The Olympiad Algebra Book

VOLUME I + Complementary:

1220 Polynomials and Trigonometry Problems

+407 Review Problems

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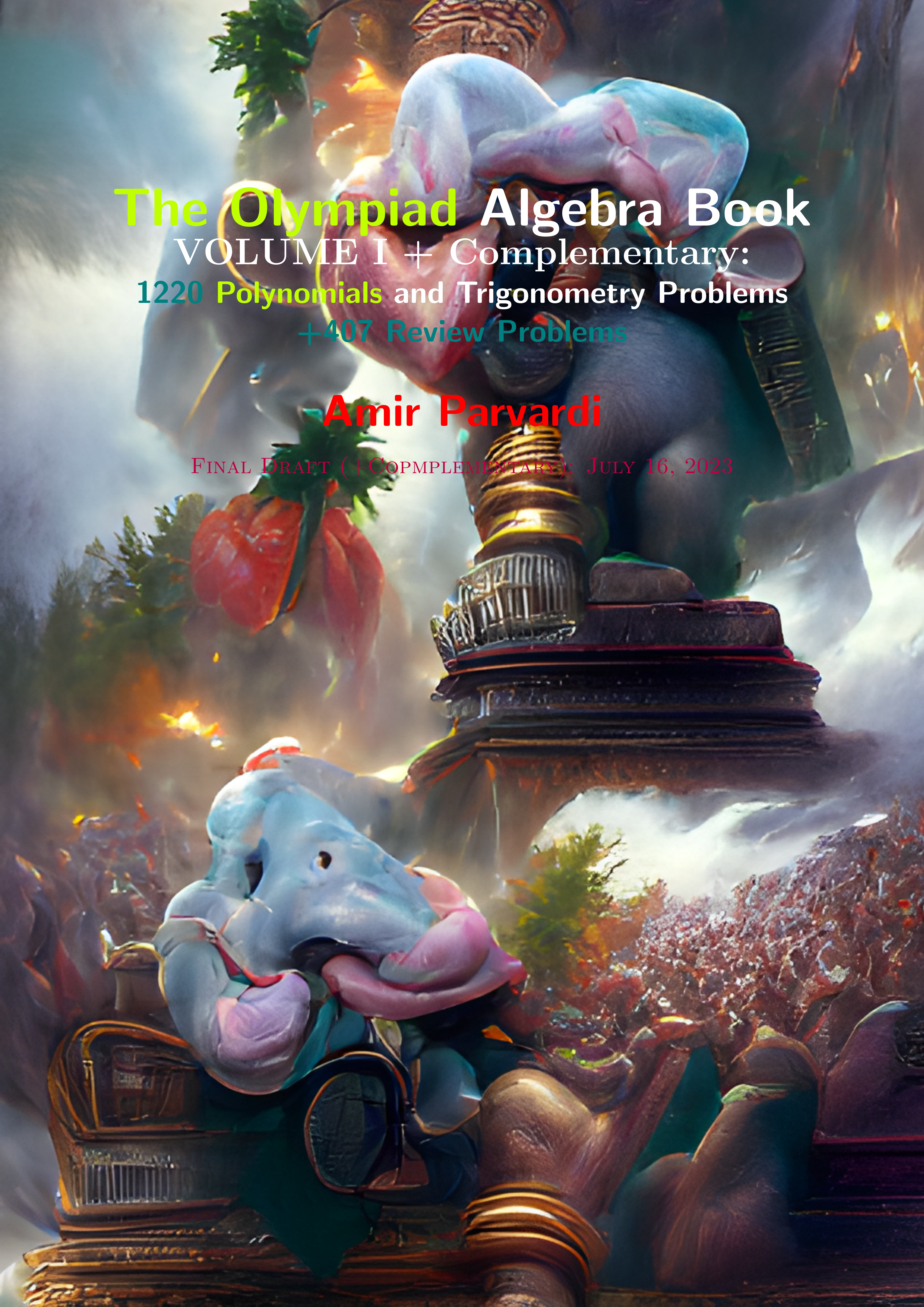

Preface

Foreword

Azermalohg speaks to **Rima**:

> Why are you so afraid of the IllLyrans?
> What has your fear had you achieved?
> You have made yourself weary for lack of sleep,
> You only fill your flesh with grief,
> You only bring the distant days closer.
> Humankind's fame is cut down
> like reeds in a reed-bed.
> A fine young man, a fine young girl... at grip of Death.
> You have seen Death,
> You have touched the face of Death,
> You hear the voice of Death lamenting in your ears,
> Savage Death just cuts humankind down.
> Sometimes we have hope,
> sometimes we make a wish,
> but then our airplanes are shot in the air.
> Sometimes there is hostility in the land,
> but in the end, only the most benevolent will remain.
> The ruthless IllLyrans bring Death with themselves;
> but the merciful Lyrans will always prevail.
> Remember, the night is darkest just before dawn.

Synopsis

The Olympiad Algebra Book comes in two volumes. The first volume, dedicated to Polynomials and Trigonometry, is a collection of lesson plans containing 1220 beautiful problems, around two-thirds of which are polynomial problems and one-third are trigonometry problems. The second volume of The Olympiad Algebra Book contains 1220 Problems on Functional Equations and Inequalities, and I hope to finish it before the end of Summer 2023. I hope I can finish collecting the FE and INEQ problems by June 29^{th}, as a reminder of the 1220 Number Theory Problems published as the first 1220 set of J29 Project. The current volumes has 843 Polynomial problems and 377 Trigonometry questions, the last 63 of which are bizarre spherical geometry problems! I was hoping to add a small section for the review of polynomials and trigonometry problems in the second volume of the Olympiad Algebra Book, but this section grew too large (407 problems) and I decided to publish it on its own, but also added it as the final chapter on the first volume. The majority of the review questions are chosen from American competitions such as AIME (American Invitational Mathematics Examination), HMMT (Harvard-MIT Math Tournament), CHMMC (Caltech Harvey Mudd Math Competition), and PUMaC (Princeton University Math Competition). All of the AIME problems are copyright © Mathematical Association of America, and they can be found on the Contests page on the Art of Problem Solving website. In this document, the links to the problems posted on AoPS forums are embedded (if existent).

This book is supposed to be a problem bank for Algebra, and it forms the resource for the first series of the KAYWAÑAN Algebra Contest. I suggest you start with Polynomials, and before you get bored or exhausted, also start solving Trigonometry problems. If you find these problems easy and not challenging enough, the Spherical Trigonometry lessons and problems are definitely going to be a must try!

Amir Parvardi,
Vancouver, British Columbia,
July 16, 2023

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Part I

KAYWAÑAN Algebra Contest Olympiad Algebra Book (Vol. I): Polynomials & Trigonometry

Chapter 1

Olympiad Algebra 101: Polynomials

1.1 Olympiad Algebra 001: Definitions & Corollaries of Pre–Algebra

We begin by reminding ourselves of why polynomials are the first topic we need to study in Pre–Algebra. The study of the relationships between the roots of polynomials, a field whose master is undoubtedly Évariste Galois, is the hidden root of the stout tree of Algebra. We may begin stating the definitions of polynomials and how their roots are related to each other, involving the Fundamental Theorem of Algebra, which states that each polynomial $P(x)$ with complex coefficients has exactly n roots in the complex plane. This ferocious fact about the roots of any polynomial is, indeed, the Most Fundamental Theorem in Algebra.

1.1.1 Introduction to Olympiad Algebra

Introduction to Olympiad Pre–Algebra

In olympiad Algebra, starting from polynomials, we seek to find special examples of equations that have solutions that seem interesting in some way. For instance, regarding the Fundamental Theorem of Algebra just stated, we my ask special questions, for instance, *Casus irreducibilis* (Latin for "the irrational case"): can we solve all third-degree polynomials with real radicals? And the answer is no. For example,

Problem 1. Prove that the cubic equation $2x^3 - 9x^2 - 6x + 3 = 0$ has three real roots. You can check this by finding the discriminant Δ, which is given by

$$\Delta := \big((x_1 - x_2)(x_1 - x_3)(x_2 - x_3)\big)^2 = 18abcd - 4ac^3 - 27a^2d^2 + b^2c^2 - 4b^3d,$$

where a, b, c, d must be replaced with the coefficients of our polynomial. Prove that if $\Delta > 0$, then x_1, x_2, x_3 would be three real roots, but, in the case of the three roots of our polynomial $2x^3 - 9x^2 - 6x + 3 = 0$, they are not presentable in any real radical form and we require imaginary radicals to solve this specific equation [from Wikipedia] "are given by:

$$t_k = \frac{3 - \omega_k \sqrt[3]{39 - 26i} - \omega_k^2 \sqrt[3]{39 + 26i}}{2},$$

for $k = 1, 2, 3$. The solutions are in radicals and involve the cube roots of complex conjugate numbers."

Definition of Polynomial and Roots

We state the definition of a "polynomial" in the broadest form:

Definition. A function $P(x)$ defined over complex numbers by

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \qquad \text{for} \quad n \geq 0,$$

where $a_0, a_1, \ldots, a_n$ are complex numbers, is called a **polynomial of degree** n with complex **coefficients** $a_0, a_1, \ldots, a_n$. We also write $\deg P := \deg(P(x)) = n$.

However, in most cases, we are interested in polynomials with real, rational, integer, or natural coefficients. For the sake of completeness and self-containment of Kaywañan, we need to discuss the definition of complex numbers in details, though, and we will mention the most important definitions, theorems, and identities for complex numbers. Start by studying the different representations of a complex number in the complex plane, once assuming the plane is Cartesian, and once in the Polar Plane. The most important result, then, would be the **De Moivre's Formula**, which will be mentioned just enough not to spoil the fun for later trigonometry lessons in Olympiad Algebra 401.

Equivalent Polynomials, Monic Polynomials

Definition. Two polynomials $P(x)$ and $Q(x)$ are **equivalent** if and only if

a) They have equal degrees, i.e., $\deg(P(x)) = \deg(Q(x))$; and

b) All their corresponding coefficients are the same.

In other words, assuming

$$P(x) = a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0 \quad \text{and} \quad Q(x) = b_mx^m+b_{m-1}x^{m-1}\cdots+b_1x+b_0,$$

we have $P(x) \equiv Q(x)$ if and only if $m = n$ and $a_i = b_i$ for all $i = 1, 2, \ldots, n$.

Definition. Let $P(x)$ be a polynomial of degree n, defined by

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0,$$

Then we say $P(x)$ is a **monic polynomial** if and only if $a_n = 1$.

Definition. We may introduce the derivative of polynomial $P(x)$ usually denoted $P'(x)$, where

$$\begin{aligned} P(x) &= a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0, \\ P'(x) &= na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2x + a_1. \end{aligned}$$

Theorem 1 (Fundamental Theorem of Algebra)**.** Any polynomial $P(x)$ with complex coefficients has precisely $\deg P$ complex roots.

Corollary 1. Let $P(x)$ be a polynomial of degree n with complex coefficients, defined by

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0.$$

If the equation $P(x) = 0$ has $n + 1$ solutions in $\mathbb{C}$, then $P(x) \equiv 0$.

1.1.2 Essential Polynomial Theorems

Here are some theorems that you really need to prove on your own.

Theorem 2. For two polynomials

$$A(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \quad \text{and} \quad B(x) = b_mx^m + b_{m-1}x^{m-1}\cdots + b_1x + b_0,$$

assuming $n \geq m$, we have

a) The polynomial $S(x) = A(x) + B(x)$ is a polynomial of degree at most n, whose coefficients are the sum of the corresponding coefficients of $A(x)$ and $B(x)$.

b) The polynomial $\Pi(x) = A(x)\cdot B(x)$ is a polynomial of degree $m+n$ whose coefficients $\pi_0, \pi_1, \dots, \pi_{m+n}$, where

$$\Pi(x) = \pi_{m+n}x^{m+n} + \cdots + \pi_1 x + \pi_0,$$

are calculated by

$$\begin{aligned}
\pi_0 &= a_0 b_0,\\
\pi_1 &= a_0 b_1 + a_1 b_0,\\
\pi_2 &= a_0 b_2 + a_1 b_1 + a_2 b_0,\\
\vdots \quad & \quad \vdots\\
\pi_{m+n-1} &= a_{n-1}b_m + a_n b_{m-1},\\
\pi_{m+n} &= a_n b_m.
\end{aligned}$$

Theorem 3 (Polynomial Division Theorem)**.** For two polynomials $A(x)$ and $B(x)$ with

$$A(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \quad \text{and} \quad B(x) = b_m x^m + b_{m-1}x^{m-1} \cdots + b_1 x + b_0,$$

we can define the **quotient polynomial** $Q(x)$ and the **remainder polynomial** $R(x)$, assuming $n \geq m$, by

$$A(x) = B(x)\cdot Q(x) + R(x), \qquad \text{and} \quad \deg R < \deg B.$$

In the special case when the remainder is the zero polynomial, $R(x) \equiv 0$, we say $A(x)$ is divisible by $B(x)$.

Theorem 4 (Bézout's Theorem for Polynomials AKA Factor Theorem)**.** As a special case of the Polynomial Division Theorem, in the polynomial division $A(x) = B(x)\cdot Q(x)+R(x)$, let $B(x) = x - x_0$, where $x_0 \in \mathbb{R}$. Then, $A(x_0) = R(x_0)$ and we may write:

$$A(x) = (x - x_0)\cdot Q(x) + A(x_0).$$

The factor theorem says $x - x_0$ is a factor of $A(x)$ if and only if $A(x_0) = 0$.

Theorem 5 (Unique Factorization Theorem)**.** According to the Fundamental Theorem of Algebra, all polynomials $P(x)$ with complex coefficients in the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad \text{with} \quad a_n \neq 0.$$

Let $x_0, x_1, \dots, x_n$ be the n complex roots of $P(x) = 0$. Then,

$$P(x) = a_n(x - x_1)(x - x_2)\cdots(x - x_n).$$

1.1.3 In Search of Rational Roots

Rational Root Theorem

Theorem 6. Let $P(x)$ be a polynomial with integer coefficients written as

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad \text{with} \quad a_n \neq 0.$$

. Show that $P(x)$ has a rational root $r = p/q$, where p and q are relatively prime positive integers, then p is a divisor of a_0 and q is a divisor of a_n.

Problem 2. We call a polynomial **monic** if the coefficient of the highest exponent in the polynomial equals 1. Consider the monic polynomial $P(x)$ with integer coefficients:

$$P(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0.$$

1. Prove that the equation $P(x) = 0$ does not have any roots in the form $x = p/q$ where p and q are coprime integers.
2. If the equation $P(x) = 0$ has a rational root, then this root is an integer and it divides a_0.
3. Let $x = \alpha$ be an integer root of the equation $P(x) = 0$. Prove that α divides $a_1 + \frac{a_0}{\alpha}$.
4. Let $x = \alpha$ be an integer root of the equation $P(x) = 0$. Prove that the numbers $\alpha, \alpha^2, \ldots, \alpha^n$ divide the following numbers, respectively:

$$a_0, \quad a_0 + \alpha a_1, \quad a_0 + \alpha a_1 + \alpha^2 a_2, \quad \ldots, \quad a_0 + \alpha a_1 + \cdots + \alpha^{n-1}a_{n-1}.$$

Problem 3. Solve for x:

$$x^4 + 5x^3 - 2x^2 - 9x + 5 = 0.$$

Problem 4. Solve for x:

$$2x^4 + 3x^3 - 10x^2 - 2x + 3 = 0.$$

Problem 5. Solve for x:

$$x^4 - 5x^3 + 2x^2 + 20x - 24 = 0.$$

Problem 6. Solve for x:

$$x^4 - 3x^3 - 8x^2 + 12x + 16 = 0.$$

2002 Croatia 7. Solve the equation

$$\left(x^2 + 3x - 4\right)^3 + \left(2x^2 - 5x + 3\right)^3 = \left(3x^2 - 2x - 1\right)^3.$$

2019 Greece 8. Solve in $\mathbb{R}$ the following equation

$$108(x-2)^4 + (4-x^2)^3 = 0.$$

2018 Romanian District 9. Show that the number

$$\sqrt[n]{\sqrt{2019}+\sqrt{2018}} + \sqrt[n]{\sqrt{2019}-\sqrt{2018}}$$

is irrational for any $n \geq 2$.

2017 Thailand 10. Let p be a prime. Show that $\sqrt[3]{p} + \sqrt[3]{p^5}$ is irrational.

2006 All–Russian 11. The sum and the product of two purely periodic decimal fractions a and b are purely periodic decimal fractions of period length T. Show that the lengths of the periods of the fractions a and b are not greater than T.
Note. A purely periodic decimal fraction is a periodic decimal fraction without a non-periodic starting part.

2006 Pan–African 12. Let a, b, c be three non-zero integers. It is known that the sums

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \qquad \text{and} \qquad \frac{b}{a} + \frac{c}{b} + \frac{a}{c},$$

are integers. Find these sums.

Problem 13. Let p be a prime number. Prove that the polynomial

$$P(x) = x^{p-1} + 2x^{p-2} + \ldots + (p-1)x + p,$$

is irreducible over $\mathbb{Z}[x]$.

Problem 14. If a and b are real numbers such that

$$\sqrt[3]{a} - \sqrt[3]{b} = 12 \quad \text{and} \quad ab = \left(\frac{a+b+8}{6}\right)^3,$$

find the value of $a - b$.

2014 Poland 15. Let x, y be positive integers such that

$$\frac{x^2}{y} + \frac{y^2}{x},$$

is an integer. Prove that $y \mid x^2$.

2022 Thailand 16. Determine all possible values of a_1 for which there exists a sequence $a_1, a_2, \ldots$ of rational numbers satisfying

$$a_{n+1}^2 - a_{n+1} = a_n,$$

for all positive integers n.

2017 Romania TST 17. Determine all integers $n \geq 2$ such that $a + \sqrt{2}$ and $a^n + \sqrt{2}$ are both rational for some real number a depending on n.

2006 Romania TST 18. Let p a prime number, $p \geq 5$. Find the number of polynomials of the form

$$x^p + px^k + px^l + 1, \quad k > l, \quad k, l \in \{1, 2, \cdots, p-1\},$$

which are irreducible in $\mathbb{Z}[X]$.

2021 Saudi Arabia TST 19. For a non-empty set T denote by $p(T)$ the product of all elements of T. Does there exist a set T of 2021 elements such that for any $a \in T$ one has that $P(T) - a$ is an odd integer? Consider two cases:

a) All elements of T are irrational numbers.

b) At least one element of T is a rational number.

2012 IMO Shortlist 20. Let f and g be two nonzero polynomials with integer coefficients and $\deg f > \deg g$. Suppose that for infinitely many primes p the polynomial $pf + g$ has a rational root. Prove that f has a rational root.

2003 Spain 21. Let x be a real number such that $x^3 + 2x^2 + 10x = 20$. Demonstrate that both x and x^2 are irrational.

1993 Italy 22. Find all pairs (p, q) of positive primes such that the equation $3x^2 - px + q = 0$ has two distinct rational roots.

2017 Romania 23. Define

$$P(x) = x^2 + \frac{x}{2} + b \quad \text{and} \quad Q(x) = x^2 + cx + d,$$

be two polynomials with real coefficients such that $P(x)Q(x) = Q(P(x))$ for all real x. Find all real roots of $P(Q(x)) = 0$.

2008 Iran Third Round 24. Let (b_0, b_1, b_2, b_3) be a permutation of the set $\{54, 72, 36, 108\}$. Prove that

$$x^5 + b_3x^3 + b_2x^2 + b_1x + b_0,$$

is irreducible in $\mathbb{Z}[x]$.

1.1.4 Viète's Formulas

Vieta's Formulas

Theorem 7 (François Viète's Formulas AKA Vieta's Formulas)**.** For any polynomial $P(x)$ with complex coefficients, written as

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

imagine the n roots are $x_1, x_2, \ldots, x_n$. Prove that

$$\begin{cases} \sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n & = -\dfrac{a_{n-1}}{a_n}, \\ \sum_{i \neq j} x_i x_j = x_1 x_2 + \cdots + x_{n-1}x_n & = +\dfrac{a_{n-2}}{a_n}, \\ \quad\vdots & \quad\vdots \\ \prod_{i=1}^{n} x_i = x_1 x_2 \cdots x_{n-1} x_n & = (-1)^n \dfrac{a_0}{a_n}. \end{cases}$$

Binomial Theorem

Theorem 8 (Binomial Theorem)**.** Prove that for all $x, y \in \mathbb{C}$, and positive integer n,

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{1}xy^{n-1} + y^n.$$

1993 AIME 25. Let $P_0(x) = x^3 + 313x^2 - 77x - 8$. For integers $n \geq 1$, define $P_n(x) = P_{n-1}(x-n)$. What is the coefficient of x in $P_{20}(x)$?

1996 AIME 26. Suppose that the roots of $x^3 + 3x^2 + 4x - 11 = 0$ are a, b, and c, and that the roots of $x^3 + rx^2 + sx + t = 0$ are $a+b, b+c$, and $c+a$. Find t.

2001 AIME 27. Find the sum of the roots, real and non-real, of the equation

$$x^{2001} + \left(\frac{1}{2} - x\right)^{2001} = 0,$$

given that there are no multiple roots.

2005 AIME 28. The equation

$$2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$$

has three real roots. Given that their sum is m/n where m and n are relatively prime positive integers, find $m+n$.

2008 AIME 29. Let r, s, and t be the three roots of the equation

$$8x^3 + 1001x + 2008 = 0.$$

Find $(r+s)^3 + (s+t)^3 + (t+r)^3$.

2014 AIME 30. Real numbers r and s are roots of $p(x) = x^3 + ax + b$, and $r + 4$ and $s - 3$ are roots of $q(x) = x^3 + ax + b + 240$. Find the sum of all possible values of $|b|$.

2015 AIME 31. Steve says to Jon, "I am thinking of a polynomial whose roots are all positive integers. The polynomial has the form

$$P(x) = 2x^3 - 2ax^2 + (a^2 - 81)x - c,$$

for some positive integers a and c. Can you tell me the values of a and c?" After some calculations, Jon says, "There is more than one such polynomial." Steve says, "You're right. Here is the value of a." He writes down a positive integer and asks, "Can you tell me the value of c?" Jon says, "There are still two possible values of c." Find the sum of the two possible values of c.

Problem 32. Define four real numbers A, B, C, D by

$$\begin{cases} A & = +\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4}, \\ B & = -\sqrt{1} + \sqrt{2} + \sqrt{3} - \sqrt{4}, \\ C & = +\sqrt{1} - \sqrt{2} + \sqrt{3} + \sqrt{4}, \\ D & = +\sqrt{1} + \sqrt{2} - \sqrt{3} + \sqrt{4}. \end{cases}$$

Prove that the product $ABCD$ of these four reals equals 8.

Problem 33. How many numbers in the 100^{th} row of the Pascal triangle (the one starting with $1, 100, \dots$) are not divisible by 3?

2012 Serbia TST 34. Let $P(x)$ be a polynomial of degree 2012 with real coefficients satisfying the condition

$$P(a)^3 + P(b)^3 + P(c)^3 \geq 3P(a)P(b)P(c),$$

for all real numbers a, b, c such that $a + b + c = 0$. Is it possible for $P(x)$ to have exactly 2012 distinct real roots?

2014 USA TST 35. Let n be a positive even integer, and let $c_1, c_2, \dots, c_{n-1}$ be real numbers satisfying

$$\sum_{i=1}^{n-1} |c_i - 1| < 1.$$

Prove that

$$2x^n - c_{n-1}x^{n-1} + c_{n-2}x^{n-2} - \cdots - c_1x^1 + 2$$

has no real roots.

2014 India Regional 36. The roots of the equation

$$x^3 - 3ax^2 + bx + 18c = 0,$$

form a non-constant arithmetic progression and the roots of the equation

$$x^3 + bx^2 + x - c^3 = 0,$$

form a non-constant geometric progression. Given that a, b, c are real numbers, find all positive integral values a and b.

1.2 Factorization

1.2.1 One–Variable Identities

Problem 37. Factorize $2x^4 + x^3 + 4x^2 + x + 2$.

Problem 38. Factorize $x^5 + x^4 + x^3 + x^2 + x + 1$.

Problem 39. Factorize $x^4 - x^2 + 7$.

Problem 40. Factorize $x^4 + 4x - 1$.

Problem 41. Factorize $(1 + x + x^2 + \cdots + x^n)^2 - x^n$.

Problem 42. Factorize $2x^4 + x^3 + 3x^2 + x + 2$.

Problem 43. Factorize $x^6 + 2x^5 + 3x^4 + 24x^3 + 23x^2 + 22x + 21$.

Problem 44. Factorize $x^4 + 6x^2 + 18$.

Problem 45. If n is a positive integer, factorize $a^{5n} + a^n + 1$.

Problem 46. Factorize $(x + 1)(x + 3)(x + 5)(x + 7) + 15$.

Problem 47. Factorize $x^3 + 5x^2 + 3x - 9$.

Problem 48. Factorize $x^3 + 9x^2 + 11x - 21$.

Problem 49. Factorize $x^3(x^2 - 7)^2 - 36x$.

1.2.2 Two–Variable Identities

(Positive Double–Variable Identity) 50. Factorize $x^2 + 2xy + y^2$.

(Negative Double–Variable Identity) 51. Factorize $x^2 - 2xy + y^2$.

(Difference of Squares Identity) 52. Factorize $x^2 - y^2$.

(n^{th} Positive Double–Variable Identity) 53. Factorize $x^n + y^n$ for odd n.

(n^{th} Negative Double–Variable Identity) 54. Factorize $x^n - y^n$ for all n.

($2^{k^{th}}$ Negative Double–Variable Identity) 55. Factorize $x^{2^k} - y^{2^k}$ for all k.

(Sophie Germain Identity) 56. Factorize $x^4 + 4y^4$.

(Sophie Parker Identity) 57. Factorize $x^4 + x^2y^2 + y^4$.

Problem 58. Factorize $x^4 + y^4 + (x + y)^4$.

Problem 59. Factorize $x^4 + y^4 + (x - y)^4$.

Problem 60. Factorize $(x + y)^3 - x^3 - y^3$.

Problem 61. Factorize $(x + y)^5 - x^5 - y^5$.

Problem 62. Factorize $(x + y)^7 - x^7 - y^7$.

Problem 63. Show that $(x + y)^n - x^n - y^n$ always has a factor of

$$nxy(x + y)(x^2 + xy + y^2)^2,$$

if $n = 6k + 1$ for some integer $k \geq 1$.

Problem 64. Factorize $4(x^2 + xy + y^2)^3 - 27x^2y^2(x + y)^2$.

1.2.3 Three–Variable Identities

Problem 65. Factorize $x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$.

Problem 66. Factorize $(xy^3 + yz^3 + zx^3) - (x^3y + y^3z + z^3x)$.

Problem 67. Factorize $(x^2y^3 + y^2z^3 + z^2x^3) - (x^3y^2 + y^3z^2 + z^3x^2)$.

Problem 68. Factorize $x^3 + y^3 + z^3 + (x + y)^3 + (y + z)^3 + (z + x)^3$.

Problem 69. Factorize $(x + y)(y + z)(z + x) + xyz$.

Problem 70. Factorize $xy(x + y) + yz(y - z) - xz(x + z)$.

Problem 71. Factorize $2a^2b + 4ab^2 - a^2c + ac^2 - 4b^2c + 2bc^2 - 4abc$.

Problem 72. Factorize $a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2$.

Problem 73. Factorize $x^2y^2z^2 + (x^2 + yz)(y^2 + zx)(z^2 + xy)$.

Problem 74. Factorize $y(x - 2z)^2 + 8xyz + x(y - 2z)^2 - 2z(x + y)^2$.

Problem 75. Factorize $(a + b + c)^3 - a^3 - b^3 - c^3$.

Problem 76. Factorize $(ab + bc + ca)(a + b + c) - abc$.

Problem 77. Factorize $(xy^2 + yz^2 + zx^2) - (x^2y + y^2z + z^2x)$.

Problem 78. Factorize $(x - y)^3 + (y - z)^3 + (z - x)^3$.

Problem 79. Define $g(x, y, z) = x^2 + y^2 + z^2 - xy - yz - zx$. Show that

$$(x - y)^2 + (y - z)^2 + (z - x)^2,$$

is divisible by $g(x, y, z)$ and that

$$(x - y)^4 + (y - z)^4 + (z - x)^4,$$

is divisible by $(g(x, y, z))^2$.

Problem 80. Factorize $(x - y)^5 + (y - z)^5 + (z - x)^5$.

Problem 81. Factorize $(x - y)^7 + (y - z)^7 + (z - x)^7$.

Problem 82. Factorize $(x^2 + y^2 + z^2)^2 - 2(x^4 + y^4 + z^4)$.

Problem 83. Factorize $x^3 + y^3 + z^3 - 3xyz$.

Problem 84. Factorize

$$(a^2 - bc)^3 + (b^2 - ac)^3 + (c^2 - ab)^3 - 3(a^2 - bc)(b^2 - ac)(c^2 - ab).$$

Problem 85. Factorize $(x + y + z)^5 - x^5 - y^5 - z^5$.

Problem 86. Factorize $a^3(x - y) + x^3(a - y) + y^3(a - x)$.

Problem 87. Factorize $a^2b^2(b - a) + b^2c^2(c - b) + c^2a^2(a - c)$.

Problem 88. Factorize $8x^3(y + z) - y^3(z + 2x) - z^3(2x - y)$.

Problem 89. Factorize $x^2y + xy^2 + x^2z + xz^2 + y^2z + yz^2 + 2xyz$.

Problem 90. Factorize $x^2y + xy^2 + x^2z + xz^2 + y^2z + yz^2 + 3xyz$.

Problem 91. Factorize $(x^2+y^2)^3+(z^2-x^2)^3-(y^2+z^2)^3$.

Problem 92. Factorize $x^3(y-z)+y^3(z-x)+z^3(x-y)$.

Problem 93. Factorize $x^3(z-y^2)+y^3(x-z^2)+z^3(y-x^2)+xyz(xyz-1)$.

2006 Korea 94. Find the number of positive integer triples (a,b,c) such that

$$\frac{a^2+b^2-c^2}{ab}+\frac{b^2+c^2-a^2}{bc}+\frac{c^2+a^2-b^2}{ca}=2+\frac{15}{abc}.$$

Problem 95. Factorize

$$[(x^2+y^2)(a^2+b^2)+4abxy]^2-4[xy(a^2+b^2)+ab(x^2+y^2)]^2.$$

1.3 Exercises in Expressions

Here, we study algebraic expressions in the general form of Functional Expressions first, and then discuss problems of the special case of Polynomial Expressions.

1.3.1 Functional Expressions

Remember that a function $f : A \to B$ takes elements of the set A as input and assigns to them elements of the set B as outputs. We study simple functional expressions here and reserve the more advanced functional equations for a later chapter.

Problem 96. If $f(x) = x^2 - 2x$, find $f(2x+1)$.

Problem 97. If

$$f(x) = x + \frac{1}{x},$$

find $f(f(x))$.

Problem 98. If

$$f(x) = \frac{x-1}{x+1},$$

find $f(f(f(x))) \cdot f(x)$.

Problem 99. If

$$f\left(\frac{x}{x+1}\right) = x^2,$$

find $f(x)$.

Problem 100. Write $x^3 - 3x + 4$ as a sum in terms of exponents of $(x+2)$.

Problem 101. If $f(x) = ax^2+bx+c$, what is the value of the following expression?

$$g(x) = f(x+3) - 3f(x+2) + 3f(x+1) - f(x).$$

Problem 102. Find a function in the form of $f(x) = a + bc^x$ such that

$$f(0) = 15, f(2) = 30, f(4) = 90.$$

Problem 103.

(a) Consider a linear function $f(x) = ax + b$. If the inputs $x = x_n$ (for $n = 1, 2, 3, \dots$) of the function form an arithmetic progression, then what kind of progression do the outputs $y_n = f(x_n)$ form?

(b) For $a > 0$, consider an exponential function $f(x) = a^x$. If the inputs $x = x_n$ (for $n = 1, 2, 3, \dots$) of the function form an arithmetic progression, then what kind of progression do the outputs $y_n = f(x_n)$ form?

Problem 104. For $a > 0$, define

$$f(x) = \frac{1}{2}\left(a^x + a^{-x}\right).$$

Find an alternative form for $f(x) + f(y)$ as a product.

Problem 105. If $f(x) + f(y) = f(z)$, find z in terms of x and y so that:

(a) $f(x) = ax$;

(b) $f(x) = \frac{1}{x}$;

(c) $f(x) = \arctan x$, where $|x| < 1$;

(d) $f(x) = \log \frac{1+x}{1-x}$.

Problem 106. Assuming

$$f(x) = \frac{1}{1-x},$$

Find $f(f(x))$ and $f(f(f(x)))$.

Problem 107. Assuming

$$f(x+1) = x^2 - 3x + 2,$$

Find $f(x)$.

Problem 108. Assuming

$$f\left(x+\frac{1}{x}\right)=x^2+\frac{1}{x^2},$$

Find $f(x)$ for $|x|\geq 2$.

Problem 109. If for $x>0$,

$$f\left(\frac{1}{x}\right)=x+\sqrt{1+x^2},$$

Find $f(x)$.

Problem 110. If

$$f\left(\frac{2x-1}{x+2}\right)=\frac{3x^2-3x+7}{(x+2)^2},$$

Find $f(x)$.

Problem 111. If we define

$$f_n(x)=\underbrace{f(f(f(\dots(f}_{n\text{ times}}(x))\dots))),$$

find $f_n(x)$ given that

$$f(x)=\frac{x}{\sqrt{1+x^2}}.$$

Problem 112. The function $f(x)$ is defined for $x>1$ as

$$f(x)=\log(x+\sqrt{x^2-1}).$$

Find $f(2x^2-1)$ and $f(4x^3-3x)$ in terms of $f(x)$.

Problem 113. If we know that

$$f\left(\frac{x+2}{x-2}\right)=\frac{x^2+4x+4}{8x},$$

Find $f(x)$.

Problem 114. If we know that

$$f(x) = \frac{4-x}{2x-4}, \text{ and}$$
$$f(\alpha + x) \cdot f(\alpha - x) = \text{constant},$$

Find α and the constant.

Problem 115. Consider the function

$$f(x) = \frac{a(x-b)(x-c)}{(a-b)(a-c)} + \frac{b(x-c)(x-a)}{(b-c)(b-a)} + \frac{c(x-a)(x-b)}{(c-a)(c-b)}.$$

Find the roots of $f(x) - x = 0$ and conclude that $f(x) = x$ for all x.

Problem 116. If n is an odd integer, $a^2 \neq 1$, and $f(x)$ is defined for all x by

$$af(x^n) + f(-x^n) = bx,$$

Find $f(x)$.

Problem 117.

(a) Find two roots for the following equation:

$$f(x) = f\left(\frac{x+8}{x-1}\right).$$

(b) If $f(x) = x^2 - 12x + 3$, find all the roots of the equation given in (a).

Problem 118. Find $f(x, y)$ given that

$$f\left(x + y, \frac{y}{x}\right) = x^2 - y^2.$$

Problem 119. For a real x and positive integer n, the function $F_n(x)$ is recursively defined by $F_1(x) = \cos x$ and

$$F_{n+1}(x) + F_n(x+1) = F_n(x).$$

Find $F_n(x)$ for different values of n modulo 4.

Problem 120. If for all $-\frac{1}{2} < x < \frac{1}{2}$, we have

$$f\left(\frac{x}{x^2+1}\right) = \frac{x^4+1}{x^2},$$

find $f(x)$.

1.3.2 Polynomial Expressions

We are now going to study special types of functional expressions called **polynomial expressions**, which are so important that the whole of Chapter 1 is dedicated to them.

Problem 121. What is the sum of coefficients of the following polynomial after expansion?

$$p(x) = (12x^3 - 54x^2 + 19x + 22)^{71}.$$

Problem 122. Given a polynomial

$$p(x) = (x+a)(x+a^2)\cdots(x+a^n),$$

find an alternative factorization for $a^n(x+1)p(x)$.

Problem 123. Given

$$p(x) = (x-a)(x-b)(x-c),$$

Find alternative expressions for $p(a+b)\cdot p(b+c)\cdot p(c+a)$ and $p(-a)\cdot p(-b)\cdot p(-c)$.

Problem 124. (a) Write the given polynomial $p(x)$ as a sum of descending exponents of $(x-1)$:

$$p(x) = 6x^4 + 19x^3 - 17x^2 - 72x - 36.$$

(b) Solve $f(x) = 0$.

Problem 125. If $a_0, a_1, \ldots, a_{50}$ are the coefficients of the polynomial

$$p(x) = (1+x+x^2)^{50},$$

determine whether the sum $a_0 + a_2 + \cdots + a_{50}$ is odd or even.

Problem 126. Let $p(x) = x^2 + ax + b$ be a quadratic polynomial in which a and b are integers. Find all integers n for which there exists an integer m such that $p(n)p(n+1) = p(m)$.

Problem 127. Let $p(x) = x^3 + ax^2 + bx + c$ and $q(x) = x^3 + bx^2 + cx + a$ be polynomials with integer coefficients and $c \neq 0$. If we know that $p(1) = 0$ and that the roots of $q(x)$ are squares of roots of $p(x)$, find $a^{2023} + b^{2023} + c^{2023}$.

Problem 128. Let $p_k(x) = x^k + 1/x^k$. If x is a non-zero real number such that both $p_4(x)$ and $p_5(x)$ are rational numbers, prove that $p_1(x)$ is also rational.

Problem 129. Let $p(x) = x^2 + ax + b$ and $q(x) = x^2 + cx + d$ be quadratic polynomials with integer coefficients such that $a \neq c$ and there exist integers $m \neq n$ for which $p(m) = q(n)$ and $p(n) = q(m)$. What is the parity of $a - c$?

Problem 130. Given three real numbers x, y, z such that $x + y + z = 0$ and $xy + yz + zx = -3$, find the value of $x^3y + y^3z + z^3x$.

1.4 Polynomial Division

1.4.1 Remainder of Polynomial Division

Problem 131. Find a and b such that $x^4 - 3x^3 + ax + b$ is divisible by $x^2 - 2x + 4$.

Problem 132. What is the quotient of division of $nx^{n+1} - (n+1)x^n + 1$ by $(x-1)^2$?

Problem 133. Find m such that $x^4 + ma^2x^2 + a^4$ is divisible by $x^2 - ax + a^2$, and find the quotient of the division.

Problem 134. Find a and b such that $a(x-2)^n + b(x-1)^n - 1$ is divisible by $x^2 - 3x + 2$, and find the quotient of the division.

Problem 135.

(a) If $p(1) = 1$ and $p(3) = -4$, what is the remainder of the division of $p(x)$ by $(x-1)(x-3)$?

(b) If $p(a) = A$ and $p(b) = B$, find the remainder of the division of $p(x)$ by $(x-a)(x-b)$.

Problem 136.

(a) If $p(-1) = 1$ and $p(2) = -3$, and $p(-2) = 2$, what is the remainder of the division of $p(x)$ by $(x+1)(x^2-4)$?

(b) If $p(a) = A$ and $p(b) = B$, and $p(c) = C$, find the remainder of the division of $p(x)$ by $(x-a)(x-b)(x-c)$.

Problem 137.

(a) Find the polynomial $p(x)$ of degree 4 such that it is divisible by $x+2$, the sum of its coefficients is 15, and it has a remainder of 5, -13, and 92 upon division by $x+1$, $x+3$, and $x-2$, respectively.

(b) Find the polynomial $q(x)$ of degree 3 such that it is divisible by $x+1$, the sum of its coefficients is 2, and it has a remainder of $1-x$ upon division by x^2+1.

Problem 138.

(a) Find the remainder of the division of $p(x) = x^{4a} + x^{4b+1} + x^{4c+2} + x^{4d+3}$ by $x^3 + x^2 + x + 1$, where a, b, c, d, are positive integers.

(b) Find the remainder of the division of $q(x) = x^{na_1} + x^{na_2+1} + x^{na_3+2} + \cdots + x^{na_n+(n-1)}$ by $x^{n-1} + x^{n-2} + x + 1$, where $a_1, a_2, \ldots, a_n$, are positive integers.

Problem 139. If $p(x) = x^4 + px^2 + qx + a^2$ is divisible by $x^2 - 1$, find the remainder of the division of $p(x)$ by $x^2 - a^2$.

Problem 140. If $p(x) = x^3 + px + q$ has a remainder of β and α upon division by $x - \alpha$ and $x - \beta$, respectively,

(a) find $(\alpha + \beta)(\alpha\beta + 1)$ and $\alpha^2 + \beta^2 + \alpha\beta$ in terms of p and q.

(b) find α and β if $p = -22$ and $q = -19$.

Problem 141.

(a) For each positive integer n, define $p_n(x, y, z) = (x + y + z)^n - x^n - y^n - z^n$. Find all positive integers m such that $p_m(x, y, z)$ is divisible by $p_3(x, y, z)$.

(b) For each positive integer n, define $q_n(x, y) = x^n - y^n$. For what values of a and b, is $q_a(x, y)$ divisible by $q_b(x, y)$?

Problem 142. Define

$$p_n(x) = x^{2n-2} + x^{2n-4} + \cdots + x^4 + x^2 + 1,$$
$$q_n(x) = x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1.$$

Find all n for which $p_n(x)$ is divisible by $q_n(x)$.

Problem 143. If $p(x, y)$ is a polynomial divisible by $x - y$ such that $p(x, y) = p(y, x)$, then find the remainder of division of $p(x, y)$ by $(x - y)^2$.

Problem 144.

(a) For which positive integers m is $x^{2m} + x^m + 1$ divisible by $x^2 + x + 1$?

(b) Find positive integers m and n such that $x^m + x^n + 1$ divisible by $x^2 + x + 1$.

Problem 145. Let a, b, x, y be integers. If $p(x, y) = a^n b^n (x^{2n} + y^{2n})$ is divisible by $q(x, y) = xy(a^2 + b^2) - ab(x^2 + y^2)$, then find the remainder of division of $s(x, y) = x^n y^n (a^{2n} + b^{2n})$ by $q(x, y)$.

Problem 146. Find the polynomial $p(x)$ of degree 4 such that $p(x+1)$ is divisible by $(x-1)^2$ and $p(x-1)$ is divisible by $(x+1)^2$, and also $p(1) = 1$.

Problem 147. Find all polynomials $p(x)$ of degree 3 such that $p(x)+2$ is divisible by $(x-1)^2$ and $p(x) - 2$ is divisible by $(x+1)^2$.

Problem 148. Find all polynomials $p(x)$ of degree 3 such that $p(x)+2$ is divisible by $(x-1)^2$ and $p(x) - 2$ is divisible by $(x+1)^2$.

Problem 149. If p, q, r are the roots of the cubic equation $x^3-3px^2+3q^2x-r^3 = 0$, then what is $p + q - 2r$?

Problem 150. Let $P_1(x) = ax^2 - bx - c$, $P_2(x) = bx^2 - cx - a$, and $P_3(x) = cx^2 - ax - b$ be three quadratic polynomials where a, b, c are non-zero real numbers. Suppose there exists a real number α such that $P_1(\alpha) = P_2(\alpha) = P_3(\alpha)$. What is $a + b - 2c$?

Problem 151. Let $P(x) = x^2 + ax + b$ be a quadratic polynomials with real coefficients. Suppose there exist real numbers α and β such that $P(\alpha) = \beta$ and $P(\beta) = \alpha$. Find the remainder of the division of $x^2 + ax + b - \alpha\beta$ by $x - b + \alpha\beta$.

Problem 152. Define a sequence $\langle f_0(x), f_1(x), f_2(x), \dots \rangle$ of functions by $f_0(x) = 1$, $f_1(x) = x$, and for $n \geq 1$,

$$(f_n(x))^2 - 1 = f_{n+1}(x) f_{n-1}(x).$$

Show that $f_n(x)$ is a polynomial with integer coefficients for all n.

Problem 153. Find all real values of a for which the equation $x^4 - 2ax^2 + x + a^2 - a = 0$ has all its roots real.

Problem 154. Find all real values of a for which the equation $x^2 + (a-5)x + 1 = 3|x|$ has exactly three distinct real solutions in x.

Problem 155. For positive reals a, b, c, which one of the following statements necessarily implies $a = b = c$? Justify your answer.

(I) $a(b^3 + c^3) = b(c^3 + a^3) = c(a^3 + b^3)$,

(II) $a(a^3 + b^3) = b(b^3 + c^3) = c(c^3 + a^3)$,

Problem 156. If a, b, c are non-zero real numbers such that

$$(ab + bc + ca)^3 = abc(a + b + c)^3,$$

prove that a, b, c are terms of a geometric progression.

2017 Ecuador 157. If we know that $x^2 - x - 1$ is a factor of the polynomial $ax^7 + bx^6 + 1$, where a and b are integers, find the value of $a - b$.

1.4.2 Greatest Common Factor of Polynomials

Problem 158. Find the greatest common factor of

a) $x^{91} + 1$ and $x^{65} + 1$;

b) $x^5 - 3x^3 + x^2 + 2x - 1$ and $x^6 - 2x^5 + x^4 - x^2 + 2x - 1$.

Problem 159. Write the following fraction in its simplest form:

$$\frac{x^6 + 3x^5 + x^4 + 4x^3 - 5x^2 - x + 1}{x^7 + 6x^5 + 7x^4 - 8x^3 - 5x^2 + 2x + 1}.$$

Problem 160. Solve the following equation for real x:

$$(x^2 + x - 2)^3 + (2x^2 - x - 1)^3 = 27(x^2 - 1)^3.$$

Problem 161. Let $P(x) = x^3 + ax^2 + b$ and $Q(x) = x^3 + bx + a$, where a and b are non-zero real numbers. Suppose that if x is a root of $P(x)$, then $1/x$ is a root of $Q(x)$.

a) Find a and b.

b) For a positive integer n, if the greatest common factor of $P(n)$ and $Q(n)$ is the same as the greatest common factor of 3 and $R(n)$, find $R(n)$.

2018 Romanian Masters in Mathematics 162. Determine whether there exist non-constant polynomials $P(x)$ and $Q(x)$ with real coefficients satisfying

$$P(x)^{10} + P(x)^9 = Q(x)^{21} + Q(x)^{20}.$$

2020 Balkan TST 163. Let $P(x), Q(x)$ be distinct polynomials of degree 2020 with non-zero coefficients. Suppose that they have r common real roots counting multiplicity and s common coefficients. Determine the maximum possible value of $r + s$.

2013 IMO Shortlist 164. Let $m \neq 0$ be an integer. Find all polynomials $P(x)$ with real coefficients such that

$$(x^3 - mx^2 + 1)P(x + 1) + (x^3 + mx^2 + 1)P(x - 1) = 2(x^3 - mx + 1)P(x)$$

for all real number x.

2022 ELMO 165. Find all monic non–constant polynomials P with integer coefficients for which there exist positive integers a and m such that for all positive integers $n \equiv a \pmod{m}$, $P(n)$ is non–zero and

$$2022 \cdot \frac{(n+1)^{n+1} - n^n}{P(n)},$$

is an integer.

2004 Russia 166. The polynomials $P(x)$ and $Q(x)$ are given. It is known that for a certain polynomial $R(x, y)$ the following identity holds for all x, y:

$$P(x) - P(y) = R(x, y)(Q(x) - Q(y)).$$

Prove that there is a polynomial $S(x)$ so that $P(x) = S(Q(x)) \quad \forall x$.

2016 USA TST 167. Let p be a prime number. Let $\mathbb{F}_p$ denote the integers modulo p, and let $\mathbb{F}_p[x]$ be the set of polynomials with coefficients in $\mathbb{F}_p$. Define $\Psi : \mathbb{F}_p[x] \to \mathbb{F}_p[x]$ by

$$\Psi\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} a_i x^{p^i}.$$

Prove that for nonzero polynomials $F, G \in \mathbb{F}_p[x]$,

$$\Psi(\gcd(F, G)) = \gcd(\Psi(F), \Psi(G)).$$

Here, a polynomial Q divides P if there exists $R \in \mathbb{F}_p[x]$ such that $P(x) - Q(x)R(x)$ is the polynomial with all coefficients 0 (with all addition and multiplication in the coefficients taken modulo p), and the gcd of two polynomials is the highest degree polynomial with leading coefficient 1 which divides both of them. A non-zero polynomial is a polynomial with not all coefficients 0. As an example of multiplication, $(x + 1)(x + 2)(x + 3) = x^3 + x^2 + x + 1$ in $\mathbb{F}_5[x]$.

2016 USA TSTST 168. Let $A = A(x, y)$ and $B = B(x, y)$ be two-variable polynomials with real coefficients. Suppose that $A(x, y)/B(x, y)$ is a polynomial in x for infinitely many values of y, and a polynomial in y for infinitely many values of x. Prove that B divides A, meaning there exists a third polynomial C with real coefficients such that $A = B \cdot C$.

1.5 Advanced Polynomial Problems & Theorems

1.5.1 Symmetric Sums

Symmetries of Things

Definition (Symmetric Sums σ_k)**.** Define σ_k, known as the k^{th} **symmetric sum** of the n numbers $x_1, x_2, \ldots, x_n$ by

$$\sigma_k(x_1, x_2, \ldots, x_n) = \sum_{1 \le n_1 < n_2 < \cdots < n_k \le n} x_{n_1} \cdot x_{n_2} \cdots x_{n_k}.$$

Then Vieta's Formulas for a polynomial $P(x) = a_n x^n + \cdots + a_0$ reduce to:

$$\sigma_k(x_1, x_2, \ldots, x_n) = (-1)^k \frac{a_{n-k}}{a_n}, \qquad \text{for} \quad k = 1, 2, \ldots, n.$$

Definition (Elementary Symmetric Polynomial)**.** An elementary symmetric polynomial is any multivariate (in more than one variable, like $x_1, x_2, \ldots$) polynomial defined as an equivalent polynomial to symmetric sums σ_k.

Definition (n–Variable Symmetric Polynomial)**.** A polynomial in n variables is called an n–**Variable Symmetric Polynomial** if switching any two of the variables leaves the polynomial unchanged.

Theorem 9 (Fundamental Theorem of Symmetric Polynomials)**.** Any symmetric polynomial can be expressed as the sum and product of multiple (not necessarily different) symmetric polynomials.

1973 USAMO 169. Determine all roots, real or complex, of the system of simultaneous equations

$$\begin{cases} x + y + z = 3, \\ x^2 + y^2 + z^2 = 3, \\ x^3 + y^3 + z^3 = 3. \end{cases}$$

1983 AIME 170. Suppose that the sum of the squares of two complex numbers x and y is 7 and the sum of the cubes is 10. What is the largest real value that $x + y$ can have?

2003 AIME 171. Consider the polynomials

$$P(x) = x^6 - x^5 - x^3 - x^2 - x,$$

and

$$Q(x) = x^4 - x^3 - x^2 - 1.$$

Given that z_1, z_2, z_3, and z_4 are the roots of $Q(x) = 0$, find $P(z_1) + P(z_2) + P(z_3) + P(z_4)$.

2015 AIME 172. Let x and y be real numbers satisfying $x^4y^5 + y^4x^5 = 810$ and $x^3y^6 + y^3x^6 = 945$. Evaluate $2x^3 + (xy)^3 + 2y^3$.

2018 AIME 173. A real number a is chosen randomly and uniformly from the interval $[-20, 18]$. The probability that the roots of the polynomial

$$x^4 + 2ax^3 + (2a-2)x^2 + (-4a+3)x - 2$$

are all real can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

2019 AIME 174. Let x be a real number such that $\sin^{10} x + \cos^{10} x = \frac{11}{36}$. Then

$$\sin^{12} x + \cos^{12} x = \frac{m}{n},$$

where m and n are relatively prime positive integers. Find $m+n$.

1988 Canada 175. For what real values of k do $1988x^2 + kx + 8891$ and $8891x^2 + kx + 1988$ have a common zero?

2011 AIME I #9 176. Suppose x is in the interval $[0, \pi/2]$ and

$$\log_{24 \sin x}(24 \cos x) = \frac{3}{2}.$$

Find $24 \cot^2 x$.

2016 AIME 177. For $1 \le i \le 215$ let $a_i = \frac{1}{2^i}$ and $a_{216} = \frac{1}{2^{215}}$. Let $x_1, x_2, \ldots, x_{216}$ be positive real numbers such that

$$\sum_{i=1}^{216} x_i = 1 \quad \text{and} \quad \sum_{1 \le i < j \le 216} x_i x_j = \frac{107}{215} + \sum_{i=1}^{216} \frac{a_i x_i^2}{2(1-a_i)}.$$

The maximum possible value of $x_2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

2019 AIME 178. For distinct complex numbers $z_1, z_2, \ldots, z_{673}$, the polynomial

$$(x-z_1)^3(x-z_2)^3 \cdots (x-z_{673})^3,$$

can be expressed as

$$x^{2019} + 20x^{2018} + 19x^{2017} + g(x),$$

where $g(x)$ is a polynomial with complex coefficients and with degree at most 2016. The value of

$$\left| \sum_{1 \le j < k \le 673} z_j z_k \right|,$$

can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 179. Let n be an integer and let x_1, x_2, x_3 be the roots of $x^3 + px + q = 0$, where p and q are two real numbers. Find the expression $x_1^n + x_2^n + x_3^n$ in terms of p and q.

Problem 180. Find all solutions (x, y, z) of

$$\begin{cases} 10 &= x+y+z+w, \\ 30 &= x^2+y^2+z^2+w^2, \\ 100 &= x^3+y^3+z^3+w^3 = 100, \\ 24 &= xyzw. \end{cases}$$

Problem 181. Prove that the sum of the reciprocals of the 5 solutions to the following equation is $1/2001$:

$$5x^5+4x^4-3x^3+2x^2+x-1=2000.$$

Problem 182. The roots of the equation $x^3+ax+b=0$ are α, β and γ. Find the equation with roots

$$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}, \frac{\alpha}{\gamma}+\frac{\gamma}{\alpha} \qquad \text{and} \qquad \frac{\beta}{\gamma}+\frac{\gamma}{\beta}.$$

Problem 183. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two distinct collections of n positive integers, where each collection may contain repetitions. If the two collections of integers $a_i + a_j$ (where $1 \leq i < j \leq n$) and $b_i + b_j$ (where $1 \leq i < j \leq n$) are the same, then show that n is a power of 2.

Problem 184. Assume that a, b, c, d are roots of the equation

$$x^4+120x^3+1279x^2+11x+9=0.$$

Also assume that

$$\frac{abc}{d}, \quad \frac{abd}{c}, \quad \frac{acd}{b}, \quad \text{and} \quad \frac{bcd}{a},$$

are the roots of

$$x^4+a_1x^3+a_2x^2+a_3x+a_4=0.$$

Find $a_1+a_2+a_3+a_4$.

Problem 185. Let a, b, c be non–zero real numbers such that $a+b+c=0$. Prove that

$$\frac{(a^3+b^3+c^3)^2(a^4+b^4+c^4)}{(a^5+b^5+c^5)^2}=\frac{18}{25}.$$

2017 PUMaC 186. Together, Kenneth and Ellen pick a real number a. Kenneth subtracts a from every thousandth root of unity (that is, the thousand complex numbers ω for which $\omega^{1000}=1$) then inverts each, then sums the results. Ellen inverts every thousandth root of unity, then subtracts a from each, and then sums the results. They are surprised to find that they actually got the same answer! How many possible values of a are there?

2019 IMO Shortlist 187. Let $x_1, x_2, \ldots, x_n$ be different real numbers. Prove that

$$\sum_{1 \leqslant i \leqslant n} \prod_{j \neq i} \frac{1-x_ix_j}{x_i-x_j} = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

2014 HMIC 188. Let ω be a root of unity and f be a polynomial with integer coefficients. Show that if $|f(\omega)|=1$, then $f(\omega)$ is also a root of unity.

1.5.2 Pool of Advanced Polynomial Theorems

Newton's Sums

Theorem 10 (Newton's Formulas on Symmetric Sums)**.** For a polynomial $P(x)$ of degree n, where

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

let $x_1, x_2, \ldots, x_n$ be the roots of $P(x) = 0$. Define the sum:

$$P_k = x_1^k + x_2^k + \cdots + x_n^k.$$

According to Newton's Formulas, and assuming $a_j = 0$ for $j < 0$,

$$\begin{aligned}
0 &= a_n P_1 + a_{n-1}, \\
0 &= a_n P_2 + a_{n-1}P_1 + 2a_{n-2}, \\
0 &= a_n P_3 + a_{n-1}P_2 + a_{n-2}P_1 + 3a_{n-3}, \\
&\ \vdots \\
0 &= a_n P_k + a_{n-1}P_{k-1} + \cdots + a_{n-k+1}P_1 + k \cdot a_{n-k}.
\end{aligned}$$

We also can write:

$$\begin{aligned}
P_1 &= \sigma_1, \\
P_2 &= \sigma_1 P_1 - 2\sigma_2, \\
P_3 &= \sigma_1 P_2 - \sigma_2 P_1 + 3\sigma_3, \\
P_4 &= \sigma_1 P_3 - \sigma_2 P_2 + \sigma_3 P_1 - 4\sigma_4, \\
P_5 &= \sigma_1 P_4 - \sigma_2 P_3 + \sigma_3 P_2 - \sigma_4 P_1 + 5\sigma_5, \\
&\ \vdots
\end{aligned}$$

Here, σ_n denotes the n^{th} elementary symmetric sum as defined before.

Theorem 11 (Intermediate Value Theorem)**.** Consider a continuous function $f : I \to \mathbb{R}$ for some interval $I = [a, b]$ (with $a < b$). Then, for all $c \in (f(a), f(b))$, we can find some real number k with $a < k < b$, such that $f(k) = c$.

Theorem 12 (Descartes' Rule of Signs)**.** Consider a polynomial $P(x)$ of degree $n \geq 1$, and write

$$P(x) = a_n\epsilon_n x^n + a_{n-1}\epsilon_{n-1}x^{n-1} + \cdots + a_1\epsilon_1 x + a_0\epsilon_0,$$

where $a_n > 0$ and $\epsilon_n \in \{-1, 0, 1\}$. Let m be the number of times $\epsilon_k\epsilon_{k-1} = -1$. Then, the number of positive roots, say p, (counting multiplicities) is at most m, and furthermore leaves the same remainder as m when divided by 2.

Corollary 2 (Number of Negative Roots)**.** The number of negative roots can be found by applying Descartes' Rule of Signs on $f(-x)$ instead.

Definition (Discriminant of a Polynomial)**.** For a polynomial $P(x)$ of degree n, where

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

let $x_1, x_2, \ldots, x_n$ be the roots of $P(x) = 0$. Define the discriminant Δ of $P(x)$ by

$$\Delta(P(x)) = a_n^{2n-2} \prod_{1 \le i < j \le n} (x_i - x_j)^2.$$

Definition (Matrix of Two Polynomials)**.** For any two polynomials $f(x)$ and $g(x)$, with $\deg f = m$ and $\deg g = n$, **the resultant** is the discriminant of the $(m+n) \times (m+n)$–matrix formed by writing n times the coefficients of $f(x)$ and m times the coefficients of $g(x)$.

Theorem 13 (Discriminant from Resultant)**.** The **discriminant** Δ of any polynomial $P(x)$ of degree n, where

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

is equal to

$$\Delta(P(x)) = \frac{(-1)^{\binom{n}{2}}}{a_n} R(P(x), P'(x)),$$

where $R(P(x), P'(x))$ is the **resultant** of the two polynomials $P(x)$ and its derivative $P'(x)$.

Corollary 3 (Quadratic Discriminant)**.** Verify, by using the Discriminant from Resultant formula for $P(x) = ax^2 + bx + c$ and $P'(x) = 2ax + b$:

$$\frac{-1}{a} \begin{vmatrix} a & b & c \\ 2a & b & 0 \\ 0 & 2a & b \end{vmatrix} = -\frac{ab^2 + 4a^2c - 2ab^2}{a} = b^2 - 4ac,$$

that the discriminant of the quadratic polynomial $ax^2 + bx + c$ is equal to $\Delta = b^2 - 4ac$.

Corollary 4 (Cubic Discriminant)**.** Prove that the discriminant of the cubic polynomial $ax^3 + bx^2 + cx + d$ is equal to

$$\Delta = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd.$$

Theorem 14 (Lagrange's Interpolation Theorem)**.** For any distinct complex numbers $x_0, x_1, \ldots, x_n$ and any complex numbers $y_0, y_1, \ldots, y_n$, there exists a unique polynomial $P(x)$ of degree less than or equal to n such that for all integers $0 \le i \le n$, with $P(x_i) = y_i$, and this polynomial is

$$P(x) = \sum_{i=0}^{n} y_i \frac{(x - x_0) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)}{(x_i - x_0) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)}.$$

Definition (Finite Differences of Polynomials)**.** The finite difference of a polynomial $P(x)$ is $\Delta^1 P(x) = P(x+1) - P(x)$.

Theorem 15 (Finite Difference Δ^1 as a Linear Operator)**.** If $\Delta^1(P(x))$ is defined as the finite difference operator of polynomial $P(x)$, that is, $\Delta^1 P(x) = P(x+1) - P(x)$, then Δ^1 is a linear operator. That is,

$$\begin{cases} \Delta^1 \left(P_1(x) + P_2(x)\right) & = \Delta^1 \left(P_1(x)\right) + \Delta^1 \left(P_2(x)\right), \\ \Delta^1 \left(k \cdot P(x)\right) & = k \cdot \Delta^1 \left(P(x)\right). \end{cases}$$

Definition (n^{th}–degree Finite Differences of Polynomials)**.** The n^{th}–degree finite difference of a polynomial $P(x)$ is

$$\Delta^n(P(x)) = \Delta^1\left(\Delta^{n-1}(P(x))\right) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}P(x+k).$$

Theorem 16 (Finite Differences with Degrees)**.** Let n be the degree of the polynomial $P(x)$. Then,

$$\Delta^n(P(x)) = n!,$$
$$\Delta^{n+1}(P(x)) = 0.$$

Theorem 17 (Finite Difference Representation)**.** Every polynomial can be represented by each degree of its finite difference, that is, we can write every polynomial $P(x)$ as

$$P(x) = \sum_{m=0}^{\deg P}\binom{x-a}{m}\Delta^m(P(a)).$$

Theorem 18 (Polynomial Summation)**.** For all polynomials $P(x)$,

$$\sum_{k=1}^{n}P(k) = \sum_{m=0}^{\deg P}\binom{n}{m+1}\Delta^m(P(1)).$$

Theorem 19 (Geometric Series Polynomial Summation)**.** For any polynomial $P(x)$ and constant q

$$\sum_{k=1}^{n}P(k)q^{k-1} = f(n)q^n - f(0),$$

where

$$\begin{aligned} f(n) &= \frac{P(n)}{q-1} + \frac{1}{(q-1)^2}\sum_{k=1}^{\deg P}\frac{(-1)^k q^{k-1}}{(q-1)^{k-1}}\Delta^k(P(n)) \\ &= \frac{1}{q-1}\sum_{k=1}^{\deg P}\left(\frac{-q}{q-1}\right)^k\Delta^k(P(n+1)). \end{aligned}$$

2022 Brazil 189. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of integers numbers. Let $\Delta^1 a_n = a_{n+1} - a_n$ for a non-negative integer n. Define $\Delta^M a_n = \Delta^{M-1}a_{n+1} - \Delta^{M-1}a_n$. A sequence is *patriota* if there are positive integers k, l such that $a_{n+k} = \Delta^M a_{n+l}$ for all non-negative integers n. Determine, with proof, whether exists a sequence that the last value of M for which the sequence is *patriota* is 2022.

1999 Brazil TST 190. A sequence a_n is defined initially by $a_0 = 0$ and $a_1 = 3$, and then recursively for $n \geq 2$:

$$a_n = 8a_{n-1} + 9a_{n-2} + 16.$$

Find the least positive integer h such that $a_{n+h} - a_n$ is divisible by 1999 for all $n \geq 0$.

1983 IMO Shortlist 191. Let $(F_1, F_2, F_3, \dots)$ be the Fibonacci sequence, defined by the starting values $F_1 = 1$ and $F_2 = 1$ and the recurrence equation $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 1$. Let $P(x)$ be a polynomial of degree 990 such that $P(k) = F_k$ for all $k \in \{992, 993, \dots, 1981, 1982\}$. Show that

$$P(1983) = F_{1983} - 1.$$

2004 Fourth Mathlinks Contest 192. Let $m \geq 2n$ be two positive integers. Find a closed form for the following expression:

$$E(m,n) = \sum_{k=0}^{n} (-1)^k \frac{(m-k)!}{n!(m-k-n)!} \frac{n!}{k!(n-k)!}.$$

Problem 193. Assume that m and n are positive integers, and $1 \leq m \leq \phi(m) + n$. Prove that

$$m \mid \sum_{i=0}^{n} (-1)^i \binom{n}{i} i^m.$$

Problem 194. Assume $i \geq j \geq 1$. Prove that

$$\sum_{k=i}^{i+j} (-1)^k \frac{(k-1)!}{(k-i)!(k-j)!(i+j-k)!} = 0.$$

Problem 195. Prove that for all positive integers n, we have

$$\binom{n}{0} n^n - \binom{n}{1} (n-1)^n + \binom{n}{2} (n-2)^n - \ldots + (-1)^{n-1} \binom{n}{n-1} 1^n = n!\ .$$

Problem 196. Let n be a natural number (i.e., a non-negative integer). Prove that

$$\sum_{k=1}^{n} \frac{(-1)^k \cdot k}{2k-1} \binom{n}{k} \binom{n+k-1}{k-1} = -(n \mod 2).$$

Here, for any integer a, the expression $a \mod 2$ means the remainder of a upon division by 2 (so, $a \mod 2 = 0$ if a is even, and $a \mod 2 = 1$ if a is odd).

Problem 197. Prove that for any polynomial $P(x)$ with degree $< n$, we have

$$\sum_{k=0}^{n} (-1)^n \frac{n!}{k!(n-k)!} P(k) = 0.$$

10490 AMM 198. Prove that for all $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \cdot \binom{n}{k} \sum_{j=1}^{k} \frac{H_j}{j} = \sum_{k=1}^{n} \frac{1}{k^3},$$

where, for every $j \in \mathbb{N}$,

$$H_j = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{j}.$$

Problem 199. Let P be a polynomial of degree n satisfying

$$P(k) = \binom{n+1}{k}^{-1} \qquad \text{for } k = 0, 1, ..., n.$$

Determine $P(n+1)$.

2011 Ibero American 200. This problem has two parts:

a) Prove that, for any positive integers $m \leq \ell$ given, there is a positive integer n and positive integers $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that the equality

$$\sum_{i=1}^{n} x_i^k = \sum_{i=1}^{n} y_i^k,$$

holds for every $k = 1, 2, \ldots, m-1, m+1, \ldots, \ell$, but does not hold for $k = m$.

b) Prove that there is a solution of the problem, where all numbers $x_1, \ldots, x_n, y_1, \ldots, y_n$ are distinct.

Komal 201. Prove that, for infinitely many positive integers n, there exists a polynomial $P(x)$ of degree n with real coefficients such that $P(1), P(2), \ldots, P(n+2)$ are different whole powers of 2.

Dadgarnia Finite Difference Identities 202. These problems were posted on AoPS as proposed by Alireza Dadgarnia:

a) Let $n > 1$ and $0 \leq m \leq n-2$ be integers. Prove that

$$\sum_{i=1}^{n} (-1)^{n-i} \binom{n-1}{i-1} i^m = 0.$$

b) For all positive integers n, prove that

$$\sum_{i=1}^{n} (-1)^{n-i} \binom{n-1}{i-1} i^{n-1} = (n-1)!.$$

c) For all positive integers n, prove that

$$\sum_{i=1}^{n} (-1)^{n-i} \binom{n-1}{i-1} i^{n} = \frac{(n+1)!}{2}.$$

d) For all positive integers n, prove that

$$\sum_{i=1}^{n}(-1)^{n-i}\binom{n-1}{i-1}i^{n+1}=\frac{(3n+1)(n+2)!}{24}.$$

e) For all positive integers n, prove that

$$\sum_{i=1}^{n}(-1)^{n-i}\binom{n-1}{i-1}i^{n+2}=\frac{n(n+1)(n+3)!}{48}.$$

Crux, by Max A. Alekseyev 203. For all integers $n>m\geq 0$, prove that:

$$\sum_{k=0}^{n}(-1)^k\cdot\tbinom{2n+1}{n-k}\cdot(2k+1)^{2m+1}=0.$$

2020 Taiwan TST 204. Alice and Bob are stuck in quarantine, so they decide to play a game. Bob will write down a polynomial $f(x)$ with the following properties:

a) for any integer n, $f(n)$ is an integer;

b) the degree of $f(x)$ is less than 187.

Alice knows that $f(x)$ satisfies (a) and (b), but she does not know $f(x)$. In every turn, Alice picks a number k from the set $\{1,2,\dots,187\}$, and Bob will tell Alice the value of $f(k)$. Find the smallest positive integer N so that Alice always knows for sure the parity of $f(0)$ within N turns.

2020 Tuymaada 205. The degrees of polynomials P and Q with real coefficients do not exceed n. These polynomials satisfy the identity

$$P(x)x^{n+1}+Q(x)(x+1)^{n+1}=1.$$

Determine all possible values of $Q\left(-\frac{1}{2}\right)$.

2020 Indonesia 206. Determine all real-coefficient polynomials $P(x)$ such that

$$P(\lfloor x\rfloor)=\lfloor P(x)\rfloor,$$

for all real numbers x.

2019 Latvian TST for Balkan 207. Let $P(x)$ be a polynomial of degree n with real coefficients. For all $0\leq y\leq 1$, we know that $|P(y)|\leq 1$. Prove that

$$P\left(-\frac{1}{n}\right)\leq 2^{n+1}-1.$$

2021 USA TSTST 208. Let $q = p^r$ for a prime number p and positive integer r. Let $\zeta = e^{\frac{2\pi i}{q}}$. Find the least positive integer n such that

$$\sum_{\substack{1 \leq k \leq q \\ \gcd(k,p)=1}} \frac{1}{(1-\zeta^k)^n},$$

is not an integer the sum is over all $1 \leq k \leq q$ with p not dividing k).

1.6 Complex Numbers

Introduction to Complex Numbers

It is assumed that a high school student is aware of the set of positive integers (also called natural numbers) $\mathbb{N}$, the set of integers $\mathbb{Z}$, the set of rational numbers $\mathbb{Q}$, and the of real numbers $\mathbb{R}$. It is unfortunate that all the numbers in these sets, even all real numbers, are not enough to solve all polynomial equations. For instance, the simple equation $x^2 + 1 = 0$ does not have any real roots and that is where the imaginary unit $i = \sqrt{-1}$ comes from.

Definition (The Set of Complex Numbers $\mathbb{C}$)**.** If we allow $i = \sqrt{-1}$, then the set

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\},$$

is the set of complex numbers. Realize that we can represent each complex number as a pair (x, y) of real numbers, and that $\mathbb{R}$ (attained by plugging $y = 0$) is a subset of $\mathbb{C}$.

Definition (Real and Imaginary Parts of a Complex Number)**.** Let $z = x+iy$ be a complex number, $z \in \mathbb{C}$ for short. We call x the real part of z and y the imaginary part of z, and denote $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$.

Definition (Sum and Product of Complex Numbers)**.** For two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, define

$$\begin{aligned} z_1 + z_2 &= (x_1 + x_2) + i(y_1 + y_2), \\ z_1 - z_2 &= (x_1 - x_2) + i(y_1 - y_2), \\ z_1 z_2 &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1). \end{aligned}$$

These can be written in terms of $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ as

$$\begin{aligned} \mathrm{Re}(z_1 \pm z_2) &= \mathrm{Re}(z_1) \pm \mathrm{Re}(z_2), \\ \mathrm{Im}(z_1 \pm z_2) &= \mathrm{Im}(z_1) \pm \mathrm{Im}(z_2), \\ \mathrm{Re}(z_1 \cdot z_2) &= \mathrm{Re}(z_1) \cdot \mathrm{Re}(z_2) - \mathrm{Im}(z_1) \cdot \mathrm{Im}(z_2), \\ \mathrm{Im}(z_1 \cdot z_2) &= \mathrm{Re}(z_1) \cdot \mathrm{Im}(z_2) + \mathrm{Re}(z_2) \cdot \mathrm{Im}(z_1). \end{aligned}$$

In order to divide $z_1 = x_1 + iy_1$ by $z_2 = x_2 + iy_2 \neq 0$, we can write

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2},$$

or in terms of $\mathrm{Re}(z_{1,2})$ and $\mathrm{Im}(z_{1,2})$,

$$\begin{aligned} \mathrm{Re}\left(\frac{\mathrm{Re}(z_1) + \mathrm{Im}(z_1)}{\mathrm{Re}(z_2) + \mathrm{Im}(z_2)}\right) &= \frac{\mathrm{Re}(z_1) \cdot \mathrm{Re}(z_2) + \mathrm{Im}(z_1) \cdot \mathrm{Im}(z_2)}{(\mathrm{Re}(z_2))^2 + (\mathrm{Im}(z_2))^2}, \\ \mathrm{Im}\left(\frac{\mathrm{Re}(z_1) + \mathrm{Im}(z_1)}{\mathrm{Re}(z_2) + \mathrm{Im}(z_2)}\right) &= \frac{\mathrm{Im}(z_1) \cdot \mathrm{Re}(z_2) - \mathrm{Re}(z_1) \cdot \mathrm{Im}(z_2)}{(\mathrm{Re}(z_2))^2 + (\mathrm{Im}(z_2))^2}. \end{aligned}$$

1.6.1 Polar Representation of Complex Numbers

Complex Conjugate Definitions

Definition. The **polar representation** of a complex number $z = x + iy$ is given by $z = r(\cos\theta + i\sin\theta)$, so that

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta,$$

where θ is the angle between the x–axis and the vector formed by connecting the point z in the complex plane to the origin. In this definition, r is the absolute value of z, and θ is the angle or argument of z. Finally,

$$r = |z| = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \theta = \arg(z) = \arctan\left(\frac{y}{x}\right).$$

Definition (Complex Conjugates)**.** The two complex numbers $x+iy$ and $x-iy$ are **conjugate** of one another in the complex plane. We usually denote the complex conjugate of z by $\overline{z}$.

Theorem 20 (Complex Conjugate Theorem)**.** Prove that z is a root of a polynomial with real coefficients if and only if $\overline{z}$ is also the root of the same polynomial.

Theorem 21. For any three complex numbers z_1, z_2, z_3, prove the following:

a) The real and imaginary part of z_1 are

$$\operatorname{Re}(z_1) = \frac{z_1 + \overline{z_1}}{2} \qquad \text{and} \qquad \operatorname{Im}(z_1) = \frac{z_1 - \overline{z_1}}{2i}.$$

b) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$, and if $z_2 \neq 0$,

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

c) the triangle inequality for absolute values:

$$|z_1| - |z_2| \leq |z_1 - z_2| \qquad \text{and} \qquad |z_1 + z_2| \leq |z_1| + |z_2|.$$

d) $\overline{\overline{z_1}} = z_1$,

e) $\overline{z_1} = z_1 \iff z_1 \in \mathbb{R}$,

f) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$,

g) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$,

h) $|z_1|^2 = z_1 \cdot \overline{z_1}$.

Theorem 22. If we have

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad \text{and} \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2),$$

prove the following polar identities:

a) $z_1z_2 = r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$.

b) If $z_2 \neq 0$,
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)).$$

c) $z_1^n = (r_1^n)(\cos(n\theta_1) + i\sin(n\theta_1))$.

Theorem 23 (De Moivre's Theorem)**.** Prove that for all $\theta \in \mathbb{R}$ and positive integer n,
$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n.$$

Theorem 24 (Euler's Formula)**.** For all reals θ,
$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Definition (Roots of Unity)**.** The n^{th} roots of unity are roots of $z^n = 1$.

1995 AIME #5 209. For certain real values of a, b, c, and d, the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ has four non-real roots. The product of two of these roots is $13+i$ and the sum of the other two roots is $3+4i$, where $i = \sqrt{-1}$. Find b.

2013 AIME I #10 210. There are nonzero integers a, b, r, and s such that the complex number $r+si$ is a zero of the polynomial $P(x) = x^3 - ax^2 + bx - 65$. For each possible combination of a and b, let $p_{a,b}$ be the sum of the zeroes of $P(x)$. Find the sum of the $p_{a,b}$'s for all possible combinations of a and b.

2013 AIME II #12 211. Let S be the set of all polynomials of the form z^3+az^2+bz+c, where a, b, and c are integers. Find the number of polynomials in S such that each of its roots z satisfies either $|z| = 20$ or $|z| = 13$.

2011 AIME II #8 212. Let $z_1, z_2, z_3, \ldots, z_{12}$ be the 12 zeroes of the polynomial $z^{12} - 2^{36}$. For each j, let w_j be one of z_j or iz_j. Then the maximum possible value of the real part of $\displaystyle\sum_{j=1}^{12} w_j$ can be written as $m+\sqrt{n}$ where m and n are positive integers. Find $m+n$.

2019 AIME II #8 213. The polynomial $f(z) = az^{2018} + bz^{2017} + cz^{2016}$ has real coefficients not exceeding 2019, and
$$f\left(\frac{1+\sqrt{3}i}{2}\right) = 2015 + 2019\sqrt{3}i.$$

Find the remainder when $f(1)$ is divided by 1000.

1996 AIME #11 214. Let P be the product of the roots of $z^6 + z^4 + z^3 + z^2 + 1 = 0$ that have positive imaginary part, and suppose that $P = r(\cos\theta^\circ + i\sin\theta^\circ)$, where $0 < r$ and $0 \leq \theta < 360$. Find θ.

1.7 Number Theoretic Study of Polynomials

1.7.1 Essential Number Theoretic Theorems

Theorem 25 (Difference of Polynomials)**.** Let $P(x)$ be a polynomial with integer coefficients. Then, for all integers a and b with $a \neq b$, we have

$$a - b | P(a) - P(b).$$

1962 Russia 215. Prove that there does not exist a polynomial $P(x) = ax^3+bx^2+cx+d$ with integer coefficients such that $P(19) = 1$ and $P(62) = 2$.

1974 USA 216. For three distinct integers a, b, c and a polynomial $P(x)$ with integer coefficients, prove that not all three of the following relations can hold at the same time:

$$P(a) = b, \quad P(b) = c, \quad P(c) = a.$$

1975 USA 217. Find all polynomials $P(x)$ such that $P(0) = 0$ and for all x,

$$P(x) = \frac{1}{2}\left(P(x + 1) + P(x - 1)\right).$$

Problem 218. The polynomial $P(x)$ has degree n and for the values of $x = 0^2, 1^2, \ldots, n^2$, we know that $f(x)$ is an integer. Prove that $P(x)$ takes infinitely many integer values.

Eisenstein's Criterion & Extension

Theorem 26 (Eisenstein's Criterion)**.** Let $a_0, a_1, \ldots, a_n$ be integers. The Eisenstein's Criterion states that the polynomial

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

cannot be factored into the product of two non–constant polynomials if all the following three conditions hold:

a) p is a prime which divides each of $a_0, a_1, a_2, \ldots, a_{n-1}$;

b) a_n is not divisible by p; and

c) a_0 is not divisible by p^2.

Theorem 27 (Extended Eisenstein's Criterion)**.** Let $a_0, a_1, \ldots, a_n$ be integers. The Extended Eisenstein's Criterion states that the polynomial

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

has an irreducible factor of degree more than k if:

a) p is a prime which divides each of $a_0, a_1, a_2, \ldots, a_k$;

b) a_{k+1} is not divisible by p; and

c) a_0 is not divisible by p^2.

Problem 219. Prove that $x^{1383} + 2003$ is irreducible in $\mathbb{Q}[x]$.

Problem 220. Prove that the polynomial $1+x+x^2+\cdots+x^{p-1}$, where $p>2$ is a prime number, is irreducible over $\mathbb{Z}[x]$.

1997 Iran Third Round 221. Let $P(x)$ be a polynomial with integer coefficients such that for two distinct integers a and b, we have

$$P(a) \cdot P(b) = -(a-b)^2.$$

Prove that $P(a) + P(b) = 0$.

1998 Romanian TST 222. Let

$$f_n(X) = (X^2 + X)^{2^n} + 1.$$

Prove, for all n, that $f_n(X)$ is irreducible over $\mathbb{Z}[X]$. If you can, prove Harazi's generalization as well: for any two integers a and b such that

$$\left(b - \frac{a^2}{4}\right)^{2^n} + 1 \qquad \text{is not a perfect square in } \mathbb{Q},$$

then $(X^2 + aX + b)^{2^n} + 1$ is irreducible in $\mathbb{Q}[X]$.

2014 Putnam 223. Show that for each positive integer n, all the roots of the polynomial

$$\sum_{k=0}^{n} 2^{k(n-k)} x^k$$

are real numbers.

Content of a Polynomial & Gauss's Lemma

Definition. The **content** of a polynomial with integer coefficients is the greatest common divisor of the polynomial's coefficients. In other words, when $a_0, a_1, \dots, a_n$ are integers and

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

the content $c(P(x))$ is defined by

$$c(P) := \gcd(a_n, a_{n-1}, \dots, a_1, a_0).$$

As an example, all monic polynomials have content equal to 1 because their leading coefficient is 1. Such polynomials $P(x)$ for which $c(P) = 1$ are called **primitive polynomials**.

Theorem 28. Let $P(x)$ and $Q(x)$ be polynomials with integer coefficients. Prove that the content of $P(x)Q(x)$ is equal to the product of contents of $P(x)$ and $Q(x)$.

Theorem 29 (Gauss's Primitive Polynomial Lemma)**.** If $P(x)$ and $Q(x)$ are primitive polynomials with integer coefficients, their product $P(x)Q(x)$ is also a primitive polynomial.

Theorem 30 (Gauss's Lemma)**.** Let $P(x)$ be a polynomial with integer coefficients which cannot be factorized into a product of two polynomials with integer coefficients. Prove that $P(x)$ cannot be decomposed into a product of two polynomials with rational coefficients either. In other words, $P(x)$ is irreducible over $\mathbb{Z}[x]$ if and only if it is irreducible over $\mathbb{Q}[x]$.

2019 Iran Third Round 224. Call a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with integer coefficients *primitive* if and only if $\gcd(a_n, a_{n-1}, \dots a_1, a_0) = 1$.

a) Let $P(x)$ be a primitive polynomial with degree less than 1398 and S be a set of primes greater than 1398.Prove that there is a positive integer n so that $P(n)$ is not divisible by any prime in S.

b) Prove that there exist a primitive polynomial $P(x)$ with degree less than 1398 so that for any set S of primes less than 1398 the polynomial $P(x)$ is always divisible by product of elements of S.

2010 Romania TST 225. Let p be a prime number,let $n_1, n_2, \dots, n_p$ be positive integer numbers, and let d be the greatest common divisor of the numbers $n_1, n_2, \dots, n_p$. Prove that the polynomial

$$\frac{X^{n_1} + X^{n_2} + \cdots + X^{n_p} - p}{X^d - 1},$$

is irreducible in $\mathbb{Q}[X]$.

1.7.2 Modular Arithmetic for Polynomials

Modular Arithmetic of Polynomials

Definition (Polynomial Congruency)**.** Two polynomials $f(x)$ and $g(x)$ with integer coefficients are congruent modulo positive integer $m \geq 2$ if, assuming

$$f(x) - g(x) = c_n x^n + \cdots + c_1 x + c_0,$$

we have $m \mid c_i$ for all $i = 0, 1, 2, \ldots, m$. In other words, we have $f(x) \equiv g(x) \pmod{m}$ if and only if $a_i \equiv b_i$ for all i, where a_i and b_i are corresponding coefficients in $f(x)$ and $g(x)$, respectively.

Definition (Degree of Polynomial Modulo Integer)**.** Let $f(x) = a_n x^n + \cdots + a_0$ be a polynomial with integer coefficients and let $m \geq 2$ be an integer. The **degree** of $f(x) \pmod{m}$ is the largest i such that $m \nmid a_i$.

Definition (Roots of Polynomial Congruence Equations)**.** Let $f(x)$ be a polynomial with integer coefficients and let $m \geq 2$ be an integer. We say that c is a root of $f(x) \pmod{m}$ if and only if $m \mid f(c)$.

Theorem 31. Let p be a prime number and let n be the degree of $f(x)$ modulo p. Then, the equation $f(x) \equiv 0 \pmod{p}$ has at most n incongruent solutions modulo p.

Theorem 32 (Hensel's Lemma)**.** Let $f(x) = a_n x^n + \cdots + a_0$ be a polynomial with integer coefficients and let $p > 2$ be a prime. Also, let $P'(x)$ denote the derivative of $P(x)$. Suppose that x_1 is an integer such that $P(x_1) \equiv 0 \pmod{p}$ and $P'(x_1) \not\equiv 0 \pmod{p}$. Then, for any positive integer k, there exists an unique residue $x \pmod{p^k}$ such that $P(x_1^k) \equiv 0 \pmod{p^k}$ and $x \equiv x_1 \pmod{p}$.

Problem 226. Let $f(x) = a_n x^n + \cdots + a_0$ be a polynomial with integer coefficients where $|a_0|$ is a prime and

$$|a_0| > |a_1| + |a_2| + \cdots + |a_n|.$$

Prove that $f(x)$ is irreducible.

Perron's Criterion 227. Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with integer coefficients where $a_0 \neq 0$, and

$$|a_{n-1}| > 1 + |a_{n-2}| + \cdots + |a_1| + |a_0|.$$

Prove that $P(x)$ is irreducible.

Cohn's Criterion 228. For a prime $p > 2$ and an integer $b \geq 2$, let $p = (\overline{p_n \cdots p_1 p_0})_b$ be the base–b representation of p (with $0 \leq p_i < b$ for $i = 0, 1, \ldots, n$, and $p_n \neq 0$). Prove that the polynomial

$$f(x) = p_n x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0,$$

is irreducible.

2003 Serbia TST 229. If $p(x)$ is a polynomial, denote by $p^n(x)$ the polynomial

$$p(\dots(p(x))\dots),$$

where p is iterated n times. Prove that the polynomial $p^{2003}(x) - 2p^{2002}(x) + p^{2001}(x)$ is divisible by $p(x) - x$.

2013 Serbia TST 230. Let $A(x)$ and $B(x)$ be the polynomials

$$A(x) = a_m x^m + \dots + a_1 x + a_0 \qquad \text{and} \qquad B(x) = b_n x^n + \dots + b_1 x + b_0,$$

where $a_m b_n \neq 0$. We say $A(x)$ and $B(x)$ are similar if the following conditions hold:

a) $n = m$,

b) There is a permutation π of the set $\{0, 1, \dots, n\}$ such that $b_i = a_{\pi(i)}$ for each $i \in 0, 1, \dots, n$.

Let $P(x)$ and $Q(x)$ be similar polynomials with integer coefficients. Given that $P(16) = 3^{2012}$, find the smallest possible value of $|Q(3^{2012})|$.

2007 China TST 231. Prove that for any positive integer n, there exists only n degree polynomial $f(x)$, satisfying $f(0) = 1$ and $(x+1)[f(x)]^2 - 1$ is an odd function.

2003 Romania TST 232. Let $f \in \mathbb{Z}[X]$ be an irreducible polynomial over the ring of integer polynomials, such that $|f(0)|$ is not a perfect square. Prove that if the leading coefficient of f is 1 (the coefficient of the term having the highest degree in f) then $f(X^2)$ is also irreducible in the ring of integer polynomials.

2000 Putnam 233. Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \cdots$ of integers such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for all $n \geq 0$. Prove that if there exists a positive integer m for which $a_m = 0$ then either $a_1 = 0$ or $a_2 = 0$.

Komal 234. Consider the coefficient x_n of x^n in $(x^2+x+1)^n$. Prove that for all primes p, we have p^2 dividing $x_p - 1$.

2007 MOP 235. Let $p(x)$ be a monic polynomial with integer coefficients. Show that there exist infinitely many positive integers k such that $p(x) - k$ is irreducible.

2007 Iran TST 236. Does there exist a a sequence $a_0, a_1, a_2, \dots$ of positive integer, such that for each $i \neq j$, we have $\gcd(a_i, a_j) = 1$, and for each n, the polynomial $\sum_{i=0}^{n} a_i x^i$ is irreducible over $\mathbb{Z}[x]$?

1997 IMO Shortlist 237. Let p be a prime number and f an integer polynomial of degree d such that $f(0) = 0, f(1) = 1$ and $f(n)$ is congruent to 0 or 1 modulo p for every integer n. Prove that $d \geq p - 1$.

2005 IMO Shortlist 238. Let a, b, c, d, e, f be positive integers and let $S = a+b+c+d+e+f$. Suppose that the number S divides $abc+def$ and $ab+bc+ca-de-ef-df$. Prove that S is composite.

2002 IMO 239. Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers a such that

$$\frac{a^m + a - 1}{a^n + a^2 - 1},$$

is itself an integer.

2002 IMO Shortlist 240. Let P be a cubic polynomial given by $P(x) = ax^3 + bx^2 + cx + d$, where a, b, c, d are integers and $a \neq 0$. Suppose that $xP(x) = yP(y)$ for infinitely many pairs x, y of integers with $x \neq y$. Prove that the equation $P(x) = 0$ has an integer root.

2006 IMO 241. Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x))\ldots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

2005 USA TST 242. We choose random a unitary polynomial of degree n and coefficients in the set $1, 2, \ldots, n!$. Prove that the probability for this polynomial to be special is between 0.71 and 0.75, where a polynomial g is called special if for every $k > 1$ in the sequence $f(1), f(2), f(3), \ldots$ there are infinitely many numbers relatively prime with k.

2008 USA TST 243. Let n be a positive integer. Given an integer coefficient polynomial $f(x)$, define its signature modulo n to be the (ordered) sequence $f(1), \ldots, f(n)$ modulo n. Of the n^n such n-term sequences of integers modulo n, how many are the signature of some polynomial $f(x)$ if

a) n is a positive integer not divisible by the square of a prime.

b) n is a positive integer not divisible by the cube of a prime.

2009 China TST 244. Prove that for any odd prime number p, the number of positive integer n satisfying $p \mid n!+1$ is less than or equal to $cp^{\frac{2}{3}}$, where c is a constant independent of p.

2002 USA TST 245. Let $p > 5$ be a prime number. For any integer x, define

$$f_p(x) = \sum_{k=1}^{p-1} \frac{1}{(px+k)^2}$$

Prove that for any pair of positive integers x, y, the numerator of $f_p(x) - f_p(y)$, when written as a fraction in lowest terms, is divisible by p^3.

2006 APMO 246. Let $p \geq 5$ be a prime and let r be the number of ways of placing p checkers on a $p \times p$ checkerboard so that not all checkers are in the same row (but they may all be in the same column). Show that r is divisible by p^5. Here, we assume that all the checkers are identical.

2004 Bay Area Math Olympiad 247. Find (with proof) all monic polynomials $f(x)$ with integer coefficients that satisfy the following two conditions:

a) $f(0) = 2004$;

b) If x is irrational, then $f(x)$ is irrational.

1997 USAMO 248. Prove that for any integer n, there exists a unique polynomial Q with coefficients in $\{0, 1, \ldots, 9\}$ such that $Q(-2) = Q(-5) = n$.

2007 USA TST 249. For a polynomial $P(x)$ with integer coefficients, $r(2i-1)$ (for $i = 1, 2, 3, \ldots, 512$) is the remainder obtained when $P(2i-1)$ is divided by 1024. The sequence

$$(r(1), r(3), \ldots, r(1023)),$$

is called the remainder sequence of $P(x)$. A remainder sequence is called complete if it is a permutation of $(1, 3, 5, \ldots, 1023)$. Prove that there are no more than 2^{35} different complete remainder sequences.

2008 Putnam 250. Let p be a prime number. Let $h(x)$ be a polynomial with integer coefficients such that $h(0), h(1), \ldots, h(p^2-1)$ are distinct modulo p^2. Show that $h(0), h(1), \ldots, h(p^3-1)$ are distinct modulo p^3.

2013 EGMO 251. Find all positive integers a and b for which there are three consecutive integers at which the polynomial

$$P(n) = \frac{n^5 + a}{b},$$

takes integer values.

2014 IMO Shortlist 252. Let $a_1 < a_2 < \cdots < a_n$ be pairwise coprime positive integers with a_1 being prime and $a_1 \geq n+2$. On the segment $I = [0, a_1 a_2 \cdots a_n]$ of the real line, mark all integers that are divisible by at least one of the numbers $a_1, \ldots, a_n$. These points split I into a number of smaller segments. Prove that the sum of the squares of the lengths of these segments is divisible by a_1.

2020 Iran TST 253. Let p be an odd prime number. Find all integer $\frac{p-1}{2}$–tuples $\left(x_1, x_2, \ldots, x_{\frac{p-1}{2}}\right)$ such that

$$\sum_{i=1}^{\frac{p-1}{2}} x_i \equiv \sum_{i=1}^{\frac{p-1}{2}} x_i^2 \equiv \cdots \equiv \sum_{i=1}^{\frac{p-1}{2}} x_i^{\frac{p-1}{2}} \pmod{p}.$$

1.8 Algebraic Systems

1.8.1 Polynomial Equations

Problem 254.

(a) Find a way to solve $(x+a)^4+(x+b)^4=c$ for x.

(b) How many real values of x satisfy $(x+1)^6+(x+5)^6=730$?

(c) Find a method to solve $(x+a)^{2n}+(x+b)^{2n}=c$, where n is a positive integer, assuming we can solve any degree–n polynomial equation.

Problem 255.

(a) If $a+b=c+d$, find a way to solve $(x+a)(x+b)(x+c)(x+d)=m$ for x.

(b) Solve $(x^2+6x+8)(x^2-8x+15)=72$.

Problem 256. In an arithmetic progression with common difference d, we add d^4 to the product of four consecutive terms $a, a+d, a+2d$, and $a+3d$ in the progression. Find the square root of the result in terms of a and d.

Problem 257. Find all k such that the following equation has four simple real roots.

$$(x+b)(x+a+b)(x+2a+b)(x+3a+b)=k.$$

Problem 258. How many real roots does this equation have? Find them.

$$x(x-4)(x-2)(x-1)^2(x+2)+66=0.$$

Problem 259. If $a+c=\alpha$ and $b+d=\beta$, solve the equation

$$(ax+b)^4+(cx+d)^4=(\alpha x+\beta)^4.$$

Problem 260. Solve for real x:

$$(2-x)^4+(2x-1)^4=(x+1)^4.$$

Problem 261. For what values of a would all the roots of this equation be unreal?

$$2x^4 + x^3 - (3a+2)x^2 + 2x + a^2 - 1 = 0.$$

Problem 262. Solve

$$x^3 - (3+\sqrt{3})x + 3 = 0.$$

Problem 263. Solve

$$(ax^2 + bx + c)^2 = x^2(x^2 + bx + c).$$

Problem 264. Solve the following nineteen polynomial equations in x:

1. $x^4 + (x+\sqrt{2})^4 = 68$,
2. $x^6 + (x+2)^6 = 2$,
3. $(x+3)^3 + (x+5)^3 = 8$,
4. $(\sqrt{x}+1)^4 + (\sqrt{x}-3)^4 = 256$,
5. $(x^2+3x+2)(x^2+6x+12) = 120$,
6. $x^4 + 2x^3 + 2x^2 + x = 42$,
7. $x^4 + 6x^3 + 7x^2 - 6x = 1$,
8. $\dfrac{\sqrt[n]{a+x}}{a} + \dfrac{\sqrt[n]{a+x}}{x} = b\sqrt[n]{x}$,
9. $\sqrt[3]{a+\sqrt{x}} + \sqrt[3]{a-\sqrt{x}} = \sqrt[3]{b}$,
10. $(x^2+2x-12)^2 = x^2(3x^2+2x-12)$,
11. $(2x^2-x-6)^2 + 3(2x^2+x-6)^2 = 4x^2$,
12. $\dfrac{(x^2+1)^2}{x(x+1)^2} = \dfrac{9}{2}$,
13. $3x^4 - 20x^3 + 45x^2 - 40x + 12 = 0$,
14. $x^3 - 3abx + a^3 + b^3 = 0$,
15. $(x^2-16)(x-3)^2 + 9x^2 = 0$.
16. $(x-2)(x+1)(x+6)(x+9) + 108 = 0$,
17. $(x^2-4)(x+1)(x+4)(x+5)(x+8) + 476 = 0$,
18. $\sqrt[m]{(x+1)^2} - \sqrt[m]{(x-1)^2} + \dfrac{3\sqrt[m]{x^2-1}}{2} = 0$,
19. $(a^2-a)^2(x^2-x+1)^3 = (a^2-a+1)^3(x^2-x)^2$,

Problem 265. Given a real number a, solve the following equation for x:

$$x^4 - 10x^3 - 2(a-11)x^2 + 2(5a+6)x + 2a + a^2 = 0.$$

Problem 266. Solve $x^3 + 2\sqrt{3}x^2 + 3x + \sqrt{3} - 1 = 0$ for x.

Problem 267. Solve the quintic equation $x^5 - 5x^3 + 5x - 1 = 0$.

Problem 268. Let a, b, c be given real numbers. Solve the following equation for x:

$$\frac{(x-a)^2 + (x-a)(x-b) + (x-b)^2}{(x-a)^2 - (x-a)(x-b) + (x-b)^2} = \frac{3c^2+1}{c^2+3}.$$

Problem 269. Given real numbers a, b, c, d, solve the following two equations:

a)

$$\frac{(x+a+b)^5 + (x+c+d)^5}{(x+a+c)^5 + (x+b+d)^5} = \frac{(a+b+c+d)^2}{(a-b+c-d)^2}.$$

b)

$$\frac{(x+a+b)^5 + (x+c+d)^5}{(x+a+c)^5 + (x+b+d)^5} = \frac{(a+b-c-d)^5}{(a-b+c-d)^5}.$$

Problem 270. Find the six roots of the sextic equation in the guise of a septic form:

$$x^7 + 7^7 = (x+7)^7.$$

Problem 271. Given real numbers a and b with $a \neq -b$, solve the following equation for x:

$$x^7 + a^7 + b^7 = (x+a+b)^7.$$

Problem 272. Find the real roots of the sextic equation

$$8x^6 - 16x^5 + 2x^4 + 12x^3 - 36x + 27 = 0.$$

Problem 273. Solve the equation $(6x+7)^3(3x+4)(x+1) = 6$ for x.

Quartic Equations & Completing the Squares 274. Solve the following three equations in x by the method of completing the squares:

1. $x^2 + \left(\dfrac{x}{x-1}\right)^2 = 8,$
2. $x^2(1+x)^2 + x^2 = 8(1+x)^2,$
3. $(1+x^2)^2 = 4x(1-x^2).$

Problem 275. Solve $(x+1)^2+(x+2)^3+(x+3)^4=2$ for x.

Problem 276. For a positive integer n, solve the following equation for x:

$$(x-1)^3+(x-2)^3+\cdots+(x-n)^3=0.$$

2005 Switzerland TST 277. Let $n\geq 2$ be a positive integer. Prove that the polynomial

$$(x^2-1^2)(x^2-2^2)(x^2-3^2)\cdots(x^2-n^2)+1,$$

cannot be written as the product of two non-constant polynomials with integer coefficients.

2008 Switzerland TST 278. Let $P(x)=x^4-2x^3+px+q$ be a polynomial with real coefficients whose roots are all real. Prove that the largest root of $P(x)=0$ lies in the interval $[1,2]$.

2009 Switzerland TST 279. For which positive integers n does there exist a polynomial $P(x)$ with integer coefficients such that $P(d)=(n/d)^2$ for all divisors d of n?

2010 Switzerland TST 280. Let $P(x)$ be a polynomial with real coefficients such that for all real x, we have

$$P(x)=P(1-x).$$

Prove that there exists a polynomial $Q(x)$ with real coefficients such that

$$P(x)=Q\left(x(1-x)\right).$$

2011 Switzerland TST 281. Find all non-zero polynomials $P(x)$ with real coefficients such that

$$P(P(k))=\left(P(k)\right)^2, \qquad \text{for} \quad k=0,1,2,\ldots,(\deg P)^2.$$

2011 Switzerland TST 282. Let $a>1$ be a positive integer, and let $f(x)$ and $g(x)$ be polynomials with integer coefficients. Suppose that there is a positive integer n_0 such that $g(n)>0$ for all $n\geq n_0$, and

$$f(n)\mid a^{g(n)}-1, \qquad \text{for all} \quad n\geq n_0.$$

Prove that f must be a constant polynomial.

2007 Ecuador TST 283. How many solutions to the equation

$$x^7-1+x^3(x-1)=0,$$

are not integers?

1.8.2 Trigonometric Tricks for Solving Algebraic Systems

Problem 284.

(a) Solve

$$\frac{3x - x^3}{1 - 3x^2} = \sqrt{3}.$$

(b) Show that

$$\begin{aligned} 3\sqrt{3} &= \tan 20^\circ - \tan 40^\circ + \tan 80^\circ, \\ 3 &= \tan 20^\circ \tan 40^\circ + \tan 40^\circ \tan 80^\circ - \tan 20^\circ \tan 80^\circ, \\ \sqrt{3} &= \tan 20^\circ \tan 40^\circ \tan 80^\circ. \end{aligned}$$

Problem 285.

(a) Solve

$$32x^5 - 40x^3 + 10x = 1.$$

(b) Show that

$$\begin{aligned} 0 &= \cos 12^\circ - \cos 24^\circ - \cos 48^\circ + \cos 60^\circ + \cos 84^\circ, \\ \frac{1}{32} &= \cos 12^\circ \cos 24^\circ \cos 48^\circ \cos 60^\circ \cos 84^\circ, \\ 10 &= \frac{1}{\cos 12^\circ} - \frac{1}{\cos 24^\circ} - \frac{1}{\cos 48^\circ} + \frac{1}{\cos 60^\circ} + \frac{1}{\cos 84^\circ}. \end{aligned}$$

Problem 286. If $0 < a < 1$ is a real number, solve

$$\left(\frac{1+a^2}{2a}\right)^x - \left(\frac{1-a^2}{2a}\right)^x = 1.$$

Problem 287. Solve the following equation:

$$\cos^2 \phi + \cos^2 2\phi - 2\cos\phi \cos 2\phi \cos 4\phi = \frac{3}{4}.$$

Problem 288. Solve the following sine equation for α:

$$\sin^2\alpha + \frac{1}{4}\sin^2 3\alpha = \sin\alpha \sin^2 3\alpha.$$

Problem 289. Let $x_1 < x_2 < x_3$ be the three roots of the cubic equation $x^3 - 3x + 1 = 0$. Prove that $x_2^2 - x_1 = 2$.

Problem 290. The equation

$$x^{10} + (13x-1)^{10} = 0$$

has 10 complex roots $r_1, \overline{r_1}, r_2, \overline{r_2}, r_3, \overline{r_3}, r_4, \overline{r_4}, r_5, \overline{r_5}$, where the bar denotes complex conjugation. Find the value of

$$\frac{1}{r_1\overline{r_1}} + \frac{1}{r_2\overline{r_2}} + \frac{1}{r_3\overline{r_3}} + \frac{1}{r_4\overline{r_4}} + \frac{1}{r_5\overline{r_5}}.$$

1.8.3 Irrational Equations

Problem 291. Solve the following equation in x:

$$x^3 - 3x + (x^2-1)\sqrt{x^2-4} = 2.$$

Problem 292. Prove that $\sqrt[5]{2} - \sqrt[5]{2}^2 + \sqrt[5]{2}^3 - \sqrt[5]{2}^4$ is a root of $x^5 + 20x^3 + 20x^2 + 30x + 10$.

1997 Switzerland TST 293. Let v and w be distinct, randomly chosen solutions to the equation $z^{1997} - 1 = 0$. Determine the probability that

$$\sqrt{2+\sqrt{3}} \leq |v+w|.$$

2009 Ecuador TST 294. Let a, b, c be distinct rational numbers. Prove that the expression

$$\sqrt{\frac{1}{(a-b)^2} + \frac{1}{(b-c)^2} + \frac{1}{(c-a)^2}},$$

is rational.

2009 Ecuador TST 295. Simplify the expression

$$\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}.$$

2008 Finland 296. Solve for x:

$$\sqrt{17+x-8\sqrt{x+1}}+\sqrt{5+x-4\sqrt{x+1}}=6.$$

2014 Finland 297. Assume that for real numbers x and y,

$$(x+\sqrt{x^2+1})(y+\sqrt{y^2+1})=1.$$

What values can the expression $x+y$ assume?

2016 Finland 298. Solve the equation

$$\sqrt{2+4x-2x^2}+\sqrt{6+6x-3x^2}=x^2-2x+6.$$

2017 Finland 299. Prove that if $a>0$, then $x=3a/4$ is a solution to

$$x=a-\sqrt{a^2-x\sqrt{x^2+a^2}}.$$

Rationalizing Irrational Equations of the Form $\sqrt[3]{u}\pm\sqrt[3]{v}=a$

Problem 300. Prove that $\sqrt[3]{u}\pm\sqrt[3]{v}=a$ is equivalent to $(u\pm v-a^3)^3=\mp 27a^3uv$.

Problem 301. Solve the irrational equation $\sqrt[3]{2x-3}+\sqrt[3]{3x+2}=3$ for x.

Review of Irrational Equations A–Z

Review of Problems 302 to 327:

Irrational Equations Collection 302. The following is a review of the 26 irrational equations in x, labeled a) to z) for the twenty six letters of the English alphabet. Question a) (Problem 302) is to solve for x the irrational equation $\sqrt{a+x} = a - \sqrt{x}$ given an appropriate real number a.

a) $\sqrt{a+x} = a - \sqrt{x}$,

b) $\dfrac{\sqrt{a+x}}{a} + \dfrac{\sqrt{a+x}}{x} = \sqrt{x}$,

c) $\sqrt{x-1} + 6\sqrt[4]{x-1} = 16$,

d) $\sqrt[3]{x} + \sqrt[3]{2x-3} = \sqrt[3]{12(x-1)}$,

e) $\sqrt[3]{a-x} + \sqrt[3]{b-x} = \sqrt[3]{a+b-2x}$,

f) $\sqrt[3]{x} + 2\sqrt[3]{x^2} = 3$,

g) $\sqrt{a+x} - \sqrt[3]{a+x} = 0$,

h) $\sqrt[5]{(7x-3)^3} + 8\sqrt[5]{(3-7x)^{-3}} = 7$,

i) $\dfrac{1-ax}{1+ax}\sqrt{\dfrac{1+bx}{1-bx}} = 1$,

j) $\sqrt[5]{16+\sqrt{x}} + \sqrt[5]{16-\sqrt{x}} = 2$,

k) $\sqrt[n]{a^k x^{n-k}} + \sqrt[n]{x^k a^{n-k}} = 2\sqrt{bx}$,

l) $\dfrac{\sqrt[n]{a-x}}{x^2} - \dfrac{\sqrt[n]{a-x}}{a^2} = \sqrt[n]{\dfrac{x^2}{a+x}}$,

m) $\sqrt{\dfrac{\sqrt[n]{a}-\sqrt[n]{x}}{\sqrt[n]{x^2}}} - \sqrt{\dfrac{\sqrt[n]{a}-\sqrt[n]{x}}{\sqrt[n]{a^2}}} = \sqrt[2n]{x}$,

n) $\sqrt{p+x} + \sqrt{p-x} = x$,

o) $3 + \sqrt{3+\sqrt{x}} = x$,

p) $\sqrt{\sqrt{5} + \sqrt{\sqrt{5}+x}} = x$,

q) $\sqrt[5]{a+\sqrt{x}} + \sqrt[5]{a-\sqrt{x}} = \sqrt[5]{2a}$,

r) $\sqrt[4]{a-x} + \sqrt[4]{b-x} = \sqrt[4]{a+b-2x}$,

s) $\sqrt{2x^2-1} + \sqrt{x^2-3x-2} = \sqrt{2x^2+2x+3} + \sqrt{x^2-x+2}$,

t) $\sqrt{2(1+x^2)} + 2(x-1) = 2a\sqrt{x}$,

u) $2\sqrt{x-1} + \sqrt{x+2} - 4 = 0$,

v) $2x - \sqrt{3-2x} - 3 = 0$,

w) $3\sqrt{x+6} - \sqrt{2-x} - 4 = 0$,

x) $2\sqrt{x-1} + \sqrt[3]{x} - 1 = 0$,

y) $\sqrt[4]{x-1} + 2\sqrt[3]{3x+2} - \sqrt{3-x} = 4$,

z) $\sqrt{2x-1} + \sqrt{x-2} - \sqrt{x+1} = 0$.

Problem 303. Given an appropriate real number a, solve the irrational equation

$$\frac{\sqrt{a+x}}{a} + \frac{\sqrt{a+x}}{x} = \sqrt{x}.$$

Problem 304. Solve the irrational equation $\sqrt{x-1} + 6\sqrt[4]{x-1} = 16$ for x.

Problem 305. Solve the irrational (third root) equation

$$\sqrt[3]{x} + \sqrt[3]{2x-3} = \sqrt[3]{12(x-1)}.$$

Problem 306. Given an appropriate real number a, solve the third root equation

$$\sqrt[3]{a-x} + \sqrt[3]{b-x} = \sqrt[3]{a+b-2x}.$$

Problem 307. Solve the irrational (third root) equation $\sqrt[3]{x} + 2\sqrt[3]{x^2} = 3$ for x.

Problem 308. Given an appropriate real number a, solve the irrational sixth root equation $\sqrt{a+x} - \sqrt[3]{a+x} = 0$ for x.

Problem 309. Solve the irrational fifth root equation $\sqrt[5]{(7x-3)^3} + 8\sqrt[5]{(3-7x)^{-3}} = 7$.

Problem 310. Given appropriate real numbers a and b, solve the irrational square root equation

$$\frac{1-ax}{1+ax}\sqrt{\frac{1+bx}{1-bx}} = 1.$$

Problem 311. Solve the irrational tenth root equation

$$\sqrt[5]{16+\sqrt{x}} + \sqrt[5]{16-\sqrt{x}} = 2,$$

for real numbers x.

Problem 312. Given appropriate real numbers a and b, and positive integers $n > k \geq 2$, solve the irrational n^{th} root equation $\sqrt[n]{a^k x^{n-k}} + \sqrt[n]{x^k a^{n-k}} = 2\sqrt{bx}$.

Problem 313. Solve the irrational n^{th} root equation

$$\sqrt{\frac{\sqrt[n]{a} - \sqrt[n]{x}}{\sqrt[n]{x^2}}} - \sqrt{\frac{\sqrt[n]{a} - \sqrt[n]{x}}{\sqrt[n]{a^2}}} = \sqrt[2n]{x}.$$

Problem 314. Solve the irrational equation

$$\sqrt{\frac{\sqrt[n]{a} - \sqrt[n]{x}}{\sqrt[n]{x^2}}} - \sqrt{\frac{\sqrt[n]{a} - \sqrt[n]{x}}{\sqrt[n]{a^2}}} = \sqrt[2n]{x}$$

for x.

Problem 315. Given an appropriate real number p, solve the square root equation

$$\sqrt{p+x} + \sqrt{p-x} = x.$$

Problem 316. Solve the irrational equation $3 + \sqrt{3+\sqrt{x}} = x$ for x.

Problem 317. Solve the irrational equation $\sqrt{\sqrt{5} + \sqrt{\sqrt{5}+x}} = x$ for x.

Problem 318. Given an appropriate real number a, solve the fifth root equation

$$\sqrt[5]{a+\sqrt{x}} + \sqrt[5]{a-\sqrt{x}} = \sqrt[5]{2a}.$$

Problem 319. Given appropriate real numbers a and b, solve the fourth root equation

$$\sqrt[4]{a-x}+\sqrt[4]{b-x}=\sqrt[4]{a+b-2x}.$$

Problem 320. Solve the following irrational square root equation in x:

$$\sqrt{2x^2-1}+\sqrt{x^2-3x-2}=\sqrt{2x^2+2x+3}+\sqrt{x^2-x+2}.$$

Problem 321. Given an appropriate real number a, solve the square root equation

$$\sqrt{2(1+x^2)}+2(x-1)=2a\sqrt{x}.$$

Problem 322. Solve the irrational square root equation $2\sqrt{x-1}+\sqrt{x+2}-4=0$.

Problem 323. Solve the irrational square root equation $2x-\sqrt{3-2x}-3=0$.

Problem 324. Solve the irrational square root equation $3\sqrt{x+6}-\sqrt{2-x}-4=0$.

Problem 325. Solve the irrational sixth root equation $2\sqrt{x-1}+\sqrt[3]{x}-1=0$.

Problem 326. Solve the twelfth root equation

$$\sqrt[4]{x-1}+2\sqrt[3]{3x+2}-\sqrt{3-x}=4.$$

Problem 327. Solve the irrational square root equation $\sqrt{2x-1}+\sqrt{x-2}-\sqrt{x+1}=0$.

1.8.4 Reciprocal Equations

Positive Reciprocal Equations

Definition. We call an equation such as $F(x) = 0$ a **reciprocal equation** in two cases: F is **positive reciprocal equation** if $F(\alpha) = F(1/\alpha) = 0$, and it is a **negative reciprocal equation** if $F(\alpha) = F(-1/\alpha) = 0$ for some α. In other words,

$$\begin{aligned}\text{Positive Reciprocal} &\iff \text{Same When } x \to \frac{1}{x},\\ \text{Negative Reciprocal} &\iff \text{Same When } x \to -\frac{1}{x},\end{aligned}$$

Problem 328. Prove that reciprocal equations of odd degree are not really interesting. In other words, prove that

1. A negative reciprocal polynomial equation cannot be of odd degree.
2. Any positive reciprocal polynomial equation of odd degree has a root of either $+1$ or -1 because ± 1 are the only reals equal to their reciprocals.

The previous question makes it clear that reciprocal equations of odd degree are not interesting, and we often mean a **reciprocal equation of even degree** when we speak of **reciprocal equations** in general.

Definition. A polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is **palindromic** if $a_i = a_{n-i}$ for $i = 0, 1, \ldots, n$ and **antipalindromic** if $a_i = -a_{n-i}$ for $i = 0, 1, \ldots, n$.

Problem 329. Prove that a positive reciprocal polynomial equation (of even degree) has its coefficients ordered in a palindromic way and the study the case for negative reciprocal equations.

Negative Reciprocal Equations

Problem 330. Prove that for a negative reciprocal equation $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ (in which the same equation is obtained by changing x to $-1/x$),

1. If n is a multiple of 4, then the coefficients of the even exponents form a palindrome, that is, $a_i = a_{n-i}$ for even i. Furthermore, prove that the coefficients of odd exponents form are antipalindromic, that is, $a_i = -a_{n-i}$ for odd i.
2. If n is not divisible by 4 (i.e. it leaves a remainder of 2 in division by 4), then the coefficients of the even exponents are antipalindromic and the coefficients of the odd exponents are palindromic.

Problem 331. Solve the following palindromic equation in x:

$$2x^4 - 13x^3 + 24x^2 - 13x + 2 = 0.$$

Problem 332. Prove that $x^n \pm \frac{1}{x^n}$ can be written as a polynomial of degree n of $x \pm \frac{1}{x}$, and that in the case of positive reciprocal equations, we need $x + \frac{1}{x}$ as the new variable, whereas for negative reciprocal equations, the variable would be $x - \frac{1}{x}$.

Problem 333. Prove, using induction, that

$$\left(x - \frac{1}{x}\right)^2 + \left(x^2 - \frac{1}{x^2}\right)^2 + \cdots + \left(x^n - \frac{1}{x^n}\right)^2 = \frac{x^{2n+1} - \frac{1}{x^{2n}}}{x^2 - 1} - 2n - 1.$$

2000 Denmark (Georg Mohr) 334. Determine all possible values of $x + \frac{1}{x}$, where the real x satisfies the equation

$$x^4 + 5x^3 - 4x^2 + 5x + 1 = 0,$$

and solve this equation.

Problem 335. For a given real, non-zero number a, solve the following antipalindromic equation in x:

$$2ax^4 - (2a^2 + 3a - 2)x^3 + (3a^2 - 4a - 3)x + (2a^2 + 3a - 2)x + 2a = 0.$$

Problem 336. Solve the following equation in x:

$$2x^4 - 15x^3 + 35x^2 - 30x + 8 = 0.$$

Problem 337. Solve the following equation in x:

$$2x^4 + 7x^3 - 34x^2 - 21x + 18 = 0.$$

Problem 338. Prove that the reciprocals of roots of the cubic equation $x^3 - x + 1 = 0$ are roots of the quintic equation $x^5 + x + 1 = 0$.

2008 Ecuador TST 339. If $z + \frac{1}{z} = 1$, find the numerical value of

$$z^{2008} + \frac{1}{z^{2008}}.$$

1.8.5 Equations Containing the Floor & Absolute Value Function

Problem 340. Solve the equation $\lfloor x + 2 \rfloor + \lfloor x \rfloor = 12$ for x.

Problem 341. Solve the equation $\lfloor 5x + 3 \rfloor + \lfloor 7x + 9 \rfloor = 18$ for x.

Problem 342. Solve the equation in x:

$$\lfloor -x^2+3x \rfloor = \left\lfloor x^2+\frac{1}{2} \right\rfloor.$$

Problem 343. Solve the following equation involving the floor function:

$$\lfloor x \rfloor = \left\lfloor \frac{x^3-2}{3} \right\rfloor.$$

Problem 344. Solve the following equation involving the floor function for x and y:

$$\frac{4x+3y}{2x} = \left\lfloor \frac{x^2+y^2}{x^2} \right\rfloor.$$

Problem 345. Solve the following equation involving the floor function for x:

$$\frac{15x-7}{5} = \left\lfloor \frac{6x+5}{8} \right\rfloor.$$

Problem 346. Solve the following equation in x:

$$\sqrt{(x+3)^2}+\sqrt{(x-2)^2}+\sqrt{(2x-8)^2}=9.$$

Problem 347. Solve the following equation in x:

$$|x^2-4|+|x|+2x=2.$$

Problem 348. Solve the absolute value equation

$$|x-2|\cdot|x+3|\cdot|x+6| = |x+1|\cdot|x+4|\cdot|x+9|.$$

1.8.6 Arithmetic of Polynomial Roots

1.8.6.1 Miscellaneous Treacheries on Polynomial Roots

Problem 349. What relationship must be happening between a, b, c so that the roots of the cubic equation

$$x^3+ax^2+bx+c=0,$$

be in an arithmetic progression.

Problem 350. Prove that if $p^2 < 3q$, the equation $x^3+px^2+qx+r=0$ has only one real root.

Problem 351. You may use the first task to prove the second task:

a) For a sequence x_n of real numbers satisfying

$$\sqrt[n+1]{n+2} < x_{n+1} < \sqrt[n]{n+1} < x_n < \sqrt[n-1]{n},$$

prove that x_n is strictly decreasing with a limit of 1.

b) Prove that the equation $x^n = x + n$, when n is a positive integer, always has a solution for x between 1 and 2, and when n is increased, the root x is decreased indefinitely with a limit of 1.

Problem 352. Prove that the quartic equation

$$x^4 - 4x^3 + 12x^2 - 24x + 24 = 0$$

does not have any real roots.

Problem 353. Let S be the area of a triangle whose heights are the roots of

$$x^3 - kx^2 + qx - z = 0.$$

Prove that if $4kqz > q^3 + 8z^2$,

$$S = \frac{z^2}{\sqrt{q(4kqz - q^3 - 8z^2)}}.$$

Problem 354. Find the angles of an isosceles triangle with base a and legs b such that

$$a^3 - 3ab^2 + b^3\sqrt{3} = 0.$$

Problem 355. Prove that if the three side-lengths a, b, c of a triangle satisfy the two equations

$$a^4 = b^4 + c^4 - b^2c^2 \qquad \text{and} \qquad b^4 = c^4 + a^4 - c^2a^2,$$

then it must also satisfy

$$c^4 = a^4 + b^4 - a^2b^2.$$

Problem 356. Let $n \geq 2$ be any integer. Prove that if a, b, c are side-lengths of a triangle, then so are $\sqrt[n]{a}$, $\sqrt[n]{b}$, $\sqrt[n]{c}$.

Problem 357. Prove that if $p^3 + q^3$ is divisible by 23, then there are two roots of $x^3 + px + q = 0$ whose square of difference is also divisible by 23.

Problem 358. Prove that the necessary and sufficient condition for the cubic equation

$$ax^3 + bx^2 + cx + d = 0$$

to have a purely imaginary root (in the form $x = \alpha i$ with $i = \sqrt{-1}$) is that $ad = bc$ and $ac > 0$.

Problem 359. Prove that for all positive integers n, there is always a root x_n in the interval $[0,1]$ for the equation

$$x^n + x^{n-1} + \cdots + x^2 + x = 1,$$

and find the limit of x_n as n increases.

Problem 360. If α, β, γ are the roots of $x^3 + px + q = 0$, and m, n are positive integers, find the sum

$$S = \frac{m\alpha + n}{m\alpha - n} + \frac{m\beta + n}{m\beta - n} + \frac{m\gamma + n}{m\gamma - n},$$

in terms of p, q, m, n.

Problem 361. Solve the following cubic equation if we know that one of its roots is double another root of the same equation:

$$x^3 + 21x^2 + 140x - 300 = 0.$$

Problem 362. Find a such that the product of two roots of the equation

$$x^4 - ax^3 + 23x^2 + ax - 168 = 0$$

is 12, and then solve the equation.

Problem 363. For a polynomial $f(x)$,

a) If $f(x) = f(a - x)$ for some a, then show that $f(x)$ may be written as a sum of even powers of $2x - a$.

b) If

$$f(x) = 16x^4 - 32x^3 - 56x^2 + 72x + 72,$$

find $f(1 - x)$ and solve $f(x) = 0$.

Problem 364. Find m such that in the equation

$$x^4 - 8x^3 + mx^2 - 8x - 3 = 0,$$

the sum of two roots equals the third root, and then solve the equation.

Problem 365. Without solving the equation, find the area of the triangle whose side-lengths are the roots of the cubic equation $x^3 - 12x^2 + 47x - 60 = 0$.

2017 Denmark (Georg Mohr) 366. Let A, B, C, D denote the digits in a four-digit number $n = \overline{ABCD}$. Determine the least n greater than 2017 satisfying that there exists an integer x such that

$$x = \sqrt{A + \sqrt{B + \sqrt{C + \sqrt{D + x}}}}.$$

1999 Switzerland TST 367. Prove that for every polynomial $P(x)$ of degree 10 with integer coefficients, there is an infinite (in both directions) arithmetic progression of integers that contains none of the values $P(k)$, where $k \in \mathbb{Z}$.

2003 Switzerland TST 368. Find all quadratic polynomials $Q(x) = ax^2 + bx + c$ such that three different prime numbers p_1, p_2, p_3 exist with

$$|Q(p_1)| = |Q(p_2)| = |Q(p_3)| = 11.$$

2006 Switzerland TST 369. The polynomial $P(x) = x^3 - 2x^2 - x + 1$ has three real roots $a > b > c$. Find the value of the expression

$$a^2b + b^2c + c^2a.$$

2007 Switzerland TST 370. A pair (r, s) of positive integers is called *good* if a polynomial $P(x)$ with integer coefficients and distinct integers $a_1, a_2, \ldots, a_r$ and $b_1, b_2, \ldots, b_s$ exist such that

$$P(a_1) = P(a_2) = \cdots = P(a_r) = 2,$$
$$P(b_1) = P(b_2) = \cdots = P(b_s) = 5.$$

a) Show that for every *good* pair (r, s) of positive integers, we have $r \leq 3$ and $s \leq 3$.

b) Find all *good* pairs.

2009 Ecuador TST 371. Let a and b be two coprime positive integers. It is known that the coefficients of x^2 and x^3 are equal in the expansion of $(ax + b)^{2009}$. Find $a + b$.

Problem 372. Let a, b, c be non-zero real numbers. Show that if the equation $ax^2 + bx + c = 0$ has a positive solution for x, then the polynomial $f(x)$ with real coefficients, defined by:

$$f(x) = 5ax^4 + mx^3 + 3bx^2 + nx + c,$$

has at least two real roots.

Problem 373. The polynomial $P(x)$ is such that the polynomials

$$P(P(x)) \qquad \text{and} \qquad P(P(P(x))),$$

are strictly monotone on the whole real x axis. Prove that $P(x)$ is also strictly monotone on $\mathbb{R}$.

1.8.6.2 Calculating Sum of Powers of the Roots

Sum S_p of Powers of Quadratic & Cubic Roots

Sum of Powers of Quadratic Roots 374. Let x_1 and x_2 be the roots of the quadratic equation $ax^2 + bx + c = 0$ and define $S_p = x_1^p + x_2^p$. Prove the recursive formula between the sum of powers of quadratic roots:

$$aS_n + bS_{n-1} + cS_{n-2} = 0.$$

Conclude that

1. The sum $S_p(x_1, x_2)$ of the p^{th} powers of quadratic roots x_1 and x_2, is calculable in terms of a, b, c, and

2. In order to find the sum of n^{th} powers of quadratic roots, one needs both $(n-1)^{th}$ and $(n-2)^{th}$ powers of the roots.

Sum of Powers of Cubic Roots 375. Let x_1, x_2 and x_3 be the roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$ and define $S_p = x_1^p + x_2^p + x_3^p$. Prove the recursive formula between the sum of powers of roots:

$$aS_n + bS_{n-1} + cS_{n-2} + cS_{n-2} + dS_{n-3} = 0.$$

Conclude that

1. The sum $S_p(x_1, x_2, x_3)$ of the p^{th} powers of the cubic roots x_1, x_2 and x_3, is calculable in terms of a, b, c, d, and

2. In order to find the sum of n^{th} powers of cubic roots, one needs all three of $(n-1)^{th}$, $(n-2)^{th}$, and $(n-3)^{th}$ powers of the roots.

Problem 376. Find the sum of the fourth powers of the roots of $2x^2 - 4x + 1 = 0$.

Problem 377. Find the sum of the sixth powers of the roots of $x^3 - 3x + 1 = 0$.

Problem 378. If x_1, x_2, x_3 are the roots of the cubic equation $x^3 - x + 1 = 0$, find the sum of fifth powers of the roots: $x_1^5 + x_2^5 + x_3^5$.

Problem 379. If x_1, x_2, x_3 are the roots of $x^3 - 1 = 0$, prove that

$$x_1^n + x_2^n + x_3^n = x_1^n x_2^n + x_2^n x_3^n + x_3^n x_1^n.$$

Problem 380. Find the values of real numbers a, b, p, q such that the equation

$$(2x-1)^{20} - (ax+b)^{20} = (x^2 + px + q)^{10}$$

becomes an identity (true for all x).

Problem 381. Find the sum of the eleventh powers of the roots of the equation

$$x^3 + x + 1 = 0.$$

Problem 382. This was the seventh problem on 2023 Indian Statistical Institute UGB 2023 and it comes in two parts:

a) Let $n \geq 1$ be an integer. Prove that $X^n + Y^n + Z^n$ can be written as a polynomial with integer coefficients in the variables $\alpha = X + Y + Z$, $\beta = XY + YZ + ZX$ and $\gamma = XYZ$.

b) Let $G_n = x^n \sin(nA) + y^n \sin(nB) + z^n \sin(nC)$, where x, y, z, A, B, C are real numbers such that $A + B + C$ is an integral multiple of π. Using (a) or otherwise show that if $G_1 = G_2 = 0$, then $G_n = 0$ for all positive integers n.

1.8.6.3 Forming Equations Given the Roots

Forming Equations A.K.A. Reverse Viète

Problem 383. Concerning the polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

we can divide everything by $a_n \neq 0$ and apply Viète's Formula's in reverse to write the equation as

$$x^n - \left(\sum x_1\right) x^{n-1} + \left(\sum x_1 x_2\right) x^{n-2} - \left(\sum x_1 x_2 x_3\right) x^{n-3} + \cdots + (-1)^n (x_1 x_2 \cdots x_n) = 0.$$

Problem 384. Form the polynomial equation whose roots are the squares of the roots of the following equation:

$$x^3 + 2x^2 - x + 5 = 0.$$

Problem 385. Let x_1, x_2, x_3, and x_4 be the roots of the equation $x^4 - 4x^2 + x + 3 = 0$. Find the sum

$$S = \frac{1}{2x_1 - 1} + \frac{1}{2x_2 - 1} + \frac{1}{2x_3 - 1} + \frac{1}{2x_4 - 1}.$$

Problem 386. Let $x_1, x_2, \ldots, x_n$ be the roots of the equation

$$x^n + x^{n-1} + x^{n-2} + \cdots + x + 1 = 0.$$

Find the sum

$$S = \frac{1}{x_1 - 1} + \frac{1}{x_2 - 1} + \cdots + \frac{1}{x_n - 1}.$$

Problem 387. Let x_1, x_2, and x_3 be the roots of the equation $x^3 - 3x^2 - x - 7 = 0$. Find the sum

$$S = \frac{1}{x_1^2 - 1} + \frac{1}{x_2^2 - 1} + \frac{1}{x_3^2 - 1}.$$

Problem 388. Let α be a root of the equation $x^{13} - 1 = 0$. Find a polynomial with rational coefficients such that it has $\alpha^4 + \alpha^6 + \alpha^7 + \alpha^9$ among its roots.

Problem 389. Find the quadratic equation whose roots are fourth powers of the roots of $ax^2 + bx + c = 0$.

Problem 390. Find a cubic polynomial whose roots are

$$\cos\frac{\pi}{7}, \qquad \cos\frac{3\pi}{7}, \qquad \cos\frac{5\pi}{7}.$$

Problem 391. Find a quartic polynomial whose roots are the square of the roots of the following equation:

$$x^4 + 2x^3 + x^2 - 3x + 5 = 0.$$

Problem 392. Let $x_1 = 1$, $x_2, \ldots, x_n$ be the roots of $x^n - 1 = 0$. Find $(1 - x_2)(1 - x_3) \cdots (1 - x_n)$.

1.8.6.4 Common Roots of Equations

Problem 393. Find the relationship between p, q, p', q', such that the two equations

$$\begin{cases} x^2 + px + q &= 0, \\ x^2 + p'x + q' &= 0. \end{cases}$$

have a common root $x = \alpha$, and then find the quadratic equation that has uncommon roots $x = \beta$ and $x = \beta'$ of the equations.

Problem 394. Find the condition for existence of a common root of these equations:

$$\begin{cases} x^3 + px + q &= 0, \\ x^3 + p'x + q &= 0. \end{cases}$$

Problem 395. Find m such that one of the roots of the equation $x^2 - x - m = 0$ is double one of the roots of the equation $x^2 - (m + 2)x + 3 = 0$.

Problem 396. If the greatest common factor of polynomials $f(x)$ and $g(x)$ is a polynomial of degree n, prove that it means that the equations $f(x) = 0$ and $g(x) = 0$ have n common roots.

Problem 397. Find the common roots of these two equations:

$$\begin{cases} 2x^4 - x^3 - 4x^2 + 1 &= 0, \\ x^4 + x^3 - 4x^2 - x + 1 &= 0. \end{cases}$$

Furthermore, solve each equation separately.

Problem 398. Find m such that the following two equations have three roots in common, and then solve them separately.

$$\begin{cases} f(x) = 2x^4 - 7x^3 - 2x^2 + (7m-2)x + 2 &= 0, \\ g(x) = x^4 - 7x^3 + 13x^2 - x - 6 &= 0. \end{cases}$$

Problem 399. Find the common roots of the following quartic equations:

$$\begin{cases} x^4 - 3x^3 + 4x^2 - 5x - 3 &= 0, \\ x^4 + x^3 - 5x^2 - 7x - 2 &= 0. \end{cases}$$

1.8.6.5 Number Theoretic Wizardry on Polynomial Roots

Problem 400. Prove that if the three-digit decimal $\overline{abc}$ is a prime number, then the roots of the quadratic $ax^2 + bx + c = 0$ are irrational.

Problem 401. Prove that the sum of cubes of the roots of the equation $x^3 + px + q = 0$ with integer coefficients p and q is an integer divisible by 3.

Problem 402. For what positive integer values of n is $n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2$ divisible by 10?

1.8.7 Using Viète's Formulas to Solve Systems of Equations

Problem 403. Given real numbers a, b, c, solve the following system of equations for x, y, z:

$$\begin{cases} (a-1)x - a(a-1)y - (a^2+1)z + a^3 &= 0, \\ (b-1)x - b(b-1)y - (b^2+1)z + b^3 &= 0, \\ (c-1)x - c(c-1)y - (c^2+1)z + c^3 &= 0. \end{cases}$$

Definition. We call a system of equations **symmetric** if the swapping of any of its two variables with each other would not change the system. For instance, these systems are symmetric:

$$\begin{cases} x^2 + y^2 &= \frac{7}{3}, \\ x^3 + y^3 &= -3. \end{cases} ; \qquad \begin{cases} x + y + z &= 2a, \\ x^2 + y^2 + z^2 &= 6a^2, \\ x^3 + y^3 + z^3 &= 8a^3. \end{cases}$$

Problem 404. Solve the symmetric equation

$$\begin{cases} x^2 + y^2 = \dfrac{7}{3}, \\ x^3 + y^3 = -3. \end{cases}$$

for x and y.

Problem 405. Solve the symmetric equation

$$\begin{cases} x + y + z & = 2a, \\ x^2 + y^2 + z^2 & = 6a^2, \\ x^3 + y^3 + z^3 & = 8a^3. \end{cases}$$

for x, y and z.

1.8.8 Homogeneous Equations

Homogeneous Equations

Definition. We call a system of equations **homogeneous** if each of its equations is a **homogeneous polynomial** in the unknowns. In simpler words, that is, the the terms containing unknowns are of the same degree. For instance, the system of equations

$$\begin{cases} ax^2 + by^2 + cxy = d, \\ a'x^2 + b'y^2 + c'xy = d'. \end{cases}$$

is homogeneous with respect to x and y since all the terms containing x, y in the system are of degree 2.

Problem 406. In order to solve the following homogeneous system of equations in x and y,

$$\begin{cases} ax^2 + by^2 + cxy & = d, \\ a'x^2 + b'y^2 + c'xy & = d'. \end{cases}$$

assume that the ratio of y over x equals λ, or simply $y = \lambda x$, then solve for λ.

Definition. We call a system of linear equations **linearly homogeneous** if there are no constant terms and the degree of all terms containing unknowns is 1. For instance, the two systems of equations

$$\begin{cases} ax + by & = 0, \\ a'x + b'y & = 0. \end{cases} \qquad \begin{cases} ax + by + cz & = 0, \\ a'x + b'y + c'z & = 0, \\ a''x + b''y + c''z & = 0. \end{cases}$$

are **linearly homogeneous** systems of equations with respect to x and y since all right sides are zero and all the terms containing x, y in the system are of degree 1.

Problem 407. Solve the following homogeneous equations in x and y:

1.

$$\begin{cases} 2x^2 + 5y^2 - 3xy &= 7, \\ 3x^2 - 2y^2 + xy &= 12. \end{cases}$$

2.

$$\begin{cases} x^3 + 3x^2y - y^3 &= 3, \\ 2x^3 - xy^2 + y^3 &= 2. \end{cases}$$

Problem 408. Find m such that the following system of equations has a non-zero solution:

$$\begin{cases} (m-2)x + (m-1)y &= 0, \\ mx + 2(2m-3)y &= 0. \end{cases}$$

Problem 409. Solve the following linearly homogeneous system of equations in x, y, z:

$$\begin{cases} x + ay + a^2z = 0, \\ x + by + b^2z = 0, \\ x + cy + c^2z = 0. \end{cases}$$

1.8.9 Miscellaneous Systems of Equations

Problem 410. Find all solutions x, y that satisfy both $x^{x-y} = 2y - 1$ and

$$\sqrt{x^2 + 5x + 2y - 3} + \sqrt{x^2 + x + y + 2} = \sqrt{x^2 + 4x + 3y - 2} + \sqrt{x^2 + 2y + 3}.$$

Problem 411. Given real numbers a, b, solve the system of equations for x, y:

$$\begin{cases} x^3 = ax + by, \\ y^3 = bx + ay. \end{cases}$$

Problem 412. If $x = y^3 - y$ and $y = 3x - x^3$, find all such x and y.

Problem 413. Remove x and y from the equations of the following system:

$$\begin{cases} x + y &= p + qxy, \\ 2x &= s + tx^2, \\ 2y &= s + ty^2. \end{cases}$$

Problem 414. Prove that if $a^3 \neq 3ab - 2c$, then the following system of equations is inconsistent (does not have solutions):

$$\begin{cases} x+y &= a, \\ x^2+y^2 &= b, \\ x^3+y^3 &= c. \end{cases}$$

Problem 415. Let $N > 1$ be a positive integer. Define

$$p = \sqrt[3]{\log N^{p-3}}, \qquad q = \sqrt[3]{\log N^{q-3}}, \qquad r = \sqrt[3]{\log N^{r-3}}.$$

Find $\dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r}$.

Problem 416. Solve the following systems of equations for x, y, z, t, assuming a, b, c, d are appropriate given real numbers. Problems a) through j) are listed separately in from Problem 416 to Problem 425 to give them proper importance in Kaywañan.

a) $\begin{cases} x+xy+y &= 11, \\ xy^2+x^2y &= 30. \end{cases}$

b) $\begin{cases} x^2-xy+y^2 &= 7, \\ x^3+y^3 &= 35. \end{cases}$

c) $\begin{cases} x^3+y^3 &= 7, \\ xy(x+y) &= -2. \end{cases}$

d) $\begin{cases} x^4+y^4 &= a^4, \\ x+y &= b. \end{cases}$

e) $\begin{cases} x^5+y^5 &= a^5, \\ x+y &= a. \end{cases}$

f) $\begin{cases} x+y+z &= 0, \\ \dfrac{x^2+y^2+z^2}{x^3+y^3+z^3} &= 1, \\ xyz &= 2. \end{cases}$

g) $\begin{cases} x+y+z &= a, \\ x^2+y^2+z^2 &= a^2, \\ x^3+y^3+z^3 &= a^3. \end{cases}$

h) $\begin{cases} x-ay-a^2z-a^3t &= a^4, \\ x-by-b^2z-b^3t &= b^4, \\ x-cy-c^2z-c^3t &= c^4, \\ x-dy-d^2z-d^3t &= d^4. \end{cases}$

i) $\begin{cases} x\sin a + y\sin 2a + z\sin 3a = \sin 4a, \\ x\sin b + y\sin 2b + z\sin 3b = \sin 4b, \\ x\sin c + y\sin 2c + z\sin 3c = \sin 4c. \end{cases}$

j) $\begin{cases} \dfrac{x^2}{a^2} + \dfrac{xy}{ab} + \dfrac{y^2}{b^2} &= 3, \\ b^2x^2+xy-a^2y^2 &= ab. \end{cases}$

Problem 417. Solve the symmetric system of equations

$$\begin{cases} x^2-xy+y^2 &= 7, \\ x^3+y^3 &= 35. \end{cases}$$

for x and y.

Problem 418. Solve the homogeneous system of equations

$$\begin{cases} x^3 + y^3 & = 7, \\ xy(x+y) & = -2. \end{cases}$$

for x and y.

Problem 419. Solve the symmetric system of equations

$$\begin{cases} x^4 + y^4 & = a^4, \\ x + y & = b. \end{cases}$$

for x and y, where a and b are appropriate coefficients.

Problem 420. Solve the symmetric system of equations

$$\begin{cases} x^5 + y^5 & = a^5, \\ x + y & = a. \end{cases}$$

for x and y.

Problem 421. Solve the symmetric system of equations

$$\begin{cases} x + y + z & = 0, \\ \dfrac{x^2 + y^2 + z^2}{x^3 + y^3 + z^3} & = 1, \\ xyz & = 2. \end{cases}$$

for x, y, z.

Problem 422. For an appropriate real number a, solve the symmetric system of equations

$$\begin{cases} x + y + z & = a, \\ x^2 + y^2 + z^2 & = a^2, \\ x^3 + y^3 + z^3 & = a^3. \end{cases}$$

for x, y, and z.

Problem 423. Given real numbers a, b, c, d, solve the system of equations

$$\begin{cases} x - ay - a^2z - a^3t & = a^4, \\ x - by - b^2z - b^3t & = b^4, \\ x - cy - c^2z - c^3t & = c^4, \\ x - dy - d^2z - d^3t & = d^4. \end{cases}$$

for x, y, z, and t.

Problem 424. Given three real numbers a, b, c, solve the trigonometric system of equations

$$\begin{cases} x\sin a + y\sin 2a + z\sin 3a &= \sin 4a, \\ x\sin b + y\sin 2b + z\sin 3b &= \sin 4b, \\ x\sin c + y\sin 2c + z\sin 3c &= \sin 4c. \end{cases}$$

for x, y, and z.

Problem 425. Let a and b be appropriate given real numbers. Solve the following system of equations for x and y:

$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{xy}{ab} + \dfrac{y^2}{b^2} &= 3, \\ \\ b^2x^2 + xy - a^2y^2 &= ab. \end{cases}$$

Problem 426. Solve the following systems of equations for x, y, z, assuming a, b, c are appropriate given real numbers.

a) $\begin{cases} x + y + z &= 9, \\ \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} &= 1, \\ xy + yz + zx &= 27. \end{cases}$

b) $\begin{cases} x^2 + y^2 + xy &= 37, \\ x^2 + z^2 + xz &= 28, \\ y^2 + z^2 + yz &= 19. \end{cases}$

c) $\begin{cases} x + y + z &= 6, \\ x^2 + y^2 + z^2 &= 18, \\ \sqrt{x} + \sqrt{y} + \sqrt{z} &= 4. \end{cases}$

d) $\begin{cases} x &= \dfrac{2yz}{y^2 + z^2}, \\ \\ y &= \dfrac{2zx}{z^2 + x^2}, \\ \\ z &= \dfrac{2xy}{x^2 + y^2}. \end{cases}$

e) $\begin{cases} ax - cy + bz &= x^2 + z^2, \\ -bx + ay + cz &= y^2 + z^2, \\ cx + by - az &= x^2 + y^2. \end{cases}$

f) $\begin{cases} \sqrt[4]{1 + 5x} + \sqrt[4]{6 - y} &= 3, \\ \\ 5x - y &= 18. \end{cases}$

2004 Denmark (Georg Mohr) 427. Find all sets (x, y, z) of real numbers which satisfy

$$\begin{cases} x^3 - y^2 &= z^2 - x, \\ y^3 - z^2 &= x^2 - y, \\ z^3 - x^2 &= y^2 - z. \end{cases}$$

2005 Denmark (Georg Mohr) 428. For any positive real number a determine the number of solutions (x, y) of the system of equations

$$\begin{cases} |x|+|y| &= 1, \\ x^2+y^2 &= a, \end{cases}$$

where x and y are real numbers.

2006 Denmark (Georg Mohr) 429. Determine all triplets (x, y, z) of real numbers which satisfy

$$\begin{cases} x+y &= 2, \\ xy-z^2 &= 1. \end{cases}$$

2009 Denmark (Georg Mohr) 430. Solve the following system of equations over reals:

$$\begin{cases} \dfrac{1}{x+y}+x &= 3, \\ \\ \dfrac{x}{x+y} &= 2. \end{cases}$$

2013 Denmark (Georg Mohr) 431. A sequence $\{x_n\}_{n=0}^{\infty}$ is given by $x_0 = 8$ and

$$x_{n+1} = \frac{1+x_n}{1-x_n}, \qquad \text{for} \quad n = 0, 1, 2, \dots$$

Determine the number x_{2013}.

2015 Denmark (Georg Mohr) 432. Find all sets (x, y, z) of real numbers which satisfy

$$\begin{cases} x^2+yz &= 1, \\ y^2-xz &= 0, \\ z^2+xy &= 1. \end{cases}$$

2017 Denmark (Georg Mohr) 433. The system of equations

$$\begin{cases} x^2\square z^2 &= -8, \\ y^2\square z^2 &= 7, \end{cases}$$

is written on a piece of paper, but unfortunately two of the symbols are a little blurred. However, it is known that the system has at least one solution, and that each of the two squares ($\square$) stands for either $+$ or $-$. What are the two symbols?

1999 Switzerland TST 434. Solve the system of equations over reals:

$$\begin{cases} \dfrac{4x^2}{1+4x^2} &= y, \\ \dfrac{4y^2}{1+4y^2} &= z, \\ \dfrac{4z^2}{1+4z^2} &= x. \end{cases}$$

2003 Switzerland TST 435. For real values of x, y, and a, we have

$$\begin{cases} x+y &= a, \\ x^3+y^3 &= a, \\ x^5+y^5 &= a. \end{cases}$$

Find all possible values of a.

2004 Switzerland TST 436. For real values of a, b, c, d, we have

$$\begin{cases} a &= \sqrt{45-\sqrt{21-a}}, \\ b &= \sqrt{45+\sqrt{21-b}}, \\ c &= \sqrt{45-\sqrt{21+c}}, \\ d &= \sqrt{45+\sqrt{21+d}}. \end{cases}$$

Prove that $abcd = 2004$.

2007 Ecuador TST 437. Let a, b, c, and x, y, z be the solutions to the system of equations

$$\begin{cases} x^2+y^2+z^2 &= 7+2\sqrt{3}, \\ xy+yz+zx &= -3\sqrt{3}, \\ a^2+b^2+c^2 &= 7, \\ ab+bc+ca &= 2\sqrt{3}. \end{cases}$$

Find the value of $|a+b+c| + |x+y+z|$.

2008 Ecuador TST 438. Solve the system of equations

$$\begin{cases} x+y^2 &= 1, \\ x^2+y^3 &= 1. \end{cases}$$

2009 Ecuador TST 439. Let x, y, z be real numbers such that $abc = 1$, and

$$\begin{cases} x+\dfrac{1}{y} &= 5, \\ \\ y+\dfrac{1}{z} &= 29, \\ \\ z+\dfrac{1}{x} &= \dfrac{m}{n}, \end{cases}$$

where m and n are coprime positive integers. Find the value of $m+n$.

2009 Ecuador TST 440. Solve the system of equations over reals:

$$\begin{cases} x+y+z &= 2, \\ (x+y)(y+z)+(y+z)(z+x)+(z+x)(x+y) &= 1, \\ x^2(y+z)+y^2(z+x)+z^2(x+y) &= -6. \end{cases}$$

2010 Ecuador TST 441. A sequence $\{a_n\}_{n=1}^{\infty}$ is defined initially by $a_1 = 1/2$ and recursively for $n \geq 1$ by

$$a_n = \frac{a_{n-1}}{2na_{n-1} + 1}.$$

Find the sum $a_1 + a_2 + \cdots + a_{2010}$.

2011 Ecuador TST 442. Solve the system of equations over reals:

$$\begin{cases} x_1 + x_2 + \cdots + x_{2011} & = 2011, \\ x_1^4 + x_2^4 + \cdots + x_{2011}^4 & = x_1^3 + x_2^3 + \cdots + x_{2011}^3. \end{cases}$$

2016 Ecuador 443. Let a, b, and x, y be real numbers satisfying:

$$\begin{cases} ax \ + by & = 3, \\ ax^2 + by^2 & = 7, \\ ax^3 + by^3 & = 16, \\ ax^4 + by^4 & = 42. \end{cases}$$

Find $ax^5 + by^5$.

1.9 Nice Polynomial Problems

We present 400 polynomial problems in this section. The first series of these nice polynomial problems is a Collection of 100 polynomial problems that I collected from AoPS around 2011. The second series is a collection of 300 ancient polynomial problems, which, I believe, should make the most complete and extensive resource for studying polynomials.

1.9.1 100 Nice Polynomial Problems

Problem 444. Find all polynomials $P(x)$ with real coefficient such that:

$$P(0) = 0, \quad \text{and} \quad \lfloor P\lfloor P(n)\rfloor\rfloor + n = 4\lfloor P(n)\rfloor \quad \forall n \in \mathbb{N}.$$

Problem 445. Find all functions $f : \mathbb{R} \to R$ such that

$$f(x^n + 2f(y)) = (f(x))^n + y + f(y) \quad \forall x, y \in \mathbb{R}, \quad n \in \mathbb{Z}_{\geq 2}.$$

Problem 446. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$x^2y^2\left(f(x+y) - f(x) - f(y)\right) = 3(x+y)f(x)f(y).$$

Problem 447. Find all polynomials $P(x)$ with real coefficients such that

$$P(x)P(x+1) = P(x^2) \quad \forall x \in \mathbb{R}.$$

Problem 448. Find all polynomials $P(x)$ with real coefficient such that

$$P(x)Q(x) = P(Q(x)) \quad \forall x \in \mathbb{R}.$$

Problem 449. Find all polynomials $P(x)$ with real coefficients such that if $P(a)$ is an integer, then so is a, where a is any real number.

Problem 450. Find all the polynomials $f \in \mathbb{R}[X]$ such that

$$\sin f(x) = f(\sin x), \ (\forall) x \in \mathbb{R}.$$

Problem 451. Find all polynomial $f(x) \in \mathbb{R}[x]$ such that

$$f(x)f(2x^2) = f(2x^3 + x^2) \quad \forall x \in \mathbb{R}.$$

Problem 452. Find all real polynomials f and g, such that:

$$(x^2 + x + 1) \cdot f(x^2 - x + 1) = (x^2 - x + 1) \cdot g(x^2 + x + 1),$$

for all $x \in \mathbb{R}$.

Problem 453. Find all polynomials $P(x)$ with integral coefficients such that $P(P'(x)) = P'(P(x))$ for all real numbers x.

Problem 454. Find all polynomials with integer coefficients f such that for all $n > 2005$ the number $f(n)$ is a divisor of $n^{n-1} - 1$.

Problem 455. Find all polynomials with complec coefficients f such that we have the equivalence: for all complex numbers z, $z \in [-1, 1]$ if and only if $f(z) \in [-1, 1]$.

Problem 456. Suppose f is a polynomial in $\mathbb{Z}[X]$ and m is integer .Consider the sequence a_i like this $a_1 = m$ and $a_{i+1} = f(a_i)$ find all polynomials f and alll integers m that for each i:

$$a_i|a_{i+1}.$$

Problem 457. $P(x), Q(x) \in \mathbb{R}[x]$ and we know that for real r we have $p(r) \in \mathbb{Q}$ if and only if $Q(r) \in \mathbb{Q}$ I want some conditions between P and Q.My conjecture is that there exist ratinal a, b, c that $aP(x) + bQ(x) + c = 0$

Problem 458. Find the gcd of the polynomials $X^n + a^n$ and $X^m + a^m$, where a is real.

Problem 459. Find all polynomials p with real coefficients that if for a real a,$p(a)$ is integer then a is integer.

Problem 460. $\mathfrak{question}$ is a real polynomial such that if α is irrational then $\mathfrak{question}(\alpha)$ is irrational. Prove that $\deg[\mathfrak{question}] \leq 1$

Problem 461. Show that the odd number n is a prime number if and only if the polynomial $T_n(x)/x$ is irreducible over the integers.

Problem 462. P, Q, R are non-zero polynomials that for each $z \in \mathbb{C}$, $P(z)Q(\bar{z}) = R(z)$. a) If $P, Q, R \in \mathbb{R}[x]$, prove that Q is constant polynomial. b) Is the above statement correct for $P, Q, R \in \mathbb{C}[x]$?

Problem 463. Let P be a polynomial such that $P(x)$ is rational if and only if x is rational. Prove that $P(x) = ax + b$ for some rational a and b.

Problem 464. Prove that any polynomial $\in \mathbb{R}[X]$ can be written as a difference of two strictly increasing polynomials.

Problem 465. Consider the polynomial $W(x) = (x - a)^k Q(x)$, where $a \neq 0$, Q is a nonzero polynomial, and k a natural number. Prove that W has at least $k + 1$ nonzero coefficients.

Problem 466. Find all polynomials $p(x) \in \mathbb{R}[x]$ such that the equation

$$f(x) = n$$

has at least one rational solution, for each positive integer n.

Problem 467. Let $f \in \mathbb{Z}[X]$ be an irreducible polynomial over the ring of integer polynomials, such that $|f(0)|$ is not a perfect square. Prove that if the leading coefficient of f is 1 (the coefficient of the term having the highest degree in f) then $f(X^2)$ is also irreducible in the ring of integer polynomials.

Problem 468. Let p be a prime number and f an integer polynomial of degree d such that $f(0) = 0, f(1) = 1$ and $f(n)$ is congruent to 0 or 1 modulo p for every integer n. Prove that $d \geq p - 1$.

Problem 469. Let

$$P(x) := x^n + \sum_{k=1}^{n} a_k x^{n-k},$$

with $0 \le a_n \le a_{n-1} \le \cdots \le a_2 \le a_1 \le 1$. Suppose that there exists $r \ge 1$, $\varphi \in \mathbb{R}$ such that $P(re^{i\varphi}) = 0$. Find r.

Problem 470. Let $\mathcal{P}$ be a polynomial with rational coefficients such that

$$\mathcal{P}^{-1}(\mathbb{Q}) \subseteq \mathbb{Q}.$$

Prove that $\deg \mathcal{P} \le 1$.

Problem 471. Let f be a polynomial with integer coefficients such that $|f(x)| < 1$ on an interval of length at least 4. Prove that $f = 0$.

Problem 472. prove that $x^n - x - 1$ is irreducible over $\mathbb{Q}$ for all $n \ge 2$.

Problem 473. Find all real polynomials $p(x)$ such that

$$p^2(x) + 2p(x)p\left(\frac{1}{x}\right) + p^2\left(\frac{1}{x}\right) = p(x^2)p\left(\frac{1}{x^2}\right),$$

for all non-zero real x.

Problem 474. Find all polynomials $P(x)$ with odd degree such that

$$P(x^2 - 2) = P^2(x) - 2.$$

Problem 475. Find all real polynomials that

$$p(x + p(x)) = p(x) + p(p(x)),$$

for all reals x.

Problem 476. Find all polynomials $P \in \mathbb{C}[X]$ such that

$$P(X^2) = P(X)^2 + 2P(X).$$

Problem 477. Find all polynomials of two variables $P(x, y)$ which satisfy

$$P(a, b)P(c, d) = P(ac + bd, ad + bc), \qquad \text{for all} \quad a, b, c, d \in \mathbb{R}.$$

Problem 478. Find all real polynomials $f(x)$ satisfying

$$f(x^2) = f(x)f(x - 1), \qquad \text{for all} \quad x \in \mathbb{R}.$$

Problem 479. Find all polynomials of degree 3, such that for all $x, y \ge 0$,

$$p(x + y) \ge p(x) + p(y).$$

Problem 480. Find all polynomials $P(x) \in \mathbb{Z}[x]$ such that for any $n \in \mathbb{N}$, the equation $P(x) = 2^n$ has an integer root.

Problem 481. Let f and g be polynomials such that $f(Q) = g(Q)$ for all rationals Q. Prove that there exist reals a and b such that $f(X) = g(aX + b)$, for all real numbers X.

Problem 482. Find all positive integers $n \geq 3$ such that there exists an arithmetic progression $a_0, a_1, \ldots, a_n$ such that the equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ has n roots setting an arithmetic progression.

Problem 483. Given non-constant linear functions $p_1(x), p_2(x), \ldots p_n(x)$. Prove that at least $n-2$ of polynomials $p_1 p_2 \ldots p_{n-1} + p_n, p_1 p_2 \ldots p_{n-2} p_n + p_{n-1}, \ldots p_2 p_3 \ldots p_n + p_1$ have a real root.

Problem 484. Find all positive real numbers $a_1, a_2, \ldots, a_k$ such that the number

$$a_1^{\frac{1}{n}} + \cdots + a_k^{\frac{1}{n}},$$

is rational for all positive integers n, where k is a fixed positive integer.

Problem 485. Let f, g be real non-constant polynomials such that $f(\mathbb{Z}) = g(\mathbb{Z})$. Show that there exists an integer A such that $f(X) = g(A + x)$ or $f(x) = g(A - x)$.

Problem 486. Does there exist a polynomial $f \in \mathbb{Q}[x]$ with rational coefficients such that $f(1) \neq -1$, and $x^n f(x) + 1$ is a reducible polynomial for every $n \in \mathbb{N}$?

Problem 487. Suppose that f is a polynomial of exact degree p. Find a rigorous proof that $S(n)$, where $S(n) = \sum\limits_{k=0}^{n} f(k)$, is a polynomial function of (exact) degree $p + 1$ in variable n .

Problem 488. The polynomials P, Q are such that $\deg P = n$,$\deg Q = m$, have the same leading coefficient, and $P^2(x) = (x^2 - 1)Q^2(x) + 1$. Prove that $P'(x) = nQ(x)$

Problem 489. Given distinct prime numbers p and q and a natural number $n \geq 3$, find all $a \in \mathbb{Z}$ such that the polynomial $f(x) = x^n + ax^{n-1} + pq$ can be factored into 2 integral polynomials of degree at least 1.

Problem 490. Let F be the set of all polynomials Γ such that all the coefficients of $\Gamma(x)$ are integers and $\Gamma(x) = 1$ has integer roots. Given a positive integer k, find the smallest integer $m(k) > 1$ such that there exist $\Gamma \in F$ for which $\Gamma(x) = m(k)$ has exactly k distinct integer roots.

Problem 491. Find all polynomials $P(x)$ with integer coefficients such that the polynomial

$$Q(x) = (x^2 + 6x + 10) \cdot P^2(x) - 1,$$

is the square of a polynomial with integer coefficients.

Problem 492. Find all polynomials p with real coefficients such that for all reals a, b, c such that $ab + bc + ca = 1$ we have the relation

$$p(a)^2 + p(b)^2 + p(c)^2 = p(a + b + c)^2.$$

Problem 493. Find all real polynomials f with $x, y \in \mathbb{R}$ such that

$$2yf(x+y) + (x-y)(f(x) + f(y)) \geq 0.$$

Problem 494. Find all polynomials such that $P(x^3+1) = P((x+1)^3)$.

Problem 495. Find all polynomials $P(x) \in \mathbb{R}[x]$ such that $P(x^2+1) = P(x)^2+1$ holds for all $x \in \mathbb{R}$.

Problem 496. Find all polynomials $p(x)$ with real coefficients such that

$$(x+1)p(x-1) + (x-1)p(x+1) = 2xp(x),$$

for all real x.

Problem 497. Find all polynomials $P(x)$ that have only real roots, such that

$$P(x^2-1) = P(x)P(-x).$$

Problem 498. Find all polynomials $P(x) \in \mathbb{R}[x]$such that:

$$P(x^2) + x \cdot (3P(x) + P(-x)) = (P(x))^2 + 2x^2, \qquad \text{for all} \quad x \in \mathbb{R}.$$

Problem 499. Find all polynomials f, g which are both monic and have the same degree and

$$f(x)^2 - f(x^2) = g(x).$$

Problem 500. Find all polynomials $P(x)$ with real coefficients such that there exists a polynomial $Q(x)$ with real coefficients that satisfy

$$P(x^2) = Q(P(x)).$$

Problem 501. Find all polynomials $p(x, y) \in \mathbb{R}[x, y]$ such that for each $x, y \in \mathbb{R}$ we have

$$p(x+y, x-y) = 2p(x, y).$$

Problem 502. Find all couples of polynomials (P, Q) with real coefficients, such that for infinitely many $x \in \mathbb{R}$ the condition

$$\frac{P(x)}{Q(x)} - \frac{P(x+1)}{Q(x+1)} = \frac{1}{x(x+2)},$$

holds.

Problem 503. Find all polynomials $P(x)$ with real coefficients, such that

$$P(P(x)) = P(x)^k,$$

for any given positive integer k.

Problem 504. Find all polynomials

$$P_n(x) = n!x^n + a_{n-1}x^{n-1} + \cdots + a_1x + (-1)^n(n+1)n$$

with integers coefficients and with n real roots $x_1, x_2, \ldots, x_n$, such that $k \le x_k \le k+1$, for $k = 1, 2, \ldots, n$.

Problem 505. The function $f(n)$ satisfies $f(0) = 0$ and

$$f(n) = n - f\left(f(n-1)\right),$$

for $n = 1, 2, 3, \ldots$. Find all polynomials $g(x)$ with real coefficient such that

$$f(n) = [g(n)], \qquad n = 0, 1, 2, \ldots$$

Where $[g(n)]$ denote the greatest integer that does not exceed $g(n)$.

Problem 506. Find all pairs of integers a, b for which there exists a polynomial $P(x) \in \mathbb{Z}[X]$ such that product $(x^2 + ax + b) \cdot P(x)$ is a polynomial of a form

$$x^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0,$$

where each of $c_0, c_1, \ldots, c_{n-1}$ is equal to 1 or -1.

Problem 507. There exists a polynomial P of degree 5 with the following property: if z is a complex number such that $z^5 + 2004z = 1$, then $P(z^2) = 0$. Find all such polynomials P.

Problem 508. Find all polynomials $P(x)$ with real coefficients satisfying the equation

$$(x+1)^3P(x-1) - (x-1)^3P(x+1) = 4(x^2-1)P(x),$$

for all real numbers x.

Problem 509. Find all polynomials $P(x, y)$ with real coefficients such that:

$$P(x, y) = P(x+1, y) = P(x, y+1) = P(x+1, y+1).$$

Problem 510. Find all polynomials $P(x)$ with reals coefficients such that

$$(x-8)P(2x) = 8(x-1)P(x).$$

Problem 511. Find all reals α for which there is a nonzero polynomial P with real coefficients such that

$$\frac{P(1) + P(3) + P(5) + \cdots + P(2n-1)}{n} = \alpha P(n), \qquad \text{for all} \quad n \in \mathbb{N},$$

and find all such polynomials for $\alpha = 2$.

Problem 512. Find all polynomials $P(x) \in \mathbb{R}[X]$ satisfying

$$(P(x))^2 - (P(y))^2 = P(x+y) \cdot P(x-y), \quad \forall x, y \in \mathbb{R}.$$

Problem 513. Find all $n \in \mathbb{N}$ such that polynomial

$$P(x) = (x-1)(x-2)\cdots(x-n),$$

can be represented as $Q(R(x))$, for some polynomials $Q(x)$ and $R(x)$ with degree greater than 1.

Problem 514. Find all polynomials $P(x) \in \mathbb{R}[x]$ such that

$$P(x^2-2x) = (P(x)-2)^2.$$

Problem 515. Find all non-constant real polynomials $f(x)$ such that for any real x the following equality holds

$$f(\sin x + \cos x) = f(\sin x) + f(\cos x).$$

Problem 516. Find all polynomials $W(x) \in \mathbb{R}[x]$ such that

$$W(x^2)W(x^3) = W(x)^5, \qquad \text{for all} \quad x \in \mathbb{R}.$$

Problem 517. Find all the polynomials $f(x)$ with integer coefficients such that $f(p)$ is prime for every prime p.

Problem 518. Let $n \geq 2$ be a positive integer. Find all polynomials $P(x) = a_0 + a_1x + \cdots + a_nx^n$ having exactly n roots not greater than -1 and satisfying

$$a_0^2 + a_1a_n = a_n^2 + a_0a_{n-1}.$$

Problem 519. Find all polynomials $P(x), Q(x)$ such that

$$P(Q(X)) = Q(P(x)), \qquad \text{for all} \quad x \in \mathbb{R}.$$

Problem 520. Find all integers k such that for infinitely many integers $n \geq 3$ the polynomial

$$P(x) = x^{n+1} + kx^n - 870x^2 + 1945x + 1995,$$

can be reduced into two polynomials with integer coefficients.

Problem 521. Find all polynomials $P(x), Q(x), R(x)$ with real coefficients such that

$$\sqrt{P(x)} - \sqrt{Q(x)} = R(x), \qquad \text{for all} \quad x \in \mathbb{R}.$$

Problem 522. Let $k = \sqrt[3]{3}$. Find a polynomial $p(x)$ with rational coefficients and degree as small as possible such that $p(k+k^2) = 3+k$. Does there exist a polynomial $q(x)$ with integer coefficients such that $q(k+k^2) = 3+k$?

Problem 523. Find all values of the positive integer m such that there exists polynomials $P(x), Q(x), R(x,y)$ with real coefficient satisfying the condition: For every real numbers a, b which satisfying $a^m - b^2 = 0$, we always have that $P(R(a,b)) = a$ and $Q(R(a,b)) = b$.

Problem 524. Find all polynomials $p(x) \in \mathbb{R}[x]$ such that

$$p(x^{2008} + y^{2008}) = (p(x))^{2008} + (p(y))^{2008},$$

for all real numbers x and y.

Problem 525. Find all Polynomials $P(x)$ satisfying $P(x)^2 - P(x^2) = 2x^4$.

Problem 526. Find all polynomials p of one variable with integer coefficients such that if a and b are natural numbers such that $a + b$ is a perfect square, then $p(a) + p(b)$ is also a perfect square.

Problem 527. Find all polynomials $P(x) \in \mathbb{Q}[x]$ such that

$$P(x) = P\left(\frac{-x + \sqrt{3 - 3x^2}}{2}\right), \quad \text{for all} \quad |x| \leq 1.$$

Problem 528. Find all polynomials f with real coefficients such that for all reals a, b, c such that $ab + bc + ca = 0$ we have the following relations

$$f(a - b) + f(b - c) + f(c - a) = 2f(a + b + c).$$

Problem 529. Find All Polynomials $P(x, y)$ such that for all reals x, y we have

$$P(x^2, y^2) = P\left(\frac{(x + y)^2}{2}, \frac{(x - y)^2}{2}\right).$$

Problem 530. Let n and k be two positive integers. Determine all monic polynomials $f \in \mathbb{Z}[X]$ of degree n, having the property that

$$f(n) \text{ divides } f\left(2^k \cdot a\right), \qquad \text{for all} \quad a \in \mathbb{Z} \text{ with } f(a) \neq 0.$$

Problem 531. Find all polynomials $P(x)$ such that

$$P(x^2 - y^2) = P(x + y)P(x - y).$$

Problem 532. Let $f(x) = x^4 - x^3 + 8ax^2 - ax + a^2$. Find all real number a such that $f(x) = 0$ has four different positive solutions.

Problem 533. Find all polynomial $P \in \mathbb{R}[x]$ such that: $P(x^2 + 2x + 1) = (P(x))^2 + 1$.

Problem 534. Let $n \geq 3$ be a natural number. Find all non-constant polynomials with real coefficients $f_1(x), f_2(x), \ldots, f_n(x)$, for which

$$f_k(x) f_{k+1}(x) = f_{k+1}(f_{k+2}(x)), \quad 1 \leq k \leq n,$$

for every real x (with $f_{n+1}(x) \equiv f_1(x)$ and $f_{n+2}(x) \equiv f_2(x)$).

Problem 535. Find all integers n such that the polynomial $p(x) = x^5 - nx - n - 2$ can be written as product of two non-constant polynomials with integral coefficients.

Problem 536. Find all polynomials $p(x)$ that satisfy

$$(p(x))^2 - 2 = 2p(2x^2 - 1), \qquad \text{for all} \quad x \in \mathbb{R}.$$

Problem 537. Find all polynomials $p(x)$ that satisfy

$$(p(x))^2 - 1 = 4p(x^2 - 4X + 1), \qquad \text{for all} \quad x \in \mathbb{R}.$$

Problem 538. Determine the polynomials P of two variables so that:

a) for any real numbers t, x, y we have $P(tx, ty) = t^n P(x, y)$ where n is a positive integer, the same for all t, x, y;

b) for any real numbers a, b, c we have

$$P(a + b, c) + P(b + c, a) + P(c + a, b) = 0;$$

c) $P(1, 0) = 1$.

Problem 539. Find all polynomials $P(x)$ satisfying the equation

$$(x + 1)P(x) = (x - 2010)P(x + 1).$$

Problem 540. Find all polynomials of degree 3 such that for all non-negative reals x and y we have

$$p(x + y) \leq p(x) + p(y).$$

Problem 541. Find all polynomials $p(x)$ with real coefficients such that

$$p(a + b - 2c) + p(b + c - 2a) + p(c + a - 2b) = 3p(a - b) + 3p(b - c) + 3p(c - a),$$

for all $a, b, c \in \mathbb{R}$.

Problem 542. Find all polynomials $P(x)$ with real coefficients such that

$$P(x^2 - 2x) = (P(x - 2))^2.$$

Problem 543. Find all two-variable polynomials $p(x, y)$ such that for each $a, b, c \in \mathbb{R}$:

$$p(ab, c^2 + 1) + p(bc, a^2 + 1) + p(ca, b^2 + 1) = 0.$$

1.9.2 300 Ancient Polynomial Problems

The following problems are taken from a resource of Olympiad Algebra in Iran, the book "*Topics and Discussions in of Algebra in Math Olympiads*" by *Mehdi Safa*, published by *Khoshkhan* publishing, written in Farsi (titles have been translated to English). Some of the problems are really old, for example the first problem in *Chapter 22: Various Problems on Polynomials*) of the book is dated 1907, from Hungary. Most problems, however, come from the 1990's and not so old when these lines are being written (May 2023). However, in the scope of math olympiad preparation, even twenty years is a lifetime, and new problems are a much more popular choice for students to solve. As a result, some of the classic, "old" and forgotten problems here are like gems to those who run out of "new" problems! The last hundred problems or so are taken from well-known competitions such as China TST, Austrian–Polish Mathematical Competition (APMC), Czech and Slovak Competition, IMO Shortlist, and IMO Longlist.

1907 Hungary 544. Prove that if p and q are two odd integers, then the equation $x^2 + 2px + 2q = 0$ does not have a rational root.

1907 Hungary 545. Prove that the polynomial $P(x) = x^4 + 2x^2 + 2x + 2 = 0$ cannot be written as a product of two polynomials $x^2 + ax + b$ and $x^2 + cx + d$ where a, b, c, d are integers.

1996 Bulgaria 546. Let a, b, c be real numbers and define M as the maximum value of the expression

$$|4x^3 + ax^2 + bx + c| \text{ for } x \in [-1, 1].$$

Prove that $M \geq 1$ and find all cases when $M = 1$.

1994 Romania 547. Let m, n be given positive integers. Find all common roots of

$$P(x) = x^{m+1} - x^n + 1 \qquad \text{and} \qquad Q(x) = x^{n+1} - x^m + 1.$$

1990 Iran 548. Can we find four real numbers such that for each two of them like x and y,

$$x^{10} + x^9y + \cdots + xy^9 + y^{10} = 1?$$

1969 USSR 549. Find the smallest positive integer a for which there exists a quadratic polynomial $P(x) = ax^2 + bx + c$ such that its roots are distinct and smaller than 1.

1987 Iran 550. Find all polynomials $P(x)$ such that for all x,

$$xP(x-1) = (x-12)P(x).$$

1997 Bulgaria 551. Find all real numbers m such that the polynomial

$$P(x) = (x^2 - 2mx - 4(m^2 + 1)) \cdot (x^2 - 4x - 2m(m^2 + 1)),$$

has exactly three distinct roots.

1997 Austrian–Polish 552. Let p_1, p_2, p_3, p_4 be distinct prime numbers. Prove that there does not exist a cubic polynomial $Q(x)$ with integer coefficients such that

$$|Q(p_1)| = |Q(p_2)| = |Q(p_3)| = |Q(p_4)| = 3.$$

1996 Iran 553. Define $f(x) = ax^2 + bx + c$ and assume that for $0 \leq x \leq 1$, we have $|f(x)| \leq 1$. Find the maximum value of $2a + b$.

Problem 554. Let P, Q, R, S are polynomials satisfying the equation

$$P(x^4) + xR(x^8) + x^2Q(x^{12}) = (1 + x + x^2 + x^3)S(x).$$

Prove that $x - 1$ is a factor of $P(x)$.

1998 Iran 555. Let $P(x)$ be a polynomial with real coefficients such that for all $x \geq 0$ we have $P(x) > 0$. Prove that there exists a positive integer m such that all coefficients of the polynomial $(1 + x)^m P(x)$ are non-negative.

Problem 556. Prove that there do not exist non-constant polynomials f, g, h such that

$$\frac{f(x+1)}{g(x+1)} - \frac{f(x)}{g(x)} = h\left(\frac{1}{x}\right).$$

Problem 557. Does there exist an integer c such that all the roots of the polynomial

$$P(x) = x^3 - 87x^2 + 181x + c,$$

are integers.

Problem 558. For a positive integer k, let $P(x)$ be a polynomial with integer coefficients such that the numbers $P(1), P(2), \dots, P(k)$ are not divisible by k. Prove that $P(x)$ cannot have any integer roots.

1978 Romania 559. Prove that for any polynomial $P(x) \neq x$ and any positive integer n, the polynomial $Q_n(x)$, defined by

$$Q_n(x) = \underbrace{P(P(\dots P(x)))}_{n \text{ times}} - x,$$

is divisible by $Q_1(x) = P(x) - x$.

Problem 560. For two integers a and b such that $x^2 - x - 1$ is a factor of $ax^{17} + bx^{16} + 1$. Find a.

Problem 561. For any positive integer n, prove that the polynomial

$$P_n(x) = x^{n+2} - 2x + 1,$$

has exactly one root in the interval $[0, 1]$.

1999 Iran 562. Let $P(x)$ be a polynomial of degree n such that for integer x, we know that $P(x)$ is integer. Prove that there exist integers $a_0, a_1, \ldots, a_n$ such that

$$P(x) = a_n \binom{x}{n} + \cdots + a_1 \binom{x}{1} + a_0.$$

Problem 563. For two distinct real numbers a and b, prove that the polynomial

$$P(x) = (a - b)x^n + (a^2 - b^2)x^{n-1} + \cdots + (a^{n+1} - b^{n+1}),$$

has at most one real root.

1995 Iran First Round 564. Let $F(x)$ and $G(x)$ be two polynomials with integer coefficients such that $F(x)/G(x)$ is an integer for values of $x = 1, 2, 3, \ldots$. Prove that $F(x)$ is divisible by $G(x)$.

1989 Iran Second Round 565. Prove that for all positive integers $n > 1$, the polynomial

$$P(x) = \frac{x^N}{n!} + \frac{x^{n-1}}{(n-1)!} + \cdots + \frac{x}{1} + 1,$$

does not have integer roots.

1997 Bulgaria 566. Find all values of a such that for all $x \in [0, 1]$, we have $|f(x)| \le 1$, where

$$f(x) = x^2 - 2ax - a^2 - \frac{3}{4}.$$

1974 International Mathematics Olympiad 567. Let $P(x)$ be a non-constant polynomial with integer coefficients and denote by $n(P(x))$ the number of integers k such that $(P(k))^2 = 1$. Prove that

$$n(P(x)) - \deg(P(x)) \le 2.$$

1996 Poland 568. Find all pairs (n, r) of positive integer n and real number r such that the polynomial $(x + 1)^n - r$ is divisible by the quadratic $2x^2 + 2x + 1$.

1914 Hungary 569. Let $P(x)$ be a quadratic polynomial such that for $x \in [-1, 1]$, we have $P(x) \in [-1, 1]$. Prove that $P'(x) \in [-4, 4]$.

1918 Hungary 570. If p, q, r, and a, b, c are real numbers such that for all reals x, we have

$$ax^2 + 2bx + c \ge 0 \qquad \text{and} \qquad px^2 + 2qx + r \ge 0,$$

then prove that $apx^2 + bqx + cr \ge 0$.

1988 Iran 571. If α is a root of the cubic polynomial $x^3 + x^2 + 2x - 1$, find the other two roots in terms of α.

1997 Iran 572. Consider all quadratic polynomials $x^2 + px + q$ in which p and q are integers with $1 \leq p, q \leq 1997$. Determine the number of which of the two kinds of polynomials is larger: those quadratics that have integer roots, or those quadratics with no integer roots?

Problem 573. Does there exist a sequence $\{a_n\}_{n=0}^{\infty}$ of real non-zero numbers such that for any positive integer n, the polynomial $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ has exactly n distinct real roots?

1999 Iran 574. Let $P(x)$ be a polynomial with degree less than n. Prove that

$$\sum_{i=0}^{n} P(i)(-1)^i \binom{n}{i} = 0.$$

1996 Bulgaria 575. Let $f(x)$ and $g(x)$ be quadratic polynomials with real coefficients such that if $g(x)$ is an integer for some $x > 0$, then $f(x)$ is also an integer. Prove that there exist integers m and n such that

$$f(x) = mg(x) + n.$$

1996 Austrian–Polish 576. Prove that there does not exist a polynomial $P(x)$ of degree 998 such that all its coefficients are real and for all $x \in \mathbb{R}$, we have

$$(P(x))^2 - 1 = P(x^2 + 1).$$

Problem 577. For all positive integers $n > 1$, prove that the following polynomial does not have rational roots:

$$P(x) = (2n + 1)x^n + \cdots + 5x^2 + 3x + 1.$$

1997 IMO Shortlist 578. Find all positive integers k such that the following statement holds true: for all polynomials $F(x)$ with integer coefficients such that $0 \leq F(c) \leq k$ for all $c \in \{0, 1, 2, \ldots, k\}$, then

$$F(0) = F(1) = \cdots = F(k + 1).$$

1996 Romania 579. Let $n \geq 2$ be a given integer. Find all polynomials $P(x) = a_nx^n + \cdots + a_1x + a_0$ with real non-zero coefficients such that the polynomial

$$P(x) - [P_1(x) \cdot P_2(x) \cdots P_{n-1}(x)] \text{ is the constant polynomial,}$$

where the polynomials $P_1, P_2, \ldots, P_{n-1}$ are defined by

$$\begin{aligned}
P_1(x) &= a_1x + a_0, \\
P_2(x) &= a_2x^2 + a_1x + a_0, \\
&\vdots \qquad \vdots \\
P_{n-1}(x) &= a_{n-1}x^{n-1} + \cdots + a_1x + a_0.
\end{aligned}$$

1999 Iran TST 580. Given a polynomial $P(x)$ of degree $n \geq 1$ with integer coefficients and n distinct integer roots such that $P(0) = 0$, find all integer roots of $P(P(x)) = 0$.

1999 Iran TST 581. Given a real number $r \geq 0$, find all polynomials $P(x)$ with real non-negative coefficients such that

a) For all $x \geq 0$, we have $P(x) \leq x^r$, and also $P(0) = 0, P(1) = 1$.

b) For all $x \geq 0$, we have $P(x) \geq x^r$, and also $P(0) = 0, P(1) = 1$.

1990 Iran 582. Let α be a root of the cubic equation $x^3 - 5x + 3 = 0$ and let $f(x)$ be a polynomial with rational coefficients. Prove that if $f(\alpha)$ is a root of the same mentioned cubic equation, then $f(f(\alpha))$ is also a root of the same cubic $x^3 - 5x + 3 = 0$.

Problem 583. Prove that the following polynomial does not have real roots:

$$P(x) = x^6 - x^5 + x^4 - x^3 + x^2 - x + \frac{3}{4}.$$

1976 Bulgaria 584. Find all polynomials $P(x)$ such that

$$P(x^2 - 2x) = (P(x - 2))^2.$$

Problem 585. If $f(x)$ is a non-constant polynomial with integer coefficients, prove that one can find infinitely many prime numbers p such that the modular arithmetic equation $f(x) \equiv 0 \pmod{p}$ has integer solutions for x.

1998 Bulgaria 586. Let $f(x) = x^3 - 3x + 1$. Find the real and distinct roots of $f(f(x)) = 0$.

1998 India 587. For all positive integers $m \geq n \geq 2$ prove that the number of polynomials of degree $2n - 1$ whose coefficients are distinct and chosen from $\{1, 2, \ldots, m\}$, and are also divisible by the polynomial $x^{n-1} + \cdots + x + 1$, is equal to:

$$2^n n! \left(4\binom{m+1}{n+1} - 3\binom{m}{n}\right).$$

1999 Hungary 588. Does there exist a polynomial $P(x)$ with integer coefficients so that

$$P(10) = 400, \quad P(14) = 440, \quad P(18) = 520?$$

1997 IMO Shortlist 589. Let p be a prime number and $f(x)$ a polynomial with integer coefficients and of degree n such that:

(i) $f(0) = 0$ and $f(1) = 1$; and

(ii) For all positive integers n, $f(n) \equiv 0$ or $1 \pmod{p}$.

Prove that $d \geq p - 1$.

1997 Romania 590. Let $n \geq 2$ be a given integer. Find all polynomials $P(x) = a_n x^n + \cdots + a_1 x + a_0$ with positive integer coefficients such that for each $k = 1, 2, \ldots, n-1$, we have $a_k = a_{n-k}$. Prove that there exist infinitely many pairs (x, y) of positive integers for which

$$x \mid P(y) \qquad \text{and} \qquad y \mid P(x).$$

1998 Austrian–Polish 591. Find all pairs (a, b) of positive integers such that the polynomial $x^3 - 17x^2 + ax - b^2 = 0$ has three (not necessarily distinct) integer roots.

Problem 592. Let $\{a_i\}_{i=0}^n$ be a sequence of $n \geq 2$ real numbers with $a_n \neq 0$ and

$$a_{n-1}^2 - \frac{2n}{n-1} a_n a_{n-2} < 0.$$

Prove that the polynomial $P(x)$ defined below has at most $n-2$ distinct real roots:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0.$$

2000 IMO Shortlist 593. For a polynomial P of degree 2000 with distinct real coefficients let $M(P)$ be the set of all polynomials that can be produced from P by permutation of its coefficients. A polynomial P will be called n-independent if $P(n) = 0$ and we can get from any $Q \in M(P)$ a polynomial Q_1 such that $Q_1(n) = 0$ by interchanging at most one pair of coefficients of Q. Find all integers n for which n-independent polynomials exist.

1995 Czech And Slovak Mathematical Olympiad 594. Find all real numbers p for which the equation

$$x^3 - 2p(p+1)x^2 + (p^4 + 4p^3 - 1)x - 3p^3 = 0,$$

has three distinct real roots which are sides of a right triangle.

1995 Greece 595. If the equation $ax^2 + (c-b)x + (e-d) = 0$ has real roots greater than 1, prove that the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ has at least one real root.

Problem 596. Let m be a non-negative integer and let n be an even positive integer. Prove that the polynomial

$$P(x) = \frac{x^n}{(n+1)^m} + \frac{x^{n-1}}{n^m} + \cdots + \frac{x}{2^m} + 1,$$

does not have any real roots, but if n is odd, then this polynomial has precisely one real root.

1996 Austrian–Polish 597. The sequence of P_n of polynomials is defined initially by $P_0(x) = 0$ and $P_1(x) = x$, and then recursively for $n \geq$,

$$P_n(x) = xP_{n-1}(x) + (1-x)P_{n-2}(x).$$

For any given positive integer n, find all x such that $P_n(x) = 0$.

2000 Iran 598. Does there exist a polynomial $f(x)$ of degree 1999 with integer coefficients such that for all integers n, the numbers $f(n), f(f(n)), f(f(f(n))), \ldots$ are pairwise coprime. That is, no two of them share integer divisors.

Problem 599. Find all polynomials $P(x)$ with real coefficients such that for all $x \in \mathbb{R}$,

$$P(x) \cdot P(x+1) = P(x^2).$$

1999 Iran 600. Find all polynomials $P(x)$ with real coefficients for which there exists a positive integer n such that for all $x \in \mathbb{R}$,

$$xP(x-n) = (x-1)P(x).$$

Problem 601. Let $P(x)$ be a polynomial of degree n with rational coefficients and n roots $\alpha_1, \alpha_2, \ldots, \alpha_n$. Prove that for all positive integers m, the expression $\alpha_1^m + \alpha_2^m + \cdots + \alpha_n^m$ is a rational number.

1997 Iran 602. For three integers a, b, c with $a \neq 1$, we know that one of the roots of the polynomial $P(x) = x^3 + ax^2 + bx + c$ equals the product of the other two roots. Prove that $2P(-1)$ is divisible by $P(1) + P(-1) - 2(1 + P(0))$.

2001 Iran Second Round 603. Find all polynomials $P(x)$ with real coefficients such that for all $x \in \mathbb{R}$,

$$P(2P(x)) = 2P(P(x)) + 2(P(x))^2.$$

1997 Iran 604. Let $P(z)$ be a polynomial with real coefficients such that $P(0) = 1$ and for all complex numbers z with $|z| = 1$, we have $|P(z)| = 1$. Prove that $P(z) \equiv 1$.

1997 Iran 605. For two monic polynomials $P(x)$ and $Q(x)$ with rational coefficients which are both irreducible, prove that if α is a root of $P(x)$ and β is a root of $Q(x)$, where $\alpha + \beta$ is rational, then the polynomial $(P(x))^2 - (Q(x))^2$ has a rational root.

1993 Iran Second Round 606. Let $f(x)$ and $g(x)$ be two polynomials with real coefficients such that for infinitely many rational values of x, the fraction $\frac{f(x)}{g(x)}$ is rational. Prove that $\frac{f(x)}{g(x)}$ can be written as the ratio of two polynomials with rational coefficients.

1994 Iran Third Round 607. Find all polynomials $f(x)$ with real roots such that for all $x \in \mathbb{R}$,

$$f(x^2 - 1) = f(x) \cdot f(-x).$$

1979 Bulgaria 608. Find all polynomials $P(x)$ with real coefficients such that for all $x \in \mathbb{R}$,

$$P(x) \cdot P(2x^2) = P(2x^3 + x).$$

1979 Hungary 609. Prove that if the polynomial $P(x)$ with real coefficients is always non-negative for all $x \in \mathbb{R}$, then we can write

$$P(x) = (Q_1(x))^2 + (Q_2(x))^2 + \cdots + (Q_n(x))^2,$$

where $Q_1(x), Q_2(x), \ldots, Q_n(x)$ are polynomials with real coefficients.

1998 Bulgaria 610. For all positive integers n, the two-variable polynomial $P_n(x, y)$ is defined initially by $P_1(x, y) = 1$ and recursively for $n \geq 1$ by

$$P_{n+1}(x, y) = (x + y - 1)(y + 1)P_n(x, y + 2) + (y - y^2)P_n(x, y).$$

Prove that for each $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$,

$$P_n(x, y) = P_n(y, x).$$

1998 Canada 611. Find all real roots of the following equation:

$$x = \sqrt{x - \frac{1}{x}} + \sqrt{1 - \frac{1}{x}}.$$

1999 Japan 612. For each positive integer n, prove that the polynomial

$$f(x) = (x^2 + 1^2)(x^2 + 2^2) \cdots (x^2 + n^2) + 1,$$

cannot be written as a product of two non-constant polynomials with real coefficients.

Problem 613. Define a sequence $\{P_n\}_{n=0}^{\infty}$ of polynomials initially by $P_0(x) = 1$ and $P_1(x) = x + 1$, and recursively for $n \geq 1$ by

$$P_{n+1}(x) = P_n(x) + xP_{n-1}(x).$$

Prove that for all $n \in \mathbb{N}$, all the roots of $P_n(x)$ are real.

1996 Romania 614. For real numbers a, b, c with $a \neq 0$, we know that a and $4a+3b+2c$ have the same sign. Prove that the polynomial $ax^2 + bx + c$ cannot have two roots in the interval $(1, 2)$.

1997 Iran 615. Let $P(x) = ax^3 + bx^2 + cx + d$ be a polynomial with rational coefficients. If the three roots of $P(x)$ are x_1, x_2, x_3 such that x_1/x_2 is a rational number not equal to 0 or 1, prove that all three roots are rational.

1997 Iran 616. Find all polynomials $P(x)$ with real coefficients such that for all $x \in \mathbb{R}$,

$$xP(x)P(1 - x) + x^3 + 100 \geq 0.$$

1981 USSR 617. Consider the two-variable polynomial

$$P(x, y) = 4 + x^2y^4 + x^4y^2 - 3x^2y^2.$$

a) Find the smallest value that this polynomial can take.

b) Prove that $P(x, y)$ cannot be written as a sum of squares of two-variable polynomials in x and y.

Problem 618. Can we find a polynomial $f(x)$ such that

$$f(f'(x)) = 27x^6 - 27x^4 + 6x^2 + 2?$$

1976 International Mathematics Olympiad 619. Let $P_1(x) = x^2 - 2$ and $P_j(x) = P_1(P_{j-1}(x))$ for $j = 2, 3, \ldots$ Prove that for any positive integer n the roots of the equation $P_n(x) = x$ are all real and distinct.

Problem 620. Let $P(x)$ be a polynomial of degree 7 such that for seven distinct integer values of x, we have $P(x)$ equal to either $+1$ or -1. Prove that $P(x)$ cannot be factorized as a product of two polynomials with integer coefficients.

Problem 621. Find all polynomials $P(x)$ of the form

$$P(x) = x^n + nx^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n,$$

such that if $r_1, r_2, \ldots, r_n$ are the roots of $P(x)$, then we have

$$r_1^{16} + r_2^{16} + \cdots + r_n^{16} = n.$$

1994 Romania 622. Let a, b, c and A, B, C be positive real numbers such that the quadratic polynomials $p(x) = ax^2 + bx + c$ and $P(x) = Ax^2 + Bx + C$ have real roots. Prove that for any u that lies between the roots of $p(x)$ and for any U that lies between the roots of $P(x)$, we have

$$(au + AU)\left(\frac{c}{u} + \frac{C}{U}\right) \leq \left(\frac{b+B}{2}\right).$$

1998 Vietnam 623. Prove that for all odd positive integers n, there exists a unique polynomial $P(x)$ of degree n and with real coefficients such that for all real $x \neq 0$,

$$P\left(x - \frac{1}{x}\right) = x^n - \frac{1}{x^n}.$$

Moreover, find out when the given statement is true for even n.

1998 Czech And Slovak 624. Let $P(x)$ be a polynomial of degree $n \geq 5$ with integer coefficients which has n distinct integer roots. If we assume that $P(0) = 0$, find all integer roots of $P(P(x))$.

1998 Russia 625. Find all two-variable polynomials $P(x, y)$ such that for all $x, y \in \mathbb{R}$,

$$P(x + y, x - y) = P(x, y).$$

1998 Russia 626. Does there exist a polynomial $P(x)$ with integer coefficients and a positive integer $k > 1$ such that the numbers $P(k), P(k^2), P(k^3), \ldots$ are pairwise coprime?

Problem 627. If $f(x)$ is a non-constant polynomial with integer coefficients such that for all primes p, we know that $f(p)$ is a power of a prime number, prove that there exists a positive integer n such that $f(x) = x^n$.

1979 Hungary 628. Let $a \neq 0$ be a real number and define $P(x) = ax^2 + bx + c$. Prove that for all positive integers n, there cannot be more than one polynomial $Q(x)$ of degree n for which

$$Q(P(x)) = P(Q(x)).$$

1983 Romania 629. Let $\{F_n\}_{n=1}^{\infty}$ denote the Fibonacci sequence defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 2$. We know that for the polynomial $P(x)$ of degree 990, we have $P(k) = F_k$ for $k = 992, 993, \ldots, 1982$. Prove that

$$P(1983) = F_{1983} - 1.$$

1977 Bulgaria 630. Let $Q(x)$ be a non-zero polynomial. Prove that for all positive integers n, the polynomial $P(x) = (x-1)^n Q(x)$ has at least $n+1$ non-zero coefficients.

1985 Sweden 631. Let $P(x)$ be a polynomial of degree n such that for all $x \in \mathbb{R}$, we have $P(x) \geq 0$. Prove, for all $x \in \mathbb{R}$, that

$$P(x) + P'(x) + P''(x) + \cdots + P^{(n)}(x) \geq 0.$$

2000 Iran 632. Let $P(x)$ be a polynomial with integer coefficients. Prove that the polynomial

$$Q(x) = P(x^4) \cdot P(x^3) \cdot P(x^2) \cdot P(x) + 1,$$

does not have any integer roots.

2003 Poland 633. Define $W(x) = x^4 - 3x^3 + 5x^2 - 9x$. Find all pairs (a, b) of distinct integers such that $W(a) = W(b)$.

2000 Austrian–Polish 634. Find all polynomials $P(x)$ with real coefficients that satisfy the following condition: there exists a positive integer n such that the following equation holds for infinitely many real values of x:

$$\sum_{k=1}^{2n+1} (-1)^k \left\lfloor \frac{k}{2} \right\rfloor P(x+k) = 0.$$

2000 Poland 635. Let $P(x)$ be a polynomial of odd degree such that

$$P(x^2 - 1) = (P(x))^2 - 1.$$

Prove that $P(x) = x$ for all $x \in \mathbb{R}$.

1996 Russia 636. Prove that for all polynomials $P(x)$ of degree 10 with integer coefficients, there exists an infinite arithmetic progression which does not contain the following numbers:

$$\ldots, P(-1), P(0), P(1), P(2), \ldots$$

1997 Romania 637. Let $a_0, a_1, \ldots, a_n$ be complex numbers such that for any complex z with $|z| \leq 1$, we have

$$|a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0| \leq 1.$$

Prove that for all $k = 0, 1, \ldots, n$, we have $|a_k| \leq 1$ and

$$|a_0 + a_1 + \cdots + a_n - (n+1)a_k| \leq n.$$

1994 Vietnam 638. Let $P(x)$ be a polynomial of degree 4 with 4 positive real roots. Prove that the polynomial

$$\frac{1-4x}{x^2}P(x)+\left(1-\frac{1-4x}{x^2}\right)P'(x)-P''(x)$$

also has 4 positive real roots.

Problem 639. Let $P(x)$ be a quadratic polynomial such that for a sequence of rational numbers $q_0, q_1, q_2, \dots$ we have $q_n = P(q_{n+1})$ for all $n \geq 1$. Prove that there exists a positive integer k such that for all $n \geq 1$, we have $q_{n+k} = q_n$.

1994 China 640. For all polynomials $f(x) = x_0x^n + c_1x^{n-1} + \cdots + c_n$ of degree n with complex coefficients, prove that there exists a complex number x_0 such that $|x_0| \leq 1$ and

$$|f(x_0)| \geq |c_0| + |c_n|.$$

Problem 641. Let $f(x)$ be a polynomial with rational coefficients such that for some $\alpha \in \mathbb{R}$,

$$\alpha^3 - 1992\alpha + 33 = (f(\alpha))^3 - 1992f(\alpha) + 33 = 0.$$

Prove that for all $n \geq 1$, we have

$$(f^n(\alpha))^3 - 1992f^n(\alpha) + 33 = 0,$$

where $f^n(\alpha) = \underbrace{f(f(\cdots f(\alpha)))}_{n \text{ times}}$.

Problem 642. Prove that if a symmetric two-variable polynomial $P(x, y)$ is divisible by $x - y$, then $P(x, y)$ is also divisible by $(x - y)^2$.

Problem 643. Find all polynomials $P(x)$ with $P(0) = 0$ and

$$P(x^2 + 1) = (P(x))^2 + 1.$$

1986 Czech And Slovak 644. Let $P(x)$ be a polynomial with integer coefficients of degree $n \geq 3$. If $x_1, x_2, \dots, x_m$ (with $m \geq 3$) are different integers such that

$$P(x_1) = P(x_2) = \cdots = P(x_m) = 1,$$

prove that P cannot have integer roots.

Problem 645. Define polynomial $P(x)$ of degree n with integer coefficients by

$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n.$$

If for two real numbers $\alpha > \beta$ we have $|P(\alpha)| = |P(\beta)| = 1$, and $P(x)$ has a rational root r, then prove that $\alpha - \beta$ equals either 1 or 2, and that $r = (\alpha + \beta)/2$.

Problem 646. Prove that there does not exist a polynomial P such that $P(x)$ is prime for all $x \in \{0, 1, 2, \dots\}$.

Problem 647. Let $f(x)$ be a non-constant polynomial with integer coefficients such that $f(0) > 0$. Prove that there exists a sequence $p_1, p_2, p_3, \ldots$ of prime numbers and a sequence $a_1, a_2, a_3, \ldots$ of pairwise coprime positive integers such that for all $n = 1, 2, 3, \ldots,$ we have

$$f(p_n) = a_1 a_2 \cdots a_n.$$

1994 Iran First Round 648. Let a, b, c be real numbers such that $9a + 11b + 29c = 0$. Prove that the cubic polynomial $ax^3 + bx + c$ has a root in the interval $[0, 2]$.

1998 Bulgaria 649. Find all positive integers n such that $x^n + 64$ can be factorized into a product of two polynomials with integer coefficients.

1998 India 650. Let N be a positive integer such that $N + 1$ is a prime. Assume that for $i = 0, 1, 2, \ldots, N$, we have $a_i \in \{0, 1\}$ and not all a_i are equal to each other. Define the polynomial $f(x)$ so that for each $i = 0, 1, 2, \ldots, N$, we have $f(i) = a_i$. Show that the degree of $f(x)$ is at least N.

1998 Romania 651. For all positive integers n, prove that the polynomial

$$f(x) = (x^2 + x)^{2^n} + 1,$$

cannot be factorized into a product of two non-constant polynomials with integer coefficients.

1999 China 652. Let a be a real number. Let $(f_n(x))_{n\geq 0}$ be a sequence of polynomials such that $f_0(x) = 1$ and $f_{n+1}(x) = xf_n(x) + f_n(ax)$ for all non-negative integers n.

a) Prove that
$$f_n(x) = x^n f_n\left(x^{-1}\right),$$
for all non-negative integers n.

b) Find an explicit expression for $f_n(x)$.

1999 Hungary 653. If the polynomial $x^4 - 2x^2 + ax + b$ has four distinct real roots, prove that the absolute value of each of its roots is less than $\sqrt{3}$.

1999 Romania 654. Let a and n be integers and p be a prime number such that $p > |a| + 1$. Prove that the polynomial $f(x) = x^n + ax + p$ cannot be factorized into a product of two polynomials with integer coefficients.

1999 Ukraine 655. Let $P(x)$ be a polynomial with integer coefficients and assume that the sequence $\{x_n\}_{n=1}^{n=2000}$ satisfies $x_1 = x_{2000} = 1999$. If we know that $x_{n+1} = P(x_n)$, then find the value of the following sum:

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{1999}}{x_{2000}}.$$

1975 International Mathematics Olympiad 656. Determine all two-variable polynomials $P(x, y)$ so that:

a) For any real numbers t, x, y we have $P(tx, ty) = t^n P(x, y)$ where n is a positive integer, the same for all t, x, y;

b) For any real numbers a, b, c we have

$$P(a+b, c) + P(b+c, a) + P(c+a, b) = 0;$$

c) $P(1, 0) = 1$.

Problem 657. Let p be an odd prime number. Prove that the polynomial

$$\sum_{1 \le m,n \le p-1} x^{mn} + p - 1,$$

is divisible by $x^{p-1} + x^{p-2} + \cdots + x + 1$.

Problem 658. Find all polynomials $P(x)$ such that

$$P(x^2) + P(x)P(x+1) = 0.$$

1997 Germany 659. Define $f(x)$ and $g(x)$ by

$$f(x) = x^5 + 5x^4 + 5x^3 + 5x^2 + 1,$$
$$g(x) = x^5 + 5x^4 + 3x^3 - 5x^2 - 1.$$

Find all prime numbers p for which there exists an integer x with $0 \le x \le p$ such that both $f(x)$ and $g(x)$ are both divisible by p. Moreover, for each such p, find all x that satisfy the condition.

1997 Ukraine 660. If we know that $ax^3 + bx^2 + cx + d$ has three distinct real roots, then how many root does the following equation have?

$$4(ax^3 + bx^2 + cx + d)(3ax + b) = (3ax^2 + 2bx + c)^2.$$

1997 British Math Olympiad 661. Find all polynomials $P(x)$ of degree 5 with distinct coefficients chosen from the set $\{1, 2, 3, \ldots, 9\}$ such that $P(x)$ is divisible by $x^2 - x + 1$.

Problem 662. Let $f(x)$ and $g(x)$ be single-variable polynomials with real coefficients and let $P(x, y)$ be a two-variable polynomial with real coefficients such that for all real numbers x, y,

$$f(x) - f(y) = (g(x) - g(y)) \cdot P(x, y).$$

Prove that there exists a single-variable polynomial $h(x)$ with real coefficients such that $f(x) = h(g(x))$ for all real numbers x.

Problem 663. Let $P(x)$ be a polynomial with real coefficients which satisfies the following inequalities:

$$P(0) > 0, \quad P(1) > P(0), \quad P(2) > 2P(1) - P(0), \quad P(3) > 3P(2) - 3P(1) + P(0).$$

Moreover, for each positive integer n, we know that

$$P(n+4) > 4P(n+3) - 6P(n+2) + 4P(n+1) - P(n).$$

Prove that $P(n) > 0$ for all positive integers n.

1995 Ireland 664. Let a, b, c be complex numbers such that all roots z of the polynomial

$$P(x) = x^3 + ax^2 + bx + c,$$

satisfy the equation $|z| = 1$. Prove that all roots ω of the polynomial

$$Q(x) = x^3 + |a|x^2 + |b|x + |c|,$$

also satisfy $|\omega| = 1$.

1995 Japan 665. Let k, n be integers such that $1 \leq k \leq n$, and let $a_1, a_2, \ldots, a_k$ be numbers satisfying the following equations:

$$\begin{cases} a_1 + a_2 + \cdots + a_k & = n, \\ a_1^2 + a_2^2 + \cdots + a_k^2 & = n, \\ \quad \vdots & \quad \vdots \\ a_1^k + a_2^k + \cdots + a_k^k & = n. \end{cases}$$

Prove that

$$(x + a_1)(x + a_2) \cdots (x + a_k) = x^k + \binom{n}{1} x^{k-1} + \binom{n}{2} x^{k-2} + \cdots + \binom{n}{k}.$$

Problem 666. For each positive integer n, define

$$f(n) = 1! + 2! + \cdots + n!.$$

Find polynomials $P(x)$ and $Q(x)$ such that for all positive integers n,

$$f(n+2) = P(n)f(n+1) + Q(n)f(n).$$

Problem 667. Find all polynomials $P(x)$ such that

$$1 + P(x) = \frac{P(x-1) + P(x+1)}{2}.$$

Problem 668. Is it possible to factorize $P(x) = x^{100} + 5x^{99} + 2x + 2$ into a product of two polynomials with integer coefficients?

Problem 669. If P and Q are two polynomials such that for all $x \in \mathbb{R}$,

$$P(x^2 + x + 1) = Q(x^2 - x + 1),$$

prove that P and Q are constant polynomials.

Problem 670. Find all polynomials $P(x)$ with real coefficients such that $P(x)$ has distinct real roots $r_1 > r_2 > \cdots > r_n$ and also, $(r_i + r_{i+1})/2$ are roots of $P'(x)$ for $i = 1, 2, \ldots, n-1$.

1997 Bulgaria 671. For integer $n \geq 2$, consider the polynomial

$$P_n(x) = \binom{n}{2} + \binom{n}{5}x + \binom{n}{8}x^2 + \cdots + \binom{n}{3k+2}x^{3k}, \quad \text{where} \quad k = \left\lfloor \frac{n-2}{3} \right\rfloor.$$

a) Prove that $P_{n+3}(x) = 3P_{n+2}(x) - 3P_{n+1}(x) + (x+1)P_n(x)$.

b) Find all integers a such that $P_n(a^3)$ is divisible by $3^{\lfloor \frac{n-1}{2} \rfloor}$ for all $n \geq 3$.

Problem 672. Find all two-variable polynomials $P(x, y)$ with real coefficients such that for all $x, y \in \mathbb{R}$,

$$P(x, y) = P(x+1, y+1).$$

Problem 673. Let $p(x)$ and $q(x)$ be non-zero polynomials such that for all $x \in \mathbb{R}$,

$$p(x^2 + x + 1) = p(x) \cdot q(x).$$

Prove that the degree of $p(x)$ is even.

1999 China 674. Determine the maximum value of λ such that if $f(x) = x^3 + ax^2 + bx + c$ is a cubic polynomial with all its roots non-negative, then

$$f(x) \geq \lambda(x-a)^3,$$

for all $x \geq 0$. Find the equality condition.

1999 Poland 675. Let $P(x) = 2x^3 - 3x^2 + 2$ and define the sets

$$S = \{P(n) \mid n \in \mathbb{N}, n \leq 999\},$$
$$T = \{n^2 + 1 \mid n \in \mathbb{N}\},$$
$$U = \{n^2 + 2 \mid n \in \mathbb{N}\}.$$

Prove that the sets $S \cap T$ and $S \cap U$ have the same number of elements.

1999 Vietnam 676. Let a and b be real numbers such that all the roots of the following polynomial are positive real numbers:

$$P(x) = ax^3 - x^2 + bx - 1.$$

Find the least value of the fraction

$$\frac{5a^2 - 3ab + 2}{a^2(b-a)}.$$

1995 Korea 677. Let a and b be integers and p be a prime number such that:

(i) p is the greatest common divisor of a and b; and

(ii) p^2 divides a.

Prove that the polynomial $x^{n+2} + ax^{n+1} + bx^n + a + b$ cannot be decomposed into the product of two polynomials with integer coefficients and degree greater than 1.

Problem 678. For a monic polynomial $P(x)$ of degree n with non-negative coefficients and n real roots, we have $P(0) = 1$. Prove that for all integers k,

$$P(k) \geq (k+1)^n.$$

Problem 679. Let $P(x)$ be a polynomial with integer coefficients such that for all primes q, we know that $P(q)$ is a power of 2. Prove that $P(x)$ must be a constant polynomial.

Problem 680. Let $P(x)$ and $Q(x)$ be polynomials with real coefficients such that either $P(x)$ and $Q(x)$ are both integers or they are both non-integers. Prove that $P(x) = \pm Q(x)$.

2000 Romania TST 681. Let P, Q be two monic polynomials with complex coefficients such that $P(P(x)) = Q(Q(x))$ for all x. Prove that $P = Q$.

1986 USSR 682. If the roots of the quadratic polynomial

$$P(x) = x^2 + ax + b + 1,$$

are positive integers, prove that $a^2 + b^2$ is a composite number.

Problem 683. The value of polynomial $P(x)$ is a perfect square for all positive integers x. Prove that there must exist a polynomial $Q(x)$ such that $P(x) = (Q(x))^2$.

1998 Iran 684. Prove that for any non-constant polynomial $f(x)$ with integer coefficients, there exists a sequence $p_1 < p_2 < p_3 < \cdots$ of primes and a sequence $n_1 < n_2 < n_3 < \cdots$ of positive integers such that $p_k \mid f(n_k)$ for all $k \in \mathbb{N}$.

1996 Taiwan 685. Show that for any real numbers $a_3, a_4, \ldots, a_{85}$, not all the roots of the equation

$$a_{85}x^{85} + a_{84}x^{84} + \cdots + a_3x^3 + 3x^2 + 2x + 1 = 0,$$

are real roots.

1998 Iran 686. The sequence $a_0, a_1, a_2, \ldots$ satisfies $2a_i = a_{i-1} + a_{i+1}$. Define the polynomial $P_n(x)$ for each $n \in \mathbb{N}$ by

$$P_n(x) = \sum_{i=0}^{n} a_i \binom{n}{i} x^i (1-x)^{n-i}.$$

Prove that $P_n(x)$ is linear for all $n \in \mathbb{N}$.

1995 Austrian–Polish 687. Let $P(x) = x^4 + x^3 + x^2 + x + 1$ and prove that there exist non-constant polynomials $Q(x)$ and $R(x)$ with integer coefficients such that for all $x \in \mathbb{R}$,

$$Q(x) \cdot R(x) = P(5x^2).$$

1995 Austrian–Polish 688. Find all polynomials $P(x)$ with real coefficients such that for all $x \neq 0$,

$$(P(x))^2 + \left(P\left(\frac{1}{x}\right)\right)^2 = P(x^2) \cdot P\left(\frac{1}{x^2}\right).$$

1995 Balkan 689. Let a and b be positive integers with $a > b$ and having the same parity. Prove that the solutions of the equation

$$x^2 - (a^2 - a + 1)(x - b^2 - 1) - (b^2 + 1)^2 = 0,$$

are positive integers, none of which is a perfect square.

1998 Iran 690. The determinant of the cubic polynomial $P(x) = x^3 + ax^2 + bx + c$ is defined by

$$\Delta = 18abc - 4a^3c + a^2b^2 - 4b^3 - 27c^3.$$

Prove that if $\Delta \geq 0$, then $P(x)$ will have real roots.

1998 Iran 691. Find the smallest positive integer d for which there exists a monic polynomial of degree d such that for all $n \in \mathbb{N}$, we have $100 \mid f(n)$.

Problem 692. Let $P(x)$ be a polynomial with integer coefficients such that $P(0) = P(1) = 1$. For an arbitrary integer a_0, define $a_{n+1} = P(a_n)$ for all integers $n \geq 0$. Prove that the elements of the sequence $\{a_i\}_{i=0}^{\infty}$ are pairwise coprime.

1977 USSR 693. Two monic polynomials $P(x)$ and $Q(x)$ are *commutable* if we have $P(Q(x)) = Q(P(x))$ for all real x.

a) For any real α, find all monic polynomials $Q(x)$ of maximum degree 3 which are commutable with the polynomial $P(x) = x^2 - \alpha$.

b) For an arbitrary quadratic polynomial $P(x)$ and a positive integer k, prove that there exists at most one polynomial of degree k which is commutable with $P(x)$.

c) Find polynomials of degree 4 and 8 which are commutable with a given quadratic polynomial.

d) Let $Q(x)$ and $R(x)$ be polynomials that are both commutable with a given quadratic polynomial $P(x)$. Prove that $Q(x)$ and $R(x)$ are commutable.

e) Let $P_2(x) = x^2 - 2$ and for all positive integers k, let $P_k(x)$ be a polynomial of degree k. Prove that there exist an infinite sequence $P_2(x), P_3(x), P_4(x), \ldots$ of polynomials each two of which are commutable.

1996 IMO Shortlist 694. Let $P(x)$ be the cubic real-coefficient polynomial

$$P(x) = ax^3 + bx^2 + cx + d.$$

Prove that if $|P(x)| \leq 1$ for all x such that $|x| \leq 1$, then,

$$|a| + |b| + |c| + |d| \leq 7.$$

1996 Poland 695. The polynomial $P(x)$ of degree n satisfies the following equation:

$$P(k) = \frac{1}{k}, \qquad \text{for} \quad k = 2^0, 2^1, 2^2, \ldots, 2^n.$$

Find $P(0)$.

1996 Poland 696. Let $P(x)$ be a non-constant polynomial with integer coefficients, and let $m \geq 1$ be a given integer. Prove that if $P(x)$ has at least three distinct integer roots, then $P(x) + 5^m$ will have at least one integer root.

1998 Baltic Way 697. Let P be a polynomial of degree 6 and let a, b be real numbers such that $0 < a < b$. Suppose that $P(a) = P(-a), P(b) = P(-b), P'(0) = 0$. Prove that $P(x) = P(-x)$ for all real x.

1995 UNESCO 698. Let $p \geq 2$ and $a_0, a_1, \ldots, a_n$ be non-negative integers and define $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. Prove that if the numbers

$$\sqrt[p]{f(0)}, \sqrt[p]{f(1)}, \sqrt[p]{f(2)}, \ldots$$

are all rational, then there exists a polynomial $g(x)$ with integer coefficients such that $f(x) = (g(x))^p$.

1995 Russia 699. Let f, g, h be quadratic polynomials. Is it possible for $x = 1, 2, \ldots, 8$ to be the roots of the equation $f(g(h(x))) = 0$?

1998 Iran 700. Let $P(x)$ and $Q(x)$ be two polynomials with complex coefficients and let $a, b \geq 2$ be integers such that for all $x \in \mathbb{R}$,

$$(P(x))^a - (Q(x))^b = x.$$

Prove that $a = b = 2$.

1983 IMO Longlist 701. Let p and q be integers. Show that there exists an interval I of length $1/q$ and a polynomial P with integral coefficients such that

$$\left|P(x) - \frac{p}{q}\right| < \frac{1}{q^2},$$

for all $x \in I$.

1998 Poland 702. Let $n \geq 2$ be a positive integer. Find all polynomials

$$P(x) = a_0 + a_1x + \cdots + a_nx^n,$$

with n real roots all less than or equal to -1, and such that

$$a_0^2 + a_1a_n = a_n^2 + a_0a_{n-1}.$$

1998 Baltic Way 703. Let P be a polynomial with integer coefficients. Suppose that for $n = 1, 2, 3, \ldots, 1998$ the number $P(n)$ is a three-digit positive integer. Prove that the polynomial P has no integer roots.

1996 IMO Shortlist 704. Let $a_1, a_2, \ldots, a_n$ be non-negative reals, not all zero. Show that that

a) The polynomial
$$p(x) = x^n - a_1x^{n-1} + \cdots - a_{n-1}x - a_n,$$
has precisely 1 positive real root R.

b) Let
$$A = \sum_{i=1}^{n} a_i \qquad \text{and} \qquad B = \sum_{i=1}^{n} ia_i.$$
Show that $A^A \leq R^B$.

1997 Romania 705. Find all polynomials $f(x)$ with integer coefficients such that $f(x)$ is bijective (that is, both injective and surjective) and for some real constant a and all x,

$$(f(x))^2 = f(x^2) - 2f(x) + a.$$

1995 Romania 706. Let $m, n \geq 2$ be integers. Find the number of polynomials of degree $2n-1$ with distinct coefficients from the set $\{1, 2, \ldots, m\}$ which are divisible by $x^{n-1} + \cdots + x + 1$.

1995 Romania 707. Let $f(x)$ be an irreducible monic polynomial with integer coefficients and of an odd degree greater than 3. Assume that the absolute value of the roots of $f(x)$ are greater than 1 and $f(0)$ is a square-free number (that is, not divisible by square of anything). Prove that the polynomial $g(x) = f(x^3)$ cannot be decomposed into a product of two polynomials with integer coefficients.

1998 Iran 708. Find all polynomials $P(x)$ with complex coefficients such that

$$P(2x^2 - 1) = \frac{(P(x))^2}{2} - 1.$$

1998 Iran 709. Let $a_0, a_1, \ldots, a_n$ be real numbers such that

$$0 < a_0 < -\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{a_{2k}}{2k+1}.$$

Prove that the polynomial $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ has a real root in the interval $[-1, 1]$.

Problem 710. Let $P(x, y)$ be a two-variable polynomial. Prove or disprove the following statement: the inequality

$$|x^y - y^x| \leq |P(x, y)|,$$

has only a finite number of solutions (x, y) in which x and y are distinct integers with $x, y \geq 2$.

1988 IMO Longlist 711. This problem comes in four questions:

a) The polynomial
$$x^{2 \cdot k} + 1 + (x+1)^{2 \cdot k},$$
is not divisible by $x^2 + x + 1$. Find the value of k.

b) If p, q and r are distinct roots of $x^3 - x^2 + x - 2 = 0$ the find the value of $p^3 + q^3 + r^3$.

c) If r is the remainder when each of the numbers 1059, 1417 and 2312 is divided by d, where d is an integer greater than one, then find the value of $d - r$.

d) What is the smallest positive odd integer n such that the product of
$$2^{\frac{1}{7}}, 2^{\frac{3}{7}}, \ldots, 2^{\frac{2 \cdot n+1}{7}},$$
is greater than 1000?

1988 IMO Longlist 712. Let n be a positive integer. Find the number of odd coefficients of the polynomial
$$u_n(x) = (x^2 + x + 1)^n.$$

1998 Baltic Way 713. Let $P_k(x) = 1 + x + x^2 + \ldots + x^{k-1}$. Show that
$$\sum_{k=1}^{n} \binom{n}{k} P_k(x) = 2^{n-1} P_n\left(\frac{x+1}{2}\right),$$
for every real number x and every positive integer n.

1998 Iran 714. Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ with $n \geq 2$ and positive real coefficients a_i such that all the roots of $P(x)$ are positive real numbers in the interval $(0, 1)$. Prove that if $0 \leq k \leq n-2$, then
$$\sum_{i=k}^{n-2} \binom{i}{k} a_i > 0.$$

1998 Iran 715. Let $P(x)$ be a polynomial with rational coefficients such that for any rational r, there exists rational s such that $P(s) = r$. Prove that $P(x)$ is linear.

1998 Iran 716. For two polynomials f and g with rational coefficients, if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of f, and we have
$$g(\alpha_1) = g(\alpha_2) = \cdots = g(\alpha_n) = A,$$
then prove that A is a rational number.

1998 Iran 717. Define
$$f(x) = 1 - x + x^2 - x^3 + \cdots + x^{16} - x^{17}.$$
Let $y = x + 1$ and $f(y) = a_0 + a_1 y + \cdots + a_{17} y^{17}$. Find the coefficients $a_0, a_1, \ldots, a_{17}$.

1998 Iran 718. Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial with integer coefficients and $a_0 \neq 0$. If we know that
$$|a_{n-1}| > 1 + |a_{n-2} + \cdots + |a_1| + |a_0|,$$
prove that $P(x)$ is irreducible.

1998 Iran 719. Prove that for any prime p with decimal representation
$$p = (a_n a_{n-1} \cdots a_1 a_0)_{10},$$
the polynomial $f(x) = a_n x^n + \cdots + a_1 x + a_0$, is irreducible.

1998 Iran 720. Let f be a polynomial with real coefficients among which $2m$ consecutive coefficients (except for the first and last ones) are zero. Prove that f has at least $2m$ real roots.

1998 Iran 721. Let f be a polynomial with real coefficients and four of its consecutive coefficients form an arithmetic progression. Prove that f has at least one non-real root.

1995 Taiwan 722. Let $P(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{C}[x]$, where $a_n = 1$. The roots of $P(x)$ are $b_1, b_2, \ldots, b_n$, where $|b_1|, |b_2|, \ldots, |b_j| > 1$ and $|b_{j+1}|, \ldots, |b_n| \leq 1$. Prove that

$$\prod_{i=1}^{j} |b_i| \leq \sqrt{|a_0|^2 + |a_1|^2 + \cdots + |a_n|^2}.$$

1995 Taiwan 723. Let $m_1, m_2, \ldots, m_n$ be mutually distinct integers. Prove that there exists a $f(x) \in \mathbb{Z}[x]$ of degree n satisfying the following two conditions:

a) $f(m_i) = -1$, for all $i = 1, 2, \ldots, n$; and

b) $f(x)$ is irreducible.

2003 APMO 724. Let a, b, c, d, e, f be real numbers such that the polynomial

$$p(x) = x^8 - 4x^7 + 7x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f,$$

factorises into eight linear factors $x - x_i$, with $x_i > 0$ for $i = 1, 2, \ldots, 8$. Determine all possible values of f.

1992 Baltic Way 725. A polynomial $f(x) = x^3 + ax^2 + bx + c$ is such that $b < 0$ and $ab = 9c$. Prove that the polynomial f has three different real roots.

1992 Baltic Way 726. Find all quartic (fourth-degree) polynomial $p(x)$ such that the following four conditions are satisfied:

(i) $p(x) = p(-x)$ for all x,

(ii) $p(x) \geq 0$ for all x,

(iii) $p(0) = 1$,

(iv) $p(x)$ has exactly two local minimum points x_1 and x_2 such that $|x_1 - x_2| = 2$.

1996 Baltic Way 727. Real numbers $x_1, x_2, \ldots, x_{1996}$ have the following property: For any polynomial W of degree 2 at least three of the numbers $W(x_1), W(x_2), \ldots, W(x_{1996})$ are equal. Prove that at least three of the numbers $x_1, x_2, \ldots, x_{1996}$ are equal.

1997 Baltic Way 728. Let P and Q be polynomials with integer coefficients. Suppose that the integers a and $a + 1997$ are roots of P, and that $Q(1998) = 2000$. Prove that the equation $Q(P(x)) = 1$ has no integer solutions.

Problem 729. Find all polynomials P for which $P(x^2) = P(x) \cdot P(x - 1)$.

1963 Dutch Mathematical Olympiad 730. One considers for $n > 2$ the polynomial:

$$(x^2 - x + 1)^n - (x^2 - x + 2)^n + (1 + x)^n + (2 - x)^n.$$

Show that the degree of this polynomial is $2n - 2$. Moreover, assume that the polynomial is written in the form

$$a_0 + a_1x + a_2x^2 + \cdots + a_{2n-2}x^{2n-2}.$$

Prove that $a_2 + a_3 + \cdots + a_{2n-2} = 0$

1970 Dutch Mathematical Olympiad 731. The equation$x^3 - x^2 + ax - 2^n = 0$ has three integer roots. Determine a and n.

1990 Dutch Mathematical Olympiad 732. A polynomial $f(x) = ax^4+bx^3+cx^2+dx$ with $a, b, c, d > 0$ is such that $f(x)$ is an integer for $x \in \{-2, -1, 0, 1, 2\}$ and $f(1) = 1$ and $f(5) = 70$.

a) Show that

$$a = \frac{1}{24}, \quad b = \frac{1}{4}, \quad c = \frac{11}{24}, \quad d = \frac{1}{4}.$$

b) Prove that $f(x)$ is an integer for all $x \in \mathbb{Z}$.

2001 Dutch Mathematical Olympiad 733. The function is given

$$f(x) = \frac{2x^3 - 6x^2 + 13x + 10}{2x^2 - 9x}.$$

Determine all positive integers x for which $f(x)$ is an integer.

1996 Belgium Flanders 734. Consider a real polynomial $p(x) = a_nx^n + \cdots + a_1x + a_0$.

a) If $\deg(p(x)) > 2$ prove that $\deg(p(x)) = 2 + \deg(p(x+1) + p(x-1) - 2p(x))$.

b) Let $p(x)$ a polynomial for which there are real constants r, s so that for all real x we have

$$p(x+1) + p(x-1) - rp(x) - s = 0.$$

Prove that $\deg(p(x)) \leq 2$.

c) Show, with the notation of the second part, that $s = 0$ implies $a_2 = 0$.

1996 Germany 735. Prove the following statement: if a polynomial $p(x) = x^3 + Ax^2 + Bx + C$ has three real roots at least two of which are distinct, then $A^2 + B^2 + 18C > 0$.

1998 Germany 736. Let a be a positive real number. Then prove that the polynomial

$$p(x) = a^3x^3 + a^2x^2 + ax + a,$$

has integer roots if and only if $a = 1$ and determine those roots.

1979 Brazil 737. The remainder on dividing the polynomial $p(x)$ by $x^2 - (a+b)x + ab$ (where $a \neq b$) is $mx + n$. Find the coefficients m, n in terms of a, b. Find m, n for the case $p(x) = x^{200}$ divided by $x^2 - x - 2$ and show that they are integral.

1985 Brazil 738. a, b, c, d are integers. Show that $x^2+ax+b = y^2+cy+d$ has infinitely many integer solutions if and only if $a^2 - 4b = c^2 - 4d$.

1987 Brazil 739. Let $p(x_1, x_2, \ldots, x_n)$ be a polynomial with integer coefficients. For each positive integer $r, k(r)$ is the number of n-tuples $(a_1, a_2, \ldots, a_n)$ such that $0 \leq a_i \leq r - 1$ and $p(a_1, a_2, \ldots, a_n)$ is prime to r. Show that if u and v are coprime then $k(u \cdot v) = k(u) \cdot k(v)$, and if p is prime then $k(p^s) = p^{n(s-1)}k(p)$.

1991 Brazil 740. Given $k > 0$, the sequence a_n is defined by its first two members and

$$a_{n+2} = a_{n+1} + \frac{k}{n} a_n.$$

a) For which k can we write a_n as a polynomial in n?

b) For which k can we write

$$\frac{a_{n+1}}{a_n} = \frac{p(n)}{q(n)},$$

where p, q are polynomials in $\mathbb{R}[X]$?

1992 Brazil 741. The equation $x^3 + px + q = 0$ has three distinct real roots. Show that $p < 0$.

1994 Brazil 742. Let $a, b > 0$ be reals such that

$$a^3 = a + 1 \qquad \text{and} \qquad b^6 = b + 3a.$$

Show that $a > b$.

1994 Brazil 743. Show that no one n-th root of a rational (for n a positive integer) can be a root of the polynomial $x^5 - x^4 - 4x^3 + 4x^2 + 2$.

1996 Brazil 744. Let $p(x)$ be the polynomial $x^3 + 14x^2 - 2x + 1$. Let $p^n(x)$ denote $p(p^(n-1)(x))$. Show that there is an integer N such that $p^N(x) - x$ is divisible by 101 for all integers x.

1997 Brazil 745. Let $f(x) = x^2 - C$ where C is a rational constant. Show that exists only finitely many rationals x such that $\{x, f(x), f(f(x)), \dots\}$ is finite.

2007 Brazil 746. Let $f(x) = x^2 + 2007x + 1$. Prove that for every positive integer n, the equation

$$\underbrace{f(f(\dots(f}_{n \text{ times}}(x))\dots)) = 0,$$

has at least one real solution.

2010 Brazil 747. Let $P(x)$ be a polynomial with real coefficients. Prove that there exist positive integers n and k such that k has n digits and more than $P(n)$ positive divisors.

2000 Czech and Slovak 748. Let $P(x)$ be a polynomial with integer coefficients. Prove that the polynomial $Q(x) = P(x^4)P(x^3)P(x^2)P(x) + 1$ has no integer roots.

2002 Czech and Slovak 749. Let $n \geq 2$ be a fixed even integer. We consider polynomials of the form

$$P(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1 x + 1,$$

with real coefficients, having at least one real roots. Find the least possible value of $a_1^2 + a_2^2 + \dots + a_{n-1}^2$.

2004 Czech and Slovak 750. Show that real numbers, p, q, r satisfy the condition $p^4(q-r)^2 + 2p^2(q+r) + 1 = p^4$ if and only if the quadratic equations $x^2 + px + q = 0$ and $y^2 - py + r = 0$ have real roots (not necessarily distinct) which can be labeled by x_1, x_2 and y_1, y_2, respectively, in such a way that $x_1y_1 - x_2y_2 = 1$.

2005 Czech and Slovak 751. Find all integers $n \geq 3$ for which the polynomial

$$W(x) = x^n - 3x^{n-1} + 2x^{n-2} + 6,$$

can be written as a product of two non-constant polynomials with integer coefficients.

2007 Czech and Slovak 752. Find all polynomials P with real coefficients satisfying $P(x^2) = P(x) \cdot P(x+2)$ for all real numbers x.

2008 Czech and Slovak 753. Determine all triples (x, y, z) of positive real numbers which satisfies the following system of equations

$$\begin{cases} 2x^3 &= 2y(x^2+1) - (z^2+1), \\ 2y^4 &= 3z(y^2+1) - 2(x^2+1), \\ 2z^5 &= 4x(z^2+1) - 3(y^2+1). \end{cases}$$

2011 Czech and Slovak 754. A polynomial $P(x)$ with integer coefficients satisfies the following: if $F(x)$, $G(x)$, and $Q(x)$ are polynomials with integer coefficients satisfying $P\Big(Q(x)\Big) = F(x) \cdot G(x)$, then $F(x)$ or $G(x)$ is a constant polynomial. Prove that $P(x)$ is a constant polynomial.

2012 Czech and Slovak 755. Let a, b, c, d be positive real numbers such that $abcd = 4$ and

$$a^2 + b^2 + c^2 + d^2 = 10.$$

Find the maximum possible value of $ab + bc + cd + da$.

2013 Czech and Slovak 756. Let a and b be integers, where b is not a perfect square. Prove that $x^2 + ax + b$ may be the square of an integer only for finite number of integer values of x.

2014 Czech and Slovak 757. Prove that if the positive real numbers a, b, c satisfy the equation

$$a^4 + b^4 + c^4 + 4a^2b^2c^2 = 2(a^2b^2 + a^2c^2 + b^2c^2),$$

then there is a triangle ABC with internal angles α, β, γ such that

$$\sin\alpha = a, \qquad \sin\beta = b, \qquad \sin\gamma = c.$$

1991 China TST 758. Let real coefficient polynomial $f(x) = x^n + a_1 \cdot x^{n-1} + \ldots + a_n$ has real roots $b_1, b_2, \ldots, b_n$, $n \geq 2$, prove that $\forall x \geq \max\{b_1, b_2, \ldots, b_n\}$, we have

$$f(x+1) \geq \frac{2 \cdot n^2}{\dfrac{1}{x-b_1} + \dfrac{1}{x-b_2} + \cdots + \dfrac{1}{x-b_n}}.$$

1995 China TST 759. A and B play the following game with a polynomial of degree at least 4:

$$x^{2n} + \Box x^{2n-1} + \Box x^{2n-2} + \cdots + \Box x + 1 = 0.$$

A and B take turns to fill in one of the blanks with a real number until all the blanks are filled up. If the resulting polynomial has no real roots, A wins. Otherwise, B wins. If A begins, which player has a winning strategy?

1995 China TST 760. Prove that the interval $[0, 1]$ can be split into black and white intervals for any quadratic polynomial $P(x)$, such that the sum of weights of the black intervals is equal to the sum of weights of the white intervals. Define the weight of the interval $[a, b]$ as $P(b) - P(a)$. Does the same result hold with a degree 3 or degree 5 polynomial?

1996 China TST 761. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$, and $\beta_1, \beta_2, \ldots, \beta_n$, where $n \geq 4$, be 2 sets of real numbers such that

$$\sum_{i=1}^{n} \alpha_i^2 < 1 \qquad \text{and} \qquad \sum_{i=1}^{n} \beta_i^2 < 1.$$

Define

$$A^2 = 1 - \sum_{i=1}^{n} \alpha_i^2,$$
$$B^2 = 1 - \sum_{i=1}^{n} \beta_i^2,$$
$$W = \frac{1}{2}(1 - \sum_{i=1}^{n} \alpha_i\beta_i)^2.$$

Find all real numbers λ such that the polynomial

$$x^n + \lambda(x^{n-1} + \cdots + x^3 + Wx^2 + ABx + 1) = 0,$$

only has real roots.

1997 China TST 762. Find all real-coefficient polynomials $f(x)$ which satisfy the following conditions:

(i) $f(x) = a_0x^{2n} + a_2x^{2n-2} + \cdots + a_{2n-2}x^2 + a_{2n}, a_0 > 0$;

(ii) $\displaystyle\sum_{j=0}^{n} a_{2j}a_{2n-2j} \leq \binom{2n}{n} a_0a_{2n}$;

(iii) All the roots of $f(x)$ are imaginary numbers with no real part.

2000 China TST 763. Let F be the set of all polynomials Γ such that all the coefficients of $\Gamma(x)$ are integers and $\Gamma(x) = 1$ has integer roots. Given a positive integer k, find the smallest integer $m(k) > 1$ such that there exist $\Gamma \in F$ for which $\Gamma(x) = m(k)$ has exactly k distinct integer roots.

2002 China TST 764. Let

$f(x_1, x_2, x_3) = -2\cdot(x_1^3+x_2^3+x_3^3)+3\cdot(x_1^2(x_2+x_3)+x_2^2\cdot(x_1+x_3)+x_3^2\cdot(x_1+x_2)-12x_1x_2x_3.$

For any reals r, s, t, we denote

$$g(r, s, t) = \max_{t\leq x_3\leq t+2} |f(r, r+2, x_3) + s|.$$

Find the minimum value of $g(r, s, t)$.

2002 China TST 765. Let $P_n(x) = a_0 + a_1x + \cdots + a_nx^n$, with $n \geq 2$, be a real-coefficient polynomial. Prove that if there exists $a > 0$ such that

$$P_n(x) = (x+a)^2 \left(\sum_{i=0}^{n-2} b_i x^i\right),$$

where b_i are positive real numbers, then there exists some i, with $1 \leq i \leq n-1$, such that

$$a_i^2 - 4a_{i-1}a_{i+1} \leq 0.$$

2002 China TST 766. For positive integers a, b, c let α, β, γ be pairwise distinct positive integers such that

$$\begin{cases} ca &= \alpha + \beta + \gamma, \\ b &= \alpha\beta + \beta\gamma + \gamma\alpha, \\ c^2 &= \alpha\beta\gamma. \end{cases}$$

Also, let λ be a real number that satisfies the condition

$$\lambda^4 - 2a\lambda^2 + 8c\lambda + a^2 - 4b = 0.$$

Prove that λ is an integer if and only if α, β, γ are all perfect squares.

2003 China TST 767. The n roots of a complex-coefficient polynomial

$$f(z) = z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n,$$

are $z_1, z_2, \ldots, z_n$. If $\sum_{k=1}^n |a_k|^2 \leq 1$, then prove that $\sum_{k=1}^n |z_k|^2 \leq n$.

2003 China TST 768. Can we find positive reals $a_1, a_2, \ldots, a_{2002}$ such that for any positive integer k, with $1 \leq k \leq 2002$, every complex root z of the following polynomial $f(x)$ satisfies the condition $|\text{Im } z| \leq |\text{Re } z|$,

$$f(x) = a_{k+2001}x^{2001} + a_{k+2000}x^{2000} + \cdots + a_{k+1}x + a_k,$$

where $a_{2002+i} = a_i$, for $i = 1, 2, \ldots, 2001$.

2004 China TST 769. Given integer n larger than 5, solve the system of equations (assuming $x_i \geq 0$, for $i = 1, 2, \ldots n$):

$$\begin{cases} x_1 + \ \ x_2 + \ \ x_3 + \cdots + \ \ x_n &= n+2, \\ x_1 + 2\ x_2 + 3\ x_3 + \cdots + n\ x_n &= 2n+2, \\ x_1 + 2^2x_2 + 3^2x_3 + \cdots + n^2x_n &= n^2+n+4, \\ x_1 + 2^3x_2 + 3^3x_3 + \cdots + n^3x_n &= n^3+n+8. \end{cases}$$

2005 China TST 770. Let $a_1, a_2 \ldots a_n$ and $x_1, x_2 \ldots x_n$ be integers and $r \geq 2$ be an integer. It is known that

$$\sum_{j=0}^{n} a_j x_j^k = 0 \qquad \text{for} \quad k = 1, 2, \ldots r.$$

Prove that

$$\sum_{j=0}^{n} a_j x_j^m \equiv 0 \pmod{m}, \qquad \text{for all} \quad m \in \{r+1, r+2, \cdots, 2r+1\}.$$

Problem 771. Prove the following statements on irrationality.

a) Show that $\sqrt{2} + \sqrt{3} + \sqrt{5} + \sqrt{7}$ is irrational.

b) Suppose a_i, with $i = 1, 2, \ldots, n$ are rationals such that $\sqrt{a_i}$ is irrational for at least one value of i. Prove that

$$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n}$$

is irrational.

2005 China TST 772. Determine whether $\sqrt{1001^2+1} + \sqrt{1002^2+1} + \cdots + \sqrt{2000^2+1}$ be a rational number or not?

Zhaobin vs Vess 773. Let $a_1, a_2, \ldots, a_n$ be n positive rational numbers and assume that $k_1, k_2, \ldots, k_n$ are n positive integers such that $\sqrt[k_i]{a_i}$ is irrational. Prove that the sum $\sum_{i=1}^{n} \sqrt[k_i]{a_i}$ is irrational.

2006 China TST 774. Let a_i and b_i (for $i = 1, 2, \ldots, n$) be rational numbers such that for any real number x, we have:

$$x^2 + x + 4 = \sum_{i=1}^{n} (a_i x + b)^2.$$

Find the least possible value of n.

2003 China TST 775. Find all second degree polynomial $d(x) = x^2 + ax + b$ with integer coefficients, so that there exists an integer-coefficient polynomial $p(x)$ and a non-zero integer-coefficient polynomial $q(x)$ that satisfy:

$$(p(x))^2 - d(x)\,(q(x))^2 = 1, \qquad \text{for all} \quad x \in \mathbb{R}.$$

2006 China TST 776. Let k be an odd number that is greater than or equal to 3. Prove that there exists a k^{th}-degree integer-valued polynomial with non-integer-coefficients that has the following properties:

1. $f(0) = 0$ and $f(1) = 1$; and
2. There exist infinitely many positive integers n so that if the following equation:

$$n = f(x_1) + \cdots + f(x_s),$$

has integer solutions $x_1, x_2, \ldots, x_s$, then $s \geq 2^k - 1$.

2008 China TST 777. After multiplying out and simplifying polynomial

$$(x-1)(x^2-1)(x^3-1)\cdots(x^{2007}-1),$$

getting rid of all terms whose powers are greater than 2007, we acquire a new polynomial $f(x)$. Find its degree and the coefficient of the term having the highest power. In other words, if

$$P(x) = (1-x)(1-x^2)\cdots(1-x^{2007}),$$

find the degree of $P(x) \pmod{x^{2008}}$.

2008 China TST 778. Let z_1, z_2, z_3 be three complex numbers of modulii less than or equal to 1. Let w_1, w_2 be two roots of the equation

$$(z-z_1)(z-z_2)+(z-z_2)(z-z_3)+(z-z_3)(z-z_1)=0.$$

Prove that, for $j=1,2,3$, we have

$$\min\{|z_j-w_1|, |z_j-w_2|\} \leq 1.$$

2008 China TST 779. Let $n > m > 1$ be odd integers, and define

$$f(x) = x^n + x^m + x + 1.$$

Prove that $f(x)$ can't be expressed as the product of two polynomials having integer coefficients and positive degrees.

2009 China TST 780. Find all complex polynomial $P(x)$ such that for any three integers a, b, c satisfying $a+b+c \neq 0$, and

$$\frac{P(a)+P(b)+P(c)}{a+b+c} \quad \text{is an integer.}$$

2010 China TST 781. Given positive integer n, find the largest real number $\lambda = \lambda(n)$, such that for any degree-n polynomial with complex coefficients

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0,$$

and any permutation $x_0, x_1, \ldots, x_n$ of $0, 1, \ldots, n$, the following inequality holds:

$$\sum_{k=0}^{n} |f(x_k) - f(x_{k+1})| \geq \lambda |a_n|,$$

where $x_{n+1} = x_0$.

2012 China TST 782. Find the smallest possible value of a real number c such that for any 2012^{th}-degree monic polynomial

$$P(x) = x^{2012} + a_{2011}x^{2011} + \cdots + a_1 x + a_0,$$

with real coefficients, we can obtain a new polynomial $Q(x)$ by multiplying some of its coefficients by -1 such that every root z of $Q(x)$ satisfies the inequality

$$|\mathrm{Im}\ z| \leq c\,|\mathrm{Re}\ z|.$$

1981 Austrian–Polish 783. Let $P(x) = x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$ be a polynomial with rational coefficients. Show that if $P(x)$ has exactly one real root ξ, then ξ is a rational number.

1986 Austrian–Polish 784. The monic polynomial $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ of degree $n > 1$ has n distinct negative roots. Prove that $a_1P(1) > 2n^2a_o$.

1986 Austrian–Polish 785. Find all real solutions x, y, u, v of the system of equations

$$\begin{cases} x^2 + y^2 + u^2 + v^2 & = 4, \\ xu + yv + xv + yu & = 0, \\ xyu + yuv + uvx + vxy & = -2, \\ xyuv = -1 \end{cases}$$

1987 Austrian–Polish 786. Let n be the square of an integer whose each prime divisor has an even number of decimal digits. Consider $P(x) = x^n - 1987x$. Show that if x, y are rational numbers with $P(x) = P(y)$, then $x = y$.

1988 Austrian–Polish 787. Let $P(x)$ be a polynomial with integer coefficients. Show that if $Q(x) = P(x) + 12$ has at least six distinct integer roots, then $P(x)$ has no integer roots.

1988 Austrian–Polish 788. If $a_1 \leq a_2 \leq \cdots \leq a_n$ are natural numbers ($n \geq 2$), show that the inequality

$$\sum_{i=1}^{n} a_ix_i^2 + 2\sum_{i=1}^{n-1} x_ix_{i+1} > 0,$$

holds for all n-tuples $(x_1, \ldots, x_n) \neq (0, \ldots, 0)$ of real numbers if and only if $a_2 \geq 2$.

1990 Austrian–Polish 789. Show that there are two real solutions (x, y, z) to:

$$\begin{cases} x + y^2 + z^4 & = 0, \\ y + z^2 + x^4 & = 0, \\ z + x^2 + y^5 & = 0. \end{cases}$$

1990 Austrian–Polish 790. Given a positive integer $n \geq 2$, find all solutions (x_i, y_i) to the following system of equations, where $1 \leq i \leq n$:

$$\begin{cases} x_1^4 + 14x_1x_2 + 1 & = y_1^4, \\ x_2^4 + 14x_2x_3 + 1 & = y_2^4, \\ \quad\vdots & \quad\vdots \\ x_n^4 + 14x_nx_1 + 1 & = y_n^4. \end{cases}$$

1990 Austrian–Polish 791. Let $p(x)$ be a polynomial with integer coefficients. The sequence of integers $a_1, a_2, \ldots, a_n$ (where $n > 2$) satisfies

$$a_2 = p(a_1), \quad a_3 = p(a_2), \quad \ldots, \quad a_n = p(a_{n-1}), \quad a_1 = p(a_n).$$

Show that $a_1 = a_3$.

1991 Austrian–Polish 792. Let $P(x)$ be a real polynomial with $P(x) \geq 0$ for $0 \leq x \leq 1$. Show that there exist polynomials $P_i(x)$ (for $i = 0, 1, 2$) with $P_i(x) \geq 0$ for all real x such that

$$P(x) = P_0(x) + xP_1(x)(1-x)P_2(x).$$

1991 Austrian–Polish 793. For a given positive integer n determine the maximum value of the function

$$f(x) = \frac{x + x^2 + \cdots + x^{2n-1}}{(1+x^n)^2}, \qquad \text{for all} \quad x \geq 0,$$

and find all positive x for which the maximum is attained.

1992 Austrian–Polish 794. Let k be a positive integer and u, v be real numbers, and

$$P(x) = (x - u^k)(x - uv)(x - v^k) = x^3 + ax^2 + bx + c.$$

a) For $k = 2$ prove that if a, b, c are rational then so is uv.

b) Is that also true for $k = 3$?

1993 Austrian–Polish 795. Solve in real numbers the system

$$\begin{cases} x^3 + y &= 3x + 4, \\ 2y^3 + z &= 6y + 6, \\ 3z^3 + x &= 9z + 8. \end{cases}$$

1993 Austrian–Polish 796. Determine all real polynomials $P(z)$ for which there exists a unique real polynomial $Q(x)$ satisfying the conditions $Q(0) = 0$, and

$$x + Q(y + P(x)) = y + Q(x + P(y)),$$

for all $x, y \in mathbbR$.

1994 Austrian–Polish 797. Let $n > 1$ be an odd positive integer. Assume that positive integers $x_1, x_2, \ldots, x_n \geq 0$ satisfy:

$$\begin{cases} (x_2 - x_1)^2 + 2(x_2 + x_1) + 1 &= n^2, \\ (x_3 - x_2)^2 + 2(x_3 + x_2) + 1 &= n^2, \\ \quad\vdots & \quad\vdots \\ (x_1 - x_n)^2 + 2(x_1 + x_n) + 1 &= n^2. \end{cases}$$

Show that there exists j with $1 \leq j \leq n$, such that $x_j = x_{j+1}$. Here, assume $x_{n+1} = x_1$.

1994 Austrian–Polish 798. Solve in integers the following equation

$$\frac{1}{2}(x+y)(y+z)(z+x) + (x+y+z)^3 = 1 - xyz.$$

1995 Austrian–Polish 799. Consider the equation $3y^4 + 4cy^3 + 2xy + 48 = 0$, where c is an integer parameter. Determine all values of c for which the number of integral solutions (x, y) satisfying the conditions (i) and (ii) is maximal:

(i) $|x|$ is a square of an integer;

(ii) y is a square-free number.

Remember that a square-free number is an integer which is not divisible by the square of any prime.

1996 Austrian–Polish 800. The polynomials $P_n(x)$ are defined initially by $P_0(x) = 0$ and $P_1(x) = x$, and then recursively, for $n \geq 2$, by

$$P_n(x) = xP_{n-1}(x) + (1-x)P_{n-2}(x).$$

For every natural number $n \geq 1$, find all real numbers x satisfying the equation $P_n(x) = 0$.

1996 Austrian–Polish 801. Given natural numbers $n > k > 1$, find all real solutions $x_1, \dots, x_n$ of the system

$$x_i^3(x_i^2 + x_{i+1}^2 + \cdots + x_{i+k-1}^2) = x_{i-1}^2,$$

for $1 \leq i \leq n$. Here $x_{n+i} = x_i$ for all$i$.

1999 Austrian–Polish 802. Solve in the non-negative real numbers the system of equations

$$x_n^2 + x_n x_{n-1} + x_{n-1}^4 = 1,$$

for $n = 1, 2, \dots, 1999$, assuming $x_0 = x_{1999}$.

2000 Austrian–Polish 803. For each integer $n \geq 3$ solve in real numbers the system of equations:

$$\begin{cases} x_1^3 & = x_2 + x_3 + 1, \\ x_2^3 & = x_3 + x_4 + 1, \\ \vdots & \qquad \vdots \\ x_{n-1}^3 & = x_n + x_1 + 1, \\ x_n^3 & = x_1 + x_2 + 1. \end{cases}$$

2003 Austrian–Polish 804. Find all real polynomials $p(x)$ such that

$$p(x-1)p(x+1) = p(x^2 - 1).$$

2003 Austrian–Polish 805. For each positive integer $n > 1$, define

$$f(n) = \frac{n^n - 1}{n - 1}.$$

a) Show that $n!^{f(n)}$ divides $(n^n)!$.

b) Find as many positive integers as possible for which $n!^{f(n)+1}$ does not divide $(n^n)!$.

2004 Austrian–Polish 806. Solve the following system of equations in $\mathbb{R}$ where all square roots are non-negative:

$$\begin{cases} a - \sqrt{1-b^2} + \sqrt{1-c^2} & = d, \\ b - \sqrt{1-c^2} + \sqrt{1-d^2} & = a, \\ c - \sqrt{1-d^2} + \sqrt{1-a^2} & = b, \\ d - \sqrt{1-a^2} + \sqrt{1-b^2} & = c. \end{cases}$$

2004 Austrian–Polish 807. Determine all n for which the system with of equations can be solved in $\mathbb{R}$:

$$\sum_{k=1}^{n} x_k = 27,$$
$$\prod_{k=1}^{n} x_k = \left(\frac{3}{2}\right)^{24}.$$

2004 Austrian–Polish 808. For each polynomial $Q(x)$ let $M(Q)$ be the set of non-negative integers x with $0 < Q(x) < 2004$. We consider polynomials $P_n(x)$ of the form

$$P_n(x) = x^n + a_1 \cdot x^{n-1} + \cdots + a_{n-1} \cdot x + 1,$$

with coefficients $a_i \in \{\pm 1\}$ for $i = 1, 2, \ldots, n-1$. For each $n = 3^k$, with $k > 0$, determine:

a) m_n, which represents the maximum of elements in $M(P_n)$ for all such polynomials $P_n(x)$; and

b) all polynomials $P_n(x)$ for which $|M(P_n)| = m_n$.

2005 Austrian–Polish 809. Determine all polynomials P with integer coefficients satisfying

$$P(P(P(P(P(x))))) = x^{28} \cdot P(P(x)), \qquad \text{for all} \quad x \in \mathbb{R}.$$

2005 Austrian–Polish 810. For each natural number $n \geq 2$, solve the following system of equations in the integers $x_1, x_2, \ldots, x_n$:

$$(n^2 - n)x_i + \left(\prod_{j \neq i} x_j\right) S = n^3 - n^2, \qquad \text{for} \quad i = 1, 2, \ldots, n,$$

where,

$$S = x_1^2 + x_2^2 + \cdots + x_n^2.$$

2006 Austrian–Polish 811. Find all polynomials $P(x)$ with real coefficients satisfying the equation

$$(x+1)^3 P(x-1) - (x-1)^3 P(x+1) = 4(x^2 - 1)P(x),$$

for all real numbers x.

2000 Austria 812. For any real number a, find all real numbers x that satisfy the following equation:

$$(2x+1)^4 + ax(x+1) - \frac{x}{2} = 0.$$

2002 Austria 813. Solve the following system of equations over the real numbers:

$$\begin{cases} 2x_1 = x_5^2 - 23, \\ 4x_2 = x_1^2 + 7, \\ 6x_3 = x_2^2 + 14, \\ 8x_4 = x_3^2 + 23, \\ 10x_5 = x_4^2 + 34. \end{cases}$$

2004 Austria 814. Solve the following equation for real numbers (all square roots are non negative):

$$\sqrt{4 - x\sqrt{4 - (x-2)\sqrt{1 + (x-5)(x-7)}}} = \frac{5x - 6 - x^2}{2}.$$

2010 Austria 815. Solve the following in equation in $\mathbb{R}^3$:

$$4x^4 - x^2(4y^4 + 4z^4 - 1) - 2xyz + y^8 + 2y^4z^4 + y^2z^2 + z^8 = 0.$$

2010 Donova (Danube) 816. Let $n \geq 3$ be a positive integer. Find non-negative real numbers $x_1, x_2, \ldots, x_n$, with $x_1 + x_2 + \cdots + x_n = n$, for which the expression

$$(n-1)(x_1^2 + x_2^2 + \cdots + x_n^2) + nx_1x_2\cdots x_n,$$

takes a minimal value.

2017 Donova (Danube) 817. Find all polynomials $P(x)$ with integer coefficients such that $a^2 + b^2 - c^2$ divides $P(a) + P(b) - P(c)$, for all integers a, b, c.

1959–1966 IMO Longlist 818. If a, b, c, d are integers such that ad is odd and bc is even, prove that at least one root of the polynomial $ax^3 + bx^2 + cx + d$ is irrational.

1968 IMO Shortlist 819. A polynomial $p(x) = a_0x^k + a_1x^{k-1} + \cdots + a_k$ with integer coefficients is said to be divisible by an integer m if $p(x)$ is divisible by m for all integers x. Prove that if $p(x)$ is divisible by m, then $k!a_0$ is also divisible by m. Also prove that if a_0, k, and m are non-negative integers for which $k!a_0$ is divisible by m, then there exists a polynomial $p(x) = a_0x^k + \cdots + a_k$ divisible by m.

1968 IMO Shortlist 820. Find all complex numbers m such that polynomial

$$x^3 + y^3 + z^3 + mxyz,$$

can be represented as the product of three linear trinomials.

1969 IMO Longlist 821. Let us define $u_0 = 0, u_1 = 1$ and for $n \geq 0$,

$$u_{n+2} = au_{n+1} + bu_n,$$

where a and b are positive integers. Express u_n as a polynomial in a and b. Prove the result. Given that b is prime, prove that b divides $a(u_b - 1)$.

1969 IMO Longlist 822. Given a polynomial $f(x)$ with integer coefficients whose value is divisible by 3 for three integers $k, k+1$, and $k+2$, prove that $f(m)$ is divisible by 3 for all integers m.

1969 IMO Longlist 823. Prove that if $0 \leq a_0 \leq a_1 \leq a_2$, then

$$(a_0 + a_1 x - a_2 x^2)^2 \leq (a_0 + a_1 + a_2)^2 \left(1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{2} + x^4\right),$$

and formulate and prove the analogous result for polynomials of third degree.

1970 IMO Longlist 824. Given a polynomial

$$P(x) = ab(a-c)x^3 + (a^3 - a^2c + 2ab^2 - b^2c + abc)x^2 + \\ (2a^2b + b^2c + a^2c + b^3 - abc)x + ab(b+c),$$

where $a, b, c \neq 0$, prove that $P(x)$ is divisible by $Q(x) = abx^2 + (a^2 + b^2)x + ab$ and conclude that $P(x_0)$ is divisible by $(a+b)^3$ for $x_0 = (a+b+1)^n$, for all $n \in \mathbb{N}$.

1970 IMO Longlist 825. Let a polynomial $p(x)$ with integer coefficients take the value 5 for five different integer values of x. Prove that $p(x)$ does not take the value 8 for any integer x.

1970 IMO Shortlist 826. Let P, Q, R be polynomials and let $S(x) = P(x^3) + xQ(x^3) + x^2R(x^3)$ be a polynomial of degree n whose roots $x_1, \ldots, x_n$ are distinct. Construct with the aid of the polynomials P, Q, R a polynomial T of degree n that has the roots $x_1^3, x_2^3, \ldots, x_n^3$.

1971 IMO Shortlist 827. Consider a sequence of polynomials $\{P_i(x)\}_{i=0}^{\infty}$, where $P_0(x) = 2, P_1(x) = x$ and for every $n \geq 1$ the following equality holds:

$$P_{n+1}(x) + P_{n-1}(x) = xP_n(x).$$

Prove that there exist three real numbers a, b, c such that for all $n \geq 1$,

$$(x^2 - 4)[P_n^2(x) - 4] = [aP_{n+1}(x) + bP_n(x) + cP_{n-1}(x)]^2.$$

1971 IMO Shortlist 828. Prove that the polynomial $x^4 + \lambda x^3 + \mu x^2 + \nu x + 1$ has no real roots if λ, μ, ν are real numbers satisfying

$$|\lambda| + |\mu| + |\nu| \leq \sqrt{2}.$$

1976 IMO Longlist 829. Prove that if for a polynomial $P(x, y)$, we have

$$P(x-1, y-2x+1) = P(x, y),$$

then there exists a polynomial $\Phi(x)$ with $P(x, y) = \Phi(y - x^2)$.

1976 IMO Longlist 830. The polynomial $1976(x + x^2 + \cdots + x^n)$ is decomposed into a sum of polynomials of the form $a_1x + a_2x^2 + \cdots + a_nx^n$, where $a_1, a_2, \ldots, a_n$ are distinct positive integers not greater than n. Find all values of n for which such a decomposition is possible.

1976 IMO Longlist 831. Let $g(x)$ be a fixed polynomial with real coefficients and define $f(x)$ by $f(x) = x^2 + xg(x^3)$. Show that $f(x)$ is not divisible by $x^2 - x + 1$.

1976 IMO Longlist 832. Let P be a polynomial with real coefficients such that $P(x) > 0$ if $x > 0$. Prove that there exist polynomials Q and R with non-negative coefficients such that

$$P(x) = \frac{Q(x)}{R(x)} \qquad \text{if} \quad x > 0.$$

1976 IMO Longlist 833. Prove that if $P(x) = (x-a)^k Q(x)$, where k is a positive integer, a is a nonzero real number, $Q(x)$ is a nonzero polynomial, then $P(x)$ has at least $k+1$ nonzero coefficients.

1978 IMO Longlist 834. Given the expression

$$P_n(x) = \frac{1}{2^n}\left[(x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n\right],$$

prove that:

a) $P_n(x)$ satisfies the identity

$$P_n(x) - xP_{n-1}(x) + \frac{1}{4}P_{n-2}(x) \equiv 0.$$

b) $P_n(x)$ is a polynomial in x of degree n.

1982 IMO Longlist 835. Determine all real values of the parameter a for which the equation

$$16x^4 - ax^3 + (2a+17)x^2 - ax + 16 = 0,$$

has exactly four distinct real roots that form a geometric progression.

1984 IMO Longlist 836. Let $f_1(x) = x^3 + a_1x^2 + b_1x + c_1 = 0$ be an equation with three positive roots $\alpha > \beta > \gamma > 0$. From the equation $f_1(x) = 0$, one constructs the equation $f_2(x) = x^3 + a_2x^2 + b_2x + c_2 = x(x+b_1)^2 - (a_1x + c_1)^2 = 0$. Continuing this process, we get equations $f_3, \cdots, f_n$. Prove that

$$\lim_{n\to\infty} \sqrt[2^{n-1}]{-a_n} = \alpha.$$

1984 IMO Longlist 837. Let P, Q, R be the polynomials with real or complex coefficients such that at least one of them is not constant. If $P^n + Q^n + R^n = 0$, prove that $n < 3$.

1985 IMO Longlist 838. Find a method by which one can compute the coefficients of $P(x) = x^6 + a_1x^5 + \cdots + a_6$ from the roots of $P(x) = 0$ by performing not more than 15 additions and 15 multiplications.

1987 IMO Longlist 839. Let P, Q, R be polynomials with real coefficients, satisfying $P^4 + Q^4 = R^2$. Prove that there exist real numbers p, q, r and a polynomial S such that $P = pS, Q = qS$ and $R = rS^2$.

1989 IMO Longlist 840. Let $f(x) = \prod_{k=1}^{n}(x - a_k) - 2$, where $n \geq 3$ and $a_1, a_2, \ldots,$ an are distinct integers. Suppose that $f(x) = g(x)h(x)$, where $g(x), h(x)$ are both nonconstant polynomials with integer coefficients. Prove that $n = 3$.

1989 IMO Longlist 841. Let $P_1(x), P_2(x), \ldots, P_n(x)$ be real polynomials, i.e., they have real coefficients. Show that there exist real polynomials $A_r(x), B_r(x) \quad (r = 1, 2, 3)$ such that

$$\sum_{s=1}^{n} \{P_s(x)\}^2 \equiv (A_1(x))^2 + (B_1(x))^2,$$
$$\sum_{s=1}^{n} \{P_s(x)\}^2 \equiv (A_2(x))^2 + x\,(B_2(x))^2,$$
$$\sum_{s=1}^{n} \{P_s(x)\}^2 \equiv (A_3(x))^2 - x\,(B_3(x))^2.$$

1992 IMO Longlist 842. Let $P_1(x, y)$ and $P_2(x, y)$ be two relatively prime polynomials with complex coefficients. Let $Q(x, y)$ and $R(x, y)$ be polynomials with complex coefficients and each of degree not exceeding d. Prove that there exist two integers A_1, A_2 not simultaneously zero with $|A_i| \leq d + 1$ $(i = 1, 2)$ and such that the polynomial $A_1P_1(x, y) + A_2P_2(x, y)$ is coprime to $Q(x, y)$ and $R(x, y)$.

1992 IMO Longlist 843. Let $f(x) = x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m$ and $g(x) = x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n$ be two polynomials with real coefficients such that for each real number $x, f(x)$ is the square of an integer if and only if so is $g(x)$. Prove that if $n + m > 0$, then there exists a polynomial $h(x)$ with real coefficients such that $f(x) \cdot g(x) = (h(x))^2$.

Chapter 2

Olympiad Algebra 201: Trigonometry 101, 201, 301, 401, and Beyond

2.1 Math Olympiad Trigonometry 101: Identities & Half-angled Trigonometry

We start with the most basic identities and move on to the more advanced ones.

2.1.1 Trigonometric Identities

Identity 1 (Fundamental Identity of Trigonometry)**.** For any real number x, we have $\sin^2 x + \cos^2 x = 1$.

Identity 2 (Special Trig Values)**.**

$$\sin\frac{\pi}{2} = 1, \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}, \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \sin\frac{\pi}{6} = \frac{1}{2}, \tag{2.1}$$

$$\cos\frac{\pi}{2} = 0, \cos\frac{\pi}{3} = \frac{1}{2}, \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}. \tag{2.2}$$

Identity 3. Prove the formulas for Sum and Difference of Two Angles:

1. Sine and cosine of the sum and difference of two angles:

$$\cos(x+y) = \cos x\cos y - \sin x \sin y, \tag{2.3}$$

$$\cos(x-y) = \cos x\cos y + \sin x \sin y, \tag{2.4}$$

$$\sin(x+y) = \sin x\cos y + \cos x \sin y, \tag{2.5}$$

$$\sin(x-y) = \sin x\cos y - \cos x \sin y. \tag{2.6}$$

$$\tag{2.7}$$

2. Tangent and cotangent of the sum and difference of two angles:

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}, \tag{2.8}$$

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}, \tag{2.9}$$

$$\cot(x+y) = \frac{\cot x \cot y - 1}{\cot x + \cot y}, \tag{2.10}$$

$$\cot(x-y) = \frac{\cot x \cot y + 1}{\cot x - \cot y}. \tag{2.11}$$

$$\tag{2.12}$$

Identity 4 (Complementary and Supplementary angles)**.**

$$\sin(\pi - x) = \sin x, \tag{2.13}$$

$$\cos(\pi - x) = -\cos x, \tag{2.14}$$

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x, \tag{2.15}$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x, \tag{2.16}$$

$$\tan\left(\frac{\pi}{2} - x\right) = \frac{\sin\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right)} = \frac{\cos x}{\sin x} = \cot x. \tag{2.17}$$

Identity 5 (Half-angles)**.** Using the trigonometric identities for sum and difference of angles, prove that

$$\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = 2\cos^2 \frac{x}{2} - 1 = 1 - 2\sin^2 \frac{x}{2}, \tag{2.18}$$

$$\sin x = 2\sin \frac{x}{2} \cos \frac{x}{2}. \tag{2.19}$$

Identity 6 ($\tan(x/2)$)**.** Using the trigonometric identities for half-angles, prove that

$$\sin x = \frac{2\tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}, \tag{2.20}$$

$$\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}. \tag{2.21}$$

Problem 844. Using the previous identities, prove that

$$\begin{aligned}
\sin 15^\circ = \cos 75^\circ &= \frac{\sqrt{6} - \sqrt{2}}{4}, \\
\cos 15^\circ = \sin 75^\circ &= \frac{\sqrt{6} + \sqrt{2}}{4}, \\
\tan 15^\circ = \cot 75^\circ &= 2 - \sqrt{3}, \\
\cot 15^\circ = \tan 75^\circ &= 2 + \sqrt{3},
\end{aligned}$$

and

$$\begin{aligned}
\sin 22.5^\circ = \cos 67.5^\circ &= \frac{\sqrt{2 - \sqrt{2}}}{2}, \\
\cos 22.5^\circ = \sin 67.5^\circ &= \frac{\sqrt{2 + \sqrt{2}}}{2}, \\
\tan 22.5^\circ = \cot 67.5^\circ &= \sqrt{2} - 1, \\
\cot 22.5^\circ = \tan 67.5^\circ &= \sqrt{2} + 1.
\end{aligned}$$

Identity 7 (Double-angles)**.** Using the trigonometric identities for sum and difference of angles, prove that

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x, \tag{2.22}$$

$$\sin 2x = 2\sin x \cos x, \tag{2.23}$$

$$1 - \cos 2x = 2\sin^2 x, \tag{2.24}$$

$$1 + \cos 2x = 2\cos^2 x, \tag{2.25}$$

$$\cos 4x = 1 - 2\sin^2 2x, \tag{2.26}$$

$$\sin 4x = 4\sin x \cos x \cos 2x. \tag{2.27}$$

Identity 8 ($1 \pm \cos 2x$)**.** Using the trigonometric identities for double-angles, prove that

$$1 - \cos 2x = 2\sin^2 x, \tag{2.28}$$

$$1 + \cos 2x = 2\cos^2 x, \tag{2.29}$$

and imply that

$$\sin x = \pm\sqrt{\frac{1-\cos 2x}{2}}, \tag{2.30}$$
$$\cos x = \pm\sqrt{\frac{1+\cos 2x}{2}}, \tag{2.31}$$
$$\tan x = \pm\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}. \tag{2.32}$$

($\cot x - \cot 2x$) Exercise Identity 845. Show that

$$\frac{1}{\sin 2\theta} - \cot 2\theta = \tan\theta \qquad \text{and} \qquad \frac{1}{\sin 2\theta} + \cot 2\theta = \cot\theta.$$

Sum of Three Angles Identity 846. Prove that for all $x, y, z \in \mathbb{R}$,

$$\sin(x+y+z) = \sin x\cos y\cos c + \sin y\cos x\cos z + \sin z\cos x\cos y - \sin x\sin y\sin z, \tag{2.33}$$
$$\cos(x+y+z) = \cos x\cos y\cos c - \cos x\sin y\sin z - \cos y\sin x\sin z - \cos z\sin x\sin y, \tag{2.34}$$
$$\tan(x+y+z) = \frac{\tan x + \tan y + \tan z - \tan x\tan y\tan z}{1 - \tan x\tan y - \tan y\tan z - \tan z\tan x}. \tag{2.35}$$

Find a formula for $\sin, \cos, \tan$ for triple-angles $3x, 6x, 9x, \ldots$

Triple–Angle Identities

Identity 9 (Triple-angles)**.** For any real number x,

$$\sin 3x = 3\sin x - 4\sin^3 x, \tag{2.36}$$
$$\sin 3x = 4\cos^3 x - 3\cos x, \tag{2.37}$$
$$\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}. \tag{2.38}$$

Problem 847. Prove the following identities for any three angles α, β, γ:

a)

$$\cos(\alpha+\beta+\gamma) + \cos(\alpha+\beta-\gamma) + \cos(\alpha-\beta+\gamma) + \cos(-\alpha+\beta+\gamma) = 4\cos\alpha\cos\beta\cos\gamma,$$

b)

$$\cos(\alpha+\beta-\gamma)+\cos(\alpha-\beta+\gamma)-\cos(\alpha+\beta+\gamma) \\ -\cos(-\alpha+\beta+\gamma)=4\cos\alpha\sin\beta\sin\gamma,$$

c)

$$\sin(\alpha+\beta-\gamma)+\sin(\alpha-\beta+\gamma)+\sin(-\alpha+\beta+\gamma) \\ =\sin(\alpha+\beta+\gamma)+4\sin\alpha\sin\beta\sin\gamma,$$

d)

$$\sin(\alpha+\beta+\gamma)+\sin(\alpha-\beta+\gamma)+\sin(-\alpha+\beta+\gamma) \\ =\sin(\alpha+\beta-\gamma)+4\cos\alpha\cos\beta\sin\gamma.$$

e)

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2(\alpha+\beta+\gamma) \\ =2\left(1+\cos(\alpha+\beta)+\cos(\beta+\gamma)+\cos(\gamma+\alpha)\right),$$

f)

$$\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)=4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}.$$

Problem 848. For any three angles A, B, C, prove that

$$\begin{aligned} 0 &= \tan(A-B)+\tan(B-C)+\tan(C-A)-\tan(A-B)\tan(B-C)\tan(C-A), \\ -3 &= \tan(A+60^\circ)\tan(A-60^\circ)+\tan A\tan(A+60^\circ)+\tan A\tan(A-60^\circ), \\ 1 &= \cos^2 A+\cos^2 B+\cos^2(A+B)-2\cos A\cos B\cos(A+B), \\ \cot A &= \tan A+2\tan 2A+4\cot 4A. \end{aligned}$$

Problem 849. Prove that for any real number x,

$$\begin{aligned} 4\cos x\cos\left(\frac{\pi}{3}-x\right)\cos\left(\frac{\pi}{3}+x\right) &= \cos 3x, \\ 4\sin x\sin\left(\frac{\pi}{3}-x\right)\sin\left(\frac{\pi}{3}+x\right) &= \sin 3x, \\ \tan x+\tan\left(\frac{\pi}{3}+x\right)+\tan\left(\frac{2\pi}{3}+x\right) &= 3\tan 3x. \end{aligned}$$

2.1.2 Sum-to-Product and Product-to-Sum

Sine and Cosine Sum-to-Product 850. Prove that for all $x, y \in \mathbb{R}$,

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}, \tag{2.39}$$

$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}, \tag{2.40}$$

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}, \tag{2.41}$$

$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}. \tag{2.42}$$

Sine and Cosine Product-to-Sum 851. Prove that for all $x, y \in \mathbb{R}$,

$$\sin x\cos y = \frac{1}{2}\left(\sin(x+y) + \sin(x-y)\right), \tag{2.43}$$

$$\sin x\sin y = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right), \tag{2.44}$$

$$\cos x\cos y = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right). \tag{2.45}$$

Tangent and Cotangent Sum-to-Product 852. Prove that for all $x, y \in \mathbb{R}$,

$$\tan x + \tan y = \frac{\sin(x+y)}{\cos x\cos y}, \tag{2.46}$$

$$\tan x - \tan y = \frac{\sin(x-y)}{\cos x\cos y}, \tag{2.47}$$

$$\cot x + \cot y = \frac{\sin(x+y)}{\sin x\sin y}, \tag{2.48}$$

$$\cot x - \cot y = \frac{\sin(x-y)}{\sin x\sin y}. \tag{2.49}$$

2.1.3 Half-angled and Double-angled Trigonometry of Triangle

Problem 853. If A, B, C are any three angles whose sum is π (thus possibly being the three angles of a triangle),

$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}, \tag{2.50}$$

$$\tan A + \tan B + \tan C = \tan A\tan B\tan C, \tag{2.51}$$

$$\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}. \tag{2.52}$$

Problem 854. If A, B, C are the three angles of a triangle ABC with three sides a, b, c and semi-perimeter $p = (a + b + c)/2$,

$$\cos\frac{A}{2} = \sqrt{\frac{p(p-a)}{bc}}, \tag{2.53}$$

$$\sin\frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{bc}}, \tag{2.54}$$

$$\tan\frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{p(p-a)}}, \tag{2.55}$$

$$\frac{\tan\frac{A}{2} - \tan\frac{B}{2}}{\tan\frac{A}{2} + \tan\frac{B}{2}} = \frac{a-b}{c}, \tag{2.56}$$

$$\frac{\tan\frac{B+C}{2}}{\tan\frac{B-C}{2}} = \frac{b+c}{b-c}. \tag{2.57}$$

2.1.3.1 Pool of Half– and Double–angled Identities in Triangle

Problem 855. If A, B, C are any three angles whose sum is π (thus possibly being the three angles of a triangle),

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C, \tag{2.58}$$

$$\cos 2A + \cos 2B + \cos 2C = -4\cos A \cos B \cos C - 1. \tag{2.59}$$

Problem 856. Prove that for any three angles α, β, γ,

$$\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha+\beta+\gamma) = 4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2},$$
$$\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma) = 4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}.$$

Problem 857. For any three angles A, B, C that add up to $180°$, prove the following identities:

$$\cos A + \cos B - \cos C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} - 1,$$
$$\sin 2A + \sin 2B - \sin 2C = 4\cos A \cos B \sin C,$$
$$\cos 2A + \cos 2B - \cos 2C = 1 - 4\sin A \sin B \cos C.$$

Problem 858. For any three angles A, B, C that add up to $180°$, prove the following half-angle identities:

$$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} = 1 - 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$
$$\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = 2\left(1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right),$$

and

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}=1+4\cos\frac{\pi+A}{4}\cos\frac{\pi+B}{4}\cos\frac{\pi+C}{4},$$
$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}=4\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}.$$

Problem 859. For any three angles A, B, C that add up to $180°$, prove the following identities on sine of double– and quadruple–angles:

$$\sin^2 A+\sin^2 B+\sin^2 C=2(1+\cos A\cos B\cos C),$$
$$\sin^2 A+\sin^2 B-\sin^2 C=2\sin A\sin B\sin C,$$
$$\sin^2 2A+\sin^2 2B+\sin^2 2C=2(1-\cos 2A\cos 2B\cos 2C),$$
$$\sin 4A+\sin 4B+\sin 4C=-4\sin 2A\sin 2B\sin 2C,$$
$$\sin 4A+\sin 4B-\sin 4C=-4\cos 2A\cos 2B\sin 2C,$$

and prove the similar identities on cosine of double– and quadruple–angles:

$$\cos^2 A+\cos^2 B+\cos^2 C=1-2\cos A\cos B\cos C,$$
$$\cos^2 A+\cos^2 B-\cos^2 C=1-2\sin A\sin B\sin C,$$
$$\cos^2 2A+\cos^2 2B+\cos^2 2C=1+2\cos 2A\cos 2B\cos 2C,$$
$$\cos 4A+\cos 4B+\cos 4C=4\cos 2A\cos 2B\cos 2C-1,$$
$$\cos 4A+\cos 4B-\cos 4C=4\sin 2A\sin 2B\cos 2C+1.$$

Problem 860. For any three angles A, B, C that add up to $180°$, prove the following identities on tangent and cotangent of half– and double–angles:

$$\tan\frac{B}{2}\cot\frac{C}{2}=\frac{\sin A+\sin B-\sin C}{\sin A-\sin B+\sin C},$$
$$\tan A\tan B=\frac{\sin 2A+\sin 2B+\sin 2C}{\sin 2A+\sin 2B-\sin 2C},$$
$$\tan\frac{A}{2}\tan\frac{B}{2}=\frac{\cos A+\cos B+\cos C-1}{\cos A+\cos B-\cos C+1},$$
$$1=\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2},$$
$$0=\tan 2A+\tan 2B+\tan 2C-\tan 2A\tan 2B\tan 2C.$$

Problem 861. In triangle ABC with side-lengths a, b, c, perimeter $2p$ and area S, prove the following identities:

$$0=(b-c)\cot\frac{A}{2}+(c-a)\cot\frac{B}{2}+(a-b)\cot\frac{C}{2},$$
$$0=(a+b+c)\sin\frac{A}{2}-2a\cos\frac{B}{2}\cos\frac{C}{2},$$
$$0=(-a+b+c)\sin\frac{A}{2}-2a\sin\frac{B}{2}\sin\frac{C}{2},$$
$$p=b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2},$$
$$\frac{p}{abc}=\frac{1}{a}\cos^2\frac{A}{2}+\frac{1}{b}\cos^2\frac{B}{2}+\frac{1}{c}\cos^2\frac{C}{2}.$$

Problem 862. In triangle ABC, show that

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{8}. \tag{2.60}$$

Problem 863. Prove in a triangle ABC that

$$\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \le \left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{8}.$$

Problem 864. Prove in a triangle ABC that

$$\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}. \tag{2.61}$$

Problem 865. Prove the following identities:

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2},$$
$$\cos\frac{A-B}{2} - \cos\frac{A+B}{2} = 2\sin\frac{A}{2}\sin\frac{B}{2}.$$

Problem 866. In triangle ABC, show that

$$\left(\sin\frac{A}{2}\right)^2 + \left(\sin\frac{B}{2}\right)^2 + \left(\sin\frac{C}{2}\right)^2 + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 1. \tag{2.62}$$

Problem 867. Show that

$$\tan^2\frac{\pi}{16} + \tan^2\frac{3\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{7\pi}{16} = 28.$$

2.1.4 Trigonometry of Geometrical Quantities

We shall find the magnitude of several important geometrical quantities related to triangles, including triangle's area, perimeter, heights, internal and external angle bisectors, circumradius, inradius and exradii, etc.

2.1.4.1 Law of Sines, Law of Cosines, and Law of Tangents

For the sake of self-containment, we remember:

Law of Sines 868. In a triangle with side lengths a, b, c, and angles A, B, C,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R,$$

where R is the circumradius (radius of the circumcircle) of triangle ABC.

Law of Cosines 869. In a triangle with side lengths a, b, c, and angles A, B, C,

$$a^2 = b^2 + c^2 - 2bc\cos A,$$
$$b^2 = c^2 + a^2 - 2ca\cos B,$$
$$c^2 = a^2 + b^2 - 2ab\cos C,$$

and conclude the Pythagorean theorem.

Projection Rule in Triangle 870. In a triangle ABC with side lengths a, b, c, and angles A, B, C, show that

$$a = b\cos C + c\cos A,$$
$$b = c\cos A + a\cos C,$$
$$c = a\cos B + b\cos A$$

Law of Tangents 871. In a triangle with side lengths a, b, c, and angles A, B, C,

$$\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2},$$
$$\tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a}\cot\frac{B}{2},$$
$$\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}.$$

Problem 872. In triangle ABC with perimeter $2p$, show that

$$p\tan\frac{A}{2} = (p-b)\cot\frac{C}{2} = (p-c)\cot\frac{B}{2}.$$

2.1.4.2 Trigonometry of Perimeter and Area

Area Using Law of Sines 873. In triangle ABC with angles A, B, C and side-lengths a, b, c, prove that

$$S = \frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B = \frac{1}{2}bc\sin A,$$

where S denotes the area of triangle ABC.

Heron's Formula 874. In a triangle with side lengths a, b, c, perimeter $2p$, and area S,

$$S = \sqrt{p(p-a)(p-b)(p-c)}.$$

Semi-perimeter and Cosine of Half-angles 875. In a triangle with angles A, B, C and perimeter $2p$, show that

$$p = 4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},$$

where R is the circumradius (radius of the circumcircle) of triangle ABC.

Area in Terms of Sine of Angles 876. In triangle ABC with angles A, B, C and radius of the circumcircle R, prove that

$$S = 2R^2\sin A\sin B\sin C,$$

where S denotes the area of triangle ABC. Conclude that

$$S = \frac{a^2\sin B\sin C}{2\sin A} = \frac{b^2\sin A\sin C}{2\sin B} = \frac{c^2\sin A\sin B}{2\sin C}.$$

Problem 877. If the angle bisectors of angles A, B, C in triangle ABC with inradius r intersect the opposite sides at D, E, F, respectively, show that:

$$\frac{4S_{\triangle ABC}\cdot S_{\triangle DEF}}{AD\cdot BE\cdot CF} = r.$$

Problem 878. In any triangle ABC with side-lengths a, b, c, and angles A, B, C, if the perimeter is $2p$, prove the following half-angled formulas:

$$\sin\frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{bc}},$$
$$\cos\frac{A}{2} = \sqrt{\frac{p(p-a)}{bc}},$$
$$\tan\frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{p(p-a)}}.$$

2.1.4.3 Trigonometry of Heights

Calculating the Height of Triangle 879. In triangle ABC with angles A, B, C and radius of the circumcircle R, prove that

$$h_a = 2R\sin B\sin C = \frac{r\sqrt{(1+\cos B)(1+\cos C)}}{\cos\frac{A}{2}},$$
$$h_b = 2R\sin C\sin A = \frac{r\sqrt{(1+\cos C)(1+\cos A)}}{\cos\frac{B}{2}},$$
$$h_c = 2R\sin A\sin B = \frac{r\sqrt{(1+\cos A)(1+\cos B)}}{\cos\frac{C}{2}}.$$

where h_a, h_b, h_c are the heights drawn from vertices A, B, C, respectively.

Problem 880. Show that

$$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c} = \frac{1}{r},$$
$$\frac{\cos A}{h_a}+\frac{\cos B}{h_b}+\frac{\cos C}{h_c} = \frac{1}{R},$$
$$h_a h_b h_c = \frac{a^2b^2c^2}{8R^3}.$$

Identities on Lengths of Heights 881. In a triangle ABC with side-lengths a, b, c, angles A, B, C and corresponding lengths of altitudes h_a, h_b, h_c, we know that the area is S, the inradius is r, and circumradius is R. Prove the following identities:

$$h_a\cos A + h_b\cos B + h_c\cos C = 2R(1+\cos A\cos B\cos C),$$
$$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c} = \frac{2ab}{(a+b+c)\cdot S}\cdot\cos^2\left(\frac{C}{2}\right),$$
$$\frac{ah_c}{b}+\frac{bh_a}{c}+\frac{ch_b}{a} = \frac{a^2+b^2+c^2}{2R},$$
$$\frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2} = \frac{\cot A+\cot B+\cot C}{S}$$

Calculating Distance from Orthocenter 882. In triangle ABC with angles A, B, C and radius of the circumcircle R, assume that H is the orthocenter (intersection of heights). If we let A', B', C' be the foot of the heights drawn from A, B, C, respectively, then

$$HA = 2R\cos A, \qquad HA' = 2R\cos B\cos C,$$
$$HB = 2R\cos B, \qquad HB' = 2R\cos C\cos A,$$
$$HC = 2R\cos C, \qquad HC' = 2R\cos A\cos B.$$

Trigonometry of the Orthic Triangle 883. In triangle ABC with sides a, b, c, angles A, B, C, and radius of the circumcircle R, assume that H is the orthocenter (intersection of heights). If we let A', B', C' be the foot of the heights drawn from A, B, C, respectively, then we call triangle $A'B'C'$ the **Orthic Triangle** of triangle ABC.
Prove the following:

1. Heights of ABC are the internal angle bisectors of $A'B'C'$.
2. The circumradius of $A'B'C'$ is half of circumradius of ABC.
3. $HA \cdot HA' = HB \cdot HB' = HC \cdot HC'$.
4. $\angle BHC = \pi - \angle A$, $\angle AHC = \pi - \angle B$, and $\angle AHB = \pi - \angle C$.
5. The circumcircles of triangles AHB, AHC, and BHC are the same as the circumcircle of ABC.
6. The reflection of H with respect to each side lies on the circumcircle of ABC.
7. The sides of triangle ABC are the external angle bisectors of $A'B'C'$.
8. The circumcircle of the orthic triangle $A'B'C'$ passes through the midpoints of HA, HB, and HC.
9. The angles of $A'B'C'$ are equal to $\pi - 2\angle A$, $\pi - 2\angle B$, and $\pi - 2\angle C$.
10. $A'B' = R\sin 2C$, $A'C' = R\sin 2B$, and $B'C' = R\sin 2A$.
11. The area of the orthic triangle is
$$S_H = \frac{abc|\cos A\cos B\cos C|}{2R}.$$
12. The inradius of the orthic triangle is
$$r_H = 2R|\cos A\cos B\cos C|.$$

Problem 884. If $A'B'C'$ is the orthic triangle of triangle ABC, then show that the area of triangle $A'B'C'$ is

$$S_{\triangle A'B'C'} = \frac{1}{2}R^2 \sin 2A \sin 2B \sin 2C,$$

and that

$$\frac{1}{AA'} + \frac{1}{BB'} + \frac{1}{CC'} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}.$$

Problem 885. Imagine that the heights of triangle ABC drawn from vertices A, B, C intersect the circumcircle of triangle ABC in A'', B'', C'', respectively. Show that

$$\frac{S_{\triangle A''B''C''}}{S_{\triangle ABC}} = 8 \cos A \cos B \cos C.$$

2.1.4.4 Trigonometry of Internal and External Angle Bisectors

Calculating the Length of Internal Angle Bisectors 886. In triangle ABC with angles A, B, C and radius of the circumcircle R, let d_a, d_b, d_c be the length of the corresponding internal angle bisectors of angles A, B, C. Prove that

$$d_a = \frac{h_a}{\cos \dfrac{B-C}{2}} = \frac{2bc}{b+c} \cos \frac{A}{2},$$
$$d_b = \frac{h_b}{\cos \dfrac{C-A}{2}} = \frac{2ca}{c+a} \cos \frac{B}{2},$$
$$d_c = \frac{h_c}{\cos \dfrac{A-B}{2}} = \frac{2ab}{a+b} \cos \frac{C}{2},$$

where h_a, h_b, h_c are the heights drawn from vertices A, B, C in triangle ABC.

Calculating the Length of External Angle Bisectors 887. In triangle ABC with angles A, B, C and radius of the circumcircle R, let d'_a, d'_b, d'_c be the length of the corresponding external angle bisectors. Prove that

$$d'_a = \frac{h_a}{\left|\sin \dfrac{B-C}{2}\right|} = \left|\frac{2bc}{c-b}\right| \sin \frac{A}{2},$$
$$d'_b = \frac{h_b}{\left|\sin \dfrac{C-A}{2}\right|} = \left|\frac{2ca}{c-a}\right| \sin \frac{B}{2},$$
$$d'_c = \frac{h_c}{\left|\sin \dfrac{A-B}{2}\right|} == \left|\frac{2ab}{a-b}\right| \sin \frac{C}{2},$$

where h_a, h_b, h_c are the heights drawn from vertices A, B, C in triangle ABC.

Problem 888. In triangle ABC, we know that $b^3+c^3=a^2(b+c)$. Show that $\angle A=60°$. If we furthermore know that $4h_a^2=d_a\cdot d_a'$, find angles $\angle B$ and $\angle C$.

2.1.4.5 Trigonometry of Inradius and Exradii

Calculating the Inradius 889. In triangle ABC with angles A,B,C with circumradius R and inradius r, show that

$$r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Calculating the Exradii 890. In triangle ABC with angles A,B,C with circumradius R, assume that r_a,r_b,r_c are the exradii (radii of the excircles) corresponding to vertices A,B,C. Show that

$$\begin{aligned}r_a&=4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},\\ r_b&=4R\sin\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2},\\ r_c&=4R\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}.\end{aligned}$$

Problem 891. In triangle ABC, assume that

$$\frac{1}{r_a}+\frac{1}{r}=\frac{K}{2a}.$$

1. Find $K\sin B\sin C$ in terms of half-angles $A/2$, $B/2$, and $C/2$.
2. Show that
$$K>\frac{8}{\cos\dfrac{B-C}{2}}.$$

Feuerbach Formulas on Exradii 892. In triangle ABC with inradius r, exradii r_a,r_b,r_c, and circumradius R, prove the following identities:

$$\begin{aligned}4R&=r_a+r_b+r_c-r,\\ \frac{1}{r}&=\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}.\end{aligned}$$

Furthermore, if S is the area and p the semiperimeter of triangle ABC with side-lengths a,b,c, prove the **Feuerbach formulas on exradii**:

$$\begin{aligned}r_ar_b+r_br_c+r_cr_a+r(r_a+r_b+r_c)&=ab+bc+ca,\\ r_ar_b+r_br_c+r_cr_a-r(r_a+r_b+r_c)&=\frac{1}{2}\left(a^2+b^2+c^2\right),\\ r(r_ar_b+r_br_c+r_cr_a)&=r_ar_br_c=Sp,\\ r(r_a+r_b+r_c)&=ab+bc+ca-p^2.\end{aligned}$$

Excentral Triangle Side–Lengths 893. In triangle ABC with incenter I, excenters I_A, I_B, I_C (centers of excircles corresponding to vertices A, B, C), and circumradius R, prove the following identities regarding the side–lengths of the **Excentral Triangle** $I_AI_BI_C$:

$$I_BI_C = \frac{a}{\sin\frac{A}{2}} = 4R\cos\frac{A}{2}, \quad I_CI_A = \frac{b}{\sin\frac{B}{2}} = 4R\cos\frac{B}{2}, \quad I_AI_B = \frac{c}{\sin\frac{C}{2}} = 4R\cos\frac{C}{2},$$

and

$$II_A = 4R\sin\frac{A}{2}, \qquad II_B = 4R\sin\frac{B}{2}, \qquad II_C = 4R\sin\frac{C}{2}.$$

Furthermore, prove that I is the orthocenter of the excentral triangle $I_AI_BI_C$, and that the area of this triangle equals

$$S_{\triangle I_AI_BI_C} = 8R^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Excentral Identities 894. In triangle ABC with area S, incenter I and excentral triangle $I_AI_BI_C$, let R be the circumradius. Show that:

$$\begin{aligned}
\frac{IA\cdot IB}{IC} &= 4R\sin^2\frac{C}{2},\\
\frac{I_AA\cdot I_AB}{I_AC} &= 4R\cos^2\frac{C}{2},\\
IA\cdot IB\cdot IC &= 4SR\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2},\\
\frac{IA}{I_AA}+\frac{IB}{I_BB}+\frac{IC}{I_CC} &= 1,\\
\frac{II_A}{I_BI_C} &= \tan\frac{A}{2}.
\end{aligned}$$

Moreover, if the side-lengths of triangle ABC are a, b, c, and $S_{\triangle XYZ}$ denotes the area of triangle XYZ, show that

$$\begin{aligned}
S_{\triangle I_AI_BI_C} &= \frac{abc}{2r},\\
\frac{S_{\triangle I_AI_BI_C}}{S_{\triangle II_AI_C}} &= \frac{r_b}{r}.
\end{aligned}$$

Problem 895. In triangle ABC with excentral triangle $I_AI_BI_C$, call the incenter I and the circumcenter O. Show that

$$\begin{aligned}
IA^2 + I_AA^2 + I_BA^2 + I_CA^2 &= 16R^2,\\
IO^2 + I_AO^2 + I_BO^2 + I_CO^2 &= 12R^2,\\
IA\cdot IB\cdot IC &= 4Rr^2.
\end{aligned}$$

Problem 896. In triangle ABC we know that the perimeter is $2p$ and inradius is r. If the area of triangle ABC is S and the area of the excentral triangle $I_AI_BI_C$ is S_I, show that

$$\frac{S_I}{S} = \frac{4abc}{(a+b-c)(b+c-a)(c+a-b)} = \frac{abc}{2r^2p}.$$

Problem 897. Show that

$$(a+b+c) \cdot II_A \cdot II_B \cdot II_C = 8Rabc.$$

2.1.4.6 Trigonometry of Distance Between Triangle Centers

Distance Between Incenter and Excenters 898. In triangle ABC with excentral triangle $I_AI_BI_C$, exradii r_a, r_b, r_c, inradius r and incenter I, show that

$$II_A^2 = 4R(r_a - r), \qquad II_B^2 = 4R(r_b - r), \qquad II_C^2 = 4R(r_c - r).$$

Distance Between Circumcenter and Incenter 899. In triangle ABC with incenter I and circumcenter O, the excenters are I_A, I_B, I_C. Show that

$$AI = 4R\cos\frac{B}{2}\cos\frac{C}{2} = \frac{r}{\sin\frac{A}{2}},$$

and

$$\begin{aligned} OI_A^2 &= R(R+2r_a), \\ OI^2 &= R(R-2r). \end{aligned}$$

Distance Between Circumcenter and Orthocenter 900. In triangle ABC with circumcenter O, the orthocenter (intersection of heights) is H. Show that

$$\begin{aligned} OH^2 &= R^2(1 - 8\cos A\cos B\cos C) \\ &= 2R^2\left(\frac{3}{2} + \cos 2A + \cos 2B + \cos 2C\right). \end{aligned}$$

Distance Between Incenter and Orthocenter 901. In triangle ABC with incenter I, inradius r and circumradius R, the orthocenter (intersection of heights) is H. Show that

$$IH^2 = 2r^2 - 4R^2\cos A\cos B\cos C.$$

For an extra point, prove that if K is the circumcenter of triangle BHC, then

$$IK^2 = (R+r)^2 + r^2 - \frac{2S^2}{r_br_c}.$$

Problem 902. Prove that the orthocenter (intersection of heights) of triangle ABC lies on the incircle of the triangle if and only if

$$\cos A \cos B \cos C = 4 \sin^2 \frac{A}{2} \sin^2 \frac{B}{2} \sin^2 \frac{C}{2}.$$

Problem 903. Prove that the line connecting the incenter and circumcenter of triangle ABC is tangent to the excircle corresponding to vertex A of triangle if and only if:

$$S\,|r_b - r_c| = r_a r_b r_c \sqrt{1 - \frac{2r}{R}}.$$

Distance Between Vertex and Centroid 904. In triangle ABC, prove that the distance between vertex A and centroid G of triangle is found by:

$$9AG^2 = 2S(4\cot A + \cot B + \cot C).$$

Distance Between Circumcenter and Centroid 905. In triangle ABC, prove that the distance between circumcenter O and centroid G of triangle is given by:

$$9OG^2 = 9R^2 - (a^2 + b^2 + c^2).$$

Distance Between Vertex and Nine-Point Center 906. In triangle ABC, prove that the distance between vertex A and the center of nine-point circle N of triangle is:

$$AN = \frac{R}{2}\sqrt{(1 + 8\cos A \cos B \cos C)}.$$

Distance Between Incenter and Nine-Point Center 907. In triangle ABC, prove that the distance between center of the incircle I and the center of nine-point circle N of triangle is:

$$IN = \frac{R}{2} - r.$$

Identities for Area in Terms of Triangle Radii

Area in Terms of Radii 908. In triangle ABC with angles A, B, C and side-lengths a, b, c, let r and R be the radius of the triangle's incircle and circumcircle, respectively, and assume r_a, r_b, r_c represent the radius of the excircle of triangle ABC with respect to A, B, C, respectively. If S is the area of the triangle, prove the following identities.

- $S = \dfrac{abc}{4R}$,
- $S = \sqrt{r r_a r_b r_c}$,
- $S = \dfrac{a r r_a}{r_a - r}$,
- $S = r r_a \cot \dfrac{A}{2}$,
- $S = \dfrac{a r_b r_c}{r_b + r_c}$,
- $S = \dfrac{(b+c) r r_a}{r_a + r}$,
- $S = \dfrac{r r_a (r_b - r_c)}{b - c}$,
- $S = \dfrac{r r_b \sqrt{r_a + r_b}}{\sqrt{r_b - r}}$,
- $S = r\sqrt{r_a r_b + r_b r_c + r_c r_a}$,
- $S = \dfrac{r_a r_b r_c}{\sqrt{r_a r_b + r_b r_c + r_c r_a}}$,
- $S = \dfrac{r}{2\sqrt{R}}\sqrt{(r_a + r_b)(r_b + r_c)(r_c + r_a)}$,
- $S = r^2 \cot \dfrac{A}{2} \cot \dfrac{B}{2} \cot \dfrac{C}{2}$,
- $S^2 = abcp \sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2}$,
- $S^2 \left(\dfrac{1}{r^2} + \dfrac{1}{r_a^2} + \dfrac{1}{r_b^2} + \dfrac{1}{r_c^2} \right) = a^2 + b^2 + c^2$,
- $4S(\cot A + \cot B + \cot C) = a^2 + b^2 + c^2$,
- $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{a \sin A + b \sin B + c \sin C}{4S}$.

More Identities on Triangle Radii

Trigonometric Identities on Radii 909. In triangle ABC with angles A, B, C and side-lengths a, b, c, let r and R be the radius of the triangle's incircle and circumcircle, respectively, and assume r_a, r_b, r_c represent the radius of the excircle of triangle ABC with respect to A, B, C, respectively. If S and $2p$ are the area and perimeter of triangle ABC, prove the following trigonometric identities:

- $r_c = r \cot \frac{A}{2} \cot \frac{B}{2}$,
- $rr_a = r_b r_c \tan^2 \frac{A}{2}$,
- $\dfrac{1}{c \sin B} + \dfrac{1}{a \sin C} + \dfrac{1}{b \sin A} = \dfrac{1}{r}$,
- $2R(1 - \cos A) = r_a - r$,
- $\dfrac{\cos A}{c \sin B} + \dfrac{\cos B}{a \sin C} + \dfrac{\cos C}{b \sin A} = \dfrac{1}{R}$,
- $r_a(\cos B - \cos C) + r_b(\cos C - \cos A) + r_c(\cos A - \cos B) = 0$,
- $abc + (a - b)(b - c)(c - a) = 4Rr(a \cos C + b \cos A + c \cos B)$,
- $2R \sin A \sin B \sin C = r(\sin A + \sin B + \sin C)$,
- $2(R + r) = a \cot A + b \cot B + c \cot C$,
- $8R^2(1 + \cos A \cos B \cos C) = a^2 + b^2 + c^2$.

Algebraic Identities on Radii 910. Prove the following algebraic identities on exradii, inradius and circumradius:

- $rp^2 = r_a r_b r_c$,
- $4Rrp = abc$,
- $\dfrac{r_c^2}{4R - r_a - r_b} = r_c + \dfrac{r_a r_b}{r_a + r_b}$,
- $r_a r_b r_c = r^3 \cot^2 \frac{A}{2} \cot^2 \frac{B}{2} \cot^2 \frac{C}{2}$,
- $a(rr_a + r_b r_c) = b(rr_b + r_c r_a) = c(rr_c + r_a r_b)$,
- $\left(\dfrac{r_a}{r} - 1\right)\left(\dfrac{r_b}{r} - 1\right)\left(\dfrac{r_c}{r} - 1\right) = \dfrac{4R}{r}$,
- $\dfrac{r_a r_b r_c}{r^3} = \dfrac{(a + b + c)^3}{(a + b - c)(b + c - a)(c + a - b)}$,
- $(b - c)r_b r_c + (c - a)r_c r_a + (a - b)r_a r_b = 0$,
- $4Rr + r^2 = ab + bc + ca - p^2$.

2.1.4.7 Trigonometry of Quadrilaterals

Brahmagupta's (Generalized) Formula & Pitot's Theorem

Brahmagupta's Formula on Area of Cyclic Quadrilaterals 911. If the cyclic quadrilateral $ABCD$ has side-lengths a, b, c, d and perimeter $2p$, then its area S is found by **Brahmagupta's formula**:

$$S = \sqrt{(p-a)(p-b)(p-c)(p-d)}.$$

Moreover,

$$\sin B = \frac{2S}{ab+cd}.$$

Generalized Brahmagupta's Formula 912. If the sum of angles $\angle B$ and $\angle D$ in any quadrilateral $ABCD$ is 2α, prove that the area of the quadrilateral is equal to

$$S = \sqrt{(p-a)(p-b)(p-c)(p-d) - abcd\cos^2\alpha}.$$

Pitot's Theorem on Area of Tangential Quadrilaterals 913. If the tangential quadrilateral $ABCD$ has side-lengths a, b, c, d and perimeter $2p$, and the sum of angles $\angle B$ and $\angle D$ is 2α, then the area S of the quadrilateral is found by **Pitot's formula**:

$$S = \sqrt{abcd\sin^2\alpha}.$$

Calculating Diagonals and Circumradius of Cyclic Quadrilaterals

Trigonometry of Quadrilateral Diagonals 914. In a cyclic quadrilateral $ABCD$ with side-lengths a, b, c, d, the length of the two diagonals AC and BD can be calculated by

$$AC^2 = \frac{(ac+bd)(ad+bc)}{ab+cd},$$
$$BD^2 = \frac{(ab+cd)(ac+bd)}{ad+bc}.$$

Calculating Circumradius of Quadrilateral 915. In a cyclic quadrilateral $ABCD$ of area S with side-lengths a, b, c, d, the circumradius R can be calculated by

$$R = \frac{AC}{2\sin B} = \frac{\sqrt{(ab+cd)(ac+bd)(ad+bc)}}{4S}$$

Problem 916. Prove that the area of any quadrilateral equals half the product of quadrilateral's two diagonals times sine of the angle between the diagonals.

Problem 917. If a, b, c, d are the four sides of a quadrilateral and α is the angle between its diagonals (facing side b), prove that the area of the quadrilateral is equal to

$$\frac{1}{4}\left(a^2+b^2+c^2+d^2\right)\tan\alpha.$$

Problem 918. If the angle between the diagonals of a cyclic quadrilateral is α, show that

$$\sin\alpha = \frac{2\sqrt{(p-a)(p-b)(p-c)(p-d)}}{ac+bd},$$

where a, b, c, d are the side-lengths of the quadrilateral and p is its semiperimeter.

Problem 919. We know that a quadrilateral with side-lengths a, b, c, d is both cyclic and tangential, and that the angle between its diagonals is α. Prove that

$$\cos\alpha = \frac{ac-bd}{ac+bd}.$$

Problem 920. In the tangential quadrilateral $ABCD$ (whose sides are tangent to an inscribed circle) with sides a, b, c, d, if two opposite angles add up to 2α, show that the area of the quadrilateral is equal to $\sin\alpha\cdot\sqrt{abcd}$.

Problem 921. In the tangential quadrilateral $ABCD$ with sides a, b, c, d, if two opposite angles add up to 2α and the angle between the diagonals is Φ, prove that:

$$\tan^2\Phi = \frac{4abcd\sin^2\alpha}{(ac-bd)^2}.$$

Problem 922. We know that the quadrilateral $ABCD$ with side-lengths a, b, c, d has the two special following properties: a) its vertices lie on a circumcircle, and b) its sides are tangent to an inscribed circle. Prove that its area is $\sqrt{abcd}$. Moreover, show that the radius of the inscribed circle inside quadrilateral $ABCD$ equals

$$\frac{2\sqrt{abcd}}{a+b+c+d}.$$

Problem 923. If x and y are the length of diagonals of a tangential quadrilateral $ABCD$ with side-lengths a, b, c, d, show that the area of $ABCD$ equals

$$\frac{1}{2}\sqrt{x^2y^2-(ac-bd)^2}.$$

Problem 924. If x and y are the length of diagonals of any quadrilateral $ABCD$ with side-lengths a, b, c, d, show that the area of $ABCD$ equals

$$\frac{1}{4}\sqrt{4x^2y^2 - (b^2 + d^2 - a^2 - c^2)^2}.$$

Problem 925. If x and y are the length of diagonals of any quadrilateral $ABCD$ with side-lengths a, b, c, d, and θ is the sum of two opposite angles in the quadrilateral, show that

$$x^2y^2 = a^2c^2 + b^2d^2 - 2abcd\cos\theta.$$

Problem 926. Let a, b, c, d be fixed side-lengths of a variable quadrilateral $ABCD$. The area of this quadrilateral when the angle between a and d is 90° is the same as the area of the quadrilateral when the angle between c and d is 90°. Prove that either $a^2+b^2 = c^2+d^2$ or $ab = cd$.

Problem 927. In a parallelogram, a and b are the length of adjacent sides, and α and Φ are acute angles between the two sides and the two diagonals of the parallelogram, respectively. Prove that

$$\frac{a}{b}\sin\Phi = \sin\alpha\cos\Phi \pm \sqrt{1 - \cos^2\alpha\cos^2\Phi}.$$

Problem 928. If p is the semiperimeter of the cyclic quadrilateral $ABCD$ (whose vertices lie on the circumcircle) with side-lengths a, b, c, d, prove that

$$(p-c)(p-d)\tan^2\frac{B}{2} = (p-a)(p-b).$$

Problem 929. If θ is the angle between the diagonals of a cyclic quadrilateral $ABCD$ with side-lengths a, b, c, d, prove that

$$\begin{aligned}(ac+bd)\sin\theta &= 2\sqrt{(p-a)(p-b)(p-c)(p-d)},\\ 2(ac+bd)\cos\theta &= (a^2+c^2) - (b^2+d^2).\end{aligned}$$

Problem 930. For quadrilateral $ABCD$, it is possible to draw two circles, one tangent to the sides AB, BC, CD, and another tangent to the sides CD, DA, AB. If we also know that these two circles touch each other at exactly one point, prove that

$$(a-b+c-d)\sin\frac{A+D}{2} = 4\sqrt{bd\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}}.$$

2.2 Math Olympiad Trigonometry 201: JEE, Roorkee, ISM

Some trigonometric problems from everywhere, mainly Indian problems taken from casual trigonometry books by Agarwal, Pearson, and Rejaul Makshud.

2.2.1 Agarwal's and Pearson's Problems

Problem 931. Show the identity

$$2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1 = 0. \tag{2.63}$$

Also, prove that

$$\sin^8 x - \cos^8 x = (\sin^2 x - \cos^2 x)(1 - 2\sin^2 x \cdot \cos^2 x). \tag{2.64}$$

Problem 932. If we know that

$$\frac{\cos^4 x}{\cos^2 y} + \frac{\sin^4 x}{\sin^2 y} = 1,$$

then prove that

$$\sin^4 x + \sin^4 y = 2\sin^2 x \cdot \sin^2 y. \tag{2.65}$$

Also, find the value of

$$\frac{\cos^4 y}{\cos^2 x} + \frac{\sin^4 y}{\sin^2 x}. \tag{2.66}$$

Problem 933. If $\tan^2\theta = 1-e^2$, and we write $\sec\theta + \tan^3\theta \csc\theta$ in the form of $(2-e^p)^{q/p}$, where p and q are prime numbers, what is $p+q$?

Problem 934. For what values of x and y can the following equation hold true?

$$\sec^2\theta = \frac{4xy}{(x+y)^2}.$$

Problem 935. Solve the equation $\cos\theta + \cos 3\theta - 2\cos 2\theta = 0$.

Problem 936. Solve the equation $\sin m\theta + \sin n\theta = 0$.

Problem 937. Solve the equation $\tan^2\theta + (1-\sqrt{3})\tan\theta - \sqrt{3} = 0$.

Problem 938. Solve the equation $\tan\theta + \tan 2\theta + \tan\theta \tan 2\theta = 1$.

Problem 939. Solve the equation $4\sin x \sin(2x) + \sin(4x) = \sin(3x)$.

Problem 940. Solve the equation $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$.

1989 Roorkee 941. If $4\sin^4 x + \cos^4 x = 1$, then x is

a) $n\pi$ b) $n\pi \pm \sin^{-1}\dfrac{2}{5}$ c) $n\pi + \dfrac{\pi}{6}$ d) NONE

2004 Orissa JEE 942. Which of the following represent the roots of the equation

$$1 - \cos\theta = \sin\theta \cdot \sin\frac{\theta}{2}?$$

a) $k\pi$ b) $2k\pi$ c) $k\dfrac{\pi}{2}$ d) NONE

1989 IIT 943. Which option shows the general solution of the following equation?

$$\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x.$$

a) $n\pi + \dfrac{\pi}{8}$ b) $\dfrac{n\pi}{2} + \dfrac{\pi}{8}$ c) $(-1)^n\dfrac{n\pi}{2} + \dfrac{\pi}{8}$ d) $2n\pi + \cos^{-1}\dfrac{2}{3}$

2003 Orissa JEE 944. The equation $\sin x + \sin y + \sin z = -3$, where x, y, z are real numbers in $[0, 2\pi]$, has

a) One solution b) Two solutions c) Three solutions d) Four solutions

1989 ISM Dhanbad 945. If $2\sin^2 x + \sin^2 2x = 2$, where $-\pi < x < \pi$, then x can be equal to

a) $\pm\dfrac{\pi}{6}$ b) $\pm\dfrac{\pi}{4}$ c) $\dfrac{3\pi}{2}$ d) NONE

1984 Roorkee 946. If $5\cos 2\theta + 2\cos^2\frac{\theta}{2} + 1 = 0$, where $-\pi < \theta < \pi$, then θ can be equal to

a) $\dfrac{\pi}{3}$ b) $\dfrac{\pi}{3}, \cos^{-1}\dfrac{3}{5}$ c) $\cos^{-1}\dfrac{3}{5}$ d) $\dfrac{\pi}{3}, \pi - \cos^{-1}\dfrac{3}{5}$

2004 Karnataka CET 947. If $81^{\sin^2 x} + 81^{\cos^2 x} = 30$, where $0 < x < \pi$, then x is equal to

a) $\dfrac{\pi}{6}$ b) $\dfrac{\pi}{2}$ c) $\dfrac{\pi}{4}$ d) $\dfrac{3\pi}{4}$

Problem 948. If $\cos x + \cos 3x + \cos 5x + \cos 7x = 0$, then x is equal to

a) $n\dfrac{\pi}{4}$ b) $n\dfrac{\pi}{2}$ c) $n\dfrac{\pi}{8}$ d) NONE

Problem 949. If $a\cos x + b\sin x = c$, where a, b, c are non-zero constants, then x is equal to

a) $n\pi + \cos^{-1}\left(\dfrac{c}{\sqrt{a^2+b^2}}\right)$

b) $2n\pi - \tan^{-1}\left(\dfrac{b}{a}\right)$

c) $2n\pi - \tan^{-1}\left(\dfrac{b}{a}\right) \pm \cos^{-1}\left(\dfrac{c}{\sqrt{a^2+b^2}}\right)$

d) $2n\pi + \tan^{-1}\left(\dfrac{b}{a}\right) \pm \cos^{-1}\left(\dfrac{c}{\sqrt{a^2+b^2}}\right)$

2.2.2 Rejaul Makshud's Problems

These problems are taken from Rejaul Makshud's book "Trigonometry Booster with Problems and Solutions," the 2019 edition published by McGraw-Hill. The problems in the book come in multiple levels, and I have chosen the important ones in ascending order of difficulty to be printed here.

Exercises in Law of Sines and Law of Cosines 950. In a triangle ABC with side–lengths a, b, c and angles A, B, C, prove the following identities:

a)

$$c^2 = (a-b)^2\cos^2\left(\frac{C}{2}\right) + (a+b)^2\sin^2\left(\frac{C}{2}\right);$$

b)

$$0 = \frac{a^2\sin(B-C)}{\sin B + \sin C} + \frac{b^2\sin(C-A)}{\sin C + \sin A} + \frac{c^2\sin(A-B)}{\sin A + \sin B};$$

c)

$$0 = \frac{a^2-b^2}{\cos A + \cos B} + \frac{b^2-c^2}{\cos B + \cos C} + \frac{c^2-a^2}{\cos A + \cos B};$$

d)

$$0 = \left(\frac{a^2-b^2}{c^2}\right)\sin 2C + \left(\frac{b^2-c^2}{a^2}\right)\sin 2A + \left(\frac{c^2-a^2}{b^2}\right)\sin 2B;$$

e)

$$27 < \left(\frac{\sin^2 A + \sin A + 1}{\sin A}\right)\left(\frac{\sin^2 B + \sin B + 1}{\sin B}\right)\left(\frac{\sin^2 C + \sin C + 1}{\sin C}\right);$$

f)

$$\frac{a\sin(B-C)}{b^2-c^2} = \frac{b\sin(C-A)}{c^2-a^2} = \frac{c\sin(A-B)}{a^2-b^2};$$

g)

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2+b^2+c^2}{2abc};$$

h)

$$\frac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B} = \frac{a^2+b^2}{a^2+c^2}.$$

1984 IIT JEE 951. Prove that triangle ABC is equilateral if and only if

$$\cos A + \cos B + \cos C = \frac{3}{2}.$$

1984 IIT JEE 952. In triangle ABC with side-lengths a, b, c, it is known that

$$\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}.$$

Prove that

$$\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}.$$

Makshud's Exercises on Progressions in Triangles

Law of Sines and Law of Cosines in Progressions 953. In triangle ABC with side lengths a, b, c and angles A, B, C, prove the following statements. Note that **AP** and **GP** are short for Arithmetic & Geometric Progressions, respectively.

1. a, b, c form an AP if and only if $(p-c)/(p-a) = (b-c)/(a-b)$, where p is the semiperimeter of $\triangle ABC$.
2. a^2, b^2, c^2 form an AP if and only if $\cot A, \cot B, \cot C$ form an AP.
3. If A, B, C form an AP while a, b, c form a GP, then a^2, b^2, c^2 form an AP.
4. If $\cos A + 2\cos B + \cos C = 2$, then a, b, c are in an AP.
5. If
$$c\cos^2\left(\frac{A}{2}\right) + a\cos^2\left(\frac{C}{2}\right) = \frac{3b}{2},$$
then a, b, c are in an AP.
6. If A, B, C form an AP, then
$$2\cos\left(\frac{A-C}{2}\right) = \frac{a+c}{\sqrt{a^2-ac+c^2}}.$$
7. If
$$\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-c)},$$
then a^2, b^2, c^2 form an AP.
8. If a, b, c form an AP, then so do
$$\cos A \cdot \cot\left(\frac{A}{2}\right), \cos B \cdot \cot\left(\frac{B}{2}\right), \cos C \cdot \cot\left(\frac{C}{2}\right).$$
9. If a, b, c form an AP and the difference between the largest and smallest angles in triangle ABC is α, show that the sides are in ratio $(1-x) : 1 : (1+x)$, where
$$x = \sqrt{\frac{1-\cos\alpha}{7-\cos\alpha}}.$$
10. Imagine that a, b, c form an AP. Let θ and Φ be the largest and smallest angles of triangle ABC. Show that $4(1+\cos\theta)(1-\cos\theta) = \cos\theta + \cos\Phi$.

Problem 954. In a triangle ABC with side lengths a, b, c and angles A, B, C, we know that

$$\frac{a^2 - b^2}{a^2 + b^2} = \frac{\sin(A - B)}{\sin(A + B)}.$$

Is this triangle right-angled or isosceles?

Problem 955. In a triangle ABC with side lengths a, b, c and angles A, B, C, we know that

$$\cot \frac{A}{2} = \frac{b + c}{a}.$$

Prove that the triangle is right-angled.

Problem 956. In a triangle ABC with side lengths a, b, c and angles A, B, C, we know that

$$2 \cos B = \frac{a}{c}.$$

Prove that the triangle is isosceles.

Problem 957. In a triangle ABC with side lengths a, b, c and angles A, B, C, if $a \cos A = b \cos B$, then the triangle is right-angled isosceles.

Problem 958. If the angle A in triangle ABC equals $60°$, find the value of

$$\left(1 + \frac{a + b}{c}\right)\left(1 + \frac{c - a}{b}\right).$$

Problem 959. Find the value of angle C in triangle ABC if the three sides a, b, c of the triangle satisfy the equation

$$\frac{1}{a + c} + \frac{1}{b + c} = \frac{3}{a + b + c}.$$

Problem 960. Find the value of angle A (in degrees) of triangle ABC if the three sides a, b, c and three angles A, B, C of the triangle satisfy the equation

$$\frac{2 \cos A}{a} + \frac{2 \cos B}{b} + \frac{2 \cos C}{c} = \frac{1}{bc} + \frac{b}{ca}.$$

Exercises in Law of Tangents

Identities in Laws of Tangents & Cotangents 961. In triangle ABC with side lengths a, b, c and angles A, B, C, let S be the area and r be the inradius. Prove the following identities:

1. $1 - \tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right) = \frac{2c}{a+b+c}$.

2. If we define x, y, z by
$$x = \tan\left(\frac{B-C}{2}\right)\tan\left(\frac{A}{2}\right),$$
$$y = \tan\left(\frac{C-A}{2}\right)\tan\left(\frac{B}{2}\right),$$
$$z = \tan\left(\frac{A-B}{2}\right)\tan\left(\frac{C}{2}\right),$$
show that $x + y + z + xyz = 0$.

3. If we define u, v, w by
$$u = \cot\left(\frac{A}{2}\right), \qquad v = \cot\left(\frac{B}{2}\right), \qquad w = \cot\left(\frac{C}{2}\right),$$
show that
$$u + v + w = u \cdot \left(\frac{b+c+a}{b+c-a}\right),$$
and
$$\frac{u+v+w}{\cot A + \cot B + \cot C} = \frac{(a+b+c)^2}{a^2+b^2+c^2}.$$

4. If D is the midpoint of side BC of triangle ABC with area S, let $\theta = \angle ADB$. Then, $\cot\theta = (b^2 - c^2)/(4S)$.

5. If h_a, h_b, h_c are the length of altitudes drawn from vertices A, B, C of triangle ABC with area S, respectively, then
$$\frac{1}{h_a^2} + \frac{1}{h_b^2} + \frac{1}{h_c^2} = \frac{\cot A + \cot B + \cot C}{S}.$$

6. Let A' be the foot of the altitude drawn from vertex A of triangle ABC, and imagine that ρ_1 and ρ_2 are the inradii of triangles ABA' and ACA', respectively. Show that
$$\frac{\cot B}{\rho_1} + \frac{\cot C}{\rho_2} = (\cot B + \cot C)\left(\frac{1}{r} + \frac{2}{a}\right).$$

1986 IIT JEE 962. In triangle ABC with side-lengths a, b, c and angles A, B, C, it is known that $\cos A \cos B + \sin A \sin B \sin C = 1$. Prove that $a : b : c = 1 : 1 : \sqrt{2}$.

Problem 963. Let θ be the angle such that in triangle ABC,

$$\sin^3 \theta = \sin(A - \theta)\sin(B - \theta)\sin(C - \theta).$$

Prove that $\cot\theta = \cot A + \cot B + \cot C$.

Problem 964. In triangle ABC, prove that

$$\tan^2\left(\frac{A}{2}\right) + \tan^2\left(\frac{B}{2}\right) + \tan^2\left(\frac{C}{2}\right) \geq 1.$$

Problem 965. In triangle ABC, prove that

$$\frac{\cot^2\left(\frac{A}{2}\right)\cot^2\left(\frac{B}{2}\right) + \cot^2\left(\frac{B}{2}\right)\cot^2\left(\frac{C}{2}\right) + \cot^2\left(\frac{C}{2}\right)\cot^2\left(\frac{A}{2}\right)}{\cot^2\left(\frac{A}{2}\right)\cot^2\left(\frac{B}{2}\right)\cot^2\left(\frac{C}{2}\right)} \geq 1.$$

Problem 966. In triangle ABC, prove that

$$\left(\cot\left(\frac{A}{2}\right) + \cot\left(\frac{B}{2}\right)\right)\left(a\sin^2\left(\frac{B}{2}\right) + b\sin^2\left(\frac{A}{2}\right)\right) = c\cot\left(\frac{C}{2}\right).$$

Problem 967. If the three sides a, b, c of triangle ABC form an arithmetic progression, prove that

$$\tan\left(\frac{A}{2}\right) + \tan\left(\frac{C}{2}\right) = \frac{2}{3}\cdot\cot\left(\frac{B}{2}\right).$$

Problem 968. In triangle ABC, if α, β, γ are the three angles that medians make with each other, then

$$\cot\alpha + \cot\beta + \cot\gamma + \cot A + \cot B + \cot C = 0.$$

Problem 969. In a triangle ABC, we know that

$$a\tan A + b\tan B = (a + b)\tan\left(\frac{A + B}{2}\right).$$

Prove that ABC is isosceles.

Problem 970. In triangle ABC with side-lengths a, b, c,

1. If $a = 2b$ and $\cos(A - B) = 4/5$, prove that $\angle C = 90°$.
2. If $3a = b + c$, show that

$$\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right) = 2.$$

Makshud's Exercises on Area & Perimeter

Makshud's Identities on Area and Semiperimeter 971. In triangle ABC with side-lengths a, b, c and angles A, B, C,

1. Prove the following identities on semiperimeter p of the triangle:

$$p = b\cos^2\left(\frac{C}{2}\right) + c\cos^2\left(\frac{B}{2}\right),$$
$$p^2 = bc\cos^2\left(\frac{A}{2}\right) + ca\cos^2\left(\frac{B}{2}\right) + ab\cos^2\left(\frac{C}{2}\right).$$

2. Prove the following identities on area S of the triangle:

$$S = \frac{a^2 - b^2}{2} \cdot \frac{\sin A \sin B}{\sin(A - B)},$$

and the next one containing both S and p:

$$a^2(p-a) + b^2(p-b) + c^2(p-c)$$
$$= 4RS\left(1 - 4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\right).$$

3. Let T be the area of the triangle whose vertices are the intersections of angle bisectors of triangle ABC. Then prove that the ratio T/S, where S is the area of triangle ABC, is equal to

$$\frac{2abc}{(a+b)(b+c)(c+a)}.$$

4. If the sides a, b, c are roots of $x^3 - ux^2 + vx - w = 0$, then show that the area S of triangle ABC equals

$$\frac{1}{4}\sqrt{u(4uv - u^3 - 8w)}.$$

5. Prove that the ratio of the area of the incircle to the area of the triangle is

$$\pi : \tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right).$$

Makshud's Exercises on Circumcircle and Incircle

Makshud's Identities on Circumradius and Inradius 972. In triangle ABC with side-lengths a, b, c and angles A, B, C,

1. Prove the following identities regarding the inradius r and circumradius R:

$$\begin{aligned}
\frac{1}{2rR} &= \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}, \\
\frac{3}{2R} &= \frac{\sin A}{a} + \frac{\sin B}{b} + \frac{\sin C}{c}, \\
\frac{r}{R} &= \frac{a\cos A + b\cos B + c\cos C}{a+b+c}, \\
4R &= \frac{a\cos A + b\cos B + c\cos C}{\sin A \sin B \sin C}, \\
2R &= \frac{a\sec A + b\sec B + c\sec C}{\tan A \tan B \tan C} \\
2(r+R) &= a\cot A + b\cot B + c\cot C, \\
2 + \frac{r}{2R} &= \cos^2\left(\frac{A}{2}\right) + \cos^2\left(\frac{B}{2}\right) + \cos^2\left(\frac{C}{2}\right), \\
4(r+R) &= (b+c)\tan\left(\frac{A}{2}\right) + (c+a)\tan\left(\frac{B}{2}\right) + (a+b)\tan\left(\frac{C}{2}\right).
\end{aligned}$$

2. Let O be the circumcenter of triangle ABC. If R_A, R_B, R_C are circumradii of triangles OBC, OCA, OAB, respectively, then show that

$$\frac{a}{R_A} + \frac{b}{R_B} + \frac{c}{R_C} = \frac{abc}{R^3}.$$

3. If O is the circumcenter of triangle ABC and D, E, F are closest points on the sides of the triangle to O, we have

$$\frac{a}{OD} + \frac{b}{OE} + \frac{c}{OF} = \frac{abc}{4OD \cdot OE \cdot OF}.$$

4. If X, Y, Z are the points where the incircle touches the sides BC, CA, AB of triangle ABC, respectively, we have

$$r^2 = \frac{AX \cdot BY \cdot CZ}{AX + BY + CZ}.$$

Problem 973. In a regular polygon with n sides, each side has a length of $2a$. Let R and r be the circumradius and inradius of the polygon, respectively. Prove that

$$r + R = a\cot\left(\frac{\pi}{2n}\right).$$

Makshud's Identities on Regular Polygons

Inradius, Circumradius, and Area of Regular Polygons 974. In a regular polygon with n sides each of length a, if r_n and R_n denote the inradius and circumradius of the polygon, respectively, then, show that

$$R_n = \frac{\frac{a}{2}}{\sin\left(\frac{a}{n}\right)} \qquad \text{and} \qquad r_n = \frac{\frac{a}{2}}{\tan\left(\frac{a}{n}\right)},$$

and if S_n denotes the area of the polygon, then prove that S_n equals

$$S_n = \frac{a^2 n}{4}\cot\left(\frac{\pi}{n}\right) = \frac{nR_n^2}{2}\sin\left(\frac{2\pi}{n}\right) = nr_n^2\tan\left(\frac{2\pi}{n}\right).$$

Problem 975. If the perimeter of a circle and the perimeter of a regular polygon with n sides are equal, then prove that the ratio of the area of the circle and the area of the polygon is equal to

$$\tan\left(\frac{\pi}{n}\right) : \left(\frac{\pi}{n}\right).$$

Problem 976. If the perimeter of two regular polygons, one with n sides and the other with $2n$ sides, is equal, prove that their areas are in the ratio

$$2\cos\left(\frac{\pi}{n}\right) : \left(1 + \cos\left(\frac{\pi}{n}\right)\right).$$

Problem 977. Let $A_1, A_2, \ldots, A_n$ be the vertices of an n-sided regular polygon such that

$$\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}.$$

Prove that $n = 7$.

2003 IIT JEE 978. If I_n is the area of the n-sided regular polygon inscribed in a circle of unit radius and O_n is the area of the n-sided polygon circumscribing the given circle, prove that

$$I_n = \frac{O_n}{2}\sqrt{1 + \sqrt{1 - \left(\frac{2I_n}{n}\right)^2}}.$$

Makshud's Excentral Identities 979. In triangle ABC of semiperimeter p and area S with side-lengths a, b, c,

1. Prove the following identities regarding the exradii r_a, r_b, r_c and inradius r:

$$\frac{b-c}{r_a}+\frac{c-a}{r_b}+\frac{a-b}{r_c}=0,$$
$$\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}+\frac{1}{r^2}=\frac{a^2+b^2+c^2}{S^2},$$
$$\frac{r_a+r_b}{1+\cos C}=\frac{r_b+r_c}{1+\cos A}=\frac{r_c+r_a}{1+\cos B}.$$

2. Show that

   - $S^2 = r \cdot r_a \cdot r_b \cdot r_c$,
   - $p^2 = r_a r_b + r_b r_c + r_c r_a$,
   - $4r^2 R = (r_a - r)(r_b - r)(r_c - r)$,
   - $4R = r_a + r_b + r_c - r$,
   - $4R\cos C = r_a + r_b - r_c + r$,
   - $16R^2 - (a^2+b^2+c^2) = r^2 + r_a^2 + r_b^2 + r_c^2$.

3. Prove the following more advanced identities

$$\frac{16R}{r^2(a+b+c)^2}=\left(\frac{1}{r}-\frac{1}{r_a}\right)\left(\frac{1}{r}-\frac{1}{r_b}\right)\left(\frac{1}{r}-\frac{1}{r_c}\right),$$
$$\frac{64R^3}{(abc)^2}=\left(\frac{1}{r_a}+\frac{1}{r_b}\right)\left(\frac{1}{r_b}+\frac{1}{r_c}\right)\left(\frac{1}{r_c}+\frac{1}{r_a}\right),$$
$$4R=\frac{(r_a+r_b)(r_b+r_c)(r_c+r_a)}{r_a r_b + r_b r_c + r_c r_a},$$
$$\frac{1}{r}-\frac{1}{2R}=\frac{r_a}{bc}+\frac{r_b}{ca}+\frac{r_c}{ab}.$$

4. If $I_A I_B I_C$ is the excentral triangle of ABC, and u, v, w are the lengths of tangents drawn from excenters I_A, I_B, I_C to the circumcircle of triangle ABC, respectively, prove that

$$\frac{1}{u^2}+\frac{1}{v^2}+\frac{1}{w^2}=\frac{abc}{a+b+c}.$$

5. If the exradii r_a, r_b, r_c are in a Harmonic Progression (**HP**), then the sides a, b, c are in an Arithmetic Progression (**AP**).

6. If the sides a, b, c form both a Geometric Progression (**GP**) as well as an Arithmetic Progression (**AP**), then r_a, r_b, r_c form a **GP**.

Problem 980. In any triangle ABC with side-lengths a, b, c and exradii r_a, r_b, r_c, let R be the circumradius. Prove that

$$\frac{bc}{r_a} + \frac{ca}{r_b} + \frac{ab}{r_c} = 2R\left[\left(\frac{a}{b} + \frac{b}{a}\right)\left(\frac{b}{c} + \frac{c}{b}\right)\left(\frac{c}{a} + \frac{a}{c}\right) - 3\right].$$

Problem 981. In any triangle ABC with exradii r_a, r_b, r_c and inradius r, prove that

$$(r + r_a)\tan\left(\frac{B-C}{2}\right) + (r + r_b)\tan\left(\frac{C-A}{2}\right) + (r + r_c)\tan\left(\frac{A-B}{2}\right) = 0.$$

Problem 982. Let T be the area of the incircle of triangle ABC and assume T_1, T_2, T_3 are the areas of the excircles of triangle ABC. Prove that

$$\frac{1}{\sqrt{T_1}} + \frac{1}{\sqrt{T_2}} + \frac{1}{\sqrt{T_3}} = \frac{1}{T}.$$

2.3 Math Olympiad Trigonometry 301: Series, Graphing & Logarithms

In this section, we investigate more advanced trigonometric topics. We start by some exercises containing trigonometric series and then move on to graphing trigonometric and logarithmic functions.

2.3.1 Trigonometric Series

Problem 983. For any positive integer n, prove that

$$\frac{1}{\sin\alpha\sin 3\alpha}+\frac{1}{\sin 3\alpha\sin 5\alpha}+\cdots+\frac{1}{\sin(2n-1)\alpha\sin(2n+1)\alpha}=\frac{\cot\alpha-\cot(2n+1)\alpha}{\sin 2\alpha}.$$

Problem 984. For any positive integer n, prove that

$$\frac{1}{\sin 2\alpha\sin 3\alpha}+\frac{1}{\sin 3\alpha\sin 4\alpha}+\cdots+\frac{1}{\sin(n+1)\alpha\sin(n+2)\alpha}=\frac{\cot 2\alpha-\cot(n+2)\alpha}{\sin\alpha}.$$

Problem 985. For any positive integer n, prove that

$$\frac{1}{\cos\alpha\cos 3\alpha}+\frac{1}{\cos 3\alpha\cos 5\alpha}+\cdots+\frac{1}{\cos(2n-1)\alpha\cos(2n+1)\alpha}=\frac{\tan(2n+1)\alpha-\tan\alpha}{\sin 2\alpha}.$$

Problem 986. For any positive integer n, prove that

$$\frac{1}{\cos\alpha\cos 2\alpha}+\frac{1}{\cos 2\alpha\cos 3\alpha}+\cdots+\frac{1}{\cos n\alpha\cos(n+1)\alpha}=\frac{\tan(n+1)\alpha-\tan\alpha}{\sin\alpha}.$$

Problem 987. For any positive integer n, prove that

$$\frac{1}{\cos\alpha+\cos 3\alpha}+\frac{1}{\cos\alpha+\cos 5\alpha}+\cdots+\frac{1}{\cos\alpha+\cos(2n+1)\alpha}=\frac{\tan(n+1)\alpha-\tan\alpha}{2\sin\alpha}.$$

Problem 988. For any positive integer n, prove that

$$\frac{1}{\sin\alpha}+\frac{1}{\sin 2\alpha}+\cdots+\frac{1}{\sin 2^{n-1}\alpha}=\cot\left(\frac{\alpha}{2}\right)-\cot\left(2^{n-1}\alpha\right),$$

then conclude

$$\csc(\alpha)+\csc\left(\frac{\alpha}{2}\right)+\cdots+\csc\left(\frac{\alpha}{2^{n-1}}\right)=\cot\left(\frac{\alpha}{2^n}\right)-\cot(\alpha).$$

Problem 989. For any positive integer n, prove that

$$\tan\alpha + \frac{1}{2}\tan\left(\frac{\alpha}{2}\right) + \cdots + \frac{1}{2^{n-1}}\tan\left(\frac{\alpha}{2^{n-1}}\right) = \frac{1}{2^{n-1}}\cot\left(\frac{\alpha}{2^{n-1}}\right) - 2\cot 2\alpha.$$

Sum of Powers of Cosine of Angles Forming an AP

Problem 990. Fix any positive integer n and angles α and β. Define $f_k(\alpha, \beta, n)$ for $k \geq 1$ by

$$f_k(\alpha, \beta, n) = \cos^k(\alpha) + \cos^k(\alpha+\beta) + \cdots + \cos^k(\alpha + (n-1)\beta).$$

Prove that

$$f_1(\alpha, \beta, n) = \frac{\cos\left(\alpha + \dfrac{(n-1)\beta}{2}\right)\sin\left(\dfrac{n\beta}{2}\right)}{\sin\left(\dfrac{\beta}{2}\right)},$$

$$f_2(\alpha, \beta, n) = \frac{1}{2}\left(n + \frac{\cos[2\alpha + (n-1)\beta]\sin(n\beta)}{\sin\beta}\right),$$

and

$$f_3(\alpha, \beta, n) = \frac{3\cos\left(\alpha + \dfrac{(n-1)\beta}{2}\right)\sin\left(\dfrac{n\beta}{2}\right)}{4\sin\left(\dfrac{\beta}{2}\right)} + \frac{\cos 3\left(\alpha + \dfrac{(n-1)\beta}{2}\right)\sin\left(\dfrac{3n\beta}{2}\right)}{4\sin\left(\dfrac{3\beta}{2}\right)}.$$

Problem 991. For all natural number n, prove that

$$\cos\left(\frac{\alpha}{3}\right) + \cos\left(\frac{4\alpha}{3}\right) + \cdots + \cos\left(\frac{(3n-2)\alpha}{3}\right) = \frac{\sin\left(\dfrac{(3n-1)\alpha}{6}\right)\sin\left(\dfrac{n\alpha}{2}\right)}{\sin\left(\dfrac{\alpha}{2}\right)}.$$

Problem 992. For all natural numbers n, prove that

$$\cos\alpha + \cos\left(\alpha + \frac{\pi}{n}\right) + \cdots + \cos\left(\alpha + \frac{(n-1)\pi}{n}\right) = \frac{\cos\left(\alpha + \dfrac{(n-1)\pi}{2n}\right)}{\sin\left(\dfrac{\pi}{2n}\right)}.$$

Problem 993. Prove that for every positive integer n,

$$\cos 3\alpha - \cos\left(3\alpha - \frac{\pi}{n}\right) + \cos\left(3\alpha - \frac{2\pi}{n}\right) - \cdots + (-1)^{n-1}\cos\left(3\alpha - \frac{(n-1)\pi}{n}\right)$$

is equal to

$$\frac{\cos\left(3\alpha + \dfrac{(n-1)^2\pi}{2n}\right)\sin\left(\dfrac{(n-1)\pi}{2}\right)}{\sin\left(\dfrac{(n-1)\pi}{2n}\right)}.$$

Problem 994. Prove that for every positive integer n,

$$\cos^4\alpha + \cos^4 3\alpha + \cos^4 5\alpha + \cdots + \cos^4(2n-1)\alpha$$

is equal to

$$\frac{1}{8}\left(3n + \frac{4\cos(2n\alpha)\sin(2n\alpha)}{\sin 2\alpha} + \frac{\cos(4n\alpha)\sin(4n\alpha)}{\sin 4\alpha}\right).$$

(Alternating) Sum of Sine of Angles in an AP

Sum of Sines of Angles in an AP 995. Fix any positive integer n and angles α and β. Define $g(\alpha,\beta,n)$ and $h(\alpha,\beta,n)$ by

$$g(\alpha,\beta,n) = \sin(\alpha) + \sin(\alpha+\beta) + \cdots + \sin(\alpha+(n-1)\beta),$$
$$h(\alpha,\beta,n) = \sin(\alpha) - \sin(\alpha+\beta) + \cdots + (-1)^{n-1}\sin(\alpha+(n-1)\beta).$$

Show that

$$g(\alpha,\beta,n) = \frac{\sin\left(\alpha + \dfrac{(n-1)\beta}{2}\right)\sin\left(\dfrac{n\beta}{2}\right)}{\sin\left(\dfrac{\beta}{2}\right)},$$

and

$$h(\alpha,\beta,n) = \frac{\sin\left(\alpha + \dfrac{(n-1)(\beta+\pi)}{2}\right)\sin\left(\dfrac{n(\beta+\pi)}{2}\right)}{\cos\left(\dfrac{\beta}{2}\right)}.$$

Problem 996. For any positive integer n and non-zero angle α, prove that

$$\sin^3\alpha + \sin^3 3\alpha + \cdots + \sin^3(2n-1)\alpha = \frac{1}{4}\left(\frac{3\sin^2(n\alpha)}{\sin\alpha} - \frac{\sin^2(3n\alpha)}{\sin 3\alpha}\right).$$

Problem 997. Prove for all integer $n \geq 1$ that

$$\sin\alpha + \sin\left(\alpha + \frac{2\pi}{n}\right) + \sin\left(\alpha + \frac{4\pi}{n}\right) + \cdots + \sin\left(\alpha + \frac{2(n-1)\pi}{n}\right) = 0.$$

Problem 998. For any positive integer n, prove the following identities:

$$\sin 2\alpha + \sin 5\alpha + \sin 8\alpha + \cdots + \sin(3n-1)\alpha = \frac{\sin\left(\dfrac{(3n+1)\alpha}{2}\right)\sin\left(\dfrac{3n\alpha}{2}\right)}{\sin\left(\dfrac{3\alpha}{2}\right)},$$

and

$$\cos\alpha + \cos 3\alpha + \cos 5\alpha + \cdots + \cos(2n-1)\alpha = \frac{\cos(n\alpha)\sin(n\alpha)}{\sin\alpha}.$$

Problem 999. For a positive integer n and an angle α, define $S(\alpha, n)$ and $C(\alpha, n)$ by

$$\begin{aligned} S(\alpha,n) &= \sin\ \alpha - \ \sin 2\alpha + \sin 3\alpha - \sin 4\alpha + \cdots + (-1)^{n-1}\sin(n\alpha), \\ C(\alpha,n) &= \cos 2\alpha - \cos 4\alpha + \cos 6\alpha - \cos 8\alpha + \cdots + (-1)^{n-1}\cos(2n\alpha). \end{aligned}$$

Prove that

$$\begin{aligned} S(\alpha,n) &= \frac{\sin\left(\dfrac{(n+1)\alpha}{2} + \dfrac{(n-1)\pi}{2}\right)\sin\left(\dfrac{n(\alpha+\pi)}{2}\right)}{\sin\left(\dfrac{\alpha+\pi}{2}\right)}, \\ C(\alpha,n) &= \frac{\cos\left((n+1)\alpha + \dfrac{(n-1)\pi}{2}\right)\sin\left(\dfrac{n(2\alpha+\pi)}{2}\right)}{\sin\left(\dfrac{2\alpha+\pi}{2}\right)}. \end{aligned}$$

Problem 1000. For all positive integers n, show that

$$\sin^4\alpha + \sin^4 2\alpha + \sin^4 3\alpha + \cdots + \sin^4 n\alpha$$

is equal to

$$\frac{1}{8}\left(3n - \frac{4\cos(n+1)\alpha\sin n\alpha}{\sin\alpha} + \frac{\cos(2n+2)\alpha\sin 2n\alpha}{\sin\alpha}\right).$$

Problem 1001. Prove that

$$\frac{\cos 2\alpha}{\sin 3\alpha} + \frac{\cos 6\alpha}{\sin 9\alpha} + \cdots + \frac{\cos\left(2\cdot 3^{n-1}\alpha\right)}{\sin\left(3^n\alpha\right)} = \frac{1}{2}\left(\frac{1}{\sin\alpha} - \frac{1}{\sin(3^n\alpha)}\right).$$

Problem 1002. Prove that

$$(2\cos\alpha - 1)(2\cos 2\alpha - 1)(2\cos 3\alpha - 1)\cdots(2\cos(n-1)\alpha - 1) = \frac{2\cos 2^n\alpha + 1}{2\cos\alpha + 1}.$$

Problem 1003. Prove that for all integers $n > 1$,

$$\sin\left(\frac{\pi}{n} - \alpha\right)\sin\left(\frac{2\pi}{n} - \alpha\right)\cdots\sin\left(\frac{(n-1)\pi}{n} - \alpha\right) = \frac{\sin n\alpha}{2^{n-1}\sin\alpha}$$

Problem 1004. Let $x_k = \sin(k\pi/n)$ for $k = 1, 2, \ldots, n-1$, where n is a given positive integer. Prove the following identities:

$$\begin{aligned}\cot\left(\frac{\pi}{2n}\right) &= x_1 + x_2 + \cdots + x_{n-1},\\ n &= (2x_1)\cdot(2x_2)\cdots(2x_{n-1}),\\ n^{n/2} &= (2x_1)\cdot(2x_2)^2\cdot(2x_3)^3\cdots(2x_{n-1})^{n-1}.\end{aligned}$$

2.3.2 Advanced Trigonometry: Limits, Infinite Sums, & Roots

Begin with some tough trigonometric identities to prove. These will be much more difficult to prove if you do not know the answer (the right side of identity's equation) prior to coming up with an idea on how to simplify the sum.

2.3.2.1 Advanced Trigonometric Identities: Sums

Problem 1005. Prove for all integers $n > 1$ that

$$\sin^4\alpha + \frac{\sin^4 2\alpha}{4} + \frac{\sin^4 4\alpha}{4^2} + \cdots + \frac{\sin^4 2^{n-1}\alpha}{4^{n-1}} = \sin^2\alpha - \frac{\sin^2 2^n\alpha}{4^n}.$$

Problem 1006. Prove for all integers $n \geq 1$ that

$$\sin^2\alpha + \sin^2 2\alpha + \sin^2 3\alpha + \cdots + \sin^2 n\alpha = \frac{n\sin\alpha - \sin n\alpha\cos(n+1)\alpha}{2\sin\alpha}.$$

Problem 1007. Prove for all integers $n \geq 1$ that

$$\begin{aligned}\cos\alpha + 2\cos 2\alpha + 3\cos 3\alpha + \cdots + n\cos n\alpha &= \frac{(n+1)\cos n\alpha - n\cos(n+1)\alpha - 1}{4\sin^2\left(\frac{\alpha}{2}\right)}\\ &= \frac{n\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{(2n+1)\alpha}{2}\right) - \sin^2\left(\frac{n\alpha}{2}\right)}{2\sin^2\left(\frac{\alpha}{2}\right)}.\end{aligned}$$

Problem 1008. For any positive integer n, prove that

$$\frac{\sin\alpha}{2\cos\alpha - 1} + \frac{2\sin 2\alpha}{2\cos 2\alpha - 1} + \frac{4\sin 4\alpha}{2\cos 4\alpha - 1} + \cdots + \frac{2^{2n-1}\sin 2^{2n-1}\alpha}{2\cos 2^{2n-1}\alpha - 1}$$

is equal to

$$\frac{2^{2n}\sin 2^{2n}\alpha}{2\cos 2^{2n}\alpha - 1} - \frac{\sin\alpha}{2\cos\alpha + 1}.$$

Problem 1009. In order to be able to solve Problem 1024, prove the following identity for all positive integers n:

$$\tan^2\left(\frac{\alpha}{2}\right)\tan\alpha + 2\tan^2\left(\frac{\alpha}{4}\right)\tan\left(\frac{\alpha}{2}\right) + \cdots + 2^{n-1}\tan^2\left(\frac{\alpha}{2^n}\right)\tan\left(\frac{\alpha}{2^{n-1}}\right)$$
$$= \tan\alpha - 2^n\tan\left(\frac{\alpha}{2^n}\right).$$

Compare with

$$\frac{\tan\left(\frac{\alpha}{2}\right)}{\cos\alpha} + \frac{\tan\left(\frac{\alpha}{4}\right)}{\cos\left(\frac{\alpha}{2}\right)} + \cdots + \frac{\tan\left(\frac{\alpha}{2^n}\right)}{\cos\left(\frac{\alpha}{2^{n-1}}\right)} = \tan\alpha - \tan\left(\frac{\alpha}{2^n}\right).$$

Problem 1010. In order to be able to solve Problem 1025, prove the identity

$$\frac{1}{4\cos^2\left(\frac{\alpha}{2}\right)} + \frac{1}{4^2\cos^2\left(\frac{\alpha}{2^2}\right)} + \cdots + \frac{1}{4^n\cos^2\left(\frac{\alpha}{2^n}\right)} = \frac{1}{\sin^2\alpha} - \frac{1}{2^n\sin^2\left(\frac{\alpha}{2^n}\right)},$$

for all positive integers n.

Problem 1011. Prove that

$$\sin^3\left(\frac{\alpha}{3}\right) + 3\sin^3\left(\frac{\alpha}{9}\right) + \cdots + 3^{n-1}\sin^3\left(\frac{\alpha}{3^n}\right) = \frac{1}{4}\left(3^n\sin\left(\frac{\alpha}{3^n}\right) - \sin\alpha\right).$$

Problem 1012. Prove that the expression

$$\frac{1}{\sin\alpha} + \frac{1}{\sin\alpha + \sin 2\alpha} + \frac{1}{\sin\alpha + \sin 2\alpha + \sin 3\alpha} + \cdots + \frac{1}{\sin\alpha + \sin 2\alpha + \cdots + \sin n\alpha}$$

for any positive integer n and non-zero angle α, is equal to

$$\cot\left(\frac{\alpha}{2}\right) - \cot\left(\frac{(n+1)\alpha}{2}\right).$$

Problem 1013. For all integers $n > 1$, prove that

$$\tan\alpha\tan 2\alpha + \tan 2\alpha\tan 3\alpha + \cdots + \tan(n-1)\alpha\tan n\alpha = \frac{\tan n\alpha}{\tan\alpha} - n.$$

Problem 1014. Find the sum of sines of all integer angles in degrees from $1°$ to $90°$. In trigonometric words, prove that

$$\sin 1° + \sin 2° + \cdots + \sin 90° = \frac{\sqrt{2}\sin 45.5°}{\cos 0.5°}.$$

Problem 1015. For all positive integers n, show that

$$\sin^2\alpha\sin 2\alpha + \frac{\sin^2 2\alpha\sin 4\alpha}{2} + \cdots + \frac{\sin^2 2^{n-1}\alpha\sin 2^n\alpha}{2^{n-1}} = \frac{\sin 2\alpha}{2} - \frac{\sin^2 2^{n+1}\alpha}{2^{n+1}}.$$

Problem 1016. Show that for any $n \geq 1$,

$$\frac{\sin\alpha}{\cos 3\alpha} + \frac{\sin 3\alpha}{\cos 9\alpha} + \cdots + \frac{\sin 3^{n-1}\alpha}{\cos 3^n\alpha} = \frac{\tan 3^n\alpha - \tan\alpha}{2}.$$

Problem 1017. Prove for all positive integers n that

$$\sin\alpha - \cos 2\alpha + \sin 3\alpha - \cos 4\alpha + \cdots + \sin(2n-1)\alpha - \cos 2n\alpha = \frac{(\sin n\alpha - \cos(n+1)\alpha)\sin n\alpha}{\sin\alpha}.$$

2.3.2.2 Advanced Trigonometric Identities: Products

Problem 1018. Show that for all positive integers n,

$$\left[1 + \frac{1}{\cos\alpha}\right] \cdot \left[1 + \frac{1}{\cos 2\alpha}\right] \cdots \left[1 + \frac{1}{\cos 2^{n-1}\alpha}\right] = \frac{\tan 2^{n-1}\alpha}{\tan\left(\frac{\alpha}{2}\right)}.$$

Problem 1019. Show that for all positive integers n,

$$\cos\frac{\pi}{2n+1} \cdot \cos\frac{2\pi}{2n+1} \cdots \cos\frac{n\pi}{2n+1} = \frac{1}{2^n}$$

Problem 1020. Find the product of half-angled cosines up to n terms:

$$\cos\alpha \cdot \cos\left(\frac{\alpha}{2}\right) \cdot \cos\left(\frac{\alpha}{4}\right) \cdots \cos\left(\frac{\alpha}{2^{n-1}}\right) = \frac{\sin 2\alpha}{2^n \sin\left(\frac{\alpha}{2^{n-1}}\right)}.$$

Problem 1021. Prove that

$$[2\cos(\alpha) - 1] \cdot \left[2\cos\left(\frac{\alpha}{2}\right) - 1\right] \cdots \left[2\cos\left(\frac{\alpha}{2^n}\right) - 1\right] = \frac{2\cos 2\alpha + 1}{2\cos\left(\frac{\alpha}{2^n}\right) + 1}.$$

Problem 1022. For any two angles α and β, and $n \geq 1$, define the angles α_n and β_n recursively as half of the angles α_{n-1} and β_{n-1}, initiated with $\alpha_0 = \alpha$ and $\beta_0 = \beta$. Find the product of the following n sums: the sum of cosines of α_0 and β_0, the sum of cosines of α_1 and β_1, and so on up to the sum of cosines of α_n and β_n. In other words, show that

$$[\cos\alpha + \cos\beta] \cdot \left[\cos\left(\frac{\alpha}{2}\right) + \cos\left(\frac{\beta}{2}\right)\right] \cdots \left[\cos\left(\frac{\alpha}{2^{n-1}}\right) + \cos\left(\frac{\beta}{2^{n-1}}\right)\right] = \frac{\cos 2\alpha - \cos 2\beta}{2^n\left[\cos\left(\frac{\alpha}{2^{n-1}}\right) - \cos\left(\frac{\beta}{2^{n-1}}\right)\right]}.$$

Problem 1023. Prove, for integers $n \geq 1$, that

$$\left[1 - \tan^2\left(\frac{\alpha}{2}\right)\right] \cdot \left[1 - \tan^2\left(\frac{\alpha}{4}\right)\right] \cdots \left[1 - \tan^2\left(\frac{\alpha}{2^n}\right)\right] = 2^n \tan\left(\frac{\alpha}{2^n}\right)\cot\alpha.$$

2.3.2.3 Trigonometric Limits and Infinite Sums

Problem 1024. Prove that the limit of the sum

$$\tan^2\left(\frac{\alpha}{2}\right)\tan\alpha + 2\tan^2\left(\frac{\alpha}{4}\right)\tan\left(\frac{\alpha}{2}\right) + \cdots + 2^{n-1}\tan^2\left(\frac{\alpha}{2^n}\right)\tan\left(\frac{\alpha}{2^{n-1}}\right) + \cdots$$

when n approaches infinity, equals $-\alpha + \tan\alpha$. You can use Problem 1009 as a hint.

Problem 1025. Using Problem 1010 as a hint, prove that the limit of the sum

$$\frac{1}{4\cos^2\left(\frac{\alpha}{2}\right)} + \frac{1}{4^2\cos^2\left(\frac{\alpha}{2^2}\right)} + \cdots + \frac{1}{4^n\cos^2\left(\frac{\alpha}{2^n}\right)} + \cdots$$

when n approaches infinity, equals

$$\frac{1}{\sin^2\alpha} - \frac{1}{\alpha^2}.$$

Problem 1026. First, find the function $f(x)$ such that $f(x) \equiv f(x-1)+x^3$, and second, find the limit of the following sum when x approaches zero:

$$S_1 = \cos^3\alpha + \left(2\cos\left(\frac{\alpha}{2}\right)\right)^3 + \left(3\cos\left(\frac{\alpha}{3}\right)\right)^3 + \cdots + \left(n\cos\left(\frac{\alpha}{n}\right)\right)^3.$$

Third, if we define

$$S_2 = \left[\cos\alpha + 2\cos\left(\frac{\alpha}{2}\right) + 3\cos\left(\frac{\alpha}{3}\right) + \cdots + n\cos\left(\frac{\alpha}{n}\right)\right]^2,$$

Prove that the limit of S_2 when α approaches zero is the same as limit of S_1 when α approaches zero.

2.3.2.4 Trigonometric Roots of Equations

Problem 1027. Let α be a real number. Prove that the roots of the equation

$$x^2\tan^2\alpha - 2(2+\tan^2\alpha)x + \tan^2\alpha = 0$$

are $\tan^2(\alpha/2)$ and $\cot^2(\alpha/2)$.

Problem 1028. Prove that one of the roots of the equation

$$x^3 - 6x^2 + 9x - 3 = 0$$

is equal to $2(1-\sin(\pi/18))$, and find the other roots.

Problem 1029. Prove that one of the roots of the equation

$$x^4 - 10x^2 + 5 = 0$$

is equal to $\tan 36°$, and find the other roots.

Problem 1030. Let $x = 2\cos\alpha$. If we know that

$$\frac{1+\cos 9\alpha}{1+\cos\alpha} = (x^4 - x^3 - 3x^2 + 2x + 1)^2,$$

find the roots of the equation $x^4 - x^3 - 3x^2 + 2x + 1 = 0$.

2.3.2.5 Various Trigonometric Inequalities

Problem 1031. If $0 < \alpha \leq 60°$, prove that

$$\frac{1}{\sin(60° + \alpha)} + \frac{1}{\sin(60° - \alpha)} \geq \frac{4\sqrt{3}}{3}.$$

Problem 1032. If $0 < \alpha < 90°$, prove that $(\sin \alpha)/\alpha > \sqrt{\cos \alpha}$.

Problem 1033. If $0 < \alpha < 90°$, prove that

$$\left(1 + \frac{1}{\sin \alpha}\right)\left(1 + \frac{1}{\cos \alpha}\right) \geq 3 + 2\sqrt{2}.$$

Problem 1034. If α, β, γ, are angles with $\alpha + \beta + \gamma = 90°$, prove that

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma + 3 \sin \alpha \sin \beta \sin \gamma \leq \frac{9}{8}.$$

Problem 1035. For all positive integers n, show that

$$\left(\frac{1}{\cos^{2n} \alpha} - 1\right)\left(\frac{1}{\sin^{2n} \alpha} - 1\right) \geq (1 + 2 + \cdots + n)^2.$$

Problem 1036. If the sum of angles A and C in a convex quadrilateral $ABCD$ exceeds $180°$, prove that

$$\frac{AC}{BD} < \frac{AB \cdot DA + BC \cdot CD}{AB \cdot BC + CD \cdot DA}.$$

2.3.2.6 Triangle Trigonometric Inequalities

Problem 1037. If $A + B + C = 180°$, where A, B, C are acute angles, prove that

$$\frac{1}{\cos A} + \frac{1}{\cos B} + \frac{1}{\cos C} \geq 6.$$

Problem 1038. In a triangle ABC with acute angles, prove that

$$\tan^2 A + \tan^2 B + \tan^2 C \geq 9.$$

Problem 1039. Let ABC be an acute-angled triangle. Prove that for all non-negative integers n,

$$\tan^n A + \tan^n B + \tan^n C \geq 3 + \frac{3n}{2}.$$

Problem 1040. In any triangle ABC, prove that

$$\cot^2\left(\frac{A}{2}\right)+\cot^2\left(\frac{B}{2}\right)+\cot^2\left(\frac{C}{2}\right)\geq 9.$$

Problem 1041. In any triangle ABC, prove that

$$\frac{1}{\sin^2\left(\frac{A}{2}\right)}+\frac{1}{\sin^2\left(\frac{B}{2}\right)}+\frac{1}{\sin^2\left(\frac{C}{2}\right)}\geq 12.$$

Problem 1042. In a triangle ABC with acute angles, prove that

$$\sin A+\sin B+\sin C+\tan A+\tan B+\tan C>2\pi.$$

Problem 1043. Prove that for any triangle ABC with side-lengths a, b, c, area S, and circumradius R, we have

$$a\sin A+b\sin B+c\sin C\geq \frac{2S\sqrt{3}}{R}.$$

Problem 1044. In a right triangle ABC, the angle between the medians drawn from non-right angles is α, prove that $\cos\alpha \geq 4/5$.

Problem 1045. Prove that if the angles of triangle ABC satisfy the equation $\sin^2 A+\sin^2 B=5\sin^2 C$, then $\sin C\leq 3/5$.

Problem 1046. There is a circle inscribed in a right triangle with hypotenuse of length c that touches the other two sides of the triangle at points M and N. Prove that

$$MN\leq \frac{2c\sqrt{3}}{9}.$$

Problem 1047. The circumcircle of triangle ABC has radius R, and the three circles touching the circumcircle and two sides of ABC have radii r_1, r_2, r_3. Prove that $4r\leq r_1+r_2+r_3\leq 2R$.

Problem 1048. Let I be the incenter of triangle ABC and r its inradius. Three line segments are formed by drawing three parallel lines to the three sides of triangle ABC from I. Prove that the sum of squares of lengths of these segments is not smaller than $16r^2$.

Problem 1049. In triangle ABC with side-lengths a, b, c, semiperimeter p and circumradius R, show that

$$\frac{1}{(p-a)^2}+\frac{1}{(p-b)^2}+\frac{1}{(p-c)^2}\geq \frac{4}{R^2}.$$

Problem 1050. In triangle ABC with side-lengths a, b, c, semiperimeter p, inradius r, and circumradius R, show that $5R - r \geq p\sqrt{3}$.

Problem 1051. Let O be the circumcenter of triangle ABC with circumradius R, and let R_1, R_2, R_3 be the radii of circumcircles of BOC, COA, AOB, respectively. Prove that $R_1^2 + R_2^2 + R_3^2 \geq 3R^2$.

Problem 1052. If a, b, c are side-lengths of a triangle and x, y, z are the sides of its orthic triangle, prove that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \geq \frac{3}{4}.$$

Problem 1053. If A, B, C are angles of a triangle, prove that the two triples of segments of length

$$\left\{\cos\left(\frac{A}{2}\right), \cos\left(\frac{B}{2}\right), \cos\left(\frac{C}{2}\right)\right\} \qquad \text{and} \qquad \left\{\cos^2\left(\frac{A}{2}\right), \cos^2\left(\frac{B}{2}\right), \cos^2\left(\frac{C}{2}\right)\right\},$$

each can be side-lengths of a triangle.

Problem 1054. In triangle ABC, the angle A, the perimeter $2p$, and the inradius r are known. Prove that we can solve the triangle and find its other two angles B and C using the identity

$$\cos\left(\frac{B-C}{2}\right) = \frac{\left[p\tan\left(\frac{A}{2}\right) + r\right]\sin\left(\frac{A}{2}\right)}{p\tan\left(\frac{A}{2}\right) - r},$$

and conclude that the following inequality must hold in the triangle:

$$p\tan\left(\frac{A}{2}\right) \cdot \tan^2\left(\frac{\pi}{4} - \frac{A}{4}\right) \geq r.$$

2.3.3 Logarithms, Graphing and Nonroutine Problems

The seven problems in this section are taken from "Nonroutine problems in algebra, geometry, and trigonometry (McGraw-Hill, 1965)."

Problem 1055. Graph the following functions:

a) $f(x) = x\sin x$,

b) $g(x) = x\sin(1/x)$,

c) $h(x) = x\log x$,

d) $j(x) = (\log x)/x$,

e) $k(x) = 2^{1/x}$.

Problem 1056. How many solutions are there to $\log_{10} x = \sin x$?

Problem 1057. Prove that $|\sin kx| \leq k|\sin x|$ for all reals x and all positive integers k.

Problem 1058. Prove that for $\alpha \neq n\pi$, where n is an integer,

$$\cos\alpha \cdot \cos 2\alpha \cdot \cos 4\alpha \cdot \cos 8\alpha \cdots \cos 2^n\alpha = \frac{\sin^{2n+1}\alpha}{2^{n+1}\sin\alpha}.$$

Problem 1059. Given

$$\sin\alpha = \frac{a^2-b^2}{a^2+b^2} \qquad \text{and} \qquad \cos\alpha = \frac{2ab}{a^2+b^2},$$

find $\tan\alpha/2$.

Problem 1060. Given $0 < \alpha < \beta < \pi/2$, show that

$$-\alpha + \tan\alpha < -\beta + \tan\beta.$$

Problem 1061. Find all real numbers x such that

$$\log(\sqrt{3}\sin x) + \log(-1+\sqrt{3}\sin x) = \log 6.$$

2.4 Math Olympiad Trigonometry 401 and Beyond: 50 Unsolved Math Olympiad Trigonometry Problems

Problem 1062. Prove that:

$$\cos\frac{2\pi}{13}+\cos\frac{6\pi}{13}+\cos\frac{8\pi}{13}=\frac{\sqrt{13}-1}{4}.$$

Problem 1063. Prove that

$$x=2\left(\cos\frac{4\pi}{19}+\cos\frac{6\pi}{19}+\cos\frac{10\pi}{19}\right)$$

is a root of the equation:

$$\sqrt{4+\sqrt{4+\sqrt{4-x}}}=x.$$

Problem 1064. Prove that

$$\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos 8\theta}}}=\cos\theta.$$

Problem 1065. Prove that

$$\sin^4\left(\frac{\pi}{8}\right)+\sin^4\left(\frac{3\pi}{8}\right)+\sin^4\left(\frac{5\pi}{8}\right)+\sin^4\left(\frac{7\pi}{8}\right)=\frac{3}{2}.$$

Problem 1066. Prove that

$$\cos x\cdot\cos\left(\frac{x}{2}\right)\cdot\cos\left(\frac{x}{4}\right)\cdot\cos\left(\frac{x}{8}\right)=\frac{\sin 2x}{16\sin\left(\frac{x}{8}\right)}.$$

Problem 1067. Prove that

$$64\cdot\sin 10^\circ\cdot\sin 20^\circ\cdot\sin 30^\circ\cdot\sin 40^\circ\cdot\sin 50^\circ\cdot\sin 60^\circ\cdot\sin 70^\circ\cdot\sin 80^\circ\cdot\sin 90^\circ=\frac{3}{4}.$$

Problem 1068. Find x if

$$\sin x=\tan 12^\circ\cdot\tan 48^\circ\cdot\tan 54^\circ\cdot\tan 72^\circ\cdot$$

Problem 1069. Solve the following equations in $\mathbb{R}$:

- $\sin 9x + \sin 5x + 2\sin^2 x = 1$.
- $\cos 5x \cdot \cos 3x - \sin 3x \cdot \sin x = \cos 2x$.
- $\cos 5x + \cos 3x + \sin 5x + \sin 3x = 2 \cdot \cos\left(\frac{\pi}{4} - 4x\right)$.
- $\sin x + \cos x - \sin x \cdot \cos x = -1$.
- $\sin 2x - \sqrt{3}\cos 2x = 2$.

Problem 1070. Prove the following equations:

- $\sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right) - \sin\left(\frac{6\pi}{7}\right) = 4\sin\left(\frac{\pi}{7}\right) \cdot \sin\left(\frac{3\pi}{7}\right) \cdot \sin\left(\frac{5\pi}{7}\right)$,
- $\cos\left(\frac{\pi}{13}\right) + \cos\left(\frac{3\pi}{13}\right) + \cos\left(\frac{5\pi}{13}\right) + \cos\left(\frac{7\pi}{13}\right) + \cos\left(\frac{9\pi}{13}\right) + \cos\left(\frac{11\pi}{13}\right) = \frac{1}{2}$,
- $\cos\left(\frac{\pi}{2k+1}\right) + \cos\left(\frac{3\pi}{2k+1}\right) + \cdots + \cos\left(\frac{(2k-1)\pi}{2k+1}\right) = \frac{1}{2}$, for all natural k,
- $\sin\left(\frac{\pi}{7}\right) + \sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{3\pi}{7}\right) = \frac{1}{4} \cdot \cot\left(\frac{\pi}{4}\right)$.

Problem 1071. Show that

$$\cos\frac{\pi}{n} + \cos\frac{2\pi}{n} + \cdots + \cos\frac{n\pi}{n} = -1.$$

Problem 1072. Show that

$$\cos a + \cos 3a + \cos 5a + \cdots + \cos(2n-1)a = \frac{\sin 2na}{2\sin a}.$$

Problem 1073. Show that

$$\sin a + \sin 3a + \sin 5a + \cdots + \sin(2n-1)a = \frac{\sin^2 na}{\sin a}.$$

Problem 1074. Calculate

$$(\tan 1^\circ)^2 + (\tan 2^\circ)^2 + (\tan 3^\circ)^2 + \cdots + (\tan 89^\circ)^2.$$

Problem 1075. Prove that

$$\cot^2\frac{\pi}{7} + \cot^2\frac{2\pi}{7} + \cot^2\frac{3\pi}{7} = 5.$$

Problem 1076. Show that

$$\tan\frac{\pi}{7}\tan\frac{2\pi}{7}\tan\frac{3\pi}{7}=\sqrt{7}.$$

Problem 1077. $\cos\left(\frac{2\pi}{7}\right)$, $\cos\left(\frac{4\pi}{7}\right)$ and $\cos\left(\frac{6\pi}{7}\right)$ are the roots of an equation of the form $ax^3+bx^2+cx+d=0$ where a,b,c,d are integers. Determine a,b,c and d.

Problem 1078. Find the value of the sum

$$\sqrt[3]{\cos\frac{2\pi}{7}}+\sqrt[3]{\cos\frac{4\pi}{7}}+\sqrt[3]{\cos\frac{6\pi}{7}}.$$

Problem 1079. Solve the equation

$$2\sin^4 x(\sin 2x-3)-2\sin^2 x(\sin 2x-3)-1=0.$$

Problem 1080. Express the sum of the following series in terms of $\sin x$ and $\cos x$.

$$\sum_{k=0}^{n}(2k+1)\sin^2\left(x+\frac{k}{2}\pi\right).$$

Problem 1081. Find the smallest positive integer N for which

$$\frac{1}{\sin 45^\circ\cdot\sin 46^\circ}+\frac{1}{\sin 47^\circ\cdot\sin 48^\circ}+\cdots+\frac{1}{\sin 133^\circ\cdot\sin 134^\circ}=\frac{1}{\sin N^\circ}.$$

Problem 1082. Find the value of

$$\frac{\sin 40^\circ+\sin 80^\circ}{\sin 110^\circ}.$$

Problem 1083. Evaluate the sum

$$S=\tan 1^\circ\cdot\tan 2^\circ+\tan 2^\circ\cdot\tan 3^\circ+\tan 3^\circ\cdot\tan 4^\circ+\cdots+\tan 2004^\circ\cdot\tan 2005^\circ.$$

Problem 1084. Solve the equation:

$$\sqrt{3}\sin x(\cos x-\sin x)+(2-\sqrt{6})\cos x+2\sin x+\sqrt{3}-2\sqrt{2}=0.$$

Problem 1085. Let

$$f(x)=\frac{1}{\sin\frac{\pi x}{7}}.$$

Prove that $f(3)+f(2)=f(1)$.

Problem 1086. Suppose that real numbers x, y, z satisfy

$$\frac{\cos x + \cos y + \cos z}{\cos(x+y+z)} = \frac{\sin x + \sin y + \sin z}{\sin(x+y+z)} = p.$$

Prove that

$$\cos(x+y) + \cos(y+z) + \cos(x+z) = p.$$

Problem 1087. Solve for $\theta, 0 \leq \theta \leq \frac{\pi}{2}$:

$$\sin^5\theta + \cos^5\theta = 1.$$

Problem 1088. For $x, y \in [0, \frac{\pi}{3}]$ prove that $\cos x + \cos y \leq 1 + \cos xy$.

Problem 1089. Prove that among any four distinct numbers from the interval $(0, \frac{\pi}{2})$ there are two, say x, y, such that:

$$8\cos x \cos y \cos(x-y) + 1 > 4(\cos^2 x + \cos^2 y).$$

Problem 1090. Let $B = \frac{\pi}{7}$. Prove that

$$\tan B \cdot \tan 2B + \tan 2B \cdot \tan 4B + \tan 4B \cdot \tan B = -7.$$

Problem 1091. a) Calculate

$$\frac{1}{\cos\frac{6\pi}{13}} - 4\cos\frac{4\pi}{13} - 4\cos\frac{5\pi}{13} = ?$$

b) Prove that

$$\tan\frac{\pi}{13} + 4\sin\frac{4\pi}{13} = \tan\frac{3\pi}{13} + 4\sin\frac{3\pi}{13}$$

c) Prove that

$$\tan\frac{2\pi}{13} + 4\sin\frac{6\pi}{13} = \tan\frac{5\pi}{13} + 4\sin\frac{2\pi}{13}$$

Problem 1092. Prove that if α, β are angles of a triangle and $(\cos^2\alpha + \cos^2\beta)(1 + \tan\alpha \cdot \tan\beta) = 2$, then $\alpha + \beta = 90°$.

Problem 1093. Let $a, b, c, d \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ be real numbers such that $\sin a + \sin b + \sin c + \sin d = 1$ and $\cos 2a + \cos 2b + \cos 2c + \cos 2d \geq \frac{10}{3}$. Prove that $a, b, c, d \in [0, \frac{\pi}{6}]$.

Problem 1094. Find all integers m, n for which we have $\sin^m x + \cos^n x = 1$, for all x.

Problem 1095. Prove that $\tan 55° \cdot \tan 65° \cdot \tan 75° = \tan 85°$.

Problem 1096. Prove that $\frac{4\cos 12° + 4\cos 36° + 1}{\sqrt{3}} = \tan 78°$.

Problem 1097. Prove that

$$\sqrt{4+\sqrt{4+\sqrt{4-\sqrt{4+\sqrt{4+\sqrt{4-\cdots}}}}}}=2\left(\cos\frac{4\pi}{19}+\cos\frac{6\pi}{19}+\cos\frac{10\pi}{19}\right).$$

The signs: $++-++-++-++-\cdots$

Problem 1098. For reals x,y Prove that $\cos x+\cos y+\sin x\sin y\leq 2$.

Problem 1099. Solve the equation in real numbers

$$\sqrt{7+2\sqrt{7-2\sqrt{7-2x}}}=x.$$

Problem 1100. Let A,B,C be three angles of triangle ABC. Prove that

$$(1-\cos A)(1-\cos B)(1-\cos C)\geq\cos A\cos B\cos C.$$

Problem 1101. Solve the equation

$$\sin^3(x)-\cos^3(x)=\sin^2(x).$$

Problem 1102. Find $S_n=\sum_{k=1}^{n}\sin^2 k\theta$ for $n\geqslant 1$.

Problem 1103. Prove the following without using induction:

$$\cos x+\cos 2x+\cdots+\cos nx=\frac{\cos\frac{n+1}{2}x\cdot\sin\frac{n}{2}x}{\sin\frac{x}{2}}.$$

Problem 1104. Evaluate:

$$\sin\theta+\frac{1}{2}\cdot\sin 2\theta+\frac{1}{2^2}\cdot\sin 3\theta+\frac{1}{2^3}\cdot\sin 4\theta+\cdots$$

Problem 1105. Compute

$$\sum_{k=1}^{n-1}\csc^2\left(\frac{k\pi}{n}\right).$$

Problem 1106. Prove that

$$\tan\theta+\tan\left(\theta+\frac{\pi}{n}\right)+\tan\left(\theta+\frac{2\pi}{n}\right)+\cdots+\tan\left[\theta+\frac{(n-1)\pi}{n}\right]=-n\cot\left(n\theta+\frac{n\pi}{2}\right),$$

$$\cot\theta+\cot\left(\theta+\frac{\pi}{n}\right)+\cot\left(\theta+\frac{2\pi}{n}\right)+\cdots+\cot\left[\theta+\frac{(n-1)\pi}{n}\right]=n\cot n\theta.$$

Problem 1107. Calculate

$$\sum_{n=1}^{\infty} 2^{2n} \sin^4 \frac{a}{2^n}.$$

Problem 1108. Compute the following sum:

$$\tan 1^\circ + \tan 5^\circ + \tan 9^\circ + \cdots + \tan 177^\circ.$$

Problem 1109. Show that for any positive integer $n > 1$,

- $\displaystyle\sum_{k=0}^{n-1} \cos \frac{2\pi k^2}{n} = \frac{\sqrt{n}}{2}\left(1 + \cos \frac{n\pi}{2} + \sin \frac{n\pi}{2}\right),$
- $\displaystyle\sum_{k=0}^{n-1} \sin \frac{2\pi k^2}{n} = \frac{\sqrt{n}}{2}\left(1 + \cos \frac{n\pi}{2} - \sin \frac{n\pi}{2}\right).$

Problem 1110. Evaluate the product

$$\prod_{k=1}^{n} \tan \frac{k\pi}{2(n+1)}.$$

Problem 1111. Prove, for even n, that

$$\sum_{k=1}^{n} (-1)^{k-1} \cot \frac{(2k-1)\pi}{4n} = n.$$

Problem 1112. Prove that

$$\sum_{k=1}^{n} \cot^2 \left\{ \frac{(2k-1)\pi}{2n} \right\} = n(2n-1).$$

Problem 1113. Prove that

$$\sum_{k=1}^{n} \cot^4 \left(\frac{k\pi}{2n+1} \right) = \frac{n(2n-1)(4n^2+10n-9)}{45}.$$

Problem 1114. Let x be a real number with $0 < x < \pi$. Prove that, for all natural numbers n, the sum

$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots + \frac{\sin(2n-1)x}{2n-1}$$

is positive.

2.5 Math Olympiad Trigonometry 501: Beyond Plane Euclidean Trigonometry (Egregia Introductio ad Monstruosam Trigonometriam)

We now summon geometric monsters from non-Euclidean spaces in order to do algebraic operations on them. In general, a geometry in which the assumption of flatness of space is disregarded would be a non-Euclidean geometry. This means that spaces studied here are curved, either positively or negatively.

We start by studying the simplest case of a positively-curved geometry called elliptic geometry, and we initially assume that the positively-curved space has a uniform curvature. The most accessible such geometry would be the spherical geometry whose trigonometric calculations are on the way.

Great Circular Arcs on a Sphere

Problem 1115. The shortest path connecting two points on a sphere is always part of a **great circular arc** whose center is the same as sphere's center. On a given sphere, imagine the lines of latitude as you would see them on a globe. Each line of latitude, except for the equator, is a **small circle** of the sphere. The equator, which splits the sphere into two equal-sized pieces, is a **great circle** of the sphere (see Figure 2.1).

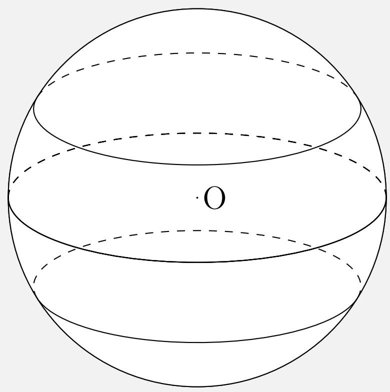

Figure 2.1: Two small circles and a great circle on a sphere.

Any two points on the surface of the sphere divide the great circle joining them into two parts. These two parts will be equal to each other if the two points are **antipodal** (diametrically opposite to one another). Otherwise, one of the two parts will be smaller than the other. Show that:

(a) If A and B are antipodal points, meaning they are the two ends of a diameter of the sphere, there are infinitely many great cirles passing through them.

(b) If A and B are not antipodal, there is exactly one **great circle** passing through them. The smaller arc connecting A to B is associated with a **central angle** connecting the center of the sphere to A & B.

Length of Shortest Arc Between Two Points on Sphere

Problem 1116. On a sphere of radius R, there are two points A and B (Figure 2.2). Prove that the **central angle** associated with the shortest arc connecting A and B is α, then the length of the arc is αR.

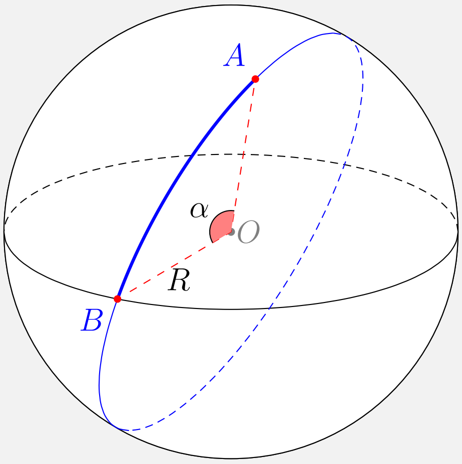

Figure 2.2: The great circle joining A to B

Main Elements of Euler's Spherical Triangle

Problem 1117. Let A, B, C be points on a unit sphere such that no two of them are antipodals. We draw the three great circles joining the three vertices A, B, C to envision **Euler's Spherical Triangle**. Each angle A, B, C and the corresponding side a, b, c facing it is a **main element** of spherical triangle ABC (Figure 2.3). Prove that the value of each of the main elements of Euler's spherical triangle lies between 0 and π.

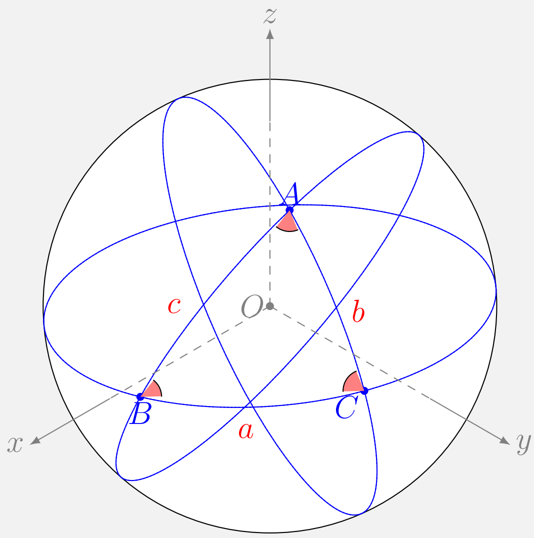

Figure 2.3: Angles A, B, C and sides a, b, c of a spherical triangle.

Trihedral Corner

Problem 1118. Each spherical triangle ABC corresponds to a **trihedral corner** whose vertex is at the center of the sphere and whose edges are the radii of the sphere connecting the center to the points A, B, C on the surface of sphere. Moreover, each trihedral corner with its vertex at the center of the sphere corresponds to a spherical triangle formed by the edges of the trihedral corner. Figure 2.4 demonstrates a **trirectangle** (a spherical triangle with three right angles) and its associated trihedral corner. Prove that:

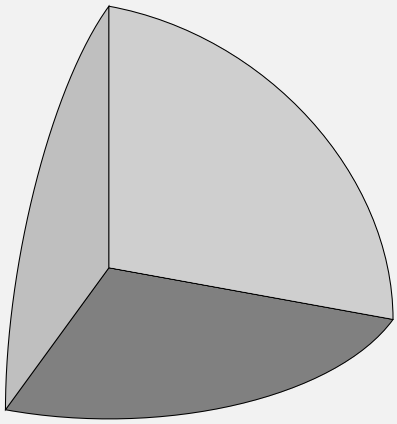

Figure 2.4: A Trihedral corner depicting an eighth of a sphere.

(a) The angles A, B, C of the spherical triangle are equal to the corresponding three dihedral angles of the trihedral corner (angles between the three planes that make the trihedral corner), and (b) The sides a, b, c of the spherical triangle are equal to the three angles formed at the vertex of the trihedral corner.

Definition. For any arc AB of any great circular arc on the sphere, if the diameter of the sphere that is perpendicular to the plane of the great circle intersects the surface of the sphere at points C_1 and C_2, we call C_1 and C_2 the **poles** of the arc AB.

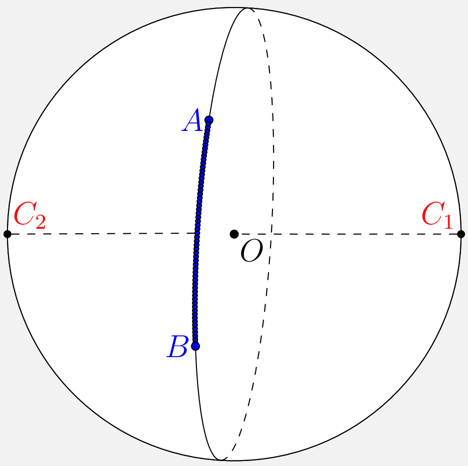

Figure 2.5: The poles of the arc AB.

Distances and Great Circles

Problem 1119. Let ℓ be a great circular arc on the unit sphere, and P a point not on ℓ. Prove that

1. If P is a pole of ℓ, then for any point Q on ℓ, PQ is a quadrant.
2. If for Q_1 and Q_2 on ℓ we have $PQ_1 = PQ_2 = \pi/2$, then P is a pole of ℓ.
3. If P is a pole of ℓ and Q_1 and Q_2 are on ℓ, then the distance between Q_1 and Q_2 equals the spherical angle between Q_1P and PQ_2: $Q_1Q_2 = \angle Q_1PQ_2$.

Polar Triangles

Problem 1120. For any triangle ABC, if the poles of the sides of ABC are vertices of triangle $A'B'C'$, then we call $A'B'C'$ the **polar triangle** of ABC. The assumption is that vertex A' is a pole of BC, vertex B' is a pole of CA, and C' is a pole of AB. Prove that:

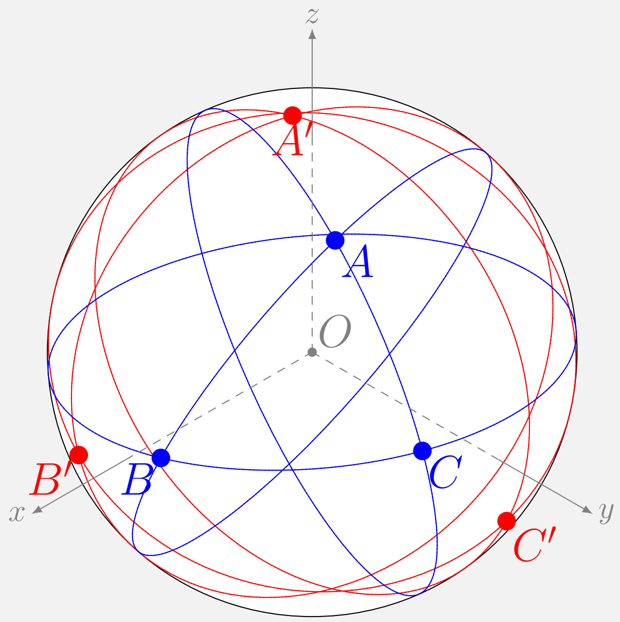

Figure 2.6: Polar triangles ABC and $A'B'C'$ on a sphere.

(a) If $A'B'C'$ is the polar triangle of ABC, then ABC is also the polar triangle of $A'B'C'$, so that vertex A is the pole of side a' of $A'B'C'$, vertex B is the pole of side b', and vertex C is the pole of side c'.

(b) If $\triangle ABC$ and $\triangle A'B'C'$ are polar triangles of one another, then the sum of each angle and its associated side in the polar triangle equals π:

$$A + a' = \pi, \quad B + b' = \pi, \quad C + c' = \pi,$$
$$A' + a = \pi, \quad B' + b = \pi, \quad C' + c = \pi.$$

Classification of Spherical Triangles

Definition. Concerning the angles of spherical triangles,

- A spherical triangle could be **acute**, **right**, or **obtuse**, like plane triangles.
- Spherical triangles may have two or three right or obtuse angles, and each angle can be close to π. We can thus see that the sum of angles of a spherical triangle cannot exceed 3π.

Regarding the sides,

- A spherical triangle may be **scalene**, **isosceles**, or **equilateral**.
- A spherical triangle that has one or more of its **sides** equal to a **quadrant** ($\pi/2$) is called a **quadrantal triangle**.
- A triangle in which one of the vertices is a pole of the opposing side is called a **semilunar** triangle, or a **semilune**.

Problem 1121. Prove the following statements:

1. It is known in Euclidean geometry that for plane triangles, being equilateral (having equal sides) is equivalent to being equiangular (having equal angles). Prove the same statement for spherical triangles.
2. Furthermore, if a plane triangle is isosceles (has two equal sides), then the angles facing those sides are equal to each other. Show that the same things happens for spherical triangles.
3. Prove the Pythagorean Theorem for Spherical Triangles: a, b, c are side-lengths of a spherical triangle ABC with right angle at A on a unit sphere if and only if

$$\cos a = \cos b \cos c.$$

   Remark. Remember that for a spherical triangle on a unit sphere, all main elements are smaller than π; and note that in the Spherical Pythagorean Formula the cosine function is applied to the side-lengths a, b, c rather than angles A, B, C. Show that we could, however, find sines of non-right angles:

$$\sin B = \frac{\sin b}{\sin a}, \cos B = \frac{\cos b \sin c}{\sin a}, \quad \text{and} \quad \sin C = \frac{\sin c}{\sin a}, \cos C = \frac{\cos c \sin b}{\sin a}.$$

4. If a spherical triangle has three right angles, all its sides are quadrants.
5. If a spherical triangle has two right angles, the sides facing those angles are quadrants and the third angle is measured by its opposite side.
6. If any two parts, a part being a side or an angle, of a spherical triangle measure $\pi/2$ radians, the triangle is a semilune. Also, the angle at the pole has the same measure as the opposing side. All of the other sides and angles measure $\pi/2$ radians.

Lunes on a Sphere

Problem 1122. Two great circles passing through antipodal points on a sphere divide the sphere into four parts like orange slices, each being a **lune** on the sphere (Figure 2.7). Prove that the area of a lune with angle α is 2α.

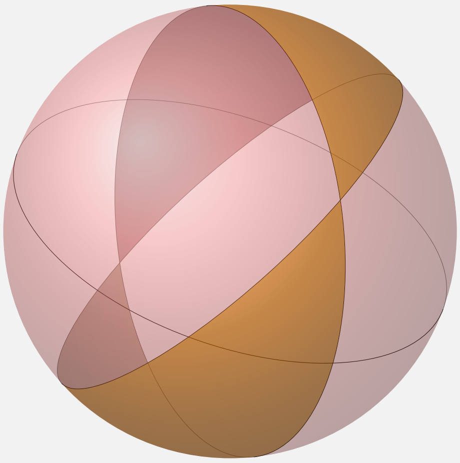

Figure 2.7: Lunes on a sphere divide it into four parts.

Spherical Triangle Inequalities

Problem 1123. Prove that for any spherical triangle ABC on a unit sphere with sides a, b, c and angles A, B, C,

1. The triangle inequality holds: each side is smaller than the sum of the other two sides:
$$a < b + c, \quad b < c + a, \quad c < a + b.$$
2. Each side is larger than the difference of the other two sides.
3. The sum of the sides of the triangle is positive and smaller than 2π:
$$0 < a + b + c < 2\pi.$$
4. The sum of the angles of the triangle is larger than π and smaller than 3π:
$$\pi < A + B + C < 3\pi.$$
5. The larger side of the triangle faces the larger angle of the spherical triangle.
6. The following inequalities hold true for triangle's angles:
$$A + B - C < \pi, \quad A - B + C < \pi, \quad -A + B + C < \pi.$$

Congruent Spherical Triangles & Gauss–Bonnet Theorem

Problem 1124. Remember from the Euclidean geometry that two plane triangles are **congruent** (having equal sides and angles) if (a) all three sides are equal (**SSS**), (b) two sides and the angle between them are equal (**SAS**), or (c) two angles and the side joining them are equal (**ASA**). However, two **incongruent** plane triangles may have all three angles equal to one another, simply because we can scale all sides of a plane triangle equally to get a triangle with larger/smaller sides but the same angles. Prove that for spherical triangles,

1. The three Euclidean criteria for congruent plane triangles (**SSS**, **SAS**, **ASA**) also hold true for spherical triangles.
2. Two spherical triangles with equal angles (**AAA**) are congruent.
3. Two congruent spherical triangles have equal areas.
4. On a unit sphere, sum of triangle's angles equals π plus the area of triangle:

$$A + B + C = \pi + (\text{area of } \triangle ABC).$$

Inverse Spherical Triangles

Problem 1125. Two spherical triangles ABC and $A'B'C'$ have all their corresponding main elements equal to one another, that is,

$$A = A', \quad B = B', \quad C = C', \qquad a = a', \quad b = b', \quad c = c'.$$

Prove that either the two triangles are **directly equal**, meaning one can be moved in space so that its vertices matches the vertices of the other triangle, or they are **inversely equal**, or simply **inverse** of each other, which means that they are reflections of each other with respect to a plane that passes through sphere's center.

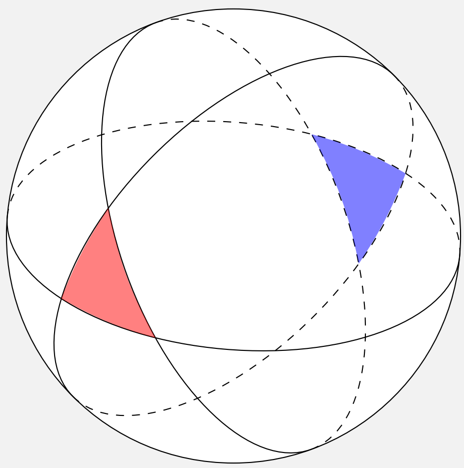

Figure 2.8: Inverse spherical triangles.

Spherical Law of Cosines for Sides

Problem 1126. The cosine of one side of a spherical triangle is equal to the product of the cosine of the other two sides plus the product of sines of these two sides times the cosine of the angle between them:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$
$$\cos b = \cos c \cos a + \sin c \sin a \cos B,$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

Problem 1127. Figure 2.9 shows a spherical triangle ABC with sides a, b, c on a sphere centered at O. The tangent at A to the arc AB meets OB at D and the tangent at A to the arc AC meets OC at E.

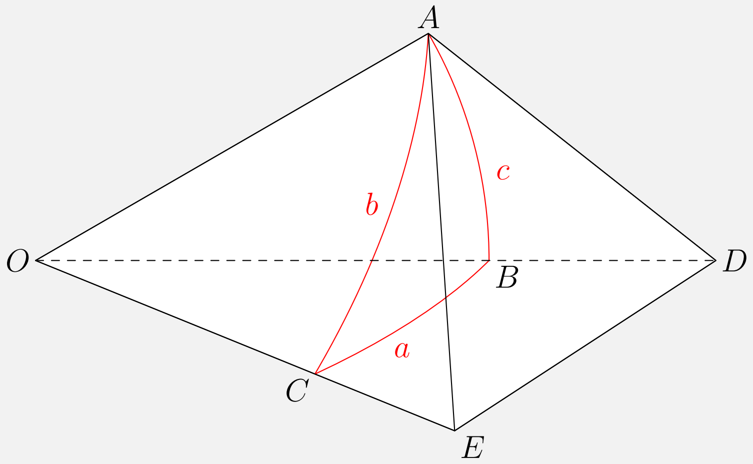

Figure 2.9: Spherical triangle ABC with tangents to arcs at vertex A.

1. Using the Law of Cosines in the plane, show that

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

2. Using the Pythagorean identity $\sin^2 A + \cos^2 A = 1$, prove that

$$\sin A = \frac{\sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}}{\sin b \sin c}.$$

Spherical Law of Sines

Consider the ratio between the sine of an angle and the sine of its opposite side in a spherical triangle. Prove that this ratio is the same for all three angles:

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

Five–Piece Spherical Trigonometric Identities

In a spherical triangle ABC with side-lengths a, b, c, prove the five-piece identities:

Side's Sine Times Adjacent Angle's Cosine 1128. For any side x and angle Y adjacent to it, the product $\sin x \cos Y$ equals $\cos y \sin z$ **minus** $\sin y \cos z \cos X$:

$$\text{Side } a: \begin{cases} \sin a \cos B = \cos b \sin c - \sin b \cos c \cos A, \\ \sin a \cos C = \cos c \sin b - \sin c \cos b \cos A, \end{cases}$$
$$\text{Side } b: \begin{cases} \sin b \cos C = \cos c \sin a - \sin c \cos a \cos B, \\ \sin b \cos A = \cos a \sin c - \sin a \cos c \cos B, \end{cases}$$
$$\text{Side } c: \begin{cases} \sin c \cos A = \cos a \sin b - \sin a \cos b \cos C, \\ \sin c \cos B = \cos b \sin a - \sin b \cos a \cos C. \end{cases}$$

Angle's Sine Times Adjacent Side's Cosine 1129. For any angle X and side y adjacent to it, the product $\sin X \cos y$ equals $\cos Y \sin Z$ **plus** $\sin Y \cos Z \cos x$:

$$\text{Angle } A: \begin{cases} \sin A \cos b = \cos B \sin C + \sin B \cos C \cos a, \\ \sin A \cos c = \cos C \sin B + \sin C \cos B \cos a, \end{cases}$$
$$\text{Angle } B: \begin{cases} \sin B \cos c = \cos C \sin A + \sin C \cos A \cos b, \\ \sin B \cos a = \cos A \sin C + \sin A \cos C \cos b, \end{cases}$$
$$\text{Angle } C: \begin{cases} \sin C \cos a = \cos A \sin B + \sin A \cos B \cos c, \\ \sin C \cos b = \cos B \sin A + \sin B \cos A \cos c. \end{cases}$$

Spherical Law of Cosines for Angles

Problem 1130. The cosine of an angle of a spherical triangle equals the product of sines of the other two angles and the cosine of the side between them **minus** the product of cosines of the other two angles:

$$\begin{aligned} \cos A &= \sin B \sin C \cos a - \cos B \cos C, \\ \cos B &= \sin C \sin A \cos b - \cos C \cos A, \\ \cos C &= \sin A \sin B \cos c - \cos A \cos B. \end{aligned}$$

Problem 1131. Imply that

$$\begin{aligned} \cos a &= \frac{\cos A + \cos B \cos C}{\sin B \sin C}, \\ \sin^2 \frac{a}{2} &= -\frac{\cos A + \cos(B + C)}{2 \sin B \sin C}. \end{aligned}$$

Half–Angle and Half–Side Spherical Formulas

Problem 1132. For every spherical triangle ABC with angles A, B, C and a, b, c, let p be the semiperimeter: $p = (a+b+c)/2$. Prove the following half–angle trigonometric formulas:

1. $\sin^2 \dfrac{A}{2} = \dfrac{\sin\left(\frac{a+b-c}{2}\right)\sin\left(\frac{a-b+c}{2}\right)}{\sin b \sin c}$,
2. $\sin^2 \dfrac{A}{2} = \dfrac{\sin(p-b)\sin(p-c)}{\sin b \sin c}$,
3. $\cos^2 \dfrac{A}{2} = \dfrac{\sin p \sin(p-a)}{\sin b \sin c}$,
4. $\tan^2 \dfrac{A}{2} = \dfrac{\sin(p-b)\sin(p-c)}{\sin p \sin(p-a)}$,
5. $\sin A = \dfrac{2\sqrt{\sin p \sin(p-a)\sin(p-b)\sin(p-c)}}{\sin b \sin c}$.

Problem 1133. For every spherical triangle ABC with angles A, B, C and a, b, c, define $P = (a+b+c)/2$. Prove the following half–side trigonometric identities:

1. $\sin^2 \dfrac{a}{2} = -\dfrac{\cos P \cos(P-A)}{\sin B \sin C}$,
2. $\cos^2 \dfrac{a}{2} = \dfrac{\cos(P-B)\cos(P-C)}{\sin B \sin C}$,
3. $\tan^2 \dfrac{a}{2} = -\dfrac{\cos P \cos(P-A)}{\cos(P-B)\cos(P-c)}$,
4. $\sin a = \dfrac{2\sqrt{-\cos P \cos(P-A)\cos(P-B)\cos(P-C)}}{\sin B \sin C}$.

Spherical Law of Haversines

Definition. Some hundreds of years ago when spherical trigonometry was a hot-topic for mathematicians, there was another periodic function besides cosine and sine, called **versine**, short for **versed sine**, defined by $\operatorname{versin}(\theta) = 1 - \cos\theta$.

Definition. Since $1 - \cos\theta = 2\sin^2\theta$, it makes sense to define a **halved versed sine**, or shortly, **haversine**, by $\operatorname{hav}(\theta) = \sin^2\left(\frac{\theta}{2}\right)$.

Problem 1134. Prove the **Haversine Formula** in a spherical triangle ABC with side-lengths a, b, c:

$$\operatorname{hav}(c) = \operatorname{hav}(a-b) + \sin(a)\sin(b)\ \operatorname{hav}(C).$$

Swimming the Depths of the Algebraic Ocean

Kaywañan is an Algebra Competition, and you may say its motto is “Let No One Ignorant of Algebra Enter.” So far, we have been vigorously forging algebraic equations and definitions that are deeply rooted within their applications. It is both an intention and a purpose of Kaywañan to be defined as the collection of most important algebraic equations and identities that one may encounter in dealing with in ordinary, Euclidean flatland geometry, as well as non-Euclidean monsters and witches that might appear in hyperbolic geometry.

We have not even started to discuss the geometry of hyperbola and its associated hyperboloid. The most advanced formula, I would say, in Kaywañan so far is that of Problem 1134, the Spherical Law of Haversines, which has absolutely fascinating applications in astronomy. The haversine formula for spherical triangles is just an example of myriads of unknown equations in Spherical Geometry, the most special type of Elliptic Geometry. You can only imagine how many more of such algebraic equations may be found, written down, and added to Kaywañan if we consider other elliptic structures than the Sphere, such as the Ellipsoid.

There are other types of non-Euclidean geometries that some might say are even more surprising than the fact that angles of a spherical triangle add up to more than π. If the Euclidean plane is not infinite in all directions, and in the special case when the plane is limited to a 1×1 Square of the Euclidean Plane, you can imagine that the corresponding points on opposite sides of the square are actually the same point, as if you map the surface of a doughnut to the 1×1 Euclidean Square. If we start walking from a certain point in any of the two directions of the Euclidean Square on the surface of a doughnut, we would reach the same point. The same would happen if we map the sphere to the Euclidean plane, but the surface of the doughnut and that of a sphere are clearly distinct to us. If we had no idea of the third dimension, as if we were ants walking around in two-dimensional Euclidean plane, we would never even be able to know whether the surface on which we walk is a doughnut or a sphere.

Here, at the depths of the Algebraic Ocean of Kaywañan, Titan, Moon of Saturn, where we can see the positive curvature of the surface of the core of Titan, nobody doubts the spherical shape of Planet of Algebra, Kaywan. There are rumours that in earlier Eons, Kaywañans (those who live around Saturn) believed that Titan is the most special place in the Sölar System, and after one of them dreamed of Maurits Cornelis Escher’s “Angels and Devils” painting, they started to preach a certain belief in a Hyperbolic Geometry of Titan to emphasize their uniqueness among moons of Saturn and other celestial objects.

It is now, however, a ridiculous claim to believe in a hyperbolic geometry for the actually spherical surface of Titan, maybe as ridiculous as believing in a Flat Earth once you have traveled to the moon and seen the biosphere of the Earth from afar. Now that we can swim the depths of Kaywañan and measure the spherical arcs close to the core of Titan, hyperbolic beliefs are but a joke. We may study mythology of such treacheries in the advanced levels of Napirañan Geometry Contest, but here in Kaywañan we stick to the algebra.

Maclaurin Series of Versine & Haversine

Problem 1135. Prove that for any complex number z,

$$\operatorname{versin}(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k}}{(2k)!} \quad \text{and} \quad \operatorname{hav}(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k}}{2(2k)!}.$$

Then show that the limit of both $\operatorname{versin}(\theta)/\theta$ and $\operatorname{hav}(\theta)/\theta$ when $\theta \to 0$ is 0.

The Sine Formulae (from "Spherical Astronomy")

The following method of notation is quoted from "Textbook on Spherical Astronomy" by W. M. Smart and R. M. Green, given in the first chapter as one of the main formulas (formulae **A** and **D**) in Spherical Trigonometry.

Definition (Laterangular Function of a Spherical Triangle)**.** For any spherical triangle ABC, the **Laterangular Function of** A, denoted $X(a, A)$, is defined by

$$(X(a, A))^2 \cdot \sin^2 a \sin^2 b \sin^2 c = 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c.$$

Problem 1136. In a spherical triangle ABC with side-lengths a, b, c,

1. Prove the **Astronomical Sine** formula:

$$\sin^2 b \cdot \sin^2 c \cdot \sin^2 A = 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c.$$

2. Prove that

$$(X(a, A))^2 = \left(\frac{\sin A}{\sin a}\right)^2,$$

and imply that the Laterangular Function must be symmetric, so that $X(a, A) = X(b, B) = X(c, C)$.

3. If all main elements of triangle ABC are smaller than π, then the **Spherical Law of Sines** holds true:

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{\sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}}{\sin a \sin b \sin c}.$$

Definition (Four–Piece Spherical Trigonometric Terminology)**.** In the spherical triangle ABC consider the four consecutive parts B, a, C, b. The angle C is contained by the two sides a and b and is called the **inner angle**. The side a is flanked by the two angles B and C and is called the **inner side**.

Four-Part Identity 1137. Prove that cos(inner side) · cos(inner angle) equals

$$\sin(\text{inner side}) \cdot \cot(\text{other side}) - \sin(\text{inner angle}) \cdot \cot(\text{other angle}).$$

Delambre's and Napier's Analogies

Delambre's Aanlogies 1138. In a spherical triangle ABC with sides a, b, c,

$$\sin\frac{c}{2}\sin\frac{A-B}{2} = \cos\frac{C}{2}\sin\frac{a-b}{2} \quad \text{and} \quad \sin\frac{c}{2}\cos\frac{A-B}{2} = \sin\frac{C}{2}\sin\frac{a+b}{2},$$
$$\cos\frac{c}{2}\sin\frac{A+B}{2} = \cos\frac{C}{2}\cos\frac{a-b}{2} \quad \text{and} \quad \cos\frac{c}{2}\cos\frac{A+B}{2} = \sin\frac{C}{2}\cos\frac{a+b}{2}.$$

Taking Delambre's equations, which are also called **Gauss's Spherical Equations**, in pairs, we obtain Napier's Analogies:

Napier's Analogies

Napier's Aanlogies 1139. In a spherical triangle ABC with sides a, b, c,

$$\tan\frac{a+b}{2} = \frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}\tan\frac{c}{2} \quad \text{and} \quad \tan\frac{a-b}{2} = \frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}}\tan\frac{c}{2},$$
$$\tan\frac{A+B}{2} = \frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}\cot\frac{C}{2} \quad \text{and} \quad \tan\frac{A-B}{2} = \frac{\sin\frac{a-b}{2}}{\sin\frac{a+b}{2}}\cot\frac{C}{2}.$$

Napier's Rules

Napier's Rules for Right Spherical Triangles 1140. If one main element among a, b, c, A, B, C is $\pi/2$, there would be five remaining unknown parts. John Napier suggested Five–Piece Mnemonics shown in Figures 2.10 and 2.11 [from Wikipedia] to prove:

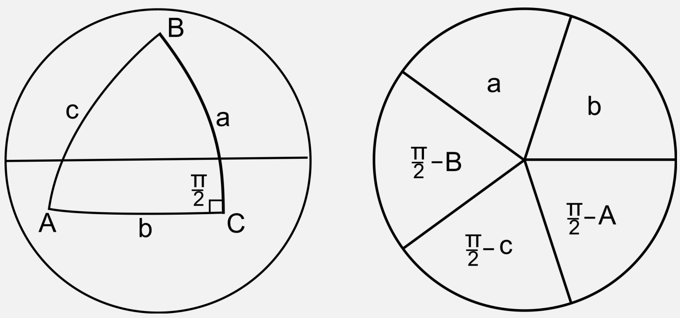

Figure 2.10: Napier's Mnemonics (When One Angle is Right)

1. The sine of any middle part equals the product of tangents of adjacent parts.
2. The sine of any middle part equals the product of cosines of opposite parts.

Napier's Ten Right Spherical Triangle Commandments

Napier's Ten Commandments 1141. For any spherical triangle ABC with a right angle at A, Napier's Ten Rules are the ten equations derived from various spherical trigonometric identities studied in previous problems [Wikipedia]:

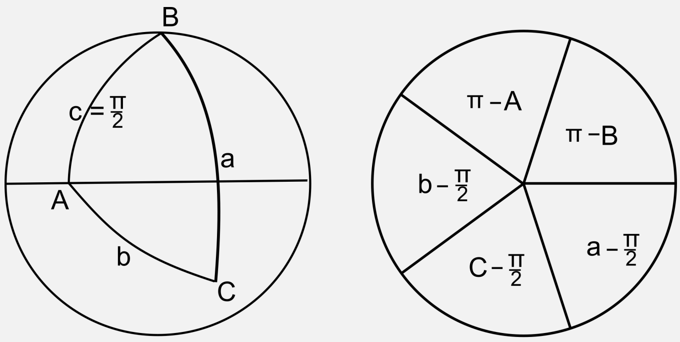

Figure 2.11: Napier's Mnemonics (When One Side is a Quadrant)

(I) $\cos a = \cos b \cos c$, A.K.A. *Spherical Pythagorean Theorem*,

(II) $\sin b = \sin a \sin B$, derived from *Spherical Law of Sines*,

(III) $\sin c = \sin a \sin C$, also from *Spherical Law of Sines*,

(IV) $\cos B = \cos b \sin C$, by *Spherical Law of Cosines for Angle B*,

(V) $\cos C = \cos c \sin B$, also by *Spherical Law of Cosines*, but for Angle C,

(VI) $\cos a = \cot B \cot C$, by *Spherical Law of Cosines for Angle A*,

(VII) $\cos B = \cot a \tan c$, by *Napier's Four–Part Identity* for $\cos c \cos B$,

(VIII) $\cos C = \tan b \cot a$, by *Napier's Four–Part Identity* for $\cos b \cos C$,

(IX) $\sin b = \tan c \cot C$, by *Napier's Four–Part Identity* for $\cos b \cos A$,

(X) $\sin c = \cos c \sin B$, by *Napier's Four–Part Identity* for $\cos c \cos A$.

Problem 1142. Prove that Napier's Ten Commandments can be reduced to the following cosine formulae by changing b and c to $\frac{\pi}{2} - b$ and $\frac{\pi}{2} - c$, respectively:

a) $\cos a = \sin\left(\frac{\pi}{2} - b\right) \sin\left(\frac{\pi}{2} - c\right)$,

b) $\cos a = \cot B \cot C$,

c) $\cos\left(\frac{\pi}{2} - b\right) = \sin a \sin B$,

d) $\cos\left(\frac{\pi}{2} - b\right) = \cot\left(\frac{\pi}{2} - c\right) \cot C$,

e) $\cos\left(\frac{\pi}{2} - c\right) = \sin a \sin C$,

f) $\cos\left(\frac{\pi}{2} - c\right) = \cot\left(\frac{\pi}{2} - b\right) \cot B$,

g) $\cos B = \sin\left(\frac{\pi}{2} - b\right) \sin C$,

h) $\cos B = \cot a \cot\left(\frac{\pi}{2} - c\right)$,

i) $\cos C = \sin\left(\frac{\pi}{2} - c\right) \sin B$,

j) $\cos C = \cot\left(\frac{\pi}{2} - b\right) \cot a$.

Napier's Ten Commandments and their ten cosine forms formulae inspired Napier to make his **Napier Rules for Right Spherical Triangles**.

The Global Half–Side Identities

Definition (Half–Side Spherical Triangle Identities)**.** For any spherical triangle ABC, the **Halveside Function of** A, denoted $H(a, A)$, is defined by

$$H(a, A) \cdot \cos(P - A) = \tan\left(\frac{a}{2}\right).$$

where $P = (A + B + C)/2$.

Problem 1143. In a spherical triangle ABC with side-lengths a, b, c,

1. Prove the **Astronomical Half–Side Tangent** formula:

$$\tan\left(\frac{a}{2}\right) = \sqrt{\frac{-\cos P}{\cos(P - A)\cos(P - B)\cos(P - C)}}\cos(P - A),$$

2. Prove that

$$(H(a, A))^2 = \frac{-\cos P}{\cos(P - A)\cos(P - B)\cos(P - C)},$$

and imply that the Laterangular Function must be symmetric, so that $H(a, A) = H(b, B) = H(c, C)$.

3. Prove **Mollweide's** formula in Euclidean plane:

$$\left(\tan\frac{A}{2}\right)^2 = \frac{(a + b - c)(a - b + c)}{(a + b + c)(-a + b + c)}.$$

Spherical Half–Side Tangent–Cotangent Formulae 1144. In the spherical triangle ABC with sides a, b, c that subtend angles A, B, C, prove that

$$\tan\left(\frac{a - b}{2}\right)\cot\left(\frac{a + b}{2}\right) = \tan\left(\frac{A - B}{2}\right)\cot\left(\frac{A + B}{2}\right).$$

L'Huilier's & Cagnoli's Halveside Theorems

L'Huilier's Theorem 1145. Let a spherical triangle have sides of length a, b, and c, and semiperimeter p. Then the spherical excess $E = (A + B + C) - \pi$ is given by

$$\tan\left(\frac{E}{4}\right) = \sqrt{\tan\left(\frac{p}{2}\right)\tan\left(\frac{p - a}{2}\right)\tan\left(\frac{p - b}{2}\right)\tan\left(\frac{p - c}{2}\right)}.$$

Cagnoli's Theorem 1146. $\sin\left(\frac{E}{2}\right) = \dfrac{\sqrt{\sin p \sin(p - a)\sin(p - b)\sin(p - c)}}{2\cos\left(\frac{a}{2}\right)\cos\left(\frac{b}{2}\right)\cos\left(\frac{c}{2}\right)}.$

Exercises on Spherical Excess E

Problem 1147. Prove that the area of any spherical triangle ABC equals the Spherical Excess $E = A + B + C - \pi$ times the square of sphere's radius.

Problem 1148. In a spherical triangle if $A = B = 2C$, show that

$$8\sin\left(a + \frac{c}{2}\right)\sin^2\left(\frac{c}{2}\right)\cos\left(\frac{c}{2}\right) = \sin^3 a.$$

Problem 1149. If $A + B + C = 2\pi$, show that

$$\cos^2\left(\frac{a}{2}\right) + \cos^2\left(\frac{b}{2}\right) + \cos^2\left(\frac{c}{2}\right) = 1.$$

Problem 1150. In spherical triangle ABC, if C is a right angle, prove

$$\frac{\sin^2 c}{\cos c}\cos E = \frac{\sin^2 a}{\cos a} + \frac{\sin^2 b}{\cos b}.$$

Problem 1151. Show that

$$\sin\left(\frac{E}{2}\right) = \sin\left(\frac{a}{2}\right)\sin\left(\frac{b}{2}\right)\sec\left(\frac{c}{2}\right),$$
$$\cos\left(\frac{E}{2}\right) = \cos\left(\frac{a}{2}\right)\cos\left(\frac{b}{2}\right)\sec\left(\frac{c}{2}\right).$$

Problem 1152. Prove that

$$\sin^2\left(\frac{C}{2} - \frac{E}{4}\right) = \frac{\cos\left(\frac{p}{2}\right)\sin\left(\frac{p-a}{2}\right)\sin\left(\frac{p-b}{2}\right)\sin\left(\frac{p-c}{2}\right)}{\sin\left(\frac{a}{2}\right)\sin\left(\frac{b}{2}\right)\cos\left(\frac{c}{2}\right)},$$
$$\cos^2\left(\frac{C}{2} - \frac{E}{4}\right) = \frac{\sin\left(\frac{p}{2}\right)\cos\left(\frac{p-a}{2}\right)\cos\left(\frac{p-b}{2}\right)\cos\left(\frac{p-c}{2}\right)}{\sin\left(\frac{a}{2}\right)\sin\left(\frac{b}{2}\right)\cos\left(\frac{c}{2}\right)}.$$

Problem 1153. If $p = (a + b + c)/2$ is the semiperimeter, show that

$$\sin p = \frac{\sqrt{\sin\left(\frac{E}{2}\right)\sin\left(A - \frac{E}{2}\right)\sin\left(B - \frac{E}{2}\right)\sin\left(C - \frac{E}{2}\right)}}{2\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}.$$

Radii of the Spherical Triangle

Inradius of the Spherical Triangle 1154. In a spherical triangle ABC with sides a, b, c, and semiperimeter $p = (a + b + c)/2$, the inradius r of the incircle of triangle may be calculated from

$$\tan r = \tan\left(\frac{A}{2}\right) \cdot \sin(p - a) = \sqrt{\frac{\sin(p-a)\sin(p-b)\sin(p-c)}{\sin p}}.$$

Circumradius of the Spherical Triangle 1155. In a spherical triangle ABC with sides a, b, c, and angles A, B, C, define $P = (A + B + C)/2$, the circumradius R of the circumscribed circle of ABC can be calculated from

$$\cot R = \cot\left(\frac{A}{2}\right) \cdot \cos(P - A) = \sqrt{\frac{\cos(P-A)\cos(P-B)\cos(P-C)}{-\cos P}}.$$

Medians of Spherical Triangles

Problem 1156. In a spherical triangle ABC with sides a, b, c the length of the median m_C drawn from C (length of CF in Figure 2.12) [from Wikipedia] satisfies

$$\cos m_C = \cos b \cdot \cos\left(\frac{c}{2}\right) + \sin b \cdot \sin\left(\frac{c}{2}\right) \cdot \cos A.$$

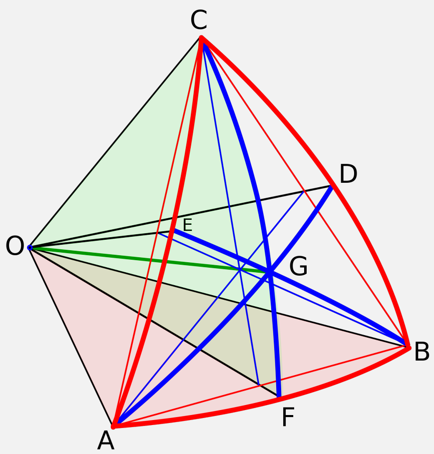

Figure 2.12: Spherical Medians

Problem 1157. Let O be the center of the sphere. If G is the centroid of spherical triangle ABC and $A'B'C'$ is the polar triangle of ABC, show that

$$\overrightarrow{OG} = \frac{1}{2E} \cdot \left(\overrightarrow{OC'} \cdot |\overline{AB}| + \overrightarrow{OA'} \cdot |\overline{BC}| + \overrightarrow{OB'} \cdot |\overline{CA}|\right).$$

2.5.1 63 Ancient Spherical Geometry Problems

1960 International Mathematical Olympiad 1158. Consider a cone of revolution with an inscribed sphere tangent to the base of the cone. A cylinder is circumscribed about this sphere so that one of its bases lies in the base of the cone. let V_1 be the volume of the cone and V_2 be the volume of the cylinder.

a) Prove that $V_1 \neq V_2$;

b) Find the smallest number k for which $V_1 = kV_2$; for this case, construct the angle subtended by a diamter of the base of the cone at the vertex of the cone.

1962 International Mathematical Olympiad 1159. The tetrahedron $SABC$ has the following property: there exist five spheres, each tangent to the edges SA, SB, SC, BC, CA, AB, or to their extensions.

a) Prove that the tetrahedron $SABC$ is regular.

b) Prove conversely that for every regular tetrahedron five such spheres exist.

1966 International Mathematical Olympiad 1160. Prove that the sum of the distances of the vertices of a regular tetrahedron from the center of its circumscribed sphere is less than the sum of the distances of these vertices from any other point in space.

1959–1966 IMO Longlist 1161. In a tetrahedron, all three pairs of opposite (skew) edges are mutually perpendicular. Prove that the midpoints of the six edges of the tetrahedron lie on one sphere.

1967 IMO Longlist 1162. Determine the volume of the body obtained by cutting the ball of radius R by the trihedron with vertex in the center of that ball, it its dihedral angles are α, β, γ.

1967 IMO Longlist 1163. Prove this proposition: Center the sphere circumscribed around a tetrahedron which coincides with the center of a sphere inscribed in that tetrahedron if and only if the skew edges of the tetrahedron are equal.

1970 Bulgaria 1164. In space, we are given the points A, B, C and a sphere with center O and radius 1. Find the point X from the sphere for which the sum $f(X) = |XA|^2 + |XB|^2 + |XC|^2$ attains its maximal and minimal value. Prove that if the segments OA, OB, OC are pairwise perpendicular and d is the distance from the center O to the centroid of the triangle ABC then:

a) the maximum of $f(X)$ is equal to $9d^2 + 3 + 6d$;

b) the minimum of $f(X)$ is equal to $9d^2 + 3 - 6d$.

1979 USAMO 1165. Let S be a great circle with pole P. On any great circle through P, two points A and B are chosen equidistant from P. For any spherical triangle ABC (the sides are great circles ares), where C is on S, prove that the great circle are CP is the angle bisector of angle C.

Note: A great circle on a sphere is one whose center is the center of the sphere. A pole of the great circle S is a point P on the sphere such that the diameter through P is perpendicular to the plane of S.

1981 USAMO 1166. The sum of the measures of all the face angles of a given complex polyhedral angle is equal to the sum of all its dihedral angles. Prove that the polyhedral angle is a trihedral angle. **Note**: A convex polyhedral angle may be formed by drawing rays from an exterior point to all points of a convex polygon.

1982 IMO Longlist 1167. Let $r_1, \ldots, r_n$ be the radii of n spheres. Call $S_1, S_2, \ldots, S_n$ the areas of the set of points of each sphere from which one cannot see any point of any other sphere. Prove that

$$\frac{S_1}{r_1^2} + \frac{S_2}{r_2^2} + \cdots + \frac{S_n}{r_n^2} = 4\pi.$$

1982 IMO Longlist 1168. A regular n-gonal truncated pyramid is circumscribed around a sphere. Denote the areas of the base and the lateral surfaces of the pyramid by S_1, S_2, and S, respectively. Let σ be the area of the polygon whose vertices are the tangential points of the sphere and the lateral faces of the pyramid. Prove that

$$\sigma S = 4 S_1 S_2 \cos^2 \frac{\pi}{n}.$$

1983 IMO Longlist 1169. Four faces of tetrahedron $ABCD$ are congruent triangles whose angles form an arithmetic progression. If the lengths of the sides of the triangles are $a < b < c$, determine the radius of the sphere circumscribed about the tetrahedron as a function on a, b, and c. What is the ratio c/a if $R = a$?

1984 IMO Longlist 1170. A tetrahedron is inscribed in a sphere of radius 1 such that the center of the sphere is inside the tetrahedron. Prove that the sum of lengths of all edges of the tetrahedron is greater than 6.

1985 IMO Longlist 1171. This problem comes in two parts:

a) The solid S is defined as the intersection of the six spheres with the six edges of a regular tetrahedron T, with edge length 1, as diameters. Prove that S contains two points at a distance $1/\sqrt{6}$.

b) Using the same assumptions in a), prove that no pair of points in S has a distance larger than $1/\sqrt{6}$.

1985 IMO Longlist 1172. Determine the radius of a sphere S that passes through the centroids of each face of a given tetrahedron T inscribed in a unit sphere with center O. Also, determine the distance from O to the center of S as a function of the edges of T.

1987 IMO Longlist 1173. Let S_1 and S_2 be two spheres with distinct radii that touch externally. The spheres lie inside a cone C, and each sphere touches the cone in a full circle. Inside the cone there are n additional solid spheres arranged in a ring in such a way that each solid sphere touches the cone C, both of the spheres S_1 and S_2 externally, as well as the two neighboring solid spheres. What are the possible values of n?

1987 Vietnam 1174. Prove that among any five distinct rays Ox, Oy, Oz, Ot, Or in space there exist two which form an angle less than or equal to $90°$.

1992 Putnam 1175. On a sphere, 4 points are randomly chosen. What is the probability that the center of the sphere is contained in the tetrahedron in the 4 points.

2006 Baltic Way 1176. There are 2006 points marked on the surface of a sphere. Prove that the surface can be partitioned into 2006 congruent pieces, so that each piece contains exactly one of these points inside it.

2016 Brazil Cono Sur Training 1177. Let $ABCD$ be a tetrahedron and let E, F, G, H, K, and L be points on the segments AB, BC, CA, DA, DB and DC, respectively, so that

$$AE \cdot BE = BF \cdot CF = CG \cdot AG = DH \cdot AH = DK \cdot BK = DL \cdot CL.$$

Prove that the six points marked on the sides of the tetrahedron are on the same sphere.

2000 Belarus TST 1178. A closed pentagonal line is inscribed in a sphere of the diameter 1, and has all edges of length ℓ. Prove that

$$\ell \leq \sin \frac{2\pi}{5}.$$

1995 Romania TST 1179. A cube is partitioned into finitely many rectangular parallelepipeds with the edges parallel to the edges of the cube. Prove that if the sum of the volumes of the circumspheres of these parallelepipeds equals the volume of the circumscribed sphere of the cube, then all the parallelepipeds are cubes.

1998 Czech and Slovak 1180. A sphere is inscribed in a tetrahedron $ABCD$. The tangent planes to the sphere parallel to the faces of the tetrahedron cut off four smaller tetrahedra. Prove that sum of all the 24 edges of the smaller tetrahedra equals twice the sum of edges of the tetrahedron $ABCD$.

1993 Brazil 1181. Let $P_1P_2\ldots P_n$ a polygon inscribed on a circunference and contained in a plane α. Let Q be a point outside α. Consider, for each $i = 1, 2, \ldots, n$, the plane β_i passing through P_i and perpendicular to QP_i. Prove that all the planes β_i intersect at one point.

2015 Kurchatov Olympiad 1182. To prepare mashed potatoes, Kolya, the chef, needs to get the specified amount of peeled potatoes as soon as possible. Not caring about saving peelings, he cuts cubes from spherical potatoes, with each stroke clearing one edge of the knife. Can he complete the task faster when the same frequency of knife strokes if you cut out any other polyhedra? (formally: is it true that of all polyhedra cut from of the given sphere, the largest ratio of volume to the number of faces is for the inscribed cube?)

2015 Miklos Schweitzer 1183. We call a bar of width w on the surface of the unit sphere $\mathbb{S}^2$, a spherical segment, centered at the origin, which has width w and is symmetric with respect to the origin. Prove that there exists a constant $c > 0$, such that for any positive integer n the surface $\mathbb{S}^2$ can be covered with n bars of the same width so that any point is contained in no more than $c\sqrt{n}$ bars.

2017 St. Petersburg 1184. Given a tetrahedron $PABC$, draw the height PH from vertex P to ABC. From point H, draw perpendiculars HA', HB', HC' to the lines PA, PB, PC. Suppose the planes ABC and $A'B'C'$ intersects at line ℓ. Let O be the circumcenter of triangle ABC. Prove that $OH \perp \ell$.

2018 Spain 1185. Points on a spherical surface with radius 4 are colored in 4 different colors. Prove that there exist two points with the same color such that the distance between them is either $4\sqrt{3}$ or $2\sqrt{6}$ (distance is Euclidean, that is, the length of the straight segment between the points).

Problem 1186. Given reals a_i, b_i, c_i, for $i = 1, 2, \dots, n$, such that

$$\sum_{i=1}^{n} a_i^2 = 1, \qquad \sum_{i=1}^{n} b_i^2 = 1, \qquad \sum_{i=1}^{n} c_i^2 = 1, \qquad \sum_{i=1}^{n} b_i c_i = 0.$$

Prove that

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 + \left(\sum_{i=1}^{n} a_i c_i\right)^2 \leq 1.$$

Spherical Inequality by Puuhikki 1187. Let x, y, z be positive real numbers such that $x^2 + y^2 + z^2 = 1$. Prove that

$$8xyz < \frac{4}{3}\pi.$$

2002 Iran TST 1188. There is a closed curve on the surface of the unit sphere. We know every big circle of the sphere has non-empty intersection with the curve. Prove that perimeter of the curve is at least 2π.

2003 China 1189. 8 spherical balls of radius 1 are placed in a cylinder in two layers. with each layer containing 4 balls. Each ball is tangent to 2 balls in the same layer, 2 balls in another layer, one base, and the lateral surface of the cylinder. What is the height of the cylinder?

Problem 1190. On a circle of positive integer radius n, there are m chords such that for any point in the circle there is a chord such that the distance from that point to the chord is ≤ 1. Prove that $m \geq n$.

Problem 1191. Four distinct points A, B, C, and M are given on a sphere, none of which is opposite to any other. Prove that if the great circles AM and BM are orthogonal to great circles BC and AC, respectively, then also the great circle CM is orthogonal to great circle AB.

1992 Tokyo University Entrance Exam 1192. Let a and b be positive reals. Four points $P(0,0,0), Q(a,0,0), R(0,1,0), S(0,1,b)$ lie on a same spherical surface with radius 1. Let r be the radius of the sphere which is inscribed in tetrahedron $PQRS$. Find the maximum value of r.

1993 Kyoto University Entrance Exam 1193. Assume that 3 Points A, B, C lie on the spherical surface with radius 1, centered at O. Let P, Q, R be the mid points of BC, CA, AB, respectively. Prove that at least one segment of the segments OP, OQ, OR in length is greater than or equal to $\frac{1}{2}$.

1999 Australian Math Competition 1194. When three spherical balls, each of radius 10 cm, are placed in a hemispherical dish, it is noticed that the tops of the balls are all exactly level with the top of the dish. What is the radius, in centimeters, of the dish?

1981 IMO Shortlist and Beyond 1195. The version b) is an ISL problem from 1981. The version a) is the easier 2-dimensional counterpart of the 3-dimensional version b).

a) If a circular planet lies completely within the convex hull H of a similarly defined system of circular planets, its entire circumference is visible from the other planets.

b) Consider a cluster of n spherical planets, all the same size, which drift rigidly together through outer space, that is, each with no motion relative to the others in the cluster. In general, there is some region R on the surface of a planet which cannot be seen from any point on any of the other planets (if a planet is in the midst of the others, then its surface is completely visible and this region is empty). Prove that the sum of the areas of these visible regions R is exactly equal to the area of the surface of one planet.

Problem 2814 in Gazeta Matematica (Bucuresti) 1196. Let A', B', C' be the points where the arc bisectors of the spherical triangle ABC meet the opposite sides. Is it possible to have $A'B' = A'C'$ without ABC being an isosceles triangle?

Problem 1197. Prove that the following two statements are equivalent, and then prove them separately.

a) For $a, b, c \geq 0$ satisfying $a^2 + b^2 + c^2 = 1$, we have

$$\frac{a}{a^3 + 2bc} + \frac{b}{b^3 + 2ca} + \frac{c}{c^3 + 2ab} \geq 2.$$

b) Using spherical coordinates, where α is the $x - y$ angle, and β is the $y - z$ angle, define $a = \cos\beta\cos\alpha$ (x-coordinate), $b = \cos\beta\sin\alpha$ (y-coordinate), and finally $c = \sin\beta$. Then, we have

$$\frac{\cos\alpha}{\cos^2\beta\cos^3\alpha + 2\sin\beta\sin\alpha} + \frac{\sin\alpha}{\cos^2\beta\sin^3\alpha + 2\sin\beta\cos\alpha} + \frac{\sin\beta}{\sin^3\beta + 2\sin\alpha\cos\alpha\cos^2\beta} \geq 2.$$

Cauchy–like Spherical Inequality by Fuzzylogic 1198. Let a_i, b_i, x_i be reals for $i = 1, 2, \ldots, n$, such that $\sum_{i=1}^n a_i x_i = 0$. Prove that

$$\left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i\right)^2\right) \geq \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i x_i\right)^2.$$

Hanging Sector by Farenhajt 1199. A circular sector with the center O and the radii OA, OB is cut out of the homogenous thin cardboard and hung freely by the point A. Find the angle between AO and the vertical if $\angle AOB = \theta \leq \pi$.

Hanging Sector by Atan 1200. Through a point in the interior of a sphere we put three planes, standing perpendicular on each other. These planes cut up the spherical surface into 8 curvilinear (spherical) triangles. The triangles are colored alternately black and white (Chessboard). Prove that exactly then half of the spherical surface is white.

Sphere Coloring by Goutham 1201. A sphere is colored in two colors. Prove that there exists an equilateral triangle whose vertices are of same colour and whose vertices lie on the sphere.

Toronto Junior Math Battle 1202. Given a solid sphere, construct its diameter using compass and straightedge.

Problem 1203. Let $ABCD$ be a tetrahedron and O its incenter, and let the line OD be perpendicular to AD. Find the angle between the planes DOB and DOC.

Problem 1204. The colonizers of a spherical planet have decided to build N towns, each having area $1/1000$ of the total area of the planet. They also decided that any two points belonging to different towns will have different latitude and different longitude. What is the maximal value of N?

Problem 1205. Find the minimum and maximum value of $f(x, y)$, where x and y are real numbers, and

$$f(x, y) = 2\sin x \cos y + 3\sin x \sin y + 6\cos x.$$

Problem 1206. Let a tetrahedron $ABCD$ be inscribed in a sphere S. Find the locus of points P inside the sphere S for which the equality

$$\frac{AP}{PA_1} + \frac{BP}{PB_1} + \frac{CP}{PC_1} + \frac{DP}{PD_1} = 4,$$

holds, where A_1, B_1, C_1, and D_1 are the intersection points of S with the lines AP, BP, CP, and DP, respectively.

Problem 1207. In the tetrahedron $OABC$ with volume V, we denote by α, β, γ the measures of the angles $\angle BOC, \angle COA$, and $\angle AOB$, respectively. Prove that:

$$36 \cdot V^2 = |OA|^2 \cdot |OB|^2 \cdot |OC|^2 \cdot \begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix}.$$

Problem 1208. Given a pyramid whose base is an n-gon inscribable in a circle, let H be the projection of the top vertex of the pyramid to its base. Prove that the projections of H to the lateral edges of the pyramid lie on a circle.

Spherical Mirrors by Amir 1209. In a curved mirror with a focal length f, we define $m = \frac{A'B'}{AB}$, where AB is the length of the object and $A'B'$ is the length of the image. Also, let d be the distance between the object and its image. Then, we have

$$f = \frac{md}{|m^2 - 1|}.$$

Find all pairs of positive integers (m, d) such that the number $f = md/(m^2 - 1)$ is an integer.

2012 Gulf Math Olympiad 1210. Fawzi cuts a spherical cheese completely into (at least three) slices of equal thickness. He starts at one end, making successive parallel cuts, working through the cheese until the slicing is complete. The discs exposed by the first two cuts have integral areas.

a) Prove that all the discs that he cuts have integral areas.

b) Prove that the original sphere had integral surface area if, and only if, the area of the second disc that he exposes is even.

Cylindrical and Spherical Coordinates by Mathwizarddude 1211. Use both cylindrical and spherical coordinates to find the volume of the solid E that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 16$.

Wisheskernel's Spherical Coordinates 1212. Determine the volume of the region that bounded by

$$\left(x^2 + y^2 + z^2\right)^2 = 2z \cdot \left(x^2 + y^2\right).$$

2010 Olympic Revenge 1213. Prove that there exists a set S of lines in the three dimensional space satisfying the following conditions:

a) For each point P in the space, there exist a unique line of S containing P.

b) There are no two lines of S which are parallel.

Problem 1214. If x and y are real numbers such that

$$2 \sin x \sin y + 3 \cos y + 6 \cos x \sin y = 7,$$

then find the value of

$$\tan^2 x + 2 \tan^2 y.$$

2018 Deerfield Math Competition (Extra) 1215. Evaluate the existence of the following limit:

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2y^2z^2}{x^2 + y^2 + z^2}.$$

Problem 1216. Describe What is the graph of the equation $x^2 + y^2 = 3z^2$ in the xyz plane?

Maximization on Sphere by WeakMathemetician 1217. Given three real numbers x, y, z, such that $x^2 + y^2 + z^2 = 1$, maximize

$$x^4 + y^4 - 2z^4 - 3\sqrt{2}xyz.$$

Problem 1218. A spherical, three dimensional planet has center at $(0, 0, 0)$ and radius 20. At any point (x, y, z), on the surface of this planet, the temperature $T := (x + y)^2 + (y - z)^2$ degrees. What is the average temperature of the surface of this planet?

Area of Spherical Polygon 1219. The radius of the sphere is 1. Spherical polygon has n edges and k inner angles: $\alpha_1, \alpha_2, \ldots, \alpha_k$. Prove that the area of spherical polygon is

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k - (n - 2)\pi.$$

2010 USAMTS Round III 1220. The sequences $(a_n), (b_n)$, and (c_n) are defined by $a_0 = 1, b_0 = 0, c_0 = 0$, and

$$a_n = a_{n-1} + \frac{c_{n-1}}{n}, b_n = b_{n-1} + \frac{a_{n-1}}{n}, c_n = c_{n-1} + \frac{b_{n-1}}{n}$$

for all $n \geq 1$. Prove that

$$\left| a_n - \frac{n+1}{3} \right| < \frac{2}{\sqrt{3n}}$$

for all $n \geq 1$. It is still a myth if anyone can solve this problem using spherical coordinates... There is a reason this problem is notorious for being the most difficult USAMTS problem ever!

Part II

Olympiad Algebra 299 (Vol I. Complementary): 407 Polynomials & Trigonometry Review Problems

.1 Systems of Equations and Complex Numbers

1983 AIME, Problem 5 1221. Suppose that the sum of the squares of two complex numbers x and y is 7 and the sum of the cubes is 10. What is the largest real value that $x+y$ can have?

1984 AIME, Problem 8 1222. The equation z^6+z^3+1 has one complex root with argument θ between $90°$ and $180°$ in the complex plane.

Determine the degree measure of θ.

1984 AIME, Problem 15 1223. Determine $w^2+x^2+y^2+z^2$ if

$$\begin{aligned}
\frac{x^2}{2^2-1}+\frac{y^2}{2^2-3^2}+\frac{z^2}{2^2-5^2}+\frac{w^2}{2^2-7^2}&=1,\\
\frac{x^2}{4^2-1}+\frac{y^2}{4^2-3^2}+\frac{z^2}{4^2-5^2}+\frac{w^2}{4^2-7^2}&=1,\\
\frac{x^2}{6^2-1}+\frac{y^2}{6^2-3^2}+\frac{z^2}{6^2-5^2}+\frac{w^2}{6^2-7^2}&=1,\\
\frac{x^2}{8^2-1}+\frac{y^2}{8^2-3^2}+\frac{z^2}{8^2-5^2}+\frac{w^2}{8^2-7^2}&=1.
\end{aligned}$$

1986 AIME, Problem 11 1224. The polynomial $1-x+x^2-x^3+\cdots+x^{16}-x^{17}$ may be written in the form $a_0+a_1y+a_2y^2+\cdots+a_{16}y^{16}+a_{17}y^{17}$, where $y=x+1$ and thet a_i's are constants. Find the value of a_2.

1988 AIME, Problem 11 1225. Let $w_1,w_2,\dots,w_n$ be complex numbers. A line L in the complex plane is called a mean line for the points $w_1,w_2,\dots,w_n$ if L contains points (complex numbers) $z_1,z_2,\dots,z_n$ such that

$$\sum_{k=1}^{n}(z_k-w_k)=0.$$

For the numbers $w_1=32+170i$, $w_2=-7+64i$, $w_3=-9+200i$, $w_4=1+27i$, and $w_5=-14+43i$, there is a unique mean line with y-intercept 3. Find the slope of this mean line.

1988 AIME, Problem 13 1226. Find a if a and b are integers such that x^2-x-1 is a factor of $ax^{17}+bx^{16}+1$.

1989 AIME, Problem 8 1227. Assume that $x_1,x_2,\dots,x_7$ are real numbers such that

$$\begin{aligned}
x_1+4x_2+9x_3+16x_4+25x_5+36x_6+49x_7&=1,\\
4x_1+9x_2+16x_3+25x_4+36x_5+49x_6+64x_7&=12,\\
9x_1+16x_2+25x_3+36x_4+49x_5+64x_6+81x_7&=123.
\end{aligned}$$

Find the value of

$$16x_1+25x_2+36x_3+49x_4+64x_5+81x_6+100x_7.$$

1989 AIME, Problem 14 1228. Given a positive integer n, it can be shown that every complex number of the form $r+si$, where r and s are integers, can be uniquely expressed in the base $-n+i$ using the integers $1, 2, \ldots, n^2$ as digits. That is, the equation

$$r+si = a_m(-n+i)^m + a_{m-1}(-n+i)^{m-1} + \cdots + a_1(-n+i) + a_0,$$

is true for a unique choice of non-negative integer m and digits $a_0, a_1, \ldots, a_m$ chosen from the set $\{0, 1, 2, \ldots, n^2\}$, with $a_m \neq 0$. We write

$$r+si = (a_m a_{m-1} \ldots a_1 a_0)_{-n+i},$$

to denote the base $-n+i$ expansion of $r+si$. There are only finitely many integers $k+0i$ that have four-digit expansions

$$k = (a_3a_2a_1a_0)_{-3+i}, \qquad a_3 \neq 0.$$

Find the sum of all such k.

1990 AIME, Problem 10 1229. The sets $A = \{z : z^{18} = 1\}$ and $B = \{w : w^{48} = 1\}$ are both sets of complex roots of unity. The set $C = \{zw : z \in A \text{ and } w \in B\}$ is also a set of complex roots of unity. How many distinct elements are in C?

1990 AIME, Problem 15 1230. Find $ax^5 + by^5$ if the real numbers a, b, x, and y satisfy the equations

$$\begin{aligned} ax+by &= 3, \\ ax^2+by^2 &= 7, \\ ax^3+by^3 &= 16, \\ ax^4+by^4 &= 42. \end{aligned}$$

1991 AIME, Problem 7 1231. Find A^2, where A is the sum of the absolute values of all roots of the following equation:

$$x = \sqrt{19} + \cfrac{91}{\sqrt{19} + \cfrac{91}{\sqrt{19} + \cfrac{91}{\sqrt{19} + \cfrac{91}{\sqrt{19} + \cfrac{91}{x}}}}}.$$

1992 AIME, Problem 8 1232. For any sequence of real numbers $A = (a_1, a_2, a_3, \ldots)$, define ΔA to be the sequence $(a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots)$, whose n^{th} term is $a_{n+1} - a_n$. Suppose that all of the terms of the sequence $\Delta(\Delta A)$ are 1, and that $a_{19} = a_{92} = 0$. Find a_1.

1992 AIME, Problem 10 1233. Consider the region A in the complex plane that consists of all points z such that both $z/40$ and $40/\overline{z}$ have real and imaginary parts between 0 and 1, inclusive. What is the integer that is nearest the area of A?

1993 AIME, Problem 5 1234. Let $P_0(x) = x^3 + 313x^2 - 77x - 8$. For integers $n \geq 1$, define $P_n(x) = P_{n-1}(x - n)$. What is the coefficient of x in $P_{20}(x)$?

1994 AIME, Problem 13 1235. The equation

$$x^{10} + (13x - 1)^{10} = 0$$

has 10 complex roots $r_1, \overline{r_1}, r_2, \overline{r_2}, r_3, \overline{r_3}, r_4, \overline{r_4}, r_5, \overline{r_5}$, where the bar denotes complex conjugation. Find the value of

$$\frac{1}{r_1\overline{r_1}} + \frac{1}{r_2\overline{r_2}} + \frac{1}{r_3\overline{r_3}} + \frac{1}{r_4\overline{r_4}} + \frac{1}{r_5\overline{r_5}}.$$

1995 AIME, Problem 5 1236. For certain real values of a, b, c, and d, the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ has four non-real roots. The product of two of these roots is $13 + i$ and the sum of the other two roots is $3 + 4i$, where $i = \sqrt{-1}$. Find b.

1995 AIME, Problem 13 1237. Let $f(n)$ be the integer closest to $\sqrt[4]{n}$. Find

$$\sum_{k=1}^{1995} \frac{1}{f(k)}.$$

1994 Turkey, Second Round, Problem 1 1238. For a positive integer n, let a_n denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{a_n^3}.$$

1996 AIME, Problem 5 1239. Suppose that the roots of $x^3 + 3x^2 + 4x - 11 = 0$ are a, b, and c, and that the roots of $x^3 + rx^2 + sx + t = 0$ are $a + b, b + c$, and $c + a$. Find t.

1996 AIME, Problem 11 1240. Let P be the product of the roots of $z^6 + z^4 + z^3 + z^2 + 1 = 0$ that have positive imaginary part, and suppose that $P = r(\cos\theta^\circ + i\sin\theta^\circ)$, where $0 < r$ and $0 \leq \theta < 360$. Find θ.

1997 AIME, Problem 14 1241. Let v and w be distinct, randomly chosen roots of the equation $z^{1997} - 1 = 0$. Let m/n be the probability that $\sqrt{2 + \sqrt{3}} \leq |v + w|$, where m and n are relatively prime positive integers. Find $m + n$.

1998 AIME, Problem 13 1242. If $\{a_1, a_2, a_3, \ldots, a_n\}$ is a set of real numbers, indexed so that $a_1 < a_2 < a_3 < \cdots < a_n$, its complex power sum is defined to be

$$a_1 i + a_2 i^2 + a_3 i^3 + \cdots + a_n i^n,$$

where $i^2 = -1$. Let S_n be the sum of the complex power sums of all nonempty subsets of $\{1, 2, \ldots, n\}$. Given that $S_8 = -176 - 64i$ and $S_9 = p + qi$, were p and q are integers, find $|p| + |q|$.

1999 AIME, Problem 9 1243. A function f is defined on the complex numbers by $f(z) = (a + bi)z$, where a and b are positive numbers. This function has the property that the image of each point in the complex plane is equidistant from that point and the origin. Given that $|a + bi| = 8$ and that $b^2 = m/n$, where m and n are relatively prime positive integers. Find $m + n$.

2000 AIME I, Problem 9 1244. The system of equations

$$\begin{aligned} \log_{10}(2000xy) - (\log_{10} x)(\log_{10} y) &= 4, \\ \log_{10}(2yz) - (\log_{10} y)(\log_{10} z) &= 1, \\ \log_{10}(zx) - (\log_{10} z)(\log_{10} x) &= 0. \end{aligned}$$

has two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2). Find $y_1 + y_2$.

2000 AIME II, Problem 1 1245. The number

$$\frac{2}{\log_4 2000^6} + \frac{3}{\log_5 2000^6}$$

can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

2000 AIME II, Problem 9 1246. Given that z is a complex number such that

$$z + \frac{1}{z} = 2\cos 3^\circ,$$

find the least integer that is greater than

$$z^{2000} + \frac{1}{z^{2000}}.$$

2000 AIME II, Problem 13 1247. The equation

$$2000x^6 + 100x^5 + 10x^3 + x - 2 = 0$$

has exactly two real roots, one of which is $(m + \sqrt{n})/r$, where m, n and r are integers, m and r are relatively prime, and $r > 0$. Find $m + n + r$.

2001 AIME I, Problem 3 1248. Find the sum of the roots, real and non-real, of the equation

$$x^{2001} + \left(\frac{1}{2} - x\right)^{2001} = 0,$$

given that there are no multiple roots.

2001 AIME II, Problem 8 1249. A certain function f has the properties that $f(3x) = 3f(x)$ for all positive real values of x, and that $f(x) = 1- \mid x - 2 \mid$ for $1 \le x \le 3$. Find the smallest x for which $f(x) = f(2001)$.

2001 AIME II, Problem 14 1250. There are $2n$ complex numbers that satisfy both

$$z^{28} - z^8 - 1 = 0 \qquad \text{and} \qquad |z| = 1.$$

These numbers have the form $z_m = \cos\theta_m + i\sin\theta_m$, where $0 \le \theta_1 < \theta_2 < \cdots < \theta_{2n} < 360$ and angles are measured in degrees. Find the value of $\theta_2 + \theta_4 + \cdots + \theta_{2n}$.

2002 AIME I, Problem 6 1251. The solutions to the system of equations

$$\begin{aligned} \log_{225} x + \log_{64} y &= 4, \\ \log_x 225 - \log_y 64 &= 1, \end{aligned}$$

are (x_1, y_1) and (x_2, y_2). Find $\log_{30}(x_1 y_1 x_2 y_2)$.

2002 AIME I, Problem 7 1252. The Binomial Expansion is valid for exponents that are not integers. That is, for all real numbers x, y, and r with $|x| > |y|$,

$$(x+y)^r = x^r + r x^{r-1} y + \frac{r(r-1)}{2} x^{r-2} y^2 + \frac{r(r-1)(r-2)}{3!} x^{r-3} y^3 + \cdots$$

What are the first three digits to the right of the decimal point in the decimal representation of $(10^{2002}+1)^{10/7}$?

2002 AIME I, Problem 12 1253. Let

$$F(z) = \frac{z+i}{z-i},$$

for all complex numbers $z \neq i$, and let $z_n = F(z_{n-1})$ for all positive integers n. Given that

$$z_0 = \frac{1}{137} + i \qquad \text{and} \qquad z_{2002} = a + bi,$$

where a and b are real numbers, find $a + b$.

2002 AIME II, Problem 3 1254. It is given that $\log_6 a + \log_6 b + \log_6 c = 6$, where a, b, and c are positive integers that form an increasing geometric sequence and $b - a$ is the square of an integer. Find $a + b + c$.

2002 AIME II, Problem 6 1255. Find the integer that is closest to

$$1000 \sum_{n=3}^{10000} \frac{1}{n^2 - 4}.$$

2003 AIME I, Problem 4 1256. Given that

$$\log_{10} \sin x + \log_{10} \cos x = -1,$$

and that

$$\log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10} n - 1),$$

find n.

2003 AIME II, Problem 9 1257. Consider the polynomials

$$P(x) = x^6 - x^5 - x^3 - x^2 - x,$$

and $Q(x) = x^4 - x^3 - x^2 - 1$. Given that z_1, z_2, z_3, and z_4 are the roots of $Q(x) = 0$, find

$$P(z_1) + P(z_2) + P(z_3) + P(z_4).$$

2003 AIME II, Problem 15 1258. Let

$$P(x) = 24x^{24} + \sum_{j=1}^{23}(24-j)(x^{24-j} + x^{24+j}).$$

Let $z_1, z_2, \ldots, z_r$ be the distinct zeros of $P(x)$, and let $z_k^2 = a_k + b_k i$ for $k = 1, 2, \ldots, r$, where $i = \sqrt{-1}$, and a_k and b_k are real numbers. Let

$$\sum_{k=1}^{r} |b_k| = m + n\sqrt{p},$$

where m, n, and p are integers and p is not divisible by the square of any prime. Find $m + n + p$.

2004 AIME I, Problem 7 1259. Let C be the coefficient of x^2 in the expansion of the product

$$(1-x)(1+2x)(1-3x)\cdots(1+14x)(1-15x).$$

Find $|C|$.

2004 AIME I, Problem 13 1260. The polynomial

$$P(x) = (1 + x + x^2 + \cdots + x^{17})^2 - x^{17}$$

has 34 complex roots of the form $z_k = r_k[\cos(2\pi a_k) + i\sin(2\pi a_k)], k = 1, 2, 3, \ldots, 34$, with $0 < a_1 \le a_2 \le a_3 \le \cdots \le a_{34} < 1$ and $r_k > 0$. Given that $a_1 + a_2 + a_3 + a_4 + a_5 = m/n$, where m and n are relatively prime positive integers, find $m + n$.

2005 AIME I, Problem 6 1261. Let P be the product of the non-real roots of $x^4 - 4x^3 + 6x^2 - 4x = 2005$. Find $\lfloor P \rfloor$.

2005 AIME I, Problem 8 1262. The equation

$$2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$$

has three real roots. Given that their sum is m/n where m and n are relatively prime positive integers, find $m + n$.

2005 AIME II, Problem 7 1263. Let

$$x = \frac{4}{(\sqrt{5}+1)(\sqrt[4]{5}+1)(\sqrt[8]{5}+1)(\sqrt[16]{5}+1)}.$$

Find $(x+1)^{48}$.

2005 AIME II, Problem 9 1264. For how many positive integers n less than or equal to 1000 is

$$(\sin t + i\cos t)^n = \sin nt + i\cos nt$$

true for all real t?

2005 AIME II, Problem 13 1265. Let $P(x)$ be a polynomial with integer coefficients that satisfies $P(17) = 10$ and $P(24) = 17$. Given that $P(n) = n + 3$ has two distinct integer solutions n_1 and n_2, find the product $n_1 \cdot n_2$.

2006 AIME II, Problem 15 1266. Given that x, y, and z are real numbers that satisfy:

$$\begin{aligned} x &= \sqrt{y^2 - \frac{1}{16}} + \sqrt{z^2 - \frac{1}{16}}, \\ y &= \sqrt{z^2 - \frac{1}{25}} + \sqrt{x^2 - \frac{1}{25}}, \\ z &= \sqrt{x^2 - \frac{1}{36}} + \sqrt{y^2 - \frac{1}{36}}, \end{aligned}$$

and that $x + y + z = m/\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime, find $m + n$.

2007 AIME I, Problem 3 1267. The complex number z is equal to $9 + bi$, where b is a positive real number and $i^2 = -1$. Given that the imaginary parts of z^2 and z^3 are equal, find b.

2007 AIME I, Problem 8 1268. The polynomial $P(x)$ is cubic. What is the largest value of k for which the polynomials

$$Q_1(x) = x^2 + (k - 29)x - k \qquad \text{and} \qquad Q_2(x) = 2x^2 + (2k - 43)x + k,$$

are both factors of $P(x)$?

2007 AIME II, Problem 7 1269. Given a real number x, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. For a certain integer k, there are exactly 70 positive integers $n_1, n_2, \ldots, n_{70}$ such that

$$k = \lfloor \sqrt[3]{n_1} \rfloor = \lfloor \sqrt[3]{n_2} \rfloor = \cdots = \lfloor \sqrt[3]{n_{70}} \rfloor,$$

and k divides n_i for all i such that $1 \le i \le 70$. Find the maximum value of n_i/k for $1 \le i \le 70$.

2007 AIME II, Problem 14 1270. Let $f(x)$ be a polynomial with real coefficients such that $f(0) = 1$, $f(2) + f(3) = 125$, and for all x, $f(x)f(2x^2) = f(2x^3 + x)$. Find $f(5)$.

2008 AIME I, Problem 13 1271. Let

$$p(x, y) = a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + a_6x^3 + a_7x^2y + a_8xy^2 + a_9y^3.$$

Suppose that

$$\begin{aligned} p(0,0) &= p(1,0) = p(-1,0) = p(0,1) = p(0,-1) \\ &= p(1,1) = p(1,-1) = p(2,2) = 0. \end{aligned}$$

There is a point $(a/c, b/c)$ for which $p\,(a/c, b/c) = 0$ for all such polynomials, where a, b, and c are positive integers, a and c are relatively prime, and $c > 1$. Find $a + b + c$.

2008 AIME II, Problem 7 1272. Let r, s, and t be the three roots of the equation

$$8x^3 + 1001x + 2008 = 0.$$

Find $(r+s)^3 + (s+t)^3 + (t+r)^3$.

2009 AIME I, Problem 2 1273. There is a complex number z with imaginary part 164 and a positive integer n such that

$$\frac{z}{z+n} = 4i.$$

Find n.

2009 AIME II, Problem 2 1274. Suppose that a, b, and c are positive real numbers such that $a^{\log_3 7} = 27$, $b^{\log_7 11} = 49$, and $c^{\log_{11} 25} = \sqrt{11}$. Find

$$a^{(\log_3 7)^2} + b^{(\log_7 11)^2} + c^{(\log_{11} 25)^2}.$$

2020 USAMTS, Year 32, Round 1, Problem 5 1275. Find all pairs of rational numbers (a, b) such that $0 < a < b$ and

$$a^a = b^b.$$

2010 AIME I, Problem 3 1276. Suppose that

$$y = \frac{3}{4}x \qquad \text{and} \qquad x^y = y^x.$$

The quantity $x+y$ can be expressed as a rational number r/s, where r and s are relatively prime positive integers. Find $r+s$.

2010 AIME I, Problem 6 1277. Let $P(x)$ be a quadratic polynomial with real coefficients satisfying

$$x^2 - 2x + 2 \leq P(x) \leq 2x^2 - 4x + 3$$

for all real numbers x, and suppose $P(11) = 181$. Find $P(16)$.

2010 AIME I, Problem 8 1278. For a real number a, let $\lfloor a \rfloor$ denominate the greatest integer less than or equal to a. Let $\mathcal{R}$ denote the region in the coordinate plane consisting of points (x, y) such that

$$\lfloor x \rfloor^2 + \lfloor y \rfloor^2 = 25.$$

The region $\mathcal{R}$ is completely contained in a disk of radius r (a disk is the union of a circle and its interior). The minimum value of r can be written as $\sqrt{m}/n$, where m and n are integers and m is not divisible by the square of any prime. Find $m+n$.

2010 AIME I, Problem 9 1279. Let (a, b, c) be the real solution of the system of equations $x^3 - xyz = 2$, $y^3 - xyz = 6$, $z^3 - xyz = 20$. The greatest possible value of $a^3 + b^3 + c^3$ can be written in the form m/n, where m and n are relatively prime positive integers. Find $m+n$.

2010 AIME II, Problem 5 1280. Positive numbers x, y, and z satisfy $xyz = 10^{81}$ and $(\log_{10} x)(\log_{10} yz) + (\log_{10} y)(\log_{10} z) = 468$. Find

$$\sqrt{(\log_{10} x)^2 + (\log_{10} y)^2 + (\log_{10} z)^2}.$$

2010 AIME II, Problem 6 1281. Find the smallest positive integer n with the property that the polynomial $x^4 - nx + 63$ can be written as a product of two nonconstant polynomials with integer coefficients.

2010 AIME II, Problem 7 1282. Let $P(z) = z^3 + az^2 + bz + c$, where a, b, and c are real. There exists a complex number w such that the three roots of $P(z)$ are $w + 3i$, $w + 9i$, and $2w - 4$, where $i^2 = -1$. Find $|a + b + c|$.

2010 AIME II, Problem 10 1283. Find the number of second-degree polynomials $f(x)$ with integer coefficients and integer zeros for which $f(0) = 2010$.

2011 AIME I, Problem 6 1284. Suppose that a parabola has vertex $(1/4, -9/8)$, and equation $y = ax^2 + bx + c$, where $a > 0$ and $a + b + c$ is an integer. The minimum possible value of a can be written as p/q, where p and q are relatively prime positive integers. Find $p + q$.

2011 AIME I, Problem 9 1285. Suppose x is in the interval $[0, \pi/2]$ and

$$\log_{24 \sin x}(24 \cos x) = \frac{3}{2}.$$

Find $24 \cot^2 x$.

2011 AIME I, Problem 15 1286. For some integer m, the polynomial $x^3 - 2011x + m$ has the three integer roots a, b, and c. Find $|a| + |b| + |c|$.

2011 AIME II, Problem 8 1287. Let $z_1, z_2, z_3, \dots, z_{12}$ be the 12 zeroes of the polynomial $z^{12} - 2^{36}$. For each j, let w_j be one of z_j or iz_j. Then the maximum possible value of the real part of $w_1 + w_2 + \cdots + w_{12}$ can be written as $m + \sqrt{n}$ where m and n are positive integers. Find $m + n$.

2011 AIME II, Problem 11 1288. Let M_n be the $n \times n$ matrix with entries as follows: for $1 \le i \le n$, $m_{i,i} = 10$; for $1 \le i \le n - 1, m_{i+1,i} = m_{i,i+1} = 3$; all other entries in M_n are zero. Let D_n be the determinant of matrix M_n. Then,

$$\sum_{n=1}^{\infty} \frac{1}{8D_n + 1}$$

can be represented as p/q, where p and q are relatively prime positive integers. Find $p + q$. *Note*: The determinant of the 1×1 matrix $[a]$ is a, and the determinant of the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$; for $n \ge 2$, the determinant of an $n \times n$ matrix with first row or first column $a_1\ a_2\ a_3 \dots\ a_n$ is equal to

$$a_1C_1 - a_2C_2 + a_3C_3 - \cdots + (-1)^{n+1}a_nC_n,$$

where C_i is the determinant of the $(n - 1) \times (n - 1)$ matrix found by eliminating the row and column containing a_i.

2011 HMMT, Guts, Problem 17 1289. Given positive real numbers x, y, and z that satisfy the following system of equations:

$$x^2 + y^2 + xy = 1,$$
$$y^2 + z^2 + yz = 4,$$
$$z^2 + x^2 + zx = 5,$$

find $x + y + z$.

2011 HMMT, Guts, Problem 27 1290. Find the number of polynomials $p(x)$ with integer coefficients satisfying

$$p(x) \geq \min\{2x^4 - 6x^2 + 1, 4 - 5x^2\},$$

and

$$p(x) \leq \max\{2x^4 - 6x^2 + 1, 4 - 5x^2\},$$

for all real numbers x.

2011 HMMT, Algebra & Geometry, Problem 27 1291. Let $f(x) = x^2 + 6x + c$ for all real numbers x, where c is some real number. For what values of c does $f(f(x))$ have exactly 3 distinct real roots?

2012 AIME I, Problem 6 1292. The complex numbers z and w satisfy

$$z^{13} = w, w^{11} = z,$$

and the imaginary part of z is $\sin(m\pi/n)$ for relatively prime positive integers m and n with $m < n$. Find n.

2012 AIME I, Problem 9 1293. Let x, y, and z be positive real numbers that satisfy

$$2\log_x(2y) = 2\log_{2x}(4z) = \log_{2x^4}(8yz) \neq 0.$$

The value of xy^5z can be expressed in the form $1/2^{p/q}$, where p and q are relatively prime integers. Find $p + q$.

2012 AIME I, Problem 14 1294. Complex numbers a, b and c are the zeros of a polynomial $P(z) = z^3 + qz + r$, and $|a|^2 + |b|^2 + |c|^2 = 250$. The points corresponding to a, b, and c in the complex plane are the vertices of a right triangle with hypotenuse h. Find h^2.

2012 AIME II, Problem 6 1295. Let $z = a + bi$ be the complex number with $|z| = 5$ and $b > 0$ such that the distance between $(1 + 2i)z^3$ and z^5 is maximized, and let $z^4 = c + di$.

2012 AIME II, Problem 8 1296. The complex numbers z and w satisfy the system

$$z + \frac{20i}{w} = 5 + i, \text{ and}$$
$$w + \frac{12i}{z} = -4 + 10i.$$

Find the smallest possible value of $|zw|^2$.

2013 AIME I, Problem 5 1297. The real root of $8x^3-3x^2-3x-1=0$ can be written in the form

$$\frac{\sqrt[3]{a}+\sqrt[3]{b}+1}{c},$$

where a, b, and c are positive integers. Find $a+b+c$.

2013 AIME I, Problem 10 1298. There are nonzero integers a, b, r, and s such that the complex number $r+si$ is a zero of the polynomial $P(x)=x^3-ax^2+bx-65$. For each possible combination of a and b, let $p_{a,b}$ be the sum of the zeroes of $P(x)$. Find the sum of the $p_{a,b}$'s for all possible combinations of a and b.

2013 AIME II, Problem 2 1299. Positive integers a and b satisfy the condition

$$\log_2(\log_{2^a}(\log_{2^b}(2^{1000})))=0.$$

Find the sum of all possible values of $a+b$.

2013 AIME II, Problem 12 1300. Let S be the set of all polynomials of the form z^3+az^2+bz+c, where a, b, and c are integers. Find the number of polynomials in S such that each of its roots z satisfies either $|z|=20$ or $|z|=13$.

2013 HMMT, Algebra, Problem 4 1301. Determine all real values of A for which there exist distinct complex numbers x_1, x_2 such that the following three equations hold:

$$\begin{aligned} x_1(x_1+1)&=A,\\ x_2(x_2+1)&=A,\\ x_1^4+3x_1^3+5x_1&=x_2^4+3x_2^3+5x_2. \end{aligned}$$

2013 HMMT, Algebra, Problem 5 1302. Let a and b be real numbers, and let r, s, and t be the roots of $f(x)=x^3+ax^2+bx-1$. Also, $g(x)=x^3+mx^2+nx+p$ has roots r^2, s^2, and t^2. If $g(-1)=-5$, find the maximum possible value of b.

2013 HMMT, Algebra, Problem 7 1303. Compute

$$\sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\cdots\sum_{a_7=0}^{\infty}\frac{a_1+a_2+\cdots+a_7}{3^{a_1+a_2+\cdots+a_7}}.$$

2013 HMMT, Algebra, Problem 8 1304. Let x,y be complex numbers such that

$$\frac{x^2+y^2}{x+y}=4 \qquad \text{and} \qquad \frac{x^4+y^4}{x^3+y^3}=2.$$

Find all possible values of

$$\frac{x^6+y^6}{x^5+y^5}.$$

2013 HMMT, Algebra, Problem 9 1305. Let z be a non-real complex number with $z^{23}=1$. Compute

$$\sum_{k=0}^{22}\frac{1}{1+z^k+z^{2k}}.$$

2013 HMMT, Guts, Problem 11 1306. Compute the prime factorization of

$$1007021035035021007001.$$

You should write your answer in the form $p_1^{e_1}p_2^{e_2}\dots p_k^{e_k}$ where $p_1,\dots,p_k$ are distinct prime numbers and $e_1,\dots,e_k$ are positive integers.

2013 HMMT, Guts, Problem 20 1307. The polynomial $f(x)=x^3-3x^2-4x+4$ has three real roots r_1, r_2, and r_3. Let $g(x)=x^3+ax^2+bx+c$ be the polynomial which has roots s_1, s_2, and s_3, where

$$\begin{aligned}s_1&=r_1+r_2z+r_3z^2,\\ s_2&=r_1z+r_2z^2+r_3,\\ s_3&=r_1z^2+r_2+r_3z,\end{aligned}$$

and $z=(-1+i\sqrt{3})/2$. Find the real part of the sum of the coefficients of $g(x)$.

2013 HMMT, Guts, Problem 28 1308. Let $z_0+z_1+z_2+\cdots$ be an infinite complex geometric series such that $z_0=1$ and $z_{2013}=\dfrac{1}{2013^{2013}}$. Find the sum of all possible sums of this series.

2013 HMMT, Guts, Problem 33 1309. Compute the value of $1^{25}+2^{24}+3^{23}+\ldots+24^2+25^1$. If your answer is A and the correct answer is C, then your score on this problem (out of 25) will be

$$\left\lfloor 25\min\left(\left(\frac{A}{C}\right)^2,\left(\frac{C}{A}\right)^2\right)\right\rfloor.$$

2012 HMMT, Algebra, Problem 8 1310. Let $x_1=y_1=x_2=y_2=1$, then for $n\geq 3$ let

$$x_n=x_{n-1}y_{n-2}+x_{n-2}y_{n-1}\qquad\text{and}\qquad y_n=y_{n-1}y_{n-2}-x_{n-1}x_{n-2}.$$

What are the last two digits of $|x_{2012}|$?

2012 HMMT, Algebra, Problem 9 1311. How many real triples (a,b,c) are there such that the polynomial

$$p(x)=x^4+ax^3+bx^2+ax+c,$$

has exactly three distinct roots, which are equal to $\tan y$, $\tan 2y$, and $\tan 3y$ for some real number y?

2012 HMMT, Guts, Problem 18 1312. Let x and y be positive real numbers such that $x^2+y^2=1$ and

$$(3x-4x^3)(3y-4y^3)=-\frac{1}{2}.$$

Compute $x+y$.

2014 AIME I, Problem 6 1313. The graphs of $y = 3(x-h)^2+j$ and $y = 2(x-h)^2+k$ have y-intercepts of 2013 and 2014, respectively, and each graph has two positive integer x-intercepts. Find h.

2014 AIME I, Problem 7 1314. Let w and z be complex numbers such that $|w| = 1$ and $|z| = 10$. Let

$$\theta = \arg\left(\frac{w-z}{z}\right).$$

The maximum possible value of $\tan^2\theta$ can be written as p/q, where p and q are relatively prime positive integers. Find $p+q$.

Note that $\arg(w)$, for $w \neq 0$, denotes the measure of the angle that the ray from 0 to w makes with the positive real axis in the complex plane.

2014 AIME I, Problem 9 1315. Let $x_1 < x_2 < x_3$ be three real roots of equation

$$\sqrt{2014}x^3 - 4029x^2 + 2 = 0.$$

Find $x_2(x_1+x_3)$.

2014 AIME I, Problem 14 1316. Let m be the largest real solution to the equation

$$\frac{3}{x-3} + \frac{5}{x-5} + \frac{17}{x-17} + \frac{19}{x-19} = x^2 - 11x - 4.$$

There are positive integers a, b, c such that $m = a + \sqrt{b + \sqrt{c}}$. Find $a+b+c$.

2014 AIME II, Problem 5 1317. Real numbers r and s are roots of $p(x) = x^3+ax+b$, and $r+4$ and $s-3$ are roots of $q(x) = x^3 + ax + b + 240$. Find the sum of all possible values of $|b|$.

2014 AIME II, Problem 7 1318. Let

$$f(x) = (x^2 + 3x + 2)^{\cos(\pi x)}.$$

Find the sum of all positive integers n for which

$$\left|\sum_{k=1}^{n} \log_{10} f(k)\right| = 1.$$

2014 AIME II, Problem 10 1319. Let z be a complex number with $|z| = 2014$. Let P be the polygon in the complex plane whose vertices are z and every w such that

$$\frac{1}{z+w} = \frac{1}{z} + \frac{1}{w}.$$

Then the area enclosed by P can be written in the form $n\sqrt{3}$, where n is an integer. Find the remainder when n is divided by 1000.

2014 HMMT, Algebra, Problem 3 1320. Let

$$A = \frac{1}{6}((\log_2(3))^3 - (\log_2(6))^3 - (\log_2(12))^3 + (\log_2(24))^3).$$

Compute 2^A.

2014 HMMT, Algebra, Problem 4 1321. Let b and c be real numbers and define the polynomial $P(x) = x^2 + bx + c$. Suppose that $P(P(1)) = P(P(2)) = 0$, and that $P(1) \neq P(2)$. Find $P(0)$.

2014 HMMT, Algebra, Problem 5 1322. Find the sum of all real numbers x such that $5x^4 - 10x^3 + 10x^2 - 5x - 11 = 0$.

2014 HMMT, Algebra, Problem 6 1323. Given w and z are complex numbers such that $|w + z| = 1$ and $|w^2 + z^2| = 14$, find the smallest possible value of $|w^3 + z^3|$. Here $|\cdot|$ denotes the absolute value of a complex number, given by $|a + bi| = \sqrt{a^2 + b^2}$ whenever a and b are real numbers.

2014 HMMT, Algebra, Problem 8 1324. Find all real numbers k such that $r^4 + kr^3 + r^2 + 4kr + 16 = 0$ is true for exactly one real number r.

2014 HMMT, Algebra, Problem 1325. Given a, b, and c are complex numbers satisfying

$$\begin{aligned} a^2 + ab + b^2 &= 1 + i, \\ b^2 + bc + c^2 &= -2, \\ c^2 + ca + a^2 &= 1, \end{aligned}$$

compute $(ab + bc + ca)^2$. Here, $i = \sqrt{-1}$ is the imaginary unit.

2014 HMMT, Algebra, Problem 10 1326. For an integer n, let $f_9(n)$ denote the number of positive integers $d \leq 9$ dividing n. Suppose that m is a positive integer and $b_1, b_2, \ldots, b_m$ are real numbers such that

$$f_9(n) = \textstyle\sum_{j=1}^{m} b_j f_9(n - j),$$

for all $n > m$. Find the smallest possible value of m.

2014 HMMT, Guts, Problem 26 1327. For $1 \leq j \leq 2014$, define

$$b_j = j^{2014} \prod_{i=1, i \neq j}^{2014} (i^{2014} - j^{2014}),$$

where the product is over all $i \in \{1, \ldots, 2014\}$ except $i = j$. Evaluate

$$\frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_{2014}}.$$

2014 HMMT, Guts, Problem 28 1328. Let $f(n)$ and $g(n)$ be polynomials of degree 2014 such that $f(n) + (-1)^n g(n) = 2^n$ for $n = 1, 2, \ldots, 4030$. Find the coefficient of x^{2014} in $g(x)$.

2014 HMMT, Guts, Problem 32 1329. Find all ordered pairs (a, b) of complex numbers with $a^2 + b^2 \neq 0$,

$$a + \frac{10b}{a^2 + b^2} = 5,$$

and

$$b + \frac{10a}{a^2 + b^2} = 4.$$

2014 HMMT, Team, Problem 5 1330. Prove that there exists a nonzero complex number c and a real number d such that

$$\left| \left| \frac{1}{1 + z + z^2} \right| - \left| \frac{1}{1 + z + z^2} - c \right| \right| = d,$$

for all z with $|z| = 1$ and $1 + z + z^2 \neq 0$.

2014 HMIC, Problem 4 1331. Let ω be a root of unity and f be a polynomial with integer coefficients. Show that if $|f(\omega)| = 1$, then $f(\omega)$ is also a root of unity.

2015 AIME I, Problem 10 1332. Let $f(x)$ be a third-degree polynomial with real coefficients satisfying

$$|f(1)| = |f(2)| = |f(3)| = |f(5)| = |f(6)| = |f(7)| = 12.$$

Find $|f(0)|$.

2015 AIME II, Problem 6 1333. Steve says to Jon, "I am thinking of a polynomial whose roots are all positive integers. The polynomial has the form $P(x) = 2x^3 - 2ax^2 + (a^2 - 81)x - c$ for some positive integers a and c. Can you tell me the values of a and c?" After some calculations, Jon says, "There is more than one such polynomial." Steve says, "You're right. Here is the value of a." He writes down a positive integer and asks, "Can you tell me the value of c?" Jon says, "There are still two possible values of c." Find the sum of the two possible values of c.

2015 AIME II, Problem 14 1334. Let x and y be real numbers satisfying

$$x^4y^5 + y^4x^5 = 810 \qquad \text{and} \qquad x^3y^6 + y^3x^6 = 945.$$

Evaluate $2x^3 + (xy)^3 + 2y^3$.

2015 HMMT, Algebra, Problem 1 1335. Let Q be a polynomial

$$Q(x) = a_0 + a_1x + \cdots + a_nx^n,$$

where $a_0, \ldots, a_n$ are non-negative integers. Given that $Q(1) = 4$ and $Q(5) = 152$, find $Q(6)$.

2015 HMMT, Algebra, Problem 7 1336. Suppose (a_1, a_2, a_3, a_4) is a 4-term sequence of real numbers satisfying the following two conditions:

a) $a_3 = a_2 + a_1$ and $a_4 = a_3 + a_2$;

b) there exist real numbers a, b, c such that

$$an^2 + bn + c = \cos(a_n),$$

for all $n \in \{1, 2, 3, 4\}$.

Compute the maximum possible value of

$$\cos(a_1) - \cos(a_4),$$

over all such sequences (a_1, a_2, a_3, a_4).

2015 HMMT, Algebra, Problem 8 1337. Find the number of ordered pairs of integers $(a, b) \in \{1, 2, \ldots, 35\}^2$ (not necessarily distinct) such that $ax + b$ is a "quadratic residue modulo x^2+1 and 35", i.e. there exists a polynomial $f(x)$ with integer coefficients such that either of the following *equivalent* conditions holds:

a) there exist polynomials P, Q with integer coefficients such that

$$f(x)^2 - (ax + b) = (x^2 + 1)P(x) + 35Q(x);$$

b) or more conceptually, the remainder when (the polynomial) $f(x)^2 - (ax + b)$ is divided by (the polynomial) $x^2 + 1$ is a polynomial with integer coefficients all divisible by 35.

2015 HMMT, Algebra, Problem 10 1338. Find all ordered 4-tuples of integers (a, b, c, d) (not necessarily distinct) satisfying the following system of equations:

$$\begin{aligned} a^2 - b^2 - c^2 - d^2 &= c - b - 2 \\ 2ab &= a - d - 32 \\ 2ac &= 28 - a - d \\ 2ad &= b + c + 31. \end{aligned}$$

2015 HMMT, Team, Problem 3 1339. Let $z = a + bi$ be a complex number with integer real and imaginary parts $a, b \in \mathbb{Z}$, where $i = \sqrt{-1}$ is the imaginary unit. If p is an odd prime number, show that the real part of $z^p - z$ is an integer divisible by p.

2015 HMMT, Team, Problem 9 1340. Let

$$z = e^{2\pi i/101} \qquad \text{and} \qquad w = e^{2\pi i/10}.$$

Prove that

$$\prod_{a=0}^{9}\prod_{b=0}^{100}\prod_{c=0}^{100}(w^a + z^b + z^c)$$

is an integer and find (with proof) its remainder upon division by 101.

2015 HMMT, Guts, Problem 13 1341. Let $P(x) = x^3 + ax^2 + bx + 2015$ be a polynomial all of whose roots are integers. Given that $P(x) \geq 0$ for all $x \geq 0$, find the sum of all possible values of $P(-1)$.

2015 HMMT, Guts, Problem 25 1342. Let $r_1, \ldots, r_n$ be the distinct real zeroes of the equation

$$x^8 - 14x^4 - 8x^3 - x^2 + 1 = 0.$$

Evaluate $r_1^2 + \cdots + r_n^2$.

2015 HMMT, Guts, Problem 26 1343. Let $a = \sqrt{17}$ and $b = i\sqrt{19}$, where $i = \sqrt{-1}$. Find the maximum possible value of the ratio $|a - z|/|b - z|$ over all complex numbers z of magnitude 1 (i.e. over the unit circle $|z| = 1$).

2015 HMMT, Guts, Problem 30 1344. Find the sum of squares of all **distinct** complex numbers x satisfying the equation

$$0 = 4x^{10} - 7x^9 + 5x^8 - 8x^7 + 12x^6 - 12x^5 + 12x^4 - 8x^3 + 5x^2 - 7x + 4.$$

2015 HMIC, Problem 5 1345. Let $\omega = e^{2\pi i/5}$ be a primitive fifth root of unity. Prove that there do not exist integers a, b, c, d, k with $k > 1$ such that

$$(a + b\omega + c\omega^2 + d\omega^3)^k = 1 + \omega.$$

2016 AIME I, Problem 7 1346. For integers a and b consider the complex number

$$\frac{\sqrt{ab + 2016}}{ab + 100} - \left(\frac{\sqrt{|a + b|}}{ab + 100}\right) i.$$

Find the number of ordered pairs of integers (a, b) such that this complex number is a real number.

2016 AIME I, Problem 11 1347. Let $P(x)$ be a nonzero polynomial such that

$$(x - 1)P(x + 1) = (x + 2)P(x),$$

for every real x, and $\left(P(2)\right)^2 = P(3)$. Then $P(7/2) = m/n$, where m and n are relatively prime positive integers. Find $m + n$.

2016 AIME II, Problem 3 1348. Let x, y and z be real numbers satisfying the system

$$\begin{aligned}\log_2(xyz - 3 + \log_5 x) &= 5,\\ \log_3(xyz - 3 + \log_5 y) &= 4,\\ \log_4(xyz - 3 + \log_5 z) &= 4.\end{aligned}$$

Find the value of $|\log_5 x| + |\log_5 y| + |\log_5 z|$.

2016 AIME II, Problem 6 1349. For polynomial

$$P(x) = 1 - \frac{1}{3}x + \frac{1}{6}x^2,$$

define

$$Q(x) = P(x)P(x^3)P(x^5)P(x^7)P(x^9) = \sum_{i=0}^{50} a_i x^i.$$

Then,

$$\sum_{i=0}^{50} |a_i| = \frac{m}{n},$$

where m and n are relatively prime positive integers. Find $m+n$.

2016 HMMT, Algebra, Problem 10 1350. Let a, b and c be real numbers such that

$$a^2+ab+b^2=9,$$
$$b^2+bc+c^2=52,$$
$$c^2+ca+a^2=49.$$

Compute the value of

$$\frac{49b^2+39bc+9c^2}{a^2}.$$

2016 HMMT, Guts, Problem 1 1351. Let x and y be complex numbers such that $x+y=\sqrt{20}$ and $x^2+y^2=15$. Compute $|x-y|$.

2016 HMMT, Guts, Problem 23 1352. Let $t=2016$ and $p=\ln 2$. Evaluate in closed form the sum

$$\sum_{k=1}^{\infty}\left(1-\sum_{n=0}^{k-1}\frac{e^{-t}t^n}{n!}\right)(1-p)^{k-1}p.$$

2016 HMMT, Team, Problem 7 1353. Let $q(x)=q^1(x)=2x^2+2x-1$, and let $q^n(x)=q(q^{n-1}(x))$ for $n>1$. How many negative real roots does $q^{2016}(x)$ have?

2016 HMMT, November Theme, Problem 6 1354. Let $P_1, P_2, \ldots, P_6$ be points in the complex plane, which are also roots of the equation $x^6+6x^3-216=0$. Given that $P_1P_2P_3P_4P_5P_6$ is a convex hexagon, determine the area of this hexagon.

2016 HMIC, Problem 4 1355. Let P be an odd-degree integer-coefficient polynomial. Suppose that $xP(x)=yP(y)$ for infinitely many pairs x, y of integers with $x \neq y$. Prove that the equation $P(x)=0$ has an integer root.

2017 AIME I, Problem 10 1356. Let $z_1=18+83i$, $z_2=18+39i$, and $z_3=78+99i$, where $i=\sqrt{-1}$. Let z be the unique complex number with the properties that

$$\frac{z_3-z_1}{z_2-z_1}\cdot\frac{z-z_2}{z-z_3},$$

is a real number and the imaginary part of z is the greatest possible. Find the real part of z.

2017 AIME I, Problem 14 1357. Let $a>1$ and $x>1$ satisfy

$$\log_a(\log_a(\log_a 2)+\log_a 24-128)=128,$$

and $\log_a(\log_a x)=256$. Find the remainder when x is divided by 1000.

2017 AIME II, Problem 7 1358. Find the number of integer values of k in the closed interval $[-500, 500]$ for which the equation $\log(kx) = 2\log(x+2)$ has exactly one real solution.

2017 HMMT, Team, Problem 1 1359. Let $P(x)$, $Q(x)$ be nonconstant polynomials with real number coefficients. Prove that if

$$\lfloor P(y) \rfloor = \lfloor Q(y) \rfloor,$$

for all real numbers y, then $P(x) = Q(x)$ for all real numbers x.

2017 HMMT, Team, Problem 2 1360. Does there exist a two-variable polynomial $P(x, y)$ with real number coefficients such that $P(x, y)$ is positive exactly when x and y are both positive?

2017 HMMT, Algebra & Number Theory, Problem 1 1361. Let $Q(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial with integer coefficients, and $0 \leq a_i < 3$ for all $0 \leq i \leq n$.

Given that $Q(\sqrt{3}) = 20 + 17\sqrt{3}$, compute $Q(2)$.

2017 HMMT, Algebra & Number Theory, Problem 6 1362. A polynomial P of degree 2015 satisfies the equation $P(n) = \frac{1}{n^2}$ for $n = 1, 2, \ldots, 2016$. Find

$$\lfloor 2017P(2017) \rfloor.$$

2017 HMMT, November General, Problem 5 1363. Given that a, b, c are integers with $abc = 60$, and that complex number $\omega \neq 1$ satisfies $\omega^3 = 1$, find the minimum possible value of

$$|a + b\omega + c\omega^2|.$$

2017 HMMT, November Guts, Problem 11 1364. Consider the graph in 3-space of

$$0 = xyz(x+y)(y+z)(z+x)(x-y)(y-z)(z-x).$$

This graph divides 3-space into N connected regions. What is N?

2017 HMMT, November Guts, Problem 16 1365. Let a and b be complex numbers satisfying the two equations

$$a^3 - 3ab^2 = 36,$$
$$b^3 - 3ba^2 = 28i.$$

Let M be the maximum possible magnitude of a. Find all a such that $|a| = M$.

2017 HMMT, November Guts, Problem 25 1366. Find all real numbers x satisfying the equation

$$x^3 - 8 = 16\sqrt[3]{x+1}.$$

2018 AIME I, Problem 5 1367. For each ordered pair of real numbers (x, y) satisfying

$$\log_2(2x + y) = \log_4(x^2 + xy + 7y^2),$$

there is a real number K such that

$$\log_3(3x + y) = \log_9(3x^2 + 4xy + Ky^2).$$

Find the product of all possible values of K.

2018 AIME I, Problem 6 1368. Let N be the number of complex numbers z with the properties that $|z| = 1$ and $z^{6!} - z^{5!}$ is a real number. Find the remainder when N is divided by 1000.

2018 AIME II, Problem 5 1369. Suppose that x, y, and z are complex numbers such that $xy = -80 - 320i$, $yz = 60$, and $zx = -96 + 24i$, where $i = \sqrt{-1}$. Then there are real numbers a and b such that $x + y + z = a + bi$. Find $a^2 + b^2$.

2018 AIME II, Problem 6 1370. A real number a is chosen randomly and uniformly from the interval $[-20, 18]$. The probability that the roots of the polynomial

$$x^4 + 2ax^3 + (2a - 2)x^2 + (-4a + 3)x - 2,$$

are all real can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

2019 AIME I, Problem 7 1371. There are positive integers x and y that satisfy the system of equations

$$\log_{10} x + 2\log_{10}(\gcd(x, y)) = 60,$$
$$\log_{10} y + 2\log_{10}(\text{lcm}(x, y)) = 570.$$

Let m be the number of (not necessarily distinct) prime factors in the prime factorization of x, and let n be the number of (not necessarily distinct) prime factors in the prime factorization of y. Find $3m + 2n$.

2019 AIME I, Problem 10 1372. For distinct complex numbers $z_1, z_2, \dots, z_{673}$, the polynomial

$$(x - z_1)^3(x - z_2)^3 \cdots (x - z_{673})^3,$$

can be expressed as $x^{2019} + 20x^{2018} + 19x^{2017} + g(x)$, where $g(x)$ is a polynomial with complex coefficients and with degree at most 2016. The value of

$$\left| \sum_{1 \le j < k \le 673} z_j z_k \right|,$$

can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

2019 AIME I, Problem 12 1373. Given $f(z) = z^2 - 19z$, there are complex numbers z with the property that z, $f(z)$, and $f(f(z))$ are the vertices of a right triangle in the complex plane with a right angle at $f(z)$. There are positive integers m and n such that one such value of z is $m + \sqrt{n} + 11i$. Find $m + n$.

2019 AIME II, Problem 6 1374. In a Martian civilization, all logarithms whose bases are not specified are assumed to be base b, for some fixed $b \geq 2$. A Martian student writes down

$$3\log(\sqrt{x}\log x) = 56,$$
$$\log_{\log(x)}(x) = 54,$$

and finds that this system of equations has a single real number solution $x > 1$. Find b.

2019 AIME II, Problem 8 1375. The polynomial $f(z) = az^{2018} + bz^{2017} + cz^{2016}$ has real coefficients not exceeding 2019, and $f(\frac{1+\sqrt{3}i}{2}) = 2015 + 2019\sqrt{3}i$. Find the remainder when $f(1)$ is divided by 1000.

2019 PUMaC, Algebra, Problem 2 1376. Let $f(x) = x^2 + 4x + 2$. Let r be the difference between the largest and smallest real solutions of the equation $f(f(f(f(x)))) = 0$. Then $r = a^{\frac{p}{q}}$ for some positive integers a, p, q so a is square-free and p, q are relatively prime positive integers. Compute $a + p + q$.

2019 PUMaC, Algebra, Problem 5 1377. Let

$$\omega = e^{2\pi i/2017} \qquad \text{and} \qquad \zeta = e^{2\pi i/2019}.$$

Define

$$S = \{(a, b) \in \mathbb{Z} \,|\, 0 \leq a \leq 2016, 0 \leq b \leq 2018, (a, b) \neq (0, 0)\}.$$

Compute

$$\prod_{(a,b)\in S} (\omega^a - \zeta^b).$$

2019 PUMaC, Team Round, Problem 7 1378. For all sets A of complex numbers, let $P(A)$ be the product of the elements of A. Let

$$S_z = \left\{1, 2, 9, 99, 999, \frac{1}{z}, \frac{1}{z^2}\right\},$$

and let T_z be the set of nonempty subsets of S_z (including S_z), and let

$$f(z) = 1 + \sum_{s\in T_z} P(s).$$

Suppose $f(z) = 6125000$ for some complex number z. Compute the product of all possible values of z.

2019 PUMaC, Team Round, Problem 13 1379. Let $e_1, e_2, \ldots, e_{2019}$ be independently chosen from the set $\{0, 1, \ldots, 20\}$ uniformly at random. Let $\omega = e^{\frac{2\pi}{i}2019}$. Determine the expected value of

$$|e_1\omega + e_2\omega^2 + \cdots + e_{2019}\omega^{2019}|.$$

2019 HMMT, Team, Problem 3 1380. Alan draws a convex 2020-gon

$$\mathcal{A} = A_1A_2 \cdots A_{2020},$$

with vertices in clockwise order and chooses 2020 angles $\theta_1, \theta_2, \ldots, \theta_{2020} \in (0, \pi)$ in radians with sum 1010π. He then constructs isosceles triangles $\triangle A_iB_iA_{i+1}$ on the exterior of $\mathcal{A}$ with $B_iA_i = B_iA_{i+1}$ and $\angle A_iB_iA_{i+1} = \theta_i$. (Here, $A_{2021} = A_1$.) Finally, he erases $\mathcal{A}$ and the point B_1. He then tells Jason the angles $\theta_1, \theta_2, \ldots, \theta_{2020}$ he chose. Show that Jason can determine where B_1 was from the remaining 2019 points, i.e. show that B_1 is uniquely determined by the information Jason has.

2019 HMMT, Team, Problem 9 1381. Let $p > 2$ be a prime number. $\mathbb{F}_p[x]$ is defined as the set of polynomials in x with coefficients in $\mathbb{F}_p$ (the integers modulo p with usual addition and subtraction), so that two polynomials are equal if and only if the coefficients of x^k are equal in $\mathbb{F}_p$ for each non-negative integer k. For example,

$$(x+2)(2x+3) = 2x^2 + 2x + 1 \quad \text{in} \quad \mathbb{F}_5[x],$$

because the corresponding coefficients are equal modulo 5. Let $f, g \in \mathbb{F}_p[x]$. The pair (f, g) is called *compositional* if

$$f(g(x)) \equiv x^{p^2} - x \quad \text{in} \quad \mathbb{F}_p[x].$$

Find, with proof, the number of *compositional* pairs.

2019 HMMT, Team, Problem 10 1382. Prove that for all positive integers n, all complex roots r of the polynomial

$$P(x) = (2n)x^{2n} + (2n-1)x^{2n-1} + \cdots + (n+1)x^{n+1} + nx^n + (n+1)x^{n-1} + \cdots + (2n-1)x + 2n$$

lie on the unit circle (i.e. $|r| = 1$).

2019 HMMT, Algebra & Number Theory, Problem 5 1383. Let $a_1, a_2, \ldots$ be an arithmetic sequence and $b_1, b_2, \ldots$ be a geometric sequence. Suppose that $a_1b_1 = 20$, $a_2b_2 = 19$, and $a_3b_3 = 14$. Find the greatest possible value of a_4b_4.

2019 HMMT, Algebra & Number Theory, Problem 7 1384. Find the value of

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\sum_{c=1}^{\infty} \frac{ab(3a+c)}{4^{a+b+c}(a+b)(b+c)(c+a)}.$$

2018-2019 San Diego Power Contest, Winter, Problem 1 1385. Let r_1, r_2, r_3 be the distinct real roots of $x^3 - 2019x^2 - 2020x + 2021 = 0$. Prove that $r_1^3 + r_2^3 + r_3^3$ is an integer multiple of 3.

2018-2019 San Diego Power Contest, Winter, Problem 5 1386. Prove that there exists a positive integer N such that for every polynomial $P(x)$ of degree 2019, there exist N linear polynomials $p_1, p_2, \ldots p_N$ such that $P(x) = p_1(x)^{2019} + p_2(x)^{2019} + \ldots + p_N(x)^{2019}$. (Assume all polynomials in this problem have real coefficients, and leading coefficients cannot be zero.)

2019-2020 San Diego Power Contest, Fall, Problem 3 1387. Find all polynomials P with integer coefficients such that for all positive integers x, y,

$$\frac{P(x) - P(y)}{x^2 + y^2}$$

evaluates to an integer (in particular, it can be zero).

2020 AIME I, Problem 2 1388. There is a unique positive real number x such that the three numbers $\log_8(2x), \log_4 x$, and $\log_2 x$, in that order, form a geometric progression with positive common ratio. The number x can be written as m/n, where m and n are relatively prime positive integers. Find $m + n$.

2020 AIME I, Problem 11 1389. For integers a, b, c, and d, let $f(x) = x^2+ax+b$ and $g(x) = x^2 + cx + d$. Find the number of ordered triples (a, b, c) of integers with absolute values not exceeding 10 for which there is an integer d such that $g(f(2)) = g(f(4)) = 0$.

2020 AIME I, Problem 14 1390. Let $P(x)$ be a quadratic polynomial with complex coefficients whose x^2 coefficient is 1. Suppose the equation $P(P(x)) = 0$ has four distinct solutions, $x = 3, 4, a, b$. Find the sum of all possible values of $(a + b)^2$.

2020 AIME II, Problem 3 1391. The value of x that satisfies

$$\log_{2^x} 3^{20} = \log_{2^{x+3}} 3^{2020},$$

can be written as m/n, where m and n are relatively prime positive integers. Find $m+n$.

2020 AIME II, Problem 8 1392. Define a sequence of functions recursively by $f_1(x) = |x - 1|$ and $f_n(x) = f_{n-1}(|x - n|)$ for integers $n > 1$. Find the least value of n such that the sum of the zeros of f_n exceeds 500,000.

2020 AIME II, Problem 11 1393. Let $P(x) = x^2 - 3x - 7$, and let $Q(x)$ and $R(x)$ be two quadratic polynomials also with the coefficient of x^2 equal to 1. David computes each of the three sums $P + Q$, $P + R$, and $Q + R$ and is surprised to find that each pair of these sums has a common root, and these three common roots are distinct. If $Q(0) = 2$, then $R(0) = m/n$, where m and n are relatively prime positive integers. Find $m + n$.

2020 AIME II, Problem 14 1394. For real number x let $\lfloor x\rfloor$ be the greatest integer less than or equal to x, and define $\{x\} = x - \lfloor x\rfloor$ to be the fractional part of x. For example, $\{3\} = 0$ and $\{4.56\} = 0.56$. Define $f(x) = x\{x\}$, and let N be the number of real-valued solutions to the equation $f(f(f(x))) = 17$ for $0 \le x \le 2020$. Find the remainder when N is divided by 1000.

2020 PUMaC, Algebra, Problem A4/B6 1395. Let P be a 10-degree monic polynomial with roots $r_1, r_2, \ldots, r_{10} \neq 0$, and let Q be a 45-degree monic polynomial with roots

$$\frac{1}{r_i} + \frac{1}{r_j} - \frac{1}{r_i r_j},$$

where $i < j$ and $i, j \in \{1, \ldots, 10\}$. If $P(0) = Q(1) = 2$, then $\log_2(|P(1)|)$ can be written as a/b for relatively prime integers a, b. Find $a + b$.

2020 PUMaC, Algebra, Problem A6/B8 1396. Given integer n, let W_n be the set of complex numbers of the form $re^{2qi\pi}$, where q is a rational number so that $q_n \in \mathbb{Z}$ and r is a real number. Suppose that p is a polynomial of degree ≥ 2 such that there exists a non-constant function $f : W_n \to C$ so that

$$p(f(x))p(f(y)) = f(xy),$$

for all $x, y \in W_n$. If p is the unique monic polynomial of lowest degree for which such an f exists for $n = 65$, find $p(10)$.

2020 PUMaC, Algebra, Problem A7 1397. Suppose that p is the unique monic polynomial of minimal degree such that its coefficients are rational numbers and one of its roots is

$$\sin\frac{2\pi}{7} + \cos\frac{4\pi}{7}.$$

If $p(1) = a/b$, where a, b are relatively prime integers, find $|a + b|$.

2020 HMMT, Algebra & Number Theory, Problem 1 1398. Let

$$P(x) = x^3 + x^2 - r^2x - 2020,$$

be a polynomial with roots r, s, t. What is $P(1)$?

2020 HMMT, Algebra & Number Theory, Problem 3 1399. Let $a = 256$. Find the unique real number $x > a^2$ such that

$$\log_a \log_a \log_a x = \log_{a^2} \log_{a^2} \log_{a^2} x.$$

2020 HMMT, Algebra & Number Theory, Problem 6 1400. A polynomial $P(x)$ is a *base-n polynomial* if it is of the form $a_dx^d + a_{d-1}x^{d-1} + \cdots + a_1x + a_0$, where each a_i is an integer between 0 and $n-1$ inclusive and $a_d > 0$. Find the largest positive integer n such that for any real number c, there exists at most one base-n polynomial $P(x)$ for which $P(\sqrt{2} + \sqrt{3}) = c$.

2020 HMMT, Algebra & Number Theory, Problem 8 1401. $P(x)$ is the unique polynomial of degree at most 2020 satisfying $P(k^2) = k$ for $k = 0, 1, 2, \ldots, 2020$. Compute $P(2021^2)$.

2020 HMMT, Algebra & Number Theory, Problem 9 1402. Let

$$P(x) = x^{2020} + x + 2,$$

which has 2020 distinct roots. Let $Q(x)$ be the monic polynomial of degree $\binom{2020}{2}$ whose roots are the pairwise products of the roots of $P(x)$. Let α satisfy $P(\alpha) = 4$. Compute the sum of all possible values of $Q(\alpha^2)^2$.

2020 HMMT, Algebra & Number Theory, Problem 10 1403. We define $\mathbb{F}_{101}[x]$ as the set of all polynomials in x with coefficients in $\mathbb{F}_{101}$ (the integers modulo 101 with usual addition and subtraction), so that two polynomials are equal if and only if the coefficients of x^k are equal in $\mathbb{F}_{101}$ for each non-negative integer k. For example,

$$(x+3)(100x+5) = 100x^2 + 2x + 15 \quad \text{in} \quad \mathbb{F}_{101}[x],$$

because the corresponding coefficients are equal modulo 101.

We say that $f(x) \in \mathbb{F}_{101}[x]$ is *lucky* if it has degree at most 1000 and there exist $g(x), h(x) \in \mathbb{F}_{101}[x]$ such that

$$f(x) = g(x)(x^{1001} - 1) + h(x)^{101} - h(x) \quad \text{in} \quad \mathbb{F}_{101}[x].$$

Find the number of *lucky* polynomials.

2020 HMMT, Team, Problem 7 1404. Positive real numbers x and y satisfy

$$\left|\left|\cdots\left|\left||x| - y\right| - x\right|\cdots - y\right| - x\right| = \left|\left|\cdots\left|\left||y| - x\right| - y\right|\cdots - x\right| - y\right|,$$

where there are 2019 absolute value signs $|\cdot|$ on each side. Determine, with proof, all possible values of x/y.

2020 HMIC, Problem 4 1405. Let

$$C_k = \frac{1}{k+1}\binom{2k}{k}, \qquad k = 1, 2, 3, \ldots$$

denote the k^{th} Catalan number and p be an odd prime. Prove that exactly half of the numbers in the set

$$\left\{ \sum_{k=1}^{p-1} C_k n^k \,\middle|\, n \in \{1, 2, \ldots, p-1\} \right\},$$

are divisible by p.

2020-2021 San Diego Power Contest, Winter, Day 1, Problem 4 1406. Find all polynomials $P(x)$ with integer coefficients such that for all positive integers n, we have that $P(n)$ is not zero and $P(\overline{nn})/P(n)$ is an integer, where $\overline{nn}$ is the integer obtained upon concatenating n with itself.

2020-21 CHMMC Winter, Individual Round, Problem 4 1407. Let $P(x) = x^3 - 6x^2 - 5x + 4$. Suppose that y and z are real numbers such that

$$zP(y) = P(y-n) + P(y+n),$$

for all reals n. Evaluate $P(y)$.

2020-21 CHMMC Winter, Individual Round, Problem 8 1408. Define

$$S = \tan^{-1}(2020) + \sum_{j=0}^{2020} \tan^{-1}(j^2 - j + 1).$$

Then S can be written as $\frac{m\pi}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

2020-21 CHMMC Winter, Team Round, Problem 6 1409. Suppose that

$$\prod_{n=1}^{\infty} \left(\frac{1 + i\cot\left(\frac{n\pi}{2n+1}\right)}{1 - i\cot\left(\frac{n\pi}{2n+1}\right)} \right)^{\frac{1}{n}} = \left(\frac{p}{q}\right)^{i\pi},$$

where p and q are relatively prime positive integers. Find $p+q$.

Note: for a complex number $z = re^{i\theta}$ for reals $r > 0, 0 \leq \theta < 2\pi$, we define $z^n = r^n e^{i\theta n}$ for all positive reals n.

2020-21 CHMMC Winter, Team Round, Problem 10 1410. Let ω be a non-real 47^{th} root of unity. Suppose that $\mathcal{S}$ is the set of polynomials of degree at most 46 and coefficients equal to either 0 or 1. Let N be the number of polynomials $Q \in \mathcal{S}$ such that

$$\sum_{j=0}^{46} \frac{Q(\omega^{2j}) - Q(\omega^j)}{\omega^{4j} + \omega^{3j} + \omega^{2j} + \omega^j + 1} = 47.$$

The prime factorization of N is $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ where $p_1, \ldots, p_s$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_s$ are positive integers. Compute $\sum_{j=1}^{s} p_j \alpha_j$.

2020-21 CHMMC Winter, Tiebreaker Round, Problem 2 1411. Find the sum of all positive integers $x < 241$ such that both $x^{24} + x^{18} + x^{12} + x^6 + 1$ and $x^{20} + x^{10} + 1$ are multiples of 241.

2020-21 CHMMC Winter, Tiebreaker Round, Problem 7 1412. Consider the polynomial $x^3 - 3x^2 + 10$. Let a, b, c be its roots. Compute

$$a^2b^2c^2 + a^2b^2 + b^2c^2 + c^2a^2 + a^2 + b^2 + c^2.$$

2021 AIME I, Problem 8 1413. Find the number of integers c such that the equation

$$\Big||20|x| - x^2| - c\Big| = 21,$$

has 12 distinct real solutions.

2021 AIME I, Problem 15 1414. Let S be the set of positive integers k such that the two parabolas

$$y = x^2 - k \ \text{ and } \ x = 2(y-20)^2 - k$$

intersect in four distinct points, and these four points lie on a circle with radius at most 21. Find the sum of the least element of S and the greatest element of S.

2021 AIME II, Problem 4 1415. There are real numbers $a, b, c,$ and d such that -20 is a root of $x^3 + ax + b$ and -21 is a root of $x^3 + cx^2 + d$. These two polynomials share a complex root $m + \sqrt{n} \cdot i$, where m and n are positive integers and $i = \sqrt{-1}$. Find $m + n$.

2021 AIME II, Problem 7 1416. Let $a, b, c,$ and d be real numbers that satisfy the system of equations

$$\begin{aligned} a + b &= -3, \\ ab + bc + ca &= -4, \\ abc + bcd + cda + dab &= 14, \\ abcd &= 30. \end{aligned}$$

There exist relatively prime positive integers m and n such that

$$a^2 + b^2 + c^2 + d^2 = \frac{m}{n}.$$

Find $m + n$.

2021 PUMaC, Team Round, Problem 7 1417. The roots of the polynomial $f(x) = x^8 + x^7 - x^5 - x^4 - x^3 + x + 1$ are all roots of unity. We say that a real number $r \in [0, 1)$ is nice if $e^{2i\pi r} = \cos 2\pi r + i \sin 2\pi r$ is a root of the polynomial f and if $e^{2i\pi r}$ has positive imaginary part. Let S be the sum of the values of nice real numbers r. If $S = \frac{p}{q}$ for relatively prime positive integers p, q, find $p + q$.

2021 PUMaC, Algebra, Problem A3/B5 1418. Let $f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4$ and let

$$\zeta = e^{2\pi i/5} = \cos\frac{2\pi}{5} + i \sin\frac{2\pi}{5}.$$

Find the value of the following expression:

$$f(\zeta)f(\zeta^2)f(\zeta^3)f(\zeta^4).$$

2021 PUMaC, Algebra, Problem A4/B6 1419. The roots of a monic cubic polynomial p are positive real numbers forming a geometric sequence. Suppose that the sum of the roots is equal to 10. Under these conditions, the largest possible value of $|p(-1)|$ can be written as m/n, where m, n are relatively prime integers. Find $m+n$.

2021 PUMaC, Algebra, Problem A6/B8 1420. Let f be a polynomial. We say that a complex number p is a double *attractor* if there exists a polynomial $h(x)$ so that $f(x)-f(p)=h(x)(x-p)^2$ for all $x\in\mathbb{R}$. Now, consider the polynomial

$$f(x)=12x^5-15x^4-40x^3+540x^2-2160x+1,$$

and suppose that its double *attractors* are $a_1,a_2,\ldots,a_n$. If the sum

$$\sum_{i=1}^{n}|a_i|=\sqrt{a}+\sqrt{b},$$

where a,b are positive integers, find $a+b$.

2021 PUMaC, Algebra, Problem B1 1421. Let x,y be distinct positive real numbers satisfying

$$\frac{1}{\sqrt{x+y}-\sqrt{x-y}}+\frac{1}{\sqrt{x+y}+\sqrt{x-y}}=\frac{x}{\sqrt{y^3}}.$$

If $x/y=(a+\sqrt{b})/c$ for positive integers a,b,c with $gcd(a,c)=1$, find $a+b+c$.

2021 HMMT, Algebra & Number Theory, Problem 2 1422. Compute the number of ordered pairs of integers (a,b), with $2\le a,b\le 2021$, that satisfy the equation

$$a^{\log_b\left(a^{-4}\right)}=b^{\log_a\left(ba^{-3}\right)}.$$

2021 HMMT, Algebra & Number Theory, Problem 3 1423. Among all polynomials $P(x)$ with integer coefficients for which $P(-10)=145$ and $P(9)=164$, compute the smallest possible value of $|P(0)|$.

2021 HMMT, Algebra & Number Theory, Problem 4 1424. Suppose that $P(x,y,z)$ is a homogeneous degree 4 polynomial in three variables such that $P(a,b,c)=P(b,c,a)$ and $P(a,a,b)=0$ for all real a, b, and c. If $P(1,2,3)=1$, compute $P(2,4,8)$.

Note: $P(x,y,z)$ is a homogeneous degree 4 polynomial if it satisfies

$$P(ka,kb,kc)=k^4P(a,b,c),$$

for all real k,a,b,c.

2021 HMMT, Algebra & Number Theory, Problem 7 1425. Suppose that x, y, and z are complex numbers of equal magnitude that satisfy

$$x+y+z=-\frac{\sqrt{3}}{2}-i\sqrt{5}\qquad\text{and}\qquad xyz=\sqrt{3}+i\sqrt{5}.$$

If $x=x_1+ix_2, y=y_1+iy_2$, and $z=z_1+iz_2$ for real x_1,x_2,y_1,y_2,z_1 and z_2 then $(x_1x_2+y_1y_2+z_1z_2)^2$ can be written as a/b for relatively prime positive integers a and b. Compute $100a+b$.

2021 HMMT, Algebra & Number Theory, Problem 9 1426. Let f be a monic cubic polynomial satisfying $f(x)+f(-x)=0$ for all real numbers x. For all real numbers y, define $g(y)$ to be the number of distinct real solutions x to the equation $f(f(x))=y$. Suppose that the set of possible values of $g(y)$ over all real numbers y is exactly $\{1,5,9\}$. Compute the sum of all possible values of $f(10)$.

2021 HMIC, Problem 3 1427. Let A be a set of $n\geq 2$ positive integers, and let $f(x)=\sum_{a\in A}x^a$. Prove that there exists a complex number z with $|z|=1$ and $|f(z)|=\sqrt{n-2}$.

2021-22 CHMMC Winter, Team Round, Problem 3 1428. Suppose a,b,c are complex numbers with $a+b+c=0$, $a^2+b^2+c^2=0$, and $|a|,|b|,|c|\leq 5$. Suppose further at least one of a,b,c have real and imaginary parts that are both integers. Find the number of possibilities for such ordered triples (a,b,c).

2021-22 CHMMC Winter, Team Round, Problem 6 1429. There is a unique degree-10 monic polynomial with integer coefficients $f(x)$ such that

$$f\left(\sum_{j=0}^{9}\sqrt[10]{2021^j}\right)=0.$$

Find the remainder when $f(1)$ is divided by 1000.

2021-22 CHMMC Winter, Proof Round, Problem 3 1430. Let $F(x_1,\ldots,x_n)$ be a polynomial with real coefficients in$n>1$ "indeterminate" variables $x_1,\ldots,x_n$. We say that F is n-alternating if for all integers $1\leq i<j\leq n$,

$$F(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_n)=F(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_n),$$

i.e., swapping the order of indeterminates x_i,x_j flips the sign of the polynomial. For example, $x_1^2x_2-x_2^2x_1$ is 2-alternating, whereas $x_1x_2x_3+2x_2x_3$ is not 3-alternating.

Note: two polynomials $P(x_1,\ldots,x_n)$ and $Q(x_1,\ldots,x_n)$ are considered equal if and only if each monomial constituent $ax_1^{k_1}\cdots x_n^{k_n}$ of P appears in Q with the same coefficient a, and vice versa. This is equivalent to saying that $P(x_1,\ldots,x_n)=0$ if and only if every possible monomial constituent of P has coefficient 0.

a) Compute a 3-alternating polynomial of degree 3.

b) Prove that the degree of any nonzero n-alternating polynomial is at least $\binom{n}{2}$.

2022 AIME I, Problem 1 1431. Quadratic polynomials $P(x)$ and $Q(x)$ have leading coefficients of 2 and -2, respectively. The graphs of both polynomials pass through the two points $(16,54)$ and $(20,53)$. Find $P(0)+Q(0)$.

2022 AIME I, Problem 4 1432. Let

$$w=\frac{\sqrt{3}+i}{2}\qquad\text{and}\qquad z=\frac{-1+i\sqrt{3}}{2},$$

where $i=\sqrt{-1}$. Find the number of ordered pairs (r,s) of positive integers not exceeding 100 that satisfy the equation $i\cdot w^r=z^s$.

2022 AIME I, Problem 15 1433. Let x, y, and z be positive real numbers satisfying the system of equations

$$\begin{aligned}\sqrt{2x-xy}+\sqrt{2y-xy}&=1,\\ \sqrt{2y-yz}+\sqrt{2z-yz}&=\sqrt{2},\\ \sqrt{2z-zx}+\sqrt{2x-zx}&=\sqrt{3}.\end{aligned}$$

Then $\left[(1-x)(1-y)(1-z)\right]^2$ can be written as m/n, where m and n are relatively prime positive integers. Find $m+n$.

2022 AIME II, Problem 4 1434. There is a positive real number x not equal to either $1/20$ or $1/2$ such that

$$\log_{20x}(22x)=\log_{2x}(202x).$$

The value $\log_{20x}(22x)$ can be written as $\log_{10}(m/n)$, where m and n are relatively prime positive integers. Find $m+n$.

2022 AIME II, Problem 12 1435. Let a, b, x, and y be real numbers with $a>4$ and $b>1$ such that

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-16}=\frac{(x-20)^2}{b^2-1}+\frac{(y-11)^2}{b^2}=1.$$

Find the least possible value of $a+b$.

2022 AIME II, Problem 13 1436. There is a polynomial $P(x)$ with integer coefficients such that

$$P(x)=\frac{(x^{2310}-1)^6}{(x^{105}-1)(x^{70}-1)(x^{42}-1)(x^{30}-1)},$$

holds for every $0<x<1$. Find the coefficient of x^{2022} in $P(x)$.

2022 HMMT, Algebra & Number Theory, Problem 1 1437. Positive integers a, b, and c are all powers of k for some positive integer k. It is known that the equation $ax^2-bx+c=0$ has exactly one real solution r, and this value r is less than 100. Compute the maximum possible value of r.

2022 HMMT, Algebra & Number Theory, Problem 9 1438. Suppose $P(x)$ is a monic polynomial of degree 2023 such that

$$P(k)=k^{2023}P\left(1-\frac{1}{k}\right),$$

for every positive integer $1\le k\le 2023$. Then $P(-1)=a/b$ where a and b are relatively prime integers. Compute the unique integer $0\le n<2027$ such that $bn-a$ is divisible by the prime 2027.

2022 HMMT, Team, Problem 6 1439. Let $P(x)=x^4+ax^3+bx^2+x$ be a polynomial with four distinct roots that lie on a circle in the complex plane. Prove that $ab\neq 9$.

2022 HMIC, Problem 1 1440. Is

$$\prod_{k=0}^{\infty}\left(1-\frac{1}{2022^{k!}}\right)$$

rational?

2022 Stanford Math Tournament, Team Round #5 1441. Let a, b, and c be the roots of the polynomial $x^3 - 3x^2 - 4x + 5$. Compute $\frac{a^4+b^4}{a+b} + \frac{b^4+c^4}{b+c} + \frac{c^4+a^4}{c+a}$.

2022 Stanford Math Tournament, Algebra #9 1442. Let $P(x) = 8x^3 + ax + b + 1$ for $a, b \in \mathbb{Z}$. It is known that P has a root $x_0 = p + \sqrt{q} + \sqrt[3]{r}$, where $p, q, r \in \mathbb{Q}$, $q \geq 0$; however, P has no *rational* roots. Find the smallest possible value of $a + b$.

2022 Stanford Math Tournament, Algebra #10 1443. Let $f^1(x) = x^3 - 3x$. Let $f^n(x) = f(f^{n-1}(x))$. Let $\mathcal{R}$ be the set of roots of $\frac{f^{2022}(x)}{x}$. If

$$\sum_{r \in \mathcal{R}} \frac{1}{r^2} = \frac{a^b - c}{d},$$

for positive integers a, b, c, d, where b is as large as possible and c and d are relatively prime, find $a + b + c + d$.

2022 Baltic Way, Problem 3 1444. We call a two-variable polynomial $P(x, y)$ secretly one-variable, if there exist polynomials $Q(x)$ and $R(x, y)$ such that $\deg(Q) \geq 2$ and $P(x, y) = Q(R(x, y))$ (e.g., $x^2 + 1$ and $x^2y^2 + 1$ are secretly one-variable, but $xy + 1$ is not). Prove or disprove the following statement: If $P(x, y)$ is a polynomial such that both $P(x, y)$ and $P(x, y) + 1$ can be written as the product of two non-constant polynomials, then P is secretly one-variable.

Note: All polynomials are assumed to have real coefficients.

2022-23 CHMMC Winter, Individual Round, Problem 10 1445. Find the number of pairs of positive integers (m, n) such that $n < m \leq 100$ and the polynomial $x^m + x^n + 1$ has a root on the unit circle.

2022-23 CHMMC Winter, Team Round, Problem 8 1446. Suppose $a_3x^3 - x^2 + a_1x - 7 = 0$ is a cubic polynomial in x whose roots α, β, γ are positive real numbers satisfying

$$\frac{225\alpha^2}{\alpha^2 + 7} = \frac{144\beta^2}{\beta^2 + 7} = \frac{100\gamma^2}{\gamma^2 + 7}.$$

Find a_1.

2022-23 CHMMC Winter, Team Round, Problem 10 1447. Suppose that $\xi \neq 1$ is a root of the polynomial $f(x) = x^{167} - 1$. Compute

$$\left| \sum_{0<a<b<167} \xi^{a^2+b^2} \right|.$$

2023 AIME I, Problem 2 1448. If $\sqrt{\log_b n} = \log_b \sqrt{n}$ and $b \log_b n = \log_b bn$, then the value of n is equal to j/k, where j and k are relatively prime. What is $j + k$?

2023 AIME I, Problem 9 1449. Find the number of cubic polynomials $p(x) = x^3 + ax^2 + bx + c$, where a, b, and c are integers in

$$\{-20, -19, -18, \ldots, 18, 19, 20\},$$

such that there is a unique integer $m \neq 2$ with $p(m) = p(2)$.

2023 AIME I, Problem 15 1450. Find the largest prime number $p < 1000$ for which there exists a complex number z satisfying

a) the real and imaginary part of z are both integers;
b) $|z| = \sqrt{p}$, and
c) there exists a triangle whose three side lengths are p, the real part of z^3, and the imaginary part of z^3.

2023 AIME II, Problem 2 1451. Let x, y, and z be real numbers satisfying the system of equations

$$\begin{aligned} xy + 4z &= 60, \\ yz + 4x &= 60, \\ zx + 4y &= 60. \end{aligned}$$

Let S be the set of possible values of x. Find the sum of the squares of the elements of S.

2023 AIME II, Problem 8 1452. Let

$$\omega = \cos\frac{2\pi}{7} + i \cdot \sin\frac{2\pi}{7},$$

where $i = \sqrt{-1}$. Find

$$\prod_{k=0}^{6}(\omega^{3k} + \omega^k + 1).$$

2023 AIME II, Problem 13 1453. Let A be an acute angle such that $\tan A = 2\cos A$. Find the number of positive integers n less than or equal to 1000 such that $\sec^n A + \tan^n A$ is a positive integer whose units digit is 9.

2023 Stanford Math Tournament, Algebra #10 1454. Suppose that $p(x), q(x)$ are monic polynomials with non-negative integer coefficients such that

$$\frac{1}{5x} \geq \frac{1}{q(x)} - \frac{1}{p(x)} \geq \frac{1}{3x^2},$$

for all integers $x \geq 2$. Compute the minimum possible value of $p(1) \cdot q(1)$.

2023 Stanford Math Tournament, Algebra Tiebreaker #1 1455. Compute the area of the polygon formed by connecting the roots of

$$x^{10} + x^9 + x^8 + x^6 + x^5 + x^4 + x^2 + x + 1,$$

graphed in the complex plane with line segments in counterclockwise order.

2023 Stanford Math Tournament, Algebra Tiebreaker #2 1456. $f(x)$ is a non-constant polynomial. Given that $f(f(x)) + f(x) = f(x)^2$, compute $f(3)$.

2023 Stanford Math Tournament, Algebra Tiebreaker #3 1457. Define

$$f(x) = x^3 - 6x^2 + \frac{25}{2}x - 7.$$

There is an interval $[a, b]$ such that for any real number x, the sequence $x, f(x), f(f(x)), \ldots$ is bounded (i.e., has a lower and upper bound) if and only if $x \in [a, b]$. Compute $(a-b)^2$.

2023 Bulgaria National Olympiad, Problem 3 1458. Let $f(x)$ be a polynomial with positive integer coefficients. For every $n \in \mathbb{N}$, let $a_1^{(n)}, a_2^{(n)}, \ldots, a_n^{(n)}$ be fixed positive integers that give pairwise different residues modulo n and let

$$g(n) = \sum_{i=1}^{n} f(a_i^{(n)}) = f(a_1^{(n)}) + f(a_2^{(n)}) + \cdots + f(a_n^{(n)}).$$

Prove that there exists a constant M such that for all integers $m > M$ we have

$$\gcd(m, g(m)) > 2023^{2023}.$$

2023 Canada National Olympiad, Problem 4 1459. Let $f(x)$ be a non-constant polynomial with integer coefficients such that $f(1) \neq 1$. For a positive integer n, define $\operatorname{divs}(n)$ to be the set of positive divisors of n.

A positive integer m is f-cool if there exists a positive integer n for which

$$f[\operatorname{divs}(m)] = \operatorname{divs}(n).$$

Prove that for any such f, there are finitely many f-cool integers.

(The notation $f[S]$ for some set S denotes the set $\{f(s) : s \in S\}$.)

2023 Romanian District Olympiad, Problem 10.3 1460. Let $n \geqslant 2$ be an integer. Determine all complex numbers z which satisfy

$$|z^{n+1} - z^n| \geqslant |z^{n+1} - 1| + |z^{n+1} - z|.$$

2023 Greece National Olympiad, Problem 1 1461. Find all quadruplets (x, y, z, w) of positive real numbers that satisfy the following system:

$$\begin{cases} \dfrac{xyz+1}{x+1} = \dfrac{yzw+1}{y+1} & = \dfrac{zwx+1}{z+1} = \dfrac{wxy+1}{w+1}, \\ \\ x+y+z+w & = 48. \end{cases}$$

2023 India National Olympiad, Problem 2 1462. Suppose $a_0, \ldots, a_{100}$ are positive reals. Consider the following polynomial for each k in $\{0, 1, \ldots, 100\}$:

$$a_{100+k}x^{100} + 100a_{99+k}x^{99} + a_{98+k}x^{98} + a_{97+k}x^{97} + \cdots + a_{2+k}x^2 + a_{1+k}x + a_k,$$

where indices are taken modulo 101, i.e., $a_{100+i} = a_{i-1}$ for any i in $\{1, 2, \ldots, 100\}$. Show that it is impossible that each of these 101 polynomials has all its roots real.

2023 Romanian Masters in Mathematics, Problem 3 1463. Let $n \geq 2$ be an integer and let f be a $4n$-variable polynomial with real coefficients. Assume that, for any $2n$ points $(x_1, y_1), \ldots, (x_{2n}, y_{2n})$ in the Cartesian plane, $f(x_1, y_1, \ldots, x_{2n}, y_{2n}) = 0$ if and only if the points form the vertices of a regular $2n$-gon in some order, or are all equal.

Determine the smallest possible degree of f. Note, for example, that the degree of the polynomial

$$g(x, y) = 4x^3y^4 + yx + x - 2$$

is 7 because $7 = 3 + 4$.

2023 Macedonian Team Selection Test, Problem 5 1464. Let $Q(x) = a_{2023}x^{2023} + a_{2022}x^{2022} + \cdots + a_1x + a_0 \in \mathbb{Z}[x]$ be a polynomial with integer coefficients. For an odd prime number p we define the polynomial $Q_p(x) = a_{2023}^{p-2}x^{2023} + a_{2022}^{p-2}x^{2022} + \cdots + a_1^{p-2}x + a_0^{p-2}$. Assume that there exist infinitely primes p such that

$$\frac{Q_p(x) - Q(x)}{p}$$

is an integer for all $x \in \mathbb{Z}$. Determine the largest possible value of $Q(2023)$ over all such polynomials Q.

2023 German TST 4, Problem 3 1465. Let $f(x)$ be a monic polynomial of degree 2023 with integer coefficients. Show that for any sufficiently large integer N and any prime number $p > 2023N$, the product

$$f(1)f(2)\ldots f(N)$$

is at most $\binom{2023}{2}$ times divisible by p.

2023 Purple Comet, Problem 16 1466. The polynomial $P(x) = x^4 - ax^2 + 2023$ has roots

$$\alpha, -\alpha, \alpha\sqrt{\alpha^2 - 10}, -\alpha\sqrt{\alpha^2 - 10},$$

for some positive real number α. Find the value of a.

2023 British Math Olympiad, Problem 3 1467. For each positive integer n, denote by $\omega(n)$ the number of distinct prime divisors of n (for example, $\omega(1) = 0$ and $\omega(12) = 2$). Find all polynomials $P(x)$ with integer coefficients, such that whenever n is a positive integer satisfying $\omega(n) > 2023^{2023}$, then $P(n)$ is also a positive integer with

$$\omega(n) \geq \omega(P(n)).$$

2023 Thailand Mock IMO, Problem 4 1468. Find all polynomials $P(x)$ with integer coefficients for which there exists an integer M such that $P(n)$ divides $(n+2023)!$ for all positive integers $n > M$.

2023 Pan African Math Olympiad, Problem 5 1469. Let a, b be reals with $a \neq 0$ and let

$$P(x) = ax^4 - 4ax^3 + (5a + b)x^2 - 4bx + b.$$

Show that all roots of $P(x)$ are real and positive if and only if $a = b$.

.2 Trigonometric and Exponential Functions; Algebra versus Geometry and Combinatorics

1984 AIME, Problem 13 1470. Find the value of

$$10\cot(\cot^{-1}3+\cot^{-1}7+\cot^{-1}13+\cot^{-1}21).$$

1989 AIME, Problem 10 1471. Let a, b, c be the three sides of a triangle, and let α, β, γ, be the angles opposite them. If $a^2+b^2=1989c^2$, find

$$\frac{\cot\gamma}{\cot\alpha+\cot\beta}.$$

1991 AIME, Problem 4 1472. How many real numbers x satisfy the equation

$$\frac{1}{5}\log_2 x=\sin(5\pi x)?$$

1991 AIME, Problem 9 1473. Suppose that

$$\sec x+\tan x=\frac{22}{7},$$

and that

$$\csc x+\cot x=\frac{m}{n},$$

where m/n is in lowest terms. Find $m+n$.

1993 AIME, Problem 14 1474. A rectangle that is inscribed in a larger rectangle (with one vertex on each side) is called unstuck if it is possible to rotate (however slightly) the smaller rectangle about its center within the confines of the larger. Of all the rectangles that can be inscribed unstuck in a 6 by 8 rectangle, the smallest perimeter has the form $\sqrt{N}$, for a positive integer N. Find N.

1994 AIME, Problem 4 1475. Find the positive integer n for which

$$\lfloor\log_2 1\rfloor+\lfloor\log_2 2\rfloor+\lfloor\log_2 3\rfloor+\cdots+\lfloor\log_2 n\rfloor=1994.$$

(For real x, $\lfloor x\rfloor$ is the greatest integer $\le x$.)

1994 AIME, Problem 7 1476. For certain ordered pairs (a,b) of real numbers, the system of equations

$$\begin{aligned}ax+by&=1,\\x^2+y^2&=50,\end{aligned}$$

has at least one solution, and each solution is an ordered pair (x,y) of integers. How many such ordered pairs (a,b) are there?

1995 AIME, Problem 2 1477. Find the last three digits of the product of the positive roots of

$$\sqrt{1995}x^{\log_{1995} x} = x^2.$$

1995 AIME, Problem 7 1478. Given that $(1 + \sin t)(1 + \cos t) = 5/4$ and

$$(1 - \sin t)(1 - \cos t) = \frac{m}{n} - \sqrt{k},$$

where $k, m,$ and n are positive integers with m and n relatively prime, find $k + m + n$.

1995 AIME, Problem 12 1479. Pyramid $OABCD$ has square base $ABCD$, congruent edges $\overline{OA}, \overline{OB}, \overline{OC}$, and $\overline{OD}$, and $\angle AOB = 45°$. Let θ be the measure of the dihedral angle formed by faces OAB and OBC. Given that $\cos\theta = m + \sqrt{n}$, where m and n are integers, find $m + n$.

1996 AIME, Problem 10 1480. Find the smallest positive integer solution to

$$\tan 19x° = \frac{\cos 96° + \sin 96°}{\cos 96° - \sin 96°}.$$

1997 AIME, Problem 11 1481. Let

$$x = \frac{\displaystyle\sum_{n=1}^{44} \cos n°}{\displaystyle\sum_{n=1}^{44} \sin n°}.$$

What is the greatest integer that does not exceed $100x$?

1997 AIME, Problem 13 1482. Let S be the set of points in the Cartesian plane that satisfy

$$\Big|\big||x| - 2\big| - 1\Big| + \Big|\big||y| - 2\big| - 1\Big| = 1.$$

If a model of S were built from wire of negligible thickness, then the total length of wire required would be $a\sqrt{b}$, where a and b are positive integers and b is not divisible by the square of any prime number. Find $a + b$.

1998 AIME, Problem 3 1483. The graph of $y^2 + 2xy + 40|x| = 400$ partitions the plane into several regions. What is the area of the bounded region?

1998 AIME, Problem 5 1484. Given that

$$A_k = \frac{k(k-1)}{2} \cos \frac{k(k-1)\pi}{2},$$

find

$$|A_{19} + A_{20} + \cdots + A_{98}|.$$

1999 AIME, Problem 6 1485. A transformation of the first quadrant of the coordinate plane maps each point (x, y) to the point $(\sqrt{x}, \sqrt{y})$. The vertices of quadrilateral $ABCD$ are $A = (900, 300), B = (1800, 600), C = (600, 1800)$, and $D = (300, 900)$. Let k be the area of the region enclosed by the image of quadrilateral $ABCD$. Find the greatest integer that does not exceed k.

1999 AIME, Problem 11 1486. Given that

$$\sum_{k=1}^{35} \sin 5k = \tan \frac{m}{n},$$

where angles are measured in degrees, and m and n are relatively prime positive integers that satisfy $m < 90n$, find $m + n$.

2000 AIME II, Problem 15 1487. Find the least positive integer n such that

$$\frac{1}{\sin 45^\circ \sin 46^\circ} + \frac{1}{\sin 47^\circ \sin 48^\circ} + \cdots + \frac{1}{\sin 133^\circ \sin 134^\circ} = \frac{1}{\sin n^\circ}.$$

2001 AIME I, Problem 5 1488. An equilateral triangle is inscribed in the ellipse whose equation is $x^2 + 4y^2 = 4$. One vertex of the triangle is $(0, 1)$, one altitude is contained in the y-axis, and the length of each side is $\sqrt{m/n}$, where m and n are relatively prime positive integers. Find $m + n$.

2001 AIME I, Problem 9 1489. In triangle ABC, $AB = 13$, $BC = 15$ and $CA = 17$. Point D is on $\overline{AB}$, E is on $\overline{BC}$, and F is on $\overline{CA}$. Let $AD = p \cdot AB$, $BE = q \cdot BC$, and $CF = r \cdot CA$, where p, q, and r are positive and satisfy $p + q + r = 2/3$ and $p^2 + q^2 + r^2 = 2/5$. The ratio of the area of triangle DEF to the area of triangle ABC can be written in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.

2001 AIME II, Problem 4 1490. Let $R = (8, 6)$. The lines whose equations are $8y = 15x$ and $10y = 3x$ contain points P and Q, respectively, such that R is the midpoint of $\overline{PQ}$. The length of PQ equals m/n, where m and n are relatively prime positive integers. Find $m + n$.

2001 AIME II, Problem 6 1491. Square $ABCD$ is inscribed in a circle. Square $EFGH$ has vertices E and F on $\overline{CD}$ and vertices G and H on the circle. The ratio of the area of square $EFGH$ to the area of square $ABCD$ can be expressed as m/n where m and n are relatively prime positive integers and $m < n$. Find $10n + m$.

2001 AIME II, Problem 12 1492. Given a triangle, its midpoint triangle is obtained by joining the midpoints of its sides. A sequence of polyhedra P_i is defined recursively as follows: P_0 is a regular tetrahedron whose volume is 1. To obtain P_{i+1}, replace the midpoint triangle of every face of P_i by an outward-pointing regular tetrahedron that has the midpoint triangle as a face. The volume of P_3 is m/n, where m and n are relatively prime positive integers. Find $m + n$.

2002 AIME I, Problem 5 1493. Let $A_1, A_2, A_3, \ldots, A_{12}$ be the vertices of a regular dodecagon. How many distinct squares in the plane of the dodecagon have at least two vertices in the set $\{A_1, A_2, A_3, \ldots, A_{12}\}$?

2002 AIME I, Problem 1494. Let $ABCD$ and $BCFG$ be two faces of a cube with $AB = 12$. A beam of light emanates from vertex A and reflects off face $BCFG$ at point P, which is 7 units from $\overline{BG}$ and 5 units from $\overline{BC}$. The beam continues to be reflected off the faces of the cube. The length of the light path from the time it leaves point A until it next reaches a vertex of the cube is given by $m\sqrt{n}$, where m and n are integers and n is not divisible by the square of any prime. Find $m + n$.

2002 AIME I, Problem 13 1495. In triangle ABC the medians $\overline{AD}$ and $\overline{CE}$ have lengths 18 and 27, respectively, and $AB = 24$. Extend $\overline{CE}$ to intersect the circumcircle of ABC at F. The area of triangle AFB is $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

2002 AIME I, Problem 15 1496. Polyhedron $ABCDEFG$ has six faces. Face $ABCD$ is a square with $AB = 12$; face $ABFG$ is a trapezoid with $\overline{AB}$ parallel to $\overline{GF}$, $BF = AG = 8$, and $GF = 6$; and face CDE has $CE = DE = 14$. The other three faces are $ADEG, BCEF$, and EFG. The distance from E to face $ABCD$ is 12. Given that $EG^2 = p - q\sqrt{r}$, where p, q, and r are positive integers and r is not divisible by the square of any prime, find $p + q + r$.

2002 AIME II, Problem 2 1497. Three vertices of a cube are $P = (7, 12, 10)$, $Q = (8, 8, 1)$, and $R = (11, 3, 9)$. What is the surface area of the cube?

2002 AIME II, Problem 10 1498. While finding the sine of a certain angle, an absent-minded professor failed to notice that his calculator was not in the correct angular mode. He was lucky to get the right answer. The two least positive real values of x for which the sine of x degrees is the same as the sine of x radians are $(m\pi)/(n - \pi)$ and $(p\pi)/(q + \pi)$, where m, n, p and q are positive integers. Find $m + n + p + q$.

2002 AIME II, Problem 15 1499. Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at two points, one of which is $(9, 6)$, and the product of the radii is 68. The x-axis and the line $y = mx$, where $m > 0$, are tangent to both circles. It is given that m can be written in the form $a\sqrt{b}/c$, where a, b, and c are positive integers, b is not divisible by the square of any prime, and a and c are relatively prime. Find $a + b + c$.

2003 AIME I, Problem 6 1500. The sum of the areas of all triangles whose vertices are also vertices of a $1 \times 1 \times 1$ cube is $m + \sqrt{n} + \sqrt{p}$, where m, n, and p are integers. Find $m + n + p$.

2003 AIME I, Problem 11 1501. An angle x is chosen at random from the interval $0^\circ < x < 90^\circ$. Let p be the probability that the numbers $\sin^2 x$, $\cos^2 x$, and $\sin x \cos x$ are not the lengths of the sides of a triangle. Given that $p = d/n$, where d is the number of degrees in $\arctan m$ and m and n are positive integers with $m + n < 1000$, find $m + n$.

2003 AIME II, Problem 4 1502. In a regular tetrahedron the centers of the four faces are the vertices of a smaller tetrahedron. The ratio of the volume of the smaller tetrahedron to that of the larger is m/n, where m and n are relatively prime positive integers. Find $m + n$.

2003 AIME II, Problem 11 1503. Triangle ABC is a right triangle with $AC = 7$, $BC = 24$, and right angle at C. Point M is the midpoint of AB, and D is on the same side of line AB as C so that $AD = BD = 15$. Given that the area of triangle CDM may be expressed as $(m\sqrt{n})/p$, where m, n, and p are positive integers, m and p are relatively prime, and n is not divisible by the square of any prime, find $m + n + p$.

2003 AIME II, Problem 13 1504. A bug starts at a vertex of an equilateral triangle. On each move, it randomly selects one of the two vertices where it is not currently located, and crawls along a side of the triangle to that vertex. Given that the probability that the bug moves to its starting vertex on its tenth move is m/n, where m and n are relatively prime positive integers, find $m+n$.

2004 AIME I, Problem 11 1505. A solid in the shape of a right circular cone is 4 inches tall and its base has a 3-inch radius. The entire surface of the cone, including its base, is painted. A plane parallel to the base of the cone divides the cone into two solids, a smaller cone-shaped solid C and a frustum-shaped solid F, in such a way that the ratio between the areas of the painted surfaces of C and F and the ratio between the volumes of C and F are both equal to k. Given that $k = m/n$, where m and n are relatively prime positive integers, find $m+n$.

2004 AIME I, Problem 12 1506. Let S be the set of ordered pairs (x, y) such that $0 < x \leq 1$, $0 < y \leq 1$, and

$$\left[\log_2\left(\frac{1}{x}\right)\right] \qquad \text{and} \qquad \left[\log_5\left(\frac{1}{y}\right)\right]$$

are both even. Given that the area of the graph of S is m/n, where m and n are relatively prime positive integers, find $m+n$. The notation $[z]$ denotes the greatest integer that is less than or equal to z.

2004 AIME II, Problem 1 1507. A chord of a circle is perpendicular to a radius at the midpoint of the radius. The ratio of the area of the larger of the two regions into which the chord divides the circle to the smaller can be expressed in the form $(a\pi + b\sqrt{c})/(d\pi - e\sqrt{f})$, where a, b, c, d, e, and f are positive integers, a and e are relatively prime, and neither c nor f is divisible by the square of any prime. Find the remainder when the product $abcdef$ is divided by 1000.

2004 AIME II, Problem 12 1508. Let $ABCD$ be an isosceles trapezoid, whose dimensions are $AB = 6$, $BC = 5 = DA$, and $CD = 4$. Draw circles of radius 3 centered at A and B, and circles of radius 2 centered at C and D. A circle contained within the trapezoid is tangent to all four of these circles. Its radius is $(-k + m\sqrt{n})/p$, where k, m, n, and p are positive integers, n is not divisible by the square of any prime, and k and p are relatively prime. Find $k+m+n+p$.

2005 AIME I, Problem 7 1509. In quadrilateral $ABCD$, $BC = 8$, $CD = 12$, $AD = 10$, and $m\angle A = m\angle B = 60^\circ$. Given that $AB = p + \sqrt{q}$, where p and q are positive integers, find $p+q$.

2005 AIME I, Problem 10 1510. Triangle ABC lies in the Cartesian Plane and has an area of 70. The coordinates of B and C are $(12, 19)$ and $(23, 20)$, respectively, and the coordinates of A are (p, q). The line containing the median to side BC has slope -5. Find the largest possible value of $p+q$.

2005 AIME I, Problem 11 1511. A semicircle with diameter d is contained in a square whose sides have length 8. Given the maximum value of d is $m - \sqrt{n}$, find $m+n$.

2005 AIME I, Problem 14 1512. Consider the points $A(0,12)$, $B(10,9)$, $C(8,0)$, and $D(-4,7)$. There is a unique square S such that each of the four points is on a different side of S. Let K be the area of S. Find the remainder when $10K$ is divided by 1000.

2005 AIME I, Problem 15 1513. Triangle ABC has $BC = 20$. The incircle of the triangle evenly trisects the median AD. If the area of the triangle is $m\sqrt{n}$ where m and n are integers and n is not divisible by the square of a prime, find $m + n$.

2005 AIME II, Problem 8 1514. Circles C_1 and C_2 are externally tangent, and they are both internally tangent to circle C_3. The radii of C_1 and C_2 are 4 and 10, respectively, and the centers of the three circles are all collinear. A chord of C_3 is also a common external tangent of C_1 and C_2. Given that the length of the chord is $(m\sqrt{n})/p$ where m, n, and p are positive integers, m and p are relatively prime, and n is not divisible by the square of any prime, find $m + n + p$.

2005 AIME II, Problem 10 1515. Given that O is a regular octahedron, that C is the cube whose vertices are the centers of the faces of O, and that the ratio of the volume of O to that of C is m/n, where m and n are relatively prime integers, find $m + n$.

2005 AIME II, Problem 14 1516. In triangle ABC, $AB = 13$, $BC = 15$, and $CA = 14$. Point D is on $\overline{BC}$ with $CD = 6$. Point E is on $\overline{BC}$ such that $\angle BAE \cong \angle CAD$. Given that $BE = p/q$ where p and q are relatively prime positive integers, find q.

2005 AIME II, Problem 15 1517. Let w_1 and w_2 denote the circles $x^2 + y^2 + 10x - 24y - 87 = 0$ and $x^2 + y^2 - 10x - 24y + 153 = 0$, respectively. Let m be the smallest positive value of a for which the line $y = ax$ contains the center of a circle that is externally tangent to w_2 and internally tangent to w_1. Given that $m^2 = p/q$, where p and q are relatively prime integers, find $p + q$.

2006 AIME I, Problem 12 1518. Find the sum of the values of x such that

$$\cos^3 3x + \cos^3 5x = 8\cos^3 4x \cos^3 x,$$

where x is measured in degrees and $100 < x < 200$.

2006 AIME I, Problem 14 1519. A tripod has three legs each of length 5 feet. When the tripod is set up, the angle between any pair of legs is equal to the angle between any other pair, and the top of the tripod is 4 feet from the ground. In setting up the tripod, the lower 1 foot of one leg breaks off. Let h be the height in feet of the top of the tripod from the ground when the broken tripod is set up. Then h can be written in the form $m/\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $\lfloor m + \sqrt{n} \rfloor$. (The notation $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x.)

2006 AIME II, Problem 12 1520. Equilateral $\triangle ABC$ is inscribed in a circle of radius 2. Extend $\overline{AB}$ through B to point D so that $AD = 13$, and extend $\overline{AC}$ through C to point E so that $AE = 11$. Through D, draw a line l_1 parallel to $\overline{AE}$, and through E, draw a line l_2 parallel to $\overline{AD}$. Let F be the intersection of l_1 and l_2. Let G be the point on the circle that is collinear with A and F and distinct from A. Given that the area of $\triangle CBG$ can be expressed in the form $(p\sqrt{q})/r$, where p, q, and r are positive integers, p and r are relatively prime, and q is not divisible by the square of any prime, find $p+q+r$.

2007 AIME II, Problem 11 1521. Two long cylindrical tubes of the same length but different diameters lie parallel to each other on a flat surface. The larger tube has radius 72 and rolls along the surface toward the smaller tube, which has radius 24. It rolls over the smaller tube and continues rolling along the flat surface until it comes to rest on the same point of its circumference as it started, having made one complete revolution. If the smaller tube never moves, and the rolling occurs with no slipping, the larger tube ends up a distance x from where it starts. The distance x can be expressed in the form $a\pi + b\sqrt{c}$, where a, b, and c are integers and c is not divisible by the square of any prime. Find $a + b + c$.

2007 AIME II, Problem 15 1522. Four circles ω, ω_A, ω_B, and ω_C with the same radius are drawn in the interior of triangle ABC such that ω_A is tangent to sides AB and AC, ω_B to BC and BA, ω_C to CA and CB, and ω is externally tangent to ω_A, ω_B, and ω_C. If the sides of triangle ABC are 13, 14, and 15, the radius of ω can be represented in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

2008 AIME I, Problem 8 1523. Find the positive integer n such that

$$\arctan\frac{1}{3} + \arctan\frac{1}{4} + \arctan\frac{1}{5} + \arctan\frac{1}{n} = \frac{\pi}{4}.$$

2008 AIME I, Problem 14 1524. Let $\overline{AB}$ be a diameter of circle ω. Extend $\overline{AB}$ through A to C. Point T lies on ω so that line CT is tangent to ω. Point P is the foot of the perpendicular from A to line CT. Suppose $AB = 18$, and let m denote the maximum possible length of segment BP. Find m^2.

2008 AIME I, Problem 15 1525. A square piece of paper has sides of length 100. From each corner a wedge is cut in the following manner: at each corner, the two cuts for the wedge each start at distance $\sqrt{17}$ from the corner, and they meet on the diagonal at an angle of 60° (see the figure below). The paper is then folded up along the lines joining the vertices of adjacent cuts. When the two edges of a cut meet, they are taped together. The result is a paper tray whose sides are not at right angles to the base. The height of the tray, that is, the perpendicular distance between the plane of the base and the plane formed by the upper edges, can be written in the form $\sqrt[n]{m}$, where m and n are positive integers, $m < 1000$, and m is not divisible by the nth power of any prime. Find $m + n$.

2008 AIME II, Problem 8 1526. Let $a = \pi/2008$. Find the smallest positive integer n such that

$$2[\cos(a)\sin(a) + \cos(4a)\sin(2a) + \cos(9a)\sin(3a) + \cdots + \cos(n^2a)\sin(na)],$$

is an integer.

2008 AIME II, Problem 9 1527. A particle is located on the coordinate plane at $(5, 0)$. Define a move for the particle as a counterclockwise rotation of $\pi/4$ radians about the origin followed by a translation of 10 units in the positive x-direction. Given that the particle's position after 150 moves is (p, q), find the greatest integer less than or equal to $|p| + |q|$.

2008 AIME II, Problem 13 1528. A regular hexagon with center at the origin in the complex plane has opposite pairs of sides one unit apart. One pair of sides is parallel to the imaginary axis. Let R be the region outside the hexagon, and let

$$S = \left\{ \frac{1}{z} \mid z \in R \right\}.$$

Then the area of S has the form $a\pi + \sqrt{b}$, where a and b are positive integers. Find $a + b$.

2009 AIME I, Problem 11 1529. Consider the set of all triangles OPQ where O is the origin and P and Q are distinct points in the plane with nonnegative integer coordinates (x, y) such that $41x + y = 2009$. Find the number of such distinct triangles whose area is a positive integer.

2009 AIME I, Problem 12 1530. In right $\triangle ABC$ with hypotenuse $\overline{AB}$, $AC = 12$, $BC = 35$, and $\overline{CD}$ is the altitude to $\overline{AB}$. Let ω be the circle having $\overline{CD}$ as a diameter. Let I be a point outside $\triangle ABC$ such that $\overline{AI}$ and $\overline{BI}$ are both tangent to circle ω. The ratio of the perimeter of $\triangle ABI$ to the length AB can be expressed in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.

2009 AIME II, Problem 10 1531. Four lighthouses are located at points A, B, C, and D. The lighthouse at A is 5 kilometers from the lighthouse at B, the lighthouse at B is 12 kilometers from the lighthouse at C, and the lighthouse at A is 13 kilometers from the lighthouse at C. To an observer at A, the angle determined by the lights at B and D and the angle determined by the lights at C and D are equal. To an observer at C, the angle determined by the lights at A and B and the angle determined by the lights at D and B are equal. The number of kilometers from A to D is given by $p\sqrt{r}/q$, where p, q, and r are relatively prime positive integers, and r is not divisible by the square of any prime. Find $p + q + r$.

2009 AIME II, Problem 13 1532. Let A and B be the endpoints of a semicircular arc of radius 2. The arc is divided into seven congruent arcs by six equally spaced points $C_1, C_2, \ldots, C_6$. All chords of the form $\overline{AC_i}$ or $\overline{BC_i}$ are drawn. Let n be the product of the lengths of these twelve chords. Find the remainder when n is divided by 1000.

2009 AIME II, Problem 15 1533. Let $\overline{MN}$ be a diameter of a circle with diameter 1. Let A and B be points on one of the semicircular arcs determined by $\overline{MN}$ such that A is the midpoint of the semicircle and $MB = 3/5$. Point C lies on the other semicircular arc. Let d be the length of the line segment whose endpoints are the intersections of diameter $\overline{MN}$ with the chords $\overline{AC}$ and $\overline{BC}$. The largest possible value of d can be written in the form $r - s\sqrt{t}$, where r, s, and t are positive integers and t is not divisible by the square of any prime. Find $r + s + t$.

2010 AIME I, Problem 13 1534. Rectangle $ABCD$ and a semicircle with diameter AB are coplanar and have nonoverlapping interiors. Let $\mathcal{R}$ denote the region enclosed by the semicircle and the rectangle. Line ℓ meets the semicircle, segment AB, and segment CD at distinct points N, U, and T, respectively. Line ℓ divides region $\mathcal{R}$ into two regions with areas in the ratio $1 : 2$. Suppose that $AU = 84$, $AN = 126$, and $UB = 168$. Then DA can be represented as $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

2010 AIME I, Problem 15 1535. In $\triangle ABC$ with $AB = 12$, $BC = 13$, and $AC = 15$, let M be a point on $\overline{AC}$ such that the incircles of $\triangle ABM$ and $\triangle BCM$ have equal radii. Let p and q be positive relatively prime integers such that

$$\frac{AM}{CM} = \frac{p}{q}.$$

Find $p + q$.

2010 AIME II, Problem 12 1536. Two noncongruent integer-sided isosceles triangles have the same perimeter and the same area. The ratio of the lengths of the bases of the two triangles is $8 : 7$. Find the minimum possible value of their common perimeter.

2010 AIME II, Problem 14 1537. In right triangle ABC with right angle at C, we know that $\angle BAC < 45^\circ$ and $AB = 4$. Point P on AB is chosen such that $\angle APC = 2\angle ACP$ and $CP = 1$. The ratio AP/BP can be represented in the form $p + q\sqrt{r}$, where p, q, r are positive integers and r is not divisible by the square of any prime. Find $p+q+r$.

2010 AIME II, Problem 15 1538. In triangle ABC, $AC = 13, BC = 14$, and $AB = 15$. Points M and D lie on AC with $AM = MC$ and $\angle ABD = \angle DBC$. Points N and E lie on AB with $AN = NB$ and $\angle ACE = \angle ECB$. Let P be the point, other than A, of intersection of the circumcircles of $\triangle AMN$ and $\triangle ADE$. Ray AP meets BC at Q. The ratio BQ/CQ can be written in the form m/n, where m and n are relatively prime positive integers. Find $m - n$.

2011 AIME I, Problem 2 1539. In rectangle $ABCD$, $AB = 12$ and $BC = 10$. Points E and F lie inside rectangle $ABCD$ so that $BE = 9$, $DF = 8$, $\overline{BE} \parallel \overline{DF}$, $\overline{EF} \parallel \overline{AB}$, and line BE intersects segment $\overline{AD}$. The length EF can be expressed in the form $m\sqrt{n} - p$, where m, n, and p are positive integers and n is not divisible by the square of any prime. Find $m + n + p$.

2011 AIME I, Problem 3 1540. Let L be the line with slope $5/12$ that contains the point $A = (24, -1)$, and let M be the line perpendicular to line L that contains the point $B = (5, 6)$. The original coordinate axes are erased, and line L is made the x-axis, and line M the y-axis. In the new coordinate system, point A is on the positive x-axis, and point B is on the positive y-axis. The point P with coordinates $(-14, 27)$ in the original system has coordinates (α, β) in the new coordinate system. Find $\alpha + \beta$.

2011 AIME I, Problem 4 1541. In triangle ABC, $AB = 125, AC = 117$, and $BC = 120$. The angle bisector of angle A intersects $\overline{BC}$ at point L, and the angle bisector of angle B intersects $\overline{AC}$ at point K. Let M and N be the feet of the perpendiculars from C to $\overline{BK}$ and $\overline{AL}$, respectively. Find MN.

2011 AIME I, Problem 13 1542. A cube with side length 10 is suspended above a plane. The vertex closest to the plane is labelled A. The three vertices adjacent to vertex A are at heights 10, 11, and 12 above the plane. The distance from vertex A to the plane can be expressed as $(r-\sqrt{s})/t$, where r, s, and t are positive integers, and $r+s+t < 1000$. Find $r + s + t$.

2011 AIME I, Problem 14 1543. Let $A_1A_2A_3A_4A_5A_6A_7A_8$ be a regular octagon. Let M_1, M_3, M_5, and M_7 be the midpoints of sides $\overline{A_1A_2}$, $\overline{A_3A_4}$, $\overline{A_5A_6}$, and $\overline{A_7A_8}$, respectively. For $i = 1, 3, 5, 7$, ray R_i is constructed from M_i towards the interior of the octagon such that $R_1 \perp R_3$, $R_3 \perp R_5$, $R_5 \perp R_7$, and $R_7 \perp R_1$. Pairs of rays R_1 and R_3, R_3 and

R_5, R_5 and R_7, and R_7 and R_1 meet at B_1, B_3, B_5, B_7 respectively. If $B_1B_3 = A_1A_2$, then $\cos 2\angle A_3M_3B_1$ can be written in the form $m - \sqrt{n}$, where m and n are positive integers. Find $m + n$.

2011 AIME II, Problem 4 1544. In triangle ABC, $11AB = 20AC$. The angle bisector of $\angle A$ intersects BC at point D, and point M is the midpoint of AD. Let P be the point of the intersection of AC and BM. The ratio of CP to PA can be expressed in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.

2011 AIME II, Problem 10 1545. A circle with center O has radius 25. Chord $\overline{AB}$ of length 30 and chord $\overline{CD}$ of length 14 intersect at point P. The distance between the midpoints of the two chords is 12. The quantity OP^2 can be represented as m/n, where m and n are relatively prime positive integers. Find the remainder where $m+n$ is divided by 1000.

2011 AIME II, Problem 13 1546. Point P lies on the diagonal AC of square $ABCD$ with $AP > CP$. Let O_1 and O_2 be the circumcenters of triangles ABP and CDP respectively. Given that $AB = 12$ and $\angle O_1PO_2 = 120°$, then $AP = \sqrt{a} + \sqrt{b}$ where a and b are positive integers. Find $a + b$.

2011 AIME II, Problem 15 1547. Let $P(x) = x^2 - 3x - 9$. A real number x is chosen at random from the interval $5 \le x \le 15$. The probability that

$$\lfloor \sqrt{P(x)} \rfloor = \sqrt{P(\lfloor x \rfloor)},$$

is equal to

$$\frac{\sqrt{a} + \sqrt{b} + \sqrt{c} - d}{e},$$

where a, b, c, d and e are positive integers and none of a, b, or c is divisible by the square of a prime. Find $a + b + c + d + e$.

2011 HMMT, Algebra, Problem 8 1548. Let $z = \cos\frac{2\pi}{2011} + i\sin\frac{2\pi}{2011}$, and let

$$P(x) = x^{2008} + 3x^{2007} + 6x^{2006} + \cdots + \frac{2008 \cdot 2009}{2}x + \frac{2009 \cdot 2010}{2}$$

for all complex numbers x. Evaluate $P(z)P(z^2)P(z^3)\cdots P(z^{2010})$.

2012 AIME I, Problem 13 1549. Three concentric circles have radii 3, 4, and 5. An equilateral triangle with one vertex on each circle has side length s. The largest possible area of the triangle can be written as

$$a + \frac{b}{c}\sqrt{d},$$

where a, b, c and d are positive integers, b and c are relatively prime, and d is not divisible by the square of any prime. Find $a + b + c + d$.

2012 AIME II, Problem 9 1550. Let x and y be real numbers such that

$$\frac{\sin x}{\sin y} = 3 \qquad \text{and} \qquad \frac{\cos x}{\cos y} = \frac{1}{2}.$$

The value of

$$\frac{\sin 2x}{\sin 2y} + \frac{\cos 2x}{\cos 2y}$$

can be expressed in the form p/q, where p and q are relatively prime positive integers. Find $p+q$.

2012 AIME II, Problem 13 1551. Equilateral $\triangle ABC$ has side length $\sqrt{111}$. There are four distinct triangles AD_1E_1, AD_1E_2, AD_2E_3, and AD_2E_4, each congruent to $\triangle ABC$, with $BD_1 = BD_2 = \sqrt{11}$. Find

$$\sum_{k=1}^{4}(CE_k)^2.$$

2012 AIME II, Problem 15 1552. Triangle ABC is inscribed in circle ω with $AB = 5$, $BC = 7$, and $AC = 3$. The bisector of angle A meets side BC at D and circle ω at a second point E. Let γ be the circle with diameter DE. Circles ω and γ meet at E and a second point F. Then $AF^2 = m/n$, where m and n are relatively prime positive integers. Find $m+n$.

2012 HMMT, Algebra, Problem 4 1553. During the weekends, Eli delivers milk in the complex plane. On Saturday, he begins at z and delivers milk to houses located at $z^3, z^5, z^7, \ldots, z^{2013}$ in that order; on Sunday, he begins at 1 and delivers milk to houses located at $z^2, z^4, z^6, \ldots, z^{2012}$ in that order. Eli always walks directly (in a straight line) between two houses. If the distance he must travel from his starting point to the last house is $\sqrt{2012}$ on both days, find the real part of z^2.

2013 AIME I, Problem 3 1554. Let $ABCD$ be a square, and let E and F be points on $\overline{AB}$ and $\overline{BC}$, respectively. The line through E parallel to $\overline{BC}$ and the line through F parallel to $\overline{AB}$ divide $ABCD$ into two squares and two non square rectangles. The sum of the areas of the two squares is $9/10$ of the area of square $ABCD$. Find

$$\frac{AE}{EB} + \frac{EB}{AE}.$$

2013 AIME I, Problem 7 1555. A rectangular box has width 12 inches, length 16 inches, and height m/n inches, where m and n are relatively prime positive integers. Three faces of the box meet at a corner of the box. The center points of those three faces are the vertices of a triangle with an area of 30 square inches. Find $m+n$.

2013 AIME I, Problem 8 1556. The domain of the function $f(x) = \arcsin(\log_m(nx))$ is a closed interval of length $1/2013$, where m and n are positive integers and $m > 1$. Find the remainder when the smallest possible sum $m+n$ is divided by 1000.

2013 AIME I, Problem 9 1557. A paper equilateral triangle ABC has side length 12. The paper triangle is folded so that vertex A touches a point on side $\overline{BC}$ a distance 9 from point B. The length of the line segment along which the triangle is folded can be written as $(m\sqrt{p})/n$, where m, n, and p are positive integers, m and n are relatively prime, and p is not divisible by the square of any prime. Find $m+n+p$.

2013 AIME I, Problem 13 1558. Triangle AB_0C_0 has side lengths $AB_0 = 12$, $B_0C_0 = 17$, and $C_0A = 25$. For each positive integer n, points B_n and C_n are located on $\overline{AB_{n-1}}$ and $\overline{AC_{n-1}}$, respectively, creating three similar triangles $\triangle AB_nC_n \sim \triangle B_{n-1}C_nC_{n-1} \sim \triangle AB_{n-1}C_{n-1}$. The area of the union of all triangles $B_{n-1}C_nB_n$ for $n \geq 1$ can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find q.

2013 AIME I, Problem 14 1559. For $\pi \le \theta < 2\pi$, let

$$P = \frac{1}{2}\cos\theta - \frac{1}{4}\sin 2\theta - \frac{1}{8}\cos 3\theta + \frac{1}{16}\sin 4\theta + \frac{1}{32}\cos 5\theta - \frac{1}{64}\sin 6\theta - \frac{1}{128}\cos 7\theta + \cdots,$$

and

$$Q = 1 - \frac{1}{2}\sin\theta - \frac{1}{4}\cos 2\theta + \frac{1}{8}\sin 3\theta + \frac{1}{16}\cos 4\theta - \frac{1}{32}\sin 5\theta - \frac{1}{64}\cos 6\theta + \frac{1}{128}\sin 7\theta + \cdots,$$

so that $P/Q = (2\sqrt{2})/7$. Then, $\sin\theta = -m/n$ where m and n are relatively prime positive integers. Find $m+n$.

2013 AIME II, Problem 4 1560. In the Cartesian plane let $A = (1,0)$ and $B = \left(2, 2\sqrt{3}\right)$. Equilateral triangle ABC is constructed so that C lies in the first quadrant. Let $P = (x,y)$ be the center of $\triangle ABC$. Then $x \cdot y$ can be written as $(p\sqrt{q})/r$, where p and r are relatively prime positive integers and q is an integer that is not divisible by the square of any prime. Find $p+q+r$.

2013 AIME II, Problem 5 1561. In equilateral $\triangle ABC$ let points D and E trisect $\overline{BC}$. Then $\sin(\angle DAE)$ can be expressed in the form $(a\sqrt{b})/c$, where a and c are relatively prime positive integers, and b is an integer that is not divisible by the square of any prime. Find $a+b+c$.

2013 AIME II, Problem 13 1562. In $\triangle ABC$, $AC = BC$, and point D is on $\overline{BC}$ so that $CD = 3 \cdot BD$. Let E be the midpoint of $\overline{AD}$. Given that $CE = \sqrt{7}$ and $BE = 3$, the area of $\triangle ABC$ can be expressed in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m+n$.

2013 AIME II, Problem 15 1563. Let A, B, C be angles of an acute triangle with

$$\cos^2 A + \cos^2 B + 2\sin A \sin B \cos C = \frac{15}{8}, \text{ and}$$
$$\cos^2 B + \cos^2 C + 2\sin B \sin C \cos A = \frac{14}{9}.$$

There are positive integers p, q, r, and s for which

$$\cos^2 C + \cos^2 A + 2\sin C \sin A \cos B = \frac{p - q\sqrt{r}}{s},$$

where $p+q$ and s are relatively prime and r is not divisible by the square of any prime. Find $p+q+r+s$. *Note*: due to an oversight by the exam-setters, there is no acute triangle satisfying these conditions. You should instead assume ABC is obtuse with $\angle B > 90°$.

2013 HMIC, Problem 5 1564. This problem has two parts:

a) Given a set X of points in the plane, let $f_X(n)$ be the largest possible area of a polygon with at most n vertices, all of which are points of X. Prove that if m, n are integers with $m \ge n > 2$ then $f_X(m) + f_X(n) \ge f_X(m+1) + f_X(n-1)$.

b) Let P_0 be a 1×2 rectangle (including its interior) and inductively define the polygon P_i to be the result of folding P_{i-1} over some line that cuts P_{i-1} into two connected parts. The diameter of a polygon P_i is the maximum distance between two points of P_i. Determine the smallest possible diameter of P_{2013}.

2013 HMMT, Guts, Problem 34 1565. For how many unordered sets $\{a, b, c, d\}$ of positive integers, none of which exceed 168, do there exist integers w, x, y, z such that $(-1)^w a + (-1)^x b + (-1)^y c + (-1)^z d = 168$? If your answer is A and the correct answer is C, then your score (out of 25) on this problem will be

$$\left\lfloor 25e^{-3\frac{|C-A|}{C}} \right\rfloor.$$

2014 AIME I, Problem 10 1566. A disk with radius 1 is externally tangent to a disk with radius 5. Let A be the point where the disks are tangent, C be the center of the smaller disk, and E be the center of the larger disk. While the larger disk remains fixed, the smaller disk is allowed to roll along the outside of the larger disk until the smaller disk has turned through an angle of 360°. That is, if the center of the smaller disk has moved to the point D, and the point on the smaller disk that began at A has now moved to point B, then $\overline{AC}$ is parallel to $\overline{BD}$. Then $\sin^2(\angle BEA) = m/n$, where m and n are relatively prime positive integers. Find $m + n$.

2014 AIME II, Problem 11 1567. In $\triangle RED, RD = 1, \angle DRE = 75°$ and $\angle RED = 45°$. Let M be the midpoint of segment $\overline{RD}$. Point C lies on side $\overline{ED}$ such that $\overline{RC} \perp \overline{EM}$. Extend segment $\overline{DE}$ through E to point A such that $CA = AR$. Then $AE = (a - \sqrt{b})/c$, where a and c are relatively prime positive integers, and b is a positive integer. Find $a + b + c$.

2014 AIME II, Problem 12 1568. Suppose that the angles of $\triangle ABC$ satisfy $\cos(3A) + \cos(3B) + \cos(3C) = 1$. Two sides of the triangle have lengths 10 and 13. There is a positive integer m so that the maximum possible length for the remaining side of $\triangle ABC$ is $\sqrt{m}$. Find m.

2014 AIME II, Problem 14 1569. In $\triangle ABC$, $AB = 10$, $\angle A = 30°$, and $\angle C = 45°$. Let H, D, and M be points on line $\overline{BC}$ such that $\overline{AH} \perp \overline{BC}$, $\angle BAD = \angle CAD$, and $BM = CM$. Point N is the midpoint of segment $\overline{HM}$, and point P is on ray AD such that $\overline{PN} \perp \overline{BC}$. Then $AP^2 = m/n$, where m and n are relatively prime positive integers. Find $m + n$.

2014 HMMT, Guts, Problem 31 1570. Compute

$$\sum_{k=1}^{1007} \left(\cos\left(\frac{\pi k}{1007} \right) \right)^{2014}.$$

2014 HMMT, Team, Problem 10 1571. Fix a positive real number $c > 1$ and positive integer n. Initially, a blackboard contains the numbers $1, c, \ldots, c^{n-1}$. Every minute, Bob chooses two numbers a, b on the board and replaces them with $ca + c^2 b$. Prove that after $n - 1$ minutes, the blackboard contains a single number no less than

$$\left(\frac{c^{n/L} - 1}{c^{1/L} - 1} \right)^L,$$

where $\phi = \frac{1+\sqrt{5}}{2}$ and $L = 1 + \log_\phi(c)$.

2015 AIME I, Problem 11 1572. Triangle ABC has positive integer side lengths with $AB = AC$. Let I be the intersection of the bisectors of $\angle B$ and $\angle C$. Suppose $BI = 8$. Find the smallest possible perimeter of $\triangle ABC$.

2015 AIME I, Problem 13 1573. With all angles measured in degrees, the product

$$\prod_{k=1}^{45} \csc^2(2k-1)^\circ = m^n,$$

where m and n are integers greater than 1. Find $m + n$.

2015 AIME II, Problem 4 1574. In an isosceles trapezoid, the parallel bases have lengths $\log 3$ and $\log 192$, and the altitude to these bases has length $\log 16$. The perimeter of the trapezoid can be written in the form $\log 2^p 3^q$, where p and q are positive integers. Find $p + q$.

2015 AIME II, Problem 7 1575. Triangle ABC has side lengths $AB = 12$, $BC = 25$, and $CA = 17$. Rectangle $PQRS$ has vertex P on $\overline{AB}$, vertex Q on $\overline{AC}$, and vertices R and S on $\overline{BC}$. In terms of the side length $PQ = w$, the area of $PQRS$ can be expressed as the quadratic polynomial

$$\text{Area}(PQRS) = \alpha w - \beta \cdot w^2$$

Then the coefficient $\beta = m/n$, where m and n are relatively prime positive integers. Find $m + n$.

2015 AIME II, Problem 13 1576. Define the sequence $a_1, a_2, a_3, \ldots$ by

$$a_n = \sum_{k=1}^{n} \sin(k),$$

where k represents radian measure. Find the index of the 100th term for which $a_n < 0$.

2015 HMMT, Geometry, Problem 10 1577. Let $\mathcal{G}$ be the set of all points (x, y) in the Cartesian plane such that $0 \le y \le 8$ and

$$(x-3)^2 + 31 = (y-4)^2 + 8\sqrt{y(8-y)}.$$

There exists a unique line ℓ of negative slope tangent to $\mathcal{G}$ and passing through the point $(0, 4)$. Suppose ℓ is tangent to $\mathcal{G}$ at a unique point P. Find the coordinates (α, β) of P.

2015 HMMT, Guts, Problem 4 1578. Consider the function $z(x, y)$ describing the paraboloid

$$z = (2x - y)^2 - 2y^2 - 3y.$$

Archimedes and Brahmagupta are playing a game. Archimedes first chooses x. Afterwards, Brahmagupta chooses y. Archimedes wishes to minimize z while Brahmagupta wishes to maximize z. Assuming that Brahmagupta will play optimally, what value of x should Archimedes choose?

2015 HMMT, Guts, Problem 20 1579. What is the largest real number θ less than π such that

$$\prod_{k=0}^{10} \cos\left(2^k\theta\right),$$

and

$$\prod_{k=0}^{10}\left(1+\frac{1}{\cos(2^k\theta)}\right)?$$

2015 HMMT, Guts, Problem 28 1580. Let $w, x, y,$ and z be positive real numbers such that

$$\begin{aligned} 0 &\neq \cos w \cos x \cos y \cos z, \\ 2\pi &= w+x+y+z, \\ 3\tan w &= k(1+\sec w), \\ 4\tan x &= k(1+\sec x), \\ 5\tan y &= k(1+\sec y), \\ 6\tan z &= k(1+\sec z). \end{aligned}$$

Here, $\sec t$ denotes $1/\cos t$ when $\cos t \neq 0$. Find k.

2016 AIME I, Problem 9 1581. Triangle ABC has $AB = 40$, $AC = 31$, and $\sin A = 1/5$. This triangle is inscribed in rectangle $AQRS$ with B on $\overline{QR}$ and C on $\overline{RS}$. Find the maximum possible area of $AQRS$.

2016 AIME II, Problem 5 1582. Triangle ABC_0 has a right angle at C_0. Its side lengths are pairwise relatively prime positive integers, and its perimeter is p. Let C_1 be the foot of the altitude to $\overline{AB}$, and for $n \geq 2$, let C_n be the foot of the altitude to $\overline{C_{n-2}B}$ in $\triangle C_{n-2}C_{n-1}B$. Given the sum

$$\sum_{n=1}^{\infty} C_{n-1}C_n = 6p,$$

find p.

2016 HMMT, Guts, Problem 15 1583. Compute

$$\tan\left(\frac{\pi}{7}\right)\tan\left(\frac{2\pi}{7}\right)\tan\left(\frac{3\pi}{7}\right).$$

2016 HMMT, November Theme, Problem 7 1584. Seven lattice points form a convex heptagon with all sides having distinct lengths. Find the minimum possible value of the sum of the squares of the sides of the heptagon.

2017 AIME I, Problem 4 1585. A pyramid has a triangular base with side lengths 20, 20, and 24. The three edges of the pyramid from the three corners of the base to the fourth vertex of the pyramid all have length 25. The volume of the pyramid is $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m+n$.

2017 AIME II, Problem 13 1586. For each integer $n \geq 3$, let $f(n)$ be the number of 3-element subsets of the vertices of a regular n-gon that are the vertices of an isosceles triangle (including equilateral triangles). Find the sum of all values of n such that

$$f(n+1) = f(n) + 78.$$

2017 AIME II, Problem 15 1587. Tetrahedron $ABCD$ has $AD = BC = 28$, $AC = BD = 44$, and $AB = CD = 52$. For any point X in space, define

$$f(X) = AX + BX + CX + DX.$$

The least possible value of $f(X)$ can be expressed as $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m+n$.

2017 HMMT, November Team, Problem 10 1588. Denote $\phi = \frac{\sqrt{5}+1}{2}$ and consider the set of all finite binary strings without leading zeroes. Each string S has a "base-ϕ" value $p(S)$. For example, $p(1101) = \phi^3 + \phi^2 + 1$. For any positive integer n, let $f(n)$ be the number of such strings S that satisfy $p(S) = \frac{\phi^{48n}-1}{\phi^{48}-1}$. The sequence of fractions $\frac{f(n+1)}{f(n)}$ approaches a real number c as n goes to infinity. Determine the value of c.

2017 HMMT, November Guts, Problem 26 1589. Kelvin the Frog is hopping on a number line (extending to infinity in both directions). Kelvin starts at 0. Every minute, he has a $\frac{1}{3}$ chance of moving 1 unit left, a $\frac{1}{3}$ chance of moving 1 unit right, and $\frac{1}{3}$ chance of getting eaten. Find the expected number of times Kelvin returns to 0 (not including the start) before he gets eaten.

2017 HMIC, Problem 3 1590. Let $v_1, v_2, \ldots, v_m$ be vectors in $\mathbb{R}^n$, such that each has a strictly positive first coordinate. Consider the following process. Start with the zero vector $w = (0, 0, \ldots, 0) \in \mathbb{R}^n$. Every round, choose an i such that $1 \leq i \leq m$ and $w \cdot v_i \leq 0$, and then replace w with $w + v_i$.

Show that there exists a constant C such that regardless of your choice of i at each step, the process is guaranteed to terminate in (at most) C rounds. The constant C may depend on the vectors $v_1, \ldots, v_m$.

2018 AIME I, Problem 13 1591. Let $\triangle ABC$ have side lengths $AB = 30$, $BC = 32$, and $AC = 34$. Point X lies in the interior of $\overline{BC}$, and points I_1 and I_2 are the incenters of $\triangle ABX$ and $\triangle ACX$, respectively. Find the minimum possible area of $\triangle AI_1I_2$ as X varies along $\overline{BC}$.

2018 AIME I, Problem 15 1592. David found four sticks of different lengths that can be used to form three non-congruent convex cyclic quadrilaterals, A, B, C, which can each be inscribed in a circle with radius 1. Let φ_A denote the measure of the acute angle made by the diagonals of quadrilateral A, and define φ_B and φ_C similarly. Suppose that $\sin\varphi_A = 2/3$, $\sin\varphi_B = 3/5$, and $\sin\varphi_C = 6/7$. All three quadrilaterals have the same area K, which can be written in the form m/n, where m and n are relatively prime positive integers. Find $m+n$.

2018 AIME II, Problem 12 1593. Let $ABCD$ be a convex quadrilateral with $AB = CD = 10$, $BC = 14$, and $AD = 2\sqrt{65}$. Assume that the diagonals of $ABCD$ intersect at point P, and that the sum of the areas of $\triangle APB$ and $\triangle CPD$ equals the sum of the areas of $\triangle BPC$ and $\triangle APD$. Find the area of quadrilateral $ABCD$.

2018 HMMT, Algebra & Number Theory, Problem 6 1594. Let α, β, and γ be three real numbers. Suppose that

$$\cos\alpha + \cos\beta + \cos\gamma = 1$$

$$\sin\alpha + \sin\beta + \sin\gamma = 1.$$

Find the smallest possible value of $\cos\alpha$.

2018 HMMT, Algebra & Number Theory, Problem 10 1595. Let S be a randomly chosen 6-element subset of the set $\{0, 1, 2, \ldots, n\}$. Consider the polynomial

$$P(x) = \sum_{i \in S} x^i.$$

Let X_n be the probability that $P(x)$ is divisible by some non-constant polynomial $Q(x)$ of degree at most 3 with integer coefficients satisfying $Q(0) \neq 0$. Find the limit of X_n as n goes to infinity.

2018 HMMT, Team, Problem 5 1596. Is it possible for the projection of the set of points (x, y, z) with $0 \leq x, y, z \leq 1$ onto some two-dimensional plane to be a simple convex pentagon?

2018 HMMT, November General, Problem 10 1597. Real numbers x, y, and z are chosen from the interval $[-1, 1]$ independently and uniformly at random. What is the probability that

$$|x| + |y| + |z| + |x + y + z| = |x + y| + |y + z| + |z + x|?$$

2019 AIME I, Problem 8 1598. Let x be a real number such that

$$\sin^{10} x + \cos^{10} x = \frac{11}{36}.$$

Then,

$$\sin^{12} x + \cos^{12} x = \frac{m}{n},$$

where m and n are relatively prime positive integers. Find $m + n$.

2019 AIME I, Problem 11 1599. In $\triangle ABC$, the sides have integers lengths and $AB = AC$. Circle ω has its center at the incenter of $\triangle ABC$. An excircle of $\triangle ABC$ is a circle in the exterior of $\triangle ABC$ that is tangent to one side of the triangle and tangent to the extensions of the other two sides. Suppose that the excircle tangent to $\overline{BC}$ is internally tangent to ω, and the other two excircles are both externally tangent to ω. Find the minimum possible value of the perimeter of $\triangle ABC$.

2019 AIME I, Problem 15 1600. Let $\overline{AB}$ be a chord of a circle ω, and let P be a point on the chord $\overline{AB}$. Circle ω_1 passes through A and P and is internally tangent to ω. Circle ω_2 passes through B and P and is internally tangent to ω. Circles ω_1 and ω_2 intersect at points P and Q. Line PQ intersects ω at X and Y. Assume that $AP = 5$, $PB = 3$, $XY = 11$, and $PQ^2 = m/n$, where m and n are relatively prime positive integers. Find $m + n$.

2019 AIME II, Problem 10 1601. There is a unique angle θ between 0° and 90° such that for non-negative integers n, the value of $\tan(2^n\theta)$ is positive when n is a multiple of 3, and negative otherwise. The degree measure of θ is p/q, where p and q are relatively prime integers. Find $p + q$.

2019 AIME II, Problem 15 1602. In acute triangle ABC points P and Q are the feet of the perpendiculars from C to $\overline{AB}$ and from B to $\overline{AC}$, respectively. Line PQ intersects the circumcircle of $\triangle ABC$ in two distinct points, X and Y. Suppose $XP = 10$, $PQ = 25$, and $QY = 15$. The value of $AB \cdot AC$ can be written in the form $m\sqrt{n}$ where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.

2019 HMMT, Combinatorics, Problem 10 1603. Fred the Four-Dimensional Fluffy Sheep is walking in 4-dimensional space. He starts at the origin. Each minute, he walks from his current position (a_1, a_2, a_3, a_4) to some position (x_1, x_2, x_3, x_4) with integer coordinates satisfying

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2 + (x_4 - a_4)^2 = 4 \quad \text{and}$$
$$|(x_1 + x_2 + x_3 + x_4) - (a_1 + a_2 + a_3 + a_4)| = 2.$$

In how many ways can Fred reach $(10, 10, 10, 10)$ after exactly 40 minutes, if he is allowed to pass through this point during his walk?

2019 HMIC, Problem 3 1604. Do there exist four points $P_i = (x_i, y_i) \in \mathbb{R}^2$ $(1 \leq i \leq 4)$ on the plane such that:

a) for all $i = 1, 2, 3, 4$, the inequality $x_i^4 + y_i^4 \leq x_i^3 + y_i^3$ holds, and

b) for all $i \neq j$, the distance between P_i and P_j is greater than 1?

2020 AIME II, Problem 13 1605. Convex pentagon $ABCDE$ has side lengths $AB = 5$, $BC = CD = DE = 6$, and $EA = 7$. Moreover, the pentagon has an inscribed circle (a circle tangent to each side of the pentagon). Find the area of $ABCDE$.

2020 AIME II, Problem 15 1606. Let $\triangle ABC$ be an acute scalene triangle with circumcircle ω. The tangents to ω at B and C intersect at T. Let X and Y be the projections of T onto lines AB and AC, respectively. Suppose $BT = CT = 16$, $BC = 22$, and

$$TX^2 + TY^2 + XY^2 = 1143.$$

Find XY^2.

2020 HMMT, Geometry, Problem 7 1607. Let Γ be a circle, and ω_1 and ω_2 be two non-intersecting circles inside Γ that are internally tangent to Γ at X_1 and X_2, respectively. Let one of the common internal tangents of ω_1 and ω_2 touch ω_1 and ω_2 at T_1 and T_2, respectively, while intersecting Γ at two points A and B. Given that

$2X_1T_1 = X_2T_2$ and that ω_1, ω_2, and Γ have radii 2, 3, and 12, respectively, compute the length of AB.

2020 HMMT, Team, Problem 4 1608. Alan draws a convex 2020-gon

$$\mathcal{A} = A_1A_2 \cdots A_{2020}$$

with vertices in clockwise order and chooses 2020 angles $\theta_1, \theta_2, \ldots, \theta_{2020} \in (0, \pi)$ in radians with sum 1010π. He then constructs isosceles triangles $\triangle A_iB_iA_{i+1}$ on the exterior of $\mathcal{A}$ with $B_iA_i = B_iA_{i+1}$ and $\angle A_iB_iA_{i+1} = \theta_i$. (Here, $A_{2021} = A_1$.) Finally, he erases $\mathcal{A}$ and the point B_1. He then tells Jason the angles $\theta_1, \theta_2, \ldots, \theta_{2020}$ he chose. Show that Jason can determine where B_1 was from the remaining 2019 points, i.e. show that B_1 is uniquely determined by the information Jason has.

2021 AIME I, Problem 6 1609. Segments $\overline{AB}, \overline{AC}$, and $\overline{AD}$ are edges of a cube and $\overline{AG}$ is a diagonal through the center of the cube. Point P satisfies $BP = 60\sqrt{10}$, $CP = 60\sqrt{5}$, $DP = 120\sqrt{2}$, and $GP = 36\sqrt{7}$. Find AP.

2021 AIME I, Problem 7 1610. Find the number of pairs (m, n) of positive integers with $1 \le m < n \le 30$ such that there exists a real number x satisfying

$$\sin(mx) + \sin(nx) = 2.$$

2021 AIME II, Problem 5 1611. For positive real numbers s, let $\tau(s)$ denote the set of all obtuse triangles that have area s and two sides with lengths 4 and 10. The set of all s for which $\tau(s)$ is nonempty, but all triangles in $\tau(s)$ are congruent, is an interval $[a, b)$. Find $a^2 + b^2$.

2021 PUMaC, Team Round, Problem 5 1612. Given a real number t with $0 < t < 1$, define the real-valued function

$$f(t, \theta) = \sum_{n=-\infty}^{\infty} t^{|n|}\omega^n,$$

where $\omega = e^{i\theta} = \cos\theta + i\sin\theta$. For $\theta \in [0, 2\pi)$, the polar curve $r(\theta) = f(t, \theta)$ traces out an ellipse E_t with a horizontal major axis whose left focus is at the origin. Let $A(t)$ be the area of the ellipse E_t. Let

$$A\left(\frac{1}{2}\right) = \frac{a\pi}{b},$$

where a, b are relatively prime positive integers. Find $100a + b$.

2021 PUMaC, Algebra, Problem 5 1613. Consider the sum

$$S = \sum_{j=1}^{2021} \left|\sin\frac{2\pi j}{2021}\right|.$$

The value of S can be written as $\tan(c\pi/d)$ for some relatively prime positive integers c, d, satisfying $2c < d$. Find the value of $c + d$.

2021 PUMaC, Algebra, Problem 7 1614. Consider the following expression

$$S = \log_2 \left(\sum_{k=1}^{2019} \sum_{j=2}^{2020} \log_{2^{1/k}}(j) \log_{j^2} \left(\sin \frac{\pi k}{2020} \right) \right).$$

Find the smallest integer n which is bigger than S (i.e. find $\lceil S \rceil$).

2021 HMIC, Problem 4 1615. Let $A_1A_2A_3A_4$, $B_1B_2B_3B_4$, and $C_1C_2C_3C_4$ be three regular tetrahedra in 3-dimensional space, no two of which are congruent. Suppose that, for each $i \in \{1, 2, 3, 4\}$, C_i is the midpoint of the line segment A_iB_i. Determine whether the four lines A_1B_1, A_2B_2, A_3B_3, and A_4B_4 must concur.

2022 AIME II, Problem 11 1616. Let $ABCD$ be a convex quadrilateral with $AB = 2, AD = 7$, and $CD = 3$ such that the bisectors of acute angles $\angle DAB$ and $\angle ADC$ intersect at the midpoint of $\overline{BC}$. Find the square of the area of $ABCD$.

2022 HMMT, Algebra & Number Theory, Problem 7 1617. Let (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), and (x_5, y_5) be the vertices of a regular pentagon centered at $(0, 0)$. Compute the product of all positive integers k such that the equality

$$x_1^k + x_2^k + x_3^k + x_4^k + x_5^k = y_1^k + y_2^k + y_3^k + y_4^k + y_5^k,$$

must hold for all possible choices of the pentagon.

2022 Stanford Math Tournament, Algebra #6 1618. Compute

$$\cot \left(\sum_{n=1}^{23} \cot^{-1} \left(1 + \sum_{k=1}^{n} 2k \right) \right).$$

2022 Stanford Math Tournament, Algebra Tiebreaker #2 1619. What is the area of the region in the complex plane consisting of all points z satisfying both

$$\left| \frac{1}{z} - 1 \right| < 1$$

and $|z-1| < 1$? ($|z|$ denotes the magnitude of a complex number, i.e., $|a+bi| = \sqrt{a^2 + b^2}$.)

2023 AIME I, Problem 5 1620. Let P be a point on the circumcircle of square $ABCD$ such that $PA \cdot PC = 56$ and $PB \cdot PD = 90$. What is the area of square $ABCD$?

2023 AIME I, Problem 8 1621. Rhombus $ABCD$ has $\angle BAD < 90°$. There is a point P on the incircle of the rhombus such that the distances from P to lines DA, AB, and BC are 9, 5, and 16, respectively. Find the perimeter of $ABCD$.

2023 AIME I, Problem 13 1622. Each face of two noncongruent parallelepipeds is a rhombus whose diagonals have lengths $\sqrt{21}$ and $\sqrt{31}$. The ratio of the volume of the larger of the two polyhedra to the volume of the smaller is m/n, where m and n are relatively prime positive integers. Find $m + n$.

2023 Bulgaria National Olympiad, Problem 5 1623. For every positive integer n determine the least possible value of the expression

$$|x_1| + |x_1 - x_2| + |x_1 + x_2 - x_3| + \cdots + |x_1 + x_2 + \cdots + x_{n-1} - x_n|,$$

given that $x_1, x_2, \ldots, x_n$ are real numbers satisfying $|x_1| + |x_2| + \cdots + |x_n| = 1$.

2023 British Mathematical Olympiad, Problem 4 1624. Find the greatest integer $k \leq 2023$ for which the following holds: whenever Alice colours exactly k numbers of the set $\{1, 2, \ldots, 2023\}$ in red, Bob can colour some of the remaining uncoloured numbers in blue, such that the sum of the red numbers is the same as the sum of the blue numbers.

2023 Benelux, Problem 2 1625. Determine all integers $k \geqslant 1$ with the following property: given k different colours, if each integer is coloured in one of these k colours, then there must exist integers $a_1 < a_2 < \cdots < a_{2023}$ of the same colour such that the differences $a_2 - a_1, a_3 - a_2, \ldots, a_{2023} - a_{2022}$ are all powers of 2.

2023 EGMO, Problem 2 1626. We are given an acute triangle ABC. Let D be the point on its circumcircle such that AD is a diameter. Suppose that points K and L lie on segments AB and AC, respectively, and that DK and DL are tangent to circle AKL. Show that line KL passes through the orthocenter of triangle ABC.

2023 Italy Math Olympiad, Problem 6 1627. Dedalo buys a finite number of binary strings, each of finite length and made up of the binary digits 0 and 1. For each string, he pays $(\frac{1}{2})^L$ drachmas, where L is the length of the string. The Minotaur is able to escape the labyrinth if he can find an infinite sequence of binary digits that does not contain any of the strings Dedalo bought. Dedalo's aim is to trap the Minotaur. For instance, if Dedalo buys the strings 00 and 11 for a total of half a drachma, the Minotaur is able to escape using the infinite string 01010101 .... On the other hand, Dedalo can trap the Minotaur by spending 75 cents of a drachma: he could for example buy the strings 0 and 11, or the strings $00, 11, 01$. Determine all positive integers c such that Dedalo can trap the Minotaur with an expense of at most c cents of a drachma.

Part III

Hints & Select Solutions

Solution 3. Simplifies to $(x-1)(x+5)(x^2+x-1)=0$ which has rational solutions $x=1,-5$ and irrational solutions $(-1\pm\sqrt{5})/2$.

Solution 4. Simplifies to $(2x-1)(x+3)(x^2-x-1)=0$ which has rational solutions $x=1/2,-3$ and irrational solutions $(1\pm\sqrt{5})/2$.

Solution 5. Simplifies to $(x-2)^2(x+2)(x-3)=0$ which has solutions $x=\pm 2,3$.

Solution 6. Simplifies to $(x-4)(x-2)(x+1)(x+2)=0$ which has solutions $x=-1,\pm 2,4$.

Solution by Boris 33. The answer is 12. To find the numbers in the 100^{th} row of the Pascal triangle (the one starting with $1,100,\dots$) that are not divisible by 3, we need to find the number of coefficients in the polynomial

$$(1+x)^{100}=1+\binom{100}{1}x+\binom{100}{2}x^2+\cdots+x^{100},$$

which are not equal to 0 modulo 3. Note that by Binomial Theorem, and taking modulo 3, one has,

$$(1+x)^3=1+3x+3x^2+x^3\equiv 1+x^3 \pmod 3.$$

and so also

$$(1+x)^9\equiv(1+x^3)^3\equiv 1+x^9 \pmod 3,$$

and so on, for any power of 3. Now, $100=81+2\cdot 9+1$. Therefore, modulo 3 one has

$$(1+x)^{100}=(1+x)^{81}\left((1+x)^9\right)^2(1+x)=(1+x^{81})(1+2x^9+x^{18})(1+x).$$

In this product all $2\cdot 3\cdot 2=12$ powers of x are different (because every integer can be written in base 3 in a unique way), and the coefficients are all nonzero modulo 3. So, the answer is 12.

Solution by Parviz Shahriari 37. Answer: $(x^2+1)(2x^2+x+2)$.

Solution by Parviz Shahriari 38. Answer: $(x+1)(x^2+x+1)(x^2-x+1)$.

Solution by Parviz Shahriari 39. Answer: $(x^2+\sqrt{2\sqrt{7}+1}x+\sqrt{7})(x^2-\sqrt{2\sqrt{7}+1}x+\sqrt{7})$.

Solution by Parviz Shahriari 40. Answer: $(x^2+\sqrt{2}x+1-\sqrt{2})(x^2-\sqrt{2}x+1+\sqrt{2})$.

Solution by Parviz Shahriari 41. Answer: $(x^{n+1}+x^n+\cdots+x+1)(x^{n-1}+x^{n-2}+\cdots+x+1)$.

Solution by Parviz Shahriari 42. Answer: $(x^2+x+1)(2x^2-x+2)$.

Solution by Parviz Shahriari 43. Begin with calculating the square root of the expression, equal to x^3+x^2+x+11 with a remainder of -100, then use difference of squares to arrive at $(x^3+x^2+x+1)(x^3+x^2+x+21)$. The final answer is $(x+1)(x+3)(x^2+1)(x^2-2x+7)$.

Solution by Parviz Shahriari 44. Answer: $(x^2+x\sqrt{6(\sqrt{2}-1)}+3\sqrt{2})(x^2-x\sqrt{6(\sqrt{2}-1)}+3\sqrt{2})$.

Solution by Parviz Shahriari 45. Answer: $(a^{2n} + a^n + 1)(a^{3n} - a^{2n} + 1)$.

Solution by Parviz Shahriari 46. Answer: $(x + 2)(x + 6)(x + 4 + \sqrt{6})(x + 4 - \sqrt{6})$.

Solution by Parviz Shahriari 47. Answer: $(x - 1)(x + 3)^2$.

Solution by Parviz Shahriari 48. Answer: $(x - 1)(x + 3)(x + 7)$.

Solution by Parviz Shahriari 49. Answer: $x(x+1)(x-1)(x-3)(x+2)(x+3)(x-2)$.

Solution by Amir Parvardi 50. This can be represented as $x^2 + xy + xy + y^2$, and by factoring x from the first two terms and also factoring y from the last two terms, we get $x(x + y) + (x + y)y$. This last expression would be the same as $x(x + y) + y(x + y)$ in our commutative algebra, and factoring $(x + y)$ results in the First Double–Variable Identity:

$$x^2 + 2xy + y^2 = (x + y)^2.$$

Solution by Amir Parvardi 51. This can be represented as $x^2 - xy - xy + y^2$, and by factoring x from the first two terms and also factoring y from the last two terms, we get $x(x - y) - (x - y)y$. This last expression would be the same as $x(x - y) - y(x - y)$ in our commutative algebra, and factoring $(x - y)$ results in the Second Double–Variable Identity:

$$x^2 - 2xy + y^2 = (x - y)^2.$$

Solution by Amir Parvardi 52. Add and subtract xy to find $x^2 + xy - xy - y^2$, factor x from the first two terms and y from the other two: $x(x + y) - (x + y)y$. Thus, using commutativity once again as in the Positive and Negative Double–Variable Identities, we arrive at the identity for the difference between two squares:

$$x^2 - y^2 = (x - y)(x + y).$$

Solution by Amir Parvardi 53. It is necessary for n to be odd for $x^n + y^n$ to be factorizable in real numbers, hence the n^{th} Positive Double–Variable Identity is only defined for odd values of n. Since n is odd, plugging $x = -y$ in $x^n + y^n$ results in zero, meaning that $(x+y)$ is a factor of x^n+y^n. Dividing x^n+y^n by $x+y$ can be done smoothly by choosing the appropriate quotient with alternating positive and negative terms that cancel each other perfectly without leaving any remainder:

$$\frac{x^n + y^n}{x + y} = x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1}.$$

Again, the above arrangements are possible only for odd n. To finish up, this is the n^{th} Positive Double–Variable Identity:

$$x^n + y^n = (x + y)\left(x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1}\right).$$

Solution by Amir Parvardi 54. Since $x = y$ yields $x^n - y^n = 0$, we know that $(x - y)$ is a factor of $x^n - y^n$. We can easily see that the quotient of the division of $x^n - y^n$ by $x - y$ contains only positive terms:

$$\frac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}.$$

Since all the terms are positive, there would be no problem of matching alternating positive and negative terms that cancel each other, and the n^{th} Negative Double–Variable Identity holds for all positive integers n:

$$x^n - y^n = (x-y)\left(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}\right).$$

Solution by Amir Parvardi 55. Since $x = y$ yields $x^{2^k} - y^{2^k} = 0$, we know that $(x-y)$ is a factor of $x^{2^k} - y^{2^k}$. We also get the quotient as in the n^{th} Negative Double–Variable Identity:

$$\frac{x^{2^k} - y^{2^k}}{x-y} = x^{2^k-1} + x^{2^k-2}y + \cdots + xy^{2^k-2} + y^{2^k-1}.$$

We see that the degree of x in the quotient is $2^k - 1$, which happens to be equal to $1 + 2 + 2^2 + \cdots + 2^{k-1}$, meaning that the leading term in the quotient, x^{2^k-1}, is in fact a product of k terms $x \cdot x^2 \cdot x^{2^2} \cdots x^{2^{k-1}}$, and there must be an identity in this form:

$$x^{2^k-1} + x^{2^k-2}y + \cdots + xy^{2^k-2} + y^{2^k-1} = (x + \dots)(x^2 + \dots)(x^{2^2} + \dots)\cdots(x^{2^{k-1}} + \dots),$$

and the same technique could be applied on y since everything is symmetric, and the missing terms are easily found:

$$\frac{x^{2^k} - y^{2^k}}{x-y} = (x+y)(x^2+y^2)(x^{2^2} + y^{2^2})\cdots(x^{2^{k-1}} + x^{2^{k-1}}),$$

giving us the magical $2^{k^{th}}$ Negative Double–Variable Identity:

$$x^{2^k} - y^{2^k} = (x-y)(x+y)(x^2+y^2)(x^{2^2} + y^{2^2})\cdots(x^{2^{k-1}} + y^{2^{k-1}}).$$

Solution by Sophie Germain 56. Adding $4x^2y^2$ to $x^4 + 4y^4$ completes the square to $(x^2 + 2y^2)^2$. Now just subtract the added term $4x^2y^2$ from $(x^2 + 2y^2)^2$ and use the Difference of Squares Identity to finish:

$$x^4 + 4y^4 = (x^2 + 2y^2 + 2xy)(x^2 + 2y^2 - 2xy).$$

Solution by Sophie Parker 57.

- **Difference of Squares**: It is easy to see that adding x^2y^2 to the given expression completes the square, making it $(x^2 + y^2)^2 = x^4 + 2x^2y^2 + y^4$. The Difference of Squares Identity yields the final factorization:

$$\begin{aligned} x^4 + x^2y^2 + y^4 &= (x^2+y^2)^2 - (xy)^2 \\ &= (x^2 + y^2 - xy)(x^2 + y^2 + xy). \end{aligned}$$

- **Difference of Squares and Cubes**: What if we begin with $x^6 - y^6$? If I apply The Difference of Squares on this expression, I would have on one hand $x^6 - y^6 = (x^3 - y^3)(x^3 + y^3)$, and on the other hand I can apply the n^{th} Negative Double–Variable Identity for $n = 3$ on $x^6 - y^6 = (x^2 - y^2)(x^4 + x^2y^2 + y^4)$.

$$\begin{aligned} x^6 - y^6 &= (x^3 - y^3)(x^3 + y^3) \\ &= (x-y)(x^2 + xy + y^2) \cdot (x+y)(x^2 - xy + y^2) \\ x^6 - y^6 &= (x^2 - y^2)(x^4 + x^2y^2 + y^4) \\ &= (x-y)(x+y)(x^4 + x^2y^2 + y^4). \end{aligned}$$

Therefore,

$$(x-y)(x^2+xy+y^2)\cdot(x+y)(x^2-xy+y^2)=(x-y)(x+y)(x^4+x^2y^2+y^4).$$

Assuming $x \neq \pm y$, we can cancel the terms $x-y$ and $x+y$ from both sides of the equation and obtain the factorization of $x^4+x^2y^2+y^4$ as a consequence:

$$(x^2+xy+y^2)\cdot(x^2-xy+y^2)=x^4+x^2y^2+y^4.$$

Solution 58. Answer: $2(x^2+xy+y^2)^2$.

Solution 59. Answer: $2(x^2-xy+y^2)^2$.

Solution 60. Answer: $3xy(x+y)$.

Solution 61. Answer: $5xy(x+y)(x^2+xy+y^2)$.

Solution 62. Answer: $7xy(x+y)(x^2+xy+y^2)^2$.

Solution 63. Answer:

Solution by Parviz Shahriari 64. The expression is symmetric with respect to x and y, and it becomes zero by plugging $x=y$ and $x=-2y$, so the factorization is $[(x+2y)(2x+y)(x-y)]^2$.

Solution 65. The answer is $(x+y+z)^2$.

Solution 66. The answer is $(x+y+z)(x-y)(y-z)(z-x)$.

Solution 67. The answer is $(xy+yz+zx)(x-y)(y-z)(z-x)$.

Solution 68. The answer is $3(x+y+z)(x^2+y^2+z^2)$.

Solution 69. The answer is $(x+y+z)(xy+yz+zx)$.

Solution by Parviz Shahriari 70. Plug $y=z$ and observe that the result is zero, so $(y-z)$ is a factor. The answer is $(x+z)(y-z)(x+y)$.

Solution by Parviz Shahriari 71. Answer: $(a+2b)(2b-c)(a-c)$.

Solution by Parviz Shahriari 72. Answer: $(a+b+c)(a-b-c)(a+b-c)(a-b+c)$.

Solution 73. Answer: $(xy^2+yz^2+zx^2)(x^2y+y^2x+z^2x)$.

Solution by Parviz Shahriari 74. Answer: $(x-2y)(y-2z)(x+y)$.

Solution by Parviz Shahriari 75. Answer: $3(a+b)(b+c)(c+a)$.

Solution by Parviz Shahriari 76. Answer: $(a+b)(b+c)(c+a)$.

Solution 77. Answer: $(x-y)(y-z)(z-x)$.

Solution 78. Answer: $3(x-y)(y-z)(z-x)$.

Solution 79. Expanding $(x-y)^2+(y-z)^2+(z-x)^2$ results in $2(x^2+y^2+z^2-xy-yz-zx)$. Moreover,

$$(x-y)^4+(y-z)^4+(z-x)^4=2(x^2+y^2+z^2-xy-yz-zx)^2.$$

Solution by Parviz Shahriari 80. Answer: $5(x-y)(y-z)(z-x)(x^2+y^2+z^2-xy-yz-zx)$.

Solution by Parviz Shahriari 81. Answer: $7(x-y)(y-z)(z-x)(x^2+y^2+z^2-xy-yz-zx)^2$.

Solution by Parviz Shahriari 82. Answer: $(x+y+z)(-x+y+z)(x-y+z)(x+y-z)$.

Solution by Parviz Shahriari 83. Answer: $(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$.

Solution by Parviz Shahriari 84. Answer: $(a+b+c)^2(a^2+b^2+c^2-ab-bc-ca)^2$.

Solution by Parviz Shahriari 85. Answer: $5(x+y)(y+z)(z+x)(x^2+y^2+z^2+xy+yz+zx)$.

Solution by Parviz Shahriari 86. Answer: $(x-y)(a-x)(a-y)(x+y+a)$.

Solution by Parviz Shahriari 87. Answer: $(a-b)(b-c)(c-a)(ab+bc+ca)$.

Solution by Parviz Shahriari 88. Answer: $(y+z)(2x-y)(2x+z)(2x+y-z)$.

Solution by Parviz Shahriari 89. Answer: $(x+y)(y+z)(z+x)$.

Solution by Parviz Shahriari 90. Answer: $(x+y+z)(xy+yz+zx)$.

Solution by Parviz Shahriari 91. Answer: $3(y^2+z^2)(x^2+y^2)(x-z)(x+z)$.

Solution by Parviz Shahriari 92. Answer: $-(x+y+z)(x-y)(y-z)(z-x)$.

Solution by Parviz Shahriari 93. Answer: $(x^2-y)(y^2-z)(z^2-x)$.

Solution by Parviz Shahriari 95. The term in the first bracket simplifies to $(ax+by)^2+(ay+bx)^2$, and the term in the second bracket is $(ay+bx)(ax+by)$. Use the difference of squares to see that the given expression factorizes into $(a-b)^2(a+b)^2(x-y)^2(x+y)^2$.

Solution by Parviz Shahriari 96. Answer: $f(2x+1)=4x^2-1$.

Solution by Parviz Shahriari 97. Answer: $f(f(x))=\frac{x^4+3x^2+1}{x(x^2+1)}$.

Solution by Parviz Shahriari 98. Answer: $f(f(f(x)))\cdot f(x)=-1$.

Solution by Parviz Shahriari 99. Answer: $f(x)=\frac{x^2}{(x+1)^2}$.

Solution by Parviz Shahriari 100. Answer: $x^3-3x+4=(x+2)^3-6(x+2)^2+9(x+2)+2$.

Solution by Parviz Shahriari 101. Answer: $g(x)\equiv 0$.

Solution by Parviz Shahriari 102. Answer: $f(x)=10+5\cdot 2^x$.

Solution by Parviz Shahriari 103. Answer: (a) Arithmetic, (b) Geometric.

Solution by Parviz Shahriari 104. Answer: $f(x)+f(y)=2f(x)f(y)$.

Solution by Parviz Shahriari 105. Answer: (a) $z=x+y$; (b) $z=\frac{xy}{x+y}$; (c) $z=\frac{x+y}{1-xy}$; (d) $z=\frac{x+y}{1+xy}$.

Solution by Parviz Shahriari 106. Answer: $f(f(x))=\frac{x-1}{x}$ and $f(f(f(x)))=x$.

Solution by Parviz Shahriari 107. Answer: $f(x) = x^2 - 5x + 7$.

Solution by Parviz Shahriari 108. Answer: $f(x) = x^2 - 2$.

Solution by Parviz Shahriari 109. Answer: $f(x) = \frac{1}{x}\left(1 + \sqrt{1+x^2}\right)$.

Solution by Parviz Shahriari 110. Answer: $f(x) = x^2 - x + 1$.

Solution by Parviz Shahriari 111. One can easily prove by induction that $f_n(x) = \frac{x}{\sqrt{1+nx^2}}$.

Solution by Parviz Shahriari 112. Answer: $f(2x^2-1) = 2f(x)$ and $f(4x^3-3x) = 3f(x)$.

Solution by Parviz Shahriari 113. Answer: $f(x) = \frac{x^2}{x^2-1}$.

Solution by Parviz Shahriari 114. Answer: $\alpha = 3$ and $f(\alpha+x)\cdot f(\alpha-x) = \frac{1}{4}$.

Solution by Parviz Shahriari 115. It is easy to see that $f(a) = a, f(b) = b$, and $f(c) = c$, so that the equation $f(x) = x$ has at least three roots. However, the equation $f(x) - x = 0$ is quadratic and having three roots implies that it is always zero, so that $f(x) = x$ for all x.

Solution by Parviz Shahriari 116. Change x to $-x$ in the given equation to easily arrive at $f(x) = \frac{b}{a-1}\sqrt[n]{x}$.

Solution by Parviz Shahriari 117. Answer: (a) $x = -2, 4$, (b) $x = -2, 2, 4, 10$.

Solution by Parviz Shahriari 118. Answer: $f(x, y) = x^2 \cdot \frac{1-y}{1+y}$.

Solution by Parviz Shahriari 119. By induction, we arrive at:

$$F_n(x) = \begin{cases} -2^{n-1}\left(\sin\frac{1}{2}\right)^{n-1}\cdot\sin\left(x+\frac{n-1}{2}\right), & \text{if } n = 4k, \\ 2^{n-1}\left(\sin\frac{1}{2}\right)^{n-1}\cdot\cos\left(x+\frac{n-1}{2}\right), & \text{if } n = 4k+1, \\ 2^{n-1}\left(\sin\frac{1}{2}\right)^{n-1}\cdot\sin\left(x+\frac{n-1}{2}\right), & \text{if } n = 4k+2, \\ -2^{n-1}\left(\sin\frac{1}{2}\right)^{n-1}\cdot\cos\left(x+\frac{n-1}{2}\right), & \text{if } n = 4k+3. \end{cases}$$

Solution by Parviz Shahriari 120. Answer: $f(x) = \frac{1}{x^2} - 2$.

Solution by Parviz Shahriari 121. Answer: $p(1) = -1$.

Solution by Parviz Shahriari 122. Answer: $a^n(x+1)p(x) = (x+a^n)p(ax)$.

Solution by Parviz Shahriari 123. Answer:

$$p(a+b)\cdot p(b+c)\cdot p(c+a) = 8p\left(\frac{a+b+c}{2}\right)\cdot(p(0))^2,$$
$$p(-a)\cdot p(-b)\cdot p(-c) = 8\left(p(a+b+c)\right)^2\cdot p(0).$$

Solution by Parviz Shahriari 124. Answer: (a) $6(x-1)^4 + 43(x-1)^3 + 76(x-1)^2 - 25(x-1) - 100$, (b) $x = 2, -3, -\frac{2}{3}, -\frac{3}{2}$.

Solution 125. Plug in $x = \pm 1$ in $p(x)$ and observe that the given sum is even.

Solution 126. Answer: all integers n work, and m would be $m = n^2 + n(a+1) + b$.

Solution 127. Answer: $a = c = -1$ and $b = 1$, so that $a^{2023} + b^{2023} + c^{2023} = -1$.

Solution 128. Using $p_{2k}(x) = p_k(x)^2 + 2$, we deduce the rationality of $p_8(x)$ and $p_{10}(x)$ from the rationality of $p_4(x)$ and $p_5(x)$, respectively. From $p_4(x)p_2(x) = p_2(x) + p_6(x)$ and $p_8(x)p_2(x) = p_{10}(x) + p_6(x)$, it follows that $p_2(x)$ and $p_6(x)$ are also rational. Finally, $p_5(x)p_1(x) = p_4(x) + p_6(x)$ implies the rationality of $p_1(x)$.

Solution 129. Show that $4(d - b) = c^2 - a^2$, proving that $a - c$ is even.

Solution by CRMO 2012 130. From the given equations we can obtain:

$$x^3y + y^3z + z^3x - 3(xy + yz + zx) - xyz(x + y + z) = 0,$$

to get $x^3y + y^3z + z^3x = -9$.

Solution by Parviz Shahriari 131. Answer: $x^4 - 3x^3 + 8x - 24 = (x^2 - 2x + 4)(x^2 - x - 6)$.

Solution by Parviz Shahriari 132. Answer: $nx^{n+1} - (n+1)x^n + 1 = (x-1)^2\,(nx^{n-1} + (n-1)x^{n-2} +$

Solution by Parviz Shahriari 133. Answer: $m = 1$ gives $x^4 + a^2x^2 + a^4 = (x^2 - ax + a^2)(x^2 + ax + a^2$.

Solution by Parviz Shahriari 134. Answer: $a = b = 1$ gives $(x-2)^n + (x-1)^n - 1 = (x^2 - 3x + 2)\,(P(x)Q(x))$, where

$$\begin{aligned} P(x) &= (x-2)^{2n-2} - (x-2)^{2n-3} + \cdots - (x-2) + 1, \\ Q(x) &= (x-1)^{n-2} + (x-1)^{n-3} + \cdots + (x-1) + 1. \end{aligned}$$

Solution by Parviz Shahriari 135.

(a) $-\dfrac{5}{2}x + \dfrac{7}{2}$,

(b) $A\dfrac{x-b}{a-b} + B\dfrac{x-a}{b-a}$.

Solution by Parviz Shahriari 136.

(a) $-\dfrac{1}{12}x^2 - \dfrac{5}{4}x - \dfrac{1}{6}$,

(b) $A\dfrac{(x-b)(x-c)}{(a-b)(a-c)} + B\dfrac{(x-c)(x-a)}{(b-c)(b-a)} + C\dfrac{(x-a)(x-b)}{(c-a)(c-b)}$.

Solution by Parviz Shahriari 137.

(a) $p(x) = x^4 + 6x^3 + 7x^2 - x + 2$,

(b) $q(x) = x^3 + 1$.

Solution by Parviz Shahriari 138.

(a) $p(-1) = p(\pm i) = 0$, so the remainder is zero.

(b) Let $t(x) = (x-1)q(x)$ and observe that $t(\alpha) = 0$ if $\alpha^n = 1$ (and $\alpha \neq 1$), so that the remainder is zero.

Solution by Parviz Shahriari 139. Answer: $p(x) = (x^2-1)(x^2-a^2)$, so that the remainder must be zero.

Solution by Parviz Shahriari 140.

(a) $(\alpha+\beta)(\alpha\beta+1) = q$ and $\alpha^2+\beta^2+\alpha\beta = -p-1$.

(b) $\alpha = 5$ and $\beta = -4$.

Solution by Parviz Shahriari 141.

(a) Note that $p_3(x,y,z) = 3(x+y)(y+z)(z+x)$, and that $p_m(x,-x,z) = 0$ if m is odd. The answer is all $m = 6k+3$, where $k \geq 0$ is an integer.

(b) For all a and b such that $b \mid a$.

Solution by Parviz Shahriari 142. Answer: all odd n work because

$$\frac{p_n(x)}{q_n(x)} = \frac{(x^{2n}-1)(x-1)}{(x^n-1)(x^2-1)} = \frac{x^n+1}{x+1}.$$

Solution by Parviz Shahriari 143. Write $p(x,y) = (x-y)q(x,y)$ and prove that $q(x-y)$ is also divisible by $x-y$, meaning that the required remainder is zero.

Solution by Parviz Shahriari 144. Answers:

(a) $m = 3k \pm 1$.

(b) $(m,n) = (3k+1, 3k+), (3k+2, 3k+1)$.

Solution by Parviz Shahriari 145. Write $q(x,y) = (ax-by)(ay-bx)$ and note that $p(x,y) - s(x,y)$ is divisible by $q(x,y)$, so that the remainder is zero.

Solution by Parviz Shahriari 146. Answer: $p(x) = \frac{1}{9}(x^2-4)^2$.

Solution by Parviz Shahriari 148. Answer: $p(x)x^3 - 3x$.

Solution by Problem Premier for the Olympiad 149. Answer: $p = q = r$, so $p + q - 2r = 0$.

Solution by Regional Math Olympiad 2010 150. Answer: $a = b = c$, so $a + b - 2c = 0$.

Solution by CRMO 2015 151. Answer: $b - st$ is a root of $x^2 + ax + b - \alpha\beta = 0$, so the remainder of division is zero.

Solution by INMO 2012 152. Reduce the equation to $f_{n+1}(x) = xf_n(x) - f_{n-1}(x)$ which makes it obvious to see that f_n must be a polynomial:

$$f_n(x) = x^n - \binom{n-1}{1}x^{n-2} + \binom{n-2}{2}x^{n-4} - \binom{n-3}{3}x^{n-6} + \cdots$$

Solution by RMO 2000 153. Factorize the expression into $(a-x^2-x)(a-x^2+x-1)$ which has all real roots if and only if $a \geq 3/4$.

Solution by CRMO 2003 154. Answer: $a = 1$ and $a = 3$ are the only values.

Solution by INMO 2016 155. Answer: (II) always implies $a = b = c$, but (I) need not imply $a = b = c$. There are three other possibilities for a, b, c other than $a = b = c$ if we assume statement (I). One such case is $a = b = \lambda c$, where λ is the positive root of $x^2 - x - 1 = 0$.

Solution 157. The answer is $a - b = 21$.

Solution by Parviz Shahriari 158. Answers:

(a) $x^{13} + 1$.

(b) $x^3 - x^2 - x + 1$.

Solution by Parviz Shahriari 159. The denominator is a perfect square equal to $(x^3 + 3x^2 - x - 1)^2$ and the numerator is divisible by x^3+3x^2-x-1 with a quotient of x^3+2x-1, so that the fraction reduces to

$$\frac{x^3 + 2x - 1}{x^3 + 3x^2 - x - 1}.$$

Solution by CRMO 2002 160. Put away the obvious solution $x = 1$ (with multiplicity three). Since $x - 1$ is a common factor of all three terms in the equation, we can divide all the terms by $(x-1)^3$ and get the equation $(x+2)^3 + (2x+1)^3 = 27(x+1)^3$ and solve the equation by expanding to get the other three solutions of the original equation: $x = -1, -2, -1/2$.

Solution Inspired by RMO 2013 161. Answer: (a) $a = b = 1$, (b) $R(n) = n - 1$.

Solution by Parviz Shahriari 254.

(a) The equation $(x + a)^4 + (x + b)^4 = c$ can be re-written as

$$\left[\left(x + \frac{a+b}{2}\right) + \frac{a-b}{2}\right]^4 + \left[\left(x + \frac{a+b}{2}\right) - \frac{a-b}{2}\right]^4 = c,$$

which can be written as $(t + \alpha)^4 + (t - \alpha)^4 = c$, where $t = x + (a + b)/2$ and $\alpha = (a - b)/2$. The last equation expands to $2t^4 + 12\alpha^2t^2 + (2\alpha^4 - c) = 0$, which can be solved as a quadratic equation in $y = t^2$.

(b) Let $t = x + 3$, so that the equation becomes $(t-2)^6 + (t+2)^6 = 730$. Expand to get $t^6 + 60t^4 + 240t^2 - 301 = 0$, which can be written as a cubic equation in $y = t^2$: $y^3 + 60y^2 + 240y - 301 = 0$. Note that $y = 1$ is the only positive solution to this equation (why?) which leads to $t = \pm 1$ and thus $x = -2$ and $x = -4$ are the only real solutions of the original equation.

(c) Use the same parameters $t = x + (a+b)/2$ and $\alpha = (a-b)/2$ in a similar way to arrive at a degree–n polynomial equation.

Solution by Parviz Shahriari 255.

(a) The equation $(x+a)(x+b)(x+c)(x+d) = m$ can be re-written as

$$(x^2 + (a+b)x + ab)(x^2 + (c+d)x + cd) = m,$$

and since $a + b = c + d$, we can set $t = x^2 + (a+b)x = x^2 + (c+d)x$ as the variable of the equation to obtain $(t+ab)(t+cd) = m$, which is a simple quadratic equation and easy to solve.

(b) Re-write the equation as $(x+2)(x+4)(x-3)(x-5) = 72$, and since $2 - 3 = 4 - 5 = -1$, rearrange it as $[(x+2)(x-3)]\,[(x+4)(x-5)] = 72$. Write the last equation as $(x^2 - x - 6)(x^2 - x - 20) = 72$, which after setting $t = x^2 - x$ becomes $t^2 - 26t + 48 = 0$. This has two solutions $t = 2$ and $t = 24$, which in turn give the following solutions for x:

$$-1, 2, \frac{1+\sqrt{97}}{2}, \frac{1-\sqrt{97}}{2}.$$

Solution by Parviz Shahriari 256. The answer is $(a^2 + 3ad) + d^2$. In the product of four terms, group a with $a + 3d$ and $a + d$ with $a + 2d$ to find

$$\begin{aligned} a(a+d)(a+2d)(a+3d) &= (a(a+3d))\,((a+d)(a+2d)) \\ &= \left(a^2 + 3ad\right)\left(a^2 + 3ad + 2d^2\right) \\ &= \left(a^2 + 3ad\right)^2 + 2d^2\left(a^2 + 3ad\right). \end{aligned}$$

It is clear now that if we add d^4 to the product it will be the square of $(a^2 + 3ad) + d^2$:

$$\left(a^2 + 3ad\right)^2 + 2d^2\left(a^2 + 3ad\right) + d^4 = \left(a^2 + 3ad + d^2\right)^2.$$

Solution by Parviz Shahriari 257. The answer is $k > 9a^4/16$. For simplicity, let $t = x + b$ to get

$$t(t+a)(t+2a)(t+3a) = k,$$

and add a^4 to both sides of the equation to find

$$(t^2 + 3at + a^2)^2 = k + a^4,$$

which results in the following four roots:

$$x = -b + \frac{-3a \pm \sqrt{5a^2 \mp \sqrt{k + a^4}}}{2}.$$

In order for all four roots to be real and simple (of multiplicity 1), $5a^2 - \sqrt{k+a^4}$ must be positive, which leads to $k > 9a^4/16$.

Solution by Parviz Shahriari 258. There are four real roots for the equation. To expand, group x with $x-2$ and $x-4$ with $x+2$, so that we can set a new variable $t = x^2 - 2x$ and arrive at $t^3 - 7t^2 - 8t + 66 = 0$. There are three values of t that satisfy the equation: $t = -3$ leads to complex solutions for x (roots of $x^2 - 2x + 3 = 0$); $t = 5 + \sqrt{3}$ leads to $x = 1 \pm \sqrt{6+\sqrt{3}}$; and $t = 5 - \sqrt{3}$ gives $x = 1 \pm \sqrt{6-\sqrt{3}}$. Therefore, the real solutions are $x = 1 \pm \sqrt{6 \pm \sqrt{3}}$.

Solution by Parviz Shahriari 259. Add and subtract $2(ax+b)^2(cx+d)^2$ to the left side of the given equation to complete the square:

$$\left((ax+b)^2 + (cx+d)^2\right)^2 - 2(ax+b)^2(cx+d)^2 = (\alpha x + \beta)^4.$$

The first term on the left side of the above equation can be written as

$$\begin{aligned}(ax+b)^2 + (cx+d)^2 &= (a+c)x + (b+d))^2 - 2(ax+b)(cx+d)\\ &= (\alpha x+\beta)^2 - 2(ax+b)(cx+d).\end{aligned}$$

After simplification, the equation becomes

$$2(ax+b)(cx+d)\left((ax+b)(cx+d) - 2(\alpha x+\beta)^2\right) = 0,$$

which is easy to solve.

Solution by Parviz Shahriari 260. The equation can be written as $2(2-x)(2x-1)(4x^2-x+4) = 0$, which has real roots $x = 2$ and $x = 1/2$.

Solution by Parviz Shahriari 261. The answer is $a < -5/4$. Since the equation is of degree 4 in x and hard to solve, we try to solve it as a quadratic equation in a. Write it as

$$a^2 - 3x^2 \cdot a + (2x^4 + x^3 - 2x^2 + 2x - 1) = 0.$$

We can use the quadratic formula to solve the above equation. We just need the Δ:

$$\begin{aligned}\Delta &= 9x^4 - 4(2x^4 + x^3 - 2x^2 + 2x - 1)\\ &= x^4 - 4x^3 + 8x^2 - 8x + 4\\ &= (x^2 - 2x + 2)^2.\end{aligned}$$

Therefore, the solutions in terms of a can be calculated from the quadratic formula as follows:

$$a_{1,2} = \frac{3x^2 \pm (x^2 - 2x + 2)}{2}.$$

These will lead to $a_1 = x^2 + x - 1$ and $a_2 = 2x^2 - x + 1$, which can now be solved as quadratic equations in x:

$$x^2 + x - (a+1) = 0 \qquad \text{and} \qquad 2x^2 - x + (1-a) = 0.$$

Thus, we get the four solutions

$$x_{1,2} = \frac{-1 \pm \sqrt{4a+5}}{2} \qquad \text{and} \qquad x_{3,4} = \frac{1 \pm \sqrt{4a-3}}{2}.$$

If we need all four roots not to be real, we need $a < -5/4$.

Solution by Parviz Shahriari 262. Let $a = \sqrt{3}$ and write the equation as a quadratic in a:

$$(1-x)a^2 - x \cdot a + x^3 = 0.$$

The solutions will be

$$x_1 = \sqrt{3} \qquad \text{and} \qquad x_{2,3} = \frac{-\sqrt{3} \pm \sqrt{3+4\sqrt{3}}}{2}.$$

Solution by Parviz Shahriari 263. Let $bx+c = t$ and write the equation as a quadratic in t:

$$t^2 + x^2(a-1)t + x^4(a^2-1) = 0.$$

The solutions will be

$$t_{1,2} = \frac{-x^2(2a-1) \pm x^2\sqrt{5-4a}}{2},$$

which lead to

$$(1 - 2a \pm \sqrt{5-4a})x^2 - 2bx - 2c = 0.$$

Solution 264. We provide hints and final answers:

1. To solve $(x-2)(x+1)(x+6)(x+9)+108 = 0$, write it as $(x^2+7x-18)(x^2+7x+6)+108 = 0$ whose roots are $x = 0, -7, \dfrac{-7 \pm \sqrt{97}}{2}$.

2. For $x^4 + (x+\sqrt{2})^4 = 68$, write the equation as

$$\left[\left(x + \frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}}{2}\right]^2 + \left[\left(x + \frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2}\right]^2 = 68,$$

   which has two real roots $x = \sqrt{2}$ and $-2\sqrt{2}$ as well as two imaginary roots.

3. Write $x^6 + (x+2)^6 = 2$ as $[(x+1)-1]^6 + [(x+1)+1]^6 = 2$, and show that there is only one real root $x = -1$ (which happens to be a double root) and four imaginary roots.

4. To solve $(x+3)^3+(x+5)^3 = 8$, let $x+4 = t$ and the equation becomes $t^3+3t-4 = 0$, which can be written as $(t-1)(t^2+t+4) = 0$. The real solution is $x = -3$ and there are two imaginary solutions.

5. For $(\sqrt{x}+1)^4 + (\sqrt{x}-3)^4 = 256$, choose $y = \sqrt{x}-1$ to find the only real solution $x = 9$.

6. For solving $(x^2+3x+2)(x^2+6x+12) = 120$, factorize the left-hand side as $(x+1)(x+2)(x+3)(x+4)$ and rearrange the equation into $(x^2+5x+4)(x^2+5x+6) = 120$ to find real roots $x = 1$ and 6, plus two imaginary roots.

7. For solving $x^4 + 2x^3 + 2x^2 + x = 42$, write it as $(x^2+x)^2 + (x^2+x) - 42 = 0$ and treat it as a quadratic in $x^2 + x$. The solutions are $2, -3$, and $\dfrac{-1+3i\sqrt{3}}{2}$.

8. The equation $x^4 + 6x^3 + 7x^2 - 6x = 1$ may be written as $(x^2+3x-1)^2 - 2 = 0$ which has the following solutions:

$$\frac{-3 \pm \sqrt{13-4\sqrt{2}}}{2} \quad \text{and} \quad \frac{-3 \pm \sqrt{13+4\sqrt{2}}}{2}.$$

9. The common denominator may be taken in the left-hand side of the equation $\dfrac{\sqrt[n]{a+x}}{a} + \dfrac{\sqrt[n]{a+x}}{x} = b\sqrt[n]{x}$ to obtain $(a+x)\sqrt[n]{a+x} = abx\sqrt[n]{x}$. Raise both sides of the latter equation to power of n to find $(a+x)^{n+1} = a^n b^n x^{n+1}$, and then take the $(n+1)^{th}$ root from both sides to find $x = \dfrac{a}{\sqrt[n+1]{a^n b^n} - 1}$ for even n and $x = \dfrac{a}{\pm\sqrt[n+1]{a^n b^n} - 1}$ for odd n.

10. The solutions to $\sqrt[3]{a+\sqrt{x}} + \sqrt[3]{a-\sqrt{x}} = \sqrt[3]{b}$ is $x = a^2 - \dfrac{(b-2a)^3}{271}$.

11. In order to solve $(x^2+2x-12)^2 = x^2(3x^2+2x-12)$, let $t = 2x - 12$ and write the equation as a quadratic in t: $t^2 + x^2 \cdot t - 2x^4 = 0$ with roots $t = x^2$ and $t = -2x^2$. The roots of the first equation are $1 \pm i\sqrt{11}$ and the roots of the second one are 2 and -3.

12. For solving $(2x^2-x-6)^2 + 3(2x^2+x-6)^2 = 4x^2$, let $t = 2x^2 - 6$ and find the roots $-2, 3/2, \pm\sqrt{3}$.

13. The equation $\dfrac{(x^2+1)^2}{x(x+1)^2} = \dfrac{9}{2}$ may be transformed to the positive reciprocal equation $2x^4 - 9x^3 - 14x^2 - 9x + 2 = 0$ with solutions $3 \pm 2\sqrt{2}$ and $(-3+i\sqrt{7})/4$.

14. For solving $3x^4 - 20x^3 + 45x^2 - 40x + 12 = 0$, divide both sides by x^2 and let $t = x + 2/x$ to find solutions $x = 1, 2, 3, 2/3$.

15. Write $x^3 - 3abx + a^3 + b^3 = 0$ as $x^3 - 3abx + (a+b)^3 - 3ab(a+b) = 0$ to see clearly that one of the roots must be $x = -(a+b)$ with two other imaginary roots $(a+b \pm i\sqrt{3}(a-b))/2$.

16. It is clear that if $(x^2-16)(x-3)^2 + 9x^2 = 0$, then $x \neq 3$ and we can divide both sides of the equation by $(x-3)^2$ to find

$$x^2 + \frac{(3x)^2}{(x-3)^2} = 16 \implies \left(\frac{x^2}{x-3}\right)^2 - 6\left(\frac{x^2}{x-3}\right) - 16 = 0,$$

and find solutions $-1 \pm \sqrt{7}$ as well as $4 \pm i2\sqrt{2}$.

17. Write the equation $(x^2-4)(x+1)(x+4)(x+5)(x+8)+476=0$ as
$$(x^2+6x-16)(x^2+6x-5)(x^2+6x+8)+476=0,$$
and let $t=x^2+6x$ to arrive at $t^3-3t^2-168t-164=0$ which factorizes into $(t+1)(t^2-4t-164)=0$, so that the solutions must be the roots of the three equations $x^2+6x-t=0$ where $t=-1$ or $t=-2\pm\sqrt{168}$.

18. To solve $\sqrt[m]{(x+1)^2}-\sqrt[m]{(x-1)^2}+\dfrac{3\sqrt[m]{x^2-1}}{2}=0$, divide both sides by $\sqrt[m]{x^2-1}$ (assuming $m\neq\pm1$) and let $y=\sqrt[m]{(x+1)/(x-1)}$ to obtain $2y^2+3y-2=0$ with roots $y=2$ and $y=-1/2$. If $y=2$, then $x=(2^m+1)/(2^m-1)$. If $y=-1/2$, then m must be odd and we would have $x=(1-2^m)/(1+2^m)$.

19. The roots of the equation $(a^2-a)^2(x^2-x+1)^3=(a^2-a+1)^3(x^2-x)^2$, assuming $a\neq0$ and $a\neq1$ are given by
$$x\in\left\{a,\frac{1}{a},1-a,\frac{1}{1-a},\frac{a-1}{a},\frac{a}{a-1}\right\}.$$

Solution 265. The solutions are $x=3\pm\sqrt{a+9}$ and $2\pm\sqrt{a+6}$.

Solution 266. Let $a=\sqrt{3}$ and the equation becomes $x^3+2ax^2+a^2x+a-1=0$ which may be solved as a cubic equation in a, giving solutions $a=1-x$ and $-\dfrac{x^2+x+1}{x}$. The solutions for x after plugging $a=\sqrt{3}$ are $x=1-\sqrt{3}$ and $\dfrac{-(\sqrt{3}+1)\pm\sqrt{12}}{2}$.

Solution 267. The five answers are $x_1=3$, and
$$x_{2,3}=\frac{-1+\sqrt{5}\pm\sqrt{30+6\sqrt{5}}}{4}\qquad\text{and}\qquad x_{4,5}=\frac{-1-\sqrt{5}\pm\sqrt{30-6\sqrt{5}}}{4}.$$

Solution 268. Use the argument
$$\frac{u}{v}=\frac{t}{z}\iff\frac{u+v}{u-v}=\frac{t+z}{t-z}$$
on the given expression to obtain
$$\frac{(x-a)^2+(x-b)^2}{2(x-a)(x-b)}=\frac{2c^2+2}{2c^2-2},$$
and applying the same argument again implies
$$\frac{(x-a)^2+(x-b)^2+2(x-a)(x-b)}{(x-a)^2+(x-b)^2-2(x-a)(x-b)}=c^2.$$
Simplify the latter equation after taking its square root to finally find the solutions
$$x_1=\frac{ac-bc+a+b}{2}\qquad\text{and}\qquad x_2=\frac{bc-ac+a+b}{2}.$$

Solution 269. We will use three variables α, β, γ defined by:

$$\begin{aligned} a+b &= \alpha+\beta, \quad & a+c &= \alpha+\gamma, \\ c+d &= \alpha-\beta, \quad & b+d &= \alpha-\gamma. \end{aligned}$$

Then,

$$\alpha = \frac{a+b+c+d}{2}, \quad \beta = \frac{a+b-c-d}{2}, \quad \gamma = \frac{a-b+c-d}{2}.$$

a) $y = x + \alpha$, so that the equation becomes

$$\frac{(y+\beta)^5 + (y-\beta)^5}{(y+\gamma)^5 + (y-\gamma)^5} = \frac{\alpha^2}{\gamma^2}.$$

Since $y \neq 0$, this may be reduced to a quartic equation in y

$$(\gamma^2 - \beta^2)y^4 + 5(\gamma^2\beta^4 - \beta^2\gamma^4) = 0,$$

and assuming $\gamma \neq \pm\beta$, it gives $y^4 = 5\beta^2\gamma^2$. The solutions for x are

$$x = -\frac{a+b+c+d}{2} \pm \frac{\sqrt[4]{5(a+b-d-d)^2(a-b+c-d)^2}}{2}.$$

b) Let $y = x + \alpha$, so that the equation becomes

$$\frac{(y+\beta)^5 + (y-\beta)^5}{(y+\gamma)^5 + (y-\gamma)^5} = \frac{\beta^5}{\gamma^5}.$$

Since $y \neq 0$, this may be reduced to a quartic equation in y

$$(\gamma^5 - \beta^5)y^4 + 10(\gamma^3 - \beta^3)\beta^2\gamma^2 y^2 + 5(\gamma - \beta)\beta^4\gamma^4 = 0,$$

which is in fact a quadratic equation in y^2, thus easy to solve with the quadratic formula.

Solution 270. The sextic equation $x^7 + 7^7 = (x+7)^7$ factorizes into

$$x(x+7)(x^4 + 2\cdot 7x^3 + 3\cdot 7^2x^2 + 2\cdot 7^3x + 7^4) = 0,$$

whose six roots are $x_1 = 0$ and $x_2 = -7$ (real roots), as well as two double imaginary roots:

$$x_3 = x_4 = \frac{7(-1+i\sqrt{3})}{2} \qquad \text{and} \qquad x_5 = x_6 = \frac{7(-1-i\sqrt{3})}{2}.$$

Solution 271. We have seen this factorization before:

$$(x+y)^7 - x^7 - y^7 = 7xy(x+y)(x^2+xy+y^2)2.$$

Writing the original equation as

$$\left[(x+a+b)^7 - x^7 - (a+b)^7\right] + \left[(a+b)^7 - a^7 - b^7\right] = 0.$$

Applying the mentioned identity on the last equation and heavily simplifying it results in a quadratic equation in $x^2+(a+b)x$ that is solvable:

$$[x^2+(a+b)x]^2 + (2a^2+3ab+2b^2)[x^2+(a+b)x] + (a^2+ab+b^2)^2 = 0.$$

Solution 272. Divide both sides of the equation by x^3 and let $y = 2x + 3/x$ to obtain the equation $(y-3)(y-5)(y+4) = 0$. Solving for y and then for x, we find that the only real roots are $x = 1$ and $3/2$.

Solution 273. Define $y = 6x + 7$ and write the equation as a quartic equation in y: $y^4 - y^2 - 72 = 0$. The final solutions are: real roots $x_1 = -2/3$, $x_2 = -5/3$, and imaginary roots

$$x_{3,4} = \frac{-7 \pm 2i\sqrt{2}}{6}.$$

Solution 274. The solutions to the three equations are:

1. Write it as

$$\left(\frac{x^2}{x-1}\right)^2 - 2\left(\frac{x^2}{x-1}\right) - 8 = 0,$$

   whose four roots are $x_1 = x_2 = 2$ and $x_{3,4} = -1 \pm \sqrt{3}$.

2. Completing the square on the left-hand side of the equation and then factorizing it leads to $(x+2)^2(x^2-2x-2) = 0$ whose roots are $x_1 = x_2 = -2$ and $x_{3,4} = 1 \pm \sqrt{3}$.

3. Again, the left-hand side must be completed to find the solutions $x_1 = x_2 = \sqrt{2} - 1$ and $x_3 = x_4 = -(\sqrt{2}+1)$.

Solution 275. Let $y = x + 2$ and simplify the equation to find

$$y^4 + 5y^3 + 7y^2 + 2y = 0,$$

which factorizes into $y(y+2)(y^2+3y+1)$. The roots for x are:

$$x_1 = -2, \quad x_2 = -4, \quad x_{3,4} = -\frac{7 \pm \sqrt{5}}{2}.$$

Solution 276. The equation after heavy simplification becomes

$$4x^3 - 6(n+1)x^2 + 2(n+1)(2n+1)x - n(n+1)^2 = 0,$$

and by changing the variable to $y = x + (n+1)/2$, it becomes $4y^3 + (n^2-1)y = 0$. The solutions for x are (when $n > 1$):

$$x_1 = \frac{n+1}{2} \qquad \text{and} \qquad x_{2,3} = \frac{n+1 \pm \sqrt{1-n^2}}{2},$$

where x_1 is real and $x_{2,3}$ are imaginary. For $n = 1$, the equations has a triple root $x = 1$.

Solution by Parviz Shahriari 284.

(a) The fraction on the left side of the equation reminds us of the coefficients of $\tan 3\alpha$, thus making $x = \tan\alpha$ a plausible change of variables which would yield an equation for α in the form of $\alpha = k\pi/3 + \pi/9$. In short, since the function $\tan x$ is periodic with the least period π,

$$\frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha} = \sqrt{3} \implies \tan 3\alpha = \sqrt{3} = \tan\frac{\pi}{3}$$
$$3\alpha = k\pi + \frac{\pi}{3} \implies \alpha = \frac{k\pi}{3} + \frac{\pi}{9}.$$

The solutions will be

$$x = \tan\left(\frac{k\pi}{3} + \frac{\pi}{9}\right) \qquad \forall k \in \mathbb{Z}.$$

Since the original equation is a third-degree polynomial in x, it has at most three real solutions in x which will be found by plugging $k = 0, 1, 2$ in the above equation. These are

$$\begin{aligned} k = 0 &\implies x_1 = \tan 20^\circ, \\ k = 1 &\implies x_2 = \tan 80^\circ, \\ k = 2 &\implies x_3 = \tan 140^\circ = -\tan 40^\circ. \end{aligned}$$

(b) Using Vieta's formulas for sums and products related to the three roots p, q, r of the polynomial equation $x^3 - 3\sqrt{3}x^2 - 3x + \sqrt{3} = 0$, we find the equations $p+q+r = 3\sqrt{3}$, $pq + qr + rp = 3$, and $pqr = \sqrt{3}$, and the values $\{p, q, r\} = x_{1,2,3}$ as calculated in part (a) would result in the trigonometric identities on the even multiples of 20°:

$$\begin{aligned} 3\sqrt{3} &= \tan 20^\circ - \tan 40^\circ + \tan 80^\circ, \\ 3 &= \tan 20^\circ \tan 40^\circ + \tan 40^\circ \tan 80^\circ - \tan 20^\circ \tan 80^\circ, \\ \sqrt{3} &= \tan 20^\circ \tan 40^\circ \tan 80^\circ. \end{aligned}$$

Solution by Parviz Shahriari 285.

(a) The fraction on the left side of the equation reminds us of the coefficients of $\cos 5\alpha$, thus making $x = \cos\alpha$ a plausible change of variables which would yield an equation for α in the form of $\alpha = 2k\pi/5 \pm \pi/15$. In short, since the function $\cos x$ is periodic with the least period 2π,

$$2(16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha) = 1 \implies 2\cos 5\alpha = 1 = 2\cos\frac{\pm\pi}{3}$$
$$5\alpha = 2k\pi \pm \frac{\pi}{3} \implies \alpha = \frac{2k\pi}{5} \pm \frac{\pi}{15}.$$

The solutions will be

$$x = \cos\left(\frac{2k\pi}{5} \pm \frac{\pi}{15}\right) \qquad \forall k \in \mathbb{Z}.$$

Since the original equation is a fifth-degree polynomial in x, it has at most five real solutions in x which will be found by plugging $k = 0, 1, 2$ in the above equation. These are

$$k = 0 \implies x_1 = \cos 12^\circ,$$
$$k = 1 \implies \begin{cases} x_2 &= \cos 84^\circ = \sin 6^\circ, \\ x_3 &= \cos 60^\circ = \frac{1}{2}. \end{cases}$$
$$k = 2 \implies \begin{cases} x_4 &= \cos 156^\circ = -\cos 24^\circ, \\ x_5 &= \cos 132^\circ = -\cos 48^\circ. \end{cases}$$

(b) Using Vieta's formulas for sums and products related to the five roots p, q, r, s, t of the polynomial equation $32x^5 - 40x^3 + 10x - 1 = 0$, we find the equations $p+q+r+s+t = 0$, $pqrts = 1/32$, and $1/p+1/q+1/r+1/s+1/t = (qrst+prst+pqst+pqrt+pqrs)/pqrst = 10$. The values $\{p, q, r, s, t\} = x_{1,2,3,4,5}$ as calculated in part (a) would result in the trigonometric identities on the first couple even and first couple odd multiples of 12°:

$$0 = \cos 12^\circ - \cos 24^\circ - \cos 48^\circ + \cos 60^\circ + \cos 84^\circ,$$
$$\frac{1}{32} = \cos 12^\circ \cos 24^\circ \cos 48^\circ \cos 60^\circ \cos 84^\circ,$$
$$10 = \frac{1}{\cos 12^\circ} - \frac{1}{\cos 24^\circ} - \frac{1}{\cos 48^\circ} + \frac{1}{\cos 60^\circ} + \frac{1}{\cos 84^\circ}.$$

Solution 286. Divide both sides of the equation by the first term on the left hand side to find

$$\left(\frac{1-a^2}{1+a^2}\right)^x + \left(\frac{2a}{1+a^2}\right)^x = 1.$$

The trick is to set $a = \tan(\alpha/2)$ to get $(\cos\alpha)^x + (\sin\alpha)^x = 1$, whose only solution is clearly $x = 2$.

Solution 287. If we define $z = \cos\phi + i\sin\phi$ where $i = \sqrt{-1}$, then it is easy to verify that $1/z = \cos\phi - i\sin\phi$, so that we can find $\cos\phi$ in terms of z:

$$\cos\phi = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

Using double-angle formulas, we can compute $\cos 2\phi$ and $\cos 4\phi$ as well:

$$\begin{aligned}\cos 2\phi &= 2\cos^2\phi - 1 = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right),\\ \cos 4\phi &= 2\cos^2 2\phi - 1 = \frac{1}{2}\left(z^4 + \frac{1}{z^4}\right).\end{aligned}$$

Thus, the problem reduces to solving the equation

$$\left(z+\frac{1}{z}\right)^2 + \left(z^2+\frac{1}{z^2}\right)^2 - \left(z+\frac{1}{z}\right)\left(z^2+\frac{1}{z^2}\right)\left(z^4+\frac{1}{z^4}\right) = 3.$$

After simplification, this turns out to be $1 + z^{15} + (z+1)(z^{13}+z) = 0$ and finally $(1+z+z^2)(1+z^{13}) = 0$. If $1+z+z^2 = 0$, then $z = -1/2 \pm i\sqrt{3}/2$, which gives $\phi = 2k\pi \pm 2\pi/3$. Otherwise, $1+z^{13} = 0$ and by De Moivre's formula, $\cos 13\phi + i\sin 13\phi = -1$, giving $\phi = 2k\pi + (2n+1)\pi/13$ for $0 \le n < 12$ and $n \neq 6$.

Solution 288. The solutions are $\alpha = k\pi$ and $\alpha = k\pi + (-1)^k\pi/6$.

Solution 289. We may solve the equation $x^3 - 3x + 1 = 0$ in two ways:

1. First, observe that $f(0)$ and $f(2)$ are positive, whereas $f(1)$ and $f(-2)$ are negative. Therefore, all three of equation's roots are real and they lie in intervals $(-2,0), (0,1), (1,2)$. Let x_1 be the smallest root and therefore in the interval $(-2,0)$. Note that

$$\left(\frac{1}{x_1-1}\right)^3 - 3\left(\frac{1}{x_1-1}\right) + 1 = \frac{-(x_1^3 - 3x_1 + 1)}{(1-x_1)^3} = 0,$$

so $1/(1-x_1)$ would be another root of the original equation. Since $-2 < x_1 < 0$, we find that $1/3 < 1/(1-x_1) < 1$, which means that $1/(1-x_1) < x_3$, and so $x_2 = 1/(1-x_1)$ is the second largest root of the equation. Now,

$$\begin{aligned}x_2^2 - x_1 &= \frac{1}{(x_1-1)^2} - x_1 = \frac{1 - x_1 + 2x_1^2 - x_1^3}{(1-x_1)^2}\\ &= \frac{-x_1^3 + 3x_1 - 1 + 2(1-x_1)^2}{(1-x_1)^2} = 2.\end{aligned}$$

2. Let $x = 2\cos\alpha$ and the equation becomes

$$\begin{aligned}x^3 - 3x + 1 &= 8\cos^3\alpha - 6\cos\alpha + 1\\ &= 2\cos 3\alpha + 1.\end{aligned}$$

So, $x^3 - 3x + 1 = 0$ is equivalent to $\cos 3\alpha = -1/2$ with solutions $\alpha = 2k\pi/3 \pm 2\pi/9$. These will lead to the following solutions:

$$x_1 = -2\cos\frac{\pi}{9}, \qquad x_2 = 2\cos\frac{4\pi}{9}, \qquad x_3 = 2\cos\frac{2\pi}{9}.$$

Now we can calculate $x_2^2 - x_1$ is

$$4\cos^2\frac{4\pi}{9} + 2\cos\frac{\pi}{9} = 2\left(1 + \cos\frac{8\pi}{9} + \cos\frac{\pi}{9}\right) = 2.$$

Solution 291. Expand $x^2 - 1$ and $x^2 - 4$, then write the equation as

$$(x^3 - 3x - 2) + (x-1)(x+1)\sqrt{(x-2)(x+2)}.$$

Now, $x^3 - 3x - 2$ becomes zero for both $x = -1$ and $x = 2$, and we may factorize it as $(x+1)^2(x-2)$. These two numbers turn out to be the only solutions to the original equation as well.

Solution 296. The answers are $x = -1$ and $x = 35$.

Solution 297. The answer is $x + y = 0$.

Solution 300. Using the identity $(a \pm b)^3 = a^3 \pm b^3 \pm 3ab(a \pm b)$, we can raise both sides of the equation $\sqrt[3]{u} \pm \sqrt[3]{v} = a$ to the power of 3 to arrive at the given expression.

Solution 301. The two sides must be cubed and simplified to obtain $9\sqrt[3]{6x^2 - 5x - 6} = 28 - 5x$, which after being cubed again becomes $125x^3 + 2274x^2 + 8115x - 26326 = 0$, whose only real solution is $x = 2$.

Solution 302. After rationalizing the equation, the solution is $x = (a-1)^2/4$ when $a \geq 1$ and there are no solutions when $a < 1$.

Solution 303. There are no solutions for $a < -1$ or $0 \leq a \leq 1$. Otherwise, when $-1 < a < 0$ or $a > 1$, the solution is $x = a/(\sqrt[3]{a^2} - 1)$.

Solution 304. The answer is $x = 17$.

Solution 305. After rationalizing, the equation becomes $(x-1)[12x(2x-3) - 27(x-1)^2] = 0$ whose solutions are $x = 1$ and $x = 3$.

Solution 306. The solutions are $x_1 = a$, $x_2 = b$, and $x_3 = (a+b)/2$.

Solution 307. The solutions are $x_1 = 1$ and $x_2 = -27/8$.

Solution 308. The solutions are $x_1 = -a$ and $x_2 = 1 - a$.

Solution 309. Let $y = \sqrt[5]{(7x-3)^3}$ to find the solutions $x_1 = 2/7$ and $x_2 = 5$.

Solution 310. First, prove that $x^2 < 1/b^2$ and $x^2 < 1/a^2$ must happen for the equation to have a solution. In that case, the trivial solution is $x_1 = 0$, and if $1/2 \leq a/b < 1$, the other two roots are:

$$x_2 = \sqrt{\frac{2a-b}{a^2b}} \qquad \text{and} \qquad x_3 = -\sqrt{\frac{2a-b}{a^2b}}.$$

Solution 311. Let $\sqrt[5]{16 + \sqrt{x}} = u$ and $\sqrt[5]{16 - \sqrt{x}} = v$, so that $u + v = 2$ and raising the equation to power of 5 results in:

$$u^5 + v^5 + 5uv(u+v)\left[(u+v)^2 - 3uv\right] + 10u^2v^2(u+v) = 32.$$

Plug in $u+v=2$ and simplify to get $uv(4-uv)=0$, and the only real solution of the equation comes from $v=0$, giving $x=256$.

Solution 312. The trivial solution is $x_1=0$ and the other two solutions (with appropriate a and b) are

$$x_{2,3}=a^{2k/(2k-n)}\left(\sqrt{b}\pm\sqrt{b-a}\right)^{2n/(n-2k)}.$$

Solution 313. We divide the solutions in two cases:

- If n is even, we must have $a>0$ and $-a<x\le a$ and the solutions are
$$x=\pm\frac{a}{\sqrt{1+a^{2n/(n+1)}}}.$$
- If n is odd, we must have $a\neq 0$ and the solutions are the same as before:
$$x=\pm\frac{a}{\sqrt{1+a^{2n/(n+1)}}}.$$

Solution 314. The only solution is

$$x=\frac{a}{\left[1+a^{2n/3}\right]^n}.$$

Solution 315. For $p\ge 2$, the solution is $x=2\sqrt{p-1}$ and there are no solutions when $p<2$.

Solution 316. Let $a=\sqrt{3}$ and turn the equation into a quadratic in a:

$$a^2-(2x+1)a+(x^2-\sqrt{x})=0.$$

The only solution is $x=(7+\sqrt{13})/2$.

Solution 317. Let $a=\sqrt{5}$ and form a quadratic equation in a like in Problem 316. The only solution is

$$x=\frac{1+\sqrt{1+4\sqrt{5}}}{2}.$$

Solution 318. Let $u=\sqrt[5]{a+\sqrt{x}}$ and $v=\sqrt[5]{a-\sqrt{x}}$ to arrive at $u^5+v^5=2a$. Reduce the equation to $uv(uv-\sqrt[5]{4a^2})=0$, implying either $uv=0$ or $uv=\sqrt[5]{4a^2}$. There is a real solution $x=a^2$ if $uv=0$ and the other case $uv=\sqrt[5]{4a^2}$ results in imaginary solutions for x.

Solution 319. If $a<b$, the only solution is $x=a$ and if $a>b$, the only solution is $x=b$.

Solution 320. The only solution is $x=-2$.

Solution 321. Define y so that $y\sqrt{x} = x - 1$ to get the equation $x^2 + 1 = x(y^2 + 2)$. Writing this as an equation in y, we find $\sqrt{2(y^2 + 2)} = 2(a - y)$ for $a \geq y$. After rationalizing and solving the latter equation, we find that the only solution for the original equation is

$$x = \frac{1}{4}\left(2a - \sqrt{2(a^2 + 1)} + \sqrt{6a^2 + 6 - 4a\sqrt{2(a^2 + 1)}}\right).$$

Solution 322. Rationalizing the equation results in $9x^2 - 196x + 356 = 0$ whose solution $x = 2$ is the only one that works in the original equation.

Solution 323. The answer is $x = 3/2$.

Solution 324. The answer is $x = -2$.

Solution 325. Prove that the function $f(x) = 2\sqrt{x-1} + \sqrt[3]{x}$ is an increasing function, so that the equation has exactly one solution $x = 1$.

Solution 326. The answer is $x = 2$.

Solution 327. The only solution is $x = 2$.

Solution 331. Dividing both sides of the equation by x^2 gives us an expression that has $x + \dfrac{1}{x}$ as a factor:

$$2\left(x^2 + \frac{1}{x^2}\right) - 13\left(x + \frac{1}{x}\right) + 24 = 0.$$

Choose $t = x + \dfrac{1}{x}$ to find $x^2 + \dfrac{1}{x^2} = t^2 - 2$ and find $2(t^2 - 2) - 13t + 24 = 0$, which has solutions $t = 4$ and $t = 5/2$. The equation $x + \dfrac{1}{x} = 4$ has real solutions $x = 2 \pm \sqrt{3}$ and the roots of the equation $x + \dfrac{1}{x} = 5/2$ are 2 and 1/2.

Solution 335. This is a negative reciprocal equation and it must be arranged as a polynomial in $x - \dfrac{1}{x}$. Dividing both sides of the original equation by x^2 and rearranging, we find

$$2a\left(x^2 + \frac{1}{x^2}\right) - (2a^2 + 3a - 2)\left(x - \frac{1}{x}\right) + (3a^2 - 4a - 3) = 0.$$

Choose $t = x - \dfrac{1}{x}$ to find $x^2 + \dfrac{1}{x^2} = t^2 + 2$; yielding $2a(t^2+2) - (2a^2+3a-2)t + (3a^2-4a-3) = 0$, which has solutions $t = 3/2$ and $t = a - \dfrac{1}{a}$. The equation $x - \dfrac{1}{x} = \dfrac{3}{2}$ has real solutions $x = 2$ and $x = -\dfrac{1}{2}$, and the roots of the equation $x - \dfrac{1}{x} = a - \dfrac{1}{a}$ are a and $-\dfrac{1}{a}$.

Solution 336. Dividing both sides of the original equation by x^2 and rearranging, we find

$$2\left(x^2 + \frac{4}{x^2}\right) - 15\left(x + \frac{2}{x}\right) + 35 = 0.$$

Choose $t = x + \dfrac{2}{x}$ to find $x^2 + \dfrac{4}{x^2} = t^2 - 4$; yielding $2(t^2-4) - 15t + 35 = 0$, which has solutions $t = 9/2$ and $t = 3$. The equation $x + \dfrac{2}{x} = \dfrac{9}{2}$ has real solutions $x = 4$ and $x = \dfrac{1}{2}$, and the roots of the equation $x + \dfrac{2}{x} = 3$ are 1 and 2.

Solution 337. Dividing both sides of the original equation by x^2 and rearranging, we find

$$2\left(x^2 + \frac{9}{x^2}\right) + 7\left(x - \frac{3}{x}\right) - 34 = 0.$$

Choose $t = x - \dfrac{3}{x}$ to find $x^2 + \dfrac{9}{x^2} = t^2 + 6$; giving $2(t^2+6) + 7t - 34 = 0$, which has solutions $t = 2$ and $t = -11/2$. The equation $x - \dfrac{3}{x} = 2$ has real solutions $x = -1$ and $x = 3$, and the roots of the equation $x - \dfrac{3}{x} = -\dfrac{11}{2}$ are $\dfrac{1}{2}$ and -6.

Solution 338. If α is a root of $x^3 - x + 1 = 0$, then it is easy to see that $1/\alpha$ must be a root of $x^3 - x^2 + 1 = 0$, and also

$$x^5 + x + 1 = (x^3 - x^2 + 1)(x^2 + x + 1).$$

Solution 340. The answer is $5 \leq x < 6$.

Solution 341. The answer is $\dfrac{4}{7} \leq x < \dfrac{3}{5}$.

Solution 342. The answer is $\dfrac{\sqrt{6}}{2} \leq x < \dfrac{\sqrt{10}}{2}$.

Solution 343. First, we will investigate in which intervals the equation cannot happen. Define $f(x) = (x^3-2)/3$. Clearly, if $f(x) \geq x+1$ or $f(x) \leq x-1$, the equation $\lfloor x \rfloor = \lfloor f(x) \rfloor$ cannot happen. The inequality $f(x) \geq x+1$ simplifies to $x^2(x-3) \geq -3x^2 + 3x + 5$, which has $x \geq 3$ in its solutions. The inequality $f(x) \leq x - 1$ reduces to $x^2(x+2) \leq 2x^2 + 3x - 1$ and it has $x \leq -2$ among its solutions. Therefore, the original equation can have its solutions in the interval $-2 < x < 3$:

$$-4 < x < 0 \qquad \text{and} \qquad \sqrt[3]{5} < x < \sqrt[3]{11}.$$

Solution 344. Let $k = 3y/2x$ and observe that k must be an integer, and the equation becomes

$$1 + k = \left\lfloor \frac{4k^2}{9} \right\rfloor \implies 1 + k \leq \frac{4k^2}{9} < 2 + k.$$

The solutions are $3 \leq k < 3.6$ and $-1.2 < k \leq 0.75$. Since k must be an integer, $k = -1$ and $k = 3$ are the only solutions, resulting in $y = 2x$ and $y = -2x/3$.

Solution 345. The solutions are $x_1 = 7/15$ and $x_2 = 4/5$.

Solution 346. The given equation is the same as $|x+3|+|x-2|+|2x-8|=9$, and the short answer is $x \in [2,4]$.

Solution 347. The answer is $x=-1$ and $x=-3$.

Solution 348. We know that the product of absolute values of several terms is equal to the absolute value of the product of the same terms, so we multiply out the terms and expand to reach two possible equations:

$$x^3+7x^2-36=\pm(x^3+14x^2+49x+36),$$

whose five (two quadratic and three cubic) roots are

$$x_{1,2}=\frac{-49\mp\sqrt{385}}{14}, \quad x_3=0, \quad x_4=-7, \quad x_5=-\frac{7}{2}.$$

Solution 349. Let the roots be $x_1, x_2, x-3$, so that their sum is $-a$. Furthermore, since the roots are in an arithmetic progression, $2x_2=x_1+x_3$. Therefore, $3x_2=-a$ and this means that $x_2=-a/3$ is a root of the equation. Plugging $x=-a/3$ in the equation yields the needed equation for a,b,c:

$$2a^3-9ab+27c=0.$$

Solution 350. Let the three roots of the cubic $x^3+px^2+qx+r=0$ be $x=\alpha,\beta,\gamma$, so that by Viète's Formulas, $\alpha+\beta+\gamma=-p$ and $\alpha\beta+\beta\gamma+\gamma\alpha=q$. Find the difference p^2-3q, which is said to be negative in the statement, in terms of α,β,γ:

$$\begin{aligned}0>p^2-3q&=(\alpha+\beta+\gamma)^2-3(\alpha+\beta+\gamma)\\&=\alpha^2+\beta^2+\gamma^2-(\alpha+\beta+\gamma).\end{aligned}$$

Multiplying both sides of the inequality $\alpha^2+\beta^2+\gamma^2-(\alpha+\beta+\gamma)<0$ by 2, we obtain $(\alpha-\beta)^2+(\beta-\gamma)^2+(\gamma-\alpha)^2<0$, which cannot happen if α,β,γ are real numbers, so at least one of them is imaginary. Since imaginary roots of any polynomial equation are paired, the original equation has two imaginary roots and one real root.

Solution 351. The first task is just the cheat-code for the second one. For each positive integer n in task b), define $f_n(x)=x^n-x-n$, and realize that $f_n(1)=-n<0$ whereas $f_n(2)>0$, so that the original equation has exactly one root in the interval $[1,2]$, which we may call x_n. Now follow task a) to finish the proof.

Solution 352. The trick is to write the polynomial as a sum of non-negative polynomials of lesser degrees. In particular for $x^4-4x^3+12x^2-24x+24$, we have

$$x^4-4x^3+12x^2-24x+24=(x^2-2x)^2+8\left[\left(x-\frac{3}{2}\right)^2+\frac{3}{4}\right],$$

whose first term is non-negative and the second term at least $3/4$, so their sum cannot be zero if x is a real number.

Solution 353. If a, b, c are the triangle's side-lengths and h_a, h_b, h_c are the length of heights, we would have

$$\begin{cases} h_a + h_b + h_c = k, \\ h_a h_b + h_b h_c + h_c h_a = q, \\ h_a h_b h_c = z. \end{cases}$$

Since the height-lengths are positive, so must be k, q, z. Moreover, we have $2S = ah_a = bh_b = ch_c$. Calculate the semiperimeter p:

$$p = \frac{a+b+c}{2} = S\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right) = \frac{Sq}{z}.$$

Using Heron's Formula $S = \sqrt{p(p-a)(p-b)(p-c)}$ and simplifying, we arrive at the required equation.

Solution 354. If α is the angle facing the base a, we would have $a = 2b\sin(\alpha/2)$. Simplifying, we get

$$2\left(4\sin^3\alpha - 3\sin\frac{\alpha}{2}\right) + \sqrt{3} = 0,$$

or simply $\sin(3\alpha/2) = \sqrt{3}/2$, which has solutions $\alpha = 2\pi/9$ and $\alpha = 4\pi/9$.

Solution 355. Adding up the given equations, we find $2c^4 = c^2(a^2+b^2)$, or simply $c^2 = (a^2+b^2)/2$. Substitute this for c^2 in $a^4 = b^4 + c^4 - b^2c^2$ to obtain

$$a^4 = b^4 + \frac{a^4 + b^4 + 2a^2b^2}{4} - \frac{a^2b^2 + b^4}{2}.$$

Finally, the equation reduces to $a = b$ and similarly $a = c$, after which everything is trivial.

Solution 356. Given $a < b + c$, we need to prove that $\sqrt[n]{a} < \sqrt[n]{b} + \sqrt[n]{c}$. Raising both sides of the latter inequality to power of n, we need to prove $a < \left(\sqrt[n]{b} + \sqrt[n]{c}\right)^n$. Now,

$$\left(\sqrt[n]{b} + \sqrt[n]{c}\right)^n = b + c + (\text{positive terms}) > b + c > a,$$

as required.

Solution 357. The discriminant of the equation is $D = -4(p^3+q^3) - 23q^2$, which would be divisible by 23 if $p^3 + q^3$ is a multiple of 23. We know also that D equals the square of the product of the pairwise difference of the roots, and since D is divisible by the prime 23, at least one of the elements in the product (square of difference of two roots) must be divisible by 23.

Solution 358. For the necessity condition, if we plug in a purely imaginary root $x = \alpha i$ into $f(x) = ax^3 + bx^2 + cx + d$, the real values and the coefficient of $i = \sqrt{-1}$ must be zero:

$$\begin{cases} d - b\alpha^2 = 0, \\ c\alpha - a\alpha^3 = 0. \end{cases}$$

To prove that $ad = bc$ and $ac > 0$ is sufficient for $f(x)$ to have a purely imaginary root, multiply both sides of the original equation by $a \neq 0$ and use $ad = bc$ to simplify the result into $(ax^2 + c)(ax + b) = 0$. The roots of $ax^2 + c = 0$ are purely imaginary: $x = \pm i\frac{\sqrt{ac}}{a}$.

Solution 359. Define $f(x) = x^n + x^{n-1} + \cdots + x^2 + x - 1$ for $n \geq 2$ and note that $f(0) = -1 < 0$ and $f(1) = n - 1 > 0$, so there must be a real root between 0 and 1, call it x_n. Furthermore, since $x_n < 1$ for all $n \geq 2$ and the sum $x_n^n + x_n^{n-1} + \cdots + x_n^2 + x_n$ may be written as a geometric series with both initial term and the common ratio equal to $x_n < 1$. According to the formula for infinite sum of geometric series with common ratio less than 1, the limit of the series when $n \to \infty$ would be $x_n/(1 - x_n)$, which must be equal to 1 according to the original equation. This gives a limit of $1/2$ for x_n as $n \to \infty$.

Solution 360. We will form a cubic equation whose roots are

$$y_1 = \frac{m\alpha + n}{m\alpha - n}, \quad y_2 = \frac{m\beta + n}{m\beta - n}, \quad y_3 = \frac{m\gamma + n}{m\gamma - n}.$$

If $y = (mx + n)/(mx - n)$, then $x = n(y + 1)/(m(y - 1))$. Simplifying, the equation becomes

$$n^3(y + 1)^3 + pm^2n(y + 1)(y - 1)^2 + qm^3(y - 1)^3 = 0.$$

The required sum $S = y_1 + y_2 + y_3$ can then be calculated as

$$S = \frac{3qm^3 + 2pm^2n - 3n^3}{n^3 + pm^2n + qm^3}.$$

Solution 361. Define $f(x) = x^3 + 21x^2 + 140x - 300$ and assume $f(\alpha) = f(2\alpha) = 0$ for some α. Simplifying the system of equations $f(\alpha) = 0$ and $f(2\alpha) = 0$, we find $\alpha = 5$, so that the roots of the equation will be $x = 5, 6, 10$.

Solution 362. Using Viète's Formulas, we find $a = \pm 10$.

Solution 363. The first part is easy to verify, and we may use it to write the polynomial in the second part as

$$f(x) = (2x - 1)^4 - 20(2x - 1)^2 + 91 = 0,$$

whose roots are

$$x_{1,2} = \frac{1 \pm \sqrt{7}}{2} \qquad \text{and} \qquad x_{3,4} = \frac{1 \pm \sqrt{13}}{2}.$$

Solution 364. Using Viète's Formulas, we find $m = 18$, and the four roots x_1, x_2, x_3, x_4 may be calculated easily from $x_1 + x_2 = 4$, $x_1x_2 = -1$ and $x_3 + x_4 = 4$ and $x_3x_4 = 3$.

Solution 365. Let a, b, c be the side-lengths of the triangle. By Viète's Formulas, we find

$$a + b + c = 12, \quad ab + bc + ca = 47, \quad abc = 60.$$

Using Heron's formula for the area of triangle, we can find the area:

$$\begin{aligned} S^2 &= p(p - a)(p - b)(p - c) \\ &= -p^4 + (ab + bc + ca)p^2 - abcp \\ &= -6^4 + 47 \cdot 6^2 - 60 \cdot 6 = 36. \end{aligned}$$

So, the area is $S = 6$.

Solution 376. Note that $S_0 = x_1^0 + x_2^0 = 2$ and $S_1 = x_1 + x_2 = 2$, and the recursive formula for the sum of quadratic roots would be $2S_n - 4S_{n-1} + S_{n-2} = 0$. Plugging $n = 2, 3, 4$ in the latter equation, we find $S_2 = 3, S_3 = 5$, and $S_4 = 17/2$, respectively. Therefore, the answer is $S_4 = x_1^4 + x_2^4 = \frac{17}{2}$.

Soluion by Parviz Shahriari 377. The final answer is 57. Let x_1, x_2, x_3 be the roots of the equation $x^3 - 3x + 1 = 0$ and define $S_p = x_1^p + x_2^p + x_3^p$. First, multiply the equation by x^3 and sum it up when plugging $x = x_1, x_2, x_3$, to obtain $S_6 = 3S_4 - S_3$, and second, multiply the equation by x and sum it up to obtain $S_4 = 3S_2 - S_1$. This last one can be written as

$$\begin{aligned} x_1^4 + x_2^4 + x_3^4 &= 3(x_1^2 + x_2^2 + x_3^2) - (x_1 + x_2 + x_3) \\ &= 3\left[(x_1 + x_2 + x_3)^2 - 2(x_1x_2 + x_2x_3 + x_3x_1)\right] - (x_1 + x_2 + x_3). \end{aligned}$$

Since $x_1 + x_2 + x_3 = 0$ and $x_1x_2 + x_2x_3 + x_3x_1 = -3$, we find $S_4 = 18$ and finally $S_6 = 57$.

Solution 378. Multiplying the equation by x^2 will result in $x^5 - x^3 + x^2 = 0$, and if we add the three equations formed by plugging $x = x_1, x_2, x_3$ in the latter equation, we find

$$x_1^5 + x_2^5 + x_3^5 = (x_1^3 + x_2^3 + x_3^3) - (x_1^2 + x_2^2 + x_3^2).$$

It is now easy to find from the original equation that $x_1^3 + x_2^3 + x_3^3 = -3$ and $x_1^2 + x_2^2 + x_3^2 = 2$, so that the answer would be $x_1^5 + x_2^5 + x_3^5 = -5$.

Solution 379. Assuming $x_1 = 1$, we would have $x_1x_2 = x_3$, $x_2x_3 = x_1$, and $x_1x_2 = x_3$ and the given equality is obvious.

Solution 380. The answer is:

$$a = \sqrt[20]{2^{20} - 1}, \quad b = -\frac{\sqrt[20]{2^{20} - 1}}{2}, \quad p = -1, \quad q = \frac{1}{4}.$$

Solution by Parviz Shahriari 381. Using the trick of Sum of Powers of Cubic Roots (Problem 375, if we define $S_p = x_1^p + x_2^p + x_3^p$, then $S_{n+3} + S_{n+1} + S_n = 0$. In order to find S_{11}, we put $n = 0, 1, 2, \ldots, 8$ into the latter equation, initiating with $S_0 = x_1^0 + x_2^0 + x_3^0 = 3$, $S_1 = x_1 + x_2 + x_3 = 0$, and $S_2 = x_1^2 + x_2^2 + x_3^2 = -2(x_1x_2 + x_2x_3 + x_3x_1) = -2$. We can find $S_3, S_4, \ldots, S_9$ from the equations $S_{n+3} + S_{n+1} + S_n = 0$ with appropriate n, calculated below:

$$S_3 = -3, \quad S_4 = 2, \quad S_5 = 5, \quad S_6 = 1, \quad S_7 = -7, \quad S_8 = -6, \quad S_9 = 6.$$

Finally, for $n = 8$, we have $S_{11} + S_9 + S_8 = 0$. Since $S_9 + S_8 = 0$, we have $S_{11} = 0$ and we are done.

Solution 384. The easiest way is to set $y = x^2$ or $x = \pm\sqrt{y}$ into the equation:

$$(\sqrt{y})^3 + 2(\sqrt{y})^2 - \sqrt{y} + 5 = 0,$$

which simplifies to $\sqrt{y}(1 - y) = 5 + 2y$. Squaring both sides and writing in descending powers of y,

$$y^3 - 6y^2 - 19y - 25 = 0.$$

The more tiresome and time-consuming method would be to find $y_1+y_2+y_3$, $y_1y_2+y_2y_3+y_3y_1$, and $y_1y_2y_3$ in terms of the same functions in x_i, assuming $y_i = x_i^2$ for $i = 1, 2, 3$.

Solution 385. Define $y_i = 1/(2x_i - 1)$ and create a degree–4 polynomial equation in y with roots y_i for $i = 1, 2, 3, 4$. If $y = 1/(2x - 1)$, then $x = (y + 1)/(2y)$. Plugging this last expression into the given equation,

$$\frac{(y+1)^4}{16y^4} - \frac{4(y+1)^2}{4y^2} + \frac{y+1}{2y} + 3 = 0.$$

This will become, after simplification,

$$41y^4 - 20y^3 - 10y^2 + 4y + 1 = 0.$$

The required sum $S = \sum_{i=1}^{4} y_i$ equals the sum of roots of the above equation: $S = 20/41$.

Solution 386. If $y = 1/(x - 1)$, then $x = (y + 1)/y$. Plugging this change of variable into the original equation and simplifying, we arrive at

$$(y+1)^n + y(y+1)^{n-1} + y^2(y+1)^{n-2} + \cdots + y^{n-1}(y+1) + y^n = 0.$$

The coefficient of y^{n-1} in the above polynomial is $n(n+1)/2$ and the coefficient of y^n is $n + 1$, so that the required sum in question is equal to $-(n + 1)/2$.

Solution 387. We need to form a polynomial whose roots are $y_i = 1/(x_i^2 - 1)$ for $i = 1, 2, 3$, and it is easy to find the polynomial by the change of variable $x = \sqrt{(y+1)/y}$ in the original equation. The new polynomial would be

$$100y^3 + 60y^2 + 8y - 1 = 0,$$

so that the sum of the roots of the equation is $S = -3/5$.

Solution 388. The general answer is

$$(y-4)(y^3 + y^2 - 4y + 1).$$

If $\alpha = 1$, then $\alpha^4 + \alpha^6 + \alpha^7 + \alpha^9 = 4$ and the linear equation $y - 4 = 0$ is the solution. If $\alpha \neq 1$, since $\alpha^{13} - 1 = 0$, we have

$$\alpha^{12} + \alpha^{11} + \cdots + \alpha^2 + \alpha + 1 = 0.$$

Define

$$\begin{cases} y_1 &= \alpha^4 + \alpha^6 + \alpha^7 + \alpha^9, \\ y_2 &= \alpha^2 + \alpha^3 + \alpha^{10} + \alpha^{11}, \\ y_3 &= \alpha + \alpha^5 + \alpha^8 + \alpha^{12}. \end{cases}$$

It is easy to see that $y_1 + y_2 + y_3 = -1$ and that $y_1y_2 = -1 + y_1$, $y_2y_3 = -1 + y_2$, and $y_3y_1 = -1 + y_3$. Therefore, $y_1y_2 + y_2y_3 + y_3y_1 = -4$. Finally, we can find the product

$$y_1y_2y_3 = (-1 + y_1)y_3 = -y_3 + y_1y - 3 = -y_3 + (-1 + y_3) = -1.$$

Finally, the equation with y_1, y_2, y_3 as its roots is obtained:

$$y^3 + y^2 - 4y + 1 = 0.$$

Solution 389. Let x_1 and x_2 be the roots of $ax^2 + bx + c = 0$. We need $x_1^4 + x_2^4$ and $x_1^4 x_2^4$. For the sum of fourth powers of the roots,

$$\begin{aligned} x_1^4 + x_2^4 &= (x_1 + x_2)^4 - 2x_1x_2(2x_1^2 + 3x_1x_2 + 2x_2^2) \\ &= \frac{b^4}{a^4} - \frac{4b^2c}{a^3} + \frac{2c^2}{a^2}. \end{aligned}$$

Also, $x_1^4 x_2^4 = c^4/a^4$, and the quadratic equation that has x_1^4 and x_2^4 would be

$$a^4x^2 - (b^4 - 4ab^2c + 2a^2c^2)x + c^4 = 0.$$

Solution 390. We need to find the sum of the roots, sum of their pairwise product, and their product. Divide and multiply the sum by $2\sin(\pi/7)$ to find

$$\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}.$$

To find the sum of pairwise product of the roots, use the product to sum trigonometric formulas to obtain

$$\cos\frac{\pi}{7}\cos\frac{3\pi}{7} + \cos\frac{3\pi}{7}\cos\frac{5\pi}{7} + \cos\frac{5\pi}{7}\cos\frac{\pi}{7} = -\frac{1}{2}.$$

It is also not difficult to see that the product of the roots is $-1/8$. So, the cubic equation is

$$8x^3 - 4x^2 - 4x + 1 = 0.$$

Solution 391. Let y be the variable for the required quartic polynomial, so that we can assume $y = x^2$ or $x = \pm\sqrt{y}$ and plug in $x = \sqrt{y}$ in the given equation, rationalize and simplify to find

$$y^4 - 2y^3 + 23y^2 + y + 25 = 0.$$

Solution 392. If we remove $x = 1$ from the roots of $x^n - 1$, we find that $x_2, x_3, \dots, x_n$ would be the roots of

$$x^{n-1} + x^{n-2} + \cdots + x + 1 = 0,$$

which means

$$x^{n-1} + x^{n-2} + \cdots + x + 1 = (x - x_2)(x - x_3)\cdots(x - x_n).$$

Plugging $x = 1$, we find that the answer is n.

Solution 393. The main equation is $(p - p')(p'q - pq') = (q - q')^2$, and the required quadratic equation is

$$(p'q - pq')x^2 + (q^2 - q'^2)x + qq'(p - p') = 0.$$

Solution 394. The common root can be found by subtracting one equation from the other: $x = -(q-q')/(p-p')$. Plugging in this root, we find the relationship between p, q, p', q':

$$(q-q')^3 = (p-p')^2(pq' - p'q).$$

Solution 395. The answer is $m = 2$. Put $x = \alpha$ in $x^2 - (m+2)x + 3 = 0$ and its double $x = 2\alpha$ would be a root of $x^2 - x - m = 0$. So, $4\alpha^2 - 2\alpha - m = 0$ and $\alpha^2 - (m+2)\alpha + 3 = 0$. This will lead to two equations

$$\alpha = \frac{m+12}{4m+6} \quad \text{and} \quad \alpha^2 = \frac{m^2+2m+6}{4m+6}.$$

Conclude that

$$\left(\frac{m+12}{4m+6}\right)^2 = \frac{m^2+2m+6}{4m+6},$$

which leads to $4m^3 + 13m^2 + 12m - 108 = 0$, having only $m = 2$ as a real root.

Solution 397. Once the greatest common factor is found to be $x^2 - x - 1$, it is easy to solve the problem:

$$\begin{cases} (x^2 - x - 1)(2x^2 + x - 1) & = 0, \\ (x^2 - x - 1)(x^2 + 2x - 1) & = 0. \end{cases}$$

The common roots are $\frac{1\pm\sqrt{5}}{2}$ and the first equation has two more roots -1 and $1/2$, whereas the second equation has $-1 \pm \sqrt{2}$.

Solution 398. The final remainder of division of f by the greatest common factor would be $(1-m)x^2 + 3(m-1)x$, which would be zero iff $m = 1$, leading to the cubic equation $x^3 - 4x^2 + x + 2 = 0$ whose roots are the common roots of $f(x) = 0$ and $g(x) = 0$. It is not difficult to simplify the equations to find the roots:

$$\begin{cases} (x-1)(x-3)(x^2 - 3x - 2) & = 0, \\ (x-1)(2x+1)(x^2 - 3x - 2) & = 0. \end{cases}$$

Solution 399. The common roots are $1 \pm \sqrt{2}$.

Solution 400. Let us assume, in hope of reaching a contradiction, that $b^2 - 4ac = d^2$, where d is a non-negative integer. Since a and c are positive, it means $d < b$. Therefore, we can write

$$\begin{aligned} 4a \cdot \overline{abc} &= 4a(100a + 10b + c) = 400a^2 + 40ab + 4ac \\ &= (20a + b)^2 - (b^2 - 4ac) = (20a + b)^2 - d^2 \\ &= (20a + b + d)(20a + b - d). \end{aligned}$$

By Euclid's lemma in number theory if a prime p divides a product xy, then p divides either x or y (or both). Since $\overline{abc}$ is a prime that divides the product $(20a+b+d)(20a+b-d)$, we must have either $\overline{abc} \mid 20a + b + d$ or $\overline{abc} \mid 20a + b - d$. In either case, the coefficient of a in $\overline{abc}$ is 100 whereas it is 20 in both $20a + b \pm d$, and it is trivial that $\overline{abc}$ cannot divide any of those two numbers. This is the contradiction we were looking for, and the proof is complete.

Solution 401. Let α, β, γ be the three roots of $x^3 + px + q = 0$. We can use the well-known identity

$$\alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha) + 3\alpha\beta\gamma.$$

Since the x^2 is missing, the sum of the roots $\alpha + \beta + \gamma$ will be zero, and because the constant term is q, the product of the roots $\alpha\beta\gamma$ will be $-q$. As a result,

$$\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma = -3q,$$

and the sum of cubes of roots $\alpha^3 + \beta^3 + \gamma^3$ is in fact divisible by 3.

Solution 402. We may write $n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2$ in two ways:

1. Expand it as $2(2n^2 + 6n + 7)$. If this equals a multiple of 10, say $10k$, then we need to solve the quadratic $2n^2 + 6n + 7 = 5k$ in n. The determinant of this equation would be an integer only when n leaves a remainder of 1 in division by 5.

2. Complete the square and write the given sum as $(2n + 3)^2 + 5$. If this number is divisible by 10, then the last digit of $2n + 3$ must be 5, that is, $2n + 3 = 10t + 5$ for some non-negative integer t. This also simplifies to the same solution $n = 5t + 1$.

Solution 403. Write the system of equations in order of descending powers of a, b, c:

$$\begin{cases} a^3 - (y+z)a^2 + (x+y)a - (x+z) & = 0, \\ b^3 - (y+z)b^2 + (x+y)b - (x+z) & = 0, \\ c^3 - (y+z)c^2 + (x+y)c - (x+z) & = 0. \end{cases}$$

So, $u = a, b, c$ would be the roots of $u^3 - (y+z)u^2 + (x+y)u - (x+z) = 0$, and Viète's Formulas result in

$$\begin{cases} y + z & = a + b + c, \\ x + y & = ab + bc + ca, \\ z + x & = abc. \end{cases}$$

All three of these equations add up to $2(x + y + z)$, so that we find

$$x + y + z = \frac{1}{2}\left(a + b + c + ab + bc + ca + abc\right),$$

and it is easy to calculate x, y, z by subtracting $y + z, z + x, x + y$ from $x + y + z$:

$$\begin{cases} x & = \frac{1}{2}\left(-a - b - c + ab + bc + ca + abc\right), \\ y & = \frac{1}{2}\left(+a + b + c + ab + bc + ca - abc\right), \\ z & = \frac{1}{2}\left(+a + b + c - ab - bc - ca + abc\right). \end{cases}$$

Solution 404. If x and y are the roots of a quadratic equation, in order to form that equation we need $S = x + y$ and $P = xy$. Then, the given system of equations becomes

$$\begin{cases} S^2 - 2P = \dfrac{7}{3}, \\ S^3 - 3PS = -3. \end{cases}$$

Plug P from the first equation into the second equation to obtain a cubic equation in S: $S^3 - 7S - 6 = 0$, giving $S = -1, -2, 3$.

- When $S = -1$, we have $P = -\dfrac{2}{3}$ and $x, y = \dfrac{-3 \pm \sqrt{33}}{6}$,
- When $S = -2$, we have $P = \dfrac{5}{6}$ and $x, y = \dfrac{-6 \pm \sqrt{6}}{6}$,
- When $S = 3$, we have $P = \dfrac{10}{3}$ and x, y will be roots of $3v^2 - 9v + 10 = 0$ which are imaginary.

Solution 405. If x, y and z are the roots of a cubic equation, in order to form that equation we need $S = x+y+z$, $P = xy+yz+zx$, and $Q = xyz$. Then, the given system of equations becomes

$$\begin{cases} S = x + y + z & = 2a, \\ P = xy + yz + za & = -a^2, \\ Q = xyz & = -2a^3. \end{cases}$$

As a result, $u = x, y, z$ are the roots of the equation $u^3 - 2au^2 - a^2u + 2a^3 = 0$. It is then easy to find $u = \pm a, 2a$.

Solution 407. The answers are:

1. Here, $\lambda = y/x$ could be either $1/2$ or $3/37$, with $\lambda = 1/2$ resulting in $(x, y) = (2, 1), (-2, -1)$ and $\lambda = 3/37$ yielding
$$(x, y) = \left(\pm\frac{37}{2}\sqrt{\frac{7}{779}}, \pm\frac{3}{2}\sqrt{\frac{7}{779}}\right).$$
2. This would lead to $\lambda = 1$ (giving $x = y = 1$) or $\lambda = \frac{-1\pm\sqrt{21}}{5}$ (yielding $x = \frac{5}{\sqrt{170\mp\sqrt{21}}}$).

Solution 408. In case

$$\frac{m-2}{m} \neq \frac{m-1}{2(2m-3)},$$

then the only solution would be $x = y = 0$. However, if the equality is at work, we would have other solutions as well: the equation would be $3m^2 - 13m + 12 = 0$ with roots $m = 3$ and $m = 4/3$. If $m = 3$, then $x + 2y = 0$; and if $m = 4/3$, then $2x - y = 0$.

Solution 409. If $a \neq b$, $b \neq c$, and $c \neq a$, then $x = y = z = 0$. Otherwise, if two are equal, say, $a = b$, we have $x = cz$ and $y = -(a + c)z$, where z can be anything. Finally, in the case when $a = b = c$, we have $z = -(x + ay)/a^2$, where x and y are arbitrary.

Solution 410. The equation containing square roots can be written as

$$\sqrt{A + (x - y - z)} + \sqrt{B + (x - y - 1)} = \sqrt{A} + \sqrt{B},$$

where $A = x^2 + 4x + 3y - 2$ and $B = x^2 + 2y + 3$. It is easy to see that $x - y - 1 = 0$, and the only solution to the system of equations is $(x, y) = (3, 2)$.

Solution 411. Adding and subtracting the two equations, we arrive at another system of equations:

$$(x + y)(x^2 - xy + y^2 - a - b) = 0,$$
$$(x - y)(x^2 + xy + y^2 - a + b) = 0.$$

This can be turned into four systems of equations which are all easy to solve:

a) $\begin{cases} x + y = 0, \\ x - y = 0. \end{cases}$ b) $\begin{cases} x + y = 0, \\ x^2 + xy + y^2 = a - b. \end{cases}$

c) $\begin{cases} x - y = 0, \\ x^2 - xy + y^2 = a + b. \end{cases}$ d) $\begin{cases} x^2 - xy + y^2 = a + b, \\ x^2 + xy + y^2 = a - b. \end{cases}$

Solution 412. The equations are $y^3 = x + 3y$ and $x^3 = 3x - y$. The trivial solution is $x = y = 0$. When $x \neq 0$, we have $y \neq 0$, and assuming $y = tx$, the equations become

$$\frac{y^3}{x^3} = \frac{x + 3y}{3x - y} \implies t^3 = \frac{1 + 3t}{3 - t}.$$

So, we find a quartic equation in t:

$$t^4 - 3t^3 + 3t + 1 = 0,$$

which may be written as a quadratic in $t^2 - 1$:

$$(t^2 - 1)^2 - 3t(t^2 - 1) + 2t^2 = 0.$$

The four solutions for t are $t = 1 \pm \sqrt{2}$ and $t = (1 \pm \sqrt{5})/2$. Since $x^3 = 3x - tx$ but $x \neq 0$, we may find x and y in terms of t: $x = \pm\sqrt{3 - t}$ and $y = \pm t\sqrt{3 - t}$. There are a total of 8 solutions which are all described above.

Solution 415. It is easy to manipulate the given equations and see that $t = p, q, r$ are the roots of

$$t^3 - t \log N + 3 \log N = 0,$$

so that $pq + qr + rp = \log N$ and $pqr = -3 \log N$. Finally, the sum of reciprocals of p, q, r equals $(pq + qr + rp)/pqr = -1/3$.

Solution 416. Write the system as

$$\begin{aligned} xy + (x+y) &= 11, \\ xy(x+y) &= 30. \end{aligned}$$

From here one can find $x + y$ and xy and eventually x and y:

$$(x, y) = (3, 2), (2, 3), (1, 5), (5, 1).$$

Solution 417. The solutions are $(x, y) = (2, 3)$ and $(3, 2)$.

Solution 418. Add three times the second equation to the first equation to find $x + y$. The solutions are $(x, y) = (-1, 2)$ and $(2, -1)$.

Solution 419. Write the first equation as $(x^2+y^2)^2-2x^2y^2 = a^4$ and substitute $x^2+y^2 = b^2 - 2xy$ from the second equation into the first equation. The result will be a quadratic equation in xy:

$$2(xy)^2 - 4b^2 \cdot xy + (b^4 - a^4) = 0.$$

Solution 420. The solutions are

$$(x, y) = (a, 0), (0, a), \left(a\frac{1 \pm i\sqrt{3}}{2}, a\frac{1 \mp i\sqrt{3}}{2}\right).$$

Solution 421. Using the identity

$$x^3 + y^3 + z^3 = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) + 3xyz,$$

we can reduce the equations to

$$x^2 + y^2 + z^2 = 6 \implies xy + yz + zx = -3.$$

Now, $t = x, y, y$ are the roots of $t^3 - 3t - 2 = 0$:

$$(x, y, z) = (2, -1, -1), (-1, 2, -1), (-1, -1, 2).$$

Solution 422. The solutions are $(x, y, z) = (a, 0, 0), (0, a, 0), (0, 0, a)$.

Solution 423. Note that $u = a, b, c, d$ are the four roots of

$$u^4 + tu^3 + zu^2 + yu - x = 0.$$

The relationship between the roots of this polynomial gives us the solutions to the system of equations:

$$\begin{cases} x = -abcd, \\ y = -(abc + abd + bcd + acd), \\ z = ab + ac + ad + bc + bd + cd, \\ t = -(a + b + c + d). \end{cases}$$

Solution 424. Assuming $a, b, c \neq k\pi$, we can divide both sides of the first equation by $\sin a$, second equation by $\sin b$, and third by $\sin c$. As a result, we find that $u = \cos a, \cos b, \cos c$ are the roots of

$$8u^3 - 4zu^2 - 2(y+2)u + (z-x) = 0.$$

This simplifies to

$$\begin{cases} \cos a + \cos b + \cos c = \dfrac{z}{2}, \\ \cos a \cos b + \cos b \cos c + \cos c \cos a = -\dfrac{y+2}{4}, \\ \cos a \cos b \cos c = \dfrac{x-z}{8}. \end{cases}$$

and so the solutions are

$$\begin{cases} x = 8\cos a \cos b \cos c + 2(\cos a + \cos b + \cos c), \\ y = -4(\cos a \cos b + \cos b \cos c + \cos c \cos a) - 2, \\ z = 2(\cos a + \cos b + \cos c). \end{cases}$$

Solution 425. Let $y = mx$ and simplify the system into

$$\begin{cases} b^2x^2 + abmx^2 + a^2m^2x^2 = 3a^2b^2, \\ b^2x^2 + mx^2 - a^2m^2x^2 = ab. \end{cases}$$

Dividing the first equation by the second, we arrive at a quadratic equation in m:

$$a^2(3ab+1)m^2 - 2abm - b^2(3ab-1) = 0.$$

The roots for m are $m = b/a$ and $m = b(1-3ab)/(a(1+3ab))$, so that the final solutions would be

$$(x, y) = (a, b), (-a, -b), \left(\pm a\sqrt{\frac{3ab+1}{3ab-1}}, \pm b\sqrt{\frac{3ab-1}{3ab+1}}\right).$$

Solution 426.

a) Show that $t = x, y, z$ are the roots of $t^3 - 9t^2 + 27t - 27 = (t-3)^2$, so the only solution is $x = y = z = 3$.

b) Subtract the second equation from the first equation, and also subtract the third equation from the second equation and simplify. The solutions are:

$$(x, y, z) = \left(\mp\frac{10\sqrt{3}}{3}, \mp\frac{\sqrt{3}}{3}, \pm\frac{8\sqrt{3}}{3}\right), (\mp 4, \mp 3, \pm 2).$$

c) Show that $xy + yz + zx = 9$ and $xyz = 4$, so that $t = x, y, z$ are the roots of

$$t^3 - 6t^2 + 9t - 4 = 0.$$

The solutions are

$$(x, y, z) = (4, 1, 1), (1, 4, 1), (1, 1, 4).$$

d) The solutions are

$$(x, y, z) = (1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1).$$

e) The trick is to solve the system for a, b, c and assume that x, y, z are given constants. In the system of equations, remove c between the first two equations:

$$(x^2 - yz)a + (y^2 + zx)b = x^3 + y^3 + xz^2 + x^2y.$$

Also, remove c between the first and the third equation in the system:

$$(y^2 + zx)a + (z^2 - xy)b = y^3 + z^3 + x^2z + yz^2.$$

Now, remove b from the two latter equations to obtain

$$y(x^3 + y^3 + z^3 + xyz)a = y(x + y)(x^3 + y^3 + z^3 + xyz).$$

The all-zero solution is obvious: $x = y = z = 0$. Assuming $x^3 + y^3 + z^3 + xyz \neq 0$, we find $a = x + y$. Similarly, we can find $b = y + z$ and $c = z + x$. This means the non-trivial solution would be

$$(x, y, z) = \left(\frac{a + c - b}{2}, \frac{a + b - c}{2}, \frac{c + b - a}{2}\right).$$

f) Let $u = \sqrt[4]{1 + 5x}$ and $v = \sqrt[4]{6 - y}$, so that the equations become $u + v = 3$ and $u^4 + v^4 = 25$, which are easy to solve.

Solution by Amir Parvardi 594. Let a, b, c be the roots of the equation so that $a^2 + b^2 = c^2$. Since these are also the roots of $x^3 - 2p(p+1)x^2 + (p^4 + 4p^3 - 1)x - 3p^3 = 0$, we can use Viète's formulas to obtain

$$\begin{cases} a + b + c & = 2p(p + 1), \\ ab + bc + ca & = p^4 + 4p^3 - 1, \\ abc & = 3p^3. \end{cases}$$

Squaring the first equation and subtracting twice the second equation, we get $a^2+b^2+c^2 = 4p^2(p + 1)^2 - 2(p^4 + 4p^3 - 1)$, and since $a^2 + b^2 = c^2$, we may write $a^2 + b^2 + c^2 = 2c^2$ and simplify the last equation as

$$2c^2 = 2p^4 + 4p^2 + 2,$$

or simply $c^2 = p^4 + 2p^2 + 1$. This simplifies to $c = \pm(p^2 + 1)$, and since c is a side of a triangle, $c = p^2+1$. From here, it is easy to find a and b from equations $a+b = 2p(p+1)-c$ and $ab = 3p^3/c$. Write them out explicitly, having $a \neq b$ in mind:

$$a + b = 2p(p + 1) - (p^2 + 1) = p^2 + 2p - 1 \quad \text{and} \quad ab = \frac{3p^3}{p^2 + 1}.$$

Since a, b, and c are side-lengths of a triangle, the triangle inequality $a+b > c$ must hold: $p^2+2p-1 > p^2+1$, or simply $p > 1$. Moreover, since $(p^2+1)^2 = c^2 = a^2+b^2 = (a+b)^2-2ab$, we find that

$$(p^2 + 1)^2 = (p^2 + 2p - 1)^2 - \frac{6p^3}{p^2 + 1}.$$

This results in $(p^2+1)^3 = (p^2+1)(p^2+2p-1)^2-6p^3$, which simplifies to $-4p^5+6p^3+4p = 0$. Neglecting the trivial $p = 0$ solution, we can divide both sides of the latter equation by p to get $-4p^4 + 6p^2 + 4 = 0$ which factorizes into $-2(p^2 - 2)(2p^2 + 1) = 0$. Therefore, the only solution that satisfies $p > 1$ is $\boxed{p = \sqrt{2}}$.

Solution 595. Let r be the root of the first equation and $f(x)$ be the second polynomial. It is easy to verify $f(\sqrt{r}) = (br + d)(\sqrt{r} + 1)$ and $f(-\sqrt{r}) = (br + d)(-\sqrt{r} + 1)$. Hence, $f(\sqrt{r})f(-\sqrt{r}) \leq 0$. Then there is a root on $[-\sqrt{r}, \sqrt{r}]$ to $f(x) = 0$.

Solution 844. Use $15° = 45° - 30°$ and $22.5° = 45°/2$ and use the appropriate identities.

($\cot x - \cot 2x$) Exercise Identity 845. Using the identities (2.28), (2.23),

$$\frac{1}{\sin 2\theta} - \cot 2\theta = \frac{1}{\sin 2\theta} - \frac{\cos 2\theta}{\sin 2\theta} = \frac{1 - \cos 2\theta}{\sin 2\theta} = \frac{2\sin^2\theta}{\sin 2\theta} = \frac{2\sin^2\theta}{2\sin\theta\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$$

Also,

$$\frac{1}{\sin 2\theta} + \cot 2\theta = \frac{1}{\sin 2\theta} + \frac{\cos 2\theta}{\sin 2\theta} = \frac{1 + \cos 2\theta}{\sin 2\theta} = \frac{2\cos^2\theta}{\sin 2\theta} = \frac{2\cos^2\theta}{2\sin\theta\cos\theta} = \frac{\cos\theta}{\sin\theta} = \cot\theta.$$

Proof of Sum of Three Angles Identity 846. Using the identity for sum of two angles, we can write $x + y + z$ as $x + (y + z)$, so that

$$\begin{aligned} \sin(x + [y + z]) &= \sin x \cos(y + z) + \cos x \sin(y + z) \\ &= \sin x \cdot (\cos y \cos z - \sin y \sin z) + \cos x \cdot (\sin y \cos z + \sin z \cos y) \\ &= \sin x \cos y \cos c + \sin y \cos x \cos z + \sin z \cos x \cos y - \sin x \sin y \sin z, \end{aligned}$$

as required.

Solution 849.

1. Expand $\cos(60° \pm x)$ and arrive at the polynomial expression $4\cos^3 x - 3\cos x$ which equals $\cos 3x$.

2. Expand $\tan(60°+x)$ and $\tan(120°+x)$, arrive at the fraction $(9\tan x - 3\tan^3 x)/(1 - 3\tan^2 x)$, which equals $3\tan 3x$.

Proof of Sine and Cosine Sum-to-Product 850.] Let $a = (x+y)/2$ and $b = (x-y)/2$. By adding and subtracting the identities for $\cos(a+b)$ and $\cos(a-b)$, and also the identities for $\sin(a+b)$ and $\sin(a-b)$, we arrive at the identities in the question.

Proof of Sine and Cosine Product-to-Sum 851.] Let $a = (x+y)/2$ and $b = (x-y)/2$. By adding and subtracting the identities for $\cos(a+b)$ and $\cos(a-b)$, and also the identities for $\sin(a+b)$ and $\sin(a-b)$, we arrive at the identities in the question.

Proof of Tangent and Cotangent Sum-to-Product 852.] Let $a = (x+y)/2$ and $b = (x-y)/2$. By adding and subtracting the identities for $\cot(a+b)$ and $\cot(a-b)$, and also the identities for $\tan(a+b)$ and $\tan(a-b)$, we arrive at the identities in the question.

Solution 853. Start with $A+B = \pi - C$, and use the supplementary angles identity to deduce $\sin C = \sin(A+B)$. Now use the identity $\sin 2x = 2\sin x \cos x$ where $2x = A+B$ to write $\sin(A+B)$ as the product of sine and cosine of $(A+B)/2$. On the other hand, use the sum-to-product identity for sines to write $\sin A + \sin B$ as the product of sine of $(A+B)/2$ and cosine of $(A-B)/2$. Simplify $\sin A + \sin B + \sin C$ using the substitutions for $\sin A + \sin B$ and $\sin C = \sin(A+B)$, also noting that the angles $(A+B)/2$ and $C/2$ are complementary angles (adding up to a right angle), finally arriving at the sum-of-sines to product-of-cosines identity in the triangle ABC:

$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Solution 855. Start with $\cos(B+C) = -\cos A$ and $\sin(B+C) = \sin A$ and finish the calculations.

Solution 862. Start with Jensen's inequality: since $\sin x$ is a concave function in the interval $\left[0, \frac{\pi}{2}\right]$,

$$\frac{\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}}{3} \le \sin\left(\frac{\frac{A}{2} + \frac{B}{2} + \frac{C}{2}}{3}\right) = \sin\frac{A+B+C}{6} = \frac{1}{2}.$$

By AM-GM, we have

$$\sqrt[3]{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \le \frac{\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}}{3} \le \frac{1}{2}.$$

Finally,

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{8}.$$

Solution 864. Start with $\cos A + \cos B$. Using the identity (2.3), let $x = \frac{A+B}{2}$ and $y = \frac{A-B}{2}$,

$$\begin{aligned}\cos A + \cos B &= \cos\left(\frac{A+B}{2} + \frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2} - \frac{A-B}{2}\right) \\ &= \cos(x+y) - \cos(x-y) = 2\cos x \cos y \\ &= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}.\end{aligned}$$

Therefore,

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}. \tag{67}$$

Since $A+B+C = 180^\circ = \pi$, by identities (2.13) and (2.14), we find $\sin C = \sin(A+B)$ and $\cos C = -\cos(A+B)$. Similarly, because $\frac{A+B}{2} + \frac{C}{2} = 90^\circ = \frac{\pi}{2}$, using identities (2.15) and (2.16), we have $\sin\frac{C}{2} = \cos\frac{A+B}{2}$ and $\cos\frac{C}{2} = \sin\frac{A+B}{2}$. Then,

$$\begin{aligned}
\cos A + \cos B + \cos C &= \cos A + \cos B - \cos(A+B) \\
&= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} - \left(\cos^2\frac{A+B}{2} - \sin^2\frac{A+B}{2}\right) \\
&= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} - \left(2\cos^2\frac{A+B}{2} - 1\right) \\
&= 2\cos\frac{A+B}{2}\left(\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right) + 1 \\
&= 2\sin\frac{C}{2}\left(\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right) + 1.
\end{aligned}$$

Similar to equation (67), we can find

$$\cos\frac{A-B}{2} - \cos\frac{A+B}{2} = 2\sin\frac{A}{2}\sin\frac{B}{2}.$$

Hence,

$$\cos A + \cos B + \cos C = 2\sin\frac{C}{2}\left(2\sin\frac{A}{2}\sin\frac{B}{2}\right) + 1 = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} + 1.$$

Solution 866. Use equation (2.61) and identity (2.18) ($\cos x = 1 - 2\sin^2\frac{x}{2}$) to write:

$$\begin{aligned}
\cos A + \cos B + \cos C &= 1 - 2\sin^2\frac{A}{2} + 1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} \\
&= 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.
\end{aligned}$$

Thus,

$$3 - 2\left(\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2}\right) = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$

reduces to

$$\left(\sin\frac{A}{2}\right)^2 + \left(\sin\frac{B}{2}\right)^2 + \left(\sin\frac{C}{2}\right)^2 + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 1,$$

which is exactly what we want.

Solution by Amir Parvardi 867. The magical equations at work here are:

$$\frac{\pi}{16} = \frac{\pi}{2} - \frac{7\pi}{16},$$
$$\frac{3\pi}{16} = \frac{\pi}{2} - \frac{5\pi}{16},$$

We will focus on finding a formula for $\tan^2\theta + \tan^2(90^\circ - \theta)$ keeping in mind that for $\theta \in \{\frac{\pi}{16}, \frac{3\pi}{16}, \frac{5\pi}{16}, \frac{7\pi}{16}\}$, we know $\tan^2 4\theta = 1$. We are looking for an identity which involves 4θ. Using the identities for $\tan(\pi/2 - x)$,

$$\tan^2\theta + \tan^2\left(\frac{\pi}{2} - \theta\right) = \tan^2\theta + \cot^2\theta$$

By equations in Problem 845, we observe

$$\tan^2\theta + \tan^2\left(\frac{\pi}{2} - \theta\right) = \tan^2\theta + \cot^2\theta = \left(\frac{1}{\sin 2\theta} - \cot 2\theta\right)^2 + \left(\frac{1}{\sin 2\theta} + \cot 2\theta\right)^2$$
$$= 2\left(\frac{1}{\sin^2 2\theta} + \cot^2 2\theta\right) = \frac{2}{\sin^2 2\theta} + 2\cot^2 2\theta.$$

Therefore,

$$\tan^2\theta + \tan^2\left(\frac{\pi}{2} - \theta\right) = \frac{2}{\sin^2 2\theta} + 2\cot^2 2\theta.$$

From the identity for $\cos 4x$, we know $\sin^2 2\theta = (1 - \cos 4\theta)/2$ and from the Problem 845, we know $\cot 2\theta = \frac{1}{\sin 4\theta} + \cot 4\theta$. Plugging in,

$$\tan^2\theta + \tan^2\left(\frac{\pi}{2} - \theta\right) = \frac{4}{1 - \cos 4\theta} + 2\left(\frac{1}{\sin 4\theta} + \cot 4\theta\right)^2.$$

Now plug $\theta = \pi/16$ and $\theta = 3\pi/16$ in the above equation to find the final result:

$$\tan^2\frac{\pi}{16} + \tan^2\frac{7\pi}{16} = \frac{4}{1 - \cos\frac{\pi}{4}} + 2\left(\frac{1}{\sin\frac{\pi}{4}} + \cot\frac{\pi}{4}\right)^2 = \frac{4}{1 - \frac{\sqrt{2}}{2}} + 2\left(\frac{2}{\sqrt{2}} + 1\right)^2,$$
$$\tan^2\frac{3\pi}{16} + \tan^2\frac{5\pi}{16} = \frac{4}{1 - \cos\frac{3\pi}{4}} + 2\left(\frac{1}{\sin\frac{3\pi}{4}} + \cot\frac{3\pi}{4}\right)^2 = \frac{4}{1 + \frac{\sqrt{2}}{2}} + 2\left(\frac{2}{\sqrt{2}} - 1\right)^2,$$

Doing the calculations,

$$\frac{4}{1 - \frac{\sqrt{2}}{2}} + 2\left(\frac{2}{\sqrt{2}} + 1\right)^2 = \frac{8}{2 - \sqrt{2}} + 2(\sqrt{2} + 1)^2 = \frac{8(\sqrt{2} + 2)}{4 - 2} + 2(3 + 2\sqrt{2})$$
$$= 4\sqrt{2} + 8 + 6 + 4\sqrt{2} = 14 + 8\sqrt{2},$$
$$\frac{4}{1 + \frac{\sqrt{2}}{2}} + 2\left(\frac{2}{\sqrt{2}} - 1\right)^2 = \frac{8}{2 + \sqrt{2}} + 2(\sqrt{2} - 1)^2 = \frac{8(2 - \sqrt{2})}{4 - 2} + 2(3 - 2\sqrt{2})$$
$$= -4\sqrt{2} + 8 + 6 - 4\sqrt{2} = 14 - 8\sqrt{2}.$$

Finally,

$$\tan^2\frac{\pi}{16}+\tan^2\frac{3\pi}{16}+\tan^2\frac{5\pi}{16}+\tan^2\frac{7\pi}{16}=\left(\tan^2\frac{\pi}{16}+\tan^2\frac{7\pi}{16}\right)+\left(\tan^2\frac{3\pi}{16}+\tan^2\frac{5\pi}{16}\right)$$
$$=(14+8\sqrt{2})+(14-8\sqrt{2})=28.$$

Solution 875. Clearly, $2p=a+b+c$, and by the law of sines, $2p=2R\sin A+2R\sin B+2R\sin C$. Factoring out the R and canceling the factor of 2 everywhere, this leads to $p=R(\sin A+\sin B+\sin C)$, which reminds us of the sum of sines in a triangle, and we can use the expression given in Problem 853 to arrive at the "*Semi-perimeter and Half-angles*" formula:

$$p=4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},$$

Solution 888. Answer: $\angle B=75^\circ$ and $\angle C=45^\circ$.

Calculating the Inradius 889. We know that $S=pr$, and we can use the formula for the semiperimeter in terms of a product containing cosine of half-angles (Problem 875) as well as the formula for the area of triangle containing a product of sine of angles (Problem 876) to write

$$r=\frac{S}{p}=\frac{2R^2\sin A\sin B\sin C}{4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$
$$=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$

where we have used half-angle formulas to write the last line.

Calculating the Exradii 890. Use the fact that $r_a=p\tan\frac{A}{2}$, where p is the semiperimeter and use the semiperimeter formula $p=4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ to arrive at the given formulas.

Solution 891. For the first part, use the exradius formula for r_a, inradius formula for r, and the law of sines for a to simplify the given equation and arrive at

$$K\sin B\sin C=8\cos\frac{B-C}{2}\cos\frac{A}{2}.$$

For the second part, check the roots of the equation

$$K\cos^2\frac{A}{2}-8\cos\frac{B-C}{2}\cos\frac{A}{2}-K\sin^2\frac{B-C}{2}=0,$$

and conclude that $K\cos\frac{B-C}{2}-8>0$.

Solution 905. Use the fact that OH is three times OG.

Solution 932. The answer is

$$\frac{\cos^4 y}{\cos^2 x}+\frac{\sin^4 y}{\sin^2 x}=\frac{\cos^4 x}{\cos^2 y}+\frac{\sin^4 x}{\sin^2 y}=1,$$

because the given equations force $\sin^2 x=\sin^2 y$ and $\cos^2 x=\cos^2 y$.

Solution 933. The answer is $\left(2-e^2\right)^{\frac{3}{2}} = (2-e^p)^{q/p}$, so $p+q=2+3=5$.

Solution 934. The answer is $x=y$. Since $\sec^2\theta$ is the reciprocal of $\cos^2\theta$ which is always less than or equal to 1, we have $\sec^2\theta \geq 1$, and also by AM–GM, $(x+y)^2 \geq 4xy$, so that $4xy/(x+y)^2 \leq 1$ for all x and y. The equality may happen only if it is happening in both $\sec^2\theta \geq 1$ and $4xy/(x+y)^2 \leq 1$, which yields $\sec^2\theta = 1$, and the equality case of AM–GM must hold: $x=y$.

Solution 935. The obvious all-zero answer is $\theta = 2n\pi$ for some integer n. The other solution is

$$\theta = (2n+1)\frac{\pi}{4}.$$

Solution 936. In terms of the given integers m and n, the answers are

$$\theta = \frac{2r\pi}{m+n} \qquad \text{and} \qquad \theta = \frac{(2s+1)\pi}{m-n},$$

for any choice of integers r and s.

Solution 937. The answers are

$$\theta = n\pi - \frac{\pi}{4} \qquad \text{and} \qquad \theta = m\pi + \frac{\pi}{3},$$

for any choice of integers m and n.

Solution 938. The answers are

$$\theta = \frac{n\pi}{3} + \frac{\pi}{12},$$

for any choice of integer n.

Solution 939. The obvious all-zero answers are $\theta = n\pi$ for any integer n, but the non-trivial answer is

$$\theta = n\pi \pm \frac{\pi}{3},$$

for any choice of integer n.

Solution 940. The answers are

$$\theta = 2n\pi + \frac{5\pi}{12} \qquad \text{and} \qquad \theta = 2m\pi - \frac{\pi}{12},$$

for any choice of integers m and n.

1989 Roorkee 941. The correct answer is a), $n\pi$. The equation leads to $\sin^2 x(5\sin^2 x - 2) = 0$ which means either $\sin x = 0$ (thus $\theta = n\pi$) or $\sin x = \pm\sqrt{2/5}$

Solution 942. The correct answer is b), $2k\pi$. The equation is $1-\cos\theta = \sin\theta \cdot \sin\frac{\theta}{2}$, which leads to $2\sin^2\frac{\theta}{2} = 2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}\cdot\sin\frac{\theta}{2}$ which reduces to $2\sin^2\frac{\theta}{2}\cdot(1-\cos\frac{\theta}{2}) = 0$. The solutions are therefore coming from $2\sin^2\frac{\theta}{2} = 0$ and $1-\cos\frac{\theta}{2} = 0$. The first equation means $\sin\frac{\theta}{2} = 0$, which happens for $\theta = 2k\pi$ for any integer k. The second equation, $1-\cos\frac{\theta}{2} = 0$ reduces to $2\sin^2\frac{\theta}{4} = 0$ whose solutions are in the form of $\theta = 4k\pi$ for any integer k. Since the family of solutions $\theta = 4k\pi$ is a subset of the family of solutions $\theta = 2k\pi$, the solutions are precisely $\theta = 2k\pi$ for all $k \in \mathbb{Z}$.

Solution 943. The correct answer is b), $\frac{n\pi}{2}+\frac{\pi}{8}$. After simplification, the given equation becomes $(\sin 2x - \cos 2x)(2\cos x - 3) = 0$, which inevitably results in $\sin 2x = \cos 2x$ or $\tan 2x = 1$, thus $x = \frac{n\pi}{2} + \frac{\pi}{8}$.

Solution 944. The correct answer is a), one solution:

$$x = y = z = \frac{3\pi}{2}.$$

Solution 945. The correct answer is b), because the equation reduces to $\cos 2x(\cos 2x + 1) = 0$, meaning $\cos 2x$ can be either 0 or -1. The solutions to $\cos 2x = 0$ are

$$x = \frac{\pm\pi}{4}, \frac{\pm 3\pi}{4}, \frac{\pm 5\pi}{4}, \dots,$$

whereas the solutions to $\cos 2x = -1$ are

$$x = \frac{\pm\pi}{2}, \frac{\pm 3\pi}{2}, \frac{\pm 5\pi}{2}, \dots,$$

and among these, only $\pm\pi/4$ are in the interval $-\pi < x < \pi$.

Solution 946. The correct answer is d). The equation is equivalent to $(5\cos\theta + 3)(2\cos\theta - 1) = 0$, so that $\cos\theta$ can be $1/2$ or $-3/5$. Therefore, θ can be $\frac{\pi}{3}$, or $\pi - \cos^{-1}\frac{3}{5}$, both of which lie in the interval $-\pi < x < \pi$.

Solution 947. The correct answer is a), $x = \pi/6$, simply because $30 = 3 + 27 = 81^{1/4} + 81^{3/4}$, and we are looking for an angle x such that $\sin^2 x = 1/4$ and $\cos^2 x = 3/4$, the only possibility among the given options being $\dfrac{\pi}{6}$.

Solution 948. The correct answer is c), $x = n\pi/8$. To solve, start with the term $\cos 3x = \cos x(2\cos 2x - 1)$, so that $\cos x + \cos 3x = 2\cos x\cos 2x$. Moreover, it is easy to see that $2\cos 6x\cos x = \cos 5x + \cos 7x$, so that the given expression is $\cos x + \cos 3x + \cos 5x + \cos 7x = 2\cos x\cos 2x + 2\cos 6x\cos x$, which after factoring $\cos x$ becomes $2\cos x(\cos 2x + \cos 6x)$. Finally, $\cos 6x = \cos 2x(2\cos 4x - 1)$, so that $\cos 2x + \cos 6x = 2\cos 2x\cos 4x$. Putting everything together, we find

$$\begin{aligned}\cos x + \cos 3x + \cos 5x + \cos 7x &= 2\cos x(\cos 2x + \cos 6x)\\ &= 2\cos x(2\cos 2x\cos 4x)\\ &= 4\cos x\cos 2x\cos 4x.\end{aligned}$$

On the other hand, note that

$$\begin{aligned}\sin 8x &= 2\sin 4x \cdot \cos 4x\\ &= 4\sin 2x\cos 2x\cos 4x\\ &= 8\sin x\cos x\cos 2x\cos 4x,\end{aligned}$$

and we get the golden identity for this question:

$$\cos x + \cos 3x + \cos 5x + \cos 7x = \frac{\sin 8x}{2\sin x}.$$

Thus the equation $\cos x + \cos 3x + \cos 5x + \cos 7x = 0$ is equivalent to $\sin 8x = 0$, which has solutions $8x = n\pi$ for any integer n.

Solution 949. The correct answer is 949. Divide both sides of the equation by $\sqrt{a^2+b^2}$ to get

$$\frac{a}{\sqrt{a^2+b^2}}\cos x+\frac{b}{\sqrt{a^2+b^2}}\sin x=\frac{c}{\sqrt{a^2+b^2}}.$$

Let $\theta=\cos^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\right)$, so that $\cos\theta=a/\sqrt{a^2+b^2}$ and $\sin\theta=b/\sqrt{a^2+b^2}$. Therefore,

$$\cos\theta\cos x+\sin\theta\sin x=\frac{c}{\sqrt{a^2+b^2}}.$$

The left side is $\cos(x-\theta)$ and if we let $\phi=\cos^{-1}\left(\frac{c}{\sqrt{a^2+b^2}}\right)$, then the equation becomes $\cos(x-\theta)=\cos\phi$, whose solutions are given by $x-\theta=2n\pi\pm\phi$. These can be written as

$$x=2n\pi\pm\phi+\theta,$$

substituting the angles for $\theta=\cos^{-1}a/\sqrt{a^2+b^2}=\tan^{-1}b/a$ and $\phi=\cos^{-1}c/\sqrt{a^2+b^2}$,

$$\begin{aligned}x&=2n\pi\pm\cos^{-1}\left(\frac{c}{\sqrt{a^2+b^2}}\right)+\cos^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\right)\\&=2n\pi\pm\cos^{-1}\left(\frac{c}{\sqrt{a^2+b^2}}\right)+\tan^{-1}\left(\frac{b}{a}\right).\end{aligned}$$

Solution 983. Use telescoping sums and the formula

$$\frac{\sin 2\alpha}{\sin(2k-1)\alpha\sin(2k+1)\alpha}=\cot(2k-1)\alpha-\cot(2k+1)\alpha.$$

Solution 988. Use telescoping sums and the formula

$$\frac{1}{\sin\alpha}=\cot\left(\frac{\alpha}{2}\right)-\cot\alpha.$$

Solution 989. Use telescoping sums and the formula

$$\tan\alpha=\cot\alpha-2\cot 2\alpha.$$

Solution 996. Use the identity $4\sin^3\alpha=3\sin\alpha-\sin 3\alpha$ and multiply the given expression by 4. Then use the formula for sum of sine of angles in an arithmetic progression.

Solution 1024. Use the identity $\tan^2\alpha\tan 2\alpha=\tan 2\alpha-2\tan\alpha$.

Solution 1026. The answer for the first part is

$$f(x)=\frac{x^4}{4}+\frac{x^3}{2}+\frac{x^2}{4}.$$

Put $x=1,2,\ldots,n$ in the equation $f(x)-f(x-1)\equiv x^3$ and sum them up to find

$$\lim_{\alpha\to 0}S_1=1^3+2^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}.$$

The limit of S_2 as α approaches 0 is also the same:

$$\lim_{\alpha\to 0} S_1 = [1+2+\cdots+n]^2 = \left(\frac{n(n+1)}{2}\right)^2.$$

Solution 1028. Answer: the roots of the given equation are

$$4\sin^2\left(\frac{\pi}{9}\right), \quad 4\sin^2\left(\frac{2\pi}{9}\right), \quad 4\sin^2\left(\frac{4\pi}{9}\right).$$

Solution 1029. Answer: the roots of the given equation are

$$\pm\tan\left(\frac{\pi}{5}\right) \quad \text{and} \quad \pm\tan\left(\frac{3\pi}{5}\right).$$

Solution 1030. Answer: the roots of the given equation are $2\cos(k\pi/9)$, where $k = 1, 3, 5, 7$.

Solution by Boxedexe 1158. Let O, A, B, θ, and r be the cone's vertex, cone's base's center, a point on the cone's base's circle, $\angle AOB$, and the radius of the sphere, respectively. Then it follows, from basic trigonometry, that $OA = \frac{r(1+\sin\theta)}{\sin\theta}$ and $AB = \frac{r(1+\sin\theta)}{\cos\theta}$. Hence,

$$V_1 = \frac{\pi AB^2 OA}{3} = \frac{\pi\left(\frac{r(1+\sin\theta)}{\cos\theta}\right)^2\left(\frac{r(1+\sin\theta)}{\sin\theta}\right)}{3},$$

and $V_2 = 2\pi r^3$. Ergo,

$$k = \frac{\pi\left(\frac{r(1+\sin\theta)}{\cos\theta}\right)^2\left(\frac{r(1+\sin\theta)}{\sin\theta}\right)}{6\pi r^3} \implies (1+6k)\sin^2\theta + (2-6k)\sin\theta + 1 = 0.$$

However, the discriminant of the above quadratic equation must be non-negative, that is,

$$(1-3k)^2 \geq 1+6k \implies k \geq \frac{4}{3},$$

and the conclusion follows.

Solution 1159. This solution was written by **randomusername** on AoPS:

a) Let $\mathcal{S}$ be the sphere tangent to the edges from the inside, and let $\mathcal{S}_X$ be the sphere tangent to the edges of the face opposite to vertex X, and the extensions of the edges from X. Consider say $\mathcal{S}$ and $\mathcal{S}_S$. If we cut these spheres by plane ABC, in both cases we get a circle on plane ABC tangent to the segments AB, BC, CA - this is the incircle of $\triangle ABC$. Hence $\mathcal{S}$ and $\mathcal{S}_S$ touches the edges AB, BC, CA in the same points. Now cut the two spheres by plane SAB. We get the incircle and S-excircle of $\triangle SAB$. We have proven that $\mathcal{S}$ and $\mathcal{S}_S$ touch AB in the same point, so the incircle and S-excircle of $\triangle SAB$ touch AB at the same point. This implies that $SA = SB$. Similarly, we get that any two edges are the same length, therefore the tetrahedron is indeed regular.

b) Look at the regular tetrahedron $SABC$. By symmetry, the center O of the regular tetrahedron $SABC$ is of equal distance d from all its edges, so taking $\mathcal{S}(O, d)$ works. Let the incenter of say $\triangle SAB$ be I, the S-excenter of $\triangle SAB$ be J. Consider the homothety with center S that maps I to J. This homothety maps the incircle ω of $\triangle SAB$ to the S-excircle of $\triangle SAB$; since $\omega \subset \mathcal{S}$, we have $\omega' \subset \mathcal{S}'$ and therefore $\mathcal{S}'$ also touches edge AB. By rotational symmetry, $\mathcal{S}'$ also touches edges BC and CA. Moreover, because $\mathcal{S}$ touched edges SA, SB, SC, $\mathcal{S}'$ will touch the extensions of SA, SB, SC. This proves that $\mathcal{S}_S = \mathcal{S}'$ is up for the job. By symmetry, so do $\mathcal{S}_A, \mathcal{S}_B, \mathcal{S}_C$ exist.

Solution by Grobber 1161. This can basically be reduced to a plane geometry problem.

The vertex D is projected onto the orthocenter H of ABC (this follows from the conditions $AB \perp CD,\ AC \perp BD,\ AD \perp BC$). This means that the midpoints U, V, T of DA, DB, DC are projected onto the midpoints X, Y, Z of HA, HB, HC. Let M, N, P be the midpoints of BC, CA, AB. The points M, N, P, X, Y, Z liue on the nine-point circle of ABC, so the points U, V, T lie on a circle of radius equal to $\frac{R}{2}$ (R is the circumradius of ABC) which lies on a plane parallel to ABC and which has its center directly above the nine-point center of ABC. It's is now clear that M, N, P, U, V, T lie on a sphere which has its center in the midpoint of the segment formed by the centers of $(MNP), (UVT)$.

Solution by Grobber 1162. I remember solving this before the crash.[1] The ratio between this volume and the volume of the sphere is equal to the ratio between the area determined by the trihedron on the sphere and the area of the sphere. The region is a spherical triangle with angles α, β, γ, and its area is thus $(\alpha+\beta+\gamma-\pi) \cdot R^2$. The volume we are looking for must then be $\frac{\alpha+\beta+\gamma-\pi}{4\pi} \cdot \frac{4\pi \cdot R^3}{3} = \frac{(\alpha+\beta+\gamma-\pi) \cdot R^3}{3}$.

Solution by Grobber 1163. Let $ABCD$ be the tetrahedron. I, its incenter, is projected on the four planes of the faces in the circumcenters of the faces. Let O_A, O_B, O_C, O_D be the circumcenters of BCD and so on. We clearly have $d(O_D, BC) = d(O_A, BC)$, and therefore, $\angle BAC = \angle BDC$. From this equality and the like we find that the sum of the plane angles around each vertex is π (call this assertion $(*)$). Now take the faces BCD, CAD, ABD and rotate them around BC, CA, AB until they are on the same plane as ABC. It is now easy to see from $(*)$ that ABC, CBD, ADB, DAC are congruent, and the conclusion follows.

Solution by Luis González 1165. Reflection C' of C about N also lies on the equator and since there are infinitely many great circles through C, C', then we deduce that C, C', A and C, C', B lie on a great circle, respectively. $NC = NC'$, $NA = NB$ and $\sphericalangle ANC = \sphericalangle BNC'$ imply that the spherical triangles ANC and BNC' are congruent by SAS criterion. Therefore, N-spherical altitudes of ANC and BNC' are congruent, i.e., N is equidistant from the great circles AC and BC'. Finally, CN bisects the spherical lune formed by the great circles CB and CA.

Solution from Kalva 1166. Clearly, $n = 3$ is certainly possible. For example, take $\angle APB = \angle APC = \angle BPC = 90^\circ$ (so that the lines PA, PB, PC are mutually perpendicular). Then the three planes through P are also mutually perpendicular, so the two sums are both 270°.

[1] Written on December 16, 2004.

We show that $n > 3$ is not possible.

The sum of the n angles APB etc at P is less than $360°$. This is almost obvious. Take another plane which meets the lines PA, PB, PC etc at $A', B', C', \dots$ and so that the foot of the perpendicular from P to the plane lies inside the n-gon $A'B'C' \dots$ then as we move P down the perpendicular the angles $A'PB'$ etc all increase. But when it reaches the plane their sum is $360°$. However, I do not immediately see how to make that rigorous. Instead, take any point O inside the n-gon ABC. We have $\angle PBA + \angle PBC > \angle ABC$.

Adding the n such equations we get $\sum(180° - APB) > \sum ABC = (n-2)180°$. So, $\sum APB < 360°$.

The sum of the n angles between the planes is at least $(n-2) * 180^o$. If we take a sphere center P. Then the lines PA, PB intersect it at $A", B", \dots$ which form a spherical polygon. The angles of this polygon are the angles between the planes. We can divide the polygon into $n-2$ triangles. The angles in a spherical triangle sum to at least $180°$. So the angles in the spherical polygon are at least $(n-2)180°$. So we have $(n-2)180° < 360°$ and hence $n < 4$.

Solution by Luis González 1168. Let $\mathcal{C}_1(r_1), \mathcal{C}_2(r_2)$, (with $r_2 > r_1$) are the incircles of the bases of the truncated pyramid. Let $\mathcal{C}_3(\varrho)$ be the circumcircle of the regular n–gon whose vertices are the tangency points of the subject sphere $\mathcal{E}$ with the lateral faces. Thus, $\mathcal{C}_1(r_1), \mathcal{C}_2(r_2)$ and $\mathcal{C}_3(\varrho)$ are obviously cross sections of a right cone with apex A circumscribed around $\mathcal{E}$. Arbitrary plane through the axis of the cone cuts $\mathcal{E}$ into a circle (I) and the bases $\mathcal{C}_2(r_2), \mathcal{C}_1(r_1)$, into the segments $BC = 2r_2$, $MN = 2r_1$, $(M \in AB$ and $N \in AC) \Longrightarrow BCNM$ is an isosceles trapezoid with incircle (I). If (I) touches AB, AC at D, E, then $DE = 2\varrho$, $NE = r_1$ and $CE = r_2$. Therefore

$$DE = \frac{BC \cdot EN + MN \cdot EC}{NC},$$

so that

$$\varrho = \frac{2r_1r_2}{r_1 + r_2} \ (\star).$$

From the well-known formulae of the areas of regular polygons, we have:

$$\sigma = n\varrho^2 \cos\frac{\pi}{n} \sin\frac{\pi}{n},$$

and also

$$S_1 = n \cdot {r_1}^2 \tan\frac{\pi}{n} \qquad \text{and} \qquad S_2 = n \cdot {r_2}^2 \tan\frac{\pi}{2},$$

which results in

$$S_1S_2 = n^2{r_1}^2{r_2}^2 \tan^2\frac{\pi}{n}.$$

Lateral faces of the truncated pyramid are congruent isosceles trapezoids with altitude $r_1 + r_2$, whose bases are the sides of the n–gons with incircles $\mathcal{C}_1(r_1), \mathcal{C}_2(r_2)$. Therefore,

$$S = \frac{n}{2} \cdot (r_1 + r_2) \cdot \left(2r_1 \tan\frac{\pi}{n} + 2r_2 \tan\frac{\pi}{n}\right) = n(r_1 + r_2)^2 \tan\frac{\pi}{n}.$$

Substituting ϱ, r_1r_2 and $(r_1 + r_2)$ from the latter expressions into $(\star)$ yields:

$$\frac{\sigma}{n \cos\frac{\pi}{n} \sin\frac{\pi}{n}} = \frac{4S_1S_2}{n^2 \tan^2\frac{\pi}{n}} \cdot \frac{n \tan\frac{\pi}{n}}{S} \Longrightarrow \sigma S = 4S_1S_2 \cos^2\frac{\pi}{n}.$$

Solution by Luis González 1172. For the sake of generality, let $\mathcal{O} \equiv (O, R)$ be the circumsphere of the tetrahedron $ABCD$ and $\mathcal{S} \equiv (S, R')$ be the circumsphere of the tetrahedron whose vertices are the centroids A', B', C', D' of its faces against A, B, C, D. Segments AA', BB', CC', CC', DD' concur at the centroid G of $ABCD$, such that G divides them in the same ratio $1:3$, therefore, Tetrahedra $A'B'C'D'$ and $ABCD$ are homothetic through the homothety with center G and coefficient $-\frac{1}{3}$. Therefore, $R' = R/3$ and $OS = 4OG/3$. Let a, b, c, d, e, f be the edges of $ABCD$. By Leibniz theorem for the circumcenter O of $ABCD$, we have

$$\begin{aligned} OG^2 &= \frac{OA^2+OB^2+OC^2+OD^2}{4} - \frac{a^2+b^2+c^2+d^2+e^2+f^2}{16} \\ &= R^2 - \frac{a^2+b^2+c^2+d^2+e^2+f^2}{16}, \end{aligned}$$

and finally,

$$OS^2 = \frac{16R^2}{9} - \frac{a^2+b^2+c^2+d^2+e^2+f^2}{9}.$$

Solution by Spanferkel 1173. Let the radii of S_1 and S_2 be $a > b$. Then by considering a plane through the axis of the cone that intersects one of the solid spheres, say S_3, in a maximal circle, we get by Descartes' theorem for the radius r of S_3:

$$\frac{1}{\sqrt{r}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}},$$

or

$$r = \frac{ab}{(\sqrt{a}+\sqrt{b})^2}.$$

For the (half) opening angle ϕ of the cone, we have

$$\sin\phi = \frac{a-b}{a+b}.$$

Define ψ by

$$\sin\psi = \frac{b-r}{b+r},$$

(this is the corresponding angle intervening between S_2 and S_3).

Let R denote the distance of the center of S_3 from the axis of the cone. Then we have

$$\begin{aligned} R &= (b+r)\sin(\phi+\psi) \\ &= 2\frac{(a-b)\sqrt{br}+(b-r)\sqrt{ab}}{a+b} \\ &= \frac{2ab(a+b+\sqrt{ab})}{(a+b)(\sqrt{a}+\sqrt{b})^2}. \end{aligned}$$

So,

$$\frac{r}{R} = \frac{a+b}{2(a+b+\sqrt{ab})} = \frac{1}{2+\frac{2\sqrt{x}}{1+x}},$$

where $x = b/a$. This function is decreasing in $(0,1)$ with values between $\frac{1}{2}$ and $\frac{1}{3}$. As we want

$$\frac{r}{R} = \sin\frac{\pi}{n},$$

we get $n = 7, 8, 9$. The limit case $n = 6$ (for $x \to 0$) cannot be attained.

Solution by Mij 1174. We assume for the sake of contradiction that all angles determined by any two of the five rays are greater than $90°$.

Consider a sphere centered at O, and let Ox, Oy, Oz, Ot, Or intersect the sphere at x, y, z, t, r respectively. The plane perpendicular to Ox at O cuts the sphere into two hemispheres. Let A be the hemisphere containing x. y cannot lie on A, because if it did, the angle $\angle xOy$ would be less than or equal to $90°$. By the same argument z, t, r cannot lie on A. Define hemispheres B, C, D, and E similarly, to contain y, z, t, r, respectively and have similar properties.

Each of the five hemispheres has a measure of 2π radians. Because of the assumption, any two hemispheres must overlap at a solid angle of less than π radians. Let the intersection of A and B have measure $\theta < \pi$. The union of A and B must have measure $2\pi + 2\pi - \theta = 4\pi - \theta$, and a complement G of measure $4\pi - (4\pi - \theta) = \theta$, and be bounded by two semi-great-circles that meet at two points, p and q. The complement G is contained within what it would be if its measure were its upper bound of π.

The intersection of G and C has a minimum of $\frac{\theta}{2}$ as z approaches either p or q, so it is greater than $\frac{\theta}{2}$. Thus the union of A, B, C has measure greater than

$$(4\pi - \theta) + \frac{\theta}{2} = 4\pi - \frac{\theta}{2},$$

so the complement H of this union has measure less than $\theta/2 < \pi/2$, and is contained within what it would be if z were at p or q.

Finally, H is contained within what it would be if z were at p or q and, since H is a subset of G, within what H would be if θ equals π. This would be a $90°$-$90°$-$90°$ spherical triangle, so H is a subset of a $90°$–$90°$–$90°$ spherical triangle. Since t and r must lie on H, and therefore a $90°$–$90°$–$90°$ spherical triangle, $\angle tOr \le 90°$. We have contradicted our assumption, so we are done.

Solution by Dinoboy 1175. Consider three points A, B, C on a sphere and for a point X denote X' is reflection across the center of the sphere. Then note that a point D satisfies the center of the sphere is in tetrahedron $ABCD$ if and only if D lies in the spherical section bounded by the arcs $A'B', A'C', B'C'$ (let's just call this triangle $A'B'C'$ for simplicity). Thus the problem reduces to finding the expected area of a triangle on a sphere of area 1. But note that every point of the sphere is in exactly one of XYZ for $X \in \{A, A'\}, Y \in \{B, B'\}, Z \in \{C, C'\}$ so it follows the sum of those triangle's areas is 1. Then since we have established each triangle can be uniquely paired with 7 other triangles, it immediately follows the expected area is $1/8$, so we are done.

Solution by Spanferkel and Arqady 1186. Generally, for vectors $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^n$, we can write the "spherical triangle inequality" $\alpha + \beta \ge \gamma$, as

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma \le 1 + 2\cos\alpha\cos\beta\cos\gamma.$$

Equivalently

$$\sum \frac{(\vec{a}\cdot\vec{b})^2}{\vec{a}^2\vec{b}^2} \le 1 + 2\frac{(\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{c})(\vec{c}\cdot\vec{a})}{\vec{a}^2\vec{b}^2\vec{c}^2},$$

or

$$\sum \vec{c}^2(\vec{a}\cdot\vec{b})^2 \le \vec{a}^2\vec{b}^2\vec{c}^2 + 2(\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{c})(\vec{c}\cdot\vec{a}).$$

Solution by Arne 1187. Consider a box inscribed in the sphere with equation $x^2 + y^2 + z^2 = 1$, such that the coordinates of its vertices are $(\pm x, \pm y, \pm z)$. The volume of the box is $8xyz$. Of course the volume of the box is strictly smaller than the volume of the sphere, which is $4\pi/3$.

Solution by Fedor Petrov 1188. The idea of possible geometric solution is that if any four points of our curve lie in the same half-space, then the whole curve also does. (Boundary of half-space contains the centre of a sphere). It follows from Helly theorem. Then we just need to prove that if A, B, C, and D are four points on a sphere such that O lies inside $ABCD$, then the perimetr of $A{-}B{-}C{-}D{-}A$ is at least 2π. Let D' be opposite of D, then D' lies inside spherical triangle ABC, so we have $AD' + CD' < AB + BC$ (the proof is as on the plane). But,

$$AD + DC = 2\pi - (AD' + CD'),$$

so it's exactly what we need.

Solution 1189. The problem reduces to solving $\sqrt{(\sqrt{2}-1)^2 + 1^2 + (z-1)^2} = 2$ for z. The answer is $2 + \sqrt[4]{8}$.

Solution by Grobber 1190. This idea is on the verge of becoming too old, unfortunately...[2]

Consider the circle as the projection of a sphere of radius n on the plane, and in this sphere, consider some spherical segments which are projected onto bands in our circle, each band being contained between two parallel lines, each line a distance 1 from one of our chords.

The hypothesis tells us that the segments cover the sphere (the area of the sphere), so the sum of their areas must be at least $4\pi n^2$. However, the lateral area of a segment with height h is $2\pi nh$, and since all our segments have height ≤ 2, we get the desired conclusion.

Solution by Grobber 1191. In other words, the altitudes are concurrent in a spherical triangle.

Let O be the center of the sphere. We know that OM is contained in the unique plane through OA orthogonal to (OBC), and also in the unique plane through OB orthogonal to (OCA), and we want to prove that it is also contained in the plane through OC which is orthogonal to (OAB). The problem is now this:

Let α, β, γ be three planes through a point O, and let $a = \beta \cap \gamma, b = \gamma \cap \alpha, c = \alpha \cap \beta$. Prove that the planes $\pi_\alpha, \pi_\beta, \pi_\gamma$, passing through a, b, c and orthogonal to α, β, γ respectively, share a point different from O (in other words, they share a line).

Pick $A \in a$, and then $B \in b, C \in c$ such that $AB \perp c, AC \perp b$. It's easy to see that $BC \perp a$ (you can prove this just like you prove the concurrence of the altitudes in a triangle by using scalar products), which means that $\pi_\alpha, \pi_\beta, \pi_\gamma$ cut the triangle ABC along its altitudes. Since they are concurrent, the concurrence point is a point different from O belonging to all π's.

[2] Written on April 17, 2005

Solution by Grobber 1193. Well, since $\angle BOC + \angle COA + \angle AOB \leq 2\pi$, we get that at least one of them is $\leq 120^\circ$, and that's pretty much it.

Solution by Yetti 1194. Let A, B, C be the centers of the 3 congruent spherical balls $(A), (B), (C)$, touching each other and forming an equilateral triangle with the side $AB = 2r = 20$ cm. Let H be the orthocenter of this equilateral triangle. Let D be the center of the sphere (D) of the hemispherical dish centered in the common tangent plane of the spheres $(A), (B), (C)$. The distance of the center A from the vertical axis DH of the hemispherical dish is equal to

$$AH = AB\frac{\sqrt{3}}{3} = \frac{2r\sqrt{3}}{3}.$$

Using Pythagorean theorem for the right angle triangle $\triangle AHD$,

$$DA = \sqrt{DH^2 + AH^2} = \sqrt{r^2 + \frac{4r^2}{3}} = r\sqrt{\frac{7}{3}}.$$

The tangency point T of the spheres (A) and (D) is on their center line DA and the radius of the sphere (D) is

$$\begin{aligned} R = DT = DA + AT &= r\sqrt{\frac{7}{3}} + r \\ &= \left(1 + \sqrt{\frac{7}{3}}\right) r \doteq 2.5275\ r \\ &= 25.275 \text{ cm.} \end{aligned}$$

Solution by Grobber 1195. I think (a) isn't that hard (not even the 3D version).

Let's assume a certain point from the circumference of the planet isn't visible from any point from any of the other planets. We draw the tangent to the circle at that particular point. The semi–plane defined by this tangent, which doesn't contain the circle, can't contain any points from any circles, because otherwise the point would be visible, but this means that the point is on the convex hull, and not inside it, which ends the proof.

We do the same in the case of spheres, by drawing the plane which is tangent to the sphere at the point which we want to show is invisible.

This only defines the set of points which are invisible from any other points: those points which are ON the convex hull, and not INSIDE it. Proving that this region has an area equal to that of each of the spheres is another problem...

An observation: for each sphere which has invisible points, the region which is invisible is a polygon on the sphere; by this I mean a polygon on the sphere, which has arcs of large circles of the sphere as its "sides."

Solution by Spanferkel 1198. Here is a geometrical proof. Let $a, b, c \in \mathbb{R}^n$ be the corresponding vectors. Then we can write the inequality as

$$x^2(a^2b^2 - \langle a, b\rangle^2) \geq a^2\langle b, x\rangle^2.$$

This inequality is equivalent to

$$1 \geq \frac{\langle b, x\rangle^2}{b^2x^2} + \frac{\langle a, b\rangle^2}{a^2b^2} = \cos^2(b, x) + \cos^2(a, b).$$

The condition is $\cos(a, x) = 0$, i.e. a and x are orthogonal. Denote the angles $\alpha := (a, b)$, $\beta := (b, x)$ and $\gamma := (a, x) = \frac{\pi}{2}$. Now,

$$1 \geq \cos^2 \alpha + cos^2\beta \Leftrightarrow \cos(\alpha + \beta)\cos(\alpha - \beta) \leq 0.$$

So, it is clear from the triangle inequalities in the spherical triangle formed by α, β, γ that $\alpha + \beta \geq \gamma = \frac{\pi}{2}$ and

$$-\frac{\pi}{2} \leq \alpha - \beta \leq \frac{\pi}{2}.$$

The result follows.

Solution by Farenhajt 1199. If the sector rotates around AO, we get the spherical sector, which has the volume $V = \frac{2}{3}r^2\pi h$, with h being the height of the corresponding spherical cap, in this case $h = r(1 - \cos\theta)$. On the other hand, if d is the distance from centroid C to the line OA, then,

$$V = \frac{r^2\theta}{2} \cdot 2d\pi \qquad \text{and} \qquad d = \frac{2r(1 - \cos\theta)}{3\theta}.$$

Let C_1 be the foot of the perpendicular from C to AO. Then $OC_1 = d\cot\frac{\theta}{2}$, hence

$$\begin{aligned}
\tan\varphi &= \frac{d}{r - d\cot\frac{\theta}{2}} \\
&= \frac{2r(1 - \cos\theta)}{3\theta} \cdot \frac{3\theta}{3r\theta - 2r(1 - \cos\theta)\cot\frac{\theta}{2}} \\
&= \frac{2(1 - \cos\theta)}{3\theta - 2(1 - \cos\theta)\cot\frac{\theta}{2}} \\
&= \frac{1 - \cos\theta}{\frac{3}{2}\theta - \sin\theta},
\end{aligned}$$

because

$$(1 - \cos\theta)\cot\frac{\theta}{2} = 2\sin^2\frac{\theta}{2}\cot\frac{\theta}{2} = \sin\theta.$$

Therefore,

$$\varphi = \arctan\frac{1 - \cos\theta}{\frac{3}{2}\theta - \sin\theta}.$$

Solution by Yetti 1200. The z-axis cuts the sphere at U, L (for upper and lower) Plane $\mathcal{P}_K$ perpendicular to the z-axis through the sphere center K cuts the segment UL at its midpoint. Therefore, spherical caps cut off by planes $\mathcal{P}_U, \mathcal{P}_L$ perpendicular to the z-axis through U, L are congruent. The $yx-$ and $zx-$planes cut the spherical surface of the 2 caps in identical arc pattern. Since the black-white colors are exchanged on the $xy-$plane cutting the segment UL in its interior, the spherical caps have complementary colors: $S_B(U) = S_W(L)$ (upper cap black surface = lower cap white surface), and $S_W(U) = S_B(L)$ (upper cap white surface = lower cap black surface).

Summing the black color surface of the 2 caps:

$$S_B(U) + S_B(L) = S_W(L) + S_W(U).$$

Sums of black and white color surfaces of the 2 caps are equal.

For the remaining spherical frustum, integrate. R is the sphere radius. Using spherical coordinates (r, φ, θ) with origin at the sphere center. (Cylindrical coordinates (ϱ, φ, z) with the same origin would serve as well). Thus,

$$\theta \in (-\tfrac{\pi}{2}, +\tfrac{\pi}{2}),$$

is angle from the plane $\mathcal{P}_K$ perpendicular to the z-axis through the sphere center. Integration limits of θ are $\pm\theta_m$ to the upper and lower caps. φ is rotation angle around the line parallel to the z-axis through the sphere center. Integration limits of φ for the black color are $(A(\theta), B(\theta))$ and $(C(\theta), D(\theta))$ dependent on θ. These are intersections of the $yz-$ and $zx-$planes with a circle on the sphere at the angle θ.

Integrating the black color surface of the spherical frustum:

$$\begin{aligned} S_B &= R^2 \int_{-\theta_m}^{+\theta_m} \cos\theta \ \mathrm{d}\theta \left(\int_{A(\theta)}^{B(\theta)} \mathrm{d}\varphi + \int_{C(\theta)}^{D(\theta)} \mathrm{d}\varphi \right) \\ &= R^2 \int_{-\theta_m}^{+\theta_m} \cos\theta \ \mathrm{d}\theta \, (\pi) \\ &= 2\pi R^2 \sin\theta_m. \end{aligned}$$

Integrating the white color surface of the spherical frustum:

$$\begin{aligned} S_W &= R^2 \int_{-\theta_m}^{+\theta_m} \cos\theta \ \mathrm{d}\theta \left(\int_{B(\theta)}^{C(\theta)} \mathrm{d}\varphi + \int_{D(\theta)}^{A(\theta)} \mathrm{d}\varphi \right) \\ &= R^2 \int_{-\theta_m}^{+\theta_m} \cos\theta \ \mathrm{d}\theta \, (\pi) \\ &= 2\pi R^2 \sin\theta_m \end{aligned}$$

Integrals of the black and white color surfaces of the spherical frustum are also equal.

Solution by Keyree10 1201. Consider an equilateral triangle ABC such that when A is rotated about BC to hit the sphere again at A', triangles $A'BA$ and $A'BC$ are equilateral. (We shall see later that such a triangle indeed exists) Similarly rotate B about AC with its distance from AC fixed, to hit the sphere at B'. (Ditto for C). Now, we've got a lot of equilateral triangles. Without loss of generality, Let A,B be red and C be blue, so that C' is blue. Since triangles ABA' and BAB' are equilateral, A' and B' must be blue. Then we have $A'B'C'$, a monochromatic equilateral triangle. and the proof is finished.

The required triangle ABC can be found in an Icosahedron inscribed in a sphere. A solution without this knowledge is possible (I think), but it would require a painfully long and rigorous argument to show that such a triangle exists using Intermediate value theorem.

Solution by Luis González 1202. This is a classical construction. Assume we are given a solid sphere $\mathcal{E}$ and a plane π for ruler-compass constructions. Draw a circumference ω with arbitrary radius and center P on $\mathcal{E}$ and take three points A, B, C on ω. Construct the rectilinear triangle $\triangle ABC$ on π by transporting the chords BC, CA, AB with the compass. Let O' be the circumcenter of $\triangle ABC$. Then its circumradius $O'A = \varrho$ is the orthogonal projection of the spherical radius PA on the plane ABC. Therefore, if

P' is the antipode of P on $\mathcal{E}$, the right $\triangle APP'$, given its leg AP and altitude $AO' = \varrho$ on the hypotenuse PP', is constructible. This produces the diameter PP' of the spherical surface $\mathcal{E}$.

Solution by Luis González 1203. Let α, β, γ denote the planes DBC, DAC, DAB. For conveniece, let's cut the trihedron α, β, γ with the spherical surface $\mathcal{E}$ with center D and radius DO. Rays DA, DB, DC cut $\mathcal{E}$ at A', B', C', thus $A'O, B'O, C'O$ become internal angle bisectors of the spherical triangle $A'B'C'$. Great circle (D, DO) cuts $A'C', A'B'$ at M, N such that $\angle OMA' = \angle ONA' = 90°$. Therefore, M, N coincide with the projections of O on $A'C', A'B'$. If L is the projection of O on $B'C'$, then $\angle LOB' = \frac{1}{2}\angle NOL$ and $\angle LOC' = \frac{1}{2}MOL$. Thus $\angle B'OC' = \frac{1}{2}(180°) = 90°$.

Solution by Ocha 1204. Suppose the maximum and minimum latitude of some town is given by points ℓ_M and ℓ_m, then no other town can intersect the band of width $w = |\ell_M \ell_m|$. The surface area of this band is proportional to it's width, i.e. $A = 2\pi r w$, where r is the radius of the planet. If the most easterly and westerly points of the town subtend an angle θ with the center of the sphere, then they chop the latitudinal band into an area of $\frac{\theta}{2\pi}A = r\theta w$. Now the town must be completely within this square(ish) area and no town can enter the latitudinal or longitudinal bands that define the town.

Let $\{w_i\}_{i=1}^N$ be the widths of longitudinal bands made by the towns, and let $\{\theta_i\}_{i=1}^N$ be angles which represent width of the longitudinal bands of the towns. Then $\sum_i w_i \le 2r$ and $\sum_i \theta_i \le 2\pi$ and the area of town i is at most

$$A_i = r\theta_i w_i = \frac{4\pi r^2}{1000}.$$

So by Cauchy-Schwarz:

$$2r \cdot 2\pi \ge \left(\sum_i w_i\right)\left(\sum_i \theta_i\right) \ge \left(\sum_i \sqrt{w_i\theta_i}\right)^2 = \left(N\sqrt{\frac{4\pi r}{1000}}\right)^2.$$

Therefore, $N \le \sqrt{1000}$; so $\max(N) = 31$.

Solution by Facis 1205. Let $v = (\sin x \cos y, \sin x \sin y, \cos x)$. The similarity to spherical coordinates here makes it clear that the only restriction on this vector is that its length is 1. Now, we can write:

$$f(v(x, y)) = (2, 3, 6) \cdot v,$$

where the dot means the dot product. By Cauchy-Schwarz, f is maximized when

$$v = \frac{(2, 3, 6)}{\sqrt{2^2 + 3^2 + 6^2}} = \frac{(2, 3, 6)}{\sqrt{49}} = \left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right),$$

in which case $f(v) = 7$; and f is minimized when

$$v = \left(-\frac{2}{7}, -\frac{3}{7}, -\frac{6}{7}\right),$$

in which case $f(v) = -7$.

Solution by Luis González 1206. Let the circumsphere S have center O and radius R. Then we have

$$\frac{PA}{PA_1}+\frac{PB}{PB_1}+\frac{PC}{PC_1}+\frac{PD}{PD_1}=\frac{PA^2+PB^2+PC^2+PD^2}{p(P,S)}=4.$$

Therefore, we get

$$PA^2+PB^2+PC^2+PD^2=4(R^2-PO^2).$$

Let G be the centroid of $ABCD$ and a,b,c,d,e,f denote its edges. By Leibniz theorem for P,G and O,G we get

$$PA^2+PB^2+PC^2+PD^2=4PG^2+\frac{a^2+b^2+c^2+d^2+e^2+f^2}{4},$$
$$OA^2+OB^2+OC^2+OD^2=4OG^2+\frac{a^2+b^2+c^2+d^2+e^2+f^2}{4}.$$

From these last two, we get

$$PA^2+PB^2+PC^2+PD^2=4(PG^2+R^2-OG^2).$$

Combining the first and the last equations yields:

$$4(PG^2+R^2-OG^2)=4(R^2-PO^2).$$

Thus, $PG^2+PO^2=OG^2$, which means $\angle OPG=90^\circ$. Therefore, locus of P is the spherical surface with diameter $\overline{OG}$.

Solution by Luis González 1207. For convenience let the unit sphere with center O cut the trihedron O_{ABC} into a spherical triangle XYZ with sides α,β,γ. $X\in OA$, $Y\in OB$ and $Z\in OC$. Let h be the spherical altitude issuing from Z. Then $\sin h=\sin\beta\cdot\sin X$, but by cosine theorem in XYZ (Bessel Formula) we have

$$\cos X=\frac{\cos\alpha-\cos\beta\cos\gamma}{\sin\beta\sin\gamma},$$

implying

$$\sin^2 X=1-\left(\frac{\cos\alpha-\cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right)^2,$$

which means

$$\sin(O_{XYZ})=\sin\gamma\cdot\sin h=\sqrt{\sin^2\beta\sin^2\gamma-(\cos\alpha-\cos\beta\cos\gamma)^2}.$$

Substituting $\sin^2\beta=1-\cos^2\beta$ and $\sin^2\gamma=1-\cos^2\gamma$ yields

$$\sin(O_{XYZ})=\sin(O_{ABC})=\sqrt{2\cos\alpha\cos\beta\cos\gamma-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+1}.$$

Since $V=\frac{1}{6}OA\cdot OB\cdot OC\cdot\sin(O_{ABC})$, then it follows that

$$36V^2=OA^2\cdot OB^2\cdot OC^2\cdot(2\cos\alpha\cos\beta\cos\gamma-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+1).$$

P.S. The proof is substantially easier using vectors (scalar product).

Solution by Luis González 1208. Let A be the apex of the pyramid. Denote the vertices of the cyclic n-gon $P_1P_2P_3...P_n$ and the projections of H onto $AP_1, AP_2, AP_3, \ldots, AP_n$ as $H_1, H_2, H_3, \ldots, H_n$. Note that

$$\overline{AH}^2 = \overline{AP_1} \cdot \overline{AH_1} = \overline{AP_2} \cdot \overline{AH_2} = \overline{AP_3} \cdot \overline{AH_3} = \overline{AP_n} \cdot \overline{AH_n}.$$

Thus, the points $H_1, H_2, H_3, \ldots, H_n$ lie on the inverse image of the circumcircle $\mathcal{C}$ of the n-gon, under the inversion with center A and radius $\overline{AH}$. The spherical surface $\mathcal{E}$ passing through $A, \mathcal{C}$ is taken into a plane π' and the plane π containing the base of the pyramid is taken into a spherical surface $\mathcal{E}'$ passing through A. Therefore, $H_1, H_2, H_3, \ldots, H_n$ lie on the intersection (circumference) $\mathcal{C}' \equiv \mathcal{E}' \cap \pi'$. Moreover, $\mathcal{C}$ and $\mathcal{C}'$ lie on a same spherical surface.

Solution by JSGandora 1209. Write

$$f = \frac{md}{m^2-1} = \frac{md}{(m-1)(m+1)}.$$

This can be an integer if the denominator divides the numerator. However, by the Euclidean Algorithm, the denominator is relatively prime to m. Thus $(m-1)(m+1) = m^2-1|d$ so $d = (m^2-1)n, n \in \mathbb{Z}^+$. So, the solution set is

$$(m,d) = (k, nk^2 - n), n \in \mathbb{Z}^+, k \in \mathbb{Z}^+ \backslash \{1\}.$$

Solution by Mavropnevma 1210. This solution was written by Dan Schwarz who was active on AoPS with username **mavropnevma**. May he rest in peace...

a) Let R be the radius of the sphere. Then the area of the k-th disk exposed is

$$\alpha_k = \pi R^2 \left(1 - \left(1 - \frac{2k}{n}\right)^2\right) = 4\pi R^2 \frac{k(n-k)}{n^2}.$$

We are told that α_1 and α_2 are integer. But $d = \gcd(n-1, 2(n-2)) \mid 2$, and clearly also $d \mid k(n-k)$ for all k. By Bézout's relation there exist integers u, v such that $u(n-1) + v(2n-4) = d$, and then, denoting $k(n-k) = md$, we have

$$mu(n-1) + mv(2n-4) = md = k(n-k),$$

hence $mu\alpha_1 + mv\alpha_2 = \alpha_k$ is an integer.

b) The area of the sphere is $A = 4\pi R^2$. We have

$$2\alpha_1 - \alpha_2 = \frac{2}{n^2}A$$

being an integer N, so $A = \dfrac{n^2}{2}N$, while $\alpha_2 = (n-2)N$. Clearly A being integer is equivalent to

$$(n \equiv 0 \pmod 2) \vee (N \equiv 0 \pmod 2),$$

which in turn is equivalent to α_2 being even.

Harun Šiljak's Hint 1211. Find the intersection. Use the cylindrical coordinates to calculate the volume of cone under the intersection, and spherical for the volume of sphere above the intersection. Don't forget your Jacobians! If you're not well into calculations with n-integrals, note that the angular coordinates will "move" freely in their standard bounds, and ρ can "move" in terms of the surface equations, while z in cylindrical coordinates can "move" from 0 to the intersecting value.

Solution 1212. The answer is $2\pi/15$.

Solution by JBL 1213. Surely it should be possible to do with a family of hyperboloids of one sheet, right? Say, take the family of lines given by $x = x_0$, $y = tx_0$ and $z = t$ for $t \in \mathbf{R}$ and then all rotations of these lines around the z-axis.

Solution 1214. The answer is $\tan^2 x + 2\tan^2 y = \boxed{81}$.

Solution 1215. Put $z = \rho\cos\phi, x = \rho\sin\phi\cos\theta$, and $y = \rho\sin\phi\sin\theta$. The answer to the limit, after simplification, is 0.

Solution by Rchokler 1216. Even without converting to spherical coordinates, we know that it is a double cone since it is clear that at each height z, the cross section is a circle of radius $r = \sqrt{3} \cdot |z|$. However, we can see it in spherical coordinates too. After conversion to spherical coordinates, the equation $x^2 + y^2 = 3z^2$ becomes

$$(\rho\cos\theta\sin\phi)^2 + (\rho\sin\theta\sin\phi)^2 = 3(\rho\cos\phi)^2,$$

which simplifies to

$$\tan\phi = \pm\sqrt{3} \implies \phi = \boxed{\frac{\pi}{2} \pm \frac{\pi}{6}}.$$

The $|\tan\phi| = \sqrt{3}$ is actually the key: it basically shows that all points with a given ϕ are on the graph. You're allowed to vary ρ and θ, but ϕ must be those angles. This gives essentially a straight line passing through the origin, and its image as it's rotated around the z–axis. That's a cone.

Solution by WeakMathemetician 1217. Substitute $z = \sin\theta, x = \cos\theta\sin\phi, y = \cos\theta\cos\phi$. Substituting this into the equation gives:

$$\cos^4\theta(\sin^4\phi + \cos^4\phi) - 2\sin^4\theta - 3\sqrt{2}\sin\theta\cos^2\theta\sin\phi\cos\phi.$$

Letting $\sin^4\phi + \cos^4\phi = 1 - 2\sin^2 phi - 2\cos^4\phi$ and also letting $2\sin\phi\cos\phi = \sin\alpha$, where $\alpha = 2\phi$ the equation becomes:

$$\cos^4\theta(1 - \frac{1}{2}\sin^2\alpha) - 2\sin^4\theta - \frac{3\sqrt{2}}{2}\sin\theta\cos^2\theta\sin\alpha.$$

Substituting $\cos^2\theta = 1 - \sin^2\theta$ yields:

$$(1 - \sin^2\theta)^2(1 - \frac{1}{2}\sin^2\alpha) - 2\sin^4\theta - \frac{3\sqrt{2}}{2}\sin\theta(1 - \sin^2\theta)\sin\alpha.$$

Also let $\sin\theta = b$ and $\sin\alpha = a$; then:

$$(1 - b^2)^2(1 - \frac{1}{2}a^2) - 2b^4 - \frac{3\sqrt{2}}{2}b(1 - b^2)a.$$

This is a quadratic in a and rearranges to:

$$\frac{-1}{2}(1-b^2)^2a^2 - \frac{3\sqrt{2}}{2}b(1-b^2)a + (1-2b^2-b^4).$$

Maximizing quadratics is easy and gives that the maxima will happen when:

$$a = \frac{-3\sqrt{2}b}{2(1-b^2)}.$$

When you substitute this value of a into the original quadratic you get the expression:

$$\frac{65}{64} - (b^2 - \frac{1}{8})^2.$$

Solution by Rchokler 1218. We need to use spherical coordinates (ρ, ϕ, θ). Note that $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$.

The Jacobian is then

$$J = \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} = \rho^2\sin\phi.$$

Note that in the problem, $\rho = 20$ is fixed, which makes the Jacobian $400\sin\phi$. Also:

$$TJ = 160000\left(\sin^3\phi\left(\sin^2\theta + \sin 2\theta + 1\right) - 2\sin\theta\sin^2\phi\cos\phi + \sin\phi\cos^2\phi\right).$$

Thus the integral for the thermal flux is:

$$160000\int_0^{\pi}\int_0^{2\pi}\left(\sin^3\phi\left(\sin^2\theta + \sin 2\theta + 1\right) - 2\sin\theta\sin^2\phi\cos\phi + \sin\phi\cos^2\phi\right)\,d\theta\,d\phi,$$

which is equal to

$$160000\int_0^{\pi}(3\pi\sin^3\phi + 2\pi\sin\phi\cos^2\phi)\,d\phi = \frac{2560000\pi}{3}.$$

The surface area is $32000\pi/3$. This makes the average temperature is 80 degrees, quite comfortable for life if the unit is assumed to be Fahrenheit.

Solution by Yenlee 1219. This depends on what you know already / what you are allowed to use. The fastest proof is to just use the Gauss-Bonnet Theorem, but of course the Gauss-Bonnet Theorem is a much more sophisticated theorem. To prove it more directly, you'd almost certainly want to break your polygon into triangles, and then prove it for triangles.

You could probably do this explicitly by using surface integration on the sphere and spherical coordinates: Set up two sides of the triangle along longitudes coming down from the north pole, and figure out how to write the third side in spherical coordinates. But this is probably a mess (I confess I've never done this exercise).

Here's a hint for a more clever proof: Extend the sides of the triangle to great circles on the sphere. These great circles divide the sphere into several regions. You want to find the area of your triangle, which is enclosed by all 3 of the circles. On the other hand, it's easy to figure out the area of any region enclosed by just 2 of these circles. Now see what happens when you add up the areas of the these regions (you'll need to have a good picture in mind of these 3 great circles on the sphere).

Solution by peace09 1221. Let $s = x + y$ and $p = xy$, so that $x^2 + y^2 = s^2 - 2p = 7$ and $x^3 + y^3 = s(s^2 - 3p) = 10$, or, equivalently, $s^2 - 3p = \frac{10}{s}$. Seeking to create elimination in p, we multiply the two equations by 3 and 2 in that order to obtain $3s^2 - 6p = 21$ and $2s^2 - 6p = \frac{20}{s}$, which after subtracting yields $s^2 = 21 - \frac{20}{s}$ or $s^3 - 21s + 20 = 0$. Evidently the LHS is divisible by $s - 1$, and dividing by $s - 1$ gives $s^2 + s - 20$, which factors as $(s + 5)(s - 4)$. Therefore, the solutions to the equation are $s = -5$, $s = 1$, and $s = 4$, of which the largest real value is $\boxed{4}$, the requested answer.

Solution by S. Zhu 1222. We desire the roots of $P(z) = z^6 + z^3 + 1$. Notice that

$$(z^3 - 1)P(z) = z^9 - 1,$$

which just has the ninth roots of unity. However, in multiplying by $(z^3 - 1)$, the extraneous third roots of unity have also been included. Thus, the roots of $P(z)$ are just the ninth roots of unity which are not third roots of unity. The ninth roots of unity are:

$$\cos(\theta) + i \cdot \sin(\theta),$$

for $\theta = (360°/9)\,(n)$ for $n = 0, 1, 2, 3, 4, 5, 6, 7, 8$ and $i = \sqrt{-1}$. The third roots of unity are

$$\cos(\phi) + i \cdot \sin(\phi),$$

for $\phi = (360°/3)\,(m)$ for $m = 0, 1, 2$. The only ninth root of unity which has argument $90° \leq \theta \leq 180°$ which is not a third root of unity is $\cos(160°) + i \cdot \sin(160°)$. Hence, the desired answer is $\boxed{160}$.

Solution by OlympusHero 1225. This means $z_1 + z_2 + z_3 + z_4 + z_5 = 3 + 504i$. Write as points on the real plane, for simplicity. From the y intercept condition, the points are

$$(a, pa + 3), (b, pb + 3), (c, pc + 3), (d, pd + 3), (e, pe + 3).$$

Then $p(a + b + c + d + e) + 3 \cdot 5 = 504$ and $a + b + c + d + e = 3$, so the answer is

$$p = \frac{504 - 3 \cdot 5}{3} = \boxed{163}.$$

Solution by bobthegod78 1226. For simplicity, let $f(x) = ax^{17} + bx^{16} + 1$ and $g(x) = x^2 - x - 1$. Notice that the roots of $g(x)$ are also roots of $f(x)$. Let these roots be u, v. We get the system

$$au^{17} + bu^{16} + 1 = 0,$$
$$av^{17} + bv^{16} + 1 = 0.$$

If we multiply the first equation by v^{16} and the second by u^{16} we get

$$u^{17}v^{16}a + u^{16}v^{16}b + v^{16} = 0,$$
$$u^{16}v^{17}a + u^{16}v^{16}b + u^{16} = 0.$$

Now subtracting, we get

$$a(u^{17}v^{16} - u^{16}v^{17}) = u^{16} - v^{16} \implies a = \frac{u^{16} - v^{16}}{u^{17}v^{16} - u^{16}v^{17}}.$$

By Vieta's, $uv = -1$ so the denominator becomes $u - v$. By difference of squares and dividing out $u - v$ we get

$$a = (u^8 + v^8)(u^4 + v^4)(u^2 + v^2)(u + v).$$

A simple exercise of Vieta's gets us $a = \boxed{987}$.

Solution by joml88 1228. From $k = (a_3a_2a_1a_0)_{-3+i}$ we have

$$\begin{aligned} k &= a_3(-3+i)^3 + a_2(-3+i)^2 + a_1(-3+i) + a_0 \\ &= a_3\left[(-3)^3 + 3(-3)^2 i + 3(-3)i^2 + i^3\right] + a_2\left[(-3)^2 + 2(-3)i + i^2\right] + a_1(-3+i) + a_0 \\ &= (-18+26i)a_3 + (8-6i)a_2 + (-3+i)a_1 + a_0 \\ &= (-18a_3 + 8a_2 - 3a_1 + a_0) + (26a_3 - 6a_2 + a_1)i \end{aligned}$$

The imaginary part is 0, hence

$$26a_3 - 6a_2 + a_1 = 0 \Leftrightarrow 26a_3 = 6a_2 - a_1.$$

Now note that $a_i \in \{0, 1, 2, \ldots, 9\}$. So the RHS of $26a_3 = 6a_2 - a_1$ is at most $6 \cdot 9 = 54$ meaning that a_3 can only be 1 or 2.

If $a_3 = 2$, then $a_2 = 9$ and $a_1 = 2$. So then

$$k = -18(2) + 8(9) - 3(2) + a_0 = 30 + a_0.$$

But a_0 can be anything from 0 to 9. We therefore sum up

$$(30 + 0) + (30 + 1) + \cdots + (30 + 9) = 345,$$

since we're interested in the sum of all k and also to account for every possibility of k.

On the other hand, if $a_3 = 1$ then $a_2 = 5$ and $a_1 = 4$. This gives $k = -18(1) + 8(5) - 3(4) + a_0 = 10 + a_0$. So we again take the sum

$$(10 + 0) + (10 + 1) + \cdots + (10 + 9) = 145.$$

To find the total sum of all possible k, we just take $345 + 145 = \boxed{490}$.

Solution by cobbler 1229. Clearly $z = e^{\frac{2ai\pi}{18}}$ and $w = e^{\frac{2bi\pi}{48}}$, with $a \in \{1, 2, ..., 18\}$ and $b \in \{1, 2, \ldots, 48\}$. So,

$$zw = e^{\frac{2\pi i(8a+3b)}{144}},$$

meaning each zw is a 144^{th} root of unity. Furthermore, by Chicken McNugget, every integer greater than $8 \cdot 3 - 8 - 3 = 13$ can be achieved by $8a + 3b$. Now since adding a multiple of 2π to the numerator doesn't change the value of zw (i.e., roots of unity are periodic with period 2π), and since $8a + 3b$ can achieve $\equiv \{1, 2, \ldots, 144\} \pmod{144}$ (because $8a+3b$ can be any value > 13 from above), it follows that zw can be any distinct 144^{th} root of unity. So, the answer is just $\boxed{144}$.

Solution by 4everwise 1232. For starters, we see that $(a_3 - a_2) - (a_2 - a_1) = a_3 - 2a_2 + a_1 = 1$. Similarly, $a_{n+2} - 2a_{n+1} + a_n = 1$, so that $a_{n+2} = 1 + 2a_{n+1} - a_n$. Now, we let $a = a_1$ and $b = a_2$, so that we're trying to solve for a. Notice that the next few terms are the sequence are $a_3 = 1 + 2b - a$, $a_4 = 3 + 3b - 2a$, etc., where the nth term a_n is

$$a_n = \frac{(n-1)(n-2)}{2} + (n-1)b - (n-2)a.$$

Because $a_{19} = a_{92} = 0$, we get two equations involving a and b. $17a - 18b = 153$ and $90a - 91b = 4095$. Solving, $a = a_1 = \boxed{819}$

Solution by OlympusHero 1236. Let the first two roots be a_1, b_1 and let the other two be c_1, d_1. We have $a_1b_1 = 13 + i, c_1 + d_1 = 3 + 4i$, so $c_1d_1 = 13 - i, a_1 + b_1 = 3 - 4i$. From Vieta's formulas,

$$\begin{aligned} b &= a_1b_1 + a_1c_1 + a_1d_1 + b_1c_1 + b_1d_1 + c_1d_1 \\ &= 26 + (a_1c_1 + a_1d_1 + b_1c_1 + b_1d_1) \\ &= 26 + (a_1 + b_1)(c_1 + d_1) \\ &= 26 + 25 = \boxed{51}. \end{aligned}$$

Solution by Rust and Deedlit 1238.

$$\sum_{n=1}^{\infty} \frac{1}{a_n^3} = \sum_{k=1}^{\infty} \sum_{n=k^2-k+1}^{k^2+k} \frac{1}{a_n^3} = \sum_{k=1}^{\infty} \frac{2k}{k^3} = \frac{\pi^2}{3}.$$

Solution by joml88 1241. The solutions of the equation $z^{1997} = 1$ are the 1997th roots of unity and are equal to

$$\cos\left(\frac{2\pi k}{1997}\right) + i\sin\left(\frac{2\pi k}{1997}\right), \qquad \text{for } k = 0, 1, \ldots, 1996.$$

They are also located at the vertices of a regular 1997–gon that is centered at the origin in the complex plane.

WLOG, let $v = 1$. Then,

$$\begin{aligned} |v+w|^2 &= |\cos\left(\frac{2\pi k}{1997}\right) + i\sin\left(\frac{2\pi k}{1997}\right) + 1|^2 \\ &= \left|\left[\cos\left(\frac{2\pi k}{1997}\right) + 1\right] + i\sin\left(\frac{2\pi k}{1997}\right)\right|^2 \\ &= \cos^2\left(\frac{2\pi k}{1997}\right) + 2\cos\left(\frac{2\pi k}{1997}\right) + 1 + \sin^2\left(\frac{2\pi k}{1997}\right) \\ &= 2 + 2\cos\left(\frac{2\pi k}{1997}\right). \end{aligned}$$

We want $|v+w|^2 \geq 2 + \sqrt{3}$. From what we just obtained, this is equivalent to

$$\cos\left(\frac{2\pi k}{1997}\right) \geq \frac{\sqrt{3}}{2}.$$

This occurs when

$$\frac{\pi}{6} \geq \frac{2\pi k}{1997} \geq -\frac{\pi}{6},$$

which is satisfied by $k = 166, 165, \ldots, -165, -166$ (we don't include 0 because that corresponds to v). So, out of the 1996 possible k, 332 work. Thus, $m/n = 332/1996 = 83/499$. So our answer is $83 + 499 = \boxed{582}$.

Solution by joml88 1251. Let $a = \log_{225} x$ and $b = \log_{64} y$. We know that $\log_x 225 = \frac{1}{\log_{225} x} = \frac{1}{a}$ and $\log_y 64 = \frac{1}{\log_{64} y} = \frac{1}{b}$. The system is thus

$$\begin{aligned} a + b &= 4 \\ \frac{1}{a} - \frac{1}{b} &= 1. \end{aligned}$$

We can solve for a in the first equation, substitute in into the second equation, and get b to be $3 \pm \sqrt{5}$. Using the same procedure again, we find that a is $1 \pm \sqrt{5}$. Therefore

$$x_1 y_1 x_2 y_2 = \left(225^{3+\sqrt{5}}\right)\left(64^{1+\sqrt{5}}\right)\left(225^{3-\sqrt{5}}\right)\left(64^{1-\sqrt{5}}\right) = 30^{12}.$$

Therefore, our answer is $\log_{30} 30^{12} = \boxed{012}$.

Solution by OlympusHero 1252. Note that if we have a negative exponent, we are multiplying by something extremely close to zero, so the result is negligible. This means we only need to consider $10^{2860} + \frac{10}{7} \cdot 10^{858}$. The first of these does not have a decimal, so it does not contribute anything. The second of these has 428571 after the decimal, so the answer is $\boxed{428}$.

Solution by paladin8 1253. After a brilliant mass of algebra, we find that $F(F(F(z))) = z$. Then $z_i = z_j$ if and only if $i \equiv j \pmod 3$, so

$$z_{2002} = z_1 = F(z_0) = \frac{\frac{1}{137} + i + i}{\frac{1}{137} + i - i} = 1 + 274i,$$

so that $a + b = 1 + 274 = \boxed{275}$.

Solution by DottedCalculator 1271. The graph of $p(x, y) = 0$ is a curve of degree 3. Consider the equations $p(x, y) = x(x-1)(2x-3y+2)$ and $p(x, y) = (x-y)(x^2+y^2+xy-1)$. These two equations satisfy the conditions of the problem. Therefore, the point $\left(\frac{a}{c}, \frac{b}{c}\right)$ must be a zero of both equations. Therefore, this point is the intersection of $2x-3y+2 = 0$ and $x^2 + y^2 + xy - 1 = 0$ other than $(-1, 0)$. From the first equation, we get $x = \frac{3}{2}y - 1$, so substituting into the second equation, we get

$$\begin{aligned} \left(\frac{3}{2}y - 1\right)^2 + y^2 + \left(\frac{3}{2}y - 1\right) y - 1 &= 0 \\ \frac{9}{4}y^2 - 3y + 1 + y^2 + \frac{3}{2}y^2 - y - 1 &= 0 \\ \frac{19}{4}y^2 - 4y &= 0. \end{aligned}$$

Since $y \neq 0$, we must have $y = \frac{16}{19}$. Then, $x = \frac{5}{19}$. Now, suppose that the graph of $p(x, y) = 0$ passes through the eight points given in the problem. Then, by Cayley-Bacharach, it must pass through $\left(\frac{5}{19}, \frac{16}{19}\right)$. Therefore, this point works, so $a = 5$, $b = 16$, and $c = 19$, so $a + b + c = \boxed{040}$.

Solution 1289. The answer is $\boxed{\sqrt{5+2\sqrt{3}}}$.

Solution 1290. The answer is $\boxed{4}$. We have $p(x) = (2x^2 - 3)q(x) - 7/2$, where $q(x)$ is either a constant polynomial (two solutions) or a quadratic polynomial (two other solutions).

Solution 1291. The answer is $\boxed{(11-\sqrt{13})/2}$.

Solution by forthegreatergood 1306. Notice the 'main' (non-zero) digits:

$$1 - 7 - 21 - 35 - 35 - 21 - 7 - 1.$$

These are from pascal's triangle! Since there are 2 zeros in between each pairs of numbers we have 1001^7 as the answer. It is a well known fact that $1001 = 7 \cdot 11 \cdot 13$ so we have

$$1007021035035021007001 = \boxed{7^7 \cdot 11^7 \cdot 13^7}.$$

Solution 1309. The answer is $\boxed{660717728292474\,09}$.

Solution 1310. The answer is $\boxed{84}$.

Solution 1311. The answer is $\boxed{18}$.

Solution 1312. Let $x = \cos\theta$ and $y = \sin\theta$ and the equations are $\sin^2\theta + \cos^2\theta = 1$ and $\sin 3\theta \cos 3\theta = 1/2$, or simply $\sin 6\theta = 1$, or $\theta = \pi/12$. Therefore,

$$x + y = \cos\frac{\pi}{12} + \sin\frac{\pi}{12} = \boxed{\frac{\sqrt{6}}{2}}.$$

Solution by Nyash 1326. We want to find the minimal degree m such that there exists a polynomial $p(x)$ of degree m, that when multiplied to the generating function

$$F(x) = \sum_{n=0}^{\infty} f_9(n)x^n = \sum_{n=1}^{9} \frac{1}{1-x^n},$$

yields a polynomial.

Obviously, the denominator of $F(x)$ is at most $\prod_{n=1}^{9} \Phi_n(x)$, where $\Phi_n(x)$ denotes the nth cyclotomic polynomial.

By explicitly writing out the numerator as a product of cyclotomics, we can see that it does not have a root at any of the nth primitive roots of unity, where n is an integer between 1 and 9 (there should probably be a better way to do this).

Since m is at least the degree of the denominator, the minimum value is just

$$\varphi(1) + \varphi(2) + \cdots + \varphi(9) = 8 + (9-1) + (5-1) + (7-1) + 2 = \boxed{28}.$$

Solution 1336. The answer is $-9 + 3\sqrt{3}$.

Solution by pi37 1355. We prove the statement for all odd-degree polynomials P.

First, note that if $x=0$, $y \neq 0$ is a pair of distinct integers satisfying $xP(x)=yP(y)$, then $P(y)=0$, so we're done. Otherwise, assume that there are infinitely many pairs of distinct nonzero integers satisfying the problem statement.

The crucial claim is that for any x, y satisfying $xP(x)=yP(y)$, $|x+y|$ is bounded above. Let

$$P(x)=a_{2n-1}x^{2n-1}+a_{2n-2}x^{2n-2}+\cdots+a_0$$

for $a_i \in \mathbb{Z}$, $a_{2n-1} \neq 0$, $n \geq 1$. Then

$$xP(x)-yP(y)=a_{2n-1}(x^{2n}-y^{2n})+a_{2n-2}(x^{2n-1}-y^{2n-1})+\cdots+a_0(x-y),$$

which gives

$$\begin{aligned} xP(x)-yP(y)=(x-y)(&a_{2n-1}(x+y)(x^{2n-2}+x^{2n-4}y^2+\cdots y^{2n-2}) \\ &+a_{2n-2}(x^{2n-2}+x^{2n-3}y+\cdots y^{2n-2})+\cdots+a_0). \end{aligned}$$

So if $x \neq y$, then

$$a_{2n-1}(x+y)(x^{2n-2}+x^{2n-4}y^2+\cdots y^{2n-2})=-(a_{2n-2}(x^{2n-2}+x^{2n-3}y+\cdots y^{2n-2})+\cdots+a_0)$$

Now since $a_{2n-1} \neq 0$ and $a_{2n-1} \in \mathbb{Z}$, we have $|a_{2n-1}| \geq 1$. Furthermore, $x^{2k}y^{2\ell} \geq 0$ for all $k, \ell \geq 0$, so

$$|x^{2n-2}+x^{2n-4}y^2+\cdots y^{2n-2}| \geq |x^{2n-2}|$$

Now assume WLOG $|x| \geq |y|$. Noting $|x^i| \geq |x^j|$ for $i \geq j$ and using the triangle inequality, we get

$$\begin{aligned} |x+y||x^{2n-2}| &\leq |a_{2n-1}(x+y)(x^{2n-2}+x^{2n-4}y^2+\cdots+y^{2n-2}| \\ &= |a_{2n-2}(x^{2n-2}+x^{2n-3}y+\cdots y^{2n-2})+\cdots+a_0| \\ &\leq |a_{2n-2}(2n-1)x^{2n-2}|+|a_{2n-3}(2n-2)x^{2n-2}|+\cdots+|a_0(1)x^{2n-2}| \\ &\leq ((2n-1)|a_{2n-2}|+(2n-2)|a_{2n-3}|+\cdots+|a_0|)|x^{2n-2}|. \end{aligned}$$

But since $x \neq 0$,

$$|x+y| \leq (2n-1)|a_{2n-2}|+(2n-2)|a_{2n-3}|+\cdots+|a_0|.$$

Thus by the infinite Pigeonhole Principle, there exists an integer c such that there are infinitely many pairs x, y with $x+y=c$ and $xP(x)=yP(y)$. In other words, $xP(x)=(c-x)P(c-x)$ holds for infinitely many x, so it holds identically. If $c \neq 0$, plugging in $x=0$ yields $cP(c)=0$, so $P(c)=0$, as desired. If $c=0$, then $xP(x)=-xP(-x)$, so $P(x)=-P(-x)$, and 0 is a root of P.

Solution by freeman66 1385. Using the identity

$$(r_1+r_2+r_3)^3-(r_1^3+r_2^3+r_3^3)=3(r_1+r_2)(r_2+r_3)(r_3+r_1),$$

and $r_1+r_2+r_3=2019$ (by Vieta's), we get

$$\begin{aligned} 2019^3-(r_1^3+r_2^3+r_3^3) &= 3(2019-r_1)(2019-r_2)(2019-r_3) \\ &= 3(2019^3-2019^2(r_1+r_2+r_3)+2019(r_1r_2+r_2r_3+r_3r_1)-r_1r_2r_3). \end{aligned}$$

Taking mod 3 easily gives us the desired answer, since

$$2019^3-2019^2(r_1+r_2+r_3)+2019(r_1r_2+r_2r_3+r_3r_1)-r_1r_2r_3,$$

is an integer.

Solution by stroller 1406. We claim that $P(x) = x^m$ for some $m \in \mathbb{N}_0$. It's easy to see that such P are solutions.

Now suppose P satisfies the problem conditions. For each $q|P(n)$ for some n let $S_q := \{m : q|P(m), m \in \mathbb{Z}\}$, then

$$m \in S_q \iff m - q \in S_q.$$

For each q such that $S_q \neq \emptyset, S_q \cap \mathbb{Z} \neq \mathbb{Z}$ and $q > 10^{\deg P}$, we contend that S_q consists of only multiples of q. Note that

$$n \in S_q \cap \mathbb{N} \implies \overline{nn} = (10^{s(n)} + 1)n \in S_q \cap \mathbb{N},$$

where $s(n)$ is the number of digits of n. However note that

$$n \in S_q \implies n + kq \in S_q \forall k \implies (10^{s(n+kq)} + 1)(n + kq) \in S_q \implies (10^{s(n+kq)} + 1)n \in S_q.$$

Since $\{s(n + kq) : k \in \mathbb{N}\}$ contains all sufficiently large integers we have from Fermat's Little Theorem that $n(10^t + 1) \in S_q \forall t \in \mathbb{N}$. Now if $n \neq 0$ then 10^t attains at least $\log(q + 1) > \log q \geq \deg P$ values mod q, hence P has $\deg P + 1$ roots $\pmod q$ (namely $n(10^t + 1), t = 0, 1, \ldots, \deg P$), so P must be identically zero $\pmod q$, again a contradiction. Thus $n \in S_q \implies n = 0$, as desired.

Since P cannot be identically zero there exists K such that for all q such that $q > K$, S_q consists of only multiples of q or is empty. Call this $(*)$.

Write $P(x) = x^a Q(x)$ with $Q(0) \neq 0$. Then Q satisfies $(*)$ as well and $Q(0) \neq 0$. However note that $0 \in S_q$ if $S_q \neq$ it follows that the number of primes $q > K$ dividing some value of P is bounded, so P has only finitely many prime divisors, contradiction.

Solution by Assassino9931 1430. By fixing any $n-1$ of the variables and considering the result as a polynomial on a single variable, we see that $x_i - x_j$ divides F for all $i < j$, hence the result. Equality for $n = 3$ holds, e.g., for $(x_1 - x_2)(x_2 - x_3)(x_1 - x_3)$.

Solution by Tintarn 1444. Someone raised the question of whether there is a polynomial P such that both $P(x)$ and $P(x)+1$ are reducible. Quickly, we found $P(x) = x^2 - 1$ as a solution. We then asked whether we can classify all the solutions, but this turns out to be quite complicated. We then went on to ask what happens if we ask for polynomials in several variables. Soon we realized that this question was ill-posed: If we have a solution to the one-variable problem, then we can always still just take $P(x, y) = x^2 - 1$ as a solution (i.e. just let P not depend on the second variable). But even if we force $P(x, y)$ to actually depend on both variables, there are a lot of trivial solutions like $P(x, y) = x^2y^2 - 1$ or more generally $P(x, y) = Q(R(x, y))$ where $Q(x)$ is a solution to the one-variable problem and $R(x, y)$ is arbitrary. We then asked: Do all solutions for the two-variable problem arise in this way from the one-variable problem? And thus, the notion of "secretly one-variable" was born. The way we went on to solve the problem originally was like this: We realized that solutions to the problem correspond to matrices with polynomial entries and determinant 1 since $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$ and so $ad - bc = 1$ implies that we can take $p = bc$ and $p + 1 = ad$. For instance, the solution $P(x) = x^2 - 1$ corresponds to the matrix $\begin{pmatrix} x & x-1 \\ x+1 & x \end{pmatrix}$. (We are thus asking, in fancy terms, for the structure of the group $SL_2(\mathbb{R}[x, y])$.) But from this point of view, there is a natural way to produce new examples: Take two matrices and multiply them. Starting from the "secretly one-variable" solutions, we can therefore construct the new example

$$\begin{pmatrix} x & x-1 \\ x+1 & x \end{pmatrix} \cdot \begin{pmatrix} y & y-1 \\ y+1 & y \end{pmatrix} = \begin{pmatrix} 2xy+y-x-1 & 2xy-x-y \\ 2xy+x+y & 2xy+x-y-1 \end{pmatrix}.$$

And so we get the solution

$$P(x,y) = 4x^2y^2 - x^2 - y^2 - 2xy = (2xy+x+y)(2xy-x-y),$$

with

$$P(x,y)+1 = 4x^2y^2 - x^2 - y^2 - 2xy + 1 = (2xy+x-y-1)(2xy+y-x-1),$$

and all that remains to be checked is that $P(x,y)$ is indeed not secretly one-variable which is a rather boring but not very difficult thing, given that there are not so many possibilities for the degree of Q and R.

Solution 1445. The answer is $\boxed{1260}$.

Solution 1446. The answer is $\boxed{77/15}$.

Solution 1447. The answer is $\boxed{3\sqrt{798}}$.

Solution by peace09 1454. The latter half of the inequality rewrites as

$$3x^2(p(x)-q(x)) \geq p(x)q(x), \ (*)$$

which, by checking degree and sign, forces $\deg p \geq \deg q$ and $\deg q \leq 2$. Before we perform casework on $\deg q$, rearrange the former half of the inequality:

$$5x(p(x)-q(x)) \leq p(x)q(x), \ (\dagger)$$

which implies that $\deg q \geq 1$.

If $\deg q = 1$, $\deg p = 2$ gives $5x^3 + \cdots \leq x^3 + \ldots$ in $(\dagger)$, impossible. So $p(x) = x + a$ and $q(x) = x + b$; then, the original inequality becomes

$$\frac{1}{5x} \geq \frac{1}{x+a} - \frac{1}{x+b} \geq \frac{1}{3x^2}.$$

We can quickly verify that $a, b \neq 0$, which means that $p(1)q(1) \geq 4$ which rules out this case in favor of the next:

If $\deg q = 2$, $\deg p = 2$ gives $3cx^2 + \cdots \geq x^4 + \ldots$ in $(*)$, impossible; so $\deg p \geq 3$. Now, the key is that it is possible to have p or q contain a single term x^n, which beats the established bound of $p(1)q(1) \geq 4$ otherwise.

First $q(x) = x^2$ fails, because $\frac{1}{p(x)} - \frac{1}{q(x)} \geq \frac{1}{4} - \frac{1}{8} > \frac{1}{10}$ in that case.

Next $p(x) = x^n$ gives $n = 5$ and $q(x) = x^2 + 2x$ after eliminating cases $n = 3, 4$. So $p(1)q(1) \geq \boxed{3}$, and the aforementioned equality case can easily be verified directly.

Solution by gracemoon124 1455. Notice that it may be factored as

$$x^8(x^2+x+1)+x^4(x^2+x+1)+(x^2+x+1)=(x^2+x+1)(x^8+x^4+1).$$

x^8+x^4+1 may be broken down as $(x^4+1)^2-x^4=(x^4+x^2+1)(x^4-x^2+1)=(x^2+x+1)(x^2-x+1)(x^4-x^2+1)$. It only remains to factor x^4-x^2+1, but this may be done using the same method:

$$x^4-x^2+1=(x^2+1)^2-3x^2=(x^2+x\sqrt{3}+1)(x^2-x\sqrt{3}+1).$$

The roots are therefore $e^{\pi i/6}$, $e^{\pi i/3}$, $e^{2\pi i/3}$, $e^{5\pi i/6}$, and their conjugates. This forms an octagon on the unit circle. Split into a rectangle and two trapezoids, to obtain the area as

$$\sqrt{3}+\left(\frac{\sqrt{3}-1}{2}\right)(\sqrt{3}+1)=\boxed{\sqrt{3}+1}.$$

Solution by David Altizio 1460. The Triangle Inequality applied to the right hand side yields

$$\begin{aligned}|z|^n|z-1| &\geqslant |z^{n+1}-1|+|z^{n+1}-z|\\ &=|z^{n+1}-1|+|z-z^{n+1}|\geqslant |z-1|. \qquad (\dagger)\end{aligned}$$

This inequality is true when either $|z-1|=0$ or $|z|^n\geqslant 1$; either way, $|z|\geqslant 1$. Analogously, the Triangle Inequality applied to the left hand side yields

$$\begin{aligned}|z^{n+1}-1|+|z^{n+1}-z| &\leqslant |z^{n+1}-z^n|\\ &\leqslant |z^{n+1}-1|+|z^n-1|,\end{aligned}$$

so $|z||z^n-1|\leqslant|z^n-1|$. This inequality is true when either $|z^n-1|=0$ or $|z|\leqslant 1$; either way, $|z|\leqslant 1$. Combining both inequalities together yields $|z|=1$.

In particular, the inequality ($\dagger$) must be an equality, so 1, z, and z^{n+1} are collinear. But all three complex numbers have magnitude 1, so this is only possible when two of them are equal to each other.

It follows that z is either an n^{th} root of unity or an $(n+1)^{\text{st}}$ root of unity. These solutions work when plugged back in.

Solution by joml88 1470. Note that

$$\begin{aligned}\cot(\alpha+\beta) &= \frac{1}{\tan(\alpha+\beta)}\\ &= \frac{1-\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}\\ &= \frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}.\end{aligned}$$

Let $x=\cot^{-1}3+\cot^{-1}7$ and $y=\cot^{-1}13+\cot^{-1}21$. Thus

$$\begin{aligned}\cot x &= \frac{\cot(\cot^{-1}3)\cot(\cot^{-1}7)-1}{\cot(\cot^{-1}3)+\cot(\cot^{-1}7)}\\ &= \frac{3\cdot 7-1}{3+7}\\ &= 2,\end{aligned}$$

and

$$\begin{aligned}\cot y &= \frac{\cot(\cot^{-1}13)\cot(\cot^{-1}21)-1}{\cot(\cot^{-1}13)+\cot(\cot^{-1}21)}\\ &= \frac{13\cdot 21-1}{13+21}\\ &= 8\end{aligned}$$

So then,

$$\begin{aligned}\cot(x+y) &= \frac{\cot x\cot y-1}{\cot x+\cot y}\\ &= \frac{2\cdot 8-1}{2+8}\\ &= \frac{3}{2}\end{aligned}$$

So the answer is therefore $\boxed{15}$.

Solution by OlympusHero 1471. Rewrite as

$$\frac{\frac{\cos\gamma}{\sin\gamma}}{\frac{\cos\alpha}{\sin\alpha}+\frac{\cos\beta}{\sin\beta}},$$

or

$$\frac{\frac{\cos\gamma}{\sin\gamma}}{\frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}}=\frac{\cos\gamma\sin\alpha\sin\beta}{\sin(\alpha+\beta)\sin\gamma},$$

or $\cos\gamma\sin\alpha\sin\beta/\sin^2\gamma$. From Law of Cosines on our original triangle, we have

$$1989c^2-2ab\cos\gamma=c^2,$$

or $ab\cos\gamma=994c^2$. But

$$\frac{\sin\alpha\sin\beta}{\sin^2\gamma}=\frac{ab}{c^2}=\frac{994}{\cos\gamma},$$

so the answer is $\cos\gamma\cdot\frac{994}{\cos\gamma}=\boxed{994}$.

Solution by chess64 1472. The range of $\sin(x)$ is $[-1,1]$. Hence, we only need to consider $\left|\frac{1}{5}\log_2 x\right|\le 1$. This is satisfied for $\frac{1}{32}\le x\le 32$. First let's consider $\frac{1}{32}\le x<1$. In this interval, the left hand side is negative while the right hand side is negative only in $[1/5,2/5]$ and $[3/5,4/5]$, so there are 4 solutions.

When $1<x\le 32$, the left hand side is positive, and the right hand side is positive only in

$$[6/5,7/5],[8/5,9/5],\ldots,[158/5,169/5].$$

There are then 2 points of intersection in each of these intervals, and there are 77 intervals.

When $x=1$ both sides of the equation are 0, so in all we have $4+77\cdot 2+1=\boxed{159}$ solutions.

Solution by chess64 1473. Recall that $\sec^2 x - \tan^2 x = 1$, from which we find that $\sec x - \tan x = 7/22$. Adding the equations

$$\begin{aligned} \sec x + \tan x &= 22/7 \\ \sec x - \tan x &= 7/22 \end{aligned}$$

together and dividing by 2 gives $\sec x = 533/308$, and subtracting the equations and dividing by 2 gives $\tan x = 435/308$. Hence, $\cos x = 308/533$ and $\sin x = \tan x \cos x = (435/308)(308/533) = 435/533$. Thus, $\csc x = 533/435$ and $\cot x = 308/435$. Finally,

$$\csc x + \cot x = \frac{841}{435} = \frac{29}{15},$$

so, $m + n = \boxed{44}$.

Solution by Qiaochu Yuan 1476. Interpreted geometrically, we have a line $ax+by = 1$ intersecting a circle of radius $\sqrt{50}$ centered at the origin. It can either intersect tangent to a lattice point or secant to two lattice points, and the lattice points are $(x, y) = (\pm 5, \pm 5), (\pm 1 \pm 7), (\pm 7, \pm 1)$ for a total of 12 lattice points.

There are 12 tangent solutions, one for each point. A secant solution can be found by choosing any one of the twelve and drawing the line connecting that point and any of the other 11 points except the point diametrically opposite. The line cannot pass through the origin (I didn't realize this before) because then $a(0) + b(0) = 1$, which does not have real solutions.

Hence, the number of solutions really is $12 + \frac{12(10)}{2} = \boxed{72}$.

Solution by joml88 1477. Take the log base 1995 of both sides

$$\begin{aligned} \log_{1995}\left(\sqrt{1995}x^{\log_{1995} x}\right) &= \log_{1995}(x^2) \\ \log_{1995} 1995^{1/2} + (\log_{1995} x)^2 &= 2\log_{1995} x \\ 2(\log_{1995} x)^2 - 4\log_{1995} x + 1 &= 0, \end{aligned}$$

so that

$$\log_{1995} x = \frac{2 \pm \sqrt{2}}{2}.$$

Both $(2 + \sqrt{2})/2$ and $(2 - \sqrt{2})/2$ are positive, so they both produce a solution. So then $x_1 = 1995^{(2+\sqrt{2})/2}$ and $x_2 = 1995^{(2-\sqrt{2})/2}$ where x_1, x_2 are solutions to the original equation. The product x_1x_2 is therefore equal to 1995^2. We only want the last three digits, which is the same as taking it mod 1000. So, our answer is

$$1995^2 \equiv (-5)^2 \equiv \boxed{025} \pmod{1000}.$$

Solution by joml88 1483. When $x \geq 0$, $|x| = x$. This gives $y^2 + 2xy + 40x = 400 \Rightarrow 2x(y + 20) = (20 + y)(20 - y)$. Thus either $y + 20 = 0 \Rightarrow y = -20$ or $2x = 20 - y \Rightarrow y = -2x + 20$. This produces two rays.

When $x < 0$, $|x| = -x$. This gives $y^2 + 2xy - 40x = 400 \Rightarrow 2x(20 - y) = -(20 + y)(20 - y)$. So either $y = 20$ or $y = -2x - 20$. This produces two more rays.

Graphing these four rays in the coordinate plane makes a parallelogram with base 20, height 40, and, therefore, area 800.

Solution by JesusFreak197 1494. It moves 5 to the side and 7 up, and we must end up at another vertex. Every time it reflects, it will move 5 units to the side and 7 units either up or down. If it hits a face partway through a reflection, it will continue the remaining number of units in the opposite direction. We first have that both $5x$ and $7x$ are multiples of 12 for $x = 12$. The length AP is $\sqrt{5^2+7^2+12^2} = \sqrt{218}$, so we multiply that by 12, and we have $12\sqrt{218} \Rightarrow 12 + 218 = \boxed{230}$.

Solution by RminusQ 1538. Warning: This solution is brute force and ugly.

Set this triangle to the plane, $A = (0,0), B = (15,0)$. It can be seen $C = (33/5, 56/5)$. Then $M = (33/10, 28/5), N = (15/2, 0), D = (99/29, 168/29), E = (65/9, 0)$, the last two by angle bisector theorem.

Now, we know nothing about circumcircles, so we set about finding X, the center of AMN, and Y, the center of ADE. Then let $X = (p,q)$, and $Y = (r,s)$. We can find $p = 15/4$ by inspection of A and N; similarly $r = 65/18$ by looking at A and E. Now,

$$\left(\frac{15}{4}\right)^2 + q^2 = \left(\frac{9}{20}\right)^2 + \left(\frac{28}{5} - q\right)^2,$$

which leads to $14.0625 = .2025 + 31.36 - 11.2q$ and

$$q = \frac{17.5}{11.2} = \frac{25}{16}.$$

Similarly,

$$\left(\frac{65}{18}\right)^2 + s^2 = \left(\frac{99}{29} - \frac{65}{18}\right)^2 + \left(\frac{168}{29} - s\right)^2,$$

which leads to

$$s = \frac{1235}{696},$$

(no that doesn't reduce).

So our circles are $(x-p)^2 + (y-q)^2 = p^2 + q^2$ and $(x-r)^2 + (y-s)^2 = r^2 + s^2$ which reduce to $x^2 + y^2 = 2px + 2qy$ and $x^2 + y^2 = 2rx + 2sy$. Setting these equal, we get $px + qy = rx + sy$. Plugging in and multiplying through by 2088 to clear fractions, we arrive at

$$15660x + 6525y = 15080x + 7410y,$$

or $580x = 885y$. Thus, whatever point P is, ray AP is the first quadrant part of the line

$$y = \frac{580}{885}x = \frac{116}{177}x.$$

Solving for the equation of that ray and BC, we get the y coordinate of that intersection is

$$\frac{145}{22} = \frac{725}{110}.$$

That's the vertical distance between Q and B; the vertical distance between Q and C is thus

$$\frac{507}{110},$$

and our desired ratio is $725/507$, giving an answer of $725 - 507 = \boxed{218}$.

Solution 1548. The answer is $\boxed{2011^{2008}(1005^{2011} - 1004^{2011})}$.

Solution by jaymuro 1553. The sum of the Saturday distances is

$$|z^3 - z| + |z^5 - z^3| + \cdots + |z^{2013} - z^{2011}| = \sqrt{2012}.$$

The sum of the Sunday distances is

$$|z^2 - 1| + |z^4 - z^2| + \cdots + |z^{2012} - z^{2010}| = \sqrt{2012}.$$

Note that

$$|z^3 - z| + |z^5 - z^3| + \cdots + |z^{2013} - z^{2011}| = |z|(|z^2 - 1| + |z^4 - z^2| + \cdots + |z^{2012} - z^{2010}|),$$

so $|z| = 1$. Then

$$\begin{aligned} |z^2 - 1| + |z^4 - z^2| + \cdots + |z^{2012} - z^{2010}| &= |z^2 - 1| + |z^2||z^2 - 1| + \cdots + |z^{2010}||z^2 - 1| \\ &= |z^2 - 1| + |z|^2|z^2 - 1| + \cdots + |z|^{2010}|z^2 - 1| \\ &= 1006|z^2 - 1|, \end{aligned}$$

so,

$$|z^2 - 1| = \frac{\sqrt{2012}}{1006}.$$

We have that $|z^2| = |z|^2 = 1$. Let $z^2 = a + bi$, where a and b are real numbers, so $a^2 + b^2 = 1$. From the equation $|z^2 - 1| = \sqrt{2012}/1006$,

$$(a - 1)^2 + b^2 = \frac{2012}{1006^2} = \frac{1}{503}.$$

Subtracting these equations, we get

$$2a - 1 = 1 - \frac{1}{503} = \frac{502}{503},$$

so

$$a = \boxed{\frac{1005}{1006}}.$$

Solution by peelybonehead 1562. Let $BD = x$ and $AE = ED = y$. Thus, $CD = 3x$ and $AC = 4x$. Noting that $\cos\angle ADC = -\cos\angle ADB$, applying Law of Cosines to angles $\angle ADC$ and $\angle ADB$ gives

$$7 = 9x^2 + y^2 - 6xy\cos\angle ADC$$

$$9 = x^2 + y^2 + 2xy\cos\angle ADC.$$

Multiplying both sides of the second equation by 3 and adding both equations together gives $34x^2 = 12 + 4y^2$. Now, noting that $\cos\angle CED = -\cos\angle AEC$, applying Law of Cosines to angles $\angle CED$ and $\angle AEC$ gives

$$9x^2 = 7 + y^2 - 2\sqrt{7}y\cos\angle CED$$

$$16x^2 = 7 + y^2 + 2\sqrt{7}y\cos\angle CED.$$

Adding the two equations, we get $25x^2 = 14 + 2y^2$. Next, subtracting $34x^2 = 12 + 4y^2$ and $50x^2 = 28 + 4y^2$ and solving for the positive value of x gives $x = 1$. Since we know that $x = 1$, $CD = 4$ and $AC = 4$. Applying Stewart's Theorem to triangle ACD with median $\overline{CE}$, we get

$$2y^2 + 14y = 16y + 9y$$

which can be solved to get $y = \frac{\sqrt{22}}{2}$. We now apply Law of Cosines on angles $\angle ADB$ and $\angle ADC$, which gives

$$16 = 22 + 9 - 6\sqrt{22}\cos\angle ADC$$
$$AB^2 = 22 + 1 + 2\sqrt{22}\cos\angle ADC$$

Multiplying both sides of the second equation by 3 and adding both equations together gives $3AB^2 = 84 \implies AB = 2\sqrt{7}$. Since $AC = BC$, $\triangle ABC$ is an isosceles triangle and thus $h_c = \sqrt{4^2 - 7} = 3$. Therefore,

$$[ABC] = \frac{3 \cdot 2\sqrt{7}}{2} = 3\sqrt{7}.$$

Thus, the answer is $\boxed{10}$.

Solution by peelybonehead 1563. We use the identity

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1.$$

Note that

$$\cos A\cos B - \sin A\sin B = \cos(A + B) = \cos(180^\circ - C) = -\cos C.$$

To get rid of the $2\sin A\sin B\cos C$ term on the left hand side and apply the identity mentioned above, we add $\cos^2 C + 2\cos A\cos B\cos C - 2\sin A\sin B\cos C$ to both sides of

$$\cos^2 A + \cos^2 B + 2\sin A\sin B\cos C = \frac{15}{8}.$$

Since $\cos^2 C + 2\cos A\cos B\cos C - 2\sin A\sin B\cos C$ is equal to

$$\cos^2 C + 2\cos C(\cos A\cos B\cos C - \sin A\sin) = \cos^2 C - 2\cos^2 C,$$

adding it to both sides gives

$$1 = \frac{15}{8} - \cos^2 C.$$

Thus, $\cos C = \sqrt{7/8}$, and therefore $\sin C = \sqrt{\frac{1}{8}}$. Similarly, we add

$$\cos^2 A + 2\cos A\cos B\cos C - 2\sin B\sin C\cos A$$

to both sides of

$$\cos^2 B + \cos^2 C + 2\sin B\sin C\cos A = \frac{14}{9}$$

to get

$$1 = \frac{14}{9} - \cos^2 C.$$

Thus, $\cos A = \sqrt{5/9}$ and $\sin A = 2/3$. Let our desired value be x. Adding

$$\cos^2 B + 2\cos A\cos B\cos C - 2\sin C\sin A\cos B$$

to both sides of

$$\cos^2 C + \cos^2 A + 2\sin C \sin A \cos B = x,$$

we get

$$1 = x - \cos^2 B.$$

We can now solve for x using our findings from above:

$$\begin{aligned}
x &= \cos^2 B + 1 \\
&= \cos^2(A+C) + 1 \\
&= (\cos A \cos C - \sin A \sin C)^2 + 1 \\
&= \left(\sqrt{\frac{5}{9}} \cdot \sqrt{\frac{7}{8}} - \frac{2}{3} \cdot \sqrt{\frac{7}{8}}\right)^2 + 1 \\
&= \left(\frac{\sqrt{35} - 2}{\sqrt{72}}\right)^2 + 1 \\
&= \frac{35 + 4 - 2\sqrt{140}}{72} + 1 \\
&= \frac{111 - 4\sqrt{35}}{72}.
\end{aligned}$$

Therefore,

$$x = \frac{111 - 4\sqrt{35}}{72},$$

and the answer is $p + q + r + s = \boxed{222}$.

Solution 1565. The answer is equal to the coefficient of x^{168} in the generating function

$$(x^{-168} + x^{-167} + \cdots + x^{167} + x^{168})^4 = \frac{(x^{169} - x^{-169})^4}{(x-1)^4},$$

which is $\boxed{761474}$.

Solution by TheUltimate123 1571. Let N be the final number; it is sufficient to show

$$N^{1/L} \stackrel{?}{\geq} \frac{c^{n/L} - 1}{c^{1/L} - 1} = 1^{1/L} + c^{1/L} + \left(c^2\right)^{1/L} + \cdots + \left(c^{n-1}\right)^{1/L}.$$

Thus we only need the following monovariant:

Claim. For any a, b, we have $\left(ca + c^2 b\right)^{1/L} \geq a^{1/L} + b^{1/L}$.

By Hölder's inequality,

$$\left[\left(c^{1/L}a^{1/L}\right)^L + \left(c^{2/L}b^{1/L}\right)^L\right]^{1/L} \left[\left(c^{-1/L}\right)^{L/(L-1)} \left(c^{-2/L}\right)^{L/(L-1)}\right]^{(L-1)/L} \geq a^{1/L} + b^{1/L}.$$

However $c = \phi^{L-1}$ and $1 = \phi^{-1} + \phi^{-2}$, so

$$\left(c^{-1/L}\right)^{L/(L-1)} + \left(c^{-2/L}\right)^{L/(L-1)} = c^{-1/(L-1)} + c^{-2/(L-1)} = 1,$$

and we are done.

Solution by shinichiman 1583. First, we prove that

$$\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}=\frac{1}{8}.$$

We have

$$\begin{aligned}-\sin\frac{\pi}{7}=\sin\frac{8\pi}{7}&=2\cos\frac{4\pi}{7}\sin\frac{4\pi}{7},\\ &=4\cos\frac{4\pi}{7}\cos\frac{2\pi}{7}\sin\frac{2\pi}{7},\\ &=8\cos\frac{4\pi}{7}\cos\frac{2\pi}{7}\cos\frac{\pi}{7}\sin\frac{\pi}{7}.\end{aligned}$$

Therefore,

$$\cos\frac{4\pi}{7}\cos\frac{2\pi}{7}\cos\frac{\pi}{7}=\frac{-1}{8}.$$

Since

$$\cos\frac{4\pi}{7}=-\cos\frac{3\pi}{7},$$

so

$$\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}=\frac{1}{8},$$

as desired.

Next, we will prove that

$$\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}=\frac{\sqrt{7}}{8}.$$

Indeed, let $z=e^{\pi i/7}$ then we have $z^7=-1$ and $z^{14}=1$. Thus,

$$(1-z)(1-z^6)=1+z^7-z-z^6=\frac{1}{z}-z,$$

and

$$(1-z^{13})(1-z^8)=\left(1-\frac{1}{z}\right)\left(1-\frac{1}{z^6}\right)=-\frac{1}{z}-\frac{1}{z^6}=z-\frac{1}{z}.$$

Thus,

$$\sin^2\frac{\pi}{7}=\left[\frac{1}{2i}\left(z-\frac{1}{z}\right)\right]^2=\frac{1}{4}(1-z)(1-z^6)(1-z^8)(1-z^{13}).$$

Similarly, we obtain

$$\begin{aligned}\sin^2\frac{5\pi}{7}&=\frac{1}{4}(1-z^2)(1-z^5)(1-z^{12})(1-z^9),\\ \sin^2\frac{3\pi}{7}&=\frac{1}{4}(1-z^3)(1-z^4)(1-z^{10})(1-z^{11}).\end{aligned}$$

Thus, by multiplying all of the above together and with $1-z^7=2$ we obtain

$$\left(\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}\right)^2=\frac{1}{4^3\cdot 2}\prod_{i=1}^{13}(1-z^i).$$

Note that $\prod_{i=1}^{13}(1-z^i) = f(1) = 14$, where $f(x) = x^{13}+x^{12}+\cdots+x+1 = (x^{14}-1)/(x-1)$ that has 13 roots $z, z^2, \cdots, z^{13}$. Thus,

$$\left(\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}\right)^2 = \frac{7}{8^2}$$

or

$$\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7} = \frac{\sqrt{7}}{8}.$$

Combining these two identities we obtain

$$\tan\frac{\pi}{7}\tan\frac{2\pi}{7}\tan\frac{3\pi}{7} = \sqrt{7}.$$

Solution by Shaddoll 1589. We consider the probability that the frog ever returns to 0 if the frog is currently at 1(by symmetry, this probability is the same when the frog is at -1), let it be p. Now, note that p is also the probability that the frog can ever get back to 1 when it's at 2, by translation. Thus,

$$p = \frac{1}{3} + \frac{1}{3}p^2,$$

(either he gets back to 0 with probability 1/3, or he goes to 2 with probability 1/3 and has to pass by 1 on his way to 0 with probability p he'll ever do it, and another factor of p to get back to 0, or he gets eaten and obviously can't return to 0), and solving, we have

$$p = \frac{3-\sqrt{5}}{2},$$

(the other root is greater than 1, so it clearly can't work). Thus, the chance that he ever returns to 0 when he's currently at 0 is

$$\frac{2p}{3} = \frac{3-\sqrt{5}}{3},$$

denote this by q(since he avoids getting eaten off the start, then returns to 0), so the expected number of times he returns to 0 is

$$q + q^2 + q^3 + \cdots = \frac{1}{\frac{\sqrt{5}}{3-\sqrt{5}}} = \frac{3-\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{5}-5}{5}.$$

Solution by Michael Tang 1590. Denote the first coordinate of a vector v by v'. Let $\ell = \max(|v_1|, |v_2|, \dots, |v_m|)$ and $t = \min(v_1', v_2', \dots, v_m') > 0$. We will show that $C = (\ell/t)^2$ works.

Lemma: After R rounds ($R \geq 1$), we have $|w| \leq \ell\sqrt{R}$, regardless of our choices for i.

Proof of Lemma: Induct on R. For $R = 1$, we have $w \in \{v_1, \dots, v_m\}$, so by the definition of ℓ, we have $|w| \leq \ell = \ell\sqrt{1}$, as desired. Now suppose that, for some $R \geq 1$, we have $|w| \leq \ell\sqrt{R}$ after R rounds, regardless of our choices for i. Let v_j be the chosen vector in round $R+1$. Then

$$|w+v_j|^2 = |w|^2 + |v_j|^2 + 2(w \cdot v_j) \leq (\ell\sqrt{R})^2 + \ell^2 + 0 = \ell^2(R+1)$$

so $|w + v_j| \le \ell\sqrt{R+1}$, completing the induction. ■

Now suppose the process lasts R rounds ($R \ge 1$). Since in each round we add some v_i to w, after R rounds we must have $w' \ge R \cdot t$. Thus $|w| \ge w' \ge R \cdot t$. But by the Lemma, we also have $|w| \le \ell\sqrt{R}$, hence

$$R \cdot t \le |w| \le \ell\sqrt{R} \implies R \le \left(\frac{\ell}{t}\right)^2 \le C,$$

as claimed.

Solution by David Altizio 1591. First note that

$$\angle I_1AI_2 = \angle I_1AX + \angle XAI_2 = \frac{\angle BAX}{2} + \frac{\angle CAX}{2} = \frac{\angle A}{2},$$

is a constant not depending on X, so by

$$[AI_1I_2] = \tfrac{1}{2}(AI_1)(AI_2)\sin\angle I_1AI_2,$$

it suffices to minimize $(AI_1)(AI_2)$. Let $a = BC$, $b = AC$, $c = AB$, and $\alpha = \angle AXB$. Remark that

$$\angle AI_1B = 180^\circ - (\angle I_1AB + \angle I_1BA) = 180^\circ - \tfrac{1}{2}(180^\circ - \alpha) = 90^\circ + \tfrac{\alpha}{2}.$$

Applying the Law of Sines to $\triangle ABI_1$ gives

$$\frac{AI_1}{AB} = \frac{\sin\angle ABI_1}{\sin\angle AI_1B} \qquad \Rightarrow \qquad AI_1 = \frac{c\sin\frac{B}{2}}{\cos\frac{\alpha}{2}}.$$

Analogously one can derive $AI_2 = \frac{b\sin\frac{C}{2}}{\sin\frac{\alpha}{2}}$, and so

$$[AI_1I_2] = \frac{bc\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}} = \frac{bc\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\alpha} \ge bc\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$

with equality when $\alpha = 90^\circ$, that is, when X is the foot of the perpendicular from A to $\overline{BC}$. In this case the desired area is $bc\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$. To make this feasible to compute, note that

$$\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}} = \sqrt{\frac{1-\frac{b^2+c^2-a^2}{2bc}}{2}} = \sqrt{\frac{(a-b+c)(a+b-c)}{4bc}}.$$

Applying similar logic to $\sin\frac{B}{2}$ and $\sin\frac{C}{2}$ and simplifying yields a final answer of

$$\begin{aligned} bc\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} &= bc\cdot\frac{(a-b+c)(b-c+a)(c-a+b)}{8abc} \\ &= \frac{(30-32+34)(32-34+30)(34-30+32)}{8\cdot 32} = \boxed{126}. \end{aligned}$$

Solution by David Altizio 1592. We claim that in general the answer is

$$K = 2R^2 \sin\varphi_A \sin\varphi_B \sin\varphi_C,$$

where R is the common circumradius. Note that this generalizes the $K = 2R^2 \sin A \sin B \sin C$ formula for the area of a triangle. Further note that since $\sin t = \sin(\pi - t)$ for all $t \in \mathbb{R}$, the acute versus obtuse distinction on the φ_Xs is irrelevant. The key observation is that all the information given in the problem can be condensed into a single quadrilateral.

Consider points W, X, Y, and Z in this order about a circle ω such that $WXYZ$ is quadrilateral A. Now let X' be a point on minor arc $\widehat{WY}$ such that $XX' \parallel WY$. Then $WXX'Y$ is an isosceles trapezoid, so $WX'YZ$ is another quadrilateral which can be formed by the four given sticks; WLOG let it be quadrilateral B. Now a bit of angle chasing reveals that

$$\varphi_B = \angle(WY, ZX') = \angle ZX'X = \angle ZYX,$$

and so we deduce that one of the interior angles of quadrilateral A has angular measure φ_B. Similarly, one of the other two interior angles (i.e. neither $\angle ZYX$ nor $\angle ZWX$) has angular measure φ_C.

The problem is practically finished from here, as we deduce

$$\begin{aligned} K = \tfrac{1}{2}(WY)(ZX)\sin\varphi_A = \tfrac{1}{2}(2R\sin\varphi_B)(2R\sin\varphi_C)\sin\varphi_A \\ = 2R^2 \sin\varphi_A \sin\varphi_B \sin\varphi_C, \end{aligned}$$

which is what we wanted. Applying this to the particular case at hand, a quick computation gives $K = \frac{24}{35}$ for an answer of $\boxed{059}$.

(For anyone curious, the specifications in the problem are satisfied if the lengths of the sticks are approximately 0.32, 0.91, 1.06, and 1.82. I actually forgot to check this when I submitted the problem - thanks CAMC for making sure the configuration is valid!)

Solution by HrishiP 1594. Allow $a = \cos\alpha + i\sin\alpha$, and define b, c similarly. Note that on the complex plane, we require a, b, c lie on the unit circle (set of points containing complex numbers z such that $|z| = 1$). Since $a+b+c = 1+i$, we are motivated to isolate a and find $1 + i - a = b + c$. The only thing we know is that $|b + c| \leq 2$. So, we have

$$|b + c|^2 = |2\cos\beta \cdot \cos\gamma + 2\sin\beta + 2\sin\gamma + 2| = |2\cos(\beta - \gamma) + 2|.$$

It is optimal to have equality, because we are trying to minimize the real part of a, and if $\cos\beta, \cos\gamma$ are as large as possible then $\cos\alpha$ is small. Since $|b + c|^2 \leq 2^2$ we need $\cos(\beta - \gamma) = 1 \implies \beta = \gamma$. Now, we solve the system

$$\cos\alpha + 2\cos\beta = 1,$$
$$\sin\alpha + 2\sin\beta = 1.$$

Now we solve this system. If we square the equations and add them, we have

$$\begin{aligned} (\cos^2\alpha + \sin^2\alpha) + 4(\cos\alpha\cos\beta + \sin\alpha\sin\beta) + 4(\cos^2\beta + \sin^2\beta) = 2, \\ 5 + 4(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = 2, \\ \cos\alpha\cos\beta + \sin\alpha\sin\beta = -\tfrac{3}{4}. \end{aligned}$$

Note that
$$\cos\beta = \frac{1-\cos\alpha}{2} \text{ and } \sin\beta = \frac{1-\sin\alpha}{2}.$$
If we substitute these into the derived equation, and rearrange (also using Pythagorean identities) we have

$$\begin{aligned} \cos\alpha + \sin\alpha &= -\tfrac{1}{2}, \\ \sqrt{1-\cos^2\alpha} &= -\left(\tfrac{1}{2}+\cos\alpha\right), \\ 2\cos^2\alpha + \cos\alpha - \tfrac{3}{4} &= 0. \end{aligned}$$

Solving the quadratic for $\cos\alpha$ gives $\cos\alpha = \frac{-1\pm\sqrt{7}}{4}$. Since we want the smaller solution, the answer is
$$\boxed{\tfrac{-1-\sqrt{7}}{4}}.$$

Solution by MathStudent2002 1595. Assume Q is irreducible and fix Q. Now, let Y_Q be the limit of the probability that $Q \mid P$. We note that we only care about the Q so that $Y_Q \neq 0$. For size reasons this means Q's roots all must have modulus 1 (since for a root z and big n the i's are usually spread apart enough that z is just never a root), which means Q's roots are roots of unity, thus Q must be cyclotomic, but $\deg Q \leq 3$ so Q is either $X-1, X+1, X^2+X+1$, or X^2+1. Clearly the first one also has $Y_Q = 0$.

Let $Y_1 = Y_{X+1}, Y_2 = Y_{X^2+X+1}, Y_3 = Y_{X^2+1}$, and let $Y_4 = Y_1 + Y_3$. Note that if S modulo 4 has a, b, c, d of $0, 1, 2, 3$, respectively, then Y_1 occurs if and only if $a+c = b+d$, and Y_3 occurs if and only if $a = c, b = d$. These two never coincide, so Y_4 is the probability that P is divisible by one of $X+1, X^2+1$. Furthermore,
$$Y_1 = \frac{\binom{6}{3}}{2^6} = \frac{5}{16},$$
while
$$Y_3 = \frac{5}{16} \cdot \frac{5}{16},$$
so that
$$Y_4 = \frac{105}{256}.$$

Finally,
$$Y_3 = \frac{\binom{6}{2,2,2}}{3^6} = \frac{10}{81}.$$
So, since divisibility by X^2+X+1 depends on $S \pmod 3$ while divisibility by one of $X+1, X^2+1$ depends on $S \pmod 4$, by Chinese Theorem these are essentially independent for big n. So, the probability that one of the divisibilities occurs is
$$1-(1-Y_3)(1-Y_4) = 1 - \frac{71}{81}\cdot\frac{151}{256} = 1 - \frac{10721}{20736} = \boxed{\frac{10015}{20736}}.$$

Solution by CantonMathGuy 1596. No. The image must be centrally symmetric about the image of $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, so it cannot be a polygon with an odd number of vertices.

Solution by iNomOnCountdown 1597. If x, y, z are all positive, this always works. If one of them is negative, WLOG, let z be negative and x, y be positive. Casework shows that the equation holds if and only if $|z| > x + y$, which happens $\frac{1}{6}$ of the time in this case. Additionally, note that multiplying all the variables by -1 does not change the result. So, our answer is:

$$\frac{\frac{1}{8} + \frac{3}{8} \cdot \frac{1}{6}}{\frac{1}{2}} = \boxed{\frac{3}{8}}.$$

Solution by Michael Tang 1602. Let R denote the foot of the A-altitude. Recall that

$$\triangle APQ \sim \triangle RPB \sim \triangle RCQ.$$

Thus, we have $\frac{AP}{PQ} = \frac{RP}{PB}$, which yields

$$PR = \frac{AP \cdot PB}{PQ} = \frac{XP \cdot PY}{PQ} = \frac{10 \cdot 40}{25} = 16.$$

Analogous reasoning yields $QR = 21$.

Now recall that $\triangle ABC$ is the excentral triangle of $\triangle PQR$; this follows since e.g. $BQ \perp AC$ and BQ bisects $\angle PQR$. Thus, the problem is reduced to the following: if $\triangle ABC$ satisfies $BC = 25$, $AB = 21$, and $AC = 16$, and I_A, I_B, and I_C are the excenters of $\triangle ABC$, then what is $I_AI_B \cdot I_AI_C$?

Denote by A' and B' the projections of I_A and I_B onto BC and AC respectively. Compute $CA' = s - b$ and $CB' = s - a$, and so armed with the fact that $C \in \overline{I_AI_B}$ we obtain

$$I_AI_B = I_AC + CI_B = \frac{s-a}{\sin\frac{C}{2}} + \frac{s-b}{\sin\frac{C}{2}} = \frac{c}{\sin\frac{C}{2}}.$$

Similarly, $I_AI_C = b/\sin\frac{B}{2}$, so

$$I_AI_B \cdot I_AI_C = \frac{bc}{\sin\frac{B}{2}\sin\frac{C}{2}}.$$

Two applications of the Law of Cosines give $\cos B = 27/35$ and $\cos C = 11/20$. Thus,

$$\sin\frac{B}{2} = \sqrt{\frac{1-\cos B}{2}} = \sqrt{\frac{4}{35}},$$

and

$$\sin\frac{C}{2} = \sqrt{\frac{9}{40}},$$

so, finally,

$$\frac{bc}{\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{21 \cdot 16}{\sqrt{\frac{4}{35} \cdot \frac{9}{40}}} = 560\sqrt{14}.$$

The requested answer is $560 + 14 = \boxed{574}$.

Solution by CantonMathGuy 1603. The allowed displacements are permutations of $(\pm 2, 0, 0, 0)$ and $(\pm 1, \pm 1, \pm 1, \mp 1)$, corresponding to the factor

$$\sum_{\text{cyc}} (a^2 + a^{-2}) + \sum_{\text{cyc}} (a^{-1}bcd + ab^{-1}c^{-1}d^{-1}).$$

Miraculously, this factors as

$$(abcd)^{-2}(abcd+1)(ab+cd)(ac+bd)(ad+bc).$$

It follows that the desired quantity is the coefficient of $(abcd)^{90}$ in the expression

$$(abcd+1)^{40}(ab+cd)^{40}(ac+bd)^{40}(ad+bc)^{40}.$$

Now we may check that the only way to obtain $(abcd)^{90}$ is to choose $abcd$ 30 times, 1 10 times, and each of ab, cd, ac, bd, ad, and bc, 20 times. Thus, the answer is

$$\tbinom{40}{10}\tbinom{40}{20}^3.$$

Solution by pieater314159 1604.

$$(0,0), (1,1), \left(-\frac{1}{4}, \frac{31}{32}\right), \left(\frac{31}{32}, -\frac{1}{4}\right).$$

Solution by HamstPan38825 1607. Set O_1 and O_2 to be the centers of ω_1 and ω_2, respectively. Let $\theta_1 = \angle X_1O_1T_1$ and $\theta_2 = \angle T_2O_2X_2$. Then, the condition $2X_1T_1 = X_2T_2$ yields

$$2\sin\frac{\theta_1}{2} = \frac{1}{2}\left(3\sin\frac{\theta_2}{2}\right) \iff \sin\frac{\theta_2}{2} = \frac{4}{3}\sin\frac{\theta_1}{2}.$$

Now, we compute the distance from O to $\overline{T_1T_2}$ in two ways. Let ℓ be the line through O perpendicular to $\overline{T_1T_2}$.

First, drawing the line through O_1 parallel to $\overline{T_1T_2}$, the distance from O to this line is $10\cos\theta_1$, so the altitude has length $10\cos\theta_1 + 2$. Next, let $P = \overline{OX_2} \cap \overline{T_1T_2}$. As $\theta_2 > 90^\circ$, we can compute $O_2P = -\frac{3}{\cos\theta_2}$, so $OX = 9 + \frac{3}{\cos\theta_2}$ and as $\measuredangle(OX_2, \ell) = -\theta_2$, we have that the distance also equals

$$d = -\cos\theta_2\left(9 + \frac{3}{\cos\theta_2}\right) = -9\cos\theta_2 - 3.$$

Setting these equal and using double angle,

$$10\left(1 - 2\sin^2\frac{\theta_1}{2}\right) + 5 + 9\left(1 - 2\sin^2\frac{\theta_2}{2}\right) = 0,$$

so this yields $\sin\frac{\theta_1}{2} = \sqrt{\frac{6}{13}}$. Then, the altitude has length $\frac{36}{13}$, so

$$AB = 2\sqrt{12^2 - \left(\frac{36}{13}\right)^2} = \boxed{\frac{96\sqrt{10}}{13}}.$$

Solution 1618. The answer is $\boxed{(n+2)/n}$.

Solution 1619. The answer is $\boxed{\frac{2\pi}{3}+\frac{\sqrt{3}}{4}}$.

Solution by naman12 1622. Note that the volume of a parallelopiped is simply

$$V = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are the three sides. Note that this is the triple product, so

$$V = \det \mathbf{M} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}.$$

Thus, if α, β, γ are the angles $\angle(\mathbf{b}, \mathbf{c}), \angle(\mathbf{c}, \mathbf{a}), \angle(\mathbf{a}, \mathbf{b})$,

$$\begin{aligned}
V^2 &= \det(\mathbf{M} \cdot \mathbf{M}^\intercal) \\
&= \det \begin{vmatrix} a_x^2 + a_y^2 + a_z^2 & a_x b_x + a_y b_y + a_z b_z & a_x c_x + a_y c_y + a_z c_z \\ b_x a_x + b_y a_y + b_z a_z & b_x^2 + b_y^2 + b_z^2 & b_x c_x + b_y c_y + b_z c_z \\ c_x a_x + c_y a_y + c_z a_z & c_x b_x + c_y b_y + c_z b_z & c_x^2 + c_y^2 + c_z^2 \end{vmatrix} \\
&= \det \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix} \\
&= \ a^2(b^2c^2 - b^2c^2\cos^2\alpha) - ab\cos\gamma(abc^2\cos\gamma - abc^2\cos\alpha\cos\beta) \\
&\quad + ac\cos\beta(ab^2c\cos\gamma\cos\alpha - ab^2c\cos\beta) \\
&= a^2b^2c^2(1 + 2\cos\alpha\cos\beta\cos\gamma - \cos^2\alpha - \cos^2\beta - \cos^2\gamma).
\end{aligned}$$

Note that the side length of the rhombus is

$$\frac{1}{2}\sqrt{\sqrt{21}^2 + \sqrt{31}^2} = \sqrt{13}.$$

Let θ be the acute angle of the rhombus. Note

$$\sqrt{\frac{1-\cos\theta}{2}} = \sin\frac{1}{2}\theta = \frac{\sqrt{21}}{2\sqrt{13}}.$$

Thus,

$$\frac{1-\cos\theta}{2} = \frac{21}{52} \implies \cos\theta = \frac{5}{26}.$$

Thus, the ratio of the volumes is

$$\frac{1 + 2(\frac{5}{26})^3 - 3(\frac{5}{26})^2}{1 - 2(\frac{5}{26})^3 - 3(\frac{5}{26})^2} = \frac{63}{62}.$$

Therefore, the answer is $63 + 62 = \boxed{125}$.

Solution by Dragomir Grozev 1624. Different approach. It needs additional work and calculations, but I hope it could be carried out. So, you have a set X of 1431 integers in the interval $[1, 2023]$. For any subset $A \subset X$ denote $S_A := \sum_{a \in A} a$. The main idea is to show that as A runs through all subsets of X, S_A covers all integers in $[L, S_X - L]$ for some not too large integer L. For example $L := 14000$ will do the job, but it can be optimized a lot! In some sense, this is a stronger claim. This claim almost solves the original problem, so let us sketch the proof.

Since $S_{X \setminus A} = S_X$-S_A, it is enough to prove S_A can cover the interval $[L, S_X/2]$. Consider the set $X' \subset X$ that consists of the last 715 largest numbers in X. Apparently, $S_{X'} \geq S_X/2$. Let $X'' := X \setminus X'$. Since $|X'| = 715$, it follows $X'' \subset [1..1310]$. (we can optimize the sets X'' and X' here, 715 is a very rough estimate).

1 For each $k \in [1200, 1403]$, there exists $A \subset X'', |A| = 2$ with $S_{A''} = k$. Indeed, assume the opposite and consider the the appropriate pairs $(k-i, i)$. If both numbers in a pair belong to X'' we are done. But it's impossible all of these pairs to have at most one number in X'' since the density of X'' prevents it.

2 For any $k_i \in [1200, 1403], i = 1, 2, \ldots 10$ there exist distinct $x_i, y_i \in X'', i = 1, \ldots, 10$ such that $x_i + y_i = k_i$. We can prove it using 1) subsequently for $i = 1, 2, \ldots, 10$ and removing the pair x_i, y_i.

3 Thus, when the set $A \subset X'', |A| = 20$ varies, S_A covers all integers in $[10 \cdot 1400, 10 \cdot 1400 + 2030]$. Then, for any fixed $A' \subset X'$ the value $S_{A'} + S_A$ as $A \subset X''$ varies will contain interval with length 2030 and since each offset $x \in X'$ is less than 2030 it means we can cover the interval $[L, S_X/2]$ where $L = 14000$.

It remains some small cases. In case $S_X - 14000 \geq 2023 \cdot 1012 - S_X$ we are done. Consider now the case $2S_X < 2023 \cdot 1012 + 14000$. In this case X slightly differs from the first 1431 natural numbers, so it can be checked directly?

Solution by Matthew Kroesche 1626. Here is a quick complex bash! Let $\triangle AKL$ be inscribed in the unit circle, and let O and H be the circumcenter and orthocenter of $\triangle ABC$ respectively, so that

$$\begin{aligned}
|a| = |k| = |\ell| &= 1, \\
d &= \frac{2k\ell}{k+\ell}, \\
o = \frac{a+d}{2} &= \frac{ak + a\ell + 2k\ell}{2(k+\ell)}, \\
b = k + o - ak\bar{o} &= \frac{k^2 + 3k\ell + a\ell - ak}{2(k+\ell)}, \\
c = \ell + o - a\ell\bar{o} &= \frac{\ell^2 + 3k\ell + ak - a\ell}{2(k+\ell)}, \\
h = a + b + c - 2o = \frac{k^2 + \ell^2 + 2k\ell}{2(k+\ell)} &= \frac{k+\ell}{2}.
\end{aligned}$$

So in fact the orthocenter of $\triangle ABC$ is the midpoint of $\overline{KL}$.

Solution by Gabriel Goh 1627. Take a big n and use the strings with $n+1$ 1s and the string with $n+1$ 0s. These cost $1/2^n$ total. Now any possible valid string can only have at most n consecutive bits that are equal. Replace the binary string with an integer array depicting the length of consecutive equal blocks. For example 011000100... becomes 12312...

Choose a big m. We pick the binary string corresponding to every possible m length array. For example if $n = 2$ and $m = 3$ we would take 010, 0100, 0110, 01100, 0010, 00100, 00110, 001100 and their complementaries. We calculate the cost. It is equal to

$$2\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}\right)^m$$

by generating functions (the first factor of 2 is because it can start with 0 or 1). This is equal to

$$2\left(1-\frac{1}{2^n}\right)^m.$$

Hence we have found a construction with cost

$$\frac{1}{2^n}+2\left(1-\frac{1}{2^n}\right)^m,$$

and taking m, n sufficiently large finishes.