^k \frac{a_{n-k}}{a_n}, \qquad \text{for} \quad k = 1, 2, \dots, n.$$

Definition (Elementary Symmetric Polynomial)**.** An elementary symmetric polynomial is any multivariate (in more than one variable, like $x_1, x_2, \dots$) polynomial defined as an equivalent polynomial to symmetric sums σ_k.

Definition (n–Variable Symmetric Polynomial)**.** A polynomial in n variables is called an n–**Variable Symmetric Polynomial** if switching any two of the variables leaves the polynomial unchanged.

Theorem 9 (Fundamental Theorem of Symmetric Polynomials)**.** Any symmetric polynomial can be expressed as the sum and product of multiple (not necessarily different) symmetric polynomials.

KACY Summer League

1973 USAMO 169. Determine all roots, real or complex, of the system of simultaneous equations

$$\begin{cases} x + y + z = 3, \\ x^2 + y^2 + z^2 = 3, \\ x^3 + y^3 + z^3 = 3. \end{cases}$$

KACY Summer League

1983 AIME 170. Suppose that the sum of the squares of two complex numbers x and y is 7 and the sum of the cubes is 10. What is the largest real value that $x + y$ can have?

KACY Summer League

2003 AIME 171. Consider the polynomials

$$P(x) = x^6 - x^5 - x^3 - x^2 - x,$$

and

$$Q(x) = x^4 - x^3 - x^2 - 1.$$

Given that z_1, z_2, z_3, and z_4 are the roots of $Q(x) = 0$, find $P(z_1) + P(z_2) + P(z_3) + P(z_4)$.

KACY Summer League

2015 AIME 172. Let x and y be real numbers satisfying $x^4y^5 + y^4x^5 = 810$ and $x^3y^6 + y^3x^6 = 945$. Evaluate $2x^3 + (xy)^3 + 2y^3$.

KACY Summer League

2018 AIME 173. A real number a is chosen randomly and uniformly from the interval $[-20, 18]$. The probability that the roots of the polynomial

$$x^4 + 2ax^3 + (2a-2)x^2 + (-4a+3)x - 2$$

are all real can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

KACY Summer League

2019 AIME 174. Let x be a real number such that $\sin^{10} x + \cos^{10} x = \frac{11}{36}$. Then

$$\sin^{12} x + \cos^{12} x = \frac{m}{n},$$

where m and n are relatively prime positive integers. Find $m + n$.

KACY Summer League

1988 Canada 175. For what real values of k do $1988x^2 + kx + 8891$ and $8891x^2 + kx + 1988$ have a common zero?

KACY Summer League

2011 AIME I #9 176. Suppose x is in the interval $[0, \pi/2]$ and

$$\log_{24 \sin x}(24 \cos x) = \frac{3}{2}.$$

Find $24 \cot^2 x$.

KACY Summer League

2016 AIME 177. For $1 \leq i \leq 215$ let $a_i = \frac{1}{2^i}$ and $a_{216} = \frac{1}{2^{215}}$. Let $x_1, x_2, \ldots, x_{216}$ be positive real numbers such that

$$\sum_{i=1}^{216} x_i = 1 \quad \text{and} \quad \sum_{1\leq i<j\leq 216} x_i x_j = \frac{107}{215} + \sum_{i=1}^{216} \frac{a_i x_i^2}{2(1-a_i)}.$$

The maximum possible value of $x_2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

KACY Summer League

2019 AIME 178. For distinct complex numbers $z_1, z_2, \ldots, z_{673}$, the polynomial

$$(x-z_1)^3(x-z_2)^3 \cdots (x-z_{673})^3,$$

can be expressed as

$$x^{2019} + 20x^{2018} + 19x^{2017} + g(x),$$

where $g(x)$ is a polynomial with complex coefficients and with degree at most 2016. The value of

$$\left| \sum_{1\leq j<k\leq 673} z_j z_k \right|,$$

can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

KACY Summer League

KACY–I 179. Let n be an integer and let x_1, x_2, x_3 be the roots of $x^3+px+q=0$, where p and q are two real numbers. Find the expression $x_1^n + x_2^n + x_3^n$ in terms of p and q.

KACY Summer League

KACY–I 180. Find all solutions (x, y, z) of

$$\begin{cases} 10 & = x+y+z+w, \\ 30 & = x^2+y^2+z^2+w^2, \\ 100 & = x^3+y^3+z^3+w^3 = 100, \\ 24 & = xyzw. \end{cases}$$

KACY Summer League

KACY–I 181. Prove that the sum of the reciprocals of the 5 solutions to the following equation is 1/2001:

$$5x^5 + 4x^4 - 3x^3 + 2x^2 + x - 1 = 2000.$$

KACY Summer League

KACY–I 182. The roots of the equation $x^3 + ax + b = 0$ are α, β and γ. Find the equation with roots

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha}, \frac{\alpha}{\gamma} + \frac{\gamma}{\alpha} \qquad \text{and} \qquad \frac{\beta}{\gamma} + \frac{\gamma}{\beta}.$$

KACY Summer League

KACY–I 183. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two distinct collections of n positive integers, where each collection may contain repetitions. If the two collections of integers $a_i + a_j$ (where $1 \le i < j \le n$) and $b_i + b_j$ (where $1 \le i < j \le n$) are the same, then show that n is a power of 2.

KACY Summer League

KACY–I 184. Assume that a, b, c, d are roots of the equation

$$x^4 + 120x^3 + 1279x^2 + 11x + 9 = 0.$$

Also assume that

$$\frac{abc}{d}, \quad \frac{abd}{c}, \quad \frac{acd}{b}, \quad \text{and} \quad \frac{bcd}{a},$$

are the roots of

$$x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0.$$

Find $a_1 + a_2 + a_3 + a_4$.

KACY Summer League

KACY–I 185. Let a, b, c be non–zero real numbers such that $a+b+c = 0$. Prove that

$$\frac{(a^3 + b^3 + c^3)^2 (a^4 + b^4 + c^4)}{(a^5 + b^5 + c^5)^2} = \frac{18}{25}.$$

KACY Summer League

2017 PUMaC 186. Together, Kenneth and Ellen pick a real number a. Kenneth subtracts a from every thousandth root of unity (that is, the thousand complex numbers ω for which $\omega^{1000} = 1$) then inverts each, then sums the results. Ellen inverts every thousandth root of unity, then subtracts a from each, and then sums the results. They are surprised to find that they actually got the same answer! How many possible values of a are there?

KACY Summer League

2019 IMO Shortlist 187. Let $x_1, x_2, \ldots, x_n$ be different real numbers. Prove that

$$\sum_{1\leqslant i\leqslant n} \prod_{j\neq i} \frac{1-x_ix_j}{x_i-x_j} = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

KACY Summer League

2014 HMIC 188. Let ω be a root of unity and f be a polynomial with integer coefficients. Show that if $|f(\omega)| = 1$, then $f(\omega)$ is also a root of unity.

1.5.2 Pool of Advanced Polynomial Theorems

Newton's Sums

Theorem 10 (Newton's Formulas on Symmetric Sums)**.** For a polynomial $P(x)$ of degree n, where

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

let $x_1, x_2, \ldots, x_n$ be the roots of $P(x) = 0$. Define the sum:

$$P_k = x_1^k + x_2^k + \cdots + x_n^k.$$

According to Newton's Formulas, and assuming $a_j = 0$ for $j < 0$,

$$\begin{aligned}
0 &= a_n P_1 + a_{n-1}, \\
0 &= a_n P_2 + a_{n-1}P_1 + 2a_{n-2}, \\
0 &= a_n P_3 + a_{n-1}P_2 + a_{n-2}P_1 + 3a_{n-3}, \\
&\vdots \\
0 &= a_n P_k + a_{n-1}P_{k-1} + \cdots + a_{n-k+1}P_1 + k \cdot a_{n-k}.
\end{aligned}$$

We also can write:

$$\begin{aligned}
P_1 &= \sigma_1, \\
P_2 &= \sigma_1 P_1 - 2\sigma_2, \\
P_3 &= \sigma_1 P_2 - \sigma_2 P_1 + 3\sigma_3, \\
P_4 &= \sigma_1 P_3 - \sigma_2 P_2 + \sigma_3 P_1 - 4\sigma_4, \\
P_5 &= \sigma_1 P_4 - \sigma_2 P_3 + \sigma_3 P_2 - \sigma_4 P_1 + 5\sigma_5, \\
&\vdots
\end{aligned}$$

Here, σ_n denotes the n^{th} elementary symmetric sum as defined before.

Theorem 11 (Intermediate Value Theorem)**.** Consider a continuous function $f : I \to \mathbb{R}$ for some interval $I = [a, b]$ (with $a < b$). Then, for all $c \in (f(a), f(b))$, we can find some real number k with $a < k < b$, such that $f(k) = c$.

Theorem 12 (Descartes' Rule of Signs)**.** Consider a polynomial $P(x)$ of degree $n \geq 1$, and write

$$P(x) = a_n \epsilon_n x^n + a_{n-1}\epsilon_{n-1}x^{n-1} + \cdots + a_1\epsilon_1 x + a_0\epsilon_0,$$

where $a_n > 0$ and $\epsilon_n \in \{-1, 0, 1\}$. Let m be the number of times $\epsilon_k \epsilon_{k-1} = -1$. Then, the number of positive roots, say p, (counting multiplicities) is at most m, and furthermore leaves the same remainder as m when divided by 2.

Corollary 2 (Number of Negative Roots)**.** The number of negative roots can be found by applying Descartes' Rule of Signs on $f(-x)$ instead.

Definition (Discriminant of a Polynomial)**.** For a polynomial $P(x)$ of degree n, where

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

let $x_1, x_2, \ldots, x_n$ be the roots of $P(x) = 0$. Define the discriminant Δ of $P(x)$ by

$$\Delta(P(x)) = a_n^{2n-2} \prod_{1 \le i < j \le n} (x_i - x_j)^2.$$

Definition (Matrix of Two Polynomials)**.** For any two polynomials $f(x)$ and $g(x)$, with $\deg f = m$ and $\deg g = n$, **the resultant** is the discriminant of the $(m+n) \times (m+n)$–matrix formed by writing n times the coefficients of $f(x)$ and m times the coefficients of $g(x)$.

Theorem 13 (Discriminant from Resultant)**.** The **discriminant** Δ of any polynomial $P(x)$ of degree n, where

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

is equal to

$$\Delta(P(x)) = \frac{(-1)^{\binom{n}{2}}}{a_n} R(P(x), P'(x)),$$

where $R(P(x), P'(x))$ is the **resultant** of the two polynomials $P(x)$ and its derivative $P'(x)$.

Corollary 3 (Quadratic Discriminant)**.** Verify, by using the Discriminant from Resultant formula for $P(x) = ax^2 + bx + c$ and $P'(x) = 2ax + b$:

$$\frac{-1}{a} \begin{vmatrix} a & b & c \\ 2a & b & 0 \\ 0 & 2a & b \end{vmatrix} = -\frac{ab^2 + 4a^2c - 2ab^2}{a} = b^2 - 4ac,$$

that the discriminant of the quadratic polynomial $ax^2 + bx + c$ is equal to $\Delta = b^2 - 4ac$.

Corollary 4 (Cubic Discriminant)**.** Prove that the discriminant of the cubic polynomial $ax^3 + bx^2 + cx + d$ is equal to

$$\Delta = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd.$$

Theorem 14 (Lagrange's Interpolation Theorem)**.** For any distinct complex numbers $x_0, x_1, \ldots, x_n$ and any complex numbers $y_0, y_1, \ldots, y_n$, there exists a unique polynomial $P(x)$ of degree less than or equal to n such that for all integers $0 \le i \le n$, with $P(x_i) = y_i$, and this polynomial is

$$P(x) = \sum_{i=0}^{n} y_i \frac{(x - x_0) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)}{(x_i - x_0) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)}.$$

Definition (Finite Differences of Polynomials)**.** The finite difference of a polynomial $P(x)$ is $\Delta^1 P(x) = P(x+1) - P(x)$.

Theorem 15 (Finite Difference Δ^1 as a Linear Operator)**.** If $\Delta^1(P(x))$ is defined as the finite difference operator of polynomial $P(x)$, that is, $\Delta^1 P(x) = P(x+1) - P(x)$, then Δ^1 is a linear operator. That is,

$$\begin{cases} \Delta^1 (P_1(x) + P_2(x)) & = \Delta^1 (P_1(x)) + \Delta^1 (P_2(x)), \\ \Delta^1 (k \cdot P(x)) & = k \cdot \Delta^1 (P(x)). \end{cases}$$

Definition (n^{th}–degree Finite Differences of Polynomials)**.** The n^{th}–degree finite difference of a polynomial $P(x)$ is

$$\Delta^n(P(x)) = \Delta^1\left(\Delta^{n-1}(P(x))\right) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}P(x+k).$$

Theorem 16 (Finite Differences with Degrees)**.** Let n be the degree of the polynomial $P(x)$. Then,

$$\begin{aligned}\Delta^n(P(x)) &= n!,\\ \Delta^{n+1}(P(x)) &= 0.\end{aligned}$$

Theorem 17 (Finite Difference Representation)**.** Every polynomial can be represented by each degree of its finite difference, that is, we can write every polynomial $P(x)$ as

$$P(x) = \sum_{m=0}^{\deg P}\binom{x-a}{m}\Delta^m(P(a)).$$

Theorem 18 (Polynomial Summation)**.** For all polynomials $P(x)$,

$$\sum_{k=1}^{n}P(k) = \sum_{m=0}^{\deg P}\binom{n}{m+1}\Delta^m(P(1)).$$

Theorem 19 (Geometric Series Polynomial Summation)**.** For any polynomial $P(x)$ and constant q

$$\sum_{k=1}^{n}P(k)q^{k-1} = f(n)q^n - f(0),$$

where

$$\begin{aligned}f(n) &= \frac{P(n)}{q-1} + \frac{1}{(q-1)^2}\sum_{k=1}^{\deg P}\frac{(-1)^k q^{k-1}}{(q-1)^{k-1}}\Delta^k(P(n))\\ &= \frac{1}{q-1}\sum_{k=1}^{\deg P}\left(\frac{-q}{q-1}\right)^k\Delta^k(P(n+1)).\end{aligned}$$

1.5.3 Pool of Advanced Polynomial Problems

KACY Summer League

2022 Brazil 189. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of integers numbers. Let $\Delta^1 a_n = a_{n+1} - a_n$ for a non-negative integer n. Define $\Delta^M a_n = \Delta^{M-1} a_{n+1} - \Delta^{M-1} a_n$. A sequence is *patriota* if there are positive integers k, l such that $a_{n+k} = \Delta^M a_{n+l}$ for all non-negative integers n. Determine, with proof, whether exists a sequence that the last value of M for which the sequence is *patriota* is 2022.

KACY Summer League

1999 Brazil TST 190. A sequence a_n is defined initially by $a_0 = 0$ and $a_1 = 3$, and then recursively for $n \geq 2$:

$$a_n = 8a_{n-1} + 9a_{n-2} + 16.$$

Find the least positive integer h such that $a_{n+h} - a_n$ is divisible by 1999 for all $n \geq 0$.

KACY Summer League

1983 IMO Shortlist 191. Let $(F_1, F_2, F_3, \dots)$ be the Fibonacci sequence, defined by the starting values $F_1 = 1$ and $F_2 = 1$ and the recurrence equation $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 1$. Let $P(x)$ be a polynomial of degree 990 such that $P(k) = F_k$ for all $k \in \{992, 993, \dots, 1981, 1982\}$. Show that

$$P(1983) = F_{1983} - 1.$$

KACY Summer League

2004 Fourth Mathlinks Contest 192. Let $m \geq 2n$ be two positive integers. Find a closed form for the following expression:

$$E(m,n) = \sum_{k=0}^{n} (-1)^k \frac{(m-k)!}{n!(m-k-n)!} \frac{n!}{k!(n-k)!}.$$

KACY Summer League

KACY–I 193. Assume that m and n are positive integers, and $1 \leq m \leq \phi(m)+n$. Prove that

$$m \mid \sum_{i=0}^{n} (-1)^i \binom{n}{i} i^m.$$

KACY Summer League

KACY–I 194. Assume $i \geq j \geq 1$. Prove that

$$\sum_{k=i}^{i+j} (-1)^k \frac{(k-1)!}{(k-i)!(k-j)!(i+j-k)!} = 0.$$

KACY Summer League

KACY–I 195. Prove that for all positive integers n, we have

$$\binom{n}{0} n^n - \binom{n}{1} (n-1)^n + \binom{n}{2} (n-2)^n - \ldots + (-1)^{n-1} \binom{n}{n-1} 1^n = n! \,.$$

KACY Summer League

KACY–I 196. Let n be a natural number (i.e., a non-negative integer). Prove that

$$\sum_{k=1}^{n} \frac{(-1)^k \cdot k}{2k-1} \binom{n}{k} \binom{n+k-1}{k-1} = -(n \mod 2)\,.$$

Here, for any integer a, the expression $a \mod 2$ means the remainder of a upon division by 2 (so, $a \mod 2 = 0$ if a is even, and $a \mod 2 = 1$ if a is odd).

KACY Summer League

KACY–I 197. Prove that for any polynomial $P(x)$ with degree $< n$, we have

$$\sum_{k=0}^{n} (-1)^n \frac{n!}{k!\,(n-k)!} P(k) = 0.$$

KACY Summer League

10490 AMM 198. Prove that for all $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \cdot \binom{n}{k} \sum_{j=1}^{k} \frac{H_j}{j} = \sum_{k=1}^{n} \frac{1}{k^3},$$

where, for every $j \in \mathbb{N}$,

$$H_j = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{j}.$$

KACY Summer League

KACY–I 199. Let P be a polynomial of degree n satisfying

$$P(k) = \binom{n+1}{k}^{-1} \qquad \text{for } k = 0, 1, ..., n.$$

Determine $P(n+1)$.

KACY Summer League

2011 Ibero American 200. This problem has two parts:

a) Prove that, for any positive integers $m \leq \ell$ given, there is a positive integer n and positive integers $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that the equality

$$\sum_{i=1}^{n} x_i^k = \sum_{i=1}^{n} y_i^k,$$

holds for every $k = 1, 2, \ldots, m-1, m+1, \ldots, \ell$, but does not hold for $k = m$.

b) Prove that there is a solution of the problem, where all numbers $x_1, \ldots, x_n, y_1, \ldots, y_n$ are distinct.

KACY Summer League

Komal 201. Prove that, for infinitely many positive integers n, there exists a polynomial $P(x)$ of degree n with real coefficients such that $P(1), P(2), \ldots, P(n+2)$ are different whole powers of 2.

KACY Summer League

Dadgarnia Finite Difference Identities 202. These problems were posted on AoPS as proposed by Alireza Dadgarnia:

a) Let $n > 1$ and $0 \leq m \leq n-2$ be integers. Prove that

$$\sum_{i=1}^{n}(-1)^{n-i}\binom{n-1}{i-1}i^{m} = 0.$$

b) For all positive integers n, prove that

$$\sum_{i=1}^{n}(-1)^{n-i}\binom{n-1}{i-1}i^{n-1} = (n-1)!.$$

c) For all positive integers n, prove that

$$\sum_{i=1}^{n}(-1)^{n-i}\binom{n-1}{i-1}i^{n} = \frac{(n+1)!}{2}.$$

d) For all positive integers n, prove that

$$\sum_{i=1}^{n}(-1)^{n-i}\binom{n-1}{i-1}i^{n+1} = \frac{(3n+1)(n+2)!}{24}.$$

e) For all positive integers n, prove that

$$\sum_{i=1}^{n}(-1)^{n-i}\binom{n-1}{i-1}i^{n+2} = \frac{n(n+1)(n+3)!}{48}.$$

KACY Summer League

Crux, by Max A. Alekseyev 203. For all integers $n > m \geq 0$, prove that:

$$\sum_{k=0}^{n}(-1)^{k}\cdot\tbinom{2n+1}{n-k}\cdot(2k+1)^{2m+1} = 0.$$

KACY Summer League

2020 Taiwan TST 204. Alice and Bob are stuck in quarantine, so they decide to play a game. Bob will write down a polynomial $f(x)$ with the following properties:

a) for any integer n, $f(n)$ is an integer;

b) the degree of $f(x)$ is less than 187.

Alice knows that $f(x)$ satisfies (a) and (b), but she does not know $f(x)$. In every turn, Alice picks a number k from the set $\{1, 2, \ldots, 187\}$, and Bob will tell Alice the value of $f(k)$. Find the smallest positive integer N so that Alice always knows for sure the parity of $f(0)$ within N turns.

KACY Summer League

2020 Tuymaada 205. The degrees of polynomials P and Q with real coefficients do not exceed n. These polynomials satisfy the identity

$$P(x)x^{n+1} + Q(x)(x+1)^{n+1} = 1.$$

Determine all possible values of $Q\left(-\frac{1}{2}\right)$.

KACY Summer League

2020 Indonesia 206. Determine all real-coefficient polynomials $P(x)$ such that

$$P(\lfloor x \rfloor) = \lfloor P(x) \rfloor,$$

for all real numbers x.

KACY Summer League

2019 Latvian TST for Balkan 207. Let $P(x)$ be a polynomial of degree n with real coefficients. For all $0 \le y \le 1$, we know that $|P(y)| \le 1$. Prove that

$$P\left(-\frac{1}{n}\right) \le 2^{n+1} - 1.$$

KACY Summer League

2021 USA TSTST 208. Let $q = p^r$ for a prime number p and positive integer r. Let $\zeta = e^{\frac{2\pi i}{q}}$. Find the least positive integer n such that

$$\sum_{\substack{1 \leq k \leq q \\ \gcd(k,p)=1}} \frac{1}{(1-\zeta^k)^n},$$

is not an integer the sum is over all $1 \leq k \leq q$ with p not dividing k).

1.6 Complex Numbers

Introduction to Complex Numbers

It is assumed that a high school student is aware of the set of positive integers (also called natural numbers) $\mathbb{N}$, the set of integers $\mathbb{Z}$, the set of rational numbers $\mathbb{Q}$, and the of real numbers $\mathbb{R}$. It is unfortunate that all the numbers in these sets, even all real numbers, are not enough to solve all polynomial equations. For instance, the simple equation $x^2 + 1 = 0$ does not have any real roots and that is where the imaginary unit $i = \sqrt{-1}$ comes from.

Definition (The Set of Complex Numbers $\mathbb{C}$)**.** If we allow $i = \sqrt{-1}$, then the set

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\},$$

is the set of complex numbers. Realize that we can represent each complex number as a pair (x, y) of real numbers, and that $\mathbb{R}$ (attained by plugging $y = 0$) is a subset of $\mathbb{C}$.

Definition (Real and Imaginary Parts of a Complex Number)**.** Let $z = x+iy$ be a complex number, $z \in \mathbb{C}$ for short. We call x the real part of z and y the imaginary part of z, and denote $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$.

Definition (Sum and Product of Complex Numbers)**.** For two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, define

$$\begin{aligned} z_1 + z_2 &= (x_1 + x_2) + i(y_1 + y_2), \\ z_1 - z_2 &= (x_1 - x_2) + i(y_1 - y_2), \\ z_1 z_2 &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1). \end{aligned}$$

These can be written in terms of $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ as

$$\begin{aligned} \mathrm{Re}(z_1 \pm z_2) &= \mathrm{Re}(z_1) \pm \mathrm{Re}(z_2), \\ \mathrm{Im}(z_1 \pm z_2) &= \mathrm{Im}(z_1) \pm \mathrm{Im}(z_2), \\ \mathrm{Re}(z_1 \cdot z_2) &= \mathrm{Re}(z_1) \cdot \mathrm{Re}(z_2) - \mathrm{Im}(z_1) \cdot \mathrm{Im}(z_2), \\ \mathrm{Im}(z_1 \cdot z_2) &= \mathrm{Re}(z_1) \cdot \mathrm{Im}(z_2) + \mathrm{Re}(z_2) \cdot \mathrm{Im}(z_1). \end{aligned}$$

In order to divide $z_1 = x_1 + iy_1$ by $z_2 = x_2 + iy_2 \neq 0$, we can write

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2},$$

or in terms of $\mathrm{Re}(z_{1,2})$ and $\mathrm{Im}(z_{1,2})$,

$$\begin{aligned} \mathrm{Re}\left(\frac{\mathrm{Re}(z_1) + \mathrm{Im}(z_1)}{\mathrm{Re}(z_2) + \mathrm{Im}(z_2)}\right) &= \frac{\mathrm{Re}(z_1) \cdot \mathrm{Re}(z_2) + \mathrm{Im}(z_1) \cdot \mathrm{Im}(z_2)}{(\mathrm{Re}(z_2))^2 + (\mathrm{Im}(z_2))^2}, \\ \mathrm{Im}\left(\frac{\mathrm{Re}(z_1) + \mathrm{Im}(z_1)}{\mathrm{Re}(z_2) + \mathrm{Im}(z_2)}\right) &= \frac{\mathrm{Im}(z_1) \cdot \mathrm{Re}(z_2) - \mathrm{Re}(z_1) \cdot \mathrm{Im}(z_2)}{(\mathrm{Re}(z_2))^2 + (\mathrm{Im}(z_2))^2}. \end{aligned}$$

1.6.1 Polar Representation of Complex Numbers

Complex Conjugate Definitions

Definition. The **polar representation** of a complex number $z = x + iy$ is given by $z = r(\cos\theta + i\sin\theta)$, so that

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta,$$

where θ is the angle between the x–axis and the vector formed by connecting the point z in the complex plane to the origin. In this definition, r is the absolute value of z, and θ is the angle or argument of z. Finally,

$$r = |z| = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \theta = \arg(z) = \arctan\left(\frac{y}{x}\right).$$

Definition (Complex Conjugates)**.** The two complex numbers $x+iy$ and $x-iy$ are **conjugate** of one another in the complex plane. We usually denote the complex conjugate of z by $\overline{z}$.

Theorem 20 (Complex Conjugate Theorem)**.** Prove that z is a root of a polynomial with real coefficients if and only if $\overline{z}$ is also the root of the same polynomial.

Theorem 21. For any three complex numbers z_1, z_2, z_3, prove the following:

a) The real and imaginary part of z_1 are

$$\mathrm{Re}(z_1) = \frac{z_1 + \overline{z_1}}{2} \qquad \text{and} \qquad \mathrm{Im}(z_1) = \frac{z_1 - \overline{z_1}}{2i}.$$

b) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$, and if $z_2 \neq 0$,

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

c) the triangle inequality for absolute values:

$$|z_1| - |z_2| \leq |z_1 - z_2| \qquad \text{and} \qquad |z_1 + z_2| \leq |z_1| + |z_2|.$$

d) $\overline{\overline{z_1}} = z_1$,

e) $\overline{z_1} = z_1 \iff z_1 \in \mathbb{R}$,

f) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$,

g) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$,

h) $|z_1|^2 = z_1 \cdot \overline{z_1}$.

Theorem 22. If we have

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad \text{and} \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2),$$

prove the following polar identities:

a) $z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$.

b) If $z_2 \neq 0$,
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)).$$

c) $z_1^n = (r_1^n)(\cos(n\theta_1) + i\sin(n\theta_1))$.

Theorem 23 (De Moivre's Theorem)**.** Prove that for all $\theta \in \mathbb{R}$ and positive integer n,
$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n.$$

Theorem 24 (Euler's Formula)**.** For all reals θ,
$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Definition (Roots of Unity)**.** The n^{th} roots of unity are roots of $z^n = 1$.

KACY Summer League

1995 AIME #5 209. For certain real values of a, b, c, and d, the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ has four non-real roots. The product of two of these roots is $13 + i$ and the sum of the other two roots is $3 + 4i$, where $i = \sqrt{-1}$. Find b.

KACY Summer League

2013 AIME I #10 210. There are nonzero integers a, b, r, and s such that the complex number $r + si$ is a zero of the polynomial $P(x) = x^3 - ax^2 + bx - 65$. For each possible combination of a and b, let $p_{a,b}$ be the sum of the zeroes of $P(x)$. Find the sum of the $p_{a,b}$'s for all possible combinations of a and b.

KACY Summer League

2013 AIME II #12 211. Let S be the set of all polynomials of the form $z^3 + az^2 + bz + c$, where a, b, and c are integers. Find the number of polynomials in S such that each of its roots z satisfies either $|z| = 20$ or $|z| = 13$.

KACY Summer League

2011 AIME II #8 212. Let $z_1, z_2, z_3, \ldots, z_{12}$ be the 12 zeroes of the polynomial $z^{12} - 2^{36}$. For each j, let w_j be one of z_j or iz_j. Then the maximum possible value of the real part of $\sum_{j=1}^{12} w_j$ can be written as $m + \sqrt{n}$ where m and n are positive integers. Find $m + n$.

KACY Summer League

2019 AIME II #8 213. The polynomial $f(z) = az^{2018} + bz^{2017} + cz^{2016}$ has real coefficients not exceeding 2019, and

$$f\left(\frac{1+\sqrt{3}i}{2}\right) = 2015 + 2019\sqrt{3}i.$$

Find the remainder when $f(1)$ is divided by 1000.

KACY Summer League

1996 AIME #11 214. Let P be the product of the roots of $z^6+z^4+z^3+z^2+1=0$ that have positive imaginary part, and suppose that $P = r(\cos\theta^\circ + i\sin\theta^\circ)$, where $0 < r$ and $0 \leq \theta < 360$. Find θ.

1.7 Number Theoretic Study of Polynomials

1.7.1 Essential Number Theoretic Theorems

Theorem 25 (Difference of Polynomials)**.** Let $P(x)$ be a polynomial with integer coefficients. Then, for all integers a and b with $a \neq b$, we have

$$a - b | P(a) - P(b).$$

KACY Summer League

1962 Russia 215. Prove that there does not exist a polynomial $P(x) = ax^3 + bx^2 + cx + d$ with integer coefficients such that $P(19) = 1$ and $P(62) = 2$.

KACY Summer League

1974 USA 216. For three distinct integers a, b, c and a polynomial $P(x)$ with integer coefficients, prove that not all three of the following relations can hold at the same time:

$$P(a) = b, \quad P(b) = c, \quad P(c) = a.$$

KACY Summer League

1975 USA 217. Find all polynomials $P(x)$ such that $P(0) = 0$ and for all x,

$$P(x) = \frac{1}{2}\left(P(x+1) + P(x-1)\right).$$

KACY Summer League

KACY–I 218. The polynomial $P(x)$ has degree n and for the values of $x = 0^2, 1^2, \ldots, n^2$, we know that $f(x)$ is an integer. Prove that $P(x)$ takes infinitely many integer values.

Eisenstein's Criterion & Extension

Theorem 26 (Eisenstein's Criterion)**.** Let $a_0, a_1, \ldots, a_n$ be integers. The Eisenstein's Criterion states that the polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

cannot be factored into the product of two non–constant polynomials if all the following three conditions hold:

a) p is a prime which divides each of $a_0, a_1, a_2, \ldots, a_{n-1}$;

b) a_n is not divisible by p; and

c) a_0 is not divisible by p^2.

Theorem 27 (Extended Eisenstein's Criterion)**.** Let $a_0, a_1, \ldots, a_n$ be integers. The Extended Eisenstein's Criterion states that the polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

has an irreducible factor of degree more than k if:

a) p is a prime which divides each of $a_0, a_1, a_2, \ldots, a_k$;

b) a_{k+1} is not divisible by p; and

c) a_0 is not divisible by p^2.

KACY Summer League

KACY–I 219. Prove that $x^{1383} + 2003$ is irreducible in $\mathbb{Q}[x]$.

KACY Summer League

KACY–I 220. Prove that the polynomial $1 + x + x^2 + \cdots + x^{p-1}$, where $p > 2$ is a prime number, is irreducible over $\mathbb{Z}[x]$.

KACY Summer League

1997 Iran Third Round 221. Let $P(x)$ be a polynomial with integer coefficients such that for two distinct integers a and b, we have

$$P(a) \cdot P(b) = -(a-b)^2.$$

Prove that $P(a) + P(b) = 0$.

KACY Summer League

1998 Romanian TST 222. Let

$$f_n(X) = (X^2 + X)^{2^n} + 1.$$

Prove, for all n, that $f_n(X)$ is irreducible over $\mathbb{Z}[X]$. If you can, prove Harazi's generalization as well: for any two integers a and b such that

$$\left(b - \frac{a^2}{4}\right)^{2^n} + 1 \qquad \text{is not a perfect square in } \mathbb{Q},$$

then $(X^2 + aX + b)^{2^n} + 1$ is irreducible in $\mathbb{Q}[X]$.

KACY Summer League

2014 Putnam 223. Show that for each positive integer n, all the roots of the polynomial

$$\sum_{k=0}^{n} 2^{k(n-k)} x^k$$

are real numbers.

Content of a Polynomial & Gauss's Lemma

Definition. The **content** of a polynomial with integer coefficients is the greatest common divisor of the polynomial's coefficients. In other words, when $a_0, a_1, \ldots, a_n$ are integers and

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

the content $c(P(x))$ is defined by

$$c(P) := \gcd(a_n, a_{n-1}, \ldots, a_1, a_0).$$

As an example, all monic polynomials have content equal to 1 because their leading coefficient is 1. Such polynomials $P(x)$ for which $c(P) = 1$ are called **primitive polynomials**.

Theorem 28. Let $P(x)$ and $Q(x)$ be polynomials with integer coefficients. Prove that the content of $P(x)Q(x)$ is equal to the product of contents of $P(x)$ and $Q(x)$.

Theorem 29 (Gauss's Primitive Polynomial Lemma)**.** If $P(x)$ and $Q(x)$ are primitive polynomials with integer coefficients, their product $P(x)Q(x)$ is also a primitive polynomial.

Theorem 30 (Gauss's Lemma)**.** Let $P(x)$ be a polynomial with integer coefficients which cannot be factorized into a product of two polynomials with integer coefficients. Prove that $P(x)$ cannot be decomposed into a product of two polynomials with rational coefficients either. In other words, $P(x)$ is irreducible over $\mathbb{Z}[x]$ if and only if it is irreducible over $\mathbb{Q}[x]$.

KACY Summer League

2019 Iran Third Round 224. Call a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with integer coefficients *primitive* if and only if $\gcd(a_n, a_{n-1}, \ldots a_1, a_0) = 1$.

a) Let $P(x)$ be a primitive polynomial with degree less than 1398 and S be a set of primes greater than 1398.Prove that there is a positive integer n so that $P(n)$ is not divisible by any prime in S.

b) Prove that there exist a primitive polynomial $P(x)$ with degree less than 1398 so that for any set S of primes less than 1398 the polynomial $P(x)$ is always divisible by product of elements of S.

KACY Summer League

2010 Romania TST 225. Let p be a prime number,let $n_1, n_2, \ldots, n_p$ be positive integer numbers, and let d be the greatest common divisor of the numbers $n_1, n_2, \ldots, n_p$. Prove that the polynomial

$$\frac{X^{n_1} + X^{n_2} + \cdots + X^{n_p} - p}{X^d - 1},$$

is irreducible in $\mathbb{Q}[X]$.

1.7.2 Modular Arithmetic for Polynomials

Modular Arithmetic of Polynomials

Definition (Polynomial Congruency)**.** Two polynomials $f(x)$ and $g(x)$ with integer coefficients are congruent modulo positive integer $m \geq 2$ if, assuming

$$f(x) - g(x) = c_n x^n + \cdots + c_1 x + c_0,$$

we have $m \mid c_i$ for all $i = 0, 1, 2, \ldots, m$. In other words, we have $f(x) \equiv g(x) \pmod{m}$ if and only if $a_i \equiv b_i$ for all i, where a_i and b_i are corresponding coefficients in $f(x)$ and $g(x)$, respectively.

Definition (Degree of Polynomial Modulo Integer)**.** Let $f(x) = a_n x^n + \cdots + a_0$ be a polynomial with integer coefficients and let $m \geq 2$ be an integer. The **degree** of $f(x) \pmod{m}$ is the largest i such that $m \nmid a_i$.

Definition (Roots of Polynomial Congruence Equations)**.** Let $f(x)$ be a polynomial with integer coefficients and let $m \geq 2$ be an integer. We say that c is a root of $f(x) \pmod{m}$ if and only if $m \mid f(c)$.

Theorem 31. Let p be a prime number and let n be the degree of $f(x)$ modulo p. Then, the equation $f(x) \equiv 0 \pmod{p}$ has at most n incongruent solutions modulo p.

Theorem 32 (Hensel's Lemma)**.** Let $f(x) = a_n x^n + \cdots + a_0$ be a polynomial with integer coefficients and let $p > 2$ be a prime. Also, let $P'(x)$ denote the derivative of $P(x)$. Suppose that x_1 is an integer such that $P(x_1) \equiv 0 \pmod{p}$ and $P'(x_1) \not\equiv 0 \pmod{p}$. Then, for any positive integer k, there exists an unique residue $x \pmod{p^k}$ such that $P(x_1^k) \equiv 0 \pmod{p^k}$ and $x \equiv x_1 \pmod{p}$.

KACY Summer League

KACY–I 226. Let $f(x) = a_n x^n + \cdots + a_0$ be a polynomial with integer coefficients where $|a_0|$ is a prime and

$$|a_0| > |a_1| + |a_2| + \cdots + |a_n|.$$

Prove that $f(x)$ is irreducible.

KACY Summer League

Perron's Criterion 227. Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a polynomial with integer coefficients where $a_0 \neq 0$, and

$$|a_{n-1}| > 1 + |a_{n-2}| + \cdots + |a_1| + |a_0|.$$

Prove that $P(x)$ is irreducible.

KACY Summer League

Cohn's Criterion 228. For a prime $p > 2$ and an integer $b \geq 2$, let $p = (\overline{p_n \cdots p_1 p_0})_b$ be the base–b representation of p (with $0 \leq p_i < b$ for $i = 0, 1, \ldots, n$, and $p_n \neq 0$). Prove that the polynomial

$$f(x) = p_n x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0,$$

is irreducible.

KACY Summer League

2003 Serbia TST 229. If $p(x)$ is a polynomial, denote by $p^n(x)$ the polynomial

$$p(\ldots(p(x))\ldots),$$

where p is iterated n times. Prove that the polynomial $p^{2003}(x) - 2p^{2002}(x) + p^{2001}(x)$ is divisible by $p(x) - x$.

KACY Summer League

2013 Serbia TST 230. Let $A(x)$ and $B(x)$ be the polynomials

$$A(x) = a_m x^m + \cdots + a_1 x + a_0 \qquad \text{and} \qquad B(x) = b_n x^n + \cdots + b_1 x + b_0,$$

where $a_m b_n \neq 0$. We say $A(x)$ and $B(x)$ are similar if the following conditions hold:

a) $n = m$,

b) There is a permutation π of the set $\{0, 1, \ldots, n\}$ such that $b_i = a_{\pi(i)}$ for each $i \in 0, 1, \ldots, n$.

Let $P(x)$ and $Q(x)$ be similar polynomials with integer coefficients. Given that $P(16) = 3^{2012}$, find the smallest possible value of $|Q(3^{2012})|$.

KACY Summer League

2007 China TST 231. Prove that for any positive integer n, there exists only n degree polynomial $f(x)$, satisfying $f(0) = 1$ and $(x+1)[f(x)]^2 - 1$ is an odd function.

KACY Summer League

2003 Romania TST 232. Let $f \in \mathbb{Z}[X]$ be an irreducible polynomial over the ring of integer polynomials, such that $|f(0)|$ is not a perfect square. Prove that if the leading coefficient of f is 1 (the coefficient of the term having the highest degree in f) then $f(X^2)$ is also irreducible in the ring of integer polynomials.

KACY Summer League

2000 Putnam 233. Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \cdots$ of integers such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for all $n \geq 0$. Prove that if there exists a positive integer m for which $a_m = 0$ then either $a_1 = 0$ or $a_2 = 0$.

KACY Summer League

Komal 234. Consider the coefficient x_n of x^n in $(x^2 + x + 1)^n$. Prove that for all primes p, we have p^2 dividing $x_p - 1$.

KACY Summer League

2007 MOP 235. Let $p(x)$ be a monic polynomial with integer coefficients. Show that there exist infinitely many positive integers k such that $p(x) - k$ is irreducible.

KACY Summer League

2007 Iran TST 236. Does there exist a a sequence $a_0, a_1, a_2, \ldots$ of positive integer, such that for each $i \neq j$, we have $\gcd(a_i, a_j) = 1$, and for each n, the polynomial $\sum_{i=0}^{n} a_i x^i$ is irreducible over $\mathbb{Z}[x]$?

KACY Summer League

1997 IMO Shortlist 237. Let p be a prime number and f an integer polynomial of degree d such that $f(0) = 0, f(1) = 1$ and $f(n)$ is congruent to 0 or 1 modulo p for every integer n. Prove that $d \geq p - 1$.

KACY Summer League

2005 IMO Shortlist 238. Let a, b, c, d, e, f be positive integers and let $S = a + b + c + d + e + f$. Suppose that the number S divides $abc + def$ and $ab + bc + ca - de - ef - df$. Prove that S is composite.

KACY Summer League

2002 IMO 239. Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers a such that

$$\frac{a^m + a - 1}{a^n + a^2 - 1},$$

is itself an integer.

KACY Summer League

2002 IMO Shortlist 240. Let P be a cubic polynomial given by $P(x) = ax^3 + bx^2 + cx + d$, where a, b, c, d are integers and $a \neq 0$. Suppose that $xP(x) = yP(y)$ for infinitely many pairs x, y of integers with $x \neq y$. Prove that the equation $P(x) = 0$ has an integer root.

KACY Summer League

2006 IMO 241. Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x))\ldots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

KACY Summer League

2005 USA TST 242. We choose random a unitary polynomial of degree n and coefficients in the set $1, 2, \ldots, n!$. Prove that the probability for this polynomial to be special is between 0.71 and 0.75, where a polynomial g is called special if for every $k > 1$ in the sequence $f(1), f(2), f(3), \ldots$ there are infinitely many numbers relatively prime with k.

KACY Summer League

2008 USA TST 243. Let n be a positive integer. Given an integer coefficient polynomial $f(x)$, define its signature modulo n to be the (ordered) sequence $f(1), \ldots, f(n)$ modulo n. Of the n^n such n-term sequences of integers modulo n, how many are the signature of some polynomial $f(x)$ if

a) n is a positive integer not divisible by the square of a prime.

b) n is a positive integer not divisible by the cube of a prime.

KACY Summer League

2009 China TST 244. Prove that for any odd prime number p, the number of positive integer n satisfying $p \mid n! + 1$ is less than or equal to $cp^{\frac{2}{3}}$, where c is a constant independent of p.

KACY Summer League

2002 USA TST 245. Let $p > 5$ be a prime number. For any integer x, define

$$f_p(x) = \sum_{k=1}^{p-1} \frac{1}{(px+k)^2}$$

Prove that for any pair of positive integers x, y, the numerator of $f_p(x) - f_p(y)$, when written as a fraction in lowest terms, is divisible by p^3.

KACY Summer League

2006 APMO 246. Let $p \geq 5$ be a prime and let r be the number of ways of placing p checkers on a $p \times p$ checkerboard so that not all checkers are in the same row (but they may all be in the same column). Show that r is divisible by p^5. Here, we assume that all the checkers are identical.

KACY Summer League

2004 Bay Area Math Olympiad 247. Find (with proof) all monic polynomials $f(x)$ with integer coefficients that satisfy the following two conditions:

a) $f(0) = 2004$;

b) If x is irrational, then $f(x)$ is irrational.

KACY Summer League

1997 USAMO 248. Prove that for any integer n, there exists a unique polynomial Q with coefficients in $\{0, 1, \ldots, 9\}$ such that $Q(-2) = Q(-5) = n$.

KACY Summer League

2007 USA TST 249. For a polynomial $P(x)$ with integer coefficients, $r(2i-1)$ (for $i = 1, 2, 3, \ldots, 512$) is the remainder obtained when $P(2i-1)$ is divided by 1024. The sequence

$$(r(1), r(3), \ldots, r(1023)),$$

is called the remainder sequence of $P(x)$. A remainder sequence is called complete if it is a permutation of $(1, 3, 5, \ldots, 1023)$. Prove that there are no more than 2^{35} different complete remainder sequences.

KACY Summer League

2008 Putnam 250. Let p be a prime number. Let $h(x)$ be a polynomial with integer coefficients such that $h(0), h(1), \ldots, h(p^2-1)$ are distinct modulo p^2. Show that $h(0), h(1), \ldots, h(p^3-1)$ are distinct modulo p^3.

KACY Summer League

2013 EGMO 251. Find all positive integers a and b for which there are three consecutive integers at which the polynomial

$$P(n) = \frac{n^5 + a}{b},$$

takes integer values.

KACY Summer League

2014 IMO Shortlist 252. Let $a_1 < a_2 < \cdots < a_n$ be pairwise coprime positive integers with a_1 being prime and $a_1 \geq n+2$. On the segment $I = [0, a_1 a_2 \cdots a_n]$ of the real line, mark all integers that are divisible by at least one of the numbers $a_1, \ldots, a_n$. These points split I into a number of smaller segments. Prove that the sum of the squares of the lengths of these segments is divisible by a_1.

KACY Summer League

2020 Iran TST 253. Let p be an odd prime number. Find all integer $\frac{p-1}{2}$–tuples $\left(x_1, x_2, \ldots, x_{\frac{p-1}{2}}\right)$ such that

$$\sum_{i=1}^{\frac{p-1}{2}} x_i \equiv \sum_{i=1}^{\frac{p-1}{2}} x_i^2 \equiv \cdots \equiv \sum_{i=1}^{\frac{p-1}{2}} x_i^{\frac{p-1}{2}} \pmod p.$$

1.8 Algebraic Systems

KACY Summer League

KACY–I 254.

(a) Find a way to solve $(x+a)^4 + (x+b)^4 = c$ for x.

(b) How many real values of x satisfy $(x+1)^6 + (x+5)^6 = 730$?

(c) Find a method to solve $(x+a)^{2n} + (x+b)^{2n} = c$, where n is a positive integer, assuming we can solve any degree–n polynomial equation.

KACY Summer League

KACY–I 255.

(a) If $a+b=c+d$, find a way to solve $(x+a)(x+b)(x+c)(x+d) = m$ for x.

(b) Solve $(x^2+6x+8)(x^2-8x+15) = 72$.

KACY Summer League

KACY–I 256. In an arithmetic progression with common difference d, we add d^4 to the product of four consecutive terms $a, a+d, a+2d$, and $a+3d$ in the progression. Find the square root of the result in terms of a and d.

KACY Summer League

KACY–I 257. Find all k such that the following equation has four simple real roots.

$$(x+b)(x+a+b)(x+2a+b)(x+3a+b) = k.$$

KACY Summer League

KACY–I 258. How many real roots does this equation have? Find them.

$$x(x-4)(x-2)(x-1)^2(x+2)+66=0.$$

KACY Summer League

KACY–I 259. If $a+c=\alpha$ and $b+d=\beta$, solve the equation

$$(ax+b)^4+(cx+d)^4=(\alpha x+\beta)^4.$$

KACY Summer League

KACY–I 260. Solve for real x:

$$(2-x)^4+(2x-1)^4=(x+1)^4.$$

KACY Summer League

KACY–I 261. For what values of a would all the roots of this equation be unreal?

$$2x^4+x^3-(3a+2)x^2+2x+a^2-1=0.$$

KACY Summer League

KACY–I 262. Solve

$$x^3-(3+\sqrt{3})x+3=0.$$

KACY Summer League

KACY–I 263. Solve

$$(ax^2 + bx + c)^2 = x^2(x^2 + bx + c).$$

KACY Summer League

KACY–I 264. Solve the following nineteen polynomial equations in x:

1. $x^4 + (x + \sqrt{2})^4 = 68,$
2. $x^6 + (x + 2)^6 = 2,$
3. $(x + 3)^3 + (x + 5)^3 = 8,$
4. $(\sqrt{x} + 1)^4 + (\sqrt{x} - 3)^4 = 256,$
5. $(x^2 + 3x + 2)(x^2 + 6x + 12) = 120,$
6. $x^4 + 2x^3 + 2x^2 + x = 42,$
7. $x^4 + 6x^3 + 7x^2 - 6x = 1,$
8. $\dfrac{\sqrt[n]{a + x}}{a} + \dfrac{\sqrt[n]{a + x}}{x} = b\sqrt[n]{x},$
9. $\sqrt[3]{a + \sqrt{x}} + \sqrt[3]{a - \sqrt{x}} = \sqrt[3]{b},$
10. $(x^2 + 2x - 12)^2 = x^2(3x^2 + 2x - 12),$
11. $(2x^2 - x - 6)^2 + 3(2x^2 + x - 6)^2 = 4x^2,$
12. $\dfrac{(x^2 + 1)^2}{x(x + 1)^2} = \dfrac{9}{2},$
13. $3x^4 - 20x^3 + 45x^2 - 40x + 12 = 0,$
14. $x^3 - 3abx + a^3 + b^3 = 0,$
15. $(x^2 - 16)(x - 3)^2 + 9x^2 = 0.$
16. $(x - 2)(x + 1)(x + 6)(x + 9) + 108 = 0,$
17. $(x^2 - 4)(x + 1)(x + 4)(x + 5)(x + 8) + 476 = 0,$
18. $\sqrt[m]{(x + 1)^2} - \sqrt[m]{(x - 1)^2} + \dfrac{3\sqrt[m]{x^2 - 1}}{2} = 0,$
19. $(a^2 - a)^2(x^2 - x + 1)^3 = (a^2 - a + 1)^3(x^2 - x)^2,$

KACY Summer League

KACY–I 265. Given a real number a, solve the following equation for x:

$$x^4 - 10x^3 - 2(a - 11)x^2 + 2(5a + 6)x + 2a + a^2 = 0.$$

KACY Summer League

KACY–I 266. Solve $x^3 + 2\sqrt{3}x^2 + 3x + \sqrt{3} - 1 = 0$ for x.

KACY Summer League

KACY–I 267. Solve the quintic equation $x^5 - 5x^3 + 5x - 1 = 0$.

KACY Summer League

KACY–I 268. Let a, b, c be given real numbers. Solve the following equation for x:

$$\frac{(x-a)^2 + (x-a)(x-b) + (x-b)^2}{(x-a)^2 - (x-a)(x-b) + (x-b)^2} = \frac{3c^2+1}{c^2+3}.$$

KACY Summer League

KACY–I 269. Given real numbers a, b, c, d, solve the following two equations:

a) $\dfrac{(x+a+b)^5 + (x+c+d)^5}{(x+a+c)^5 + (x+b+d)^5} = \dfrac{(a+b+c+d)^2}{(a-b+c-d)^2}$.

b) $\dfrac{(x+a+b)^5 + (x+c+d)^5}{(x+a+c)^5 + (x+b+d)^5} = \dfrac{(a+b-c-d)^5}{(a-b+c-d)^5}$.

KACY Summer League

KACY–I 270. Find the six roots of the sextic equation in the guise of a septic form:

$$x^7 + 7^7 = (x+7)^7.$$

KACY Summer League

KACY–I 271. Given real numbers a and b with $a \neq -b$, solve the following equation for x:

$$x^7 + a^7 + b^7 = (x+a+b)^7.$$

KACY Summer League

KACY–I 272. Find the real roots of the sextic equation

$$8x^6 - 16x^5 + 2x^4 + 12x^3 - 36x + 27 = 0.$$

KACY Summer League

KACY–I 273. Solve the equation $(6x+7)^3(3x+4)(x+1) = 6$ for x.

KACY Summer League

Quartic Equations & Completing the Squares 274. Solve the following three equations in x by the method of completing the squares:

1. $x^2 + \left(\dfrac{x}{x-1}\right)^2 = 8$,
2. $x^2(1+x)^2 + x^2 = 8(1+x)^2$,
3. $(1+x^2)^2 = 4x(1-x^2)$.

KACY Summer League

KACY–I 275. Solve $(x+1)^2 + (x+2)^3 + (x+3)^4 = 2$ for x.

KACY Summer League

KACY–I 276. For a positive integer n, solve the following equation for x:

$$(x-1)^3 + (x-2)^3 + \cdots + (x-n)^3 = 0.$$

KACY Summer League

2005 Switzerland TST 277. Let $n \geq 2$ be a positive integer. Prove that the polynomial

$$(x^2-1^2)(x^2-2^2)(x^2-3^2)\cdots(x^2-n^2)+1,$$

cannot be written as the product of two non-constant polynomials with integer coefficients.

KACY Summer League

2008 Switzerland TST 278. Let $P(x) = x^4 - 2x^3 + px + q$ be a polynomial with real coefficients whose roots are all real. Prove that the largest root of $P(x) = 0$ lies in the interval $[1, 2]$.

KACY Summer League

2009 Switzerland TST 279. For which positive integers n does there exist a polynomial $P(x)$ with integer coefficients such that $P(d) = (n/d)^2$ for all divisors d of n?

KACY Summer League

2010 Switzerland TST 280. Let $P(x)$ be a polynomial with real coefficients such that for all real x, we have

$$P(x) = P(1-x).$$

Prove that there exists a polynomial $Q(x)$ with real coefficients such that

$$P(x) = Q\left(x(1-x)\right).$$

KACY Summer League

2011 Switzerland TST 281. Find all non-zero polynomials $P(x)$ with real coefficients such that

$$P(P(k)) = \left(P(k)\right)^2, \qquad \text{for} \quad k = 0, 1, 2, \dots, (\deg P)^2.$$

KACY Summer League

2011 Switzerland TST 282. Let $a > 1$ be a positive integer, and let $f(x)$ and $g(x)$ be polynomials with integer coefficients. Suppose that there is a positive integer n_0 such that $g(n) > 0$ for all $n \geq n_0$, and

$$f(n) \mid a^{g(n)} - 1, \qquad \text{for all} \quad n \geq n_0.$$

Prove that f must be a constant polynomial.

KACY Summer League

2007 Ecuador TST 283. How many solutions to the equation

$$x^7 - 1 + x^3(x-1) = 0,$$

are not integers?

1.8.1 Trigonometric Tricks for Solving Algebraic Systems

KACY Summer League

KACY–I 284.

(a) Solve

$$\frac{3x-x^3}{1-3x^2}=\sqrt{3}.$$

(b) Show that

$$\begin{aligned}3\sqrt{3}&=\tan 20^\circ-\tan 40^\circ+\tan 80^\circ,\\ 3&=\tan 20^\circ\tan 40^\circ+\tan 40^\circ\tan 80^\circ-\tan 20^\circ\tan 80^\circ,\\ \sqrt{3}&=\tan 20^\circ\tan 40^\circ\tan 80^\circ.\end{aligned}$$

KACY Summer League

KACY–I 285.

(a) Solve

$$32x^5-40x^3+10x=1.$$

(b) Show that

$$\begin{aligned}0&=\cos 12^\circ-\cos 24^\circ-\cos 48^\circ+\cos 60^\circ+\cos 84^\circ,\\ \frac{1}{32}&=\cos 12^\circ\cos 24^\circ\cos 48^\circ\cos 60^\circ\cos 84^\circ,\\ 10&=\frac{1}{\cos 12^\circ}-\frac{1}{\cos 24^\circ}-\frac{1}{\cos 48^\circ}+\frac{1}{\cos 60^\circ}+\frac{1}{\cos 84^\circ}.\end{aligned}$$

KACY Summer League

KACY–I 286. If $0<a<1$ is a real number, solve

$$\left(\frac{1+a^2}{2a}\right)^x-\left(\frac{1-a^2}{2a}\right)^x=1.$$

KACY Summer League

KACY–I 287. Solve the following equation:

$$\cos^2\phi + \cos^2 2\phi - 2\cos\phi\cos 2\phi\cos 4\phi = \frac{3}{4}.$$

KACY Summer League

KACY–I 288. Solve the following sine equation for α:

$$\sin^2\alpha + \frac{1}{4}\sin^2 3\alpha = \sin\alpha\sin^2 3\alpha.$$

KACY Summer League

KACY–I 289. Let $x_1 < x_2 < x_3$ be the three roots of the cubic equation $x^3 - 3x + 1 = 0$. Prove that $x_2^2 - x_1 = 2$.

KACY Summer League

KACY–I 290. The equation

$$x^{10} + (13x - 1)^{10} = 0$$

has 10 complex roots $r_1, \overline{r_1}, r_2, \overline{r_2}, r_3, \overline{r_3}, r_4, \overline{r_4}, r_5, \overline{r_5}$, where the bar denotes complex conjugation. Find the value of

$$\frac{1}{r_1\overline{r_1}} + \frac{1}{r_2\overline{r_2}} + \frac{1}{r_3\overline{r_3}} + \frac{1}{r_4\overline{r_4}} + \frac{1}{r_5\overline{r_5}}.$$

1.8.2 Irrational Equations

KACY Summer League

KACY–I 291. Solve the following equation in x:

$$x^3 - 3x + (x^2 - 1)\sqrt{x^2 - 4} = 2.$$

KACY Summer League

KACY–I 292. Prove that $\sqrt[5]{2} - \sqrt[5]{2}^2 + \sqrt[5]{2}^3 - \sqrt[5]{2}^4$ is a root of $x^5 + 20x^3 + 20x^2 + 30x + 10$.

KACY Summer League

1997 Switzerland TST 293. Let v and w be distinct, randomly chosen solutions to the equation $z^{1997} - 1 = 0$. Determine the probability that

$$\sqrt{2 + \sqrt{3}} \le |v + w|.$$

KACY Summer League

2009 Ecuador TST 294. Let a, b, c be distinct rational numbers. Prove that the expression

$$\sqrt{\frac{1}{(a-b)^2} + \frac{1}{(b-c)^2} + \frac{1}{(c-a)^2}},$$

is rational.

KACY Summer League

2009 Ecuador TST 295. Simplify the expression

$$\sqrt[3]{2 + \sqrt{5}} + \sqrt[3]{2 - \sqrt{5}}.$$

KACY Summer League

2008 Finland 296. Solve for x:

$$\sqrt{17 + x - 8\sqrt{x+1}} + \sqrt{5 + x - 4\sqrt{x+1}} = 6.$$

KACY Summer League

2014 Finland 297. Assume that for real numbers x and y,

$$(x + \sqrt{x^2+1})(y + \sqrt{y^2+1}) = 1.$$

What values can the expression $x + y$ assume?

KACY Summer League

2016 Finland 298. Solve the equation

$$\sqrt{2+4x-2x^2}+\sqrt{6+6x-3x^2}=x^2-2x+6.$$

KACY Summer League

2017 Finland 299. Prove that if $a>0$, then $x=3a/4$ is a solution to

$$x=a-\sqrt{a^2-x\sqrt{x^2+a^2}}.$$

Rationalizing Irrational Equations of the Form $\sqrt[3]{u}\pm\sqrt[3]{v}=a$

KACY Summer League

KACY–I 300. Prove that $\sqrt[3]{u}\pm\sqrt[3]{v}=a$ is equivalent to

$$(u\pm v-a^3)^3=\mp 27a^3uv.$$

KACY Summer League

KACY–I 301. Solve the irrational equation $\sqrt[3]{2x-3}+\sqrt[3]{3x+2}=3$ for x.

This is a review of Problems 302 to 327:

KACY Summer League

Irrational Equations Collection 302. The following is a review of the 26 irrational equations in x, labeled a) to z) for the twenty six letters of the English alphabet. Question a) (Problem 302) is to solve for x the irrational equation $\sqrt{a+x} = a - \sqrt{x}$ given an appropriate real number a.

a) $\sqrt{a+x} = a - \sqrt{x}$,

b) $\dfrac{\sqrt{a+x}}{a} + \dfrac{\sqrt{a+x}}{x} = \sqrt{x}$,

c) $\sqrt{x-1} + 6\sqrt[4]{x-1} = 16$,

d) $\sqrt[3]{x} + \sqrt[3]{2x-3} = \sqrt[3]{12(x-1)}$,

e) $\sqrt[3]{a-x} + \sqrt[3]{b-x} = \sqrt[3]{a+b-2x}$,

f) $\sqrt[3]{x} + 2\sqrt[3]{x^2} = 3$,

g) $\sqrt{a+x} - \sqrt[3]{a+x} = 0$,

h) $\sqrt[5]{(7x-3)^3} + 8\sqrt[5]{(3-7x)^{-3}} = 7$,

i) $\dfrac{1-ax}{1+ax}\sqrt{\dfrac{1+bx}{1-bx}} = 1$,

j) $\sqrt[5]{16+\sqrt{x}} + \sqrt[5]{16-\sqrt{x}} = 2$,

k) $\sqrt[n]{a^k x^{n-k}} + \sqrt[n]{x^k a^{n-k}} = 2\sqrt{bx}$,

l) $\dfrac{\sqrt[n]{a-x}}{x^2} - \dfrac{\sqrt[n]{a-x}}{a^2} = \sqrt[n]{\dfrac{x^2}{a+x}}$,

m) $\sqrt{\dfrac{\sqrt[n]{a}-\sqrt[n]{x}}{\sqrt[n]{x^2}}} - \sqrt{\dfrac{\sqrt[n]{a}-\sqrt[n]{x}}{\sqrt[n]{a^2}}} = \sqrt[2n]{x}$,

n) $\sqrt{p+x} + \sqrt{p-x} = x$,

o) $3 + \sqrt{3+\sqrt{x}} = x$,

p) $\sqrt{\sqrt{5}+\sqrt{\sqrt{5}+x}} = x$,

q) $\sqrt[5]{a+\sqrt{x}} + \sqrt[5]{a-\sqrt{x}} = \sqrt[5]{2a}$,

r) $\sqrt[4]{a-x} + \sqrt[4]{b-x} = \sqrt[4]{a+b-2x}$,

s) $\sqrt{2x^2-1} + \sqrt{x^2-3x-2} = \sqrt{2x^2+2x+3} + \sqrt{x^2-x+2}$,

t) $\sqrt{2(1+x^2)} + 2(x-1) = 2a\sqrt{x}$,

u) $2\sqrt{x-1} + \sqrt{x+2} - 4 = 0$,

v) $2x - \sqrt{3-2x} - 3 = 0$,

w) $3\sqrt{x+6} - \sqrt{2-x} - 4 = 0$,

x) $2\sqrt{x-1} + \sqrt[3]{x} - 1 = 0$,

y) $\sqrt[4]{x-1} + 2\sqrt[3]{3x+2} - \sqrt{3-x} = 4$,

z) $\sqrt{2x-1} + \sqrt{x-2} - \sqrt{x+1} = 0$.

KACY Summer League

KACY–I 303. Given an appropriate real number a, solve the irrational equation

$$\frac{\sqrt{a+x}}{a} + \frac{\sqrt{a+x}}{x} = \sqrt{x}.$$

KACY Summer League

KACY–I 304. Solve the irrational equation $\sqrt{x-1} + 6\sqrt[4]{x-1} = 16$ for x.

KACY Summer League

KACY–I 305. Solve the irrational (third root) equation

$$\sqrt[3]{x} + \sqrt[3]{2x-3} = \sqrt[3]{12(x-1)}.$$

KACY Summer League

KACY–I 306. Given an appropriate real number a, solve the third root equation

$$\sqrt[3]{a-x} + \sqrt[3]{b-x} = \sqrt[3]{a+b-2x}.$$

KACY Summer League

KACY–I 307. Solve the irrational (third root) equation $\sqrt[3]{x} + 2\sqrt[3]{x^2} = 3$ for x.

KACY Summer League

KACY–I 308. Given an appropriate real number a, solve the irrational sixth root equation $\sqrt{a+x} - \sqrt[3]{a+x} = 0$ for x.

KACY Summer League

KACY–I 309. Solve the irrational fifth root equation

$$\sqrt[5]{(7x-3)^3} + 8\sqrt[5]{(3-7x)^{-3}} = 7.$$

KACY Summer League

KACY–I 310. Given appropriate real numbers a and b, solve the irrational square root equation

$$\frac{1-ax}{1+ax}\sqrt{\frac{1+bx}{1-bx}} = 1.$$

KACY Summer League

KACY–I 311. Solve the irrational tenth root equation

$$\sqrt[5]{16+\sqrt{x}} + \sqrt[5]{16-\sqrt{x}} = 2,$$

for real numbers x.

KACY Summer League

KACY–I 312. Given appropriate real numbers a and b, and positive integers $n > k \geq 2$, solve the irrational n^{th} root equation $\sqrt[n]{a^k x^{n-k}} + \sqrt[n]{x^k a^{n-k}} = 2\sqrt{bx}$.

KACY Summer League

KACY–I 313. Solve the irrational n^{th} root equation

$$\sqrt{\frac{\sqrt[n]{a} - \sqrt[n]{x}}{\sqrt[n]{x^2}}} - \sqrt{\frac{\sqrt[n]{a} - \sqrt[n]{x}}{\sqrt[n]{a^2}}} = \sqrt[2n]{x}.$$

KACY Summer League

KACY–I 314. Solve the irrational equation

$$\sqrt{\frac{\sqrt[n]{a} - \sqrt[n]{x}}{\sqrt[n]{x^2}}} - \sqrt{\frac{\sqrt[n]{a} - \sqrt[n]{x}}{\sqrt[n]{a^2}}} = \sqrt[2n]{x}$$

for x.

KACY Summer League

KACY–I 315. Given an appropriate real number p, solve the square root equation

$$\sqrt{p+x} + \sqrt{p-x} = x.$$

KACY Summer League

KACY–I 316. Solve the irrational equation $3 + \sqrt{3 + \sqrt{x}} = x$ for x.

KACY Summer League

KACY–I 317. Solve the irrational equation $\sqrt{\sqrt{5} + \sqrt{\sqrt{5} + x}} = x$ for x.

KACY Summer League

KACY–I 318. Given an appropriate real number a, solve the fifth root equation

$$\sqrt[5]{a + \sqrt{x}} + \sqrt[5]{a - \sqrt{x}} = \sqrt[5]{2a}.$$

KACY Summer League

KACY–I 319. Given appropriate real numbers a and b, solve the fourth root equation

$$\sqrt[4]{a-x}+\sqrt[4]{b-x}=\sqrt[4]{a+b-2x}.$$

KACY Summer League

KACY–I 320. Solve the following irrational square root equation in x:

$$\sqrt{2x^2-1}+\sqrt{x^2-3x-2}=\sqrt{2x^2+2x+3}+\sqrt{x^2-x+2}.$$

KACY Summer League

KACY–I 321. Given an appropriate real number a, solve the square root equation

$$\sqrt{2(1+x^2)}+2(x-1)=2a\sqrt{x}.$$

KACY Summer League

KACY–I 322. Solve the irrational square root equation $2\sqrt{x-1}+\sqrt{x+2}-4=0$.

KACY Summer League

KACY–I 323. Solve the irrational square root equation $2x-\sqrt{3-2x}-3=0$.

KACY Summer League

KACY–I 324. Solve the irrational square root equation $3\sqrt{x+6}-\sqrt{2-x}-4=0$.

KACY Summer League

KACY–I 325. Solve the irrational sixth root equation $2\sqrt{x-1}+\sqrt[3]{x}-1=0$.

KACY Summer League

KACY–I 326. Solve the twelfth root equation

$$\sqrt[4]{x-1}+2\sqrt[3]{3x+2}-\sqrt{3-x}=4.$$

KACY Summer League

KACY–I 327. Solve the irrational square root equation $\sqrt{2x-1}+\sqrt{x-2}-\sqrt{x+1}=0$.

1.8.3 Reciprocal Equations

Definition. We call an equation such as $F(x)=0$ a **reciprocal equation** in two cases: F is **positive reciprocal equation** if $F(\alpha)=F(1/\alpha)=0$, and it is a **negative reciprocal equation** if $F(\alpha)=F(-1/\alpha)=0$ for some α. In other words,

$$\begin{aligned}\text{Positive Reciprocal} &\iff f(x) \text{ Is the Same When } x \to \frac{1}{x},\\ \text{Negative Reciprocal} &\iff f(x) \text{ Is the Same When } x \to -\frac{1}{x},\end{aligned}$$

1.8.3.1 Positive Reciprocal Equations

KACY Summer League

KACY–I 328. Prove that reciprocal equations of odd degree are not really interesting. In other words, prove that

1. A negative reciprocal polynomial equation cannot be of odd degree.
2. Any positive reciprocal polynomial equation of odd degree has a root of either $+1$ or -1 because ±1 are the only reals equal to their reciprocals.

The previous question makes it clear that reciprocal equations of odd degree are not interesting, and we often mean a **reciprocal equation of even degree** when we speak of **reciprocal equations** in general.

Definition. A polynomial $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ is **palindromic** if $a_i=a_{n-i}$ for $i=0,1,\ldots,n$ and **antipalindromic** if $a_i=-a_{n-i}$ for $i=0,1,\ldots,n$.

KACY Summer League

KACY–I 329. Prove that a positive reciprocal polynomial equation (of even degree) has its coefficients ordered in a palindromic way and the study the case for negative reciprocal equations.

1.8.3.2 Negative Reciprocal Equations

KACY Summer League

KACY–I 330. Prove that for a negative reciprocal equation $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ (in which the same equation is obtained by changing x to $-1/x$),

1. If n is a multiple of 4, then the coefficients of the even exponents form a palindrome, that is, $a_i = a_{n-i}$ for even i. Furthermore, prove that the coefficients of odd exponents form are antipalindromic, that is, $a_i = -a_{n-i}$ for odd i.

2. If n is not divisible by 4 (i.e. it leaves a remainder of 2 in division by 4), then the coefficients of the even exponents are antipalindromic and the coefficients of the odd exponents are palindromic.

KACY Summer League

KACY–I 331. Solve the following palindromic equation in x:

$$2x^4 - 13x^3 + 24x^2 - 13x + 2 = 0.$$

KACY Summer League

KACY–I 332. Prove that $x^n \pm \dfrac{1}{x^n}$ can be written as a polynomial of degree n of $x \pm \dfrac{1}{x}$, and that in the case of positive reciprocal equations, we need $x + \dfrac{1}{x}$ as the new variable, whereas for negative reciprocal equations, the variable would be $x - \dfrac{1}{x}$.

KACY Summer League

KACY–I 333. Prove, using induction, that

$$\left(x - \frac{1}{x}\right)^2 + \left(x^2 - \frac{1}{x^2}\right)^2 + \cdots + \left(x^n - \frac{1}{x^n}\right)^2 = \frac{x^{2n+1} - \dfrac{1}{x^{2n}}}{x^2 - 1} - 2n - 1.$$

KACY Summer League

2000 Denmark (Georg Mohr) 334. Determine all possible values of $x + \frac{1}{x}$, where the real x satisfies the equation

$$x^4 + 5x^3 - 4x^2 + 5x + 1 = 0,$$

and solve this equation.

KACY Summer League

KACY–I 335. For a given real, non-zero number a, solve the following antipalindromic equation in x:

$$2ax^4 - (2a^2 + 3a - 2)x^3 + (3a^2 - 4a - 3)x + (2a^2 + 3a - 2)x + 2a = 0.$$

KACY Summer League

KACY–I 336. Solve the following equation in x:

$$2x^4 - 15x^3 + 35x^2 - 30x + 8 = 0.$$

KACY Summer League

KACY–I 337. Solve the following equation in x:

$$2x^4 + 7x^3 - 34x^2 - 21x + 18 = 0.$$

KACY Summer League

KACY–I 338. Prove that the reciprocals of roots of the cubic equation $x^3 - x + 1 = 0$ are roots of the quintic equation $x^5 + x + 1 = 0$.

KACY Summer League

2008 Ecuador TST 339. If $z + \frac{1}{z} = 1$, find the numerical value of

$$z^{2008} + \frac{1}{z^{2008}}.$$

1.8.4 Equations Containing the Floor & Absolute Value Function

KACY Summer League

KACY–I 340. Solve the equation $\lfloor x+2 \rfloor + \lfloor x \rfloor = 12$ for x.

KACY Summer League

KACY–I 341. Solve the equation $\lfloor 5x+3 \rfloor + \lfloor 7x+9 \rfloor = 18$ for x.

KACY Summer League

KACY–I 342. Solve the equation in x:

$$\lfloor -x^2+3x \rfloor = \left\lfloor x^2+\frac{1}{2} \right\rfloor .$$

KACY Summer League

KACY–I 343. Solve the following equation involving the floor function:

$$\lfloor x \rfloor = \left\lfloor \frac{x^3-2}{3} \right\rfloor .$$

KACY Summer League

KACY–I 344. Solve the following equation involving the floor function for x, y:

$$\frac{4x+3y}{2x} = \left\lfloor \frac{x^2+y^2}{x^2} \right\rfloor .$$

KACY Summer League

KACY–I 345. Solve the following equation involving the floor function for x:

$$\frac{15x-7}{5} = \left\lfloor \frac{6x+5}{8} \right\rfloor .$$

KACY Summer League

KACY–I 346. Solve the following equation in x:

$$\sqrt{(x+3)^2}+\sqrt{(x-2)^2}+\sqrt{(2x-8)^2}=9.$$

KACY Summer League

KACY–I 347. Solve the following equation in x:

$$|x^2-4|+|x|+2x=2.$$

KACY Summer League

KACY–I 348. Solve the absolute value equation

$$|x-2|\cdot|x+3|\cdot|x+6|=|x+1|\cdot|x+4|\cdot|x+9|.$$

1.8.5 Arithmetic of Polynomial Roots

1.8.5.1 Miscellaneous Treacheries on Polynomial Roots

KACY Summer League

KACY–I 349. What relationship must be happening between a, b, c so that the roots of the cubic equation

$$x^3+ax^2+bx+c=0,$$

be in an arithmetic progression.

KACY Summer League

KACY–I 350. Prove that if $p^2<3q$, the equation $x^3+px^2+qx+r=0$ has only one real root.

KACY Summer League

KACY–I 351. You may use the first task to prove the second task:

a) For a sequence x_n of real numbers satisfying

$$\sqrt[n+1]{n+2} < x_{n+1} < \sqrt[n]{n+1} < x_n < \sqrt[n-1]{n},$$

prove that x_n is strictly decreasing with a limit of 1.

b) Prove that the equation $x^n = x + n$, when n is a positive integer, always has a solution for x between 1 and 2, and when n is increased, the root x is decreased indefinitely with a limit of 1.

KACY Summer League

KACY–I 352. Prove that the quartic equation

$$x^4 - 4x^3 + 12x^2 - 24x + 24 = 0$$

does not have any real roots.

KACY Summer League

KACY–I 353. Let S be the area of a triangle whose heights are the roots of

$$x^3 - kx^2 + qx - z = 0.$$

Prove that if $4kqz > q^3 + 8z^2$,

$$S = \frac{z^2}{\sqrt{q(4kqz - q^3 - 8z^2)}}.$$

KACY Summer League

KACY–I 354. Find the angles of an isosceles triangle with base a and legs b such that

$$a^3 - 3ab^2 + b^3\sqrt{3} = 0.$$

KACY Summer League

KACY–I 355. Prove that if the three side-lengths a, b, c of a triangle satisfy the two equations

$$a^4 = b^4 + c^4 - b^2c^2 \qquad \text{and} \qquad b^4 = c^4 + a^4 - c^2a^2,$$

then it must also satisfy

$$c^4 = a^4 + b^4 - a^2b^2.$$

KACY Summer League

KACY–I 356. Let $n \geq 2$ be any integer. Prove that if a, b, c are side-lengths of a triangle, then so are $\sqrt[n]{a}$, $\sqrt[n]{b}$, $\sqrt[n]{c}$.

KACY Summer League

KACY–I 357. Prove that if $p^3 + q^3$ is divisible by 23, then there are two roots of $x^3 + px + q = 0$ whose square of difference is also divisible by 23.

KACY Summer League

KACY–I 358. Prove that the necessary and sufficient condition for the cubic equation

$$ax^3 + bx^2 + cx + d = 0$$

to have a purely imaginary root (in the form $x = \alpha i$ with $i = \sqrt{-1}$) is that $ad = bc$ and $ac > 0$.

KACY Summer League

KACY–I 359. Prove that for all positive integers n, there is always a root x_n in the interval $[0, 1]$ for the equation

$$x^n + x^{n-1} + \cdots + x^2 + x = 1,$$

and find the limit of x_n as n increases.

KACY Summer League

KACY–I 360. If α, β, γ are the roots of $x^3 + px + q = 0$, and m, n are positive integers, find the sum

$$S = \frac{m\alpha + n}{m\alpha - n} + \frac{m\beta + n}{m\beta - n} + \frac{m\gamma + n}{m\gamma - n},$$

in terms of p, q, m, n.

KACY Summer League

KACY–I 361. Solve the following cubic equation if we know that one of its roots is double another root of the same equation:

$$x^3 + 21x^2 + 140x - 300 = 0.$$

KACY Summer League

KACY–I 362. Find a such that the product of two roots of the equation

$$x^4 - ax^3 + 23x^2 + ax - 168 = 0$$

is 12, and then solve the equation.

KACY Summer League

KACY–I 363. For a polynomial $f(x)$,

a) If $f(x) = f(a - x)$ for some a, then show that $f(x)$ may be written as a sum of even powers of $2x - a$.

b) If

$$f(x) = 16x^4 - 32x^3 - 56x^2 + 72x + 72,$$

find $f(1 - x)$ and solve $f(x) = 0$.

KACY Summer League

KACY–I 364. Find m such that in the equation

$$x^4 - 8x^3 + mx^2 - 8x - 3 = 0,$$

the sum of two roots equals the third root, and then solve the equation.

KACY Summer League

KACY–I 365. Without solving the equation, find the area of the triangle whose side-lengths are the roots of the cubic equation $x^3 - 12x^2 + 47x - 60 = 0$.

KACY Summer League

2017 Denmark (Georg Mohr) 366. Let A, B, C, D denote the digits in a four-digit number $n = \overline{ABCD}$. Determine the least n greater than 2017 satisfying that there exists an integer x such that

$$x = \sqrt{A + \sqrt{B + \sqrt{C + \sqrt{D + x}}}}.$$

KACY Summer League

1999 Switzerland TST 367. Prove that for every polynomial $P(x)$ of degree 10 with integer coefficients, there is an infinite (in both directions) arithmetic progression of integers that contains none of the values $P(k)$, where $k \in \mathbb{Z}$.

KACY Summer League

2003 Switzerland TST 368. Find all quadratic polynomials $Q(x) = ax^2+bx+c$ such that three different prime numbers p_1, p_2, p_3 exist with

$$|Q(p_1)| = |Q(p_2)| = |Q(p_3)| = 11.$$

KACY Summer League

2006 Switzerland TST 369. The polynomial $P(x) = x^3 - 2x^2 - x + 1$ has three real roots $a > b > c$. Find the value of the expression

$$a^2b + b^2c + c^2a.$$

KACY Summer League

2007 Switzerland TST 370. A pair (r, s) of positive integers is called *good* if a polynomial $P(x)$ with integer coefficients and distinct integers $a_1, a_2, \ldots, a_r$ and $b_1, b_2, \ldots, b_s$ exist such that

$$P(a_1) = P(a_2) = \cdots = P(a_r) = 2,$$
$$P(b_1) = P(b_2) = \cdots = P(b_s) = 5.$$

a) Show that for every *good* pair (r, s) of positive integers, we have $r \leq 3$ and $s \leq 3$.

b) Find all *good* pairs.

KACY Summer League

2009 Ecuador TST 371. Let a and b be two coprime positive integers. It is known that the coefficients of x^2 and x^3 are equal in the expansion of $(ax + b)^{2009}$. Find $a + b$.

KACY Summer League

KACY–I 372. Let a, b, c be non-zero real numbers. Show that if the equation $ax^2 + bx + c = 0$ has a positive solution for x, then the polynomial $f(x)$ with real coefficients, defined by:

$$f(x) = 5ax^4 + mx^3 + 3bx^2 + nx + c,$$

has at least two real roots.

KACY Summer League

KACY–I 373. The polynomial $P(x)$ is such that the polynomials

$$P(P(x)) \qquad \text{and} \qquad P(P(P(x))),$$

are strictly monotone on the whole real x axis. Prove that $P(x)$ is also strictly monotone on $\mathbb{R}$.

1.8.5.2 Calculating Sum of Powers of the Roots

KACY Summer League

Sum of Powers of Quadratic Roots 374. Let x_1 and x_2 be the roots of the quadratic equation $ax^2 + bx + c = 0$ and define $S_p = x_1^p + x_2^p$. Prove the recursive formula between the sum of powers of quadratic roots:

$$aS_n + bS_{n-1} + cS_{n-2} = 0.$$

Conclude that

1. The sum $S_p(x_1, x_2)$ of the p^{th} powers of quadratic roots x_1 and x_2, is calculable in terms of a, b, c, and
2. In order to find the sum of n^{th} powers of quadratic roots, one needs both $(n-1)^{th}$ and $(n-2)^{th}$ powers of the roots.

KACY Summer League

Sum of Powers of Cubic Roots 375. Let x_1, x_2 and x_3 be the roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$ and define $S_p = x_1^p + x_2^p + x_3^p$. Prove the recursive formula between the sum of powers of roots:

$$aS_n + bS_{n-1} + cS_{n-2} + cS_{n-2} + dS_{n-3} = 0.$$

Conclude that

1. The sum $S_p(x_1, x_2, x_3)$ of the p^{th} powers of the cubic roots x_1, x_2 and x_3, is calculable in terms of a, b, c, d, and
2. In order to find the sum of n^{th} powers of cubic roots, one needs all three of $(n-1)^{th}$, $(n-2)^{th}$, and $(n-3)^{th}$ powers of the roots.

KACY Summer League

KACY–I 376. Find the sum of the fourth powers of the roots of $2x^2 - 4x + 1 = 0$.

KACY Summer League

KACY–I 377. Find the sum of the sixth powers of the roots of $x^3 - 3x + 1 = 0$.

KACY Summer League

KACY–I 378. If x_1, x_2, x_3 are the roots of the cubic equation $x^3 - x + 1 = 0$, find the sum of fifth powers of the roots: $x_1^5 + x_2^5 + x_3^5$.

KACY Summer League

KACY–I 379. If x_1, x_2, x_3 are the roots of $x^3 - 1 = 0$, prove that

$$x_1^n + x_2^n + x_3^n = x_1^n x_2^n + x_2^n x_3^n + x_3^n x_1^n.$$

KACY Summer League

KACY–I 380. Find the values of real numbers a, b, p, q such that the equation

$$(2x-1)^{20} - (ax+b)^{20} = (x^2+px+q)^{10}$$

becomes an identity (true for all x).

KACY Summer League

KACY–I 381. Find the sum of the eleventh powers of the roots of the equation

$$x^3 + x + 1 = 0.$$

KACY Summer League

KACY–I 382. This was the seventh problem on 2023 Indian Statistical Institute UGB 2023 and it comes in two parts:

a) Let $n \geq 1$ be an integer. Prove that $X^n + Y^n + Z^n$ can be written as a polynomial with integer coefficients in the variables $\alpha = X + Y + Z$, $\beta = XY + YZ + ZX$ and $\gamma = XYZ$.

b) Let $G_n = x^n \sin(nA) + y^n \sin(nB) + z^n \sin(nC)$, where x, y, z, A, B, C are real numbers such that $A + B + C$ is an integral multiple of π. Using (a) or otherwise show that if $G_1 = G_2 = 0$, then $G_n = 0$ for all positive integers n.

1.8.5.3 Forming Equations Given the Roots

Forming Equations A.K.A. Reverse Viète

KACY Summer League

KACY–I 383. Concerning the polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

we can divide everything by $a_n \neq 0$ and apply Viète's Formula's in reverse to write the equation as

$$x^n - \left(\sum x_1\right) x^{n-1} + \left(\sum x_1 x_2\right) x^{n-2} - \left(\sum x_1 x_2 x_3\right) x^{n-3} + \\ \cdots + (-1)^n (x_1 x_2 \cdots x_n) = 0.$$

KACY Summer League

KACY–I 384. Form the polynomial equation whose roots are the squares of the roots of the following equation:

$$x^3 + 2x^2 - x + 5 = 0.$$

KACY Summer League

KACY–I 385. Let x_1, x_2, x_3, and x_4 be the roots of the equation $x^4 - 4x^2 + x + 3 = 0$. Find the sum

$$S = \frac{1}{2x_1 - 1} + \frac{1}{2x_2 - 1} + \frac{1}{2x_3 - 1} + \frac{1}{2x_4 - 1}.$$

KACY Summer League

KACY–I 386. Let $x_1, x_2, \ldots, x_n$ be the roots of the equation

$$x^n + x^{n-1} + x^{n-2} + \cdots + x + 1 = 0.$$

Find the sum

$$S = \frac{1}{x_1 - 1} + \frac{1}{x_2 - 1} + \cdots + \frac{1}{x_n - 1}.$$

KACY Summer League

KACY–I 387. Let x_1, x_2, and x_3 be the roots of the equation $x^3 - 3x^2 - x - 7 = 0$. Find the sum

$$S = \frac{1}{x_1^2 - 1} + \frac{1}{x_2^2 - 1} + \frac{1}{x_3^2 - 1}.$$

KACY Summer League

KACY–I 388. Let α be a root of the equation $x^{13} - 1 = 0$. Find a polynomial with rational coefficients such that it has $\alpha^4 + \alpha^6 + \alpha^7 + \alpha^9$ among its roots.

KACY Summer League

KACY–I 389. Find the quadratic equation whose roots are fourth powers of the roots of $ax^2 + bx + c = 0$.

KACY Summer League

KACY–I 390. Find a cubic polynomial whose roots are

$$\cos\frac{\pi}{7}, \qquad \cos\frac{3\pi}{7}, \qquad \cos\frac{5\pi}{7}.$$

KACY Summer League

KACY–I 391. Find a quartic polynomial whose roots are the square of the roots of the following equation:

$$x^4 + 2x^3 + x^2 - 3x + 5 = 0.$$

KACY Summer League

KACY–I 392. Let $x_1 = 1, x_2, \ldots, x_n$ be the roots of $x^n - 1 = 0$. Find $(1 - x_2)(1 - x_3) \cdots (1 - x_n)$.

1.8.5.4 Common Roots of Equations

KACY Summer League

KACY–I 393. Find the relationship between p, q, p', q', such that the two equations

$$\begin{cases} x^2 + px + q &= 0, \\ x^2 + p'x + q' &= 0. \end{cases}$$

have a common root $x = \alpha$, and then find the quadratic equation that has uncommon roots $x = \beta$ and $x = \beta'$ of the equations.

KACY Summer League

KACY–I 394. Find the condition for existence of a common root of these equations:

$$\begin{cases} x^3 + px + q &= 0, \\ x^3 + p'x + q &= 0. \end{cases}$$

KACY Summer League

KACY–I 395. Find m such that one of the roots of the equation $x^2 - x - m = 0$ is double one of the roots of the equation $x^2 - (m+2)x + 3 = 0$.

KACY Summer League

KACY–I 396. If the greatest common factor of polynomials $f(x)$ and $g(x)$ is a polynomial of degree n, prove that it means that the equations $f(x) = 0$ and $g(x) = 0$ have n common roots.

KACY Summer League

KACY–I 397. Find the common roots of these two equations:

$$\begin{cases} 2x^4 - x^3 - 4x^2 + 1 &= 0, \\ x^4 + x^3 - 4x^2 - x + 1 &= 0. \end{cases}$$

Furthermore, solve each equation separately.

KACY Summer League

KACY–I 398. Find m such that the following two equations have three roots in common, and then solve them separately.

$$\begin{cases} f(x) = 2x^4 - 7x^3 - 2x^2 + (7m-2)x + 2 & = 0, \\ g(x) = x^4 - 7x^3 + 13x^2 - x - 6 & = 0. \end{cases}$$

KACY Summer League

KACY–I 399. Find the common roots of the following quartic equations:

$$\begin{cases} x^4 - 3x^3 + 4x^2 - 5x - 3 & = 0, \\ x^4 + x^3 - 5x^2 - 7x - 2 & = 0. \end{cases}$$

1.8.5.5 Number Theoretic Wizardry on Polynomial Roots

KACY Summer League

KACY–I 400. Prove that if the three-digit decimal $\overline{abc}$ is a prime number, then the roots of the quadratic $ax^2 + bx + c = 0$ are irrational.

KACY Summer League

KACY–I 401. Prove that the sum of cubes of the roots of the equation $x^3 + px + q = 0$ with integer coefficients p and q is an integer divisible by 3.

KACY Summer League

KACY–I 402. For what positive integer values of n is $n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2$ divisible by 10?

1.8.6 Using Viète's Formulas to Solve Systems of Equations

KACY Summer League

KACY–I 403. Given real numbers a, b, c, solve the following system of equations for x, y, z:

$$\begin{cases} (a-1)x - a(a-1)y - (a^2+1)z + a^3 & = 0, \\ (b-1)x - b(b-1)y - (b^2+1)z + b^3 & = 0, \\ (c-1)x - c(c-1)y - (c^2+1)z + c^3 & = 0. \end{cases}$$

Definition. We call a system of equations **symmetric** if the swapping of any of its two variables with each other would not change the system. For instance, these systems are symmetric:

$$\begin{cases} x^2+y^2 &= \frac{7}{3}, \\ x^3+y^3 &= -3. \end{cases} ; \quad \begin{cases} x+y+z &= 2a, \\ x^2+y^2+z^2 &= 6a^2, \\ x^3+y^3+z^3 &= 8a^3. \end{cases}$$

KACY Summer League

KACY–I 404. Solve the symmetric equation

$$\begin{cases} x^2+y^2 = \dfrac{7}{3}, \\ x^3+y^3 = -3. \end{cases}$$

for x and y.

KACY Summer League

KACY–I 405. Solve the symmetric equation

$$\begin{cases} x+y+z &= 2a, \\ x^2+y^2+z^2 &= 6a^2, \\ x^3+y^3+z^3 &= 8a^3. \end{cases}$$

for x, y and z.

1.8.7 Homogeneous Equations

Definition. We call a system of equations **homogeneous** if each of its equations is a **homogeneous polynomial** in the unknowns. In simpler words, that is, the the terms containing unknowns are of the same degree. For instance, the system of equations

$$\begin{cases} ax^2+by^2+cxy = d, \\ a'x^2+b'y^2+c'xy = d'. \end{cases}$$

is homogeneous with respect to x and y since all the terms containing x, y in the system are of degree 2.

KACY Summer League

KACY–I 406. In order to solve the following homogeneous system of equations in x and y,

$$\begin{cases} ax^2 + by^2 + cxy &= d, \\ a'x^2 + b'y^2 + c'xy &= d'. \end{cases}$$

assume that the ratio of y over x equals λ, or simply $y = \lambda x$, then solve for λ.

Definition. We call a system of linear equations **linearly homogeneous** if there are no constant terms and the degree of all terms containing unknowns is 1. For instance, the two systems of equations

$$\begin{cases} ax + by &= 0, \\ a'x + b'y &= 0. \end{cases} \qquad \begin{cases} ax + by + cz &= 0, \\ a'x + b'y + c'z &= 0, \\ a''x + b''y + c''z &= 0. \end{cases}$$

are **linearly homogeneous** systems of equations with respect to x and y since all right sides are zero and all the terms containing x, y in the system are of degree 1.

KACY Summer League

KACY–I 407. Solve the following homogeneous equations in x and y:

1. $\begin{cases} 2x^2 + 5y^2 - 3xy &= 7, \\ 3x^2 - 2y^2 + xy &= 12. \end{cases}$

2. $\begin{cases} x^3 + 3x^2y - y^3 &= 3, \\ 2x^3 - xy^2 + y^3 &= 2. \end{cases}$

KACY Summer League

KACY–I 408. Find m such that the following system of equations has a non-zero solution:

$$\begin{cases} (m-2)x + (m-1)y &= 0, \\ mx + 2(2m-3)y &= 0. \end{cases}$$

KACY Summer League

KACY–I 409. Solve the following linearly homogeneous system of equations in x, y, z:

$$\begin{cases} x + ay + a^2z = 0, \\ x + by + b^2z = 0, \\ x + cy + c^2z = 0. \end{cases}$$

1.8.8 Miscellaneous Systems of Equations

KACY Summer League

KACY–I 410. Find all solutions x, y that satisfy both $x^{x-y} = 2y - 1$ and

$$\sqrt{x^2 + 5x + 2y - 3} + \sqrt{x^2 + x + y + 2} = \sqrt{x^2 + 4x + 3y - 2} + \sqrt{x^2 + 2y + 3}.$$

KACY Summer League

KACY–I 411. Given real numbers a, b, solve the system of equations for x, y:

$$\begin{cases} x^3 = ax + by, \\ y^3 = bx + ay. \end{cases}$$

KACY Summer League

KACY–I 412. If $x = y^3 - y$ and $y = 3x - x^3$, find all such x and y.

KACY Summer League

KACY–I 413. Remove x and y from the equations of the following system:

$$\begin{cases} x + y &= p + qxy, \\ 2x &= s + tx^2, \\ 2y &= s + ty^2. \end{cases}$$

KACY Summer League

KACY–I 414. Prove that if $a^3 \neq 3ab-2c$, then the following system of equations is inconsistent (does not have solutions):

$$\begin{cases} x+y &= a, \\ x^2+y^2 &= b, \\ x^3+y^3 &= c. \end{cases}$$

KACY Summer League

KACY–I 415. Let $N>1$ be a positive integer. Define

$$p = \sqrt[3]{\log N^{p-3}}, \qquad q = \sqrt[3]{\log N^{q-3}}, \qquad r = \sqrt[3]{\log N^{r-3}}.$$

Find $\dfrac{1}{p}+\dfrac{1}{q}+\dfrac{1}{r}$.

KACY Summer League

KACY–I 416. Solve the following systems of equations for x, y, z, t, assuming a, b, c, d are appropriate given real numbers. Problems a) through j) are listed separately in from Problem 416 to Problem 425 to give them proper importance in Kaywañan.

a) $\begin{cases} x+xy+y &= 11, \\ xy^2+x^2y &= 30. \end{cases}$

b) $\begin{cases} x^2-xy+y^2 &= 7, \\ x^3+y^3 &= 35. \end{cases}$

c) $\begin{cases} x^3+y^3 &= 7, \\ xy(x+y) &= -2. \end{cases}$

d) $\begin{cases} x^4+y^4 &= a^4, \\ x+y &= b. \end{cases}$

e) $\begin{cases} x^5+y^5 &= a^5, \\ x+y &= a. \end{cases}$

f) $\begin{cases} x+y+z &= 0, \\ \dfrac{x^2+y^2+z^2}{x^3+y^3+z^3} &= 1, \\ xyz &= 2. \end{cases}$

g) $\begin{cases} x+y+z &= a, \\ x^2+y^2+z^2 &= a^2, \\ x^3+y^3+z^3 &= a^3. \end{cases}$

h) $\begin{cases} x-ay-a^2z-a^3t &= a^4, \\ x-by-b^2z-b^3t &= b^4, \\ x-cy-c^2z-c^3t &= c^4, \\ x-dy-d^2z-d^3t &= d^4. \end{cases}$

i) $\begin{cases} x\sin a+y\sin 2a+z\sin 3a = \sin 4a, \\ x\sin b+y\sin 2b+z\sin 3b = \sin 4b, \\ x\sin c+y\sin 2c+z\sin 3c = \sin 4c. \end{cases}$

j) $\begin{cases} \dfrac{x^2}{a^2}+\dfrac{xy}{ab}+\dfrac{y^2}{b^2} &= 3, \\ b^2x^2+xy-a^2y^2 &= ab. \end{cases}$

KACY Summer League

KACY–I 417. Solve the symmetric system of equations

$$\begin{cases} x^2 - xy + y^2 &= 7, \\ x^3 + y^3 &= 35. \end{cases}$$

for x and y.

KACY Summer League

KACY–I 418. Solve the homogeneous system of equations

$$\begin{cases} x^3 + y^3 &= 7, \\ xy(x+y) &= -2. \end{cases}$$

for x and y.

KACY Summer League

KACY–I 419. Solve the symmetric system of equations

$$\begin{cases} x^4 + y^4 &= a^4, \\ x + y &= b. \end{cases}$$

for x and y, where a and b are appropriate coefficients.

KACY Summer League

KACY–I 420. Solve the symmetric system of equations

$$\begin{cases} x^5 + y^5 &= a^5, \\ x + y &= a. \end{cases}$$

for x and y.

KACY Summer League

KACY–I 421. Solve the symmetric system of equations

$$\begin{cases} x + y + z &= 0, \\ \dfrac{x^2 + y^2 + z^2}{x^3 + y^3 + z^3} &= 1, \\ xyz &= 2. \end{cases}$$

for x, y, z.

KACY Summer League

KACY–I 422. For an appropriate real number a, solve the symmetric system of equations

$$\begin{cases} x+y+z & = a, \\ x^2+y^2+z^2 & = a^2, \\ x^3+y^3+z^3 & = a^3. \end{cases}$$

for x, y, and z.

KACY Summer League

KACY–I 423. Given real numbers a, b, c, d, solve the system of equations

$$\begin{cases} x-ay-a^2z-a^3t & = a^4, \\ x-by-b^2z-b^3t & = b^4, \\ x-cy-c^2z-c^3t & = c^4, \\ x-dy-d^2z-d^3t & = d^4. \end{cases}$$

for x, y, z, and t.

KACY Summer League

KACY–I 424. Given three real numbers a, b, c, solve the trigonometric system of equations

$$\begin{cases} x\sin a+y\sin 2a+z\sin 3a & = \sin 4a, \\ x\sin b+y\sin 2b+z\sin 3b & = \sin 4b, \\ x\sin c+y\sin 2c+z\sin 3c & = \sin 4c. \end{cases}$$

for x, y, and z.

KACY Summer League

KACY–I 425. Let a and b be appropriate given real numbers. Solve the following system of equations for x and y:

$$\begin{cases} \dfrac{x^2}{a^2}+\dfrac{xy}{ab}+\dfrac{y^2}{b^2} & = 3, \\ b^2x^2+xy-a^2y^2 & = ab. \end{cases}$$

KACY Summer League

KACY–I 426. Solve the following systems of equations for x, y, z, assuming a, b, c are appropriate given real numbers.

a) $$\begin{cases} x+y+z &= 9, \\ \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z} &= 1, \\ xy+yz+zx &= 27. \end{cases}$$

b) $$\begin{cases} x^2+y^2+xy &= 37, \\ x^2+z^2+xz &= 28, \\ y^2+z^2+yz &= 19. \end{cases}$$

c) $$\begin{cases} x+y+z &= 6, \\ x^2+y^2+z^2 &= 18, \\ \sqrt{x}+\sqrt{y}+\sqrt{z} &= 4. \end{cases}$$

d) $$\begin{cases} x &= \dfrac{2yz}{y^2+z^2}, \\ y &= \dfrac{2zx}{z^2+x^2}, \\ z &= \dfrac{2xy}{x^2+y^2}. \end{cases}$$

e) $$\begin{cases} ax-cy+bz &= x^2+z^2, \\ -bx+ay+cz &= y^2+z^2, \\ cx+by-az &= x^2+y^2. \end{cases}$$

f) $$\begin{cases} \sqrt[4]{1+5x}+\sqrt[4]{6-y} &= 3, \\ 5x-y &= 18. \end{cases}$$

KACY Summer League

2004 Denmark (Georg Mohr) 427. Find all sets (x, y, z) of real numbers which satisfy

$$\begin{cases} x^3-y^2 &= z^2-x, \\ y^3-z^2 &= x^2-y, \\ z^3-x^2 &= y^2-z. \end{cases}$$

KACY Summer League

2005 Denmark (Georg Mohr) 428. For any positive real number a determine the number of solutions (x, y) of the system of equations

$$\begin{cases} |x|+|y| &= 1, \\ x^2+y^2 &= a, \end{cases}$$

where x and y are real numbers.

KACY Summer League

2006 Denmark (Georg Mohr) 429. Determine all triplets (x, y, z) of real numbers which satisfy

$$\begin{cases} x + y &= 2, \\ xy - z^2 &= 1. \end{cases}$$

KACY Summer League

2009 Denmark (Georg Mohr) 430. Solve the following system of equations over reals:

$$\begin{cases} \dfrac{1}{x+y} + x &= 3, \\ \\ \dfrac{x}{x+y} &= 2. \end{cases}$$

KACY Summer League

2013 Denmark (Georg Mohr) 431. A sequence $\{x_n\}_{n=0}^{\infty}$ is given by $x_0 = 8$ and

$$x_{n+1} = \frac{1 + x_n}{1 - x_n}, \qquad \text{for} \quad n = 0, 1, 2, \ldots$$

Determine the number x_{2013}.

KACY Summer League

2015 Denmark (Georg Mohr) 432. Find all sets (x, y, z) of real numbers which satisfy

$$\begin{cases} x^2 + yz &= 1, \\ y^2 - xz &= 0, \\ z^2 + xy &= 1. \end{cases}$$

KACY Summer League

2017 Denmark (Georg Mohr) 433. The system of equations

$$\begin{cases} x^2 \square z^2 &= -8, \\ y^2 \square z^2 &= 7, \end{cases}$$

is written on a piece of paper, but unfortunately two of the symbols are a little blurred. However, it is known that the system has at least one solution, and that each of the two squares $(\square)$ stands for either $+$ or $-$. What are the two symbols?

KACY Summer League

1999 Switzerland TST 434. Solve the system of equations over reals:

$$\begin{cases} \dfrac{4x^2}{1+4x^2} &= y, \\ \dfrac{4y^2}{1+4y^2} &= z, \\ \dfrac{4z^2}{1+4z^2} &= x. \end{cases}$$

KACY Summer League

2003 Switzerland TST 435. For real values of x, y, and a, we have

$$\begin{cases} x+y &= a, \\ x^3+y^3 &= a, \\ x^5+y^5 &= a. \end{cases}$$

Find all possible values of a.

KACY Summer League

2004 Switzerland TST 436. For real values of a, b, c, d, we have

$$\begin{cases} a &= \sqrt{45-\sqrt{21-a}}, \\ b &= \sqrt{45+\sqrt{21-b}}, \\ c &= \sqrt{45-\sqrt{21+c}}, \\ d &= \sqrt{45+\sqrt{21+d}}. \end{cases}$$

Prove that $abcd = 2004$.

KACY Summer League

2007 Ecuador TST 437. Let a, b, c, and x, y, z be the solutions to the system of equations

$$\begin{cases} x^2+y^2+z^2 &= 7+2\sqrt{3}, \\ xy+yz+zx &= -3\sqrt{3}, \\ a^2+b^2+c^2 &= 7, \\ ab+bc+ca &= 2\sqrt{3}. \end{cases}$$

Find the value of $|a+b+c|+|x+y+z|$.

KACY Summer League

2008 Ecuador TST 438. Solve the system of equations

$$\begin{cases} x + y^2 &= 1, \\ x^2 + y^3 &= 1. \end{cases}$$

KACY Summer League

2009 Ecuador TST 439. Let x, y, z be real numbers such that $abc = 1$, and

$$\begin{cases} x + \dfrac{1}{y} &= 5, \\ \\ y + \dfrac{1}{z} &= 29, \\ \\ z + \dfrac{1}{x} &= \dfrac{m}{n}, \end{cases}$$

where m and n are coprime positive integers. Find the value of $m + n$.

KACY Summer League

2009 Ecuador TST 440. Solve the system of equations over reals:

$$\begin{cases} x + y + z &= 2, \\ (x+y)(y+z) + (y+z)(z+x) + (z+x)(x+y) &= 1, \\ x^2(y+z) + y^2(z+x) + z^2(x+y) &= -6. \end{cases}$$

KACY Summer League

2010 Ecuador TST 441. A sequence $\{a_n\}_{n=1}^{\infty}$ is defined initially by $a_1 = 1/2$ and recursively for $n \geq 1$ by

$$a_n = \frac{a_{n-1}}{2na_{n-1} + 1}.$$

Find the sum $a_1 + a_2 + \cdots + a_{2010}$.

KACY Summer League

2011 Ecuador TST 442. Solve the system of equations over reals:

$$\begin{cases} x_1 + x_2 + \cdots + x_{2011} &= 2011, \\ x_1^4 + x_2^4 + \cdots + x_{2011}^4 &= x_1^3 + x_2^3 + \cdots + x_{2011}^3. \end{cases}$$

KACY Summer League

2016 Ecuador 443. Let a, b, and x, y be real numbers satisfying:

$$\begin{cases} ax \quad + by &= 3, \\ ax^2 + by^2 &= 7, \\ ax^3 + by^3 &= 16, \\ ax^4 + by^4 &= 42. \end{cases}$$

Find $ax^5 + by^5$.

1.9 Nice Polynomial Problems

We present 400 polynomial problems in this section. The first series of these nice polynomial problems is a Collection of 100 polynomial problems that I collected from AoPS around 2011. The second series is a collection of 300 ancient polynomial problems, which, I believe, should make the most complete and extensive resource for studying polynomials.

1.9.1 100 Nice Polynomial Problems

KACY Summer League

KACY–I 444. Find all polynomials $P(x)$ with real coefficient such that:

$$P(0) = 0, \quad \text{and} \quad \lfloor P\lfloor P(n)\rfloor\rfloor + n = 4\lfloor P(n)\rfloor \quad \forall n \in \mathbb{N}.$$

KACY Summer League

KACY–I 445. Find all functions $f : \mathbb{R} \to R$ such that

$$f(x^n + 2f(y)) = (f(x))^n + y + f(y) \quad \forall x, y \in \mathbb{R}, \quad n \in \mathbb{Z}_{\geq 2}.$$

KACY Summer League

KACY–I 446. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$x^2y^2\left(f(x+y) - f(x) - f(y)\right) = 3(x+y)f(x)f(y).$$

KACY Summer League

KACY–I 447. Find all polynomials $P(x)$ with real coefficients such that

$$P(x)P(x+1) = P(x^2) \quad \forall x \in \mathbb{R}.$$

KACY Summer League

KACY–I 448. Find all polynomials $P(x)$ with real coefficient such that

$$P(x)Q(x) = P(Q(x)) \quad \forall x \in \mathbb{R}.$$

KACY Summer League

KACY–I 449. Find all polynomials $P(x)$ with real coefficients such that if $P(a)$ is an integer, then so is a, where a is any real number.

KACY Summer League

KACY–I 450. Find all the polynomials $f \in \mathbb{R}[X]$ such that

$$\sin f(x) = f(\sin x), \ (\forall)x \in \mathbb{R}.$$

KACY Summer League

KACY–I 451. Find all polynomial $f(x) \in \mathbb{R}[x]$ such that

$$f(x)f(2x^2) = f(2x^3 + x^2) \quad \forall x \in \mathbb{R}.$$

KACY Summer League

KACY–I 452. Find all real polynomials f and g, such that:

$$(x^2 + x + 1) \cdot f(x^2 - x + 1) = (x^2 - x + 1) \cdot g(x^2 + x + 1),$$

for all $x \in \mathbb{R}$.

KACY Summer League

KACY–I 453. Find all polynomials $P(x)$ with integral coefficients such that $P(P'(x)) = P'(P(x))$ for all real numbers x.

KACY Summer League

KACY–I 454. Find all polynomials with integer coefficients f such that for all $n > 2005$ the number $f(n)$ is a divisor of $n^{n-1} - 1$.

KACY Summer League

KACY–I 455. Find all polynomials with complec coefficients f such that we have the equivalence: for all complex numbers z, $z \in [-1, 1]$ if and only if $f(z) \in [-1, 1]$.

KACY Summer League

KACY–I 456. Suppose f is a polynomial in $\mathbb{Z}[X]$ and m is integer .Consider the sequence a_i like this $a_1 = m$ and $a_{i+1} = f(a_i)$ find all polynomials f and alll integers m that for each i:

$$a_i | a_{i+1}.$$

KACY Summer League

KACY–I 457. $P(x), Q(x) \in \mathbb{R}[x]$ and we know that for real r we have $p(r) \in \mathbb{Q}$ if and only if $Q(r) \in \mathbb{Q}$ I want some conditions between P and Q.My conjecture is that there exist ratinal a, b, c that $aP(x) + bQ(x) + c = 0$

KACY Summer League

KACY–I 458. Find the gcd of the polynomials $X^n + a^n$ and $X^m + a^m$, where a is real.

KACY Summer League

KACY–I 459. Find all polynomials p with real coefficients that if for a real a,$p(a)$ is integer then a is integer.

KACY Summer League

KACY–I 460. $\mathfrak{question}$ is a real polynomial such that if α is irrational then $\mathfrak{question}(\alpha)$ is irrational. Prove that $\deg[\mathfrak{question}] \leq 1$

KACY Summer League

KACY–I 461. Show that the odd number n is a prime number if and only if the polynomial $T_n(x)/x$ is irreducible over the integers.

KACY Summer League

KACY–I 462. P, Q, R are non-zero polynomials that for each $z \in \mathbb{C}$, $P(z)Q(\bar{z}) = R(z)$. a) If $P, Q, R \in \mathbb{R}[x]$, prove that Q is constant polynomial. b) Is the above statement correct for $P, Q, R \in \mathbb{C}[x]$?

KACY Summer League

KACY–I 463. Let P be a polynomial such that $P(x)$ is rational if and only if x is rational. Prove that $P(x) = ax + b$ for some rational a and b.

KACY Summer League

KACY–I 464. Prove that any polynomial $\in \mathbb{R}[X]$ can be written as a difference of two strictly increasing polynomials.

KACY Summer League

KACY–I 465. Consider the polynomial $W(x) = (x-a)^k Q(x)$, where $a \neq 0$, Q is a nonzero polynomial, and k a natural number. Prove that W has at least $k+1$ nonzero coefficients.

KACY Summer League

KACY–I 466. Find all polynomials $p(x) \in \mathbb{R}[x]$ such that the equation

$$f(x) = n$$

has at least one rational solution, for each positive integer n.

KACY Summer League

KACY–I 467. Let $f \in \mathbb{Z}[X]$ be an irreducible polynomial over the ring of integer polynomials, such that $|f(0)|$ is not a perfect square. Prove that if the leading coefficient of f is 1 (the coefficient of the term having the highest degree in f) then $f(X^2)$ is also irreducible in the ring of integer polynomials.

KACY Summer League

KACY–I 468. Let p be a prime number and f an integer polynomial of degree d such that $f(0) = 0, f(1) = 1$ and $f(n)$ is congruent to 0 or 1 modulo p for every integer n. Prove that $d \geq p-1$.

KACY Summer League

KACY–I 469. Let

$$P(x) := x^n + \sum_{k=1}^{n} a_k x^{n-k},$$

with $0 \leq a_n \leq a_{n-1} \leq \cdots \leq a_2 \leq a_1 \leq 1$. Suppose that there exists $r \geq 1$, $\varphi \in \mathbb{R}$ such that $P(re^{i\varphi}) = 0$. Find r.

KACY Summer League

KACY–I 470. Let $\mathcal{P}$ be a polynomial with rational coefficients such that

$$\mathcal{P}^{-1}(\mathbb{Q}) \subseteq \mathbb{Q}.$$

Prove that $\deg \mathcal{P} \leq 1$.

KACY Summer League

KACY–I 471. Let f be a polynomial with integer coefficients such that $|f(x)| < 1$ on an interval of length at least 4. Prove that $f = 0$.

KACY Summer League

KACY–I 472. prove that $x^n - x - 1$ is irreducible over $\mathbb{Q}$ for all $n \geq 2$.

KACY Summer League

KACY–I 473. Find all real polynomials $p(x)$ such that

$$p^2(x) + 2p(x)p\left(\frac{1}{x}\right) + p^2\left(\frac{1}{x}\right) = p(x^2)p\left(\frac{1}{x^2}\right),$$

for all non-zero real x.

KACY Summer League

KACY–I 474. Find all polynomials $P(x)$ with odd degree such that

$$P(x^2 - 2) = P^2(x) - 2.$$

KACY Summer League

KACY–I 475. Find all real polynomials that

$$p(x + p(x)) = p(x) + p(p(x)),$$

for all reals x.

KACY Summer League

KACY–I 476. Find all polynomials $P \in \mathbb{C}[X]$ such that

$$P(X^2) = P(X)^2 + 2P(X).$$

KACY Summer League

KACY–I 477. Find all polynomials of two variables $P(x, y)$ which satisfy

$$P(a, b)P(c, d) = P(ac + bd, ad + bc), \qquad \text{for all} \quad a, b, c, d \in \mathbb{R}.$$

KACY Summer League

KACY–I 478. Find all real polynomials $f(x)$ satisfying

$$f(x^2) = f(x)f(x-1), \qquad \text{for all} \quad x \in \mathbb{R}.$$

KACY Summer League

KACY–I 479. Find all polynomials of degree 3, such that for all $x, y \geq 0$,

$$p(x+y) \geq p(x) + p(y).$$

KACY Summer League

KACY–I 480. Find all polynomials $P(x) \in \mathbb{Z}[x]$ such that for any $n \in \mathbb{N}$, the equation $P(x) = 2^n$ has an integer root.

KACY Summer League

KACY–I 481. Let f and g be polynomials such that $f(Q) = g(Q)$ for all rationals Q. Prove that there exist reals a and b such that $f(X) = g(aX + b)$, for all real numbers X.

KACY Summer League

KACY–I 482. Find all positive integers $n \geq 3$ such that there exists an arithmetic progression $a_0, a_1, \ldots, a_n$ such that the equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ has n roots setting an arithmetic progression.

KACY Summer League

KACY–I 483. Given non-constant linear functions $p_1(x), p_2(x), \ldots p_n(x)$. Prove that at least $n-2$ of polynomials $p_1p_2\ldots p_{n-1} + p_n, p_1p_2\ldots p_{n-2}p_n + p_{n-1}, \ldots p_2p_3\ldots p_n + p_1$ have a real root.

KACY Summer League

KACY–I 484. Find all positive real numbers $a_1, a_2, \ldots, a_k$ such that the number

$$a_1^{\frac{1}{n}} + \cdots + a_k^{\frac{1}{n}},$$

is rational for all positive integers n, where k is a fixed positive integer.

KACY Summer League

KACY–I 485. Let f, g be real non-constant polynomials such that $f(\mathbb{Z}) = g(\mathbb{Z})$. Show that there exists an integer A such that $f(X) = g(A+x)$ or $f(x) = g(A-x)$.

KACY Summer League

KACY–I 486. Does there exist a polynomial $f \in \mathbb{Q}[x]$ with rational coefficients such that $f(1) \neq -1$, and $x^n f(x) + 1$ is a reducible polynomial for every $n \in \mathbb{N}$?

KACY Summer League

KACY–I 487. Suppose that f is a polynomial of exact degree p. Find a rigorous proof that $S(n)$, where $S(n) = \sum\limits_{k=0}^{n} f(k)$, is a polynomial function of (exact) degree $p+1$ in variable n .

KACY Summer League

KACY–I 488. The polynomials P, Q are such that $\deg P = n$,$\deg Q = m$, have the same leading coefficient, and $P^2(x) = (x^2 - 1)Q^2(x) + 1$. Prove that $P'(x) = nQ(x)$

KACY Summer League

KACY–I 489. Given distinct prime numbers p and q and a natural number $n \geq 3$, find all $a \in \mathbb{Z}$ such that the polynomial $f(x) = x^n + ax^{n-1} + pq$ can be factored into 2 integral polynomials of degree at least 1.

KACY Summer League

KACY–I 490. Let F be the set of all polynomials Γ such that all the coefficients of $\Gamma(x)$ are integers and $\Gamma(x) = 1$ has integer roots. Given a positive integer k, find the smallest integer $m(k) > 1$ such that there exist $\Gamma \in F$ for which $\Gamma(x) = m(k)$ has exactly k distinct integer roots.

KACY Summer League

KACY–I 491. Find all polynomials $P(x)$ with integer coefficients such that the polynomial

$$Q(x) = (x^2 + 6x + 10) \cdot P^2(x) - 1,$$

is the square of a polynomial with integer coefficients.

KACY Summer League

KACY–I 492. Find all polynomials p with real coefficients such that for all reals a, b, c such that $ab + bc + ca = 1$ we have the relation

$$p(a)^2 + p(b)^2 + p(c)^2 = p(a + b + c)^2.$$

KACY Summer League

KACY–I 493. Find all real polynomials f with $x, y \in \mathbb{R}$ such that

$$2yf(x + y) + (x - y)(f(x) + f(y)) \geq 0.$$

KACY Summer League

KACY–I 494. Find all polynomials such that $P(x^3 + 1) = P((x + 1)^3)$.

KACY Summer League

KACY–I 495. Find all polynomials $P(x) \in \mathbb{R}[x]$ such that $P(x^2+1) = P(x)^2+1$ holds for all $x \in \mathbb{R}$.

KACY Summer League

KACY–I 496. Find all polynomials $p(x)$ with real coefficients such that

$$(x + 1)p(x - 1) + (x - 1)p(x + 1) = 2xp(x),$$

for all real x.

KACY Summer League

KACY–I 497. Find all polynomials $P(x)$ that have only real roots, such that

$$P(x^2 - 1) = P(x)P(-x).$$

KACY Summer League

KACY–I 498. Find all polynomials $P(x) \in \mathbb{R}[x]$such that:

$$P(x^2) + x \cdot (3P(x) + P(-x)) = (P(x))^2 + 2x^2, \qquad \text{for all} \quad x \in \mathbb{R}.$$

KACY Summer League

KACY–I 499. Find all polynomials f, g which are both monic and have the same degree and

$$f(x)^2 - f(x^2) = g(x).$$

KACY Summer League

KACY–I 500. Find all polynomials $P(x)$ with real coefficients such that there exists a polynomial $Q(x)$ with real coefficients that satisfy

$$P(x^2) = Q(P(x)).$$

KACY Summer League

KACY–I 501. Find all polynomials $p(x, y) \in \mathbb{R}[x, y]$ such that for each $x, y \in \mathbb{R}$ we have

$$p(x + y, x - y) = 2p(x, y).$$

KACY Summer League

KACY–I 502. Find all couples of polynomials (P, Q) with real coefficients, such that for infinitely many $x \in \mathbb{R}$ the condition

$$\frac{P(x)}{Q(x)} - \frac{P(x+1)}{Q(x+1)} = \frac{1}{x(x+2)},$$

holds.

KACY Summer League

KACY–I 503. Find all polynomials $P(x)$ with real coefficients, such that

$$P(P(x)) = P(x)^k,$$

for any given positive integer k.

KACY Summer League

KACY–I 504. Find all polynomials

$$P_n(x) = n!x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + (-1)^n(n+1)n$$

with integers coefficients and with n real roots $x_1, x_2, \ldots, x_n$, such that $k \leq x_k \leq k+1$, for $k = 1, 2, \ldots, n$.

KACY Summer League

KACY–I 505. The function $f(n)$ satisfies $f(0) = 0$ and

$$f(n) = n - f(f(n-1)),$$

for $n = 1, 2, 3, \ldots$. Find all polynomials $g(x)$ with real coefficient such that

$$f(n) = [g(n)], \qquad n = 0, 1, 2, \ldots$$

Where $[g(n)]$ denote the greatest integer that does not exceed $g(n)$.

KACY Summer League

KACY–I 506. Find all pairs of integers a, b for which there exists a polynomial $P(x) \in \mathbb{Z}[X]$ such that product $(x^2 + ax + b) \cdot P(x)$ is a polynomial of a form

$$x^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0,$$

where each of $c_0, c_1, \ldots, c_{n-1}$ is equal to 1 or -1.

KACY Summer League

KACY–I 507. There exists a polynomial P of degree 5 with the following property: if z is a complex number such that $z^5 + 2004z = 1$, then $P(z^2) = 0$. Find all such polynomials P.

KACY Summer League

KACY–I 508. Find all polynomials $P(x)$ with real coefficients satisfying the equation

$$(x+1)^3P(x-1) - (x-1)^3P(x+1) = 4(x^2-1)P(x),$$

for all real numbers x.

KACY Summer League

KACY–I 509. Find all polynomials $P(x, y)$ with real coefficients such that:

$$P(x, y) = P(x+1, y) = P(x, y+1) = P(x+1, y+1).$$

KACY Summer League

KACY–I 510. Find all polynomials $P(x)$ with reals coefficients such that

$$(x-8)P(2x) = 8(x-1)P(x).$$

KACY Summer League

KACY–I 511. Find all reals α for which there is a nonzero polynomial P with real coefficients such that

$$\frac{P(1)+P(3)+P(5)+\cdots+P(2n-1)}{n} = \alpha P(n), \qquad \text{for all} \quad n \in \mathbb{N},$$

and find all such polynomials for $\alpha = 2$.

KACY Summer League

KACY–I 512. Find all polynomials $P(x) \in \mathbb{R}[X]$ satisfying

$$(P(x))^2 - (P(y))^2 = P(x+y) \cdot P(x-y), \quad \forall x, y \in \mathbb{R}.$$

KACY Summer League

KACY–I 513. Find all $n \in \mathbb{N}$ such that polynomial

$$P(x) = (x-1)(x-2)\cdots(x-n),$$

can be represented as $Q(R(x))$, for some polynomials $Q(x)$ and $R(x)$ with degree greater than 1.

KACY Summer League

KACY–I 514. Find all polynomials $P(x) \in \mathbb{R}[x]$ such that

$$P(x^2-2x) = (P(x)-2)^2.$$

KACY Summer League

KACY–I 515. Find all non-constant real polynomials $f(x)$ such that for any real x the following equality holds

$$f(\sin x + \cos x) = f(\sin x) + f(\cos x).$$

KACY Summer League

KACY–I 516. Find all polynomials $W(x) \in \mathbb{R}[x]$ such that

$$W(x^2)W(x^3) = W(x)^5, \qquad \text{for all} \quad x \in \mathbb{R}.$$

KACY Summer League

KACY–I 517. Find all the polynomials $f(x)$ with integer coefficients such that $f(p)$ is prime for every prime p.

KACY Summer League

KACY–I 518. Let $n \geq 2$ be a positive integer. Find all polynomials $P(x) = a_0 + a_1x + \cdots + a_nx^n$ having exactly n roots not greater than -1 and satisfying

$$a_0^2 + a_1a_n = a_n^2 + a_0a_{n-1}.$$

KACY Summer League

KACY–I 519. Find all polynomials $P(x), Q(x)$ such that

$$P(Q(X)) = Q(P(x)), \qquad \text{for all} \quad x \in \mathbb{R}.$$

KACY Summer League

KACY–I 520. Find all integers k such that for infinitely many integers $n \geq 3$ the polynomial

$$P(x) = x^{n+1} + kx^n - 870x^2 + 1945x + 1995,$$

can be reduced into two polynomials with integer coefficients.

KACY Summer League

KACY–I 521. Find all polynomials $P(x), Q(x), R(x)$ with real coefficients such that

$$\sqrt{P(x)} - \sqrt{Q(x)} = R(x), \qquad \text{for all} \quad x \in \mathbb{R}.$$

KACY Summer League

KACY–I 522. Let $k = \sqrt[3]{3}$. Find a polynomial $p(x)$ with rational coefficients and degree as small as possible such that $p(k + k^2) = 3 + k$. Does there exist a polynomial $q(x)$ with integer coefficients such that $q(k + k^2) = 3 + k$?

KACY Summer League

KACY–I 523. Find all values of the positive integer m such that there exists polynomials $P(x), Q(x), R(x, y)$ with real coefficient satisfying the condition: For every real numbers a, b which satisfying $a^m - b^2 = 0$, we always have that $P(R(a, b)) = a$ and $Q(R(a, b)) = b$.

KACY Summer League

KACY–I 524. Find all polynomials $p(x) \in \mathbb{R}[x]$ such that

$$p(x^{2008} + y^{2008}) = (p(x))^{2008} + (p(y))^{2008},$$

for all real numbers x and y.

KACY Summer League

KACY–I 525. Find all Polynomials $P(x)$ satisfying $P(x)^2 - P(x^2) = 2x^4$.

KACY Summer League

KACY–I 526. Find all polynomials p of one variable with integer coefficients such that if a and b are natural numbers such that $a + b$ is a perfect square, then $p\,(a) + p\,(b)$ is also a perfect square.

KACY Summer League

KACY–I 527. Find all polynomials $P(x) \in \mathbb{Q}[x]$ such that

$$P(x) = P\left(\frac{-x + \sqrt{3 - 3x^2}}{2}\right), \quad \text{for all} \quad |x| \leq 1.$$

KACY Summer League

KACY–I 528. Find all polynomials f with real coefficients such that for all reals a, b, c such that $ab + bc + ca = 0$ we have the following relations

$$f(a - b) + f(b - c) + f(c - a) = 2f(a + b + c).$$

KACY Summer League

KACY–I 529. Find All Polynomials $P(x, y)$ such that for all reals x, y we have

$$P(x^2, y^2) = P\left(\frac{(x + y)^2}{2}, \frac{(x - y)^2}{2}\right).$$

KACY Summer League

KACY–I 530. Let n and k be two positive integers. Determine all monic polynomials $f \in \mathbb{Z}[X]$ of degree n, having the property that

$$f(n) \text{ divides } f\left(2^k \cdot a\right), \qquad \text{for all} \quad a \in \mathbb{Z} \text{ with } f(a) \neq 0.$$

KACY Summer League

KACY–I 531. Find all polynomials $P(x)$ such that

$$P(x^2 - y^2) = P(x+y)P(x-y).$$

KACY Summer League

KACY–I 532. Let $f(x) = x^4 - x^3 + 8ax^2 - ax + a^2$. Find all real number a such that $f(x) = 0$ has four different positive solutions.

KACY Summer League

KACY–I 533. Find all polynomial $P \in \mathbb{R}[x]$ such that: $P(x^2+2x+1) = (P(x))^2 + 1$.

KACY Summer League

KACY–I 534. Let $n \geq 3$ be a natural number. Find all non-constant polynomials with real coefficients $f_1(x), f_2(x), \ldots, f_n(x)$, for which

$$f_k(x) f_{k+1}(x) = f_{k+1}(f_{k+2}(x)), \quad 1 \leq k \leq n,$$

for every real x (with $f_{n+1}(x) \equiv f_1(x)$ and $f_{n+2}(x) \equiv f_2(x)$).

KACY Summer League

KACY–I 535. Find all integers n such that the polynomial $p(x) = x^5 - nx - n - 2$ can be written as product of two non-constant polynomials with integral coefficients.

KACY Summer League

KACY–I 536. Find all polynomials $p(x)$ that satisfy

$$(p(x))^2 - 2 = 2p(2x^2 - 1), \qquad \text{for all} \quad x \in \mathbb{R}.$$

KACY Summer League

KACY–I 537. Find all polynomials $p(x)$ that satisfy

$$(p(x))^2 - 1 = 4p(x^2 - 4X + 1), \qquad \text{for all} \quad x \in \mathbb{R}.$$

KACY Summer League

KACY–I 538. Determine the polynomials P of two variables so that:

a) for any real numbers t, x, y we have $P(tx, ty) = t^n P(x, y)$ where n is a positive integer, the same for all t, x, y;

b) for any real numbers a, b, c we have

$$P(a + b, c) + P(b + c, a) + P(c + a, b) = 0;$$

c) $P(1, 0) = 1$.

KACY Summer League

KACY–I 539. Find all polynomials $P(x)$ satisfying the equation

$$(x + 1)P(x) = (x - 2010)P(x + 1).$$

KACY Summer League

KACY–I 540. Find all polynomials of degree 3 such that for all non-negative reals x and y we have

$$p(x + y) \leq p(x) + p(y).$$

KACY Summer League

KACY–I 541. Find all polynomials $p(x)$ with real coefficients such that

$$p(a + b - 2c) + p(b + c - 2a) + p(c + a - 2b) = 3p(a - b) + 3p(b - c) + 3p(c - a),$$

for all $a, b, c \in \mathbb{R}$.

KACY Summer League

KACY–I 542. Find all polynomials $P(x)$ with real coefficients such that

$$P(x^2 - 2x) = (P(x-2))^2 .$$

KACY Summer League

KACY–I 543. Find all two-variable polynomials $p(x, y)$ such that for each $a, b, c \in \mathbb{R}$:

$$p(ab, c^2 + 1) + p(bc, a^2 + 1) + p(ca, b^2 + 1) = 0.$$

1.9.2 300 Ancient Polynomial Problems

The following problems are taken from a resource of Olympiad Algebra in Iran, the book "*Topics and Discussions in of Algebra in Math Olympiads*" by *Mehdi Safa*, published by *Khoshkhan* publishing, written in Farsi (titles have been translated to English). Some of the problems are really old, for example the first problem in *Chapter 22: Various Problems on Polynomials*) of the book is dated 1907, from Hungary. Most problems, however, come from the 1990's and not so old when these lines are being written (May 2023). However, in the scope of math olympiad preparation, even twenty years is a lifetime, and new problems are a much more popular choice for students to solve. As a result, some of the classic, "old" and forgotten problems here are like gems to those who run out of "new" problems! The last hundred problems or so are taken from well-known competitions such as China TST, Austrian–Polish Mathematical Competition (APMC), Czech and Slovak Competition, IMO Shortlist, and IMO Longlist.

KACY Summer League

1907 Hungary 544. Prove that if p and q are two odd integers, then the equation $x^2 + 2px + 2q = 0$ does not have a rational root.

KACY Summer League

1907 Hungary 545. Prove that the polynomial $P(x) = x^4 + 2x^2 + 2x + 2 = 0$ cannot be written as a product of two polynomials $x^2 + ax + b$ and $x^2 + cx + d$ where a, b, c, d are integers.

KACY Summer League

1996 Bulgaria 546. Let a, b, c be real numbers and define M as the maximum value of the expression

$$|4x^3 + ax^2 + bx + c| \text{ for } x \in [-1, 1].$$

Prove that $M \geq 1$ and find all cases when $M = 1$.

KACY Summer League

1994 Romania 547. Let m, n be given positive integers. Find all common roots of

$$P(x) = x^{m+1} - x^n + 1 \qquad \text{and} \qquad Q(x) = x^{n+1} - x^m + 1.$$

KACY Summer League

1990 Iran 548. Can we find four real numbers such that for each two of them like x and y,

$$x^{10} + x^9y + \cdots + xy^9 + y^{10} = 1?$$

KACY Summer League

1969 USSR 549. Find the smallest positive integer a for which there exists a quadratic polynomial $P(x) = ax^2 + bx + c$ such that its roots are distinct and smaller than 1.

KACY Summer League

1987 Iran 550. Find all polynomials $P(x)$ such that for all x,

$$xP(x-1) = (x-12)P(x).$$

KACY Summer League

1997 Bulgaria 551. Find all real numbers m such that the polynomial

$$P(x) = (x^2 - 2mx - 4(m^2+1)) \cdot (x^2 - 4x - 2m(m^2+1)),$$

has exactly three distinct roots.

KACY Summer League

1997 Austrian–Polish 552. Let p_1, p_2, p_3, p_4 be distinct prime numbers. Prove that there does not exist a cubic polynomial $Q(x)$ with integer coefficients such that

$$|Q(p_1)| = |Q(p_2)| = |Q(p_3)| = |Q(p_4)| = 3.$$

KACY Summer League

1996 Iran 553. Define $f(x) = ax^2 + bx + c$ and assume that for $0 \leq x \leq 1$, we have $|f(x)| \leq 1$. Find the maximum value of $2a + b$.

KACY Summer League

KACY–I 554. Let P, Q, R, S are polynomials satisfying the equation

$$P(x^4) + xR(x^8) + x^2Q(x^{12}) = (1 + x + x^2 + x^3)S(x).$$

Prove that $x - 1$ is a factor of $P(x)$.

KACY Summer League

1998 Iran 555. Let $P(x)$ be a polynomial with real coefficients such that for all $x \geq 0$ we have $P(x) > 0$. Prove that there exists a positive integer m such that all coefficients of the polynomial $(1 + x)^m P(x)$ are non-negative.

KACY Summer League

KACY–I 556. Prove that there do not exist non-constant polynomials f, g, h such that

$$\frac{f(x+1)}{g(x+1)} - \frac{f(x)}{g(x)} = h\left(\frac{1}{x}\right).$$

KACY Summer League

KACY–I 557. Does there exist an integer c such that all the roots of the polynomial

$$P(x) = x^3 - 87x^2 + 181x + c,$$

are integers.

KACY Summer League

KACY–I 558. For a positive integer k, let $P(x)$ be a polynomial with integer coefficients such that the numbers $P(1), P(2), \ldots, P(k)$ are not divisible by k. Prove that $P(x)$ cannot have any integer roots.

KACY Summer League

1978 Romania 559. Prove that for any polynomial $P(x) \neq x$ and any positive integer n, the polynomial $Q_n(x)$, defined by

$$Q_n(x) = \underbrace{P(P(\ldots P(x)))}_{n \text{ times}} - x,$$

is divisible by $Q_1(x) = P(x) - x$.

KACY Summer League

KACY–I 560. For two integers a and b such that $x^2 - x - 1$ is a factor of $ax^{17} + bx^{16} + 1$. Find a.

KACY Summer League

KACY–I 561. For any positive integer n, prove that the polynomial

$$P_n(x) = x^{n+2} - 2x + 1,$$

has exactly one root in the interval $[0, 1]$.

KACY Summer League

1999 Iran 562. Let $P(x)$ be a polynomial of degree n such that for integer x, we know that $P(x)$ is integer. Prove that there exist integers $a_0, a_1, \ldots, a_n$ such that

$$P(x) = a_n \binom{x}{n} + \cdots + a_1 \binom{x}{1} + a_0.$$

KACY Summer League

KACY–I 563. For two distinct real numbers a and b, prove that the polynomial

$$P(x) = (a - b)x^n + (a^2 - b^2)x^{n-1} + \cdots + (a^{n+1} - b^{n+1}),$$

has at most one real root.

KACY Summer League

1995 Iran First Round 564. Let $F(x)$ and $G(x)$ be two polynomials with integer coefficients such that $F(x)/G(x)$ is an integer for values of $x = 1, 2, 3, \ldots$. Prove that $F(x)$ is divisible by $G(x)$.

KACY Summer League

1989 Iran Second Round 565. Prove that for all positive integers $n > 1$, the polynomial

$$P(x) = \frac{x^N}{n!} + \frac{x^{n-1}}{(n-1)!} + \cdots + \frac{x}{1} + 1,$$

does not have integer roots.

KACY Summer League

1997 Bulgaria 566. Find all values of a such that for all $x \in [0, 1]$, we have $|f(x)| \leq 1$, where

$$f(x) = x^2 - 2ax - a^2 - \frac{3}{4}.$$

KACY Summer League

1974 International Mathematics Olympiad 567. Let $P(x)$ be a non-constant polynomial with integer coefficients and denote by $n(P(x))$ the number of integers k such that $(P(k))^2 = 1$. Prove that

$$n(P(x)) - \deg(P(x)) \leq 2.$$

KACY Summer League

1996 Poland 568. Find all pairs (n, r) of positive integer n and real number r such that the polynomial $(x + 1)^n - r$ is divisible by the quadratic $2x^2 + 2x + 1$.

KACY Summer League

1914 Hungary 569. Let $P(x)$ be a quadratic polynomial such that for $x \in [-1, 1]$, we have $P(x) \in [-1, 1]$. Prove that $P'(x) \in [-4, 4]$.

KACY Summer League

1918 Hungary 570. If p, q, r, and a, b, c are real numbers such that for all reals x, we have

$$ax^2 + 2bx + c \geq 0 \qquad \text{and} \qquad px^2 + 2qx + r \geq 0,$$

then prove that $apx^2 + bqx + cr \geq 0$.

KACY Summer League

1988 Iran 571. If α is a root of the cubic polynomial $x^3 + x^2 + 2x - 1$, find the other two roots in terms of α.

KACY Summer League

1997 Iran 572. Consider all quadratic polynomials $x^2 + px + q$ in which p and q are integers with $1 \leq p, q \leq 1997$. Determine the number of which of the two kinds of polynomials is larger: those quadratics that have integer roots, or those quadratics with no integer roots?

KACY Summer League

KACY–I 573. Does there exist a sequence $\{a_n\}_{n=0}^{\infty}$ of real non-zero numbers such that for any positive integer n, the polynomial $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ has exactly n distinct real roots?

KACY Summer League

1999 Iran 574. Let $P(x)$ be a polynomial with degree less than n. Prove that

$$\sum_{i=0}^{n} P(i)(-1)^i \binom{n}{i} = 0.$$

KACY Summer League

1996 Bulgaria 575. Let $f(x)$ and $g(x)$ be quadratic polynomials with real coefficients such that if $g(x)$ is an integer for some $x > 0$, then $f(x)$ is also an integer. Prove that there exist integers m and n such that

$$f(x) = mg(x) + n.$$

KACY Summer League

1996 Austrian–Polish 576. Prove that there does not exist a polynomial $P(x)$ of degree 998 such that all its coefficients are real and for all $x \in \mathbb{R}$, we have

$$(P(x))^2 - 1 = P(x^2 + 1).$$

KACY Summer League

KACY–I 577. For all positive integers $n > 1$, prove that the following polynomial does not have rational roots:

$$P(x) = (2n + 1)x^n + \cdots + 5x^2 + 3x + 1.$$

KACY Summer League

1997 IMO Shortlist 578. Find all positive integers k such that the following statement holds true: for all polynomials $F(x)$ with integer coefficients such that $0 \leq F(c) \leq k$ for all $c \in \{0, 1, 2, \ldots, k\}$, then

$$F(0) = F(1) = \cdots = F(k+1).$$

KACY Summer League

1996 Romania 579. Let $n \geq 2$ be a given integer. Find all polynomials $P(x) = a_n x^n + \cdots + a_1 x + a_0$ with real non-zero coefficients such that the polynomial

$$P(x) - [P_1(x) \cdot P_2(x) \cdots P_{n-1}(x)] \text{ is the constant polynomial,}$$

where the polynomials $P_1, P_2, \ldots, P_{n-1}$ are defined by

$$\begin{aligned} P_1(x) &= a_1 x + a_0, \\ P_2(x) &= a_2 x^2 + a_1 x + a_0, \\ \vdots \quad & \quad \vdots \\ P_{n-1}(x) &= a_{n-1} x^{n-1} + \cdots + a_1 x + a_0. \end{aligned}$$

KACY Summer League

1999 Iran TST 580. Given a polynomial $P(x)$ of degree $n \geq 1$ with integer coefficients and n distinct integer roots such that $P(0) = 0$, find all integer roots of $P(P(x)) = 0$.

KACY Summer League

1999 Iran TST 581. Given a real number $r \geq 0$, find all polynomials $P(x)$ with real non-negative coefficients such that

a) For all $x \geq 0$, we have $P(x) \leq x^r$, and also $P(0) = 0, P(1) = 1$.

b) For all $x \geq 0$, we have $P(x) \geq x^r$, and also $P(0) = 0, P(1) = 1$.

KACY Summer League

1990 Iran 582. Let α be a root of the cubic equation $x^3 - 5x + 3 = 0$ and let $f(x)$ be a polynomial with rational coefficients. Prove that if $f(\alpha)$ is a root of the same mentioned cubic equation, then $f(f(\alpha))$ is also a root of the same cubic $x^3 - 5x + 3 = 0$.

KACY Summer League

KACY–I 583. Prove that the following polynomial does not have real roots:

$$P(x) = x^6 - x^5 + x^4 - x^3 + x^2 - x + \frac{3}{4}.$$

KACY Summer League

1976 Bulgaria 584. Find all polynomials $P(x)$ such that

$$P(x^2 - 2x) = (P(x-2))^2.$$

KACY Summer League

KACY–I 585. If $f(x)$ is a non-constant polynomial with integer coefficients, prove that one can find infinitely many prime numbers p such that the modular arithmetic equation $f(x) \equiv 0 \pmod{p}$ has integer solutions for x.

KACY Summer League

1998 Bulgaria 586. Let $f(x) = x^3 - 3x + 1$. Find the real and distinct roots of $f(f(x)) = 0$.

KACY Summer League

1998 India 587. For all positive integers $m \geq n \geq 2$ prove that the number of polynomials of degree $2n - 1$ whose coefficients are distinct and chosen from $\{1, 2, \ldots, m\}$, and are also divisible by the polynomial $x^{n-1} + \cdots + x + 1$, is equal to:

$$2^n n! \left(4\binom{m+1}{n+1} - 3\binom{m}{n}\right).$$

KACY Summer League

1999 Hungary 588. Does there exist a polynomial $P(x)$ with integer coefficients so that

$$P(10) = 400, \quad P(14) = 440, \quad P(18) = 520?$$

KACY Summer League

1997 IMO Shortlist 589. Let p be a prime number and $f(x)$ a polynomial with integer coefficients and of degree n such that:

(i) $f(0) = 0$ and $f(1) = 1$; and

(ii) For all positive integers n, $f(n) \equiv 0$ or $1 \pmod p$.

Prove that $d \geq p - 1$.

KACY Summer League

1997 Romania 590. Let $n \geq 2$ be a given integer. Find all polynomials $P(x) = a_n x^n + \cdots + a_1 x + a_0$ with positive integer coefficients such that for each $k = 1, 2, \ldots, n-1$, we have $a_k = a_{n-k}$. Prove that there exist infinitely many pairs (x, y) of positive integers for which

$$x \mid P(y) \qquad \text{and} \qquad y \mid P(x).$$

KACY Summer League

1998 Austrian–Polish 591. Find all pairs (a, b) of positive integers such that the polynomial $x^3 - 17x^2 + ax - b^2 = 0$ has three (not necessarily distinct) integer roots.

KACY Summer League

KACY–I 592. Let $\{a_i\}_{i=0}^{n}$ be a sequence of $n \geq 2$ real numbers with $a_n \neq 0$ and

$$a_{n-1}^2 - \frac{2n}{n-1} a_n a_{n-2} < 0.$$

Prove that the polynomial $P(x)$ defined below has at most $n-2$ distinct real roots:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0.$$

KACY Summer League

2000 IMO Shortlist 593. For a polynomial P of degree 2000 with distinct real coefficients let $M(P)$ be the set of all polynomials that can be produced from P by permutation of its coefficients. A polynomial P will be called n-independent if $P(n) = 0$ and we can get from any $Q \in M(P)$ a polynomial Q_1 such that $Q_1(n) = 0$ by interchanging at most one pair of coefficients of Q. Find all integers n for which n-independent polynomials exist.

KACY Summer League

1995 Czech And Slovak Mathematical Olympiad 594. Find all real numbers p for which the equation

$$x^3 - 2p(p+1)x^2 + (p^4 + 4p^3 - 1)x - 3p^3 = 0,$$

has three distinct real roots which are sides of a right triangle.

KACY Summer League

1995 Greece 595. If the equation $ax^2 + (c-b)x + (e-d) = 0$ has real roots greater than 1, prove that the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ has at least one real root.

KACY Summer League

KACY–I 596. Let m be a non-negative integer and let n be an even positive integer. Prove that the polynomial

$$P(x) = \frac{x^n}{(n+1)^m} + \frac{x^{n-1}}{n^m} + \cdots + \frac{x}{2^m} + 1,$$

does not have any real roots, but if n is odd, then this polynomial has precisely one real root.

KACY Summer League

1996 Austrian–Polish 597. The sequence of P_n of polynomials is defined initially by $P_0(x) = 0$ and $P_1(x) = x$, and then recursively for $n \geq$,

$$P_n(x) = xP_{n-1}(x) + (1-x)P_{n-2}(x).$$

For any given positive integer n, find all x such that $P_n(x) = 0$.

KACY Summer League

2000 Iran 598. Does there exist a polynomial $f(x)$ of degree 1999 with integer coefficients such that for all integers n, the numbers $f(n), f(f(n)), f(f(f(n))), \ldots$ are pairwise coprime. That is, no two of them share integer divisors.

KACY Summer League

KACY–I 599. Find all polynomials $P(x)$ with real coefficients such that for all $x \in \mathbb{R}$,

$$P(x) \cdot P(x+1) = P(x^2).$$

KACY Summer League

1999 Iran 600. Find all polynomials $P(x)$ with real coefficients for which there exists a positive integer n such that for all $x \in \mathbb{R}$,

$$xP(x-n) = (x-1)P(x).$$

KACY Summer League

KACY–I 601. Let $P(x)$ be a polynomial of degree n with rational coefficients and n roots $\alpha_1, \alpha_2, \ldots, \alpha_n$. Prove that for all positive integers m, the expression $\alpha_1^m + \alpha_2^m + \cdots + \alpha_n^m$ is a rational number.

KACY Summer League

1997 Iran 602. For three integers a, b, c with $a \neq 1$, we know that one of the roots of the polynomial $P(x) = x^3 + ax^2 + bx + c$ equals the product of the other two roots. Prove that $2P(-1)$ is divisible by $P(1) + P(-1) - 2(1 + P(0))$.

KACY Summer League

2001 Iran Second Round 603. Find all polynomials $P(x)$ with real coefficients such that for all $x \in \mathbb{R}$,

$$P(2P(x)) = 2P(P(x)) + 2(P(x))^2.$$

KACY Summer League

1997 Iran 604. Let $P(z)$ be a polynomial with real coefficients such that $P(0) = 1$ and for all complex numbers z with $|z| = 1$, we have $|P(z)| = 1$. Prove that $P(z) \equiv 1$.

KACY Summer League

1997 Iran 605. For two monic polynomials $P(x)$ and $Q(x)$ with rational coefficients which are both irreducible, prove that if α is a root of $P(x)$ and β is a root of $Q(x)$, where $\alpha + \beta$ is rational, then the polynomial $(P(x))^2 - (Q(x))^2$ has a rational root.

KACY Summer League

1993 Iran Second Round 606. Let $f(x)$ and $g(x)$ be two polynomials with real coefficients such that for infinitely many rational values of x, the fraction $\frac{f(x)}{g(x)}$ is rational. Prove that $\frac{f(x)}{g(x)}$ can be written as the ratio of two polynomials with rational coefficients.

KACY Summer League

1994 Iran Third Round 607. Find all polynomials $f(x)$ with real roots such that for all $x \in \mathbb{R}$,

$$f(x^2 - 1) = f(x) \cdot f(-x).$$

KACY Summer League

1979 Bulgaria 608. Find all polynomials $P(x)$ with real coefficients such that for all $x \in \mathbb{R}$,

$$P(x) \cdot P(2x^2) = P(2x^3 + x).$$

KACY Summer League

1979 Hungary 609. Prove that if the polynomial $P(x)$ with real coefficients is always non-negative for all $x \in \mathbb{R}$, then we can write

$$P(x) = (Q_1(x))^2 + (Q_2(x))^2 + \cdots + (Q_n(x))^2,$$

where $Q_1(x), Q_2(x), \ldots, Q_n(x)$ are polynomials with real coefficients.

KACY Summer League

1998 Bulgaria 610. For all positive integers n, the two-variable polynomial $P_n(x, y)$ is defined initially by $P_1(x, y) = 1$ and recursively for $n \geq 1$ by

$$P_{n+1}(x, y) = (x + y - 1)(y + 1)P_n(x, y + 2) + (y - y^2)P_n(x, y).$$

Prove that for each $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$,

$$P_n(x, y) = P_n(y, x).$$

KACY Summer League

1998 Canada 611. Find all real roots of the following equation:

$$x = \sqrt{x - \frac{1}{x}} + \sqrt{1 - \frac{1}{x}}.$$

KACY Summer League

1999 Japan 612. For each positive integer n, prove that the polynomial

$$f(x) = (x^2 + 1^2)(x^2 + 2^2) \cdots (x^2 + n^2) + 1,$$

cannot be written as a product of two non-constant polynomials with real coefficients.

KACY Summer League

KACY–I 613. Define a sequence $\{P_n\}_{n=0}^{\infty}$ of polynomials initially by $P_0(x) = 1$ and $P_1(x) = x + 1$, and recursively for $n \geq 1$ by

$$P_{n+1}(x) = P_n(x) + xP_{n-1}(x).$$

Prove that for all $n \in \mathbb{N}$, all the roots of $P_n(x)$ are real.

KACY Summer League

1996 Romania 614. For real numbers a, b, c with $a \neq 0$, we know that a and $4a + 3b + 2c$ have the same sign. Prove that the polynomial $ax^2 + bx + c$ cannot have two roots in the interval $(1, 2)$.

KACY Summer League

1997 Iran 615. Let $P(x) = ax^3 + bx^2 + cx + d$ be a polynomial with rational coefficients. If the three roots of $P(x)$ are x_1, x_2, x_3 such that x_1/x_2 is a rational number not equal to 0 or 1, prove that all three roots are rational.

KACY Summer League

1997 Iran 616. Find all polynomials $P(x)$ with real coefficients such that for all $x \in \mathbb{R}$,

$$xP(x)P(1 - x) + x^3 + 100 \geq 0.$$

KACY Summer League

1981 USSR 617. Consider the two-variable polynomial

$$P(x,y) = 4 + x^2y^4 + x^4y^2 - 3x^2y^2.$$

a) Find the smallest value that this polynomial can take.

b) Prove that $P(x,y)$ cannot be written as a sum of squares of two-variable polynomials in x and y.

KACY Summer League

KACY–I 618. Can we find a polynomial $f(x)$ such that

$$f(f'(x)) = 27x^6 - 27x^4 + 6x^2 + 2?$$

KACY Summer League

1976 International Mathematics Olympiad 619. Let $P_1(x) = x^2 - 2$ and $P_j(x) = P_1(P_{j-1}(x))$ for $j = 2, 3, \dots$ Prove that for any positive integer n the roots of the equation $P_n(x) = x$ are all real and distinct.

KACY Summer League

KACY–I 620. Let $P(x)$ be a polynomial of degree 7 such that for seven distinct integer values of x, we have $P(x)$ equal to either $+1$ or -1. Prove that $P(x)$ cannot be factorized as a product of two polynomials with integer coefficients.

KACY Summer League

KACY–I 621. Find all polynomials $P(x)$ of the form

$$P(x) = x^n + nx^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n,$$

such that if $r_1, r_2, \dots, r_n$ are the roots of $P(x)$, then we have

$$r_1^{16} + r_2^{16} + \cdots + r_n^{16} = n.$$

KACY Summer League

1994 Romania 622. Let a, b, c and A, B, C be positive real numbers such that the quadratic polynomials $p(x) = ax^2 + bx + c$ and $P(x) = Ax^2 + Bx + C$ have real roots. Prove that for any u that lies between the roots of $p(x)$ and for any U that lies between the roots of $P(x)$, we have

$$(au + AU)\left(\frac{c}{u} + \frac{C}{U}\right) \leq \left(\frac{b+B}{2}\right).$$

KACY Summer League

1998 Vietnam 623. Prove that for all odd positive integers n, there exists a unique polynomial $P(x)$ of degree n and with real coefficients such that for all real $x \neq 0$,

$$P\left(x - \frac{1}{x}\right) = x^n - \frac{1}{x^n}.$$

Moreover, find out when the given statement is true for even n.

KACY Summer League

1998 Czech And Slovak 624. Let $P(x)$ be a polynomial of degree $n \geq 5$ with integer coefficients which has n distinct integer roots. If we assume that $P(0) = 0$, find all integer roots of $P(P(x))$.

KACY Summer League

1998 Russia 625. Find all two-variable polynomials $P(x, y)$ such that for all $x, y \in \mathbb{R}$,

$$P(x + y, x - y) = P(x, y).$$

KACY Summer League

1998 Russia 626. Does there exist a polynomial $P(x)$ with integer coefficients and a positive integer $k > 1$ such that the numbers $P(k), P(k^2), P(k^3), \ldots$ are pairwise coprime?

KACY Summer League

KACY–I 627. If $f(x)$ is a non-constant polynomial with integer coefficients such that for all primes p, we know that $f(p)$ is a power of a prime number, prove that there exists a positive integer n such that $f(x) = x^n$.

KACY Summer League

1979 Hungary 628. Let $a \neq 0$ be a real number and define $P(x) = ax^2 + bx + c$. Prove that for all positive integers n, there cannot be more than one polynomial $Q(x)$ of degree n for which

$$Q(P(x)) = P(Q(x)).$$

KACY Summer League

1983 Romania 629. Let $\{F_n\}_{n=1}^{\infty}$ denote the Fibonacci sequence defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 2$. We know that for the polynomial $P(x)$ of degree 990, we have $P(k) = F_k$ for $k = 992, 993, \ldots, 1982$. Prove that

$$P(1983) = F_{1983} - 1.$$

KACY Summer League

1977 Bulgaria 630. Let $Q(x)$ be a non-zero polynomial. Prove that for all positive integers n, the polynomial $P(x) = (x-1)^n Q(x)$ has at least $n+1$ non-zero coefficients.

KACY Summer League

1985 Sweden 631. Let $P(x)$ be a polynomial of degree n such that for all $x \in \mathbb{R}$, we have $P(x) \geq 0$. Prove, for all $x \in \mathbb{R}$, that

$$P(x) + P'(x) + P''(x) + \cdots + P^{(n)}(x) \geq 0.$$

KACY Summer League

2000 Iran 632. Let $P(x)$ be a polynomial with integer coefficients. Prove that the polynomial

$$Q(x) = P(x^4) \cdot P(x^3) \cdot P(x^2) \cdot P(x) + 1,$$

does not have any integer roots.

KACY Summer League

2003 Poland 633. Define $W(x) = x^4 - 3x^3 + 5x^2 - 9x$. Find all pairs (a, b) of distinct integers such that $W(a) = W(b)$.

KACY Summer League

2000 Austrian–Polish 634. Find all polynomials $P(x)$ with real coefficients that satisfy the following condition: there exists a positive integer n such that the following equation holds for infinitely many real values of x:

$$\sum_{k=1}^{2n+1}(-1)^k\left\lfloor\frac{k}{2}\right\rfloor P(x+k)=0.$$

KACY Summer League

2000 Poland 635. Let $P(x)$ be a polynomial of odd degree such that

$$P(x^2-1)=(P(x))^2-1.$$

Prove that $P(x)=x$ for all $x\in\mathbb{R}$.

KACY Summer League

1996 Russia 636. Prove that for all polynomials $P(x)$ of degree 10 with integer coefficients, there exists an infinite arithmetic progression which does not contain the following numbers:

$$\ldots,P(-1),P(0),P(1),P(2),\ldots$$

KACY Summer League

1997 Romania 637. Let $a_0,a_1,\ldots,a_n$ be complex numbers such that for any complex z with $|z|\le 1$, we have

$$|a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0|\le 1.$$

Prove that for all $k=0,1,\ldots,n$, we have $|a_k|\le 1$ and

$$|a_0+a_1+\cdots+a_n-(n+1)a_k|\le n.$$

KACY Summer League

1994 Vietnam 638. Let $P(x)$ be a polynomial of degree 4 with 4 positive real roots. Prove that the polynomial

$$\frac{1-4x}{x^2}P(x)+\left(1-\frac{1-4x}{x^2}\right)P'(x)-P''(x)$$

also has 4 positive real roots.

KACY Summer League

KACY–I 639. Let $P(x)$ be a quadratic polynomial such that for a sequence of rational numbers $q_0, q_1, q_2, \ldots$ we have $q_n = P(q_{n+1})$ for all $n \geq 1$. Prove that there exists a positive integer k such that for all $n \geq 1$, we have $q_{n+k} = q_n$.

KACY Summer League

1994 China 640. For all polynomials $f(x) = x_0x^n + c_1x^{n-1} + \cdots + c_n$ of degree n with complex coefficients, prove that there exists a complex number x_0 such that $|x_0| \leq 1$ and
$$|f(x_0)| \geq |c_0| + |c_n|.$$

KACY Summer League

KACY–I 641. Let $f(x)$ be a polynomial with rational coefficients such that for some $\alpha \in \mathbb{R}$,
$$\alpha^3 - 1992\alpha + 33 = (f(\alpha))^3 - 1992f(\alpha) + 33 = 0.$$
Prove that for all $n \geq 1$, we have
$$(f^n(\alpha))^3 - 1992f^n(\alpha) + 33 = 0,$$
where $f^n(\alpha) = \underbrace{f(f(\cdots f(\alpha)))}_{n \text{ times}}$.

KACY Summer League

KACY–I 642. Prove that if a symmetric two-variable polynomial $P(x, y)$ is divisible by $x - y$, then $P(x, y)$ is also divisible by $(x - y)^2$.

KACY Summer League

KACY–I 643. Find all polynomials $P(x)$ with $P(0) = 0$ and
$$P(x^2 + 1) = (P(x))^2 + 1.$$

KACY Summer League

1986 Czech And Slovak 644. Let $P(x)$ be a polynomial with integer coefficients of degree $n \geq 3$. If $x_1, x_2, \ldots, x_m$ (with $m \geq 3$) are different integers such that
$$P(x_1) = P(x_2) = \cdots = P(x_m) = 1,$$
prove that P cannot have integer roots.

KACY Summer League

KACY–I 645. Define polynomial $P(x)$ of degree n with integer coefficients by

$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n.$$

If for two real numbers $\alpha > \beta$ we have $|P(\alpha)| = |P(\beta)| = 1$, and $P(x)$ has a rational root r, then prove that $\alpha - \beta$ equals either 1 or 2, and that $r = (\alpha + \beta)/2$.

KACY Summer League

KACY–I 646. Prove that there does not exist a polynomial P such that $P(x)$ is prime for all $x \in \{0, 1, 2, \dots\}$.

KACY Summer League

KACY–I 647. Let $f(x)$ be a non-constant polynomial with integer coefficients such that $f(0) > 0$. Prove that there exists a sequence $p_1, p_2, p_3, \dots$ of prime numbers and a sequence $a_1, a_2, a_3, \dots$ of pairwise coprime positive integers such that for all $n = 1, 2, 3, \dots,$ we have

$$f(p_n) = a_1a_2 \cdots a_n.$$

KACY Summer League

1994 Iran First Round 648. Let a, b, c be real numbers such that $9a + 11b + 29c = 0$. Prove that the cubic polynomial $ax^3 + bx + c$ has a root in the interval $[0, 2]$.

KACY Summer League

1998 Bulgaria 649. Find all positive integers n such that $x^n + 64$ can be factorized into a product of two polynomials with integer coefficients.

KACY Summer League

1998 India 650. Let N be a positive integer such that $N + 1$ is a prime. Assume that for $i = 0, 1, 2, \dots, N$, we have $a_i \in \{0, 1\}$ and not all a_i are equal to each other. Define the polynomial $f(x)$ so that for each $i = 0, 1, 2, \dots, N$, we have $f(i) = a_i$. Show that the degree of $f(x)$ is at least N.

KACY Summer League

1998 Romania 651. For all positive integers n, prove that the polynomial

$$f(x) = (x^2 + x)^{2^n} + 1,$$

cannot be factorized into a product of two non-constant polynomials with integer coefficients.

KACY Summer League

1999 China 652. Let a be a real number. Let $(f_n(x))_{n\geq 0}$ be a sequence of polynomials such that $f_0(x) = 1$ and $f_{n+1}(x) = xf_n(x) + f_n(ax)$ for all non-negative integers n.

a) Prove that

$$f_n(x) = x^n f_n\left(x^{-1}\right),$$

for all non-negative integers n.

b) Find an explicit expression for $f_n(x)$.

KACY Summer League

1999 Hungary 653. If the polynomial $x^4 - 2x^2 + ax + b$ has four distinct real roots, prove that the absolute value of each of its roots is less than $\sqrt{3}$.

KACY Summer League

1999 Romania 654. Let a and n be integers and p be a prime number such that $p > |a| + 1$. Prove that the polynomial $f(x) = x^n + ax + p$ cannot be factorized into a product of two polynomials with integer coefficients.

KACY Summer League

1999 Ukraine 655. Let $P(x)$ be a polynomial with integer coefficients and assume that the sequence $\{x_n\}_{n=1}^{n=2000}$ satisfies $x_1 = x_{2000} = 1999$. If we know that $x_{n+1} = P(x_n)$, then find the value of the following sum:

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{1999}}{x_{2000}}.$$

KACY Summer League

1975 International Mathematics Olympiad 656. Determine all two-variable polynomials $P(x,y)$ so that:

a) For any real numbers t,x,y we have $P(tx,ty)=t^nP(x,y)$ where n is a positive integer, the same for all t,x,y;

b) For any real numbers a,b,c we have

$$P(a+b,c)+P(b+c,a)+P(c+a,b)=0;$$

c) $P(1,0)=1$.

KACY Summer League

KACY–I 657. Let p be an odd prime number. Prove that the polynomial

$$\sum_{1\le m,n\le p-1} x^{mn}+p-1,$$

is divisible by $x^{p-1}+x^{p-2}+\cdots+x+1$.

KACY Summer League

KACY–I 658. Find all polynomials $P(x)$ such that

$$P(x^2)+P(x)P(x+1)=0.$$

KACY Summer League

1997 Germany 659. Define $f(x)$ and $g(x)$ by

$$\begin{aligned}f(x)&=x^5+5x^4+5x^3+5x^2+1,\\ g(x)&=x^5+5x^4+3x^3-5x^2-1.\end{aligned}$$

Find all prime numbers p for which there exists an integer x with $0\le x\le p$ such that both $f(x)$ and $g(x)$ are both divisible by p. Moreover, for each such p, find all x that satisfy the condition.

KACY Summer League

1997 Ukraine 660. If we know that $ax^3 + bx^2 + cx + d$ has three distinct real roots, then how many root does the following equation have?

$$4(ax^3 + bx^2 + cx + d)(3ax + b) = (3ax^2 + 2bx + c)^2.$$

KACY Summer League

1997 British Math Olympiad 661. Find all polynomials $P(x)$ of degree 5 with distinct coefficients chosen from the set $\{1, 2, 3, \ldots, 9\}$ such that $P(x)$ is divisible by $x^2 - x + 1$.

KACY Summer League

KACY–I 662. Let $f(x)$ and $g(x)$ be single-variable polynomials with real coefficients and let $P(x, y)$ be a two-variable polynomial with real coefficients such that for all real numbers x, y,

$$f(x) - f(y) = (g(x) - g(y)) \cdot P(x, y).$$

Prove that there exists a single-variable polynomial $h(x)$ with real coefficients such that $f(x) = h(g(x))$ for all real numbers x.

KACY Summer League

KACY–I 663. Let $P(x)$ be a polynomial with real coefficients which satisfies the following inequalities:

$$P(0) > 0, \quad P(1) > P(0), \quad P(2) > 2P(1) - P(0), \quad P(3) > 3P(2) - 3P(1) + P(0).$$

Moreover, for each positive integer n, we know that

$$P(n + 4) > 4P(n + 3) - 6P(n + 2) + 4P(n + 1) - P(n).$$

Prove that $P(n) > 0$ for all positive integers n.

KACY Summer League

1995 Ireland 664. Let a, b, c be complex numbers such that all roots z of the polynomial

$$P(x) = x^3 + ax^2 + bx + c,$$

satisfy the equation $|z| = 1$. Prove that all roots ω of the polynomial

$$Q(x) = x^3 + |a|x^2 + |b|x + |c|,$$

also satisfy $|\omega| = 1$.

KACY Summer League

1995 Japan 665. Let k, n be integers such that $1 \le k \le n$, and let $a_1, a_2, \dots, a_k$ be numbers satisfying the following equations:

$$\begin{cases} a_1 + a_2 + \cdots + a_k & = n, \\ a_1^2 + a_2^2 + \cdots + a_k^2 & = n, \\ \quad \vdots & \quad \vdots \\ a_1^k + a_2^k + \cdots + a_k^k & = n. \end{cases}$$

Prove that

$$(x + a_1)(x + a_2) \cdots (x + a_k) = x^k + \binom{n}{1} x^{k-1} + \binom{n}{2} x^{k-2} + \cdots + \binom{n}{k}.$$

KACY Summer League

KACY–I 666. For each positive integer n, define

$$f(n) = 1! + 2! + \cdots + n!.$$

Find polynomials $P(x)$ and $Q(x)$ such that for all positive integers n,

$$f(n+2) = P(n) f(n+1) + Q(n) f(n).$$

KACY Summer League

KACY–I 667. Find all polynomials $P(x)$ such that

$$1 + P(x) = \frac{P(x-1) + P(x+1)}{2}.$$

KACY Summer League

KACY–I 668. Is it possible to factorize $P(x) = x^{100} + 5x^{99} + 2x + 2$ into a product of two polynomials with integer coefficients?

KACY Summer League

KACY–I 669. If P and Q are two polynomials such that for all $x \in \mathbb{R}$,

$$P(x^2 + x + 1) = Q(x^2 - x + 1),$$

prove that P and Q are constant polynomials.

KACY Summer League

KACY–I 670. Find all polynomials $P(x)$ with real coefficients such that $P(x)$ has distinct real roots $r_1 > r_2 > \cdots > r_n$ and also, $(r_i + r_{i+1})/2$ are roots of $P'(x)$ for $i = 1, 2, \ldots, n-1$.

KACY Summer League

1997 Bulgaria 671. For integer $n \geq 2$, consider the polynomial

$$P_n(x) = \binom{n}{2} + \binom{n}{5}x + \binom{n}{8}x^2 + \cdots + \binom{n}{3k+2}x^{3k}, \quad \text{where} \quad k = \left\lfloor \frac{n-2}{3} \right\rfloor.$$

a) Prove that $P_{n+3}(x) = 3P_{n+2}(x) - 3P_{n+1}(x) + (x+1)P_n(x)$.

b) Find all integers a such that $P_n(a^3)$ is divisible by $3^{\lfloor \frac{n-1}{2} \rfloor}$ for all $n \geq 3$.

KACY Summer League

KACY–I 672. Find all two-variable polynomials $P(x, y)$ with real coefficients such that for all $x, y \in \mathbb{R}$,

$$P(x, y) = P(x+1, y+1).$$

KACY Summer League

KACY–I 673. Let $p(x)$ and $q(x)$ be non-zero polynomials such that for all $x \in \mathbb{R}$,

$$p(x^2 + x + 1) = p(x) \cdot q(x).$$

Prove that the degree of $p(x)$ is even.

KACY Summer League

1999 China 674. Determine the maximum value of λ such that if $f(x) = x^3 + ax^2 + bx + c$ is a cubic polynomial with all its roots non-negative, then

$$f(x) \geq \lambda(x-a)^3,$$

for all $x \geq 0$. Find the equality condition.

KACY Summer League

1999 Poland 675. Let $P(x) = 2x^3 - 3x^2 + 2$ and define the sets

$$S = \{P(n) \mid n \in \mathbb{N}, n \leq 999\},$$
$$T = \{n^2 + 1 \mid n \in \mathbb{N}\},$$
$$U = \{n^2 + 2 \mid n \in \mathbb{N}\}.$$

Prove that the sets $S \cap T$ and $S \cap U$ have the same number of elements.

KACY Summer League

1999 Vietnam 676. Let a and b be real numbers such that all the roots of the following polynomial are positive real numbers:

$$P(x) = ax^3 - x^2 + bx - 1.$$

Find the least value of the fraction

$$\frac{5a^2 - 3ab + 2}{a^2(b - a)}.$$

KACY Summer League

1995 Korea 677. Let a and b be integers and p be a prime number such that:

(i) p is the greatest common divisor of a and b; and

(ii) p^2 divides a.

Prove that the polynomial $x^{n+2} + ax^{n+1} + bx^n + a + b$ cannot be decomposed into the product of two polynomials with integer coefficients and degree greater than 1.

KACY Summer League

KACY–I 678. For a monic polynomial $P(x)$ of degree n with non-negative coefficients and n real roots, we have $P(0) = 1$. Prove that for all integers k,

$$P(k) \geq (k + 1)^n.$$

KACY Summer League

KACY–I 679. Let $P(x)$ be a polynomial with integer coefficients such that for all primes q, we know that $P(q)$ is a power of 2. Prove that $P(x)$ must be a constant polynomial.

KACY Summer League

KACY–I 680. Let $P(x)$ and $Q(x)$ be polynomials with real coefficients such that either $P(x)$ and $Q(x)$ are both integers or they are both non-integers. Prove that $P(x) = \pm Q(x)$.

KACY Summer League

2000 Romania TST 681. Let P, Q be two monic polynomials with complex coefficients such that $P(P(x)) = Q(Q(x))$ for all x. Prove that $P = Q$.

KACY Summer League

1986 USSR 682. If the roots of the quadratic polynomial

$$P(x) = x^2 + ax + b + 1,$$

are positive integers, prove that $a^2 + b^2$ is a composite number.

KACY Summer League

KACY–I 683. The value of polynomial $P(x)$ is a perfect square for all positive integers x. Prove that there must exist a polynomial $Q(x)$ such that $P(x) = (Q(x))^2$.

KACY Summer League

1998 Iran 684. Prove that for any non-constant polynomial $f(x)$ with integer coefficients, there exists a sequence $p_1 < p_2 < p_3 < \cdots$ of primes and a sequence $n_1 < n_2 < n_3 < \cdots$ of positive integers such that $p_k \mid f(n_k)$ for all $k \in \mathbb{N}$.

KACY Summer League

1996 Taiwan 685. Show that for any real numbers $a_3, a_4, \ldots, a_{85}$, not all the roots of the equation

$$a_{85}x^{85} + a_{84}x^{84} + \cdots + a_3x^3 + 3x^2 + 2x + 1 = 0,$$

are real roots.

KACY Summer League

1998 Iran 686. The sequence $a_0, a_1, a_2, \ldots$ satisfies $2a_i = a_{i-1} + a_{i+1}$. Define the polynomial $P_n(x)$ for each $n \in \mathbb{N}$ by

$$P_n(x) = \sum_{i=0}^{n} a_i \binom{n}{i} x^i (1-x)^{n-i}.$$

Prove that $P_n(x)$ is linear for all $n \in \mathbb{N}$.

KACY Summer League

1995 Austrian–Polish 687. Let $P(x) = x^4+x^3+x^2+x+1$ and prove that there exist non-constant polynomials $Q(x)$ and $R(x)$ with integer coefficients such that for all $x \in \mathbb{R}$,

$$Q(x) \cdot R(x) = P(5x^2).$$

KACY Summer League

1995 Austrian–Polish 688. Find all polynomials $P(x)$ with real coefficients such that for all $x \neq 0$,

$$(P(x))^2 + \left(P\left(\frac{1}{x}\right)\right)^2 = P(x^2) \cdot P\left(\frac{1}{x^2}\right).$$

KACY Summer League

1995 Balkan 689. Let a and b be positive integers with $a > b$ and having the same parity. Prove that the solutions of the equation

$$x^2 - (a^2 - a + 1)(x - b^2 - 1) - (b^2 + 1)^2 = 0,$$

are positive integers, none of which is a perfect square.

KACY Summer League

1998 Iran 690. The determinant of the cubic polynomial $P(x) = x^3+ax^2+bx+c$ is defined by

$$\Delta = 18abc - 4a^3c + a^2b^2 - 4b^3 - 27c^3.$$

Prove that if $\Delta \geq 0$, then $P(x)$ will have real roots.

KACY Summer League

1998 Iran 691. Find the smallest positive integer d for which there exists a monic polynomial of degree d such that for all $n \in \mathbb{N}$, we have $100 \mid f(n)$.

KACY Summer League

KACY–I 692. Let $P(x)$ be a polynomial with integer coefficients such that $P(0) = P(1) = 1$. For an arbitrary integer a_0, define $a_{n+1} = P(a_n)$ for all integers $n \geq 0$. Prove that the elements of the sequence $\{a_i\}_{i=0}^{\infty}$ are pairwise coprime.

KACY Summer League

1977 USSR 693. Two monic polynomials $P(x)$ and $Q(x)$ are *commutable* if we have $P(Q(x)) = Q(P(x))$ for all real x.

a) For any real α, find all monic polynomials $Q(x)$ of maximum degree 3 which are commutable with the polynomial $P(x) = x^2 - \alpha$.

b) For an arbitrary quadratic polynomial $P(x)$ and a positive integer k, prove that there exists at most one polynomial of degree k which is commutable with $P(x)$.

c) Find polynomials of degree 4 and 8 which are commutable with a given quadratic polynomial.

d) Let $Q(x)$ and $R(x)$ be polynomials that are both commutable with a given quadratic polynomial $P(x)$. Prove that $Q(x)$ and $R(x)$ are commutable.

e) Let $P_2(x) = x^2 - 2$ and for all positive integers k, let $P_k(x)$ be a polynomial of degree k. Prove that there exist an infinite sequence $P_2(x), P_3(x), P_4(x), \ldots$ of polynomials each two of which are commutable.

KACY Summer League

1996 IMO Shortlist 694. Let $P(x)$ be the cubic real-coefficient polynomial

$$P(x) = ax^3 + bx^2 + cx + d.$$

Prove that if $|P(x)| \leq 1$ for all x such that $|x| \leq 1$, then,

$$|a| + |b| + |c| + |d| \leq 7.$$

KACY Summer League

1996 Poland 695. The polynomial $P(x)$ of degree n satisfies the following equation:

$$P(k) = \frac{1}{k}, \qquad \text{for} \quad k = 2^0, 2^1, 2^2, \ldots, 2^n.$$

Find $P(0)$.

KACY Summer League

1996 Poland 696. Let $P(x)$ be a non-constant polynomial with integer coefficients, and let $m \geq 1$ be a given integer. Prove that if $P(x)$ has at least three distinct integer roots, then $P(x) + 5^m$ will have at least one integer root.

KACY Summer League

1998 Baltic Way 697. Let P be a polynomial of degree 6 and let a, b be real numbers such that $0 < a < b$. Suppose that $P(a) = P(-a), P(b) = P(-b), P'(0) = 0$. Prove that $P(x) = P(-x)$ for all real x.

KACY Summer League

1995 UNESCO 698. Let $p \geq 2$ and $a_0, a_1, \ldots, a_n$ be non-negative integers and define $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$. Prove that if the numbers

$$\sqrt[p]{f(0)}, \sqrt[p]{f(1)}, \sqrt[p]{f(2)}, \ldots$$

are all rational, then there exists a polynomial $g(x)$ with integer coefficients such that $f(x) = (g(x))^p$.

KACY Summer League

1995 Russia 699. Let f, g, h be quadratic polynomials. Is it possible for $x = 1, 2, \ldots, 8$ to be the roots of the equation $f(g(h(x))) = 0$?

KACY Summer League

1998 Iran 700. Let $P(x)$ and $Q(x)$ be two polynomials with complex coefficients and let $a, b \geq 2$ be integers such that for all $x \in \mathbb{R}$,

$$(P(x))^a - (Q(x))^b = x.$$

Prove that $a = b = 2$.

KACY Summer League

1983 IMO Longlist 701. Let p and q be integers. Show that there exists an interval I of length $1/q$ and a polynomial P with integral coefficients such that

$$\left| P(x) - \frac{p}{q} \right| < \frac{1}{q^2},$$

for all $x \in I$.

KACY Summer League

1998 Poland 702. Let $n \geq 2$ be a positive integer. Find all polynomials

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

with n real roots all less than or equal to -1, and such that

$$a_0^2 + a_1 a_n = a_n^2 + a_0 a_{n-1}.$$

KACY Summer League

1998 Baltic Way 703. Let P be a polynomial with integer coefficients. Suppose that for $n = 1, 2, 3, \ldots, 1998$ the number $P(n)$ is a three-digit positive integer. Prove that the polynomial P has no integer roots.

KACY Summer League

1996 IMO Shortlist 704. Let $a_1, a_2, \ldots, a_n$ be non-negative reals, not all zero. Show that that

a) The polynomial

$$p(x) = x^n - a_1 x^{n-1} + \cdots - a_{n-1} x - a_n,$$

has precisely 1 positive real root R.

b) Let

$$A = \sum_{i=1}^{n} a_i \qquad \text{and} \qquad B = \sum_{i=1}^{n} i a_i.$$

Show that $A^A \leq R^B$.

KACY Summer League

1997 Romania 705. Find all polynomials $f(x)$ with integer coefficients such that $f(x)$ is bijective (that is, both injective and surjective) and for some real constant a and all x,

$$(f(x))^2 = f(x^2) - 2f(x) + a.$$

KACY Summer League

1995 Romania 706. Let $m, n \geq 2$ be integers. Find the number of polynomials of degree $2n - 1$ with distinct coefficients from the set $\{1, 2, \ldots, m\}$ which are divisible by $x^{n-1} + \cdots + x + 1$.

KACY Summer League

1995 Romania 707. Let $f(x)$ be an irreducible monic polynomial with integer coefficients and of an odd degree greater than 3. Assume that the absolute value of the roots of $f(x)$ are greater than 1 and $f(0)$ is a square-free number (that is, not divisible by square of anything). Prove that the polynomial $g(x) = f(x^3)$ cannot be decomposed into a product of two polynomials with integer coefficients.

KACY Summer League

1998 Iran 708. Find all polynomials $P(x)$ with complex coefficients such that

$$P(2x^2 - 1) = \frac{(P(x))^2}{2} - 1.$$

KACY Summer League

1998 Iran 709. Let $a_0, a_1, \dots, a_n$ be real numbers such that

$$0 < a_0 < -\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{a_{2k}}{2k+1}.$$

Prove that the polynomial $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ has a real root in the interval $[-1, 1]$.

KACY Summer League

KACY–I 710. Let $P(x, y)$ be a two-variable polynomial. Prove or disprove the following statement: the inequality

$$|x^y - y^x| \leq |P(x, y)|,$$

has only a finite number of solutions (x, y) in which x and y are distinct integers with $x, y \geq 2$.

KACY Summer League

1988 IMO Longlist 711. This problem comes in four questions:

a) The polynomial
$$x^{2\cdot k} + 1 + (x+1)^{2\cdot k},$$
is not divisible by $x^2 + x + 1$. Find the value of k.

b) If p, q and r are distinct roots of $x^3 - x^2 + x - 2 = 0$ the find the value of $p^3 + q^3 + r^3$.

c) If r is the remainder when each of the numbers 1059, 1417 and 2312 is divided by d, where d is an integer greater than one, then find the value of $d - r$.

d) What is the smallest positive odd integer n such that the product of
$$2^{\frac{1}{7}}, 2^{\frac{3}{7}}, \ldots, 2^{\frac{2\cdot n+1}{7}},$$
is greater than 1000?

KACY Summer League

1988 IMO Longlist 712. Let n be a positive integer. Find the number of odd coefficients of the polynomial
$$u_n(x) = (x^2 + x + 1)^n.$$

KACY Summer League

1998 Baltic Way 713. Let $P_k(x) = 1 + x + x^2 + \ldots + x^{k-1}$. Show that
$$\sum_{k=1}^{n} \binom{n}{k} P_k(x) = 2^{n-1} P_n\left(\frac{x+1}{2}\right),$$
for every real number x and every positive integer n.

KACY Summer League

1998 Iran 714. Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ with $n \geq 2$ and positive real coefficients a_i such that all the roots of $P(x)$ are positive real numbers in the interval $(0, 1)$. Prove that if $0 \leq k \leq n-2$, then
$$\sum_{i=k}^{n-2} \binom{i}{k} a_i > 0.$$

KACY Summer League

1998 Iran 715. Let $P(x)$ be a polynomial with rational coefficients such that for any rational r, there exists rational s such that $P(s) = r$. Prove that $P(x)$ is linear.

KACY Summer League

1998 Iran 716. For two polynomials f and g with rational coefficients, if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of f, and we have

$$g(\alpha_1) = g(\alpha_2) = \cdots = g(\alpha_n) = A,$$

then prove that A is a rational number.

KACY Summer League

1998 Iran 717. Define

$$f(x) = 1 - x + x^2 - x^3 + \cdots + x^{16} - x^{17}.$$

Let $y = x+1$ and $f(y) = a_0+a_1y+\cdots+a_{17}y^{17}$. Find the coefficients $a_0, a_1, \ldots, a_{17}$.

KACY Summer League

1998 Iran 718. Let $P(x) = a_nx^n + \cdots + a_1x + a_0$ be a polynomial with integer coefficients and $a_0 \neq 0$. If we know that

$$|a_{n-1}| > 1 + |a_{n-2} + \cdots + |a_1| + |a_0|,$$

prove that $P(x)$ is irreducible.

KACY Summer League

1998 Iran 719. Prove that for any prime p with decimal representation

$$p = (a_na_{n-1} \cdots a_1a_0)_{10},$$

the polynomial $f(x) = a_nx^n + \cdots + a_1x + a_0$, is irreducible.

KACY Summer League

1998 Iran 720. Let f be a polynomial with real coefficients among which $2m$ consecutive coefficients (except for the first and last ones) are zero. Prove that f has at least $2m$ real roots.

KACY Summer League

1998 Iran 721. Let f be a polynomial with real coefficients and four of its consecutive coefficients form an arithmetic progression. Prove that f has at least one non-real root.

KACY Summer League

1995 Taiwan 722. Let $P(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{C}[x]$, where $a_n = 1$. The roots of $P(x)$ are $b_1, b_2, \ldots, b_n$, where $|b_1|, |b_2|, \ldots, |b_j| > 1$ and $|b_{j+1}|, \ldots, |b_n| \leq 1$. Prove that

$$\prod_{i=1}^{j} |b_i| \leq \sqrt{|a_0|^2 + |a_1|^2 + \cdots + |a_n|^2}.$$

KACY Summer League

1995 Taiwan 723. Let $m_1, m_2, \ldots, m_n$ be mutually distinct integers. Prove that there exists a $f(x) \in \mathbb{Z}[x]$ of degree n satisfying the following two conditions:

a) $f(m_i) = -1$, for all $i = 1, 2, \ldots, n$; and

b) $f(x)$ is irreducible.

KACY Summer League

2003 APMO 724. Let a, b, c, d, e, f be real numbers such that the polynomial

$$p(x) = x^8 - 4x^7 + 7x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f,$$

factorises into eight linear factors $x - x_i$, with $x_i > 0$ for $i = 1, 2, \ldots, 8$. Determine all possible values of f.

KACY Summer League

1992 Baltic Way 725. A polynomial $f(x) = x^3 + ax^2 + bx + c$ is such that $b < 0$ and $ab = 9c$. Prove that the polynomial f has three different real roots.

KACY Summer League

1992 Baltic Way 726. Find all quartic (fourth-degree) polynomial $p(x)$ such that the following four conditions are satisfied:

(i) $p(x) = p(-x)$ for all x,

(ii) $p(x) \geq 0$ for all x,

(iii) $p(0) = 1$,

(iv) $p(x)$ has exactly two local minimum points x_1 and x_2 such that $|x_1 - x_2| = 2$.

KACY Summer League

1996 Baltic Way 727. Real numbers $x_1, x_2, \ldots, x_{1996}$ have the following property: For any polynomial W of degree 2 at least three of the numbers $W(x_1), W(x_2), \ldots, W(x_{1996})$ are equal. Prove that at least three of the numbers $x_1, x_2, \ldots, x_{1996}$ are equal.

KACY Summer League

1997 Baltic Way 728. Let P and Q be polynomials with integer coefficients. Suppose that the integers a and $a + 1997$ are roots of P, and that $Q(1998) = 2000$. Prove that the equation $Q(P(x)) = 1$ has no integer solutions.

KACY Summer League

KACY–I 729. Find all polynomials P for which $P(x^2) = P(x) \cdot P(x-1)$.

KACY Summer League

1963 Dutch Mathematical Olympiad 730. One considers for $n > 2$ the polynomial:

$$(x^2 - x + 1)^n - (x^2 - x + 2)^n + (1 + x)^n + (2 - x)^n.$$

Show that the degree of this polynomial is $2n - 2$. Moreover, assume that the polynomial is written in the form

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_{2n-2} x^{2n-2}.$$

Prove that $a_2 + a_3 + \cdots + a_{2n-2} = 0$

KACY Summer League

1970 Dutch Mathematical Olympiad 731. The equation$x^3 - x^2 + ax - 2^n = 0$ has three integer roots. Determine a and n.

KACY Summer League

1990 Dutch Mathematical Olympiad 732. A polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx$ with $a, b, c, d > 0$ is such that $f(x)$ is an integer for $x \in \{-2, -1, 0, 1, 2\}$ and $f(1) = 1$ and $f(5) = 70$.

a) Show that

$$a = \frac{1}{24}, \quad b = \frac{1}{4}, \quad c = \frac{11}{24}, \quad d = \frac{1}{4}.$$

b) Prove that $f(x)$ is an integer for all $x \in \mathbb{Z}$.

KACY Summer League

2001 Dutch Mathematical Olympiad 733. The function is given

$$f(x) = \frac{2x^3 - 6x^2 + 13x + 10}{2x^2 - 9x}.$$

Determine all positive integers x for which $f(x)$ is an integer.

KACY Summer League

1996 Belgium Flanders 734. Consider a real polynomial $p(x) = a_n x^n + \cdots + a_1 x + a_0$.

a) If $\deg(p(x)) > 2$ prove that $\deg(p(x)) = 2 + \deg(p(x+1) + p(x-1) - 2p(x))$.

b) Let $p(x)$ a polynomial for which there are real constants r, s so that for all real x we have

$$p(x+1) + p(x-1) - rp(x) - s = 0.$$

Prove that $\deg(p(x)) \leq 2$.

c) Show, with the notation of the second part, that $s = 0$ implies $a_2 = 0$.

KACY Summer League

1996 Germany 735. Prove the following statement: if a polynomial $p(x) = x^3 + Ax^2 + Bx + C$ has three real roots at least two of which are distinct, then $A^2 + B^2 + 18C > 0$.

KACY Summer League

1998 Germany 736. Let a be a positive real number. Then prove that the polynomial

$$p(x) = a^3x^3 + a^2x^2 + ax + a,$$

has integer roots if and only if $a = 1$ and determine those roots.

KACY Summer League

1979 Brazil 737. The remainder on dividing the polynomial $p(x)$ by $x^2 - (a + b)x + ab$ (where $a \neq b$) is $mx + n$. Find the coefficients m, n in terms of a, b. Find m, n for the case $p(x) = x^{200}$ divided by $x^2 - x - 2$ and show that they are integral.

KACY Summer League

1985 Brazil 738. a, b, c, d are integers. Show that $x^2 + ax + b = y^2 + cy + d$ has infinitely many integer solutions if and only if $a^2 - 4b = c^2 - 4d$.

KACY Summer League

1987 Brazil 739. Let $p(x_1, x_2, \dots, x_n)$ be a polynomial with integer coefficients. For each positive integer $r, k(r)$ is the number of n-tuples $(a_1, a_2, \dots, a_n)$ such that $0 \leq a_i \leq r - 1$ and $p(a_1, a_2, \dots, a_n)$ is prime to r. Show that if u and v are coprime then $k(u \cdot v) = k(u) \cdot k(v)$, and if p is prime then $k(p^s) = p^{n(s-1)}k(p)$.

KACY Summer League

1991 Brazil 740. Given $k > 0$, the sequence a_n is defined by its first two members and

$$a_{n+2} = a_{n+1} + \frac{k}{n}a_n.$$

a) For which k can we write a_n as a polynomial in n?

b) For which k can we write

$$\frac{a_{n+1}}{a_n} = \frac{p(n)}{q(n)},$$

where p, q are polynomials in $\mathbb{R}[X]$?

KACY Summer League

1992 Brazil 741. The equation $x^3 + px + q = 0$ has three distinct real roots. Show that $p < 0$.

KACY Summer League

1994 Brazil 742. Let $a, b > 0$ be reals such that

$$a^3 = a + 1 \qquad \text{and} \qquad b^6 = b + 3a.$$

Show that $a > b$.

KACY Summer League

1994 Brazil 743. Show that no one n-th root of a rational (for n a positive integer) can be a root of the polynomial $x^5 - x^4 - 4x^3 + 4x^2 + 2$.

KACY Summer League

1996 Brazil 744. Let $p(x)$ be the polynomial $x^3 + 14x^2 - 2x + 1$. Let $p^n(x)$ denote $p(p^(n-1)(x))$. Show that there is an integer N such that $p^N(x) - x$ is divisible by 101 for all integers x.

KACY Summer League

1997 Brazil 745. Let $f(x) = x^2 - C$ where C is a rational constant. Show that exists only finitely many rationals x such that $\{x, f(x), f(f(x)), \ldots\}$ is finite.

KACY Summer League

2007 Brazil 746. Let $f(x) = x^2 + 2007x + 1$. Prove that for every positive integer n, the equation

$$\underbrace{f(f(\ldots(f}_{n \text{ times}}(x))\ldots)) = 0,$$

has at least one real solution.

KACY Summer League

2010 Brazil 747. Let $P(x)$ be a polynomial with real coefficients. Prove that there exist positive integers n and k such that k has n digits and more than $P(n)$ positive divisors.

KACY Summer League

2000 Czech and Slovak 748. Let $P(x)$ be a polynomial with integer coefficients. Prove that the polynomial $Q(x) = P(x^4)P(x^3)P(x^2)P(x) + 1$ has no integer roots.

KACY Summer League

2002 Czech and Slovak 749. Let $n \geq 2$ be a fixed even integer. We consider polynomials of the form

$$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + 1,$$

with real coefficients, having at least one real roots. Find the least possible value of $a_1^2 + a_2^2 + \cdots + a_{n-1}^2$.

KACY Summer League

2004 Czech and Slovak 750. Show that real numbers, p, q, r satisfy the condition $p^4(q-r)^2 + 2p^2(q+r) + 1 = p^4$ if and only if the quadratic equations $x^2 + px + q = 0$ and $y^2 - py + r = 0$ have real roots (not necessarily distinct) which can be labeled by x_1, x_2 and y_1, y_2, respectively, in such a way that $x_1y_1 - x_2y_2 = 1$.

KACY Summer League

2005 Czech and Slovak 751. Find all integers $n \geq 3$ for which the polynomial

$$W(x) = x^n - 3x^{n-1} + 2x^{n-2} + 6,$$

can be written as a product of two non-constant polynomials with integer coefficients.

KACY Summer League

2007 Czech and Slovak 752. Find all polynomials P with real coefficients satisfying $P(x^2) = P(x) \cdot P(x+2)$ for all real numbers x.

KACY Summer League

2008 Czech and Slovak 753. Determine all triples (x, y, z) of positive real numbers which satisfies the following system of equations

$$\begin{cases} 2x^3 &= 2y(x^2+1) - (z^2+1), \\ 2y^4 &= 3z(y^2+1) - 2(x^2+1), \\ 2z^5 &= 4x(z^2+1) - 3(y^2+1). \end{cases}$$

KACY Summer League

2011 Czech and Slovak 754. A polynomial $P(x)$ with integer coefficients satisfies the following: if $F(x)$, $G(x)$, and $Q(x)$ are polynomials with integer coefficients satisfying $P\Big(Q(x)\Big) = F(x) \cdot G(x)$, then $F(x)$ or $G(x)$ is a constant polynomial. Prove that $P(x)$ is a constant polynomial.

KACY Summer League

2012 Czech and Slovak 755. Let a, b, c, d be positive real numbers such that $abcd = 4$ and

$$a^2 + b^2 + c^2 + d^2 = 10.$$

Find the maximum possible value of $ab + bc + cd + da$.

KACY Summer League

2013 Czech and Slovak 756. Let a and b be integers, where b is not a perfect square. Prove that $x^2 + ax + b$ may be the square of an integer only for finite number of integer values of x.

KACY Summer League

2014 Czech and Slovak 757. Prove that if the positive real numbers a, b, c satisfy the equation

$$a^4 + b^4 + c^4 + 4a^2b^2c^2 = 2(a^2b^2 + a^2c^2 + b^2c^2),$$

then there is a triangle ABC with internal angles α, β, γ such that

$$\sin\alpha = a, \qquad \sin\beta = b, \qquad \sin\gamma = c.$$

KACY Summer League

1991 China TST 758. Let real coefficient polynomial $f(x) = x^n + a_1 \cdot x^{n-1} + \ldots + a_n$ has real roots $b_1, b_2, \ldots, b_n$, $n \geq 2$, prove that $\forall x \geq \max\{b_1, b_2, \ldots, b_n\}$, we have

$$f(x+1) \geq \frac{2 \cdot n^2}{\dfrac{1}{x-b_1} + \dfrac{1}{x-b_2} + \cdots + \dfrac{1}{x-b_n}}.$$

KACY Summer League

1995 China TST 759. A and B play the following game with a polynomial of degree at least 4:

$$x^{2n} + \square x^{2n-1} + \square x^{2n-2} + \cdots + \square x + 1 = 0.$$

A and B take turns to fill in one of the blanks with a real number until all the blanks are filled up. If the resulting polynomial has no real roots, A wins. Otherwise, B wins. If A begins, which player has a winning strategy?

KACY Summer League

1995 China TST 760. Prove that the interval $[0, 1]$ can be split into black and white intervals for any quadratic polynomial $P(x)$, such that the sum of weights of the black intervals is equal to the sum of weights of the white intervals. Define the weight of the interval $[a, b]$ as $P(b) - P(a)$. Does the same result hold with a degree 3 or degree 5 polynomial?

KACY Summer League

1996 China TST 761. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$, and $\beta_1, \beta_2, \ldots, \beta_n$, where $n \geq 4$, be 2 sets of real numbers such that

$$\sum_{i=1}^{n} \alpha_i^2 < 1 \qquad \text{and} \qquad \sum_{i=1}^{n} \beta_i^2 < 1.$$

Define

$$A^2 = 1 - \sum_{i=1}^{n} \alpha_i^2,$$
$$B^2 = 1 - \sum_{i=1}^{n} \beta_i^2,$$
$$W = \frac{1}{2}(1 - \sum_{i=1}^{n} \alpha_i \beta_i)^2.$$

Find all real numbers λ such that the polynomial

$$x^n + \lambda(x^{n-1} + \cdots + x^3 + Wx^2 + ABx + 1) = 0,$$

only has real roots.

KACY Summer League

1997 China TST 762. Find all real-coefficient polynomials $f(x)$ which satisfy the following conditions:

(i) $f(x) = a_0x^{2n} + a_2x^{2n-2} + \cdots + a_{2n-2}x^2 + a_{2n}, a_0 > 0$;

(ii) $\sum_{j=0}^{n} a_{2j}a_{2n-2j} \le \binom{2n}{n} a_0a_{2n}$;

(iii) All the roots of $f(x)$ are imaginary numbers with no real part.

KACY Summer League

2000 China TST 763. Let F be the set of all polynomials Γ such that all the coefficients of $\Gamma(x)$ are integers and $\Gamma(x) = 1$ has integer roots. Given a positive integer k, find the smallest integer $m(k) > 1$ such that there exist $\Gamma \in F$ for which $\Gamma(x) = m(k)$ has exactly k distinct integer roots.

KACY Summer League

2002 China TST 764. Let

$$f(x_1, x_2, x_3) = -2\cdot(x_1^3+x_2^3+x_3^3)+3\cdot(x_1^2(x_2+x_3)+x_2^2\cdot(x_1+x_3)+x_3^2\cdot(x_1+x_2)-12x_1x_2x_3.$$

For any reals r, s, t, we denote

$$g(r, s, t) = \max_{t\le x_3\le t+2} |f(r, r+2, x_3) + s|.$$

Find the minimum value of $g(r, s, t)$.

KACY Summer League

2002 China TST 765. Let $P_n(x) = a_0 + a_1x + \cdots + a_nx^n$, with $n \ge 2$, be a real-coefficient polynomial. Prove that if there exists $a > 0$ such that

$$P_n(x) = (x + a)^2 \left(\sum_{i=0}^{n-2} b_ix^i\right),$$

where b_i are positive real numbers, then there exists some i, with $1 \le i \le n-1$, such that

$$a_i^2 - 4a_{i-1}a_{i+1} \le 0.$$

KACY Summer League

2002 China TST 766. For positive integers a, b, c let α, β, γ be pairwise distinct positive integers such that

$$\begin{cases} ca &= \alpha + \beta + \gamma, \\ b &= \alpha\beta + \beta\gamma + \gamma\alpha, \\ c^2 &= \alpha\beta\gamma. \end{cases}$$

Also, let λ be a real number that satisfies the condition

$$\lambda^4 - 2a\lambda^2 + 8c\lambda + a^2 - 4b = 0.$$

Prove that λ is an integer if and only if α, β, γ are all perfect squares.

KACY Summer League

2003 China TST 767. The n roots of a complex-coefficient polynomial

$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n,$$

are $z_1, z_2, \ldots, z_n$. If $\sum_{k=1}^{n} |a_k|^2 \leq 1$, then prove that $\sum_{k=1}^{n} |z_k|^2 \leq n$.

KACY Summer League

2003 China TST 768. Can we find positive reals $a_1, a_2, \ldots, a_{2002}$ such that for any positive integer k, with $1 \leq k \leq 2002$, every complex root z of the following polynomial $f(x)$ satisfies the condition $|\text{Im } z| \leq |\text{Re } z|$,

$$f(x) = a_{k+2001} x^{2001} + a_{k+2000} x^{2000} + \cdots + a_{k+1} x + a_k,$$

where $a_{2002+i} = a_i$, for $i = 1, 2, \ldots, 2001$.

KACY Summer League

2004 China TST 769. Given integer n larger than 5, solve the system of equations (assuming $x_i \geq 0$, for $i = 1, 2, \ldots n$):

$$\begin{cases} x_1 + \quad x_2 + \quad x_3 + \cdots + \quad x_n &= n + 2, \\ x_1 + 2\ x_2 + 3\ x_3 + \cdots + n\ x_n &= 2n + 2, \\ x_1 + 2^2 x_2 + 3^2 x_3 + \cdots + n^2 x_n &= n^2 + n + 4, \\ x_1 + 2^3 x_2 + 3^3 x_3 + \cdots + n^3 x_n &= n^3 + n + 8. \end{cases}$$

KACY Summer League

2005 China TST 770. Let $a_1, a_2 \ldots a_n$ and $x_1, x_2 \ldots x_n$ be integers and $r \geq 2$ be an integer. It is known that

$$\sum_{j=0}^{n} a_j x_j^k = 0 \qquad \text{for} \quad k = 1, 2, \ldots r.$$

Prove that

$$\sum_{j=0}^{n} a_j x_j^m \equiv 0 \pmod{m}, \qquad \text{for all} \quad m \in \{r+1, r+2, \cdots, 2r+1\}.$$

KACY Summer League

KACY–I 771. Prove the following statements on irrationality.

a) Show that $\sqrt{2} + \sqrt{3} + \sqrt{5} + \sqrt{7}$ is irrational.

b) Suppose a_i, with $i = 1, 2, \ldots, n$ are rationals such that $\sqrt{a_i}$ is irrational for at least one value of i. Prove that

$$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n}$$

is irrational.

KACY Summer League

2005 China TST 772. Determine whether $\sqrt{1001^2+1} + \sqrt{1002^2+1} + \cdots + \sqrt{2000^2+1}$ be a rational number or not?

KACY Summer League

Zhaobin vs Vess 773. Let $a_1, a_2, \ldots, a_n$ be n positive rational numbers and assume that $k_1, k_2, \ldots, k_n$ are n positive integers such that $\sqrt[k_i]{a_i}$ is irrational. Prove that the sum $\sum_{i=1}^{n} \sqrt[k_i]{a_i}$ is irrational.

KACY Summer League

2006 China TST 774. Let a_i and b_i (for $i = 1, 2, \ldots, n$) be rational numbers such that for any real number x, we have:

$$x^2 + x + 4 = \sum_{i=1}^{n} (a_i x + b)^2.$$

Find the least possible value of n.

KACY Summer League

2003 China TST 775. Find all second degree polynomial $d(x) = x^2+ax+b$ with integer coefficients, so that there exists an integer-coefficient polynomial $p(x)$ and a non-zero integer-coefficient polynomial $q(x)$ that satisfy:

$$(p(x))^2 - d(x)\,(q(x))^2 = 1, \qquad \text{for all} \quad x \in \mathbb{R}.$$

KACY Summer League

2006 China TST 776. Let k be an odd number that is greater than or equal to 3. Prove that there exists a k^{th}-degree integer-valued polynomial with non-integer-coefficients that has the following properties:

1. $f(0) = 0$ and $f(1) = 1$; and
2. There exist infinitely many positive integers n so that if the following equation:

$$n = f(x_1) + \cdots + f(x_s),$$

has integer solutions $x_1, x_2, \ldots, x_s$, then $s \geq 2^k - 1$.

KACY Summer League

2008 China TST 777. After multiplying out and simplifying polynomial

$$(x-1)(x^2-1)(x^3-1)\cdots(x^{2007}-1),$$

getting rid of all terms whose powers are greater than 2007, we acquire a new polynomial $f(x)$. Find its degree and the coefficient of the term having the highest power. In other words, if

$$P(x) = (1-x)(1-x^2)\cdots(1-x^{2007}),$$

find the degree of $P(x) \pmod{x^{2008}}$.

KACY Summer League

2008 China TST 778. Let z_1, z_2, z_3 be three complex numbers of modulii less than or equal to 1. Let w_1, w_2 be two roots of the equation

$$(z-z_1)(z-z_2) + (z-z_2)(z-z_3) + (z-z_3)(z-z_1) = 0.$$

Prove that, for $j = 1, 2, 3$, we have

$$\min\{|z_j - w_1|, |z_j - w_2|\} \leq 1.$$

KACY Summer League

2008 China TST 779. Let $n > m > 1$ be odd integers, and define

$$f(x) = x^n + x^m + x + 1.$$

Prove that $f(x)$ can't be expressed as the product of two polynomials having integer coefficients and positive degrees.

KACY Summer League

2009 China TST 780. Find all complex polynomial $P(x)$ such that for any three integers a, b, c satisfying $a + b + c \neq 0$, and

$$\frac{P(a) + P(b) + P(c)}{a + b + c} \quad \text{is an integer.}$$

KACY Summer League

2010 China TST 781. Given positive integer n, find the largest real number $\lambda = \lambda(n)$, such that for any degree-n polynomial with complex coefficients

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0,$$

and any permutation $x_0, x_1, \ldots, x_n$ of $0, 1, \ldots, n$, the following inequality holds:

$$\sum_{k=0}^{n} |f(x_k) - f(x_{k+1})| \geq \lambda |a_n|,$$

where $x_{n+1} = x_0$.

KACY Summer League

2012 China TST 782. Find the smallest possible value of a real number c such that for any 2012^{th}-degree monic polynomial

$$P(x) = x^{2012} + a_{2011}x^{2011} + \cdots + a_1 x + a_0,$$

with real coefficients, we can obtain a new polynomial $Q(x)$ by multiplying some of its coefficients by -1 such that every root z of $Q(x)$ satisfies the inequality

$$|\mathrm{Im}\ z| \leq c\,|\mathrm{Re}\ z|\,.$$

KACY Summer League

1981 Austrian–Polish 783. Let $P(x) = x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$ be a polynomial with rational coefficients. Show that if $P(x)$ has exactly one real root ξ, then ξ is a rational number.

KACY Summer League

1986 Austrian–Polish 784. The monic polynomial $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ of degree $n > 1$ has n distinct negative roots. Prove that $a_1P(1) > 2n^2a_o$.

KACY Summer League

1986 Austrian–Polish 785. Find all real solutions x, y, u, v of the system of equations

$$\begin{cases} x^2 + y^2 + u^2 + v^2 & = 4, \\ xu + yv + xv + yu & = 0, \\ xyu + yuv + uvx + vxy & = -2, \\ xyuv = -1 \end{cases}$$

KACY Summer League

1987 Austrian–Polish 786. Let n be the square of an integer whose each prime divisor has an even number of decimal digits. Consider $P(x) = x^n - 1987x$. Show that if x, y are rational numbers with $P(x) = P(y)$, then $x = y$.

KACY Summer League

1988 Austrian–Polish 787. Let $P(x)$ be a polynomial with integer coefficients. Show that if $Q(x) = P(x) + 12$ has at least six distinct integer roots, then $P(x)$ has no integer roots.

KACY Summer League

1988 Austrian–Polish 788. If $a_1 \le a_2 \le \cdots \le a_n$ are natural numbers $(n \ge 2)$, show that the inequality

$$\sum_{i=1}^{n} a_i x_i^2 + 2\sum_{i=1}^{n-1} x_i x_{i+1} > 0,$$

holds for all n-tuples $(x_1, \ldots, x_n) \ne (0, \ldots, 0)$ of real numbers if and only if $a_2 \ge 2$.

KACY Summer League

1990 Austrian–Polish 789. Show that there are two real solutions (x, y, z) to:

$$\begin{cases} x + y^2 + z^4 &= 0, \\ y + z^2 + x^4 &= 0, \\ z + x^2 + y^5 &= 0. \end{cases}$$

KACY Summer League

1990 Austrian–Polish 790. Given a positive integer $n \geq 2$, find all solutions (x_i, y_i) to the following system of equations, where $1 \leq i \leq n$:

$$\begin{cases} x_1^4 + 14x_1x_2 + 1 &= y_1^4, \\ x_2^4 + 14x_2x_3 + 1 &= y_2^4, \\ \quad\vdots & \quad\vdots \\ x_n^4 + 14x_nx_1 + 1 &= y_n^4. \end{cases}$$

KACY Summer League

1990 Austrian–Polish 791. Let $p(x)$ be a polynomial with integer coefficients. The sequence of integers $a_1, a_2, \ldots, a_n$ (where $n > 2$) satisfies

$$a_2 = p(a_1), \quad a_3 = p(a_2), \quad \ldots, \quad a_n = p(a_{n-1}), \quad a_1 = p(a_n).$$

Show that $a_1 = a_3$.

KACY Summer League

1991 Austrian–Polish 792. Let $P(x)$ be a real polynomial with $P(x) \geq 0$ for $0 \leq x \leq 1$. Show that there exist polynomials $P_i(x)$ (for $i = 0, 1, 2$) with $P_i(x) \geq 0$ for all real x such that

$$P(x) = P_0(x) + xP_1(x)(1 - x)P_2(x).$$

KACY Summer League

1991 Austrian–Polish 793. For a given positive integer n determine the maximum value of the function

$$f(x) = \frac{x + x^2 + \cdots + x^{2n-1}}{(1+x^n)^2}, \qquad \text{for all} \quad x \geq 0,$$

and find all positive x for which the maximum is attained.

KACY Summer League

1992 Austrian–Polish 794. Let k be a positive integer and u, v be real numbers, and

$$P(x) = (x - u^k)(x - uv)(x - v^k) = x^3 + ax^2 + bx + c.$$

a) For $k = 2$ prove that if a, b, c are rational then so is uv.

b) Is that also true for $k = 3$?

KACY Summer League

1993 Austrian–Polish 795. Solve in real numbers the system

$$\begin{cases} x^3 + y &= 3x + 4, \\ 2y^3 + z &= 6y + 6, \\ 3z^3 + x &= 9z + 8. \end{cases}$$

KACY Summer League

1993 Austrian–Polish 796. Determine all real polynomials $P(z)$ for which there exists a unique real polynomial $Q(x)$ satisfying the conditions $Q(0) = 0$, and

$$x + Q(y + P(x)) = y + Q(x + P(y)),$$

for all $x, y \in mathbbR$.

KACY Summer League

1994 Austrian–Polish 797. Let $n > 1$ be an odd positive integer. Assume that positive integers $x_1, x_2, \ldots, x_n \geq 0$ satisfy:

$$\begin{cases} (x_2 - x_1)^2 + 2(x_2 + x_1) + 1 & = n^2, \\ (x_3 - x_2)^2 + 2(x_3 + x_2) + 1 & = n^2, \\ \quad \vdots & \quad \vdots \\ (x_1 - x_n)^2 + 2(x_1 + x_n) + 1 & = n^2. \end{cases}$$

Show that there exists j with $1 \leq j \leq n$, such that $x_j = x_{j+1}$. Here, assume $x_{n+1} = x_1$.

KACY Summer League

1994 Austrian–Polish 798. Solve in integers the following equation

$$\frac{1}{2}(x+y)(y+z)(z+x) + (x+y+z)^3 = 1 - xyz.$$

KACY Summer League

1995 Austrian–Polish 799. Consider the equation $3y^4 + 4cy^3 + 2xy + 48 = 0$, where c is an integer parameter. Determine all values of c for which the number of integral solutions (x, y) satisfying the conditions (i) and (ii) is maximal:

(i) $|x|$ is a square of an integer;

(ii) y is a square-free number.

Remember that a square-free number is an integer which is not divisible by the square of any prime.

KACY Summer League

1996 Austrian–Polish 800. The polynomials $P_n(x)$ are defined initially by $P_0(x) = 0$ and $P_1(x) = x$, and then recursively, for $n \geq 2$, by

$$P_n(x) = xP_{n-1}(x) + (1-x)P_{n-2}(x).$$

For every natural number $n \geq 1$, find all real numbers x satisfying the equation $P_n(x) = 0$.

KACY Summer League

1996 Austrian–Polish 801. Given natural numbers $n > k > 1$, find all real solutions $x_1, \ldots, x_n$ of the system

$$x_i^3(x_i^2 + x_{i+1}^2 + \cdots + x_{i+k-1}^2) = x_{i-1}^2,$$

for $1 \leq i \leq n$. Here $x_{n+i} = x_i$ for all$i$.

KACY Summer League

1999 Austrian–Polish 802. Solve in the non-negative real numbers the system of equations

$$x_n^2 + x_n x_{n-1} + x_{n-1}^4 = 1,$$

for $n = 1, 2, \ldots, 1999$, assuming $x_0 = x_{1999}$.

KACY Summer League

2000 Austrian–Polish 803. For each integer $n \geq 3$ solve in real numbers the system of equations:

$$\begin{cases} x_1^3 & = x_2 + x_3 + 1, \\ x_2^3 & = x_3 + x_4 + 1, \\ \vdots & \quad \vdots \\ x_{n-1}^3 & = x_n + x_1 + 1, \\ x_n^3 & = x_1 + x_2 + 1. \end{cases}$$

KACY Summer League

2003 Austrian–Polish 804. Find all real polynomials $p(x)$ such that

$$p(x-1)p(x+1) = p(x^2 - 1).$$

KACY Summer League

2003 Austrian–Polish 805. For each positive integer $n > 1$, define

$$f(n) = \frac{n^n - 1}{n - 1}.$$

a) Show that $n!^{f(n)}$ divides $(n^n)!$.

b) Find as many positive integers as possible for which $n!^{f(n)+1}$ does not divide $(n^n)!$.

KACY Summer League

2004 Austrian–Polish 806. Solve the following system of equations in $\mathbb{R}$ where all square roots are non-negative:

$$\begin{cases} a - \sqrt{1-b^2} + \sqrt{1-c^2} &= d, \\ b - \sqrt{1-c^2} + \sqrt{1-d^2} &= a, \\ c - \sqrt{1-d^2} + \sqrt{1-a^2} &= b, \\ d - \sqrt{1-a^2} + \sqrt{1-b^2} &= c. \end{cases}$$

KACY Summer League

2004 Austrian–Polish 807. Determine all n for which the system with of equations can be solved in $\mathbb{R}$:

$$\sum_{k=1}^{n} x_k = 27,$$
$$\prod_{k=1}^{n} x_k = \left(\frac{3}{2}\right)^{24}.$$

KACY Summer League

2004 Austrian–Polish 808. For each polynomial $Q(x)$ let $M(Q)$ be the set of non-negative integers x with $0 < Q(x) < 2004$. We consider polynomials $P_n(x)$ of the form

$$P_n(x) = x^n + a_1 \cdot x^{n-1} + \cdots + a_{n-1} \cdot x + 1,$$

with coefficients $a_i \in \{\pm 1\}$ for $i = 1, 2, \ldots, n-1$. For each $n = 3^k$, with $k > 0$, determine:

a) m_n, which represents the maximum of elements in $M(P_n)$ for all such polynomials $P_n(x)$; and

b) all polynomials $P_n(x)$ for which $|M(P_n)| = m_n$.

KACY Summer League

2005 Austrian–Polish 809. Determine all polynomials P with integer coefficients satisfying

$$P(P(P(P(P(x))))) = x^{28} \cdot P(P(x)), \qquad \text{for all} \quad x \in \mathbb{R}.$$

KACY Summer League

2005 Austrian–Polish 810. For each natural number $n \geq 2$, solve the following system of equations in the integers $x_1, x_2, \ldots, x_n$:

$$(n^2 - n)x_i + \left(\prod_{j \neq i} x_j\right) S = n^3 - n^2, \qquad \text{for} \quad i = 1, 2, \ldots, n,$$

where,

$$S = x_1^2 + x_2^2 + \cdots + x_n^2.$$

KACY Summer League

2006 Austrian–Polish 811. Find all polynomials $P(x)$ with real coefficients satisfying the equation

$$(x+1)^3 P(x-1) - (x-1)^3 P(x+1) = 4(x^2 - 1)P(x),$$

for all real numbers x.

KACY Summer League

2000 Austria 812. For any real number a, find all real numbers x that satisfy the following equation:

$$(2x+1)^4 + ax(x+1) - \frac{x}{2} = 0.$$

KACY Summer League

2002 Austria 813. Solve the following system of equations over the real numbers:

$$\begin{cases} 2x_1 = x_5^2 - 23, \\ 4x_2 = x_1^2 + 7, \\ 6x_3 = x_2^2 + 14, \\ 8x_4 = x_3^2 + 23, \\ 10x_5 = x_4^2 + 34. \end{cases}$$

KACY Summer League

2004 Austria 814. Solve the following equation for real numbers (all square roots are non negative):

$$\sqrt{4 - x\sqrt{4 - (x-2)\sqrt{1 + (x-5)(x-7)}}} = \frac{5x - 6 - x^2}{2}.$$

KACY Summer League

2010 Austria 815. Solve the following in equation in $\mathbb{R}^3$:

$$4x^4 - x^2(4y^4 + 4z^4 - 1) - 2xyz + y^8 + 2y^4z^4 + y^2z^2 + z^8 = 0.$$

KACY Summer League

2010 Donova (Danube) 816. Let $n \geq 3$ be a positive integer. Find non-negative real numbers $x_1, x_2, \ldots, x_n$, with $x_1 + x_2 + \cdots + x_n = n$, for which the expression

$$(n-1)(x_1^2 + x_2^2 + \cdots + x_n^2) + nx_1x_2\cdots x_n,$$

takes a minimal value.

KACY Summer League

2017 Donova (Danube) 817. Find all polynomials $P(x)$ with integer coefficients such that $a^2 + b^2 - c^2$ divides $P(a) + P(b) - P(c)$, for all integers a, b, c.

KACY Summer League

1959–1966 IMO Longlist 818. If a, b, c, d are integers such that ad is odd and bc is even, prove that at least one root of the polynomial $ax^3 + bx^2 + cx + d$ is irrational.

KACY Summer League

1968 IMO Shortlist 819. A polynomial $p(x) = a_0x^k + a_1x^{k-1} + \cdots + a_k$ with integer coefficients is said to be divisible by an integer m if $p(x)$ is divisible by m for all integers x. Prove that if $p(x)$ is divisible by m, then $k!a_0$ is also divisible by m. Also prove that if a_0, k, and m are non-negative integers for which $k!a_0$ is divisible by m, then there exists a polynomial $p(x) = a_0x^k + \cdots + a_k$ divisible by m.

KACY Summer League

1968 IMO Shortlist 820. Find all complex numbers m such that polynomial

$$x^3 + y^3 + z^3 + mxyz,$$

can be represented as the product of three linear trinomials.

KACY Summer League

1969 IMO Longlist 821. Let us define $u_0 = 0, u_1 = 1$ and for $n \geq 0$,

$$u_{n+2} = au_{n+1} + bu_n,$$

where a and b are positive integers. Express u_n as a polynomial in a and b. Prove the result. Given that b is prime, prove that b divides $a(u_b - 1)$.

KACY Summer League

1969 IMO Longlist 822. Given a polynomial $f(x)$ with integer coefficients whose value is divisible by 3 for three integers $k, k+1$, and $k+2$, prove that $f(m)$ is divisible by 3 for all integers m.

KACY Summer League

1969 IMO Longlist 823. Prove that if $0 \leq a_0 \leq a_1 \leq a_2$, then

$$(a_0 + a_1 x - a_2 x^2)^2 \leq (a_0 + a_1 + a_2)^2 \left(1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{2} + x^4\right),$$

and formulate and prove the analogous result for polynomials of third degree.

KACY Summer League

1970 IMO Longlist 824. Given a polynomial

$$P(x) = ab(a-c)x^3 + (a^3 - a^2c + 2ab^2 - b^2c + abc)x^2 + (2a^2b + b^2c + a^2c + b^3 - abc)x + ab(b+c),$$

where $a, b, c \neq 0$, prove that $P(x)$ is divisible by $Q(x) = abx^2 + (a^2 + b^2)x + ab$ and conclude that $P(x_0)$ is divisible by $(a+b)^3$ for $x_0 = (a+b+1)^n$, for all $n \in \mathbb{N}$.

KACY Summer League

1970 IMO Longlist 825. Let a polynomial $p(x)$ with integer coefficients take the value 5 for five different integer values of x. Prove that $p(x)$ does not take the value 8 for any integer x.

KACY Summer League

1970 IMO Shortlist 826. Let P, Q, R be polynomials and let $S(x) = P(x^3) + xQ(x^3) + x^2R(x^3)$ be a polynomial of degree n whose roots $x_1, \ldots, x_n$ are distinct. Construct with the aid of the polynomials P, Q, R a polynomial T of degree n that has the roots $x_1^3, x_2^3, \ldots, x_n^3$.

KACY Summer League

1971 IMO Shortlist 827. Consider a sequence of polynomials $\{P_i(x)\}_{i=0}^{\infty}$, where $P_0(x) = 2$, $P_1(x) = x$ and for every $n \geq 1$ the following equality holds:

$$P_{n+1}(x) + P_{n-1}(x) = xP_n(x).$$

Prove that there exist three real numbers a, b, c such that for all $n \geq 1$,

$$(x^2 - 4)[P_n^2(x) - 4] = [aP_{n+1}(x) + bP_n(x) + cP_{n-1}(x)]^2.$$

KACY Summer League

1971 IMO Shortlist 828. Prove that the polynomial $x^4 + \lambda x^3 + \mu x^2 + \nu x + 1$ has no real roots if λ, μ, ν are real numbers satisfying

$$|\lambda| + |\mu| + |\nu| \leq \sqrt{2}.$$

KACY Summer League

1976 IMO Longlist 829. Prove that if for a polynomial $P(x, y)$, we have

$$P(x - 1, y - 2x + 1) = P(x, y),$$

then there exists a polynomial $\Phi(x)$ with $P(x, y) = \Phi(y - x^2)$.

KACY Summer League

1976 IMO Longlist 830. The polynomial $1976(x + x^2 + \cdots + x^n)$ is decomposed into a sum of polynomials of the form $a_1x + a_2x^2 + \cdots + a_nx^n$, where $a_1, a_2, \ldots, a_n$ are distinct positive integers not greater than n. Find all values of n for which such a decomposition is possible.

KACY Summer League

1976 IMO Longlist 831. Let $g(x)$ be a fixed polynomial with real coefficients and define $f(x)$ by $f(x) = x^2 + xg(x^3)$. Show that $f(x)$ is not divisible by $x^2 - x + 1$.

KACY Summer League

1976 IMO Longlist 832. Let P be a polynomial with real coefficients such that $P(x) > 0$ if $x > 0$. Prove that there exist polynomials Q and R with non-negative coefficients such that
$$P(x) = \frac{Q(x)}{R(x)} \qquad \text{if} \quad x > 0.$$

KACY Summer League

1976 IMO Longlist 833. Prove that if $P(x) = (x-a)^k Q(x)$, where k is a positive integer, a is a nonzero real number, $Q(x)$ is a nonzero polynomial, then $P(x)$ has at least $k+1$ nonzero coefficients.

KACY Summer League

1978 IMO Longlist 834. Given the expression

$$P_n(x) = \frac{1}{2^n}\left[(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\right],$$

prove that:

a) $P_n(x)$ satisfies the identity

$$P_n(x) - xP_{n-1}(x) + \frac{1}{4}P_{n-2}(x) \equiv 0.$$

b) $P_n(x)$ is a polynomial in x of degree n.

KACY Summer League

1982 IMO Longlist 835. Determine all real values of the parameter a for which the equation
$$16x^4 - ax^3 + (2a+17)x^2 - ax + 16 = 0,$$
has exactly four distinct real roots that form a geometric progression.

KACY Summer League

1984 IMO Longlist 836. Let $f_1(x) = x^3 + a_1x^2 + b_1x + c_1 = 0$ be an equation with three positive roots $\alpha > \beta > \gamma > 0$. From the equation $f_1(x) = 0$, one constructs the equation $f_2(x) = x^3 + a_2x^2 + b_2x + c_2 = x(x+b_1)^2 - (a_1x+c_1)^2 = 0$. Continuing this process, we get equations $f_3, \cdots, f_n$. Prove that

$$\lim_{n\to\infty} \sqrt[2^{n-1}]{-a_n} = \alpha.$$

KACY Summer League

1984 IMO Longlist 837. Let P, Q, R be the polynomials with real or complex coefficients such that at least one of them is not constant. If $P^n + Q^n + R^n = 0$, prove that $n < 3$.

KACY Summer League

1985 IMO Longlist 838. Find a method by which one can compute the coefficients of $P(x) = x^6 + a_1x^5 + \cdots + a_6$ from the roots of $P(x) = 0$ by performing not more than 15 additions and 15 multiplications.

KACY Summer League

1987 IMO Longlist 839. Let P, Q, R be polynomials with real coefficients, satisfying $P^4 + Q^4 = R^2$. Prove that there exist real numbers p, q, r and a polynomial S such that $P = pS, Q = qS$ and $R = rS^2$.

KACY Summer League

1989 IMO Longlist 840. Let $f(x) = \prod_{k=1}^{n}(x - a_k) - 2$, where $n \geq 3$ and $a_1, a_2, \ldots,$ an are distinct integers. Suppose that $f(x) = g(x)h(x)$, where $g(x), h(x)$ are both non-constant polynomials with integer coefficients. Prove that $n = 3$.

KACY Summer League

1989 IMO Longlist 841. Let $P_1(x), P_2(x), \ldots, P_n(x)$ be real polynomials, i.e., they have real coefficients. Show that there exist real polynomials $A_r(x), B_r(x) \quad (r = 1, 2, 3)$ such that

$$\sum_{s=1}^{n} \{P_s(x)\}^2 \equiv (A_1(x))^2 + (B_1(x))^2,$$
$$\sum_{s=1}^{n} \{P_s(x)\}^2 \equiv (A_2(x))^2 + x\,(B_2(x))^2,$$
$$\sum_{s=1}^{n} \{P_s(x)\}^2 \equiv (A_3(x))^2 - x\,(B_3(x))^2.$$

KACY Summer League

1992 IMO Longlist 842. Let $P_1(x, y)$ and $P_2(x, y)$ be two relatively prime polynomials with complex coefficients. Let $Q(x, y)$ and $R(x, y)$ be polynomials with complex coefficients and each of degree not exceeding d. Prove that there exist two integers A_1, A_2 not simultaneously zero with $|A_i| \leq d + 1$ $(i = 1, 2)$ and such that the polynomial $A_1P_1(x, y) + A_2P_2(x, y)$ is coprime to $Q(x, y)$ and $R(x, y)$.

KACY Summer League

1992 IMO Longlist 843. Let $f(x) = x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m$ and $g(x) = x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n$ be two polynomials with real coefficients such that for each real number x, $f(x)$ is the square of an integer if and only if so is $g(x)$. Prove that if $n + m > 0$, then there exists a polynomial $h(x)$ with real coefficients such that $f(x) \cdot g(x) = (h(x))^2$.

Chapter 2

Olympiad Algebra 201: Trigonometry 101, 201, 301, 401, and Beyond

2.1 Math Olympiad Trigonometry 101: Identities & Half-angled Trigonometry

We start with the most basic identities and move on to the more advanced ones.

2.1.1 Trigonometric Identities

Identity 1 (Fundamental Identity of Trigonometry)**.** For any real number x, we have $\sin^2 x + \cos^2 x = 1$.

Identity 2 (Special Trig Values)**.**

$$\sin\frac{\pi}{2} = 1, \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}, \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \sin\frac{\pi}{6} = \frac{1}{2}, \tag{2.1}$$

$$\cos\frac{\pi}{2} = 0, \cos\frac{\pi}{3} = \frac{1}{2}, \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}. \tag{2.2}$$

Identity 3. Prove the formulas for Sum and Difference of Two Angles:

1. Sine and cosine of the sum and difference of two angles:

$$\cos(x+y) = \cos x \cos y - \sin x \sin y, \tag{2.3}$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y, \tag{2.4}$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y, \tag{2.5}$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y. \tag{2.6}$$

$$\tag{2.7}$$

2. Tangent and cotangent of the sum and difference of two angles:

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}, \tag{2.8}$$
$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}, \tag{2.9}$$
$$\cot(x+y) = \frac{\cot x \cot y - 1}{\cot x + \cot y}, \tag{2.10}$$
$$\cot(x-y) = \frac{\cot x \cot y + 1}{\cot x - \cot y}. \tag{2.11}$$
$$\tag{2.12}$$

Identity 4 (Complementary and Supplementary angles)**.**

$$\sin(\pi - x) = \sin x, \tag{2.13}$$
$$\cos(\pi - x) = -\cos x, \tag{2.14}$$
$$\sin\left(\frac{\pi}{2} - x\right) = \cos x, \tag{2.15}$$
$$\cos\left(\frac{\pi}{2} - x\right) = \sin x, \tag{2.16}$$
$$\tan\left(\frac{\pi}{2} - x\right) = \frac{\sin\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right)} = \frac{\cos x}{\sin x} = \cot x. \tag{2.17}$$

Identity 5 (Half-angles)**.** Using the trigonometric identities for sum and difference of angles, prove that

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = 2\cos^2\frac{x}{2} - 1 = 1 - 2\sin^2\frac{x}{2}, \tag{2.18}$$
$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}. \tag{2.19}$$

Identity 6 ($\tan(x/2)$)**.** Using the trigonometric identities for half-angles, prove that

$$\sin x = \frac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}}, \tag{2.20}$$
$$\cos x = \frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}. \tag{2.21}$$

KACY Summer League

KACY–I 844. Using the previous identities, prove that

$$\begin{aligned}\sin 15^\circ &= \cos 75^\circ = \frac{\sqrt{6}-\sqrt{2}}{4},\\ \cos 15^\circ &= \sin 75^\circ = \frac{\sqrt{6}+\sqrt{2}}{4},\\ \tan 15^\circ &= \cot 75^\circ = 2-\sqrt{3},\\ \cot 15^\circ &= \tan 75^\circ = 2+\sqrt{3},\end{aligned}$$

and

$$\begin{aligned}\sin 22.5^\circ &= \cos 67.5^\circ = \frac{\sqrt{2-\sqrt{2}}}{2},\\ \cos 22.5^\circ &= \sin 67.5^\circ = \frac{\sqrt{2+\sqrt{2}}}{2},\\ \tan 22.5^\circ &= \cot 67.5^\circ = \sqrt{2}-1,\\ \cot 22.5^\circ &= \tan 67.5^\circ = \sqrt{2}+1.\end{aligned}$$

Identity 7 (Double-angles)**.** Using the trigonometric identities for sum and difference of angles, prove that

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x, \tag{2.22}$$
$$\sin 2x = 2\sin x\cos x, \tag{2.23}$$
$$1-\cos 2x = 2\sin^2 x, \tag{2.24}$$
$$1+\cos 2x = 2\cos^2 x, \tag{2.25}$$
$$\cos 4x = 1 - 2\sin^2 2x, \tag{2.26}$$
$$\sin 4x = 4\sin x\cos x\cos 2x. \tag{2.27}$$

Identity 8 $(1\pm\cos 2x)$**.** Using the trigonometric identities for double-angles, prove that

$$1-\cos 2x = 2\sin^2 x, \tag{2.28}$$
$$1+\cos 2x = 2\cos^2 x, \tag{2.29}$$

and imply that

$$\sin x = \pm\sqrt{\frac{1-\cos 2x}{2}}, \tag{2.30}$$
$$\cos x = \pm\sqrt{\frac{1+\cos 2x}{2}}, \tag{2.31}$$
$$\tan x = \pm\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}. \tag{2.32}$$

KACY Summer League

$(\cot x - \cot 2x)$ **Exercise Identity 845.** Show that

$$\frac{1}{\sin 2\theta} - \cot 2\theta = \tan\theta \qquad \text{and} \qquad \frac{1}{\sin 2\theta} + \cot 2\theta = \cot\theta.$$

KACY Summer League

Sum of Three Angles Identity 846. Prove that for all $x, y, z \in \mathbb{R}$,

$$\sin(x+y+z) = \sin x \cos y \cos c + \sin y \cos x \cos z + \sin z \cos x \cos y - \sin x \sin y \sin z, \tag{2.33}$$

$$\cos(x+y+z) = \cos x \cos y \cos c - \cos x \sin y \sin z - \cos y \sin x \sin z - \cos z \sin x \sin y, \tag{2.34}$$

$$\tan(x+y+z) = \frac{\tan x + \tan y + \tan z - \tan x \tan y \tan z}{1 - \tan x \tan y - \tan y \tan z - \tan z \tan x}. \tag{2.35}$$

Find a formula for $\sin, \cos, \tan$ for triple-angles $3x, 6x, 9x, \ldots$

Triple–Angle Identities

Identity 9 (Triple-angles)**.** For any real number x,

$$\sin 3x = 3\sin x - 4\sin^3 x, \tag{2.36}$$

$$\sin 3x = 4\cos^3 x - 3\cos x, \tag{2.37}$$

$$\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}. \tag{2.38}$$

KACY Summer League

KACY–I 847. Prove the following identities for any three angles α, β, γ:

a)

$$\cos(\alpha+\beta+\gamma)+\cos(\alpha+\beta-\gamma)+\cos(\alpha-\beta+\gamma) \\ +\cos(-\alpha+\beta+\gamma)=4\cos\alpha\cos\beta\cos\gamma,$$

b)

$$\cos(\alpha+\beta-\gamma)+\cos(\alpha-\beta+\gamma)-\cos(\alpha+\beta+\gamma) \\ -\cos(-\alpha+\beta+\gamma)=4\cos\alpha\sin\beta\sin\gamma,$$

c)

$$\sin(\alpha+\beta-\gamma)+\sin(\alpha-\beta+\gamma)+\sin(-\alpha+\beta+\gamma) \\ =\sin(\alpha+\beta+\gamma)+4\sin\alpha\sin\beta\sin\gamma,$$

d)

$$\sin(\alpha+\beta+\gamma)+\sin(\alpha-\beta+\gamma)+\sin(-\alpha+\beta+\gamma) \\ =\sin(\alpha+\beta-\gamma)+4\cos\alpha\cos\beta\sin\gamma.$$

e)

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2(\alpha+\beta+\gamma) \\ =2\left(1+\cos(\alpha+\beta)+\cos(\beta+\gamma)+\cos(\gamma+\alpha)\right),$$

f)

$$\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)=4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}.$$

KACY Summer League

KACY–I 848. For any three angles A, B, C, prove that

$$\begin{aligned} 0 &= \tan(A-B)+\tan(B-C)+\tan(C-A)-\tan(A-B)\tan(B-C)\tan(C-A), \\ -3 &= \tan(A+60^\circ)\tan(A-60^\circ)+\tan A\tan(A+60^\circ)+\tan A\tan(A-60^\circ), \\ 1 &= \cos^2 A+\cos^2 B+\cos^2(A+B)-2\cos A\cos B\cos(A+B), \end{aligned}$$

and

$$\cot A = \tan A + 2\tan 2A + 4\cot 4A.$$

KACY Summer League

KACY–I 849. Prove that for any real number x,

$$\begin{aligned} 4\cos x\cos\left(\frac{\pi}{3}-x\right)\cos\left(\frac{\pi}{3}+x\right) &= \cos 3x, \\ 4\sin x\sin\left(\frac{\pi}{3}-x\right)\sin\left(\frac{\pi}{3}+x\right) &= \sin 3x, \\ \tan x+\tan\left(\frac{\pi}{3}+x\right)+\tan\left(\frac{2\pi}{3}+x\right) &= 3\tan 3x. \end{aligned}$$

2.1.2 Sum-to-Product and Product-to-Sum

KACY Summer League

Sine and Cosine Sum-to-Product 850. Prove that for all $x, y \in \mathbb{R}$,

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}, \tag{2.39}$$
$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}, \tag{2.40}$$
$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}, \tag{2.41}$$
$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}. \tag{2.42}$$

KACY Summer League

Sine and Cosine Product-to-Sum 851. Prove that for all $x, y \in \mathbb{R}$,

$$\sin x\cos y = \frac{1}{2}\left(\sin(x+y) + \sin(x-y)\right), \tag{2.43}$$
$$\sin x\sin y = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right), \tag{2.44}$$
$$\cos x\cos y = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right). \tag{2.45}$$

KACY Summer League

Tangent and Cotangent Sum-to-Product 852. Prove that for all $x, y \in \mathbb{R}$,

$$\tan x + \tan y = \frac{\sin(x+y)}{\cos x\cos y}, \tag{2.46}$$
$$\tan x - \tan y = \frac{\sin(x-y)}{\cos x\cos y}, \tag{2.47}$$
$$\cot x + \cot y = \frac{\sin(x+y)}{\sin x\sin y}, \tag{2.48}$$
$$\cot x - \cot y = \frac{\sin(x-y)}{\sin x\sin y}. \tag{2.49}$$

2.1.3 Half-angled and Double-angled Trigonometry of Triangle

KACY Summer League

KACY–I 853. If A, B, C are any three angles whose sum is π (thus possibly being the three angles of a triangle),

$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}, \tag{2.50}$$

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C, \tag{2.51}$$

$$\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}. \tag{2.52}$$

KACY Summer League

KACY–I 854. If A, B, C are the three angles of a triangle ABC with three sides a, b, c and semi-perimeter $p = (a + b + c)/2$,

$$\cos\frac{A}{2} = \sqrt{\frac{p(p-a)}{bc}}, \tag{2.53}$$

$$\sin\frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{bc}}, \tag{2.54}$$

$$\tan\frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{p(p-a)}}, \tag{2.55}$$

$$\frac{\tan\frac{A}{2} - \tan\frac{B}{2}}{\tan\frac{A}{2} + \tan\frac{B}{2}} = \frac{a-b}{c}, \tag{2.56}$$

$$\frac{\tan\frac{B+C}{2}}{\tan\frac{B-C}{2}} = \frac{b+c}{b-c}. \tag{2.57}$$

2.1.3.1 Pool of Half– and Double–angled Identities in Triangle

KACY Summer League

KACY–I 855. If A, B, C are any three angles whose sum is π (thus possibly being the three angles of a triangle),

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C, \tag{2.58}$$

$$\cos 2A + \cos 2B + \cos 2C = -4\cos A\cos B\cos C - 1. \tag{2.59}$$

KACY Summer League

KACY–I 856. Prove that for any three angles α, β, γ,

$$\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha+\beta+\gamma) = 4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2},$$
$$\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma) = 4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}.$$

KACY Summer League

KACY–I 857. For any three angles A, B, C that add up to $180°$, prove the following identities:

$$\cos A + \cos B - \cos C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} - 1,$$
$$\sin 2A + \sin 2B - \sin 2C = 4\cos A\cos B\sin C,$$
$$\cos 2A + \cos 2B - \cos 2C = 1 - 4\sin A\sin B\cos C.$$

KACY Summer League

KACY–I 858. For any three angles A, B, C that add up to $180°$, prove the following half-angle identities:

$$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} = 1 - 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$
$$\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = 2\left(1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right),$$

and

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} = 1 + 4\cos\frac{\pi+A}{4}\cos\frac{\pi+B}{4}\cos\frac{\pi+C}{4},$$
$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} = 4\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}.$$

KACY Summer League

KACY–I 859. For any three angles A, B, C that add up to $180°$, prove the following identities on sine of double– and quadruple–angles:

$$\begin{aligned}
\sin^2 A + \sin^2 B + \sin^2 C &= 2(1 + \cos A \cos B \cos C),\\
\sin^2 A + \sin^2 B - \sin^2 C &= 2 \sin A \sin B \sin C,\\
\sin^2 2A + \sin^2 2B + \sin^2 2C &= 2(1 - \cos 2A \cos 2B \cos 2C),\\
\sin 4A + \sin 4B + \sin 4C &= -4 \sin 2A \sin 2B \sin 2C,\\
\sin 4A + \sin 4B - \sin 4C &= -4 \cos 2A \cos 2B \sin 2C,
\end{aligned}$$

and prove the similar identities on cosine of double– and quadruple–angles:

$$\begin{aligned}
\cos^2 A + \cos^2 B + \cos^2 C &= 1 - 2 \cos A \cos B \cos C,\\
\cos^2 A + \cos^2 B - \cos^2 C &= 1 - 2 \sin A \sin B \sin C,\\
\cos^2 2A + \cos^2 2B + \cos^2 2C &= 1 + 2 \cos 2A \cos 2B \cos 2C,\\
\cos 4A + \cos 4B + \cos 4C &= 4 \cos 2A \cos 2B \cos 2C - 1,\\
\cos 4A + \cos 4B - \cos 4C &= 4 \sin 2A \sin 2B \cos 2C + 1.
\end{aligned}$$

KACY Summer League

KACY–I 860. For any three angles A, B, C that add up to $180°$, prove the following identities on tangent and cotangent of half– and double–angles:

$$\begin{aligned}
\tan \frac{B}{2} \cot \frac{C}{2} &= \frac{\sin A + \sin B - \sin C}{\sin A - \sin B + \sin C},\\
\tan A \tan B &= \frac{\sin 2A + \sin 2B + \sin 2C}{\sin 2A + \sin 2B - \sin 2C},\\
\tan \frac{A}{2} \tan \frac{B}{2} &= \frac{\cos A + \cos B + \cos C - 1}{\cos A + \cos B - \cos C + 1},
\end{aligned}$$

$$\begin{aligned}
1 &= \tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2},\\
0 &= \tan 2A + \tan 2B + \tan 2C - \tan 2A \tan 2B \tan 2C.
\end{aligned}$$

KACY Summer League

KACY–I 861. In triangle ABC with side-lengths a, b, c, perimeter $2p$ and area S, prove the following identities:

$$\begin{aligned}
0 &= (b-c)\cot\frac{A}{2} + (c-a)\cot\frac{B}{2} + (a-b)\cot\frac{C}{2},\\
0 &= (a+b+c)\sin\frac{A}{2} - 2a\cos\frac{B}{2}\cos\frac{C}{2},\\
0 &= (-a+b+c)\sin\frac{A}{2} - 2a\sin\frac{B}{2}\sin\frac{C}{2},\\
p &= b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2},\\
\frac{p}{abc} &= \frac{1}{a}\cos^2\frac{A}{2} + \frac{1}{b}\cos^2\frac{B}{2} + \frac{1}{c}\cos^2\frac{C}{2}.
\end{aligned}$$

KACY Summer League

KACY–I 862. In triangle ABC, show that

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{8}. \tag{2.60}$$

KACY Summer League

KACY–I 863. Prove in a triangle ABC that

$$\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \le \left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{8}.$$

KACY Summer League

KACY–I 864. Prove in a triangle ABC that

$$\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}. \tag{2.61}$$

KACY Summer League

KACY–I 865. Prove the following identities:

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2},$$
$$\cos\frac{A-B}{2} - \cos\frac{A+B}{2} = 2\sin\frac{A}{2}\sin\frac{B}{2}.$$

KACY Summer League

KACY–I 866. In triangle ABC, show that

$$\left(\sin\frac{A}{2}\right)^2 + \left(\sin\frac{B}{2}\right)^2 + \left(\sin\frac{C}{2}\right)^2 + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 1. \tag{2.62}$$

KACY Summer League

KACY–I 867. Show that

$$\tan^2\frac{\pi}{16} + \tan^2\frac{3\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{7\pi}{16} = 28.$$

2.1.4 Trigonometry of Geometrical Quantities

We shall find the magnitude of several important geometrical quantities related to triangles, including triangle's area, perimeter, heights, internal and external angle bisectors, circumradius, inradius and exradii, etc.

2.1.4.1 Law of Sines, Law of Cosines, and Law of Tangents

For the sake of self-containment, we remember:

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Law of Sines 868. In a triangle with side lengths a, b, c, and angles A, B, C,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R,$$

where R is the circumradius (radius of the circumcircle) of triangle ABC.

KACY Summer League

Law of Cosines 869. In a triangle with side lengths a, b, c, and angles A, B, C,

$$a^2 = b^2 + c^2 - 2bc\cos A,$$
$$b^2 = c^2 + a^2 - 2ca\cos B,$$
$$c^2 = a^2 + b^2 - 2ab\cos C,$$

and conclude the Pythagorean theorem.

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Projection Rule in Triangle 870. In a triangle ABC with side lengths a, b, c, and angles A, B, C, show that

$$a = b\cos C + c\cos A,$$
$$b = c\cos A + a\cos C,$$
$$c = a\cos B + b\cos A$$

KACY Summer League

Law of Tangents 871. In a triangle with side lengths a, b, c, and angles A, B, C,

$$\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2},$$
$$\tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a}\cot\frac{B}{2},$$
$$\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}.$$

KACY Summer League

KACY–I 872. In triangle ABC with perimeter $2p$, show that

$$p\tan\frac{A}{2} = (p-b)\cot\frac{C}{2} = (p-c)\cot\frac{B}{2}.$$

2.1.4.2 Trigonometry of Perimeter and Area

KACY Summer League

Area Using Law of Sines 873. In triangle ABC with angles A, B, C and side-lengths a, b, c, prove that

$$S = \frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B = \frac{1}{2}bc\sin A,$$

where S denotes the area of triangle ABC.

KACY Summer League

Heron's Formula 874. In a triangle with side lengths a, b, c, perimeter $2p$, and area S,

$$S = \sqrt{p(p-a)(p-b)(p-c)}.$$

KACY Summer League

Semi-perimeter and Cosine of Half-angles 875. In a triangle with angles A, B, C and perimeter $2p$, show that

$$p = 4R \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2},$$

where R is the circumradius (radius of the circumcircle) of triangle ABC.

KACY Summer League

Area in Terms of Sine of Angles 876. In triangle ABC with angles A, B, C and radius of the circumcircle R, prove that

$$S = 2R^2 \sin A \sin B \sin C,$$

where S denotes the area of triangle ABC. Conclude that

$$S = \frac{a^2 \sin B \sin C}{2 \sin A} = \frac{b^2 \sin A \sin C}{2 \sin B} = \frac{c^2 \sin A \sin B}{2 \sin C}.$$

KACY Summer League

KACY–I 877. If the angle bisectors of angles A, B, C in triangle ABC with in-radius r intersect the opposite sides at D, E, F, respectively, show that:

$$\frac{4S_{\triangle ABC} \cdot S_{\triangle DEF}}{AD \cdot BE \cdot CF} = r.$$

KACY Summer League

KACY–I 878. In any triangle ABC with side-lengths a, b, c, and angles A, B, C, if the perimeter is $2p$, prove the following half-angled formulas:

$$\sin \frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{bc}},$$
$$\cos \frac{A}{2} = \sqrt{\frac{p(p-a)}{bc}},$$
$$\tan \frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{p(p-a)}}.$$

2.1.4.3 Trigonometry of Heights

KACY Summer League

Calculating the Height of Triangle 879. In triangle ABC with angles A, B, C and radius of the circumcircle R, prove that

$$h_a = 2R\sin B\sin C = \frac{r\sqrt{(1+\cos B)(1+\cos C)}}{\cos\dfrac{A}{2}},$$
$$h_b = 2R\sin C\sin A = \frac{r\sqrt{(1+\cos C)(1+\cos A)}}{\cos\dfrac{B}{2}},$$
$$h_c = 2R\sin A\sin B = \frac{r\sqrt{(1+\cos A)(1+\cos B)}}{\cos\dfrac{C}{2}}.$$

where h_a, h_b, h_c are the heights drawn from vertices A, B, C, respectively.

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KACY–I 880. Show that

$$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r},$$
$$\frac{\cos A}{h_a}+\frac{\cos B}{h_b}+\frac{\cos C}{h_c}=\frac{1}{R},$$
$$h_ah_bh_c=\frac{a^2b^2c^2}{8R^3}.$$

KACY Summer League

Identities on Lengths of Heights 881. In a triangle ABC with side-lengths a, b, c, angles A, B, C and corresponding lengths of altitudes h_a, h_b, h_c, we know that the area is S, the inradius is r, and circumradius is R. Prove the following identities:

$$h_a\cos A+h_b\cos B+h_c\cos C=2R(1+\cos A\cos B\cos C),$$
$$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{2ab}{(a+b+c)\cdot S}\cdot\cos^2\left(\frac{C}{2}\right),$$
$$\frac{ah_c}{b}+\frac{bh_a}{c}+\frac{ch_b}{a}=\frac{a^2+b^2+c^2}{2R},$$
$$\frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2}=\frac{\cot A+\cot B+\cot C}{S}$$

KACY Summer League

Calculating Distance from Orthocenter 882. In triangle ABC with angles A, B, C and radius of the circumcircle R, assume that H is the orthocenter (intersection of heights). If we let A', B', C' be the foot of the heights drawn from A, B, C, respectively, then

$$HA = 2R\cos A, \qquad HA' = 2R\cos B\cos C,$$
$$HB = 2R\cos B, \qquad HB' = 2R\cos C\cos A,$$
$$HC = 2R\cos C, \qquad HC' = 2R\cos A\cos B.$$

KACY Summer League

Trigonometry of the Orthic Triangle 883. In triangle ABC with sides a, b, c, angles A, B, C, and radius of the circumcircle R, assume that H is the orthocenter (intersection of heights). If we let A', B', C' be the foot of the heights drawn from A, B, C, respectively, then we call triangle $A'B'C'$ the **Orthic Triangle** of triangle ABC.
Prove the following:

1. Heights of ABC are the internal angle bisectors of $A'B'C'$.
2. The circumradius of $A'B'C'$ is half of circumradius of ABC.
3. $HA \cdot HA' = HB \cdot HB' = HC \cdot HC'$.
4. $\angle BHC = \pi - \angle A$, $\angle AHC = \pi - \angle B$, and $\angle AHB = \pi - \angle C$.
5. The circumcircles of triangles AHB, AHC, and BHC are the same as the circumcircle of ABC.
6. The reflection of H with respect to each side lies on the circumcircle of ABC.
7. The sides of triangle ABC are the external angle bisectors of $A'B'C'$.
8. The circumcircle of the orthic triangle $A'B'C'$ passes through the midpoints of HA, HB, and HC.
9. The angles of $A'B'C'$ are equal to $\pi - 2\angle A$, $\pi - 2\angle B$, and $\pi - 2\angle C$.
10. $A'B' = R\sin 2C$, $A'C' = R\sin 2B$, and $B'C' = R\sin 2A$.
11. The area of the orthic triangle is $S_H = \dfrac{abc|\cos A\cos B\cos C|}{2R}$.
12. The inradius of the orthic triangle is $r_H = 2R|\cos A\cos B\cos C|$.

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KACY–I 884. If $A'B'C'$ is the orthic triangle of triangle ABC, then show that the area of triangle $A'B'C'$ is

$$S_{\triangle A'B'C'} = \frac{1}{2}R^2 \sin 2A \sin 2B \sin 2C,$$

and that

$$\frac{1}{AA'} + \frac{1}{BB'} + \frac{1}{CC'} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}.$$

KACY Summer League

KACY–I 885. Imagine that the heights of triangle ABC drawn from vertices A, B, C intersect the circumcircle of triangle ABC in A'', B'', C'', respectively. Show that

$$\frac{S_{\triangle A''B''C''}}{S_{\triangle ABC}} = 8 \cos A \cos B \cos C.$$

2.1.4.4 Trigonometry of Internal and External Angle Bisectors

KACY Summer League

Calculating the Length of Internal Angle Bisectors 886. In triangle ABC with angles A, B, C and radius of the circumcircle R, let d_a, d_b, d_c be the length of the corresponding internal angle bisectors of angles A, B, C. Prove that

$$d_a = \frac{h_a}{\cos \dfrac{B-C}{2}} = \frac{2bc}{b+c} \cos \frac{A}{2},$$

$$d_b = \frac{h_b}{\cos \dfrac{C-A}{2}} = \frac{2ca}{c+a} \cos \frac{B}{2},$$

$$d_c = \frac{h_c}{\cos \dfrac{A-B}{2}} = \frac{2ab}{a+b} \cos \frac{C}{2},$$

where h_a, h_b, h_c are the heights drawn from vertices A, B, C in triangle ABC.

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Calculating the Length of External Angle Bisectors 887. In triangle ABC with angles A, B, C and radius of the circumcircle R, let d'_a, d'_b, d'_c be the length of the corresponding external angle bisectors. Prove that

$$d'_a = \frac{h_a}{\left|\sin\dfrac{B-C}{2}\right|} = \left|\frac{2bc}{c-b}\right|\sin\frac{A}{2},$$

$$d'_b = \frac{h_b}{\left|\sin\dfrac{C-A}{2}\right|} = \left|\frac{2ca}{c-a}\right|\sin\frac{B}{2},$$

$$d'_c = \frac{h_c}{\left|\sin\dfrac{A-B}{2}\right|} == \left|\frac{2ab}{a-b}\right|\sin\frac{C}{2},$$

where h_a, h_b, h_c are the heights drawn from vertices A, B, C in triangle ABC.

KACY Summer League

KACY–I 888. In triangle ABC, we know that $b^3 + c^3 = a^2(b + c)$. Show that $\angle A = 60°$. If we furthermore know that $4h_a^2 = d_a \cdot d'_a$, find angles $\angle B$ and $\angle C$.

2.1.4.5 Trigonometry of Inradius and Exradii

KACY Summer League

Calculating the Inradius 889. In triangle ABC with angles A, B, C with circumradius R and inradius r, show that

$$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

KACY Summer League

Calculating the Exradii 890. In triangle ABC with angles A, B, C with circumradius R, assume that r_a, r_b, r_c are the exradii (radii of the excircles) corresponding to vertices A, B, C. Show that

$$r_a = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},$$

$$r_b = 4R\sin\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2},$$

$$r_c = 4R\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}.$$

KACY Summer League

KACY–I 891. In triangle ABC, assume that $\dfrac{1}{r_a}+\dfrac{1}{r}=\dfrac{K}{2a}$.

1. Find $K\sin B\sin C$ in terms of half-angles $A/2$, $B/2$, and $C/2$.
2. Show that $K>\dfrac{8}{\cos\dfrac{B-C}{2}}$.

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Feuerbach Formulas on Exradii 892. In triangle ABC with inradius r, exradii r_a, r_b, r_c, and circumradius R, prove the following identities:

$$4R=r_a+r_b+r_c-r,$$
$$\frac{1}{r}=\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}.$$

Furthermore, if S is the area and p the semiperimeter of triangle ABC with side-lengths a, b, c, prove the **Feuerbach formulas on exradii**:

$$r_ar_b+r_br_c+r_cr_a+r(r_a+r_b+r_c)=ab+bc+ca,$$
$$r_ar_b+r_br_c+r_cr_a-r(r_a+r_b+r_c)=\frac{1}{2}\left(a^2+b^2+c^2\right),$$
$$r(r_ar_b+r_br_c+r_cr_a)=r_ar_br_c=Sp,$$
$$r(r_a+r_b+r_c)=ab+bc+ca-p^2.$$

KACY Summer League

Excentral Triangle Side–Lengths 893. In triangle ABC with incenter I, excenters I_A, I_B, I_C (centers of excircles corresponding to vertices A, B, C), and circumradius R, prove the following identities regarding the side–lengths of the **Excentral Triangle** $I_AI_BI_C$:

$$I_BI_C = \frac{a}{\sin\frac{A}{2}} = 4R\cos\frac{A}{2}, \quad I_CI_A = \frac{b}{\sin\frac{B}{2}} = 4R\cos\frac{B}{2}, \quad I_AI_B = \frac{c}{\sin\frac{C}{2}} = 4R\cos\frac{C}{2},$$

and

$$II_A = 4R\sin\frac{A}{2}, \qquad II_B = 4R\sin\frac{B}{2}, \qquad II_C = 4R\sin\frac{C}{2}.$$

Furthermore, prove that I is the orthocenter of the excentral triangle $I_AI_BI_C$, and that the area of this triangle equals

$$S_{\triangle I_AI_BI_C} = 8R^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

KACY Summer League

Excentral Identities 894. In triangle ABC with area S, incenter I and excentral triangle $I_AI_BI_C$, let R be the circumradius. Show that:

$$\frac{IA\cdot IB}{IC} = 4R\sin^2\frac{C}{2},$$
$$\frac{I_AA\cdot I_AB}{I_AC} = 4R\cos^2\frac{C}{2},$$
$$IA\cdot IB\cdot IC = 4SR\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2},$$
$$\frac{IA}{I_AA} + \frac{IB}{I_BB} + \frac{IC}{I_CC} = 1,$$
$$\frac{II_A}{I_BI_C} = \tan\frac{A}{2}.$$

Moreover, if the side-lengths of triangle ABC are a, b, c, and $S_{\triangle XYZ}$ denotes the area of triangle XYZ, show that

$$S_{\triangle I_AI_BI_C} = \frac{abc}{2r},$$
$$\frac{S_{\triangle I_AI_BI_C}}{S_{\triangle II_AI_C}} = \frac{r_b}{r}.$$

KACY Summer League

KACY–I 895. In triangle ABC with excentral triangle $I_AI_BI_C$, call the incenter I and the circumcenter O. Show that

$$IA^2 + I_AA^2 + I_BA^2 + I_CA^2 = 16R^2,$$
$$IO^2 + I_AO^2 + I_BO^2 + I_CO^2 = 12R^2,$$
$$IA \cdot IB \cdot IC = 4Rr^2.$$

KACY Summer League

KACY–I 896. In triangle ABC we know that the perimeter is $2p$ and inradius is r. If the area of triangle ABC is S and the area of the excentral triangle $I_AI_BI_C$ is S_I, show that

$$\frac{S_I}{S} = \frac{4abc}{(a+b-c)(b+c-a)(c+a-b)} = \frac{abc}{2r^2p}.$$

KACY Summer League

KACY–I 897. Show that

$$(a+b+c) \cdot II_A \cdot II_B \cdot II_C = 8Rabc.$$

2.1.4.6 Trigonometry of Distance Between Triangle Centers

KACY Summer League

Distance Between Incenter and Excenters 898. In triangle ABC with excentral triangle $I_AI_BI_C$, exradii r_a, r_b, r_c, inradius r and incenter I, show that

$$II_A^2 = 4R(r_a - r), \qquad II_B^2 = 4R(r_b - r), \qquad II_C^2 = 4R(r_c - r).$$

KACY Summer League

Distance Between Circumcenter and Incenter 899. In triangle ABC with incenter I and circumcenter O, the excenters are I_A, I_B, I_C. Show that

$$AI = 4R\cos\frac{B}{2}\cos\frac{C}{2} = \frac{r}{\sin\frac{A}{2}},$$

and

$$\begin{aligned} OI_A^2 &= R(R+2r_a), \\ OI^2 &= R(R-2r). \end{aligned}$$

KACY Summer League

Distance Between Circumcenter and Orthocenter 900. In triangle ABC with circumcenter O, the orthocenter (intersection of heights) is H. Show that

$$\begin{aligned} OH^2 &= R^2(1-8\cos A\cos B\cos C) \\ &= 2R^2\left(\frac{3}{2}+\cos 2A+\cos 2B+\cos 2C\right). \end{aligned}$$

KACY Summer League

Distance Between Incenter and Orthocenter 901. In triangle ABC with incenter I, inradius r and circumradius R, the orthocenter (intersection of heights) is H. Show that

$$IH^2 = 2r^2 - 4R^2\cos A\cos B\cos C.$$

For an extra point, prove that if K is the circumcenter of triangle BHC, then

$$IK^2 = (R+r)^2 + r^2 - \frac{2S^2}{r_b r_c}.$$

KACY Summer League

KACY–I 902. Prove that the orthocenter (intersection of heights) of triangle ABC lies on the incircle of the triangle if and only if

$$\cos A\cos B\cos C = 4\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}.$$

KACY Summer League

KACY–I 903. Prove that the line connecting the incenter and circumcenter of triangle ABC is tangent to the excircle corresponding to vertex A of triangle if and only if:

$$S\,|r_b - r_c| = r_a r_b r_c \sqrt{1 - \frac{2r}{R}}.$$

KACY Summer League

Distance Between Vertex and Centroid 904. In triangle ABC, prove that the distance between vertex A and centroid G of triangle is found by:

$$9AG^2 = 2S(4\cot A + \cot B + \cot C).$$

KACY Summer League

Distance Between Circumcenter and Centroid 905. In triangle ABC, prove that the distance between circumcenter O and centroid G of triangle is given by:

$$9OG^2 = 9R^2 - (a^2 + b^2 + c^2).$$

KACY Summer League

Distance Between Vertex and Nine-Point Center 906. In triangle ABC, prove that the distance between vertex A and the center of nine-point circle N of triangle is:

$$AN = \frac{R}{2}\sqrt{(1 + 8\cos A\cos B\cos C)}.$$

KACY Summer League

Distance Between Incenter and Nine-Point Center 907. In triangle ABC, prove that the distance between center of the incircle I and the center of nine-point circle N of triangle is:

$$IN = \frac{R}{2} - r.$$

2.1.5 Identities for Area in Terms of Triangle Radii

KACY Summer League

Area in Terms of Radii 908. In triangle ABC with angles A, B, C and side-lengths a, b, c, let r and R be the radius of the triangle's incircle and circumcircle, respectively, and assume r_a, r_b, r_c represent the radius of the excircle of triangle ABC with respect to A, B, C, respectively. If S is the area of the triangle, prove the following identities.

- $S = \dfrac{abc}{4R}$,
- $S = \sqrt{rr_ar_br_c}$,
- $S = \dfrac{arr_a}{r_a - r}$,
- $S = rr_a \cot \dfrac{A}{2}$,
- $S = \dfrac{ar_br_c}{r_b + r_c}$,
- $S = \dfrac{(b+c)rr_a}{r_a + r}$,
- $S = \dfrac{rr_a(r_b - r_c)}{b - c}$,
- $S = \dfrac{rr_b\sqrt{r_a + r_b}}{\sqrt{r_b - r}}$,
- $S = r\sqrt{r_ar_b + r_br_c + r_cr_a}$,
- $S = \dfrac{r_ar_br_c}{\sqrt{r_ar_b + r_br_c + r_cr_a}}$,
- $S = r^2 \cot \dfrac{A}{2} \cot \dfrac{B}{2} \cot \dfrac{C}{2}$,
- $S^2 = abcp \sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2}$,
- $S = \dfrac{r}{2\sqrt{R}}\sqrt{(r_a + r_b)(r_b + r_c)(r_c + r_a)}$,
- $S^2 \left(\dfrac{1}{r^2} + \dfrac{1}{r_a^2} + \dfrac{1}{r_b^2} + \dfrac{1}{r_c^2} \right) = a^2 + b^2 + c^2$,
- $4S(\cot A + \cot B + \cot C) = a^2 + b^2 + c^2$,
- $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{a \sin A + b \sin B + c \sin C}{4S}$.

2.1.6 More Identities on Triangle Radii

KACY Summer League

Trigonometric Identities on Radii 909. In triangle ABC with angles A, B, C and side-lengths a, b, c, let r and R be the radius of the triangle's incircle and circumcircle, respectively, and assume r_a, r_b, r_c represent the radius of the excircle of triangle ABC with respect to A, B, C, respectively. If S and $2p$ are the area and perimeter of triangle ABC, prove the following trigonometric identities:

- $r_c = r \cot \dfrac{A}{2} \cot \dfrac{B}{2}$,
- $\dfrac{1}{c \sin B} + \dfrac{1}{a \sin C} + \dfrac{1}{b \sin A} = \dfrac{1}{r}$,
- $2R(1 - \cos A) = r_a - r$,
- $rr_a = r_b r_c \tan^2 \dfrac{A}{2}$,
- $\dfrac{\cos A}{c \sin B} + \dfrac{\cos B}{a \sin C} + \dfrac{\cos C}{b \sin A} = \dfrac{1}{R}$,
- $r_a(\cos B - \cos C) + r_b(\cos C - \cos A) + r_c(\cos A - \cos B) = 0$,
- $abc + (a - b)(b - c)(c - a) = 4Rr(a \cos C + b \cos A + c \cos B)$,
- $2R \sin A \sin B \sin C = r(\sin A + \sin B + \sin C)$,
- $2(R + r) = a \cot A + b \cot B + c \cot C$,
- $8R^2(1 + \cos A \cos B \cos C) = a^2 + b^2 + c^2$.

KACY Summer League

Algebraic Identities on Radii 910. Prove the following algebraic identities on exradii, inradius and circumradius:

- $rp^2 = r_a r_b r_c$,
- $4Rr + r^2 = ab + bc + ca - p^2$,
- $4Rrp = abc$,
- $\dfrac{r_c^2}{4R - r_a - r_b} = r_c + \dfrac{r_a r_b}{r_a + r_b}$,
- $r_a r_b r_c = r^3 \cot^2 \dfrac{A}{2} \cot^2 \dfrac{B}{2} \cot^2 \dfrac{C}{2}$,
- $a(rr_a + r_b r_c) = b(rr_b + r_c r_a) = c(rr_c + r_a r_b)$,
- $\left(\dfrac{r_a}{r} - 1\right)\left(\dfrac{r_b}{r} - 1\right)\left(\dfrac{r_c}{r} - 1\right) = \dfrac{4R}{r}$,
- $\dfrac{r_a r_b r_c}{r^3} = \dfrac{(a + b + c)^3}{(a + b - c)(b + c - a)(c + a - b)}$,
- $(b - c)r_b r_c + (c - a)r_c r_a + (a - b)r_a r_b = 0$.

2.1.6.1 Trigonometry of Quadrilaterals

KACY Summer League

Brahmagupta's Formula on Area of Cyclic Quadrilaterals 911. If the cyclic quadrilateral $ABCD$ has side-lengths a, b, c, d and perimeter $2p$, then its area S is found by **Brahmagupta's formula**:

$$S = \sqrt{(p-a)(p-b)(p-c)(p-d)}.$$

Moreover,

$$\sin B = \frac{2S}{ab+cd}.$$

KACY Summer League

Generalized Brahmagupta's Formula 912. If the sum of angles $\angle B$ and $\angle D$ in any quadrilateral $ABCD$ is 2α, prove that the area of the quadrilateral is equal to

$$S = \sqrt{(p-a)(p-b)(p-c)(p-d) - abcd\cos^2\alpha}.$$

KACY Summer League

Pitot's Theorem on Area of Tangential Quadrilaterals 913. If the tangential quadrilateral $ABCD$ has side-lengths a, b, c, d and perimeter $2p$, and the sum of angles $\angle B$ and $\angle D$ is 2α, then the area S of the quadrilateral is found by **Pitot's formula**:

$$S = \sqrt{abcd\sin^2\alpha}.$$

2.1.7 Calculating Diagonals and Circumradius of Cyclic Quadrilaterals

KACY Summer League

Trigonometry of Quadrilateral Diagonals 914. In a cyclic quadrilateral $ABCD$ with side-lengths a, b, c, d, the length of the two diagonals AC and BD can be calculated by

$$AC^2 = \frac{(ac+bd)(ad+bc)}{ab+cd},$$
$$BD^2 = \frac{(ab+cd)(ac+bd)}{ad+bc}.$$

KACY Summer League

Calculating Circumradius of Quadrilateral 915. In a cyclic quadrilateral $ABCD$ of area S with side-lengths a, b, c, d, the circumradius R can be calculated by

$$R = \frac{AC}{2\sin B} = \frac{\sqrt{(ab+cd)(ac+bd)(ad+bc)}}{4S}.$$

KACY Summer League

KACY–I 916. Prove that the area of any quadrilateral equals half the product of quadrilateral's two diagonals times sine of the angle between the diagonals.

KACY Summer League

KACY–I 917. If a, b, c, d are the four sides of a quadrilateral and α is the angle between its diagonals (facing side b), prove that the area of the quadrilateral is equal to

$$\frac{1}{4}\left(a^2+b^2+c^2+d^2\right)\tan\alpha.$$

KACY Summer League

KACY–I 918. If the angle between the diagonals of a cyclic quadrilateral is α, show that

$$\sin\alpha = \frac{2\sqrt{(p-a)(p-b)(p-c)(p-d)}}{ac+bd},$$

where a, b, c, d are the side-lengths of the quadrilateral and p is its semi–perimeter.

KACY Summer League

KACY–I 919. We know that a quadrilateral with side-lengths a, b, c, d is both cyclic and tangential, and that the angle between its diagonals is α. Prove that

$$\cos\alpha = \frac{ac - bd}{ac + bd}.$$

KACY Summer League

KACY–I 920. In the tangential quadrilateral $ABCD$ (whose sides are tangent to an inscribed circle) with sides a, b, c, d, if two opposite angles add up to 2α, show that the area of the quadrilateral is equal to $\sin\alpha \cdot \sqrt{abcd}$.

KACY Summer League

KACY–I 921. In the tangential quadrilateral $ABCD$ with sides a, b, c, d, if two opposite angles add up to 2α and the angle between the diagonals is Φ, prove that:

$$\tan^2\Phi = \frac{4abcd\sin^2\alpha}{(ac - bd)^2}.$$

KACY Summer League

KACY–I 922. We know that the quadrilateral $ABCD$ with side-lengths a, b, c, d has the two special following properties: a) its vertices lie on a circumcircle, and b) its sides are tangent to an inscribed circle. Prove that its area is $\sqrt{abcd}$. Moreover, show that the radius of the inscribed circle inside quadrilateral $ABCD$ equals

$$\frac{2\sqrt{abcd}}{a + b + c + d}.$$

KACY Summer League

KACY–I 923. If x and y are the length of diagonals of a tangential quadrilateral $ABCD$ with side-lengths a, b, c, d, show that the area of $ABCD$ equals

$$\frac{1}{2}\sqrt{x^2y^2 - (ac - bd)^2}.$$

KACY Summer League

KACY–I 924. If x and y are the length of diagonals of any quadrilateral $ABCD$ with side-lengths a, b, c, d, show that the area of $ABCD$ equals

$$\frac{1}{4}\sqrt{4x^2y^2 - (b^2 + d^2 - a^2 - c^2)^2}.$$

KACY Summer League

KACY–I 925. If x and y are the length of diagonals of any quadrilateral $ABCD$ with side-lengths a, b, c, d, and θ is the sum of two opposite angles in the quadrilateral, show that

$$x^2y^2 = a^2c^2 + b^2d^2 - 2abcd\cos\theta.$$

KACY Summer League

KACY–I 926. Let a, b, c, d be fixed side-lengths of a variable quadrilateral $ABCD$. The area of this quadrilateral when the angle between a and d is $90°$ is the same as the area of the quadrilateral when the angle between c and d is $90°$. Prove that either $a^2 + b^2 = c^2 + d^2$ or $ab = cd$.

KACY Summer League

KACY–I 927. In a parallelogram, a and b are the length of adjacent sides, and α and Φ are acute angles between the two sides and the two diagonals of the parallelogram, respectively. Prove that

$$\frac{a}{b}\sin\Phi = \sin\alpha\cos\Phi \pm \sqrt{1 - \cos^2\alpha\cos^2\Phi}.$$

KACY Summer League

KACY–I 928. If p is the semiperimeter of the cyclic quadrilateral $ABCD$ (whose vertices lie on the circumcircle) with side-lengths a, b, c, d, prove that

$$(p - c)(p - d)\tan^2\frac{B}{2} = (p - a)(p - b).$$

KACY Summer League

KACY–I 929. If θ is the angle between the diagonals of a cyclic quadrilateral $ABCD$ with side-lengths a, b, c, d, prove that

$$(ac+bd)\sin\theta = 2\sqrt{(p-a)(p-b)(p-c)(p-d)},$$
$$2(ac+bd)\cos\theta = (a^2+c^2)-(b^2+d^2).$$

KACY Summer League

KACY–I 930. For quadrilateral $ABCD$, it is possible to draw two circles, one tangent to the sides AB, BC, CD, and another tangent to the sides CD, DA, AB. If we also know that these two circles touch each other at exactly one point, prove that

$$(a-b+c-d)\sin\frac{A+D}{2} = 4\sqrt{bd\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}}.$$

2.2 Math Olympiad Trigonometry 201: JEE, Roorkee, ISM

Some trigonometric problems from everywhere, mainly Indian problems taken from casual trigonometry books by Agarwal, Pearson, and Rejaul Makshud.

2.2.1 Agarwal's and Pearson's Problems

KACY Summer League

KACY–I 931. Show the identity

$$2(\sin^6 x+\cos^6 x)-3(\sin^4 x+\cos^4 x)+1=0. \tag{2.63}$$

Also, prove that

$$\sin^8 x-\cos^8 x=(\sin^2 x-\cos^2 x)(1-2\sin^2 x\cdot\cos^2 x). \tag{2.64}$$

KACY Summer League

KACY–I 932. If we know that

$$\frac{\cos^4 x}{\cos^2 y} + \frac{\sin^4 x}{\sin^2 y} = 1,$$

then prove that

$$\sin^4 x + \sin^4 y = 2\sin^2 x \cdot \sin^2 y. \tag{2.65}$$

Also, find the value of

$$\frac{\cos^4 y}{\cos^2 x} + \frac{\sin^4 y}{\sin^2 x}. \tag{2.66}$$

KACY Summer League

KACY–I 933. If $\tan^2\theta = 1 - e^2$, and we write $\sec\theta + \tan^3\theta\csc\theta$ in the form of $(2 - e^p)^{q/p}$, where p and q are prime numbers, what is $p + q$?

KACY Summer League

KACY–I 934. For what values of x and y can the following equation hold true?

$$\sec^2\theta = \frac{4xy}{(x+y)^2}.$$

KACY Summer League

KACY–I 935. Solve the equation $\cos\theta + \cos 3\theta - 2\cos 2\theta = 0$.

KACY Summer League

KACY–I 936. Solve the equation $\sin m\theta + \sin n\theta = 0$.

KACY Summer League

KACY–I 937. Solve the equation $\tan^2\theta + (1 - \sqrt{3})\tan\theta - \sqrt{3} = 0$.

KACY Summer League

KACY–I 938. Solve the equation $\tan\theta + \tan 2\theta + \tan\theta\tan 2\theta = 1$.

KACY Summer League

KACY–I 939. Solve the equation $4\sin x\sin(2x)+\sin(4x)=\sin(3x)$.

KACY Summer League

KACY–I 940. Solve the equation $\sqrt{3}\cos\theta+\sin\theta=\sqrt{2}$.

KACY Summer League

1989 Roorkee 941. If $4\sin^4 x+\cos^4 x=1$, then x is

a) $n\pi$ b) $n\pi\pm\sin^{-1}\frac{2}{5}$ c) $n\pi+\frac{\pi}{6}$ d) NONE

KACY Summer League

2004 Orissa JEE 942. Which of the following represent the roots of the equation

$$1-\cos\theta=\sin\theta\cdot\sin\frac{\theta}{2}?$$

a) $k\pi$ b) $2k\pi$ c) $k\frac{\pi}{2}$ d) NONE

KACY Summer League

1989 IIT 943. Which option shows the general solution of the following equation?

$$\sin x-3\sin 2x+\sin 3x=\cos x-3\cos 2x+\cos 3x.$$

a) $n\pi+\frac{\pi}{8}$ b) $\frac{n\pi}{2}+\frac{\pi}{8}$ c) $(-1)^n\frac{n\pi}{2}+\frac{\pi}{8}$ d) $2n\pi+\cos^{-1}\frac{2}{3}$

KACY Summer League

2003 Orissa JEE 944. The equation $\sin x+\sin y+\sin z=-3$, where x,y,z are real numbers in $[0,2\pi]$, has

a) One solution b) Two solutions c) Three solutions d) Four solutions

KACY Summer League

1989 ISM Dhanbad 945. If $2\sin^2 x + \sin^2 2x = 2$, where $-\pi < x < \pi$, then x can be equal to

a) $\pm\frac{\pi}{6}$ b) $\pm\frac{\pi}{4}$ c) $\frac{3\pi}{2}$ d) NONE

KACY Summer League

1984 Roorkee 946. If $5\cos 2\theta + 2\cos^2\frac{\theta}{2} + 1 = 0$, where $-\pi < \theta < \pi$, then θ can be equal to

a) $\frac{\pi}{3}$ b) $\frac{\pi}{3}, \cos^{-1}\frac{3}{5}$ c) $\cos^{-1}\frac{3}{5}$ d) $\frac{\pi}{3}, \pi - \cos^{-1}\frac{3}{5}$

KACY Summer League

2004 Karnataka CET 947. If $81^{\sin^2 x} + 81^{\cos^2 x} = 30$, where $0 < x < \pi$, then x is equal to

a) $\frac{\pi}{6}$ b) $\frac{\pi}{2}$ c) $\frac{\pi}{4}$ d) $\frac{3\pi}{4}$

KACY Summer League

KACY–I 948. If $\cos x + \cos 3x + \cos 5x + \cos 7x = 0$, then x is equal to

a) $n\frac{\pi}{4}$ b) $n\frac{\pi}{2}$ c) $n\frac{\pi}{8}$ d) NONE

KACY Summer League

KACY–I 949. If $a\cos x + b\sin x = c$, where a, b, c are non-zero constants, then x is equal to

a) $n\pi + \cos^{-1}\left(\frac{c}{\sqrt{a^2+b^2}}\right)$

b) $2n\pi - \tan^{-1}\left(\frac{b}{a}\right)$

c) $2n\pi - \tan^{-1}\left(\frac{b}{a}\right) \pm \cos^{-1}\left(\frac{c}{\sqrt{a^2+b^2}}\right)$

d) $2n\pi + \tan^{-1}\left(\frac{b}{a}\right) \pm \cos^{-1}\left(\frac{c}{\sqrt{a^2+b^2}}\right)$

2.2.2 Rejaul Makshud's Problems

These problems are taken from Rejaul Makshud's book "Trigonometry Booster with Problems and Solutions," the 2019 edition published by McGraw-Hill. The problems in the book come in multiple levels, and I have chosen the important ones in ascending order of difficulty to be printed here.

KACY Summer League

Exercises in Law of Sines and Law of Cosines 950. In a triangle ABC with side–lengths a, b, c and angles A, B, C, prove the following identities:

a) $c^2 = (a-b)^2 \cos^2\left(\frac{C}{2}\right) + (a+b)^2 \sin^2\left(\frac{C}{2}\right)$;

b) $0 = \frac{a^2 \sin(B-C)}{\sin B + \sin C} + \frac{b^2 \sin(C-A)}{\sin C + \sin A} + \frac{c^2 \sin(A-B)}{\sin A + \sin B}$;

c) $0 = \frac{a^2 - b^2}{\cos A + \cos B} + \frac{b^2 - c^2}{\cos B + \cos C} + \frac{c^2 - a^2}{\cos A + \cos B}$;

d) $\left(\frac{a^2 - b^2}{c^2}\right) \sin 2C + \left(\frac{b^2 - c^2}{a^2}\right) \sin 2A + \left(\frac{c^2 - a^2}{b^2}\right) \sin 2B$;

e) $27 < \left(\frac{\sin^2 A + \sin A + 1}{\sin A}\right)\left(\frac{\sin^2 B + \sin B + 1}{\sin B}\right)\left(\frac{\sin^2 C + \sin C + 1}{\sin C}\right)$;

f) $\frac{a \sin(B-C)}{b^2 - c^2} = \frac{b \sin(C-A)}{c^2 - a^2} = \frac{c \sin(A-B)}{a^2 - b^2}$;

g) $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$;

h) $\frac{1 + \cos(A-B)\cos C}{1 + \cos(A-C)\cos B} = \frac{a^2 + b^2}{a^2 + c^2}$.

KACY Summer League

1984 IIT JEE 951. Prove that triangle ABC is equilateral if and only if

$$\cos A + \cos B + \cos C = \frac{3}{2}.$$

KACY Summer League

1984 IIT JEE 952. In triangle ABC with side-lengths a, b, c, it is known that

$$\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}.$$

Prove that

$$\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}.$$

KACY Summer League

Law of Sines and Law of Cosines in Progressions 953. In triangle ABC with side lengths a, b, c and angles A, B, C, prove the following statements. Note that **AP** and **GP** are short for Arithmetic & Geometric Progressions, respectively.

1. a, b, c form an AP if and only if $(p-c)/(p-a) = (b-c)/(a-b)$, where p is the semiperimeter of $\triangle ABC$.

2. a^2, b^2, c^2 form an AP if and only if $\cot A, \cot B, \cot C$ form an AP.

3. If A, B, C form an AP while a, b, c form a GP, then a^2, b^2, c^2 form an AP.

4. If $\cos A + 2\cos B + \cos C = 2$, then a, b, c are in an AP.

5. Imagine that a, b, c form an AP. Let θ and Φ be the largest and smallest angles of triangle ABC. Show that $4(1+\cos\theta)(1-\cos\theta) = \cos\theta + \cos\Phi$.

6. If $c\cos^2\left(\dfrac{A}{2}\right) + a\cos^2\left(\dfrac{C}{2}\right) = \dfrac{3b}{2}$, then a, b, c are in an AP.

7. If A, B, C form an AP, then $2\cos\left(\dfrac{A-C}{2}\right) = \dfrac{a+c}{\sqrt{a^2-ac+c^2}}$.

8. If $\dfrac{\sin A}{\sin C} = \dfrac{\sin(A-B)}{\sin(B-c)}$, then a^2, b^2, c^2 form an AP.

9. If a, b, c form an AP, then so do

$$\cos A \cdot \cot\left(\frac{A}{2}\right), \quad \cos B \cdot \cot\left(\frac{B}{2}\right), \quad \cos C \cdot \cot\left(\frac{C}{2}\right).$$

10. If a, b, c form an AP and the difference between the largest and smallest angles in triangle ABC is α, show that the sides are in ratio $(1-x) : 1 : (1+x)$, where $x = \sqrt{(1-\cos\alpha)/(7-\cos\alpha)}$.

KACY Summer League

KACY–I 954. In a triangle ABC with side lengths a, b, c and angles A, B, C, we know that $(a^2 - b^2)/(a^2 + b^2) = (\sin(A - B))/(\sin(A + B))$. Is this triangle right-angled or isosceles?

KACY Summer League

KACY–I 955. In a triangle ABC with side lengths a, b, c and angles A, B, C, we know that $\cot \frac{A}{2} = \frac{b + c}{a}$. Prove that the triangle is right-angled.

KACY Summer League

KACY–I 956. In a triangle ABC with side lengths a, b, c and angles A, B, C, we know that $2 \cos B = \frac{a}{c}$. Prove that the triangle is isosceles.

KACY Summer League

KACY–I 957. In a triangle ABC with side lengths a, b, c and angles A, B, C, if $a \cos A = b \cos B$, then the triangle is right-angled isosceles.

KACY Summer League

KACY–I 958. If the angle A in triangle ABC equals $60°$, find the value of

$$\left(1 + \frac{a + b}{c}\right)\left(1 + \frac{c - a}{b}\right).$$

KACY Summer League

KACY–I 959. Find the value of angle C in triangle ABC if the three sides a, b, c of the triangle satisfy the equation

$$\frac{1}{a + c} + \frac{1}{b + c} = \frac{3}{a + b + c}.$$

KACY Summer League

KACY–I 960. Find the value of angle A (in degrees) of triangle ABC if the three sides a, b, c and three angles A, B, C of the triangle satisfy the equation

$$\frac{2 \cos A}{a} + \frac{2 \cos B}{b} + \frac{2 \cos C}{c} = \frac{1}{bc} + \frac{b}{ca}.$$

KACY Summer League

Identities in Laws of Tangents & Cotangents 961. In triangle ABC with side lengths a, b, c and angles A, B, C, let S be the area and r be the inradius. Prove the following identities:

1. $1 - \tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right) = \frac{2c}{a+b+c}$.

2. If we define x by $x = \tan\left(\frac{B-C}{2}\right)\tan\left(\frac{A}{2}\right)$, and y, z by
$$y = \tan\left(\frac{C-A}{2}\right)\tan\left(\frac{B}{2}\right) \qquad \text{and} \qquad z = \tan\left(\frac{A-B}{2}\right)\tan\left(\frac{C}{2}\right),$$
show that $x + y + z + xyz = 0$.

3. If we define u, v, w by
$$u = \cot\left(\frac{A}{2}\right), \qquad v = \cot\left(\frac{B}{2}\right), \qquad w = \cot\left(\frac{C}{2}\right),$$
show that
$$u + v + w = u \cdot \left(\frac{b+c+a}{b+c-a}\right),$$
and
$$\frac{u+v+w}{\cot A + \cot B + \cot C} = \frac{(a+b+c)^2}{a^2+b^2+c^2}.$$

4. If D is the midpoint of side BC of triangle ABC with area S, let $\theta = \angle ADB$. Then, $\cot\theta = (b^2 - c^2)/(4S)$.

5. If h_a, h_b, h_c are the length of altitudes drawn from vertices A, B, C of triangle ABC with area S, respectively, then
$$\frac{1}{h_a^2} + \frac{1}{h_b^2} + \frac{1}{h_c^2} = \frac{\cot A + \cot B + \cot C}{S}.$$

6. Let A' be the foot of the altitude drawn from vertex A of triangle ABC, and imagine that ρ_1 and ρ_2 are the inradii of triangles ABA' and ACA', respectively. Show that
$$\frac{\cot B}{\rho_1} + \frac{\cot C}{\rho_2} = (\cot B + \cot C)\left(\frac{1}{r} + \frac{2}{a}\right).$$

KACY Summer League

1986 IIT JEE 962. In triangle ABC with side-lengths a, b, c and angles A, B, C, it is known that $\cos A \cos B + \sin A \sin B \sin C = 1$. Prove that

$$a : b : c = 1 : 1 : \sqrt{2}.$$

KACY Summer League

KACY–I 963. Let θ be the angle such that in triangle ABC,

$$\sin^3 \theta = \sin(A - \theta) \sin(B - \theta) \sin(C - \theta).$$

Prove that $\cot \theta = \cot A + \cot B + \cot C$.

KACY Summer League

KACY–I 964. In triangle ABC, prove that

$$\tan^2\left(\frac{A}{2}\right) + \tan^2\left(\frac{B}{2}\right) + \tan^2\left(\frac{C}{2}\right) \geq 1.$$

KACY Summer League

KACY–I 965. In triangle ABC, prove that

$$\frac{\cot^2\left(\frac{A}{2}\right)\cot^2\left(\frac{B}{2}\right) + \cot^2\left(\frac{B}{2}\right)\cot^2\left(\frac{C}{2}\right) + \cot^2\left(\frac{C}{2}\right)\cot^2\left(\frac{A}{2}\right)}{\cot^2\left(\frac{A}{2}\right)\cot^2\left(\frac{B}{2}\right)\cot^2\left(\frac{C}{2}\right)} \geq 1.$$

KACY Summer League

KACY–I 966. In triangle ABC, prove that

$$\left(\cot\left(\frac{A}{2}\right) + \cot\left(\frac{B}{2}\right)\right)\left(a\sin^2\left(\frac{B}{2}\right) + b\sin^2\left(\frac{A}{2}\right)\right) = c\cot\left(\frac{C}{2}\right).$$

KACY Summer League

KACY–I 967. If the three sides a, b, c of triangle ABC form an arithmetic progression, prove that

$$\tan\left(\frac{A}{2}\right) + \tan\left(\frac{C}{2}\right) = \frac{2}{3} \cdot \cot\left(\frac{B}{2}\right).$$

KACY Summer League

KACY–I 968. In triangle ABC, if α, β, γ are the three angles that medians make with each other, then

$$\cot\alpha + \cot\beta + \cot\gamma + \cot A + \cot B + \cot C = 0.$$

KACY Summer League

KACY–I 969. In a triangle ABC, we know that

$$a\tan A + b\tan B = (a+b)\tan\left(\frac{A+B}{2}\right).$$

Prove that ABC is isosceles.

KACY Summer League

KACY–I 970. In triangle ABC with side-lengths a, b, c,

1. If $a = 2b$ and $\cos(A - B) = 4/5$, prove that $\angle C = 90°$.
2. If $3a = b + c$, show that

$$\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right) = 2.$$

KACY Summer League

Makshud's Identities on Area and Semiperimeter 971. In triangle ABC with side-lengths a, b, c and angles A, B, C,

1. Prove the following identities on semiperimeter p of the triangle:

$$p = b\cos^2\left(\frac{C}{2}\right) + c\cos^2\left(\frac{B}{2}\right),$$
$$p^2 = bc\cos^2\left(\frac{A}{2}\right) + ca\cos^2\left(\frac{B}{2}\right) + ab\cos^2\left(\frac{C}{2}\right).$$

2. Prove the following identities on area S of the triangle:

$$S = \frac{a^2 - b^2}{2} \cdot \frac{\sin A \sin B}{\sin(A - B)},$$

and the next one containing both S and p:

$$a^2(p-a) + b^2(p-b) + c^2(p-c) = 4RS\left(1 - 4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\right).$$

3. Let T be the area of the triangle whose vertices are the intersections of angle bisectors of triangle ABC. Then prove that the ratio T/S, where S is the area of triangle ABC, is equal to

$$\frac{2abc}{(a+b)(b+c)(c+a)}.$$

4. If the sides a, b, c are roots of $x^3 - ux^2 + vx - w = 0$, then show that the area S of triangle ABC equals

$$\frac{\sqrt{u(4uv - u^3 - 8w)}}{4}.$$

5. Prove that the ratio of the area of the incircle to the area of the triangle is

$$\pi : \tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right).$$

KACY Summer League

Makshud's Identities on Circumradius and Inradius 972. In triangle ABC with side-lengths a, b, c and angles A, B, C,

1. Prove the following identities regarding the inradius r and circumradius R:

$$\begin{aligned}
\frac{1}{2rR} &= \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}, \\
\frac{3}{2R} &= \frac{\sin A}{a} + \frac{\sin B}{b} + \frac{\sin C}{c}, \\
\frac{r}{R} &= \frac{a\cos A + b\cos B + c\cos C}{a+b+c}, \\
4R &= \frac{a\cos A + b\cos B + c\cos C}{\sin A \sin B \sin C}, \\
2R &= \frac{a\sec A + b\sec B + c\sec C}{\tan A \tan B \tan C} \\
2(r+R) &= a\cot A + b\cot B + c\cot C, \\
2 + \frac{r}{2R} &= \cos^2\left(\frac{A}{2}\right) + \cos^2\left(\frac{B}{2}\right) + \cos^2\left(\frac{C}{2}\right), \\
4(r+R) &= (b+c)\tan\left(\frac{A}{2}\right) + (c+a)\tan\left(\frac{B}{2}\right) + (a+b)\tan\left(\frac{C}{2}\right).
\end{aligned}$$

2. Let O be the circumcenter of triangle ABC. If R_A, R_B, R_C are circumradii of triangles OBC, OCA, OAB, respectively, then show that

$$\frac{a}{R_A} + \frac{b}{R_B} + \frac{c}{R_C} = \frac{abc}{R^3}.$$

3. If O is the circumcenter of triangle ABC and D, E, F are closest points on the sides of the triangle to O, we have

$$\frac{a}{OD} + \frac{b}{OE} + \frac{c}{OF} = \frac{abc}{4OD \cdot OE \cdot OF}.$$

4. If X, Y, Z are the points where the incircle touches the sides BC, CA, AB of triangle ABC, respectively, we have

$$r^2 = \frac{AX \cdot BY \cdot CZ}{AX + BY + CZ}.$$

KACY Summer League

KACY–I 973. In a regular polygon with n sides, each side has a length of $2a$. Let R and r be the circumradius and inradius of the polygon, respectively. Prove that $r + R = a\cot(\pi/2n)$.

KACY Summer League

Inradius, Circumradius, and Area of Regular Polygons 974. In a regular polygon with n sides each of length a, if r_n and R_n denote the inradius and circumradius of the polygon, respectively, then, show that

$$R_n = \frac{a/2}{\sin\left(\frac{a}{n}\right)} \qquad \text{and} \qquad r_n = \frac{a/2}{\tan\left(\frac{a}{n}\right)},$$

and if S_n denotes the area of the polygon, then prove that S_n equals

$$S_n = \frac{a^2 n}{4}\cot\left(\frac{\pi}{n}\right) = \frac{nR_n^2}{2}\sin\left(\frac{2\pi}{n}\right) = nr_n^2\tan\left(\frac{2\pi}{n}\right).$$

KACY Summer League

KACY–I 975. If the perimeter of a circle and the perimeter of a regular polygon with n sides are equal, then prove that the ratio of the area of the circle and the area of the polygon is equal to

$$\tan\left(\frac{\pi}{n}\right) : \left(\frac{\pi}{n}\right).$$

KACY Summer League

KACY–I 976. If the perimeter of two regular polygons, one with n sides and the other with $2n$ sides, is equal, prove that their areas are in the ratio

$$2\cos\left(\frac{\pi}{n}\right) : \left(1+\cos\left(\frac{\pi}{n}\right)\right).$$

KACY Summer League

KACY–I 977. Let $A_1, A_2, \ldots, A_n$ be the vertices of an n-sided regular polygon such that

$$\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}.$$

Prove that $n = 7$.

KACY Summer League

2003 IIT JEE 978. If I_n is the area of the n-sided regular polygon inscribed in a circle of unit radius and O_n is the area of the n-sided polygon circumscribing the given circle, prove that

$$I_n = \frac{O_n}{2}\sqrt{1+\sqrt{1-\left(\frac{2I_n}{n}\right)^2}}.$$

KACY Summer League

Makshud's Excentral Identities 979. In triangle ABC of semiperimeter p and area S with side-lengths a, b, c,

1. Prove the following identities regarding the exradii r_a, r_b, r_c and inradius r:

$$\frac{b-c}{r_a} + \frac{c-a}{r_b} + \frac{a-b}{r_c} = 0,$$
$$\frac{1}{r_a^2} + \frac{1}{r_b^2} + \frac{1}{r_c^2} + \frac{1}{r^2} = \frac{a^2+b^2+c^2}{S^2},$$
$$\frac{r_a+r_b}{1+\cos C} = \frac{r_b+r_c}{1+\cos A} = \frac{r_c+r_a}{1+\cos B}.$$

2. Show that

   - $S^2 = r \cdot r_a \cdot r_b \cdot r_c$,
   - $p^2 = r_a r_b + r_b r_c + r_c r_a$,
   - $4r^2 R = (r_a - r)(r_b - r)(r_c - r)$,
   - $4R = r_a + r_b + r_c - r$,
   - $4R \cos C = r_a + r_b - r_c + r$,
   - $16R^2 - (a^2+b^2+c^2) = r^2 + r_a^2 + r_b^2 + r_c^2$.

3. Prove the following more advanced identities

$$\frac{16R}{r^2(a+b+c)^2} = \left(\frac{1}{r} - \frac{1}{r_a}\right)\left(\frac{1}{r} - \frac{1}{r_b}\right)\left(\frac{1}{r} - \frac{1}{r_c}\right),$$
$$\frac{64R^3}{(abc)^2} = \left(\frac{1}{r_a} + \frac{1}{r_b}\right)\left(\frac{1}{r_b} + \frac{1}{r_c}\right)\left(\frac{1}{r_c} + \frac{1}{r_a}\right),$$
$$4R = \frac{(r_a+r_b)(r_b+r_c)(r_c+r_a)}{r_a r_b + r_b r_c + r_c r_a},$$
$$\frac{1}{r} - \frac{1}{2R} = \frac{r_a}{bc} + \frac{r_b}{ca} + \frac{r_c}{ab}.$$

4. If $I_A I_B I_C$ is the excentral triangle of ABC, and u, v, w are the lengths of tangents drawn from excenters I_A, I_B, I_C to the circumcircle of triangle ABC, respectively, prove that

$$\frac{1}{u^2} + \frac{1}{v^2} + \frac{1}{w^2} = \frac{abc}{a+b+c}.$$

5. If the exradii r_a, r_b, r_c are in a Harmonic Progression (**HP**), then the sides a, b, c are in an Arithmetic Progression (**AP**).

6. If the sides a, b, c form both a Geometric Progression (**GP**) as well as an Arithmetic Progression (**AP**), then r_a, r_b, r_c form a **GP**.

KACY Summer League

KACY–I 980. In any triangle ABC with side-lengths a, b, c and exradii r_a, r_b, r_c, let R be the circumradius. Prove that

$$\frac{bc}{r_a} + \frac{ca}{r_b} + \frac{ab}{r_c} = 2R\left[\left(\frac{a}{b} + \frac{b}{a}\right)\left(\frac{b}{c} + \frac{c}{b}\right)\left(\frac{c}{a} + \frac{a}{c}\right) - 3\right].$$

KACY Summer League

KACY–I 981. In any triangle ABC with exradii r_a, r_b, r_c and inradius r, prove that

$$(r + r_a)\tan\left(\frac{B-C}{2}\right) + (r + r_b)\tan\left(\frac{C-A}{2}\right) + (r + r_c)\tan\left(\frac{A-B}{2}\right) = 0.$$

KACY Summer League

KACY–I 982. Let T be the area of the incircle of triangle ABC and assume T_1, T_2, T_3 are the areas of the excircles of triangle ABC. Prove that

$$\frac{1}{\sqrt{T_1}} + \frac{1}{\sqrt{T_2}} + \frac{1}{\sqrt{T_3}} = \frac{1}{T}.$$

2.3 Math Olympiad Trigonometry 301: Series, Graphing & Logarithms

In this section, we investigate more advanced trigonometric topics. We start by some exercises containing trigonometric series and then move on to graphing trigonometric and logarithmic functions.

2.3.1 Trigonometric Series

KACY Summer League

KACY–I 983. For any positive integer n, prove that

$$\frac{1}{\sin\alpha\sin 3\alpha} + \frac{1}{\sin 3\alpha\sin 5\alpha} + \cdots + \frac{1}{\sin(2n-1)\alpha\sin(2n+1)\alpha} = \frac{\cot\alpha - \cot(2n+1)\alpha}{\sin 2\alpha}.$$

KACY Summer League

KACY–I 984. For any positive integer n, prove that

$$\frac{1}{\sin 2\alpha \sin 3\alpha} + \frac{1}{\sin 3\alpha \sin 4\alpha} + \cdots + \frac{1}{\sin(n+1)\alpha \sin(n+2)\alpha} = \frac{\cot 2\alpha - \cot(n+2)\alpha}{\sin\alpha}.$$

KACY Summer League

KACY–I 985. For any positive integer n, prove that

$$\frac{1}{\cos\alpha \cos 3\alpha} + \frac{1}{\cos 3\alpha \cos 5\alpha} + \cdots + \frac{1}{\cos(2n-1)\alpha \cos(2n+1)\alpha} = \frac{\tan(2n+1)\alpha - \tan\alpha}{\sin 2\alpha}.$$

KACY Summer League

KACY–I 986. For any positive integer n, prove that

$$\frac{1}{\cos\alpha \cos 2\alpha} + \frac{1}{\cos 2\alpha \cos 3\alpha} + \cdots + \frac{1}{\cos n\alpha \cos(n+1)\alpha} = \frac{\tan(n+1)\alpha - \tan\alpha}{\sin\alpha}.$$

KACY Summer League

KACY–I 987. For any positive integer n, prove that

$$\frac{1}{\cos\alpha + \cos 3\alpha} + \frac{1}{\cos\alpha + \cos 5\alpha} + \cdots + \frac{1}{\cos\alpha + \cos(2n+1)\alpha} = \frac{\tan(n+1)\alpha - \tan\alpha}{2\sin\alpha}.$$

KACY Summer League

KACY–I 988. For any positive integer n, prove that

$$\frac{1}{\sin\alpha}+\frac{1}{\sin 2\alpha}+\cdots+\frac{1}{\sin 2^{n-1}\alpha}=\cot\left(\frac{\alpha}{2}\right)-\cot\left(2^{n-1}\alpha\right),$$

then conclude

$$\csc(\alpha)+\csc\left(\frac{\alpha}{2}\right)+\cdots+\csc\left(\frac{\alpha}{2^{n-1}}\right)=\cot\left(\frac{\alpha}{2^n}\right)-\cot(\alpha).$$

KACY Summer League

KACY–I 989. For any positive integer n, prove that

$$\tan\alpha+\frac{1}{2}\tan\left(\frac{\alpha}{2}\right)+\cdots+\frac{1}{2^{n-1}}\tan\left(\frac{\alpha}{2^{n-1}}\right)=\frac{1}{2^{n-1}}\cot\left(\frac{\alpha}{2^{n-1}}\right)-2\cot 2\alpha.$$

KACY Summer League

KACY–I 990. Fix any positive integer n and angles α and β. Define $f_k(\alpha,\beta,n)$ for $k\ge 1$ by

$$f_k(\alpha,\beta,n)=\cos^k(\alpha)+\cos^k(\alpha+\beta)+\cdots+\cos^k(\alpha+(n-1)\beta).$$

Prove that

$$f_1(\alpha,\beta,n)=\frac{\cos\left(\alpha+\dfrac{(n-1)\beta}{2}\right)\sin\left(\dfrac{n\beta}{2}\right)}{\sin\left(\dfrac{\beta}{2}\right)},$$

$$f_2(\alpha,\beta,n)=\frac{1}{2}\left(n+\frac{\cos[2\alpha+(n-1)\beta]\sin(n\beta)}{\sin\beta}\right),$$

and

$$f_3(\alpha,\beta,n)=\frac{3\cos\left(\alpha+\dfrac{(n-1)\beta}{2}\right)\sin\left(\dfrac{n\beta}{2}\right)}{4\sin\left(\dfrac{\beta}{2}\right)}+\frac{\cos 3\left(\alpha+\dfrac{(n-1)\beta}{2}\right)\sin\left(\dfrac{3n\beta}{2}\right)}{4\sin\left(\dfrac{3\beta}{2}\right)}.$$

KACY Summer League

KACY–I 991. For all natural number n, prove that

$$\cos\left(\frac{\alpha}{3}\right)+\cos\left(\frac{4\alpha}{3}\right)+\cdots+\cos\left(\frac{(3n-2)\alpha}{3}\right)=\frac{\sin\left(\frac{(3n-1)\alpha}{6}\right)\sin\left(\frac{n\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}.$$

KACY Summer League

KACY–I 992. For all natural numbers n, prove that

$$\cos\alpha+\cos\left(\alpha+\frac{\pi}{n}\right)+\cdots+\cos\left(\alpha+\frac{(n-1)\pi}{n}\right)=\frac{\cos\left(\alpha+\frac{(n-1)\pi}{2n}\right)}{\sin\left(\frac{\pi}{2n}\right)}.$$

KACY Summer League

KACY–I 993. Prove that for every positive integer n,

$$\cos 3\alpha-\cos\left(3\alpha-\frac{\pi}{n}\right)+\cos\left(3\alpha-\frac{2\pi}{n}\right)-\cdots+(-1)^{n-1}\cos\left(3\alpha-\frac{(n-1)\pi}{n}\right)$$

is equal to

$$\frac{\cos\left(3\alpha+\frac{(n-1)^2\pi}{2n}\right)\sin\left(\frac{(n-1)\pi}{2}\right)}{\sin\left(\frac{(n-1)\pi}{2n}\right)}.$$

KACY Summer League

KACY–I 994. Prove that for every positive integer n,

$$\cos^4\alpha+\cos^4 3\alpha+\cos^4 5\alpha+\cdots+\cos^4(2n-1)\alpha$$

is equal to

$$\frac{1}{8}\left(3n+\frac{4\cos(2n\alpha)\sin(2n\alpha)}{\sin 2\alpha}+\frac{\cos(4n\alpha)\sin(4n\alpha)}{\sin 4\alpha}\right).$$

KACY Summer League

Sum of Sines of Angles in an AP 995. Fix any positive integer n and angles α and β. Define $g(\alpha, \beta, n)$ and $h(\alpha, \beta, n)$ by

$$g(\alpha, \beta, n) = \sin(\alpha) + \sin(\alpha + \beta) + \cdots + \sin(\alpha + (n-1)\beta),$$
$$h(\alpha, \beta, n) = \sin(\alpha) - \sin(\alpha + \beta) + \cdots + (-1)^{n-1} \sin(\alpha + (n-1)\beta).$$

Show that

$$g(\alpha, \beta, n) = \frac{\sin\left(\alpha + \dfrac{(n-1)\beta}{2}\right) \sin\left(\dfrac{n\beta}{2}\right)}{\sin\left(\dfrac{\beta}{2}\right)},$$

and

$$h(\alpha, \beta, n) = \frac{\sin\left(\alpha + \dfrac{(n-1)(\beta+\pi)}{2}\right) \sin\left(\dfrac{n(\beta+\pi)}{2}\right)}{\cos\left(\dfrac{\beta}{2}\right)}.$$

KACY Summer League

KACY–I 996. For any positive integer n and non-zero angle α, prove that

$$\sin^3 \alpha + \sin^3 3\alpha + \cdots + \sin^3(2n-1)\alpha = \frac{1}{4}\left(\frac{3\sin^2(n\alpha)}{\sin\alpha} - \frac{\sin^2(3n\alpha)}{\sin 3\alpha}\right).$$

KACY Summer League

KACY–I 997. Prove for all integer $n \geq 1$ that

$$\sin\alpha + \sin\left(\alpha + \frac{2\pi}{n}\right) + \sin\left(\alpha + \frac{4\pi}{n}\right) + \cdots + \sin\left(\alpha + \frac{2(n-1)\pi}{n}\right) = 0.$$

KACY Summer League

KACY–I 998. For any positive integer n, prove the following identities:

$$\sin 2\alpha + \sin 5\alpha + \sin 8\alpha + \cdots + \sin(3n-1)\alpha = \frac{\sin\left(\frac{(3n+1)\alpha}{2}\right)\sin\left(\frac{3n\alpha}{2}\right)}{\sin\left(\frac{3\alpha}{2}\right)},$$

and

$$\cos\alpha + \cos 3\alpha + \cos 5\alpha + \cdots + \cos(2n-1)\alpha = \frac{\cos(n\alpha)\sin(n\alpha)}{\sin\alpha}.$$

KACY Summer League

KACY–I 999. For a positive integer n and an angle α, define $S(\alpha,n)$ and $C(\alpha,n)$ by

$$S(\alpha,n) = \sin\alpha - \sin 2\alpha + \sin 3\alpha - \sin 4\alpha + \cdots + (-1)^{n-1}\sin(n\alpha),$$
$$C(\alpha,n) = \cos 2\alpha - \cos 4\alpha + \cos 6\alpha - \cos 8\alpha + \cdots + (-1)^{n-1}\cos(2n\alpha).$$

Prove that

$$S(\alpha,n) = \frac{\sin\left(\frac{(n+1)\alpha}{2} + \frac{(n-1)\pi}{2}\right)\sin\left(\frac{n(\alpha+\pi)}{2}\right)}{\sin\left(\frac{\alpha+\pi}{2}\right)},$$
$$C(\alpha,n) = \frac{\cos\left((n+1)\alpha + \frac{(n-1)\pi}{2}\right)\sin\left(\frac{n(2\alpha+\pi)}{2}\right)}{\sin\left(\frac{2\alpha+\pi}{2}\right)}.$$

KACY Summer League

KACY–I 1000. For all positive integers n, show that

$$\sin^4\alpha + \sin^4 2\alpha + \sin^4 3\alpha + \cdots + \sin^4 n\alpha$$

is equal to

$$\frac{1}{8}\left(3n - \frac{4\cos(n+1)\alpha\sin n\alpha}{\sin\alpha} + \frac{\cos(2n+2)\alpha\sin 2n\alpha}{\sin\alpha}\right).$$

KACY Summer League

KACY–I 1001. Prove that

$$\frac{\cos 2\alpha}{\sin 3\alpha}+\frac{\cos 6\alpha}{\sin 9\alpha}+\cdots+\frac{\cos\left(2\cdot 3^{n-1}\alpha\right)}{\sin\left(3^{n}\alpha\right)}=\frac{1}{2}\left(\frac{1}{\sin\alpha}-\frac{1}{\sin(3^{n}\alpha)}\right).$$

KACY Summer League

KACY–I 1002. Prove that

$$(2\cos\alpha-1)(2\cos 2\alpha-1)(2\cos 3\alpha-1)\cdots(2\cos(n-1)\alpha-1)=\frac{2\cos 2^{n}\alpha+1}{2\cos\alpha+1}.$$

KACY Summer League

KACY–I 1003. Prove that for all integers $n>1$,

$$\sin\left(\frac{\pi}{n}-\alpha\right)\sin\left(\frac{2\pi}{n}-\alpha\right)\cdots\sin\left(\frac{(n-1)\pi}{n}-\alpha\right)=\frac{\sin n\alpha}{2^{n-1}\sin\alpha}.$$

KACY Summer League

KACY–I 1004. Let $x_k=\sin(k\pi/n)$ for $k=1,2,\ldots,n-1$, where n is a given positive integer. Prove the following identities:

$$\begin{aligned}\cot\left(\frac{\pi}{2n}\right)&=x_1+x_2+\cdots+x_{n-1},\\ n&=(2x_1)\cdot(2x_2)\cdots(2x_{n-1}),\\ n^{n/2}&=(2x_1)\cdot(2x_2)^2\cdot(2x_3)^3\cdots(2x_{n-1})^{n-1}.\end{aligned}$$

2.3.2 Advanced Trigonometry: Limits, Infinite Sums, & Roots

Begin with some tough trigonometric identities to prove. These will be much more difficult to prove if you do not know the answer (the right side of identity's equation) prior to coming up with an idea on how to simplify the sum.

2.3.2.1 Advanced Trigonometric Identities: Sums

KACY Summer League

KACY–I 1005. Prove for all integers $n > 1$ that

$$\sin^4\alpha + \frac{\sin^4 2\alpha}{4} + \frac{\sin^4 4\alpha}{4^2} + \cdots + \frac{\sin^4 2^{n-1}\alpha}{4^{n-1}} = \sin^2\alpha - \frac{\sin^2 2^n\alpha}{4^n}.$$

KACY Summer League

KACY–I 1006. Prove for all integers $n \geq 1$ that

$$\sin^2\alpha + \sin^2 2\alpha + \sin^2 3\alpha + \cdots + \sin^2 n\alpha = \frac{n\sin\alpha - \sin n\alpha\cos(n+1)\alpha}{2\sin\alpha}.$$

KACY Summer League

KACY–I 1007. Prove for all integers $n \geq 1$ that

$$\cos\alpha + 2\cos 2\alpha + 3\cos 3\alpha + \cdots + n\cos n\alpha = \frac{(n+1)\cos n\alpha - n\cos(n+1)\alpha - 1}{4\sin^2\left(\frac{\alpha}{2}\right)}$$

$$= \frac{n\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{(2n+1)\alpha}{2}\right) - \sin^2\left(\frac{n\alpha}{2}\right)}{2\sin^2\left(\frac{\alpha}{2}\right)}.$$

KACY Summer League

KACY–I 1008. For any positive integer n, prove that

$$\frac{\sin\alpha}{2\cos\alpha - 1} + \frac{2\sin 2\alpha}{2\cos 2\alpha - 1} + \frac{4\sin 4\alpha}{2\cos 4\alpha - 1} + \cdots + \frac{2^{2n-1}\sin 2^{2n-1}\alpha}{2\cos 2^{2n-1}\alpha - 1}$$

is equal to

$$\frac{2^{2n}\sin 2^{2n}\alpha}{2\cos 2^{2n}\alpha - 1} - \frac{\sin\alpha}{2\cos\alpha + 1}.$$

KACY Summer League

KACY–I 1009. In order to be able to solve Problem 1024, prove the following identity for all positive integers n:

$$\tan^2\left(\frac{\alpha}{2}\right)\tan\alpha + 2\tan^2\left(\frac{\alpha}{4}\right)\tan\left(\frac{\alpha}{2}\right) + \cdots + 2^{n-1}\tan^2\left(\frac{\alpha}{2^n}\right)\tan\left(\frac{\alpha}{2^{n-1}}\right) = \tan\alpha - 2^n\tan\left(\frac{\alpha}{2^n}\right).$$

Compare with

$$\frac{\tan\left(\frac{\alpha}{2}\right)}{\cos\alpha} + \frac{\tan\left(\frac{\alpha}{4}\right)}{\cos\left(\frac{\alpha}{2}\right)} + \cdots + \frac{\tan\left(\frac{\alpha}{2^n}\right)}{\cos\left(\frac{\alpha}{2^{n-1}}\right)} = \tan\alpha - \tan\left(\frac{\alpha}{2^n}\right).$$

KACY Summer League

KACY–I 1010. In order to be able to solve Problem 1025, prove the identity

$$\frac{1}{4\cos^2\left(\frac{\alpha}{2}\right)} + \frac{1}{4^2\cos^2\left(\frac{\alpha}{2^2}\right)} + \cdots + \frac{1}{4^n\cos^2\left(\frac{\alpha}{2^n}\right)} = \frac{1}{\sin^2\alpha} - \frac{1}{2^n\sin^2\left(\frac{\alpha}{2^n}\right)},$$

for all positive integers n.

KACY Summer League

KACY–I 1011. Prove that

$$\sin^3\left(\frac{\alpha}{3}\right) + 3\sin^3\left(\frac{\alpha}{9}\right) + \cdots + 3^{n-1}\sin^3\left(\frac{\alpha}{3^n}\right) = \frac{1}{4}\left(3^n\sin\left(\frac{\alpha}{3^n}\right) - \sin\alpha\right).$$

KACY Summer League

KACY–I 1012. Prove that the expression

$$\frac{1}{\sin\alpha} + \frac{1}{\sin\alpha + \sin 2\alpha} + \frac{1}{\sin\alpha + \sin 2\alpha + \sin 3\alpha} + \cdots + \frac{1}{\sin\alpha + \sin 2\alpha + \cdots + \sin n\alpha}$$

for any positive integer n and non-zero angle α, is equal to

$$\cot\left(\frac{\alpha}{2}\right) - \cot\left(\frac{(n+1)\alpha}{2}\right).$$

KACY Summer League

KACY–I 1013. For all integers $n > 1$, prove that

$$\tan\alpha\tan 2\alpha + \tan 2\alpha\tan 3\alpha + \cdots + \tan(n-1)\alpha\tan n\alpha = \frac{\tan n\alpha}{\tan\alpha} - n.$$

KACY Summer League

KACY–I 1014. Find the sum of sines of all integer angles in degrees from $1°$ to $90°$. In trigonometric words, prove that

$$\sin 1° + \sin 2° + \cdots + \sin 90° = \frac{\sqrt{2}\sin 45.5°}{\cos 0.5°}.$$

KACY Summer League

KACY–I 1015. For all positive integers n, show that

$$\sin^2\alpha\sin 2\alpha + \frac{\sin^2 2\alpha\sin 4\alpha}{2} + \cdots + \frac{\sin^2 2^{n-1}\alpha\sin 2^n\alpha}{2^{n-1}} = \frac{\sin 2\alpha}{2} - \frac{\sin^2 2^{n+1}\alpha}{2^{n+1}}.$$

KACY Summer League

KACY–I 1016. Show that for any $n \geq 1$,

$$\frac{\sin\alpha}{\cos 3\alpha} + \frac{\sin 3\alpha}{\cos 9\alpha} + \cdots + \frac{\sin 3^{n-1}\alpha}{\cos 3^n\alpha} = \frac{\tan 3^n\alpha - \tan\alpha}{2}.$$

KACY Summer League

KACY–I 1017. Prove for all positive integers n that

$$\sin\alpha - \cos 2\alpha + \sin 3\alpha - \cos 4\alpha + \cdots + \sin(2n-1)\alpha - \cos 2n\alpha$$
$$= \frac{(\sin n\alpha - \cos(n+1)\alpha)\sin n\alpha}{\sin\alpha}.$$

2.3.2.2 Advanced Trigonometric Identities: Products

KACY Summer League

KACY–I 1018. Show that for all positive integers n,

$$\left[1+\frac{1}{\cos\alpha}\right]\cdot\left[1+\frac{1}{\cos 2\alpha}\right]\cdots\left[1+\frac{1}{\cos 2^{n-1}\alpha}\right]=\frac{\tan 2^{n-1}\alpha}{\tan\left(\dfrac{\alpha}{2}\right)}.$$

KACY Summer League

KACY–I 1019. Show that for all positive integers n,

$$\cos\frac{\pi}{2n+1}\cdot\cos\frac{2\pi}{2n+1}\cdots\cos\frac{n\pi}{2n+1}=\frac{1}{2^n}$$

KACY Summer League

KACY–I 1020. Find the product of half-angled cosines up to n terms:

$$\cos\alpha\cdot\cos\left(\frac{\alpha}{2}\right)\cdot\cos\left(\frac{\alpha}{4}\right)\cdots\cos\left(\frac{\alpha}{2^{n-1}}\right)=\frac{\sin 2\alpha}{2^n\sin\left(\frac{\alpha}{2^{n-1}}\right)}.$$

KACY Summer League

KACY–I 1021. Prove that

$$[2\cos(\alpha)-1]\cdot\left[2\cos\left(\frac{\alpha}{2}\right)-1\right]\cdots\left[2\cos\left(\frac{\alpha}{2^n}\right)-1\right]=\frac{2\cos 2\alpha+1}{2\cos\left(\dfrac{\alpha}{2^n}\right)+1}.$$

KACY Summer League

KACY–I 1022. For any two angles α and β, and $n \geq 1$, define the angles α_n and β_n recursively as half of the angles α_{n-1} and β_{n-1}, initiated with $\alpha_0 = \alpha$ and $\beta_0 = \beta$. Find the product of the following n sums: the sum of cosines of α_0 and β_0, the sum of cosines of α_1 and β_1, and so on up to the sum of cosines of α_n and β_n. In other words, show that

$$[\cos\alpha + \cos\beta] \cdot \left[\cos\left(\frac{\alpha}{2}\right) + \cos\left(\frac{\beta}{2}\right)\right] \cdots \left[\cos\left(\frac{\alpha}{2^{n-1}}\right) + \cos\left(\frac{\beta}{2^{n-1}}\right)\right] = \frac{\cos 2\alpha - \cos 2\beta}{2^n\left[\cos\left(\frac{\alpha}{2^{n-1}}\right) - \cos\left(\frac{\beta}{2^{n-1}}\right)\right]}.$$

KACY Summer League

KACY–I 1023. Prove, for integers $n \geq 1$, that

$$\left[1 - \tan^2\left(\frac{\alpha}{2}\right)\right] \cdot \left[1 - \tan^2\left(\frac{\alpha}{4}\right)\right] \cdots \left[1 - \tan^2\left(\frac{\alpha}{2^n}\right)\right] = 2^n \tan\left(\frac{\alpha}{2^n}\right) \cot\alpha.$$

2.3.2.3 Trigonometric Limits and Infinite Sums

KACY Summer League

KACY–I 1024. Prove that the limit of the sum

$$\tan^2\left(\frac{\alpha}{2}\right)\tan\alpha + 2\tan^2\left(\frac{\alpha}{4}\right)\tan\left(\frac{\alpha}{2}\right) + \cdots + 2^{n-1}\tan^2\left(\frac{\alpha}{2^n}\right)\tan\left(\frac{\alpha}{2^{n-1}}\right) + \cdots$$

when n approaches infinity, equals $-\alpha + \tan\alpha$. You can use Problem 1009 as a hint.

KACY Summer League

KACY–I 1025. Using Problem 1010 as a hint, prove that the limit of the sum

$$\frac{1}{4\cos^2\left(\frac{\alpha}{2}\right)} + \frac{1}{4^2\cos^2\left(\frac{\alpha}{2^2}\right)} + \cdots + \frac{1}{4^n\cos^2\left(\frac{\alpha}{2^n}\right)} + \cdots$$

when n approaches infinity, equals

$$\frac{1}{\sin^2\alpha} - \frac{1}{\alpha^2}.$$

KACY Summer League

KACY–I 1026. First, find the function $f(x)$ such that $f(x) \equiv f(x-1) + x^3$, and second, find the limit of the following sum when x approaches zero:

$$S_1 = \cos^3 \alpha + \left(2 \cos\left(\frac{\alpha}{2}\right)\right)^3 + \left(3 \cos\left(\frac{\alpha}{3}\right)\right)^3 + \cdots + \left(n \cos\left(\frac{\alpha}{n}\right)\right)^3.$$

Third, if we define

$$S_2 = \left[\cos \alpha + 2 \cos\left(\frac{\alpha}{2}\right) + 3 \cos\left(\frac{\alpha}{3}\right) + \cdots + n \cos\left(\frac{\alpha}{n}\right)\right]^2,$$

Prove that the limit of S_2 when α approaches zero is the same as limit of S_1 when α approaches zero.

2.3.2.4 Trigonometric Roots of Equations

KACY Summer League

KACY–I 1027. Let α be a real number. Prove that the roots of the equation

$$x^2 \tan^2 \alpha - 2(2 + \tan^2 \alpha)x + \tan^2 \alpha = 0$$

are $\tan^2(\alpha/2)$ and $\cot^2(\alpha/2)$.

KACY Summer League

KACY–I 1028. Prove that one of the roots of the equation

$$x^3 - 6x^2 + 9x - 3 = 0$$

is equal to $2(1 - \sin(\pi/18))$, and find the other roots.

KACY Summer League

KACY–I 1029. Prove that one of the roots of the equation

$$x^4 - 10x^2 + 5 = 0$$

is equal to $\tan 36°$, and find the other roots.

KACY Summer League

KACY–I 1030. Let $x = 2\cos\alpha$. If we know that

$$\frac{1 + \cos 9\alpha}{1 + \cos \alpha} = (x^4 - x^3 - 3x^2 + 2x + 1)^2,$$

find the roots of the equation $x^4 - x^3 - 3x^2 + 2x + 1 = 0$.

2.3.2.5 Various Trigonometric Inequalities

KACY Summer League

KACY–I 1031. If $0 < \alpha \leq 60°$, prove that

$$\frac{1}{\sin(60° + \alpha)} + \frac{1}{\sin(60° - \alpha)} \geq \frac{4\sqrt{3}}{3}.$$

KACY Summer League

KACY–I 1032. If $0 < \alpha < 90°$, prove that $(\sin \alpha)/\alpha > \sqrt{\cos \alpha}$.

KACY Summer League

KACY–I 1033. If $0 < \alpha < 90°$, prove that

$$\left(1 + \frac{1}{\sin \alpha}\right)\left(1 + \frac{1}{\cos \alpha}\right) \geq 3 + 2\sqrt{2}.$$

KACY Summer League

KACY–I 1034. If α, β, γ, are angles with $\alpha + \beta + \gamma = 90°$, prove that

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma + 3 \sin \alpha \sin \beta \sin \gamma \leq \frac{9}{8}.$$

KACY Summer League

KACY–I 1035. For all positive integers n, show that

$$\left(\frac{1}{\cos^{2n} \alpha} - 1\right)\left(\frac{1}{\sin^{2n} \alpha} - 1\right) \geq (1 + 2 + \cdots + n)^2 .$$

KACY Summer League

KACY–I 1036. If the sum of angles A and C in a convex quadrilateral $ABCD$ exceeds $180°$, prove that

$$\frac{AC}{BD} < \frac{AB \cdot DA + BC \cdot CD}{AB \cdot BC + CD \cdot DA}.$$

2.3.2.6 Triangle Trigonometric Inequalities

KACY Summer League

KACY–I 1037. If $A+B+C=180°$, where A, B, C are acute angles, prove that

$$\frac{1}{\cos A}+\frac{1}{\cos B}+\frac{1}{\cos C}\geq 6.$$

KACY Summer League

KACY–I 1038. In a triangle ABC with acute angles, prove that

$$\tan^2 A+\tan^2 B+\tan^2 C\geq 9.$$

KACY Summer League

KACY–I 1039. Let ABC be an acute-angled triangle. Prove that for all non-negative integers n,

$$\tan^n A+\tan^n B+\tan^n C\geq 3+\frac{3n}{2}.$$

KACY Summer League

KACY–I 1040. In any triangle ABC, prove that

$$\cot^2\left(\frac{A}{2}\right)+\cot^2\left(\frac{B}{2}\right)+\cot^2\left(\frac{C}{2}\right)\geq 9.$$

KACY Summer League

KACY–I 1041. In any triangle ABC, prove that

$$\frac{1}{\sin^2\left(\frac{A}{2}\right)}+\frac{1}{\sin^2\left(\frac{B}{2}\right)}+\frac{1}{\sin^2\left(\frac{C}{2}\right)}\geq 12.$$

KACY Summer League

KACY–I 1042. In a triangle ABC with acute angles, prove that

$$\sin A+\sin B+\sin C+\tan A+\tan B+\tan C>2\pi.$$

KACY Summer League

KACY–I 1043. Prove that for any triangle ABC with side-lengths a, b, c, area S, and circumradius R, we have

$$a \sin A + b \sin B + c \sin C \geq \frac{2S\sqrt{3}}{R}.$$

KACY Summer League

KACY–I 1044. In a right triangle ABC, the angle between the medians drawn from non-right angles is α, prove that $\cos \alpha \geq 4/5$.

KACY Summer League

KACY–I 1045. Prove that if the angles of triangle ABC satisfy the equation $\sin^2 A + \sin^2 B = 5 \sin^2 C$, then $\sin C \leq 3/5$.

KACY Summer League

KACY–I 1046. There is a circle inscribed in a right triangle with hypotenuse of length c that touches the other two sides of the triangle at points M and N. Prove that

$$MN \leq \frac{2c\sqrt{3}}{9}.$$

KACY Summer League

KACY–I 1047. The circumcircle of triangle ABC has radius R, and the three circles touching the circumcircle and two sides of ABC have radii r_1, r_2, r_3. Prove that $4r \leq r_1 + r_2 + r_3 \leq 2R$.

KACY Summer League

KACY–I 1048. Let I be the incenter of triangle ABC and r its inradius. Three line segments are formed by drawing three parallel lines to the three sides of triangle ABC from I. Prove that the sum of squares of lengths of these segments is not smaller than $16r^2$.

KACY Summer League

KACY–I 1049. In triangle ABC with side-lengths a, b, c, semiperimeter p and circumradius R, show that

$$\frac{1}{(p-a)^2} + \frac{1}{(p-b)^2} + \frac{1}{(p-c)^2} \geq \frac{4}{R^2}.$$

KACY Summer League

KACY–I 1050. In triangle ABC with side-lengths a, b, c, semiperimeter p, inradius r, and circumradius R, show that $5R - r \geq p\sqrt{3}$.

KACY Summer League

KACY–I 1051. Let O be the circumcenter of triangle ABC with circumradius R, and let R_1, R_2, R_3 be the radii of circumcircles of BOC, COA, AOB, respectively. Prove that $R_1^2 + R_2^2 + R_3^2 \geq 3R^2$.

KACY Summer League

KACY–I 1052. If a, b, c are side-lengths of a triangle and x, y, z are the sides of its orthic triangle, prove that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \geq \frac{3}{4}.$$

KACY Summer League

KACY–I 1053. If A, B, C are angles of a triangle, prove that the two triples of segments of length

$$\left\{\cos\left(\frac{A}{2}\right), \cos\left(\frac{B}{2}\right), \cos\left(\frac{C}{2}\right)\right\}$$

and

$$\left\{\cos^2\left(\frac{A}{2}\right), \cos^2\left(\frac{B}{2}\right), \cos^2\left(\frac{C}{2}\right)\right\},$$

each can be side-lengths of a triangle.

KACY Summer League

KACY–I 1054. In triangle ABC, the angle A, the perimeter $2p$, and the inradius r are known. Prove that we can solve the triangle and find its other two angles B and C using the identity

$$\cos\left(\frac{B-C}{2}\right) = \frac{\left[p\tan\left(\frac{A}{2}\right)+r\right]\sin\left(\frac{A}{2}\right)}{p\tan\left(\frac{A}{2}\right)-r},$$

and conclude that the following inequality must hold in the triangle:

$$p\tan\left(\frac{A}{2}\right)\cdot\tan^2\left(\frac{\pi}{4}-\frac{A}{4}\right) \geq r.$$

2.3.3 Logarithms, Graphing and Nonroutine Problems

The seven problems in this section are taken from "Nonroutine problems in algebra, geometry, and trigonometry (McGraw-Hill, 1965)."

KACY Summer League

Nonroutine I 1055. Graph the following functions:

a) $f(x) = x \sin x$,

b) $g(x) = x \sin(1/x)$,

c) $h(x) = x \log x$,

d) $j(x) = (\log x)/x$,

e) $k(x) = 2^{1/x}$.

KACY Summer League

Nonroutine II 1056. How many solutions are there to $\log_{10} x = \sin x$?

KACY Summer League

Nonroutine III 1057. Prove that $|\sin kx| \leq k|\sin x|$ for all reals x and all positive integers k.

KACY Summer League

Nonroutine IV 1058. Prove that for $\alpha \neq n\pi$, where n is an integer,

$$\cos\alpha \cdot \cos 2\alpha \cdot \cos 4\alpha \cdot \cos 8\alpha \cdots \cos 2^n\alpha = \frac{\sin^{2n+1}\alpha}{2^{n+1}\sin\alpha}.$$

KACY Summer League

Nonroutine V 1059. Given

$$\sin\alpha = \frac{a^2 - b^2}{a^2 + b^2} \qquad \text{and} \qquad \cos\alpha = \frac{2ab}{a^2 + b^2},$$

find $\tan\alpha/2$.

KACY Summer League

Nonroutine VI 1060. Given $0 < \alpha < \beta < \pi/2$, show that

$$-\alpha + \tan\alpha < -\beta + \tan\beta.$$

KACY Summer League

Nonroutine VII 1061. Find all real numbers x such that

$$\log(\sqrt{3}\sin x) + \log(-1 + \sqrt{3}\sin x) = \log 6.$$

2.4 Math Olympiad Trigonometry 401 and Beyond: 50 Unsolved Math Olympiad Trigonometry Problems

KACY Summer League

KACY–I 1062. Prove that:

$$\cos\frac{2\pi}{13}+\cos\frac{6\pi}{13}+\cos\frac{8\pi}{13}=\frac{\sqrt{13}-1}{4}.$$

KACY Summer League

KACY–I 1063. Prove that

$$x=2\left(\cos\frac{4\pi}{19}+\cos\frac{6\pi}{19}+\cos\frac{10\pi}{19}\right)$$

is a root of the equation:

$$\sqrt{4+\sqrt{4+\sqrt{4-x}}}=x.$$

KACY Summer League

KACY–I 1064. Prove that

$$\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos 8\theta}}}=\cos\theta.$$

KACY Summer League

KACY–I 1065. Prove that

$$\sin^4\left(\frac{\pi}{8}\right)+\sin^4\left(\frac{3\pi}{8}\right)+\sin^4\left(\frac{5\pi}{8}\right)+\sin^4\left(\frac{7\pi}{8}\right)=\frac{3}{2}.$$

KACY Summer League

KACY–I 1066. Prove that

$$\cos x \cdot \cos\left(\frac{x}{2}\right) \cdot \cos\left(\frac{x}{4}\right) \cdot \cos\left(\frac{x}{8}\right) = \frac{\sin 2x}{16 \sin\left(\frac{x}{8}\right)}.$$

KACY Summer League

KACY–I 1067. Prove that

$$64 \cdot \sin 10^\circ \cdot \sin 20^\circ \cdot \sin 30^\circ \cdot \sin 40^\circ \cdot \sin 50^\circ \cdot \sin 60^\circ \cdot \sin 70^\circ \cdot \sin 80^\circ \cdot \sin 90^\circ = \frac{3}{4}.$$

KACY Summer League

KACY–I 1068. Find x if

$$\sin x = \tan 12^\circ \cdot \tan 48^\circ \cdot \tan 54^\circ \cdot \tan 72^\circ \cdot$$

KACY Summer League

KACY–I 1069. Solve the following equations in $\mathbb{R}$:

- $\sin 9x + \sin 5x + 2\sin^2 x = 1.$
- $\cos 5x \cdot \cos 3x - \sin 3x \cdot \sin x = \cos 2x.$
- $\cos 5x + \cos 3x + \sin 5x + \sin 3x = 2 \cdot \cos\left(\frac{\pi}{4} - 4x\right).$
- $\sin x + \cos x - \sin x \cdot \cos x = -1.$
- $\sin 2x - \sqrt{3}\cos 2x = 2.$

KACY Summer League

KACY–I 1070. Prove the following equations:

- $\sin\left(\frac{2\pi}{7}\right)+\sin\left(\frac{4\pi}{7}\right)-\sin\left(\frac{6\pi}{7}\right)=4\sin\left(\frac{\pi}{7}\right)\cdot\sin\left(\frac{3\pi}{7}\right)\cdot\sin\left(\frac{5\pi}{7}\right)$,
- $\cos\left(\frac{\pi}{13}\right)+\cos\left(\frac{3\pi}{13}\right)+\cos\left(\frac{5\pi}{13}\right)+\cos\left(\frac{7\pi}{13}\right)+\cos\left(\frac{9\pi}{13}\right)+\cos\left(\frac{11\pi}{13}\right)=\frac{1}{2}$,
- $\cos\left(\frac{\pi}{2k+1}\right)+\cos\left(\frac{3\pi}{2k+1}\right)+\cdots+\cos\left(\frac{(2k-1)\pi}{2k+1}\right)=\frac{1}{2}$, for all natural k,
- $\sin\left(\dfrac{\pi}{7}\right)+\sin\left(\dfrac{2\pi}{7}\right)+\sin\left(\dfrac{3\pi}{7}\right)=\dfrac{1}{4}\cdot\cot\left(\dfrac{\pi}{4}\right)$.

KACY Summer League

KACY–I 1071. Show that

$$\cos\frac{\pi}{n}+\cos\frac{2\pi}{n}+\cdots+\cos\frac{n\pi}{n}=-1.$$

KACY Summer League

KACY–I 1072. Show that

$$\cos a+\cos 3a+\cos 5a+\cdots+\cos(2n-1)a=\frac{\sin 2na}{2\sin a}.$$

KACY Summer League

KACY–I 1073. Show that

$$\sin a+\sin 3a+\sin 5a+\cdots+\sin(2n-1)a=\frac{\sin^2 na}{\sin a}.$$

KACY Summer League

KACY–I 1074. Calculate

$$(\tan 1^\circ)^2+(\tan 2^\circ)^2+(\tan 3^\circ)^2+\cdots+(\tan 89^\circ)^2.$$

KACY Summer League

KACY–I 1075. Prove that

$$\cot^2\frac{\pi}{7}+\cot^2\frac{2\pi}{7}+\cot^2\frac{3\pi}{7}=5.$$

KACY Summer League

KACY–I 1076. Show that

$$\tan\frac{\pi}{7}\tan\frac{2\pi}{7}\tan\frac{3\pi}{7}=\sqrt{7}.$$

KACY Summer League

KACY–I 1077. Imagine $\cos\left(\frac{2\pi}{7}\right)$, $\cos\left(\frac{4\pi}{7}\right)$ and $\cos\left(\frac{6\pi}{7}\right)$ are the roots of an equation of the form $ax^3+bx^2+cx+d=0$ where a,b,c,d are integers. Determine a,b,c and d.

KACY Summer League

KACY–I 1078. Find the value of the sum

$$\sqrt[3]{\cos\frac{2\pi}{7}}+\sqrt[3]{\cos\frac{4\pi}{7}}+\sqrt[3]{\cos\frac{6\pi}{7}}.$$

KACY Summer League

KACY–I 1079. Solve the equation

$$2\sin^4 x(\sin 2x-3)-2\sin^2 x(\sin 2x-3)-1=0.$$

KACY Summer League

KACY–I 1080. Express the sum of the following series in terms of $\sin x$ and $\cos x$.

$$\sum_{k=0}^{n}(2k+1)\sin^2\left(x+\frac{k}{2}\pi\right).$$

KACY Summer League

KACY–I 1081. Find the smallest positive integer N for which

$$\frac{1}{\sin 45^\circ \cdot \sin 46^\circ} + \frac{1}{\sin 47^\circ \cdot \sin 48^\circ} + \cdots + \frac{1}{\sin 133^\circ \cdot \sin 134^\circ} = \frac{1}{\sin N^\circ}.$$

KACY Summer League

KACY–I 1082. Find the value of

$$\frac{\sin 40^\circ + \sin 80^\circ}{\sin 110^\circ}.$$

KACY Summer League

KACY–I 1083. Evaluate the sum

$$S = \tan 1^\circ \cdot \tan 2^\circ + \tan 2^\circ \cdot \tan 3^\circ + \tan 3^\circ \cdot \tan 4^\circ + \cdots + \tan 2004^\circ \cdot \tan 2005^\circ.$$

KACY Summer League

KACY–I 1084. Solve the equation:

$$\sqrt{3}\sin x(\cos x - \sin x) + (2 - \sqrt{6})\cos x + 2\sin x + \sqrt{3} - 2\sqrt{2} = 0.$$

KACY Summer League

KACY–I 1085. Let

$$f(x) = \frac{1}{\sin \frac{\pi x}{7}}.$$

Prove that $f(3) + f(2) = f(1)$.

KACY Summer League

KACY–I 1086. Suppose that real numbers x, y, z satisfy

$$\frac{\cos x + \cos y + \cos z}{\cos(x+y+z)} = \frac{\sin x + \sin y + \sin z}{\sin(x+y+z)} = p.$$

Prove that

$$\cos(x+y) + \cos(y+z) + \cos(x+z) = p.$$

KACY Summer League

KACY–I 1087. Solve for θ, where $0 \leq \theta \leq \pi/2$:

$$\sin^5\theta + \cos^5\theta = 1.$$

KACY Summer League

KACY–I 1088. For $x, y \in [0, \pi/3]$, prove that $\cos x + \cos y \leq 1 + \cos xy$.

KACY Summer League

KACY–I 1089. Prove that among any four distinct numbers from the interval $(0, \pi/2)$ there are two, say x, y, such that:

$$8\cos x \cos y \cos(x-y) + 1 > 4(\cos^2 x + \cos^2 y).$$

KACY Summer League

KACY–I 1090. Let $B = \frac{\pi}{7}$. Prove that

$$\tan B \cdot \tan 2B + \tan 2B \cdot \tan 4B + \tan 4B \cdot \tan B = -7.$$

KACY Summer League

KACY–I 1091. This is a question on multiples of $\pi/13$.

a) Calculate

$$\frac{1}{\cos\frac{6\pi}{13}} - 4\cos\frac{4\pi}{13} - 4\cos\frac{5\pi}{13} = ?$$

b) Prove that

$$\tan\frac{\pi}{13} + 4\sin\frac{4\pi}{13} = \tan\frac{3\pi}{13} + 4\sin\frac{3\pi}{13}.$$

c) Prove that

$$\tan\frac{2\pi}{13} + 4\sin\frac{6\pi}{13} = \tan\frac{5\pi}{13} + 4\sin\frac{2\pi}{13}.$$

KACY Summer League

KACY–I 1092. Prove that if α and β are angles of a triangle and

$$\left(\cos^2\alpha + \cos^2\beta\right)(1 + \tan\alpha \cdot \tan\beta) = 2,$$

then $\alpha + \beta = 90^\circ$.

KACY Summer League

KACY–I 1093. Let $a, b, c, d \in [-\pi/2, \pi/2]$ be real numbers such that

$$\sin a + \sin b + \sin c + \sin d = 1,$$

and

$$\cos 2a + \cos 2b + \cos 2c + \cos 2d \geq \frac{10}{3}.$$

Prove that $a, b, c, d \in [0, \pi/6]$.

KACY Summer League

KACY–I 1094. Find all integers m and n for which we have

$$\sin^m x + \cos^n x = 1,$$

for all real numbers x.

KACY Summer League

KACY–I 1095. Prove that $\tan 55^\circ \cdot \tan 65^\circ \cdot \tan 75^\circ = \tan 85^\circ$.

KACY Summer League

KACY–I 1096. Prove that

$$\frac{4\cos 12^\circ + 4\cos 36^\circ + 1}{\sqrt{3}} = \tan 78^\circ.$$

KACY Summer League

KACY–I 1097. Prove that

$$\sqrt{4+\sqrt{4+\sqrt{4-\sqrt{4+\sqrt{4+\sqrt{4-\cdots}}}}}}=2\left(\cos\frac{4\pi}{19}+\cos\frac{6\pi}{19}+\cos\frac{10\pi}{19}\right).$$

The signs: $++-++-++-++-\cdots$

KACY Summer League

KACY–I 1098. For reals x, y Prove that $\cos x+\cos y+\sin x\sin y\leq 2$.

KACY Summer League

KACY–I 1099. Solve the equation in real numbers

$$\sqrt{7+2\sqrt{7-2\sqrt{7-2x}}}=x.$$

KACY Summer League

KACY–I 1100. Let A, B, C be three angles of triangle ABC. Prove that

$$(1-\cos A)(1-\cos B)(1-\cos C)\geq\cos A\cos B\cos C.$$

KACY Summer League

KACY–I 1101. Solve the equation

$$\sin^3(x)-\cos^3(x)=\sin^2(x).$$

KACY Summer League

KACY–I 1102. Let $n\geq 1$ be an integer. Find the sum

$$S_n=\sum_{k=1}^{n}\sin^2 k\theta.$$

KACY Summer League

KACY–I 1103. Prove the following without using induction:

$$\cos x + \cos 2x + \cdots + \cos nx = \frac{\cos \frac{n+1}{2}x \cdot \sin \frac{n}{2}x}{\sin \frac{x}{2}}.$$

KACY Summer League

KACY–I 1104. Evaluate:

$$\sin\theta + \frac{1}{2}\cdot\sin 2\theta + \frac{1}{2^2}\cdot\sin 3\theta + \frac{1}{2^3}\cdot\sin 4\theta + \cdots$$

KACY Summer League

KACY–I 1105. Compute

$$\sum_{k=1}^{n-1} \csc^2\left(\frac{k\pi}{n}\right).$$

KACY Summer League

KACY–I 1106. Prove that

$$\tan\theta + \tan\left(\theta + \frac{\pi}{n}\right) + \tan\left(\theta + \frac{2\pi}{n}\right) + \cdots + \tan\left(\theta + \frac{(n-1)\pi}{n}\right) = -n\cot\left(n\theta + \frac{n\pi}{2}\right),$$

and

$$\cot\theta + \cot\left(\theta + \frac{\pi}{n}\right) + \cot\left(\theta + \frac{2\pi}{n}\right) + \cdots + \cot\left(\theta + \frac{(n-1)\pi}{n}\right) = n\cot n\theta.$$

KACY Summer League

KACY–I 1107. Calculate

$$\sum_{n=1}^{\infty} 2^{2n}\sin^4\frac{a}{2^n}.$$

KACY Summer League

KACY–I 1108. Compute the following sum:

$$\tan 1^\circ + \tan 5^\circ + \tan 9^\circ + \cdots + \tan 177^\circ.$$

KACY Summer League

KACY–I 1109. Show that for any positive integer $n > 1$,

- $\displaystyle\sum_{k=0}^{n-1} \cos\frac{2\pi k^2}{n} = \frac{\sqrt{n}}{2}\left(1 + \cos\frac{n\pi}{2} + \sin\frac{n\pi}{2}\right),$
- $\displaystyle\sum_{k=0}^{n-1} \sin\frac{2\pi k^2}{n} = \frac{\sqrt{n}}{2}\left(1 + \cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right).$

KACY Summer League

KACY–I 1110. Evaluate the product

$$\prod_{k=1}^{n} \tan\frac{k\pi}{2(n+1)}.$$

KACY Summer League

KACY–I 1111. Prove, for even n, that

$$\sum_{k=1}^{n} (-1)^{k-1} \cot\frac{(2k-1)\pi}{4n} = n.$$

KACY Summer League

KACY–I 1112. Prove that

$$\sum_{k=1}^{n} \cot^2\left\{\frac{(2k-1)\pi}{2n}\right\} = n(2n-1).$$

KACY Summer League

KACY–I 1113. Prove that

$$\sum_{k=1}^{n} \cot^4\left(\frac{k\pi}{2n+1}\right) = \frac{n(2n-1)(4n^2+10n-9)}{45}.$$

KACY Summer League

KACY–I 1114. Let x be a real number with $0 < x < \pi$. Prove that, for all natural numbers n, the sum

$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots + \frac{\sin(2n-1)x}{2n-1}$$

is positive.

2.5 Math Olympiad Trigonometry 501: Beyond Plane Euclidean Trigonometry (Egregia Introductio ad Monstruosam Trigonometriam)

We now summon geometric monsters from non-Euclidean spaces in order to do algebraic operations on them. In general, a geometry in which the assumption of flatness of space is disregarded would be a non-Euclidean geometry. This means that spaces studied here are curved, either positively or negatively.

We start by studying the simplest case of a positively-curved geometry called elliptic geometry, and we initially assume that the positively-curved space has a uniform curvature. The most accessible such geometry would be the spherical geometry whose trigonometric calculations are on the way.

2.5.1 Great Circular Arcs on a Sphere

The shortest path connecting two points on a sphere is always part of a **great circular arc** whose center is the same as sphere's center. On a given sphere, imagine the lines of latitude as you would see them on a globe. Each line of latitude, except for the equator, is a **small circle** of the sphere. The equator, which splits the sphere into two equal-sized pieces, is a **great circle** of the sphere (see Figure 2.1).

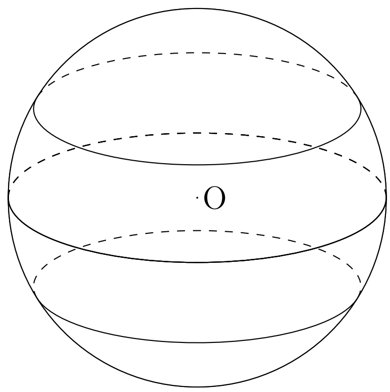

Figure 2.1: Two small circles and a great circle on a sphere.

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KACY–I 1115. Any two points on the surface of the sphere divide the great circle joining them into two parts. These two parts will be equal to each other if the two points are **antipodal** (diametrically opposite to one another). Otherwise, one of the two parts will be smaller than the other. Show that:

(a) If A and B are antipodal points, meaning they are the two ends of a diameter of the sphere, there are infinitely many great cirles passing through them.

(b) If A and B are not antipodal, there is exactly one **great circle** passing through them. The smaller arc connecting A to B is associated with a **central angle** connecting the center of the sphere to A & B.

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Length of Shortest Arc Between Two Points on Sphere 1116. On a sphere of radius R, there are two points A and B (Figure 2.2). Prove that the **central angle** associated with the shortest arc connecting A and B is α, then the length of the arc is αR.

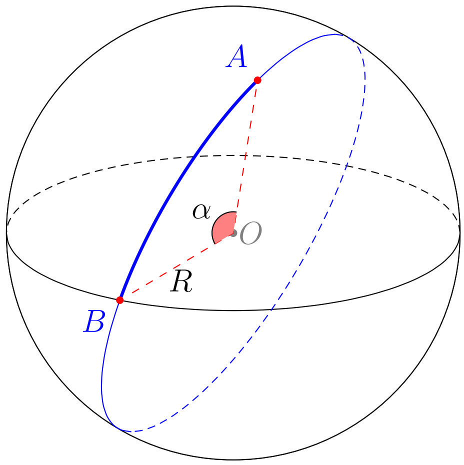

Figure 2.2: The great circle joining A to B

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Main Elements of Euler's Spherical Triangle 1117. Let A, B, C be points on a unit sphere such that no two of them are antipodals. We draw the three great circles joining the three vertices A, B, C to envision **Euler's Spherical Triangle**. Each angle A, B, C and the corresponding side a, b, c facing it is a **main element** of spherical triangle ABC (Figure 2.3). Prove that the value of each of the main elements of Euler's spherical triangle lies between 0 and π.

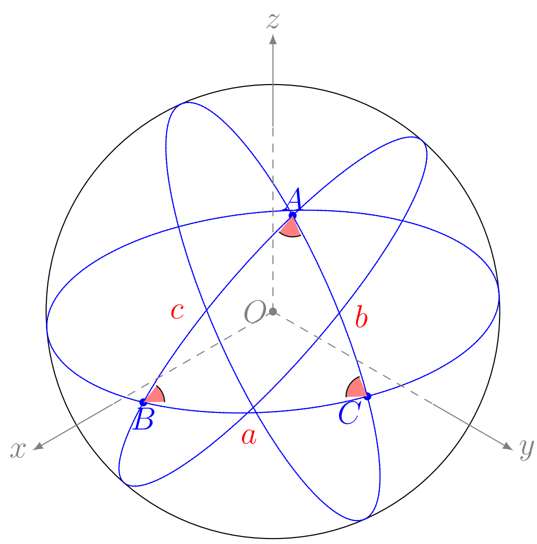

Figure 2.3: Angles A, B, C and sides a, b, c of a spherical triangle.

Each spherical triangle ABC corresponds to a **trihedral corner** whose vertex is at the center of the sphere and whose edges are the radii of the sphere connecting the center to the points A, B, C on the surface of sphere. Moreover, each trihedral corner with its vertex at the center of the sphere corresponds to a spherical triangle formed by the edges of the trihedral corner.

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Trihedral Corner 1118. Figure 2.4 demonstrates a **trirectangle** (a spherical triangle with three right angles) and its associated trihedral corner. Prove that:

a) The angles A, B, C of the spherical triangle are equal to the corresponding three dihedral angles of the trihedral corner (angles between the three planes that make the trihedral corner), and

b) The sides a, b, c of the spherical triangle are equal to the three angles formed at the vertex of the trihedral corner.

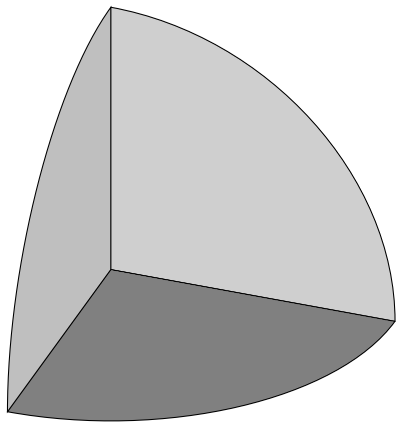

Figure 2.4: A Trihedral corner depicting an eighth of a sphere.

Great Circles, Arcs, and Their Poles

Definition (Poles of an Arc)**.** For any arc AB of any great circular arc on the sphere, if the diameter of the sphere that is perpendicular to the plane of the great circle intersects the surface of the sphere at points C_1 and C_2, we call C_1 and C_2 the **poles** of the arc AB, as shown in Figure 2.5.

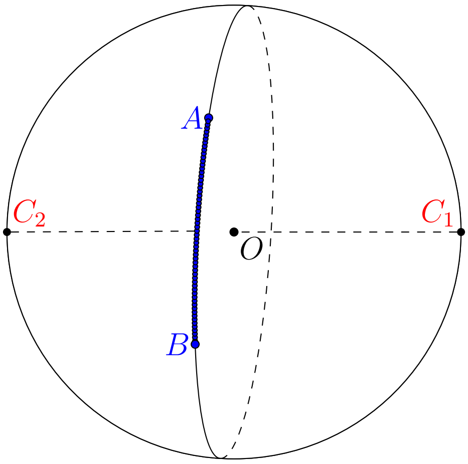

Figure 2.5: The poles of the arc AB.

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Distances and Great Circles 1119. Let ℓ be a great circular arc on the unit sphere, and P a point not on ℓ. Prove that

1. If P is a pole of ℓ, then for any point Q on ℓ, PQ is a quadrant.
2. If for Q_1 and Q_2 on ℓ we have $PQ_1 = PQ_2 = \pi/2$, then P is a pole of ℓ.
3. If P is a pole of ℓ and Q_1 and Q_2 are on ℓ, then the distance between Q_1 and Q_2 equals the spherical angle between Q_1P and PQ_2: $Q_1Q_2 = \angle Q_1PQ_2$.

2.5.2 Polar Triangles

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KACY–I 1120. For any triangle ABC, if the poles of the sides of ABC are vertices of triangle $A'B'C'$, then we call $A'B'C'$ the **polar triangle** of ABC. The assumption is that vertex A' is a pole of BC, vertex B' is a pole of CA, and C' is a pole of AB, as shown in Figure 2.6. Prove that:

(a) If $A'B'C'$ is the polar triangle of ABC, then ABC is also the polar triangle of $A'B'C'$, so that vertex A is the pole of side a' of $A'B'C'$, vertex B is the pole of side b', and vertex C is the pole of side c'.

(b) If $\triangle ABC$ and $\triangle A'B'C'$ are polar triangles of one another, then the sum of each angle and its associated side in the polar triangle equals π:

$$A + a' = \pi, \quad B + b' = \pi, \quad C + c' = \pi,$$
$$A' + a = \pi, \quad B' + b = \pi, \quad C' + c = \pi.$$

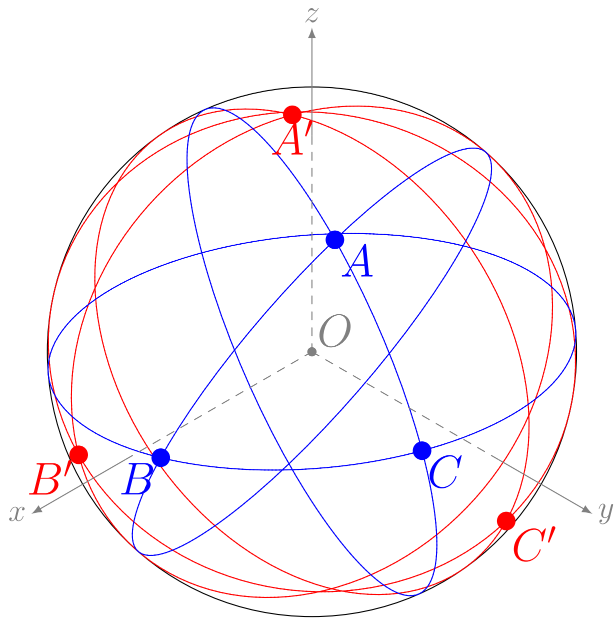

Figure 2.6: Polar triangles ABC and $A'B'C'$ on a sphere.

2.5.3 Classification of Spherical Triangles

Definition. Concerning the angles of spherical triangles,

- A spherical triangle could be **acute**, **right**, or **obtuse**, like plane triangles.
- Spherical triangles may have two or three right or obtuse angles, and each angle can be close to π. We can thus see that the sum of angles of a spherical triangle cannot exceed 3π.

Regarding the sides,

- A spherical triangle may be **scalene**, **isosceles**, or **equilateral**.
- A spherical triangle that has one or more of its **sides** equal to a **quadrant** ($\pi/2$) is called a **quadrantal triangle**.
- A triangle in which one of the vertices is a pole of the opposing side is called a **semilunar** triangle, or a **semilune**.

Definition. Concerning the sides of a spherical triangle,

- A spherical triangle may be **scalene**, **isosceles**, or **equilateral**.
- A spherical triangle that has one or more of its **sides** equal to a **quadrant** ($\pi/2$) is called a **quadrantal triangle**.
- A triangle in which one of the vertices is a pole of the opposing side is called a **semilunar** triangle, or a **semilune**.

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KACY–I 1121. Prove the following statements:

1. It is known in Euclidean geometry that for plane triangles, being equilateral (having equal sides) is equivalent to being equiangular (having equal angles). Prove the same statement for spherical triangles.

2. Furthermore, if a plane triangle is isosceles (has two equal sides), then the angles facing those sides are equal to each other. Show that the same things happens for spherical triangles.

3. Prove the Pythagorean Theorem for Spherical Triangles: a, b, c are side-lengths of a spherical triangle ABC with right angle at A on a unit sphere if and only if

$$\cos a = \cos b \cos c.$$

Remark. Remember that for a spherical triangle on a unit sphere, all main elements are smaller than π; and note that in the Spherical Pythagorean Formula the cosine function is applied to the side-lengths a, b, c rather than angles A, B, C. Show that we could, however, find sines of non-right angles:

$$\sin B = \frac{\sin b}{\sin a}, \cos B = \frac{\cos b \sin c}{\sin a}, \quad \text{and} \quad \sin C = \frac{\sin c}{\sin a}, \cos C = \frac{\cos c \sin b}{\sin a}.$$

4. If a spherical triangle has three right angles, all its sides are quadrants.

5. If a spherical triangle has two right angles, the sides facing those angles are quadrants and the third angle is measured by its opposite side.

6. If any two parts, a part being a side or an angle, of a spherical triangle measure $\pi/2$ radians, the triangle is a semilune. Also, the angle at the pole has the same measure as the opposing side. All of the other sides and angles measure $\pi/2$ radians.

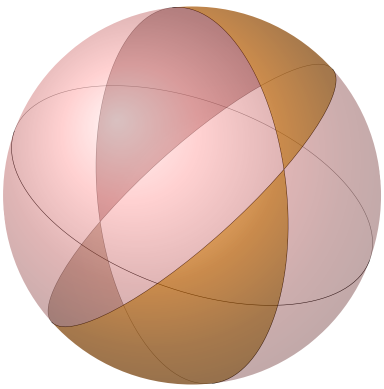

Figure 2.7: Lunes on a sphere divide it into four parts.

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Lunes on a Sphere 1122. Two great circles passing through antipodal points on a sphere divide the sphere into four parts like orange slices, each being a **lune** on the sphere (Figure 2.7). Prove that the area of a lune with angle α is 2α.

2.5.4 Spherical Triangle Inequalities

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KACY–I 1123. Prove that for any spherical triangle ABC on a unit sphere with sides a, b, c and angles A, B, C,

1. The triangle inequality holds: each side is smaller than the sum of the other two sides:

$$a < b + c, \quad b < c + a, \quad c < a + b.$$

2. Each side is larger than the difference of the other two sides.

3. The sum of the sides of the triangle is positive and smaller than 2π:

$$0 < a + b + c < 2\pi.$$

4. The sum of the angles of the triangle is larger than π and smaller than 3π:

$$\pi < A + B + C < 3\pi.$$

5. The larger side of the triangle faces the larger angle of the spherical triangle.

6. The following inequalities hold true for triangle's angles:

$$A + B - C < \pi, \quad A - B + C < \pi, \quad -A + B + C < \pi.$$

2.5.5 Congruent Spherical Triangles & Gauss–Bonnet Theorem

Remember from the Euclidean geometry that two plane triangles are **congruent** (having equal sides and angles) if (a) all three sides are equal (**SSS**), (b) two sides and the angle between them are equal (**SAS**), or (c) two angles and the side joining them are equal (**ASA**). However, two **incongruent** plane triangles may have all three angles equal to one another, simply because we can scale all sides of a plane triangle equally to get a triangle with larger/smaller sides but the same angles.

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KACY–I 1124. Prove that for spherical triangles,

1. The three Euclidean criteria for congruent plane triangles (**SSS**, **SAS**, **ASA**) also hold true for spherical triangles.

2. Two spherical triangles with equal angles (**AAA**) are congruent.

3. Two congruent spherical triangles have equal areas.

4. On a unit sphere, sum of triangle's angles equals π plus the area of triangle:

$$A + B + C = \pi + (\text{area of } \triangle ABC).$$

2.5.6 Inverse Spherical Triangles

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KACY–I 1125. Two spherical triangles ABC and $A'B'C'$ have all their corresponding main elements equal to one another, that is,

$$A = A', \quad B = B', \quad C = C', \qquad a = a', \quad b = b', \quad c = c'.$$

Prove that either the two triangles are **directly equal**, meaning one can be moved in space so that its vertices matches the vertices of the other triangle, or they are **inversely equal**, or simply **inverse** of each other, which means that they are reflections of each other with respect to a plane that passes through sphere's center. See Figure 2.8.

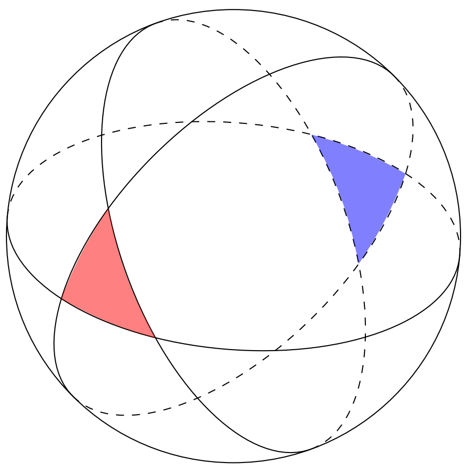

Figure 2.8: Inverse spherical triangles.

2.5.7 Spherical Law of Cosines for Sides

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KACY–I 1126. The cosine of one side of a spherical triangle is equal to the product of the cosine of the other two sides plus the product of sines of these two sides times the cosine of the angle between them:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$
$$\cos b = \cos c \cos a + \sin c \sin a \cos B,$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

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KACY–I 1127. Figure 2.9 shows a spherical triangle ABC with sides a, b, c on a sphere centered at O. The tangent at A to the arc AB meets OB at D and the tangent at A to the arc AC meets OC at E.

1. Using the Law of Cosines in the plane, show that

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

2. Using the Pythagorean identity $\sin^2 A + \cos^2 A = 1$, prove that

$$\sin A = \frac{\sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}}{\sin b \sin c}.$$

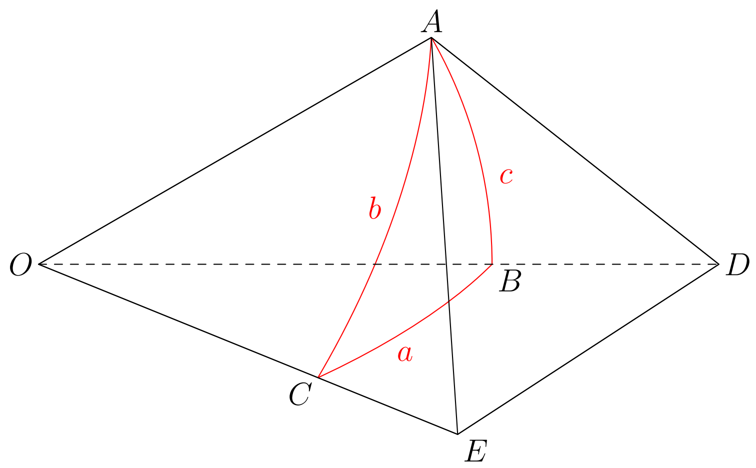

Figure 2.9: Spherical triangle ABC with tangents to arcs at vertex A.

Spherical Law of Sines

Consider the ratio between the sine of an angle and the sine of its opposite side in a spherical triangle. Prove that this ratio is the same for all three angles:

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

2.5.8 Five–Piece Spherical Trigonometric Identities

In a spherical triangle ABC with side-lengths a, b, c, prove the five-piece identities:

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Side's Sine Times Adjacent Angle's Cosine 1128. For any side x and angle Y adjacent to it, the product $\sin x \cos Y$ equals $\cos y \sin z$ **minus** $\sin y \cos z \cos X$:

$$\text{Side } a : \begin{cases} \sin a \cos B = \cos b \sin c - \sin b \cos c \cos A, \\ \sin a \cos C = \cos c \sin b - \sin c \cos b \cos A, \end{cases}$$

$$\text{Side } b : \begin{cases} \sin b \cos C = \cos c \sin a - \sin c \cos a \cos B, \\ \sin b \cos A = \cos a \sin c - \sin a \cos c \cos B, \end{cases}$$

$$\text{Side } c : \begin{cases} \sin c \cos A = \cos a \sin b - \sin a \cos b \cos C, \\ \sin c \cos B = \cos b \sin a - \sin b \cos a \cos C. \end{cases}$$

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Angle's Sine Times Adjacent Side's Cosine 1129. For any angle X and side y adjacent to it, the product $\sin X \cos y$ equals $\cos Y \sin Z$ **plus** $\sin Y \cos Z \cos x$:

$$\text{Angle } A : \begin{cases} \sin A \cos b = \cos B \sin C + \sin B \cos C \cos a, \\ \sin A \cos c = \cos C \sin B + \sin C \cos B \cos a, \end{cases}$$

$$\text{Angle } B : \begin{cases} \sin B \cos c = \cos C \sin A + \sin C \cos A \cos b, \\ \sin B \cos a = \cos A \sin C + \sin A \cos C \cos b, \end{cases}$$

$$\text{Angle } C : \begin{cases} \sin C \cos a = \cos A \sin B + \sin A \cos B \cos c, \\ \sin C \cos b = \cos B \sin A + \sin B \cos A \cos c. \end{cases}$$

2.5.9 Spherical Law of Cosines for Angles

KACY Summer League

KACY–I 1130. The cosine of an angle of a spherical triangle equals the product of sines of the other two angles and the cosine of the side between them **minus** the product of cosines of the other two angles:

$$\begin{aligned}\cos A &= \sin B \sin C \cos a - \cos B \cos C,\\ \cos B &= \sin C \sin A \cos b - \cos C \cos A,\\ \cos C &= \sin A \sin B \cos c - \cos A \cos B.\end{aligned}$$

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KACY–I 1131. Imply that

$$\begin{aligned}\cos a &= \frac{\cos A + \cos B \cos C}{\sin B \sin C},\\ \sin^2 \frac{a}{2} &= -\frac{\cos A + \cos(B+C)}{2 \sin B \sin C}.\end{aligned}$$

2.5.10 Half–Angle and Half–Side Spherical Formulas

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KACY–I 1132. For every spherical triangle ABC with angles A, B, C and a, b, c, let p be the semiperimeter: $p = (a + b + c)/2$. Prove the following half–angle trigonometric formulas:

1. $\sin^2 \frac{A}{2} = \frac{\sin\left(\frac{a+b-c}{2}\right) \sin\left(\frac{a-b+c}{2}\right)}{\sin b \sin c}$,
2. $\sin^2 \frac{A}{2} = \frac{\sin(p-b)\sin(p-c)}{\sin b \sin c}$,
3. $\cos^2 \frac{A}{2} = \frac{\sin p \sin(p-a)}{\sin b \sin c}$,
4. $\tan^2 \frac{A}{2} = \frac{\sin(p-b)\sin(p-c)}{\sin p \sin(p-a)}$,
5. $\sin A = \frac{2\sqrt{\sin p \sin(p-a)\sin(p-b)\sin(p-c)}}{\sin b \sin c}$.

KACY Summer League

KACY–I 1133. For every spherical triangle ABC with angles A, B, C and a, b, c, define $P = (a+b+c)/2$. Prove the following half–side trigonometric identities:

1. $\sin^2 \dfrac{a}{2} = -\dfrac{\cos P \cos(P-A)}{\sin B \sin C}$,
2. $\cos^2 \dfrac{a}{2} = \dfrac{\cos(P-B)\cos(P-C)}{\sin B \sin C}$,
3. $\tan^2 \dfrac{a}{2} = -\dfrac{\cos P \cos(P-A)}{\cos(P-B)\cos(P-c)}$,
4. $\sin a = \dfrac{2\sqrt{-\cos P \cos(P-A)\cos(P-B)\cos(P-C)}}{\sin B \sin C}$.

2.5.11 Spherical Law of Haversines

Definition. Some hundreds of years ago when spherical trigonometry was a hot-topic for mathematicians, there was another periodic function besides cosine and sine, called **versine**, short for **versed sine**, defined by $\operatorname{versin}(\theta) = 1 - \cos\theta$.

Definition. Since $1 - \cos\theta = 2\sin^2\theta$, it makes sense to define a **halved versed sine**, or shortly, **haversine**, by $\operatorname{hav}(\theta) = \sin^2\left(\frac{\theta}{2}\right)$.

KACY Summer League

KACY–I 1134. Prove the **Haversine Formula** in a spherical triangle ABC with side-lengths a, b, c:

$$\operatorname{hav}(c) = \operatorname{hav}(a-b) + \sin(a)\sin(b)\operatorname{hav}(C).$$

Swimming the Depths of the Algebraic Ocean

Kaywañan is an Algebra Competition, and you may say its motto is "Let No One Ignorant of Algebra Enter." So far, we have been vigorously forging algebraic equations and definitions that are deeply rooted within their applications. It is both an intention and a purpose of Kaywañan to be defined as the collection of most important algebraic equations and identities that one may encounter in dealing with in ordinary, Euclidean flatland geometry, as well as non-Euclidean monsters and witches that might appear in hyperbolic geometry.

We have not even started to discuss the geometry of hyperbola and its associated hyperboloid. The most advanced formula, I would say, in Kaywañan so far is that of Problem 1134, the Spherical Law of Haversines, which has absolutely fascinating applications in astronomy. The haversine formula for spherical triangles is just an example of myriads of unknown equations in Spherical Geometry, the most special type of Elliptic Geometry. You can only imagine how many more of such algebraic equations may be found, written down, and added to Kaywañan if we consider other elliptic structures than the Sphere, such as the Ellipsoid.

There are other types of non-Euclidean geometries that some might say are even more surprising than the fact that angles of a spherical triangle add up to more than π. If the Euclidean plane is not infinite in all directions, and in the special case when the plane is limited to a 1×1 Square of the Euclidean Plane, you can imagine that the corresponding points on opposite sides of the square are actually the same point, as if you map the surface of a doughnut to the 1×1 Euclidean Square. If we start walking from a certain point in any of the two directions of the Euclidean Square on the surface of a doughnut, we would reach the same point. The same would happen if we map the sphere to the Euclidean plane, but the surface of the doughnut and that of a sphere are clearly distinct to us. If we had no idea of the third dimension, as if we were ants walking around in two-dimensional Euclidean plane, we would never even be able to know whether the surface on which we walk is a doughnut or a sphere.

Here, at the depths of the Algebraic Ocean of Kaywañan, Titan, Moon of Saturn, where we can see the positive curvature of the surface of the core of Titan, nobody doubts the spherical shape of Planet of Algebra, Kaywan. There are rumours that in earlier Eons, Kaywañans (those who live around Saturn) believed that Titan is the most special place in the Sölar System, and after one of them dreamed of Maurits Cornelis Escher's "Angels and Devils" painting, they started to preach a certain belief in a Hyperbolic Geometry of Titan to emphasize their uniqueness among moons of Saturn and other celestial objects.

It is now, however, a ridiculous claim to believe in a hyperbolic geometry for the actually spherical surface of Titan, maybe as ridiculous as believing in a Flat Earth once you have traveled to the moon and seen the biosphere of the Earth from afar. Now that we can swim the depths of Kaywañan and measure the spherical arcs close to the core of Titan, hyperbolic beliefs are but a joke. We may study mythology of such treacheries in the advanced levels of Napirañan Geometry Contest, but here in Kaywañan we stick to the algebra.

KACY Summer League

Maclaurin Series of Versine & Haversine 1135. Prove that for any complex number z,

$$\operatorname{versin}(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k}}{(2k)!} \quad \text{and} \quad \operatorname{hav}(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k}}{2(2k)!}.$$

Then show that the limit of both $\operatorname{versin}(\theta)/\theta$ and $\operatorname{hav}(\theta)/\theta$ when $\theta \to 0$ is 0.

2.5.12 The Sine Formulae (from "Spherical Astronomy")

The following method of notation is quoted from "Textbook on Spherical Astronomy" by W. M. Smart and R. M. Green, given in the first chapter as one of the main formulas (formulae **A** and **D**) in Spherical Trigonometry.

Definition (Laterangular Function of a Spherical Triangle)**.** For any spherical triangle ABC, the **Laterangular Function of** A, denoted $X(a, A)$, is defined by

$$(X(a, A))^2 \cdot \sin^2 a \sin^2 b \sin^2 c = 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c.$$

KACY Summer League

KACY–I 1136. In a spherical triangle ABC with side-lengths a, b, c,

1. Prove the **Astronomical Sine** formula:

$$\sin^2 b \cdot \sin^2 c \cdot \sin^2 A = 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c.$$

2. Prove that

$$(X(a, A))^2 = \left(\frac{\sin A}{\sin a}\right)^2,$$

and imply that the Laterangular Function must be symmetric, so that $X(a, A) = X(b, B) = X(c, C)$.

3. If all main elements of triangle ABC are smaller than π, then the **Spherical Law of Sines** holds true:

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{\sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}}{\sin a \sin b \sin c}.$$

Definition (Four–Piece Spherical Trigonometric Terminology)**.** In the spherical triangle ABC consider the four consecutive parts B, a, C, b. The angle C is contained by the two sides a and b and is called the **inner angle**. The side a is flanked by the two angles B and C and is called the **inner side**.

KACY Summer League

Four-Part Identity 1137. Prove that cos(inner side) · cos(inner angle) equals

$$\sin(\text{inner side}) \cdot \cot(\text{other side}) - \sin(\text{inner angle}) \cdot \cot(\text{other angle}).$$

2.5.13 Delambre's and Napier's Analogies

KACY Summer League

Delambre's Aanlogies 1138. In a spherical triangle ABC with sides a, b, c,

$$\sin\frac{c}{2}\sin\frac{A-B}{2} = \cos\frac{C}{2}\sin\frac{a-b}{2} \quad \text{and} \quad \sin\frac{c}{2}\cos\frac{A-B}{2} = \sin\frac{C}{2}\sin\frac{a+b}{2},$$
$$\cos\frac{c}{2}\sin\frac{A+B}{2} = \cos\frac{C}{2}\cos\frac{a-b}{2} \quad \text{and} \quad \cos\frac{c}{2}\cos\frac{A+B}{2} = \sin\frac{C}{2}\cos\frac{a+b}{2}.$$

Taking Delambre's equations, which are also called **Gauss's Spherical Equations**, in pairs, we obtain Napier's Analogies:

KACY Summer League

Napier's Aanlogies 1139. In a spherical triangle ABC with sides a, b, c,

$$\tan\frac{a+b}{2} = \frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}\tan\frac{c}{2} \quad \text{and} \quad \tan\frac{a-b}{2} = \frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}}\tan\frac{c}{2},$$
$$\tan\frac{A+B}{2} = \frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}\cot\frac{C}{2} \quad \text{and} \quad \tan\frac{A-B}{2} = \frac{\sin\frac{a-b}{2}}{\sin\frac{a+b}{2}}\cot\frac{C}{2}.$$

2.5.14 Napier's Rules

KACY Summer League

Napier's Rules for Right Spherical Triangles 1140. If one main element among a, b, c, A, B, C is $\pi/2$, there would be five remaining unknown parts. John Napier suggested Five–Piece Mnemonics shown in Figures 2.10 and 2.11 [from Wikipedia] to prove:

1. The sine of any middle part equals the product of tangents of adjacent parts.
2. The sine of any middle part equals the product of cosines of opposite parts.

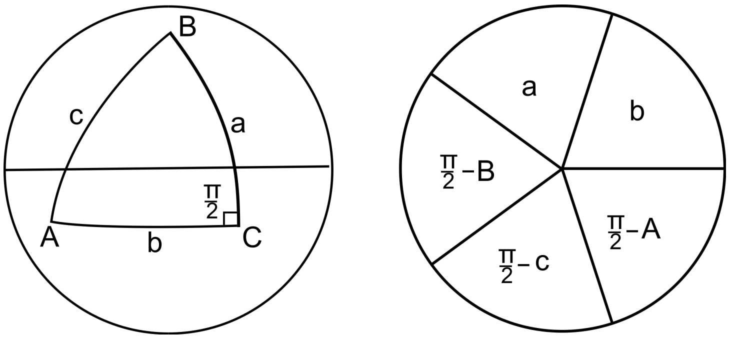

Figure 2.10: Napier's Mnemonics (When One Angle is Right)

KACY Summer League

Napier's Ten Commandments 1141. For any spherical triangle ABC with a right angle at A, Napier's Ten Rules are the ten equations derived from various spherical trigonometric identities studied in previous problems [Wikipedia]:

(I) $\cos a = \cos b \cos c$, A.K.A. *Spherical Pythagorean Theorem*,

(II) $\sin b = \sin a \sin B$, derived from *Spherical Law of Sines*,

(III) $\sin c = \sin a \sin C$, also from *Spherical Law of Sines*,

(IV) $\cos B = \cos b \sin C$, by *Spherical Law of Cosines for Angle B*,

(V) $\cos C = \cos c \sin B$, also by *Spherical Law of Cosines*, but for Angle C,

(VI) $\cos a = \cot B \cot C$, by *Spherical Law of Cosines for Angle A*,

(VII) $\cos B = \cot a \tan c$, by *Napier's Four–Part Identity* for $\cos c \cos B$,

(VIII) $\cos C = \tan b \cot a$, by *Napier's Four–Part Identity* for $\cos b \cos C$,

(IX) $\sin b = \tan c \cot C$, by *Napier's Four–Part Identity* for $\cos b \cos A$,

(X) $\sin c = \cos c \sin B$, by *Napier's Four–Part Identity* for $\cos c \cos A$.

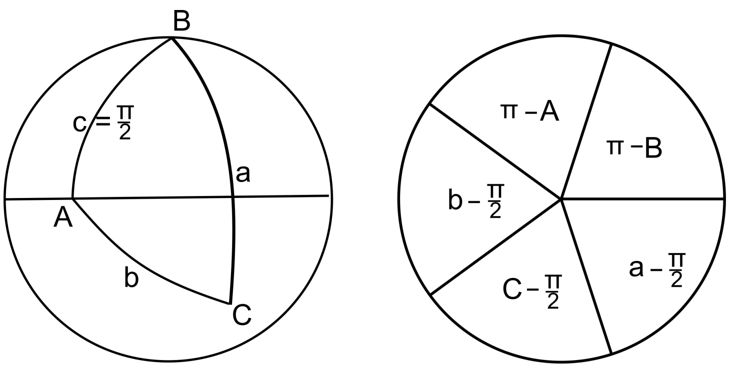

Figure 2.11: Napier's Mnemonics (When One Side is a Quadrant)

KACY Summer League

KACY–I 1142. Prove that Napier's Ten Commandments can be reduced to the following cosine formulae by changing b and c to $\frac{\pi}{2} - b$ and $\frac{\pi}{2} - c$, respectively:

a) $\cos a = \sin\left(\frac{\pi}{2} - b\right)\sin\left(\frac{\pi}{2} - c\right)$,

b) $\cos a = \cot B \cot C$,

c) $\cos\left(\frac{\pi}{2} - b\right) = \sin a \sin B$,

d) $\cos\left(\frac{\pi}{2} - b\right) = \cot\left(\frac{\pi}{2} - c\right)\cot C$,

e) $\cos\left(\frac{\pi}{2} - c\right) = \sin a \sin C$,

f) $\cos\left(\frac{\pi}{2} - c\right) = \cot\left(\frac{\pi}{2} - b\right)\cot B$,

g) $\cos B = \sin\left(\frac{\pi}{2} - b\right)\sin C$,

h) $\cos B = \cot a \cot\left(\frac{\pi}{2} - c\right)$,

i) $\cos C = \sin\left(\frac{\pi}{2} - c\right)\sin B$,

j) $\cos C = \cot\left(\frac{\pi}{2} - b\right)\cot a$.

Napier's Ten Commandments and their ten cosine forms formulae inspired Napier to make his **Napier Rules for Right Spherical Triangles**.

2.5.15 The Global Half–Side Identities

Definition (Half–Side Spherical Triangle Identities)**.** For any spherical triangle ABC, the **Halveside Function of** A, denoted $H(a, A)$, is defined by

$$H(a, A) \cdot \cos(P - A) = \tan\left(\frac{a}{2}\right).$$

where $P = (A + B + C)/2$.

KACY Summer League

KACY–I 1143. In a spherical triangle ABC with side-lengths a, b, c,

1. Prove the **Astronomical Half–Side Tangent** formula:

$$\tan\left(\frac{a}{2}\right) = \sqrt{\frac{-\cos P}{\cos(P - A)\cos(P - B)\cos(P - C)}}\cos(P - A),$$

2. Prove that

$$(H(a, A))^2 = \frac{-\cos P}{\cos(P - A)\cos(P - B)\cos(P - C)},$$

and imply that the Laterangular Function must be symmetric, so that $H(a, A) = H(b, B) = H(c, C)$.

3. Prove **Mollweide's** formula in Euclidean plane:

$$\left(\tan\frac{A}{2}\right)^2 = \frac{(a + b - c)(a - b + c)}{(a + b + c)(-a + b + c)}.$$

2.5.15.1 Spherical Half–Side Tangent–Cotangent Formulae

KACY Summer League

KACY–I 1144. In the spherical triangle ABC with sides a, b, c that subtend angles A, B, C, prove that

$$\tan\left(\frac{a-b}{2}\right)\cot\left(\frac{a+b}{2}\right) = \tan\left(\frac{A-B}{2}\right)\cot\left(\frac{A+B}{2}\right).$$

2.5.15.2 L'Huilier's & Cagnoli's Halveside Theorems

KACY Summer League

L'Huilier's Theorem 1145. Let a spherical triangle have sides of length a, b, and c, and semiperimeter p. Then the spherical excess $E = (A+B+C) - \pi$ is given by

$$\tan\left(\frac{E}{4}\right) = \sqrt{\tan\left(\frac{p}{2}\right)\tan\left(\frac{p-a}{2}\right)\tan\left(\frac{p-b}{2}\right)\tan\left(\frac{p-c}{2}\right)}.$$

KACY Summer League

Cagnoli's Theorem 1146. $\sin\left(\frac{E}{2}\right) = \dfrac{\sqrt{\sin p \sin(p-a)\sin(p-b)\sin(p-c)}}{2\cos\left(\frac{a}{2}\right)\cos\left(\frac{b}{2}\right)\cos\left(\frac{c}{2}\right)}.$

2.5.16 Exercises on Spherical Excess E

KACY Summer League

KACY–I 1147. Prove that the area of any spherical triangle ABC equals the Spherical Excess $E = A + B + C - \pi$ times the square of sphere's radius.

KACY Summer League

KACY–I 1148. In a spherical triangle if $A = B = 2C$, show that

$$8\sin\left(a+\frac{c}{2}\right)\sin^2\left(\frac{c}{2}\right)\cos\left(\frac{c}{2}\right) = \sin^3 a.$$

KACY Summer League

KACY–I 1149. If $A+B+C=2\pi$, show that

$$\cos^2\left(\frac{a}{2}\right)+\cos^2\left(\frac{b}{2}\right)+\cos^2\left(\frac{c}{2}\right)=1.$$

KACY Summer League

KACY–I 1150. In spherical triangle ABC, if C is a right angle, prove

$$\frac{\sin^2 c}{\cos c}\cos E=\frac{\sin^2 a}{\cos a}+\frac{\sin^2 b}{\cos b}.$$

KACY Summer League

KACY–I 1151. Show that

$$\sin\left(\frac{E}{2}\right)=\sin\left(\frac{a}{2}\right)\sin\left(\frac{b}{2}\right)\sec\left(\frac{c}{2}\right),$$
$$\cos\left(\frac{E}{2}\right)=\cos\left(\frac{a}{2}\right)\cos\left(\frac{b}{2}\right)\sec\left(\frac{c}{2}\right).$$

KACY Summer League

KACY–I 1152. Prove that

$$\sin^2\left(\frac{C}{2}-\frac{E}{4}\right)=\frac{\cos\left(\frac{p}{2}\right)\sin\left(\frac{p-a}{2}\right)\sin\left(\frac{p-b}{2}\right)\sin\left(\frac{p-c}{2}\right)}{\sin\left(\frac{a}{2}\right)\sin\left(\frac{b}{2}\right)\cos\left(\frac{c}{2}\right)},$$
$$\cos^2\left(\frac{C}{2}-\frac{E}{4}\right)=\frac{\sin\left(\frac{p}{2}\right)\cos\left(\frac{p-a}{2}\right)\cos\left(\frac{p-b}{2}\right)\cos\left(\frac{p-c}{2}\right)}{\sin\left(\frac{a}{2}\right)\sin\left(\frac{b}{2}\right)\cos\left(\frac{c}{2}\right)}.$$

KACY Summer League

KACY–I 1153. If $p = (a+b+c)/2$ is the semiperimeter, show that

$$\sin p = \frac{\sqrt{\sin\left(\frac{E}{2}\right)\sin\left(A-\frac{E}{2}\right)\sin\left(B-\frac{E}{2}\right)\sin\left(C-\frac{E}{2}\right)}}{2\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}.$$

2.5.17 Radii of the Spherical Triangle

KACY Summer League

Inradius of the Spherical Triangle 1154. In a spherical triangle ABC with sides a, b, c, and semiperimeter $p = (a+b+c)/2$, the inradius r of the incircle of triangle may be calculated from

$$\tan r = \tan\left(\frac{A}{2}\right)\cdot\sin(p-a) = \sqrt{\frac{\sin(p-a)\sin(p-b)\sin(p-c)}{\sin p}}.$$

KACY Summer League

Circumradius of the Spherical Triangle 1155. In a spherical triangle ABC with sides a, b, c, and angles A, B, C, define $P = (A+B+C)/2$, the circumradius R of the circumscribed circle of ABC can be calculated from

$$\cot R = \cot\left(\frac{A}{2}\right)\cdot\cos(P-A) = \sqrt{\frac{\cos(P-A)\cos(P-B)\cos(P-C)}{-\cos P}}.$$

2.5.18 Medians of Spherical Triangles

KACY Summer League

KACY–I 1156. In a spherical triangle ABC with sides a, b, c the length of the median m_C drawn from C (length of CF in Figure 2.12) [from Wikipedia] satisfies

$$\cos m_C = \cos b\cdot\cos\left(\frac{c}{2}\right) + \sin b\cdot\sin\left(\frac{c}{2}\right)\cdot\cos A.$$

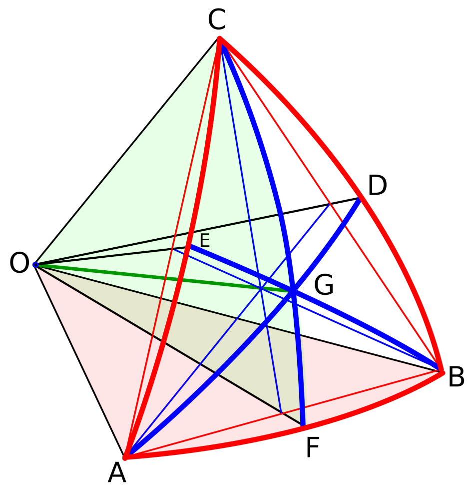

Figure 2.12: Spherical Medians

KACY Summer League

KACY–I 1157. Let O be the center of the sphere. If G is the centroid of spherical triangle ABC and $A'B'C'$ is the polar triangle of ABC, show that

$$\overrightarrow{OG} = \frac{1}{2E} \cdot \left(\overrightarrow{OC'} \cdot |\overline{AB}| + \overrightarrow{OA'} \cdot |\overline{BC}| + \overrightarrow{OB'} \cdot |\overline{CA}|\right).$$

2.5.19 63 Ancient Spherical Geometry Problems

KACY Summer League

1960 International Mathematical Olympiad 1158. Consider a cone of revolution with an inscribed sphere tangent to the base of the cone. A cylinder is circumscribed about this sphere so that one of its bases lies in the base of the cone. let V_1 be the volume of the cone and V_2 be the volume of the cylinder.

a) Prove that $V_1 \neq V_2$;

b) Find the smallest number k for which $V_1 = kV_2$; for this case, construct the angle subtended by a diamter of the base of the cone at the vertex of the cone.

KACY Summer League

1962 International Mathematical Olympiad 1159. The tetrahedron $SABC$ has the following property: there exist five spheres, each tangent to the edges $SA, SB, SC, BC, CA, AB,$ or to their extensions.

a) Prove that the tetrahedron $SABC$ is regular.

b) Prove conversely that for every regular tetrahedron five such spheres exist.

KACY Summer League

1966 International Mathematical Olympiad 1160. Prove that the sum of the distances of the vertices of a regular tetrahedron from the center of its circumscribed sphere is less than the sum of the distances of these vertices from any other point in space.

KACY Summer League

1959–1966 IMO Longlist 1161. In a tetrahedron, all three pairs of opposite (skew) edges are mutually perpendicular. Prove that the midpoints of the six edges of the tetrahedron lie on one sphere.

KACY Summer League

1967 IMO Longlist 1162. Determine the volume of the body obtained by cutting the ball of radius R by the trihedron with vertex in the center of that ball, it its dihedral angles are α, β, γ.

KACY Summer League

1967 IMO Longlist 1163. Prove this proposition: Center the sphere circumscribed around a tetrahedron which coincides with the center of a sphere inscribed in that tetrahedron if and only if the skew edges of the tetrahedron are equal.

KACY Summer League

1970 Bulgaria 1164. In space, we are given the points A, B, C and a sphere with center O and radius 1. Find the point X from the sphere for which the sum $f(X) = |XA|^2 + |XB|^2 + |XC|^2$ attains its maximal and minimal value. Prove that if the segments OA, OB, OC are pairwise perpendicular and d is the distance from the center O to the centroid of the triangle ABC then:

a) the maximum of $f(X)$ is equal to $9d^2 + 3 + 6d$;

b) the minimum of $f(X)$ is equal to $9d^2 + 3 - 6d$.

KACY Summer League

1979 USAMO 1165. Let S be a great circle with pole P. On any great circle through P, two points A and B are chosen equidistant from P. For any spherical triangle ABC (the sides are great circles ares), where C is on S, prove that the great circle are CP is the angle bisector of angle C.
Note: A great circle on a sphere is one whose center is the center of the sphere. A pole of the great circle S is a point P on the sphere such that the diameter through P is perpendicular to the plane of S.

KACY Summer League

1981 USAMO 1166. The sum of the measures of all the face angles of a given complex polyhedral angle is equal to the sum of all its dihedral angles. Prove that the polyhedral angle is a trihedral angle. **Note**: A convex polyhedral angle may be formed by drawing rays from an exterior point to all points of a convex polygon.

KACY Summer League

1982 IMO Longlist 1167. Let $r_1, \dots, r_n$ be the radii of n spheres. Call $S_1, S_2, \dots, S_n$ the areas of the set of points of each sphere from which one cannot see any point of any other sphere. Prove that

$$\frac{S_1}{r_1^2} + \frac{S_2}{r_2^2} + \cdots + \frac{S_n}{r_n^2} = 4\pi.$$

KACY Summer League

1982 IMO Longlist 1168. A regular n-gonal truncated pyramid is circumscribed around a sphere. Denote the areas of the base and the lateral surfaces of the pyramid by S_1, S_2, and S, respectively. Let σ be the area of the polygon whose vertices are the tangential points of the sphere and the lateral faces of the pyramid. Prove that

$$\sigma S = 4S_1S_2\cos^2\frac{\pi}{n}.$$

KACY Summer League

1983 IMO Longlist 1169. Four faces of tetrahedron $ABCD$ are congruent triangles whose angles form an arithmetic progression. If the lengths of the sides of the triangles are $a < b < c$, determine the radius of the sphere circumscribed about the tetrahedron as a function on a, b, and c. What is the ratio c/a if $R = a$?

KACY Summer League

1984 IMO Longlist 1170. A tetrahedron is inscribed in a sphere of radius 1 such that the center of the sphere is inside the tetrahedron. Prove that the sum of lengths of all edges of the tetrahedron is greater than 6.

KACY Summer League

1985 IMO Longlist 1171. This problem comes in two parts:

a) The solid S is defined as the intersection of the six spheres with the six edges of a regular tetrahedron T, with edge length 1, as diameters. Prove that S contains two points at a distance $1/\sqrt{6}$.

b) Using the same assumptions in a), prove that no pair of points in S has a distance larger than $1/\sqrt{6}$.

KACY Summer League

1985 IMO Longlist 1172. Determine the radius of a sphere S that passes through the centroids of each face of a given tetrahedron T inscribed in a unit sphere with center O. Also, determine the distance from O to the center of S as a function of the edges of T.

KACY Summer League

1987 IMO Longlist 1173. Let S_1 and S_2 be two spheres with distinct radii that touch externally. The spheres lie inside a cone C, and each sphere touches the cone in a full circle. Inside the cone there are n additional solid spheres arranged in a ring in such a way that each solid sphere touches the cone C, both of the spheres S_1 and S_2 externally, as well as the two neighboring solid spheres. What are the possible values of n?

KACY Summer League

1987 Vietnam 1174. Prove that among any five distinct rays Ox, Oy, Oz, Ot, Or in space there exist two which form an angle less than or equal to $90°$.

KACY Summer League

1992 Putnam 1175. On a sphere, 4 points are randomly chosen. What is the probability that the center of the sphere is contained in the tetrahedron in the 4 points.

KACY Summer League

2006 Baltic Way 1176. There are 2006 points marked on the surface of a sphere. Prove that the surface can be partitioned into 2006 congruent pieces, so that each piece contains exactly one of these points inside it.

KACY Summer League

2016 Brazil Cono Sur Training 1177. Let $ABCD$ be a tetrahedron and let E, F, G, H, K, and L be points on the segments AB, BC, CA, DA, DB and DC, respectively, so that

$$AE \cdot BE = BF \cdot CF = CG \cdot AG = DH \cdot AH = DK \cdot BK = DL \cdot CL.$$

Prove that the six points marked on the sides of the tetrahedron are on the same sphere.

KACY Summer League

2000 Belarus TST 1178. A closed pentagonal line is inscribed in a sphere of the diameter 1, and has all edges of length ℓ. Prove that

$$\ell \leq \sin \frac{2\pi}{5}.$$

KACY Summer League

1995 Romania TST 1179. A cube is partitioned into finitely many rectangular parallelepipeds with the edges parallel to the edges of the cube. Prove that if the sum of the volumes of the circumspheres of these parallelepipeds equals the volume of the circumscribed sphere of the cube, then all the parallelepipeds are cubes.

KACY Summer League

1998 Czech and Slovak 1180. A sphere is inscribed in a tetrahedron $ABCD$. The tangent planes to the sphere parallel to the faces of the tetrahedron cut off four smaller tetrahedra. Prove that sum of all the 24 edges of the smaller tetrahedra equals twice the sum of edges of the tetrahedron $ABCD$.

KACY Summer League

1993 Brazil 1181. Let $P_1P_2\dots P_n$ a polygon inscribed on a circunference and contained in a plane α. Let Q be a point outside α. Consider, for each $i = 1, 2, \dots, n$, the plane β_i passing through P_i and perpendicular to QP_i. Prove that all the planes β_i intersect at one point.

KACY Summer League

2015 Kurchatov Olympiad 1182. To prepare mashed potatoes, Kolya, the chef, needs to get the specified amount of peeled potatoes as soon as possible. Not caring about saving peelings, he cuts cubes from spherical potatoes, with each stroke clearing one edge of the knife. Can he complete the task faster when the same frequency of knife strokes if you cut out any other polyhedra? (formally: is it true that of all polyhedra cut from of the given sphere, the largest ratio of volume to the number of faces is for the inscribed cube?)

KACY Summer League

2015 Miklos Schweitzer 1183. We call a bar of width w on the surface of the unit sphere $\mathbb{S}^2$, a spherical segment, centered at the origin, which has width w and is symmetric with respect to the origin. Prove that there exists a constant $c > 0$, such that for any positive integer n the surface $\mathbb{S}^2$ can be covered with n bars of the same width so that any point is contained in no more than $c\sqrt{n}$ bars.

KACY Summer League

2017 St. Petersburg 1184. Given a tetrahedron $PABC$, draw the height PH from vertex P to ABC. From point H, draw perpendiculars HA', HB', HC' to the lines PA, PB, PC. Suppose the planes ABC and $A'B'C'$ intersects at line ℓ. Let O be the circumcenter of triangle ABC. Prove that $OH \perp \ell$.

KACY Summer League

2018 Spain 1185. Points on a spherical surface with radius 4 are colored in 4 different colors. Prove that there exist two points with the same color such that the distance between them is either $4\sqrt{3}$ or $2\sqrt{6}$ (distance is Euclidean, that is, the length of the straight segment between the points).

KACY Summer League

KACY–I 1186. Given reals a_i, b_i, c_i, for $i = 1, 2, \ldots, n$, such that

$$\sum_{i=1}^{n} a_i^2 = 1, \qquad \sum_{i=1}^{n} b_i^2 = 1, \qquad \sum_{i=1}^{n} c_i^2 = 1, \qquad \sum_{i=1}^{n} b_i c_i = 0.$$

Prove that

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 + \left(\sum_{i=1}^{n} a_i c_i\right)^2 \leq 1.$$

KACY Summer League

Spherical Inequality by Puuhikki 1187. Let x, y, z be positive real numbers such that $x^2 + y^2 + z^2 = 1$. Prove that

$$8xyz < \frac{4}{3}\pi.$$

KACY Summer League

2002 Iran TST 1188. There is a closed curve on the surface of the unit sphere. We know every big circle of the sphere has non-empty intersection with the curve. Prove that perimeter of the curve is at least 2π.

KACY Summer League

2003 China 1189. 8 spherical balls of radius 1 are placed in a cylinder in two layers. with each layer containing 4 balls. Each ball is tangent to 2 balls in the same layer, 2 balls in another layer, one base, and the lateral surface of the cylinder. What is the height of the cylinder?

KACY Summer League

KACY–I 1190. On a circle of positive integer radius n, there are m chords such that for any point in the circle there is a chord such that the distance from that point to the chord is ≤ 1. Prove that $m \geq n$.

KACY Summer League

KACY–I 1191. Four distinct points A, B, C, and M are given on a sphere, none of which is opposite to any other. Prove that if the great circles AM and BM are orthogonal to great circles BC and AC, respectively, then also the great circle CM is orthogonal to great circle AB.

KACY Summer League

1992 Tokyo University Entrance Exam 1192. Let a and b be positive reals. Four points $P(0,0,0), Q(a,0,0), R(0,1,0), S(0,1,b)$ lie on a same spherical surface with radius 1. Let r be the radius of the sphere which is inscribed in tetrahedron $PQRS$. Find the maximum value of r.

KACY Summer League

1993 Kyoto University Entrance Exam 1193. Assume that 3 Points A, B, C lie on the spherical surface with radius 1, centered at O. Let P, Q, R be the mid points of BC, CA, AB, respectively. Prove that at least one segment of the segments OP, OQ, OR in length is greater than or equal to $\frac{1}{2}$.

KACY Summer League

1999 Australian Math Competition 1194. When three spherical balls, each of radius 10 cm, are placed in a hemispherical dish, it is noticed that the tops of the balls are all exactly level with the top of the dish. What is the radius, in centimeters, of the dish?

KACY Summer League

1981 IMO Shortlist and Beyond 1195. The version b) is an ISL problem from 1981. The version a) is the easier 2-dimensional counterpart of the 3-dimensional version b).

a) If a circular planet lies completely within the convex hull H of a similarly defined system of circular planets, its entire circumference is visible from the other planets.

b) Consider a cluster of n spherical planets, all the same size, which drift rigidly together through outer space, that is, each with no motion relative to the others in the cluster. In general, there is some region R on the surface of a planet which cannot be seen from any point on any of the other planets (if a planet is in the midst of the others, then its surface is completely visible and this region is empty). Prove that the sum of the areas of these visible regions R is exactly equal to the area of the surface of one planet.

KACY Summer League

Problem 2814 in Gazeta Matematica (Bucuresti) 1196. Let A', B', C' be the points where the arc bisectors of the spherical triangle ABC meet the opposite sides. Is it possible to have $A'B' = A'C'$ without ABC being an isosceles triangle?

KACY Summer League

KACY–I 1197. Prove that the following two statements are equivalent, and then prove them separately.

a) For $a, b, c \geq 0$ satisfying $a^2 + b^2 + c^2 = 1$, we have

$$\frac{a}{a^3 + 2bc} + \frac{b}{b^3 + 2ca} + \frac{c}{c^3 + 2ab} \geq 2.$$

b) Using spherical coordinates, where α is the $x - y$ angle, and β is the $y - z$ angle, define $a = \cos\beta\cos\alpha$ (x-coordinate), $b = \cos\beta\sin\alpha$ (y-coordinate), and finally $c = \sin\beta$. Then, we have

$$\frac{\cos\alpha}{\cos^2\beta\cos^3\alpha + 2\sin\beta\sin\alpha} + \frac{\sin\alpha}{\cos^2\beta\sin^3\alpha + 2\sin\beta\cos\alpha} + \frac{\sin\beta}{\sin^3\beta + 2\sin\alpha\cos\alpha\cos^2\beta} \geq 2.$$

KACY Summer League

Cauchy–like Spherical Inequality by Fuzzylogic 1198. Let a_i, b_i, x_i be reals for $i = 1, 2, \ldots, n$, such that $\sum_{i=1}^n a_i x_i = 0$. Prove that

$$\left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i\right)^2\right) \geq \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i x_i\right)^2.$$

KACY Summer League

Hanging Sector by Farenhajt 1199. A circular sector with the center O and the radii OA, OB is cut out of the homogenous thin cardboard and hung freely by the point A. Find the angle between AO and the vertical if $\angle AOB = \theta \leq \pi$.

KACY Summer League

Hanging Sector by Atan 1200. Through a point in the interior of a sphere we put three planes, standing perpendicular on each other. These planes cut up the spherical surface into 8 curvilinear (spherical) triangles. The triangles are colored alternately black and white (Chessboard). Prove that exactly then half of the spherical surface is white.

KACY Summer League

Sphere Coloring by Goutham 1201. A sphere is colored in two colors. Prove that there exists an equilateral triangle whose vertices are of same colour and whose vertices lie on the sphere.

KACY Summer League

Toronto Junior Math Battle 1202. Given a solid sphere, construct its diameter using compass and straightedge.

KACY Summer League

KACY–I 1203. Let $ABCD$ be a tetrahedron and O its incenter, and let the line OD be perpendicular to AD. Find the angle between the planes DOB and DOC.

KACY Summer League

KACY–I 1204. The colonizers of a spherical planet have decided to build N towns, each having area $1/1000$ of the total area of the planet. They also decided that any two points belonging to different towns will have different latitude and different longitude. What is the maximal value of N?

KACY Summer League

KACY–I 1205. Find the minimum and maximum value of $f(x,y)$, where x and y are real numbers, and

$$f(x,y) = 2\sin x\cos y + 3\sin x\sin y + 6\cos x.$$

KACY Summer League

KACY–I 1206. Let a tetrahedron $ABCD$ be inscribed in a sphere S. Find the locus of points P inside the sphere S for which the equality

$$\frac{AP}{PA_1} + \frac{BP}{PB_1} + \frac{CP}{PC_1} + \frac{DP}{PD_1} = 4,$$

holds, where A_1, B_1, C_1, and D_1 are the intersection points of S with the lines AP, BP, CP, and DP, respectively.

KACY Summer League

KACY–I 1207. In the tetrahedron $OABC$ with volume V, we denote by α, β, γ the measures of the angles $\angle BOC, \angle COA$, and $\angle AOB$, respectively. Prove that:

$$36 \cdot V^2 = |OA|^2 \cdot |OB|^2 \cdot |OC|^2 \cdot \begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix}.$$

KACY Summer League

KACY–I 1208. Given a pyramid whose base is an n-gon inscribable in a circle, let H be the projection of the top vertex of the pyramid to its base. Prove that the projections of H to the lateral edges of the pyramid lie on a circle.

KACY Summer League

Spherical Mirrors by Amir 1209. In a curved mirror with a focal length f, we define $m = \frac{A'B'}{AB}$, where AB is the length of the object and $A'B'$ is the length of the image. Also, let d be the distance between the object and its image. Then, we have

$$f = \frac{md}{|m^2 - 1|}.$$

Find all pairs of positive integers (m, d) such that the number $f = md/(m^2 - 1)$ is an integer.

KACY Summer League

2012 Gulf Math Olympiad 1210. Fawzi cuts a spherical cheese completely into (at least three) slices of equal thickness. He starts at one end, making successive parallel cuts, working through the cheese until the slicing is complete. The discs exposed by the first two cuts have integral areas.

a) Prove that all the discs that he cuts have integral areas.

b) Prove that the original sphere had integral surface area if, and only if, the area of the second disc that he exposes is even.

KACY Summer League

Cylindrical and Spherical Coordinates by Mathwizarddude 1211. Use both cylindrical and spherical coordinates to find the volume of the solid E that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 16$.

KACY Summer League

Wisheskernel's Spherical Coordinates 1212. Determine the volume of the region that bounded by

$$\left(x^2 + y^2 + z^2\right)^2 = 2z \cdot \left(x^2 + y^2\right).$$

KACY Summer League

2010 Olympic Revenge 1213. Prove that there exists a set S of lines in the three dimensional space satisfying the following conditions:

a) For each point P in the space, there exist a unique line of S containing P.

b) There are no two lines of S which are parallel.

KACY Summer League

KACY–I 1214. If x and y are real numbers such that

$$2 \sin x \sin y + 3 \cos y + 6 \cos x \sin y = 7,$$

then find the value of

$$\tan^2 x + 2 \tan^2 y.$$

KACY Summer League

2018 Deerfield Math Competition (Extra) 1215. Evaluate the existence of the following limit:

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2y^2z^2}{x^2+y^2+z^2}.$$

KACY Summer League

KACY–I 1216. Describe What is the graph of the equation $x^2+y^2=3z^2$ in the xyz plane?

KACY Summer League

Maximization on Sphere by WeakMathemetician 1217. Given three real numbers x, y, z, such that $x^2+y^2+z^2=1$, maximize

$$x^4+y^4-2z^4-3\sqrt{2}xyz.$$

KACY Summer League

KACY–I 1218. A spherical, three dimensional planet has center at $(0,0,0)$ and radius 20. At any point (x,y,z), on the surface of this planet, the temperature $T := (x+y)^2+(y-z)^2$ degrees. What is the average temperature of the surface of this planet?

KACY Summer League

Area of Spherical Polygon 1219. The radius of the sphere is 1. Spherical polygon has n edges and k inner angles: $\alpha_1, \alpha_2, \ldots, \alpha_k$. Prove that the area of spherical polygon is

$$\alpha_1+\alpha_2+\cdots+\alpha_k-(n-2)\pi.$$

KACY Summer League

2010 USAMTS Round III 1220. The sequences (a_n), (b_n), and (c_n) are defined by $a_0 = 1, b_0 = 0, c_0 = 0$, and

$$a_n = a_{n-1} + \frac{c_{n-1}}{n}, b_n = b_{n-1} + \frac{a_{n-1}}{n}, c_n = c_{n-1} + \frac{b_{n-1}}{n}$$

for all $n \geq 1$. Prove that

$$\left| a_n - \frac{n+1}{3} \right| < \frac{2}{\sqrt{3n}}$$

for all $n \geq 1$. It is still a myth if anyone can solve this problem using spherical coordinates... There is a reason this problem is notorious for being the most difficult USAMTS problem ever!

Part II

Hints & Select Solutions

Solution 3. Simplifies to $(x-1)(x+5)(x^2+x-1)=0$ which has rational solutions $x=1,-5$ and irrational solutions $(-1\pm\sqrt{5})/2$.

Solution 4. Simplifies to $(2x-1)(x+3)(x^2-x-1)=0$ which has rational solutions $x=1/2,-3$ and irrational solutions $(1\pm\sqrt{5})/2$.

Solution 5. Simplifies to $(x-2)^2(x+2)(x-3)=0$ which has solutions $x=\pm 2, 3$.

Solution 6. Simplifies to $(x-4)(x-2)(x+1)(x+2)=0$ which has solutions $x=-1,\pm 2,4$.

Solution by Boris 33. The answer is 12. To find the numbers in the 100^{th} row of the Pascal triangle (the one starting with $1, 100, \dots$) that are not divisible by 3, we need to find the number of coefficients in the polynomial

$$(1+x)^{100} = 1 + \binom{100}{1}x + \binom{100}{2}x^2 + \cdots + x^{100},$$

which are not equal to 0 modulo 3. Note that by Binomial Theorem, and taking modulo 3, one has,

$$(1+x)^3 = 1+3x+3x^2+x^3 \equiv 1+x^3 \pmod 3.$$

and so also

$$(1+x)^9 \equiv (1+x^3)^3 \equiv 1+x^9 \pmod 3,$$

and so on, for any power of 3. Now, $100 = 81 + 2\cdot 9 + 1$. Therefore, modulo 3 one has

$$(1+x)^{100} = (1+x)^{81}\left((1+x)^9\right)^2(1+x) = (1+x^{81})(1+2x^9+x^{18})(1+x).$$

In this product all $2\cdot 3\cdot 2 = 12$ powers of x are different (because every integer can be written in base 3 in a unique way), and the coefficients are all nonzero modulo 3. So, the answer is 12.

Solution by Parviz Shahriari 37. Answer: $(x^2+1)(2x^2+x+2)$.

Solution by Parviz Shahriari 38. Answer: $(x+1)(x^2+x+1)(x^2-x+1)$.

Solution by Parviz Shahriari 39. Answer: $(x^2+\sqrt{2\sqrt{7}+1}x+\sqrt{7})(x^2-\sqrt{2\sqrt{7}+1}x+\sqrt{7})$.

Solution by Parviz Shahriari 40. Answer: $(x^2+\sqrt{2}x+1-\sqrt{2})(x^2-\sqrt{2}x+1+\sqrt{2})$.

Solution by Parviz Shahriari 41. Answer: $(x^{n+1}+x^n+\cdots+x+1)(x^{n-1}+x^{n-2}+\cdots+x+1)$.

Solution by Parviz Shahriari 42. Answer: $(x^2+x+1)(2x^2-x+2)$.

Solution by Parviz Shahriari 43. Begin with calculating the square root of the expression, equal to x^3+x^2+x+11 with a remainder of -100, then use difference of squares to arrive at $(x^3+x^2+x+1)(x^3+x^2+x+21)$. The final answer is $(x+1)(x+3)(x^2+1)(x^2-2x+7)$.

Solution by Parviz Shahriari 44. Answer: $(x^2+x\sqrt{6(\sqrt{2}-1)}+3\sqrt{2})(x^2-x\sqrt{6(\sqrt{2}-1)}+3\sqrt{2})$.

Solution by Parviz Shahriari 45. Answer: $(a^{2n}+a^n+1)(a^{3n}-a^{2n}+1)$.

Solution by Parviz Shahriari 46. Answer: $(x+2)(x+6)(x+4+\sqrt{6})(x+4-\sqrt{6})$.

Solution by Parviz Shahriari 47. Answer: $(x-1)(x+3)^2$.

Solution by Parviz Shahriari 48. Answer: $(x-1)(x+3)(x+7)$.

Solution by Parviz Shahriari 49. Answer: $x(x+1)(x-1)(x-3)(x+2)(x+3)(x-2)$.

Solution by Amir Parvardi 50. This can be represented as $x^2+xy+xy+y^2$, and by factoring x from the first two terms and also factoring y from the last two terms, we get $x(x+y)+(x+y)y$. This last expression would be the same as $x(x+y)+y(x+y)$ in our commutative algebra, and factoring $(x+y)$ results in the First Double–Variable Identity:

$$x^2+2xy+y^2=(x+y)^2.$$

Solution by Amir Parvardi 51. This can be represented as $x^2-xy-xy+y^2$, and by factoring x from the first two terms and also factoring y from the last two terms, we get $x(x-y)-(x-y)y$. This last expression would be the same as $x(x-y)-y(x-y)$ in our commutative algebra, and factoring $(x-y)$ results in the Second Double–Variable Identity:

$$x^2-2xy+y^2=(x-y)^2.$$

Solution by Amir Parvardi 52. Add and subtract xy to find $x^2+xy-xy-y^2$, factor x from the first two terms and y from the other two: $x(x+y)-(x+y)y$. Thus, using commutativity once again as in the Positive and Negative Double–Variable Identities, we arrive at the identity for the difference between two squares:

$$x^2-y^2=(x-y)(x+y).$$

Solution by Amir Parvardi 53. It is necessary for n to be odd for x^n+y^n to be factorizable in real numbers, hence the n^{th} Positive Double–Variable Identity is only defined for odd values of n. Since n is odd, plugging $x=-y$ in x^n+y^n results in zero, meaning that $(x+y)$ is a factor of x^n+y^n. Dividing x^n+y^n by $x+y$ can be done smoothly by choosing the appropriate quotient with alternating positive and negative terms that cancel each other perfectly without leaving any remainder:

$$\frac{x^n+y^n}{x+y}=x^{n-1}-x^{n-2}y+\cdots-xy^{n-2}+y^{n-1}.$$

Again, the above arrangements are possible only for odd n. To finish up, this is the n^{th} Positive Double–Variable Identity:

$$x^n+y^n=(x+y)\left(x^{n-1}-x^{n-2}y+\cdots-xy^{n-2}+y^{n-1}\right).$$

Solution by Amir Parvardi 54. Since $x=y$ yields $x^n-y^n=0$, we know that $(x-y)$ is a factor of x^n-y^n. We can easily see that the quotient of the division of x^n-y^n by $x-y$ contains only positive terms:

$$\frac{x^n-y^n}{x-y}=x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1}.$$

Since all the terms are positive, there would be no problem of matching alternating positive and negative terms that cancel each other, and the n^{th} Negative Double–Variable Identity holds for all positive integers n:

$$x^n - y^n = (x-y)\left(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}\right).$$

Solution by Amir Parvardi 55. Since $x = y$ yields $x^{2^k} - y^{2^k} = 0$, we know that $(x-y)$ is a factor of $x^{2^k} - y^{2^k}$. We also get the quotient as in the n^{th} Negative Double–Variable Identity:

$$\frac{x^{2^k} - y^{2^k}}{x-y} = x^{2^k-1} + x^{2^k-2}y + \cdots + xy^{2^k-2} + y^{2^k-1}.$$

We see that the degree of x in the quotient is $2^k - 1$, which happens to be equal to $1 + 2 + 2^2 + \cdots + 2^{k-1}$, meaning that the leading term in the quotient, x^{2^k-1}, is in fact a product of k terms $x \cdot x^2 \cdot x^{2^2} \cdots x^{2^{k-1}}$, and there must be an identity in this form:

$$x^{2^k-1} + x^{2^k-2}y + \cdots + xy^{2^k-2} + y^{2^k-1} = (x + \dots)(x^2 + \dots)(x^{2^2} + \dots)\cdots(x^{2^{k-1}} + \dots),$$

and the same technique could be applied on y since everything is symmetric, and the missing terms are easily found:

$$\frac{x^{2^k} - y^{2^k}}{x-y} = (x+y)(x^2+y^2)(x^{2^2}+y^{2^2})\cdots(x^{2^{k-1}} + x^{2^{k-1}}),$$

giving us the magical $2^{k^{th}}$ Negative Double–Variable Identity:

$$x^{2^k} - y^{2^k} = (x-y)(x+y)(x^2+y^2)(x^{2^2}+y^{2^2})\cdots(x^{2^{k-1}} + y^{2^{k-1}}).$$

Solution by Sophie Germain 56. Adding $4x^2y^2$ to $x^4 + 4y^4$ completes the square to $(x^2 + 2y^2)^2$. Now just subtract the added term $4x^2y^2$ from $(x^2 + 2y^2)^2$ and use the Difference of Squares Identity to finish:

$$x^4 + 4y^4 = (x^2 + 2y^2 + 2xy)(x^2 + 2y^2 - 2xy).$$

Solution by Sophie Parker 57.

- **Difference of Squares**: It is easy to see that adding x^2y^2 to the given expression completes the square, making it $(x^2 + y^2)^2 = x^4 + 2x^2y^2 + y^4$. The Difference of Squares Identity yields the final factorization:

$$\begin{aligned} x^4 + x^2y^2 + y^4 &= (x^2+y^2)^2 - (xy)^2 \\ &= (x^2 + y^2 - xy)(x^2 + y^2 + xy). \end{aligned}$$

- **Difference of Squares and Cubes**: What if we begin with $x^6 - y^6$? If I apply The Difference of Squares on this expression, I would have on one hand $x^6 - y^6 = (x^3 - y^3)(x^3 + y^3)$, and on the other hand I can apply the n^{th} Negative Double–Variable Identity for $n = 3$ on $x^6 - y^6 = (x^2 - y^2)(x^4 + x^2y^2 + y^4)$.

$$\begin{aligned} x^6 - y^6 &= (x^3 - y^3)(x^3 + y^3) \\ &= (x-y)(x^2 + xy + y^2)\cdot(x+y)(x^2 - xy + y^2) \\ x^6 - y^6 &= (x^2 - y^2)(x^4 + x^2y^2 + y^4) \\ &= (x-y)(x+y)(x^4 + x^2y^2 + y^4). \end{aligned}$$

Therefore,

$$(x-y)(x^2+xy+y^2)\cdot(x+y)(x^2-xy+y^2)=(x-y)(x+y)(x^4+x^2y^2+y^4).$$

Assuming $x\neq\pm y$, we can cancel the terms $x-y$ and $x+y$ from both sides of the equation and obtain the factorization of $x^4+x^2y^2+y^4$ as a consequence:

$$(x^2+xy+y^2)\cdot(x^2-xy+y^2)=x^4+x^2y^2+y^4.$$

Solution 58. Answer: $2(x^2+xy+y^2)^2$.

Solution 59. Answer: $2(x^2-xy+y^2)^2$.

Solution 60. Answer: $3xy(x+y)$.

Solution 61. Answer: $5xy(x+y)(x^2+xy+y^2)$.

Solution 62. Answer: $7xy(x+y)(x^2+xy+y^2)^2$.

Solution 63. Answer:

Solution by Parviz Shahriari 64. The expression is symmetric with respect to x and y, and it becomes zero by plugging $x=y$ and $x=-2y$, so the factorization is $[(x+2y)(2x+y)(x-y)]^2$.

Solution 65. The answer is $(x+y+z)^2$.

Solution 66. The answer is $(x+y+z)(x-y)(y-z)(z-x)$.

Solution 67. The answer is $(xy+yz+zx)(x-y)(y-z)(z-x)$.

Solution 68. The answer is $3(x+y+z)(x^2+y^2+z^2)$.

Solution 69. The answer is $(x+y+z)(xy+yz+zx)$.

Solution by Parviz Shahriari 70. Plug $y=z$ and observe that the result is zero, so $(y-z)$ is a factor. The answer is $(x+z)(y-z)(x+y)$.

Solution by Parviz Shahriari 71. Answer: $(a+2b)(2b-c)(a-c)$.

Solution by Parviz Shahriari 72. Answer: $(a+b+c)(a-b-c)(a+b-c)(a-b+c)$.

Solution 73. Answer: $(xy^2+yz^2+zx^2)(x^2y+y^2x+z^2x)$.

Solution by Parviz Shahriari 74. Answer: $(x-2y)(y-2z)(x+y)$.

Solution by Parviz Shahriari 75. Answer: $3(a+b)(b+c)(c+a)$.

Solution by Parviz Shahriari 76. Answer: $(a+b)(b+c)(c+a)$.

Solution 77. Answer: $(x-y)(y-z)(z-x)$.

Solution 78. Answer: $3(x-y)(y-z)(z-x)$.

Solution 79. Expanding $(x-y)^2+(y-z)^2+(z-x)^2$ results in $2(x^2+y^2+z^2-xy-yz-zx)$. Moreover,

$$(x-y)^4+(y-z)^4+(z-x)^4=2(x^2+y^2+z^2-xy-yz-zx)^2.$$

Solution by Parviz Shahriari 80. Answer: $5(x-y)(y-z)(z-x)(x^2+y^2+z^2-xy-yz-zx)$.

Solution by Parviz Shahriari 81. Answer: $7(x-y)(y-z)(z-x)(x^2+y^2+z^2-xy-yz-zx)^2$.

Solution by Parviz Shahriari 82. Answer: $(x+y+z)(-x+y+z)(x-y+z)(x+y-z)$.

Solution by Parviz Shahriari 83. Answer: $(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$.

Solution by Parviz Shahriari 84. Answer: $(a+b+c)^2(a^2+b^2+c^2-ab-bc-ca)^2$.

Solution by Parviz Shahriari 85. Answer: $5(x+y)(y+z)(z+x)(x^2+y^2+z^2+xy+yz+zx)$.

Solution by Parviz Shahriari 86. Answer: $(x-y)(a-x)(a-y)(x+y+a)$.

Solution by Parviz Shahriari 87. Answer: $(a-b)(b-c)(c-a)(ab+bc+ca)$.

Solution by Parviz Shahriari 88. Answer: $(y+z)(2x-y)(2x+z)(2x+y-z)$.

Solution by Parviz Shahriari 89. Answer: $(x+y)(y+z)(z+x)$.

Solution by Parviz Shahriari 90. Answer: $(x+y+z)(xy+yz+zx)$.

Solution by Parviz Shahriari 91. Answer: $3(y^2+z^2)(x^2+y^2)(x-z)(x+z)$.

Solution by Parviz Shahriari 92. Answer: $-(x+y+z)(x-y)(y-z)(z-x)$.

Solution by Parviz Shahriari 93. Answer: $(x^2-y)(y^2-z)(z^2-x)$.

Solution by Parviz Shahriari 95. The term in the first bracket simplifies to $(ax+by)^2+(ay+bx)^2$, and the term in the second bracket is $(ay+bx)(ax+by)$. Use the difference of squares to see that the given expression factorizes into $(a-b)^2(a+b)^2(x-y)^2(x+y)^2$.

Solution by Parviz Shahriari 96. Answer: $f(2x+1)=4x^2-1$.

Solution by Parviz Shahriari 97. Answer: $f(f(x))=\frac{x^4+3x^2+1}{x(x^2+1)}$.

Solution by Parviz Shahriari 98. Answer: $f(f(f(x)))\cdot f(x)=-1$.

Solution by Parviz Shahriari 99. Answer: $f(x)=\frac{x^2}{(x+1)^2}$.

Solution by Parviz Shahriari 100. Answer: $x^3-3x+4=(x+2)^3-6(x+2)^2+9(x+2)+2$.

Solution by Parviz Shahriari 101. Answer: $g(x)\equiv 0$.

Solution by Parviz Shahriari 102. Answer: $f(x)=10+5\cdot 2^x$.

Solution by Parviz Shahriari 103. Answer: (a) Arithmetic, (b) Geometric.

Solution by Parviz Shahriari 104. Answer: $f(x)+f(y)=2f(x)f(y)$.

Solution by Parviz Shahriari 105. Answer: (a) $z=x+y$; (b) $z=\frac{xy}{x+y}$; (c) $z=\frac{x+y}{1-xy}$; (d) $z=\frac{x+y}{1+xy}$.

Solution by Parviz Shahriari 106. Answer: $f(f(x))=\frac{x-1}{x}$ and $f(f(f(x)))=x$.

Solution by Parviz Shahriari 107. Answer: $f(x) = x^2 - 5x + 7$.

Solution by Parviz Shahriari 108. Answer: $f(x) = x^2 - 2$.

Solution by Parviz Shahriari 109. Answer: $f(x) = \frac{1}{x}\left(1 + \sqrt{1+x^2}\right)$.

Solution by Parviz Shahriari 110. Answer: $f(x) = x^2 - x + 1$.

Solution by Parviz Shahriari 111. One can easily prove by induction that $f_n(x) = \frac{x}{\sqrt{1+nx^2}}$.

Solution by Parviz Shahriari 112. Answer: $f(2x^2-1) = 2f(x)$ and $f(4x^3-3x) = 3f(x)$.

Solution by Parviz Shahriari 113. Answer: $f(x) = \frac{x^2}{x^2-1}$.

Solution by Parviz Shahriari 114. Answer: $\alpha = 3$ and $f(\alpha+x)\cdot f(\alpha-x) = \frac{1}{4}$.

Solution by Parviz Shahriari 115. It is easy to see that $f(a) = a, f(b) = b$, and $f(c) = c$, so that the equation $f(x) = x$ has at least three roots. However, the equation $f(x) - x = 0$ is quadratic and having three roots implies that it is always zero, so that $f(x) = x$ for all x.

Solution by Parviz Shahriari 116. Change x to $-x$ in the given equation to easily arrive at $f(x) = \frac{b}{a-1}\sqrt[n]{x}$.

Solution by Parviz Shahriari 117. Answer: (a) $x = -2, 4$, (b) $x = -2, 2, 4, 10$.

Solution by Parviz Shahriari 118. Answer: $f(x,y) = x^2 \cdot \frac{1-y}{1+y}$.

Solution by Parviz Shahriari 119. By induction, we arrive at:

$$F_n(x) = \begin{cases} -2^{n-1}\left(\sin\frac{1}{2}\right)^{n-1}\cdot\sin\left(x+\frac{n-1}{2}\right), & \text{if } n = 4k, \\ 2^{n-1}\left(\sin\frac{1}{2}\right)^{n-1}\cdot\cos\left(x+\frac{n-1}{2}\right), & \text{if } n = 4k+1, \\ 2^{n-1}\left(\sin\frac{1}{2}\right)^{n-1}\cdot\sin\left(x+\frac{n-1}{2}\right), & \text{if } n = 4k+2, \\ -2^{n-1}\left(\sin\frac{1}{2}\right)^{n-1}\cdot\cos\left(x+\frac{n-1}{2}\right), & \text{if } n = 4k+3. \end{cases}$$

Solution by Parviz Shahriari 120. Answer: $f(x) = \frac{1}{x^2} - 2$.

Solution by Parviz Shahriari 121. Answer: $p(1) = -1$.

Solution by Parviz Shahriari 122. Answer: $a^n(x+1)p(x) = (x+a^n)p(ax)$.

Solution by Parviz Shahriari 123. Answer:

$$p(a+b)\cdot p(b+c)\cdot p(c+a) = 8p\left(\frac{a+b+c}{2}\right)\cdot(p(0))^2,$$
$$p(-a)\cdot p(-b)\cdot p(-c) = 8\left(p(a+b+c)\right)^2\cdot p(0).$$

Solution by Parviz Shahriari 124. Answer: (a) $6(x-1)^4 + 43(x-1)^3 + 76(x-1)^2 - 25(x-1) - 100$, (b) $x = 2, -3, -\frac{2}{3}, -\frac{3}{2}$.

Solution 125. Plug in $x = \pm 1$ in $p(x)$ and observe that the given sum is even.

Solution 126. Answer: all integers n work, and m would be $m = n^2 + n(a+1) + b$.

Solution 127. Answer: $a = c = -1$ and $b = 1$, so that $a^{2023} + b^{2023} + c^{2023} = -1$.

Solution 128. Using $p_{2k}(x) = p_k(x)^2 + 2$, we deduce the rationality of $p_8(x)$ and $p_{10}(x)$ from the rationality of $p_4(x)$ and $p_5(x)$, respectively. From $p_4(x)p_2(x) = p_2(x) + p_6(x)$ and $p_8(x)p_2(x) = p_{10}(x) + p_6(x)$, it follows that $p_2(x)$ and $p_6(x)$ are also rational. Finally, $p_5(x)p_1(x) = p_4(x) + p_6(x)$ implies the rationality of $p_1(x)$.

Solution 129. Show that $4(d - b) = c^2 - a^2$, proving that $a - c$ is even.

Solution by CRMO 2012 130. From the given equations we can obtain:

$$x^3y + y^3z + z^3x - 3(xy + yz + zx) - xyz(x + y + z) = 0,$$

to get $x^3y + y^3z + z^3x = -9$.

Solution by Parviz Shahriari 131. Answer: $x^4 - 3x^3 + 8x - 24 = (x^2 - 2x + 4)(x^2 - x - 6)$.

Solution by Parviz Shahriari 132. Answer: $nx^{n+1} - (n+1)x^n + 1 = (x-1)^2\left(nx^{n-1} + (n-1)x^{n-2} + \right.$

Solution by Parviz Shahriari 133. Answer: $m = 1$ gives $x^4 + a^2x^2 + a^4 = (x^2 - ax + a^2)(x^2 + ax + a^2$.

Solution by Parviz Shahriari 134. Answer: $a = b = 1$ gives $(x-2)^n + (x-1)^n - 1 = (x^2 - 3x + 2)\left(P(x)Q(x)\right)$, where

$$P(x) = (x-2)^{2n-2} - (x-2)^{2n-3} + \cdots - (x-2) + 1,$$
$$Q(x) = (x-1)^{n-2} + (x-1)^{n-3} + \cdots + (x-1) + 1.$$

Solution by Parviz Shahriari 135.

(a) $-\frac{5}{2}x + \frac{7}{2}$,

(b) $A\frac{x-b}{a-b} + B\frac{x-a}{b-a}$.

Solution by Parviz Shahriari 136.

(a) $-\frac{1}{12}x^2 - \frac{5}{4}x - \frac{1}{6}$,

(b) $A\frac{(x-b)(x-c)}{(a-b)(a-c)} + B\frac{(x-c)(x-a)}{(b-c)(b-a)} + C\frac{(x-a)(x-b)}{(c-a)(c-b)}$.

Solution by Parviz Shahriari 137.

(a) $p(x) = x^4 + 6x^3 + 7x^2 - x + 2$,

(b) $q(x) = x^3 + 1$.

Solution by Parviz Shahriari 138.

(a) $p(-1) = p(\pm i) = 0$, so the remainder is zero.

(b) Let $t(x) = (x-1)q(x)$ and observe that $t(\alpha) = 0$ if $\alpha^n = 1$ (and $\alpha \neq 1$), so that the remainder is zero.

Solution by Parviz Shahriari 139. Answer: $p(x) = (x^2-1)(x^2-a^2)$, so that the remainder must be zero.

Solution by Parviz Shahriari 140.

(a) $(\alpha+\beta)(\alpha\beta+1) = q$ and $\alpha^2+\beta^2+\alpha\beta = -p-1$.

(b) $\alpha = 5$ and $\beta = -4$.

Solution by Parviz Shahriari 141.

(a) Note that $p_3(x,y,z) = 3(x+y)(y+z)(z+x)$, and that $p_m(x,-x,z) = 0$ if m is odd. The answer is all $m = 6k+3$, where $k \geq 0$ is an integer.

(b) For all a and b such that $b \mid a$.

Solution by Parviz Shahriari 142. Answer: all odd n work because

$$\frac{p_n(x)}{q_n(x)} = \frac{(x^{2n}-1)(x-1)}{(x^n-1)(x^2-1)} = \frac{x^n+1}{x+1}.$$

Solution by Parviz Shahriari 143. Write $p(x,y) = (x-y)q(x,y)$ and prove that $q(x-y)$ is also divisible by $x-y$, meaning that the required remainder is zero.

Solution by Parviz Shahriari 144. Answers:

(a) $m = 3k \pm 1$.

(b) $(m,n) = (3k+1, 3k+), (3k+2, 3k+1)$.

Solution by Parviz Shahriari 145. Write $q(x,y) = (ax-by)(ay-bx)$ and note that $p(x,y) - s(x,y)$ is divisible by $q(x,y)$, so that the remainder is zero.

Solution by Parviz Shahriari 146. Answer: $p(x) = \frac{1}{9}(x^2-4)^2$.

Solution by Parviz Shahriari 148. Answer: $p(x)x^3 - 3x$.

Solution by Problem Premier for the Olympiad 149. Answer: $p = q = r$, so $p + q - 2r = 0$.

Solution by Regional Math Olympiad 2010 150. Answer: $a = b = c$, so $a + b - 2c = 0$.

Solution by CRMO 2015 151. Answer: $b - st$ is a root of $x^2 + ax + b - \alpha\beta = 0$, so the remainder of division is zero.

Solution by INMO 2012 152. Reduce the equation to $f_{n+1}(x) = xf_n(x) - f_{n-1}(x)$ which makes it obvious to see that f_n must be a polynomial:

$$f_n(x) = x^n - \binom{n-1}{1}x^{n-2} + \binom{n-2}{2}x^{n-4} - \binom{n-3}{3}x^{n-6} + \cdots$$

Solution by RMO 2000 153. Factorize the expression into $(a-x^2-x)(a-x^2+x-1)$ which has all real roots if and only if $a \geq 3/4$.

Solution by CRMO 2003 154. Answer: $a = 1$ and $a = 3$ are the only values.

Solution by INMO 2016 155. Answer: (II) always implies $a = b = c$, but (I) need not imply $a = b = c$. There are three other possibilities for a, b, c other than $a = b = c$ if we assume statement (I). One such case is $a = b = \lambda c$, where λ is the positive root of $x^2 - x - 1 = 0$.

Solution 157. The answer is $a - b = 21$.

Solution by Parviz Shahriari 158. Answers:

(a) $x^{13} + 1$.

(b) $x^3 - x^2 - x + 1$.

Solution by Parviz Shahriari 159. The denominator is a perfect square equal to $(x^3 + 3x^2 - x - 1)^2$ and the numerator is divisible by x^3+3x^2-x-1 with a quotient of x^3+2x-1, so that the fraction reduces to

$$\frac{x^3 + 2x - 1}{x^3 + 3x^2 - x - 1}.$$

Solution by CRMO 2002 160. Put away the obvious solution $x = 1$ (with multiplicity three). Since $x - 1$ is a common factor of all three terms in the equation, we can divide all the terms by $(x-1)^3$ and get the equation $(x+2)^3 + (2x+1)^3 = 27(x+1)^3$ and solve the equation by expanding to get the other three solutions of the original equation: $x = -1, -2, -1/2$.

Solution Inspired by RMO 2013 161. Answer: (a) $a = b = 1$, (b) $R(n) = n - 1$.

Solution by Parviz Shahriari 254.

(a) The equation $(x + a)^4 + (x + b)^4 = c$ can be re-written as

$$\left[\left(x + \frac{a+b}{2}\right) + \frac{a-b}{2}\right]^4 + \left[\left(x + \frac{a+b}{2}\right) - \frac{a-b}{2}\right]^4 = c,$$

which can be written as $(t + \alpha)^4 + (t - \alpha)^4 = c$, where $t = x + (a + b)/2$ and $\alpha = (a - b)/2$. The last equation expands to $2t^4 + 12\alpha^2t^2 + (2\alpha^4 - c) = 0$, which can be solved as a quadratic equation in $y = t^2$.

(b) Let $t = x + 3$, so that the equation becomes $(t - 2)^6 + (t + 2)^6 = 730$. Expand to get $t^6 + 60t^4 + 240t^2 - 301 = 0$, which can be written as a cubic equation in $y = t^2$: $y^3 + 60y^2 + 240y - 301 = 0$. Note that $y = 1$ is the only positive solution to this equation (why?) which leads to $t = \pm 1$ and thus $x = -2$ and $x = -4$ are the only real solutions of the original equation.

(c) Use the same parameters $t = x + (a + b)/2$ and $\alpha = (a - b)/2$ in a similar way to arrive at a degree–n polynomial equation.

Solution by Parviz Shahriari 255.

(a) The equation $(x + a)(x + b)(x + c)(x + d) = m$ can be re-written as

$$(x^2 + (a + b)x + ab)(x^2 + (c + d)x + cd) = m,$$

and since $a + b = c + d$, we can set $t = x^2 + (a + b)x = x^2 + (c + d)x$ as the variable of the equation to obtain $(t + ab)(t + cd) = m$, which is a simple quadratic equation and easy to solve.

(b) Re-write the equation as $(x + 2)(x + 4)(x - 3)(x - 5) = 72$, and since $2 - 3 = 4 - 5 = -1$, rearrange it as $[(x + 2)(x - 3)]\,[(x + 4)(x - 5)] = 72$. Write the last equation as $(x^2 - x - 6)(x^2 - x - 20) = 72$, which after setting $t = x^2 - x$ becomes $t^2 - 26t + 48 = 0$. This has two solutions $t = 2$ and $t = 24$, which in turn give the following solutions for x:

$$-1, 2, \frac{1 + \sqrt{97}}{2}, \frac{1 - \sqrt{97}}{2}.$$

Solution by Parviz Shahriari 256. The answer is $(a^2 + 3ad) + d^2$. In the product of four terms, group a with $a + 3d$ and $a + d$ with $a + 2d$ to find

$$\begin{aligned} a(a + d)(a + 2d)(a + 3d) &= (a(a + 3d))\,((a + d)(a + 2d)) \\ &= \left(a^2 + 3ad\right)\left(a^2 + 3ad + 2d^2\right) \\ &= \left(a^2 + 3ad\right)^2 + 2d^2\left(a^2 + 3ad\right). \end{aligned}$$

It is clear now that if we add d^4 to the product it will be the square of $(a^2 + 3ad) + d^2$:

$$\left(a^2 + 3ad\right)^2 + 2d^2\left(a^2 + 3ad\right) + d^4 = \left(a^2 + 3ad + d^2\right)^2.$$

Solution by Parviz Shahriari 257. The answer is $k > 9a^4/16$. For simplicity, let $t = x + b$ to get

$$t(t + a)(t + 2a)(t + 3a) = k,$$

and add a^4 to both sides of the equation to find

$$(t^2 + 3at + a^2)^2 = k + a^4,$$

which results in the following four roots:

$$x = -b + \frac{-3a \pm \sqrt{5a^2 \mp \sqrt{k + a^4}}}{2}.$$

In order for all four roots to be real and simple (of multiplicity 1), $5a^2 - \sqrt{k+a^4}$ must be positive, which leads to $k > 9a^4/16$.

Solution by Parviz Shahriari 258. There are four real roots for the equation. To expand, group x with $x-2$ and $x-4$ with $x+2$, so that we can set a new variable $t = x^2 - 2x$ and arrive at $t^3 - 7t^2 - 8t + 66 = 0$. There are three values of t that satisfy the equation: $t = -3$ leads to complex solutions for x (roots of $x^2 - 2x + 3 = 0$); $t = 5 + \sqrt{3}$ leads to $x = 1 \pm \sqrt{6+\sqrt{3}}$; and $t = 5 - \sqrt{3}$ gives $x = 1 \pm \sqrt{6-\sqrt{3}}$. Therefore, the real solutions are $x = 1 \pm \sqrt{6 \pm \sqrt{3}}$.

Solution by Parviz Shahriari 259. Add and subtract $2(ax+b)^2(cx+d)^2$ to the left side of the given equation to complete the square:

$$\left((ax+b)^2 + (cx+d)^2\right)^2 - 2(ax+b)^2(cx+d)^2 = (\alpha x + \beta)^4.$$

The first term on the left side of the above equation can be written as

$$\begin{aligned}(ax+b)^2 + (cx+d)^2 &= (a+c)x + (b+d))^2 - 2(ax+b)(cx+d)\\ &= (\alpha x+\beta)^2 - 2(ax+b)(cx+d).\end{aligned}$$

After simplification, the equation becomes

$$2(ax+b)(cx+d)\left((ax+b)(cx+d) - 2(\alpha x+\beta)^2\right) = 0,$$

which is easy to solve.

Solution by Parviz Shahriari 260. The equation can be written as $2(2-x)(2x-1)(4x^2-x+4) = 0$, which has real roots $x = 2$ and $x = 1/2$.

Solution by Parviz Shahriari 261. The answer is $a < -5/4$. Since the equation is of degree 4 in x and hard to solve, we try to solve it as a quadratic equation in a. Write it as

$$a^2 - 3x^2 \cdot a + (2x^4 + x^3 - 2x^2 + 2x - 1) = 0.$$

We can use the quadratic formula to solve the above equation. We just need the Δ:

$$\begin{aligned}\Delta &= 9x^4 - 4(2x^4 + x^3 - 2x^2 + 2x - 1)\\ &= x^4 - 4x^3 + 8x^2 - 8x + 4\\ &= (x^2 - 2x + 2)^2.\end{aligned}$$

Therefore, the solutions in terms of a can be calculated from the quadratic formula as follows:

$$a_{1,2} = \frac{3x^2 \pm (x^2 - 2x + 2)}{2}.$$

These will lead to $a_1 = x^2 + x - 1$ and $a_2 = 2x^2 - x + 1$, which can now be solved as quadratic equations in x:

$$x^2 + x - (a+1) = 0 \qquad \text{and} \qquad 2x^2 - x + (1-a) = 0.$$

Thus, we get the four solutions

$$x_{1,2} = \frac{-1 \pm \sqrt{4a+5}}{2} \qquad \text{and} \qquad x_{3,4} = \frac{1 \pm \sqrt{4a-3}}{2}.$$

If we need all four roots not to be real, we need $a < -5/4$.

Solution by Parviz Shahriari 262. Let $a = \sqrt{3}$ and write the equation as a quadratic in a:

$$(1-x)a^2 - x \cdot a + x^3 = 0.$$

The solutions will be

$$x_1 = \sqrt{3} \qquad \text{and} \qquad x_{2,3} = \frac{-\sqrt{3} \pm \sqrt{3+4\sqrt{3}}}{2}.$$

Solution by Parviz Shahriari 263. Let $bx+c = t$ and write the equation as a quadratic in t:

$$t^2 + x^2(a-1)t + x^4(a^2-1) = 0.$$

The solutions will be

$$t_{1,2} = \frac{-x^2(2a-1) \pm x^2\sqrt{5-4a}}{2},$$

which lead to

$$(1-2a \pm \sqrt{5-4a})x^2 - 2bx - 2c = 0.$$

Solution 264. We provide hints and final answers:

1. To solve $(x-2)(x+1)(x+6)(x+9)+108 = 0$, write it as $(x^2+7x-18)(x^2+7x+6)+108 = 0$ whose roots are $x = 0, -7, \dfrac{-7 \pm \sqrt{97}}{2}$.
2. For $x^4 + (x+\sqrt{2})^4 = 68$, write the equation as
$$\left[\left(x + \frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}}{2}\right]^2 + \left[\left(x + \frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2}\right]^2 = 68,$$
which has two real roots $x = \sqrt{2}$ and $-2\sqrt{2}$ as well as two imaginary roots.
3. Write $x^6 + (x+2)^6 = 2$ as $[(x+1)-1]^6 + [(x+1)+1]^6 = 2$, and show that there is only one real root $x = -1$ (which happens to be a double root) and four imaginary roots.
4. To solve $(x+3)^3+(x+5)^3 = 8$, let $x+4 = t$ and the equation becomes $t^3+3t-4 = 0$, which can be written as $(t-1)(t^2+t+4) = 0$. The real solution is $x = -3$ and there are two imaginary solutions.
5. For $(\sqrt{x}+1)^4 + (\sqrt{x}-3)^4 = 256$, choose $y = \sqrt{x}-1$ to find the only real solution $x = 9$.

6. For solving $(x^2+3x+2)(x^2+6x+12) = 120$, factorize the left-hand side as $(x+1)(x+2)(x+3)(x+4)$ and rearrange the equation into $(x^2+5x+4)(x^2+5x+6) = 120$ to find real roots $x = 1$ and 6, plus two imaginary roots.

7. For solving $x^4+2x^3+2x^2+x = 42$, write it as $(x^2+x)^2+(x^2+x)-42 = 0$ and treat it as a quadratic in x^2+x. The solutions are $2, -3$, and $\dfrac{-1+3i\sqrt{3}}{2}$.

8. The equation $x^4+6x^3+7x^2-6x = 1$ may be written as $(x^2+3x-1)^2-2 = 0$ which has the following solutions:
$$\frac{-3\pm\sqrt{13-4\sqrt{2}}}{2} \quad \text{and} \quad \frac{-3\pm\sqrt{13+4\sqrt{2}}}{2}.$$

9. The common denominator may be taken in the left-hand side of the equation $\dfrac{\sqrt[n]{a+x}}{a}+\dfrac{\sqrt[n]{a+x}}{x} = b\sqrt[n]{x}$ to obtain $(a+x)\sqrt[n]{a+x} = abx\sqrt[n]{x}$. Raise both sides of the latter equation to power of n to find $(a+x)^{n+1} = a^nb^nx^{n+1}$, and then take the $(n+1)^{th}$ root from both sides to find $x = \dfrac{a}{\sqrt[n+1]{a^nb^n}-1}$ for even n and $x = \dfrac{a}{\pm\sqrt[n+1]{a^nb^n}-1}$ for odd n.

10. The solutions to $\sqrt[3]{a+\sqrt{x}}+\sqrt[3]{a-\sqrt{x}} = \sqrt[3]{b}$ is $x = a^2-\dfrac{(b-2a)^3}{271}$.

11. In order to solve $(x^2+2x-12)^2 = x^2(3x^2+2x-12)$, let $t = 2x-12$ and write the equation as a quadratic in t: $t^2+x^2\cdot t-2x^4 = 0$ with roots $t = x^2$ and $t = -2x^2$. The roots of the first equation are $1\pm i\sqrt{11}$ and the roots of the second one are 2 and -3.

12. For solving $(2x^2-x-6)^2+3(2x^2+x-6)^2 = 4x^2$, let $t = 2x^2-6$ and find the roots $-2, 3/2, \pm\sqrt{3}$.

13. The equation $\dfrac{(x^2+1)^2}{x(x+1)^2} = \dfrac{9}{2}$ may be transformed to the positive reciprocal equation $2x^4-9x^3-14x^2-9x+2 = 0$ with solutions $3\pm2\sqrt{2}$ and $(-3+i\sqrt{7})/4$.

14. For solving $3x^4-20x^3+45x^2-40x+12 = 0$, divide both sides by x^2 and let $t = x+2/x$ to find solutions $x = 1, 2, 3, 2/3$.

15. Write $x^3-3abx+a^3+b^3 = 0$ as $x^3-3abx+(a+b)^3-3ab(a+b) = 0$ to see clearly that one of the roots must be $x = -(a+b)$ with two other imaginary roots $(a+b\pm i\sqrt{3}(a-b))/2$.

16. It is clear that if $(x^2-16)(x-3)^2+9x^2 = 0$, then $x \neq 3$ and we can divide both sides of the equation by $(x-3)^2$ to find
$$x^2+\frac{(3x)^2}{(x-3)^2} = 16 \implies \left(\frac{x^2}{x-3}\right)^2-6\left(\frac{x^2}{x-3}\right)-16 = 0,$$
and find solutions $-1\pm\sqrt{7}$ as well as $4\pm i2\sqrt{2}$.

17. Write the equation $(x^2-4)(x+1)(x+4)(x+5)(x+8)+476=0$ as

$$(x^2+6x-16)(x^2+6x-5)(x^2+6x+8)+476=0,$$

and let $t=x^2+6x$ to arrive at $t^3-3t^2-168t-164=0$ which factorizes into $(t+1)(t^2-4t-164)=0$, so that the solutions must be the roots of the three equations $x^2+6x-t=0$ where $t=-1$ or $t=-2\pm\sqrt{168}$.

18. To solve $\sqrt[m]{(x+1)^2}-\sqrt[m]{(x-1)^2}+\dfrac{3\sqrt[m]{x^2-1}}{2}=0$, divide both sides by $\sqrt[m]{x^2-1}$ (assuming $m\neq\pm1$) and let $y=\sqrt[m]{(x+1)/(x-1)}$ to obtain $2y^2+3y-2=0$ with roots $y=2$ and $y=-1/2$. If $y=2$, then $x=(2^m+1)/(2^m-1)$. If $y=-1/2$, then m must be odd and we would have $x=(1-2^m)/(1+2^m)$.

19. The roots of the equation $(a^2-a)^2(x^2-x+1)^3=(a^2-a+1)^3(x^2-x)^2$, assuming $a\neq0$ and $a\neq1$ are given by

$$x\in\left\{a,\frac{1}{a},1-a,\frac{1}{1-a},\frac{a-1}{a},\frac{a}{a-1}\right\}.$$

Solution 265. The solutions are $x=3\pm\sqrt{a+9}$ and $2\pm\sqrt{a+6}$.

Solution 266. Let $a=\sqrt{3}$ and the equation becomes $x^3+2ax^2+a^2x+a-1=0$ which may be solved as a cubic equation in a, giving solutions $a=1-x$ and $-\dfrac{x^2+x+1}{x}$. The solutions for x after plugging $a=\sqrt{3}$ are $x=1-\sqrt{3}$ and $\dfrac{-(\sqrt{3}+1)\pm\sqrt{12}}{2}$.

Solution 267. The five answers are $x_1=3$, and

$$x_{2,3}=\frac{-1+\sqrt{5}\pm\sqrt{30+6\sqrt{5}}}{4}\qquad\text{and}\qquad x_{4,5}=\frac{-1-\sqrt{5}\pm\sqrt{30-6\sqrt{5}}}{4}.$$

Solution 268. Use the argument

$$\frac{u}{v}=\frac{t}{z}\iff\frac{u+v}{u-v}=\frac{t+z}{t-z}$$

on the given expression to obtain

$$\frac{(x-a)^2+(x-b)^2}{2(x-a)(x-b)}=\frac{2c^2+2}{2c^2-2},$$

and applying the same argument again implies

$$\frac{(x-a)^2+(x-b)^2+2(x-a)(x-b)}{(x-a)^2+(x-b)^2-2(x-a)(x-b)}=c^2.$$

Simplify the latter equation after taking its square root to finally find the solutions

$$x_1=\frac{ac-bc+a+b}{2}\qquad\text{and}\qquad x_2=\frac{bc-ac+a+b}{2}.$$

Solution 269. We will use three variables α, β, γ defined by:

$$a+b=\alpha+\beta, \quad a+c=\alpha+\gamma,$$
$$c+d=\alpha-\beta, \quad b+d=\alpha-\gamma.$$

Then,

$$\alpha=\frac{a+b+c+d}{2}, \quad \beta=\frac{a+b-c-d}{2}, \quad \gamma=\frac{a-b+c-d}{2}.$$

a) $y=x+\alpha$, so that the equation becomes

$$\frac{(y+\beta)^5+(y-\beta)^5}{(y+\gamma)^5+(y-\gamma)^5}=\frac{\alpha^2}{\gamma^2}.$$

Since $y \neq 0$, this may be reduced to a quartic equation in y

$$(\gamma^2-\beta^2)y^4+5(\gamma^2\beta^4-\beta^2\gamma^4)=0,$$

and assuming $\gamma \neq \pm\beta$, it gives $y^4=5\beta^2\gamma^2$. The solutions for x are

$$x=-\frac{a+b+c+d}{2} \pm \frac{\sqrt[4]{5(a+b-d-d)^2(a-b+c-d)^2}}{2}.$$

b) Let $y=x+\alpha$, so that the equation becomes

$$\frac{(y+\beta)^5+(y-\beta)^5}{(y+\gamma)^5+(y-\gamma)^5}=\frac{\beta^5}{\gamma^5}.$$

Since $y \neq 0$, this may be reduced to a quartic equation in y

$$(\gamma^5-\beta^5)y^4+10(\gamma^3-\beta^3)\beta^2\gamma^2y^2+5(\gamma-\beta)\beta^4\gamma^4=0,$$

which is in fact a quadratic equation in y^2, thus easy to solve with the quadratic formula.

Solution 270. The sextic equation $x^7+7^7=(x+7)^7$ factorizes into

$$x(x+7)(x^4+2\cdot 7x^3+3\cdot 7^2x^2+2\cdot 7^3x+7^4)=0,$$

whose six roots are $x_1=0$ and $x_2=-7$ (real roots), as well as two double imaginary roots:

$$x_3=x_4=\frac{7(-1+i\sqrt{3})}{2} \qquad \text{and} \qquad x_5=x_6=\frac{7(-1-i\sqrt{3})}{2}.$$

Solution 271. We have seen this factorization before:

$$(x+y)^7 - x^7 - y^7 = 7xy(x+y)(x^2+xy+y^2)2.$$

Writing the original equation as

$$\left[(x+a+b)^7 - x^7 - (a+b)^7\right] + \left[(a+b)^7 - a^7 - b^7\right] = 0.$$

Applying the mentioned identity on the last equation and heavily simplifying it results in a quadratic equation in $x^2 + (a+b)x$ that is solvable:

$$[x^2+(a+b)x]^2 + (2a^2+3ab+2b^2)[x^2+(a+b)x] + (a^2+ab+b^2)^2 = 0.$$

Solution 272. Divide both sides of the equation by x^3 and let $y = 2x + 3/x$ to obtain the equation $(y-3)(y-5)(y+4) = 0$. Solving for y and then for x, we find that the only real roots are $x = 1$ and $3/2$.

Solution 273. Define $y = 6x + 7$ and write the equation as a quartic equation in y: $y^4 - y^2 - 72 = 0$. The final solutions are: real roots $x_1 = -2/3$, $x_2 = -5/3$, and imaginary roots

$$x_{3,4} = \frac{-7 \pm 2i\sqrt{2}}{6}.$$

Solution 274. The solutions to the three equations are:

1. Write it as

$$\left(\frac{x^2}{x-1}\right)^2 - 2\left(\frac{x^2}{x-1}\right) - 8 = 0,$$

whose four roots are $x_1 = x_2 = 2$ and $x_{3,4} = -1 \pm \sqrt{3}$.

2. Completing the square on the left-hand side of the equation and then factorizing it leads to $(x+2)^2(x^2-2x-2) = 0$ whose roots are $x_1 = x_2 = -2$ and $x_{3,4} = 1 \pm \sqrt{3}$.

3. Again, the left-hand side must be completed to find the solutions $x_1 = x_2 = \sqrt{2} - 1$ and $x_3 = x_4 = -(\sqrt{2}+1)$.

Solution 275. Let $y = x + 2$ and simplify the equation to find

$$y^4 + 5y^3 + 7y^2 + 2y = 0,$$

which factorizes into $y(y+2)(y^2+3y+1)$. The roots for x are:

$$x_1 = -2, \quad x_2 = -4, \quad x_{3,4} = -\frac{7 \pm \sqrt{5}}{2}.$$

Solution 276. The equation after heavy simplification becomes

$$4x^3 - 6(n+1)x^2 + 2(n+1)(2n+1)x - n(n+1)^2 = 0,$$

and by changing the variable to $y = x + (n+1)/2$, it becomes $4y^3 + (n^2-1)y = 0$. The solutions for x are (when $n > 1$):

$$x_1 = \frac{n+1}{2} \qquad \text{and} \qquad x_{2,3} = \frac{n+1 \pm \sqrt{1-n^2}}{2},$$

where x_1 is real and $x_{2,3}$ are imaginary. For $n = 1$, the equations has a triple root $x = 1$.

Solution by Parviz Shahriari 284.

(a) The fraction on the left side of the equation reminds us of the coefficients of $\tan 3\alpha$, thus making $x = \tan\alpha$ a plausible change of variables which would yield an equation for α in the form of $\alpha = k\pi/3 + \pi/9$. In short, since the function $\tan x$ is periodic with the least period π,

$$\frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha} = \sqrt{3} \implies \tan 3\alpha = \sqrt{3} = \tan\frac{\pi}{3}$$
$$3\alpha = k\pi + \frac{\pi}{3} \implies \alpha = \frac{k\pi}{3} + \frac{\pi}{9}.$$

The solutions will be

$$x = \tan\left(\frac{k\pi}{3} + \frac{\pi}{9}\right) \qquad \forall k \in \mathbb{Z}.$$

Since the original equation is a third-degree polynomial in x, it has at most three real solutions in x which will be found by plugging $k = 0, 1, 2$ in the above equation. These are

$$\begin{aligned} k = 0 &\implies x_1 = \tan 20^\circ, \\ k = 1 &\implies x_2 = \tan 80^\circ, \\ k = 2 &\implies x_3 = \tan 140^\circ = -\tan 40^\circ. \end{aligned}$$

(b) Using Vieta's formulas for sums and products related to the three roots p, q, r of the polynomial equation $x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} = 0$, we find the equations $p+q+r = 3\sqrt{3}$, $pq + qr + rp = 3$, and $pqr = \sqrt{3}$, and the values $\{p, q, r\} = x_{1,2,3}$ as calculated in part (a) would result in the trigonometric identities on the even multiples of 20°:

$$\begin{aligned} 3\sqrt{3} &= \tan 20^\circ - \tan 40^\circ + \tan 80^\circ, \\ 3 &= \tan 20^\circ \tan 40^\circ + \tan 40^\circ \tan 80^\circ - \tan 20^\circ \tan 80^\circ, \\ \sqrt{3} &= \tan 20^\circ \tan 40^\circ \tan 80^\circ. \end{aligned}$$

Solution by Parviz Shahriari 285.

(a) The fraction on the left side of the equation reminds us of the coefficients of $\cos 5\alpha$, thus making $x = \cos\alpha$ a plausible change of variables which would yield an equation for α in the form of $\alpha = 2k\pi/5 \pm \pi/15$. In short, since the function $\cos x$ is periodic with the least period 2π,

$$2(16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha) = 1 \implies 2\cos 5\alpha = 1 = 2\cos\frac{\pm\pi}{3}$$
$$5\alpha = 2k\pi \pm \frac{\pi}{3} \implies \alpha = \frac{2k\pi}{5} \pm \frac{\pi}{15}.$$

The solutions will be

$$x = \cos\left(\frac{2k\pi}{5} \pm \frac{\pi}{15}\right) \qquad \forall k \in \mathbb{Z}.$$

Since the original equation is a fifth-degree polynomial in x, it has at most five real solutions in x which will be found by plugging $k = 0, 1, 2$ in the above equation. These are

$$k = 0 \implies x_1 = \cos 12^\circ,$$
$$k = 1 \implies \begin{cases} x_2 &= \cos 84^\circ = \sin 6^\circ, \\ x_3 &= \cos 60^\circ = \frac{1}{2}. \end{cases}$$
$$k = 2 \implies \begin{cases} x_4 &= \cos 156^\circ = -\cos 24^\circ, \\ x_5 &= \cos 132^\circ = -\cos 48^\circ. \end{cases}$$

(b) Using Vieta's formulas for sums and products related to the five roots p, q, r, s, t of the polynomial equation $32x^5 - 40x^3 + 10x - 1 = 0$, we find the equations $p+q+r+s+t = 0$, $pqrts = 1/32$, and $1/p+1/q+1/r+1/s+1/t = (qrst+prst+pqst+pqrt+pqrs)/pqrst = 10$. The values $\{p, q, r, s, t\} = x_{1,2,3,4,5}$ as calculated in part (a) would result in the trigonometric identities on the first couple even and first couple odd multiples of 12°:

$$0 = \cos 12^\circ - \cos 24^\circ - \cos 48^\circ + \cos 60^\circ + \cos 84^\circ,$$
$$\frac{1}{32} = \cos 12^\circ \cos 24^\circ \cos 48^\circ \cos 60^\circ \cos 84^\circ,$$
$$10 = \frac{1}{\cos 12^\circ} - \frac{1}{\cos 24^\circ} - \frac{1}{\cos 48^\circ} + \frac{1}{\cos 60^\circ} + \frac{1}{\cos 84^\circ}.$$

Solution 286. Divide both sides of the equation by the first term on the left hand side to find

$$\left(\frac{1-a^2}{1+a^2}\right)^x + \left(\frac{2a}{1+a^2}\right)^x = 1.$$

The trick is to set $a = \tan(\alpha/2)$ to get $(\cos\alpha)^x + (\sin\alpha)^x = 1$, whose only solution is clearly $x = 2$.

Solution 287. If we define $z = \cos\phi + i\sin\phi$ where $i = \sqrt{-1}$, then it is easy to verify that $1/z = \cos\phi - i\sin\phi$, so that we can find $\cos\phi$ in terms of z:

$$\cos\phi = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

Using double-angle formulas, we can compute $\cos 2\phi$ and $\cos 4\phi$ as well:

$$\cos 2\phi = 2\cos^2\phi - 1 = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right),$$
$$\cos 4\phi = 2\cos^2 2\phi - 1 = \frac{1}{2}\left(z^4 + \frac{1}{z^4}\right).$$

Thus, the problem reduces to solving the equation

$$\left(z + \frac{1}{z}\right)^2 + \left(z^2 + \frac{1}{z^2}\right)^2 - \left(z + \frac{1}{z}\right)\left(z^2 + \frac{1}{z^2}\right)\left(z^4 + \frac{1}{z^4}\right) = 3.$$

After simplification, this turns out to be $1 + z^{15} + (z+1)(z^{13} + z) = 0$ and finally $(1 + z + z^2)(1 + z^{13}) = 0$. If $1 + z + z^2 = 0$, then $z = -1/2 \pm i\sqrt{3}/2$, which gives $\phi = 2k\pi \pm 2\pi/3$. Otherwise, $1 + z^{13} = 0$ and by De Moivre's formula, $\cos 13\phi + i\sin 13\phi = -1$, giving $\phi = 2k\pi + (2n+1)\pi/13$ for $0 \le n < 12$ and $n \neq 6$.

Solution 288. The solutions are $\alpha = k\pi$ and $\alpha = k\pi + (-1)^k\pi/6$.

Solution 289. We may solve the equation $x^3 - 3x + 1 = 0$ in two ways:

1. First, observe that $f(0)$ and $f(2)$ are positive, whereas $f(1)$ and $f(-2)$ are negative. Therefore, all three of equation's roots are real and they lie in intervals $(-2, 0), (0, 1), (1, 2)$. Let x_1 be the smallest root and therefore in the interval $(-2, 0)$. Note that

$$\left(\frac{1}{x_1 - 1}\right)^3 - 3\left(\frac{1}{x_1 - 1}\right) + 1 = \frac{-(x_1^3 - 3x_1 + 1)}{(1 - x_1)^3} = 0,$$

so $1/(1 - x_1)$ would be another root of the original equation. Since $-2 < x_1 < 0$, we find that $1/3 < 1/(1 - x_1) < 1$, which means that $1/(1 - x_1) < x_3$, and so $x_2 = 1/(1 - x_1)$ is the second largest root of the equation. Now,

$$\begin{aligned} x_2^2 - x_1 &= \frac{1}{(x_1 - 1)^2} - x_1 = \frac{1 - x_1 + 2x_1^2 - x_1^3}{(1 - x_1)^2} \\ &= \frac{-x_1^3 + 3x_1 - 1 + 2(1 - x_1)^2}{(1 - x_1)^2} = 2. \end{aligned}$$

2. Let $x = 2\cos\alpha$ and the equation becomes

$$\begin{aligned} x^3 - 3x + 1 &= 8\cos^3\alpha - 6\cos\alpha + 1 \\ &= 2\cos 3\alpha + 1. \end{aligned}$$

So, $x^3 - 3x + 1 = 0$ is equivalent to $\cos 3\alpha = -1/2$ with solutions $\alpha = 2k\pi/3 \pm 2\pi/9$. These will lead to the following solutions:

$$x_1 = -2\cos\frac{\pi}{9}, \qquad x_2 = 2\cos\frac{4\pi}{9}, \qquad x_3 = 2\cos\frac{2\pi}{9}.$$

Now we can calculate $x_2^2 - x_1$ is

$$4\cos^2\frac{4\pi}{9} + 2\cos\frac{\pi}{9} = 2\left(1 + \cos\frac{8\pi}{9} + \cos\frac{\pi}{9}\right) = 2.$$

Solution 291. Expand $x^2 - 1$ and $x^2 - 4$, then write the equation as

$$(x^3 - 3x - 2) + (x-1)(x+1)\sqrt{(x-2)(x+2)}.$$

Now, $x^3 - 3x - 2$ becomes zero for both $x = -1$ and $x = 2$, and we may factorize it as $(x+1)^2(x-2)$. These two numbers turn out to be the only solutions to the original equation as well.

Solution 296. The answers are $x = -1$ and $x = 35$.

Solution 297. The answer is $x + y = 0$.

Solution 300. Using the identity $(a \pm b)^3 = a^3 \pm b^3 \pm 3ab(a \pm b)$, we can raise both sides of the equation $\sqrt[3]{u} \pm \sqrt[3]{v} = a$ to the power of 3 to arrive at the given expression.

Solution 301. The two sides must be cubed and simplified to obtain $9\sqrt[3]{6x^2 - 5x - 6} = 28 - 5x$, which after being cubed again becomes $125x^3 + 2274x^2 + 8115x - 26326 = 0$, whose only real solution is $x = 2$.

Solution 302. After rationalizing the equation, the solution is $x = (a-1)^2/4$ when $a \geq 1$ and there are no solutions when $a < 1$.

Solution 303. There are no solutions for $a < -1$ or $0 \leq a \leq 1$. Otherwise, when $-1 < a < 0$ or $a > 1$, the solution is $x = a/(\sqrt[3]{a^2} - 1)$.

Solution 304. The answer is $x = 17$.

Solution 305. After rationalizing, the equation becomes $(x-1)[12x(2x-3) - 27(x-1)^2] = 0$ whose solutions are $x = 1$ and $x = 3$.

Solution 306. The solutions are $x_1 = a$, $x_2 = b$, and $x_3 = (a+b)/2$.

Solution 307. The solutions are $x_1 = 1$ and $x_2 = -27/8$.

Solution 308. The solutions are $x_1 = -a$ and $x_2 = 1 - a$.

Solution 309. Let $y = \sqrt[5]{(7x-3)^3}$ to find the solutions $x_1 = 2/7$ and $x_2 = 5$.

Solution 310. First, prove that $x^2 < 1/b^2$ and $x^2 < 1/a^2$ must happen for the equation to have a solution. In that case, the trivial solution is $x_1 = 0$, and if $1/2 \leq a/b < 1$, the other two roots are:

$$x_2 = \sqrt{\frac{2a-b}{a^2b}} \qquad \text{and} \qquad x_3 = -\sqrt{\frac{2a-b}{a^2b}}.$$

Solution 311. Let $\sqrt[5]{16 + \sqrt{x}} = u$ and $\sqrt[5]{16 - \sqrt{x}} = v$, so that $u + v = 2$ and raising the equation to power of 5 results in:

$$u^5 + v^5 + 5uv(u+v)\left[(u+v)^2 - 3uv\right] + 10u^2v^2(u+v) = 32.$$

Plug in $u+v=2$ and simplify to get $uv(4-uv)=0$, and the only real solution of the equation comes from $v=0$, giving $x=256$.

Solution 312. The trivial solution is $x_1=0$ and the other two solutions (with appropriate a and b) are

$$x_{2,3}=a^{2k/(2k-n)}\left(\sqrt{b}\pm\sqrt{b-a}\right)^{2n/(n-2k)}.$$

Solution 313. We divide the solutions in two cases:

- If n is even, we must have $a>0$ and $-a<x\le a$ and the solutions are

$$x=\pm\frac{a}{\sqrt{1+a^{2n/(n+1)}}}.$$

- If n is odd, we must have $a\neq 0$ and the solutions are the same as before:

$$x=\pm\frac{a}{\sqrt{1+a^{2n/(n+1)}}}.$$

Solution 314. The only solution is

$$x=\frac{a}{\left[1+a^{2n/3}\right]^n}.$$

Solution 315. For $p\ge 2$, the solution is $x=2\sqrt{p-1}$ and there are no solutions when $p<2$.

Solution 316. Let $a=\sqrt{3}$ and turn the equation into a quadratic in a:

$$a^2-(2x+1)a+(x^2-\sqrt{x})=0.$$

The only solution is $x=(7+\sqrt{13})/2$.

Solution 317. Let $a=\sqrt{5}$ and form a quadratic equation in a like in Problem 316. The only solution is

$$x=\frac{1+\sqrt{1+4\sqrt{5}}}{2}.$$

Solution 318. Let $u=\sqrt[5]{a+\sqrt{x}}$ and $v=\sqrt[5]{a-\sqrt{x}}$ to arrive at $u^5+v^5=2a$. Reduce the equation to $uv(uv-\sqrt[5]{4a^2})=0$, implying either $uv=0$ or $uv=\sqrt[5]{4a^2}$. There is a real solution $x=a^2$ if $uv=0$ and the other case $uv=\sqrt[5]{4a^2}$ results in imaginary solutions for x.

Solution 319. If $a<b$, the only solution is $x=a$ and if $a>b$, the only solution is $x=b$.

Solution 320. The only solution is $x=-2$.

Solution 321. Define y so that $y\sqrt{x} = x - 1$ to get the equation $x^2 + 1 = x(y^2 + 2)$. Writing this as an equation in y, we find $\sqrt{2(y^2 + 2)} = 2(a - y)$ for $a \geq y$. After rationalizing and solving the latter equation, we find that the only solution for the original equation is

$$x = \frac{1}{4}\left(2a - \sqrt{2(a^2+1)} + \sqrt{6a^2 + 6 - 4a\sqrt{2(a^2+1)}}\right).$$

Solution 322. Rationalizing the equation results in $9x^2 - 196x + 356 = 0$ whose solution $x = 2$ is the only one that works in the original equation.

Solution 323. The answer is $x = 3/2$.

Solution 324. The answer is $x = -2$.

Solution 325. Prove that the function $f(x) = 2\sqrt{x-1} + \sqrt[3]{x}$ is an increasing function, so that the equation has exactly one solution $x = 1$.

Solution 326. The answer is $x = 2$.

Solution 327. The only solution is $x = 2$.

Solution 331. Dividing both sides of the equation by x^2 gives us an expression that has $x + \dfrac{1}{x}$ as a factor:

$$2\left(x^2 + \frac{1}{x^2}\right) - 13\left(x + \frac{1}{x}\right) + 24 = 0.$$

Choose $t = x + \dfrac{1}{x}$ to find $x^2 + \dfrac{1}{x^2} = t^2 - 2$ and find $2(t^2 - 2) - 13t + 24 = 0$, which has solutions $t = 4$ and $t = 5/2$. The equation $x + \dfrac{1}{x} = 4$ has real solutions $x = 2 \pm \sqrt{3}$ and the roots of the equation $x + \dfrac{1}{x} = 5/2$ are 2 and 1/2.

Solution 335. This is a negative reciprocal equation and it must be arranged as a polynomial in $x - \dfrac{1}{x}$. Dividing both sides of the original equation by x^2 and rearranging, we find

$$2a\left(x^2 + \frac{1}{x^2}\right) - (2a^2 + 3a - 2)\left(x - \frac{1}{x}\right) + (3a^2 - 4a - 3) = 0.$$

Choose $t = x - \dfrac{1}{x}$ to find $x^2 + \dfrac{1}{x^2} = t^2 + 2$; yielding $2a(t^2+2) - (2a^2+3a-2)t + (3a^2-4a-3) = 0$, which has solutions $t = 3/2$ and $t = a - \dfrac{1}{a}$. The equation $x - \dfrac{1}{x} = \dfrac{3}{2}$ has real solutions $x = 2$ and $x = -\dfrac{1}{2}$, and the roots of the equation $x - \dfrac{1}{x} = a - \dfrac{1}{a}$ are a and $-\dfrac{1}{a}$.

Solution 336. Dividing both sides of the original equation by x^2 and rearranging, we find

$$2\left(x^2 + \frac{4}{x^2}\right) - 15\left(x + \frac{2}{x}\right) + 35 = 0.$$

Choose $t = x + \dfrac{2}{x}$ to find $x^2 + \dfrac{4}{x^2} = t^2 - 4$; yielding $2(t^2 - 4) - 15t + 35 = 0$, which has solutions $t = 9/2$ and $t = 3$. The equation $x + \dfrac{2}{x} = \dfrac{9}{2}$ has real solutions $x = 4$ and $x = \dfrac{1}{2}$, and the roots of the equation $x + \dfrac{2}{x} = 3$ are 1 and 2.

Solution 337. Dividing both sides of the original equation by x^2 and rearranging, we find

$$2\left(x^2 + \frac{9}{x^2}\right) + 7\left(x - \frac{3}{x}\right) - 34 = 0.$$

Choose $t = x - \dfrac{3}{x}$ to find $x^2 + \dfrac{9}{x^2} = t^2 + 6$; giving $2(t^2 + 6) + 7t - 34 = 0$, which has solutions $t = 2$ and $t = -11/2$. The equation $x - \dfrac{3}{x} = 2$ has real solutions $x = -1$ and $x = 3$, and the roots of the equation $x - \dfrac{3}{x} = -\dfrac{11}{2}$ are $\dfrac{1}{2}$ and -6.

Solution 338. If α is a root of $x^3 - x + 1 = 0$, then it is easy to see that $1/\alpha$ must be a root of $x^3 - x^2 + 1 = 0$, and also

$$x^5 + x + 1 = (x^3 - x^2 + 1)(x^2 + x + 1).$$

Solution 340. The answer is $5 \le x < 6$.

Solution 341. The answer is $\dfrac{4}{7} \le x < \dfrac{3}{5}$.

Solution 342. The answer is $\dfrac{\sqrt{6}}{2} \le x < \dfrac{\sqrt{10}}{2}$.

Solution 343. First, we will investigate in which intervals the equation cannot happen. Define $f(x) = (x^3 - 2)/3$. Clearly, if $f(x) \ge x + 1$ or $f(x) \le x - 1$, the equation $\lfloor x \rfloor = \lfloor f(x) \rfloor$ cannot happen. The inequality $f(x) \ge x + 1$ simplifies to $x^2(x - 3) \ge -3x^2 + 3x + 5$, which has $x \ge 3$ in its solutions. The inequality $f(x) \le x - 1$ reduces to $x^2(x + 2) \le 2x^2 + 3x - 1$ and it has $x \le -2$ among its solutions. Therefore, the original equation can have its solutions in the interval $-2 < x < 3$:

$$-4 < x < 0 \qquad \text{and} \qquad \sqrt[3]{5} < x < \sqrt[3]{11}.$$

Solution 344. Let $k = 3y/2x$ and observe that k must be an integer, and the equation becomes

$$1 + k = \left\lfloor \frac{4k^2}{9} \right\rfloor \implies 1 + k \le \frac{4k^2}{9} < 2 + k.$$

The solutions are $3 \le k < 3.6$ and $-1.2 < k \le 0.75$. Since k must be an integer, $k = -1$ and $k = 3$ are the only solutions, resulting in $y = 2x$ and $y = -2x/3$.

Solution 345. The solutions are $x_1 = 7/15$ and $x_2 = 4/5$.

Solution 346. The given equation is the same as $|x+3|+|x-2|+|2x-8|=9$, and the short answer is $x \in [2,4]$.

Solution 347. The answer is $x=-1$ and $x=-3$.

Solution 348. We know that the product of absolute values of several terms is equal to the absolute value of the product of the same terms, so we multiply out the terms and expand to reach two possible equations:

$$x^3+7x^2-36=\pm(x^3+14x^2+49x+36),$$

whose five (two quadratic and three cubic) roots are

$$x_{1,2}=\frac{-49\mp\sqrt{385}}{14}, \quad x_3=0, \quad x_4=-7, \quad x_5=-\frac{7}{2}.$$

Solution 349. Let the roots be $x_1, x_2, x-3$, so that their sum is $-a$. Furthermore, since the roots are in an arithmetic progression, $2x_2=x_1+x_3$. Therefore, $3x_2=-a$ and this means that $x_2=-a/3$ is a root of the equation. Plugging $x=-a/3$ in the equation yields the needed equation for a,b,c:

$$2a^3-9ab+27c=0.$$

Solution 350. Let the three roots of the cubic $x^3+px^2+qx+r=0$ be $x=\alpha,\beta,\gamma$, so that by Viète's Formulas, $\alpha+\beta+\gamma=-p$ and $\alpha\beta+\beta\gamma+\gamma\alpha=q$. Find the difference p^2-3q, which is said to be negative in the statement, in terms of α,β,γ:

$$\begin{aligned}0>p^2-3q&=(\alpha+\beta+\gamma)^2-3(\alpha+\beta+\gamma)\\&=\alpha^2+\beta^2+\gamma^2-(\alpha+\beta+\gamma).\end{aligned}$$

Multiplying both sides of the inequality $\alpha^2+\beta^2+\gamma^2-(\alpha+\beta+\gamma)<0$ by 2, we obtain $(\alpha-\beta)^2+(\beta-\gamma)^2+(\gamma-\alpha)^2<0$, which cannot happen if α,β,γ are real numbers, so at least one of them is imaginary. Since imaginary roots of any polynomial equation are paired, the original equation has two imaginary roots and one real root.

Solution 351. The first task is just the cheat-code for the second one. For each positive integer n in task b), define $f_n(x)=x^n-x-n$, and realize that $f_n(1)=-n<0$ whereas $f_n(2)>0$, so that the original equation has exactly one root in the interval $[1,2]$, which we may call x_n. Now follow task a) to finish the proof.

Solution 352. The trick is to write the polynomial as a sum of non-negative polynomials of lesser degrees. In particular for $x^4-4x^3+12x^2-24x+24$, we have

$$x^4-4x^3+12x^2-24x+24=(x^2-2x)^2+8\left[\left(x-\frac{3}{2}\right)^2+\frac{3}{4}\right],$$

whose first term is non-negative and the second term at least $3/4$, so their sum cannot be zero if x is a real number.

Solution 353. If a, b, c are the triangle's side-lengths and h_a, h_b, h_c are the length of heights, we would have

$$\begin{cases} h_a + h_b + h_c = k, \\ h_a h_b + h_b h_c + h_c h_a = q, \\ h_a h_b h_c = z. \end{cases}$$

Since the height-lengths are positive, so must be k, q, z. Moreover, we have $2S = ah_a = bh_b = ch_c$. Calculate the semiperimeter p:

$$p = \frac{a+b+c}{2} = S\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right) = \frac{Sq}{z}.$$

Using Heron's Formula $S = \sqrt{p(p-a)(p-b)(p-c)}$ and simplifying, we arrive at the required equation.

Solution 354. If α is the angle facing the base a, we would have $a = 2b\sin(\alpha/2)$. Simplifying, we get

$$2\left(4\sin^3\alpha - 3\sin\frac{\alpha}{2}\right) + \sqrt{3} = 0,$$

or simply $\sin(3\alpha/2) = \sqrt{3}/2$, which has solutions $\alpha = 2\pi/9$ and $\alpha = 4\pi/9$.

Solution 355. Adding up the given equations, we find $2c^4 = c^2(a^2 + b^2)$, or simply $c^2 = (a^2 + b^2)/2$. Substitute this for c^2 in $a^4 = b^4 + c^4 - b^2c^2$ to obtain

$$a^4 = b^4 + \frac{a^4 + b^4 + 2a^2b^2}{4} - \frac{a^2b^2 + b^4}{2}.$$

Finally, the equation reduces to $a = b$ and similarly $a = c$, after which everything is trivial.

Solution 356. Given $a < b + c$, we need to prove that $\sqrt[n]{a} < \sqrt[n]{b} + \sqrt[n]{c}$. Raising both sides of the latter inequality to power of n, we need to prove $a < \left(\sqrt[n]{b} + \sqrt[n]{c}\right)^n$. Now,

$$\left(\sqrt[n]{b} + \sqrt[n]{c}\right)^n = b + c + (\text{positive terms}) > b + c > a,$$

as required.

Solution 357. The discriminant of the equation is $D = -4(p^3 + q^3) - 23q^2$, which would be divisible by 23 if $p^3 + q^3$ is a multiple of 23. We know also that D equals the square of the product of the pairwise difference of the roots, and since D is divisible by the prime 23, at least one of the elements in the product (square of difference of two roots) must be divisible by 23.

Solution 358. For the necessity condition, if we plug in a purely imaginary root $x = \alpha i$ into $f(x) = ax^3 + bx^2 + cx + d$, the real values and the coefficient of $i = \sqrt{-1}$ must be zero:

$$\begin{cases} d - b\alpha^2 = 0, \\ c\alpha - a\alpha^3 = 0. \end{cases}$$

To prove that $ad = bc$ and $ac > 0$ is sufficient for $f(x)$ to have a purely imaginary root, multiply both sides of the original equation by $a \neq 0$ and use $ad = bc$ to simplify the result into $(ax^2 + c)(ax + b) = 0$. The roots of $ax^2 + c = 0$ are purely imaginary: $x = \pm i\frac{\sqrt{ac}}{a}$.

Solution 359. Define $f(x) = x^n + x^{n-1} + \cdots + x^2 + x - 1$ for $n \geq 2$ and note that $f(0) = -1 < 0$ and $f(1) = n - 1 > 0$, so there must be a real root between 0 and 1, call it x_n. Furthermore, since $x_n < 1$ for all $n \geq 2$ and the sum $x_n^n + x_n^{n-1} + \cdots + x_n^2 + x_n$ may be written as a geometric series with both initial term and the common ratio equal to $x_n < 1$. According to the formula for infinite sum of geometric series with common ratio less than 1, the limit of the series when $n \to \infty$ would be $x_n/(1 - x_n)$, which must be equal to 1 according to the original equation. This gives a limit of $1/2$ for x_n as $n \to \infty$.

Solution 360. We will form a cubic equation whose roots are

$$y_1 = \frac{m\alpha + n}{m\alpha - n}, \quad y_2 = \frac{m\beta + n}{m\beta - n}, \quad y_3 = \frac{m\gamma + n}{m\gamma - n}.$$

If $y = (mx + n)/(mx - n)$, then $x = n(y + 1)/(m(y - 1))$. Simplifying, the equation becomes

$$n^3(y + 1)^3 + pm^2n(y + 1)(y - 1)^2 + qm^3(y - 1)^3 = 0.$$

The required sum $S = y_1 + y_2 + y_3$ can then be calculated as

$$S = \frac{3qm^3 + 2pm^2n - 3n^3}{n^3 + pm^2n + qm^3}.$$

Solution 361. Define $f(x) = x^3 + 21x^2 + 140x - 300$ and assume $f(\alpha) = f(2\alpha) = 0$ for some α. Simplifying the system of equations $f(\alpha) = 0$ and $f(2\alpha) = 0$, we find $\alpha = 5$, so that the roots of the equation will be $x = 5, 6, 10$.

Solution 362. Using Viète's Formulas, we find $a = \pm 10$.

Solution 363. The first part is easy to verify, and we may use it to write the polynomial in the second part as

$$f(x) = (2x - 1)^4 - 20(2x - 1)^2 + 91 = 0,$$

whose roots are

$$x_{1,2} = \frac{1 \pm \sqrt{7}}{2} \qquad \text{and} \qquad x_{3,4} = \frac{1 \pm \sqrt{13}}{2}.$$

Solution 364. Using Viète's Formulas, we find $m = 18$, and the four roots x_1, x_2, x_3, x_4 may be calculated easily from $x_1 + x_2 = 4$, $x_1x_2 = -1$ and $x_3 + x_4 = 4$ and $x_3x_4 = 3$.

Solution 365. Let a, b, c be the side-lengths of the triangle. By Viète's Formulas, we find

$$a + b + c = 12, \quad ab + bc + ca = 47, \quad abc = 60.$$

Using Heron's formula for the area of triangle, we can find the area:

$$\begin{aligned} S^2 &= p(p - a)(p - b)(p - c) \\ &= -p^4 + (ab + bc + ca)p^2 - abcp \\ &= -6^4 + 47 \cdot 6^2 - 60 \cdot 6 = 36. \end{aligned}$$

So, the area is $S = 6$.

Solution 376. Note that $S_0 = x_1^0 + x_2^0 = 2$ and $S_1 = x_1 + x_2 = 2$, and the recursive formula for the sum of quadratic roots would be $2S_n - 4S_{n-1} + S_{n-2} = 0$. Plugging $n = 2, 3, 4$ in the latter equation, we find $S_2 = 3, S_3 = 5$, and $S_4 = 17/2$, respectively. Therefore, the answer is $S_4 = x_1^4 + x_2^4 = \frac{17}{2}$.

Soluion by Parviz Shahriari 377. The final answer is 57. Let x_1, x_2, x_3 be the roots of the equation $x^3 - 3x + 1 = 0$ and define $S_p = x_1^p + x_2^p + x_3^p$. First, multiply the equation by x^3 and sum it up when plugging $x = x_1, x_2, x_3$, to obtain $S_6 = 3S_4 - S_3$, and second, multiply the equation by x and sum it up to obtain $S_4 = 3S_2 - S_1$. This last one can be written as

$$\begin{aligned} x_1^4 + x_2^4 + x_3^4 &= 3(x_1^2 + x_2^2 + x_3^2) - (x_1 + x_2 + x_3) \\ &= 3\left[(x_1 + x_2 + x_3)^2 - 2(x_1x_2 + x_2x_3 + x_3x_1)\right] - (x_1 + x_2 + x_3). \end{aligned}$$

Since $x_1 + x_2 + x_3 = 0$ and $x_1x_2 + x_2x_3 + x_3x_1 = -3$, we find $S_4 = 18$ and finally $S_6 = 57$.

Solution 378. Multiplying the equation by x^2 will result in $x^5 - x^3 + x^2 = 0$, and if we add the three equations formed by plugging $x = x_1, x_2, x_3$ in the latter equation, we find

$$x_1^5 + x_2^5 + x_3^5 = (x_1^3 + x_2^3 + x_3^3) - (x_1^2 + x_2^2 + x_3^2).$$

It is now easy to find from the original equation that $x_1^3 + x_2^3 + x_3^3 = -3$ and $x_1^2 + x_2^2 + x_3^2 = 2$, so that the answer would be $x_1^5 + x_2^5 + x_3^5 = -5$.

Solution 379. Assuming $x_1 = 1$, we would have $x_1x_2 = x_3$, $x_2x_3 = x_1$, and $x_1x_2 = x_3$ and the given equality is obvious.

Solution 380. The answer is:

$$a = \sqrt[20]{2^{20} - 1}, \quad b = -\frac{\sqrt[20]{2^{20} - 1}}{2}, \quad p = -1, \quad q = \frac{1}{4}.$$

Solution by Parviz Shahriari 381. Using the trick of Sum of Powers of Cubic Roots (Problem 375, if we define $S_p = x_1^p + x_2^p + x_3^p$, then $S_{n+3} + S_{n+1} + S_n = 0$. In order to find S_{11}, we put $n = 0, 1, 2, \ldots, 8$ into the latter equation, initiating with $S_0 = x_1^0 + x_2^0 + x_3^0 = 3$, $S_1 = x_1 + x_2 + x_3 = 0$, and $S_2 = x_1^2 + x_2^2 + x_3^2 = -2(x_1x_2 + x_2x_3 + x_3x_1) = -2$. We can find $S_3, S_4, \ldots, S_9$ from the equations $S_{n+3} + S_{n+1} + S_n = 0$ with appropriate n, calculated below:

$$S_3 = -3, \quad S_4 = 2, \quad S_5 = 5, \quad S_6 = 1, \quad S_7 = -7, \quad S_8 = -6, \quad S_9 = 6.$$

Finally, for $n = 8$, we have $S_{11} + S_9 + S_8 = 0$. Since $S_9 + S_8 = 0$, we have $S_{11} = 0$ and we are done.

Solution 384. The easiest way is to set $y = x^2$ or $x = \pm\sqrt{y}$ into the equation:

$$(\sqrt{y})^3 + 2(\sqrt{y})^2 - \sqrt{y} + 5 = 0,$$

which simplifies to $\sqrt{y}(1 - y) = 5 + 2y$. Squaring both sides and writing in descending powers of y,

$$y^3 - 6y^2 - 19y - 25 = 0.$$

The more tiresome and time-consuming method would be to find $y_1+y_2+y_3, y_1y_2+y_2y_3+y_3y_1$, and $y_1y_2y_3$ in terms of the same functions in x_i, assuming $y_i = x_i^2$ for $i = 1, 2, 3$.

Solution 385. Define $y_i = 1/(2x_i - 1)$ and create a degree–4 polynomial equation in y with roots y_i for $i = 1, 2, 3, 4$. If $y = 1/(2x - 1)$, then $x = (y + 1)/(2y)$. Plugging this last expression into the given equation,

$$\frac{(y+1)^4}{16y^4} - \frac{4(y+1)^2}{4y^2} + \frac{y+1}{2y} + 3 = 0.$$

This will become, after simplification,

$$41y^4 - 20y^3 - 10y^2 + 4y + 1 = 0.$$

The required sum $S = \sum_{i=1}^{4} y_i$ equals the sum of roots of the above equation: $S = 20/41$.

Solution 386. If $y = 1/(x - 1)$, then $x = (y + 1)/y$. Plugging this change of variable into the original equation and simplifying, we arrive at

$$(y+1)^n + y(y+1)^{n-1} + y^2(y+1)^{n-2} + \cdots + y^{n-1}(y+1) + y^n = 0.$$

The coefficient of y^{n-1} in the above polynomial is $n(n + 1)/2$ and the coefficient of y^n is $n + 1$, so that the required sum in question is equal to $-(n + 1)/2$.

Solution 387. We need to form a polynomial whose roots are $y_i = 1/(x_i^2 - 1)$ for $i = 1, 2, 3$, and it is easy to find the polynomial by the change of variable $x = \sqrt{(y + 1)/y}$ in the original equation. The new polynomial would be

$$100y^3 + 60y^2 + 8y - 1 = 0,$$

so that the sum of the roots of the equation is $S = -3/5$.

Solution 388. The general answer is

$$(y - 4)(y^3 + y^2 - 4y + 1).$$

If $\alpha = 1$, then $\alpha^4 + \alpha^6 + \alpha^7 + \alpha^9 = 4$ and the linear equation $y - 4 = 0$ is the solution. If $\alpha \neq 1$, since $\alpha^{13} - 1 = 0$, we have

$$\alpha^{12} + \alpha^{11} + \cdots + \alpha^2 + \alpha + 1 = 0.$$

Define

$$\begin{cases} y_1 &= \alpha^4 + \alpha^6 + \alpha^7 + \alpha^9, \\ y_2 &= \alpha^2 + \alpha^3 + \alpha^{10} + \alpha^{11}, \\ y_3 &= \alpha + \alpha^5 + \alpha^8 + \alpha^{12}. \end{cases}$$

It is easy to see that $y_1 + y_2 + y_3 = -1$ and that $y_1y_2 = -1 + y_1$, $y_2y_3 = -1 + y_2$, and $y_3y_1 = -1 + y_3$. Therefore, $y_1y_2 + y_2y_3 + y_3y_1 = -4$. Finally, we can find the product

$$y_1y_2y_3 = (-1 + y_1)y_3 = -y_3 + y_1y - 3 = -y_3 + (-1 + y_3) = -1.$$

Finally, the equation with y_1, y_2, y_3 as its roots is obtained:

$$y^3 + y^2 - 4y + 1 = 0.$$

Solution 389. Let x_1 and x_2 be the roots of $ax^2 + bx + c = 0$. We need $x_1^4 + x_2^4$ and $x_1^4 x_2^4$. For the sum of fourth powers of the roots,

$$\begin{aligned} x_1^4 + x_2^4 &= (x_1 + x_2)^4 - 2x_1x_2(2x_1^2 + 3x_1x_2 + 2x_2^2) \\ &= \frac{b^4}{a^4} - \frac{4b^2c}{a^3} + \frac{2c^2}{a^2}. \end{aligned}$$

Also, $x_1^4 x_2^4 = c^4/a^4$, and the quadratic equation that has x_1^4 and x_2^4 would be

$$a^4x^2 - (b^4 - 4ab^2c + 2a^2c^2)x + c^4 = 0.$$

Solution 390. We need to find the sum of the roots, sum of their pairwise product, and their product. Divide and multiply the sum by $2\sin(\pi/7)$ to find

$$\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}.$$

To find the sum of pairwise product of the roots, use the product to sum trigonometric formulas to obtain

$$\cos\frac{\pi}{7}\cos\frac{3\pi}{7} + \cos\frac{3\pi}{7}\cos\frac{5\pi}{7} + \cos\frac{5\pi}{7}\cos\frac{\pi}{7} = -\frac{1}{2}.$$

It is also not difficult to see that the product of the roots is $-1/8$. So, the cubic equation is

$$8x^3 - 4x^2 - 4x + 1 = 0.$$

Solution 391. Let y be the variable for the required quartic polynomial, so that we can assume $y = x^2$ or $x = \pm\sqrt{y}$ and plug in $x = \sqrt{y}$ in the given equation, rationalize and simplify to find

$$y^4 - 2y^3 + 23y^2 + y + 25 = 0.$$

Solution 392. If we remove $x = 1$ from the roots of $x^n - 1$, we find that $x_2, x_3, \ldots, x_n$ would be the roots of

$$x^{n-1} + x^{n-2} + \cdots + x + 1 = 0,$$

which means

$$x^{n-1} + x^{n-2} + \cdots + x + 1 = (x - x_2)(x - x_3)\cdots(x - x_n).$$

Plugging $x = 1$, we find that the answer is n.

Solution 393. The main equation is $(p - p')(p'q - pq') = (q - q')^2$, and the required quadratic equation is

$$(p'q - pq')x^2 + (q^2 - q'^2)x + qq'(p - p') = 0.$$

Solution 394. The common root can be found by subtracting one equation from the other: $x = -(q-q')/(p-p')$. Plugging in this root, we find the relationship between p, q, p', q':

$$(q-q')^3 = (p-p')^2(pq' - p'q).$$

Solution 395. The answer is $m = 2$. Put $x = \alpha$ in $x^2 - (m+2)x + 3 = 0$ and its double $x = 2\alpha$ would be a root of $x^2 - x - m = 0$. So, $4\alpha^2 - 2\alpha - m = 0$ and $\alpha^2 - (m+2)\alpha + 3 = 0$. This will lead to two equations

$$\alpha = \frac{m+12}{4m+6} \quad \text{and} \quad \alpha^2 = \frac{m^2+2m+6}{4m+6}.$$

Conclude that

$$\left(\frac{m+12}{4m+6}\right)^2 = \frac{m^2+2m+6}{4m+6},$$

which leads to $4m^3 + 13m^2 + 12m - 108 = 0$, having only $m = 2$ as a real root.

Solution 397. Once the greatest common factor is found to be $x^2 - x - 1$, it is easy to solve the problem:

$$\begin{cases} (x^2 - x - 1)(2x^2 + x - 1) &= 0, \\ (x^2 - x - 1)(x^2 + 2x - 1) &= 0. \end{cases}$$

The common roots are $\frac{1\pm\sqrt{5}}{2}$ and the first equation has two more roots -1 and $1/2$, whereas the second equation has $-1 \pm \sqrt{2}$.

Solution 398. The final remainder of division of f by the greatest common factor would be $(1-m)x^2 + 3(m-1)x$, which would be zero iff $m = 1$, leading to the cubic equation $x^3 - 4x^2 + x + 2 = 0$ whose roots are the common roots of $f(x) = 0$ and $g(x) = 0$. It is not difficult to simplify the equations to find the roots:

$$\begin{cases} (x-1)(x-3)(x^2 - 3x - 2) &= 0, \\ (x-1)(2x+1)(x^2 - 3x - 2) &= 0. \end{cases}$$

Solution 399. The common roots are $1 \pm \sqrt{2}$.

Solution 400. Let us assume, in hope of reaching a contradiction, that $b^2 - 4ac = d^2$, where d is a non-negative integer. Since a and c are positive, it means $d < b$. Therefore, we can write

$$\begin{aligned} 4a \cdot \overline{abc} &= 4a(100a + 10b + c) = 400a^2 + 40ab + 4ac \\ &= (20a + b)^2 - (b^2 - 4ac) = (20a + b)^2 - d^2 \\ &= (20a + b + d)(20a + b - d). \end{aligned}$$

By Euclid's lemma in number theory if a prime p divides a product xy, then p divides either x or y (or both). Since $\overline{abc}$ is a prime that divides the product $(20a+b+d)(20a+b-d)$, we must have either $\overline{abc} \mid 20a + b + d$ or $\overline{abc} \mid 20a + b - d$. In either case, the coefficient of a in $\overline{abc}$ is 100 whereas it is 20 in both $20a + b \pm d$, and it is trivial that $\overline{abc}$ cannot divide any of those two numbers. This is the contradiction we were looking for, and the proof is complete.

Solution 401. Let α, β, γ be the three roots of $x^3 + px + q = 0$. We can use the well-known identity

$$\alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha) + 3\alpha\beta\gamma.$$

Since the x^2 is missing, the sum of the roots $\alpha + \beta + \gamma$ will be zero, and because the constant term is q, the product of the roots $\alpha\beta\gamma$ will be $-q$. As a result,

$$\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma = -3q,$$

and the sum of cubes of roots $\alpha^3 + \beta^3 + \gamma^3$ is in fact divisible by 3.

Solution 402. We may write $n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2$ in two ways:

1. Expand it as $2(2n^2 + 6n + 7)$. If this equals a multiple of 10, say $10k$, then we need to solve the quadratic $2n^2 + 6n + 7 = 5k$ in n. The determinant of this equation would be an integer only when n leaves a remainder of 1 in division by 5.

2. Complete the square and write the given sum as $(2n+3)^2 + 5$. If this number is divisible by 10, then the last digit of $2n + 3$ must be 5, that is, $2n + 3 = 10t + 5$ for some non-negative integer t. This also simplifies to the same solution $n = 5t + 1$.

Solution 403. Write the system of equations in order of descending powers of a, b, c:

$$\begin{cases} a^3 - (y+z)a^2 + (x+y)a - (x+z) &= 0, \\ b^3 - (y+z)b^2 + (x+y)b - (x+z) &= 0, \\ c^3 - (y+z)c^2 + (x+y)c - (x+z) &= 0. \end{cases}$$

So, $u = a, b, c$ would be the roots of $u^3 - (y+z)u^2 + (x+y)u - (x+z) = 0$, and Viète's Formulas result in

$$\begin{cases} y + z &= a + b + c, \\ x + y &= ab + bc + ca, \\ z + x &= abc. \end{cases}$$

All three of these equations add up to $2(x + y + z)$, so that we find

$$x + y + z = \frac{1}{2}\left(a + b + c + ab + bc + ca + abc\right),$$

and it is easy to calculate x, y, z by subtracting $y + z, z + x, x + y$ from $x + y + z$:

$$\begin{cases} x &= \frac{1}{2}\left(-a - b - c + ab + bc + ca + abc\right), \\ y &= \frac{1}{2}\left(+a + b + c + ab + bc + ca - abc\right), \\ z &= \frac{1}{2}\left(+a + b + c - ab - bc - ca + abc\right). \end{cases}$$

Solution 404. If x and y are the roots of a quadratic equation, in order to form that equation we need $S = x + y$ and $P = xy$. Then, the given system of equations becomes

$$\begin{cases} S^2 - 2P = \dfrac{7}{3}, \\ S^3 - 3PS = -3. \end{cases}$$

Plug P from the first equation into the second equation to obtain a cubic equation in S: $S^3 - 7S - 6 = 0$, giving $S = -1, -2, 3$.

- When $S = -1$, we have $P = -\dfrac{2}{3}$ and $x, y = \dfrac{-3 \pm \sqrt{33}}{6}$,
- When $S = -2$, we have $P = \dfrac{5}{6}$ and $x, y = \dfrac{-6 \pm \sqrt{6}}{6}$,
- When $S = 3$, we have $P = \dfrac{10}{3}$ and x, y will be roots of $3v^2 - 9v + 10 = 0$ which are imaginary.

Solution 405. If x, y and z are the roots of a cubic equation, in order to form that equation we need $S = x + y + z$, $P = xy + yz + zx$, and $Q = xyz$. Then, the given system of equations becomes

$$\begin{cases} S = x + y + z & = 2a, \\ P = xy + yz + za & = -a^2, \\ Q = xyz & = -2a^3. \end{cases}$$

As a result, $u = x, y, z$ are the roots of the equation $u^3 - 2au^2 - a^2u + 2a^3 = 0$. It is then easy to find $u = \pm a, 2a$.

Solution 407. The answers are:

1. Here, $\lambda = y/x$ could be either $1/2$ or $3/37$, with $\lambda = 1/2$ resulting in $(x, y) = (2, 1), (-2, -1)$ and $\lambda = 3/37$ yielding
$$(x, y) = \left(\pm\frac{37}{2}\sqrt{\frac{7}{779}}, \pm\frac{3}{2}\sqrt{\frac{7}{779}}\right).$$
2. This would lead to $\lambda = 1$ (giving $x = y = 1$) or $\lambda = \frac{-1\pm\sqrt{21}}{5}$ (yielding $x = \frac{5}{\sqrt{170\mp\sqrt{21}}}$).

Solution 408. In case

$$\frac{m-2}{m} \neq \frac{m-1}{2(2m-3)},$$

then the only solution would be $x = y = 0$. However, if the equality is at work, we would have other solutions as well: the equation would be $3m^2 - 13m + 12 = 0$ with roots $m = 3$ and $m = 4/3$. If $m = 3$, then $x + 2y = 0$; and if $m = 4/3$, then $2x - y = 0$.

Solution 409. If $a \neq b$, $b \neq c$, and $c \neq a$, then $x = y = z = 0$. Otherwise, if two are equal, say, $a = b$, we have $x = cz$ and $y = -(a+c)z$, where z can be anything. Finally, in the case when $a = b = c$, we have $z = -(x + ay)/a^2$, where x and y are arbitrary.

Solution 410. The equation containing square roots can be written as

$$\sqrt{A + (x - y - z)} + \sqrt{B + (x - y - 1)} = \sqrt{A} + \sqrt{B},$$

where $A = x^2 + 4x + 3y - 2$ and $B = x^2 + 2y + 3$. It is easy to see that $x - y - 1 = 0$, and the only solution to the system of equations is $(x, y) = (3, 2)$.

Solution 411. Adding and subtracting the two equations, we arrive at another system of equations:

$$\begin{aligned}(x+y)(x^2 - xy + y^2 - a - b) &= 0,\\ (x-y)(x^2 + xy + y^2 - a + b) &= 0.\end{aligned}$$

This can be turned into four systems of equations which are all easy to solve:

a) $\begin{cases} x + y = 0, \\ x - y = 0. \end{cases}$ b) $\begin{cases} x + y = 0, \\ x^2 + xy + y^2 = a - b. \end{cases}$

c) $\begin{cases} x - y = 0, \\ x^2 - xy + y^2 = a + b. \end{cases}$ d) $\begin{cases} x^2 - xy + y^2 = a + b, \\ x^2 + xy + y^2 = a - b. \end{cases}$

Solution 412. The equations are $y^3 = x + 3y$ and $x^3 = 3x - y$. The trivial solution is $x = y = 0$. When $x \neq 0$, we have $y \neq 0$, and assuming $y = tx$, the equations become

$$\frac{y^3}{x^3} = \frac{x + 3y}{3x - y} \implies t^3 = \frac{1 + 3t}{3 - t}.$$

So, we find a quartic equation in t:

$$t^4 - 3t^3 + 3t + 1 = 0,$$

which may be written as a quadratic in $t^2 - 1$:

$$(t^2 - 1)^2 - 3t(t^2 - 1) + 2t^2 = 0.$$

The four solutions for t are $t = 1 \pm \sqrt{2}$ and $t = (1 \pm \sqrt{5})/2$. Since $x^3 = 3x - tx$ but $x \neq 0$, we may find x and y in terms of t: $x = \pm\sqrt{3 - t}$ and $y = \pm t\sqrt{3 - t}$. There are a total of 8 solutions which are all described above.

Solution 415. It is easy to manipulate the given equations and see that $t = p, q, r$ are the roots of

$$t^3 - t \log N + 3 \log N = 0,$$

so that $pq + qr + rp = \log N$ and $pqr = -3 \log N$. Finally, the sum of reciprocals of p, q, r equals $(pq + qr + rp)/pqr = -1/3$.

Solution 416. Write the system as

$$\begin{aligned} xy + (x+y) &= 11, \\ xy(x+y) &= 30. \end{aligned}$$

From here one can find $x+y$ and xy and eventually x and y:

$$(x, y) = (3, 2), (2, 3), (1, 5), (5, 1).$$

Solution 417. The solutions are $(x, y) = (2, 3)$ and $(3, 2)$.

Solution 418. Add three times the second equation to the first equation to find $x + y$. The solutions are $(x, y) = (-1, 2)$ and $(2, -1)$.

Solution 419. Write the first equation as $(x^2+y^2)^2-2x^2y^2 = a^4$ and substitute $x^2+y^2 = b^2 - 2xy$ from the second equation into the first equation. The result will be a quadratic equation in xy:

$$2(xy)^2 - 4b^2 \cdot xy + (b^4 - a^4) = 0.$$

Solution 420. The solutions are

$$(x, y) = (a, 0), (0, a), \left(a\frac{1 \pm i\sqrt{3}}{2}, a\frac{1 \mp i\sqrt{3}}{2}\right).$$

Solution 421. Using the identity

$$x^3 + y^3 + z^3 = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) + 3xyz,$$

we can reduce the equations to

$$x^2 + y^2 + z^2 = 6 \implies xy + yz + zx = -3.$$

Now, $t = x, y, y$ are the roots of $t^3 - 3t - 2 = 0$:

$$(x, y, z) = (2, -1, -1), (-1, 2, -1), (-1, -1, 2).$$

Solution 422. The solutions are $(x, y, z) = (a, 0, 0), (0, a, 0), (0, 0, a)$.

Solution 423. Note that $u = a, b, c, d$ are the four roots of

$$u^4 + tu^3 + zu^2 + yu - x = 0.$$

The relationship between the roots of this polynomial gives us the solutions to the system of equations:

$$\begin{cases} x = -abcd, \\ y = -(abc + abd + bcd + acd), \\ z = ab + ac + ad + bc + bd + cd, \\ t = -(a + b + c + d). \end{cases}$$

Solution 424. Assuming $a, b, c \neq k\pi$, we can divide both sides of the first equation by $\sin a$, second equation by $\sin b$, and third by $\sin c$. As a result, we find that $u = \cos a, \cos b, \cos c$ are the roots of

$$8u^3 - 4zu^2 - 2(y+2)u + (z-x) = 0.$$

This simplifies to

$$\begin{cases} \cos a + \cos b + \cos c = \dfrac{z}{2}, \\ \cos a \cos b + \cos b \cos c + \cos c \cos a = -\dfrac{y+2}{4}, \\ \cos a \cos b \cos c = \dfrac{x-z}{8}. \end{cases}$$

and so the solutions are

$$\begin{cases} x = 8\cos a \cos b \cos c + 2(\cos a + \cos b + \cos c), \\ y = -4(\cos a \cos b + \cos b \cos c + \cos c \cos a) - 2, \\ z = 2(\cos a + \cos b + \cos c). \end{cases}$$

Solution 425. Let $y = mx$ and simplify the system into

$$\begin{cases} b^2x^2 + abmx^2 + a^2m^2x^2 = 3a^2b^2, \\ b^2x^2 + mx^2 - a^2m^2x^2 = ab. \end{cases}$$

Dividing the first equation by the second, we arrive at a quadratic equation in m:

$$a^2(3ab+1)m^2 - 2abm - b^2(3ab-1) = 0.$$

The roots for m are $m = b/a$ and $m = b(1-3ab)/(a(1+3ab))$, so that the final solutions would be

$$(x, y) = (a, b), (-a, -b), \left(\pm a\sqrt{\frac{3ab+1}{3ab-1}}, \pm b\sqrt{\frac{3ab-1}{3ab+1}}\right).$$

Solution 426.

a) Show that $t = x, y, z$ are the roots of $t^3 - 9t^2 + 27t - 27 = (t-3)^2$, so the only solution is $x = y = z = 3$.

b) Subtract the second equation from the first equation, and also subtract the third equation from the second equation and simplify. The solutions are:

$$(x, y, z) = \left(\mp\frac{10\sqrt{3}}{3}, \mp\frac{\sqrt{3}}{3}, \pm\frac{8\sqrt{3}}{3}\right), (\mp 4, \mp 3, \pm 2).$$

c) Show that $xy+yz+zx=9$ and $xyz=4$, so that $t=x,y,z$ are the roots of

$$t^3-6t^2+9t-4=0.$$

The solutions are

$$(x,y,z)=(4,1,1),(1,4,1),(1,1,4).$$

d) The solutions are

$$(x,y,z)=(1,1,1),(1,-1,-1),(-1,1,-1),(-1,-1,1).$$

e) The trick is to solve the system for a,b,c and assume that x,y,z are given constants. In the system of equations, remove c between the first two equations:

$$(x^2-yz)a+(y^2+zx)b=x^3+y^3+xz^2+x^2y.$$

Also, remove c between the first and the third equation in the system:

$$(y^2+zx)a+(z^2-xy)b=y^3+z^3+x^2z+yz^2.$$

Now, remove b from the two latter equations to obtain

$$y(x^3+y^3+z^3+xyz)a=y(x+y)(x^3+y^3+z^3+xyz).$$

The all-zero solution is obvious: $x=y=z=0$. Assuming $x^3+y^3+z^3+xyz\neq 0$, we find $a=x+y$. Similarly, we can find $b=y+z$ and $c=z+x$. This means the non-trivial solution would be

$$(x,y,z)=\left(\frac{a+c-b}{2},\frac{a+b-c}{2},\frac{c+b-a}{2}\right).$$

f) Let $u=\sqrt[4]{1+5x}$ and $v=\sqrt[4]{6-y}$, so that the equations become $u+v=3$ and $u^4+v^4=25$, which are easy to solve.

Solution by Amir Parvardi 594. Let a,b,c be the roots of the equation so that $a^2+b^2=c^2$. Since these are also the roots of $x^3-2p(p+1)x^2+(p^4+4p^3-1)x-3p^3=0$, we can use Viète's formulas to obtain

$$\begin{cases} a+b+c &= 2p(p+1),\\ ab+bc+ca &= p^4+4p^3-1,\\ abc &= 3p^3.\end{cases}$$

Squaring the first equation and subtracting twice the second equation, we get $a^2+b^2+c^2=4p^2(p+1)^2-2(p^4+4p^3-1)$, and since $a^2+b^2=c^2$, we may write $a^2+b^2+c^2=2c^2$ and simplify the last equation as

$$2c^2=2p^4+4p^2+2,$$

or simply $c^2 = p^4 + 2p^2 + 1$. This simplifies to $c = \pm(p^2 + 1)$, and since c is a side of a triangle, $c = p^2+1$. From here, it is easy to find a and b from equations $a+b = 2p(p+1)-c$ and $ab = 3p^3/c$. Write them out explicitly, having $a \neq b$ in mind:

$$a + b = 2p(p + 1) - (p^2 + 1) = p^2 + 2p - 1 \quad \text{and} \quad ab = \frac{3p^3}{p^2 + 1}.$$

Since a, b, and c are side-lengths of a triangle, the triangle inequality $a+b > c$ must hold: $p^2+2p-1 > p^2+1$, or simply $p > 1$. Moreover, since $(p^2+1)^2 = c^2 = a^2+b^2 = (a+b)^2-2ab$, we find that

$$(p^2 + 1)^2 = (p^2 + 2p - 1)^2 - \frac{6p^3}{p^2 + 1}.$$

This results in $(p^2+1)^3 = (p^2+1)(p^2+2p-1)^2-6p^3$, which simplifies to $-4p^5+6p^3+4p = 0$. Neglecting the trivial $p = 0$ solution, we can divide both sides of the latter equation by p to get $-4p^4 + 6p^2 + 4 = 0$ which factorizes into $-2(p^2 - 2)(2p^2 + 1) = 0$. Therefore, the only solution that satisfies $p > 1$ is $\boxed{p = \sqrt{2}}$.

Solution 595. Let r be the root of the first equation and $f(x)$ be the second polynomial. It is easy to verify $f(\sqrt{r}) = (br + d)(\sqrt{r} + 1)$ and $f(-\sqrt{r}) = (br + d)(-\sqrt{r} + 1)$. Hence, $f(\sqrt{r})f(-\sqrt{r}) \leq 0$. Then there is a root on $[-\sqrt{r}, \sqrt{r}]$ to $f(x) = 0$.

Solution 844. Use $15° = 45°-30°$ and $22.5° = 45°/2$ and use the appropriate identities.

($\cot x - \cot 2x$) **Exercise Identity 845.** Using the identities (2.28), (2.23),

$$\frac{1}{\sin 2\theta} - \cot 2\theta = \frac{1}{\sin 2\theta} - \frac{\cos 2\theta}{\sin 2\theta} = \frac{1 - \cos 2\theta}{\sin 2\theta} = \frac{2\sin^2\theta}{\sin 2\theta} = \frac{2\sin^2\theta}{2\sin\theta\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$$

Also,

$$\frac{1}{\sin 2\theta} + \cot 2\theta = \frac{1}{\sin 2\theta} + \frac{\cos 2\theta}{\sin 2\theta} = \frac{1 + \cos 2\theta}{\sin 2\theta} = \frac{2\cos^2\theta}{\sin 2\theta} = \frac{2\cos^2\theta}{2\sin\theta\cos\theta} = \frac{\cos\theta}{\sin\theta} = \cot\theta.$$

Proof of Sum of Three Angles Identity 846. Using the identity for sum of two angles, we can write $x + y + z$ as $x + (y + z)$, so that

$$\begin{aligned}\sin(x + [y + z]) &= \sin x \cos(y + z) + \cos x \sin(y + z)\\ &= \sin x \cdot (\cos y \cos z - \sin y \sin z) + \cos x \cdot (\sin y \cos z + \sin z \cos y)\\ &= \sin x \cos y \cos c + \sin y \cos x \cos z + \sin z \cos x \cos y - \sin x \sin y \sin z,\end{aligned}$$

as required.

Solution 849.

1. Expand $\cos(60° \pm x)$ and arrive at the polynomial expression $4\cos^3 x - 3\cos x$ which equals $\cos 3x$.

2. Expand $\tan(60°+x)$ and $\tan(120°+x)$, arrive at the fraction $(9\tan x-3\tan^3 x)/(1-3\tan^2 x)$, which equals $3\tan 3x$.

Proof of Sine and Cosine Sum-to-Product 850.] Let $a = (x + y)/2$ and $b = (x - y)/2$. By adding and subtracting the identities for $\cos(a+b)$ and $\cos(a-b)$, and also the identities for $\sin(a + b)$ and $\sin(a - b)$, we arrive at the identities in the question.

Proof of Sine and Cosine Product-to-Sum 851.] Let $a = (x + y)/2$ and $b = (x - y)/2$. By adding and subtracting the identities for $\cos(a+b)$ and $\cos(a-b)$, and also the identities for $\sin(a + b)$ and $\sin(a - b)$, we arrive at the identities in the question.

Proof of Tangent and Cotangent Sum-to-Product 852.] Let $a = (x + y)/2$ and $b = (x-y)/2$. By adding and subtracting the identities for $\cot(a+b)$ and $\cot(a-b)$, and also the identities for $\tan(a+b)$ and $\tan(a-b)$, we arrive at the identities in the question.

Solution 853. Start with $A+B = \pi - C$, and use the supplementary angles identity to deduce $\sin C = \sin(A+B)$. Now use the identity $\sin 2x = 2\sin x \cos x$ where $2x = A+B$ to write $\sin(A + B)$ as the product of sine and cosine of $(A + B)/2$. On the other hand, use the sum-to-product identity for sines to write $\sin A + \sin B$ as the product of sine of $(A+B)/2$ and cosine of $(A-B)/2$. Simplify $\sin A+\sin B+\sin C$ using the substitutions for $\sin A+\sin B$ and $\sin C = \sin(A+B)$, also noting that the angles $(A+B)/2$ and $C/2$ are complementary angles (adding up to a right angle), finally arriving at the sum-of-sines to product-of-cosines identity in the triangle ABC:

$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Solution 855. Start with $\cos(B + C) = -\cos A$ and $\sin(B + C) = \sin A$ and finish the calculations.

Solution 862. Start with Jensen's inequality: since $\sin x$ is a concave function in the interval $\left[0, \frac{\pi}{2}\right]$,

$$\frac{\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}}{3} \leq \sin\left(\frac{\frac{A}{2} + \frac{B}{2} + \frac{C}{2}}{3}\right) = \sin\frac{A+B+C}{6} = \frac{1}{2}.$$

By AM-GM, we have

$$\sqrt[3]{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \leq \frac{\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}}{3} \leq \frac{1}{2}.$$

Finally,

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \leq \frac{1}{8}.$$

Solution 864. Start with $\cos A + \cos B$. Using the identity (2.3), let $x = \frac{A+B}{2}$ and $y = \frac{A-B}{2}$,

$$\begin{aligned}\cos A + \cos B &= \cos\left(\frac{A+B}{2} + \frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2} - \frac{A-B}{2}\right)\\ &= \cos(x+y) - \cos(x-y) = 2\cos x\cos y\\ &= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}.\end{aligned}$$

Therefore,

$$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}. \tag{67}$$

Since $A+B+C=180^\circ=\pi$, by identities (2.13) and (2.14), we find $\sin C=\sin(A+B)$ and $\cos C=-\cos(A+B)$. Similarly, because $\frac{A+B}{2}+\frac{C}{2}=90^\circ=\frac{\pi}{2}$, using identities (2.15) and (2.16), we have $\sin\frac{C}{2}=\cos\frac{A+B}{2}$ and $\cos\frac{C}{2}=\sin\frac{A+B}{2}$. Then,

$$\begin{aligned}\cos A+\cos B+\cos C&=\cos A+\cos B-\cos(A+B)\\&=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}-\left(\cos^2\frac{A+B}{2}-\sin^2\frac{A+B}{2}\right)\\&=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}-\left(2\cos^2\frac{A+B}{2}-1\right)\\&=2\cos\frac{A+B}{2}\left(\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right)+1\\&=2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right)+1.\end{aligned}$$

Similar to equation (67), we can find

$$\cos\frac{A-B}{2}-\cos\frac{A+B}{2}=2\sin\frac{A}{2}\sin\frac{B}{2}.$$

Hence,

$$\cos A+\cos B+\cos C=2\sin\frac{C}{2}\left(2\sin\frac{A}{2}\sin\frac{B}{2}\right)+1=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}+1.$$

Solution 866. Use equation (2.61) and identity (2.18) ($\cos x=1-2\sin^2\frac{x}{2}$) to write:

$$\begin{aligned}\cos A+\cos B+\cos C&=1-2\sin^2\frac{A}{2}+1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}\\&=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.\end{aligned}$$

Thus,

$$3-2\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\right)=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$

reduces to

$$\left(\sin\frac{A}{2}\right)^2+\left(\sin\frac{B}{2}\right)^2+\left(\sin\frac{C}{2}\right)^2+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=1,$$

which is exactly what we want.

Solution by Amir Parvardi 867. The magical equations at work here are:

$$\frac{\pi}{16} = \frac{\pi}{2} - \frac{7\pi}{16},$$
$$\frac{3\pi}{16} = \frac{\pi}{2} - \frac{5\pi}{16},$$

We will focus on finding a formula for $\tan^2\theta + \tan^2(90^\circ - \theta)$ keeping in mind that for $\theta \in \{\frac{\pi}{16}, \frac{3\pi}{16}, \frac{5\pi}{16}, \frac{7\pi}{16}\}$, we know $\tan^2 4\theta = 1$. We are looking for an identity which involves 4θ. Using the identities for $\tan(\pi/2 - x)$,

$$\tan^2\theta + \tan^2\left(\frac{\pi}{2} - \theta\right) = \tan^2\theta + \cot^2\theta$$

By equations in Problem 845, we observe

$$\tan^2\theta + \tan^2\left(\frac{\pi}{2} - \theta\right) = \tan^2\theta + \cot^2\theta = \left(\frac{1}{\sin 2\theta} - \cot 2\theta\right)^2 + \left(\frac{1}{\sin 2\theta} + \cot 2\theta\right)^2$$
$$= 2\left(\frac{1}{\sin^2 2\theta} + \cot^2 2\theta\right) = \frac{2}{\sin^2 2\theta} + 2\cot^2 2\theta.$$

Therefore,

$$\tan^2\theta + \tan^2\left(\frac{\pi}{2} - \theta\right) = \frac{2}{\sin^2 2\theta} + 2\cot^2 2\theta.$$

From the identity for $\cos 4x$, we know $\sin^2 2\theta = (1 - \cos 4\theta)/2$ and from the Problem 845, we know $\cot 2\theta = \frac{1}{\sin 4\theta} + \cot 4\theta$. Plugging in,

$$\tan^2\theta + \tan^2\left(\frac{\pi}{2} - \theta\right) = \frac{4}{1 - \cos 4\theta} + 2\left(\frac{1}{\sin 4\theta} + \cot 4\theta\right)^2.$$

Now plug $\theta = \pi/16$ and $\theta = 3\pi/16$ in the above equation to find the final result:

$$\tan^2\frac{\pi}{16} + \tan^2\frac{7\pi}{16} = \frac{4}{1 - \cos\frac{\pi}{4}} + 2\left(\frac{1}{\sin\frac{\pi}{4}} + \cot\frac{\pi}{4}\right)^2 = \frac{4}{1 - \frac{\sqrt{2}}{2}} + 2\left(\frac{2}{\sqrt{2}} + 1\right)^2,$$
$$\tan^2\frac{3\pi}{16} + \tan^2\frac{5\pi}{16} = \frac{4}{1 - \cos\frac{3\pi}{4}} + 2\left(\frac{1}{\sin\frac{3\pi}{4}} + \cot\frac{3\pi}{4}\right)^2 = \frac{4}{1 + \frac{\sqrt{2}}{2}} + 2\left(\frac{2}{\sqrt{2}} - 1\right)^2,$$

Doing the calculations,

$$\frac{4}{1 - \frac{\sqrt{2}}{2}} + 2\left(\frac{2}{\sqrt{2}} + 1\right)^2 = \frac{8}{2 - \sqrt{2}} + 2(\sqrt{2} + 1)^2 = \frac{8(\sqrt{2} + 2)}{4 - 2} + 2(3 + 2\sqrt{2})$$
$$= 4\sqrt{2} + 8 + 6 + 4\sqrt{2} = 14 + 8\sqrt{2},$$
$$\frac{4}{1 + \frac{\sqrt{2}}{2}} + 2\left(\frac{2}{\sqrt{2}} - 1\right)^2 = \frac{8}{2 + \sqrt{2}} + 2(\sqrt{2} - 1)^2 = \frac{8(2 - \sqrt{2})}{4 - 2} + 2(3 - 2\sqrt{2})$$
$$= -4\sqrt{2} + 8 + 6 - 4\sqrt{2} = 14 - 8\sqrt{2}.$$

Finally,

$$\tan^2\frac{\pi}{16}+\tan^2\frac{3\pi}{16}+\tan^2\frac{5\pi}{16}+\tan^2\frac{7\pi}{16}=\left(\tan^2\frac{\pi}{16}+\tan^2\frac{7\pi}{16}\right)+\left(\tan^2\frac{3\pi}{16}+\tan^2\frac{5\pi}{16}\right)$$
$$=(14+8\sqrt{2})+(14-8\sqrt{2})=28.$$

Solution 875. Clearly, $2p=a+b+c$, and by the law of sines, $2p=2R\sin A+2R\sin B+2R\sin C$. Factoring out the R and canceling the factor of 2 everywhere, this leads to $p=R(\sin A+\sin B+\sin C)$, which reminds us of the sum of sines in a triangle, and we can use the expression given in Problem 853 to arrive at the "*Semi-perimeter and Half-angles*" formula:

$$p=4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},$$

Solution 888. Answer: $\angle B=75^\circ$ and $\angle C=45^\circ$.

Calculating the Inradius 889. We know that $S=pr$, and we can use the formula for the semiperimeter in terms of a product containing cosine of half-angles (Problem 875) as well as the formula for the area of triangle containing a product of sine of angles (Problem 876) to write

$$r=\frac{S}{p}=\frac{2R^2\sin A\sin B\sin C}{4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$
$$=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$

where we have used half-angle formulas to write the last line.

Calculating the Exradii 890. Use the fact that $r_a=p\tan\frac{A}{2}$, where p is the semiperimeter and use the semiperimeter formula $p=4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ to arrive at the given formulas.

Solution 891. For the first part, use the exradius formula for r_a, inradius formula for r, and the law of sines for a to simplify the given equation and arrive at

$$K\sin B\sin C=8\cos\frac{B-C}{2}\cos\frac{A}{2}.$$

For the second part, check the roots of the equation

$$K\cos^2\frac{A}{2}-8\cos\frac{B-C}{2}\cos\frac{A}{2}-K\sin^2\frac{B-C}{2}=0,$$

and conclude that $K\cos\frac{B-C}{2}-8>0$.

Solution 905. Use the fact that OH is three times OG.

Solution 932. The answer is

$$\frac{\cos^4 y}{\cos^2 x}+\frac{\sin^4 y}{\sin^2 x}=\frac{\cos^4 x}{\cos^2 y}+\frac{\sin^4 x}{\sin^2 y}=1,$$

because the given equations force $\sin^2 x=\sin^2 y$ and $\cos^2 x=\cos^2 y$.

Solution 933. The answer is $\left(2-e^2\right)^{\frac{3}{2}} = (2-e^p)^{q/p}$, so $p+q = 2+3 = 5$.

Solution 934. The answer is $x = y$. Since $\sec^2\theta$ is the reciprocal of $\cos^2\theta$ which is always less than or equal to 1, we have $\sec^2\theta \geq 1$, and also by AM–GM, $(x+y)^2 \geq 4xy$, so that $4xy/(x+y)^2 \leq 1$ for all x and y. The equality may happen only if it is happening in both $\sec^2\theta \geq 1$ and $4xy/(x+y)^2 \leq 1$, which yields $\sec^2\theta = 1$, and the equality case of AM–GM must hold: $x = y$.

Solution 935. The obvious all-zero answer is $\theta = 2n\pi$ for some integer n. The other solution is

$$\theta = (2n+1)\frac{\pi}{4}.$$

Solution 936. In terms of the given integers m and n, the answers are

$$\theta = \frac{2r\pi}{m+n} \qquad \text{and} \qquad \theta = \frac{(2s+1)\pi}{m-n},$$

for any choice of integers r and s.

Solution 937. The answers are

$$\theta = n\pi - \frac{\pi}{4} \qquad \text{and} \qquad \theta = m\pi + \frac{\pi}{3},$$

for any choice of integers m and n.

Solution 938. The answers are

$$\theta = \frac{n\pi}{3} + \frac{\pi}{12},$$

for any choice of integer n.

Solution 939. The obvious all-zero answers are $\theta = n\pi$ for any integer n, but the non-trivial answer is

$$\theta = n\pi \pm \frac{\pi}{3},$$

for any choice of integer n.

Solution 940. The answers are

$$\theta = 2n\pi + \frac{5\pi}{12} \qquad \text{and} \qquad \theta = 2m\pi - \frac{\pi}{12},$$

for any choice of integers m and n.

1989 Roorkee 941. The correct answer is a), $n\pi$. The equation leads to $\sin^2 x(5\sin^2 x - 2) = 0$ which means either $\sin x = 0$ (thus $\theta = n\pi$) or $\sin x = \pm\sqrt{2/5}$

Solution 942. The correct answer is b), $2k\pi$. The equation is $1-\cos\theta = \sin\theta\cdot\sin\frac{\theta}{2}$, which leads to $2\sin^2\frac{\theta}{2} = 2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}\cdot\sin\frac{\theta}{2}$ which reduces to $2\sin^2\frac{\theta}{2}\cdot(1-\cos\frac{\theta}{2}) = 0$. The solutions are therefore coming from $2\sin^2\frac{\theta}{2} = 0$ and $1-\cos\frac{\theta}{2} = 0$. The first equation means $\sin\frac{\theta}{2} = 0$, which happens for $\theta = 2k\pi$ for any integer k. The second equation, $1-\cos\frac{\theta}{2} = 0$ reduces to $2\sin^2\frac{\theta}{4} = 0$ whose solutions are in the form of $\theta = 4k\pi$ for any integer k. Since the family of solutions $\theta = 4k\pi$ is a subset of the family of solutions $\theta = 2k\pi$, the solutions are precisely $\theta = 2k\pi$ for all $k \in \mathbb{Z}$.

Solution 943. The correct answer is b), $\frac{n\pi}{2}+\frac{\pi}{8}$. After simplification, the given equation becomes $(\sin 2x - \cos 2x)(2\cos x - 3) = 0$, which inevitably results in $\sin 2x = \cos 2x$ or $\tan 2x = 1$, thus $x = \frac{n\pi}{2} + \frac{\pi}{8}$.

Solution 944. The correct answer is a), one solution:

$$x = y = z = \frac{3\pi}{2}.$$

Solution 945. The correct answer is b), because the equation reduces to $\cos 2x(\cos 2x + 1) = 0$, meaning $\cos 2x$ can be either 0 or -1. The solutions to $\cos 2x = 0$ are

$$x = \frac{\pm\pi}{4}, \frac{\pm 3\pi}{4}, \frac{\pm 5\pi}{4}, \dots,$$

whereas the solutions to $\cos 2x = -1$ are

$$x = \frac{\pm\pi}{2}, \frac{\pm 3\pi}{2}, \frac{\pm 5\pi}{2}, \dots,$$

and among these, only $\pm\pi/4$ are in the interval $-\pi < x < \pi$.

Solution 946. The correct answer is d). The equation is equivalent to $(5\cos\theta + 3)(2\cos\theta - 1) = 0$, so that $\cos\theta$ can be $1/2$ or $-3/5$. Therefore, θ can be $\frac{\pi}{3}$, or $\pi - \cos^{-1}\frac{3}{5}$, both of which lie in the interval $-\pi < x < \pi$.

Solution 947. The correct answer is a), $x = \pi/6$, simply because $30 = 3 + 27 = 81^{1/4} + 81^{3/4}$, and we are looking for an angle x such that $\sin^2 x = 1/4$ and $\cos^2 x = 3/4$, the only possibility among the given options being $\dfrac{\pi}{6}$.

Solution 948. The correct answer is c), $x = n\pi/8$. To solve, start with the term $\cos 3x = \cos x(2\cos 2x - 1)$, so that $\cos x + \cos 3x = 2\cos x \cos 2x$. Moreover, it is easy to see that $2\cos 6x \cos x = \cos 5x + \cos 7x$, so that the given expression is $\cos x + \cos 3x + \cos 5x + \cos 7x = 2\cos x\cos 2x + 2\cos 6x \cos x$, which after factoring $\cos x$ becomes $2\cos x(\cos 2x + \cos 6x)$. Finally, $\cos 6x = \cos 2x(2\cos 4x - 1)$, so that $\cos 2x + \cos 6x = 2\cos 2x \cos 4x$. Putting everything together, we find

$$\begin{aligned}\cos x + \cos 3x + \cos 5x + \cos 7x &= 2\cos x(\cos 2x + \cos 6x)\\ &= 2\cos x(2\cos 2x\cos 4x)\\ &= 4\cos x\cos 2x\cos 4x.\end{aligned}$$

On the other hand, note that

$$\begin{aligned}\sin 8x &= 2\sin 4x \cdot \cos 4x\\ &= 4\sin 2x\cos 2x\cos 4x\\ &= 8\sin x\cos x\cos 2x\cos 4x,\end{aligned}$$

and we get the golden identity for this question:

$$\cos x + \cos 3x + \cos 5x + \cos 7x = \frac{\sin 8x}{2\sin x}.$$

Thus the equation $\cos x + \cos 3x + \cos 5x + \cos 7x = 0$ is equivalent to $\sin 8x = 0$, which has solutions $8x = n\pi$ for any integer n.

Solution 949. The correct answer is 949. Divide both sides of the equation by $\sqrt{a^2+b^2}$ to get

$$\frac{a}{\sqrt{a^2+b^2}}\cos x + \frac{b}{\sqrt{a^2+b^2}}\sin x = \frac{c}{\sqrt{a^2+b^2}}.$$

Let $\theta = \cos^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\right)$, so that $\cos\theta = a/\sqrt{a^2+b^2}$ and $\sin\theta = b/\sqrt{a^2+b^2}$. Therefore,

$$\cos\theta\cos x + \sin\theta\sin x = \frac{c}{\sqrt{a^2+b^2}}.$$

The left side is $\cos(x-\theta)$ and if we let $\phi = \cos^{-1}\left(\frac{c}{\sqrt{a^2+b^2}}\right)$, then the equation becomes $\cos(x-\theta) = \cos\phi$, whose solutions are given by $x - \theta = 2n\pi \pm \phi$. These can be written as

$$x = 2n\pi \pm \phi + \theta,$$

substituting the angles for $\theta = \cos^{-1} a/\sqrt{a^2+b^2} = \tan^{-1} b/a$ and $\phi = \cos^{-1} c/\sqrt{a^2+b^2}$,

$$\begin{aligned} x &= 2n\pi \pm \cos^{-1}\left(\frac{c}{\sqrt{a^2+b^2}}\right) + \cos^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\right) \\ &= 2n\pi \pm \cos^{-1}\left(\frac{c}{\sqrt{a^2+b^2}}\right) + \tan^{-1}\left(\frac{b}{a}\right). \end{aligned}$$

Solution 983. Use telescoping sums and the formula

$$\frac{\sin 2\alpha}{\sin(2k-1)\alpha\sin(2k+1)\alpha} = \cot(2k-1)\alpha - \cot(2k+1)\alpha.$$

Solution 988. Use telescoping sums and the formula

$$\frac{1}{\sin\alpha} = \cot\left(\frac{\alpha}{2}\right) - \cot\alpha.$$

Solution 989. Use telescoping sums and the formula

$$\tan\alpha = \cot\alpha - 2\cot 2\alpha.$$

Solution 996. Use the identity $4\sin^3\alpha = 3\sin\alpha - \sin 3\alpha$ and multiply the given expression by 4. Then use the formula for sum of sine of angles in an arithmetic progression.

Solution 1024. Use the identity $\tan^2\alpha\tan 2\alpha = \tan 2\alpha - 2\tan\alpha$.

Solution 1026. The answer for the first part is

$$f(x) = \frac{x^4}{4} + \frac{x^3}{2} + \frac{x^2}{4}.$$

Put $x = 1, 2, \ldots, n$ in the equation $f(x) - f(x-1) \equiv x^3$ and sum them up to find

$$\lim_{\alpha\to 0} S_1 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

The limit of S_2 as α approaches 0 is also the same:

$$\lim_{\alpha\to 0} S_1 = [1+2+\cdots+n]^2 = \left(\frac{n(n+1)}{2}\right)^2.$$

Solution 1028. Answer: the roots of the given equation are

$$4\sin^2\left(\frac{\pi}{9}\right),\quad 4\sin^2\left(\frac{2\pi}{9}\right),\quad 4\sin^2\left(\frac{4\pi}{9}\right).$$

Solution 1029. Answer: the roots of the given equation are

$$\pm\tan\left(\frac{\pi}{5}\right)\quad\text{and}\quad\pm\tan\left(\frac{3\pi}{5}\right).$$

Solution 1030. Answer: the roots of the given equation are $2\cos(k\pi/9)$, where $k = 1, 3, 5, 7$.

Solution by Boxedexe 1158. Let O, A, B, θ, and r be the cone's vertex, cone's base's center, a point on the cone's base's circle, $\angle AOB$, and the radius of the sphere, respectively. Then it follows, from basic trigonometry, that $OA = \frac{r(1+\sin\theta)}{\sin\theta}$ and $AB = \frac{r(1+\sin\theta)}{\cos\theta}$. Hence,

$$V_1 = \frac{\pi AB^2 OA}{3} = \frac{\pi\left(\frac{r(1+\sin\theta)}{\cos\theta}\right)^2\left(\frac{r(1+\sin\theta)}{\sin\theta}\right)}{3},$$

and $V_2 = 2\pi r^3$. Ergo,

$$k = \frac{\pi\left(\frac{r(1+\sin\theta)}{\cos\theta}\right)^2\left(\frac{r(1+\sin\theta)}{\sin\theta}\right)}{6\pi r^3} \implies (1+6k)\sin^2\theta + (2-6k)\sin\theta + 1 = 0.$$

However, the discriminant of the above quadratic equation must be non-negative, that is,

$$(1-3k)^2 \geq 1+6k \implies k \geq \frac{4}{3},$$

and the conclusion follows.

Solution 1159. This solution was written by **randomusername** on AoPS:

a) Let $\mathcal{S}$ be the sphere tangent to the edges from the inside, and let $\mathcal{S}_X$ be the sphere tangent to the edges of the face opposite to vertex X, and the extensions of the edges from X. Consider say $\mathcal{S}$ and $\mathcal{S}_S$. If we cut these spheres by plane ABC, in both cases we get a circle on plane ABC tangent to the segments AB, BC, CA - this is the incircle of $\triangle ABC$. Hence $\mathcal{S}$ and $\mathcal{S}_S$ touches the edges AB, BC, CA in the same points. Now cut the two spheres by plane SAB. We get the incircle and S-excircle of $\triangle SAB$. We have proven that $\mathcal{S}$ and $\mathcal{S}_S$ touch AB in the same point, so the incircle and S-excircle of $\triangle SAB$ touch AB at the same point. This implies that $SA = SB$. Similarly, we get that any two edges are the same length, therefore the tetrahedron is indeed regular.

b) Look at the regular tetrahedron $SABC$. By symmetry, the center O of the regular tetrahedron $SABC$ is of equal distance d from all its edges, so taking $\mathcal{S}(O, d)$ works. Let the incenter of say $\triangle SAB$ be I, the S-excenter of $\triangle SAB$ be J. Consider the homothety with center S that maps I to J. This homothety maps the incircle ω of $\triangle SAB$ to the S-excircle of $\triangle SAB$; since $\omega \subset \mathcal{S}$, we have $\omega' \subset \mathcal{S}'$ and therefore $\mathcal{S}'$ also touches edge AB. By rotational symmetry, $\mathcal{S}'$ also touches edges BC and CA. Moreover, because $\mathcal{S}$ touched edges SA, SB, SC, $\mathcal{S}'$ will touch the extensions of SA, SB, SC. This proves that $\mathcal{S}_S = \mathcal{S}'$ is up for the job. By symmetry, so do $\mathcal{S}_A, \mathcal{S}_B, \mathcal{S}_C$ exist.

Solution by Grobber 1161. This can basically be reduced to a plane geometry problem.

The vertex D is projected onto the orthocenter H of ABC (this follows from the conditions $AB \perp CD,\ AC \perp BD,\ AD \perp BC$). This means that the midpoints U, V, T of DA, DB, DC are projected onto the midpoints X, Y, Z of HA, HB, HC. Let M, N, P be the midpoints of BC, CA, AB. The points M, N, P, X, Y, Z liue on the nine-point circle of ABC, so the points U, V, T lie on a circle of radius equal to $\frac{R}{2}$ (R is the circumradius of ABC) which lies on a plane parallel to ABC and which has its center directly above the nine-point center of ABC. It's is now clear that M, N, P, U, V, T lie on a sphere which has its center in the midpoint of the segment formed by the centers of $(MNP), (UVT)$.

Solution by Grobber 1162. I remember solving this before the crash.[1] The ratio between this volume and the volume of the sphere is equal to the ratio between the area determined by the trihedron on the sphere and the area of the sphere. The region is a spherical triangle with angles α, β, γ, and its area is thus $(\alpha+\beta+\gamma-\pi) \cdot R^2$. The volume we are looking for must then be $\frac{\alpha+\beta+\gamma-\pi}{4\pi} \cdot \frac{4\pi \cdot R^3}{3} = \frac{(\alpha+\beta+\gamma-\pi) \cdot R^3}{3}$.

Solution by Grobber 1163. Let $ABCD$ be the tetrahedron. I, its incenter, is projected on the four planes of the faces in the circumcenters of the faces. Let O_A, O_B, O_C, O_D be the circumcenters of BCD and so on. We clearly have $d(O_D, BC) = d(O_A, BC)$, and therefore, $\angle BAC = \angle BDC$. From this equality and the like we find that the sum of the plane angles around each vertex is π (call this assertion $(*)$). Now take the faces BCD, CAD, ABD and rotate them around BC, CA, AB until they are on the same plane as ABC. It is now easy to see from $(*)$ that ABC, CBD, ADB, DAC are congruent, and the conclusion follows.

Solution by Luis González 1165. Reflection C' of C about N also lies on the equator and since there are infinitely many great circles through C, C', then we deduce that C, C', A and C, C', B lie on a great circle, respectively. $NC = NC'$, $NA = NB$ and $\sphericalangle ANC = \sphericalangle BNC'$ imply that the spherical triangles ANC and BNC' are congruent by SAS criterion. Therefore, N-spherical altitudes of ANC and BNC' are congruent, i.e., N is equidistant from the great circles AC and BC'. Finally, CN bisects the spherical lune formed by the great circles CB and CA.

Solution from Kalva 1166. Clearly, $n = 3$ is certainly possible. For example, take $\angle APB = \angle APC = \angle BPC = 90°$ (so that the lines PA, PB, PC are mutually perpendicular). Then the three planes through P are also mutually perpendicular, so the two sums are both $270°$.

[1]Written on December 16, 2004.

We show that $n > 3$ is not possible.

The sum of the n angles APB etc at P is less than $360°$. This is almost obvious. Take another plane which meets the lines PA, PB, PC etc at $A', B', C', \ldots$ and so that the foot of the perpendicular from P to the plane lies inside the n-gon $A'B'C'\ldots$ then as we move P down the perpendicular the angles $A'PB'$ etc all increase. But when it reaches the plane their sum is $360°$. However, I do not immediately see how to make that rigorous. Instead, take any point O inside the n-gon ABC. We have $\angle PBA + \angle PBC > \angle ABC$.

Adding the n such equations we get $\sum(180° - APB) > \sum ABC = (n-2)180°$. So, $\sum APB < 360°$.

The sum of the n angles between the planes is at least $(n-2) * 180^o$. If we take a sphere center P. Then the lines PA, PB intersect it at $A", B", \ldots$ which form a spherical polygon. The angles of this polygon are the angles between the planes. We can divide the polygon into $n-2$ triangles. The angles in a spherical triangle sum to at least $180°$. So the angles in the spherical polygon are at least $(n-2)180°$. So we have $(n-2)180° < 360°$ and hence $n < 4$.

Solution by Luis González 1168. Let $\mathcal{C}_1(r_1), \mathcal{C}_2(r_2)$, (with $r_2 > r_1$) are the incircles of the bases of the truncated pyramid. Let $\mathcal{C}_3(\varrho)$ be the circumcircle of the regular n–gon whose vertices are the tangency points of the subject sphere $\mathcal{E}$ with the lateral faces. Thus, $\mathcal{C}_1(r_1), \mathcal{C}_2(r_2)$ and $\mathcal{C}_3(\varrho)$ are obviously cross sections of a right cone with apex A circumscribed around $\mathcal{E}$. Arbitrary plane through the axis of the cone cuts $\mathcal{E}$ into a circle (I) and the bases $\mathcal{C}_2(r_2), \mathcal{C}_1(r_1)$, into the segments $BC = 2r_2$, $MN = 2r_1$, $(M \in AB$ and $N \in AC) \Longrightarrow BCNM$ is an isosceles trapezoid with incircle (I). If (I) touches AB, AC at D, E, then $DE = 2\varrho$, $NE = r_1$ and $CE = r_2$. Therefore

$$DE = \frac{BC \cdot EN + MN \cdot EC}{NC},$$

so that

$$\varrho = \frac{2r_1r_2}{r_1 + r_2} \ (\star).$$

From the well-known formulae of the areas of regular polygons, we have:

$$\sigma = n\varrho^2 \cos\frac{\pi}{n}\sin\frac{\pi}{n},$$

and also

$$S_1 = n \cdot {r_1}^2 \tan\frac{\pi}{n} \qquad \text{and} \qquad S_2 = n \cdot {r_2}^2 \tan\frac{\pi}{2},$$

which results in

$$S_1S_2 = n^2{r_1}^2{r_2}^2\tan^2\frac{\pi}{n}.$$

Lateral faces of the truncated pyramid are congruent isosceles trapezoids with altitude $r_1 + r_2$, whose bases are the sides of the n–gons with incircles $\mathcal{C}_1(r_1), \mathcal{C}_2(r_2)$. Therefore,

$$S = \frac{n}{2} \cdot (r_1 + r_2) \cdot \left(2r_1 \tan\frac{\pi}{n} + 2r_2 \tan\frac{\pi}{n}\right) = n(r_1 + r_2)^2 \tan\frac{\pi}{n}.$$

Substituting ϱ, r_1r_2 and $(r_1 + r_2)$ from the latter expressions into $(\star)$ yields:

$$\frac{\sigma}{n\cos\frac{\pi}{n}\sin\frac{\pi}{n}} = \frac{4S_1S_2}{n^2\tan^2\frac{\pi}{n}} \cdot \frac{n\tan\frac{\pi}{n}}{S} \Longrightarrow \sigma S = 4S_1S_2\cos^2\frac{\pi}{n}.$$

Solution by Luis González 1172. For the sake of generality, let $\mathcal{O} \equiv (O, R)$ be the circumsphere of the tetrahedron $ABCD$ and $\mathcal{S} \equiv (S, R')$ be the circumsphere of the tetrahedron whose vertices are the centroids A', B', C', D' of its faces against A, B, C, D. Segments AA', BB', CC', CC', DD' concur at the centroid G of $ABCD$, such that G divides them in the same ratio $1 : 3$, therefore, Tetrahedra $A'B'C'D'$ and $ABCD$ are homothetic through the homothety with center G and coefficient $-\frac{1}{3}$. Therefore, $R' = R/3$ and $OS = 4OG/3$. Let a, b, c, d, e, f be the edges of $ABCD$. By Leibniz theorem for the circumcenter O of $ABCD$, we have

$$\begin{aligned} OG^2 &= \frac{OA^2 + OB^2 + OC^2 + OD^2}{4} - \frac{a^2 + b^2 + c^2 + d^2 + e^2 + f^2}{16} \\ &= R^2 - \frac{a^2 + b^2 + c^2 + d^2 + e^2 + f^2}{16}, \end{aligned}$$

and finally,

$$OS^2 = \frac{16R^2}{9} - \frac{a^2 + b^2 + c^2 + d^2 + e^2 + f^2}{9}.$$

Solution by Spanferkel 1173. Let the radii of S_1 and S_2 be $a > b$. Then by considering a plane through the axis of the cone that intersects one of the solid spheres, say S_3, in a maximal circle, we get by Descartes' theorem for the radius r of S_3:

$$\frac{1}{\sqrt{r}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}},$$

or

$$r = \frac{ab}{(\sqrt{a} + \sqrt{b})^2}.$$

For the (half) opening angle ϕ of the cone, we have

$$\sin \phi = \frac{a - b}{a + b}.$$

Define ψ by

$$\sin \psi = \frac{b - r}{b + r},$$

(this is the corresponding angle intervening between S_2 and S_3).

Let R denote the distance of the center of S_3 from the axis of the cone. Then we have

$$\begin{aligned} R &= (b + r)\sin(\phi + \psi) \\ &= 2\frac{(a - b)\sqrt{br} + (b - r)\sqrt{ab}}{a + b} \\ &= \frac{2ab(a + b + \sqrt{ab})}{(a + b)(\sqrt{a} + \sqrt{b})^2}. \end{aligned}$$

So,

$$\frac{r}{R} = \frac{a + b}{2(a + b + \sqrt{ab})} = \frac{1}{2 + \frac{2\sqrt{x}}{1+x}},$$

where $x = b/a$. This function is decreasing in $(0, 1)$ with values between $\frac{1}{2}$ and $\frac{1}{3}$. As we want

$$\frac{r}{R} = \sin \frac{\pi}{n},$$

we get $n = 7, 8, 9$. The limit case $n = 6$ (for $x \to 0$) cannot be attained.

Solution by Mij 1174. We assume for the sake of contradiction that all angles determined by any two of the five rays are greater than $90°$.

Consider a sphere centered at O, and let Ox, Oy, Oz, Ot, Or intersect the sphere at x, y, z, t, r respectively. The plane perpendicular to Ox at O cuts the sphere into two hemispheres. Let A be the hemisphere containing x. y cannot lie on A, because if it did, the angle $\angle xOy$ would be less than or equal to $90°$. By the same argument z, t, r cannot lie on A. Define hemispheres B, C, D, and E similarly, to contain y, z, t, r, respectively and have similar properties.

Each of the five hemispheres has a measure of 2π radians. Because of the assumption, any two hemispheres must overlap at a solid angle of less than π radians. Let the intersection of A and B have measure $\theta < \pi$. The union of A and B must have measure $2\pi + 2\pi - \theta = 4\pi - \theta$, and a complement G of measure $4\pi - (4\pi - \theta) = \theta$, and be bounded by two semi-great-circles that meet at two points, p and q. The complement G is contained within what it would be if its measure were its upper bound of π.

The intersection of G and C has a minimum of $\dfrac{\theta}{2}$ as z approaches either p or q, so it is greater than $\dfrac{\theta}{2}$. Thus the union of A, B, C has measure greater than

$$(4\pi - \theta) + \frac{\theta}{2} = 4\pi - \frac{\theta}{2},$$

so the complement H of this union has measure less than $\theta/2 < \pi/2$, and is contained within what it would be if z were at p or q.

Finally, H is contained within what it would be if z were at p or q and, since H is a subset of G, within what H would be if θ equals π. This would be a 90°-90°-90° spherical triangle, so H is a subset of a 90°–90°–90° spherical triangle. Since t and r must lie on H, and therefore a 90°–90°–90° spherical triangle, $\angle tOr \le 90°$. We have contradicted our assumption, so we are done.

Solution by Dinoboy 1175. Consider three points A, B, C on a sphere and for a point X denote X' is reflection across the center of the sphere. Then note that a point D satisfies the center of the sphere is in tetrahedron $ABCD$ if and only if D lies in the spherical section bounded by the arcs $A'B', A'C', B'C'$ (let's just call this triangle $A'B'C'$ for simplicity). Thus the problem reduces to finding the expected area of a triangle on a sphere of area 1. But note that every point of the sphere is in exactly one of XYZ for $X \in \{A, A'\}, Y \in \{B, B'\}, Z \in \{C, C'\}$ so it follows the sum of those triangle's areas is 1. Then since we have established each triangle can be uniquely paired with 7 other triangles, it immediately follows the expected area is $1/8$, so we are done.

Solution by Spanferkel and Arqady 1186. Generally, for vectors $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^n$, we can write the "spherical triangle inequality" $\alpha + \beta \ge \gamma$, as

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma \le 1 + 2\cos\alpha\cos\beta\cos\gamma.$$

Equivalently

$$\sum \frac{(\vec{a}\cdot\vec{b})^2}{\vec{a}^2\vec{b}^2} \le 1 + 2\frac{(\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{c})(\vec{c}\cdot\vec{a})}{\vec{a}^2\vec{b}^2\vec{c}^2},$$

or

$$\sum \vec{c}^2(\vec{a}\cdot\vec{b})^2 \le \vec{a}^2\vec{b}^2\vec{c}^2 + 2(\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{c})(\vec{c}\cdot\vec{a}).$$

Solution by Arne 1187. Consider a box inscribed in the sphere with equation $x^2 + y^2 + z^2 = 1$, such that the coordinates of its vertices are $(\pm x, \pm y, \pm z)$. The volume of the box is $8xyz$. Of course the volume of the box is strictly smaller than the volume of the sphere, which is $4\pi/3$.

Solution by Fedor Petrov 1188. The idea of possible geometric solution is that if any four points of our curve lie in the same half-space, then the whole curve also does. (Boundary of half-space contains the centre of a sphere). It follows from Helly theorem. Then we just need to prove that if A, B, C, and D are four points on a sphere such that O lies inside $ABCD$, then the perimetr of $A-B-C-D-A$ is at least 2π. Let D' be opposite of D, then D' lies inside spherical triangle ABC, so we have $AD' + CD' < AB + BC$ (the proof is as on the plane). But,

$$AD + DC = 2\pi - (AD' + CD'),$$

so it's exactly what we need.

Solution 1189. The problem reduces to solving $\sqrt{(\sqrt{2}-1)^2 + 1^2 + (z-1)^2} = 2$ for z. The answer is $2 + \sqrt[4]{8}$.

Solution by Grobber 1190. This idea is on the verge of becoming too old, unfortunately...[2]

Consider the circle as the projection of a sphere of radius n on the plane, and in this sphere, consider some spherical segments which are projected onto bands in our circle, each band being contained between two parallel lines, each line a distance 1 from one of our chords.

The hypothesis tells us that the segments cover the sphere (the area of the sphere), so the sum of their areas must be at least $4\pi n^2$. However, the lateral area of a segment with height h is $2\pi nh$, and since all our segments have height ≤ 2, we get the desired conclusion.

Solution by Grobber 1191. In other words, the altitudes are concurrent in a spherical triangle.

Let O be the center of the sphere. We know that OM is contained in the unique plane through OA orthogonal to (OBC), and also in the unique plane through OB orthogonal to (OCA), and we want to prove that it is also contained in the plane through OC which is orthogonal to (OAB). The problem is now this:

Let α, β, γ be three planes through a point O, and let $a = \beta \cap \gamma, b = \gamma \cap \alpha, c = \alpha \cap \beta$. Prove that the planes $\pi_\alpha, \pi_\beta, \pi_\gamma$, passing through a, b, c and orthogonal to α, β, γ respectively, share a point different from O (in other words, they share a line).

Pick $A \in a$, and then $B \in b, C \in c$ such that $AB \perp c, AC \perp b$. It's easy to see that $BC \perp a$ (you can prove this just like you prove the concurrence of the altitudes in a triangle by using scalar products), which means that $\pi_\alpha, \pi_\beta, \pi_\gamma$ cut the triangle ABC along its altitudes. Since they are concurrent, the concurrence point is a point different from O belonging to all π's.

[2]Written on April 17, 2005

Solution by Grobber 1193. Well, since $\angle BOC + \angle COA + \angle AOB \leq 2\pi$, we get that at least one of them is $\leq 120°$, and that's pretty much it.

Solution by Yetti 1194. Let A, B, C be the centers of the 3 congruent spherical balls $(A), (B), (C)$, touching each other and forming an equilateral triangle with the side $AB = 2r = 20$ cm. Let H be the orthocenter of this equilateral triangle. Let D be the center of the sphere (D) of the hemispherical dish centered in the common tangent plane of the spheres $(A), (B), (C)$. The distance of the center A from the vertical axis DH of the hemispherical dish is equal to

$$AH = AB\frac{\sqrt{3}}{3} = \frac{2r\sqrt{3}}{3}.$$

Using Pythagorean theorem for the right angle triangle $\triangle AHD$,

$$DA = \sqrt{DH^2 + AH^2} = \sqrt{r^2 + \frac{4r^2}{3}} = r\sqrt{\frac{7}{3}}.$$

The tangency point T of the spheres (A) and (D) is on their center line DA and the radius of the sphere (D) is

$$\begin{aligned} R = DT = DA + AT &= r\sqrt{\frac{7}{3}} + r \\ &= \left(1 + \sqrt{\frac{7}{3}}\right) r \doteq 2.5275\ r \\ &= 25.275 \text{ cm.} \end{aligned}$$

Solution by Grobber 1195. I think (a) isn't that hard (not even the 3D version).

Let's assume a certain point from the circumference of the planet isn't visible from any point from any of the other planets. We draw the tangent to the circle at that particular point. The semi–plane defined by this tangent, which doesn't contain the circle, can't contain any points from any circles, because otherwise the point would be visible, but this means that the point is on the convex hull, and not inside it, which ends the proof.

We do the same in the case of spheres, by drawing the plane which is tangent to the sphere at the point which we want to show is invisible.

This only defines the set of points which are invisible from any other points: those points which are ON the convex hull, and not INSIDE it. Proving that this region has an area equal to that of each of the spheres is another problem...

An observation: for each sphere which has invisible points, the region which is invisible is a polygon on the sphere; by this I mean a polygon on the sphere, which has arcs of large circles of the sphere as its "sides."

Solution by Spanferkel 1198. Here is a geometrical proof. Let $a, b, c \in \mathbb{R}^n$ be the corresponding vectors. Then we can write the inequality as

$$x^2(a^2b^2 - \langle a, b\rangle^2) \geq a^2\langle b, x\rangle^2.$$

This inequality is equivalent to

$$1 \geq \frac{\langle b, x\rangle^2}{b^2x^2} + \frac{\langle a, b\rangle^2}{a^2b^2} = \cos^2(b, x) + \cos^2(a, b).$$

The condition is $\cos(a,x)=0$, i.e. a and x are orthogonal. Denote the angles $\alpha := (a,b)$, $\beta := (b,x)$ and $\gamma := (a,x) = \frac{\pi}{2}$. Now,

$$1 \geq \cos^2\alpha + cos^2\beta \Leftrightarrow \cos(\alpha+\beta)\cos(\alpha-\beta) \leq 0.$$

So, it is clear from the triangle inequalities in the spherical triangle formed by α,β,γ that $\alpha+\beta \geq \gamma = \frac{\pi}{2}$ and

$$-\frac{\pi}{2} \leq \alpha-\beta \leq \frac{\pi}{2}.$$

The result follows.

Solution by Farenhajt 1199. If the sector rotates around AO, we get the spherical sector, which has the volume $V=\frac{2}{3}r^2\pi h$, with h being the height of the corresponding spherical cap, in this case $h=r(1-\cos\theta)$. On the other hand, if d is the distance from centroid C to the line OA, then,

$$V = \frac{r^2\theta}{2}\cdot 2d\pi \qquad \text{and} \qquad d = \frac{2r(1-\cos\theta)}{3\theta}.$$

Let C_1 be the foot of the perpendicular from C to AO. Then $OC_1 = d\cot\frac{\theta}{2}$, hence

$$\begin{aligned}
\tan\varphi &= \frac{d}{r-d\cot\frac{\theta}{2}}\\
&= \frac{2r(1-\cos\theta)}{3\theta}\cdot\frac{3\theta}{3r\theta-2r(1-\cos\theta)\cot\frac{\theta}{2}}\\
&= \frac{2(1-\cos\theta)}{3\theta-2(1-\cos\theta)\cot\frac{\theta}{2}}\\
&= \frac{1-\cos\theta}{\frac{3}{2}\theta-\sin\theta},
\end{aligned}$$

because

$$(1-\cos\theta)\cot\frac{\theta}{2} = 2\sin^2\frac{\theta}{2}\cot\frac{\theta}{2} = \sin\theta.$$

Therefore,

$$\varphi = \arctan\frac{1-\cos\theta}{\frac{3}{2}\theta-\sin\theta}.$$

Solution by Yetti 1200. The z-axis cuts the sphere at U,L (for upper and lower) Plane $\mathcal{P}_K$ perpendicular to the z-axis through the sphere center K cuts the segment UL at its midpoint. Therefore, spherical caps cut off by planes $\mathcal{P}_U,\mathcal{P}_L$ perpendicular to the z-axis through U,L are congruent. The $yx-$ and $zx-$planes cut the spherical surface of the 2 caps in identical arc pattern. Since the black-white colors are exchanged on the $xy-$plane cutting the segment UL in its interior, the spherical caps have complementary colors: $S_B(U)=S_W(L)$ (upper cap black surface = lower cap white surface), and $S_W(U)=S_B(L)$ (upper cap white surface = lower cap black surface).

Summing the black color surface of the 2 caps:

$$S_B(U)+S_B(L) = S_W(L)+S_W(U).$$

Sums of black and white color surfaces of the 2 caps are equal.

For the remaining spherical frustum, integrate. R is the sphere radius. Using spherical coordinates (r, φ, θ) with origin at the sphere center. (Cylindrical coordinates (ϱ, φ, z) with the same origin would serve as well). Thus,

$$\theta \in (-\tfrac{\pi}{2}, +\tfrac{\pi}{2}),$$

is angle from the plane $\mathcal{P}_K$ perpendicular to the z-axis through the sphere center. Integration limits of θ are $\pm\theta_m$ to the upper and lower caps. φ is rotation angle around the line parallel to the z-axis through the sphere center. Integration limits of φ for the black color are $(A(\theta), B(\theta))$ and $(C(\theta), D(\theta))$ dependent on θ. These are intersections of the $yz-$ and $zx-$planes with a circle on the sphere at the angle θ.

Integrating the black color surface of the spherical frustum:

$$\begin{aligned} S_B &= R^2 \int_{-\theta_m}^{+\theta_m} \cos\theta \ \mathrm{d}\theta \left(\int_{A(\theta)}^{B(\theta)} \mathrm{d}\varphi + \int_{C(\theta)}^{D(\theta)} \mathrm{d}\varphi \right) \\ &= R^2 \int_{-\theta_m}^{+\theta_m} \cos\theta \ \mathrm{d}\theta \, (\pi) \\ &= 2\pi R^2 \sin\theta_m. \end{aligned}$$

Integrating the white color surface of the spherical frustum:

$$\begin{aligned} S_W &= R^2 \int_{-\theta_m}^{+\theta_m} \cos\theta \ \mathrm{d}\theta \left(\int_{B(\theta)}^{C(\theta)} \mathrm{d}\varphi + \int_{D(\theta)}^{A(\theta)} \mathrm{d}\varphi \right) \\ &= R^2 \int_{-\theta_m}^{+\theta_m} \cos\theta \ \mathrm{d}\theta \, (\pi) \\ &= 2\pi R^2 \sin\theta_m \end{aligned}$$

Integrals of the black and white color surfaces of the spherical frustum are also equal.

Solution by Keyree10 1201. Consider an equilateral triangle ABC such that when A is rotated about BC to hit the sphere again at A', triangles $A'BA$ and $A'BC$ are equilateral. (We shall see later that such a triangle indeed exists) Similarly rotate B about AC with its distance from AC fixed, to hit the sphere at B'. (Ditto for C). Now, we've got a lot of equilateral triangles. Without loss of generality, Let A,B be red and C be blue, so that C' is blue. Since triangles ABA' and BAB' are equilateral, A' and B' must be blue. Then we have $A'B'C'$, a monochromatic equilateral triangle. and the proof is finished.

The required triangle ABC can be found in an Icosahedron inscribed in a sphere. A solution without this knowledge is possible (I think), but it would require a painfully long and rigorous argument to show that such a triangle exists using Intermediate value theorem.

Solution by Luis González 1202. This is a classical construction. Assume we are given a solid sphere $\mathcal{E}$ and a plane π for ruler-compass constructions. Draw a circumference ω with arbitrary radius and center P on $\mathcal{E}$ and take three points A, B, C on ω. Construct the rectilinear triangle $\triangle ABC$ on π by transporting the chords BC, CA, AB with the compass. Let O' be the circumcenter of $\triangle ABC$. Then its circumradius $O'A = \varrho$ is the orthogonal projection of the spherical radius PA on the plane ABC. Therefore, if

P' is the antipode of P on $\mathcal{E}$, the right $\triangle APP'$, given its leg AP and altitude $AO' = \varrho$ on the hypotenuse PP', is constructible. This produces the diameter PP' of the spherical surface $\mathcal{E}$.

Solution by Luis González 1203. Let α, β, γ denote the planes DBC, DAC, DAB. For conveniece, let's cut the trihedron α, β, γ with the spherical surface $\mathcal{E}$ with center D and radius DO. Rays DA, DB, DC cut $\mathcal{E}$ at A', B', C', thus $A'O, B'O, C'O$ become internal angle bisectors of the spherical triangle $A'B'C'$. Great circle (D, DO) cuts $A'C', A'B'$ at M, N such that $\angle OMA' = \angle ONA' = 90°$. Therefore, M, N coincide with the projections of O on $A'C', A'B'$. If L is the projection of O on $B'C'$, then $\angle LOB' = \frac{1}{2}\angle NOL$ and $\angle LOC' = \frac{1}{2}MOL$. Thus $\angle B'OC' = \frac{1}{2}(180°) = 90°$.

Solution by Ocha 1204. Suppose the maximum and minimum latitude of some town is given by points ℓ_M and ℓ_m, then no other town can intersect the band of width $w = |\ell_M \ell_m|$. The surface area of this band is proportional to it's width, i.e. $A = 2\pi rw$, where r is the radius of the planet. If the most easterly and westerly points of the town subtend an angle θ with the center of the sphere, then they chop the latitudinal band into an area of $\frac{\theta}{2\pi}A = r\theta w$. Now the town must be completely within this square(ish) area and no town can enter the latitudinal or longitudinal bands that define the town.

Let $\{w_i\}_{i=1}^N$ be the widths of longitudinal bands made by the towns, and let $\{\theta_i\}_{i=1}^N$ be angles which represent width of the longitudinal bands of the towns. Then $\sum_i w_i \leq 2r$ and $\sum_i \theta_i \leq 2\pi$ and the area of town i is at most

$$A_i = r\theta_i w_i = \frac{4\pi r^2}{1000}.$$

So by Cauchy-Schwarz:

$$2r \cdot 2\pi \geq \left(\sum_i w_i\right)\left(\sum_i \theta_i\right) \geq \left(\sum_i \sqrt{w_i\theta_i}\right)^2 = \left(N\sqrt{\frac{4\pi r}{1000}}\right)^2.$$

Therefore, $N \leq \sqrt{1000}$; so $\max(N) = 31$.

Solution by Facis 1205. Let $v = (\sin x \cos y, \sin x \sin y, \cos x)$. The similarity to spherical coordinates here makes it clear that the only restriction on this vector is that its length is 1. Now, we can write:

$$f(v(x, y)) = (2, 3, 6) \cdot v,$$

where the dot means the dot product. By Cauchy-Schwarz, f is maximized when

$$v = \frac{(2, 3, 6)}{\sqrt{2^2 + 3^2 + 6^2}} = \frac{(2, 3, 6)}{\sqrt{49}} = \left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right),$$

in which case $f(v) = 7$; and f is minimized when

$$v = \left(-\frac{2}{7}, -\frac{3}{7}, -\frac{6}{7}\right),$$

in which case $f(v) = -7$.

Solution by Luis González 1206. Let the circumsphere S have center O and radius R. Then we have

$$\frac{PA}{PA_1}+\frac{PB}{PB_1}+\frac{PC}{PC_1}+\frac{PD}{PD_1}=\frac{PA^2+PB^2+PC^2+PD^2}{p(P,S)}=4.$$

Therefore, we get

$$PA^2+PB^2+PC^2+PD^2=4(R^2-PO^2).$$

Let G be the centroid of $ABCD$ and a,b,c,d,e,f denote its edges. By Leibniz theorem for P,G and O,G we get

$$PA^2+PB^2+PC^2+PD^2=4PG^2+\frac{a^2+b^2+c^2+d^2+e^2+f^2}{4},$$
$$OA^2+OB^2+OC^2+OD^2=4OG^2+\frac{a^2+b^2+c^2+d^2+e^2+f^2}{4}.$$

From these last two, we get

$$PA^2+PB^2+PC^2+PD^2=4(PG^2+R^2-OG^2).$$

Combining the first and the last equations yields:

$$4(PG^2+R^2-OG^2)=4(R^2-PO^2).$$

Thus, $PG^2+PO^2=OG^2$, which means $\angle OPG=90°$. Therefore, locus of P is the spherical surface with diameter $\overline{OG}$.

Solution by Luis González 1207. For convenience let the unit sphere with center O cut the trihedron O_{ABC} into a spherical triangle XYZ with sides α,β,γ. $X\in OA$, $Y\in OB$ and $Z\in OC$. Let h be the spherical altitude issuing from Z. Then $\sin h=\sin\beta\cdot\sin X$, but by cosine theorem in XYZ (Bessel Formula) we have

$$\cos X=\frac{\cos\alpha-\cos\beta\cos\gamma}{\sin\beta\sin\gamma},$$

implying

$$\sin^2 X=1-\left(\frac{\cos\alpha-\cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right)^2,$$

which means

$$\sin(O_{XYZ})=\sin\gamma\cdot\sin h=\sqrt{\sin^2\beta\sin^2\gamma-(\cos\alpha-\cos\beta\cos\gamma)^2}.$$

Substituting $\sin^2\beta=1-\cos^2\beta$ and $\sin^2\gamma=1-\cos^2\gamma$ yields

$$\sin(O_{XYZ})=\sin(O_{ABC})=\sqrt{2\cos\alpha\cos\beta\cos\gamma-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+1}.$$

Since $V=\frac{1}{6}OA\cdot OB\cdot OC\cdot\sin(O_{ABC})$, then it follows that

$$36V^2=OA^2\cdot OB^2\cdot OC^2\cdot(2\cos\alpha\cos\beta\cos\gamma-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+1).$$

P.S. The proof is substantially easier using vectors (scalar product).

Solution by Luis González 1208. Let A be the apex of the pyramid. Denote the vertices of the cyclic n-gon $P_1P_2P_3...P_n$ and the projections of H onto $AP_1, AP_2, AP_3, \ldots, AP_n$ as $H_1, H_2, H_3, \ldots, H_n$. Note that

$$\overline{AH}^2 = \overline{AP_1} \cdot \overline{AH_1} = \overline{AP_2} \cdot \overline{AH_2} = \overline{AP_3} \cdot \overline{AH_3} = \overline{AP_n} \cdot \overline{AH_n}.$$

Thus, the points $H_1, H_2, H_3, \ldots, H_n$ lie on the inverse image of the circumcircle $\mathcal{C}$ of the n-gon, under the inversion with center A and radius $\overline{AH}$. The spherical surface $\mathcal{E}$ passing through $A, \mathcal{C}$ is taken into a plane π' and the plane π containing the base of the pyramid is taken into a spherical surface $\mathcal{E}'$ passing through A. Therefore, $H_1, H_2, H_3, \ldots, H_n$ lie on the intersection (circumference) $\mathcal{C}' \equiv \mathcal{E}' \cap \pi'$. Moreover, $\mathcal{C}$ and $\mathcal{C}'$ lie on a same spherical surface.

Solution by JSGandora 1209. Write

$$f = \frac{md}{m^2-1} = \frac{md}{(m-1)(m+1)}.$$

This can be an integer if the denominator divides the numerator. However, by the Euclidean Algorithm, the denominator is relatively prime to m. Thus $(m-1)(m+1) = m^2-1|d$ so $d = (m^2-1)n, n \in \mathbb{Z}^+$. So, the solution set is

$$(m, d) = (k, nk^2 - n), n \in \mathbb{Z}^+, k \in \mathbb{Z}^+ \backslash \{1\}.$$

Solution by Mavropnevma 1210. This solution was written by Dan Schwarz who was active on AoPS with username **mavropnevma**. May he rest in peace...

a) Let R be the radius of the sphere. Then the area of the k-th disk exposed is

$$\alpha_k = \pi R^2 \left(1 - \left(1 - \frac{2k}{n}\right)^2\right) = 4\pi R^2 \frac{k(n-k)}{n^2}.$$

We are told that α_1 and α_2 are integer. But $d = \gcd(n-1, 2(n-2)) \mid 2$, and clearly also $d \mid k(n-k)$ for all k. By Bézout's relation there exist integers u, v such that $u(n-1) + v(2n-4) = d$, and then, denoting $k(n-k) = md$, we have

$$mu(n-1) + mv(2n-4) = md = k(n-k),$$

hence $mu\alpha_1 + mv\alpha_2 = \alpha_k$ is an integer.

b) The area of the sphere is $A = 4\pi R^2$. We have

$$2\alpha_1 - \alpha_2 = \frac{2}{n^2} A$$

being an integer N, so $A = \frac{n^2}{2} N$, while $\alpha_2 = (n-2)N$. Clearly A being integer is equivalent to

$$(n \equiv 0 \pmod 2) \vee (N \equiv 0 \pmod 2),$$

which in turn is equivalent to α_2 being even.

Harun Šiljak's Hint 1211. Find the intersection. Use the cylindrical coordinates to calculate the volume of cone under the intersection, and spherical for the volume of sphere above the intersection. Don't forget your Jacobians! If you're not well into calculations with n-integrals, note that the angular coordinates will "move" freely in their standard bounds, and ρ can "move" in terms of the surface equations, while z in cylindrical coordinates can "move" from 0 to the intersecting value.

Solution 1212. The answer is $2\pi/15$.

Solution by JBL 1213. Surely it should be possible to do with a family of hyperboloids of one sheet, right? Say, take the family of lines given by $x = x_0$, $y = tx_0$ and $z = t$ for $t \in \mathbf{R}$ and then all rotations of these lines around the z-axis.

Solution 1214. The answer is $\tan^2 x + 2\tan^2 y = \boxed{81}$.

Solution 1215. Put $z = \rho\cos\phi, x = \rho\sin\phi\cos\theta$, and $y = \rho\sin\phi\sin\theta$. The answer to the limit, after simplification, is 0.

Solution by Rchokler 1216. Even without converting to spherical coordinates, we know that it is a double cone since it is clear that at each height z, the cross section is a circle of radius $r = \sqrt{3}\cdot|z|$. However, we can see it in spherical coordinates too. After conversion to spherical coordinates, the equation $x^2 + y^2 = 3z^2$ becomes

$$(\rho\cos\theta\sin\phi)^2 + (\rho\sin\theta\sin\phi)^2 = 3(\rho\cos\phi)^2,$$

which simplifies to

$$\tan\phi = \pm\sqrt{3} \implies \phi = \boxed{\frac{\pi}{2} \pm \frac{\pi}{6}}.$$

The $|\tan\phi| = \sqrt{3}$ is actually the key: it basically shows that all points with a given ϕ are on the graph. You're allowed to vary ρ and θ, but ϕ must be those angles. This gives essentially a straight line passing through the origin, and its image as it's rotated around the z–axis. That's a cone.

Solution by WeakMathemetician 1217. Substitute $z = \sin\theta, x = \cos\theta\sin\phi, y = \cos\theta\cos\phi$. Substituting this into the equation gives:

$$\cos^4\theta(\sin^4\phi + \cos^4\phi) - 2\sin^4\theta - 3\sqrt{2}\sin\theta\cos^2\theta\sin\phi\cos\phi.$$

Letting $\sin^4\phi + \cos^4\phi = 1 - 2\sin^2 phi - 2\cos^4\phi$ and also letting $2\sin\phi\cos\phi = \sin\alpha$, where $\alpha = 2\phi$ the equation becomes:

$$\cos^4\theta(1 - \frac{1}{2}\sin^2\alpha) - 2\sin^4\theta - \frac{3\sqrt{2}}{2}\sin\theta\cos^2\theta\sin\alpha.$$

Substituting $\cos^2\theta = 1 - \sin^2\theta$ yields:

$$(1 - \sin^2\theta)^2(1 - \frac{1}{2}\sin^2\alpha) - 2\sin^4\theta - \frac{3\sqrt{2}}{2}\sin\theta(1 - \sin^2\theta)\sin\alpha.$$

Also let $\sin\theta = b$ and $\sin\alpha = a$; then:

$$(1 - b^2)^2(1 - \frac{1}{2}a^2) - 2b^4 - \frac{3\sqrt{2}}{2}b(1 - b^2)a.$$

This is a quadratic in a and rearranges to:

$$\frac{-1}{2}(1-b^2)^2a^2 - \frac{3\sqrt{2}}{2}b(1-b^2)a + (1-2b^2-b^4).$$

Maximizing quadratics is easy and gives that the maxima will happen when:

$$a = \frac{-3\sqrt{2}b}{2(1-b^2)}.$$

When you substitute this value of a into the original quadratic you get the expression:

$$\frac{65}{64} - (b^2 - \frac{1}{8})^2.$$

Solution by Rchokler 1218. We need to use spherical coordinates (ρ, ϕ, θ). Note that $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho\cos\phi$.

The Jacobian is then

$$J = \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} = \rho^2\sin\phi.$$

Note that in the problem, $\rho = 20$ is fixed, which makes the Jacobian $400\sin\phi$. Also:

$$TJ = 160000\left(\sin^3\phi\left(\sin^2\theta + \sin 2\theta + 1\right) - 2\sin\theta\sin^2\phi\cos\phi + \sin\phi\cos^2\phi\right).$$

Thus the integral for the thermal flux is:

$$160000\int_0^{\pi}\int_0^{2\pi}\left(\sin^3\phi\left(\sin^2\theta + \sin 2\theta + 1\right) - 2\sin\theta\sin^2\phi\cos\phi + \sin\phi\cos^2\phi\right)\,d\theta\,d\phi,$$

which is equal to

$$160000\int_0^{\pi}(3\pi\sin^3\phi + 2\pi\sin\phi\cos^2\phi)\,d\phi = \frac{2560000\pi}{3}.$$

The surface area is $32000\pi/3$. This makes the average temperature is 80 degrees, quite comfortable for life if the unit is assumed to be Fahrenheit.

Solution by Yenlee 1219. This depends on what you know already / what you are allowed to use. The fastest proof is to just use the Gauss-Bonnet Theorem, but of course the Gauss-Bonnet Theorem is a much more sophisticated theorem. To prove it more directly, you'd almost certainly want to break your polygon into triangles, and then prove it for triangles.

You could probably do this explicitly by using surface integration on the sphere and spherical coordinates: Set up two sides of the triangle along longitudes coming down from the north pole, and figure out how to write the third side in spherical coordinates. But this is probably a mess (I confess I've never done this exercise).

Here's a hint for a more clever proof: Extend the sides of the triangle to great circles on the sphere. These great circles divide the sphere into several regions. You want to find the area of your triangle, which is enclosed by all 3 of the circles. On the other hand, it's easy to figure out the area of any region enclosed by just 2 of these circles. Now see what happens when you add up the areas of the these regions (you'll need to have a good picture in mind of these 3 great circles on the sphere).