1

Complex Numbers

You, have you really understood all that stuff?
What?
The story of imaginary numbers?

Robert Musil, *The Confusions of Young Törless* (1907)[1]

Complex numbers lie at the very core of quantum mechanics and are therefore absolutely essential to a basic understanding of quantum computation. In this chapter we present this important system of numbers from both the algebraic and the geometric standpoints. Section 1.1 presents some motivation and the basic definitions. The algebraic structure and operations on complex numbers are given in Section 1.2. The chapter concludes with Section 1.3, where complex numbers are presented from a geometric point of view and advanced topics are discussed. Our hope is that this chapter will help you get a little closer to what Sir Roger Penrose has very aptly called the "magic of complex numbers" (Penrose, 2005).

Reader Tip. Many readers will find that they are already familiar with some of the material presented in this chapter. The reader who feels confident in her comprehension of the fundamental knowledge of complex numbers, the basic operations, and their properties can safely move on to later chapters. We suggest, though, that you at least skim through the following pages to see what topics are covered. Return to Chapter 1 as a reference when needed (using the index to find specific topics). ♡

[1] For the German-speaking reader, here is the original text (the translation at the beginning is ours):

Du, hast du das vorhin ganz verstanden?
Was?
Die Geschichte mit den imaginären Zahlen?

Musil's *Törless* is a remarkable book. A substantial part is dedicated to the struggle of young Törless to come to grips with mathematics, as well as with his own life. Definitely recommended!

1.1 BASIC DEFINITIONS

The original motivation for the introduction of complex numbers was the theory of algebraic equations, the part of algebra that seeks solutions of polynomial equations. It became readily apparent that there are plenty of cases in which no solution among familiar numbers can be found. Here is the simplest example:

$$x^2 + 1 = 0. \tag{1.1}$$

Indeed, any possible x^2 would be positive or zero. Adding 1 ends up with some quantity to the left that is strictly positive; hence, no solution exists.

Exercise 1.1.1 Verify that the equation $x^4 + 2x^2 + 1 = 0$ has no solution among the real numbers. (Hint: Factor the polynomial.) ■

The aforementioned argument seems to dash any hope of solving Equation (1.1). But does it?

Before building any new number system, it pays to remind ourselves of other sets of numbers that we usually work with

- positive numbers, $\mathbb{P} = \{1, 2, 3, \ldots\}$;
- natural numbers, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$;
- integers (or whole numbers), $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$;
- rational numbers, $\mathbb{Q} = \left\{\frac{m}{n} | m \in \mathbb{Z}, n \in \mathbb{P}\right\}$;
- real numbers, $\mathbb{R} = \mathbb{Q} \bigcup \{\ldots, \sqrt{2}, \ldots, e, \ldots, \pi, \ldots, \frac{e}{\pi} \ldots\}$;

In none of these familiar number systems can a valid solution to Equation (1.1) be found. Mathematics often works around difficulties by simply *postulating* that such a solution, albeit unknown, is available somewhere. Let us thus boldly assume that this enigmatic solution does indeed exist and determine what it looks like: Equation (1.1) is equivalent to

$$x^2 = -1. \tag{1.2}$$

What does this state? That the solution of Equation (1.1) is a number such that its square is -1, i.e., a number i such that

$$i^2 = -1 \quad \text{or} \quad i = \sqrt{-1}. \tag{1.3}$$

Of course we know that no such number exists among known (i.e., real) numbers, but we have already stated that this is not going to deter us. We will simply allow this new creature into the realm of well-established numbers and use it as it pleases us. Because it is *imaginary*, it is denoted i. We will impose on ourselves an important restriction: aside from its weird behavior when squared, i will behave just like an ordinary number.

Example 1.1.1 What is the value of i^3? We shall treat i as a legitimate number, so

$$i^3 = i \times i \times i = (i^2) \times i = -1 \times i = -i. \tag{1.4}$$

□

Exercise 1.1.2 Find the value of i^{15}. (Hint: Calculate i, i^2, i^3, i^4, and i^5. Find a pattern.) ■

In opening the door to our new friend i, we are now flooded with an entire universe of new numbers: to begin with, all the multiples of i by a real number, like $2 \times i$. These fellows, being akin to i, are known as **imaginary numbers**. But there is more: add a real number and an imaginary number, for instance, $3 + 5 \times i$, and you get a number that is neither a real nor an imaginary. Such a number, being a hybrid entity, is rightfully called a **complex number**.

Definition 1.1.1 *A* complex number *is an expression*

$$c = a + b \times i = a + bi, \tag{1.5}$$

where a, b are two real numbers; a is called the real part of c, whereas b is its imaginary part. The set of all complex numbers will be denoted as $\mathbb{C}$. When the $\times$ is understood, we shall omit it.

Complex numbers can be added and multiplied, as shown next.

Example 1.1.2 Let $c_1 = 3 - i$ and $c_2 = 1 + 4i$. We want to compute $c_1 + c_2$ and $c_1 \times c_2$.

$$c_1 + c_2 = 3 - i + 1 + 4i = (3 + 1) + (-1 + 4)i = 4 + 3i. \tag{1.6}$$

Multiplying is not as easy. We must remember to multiply each term of the first complex number with each term of the second complex number. Also, remember that $i^2 = -1$.

$$\begin{aligned} c_1 \times c_2 &= (3 - i) \times (1 + 4i) = (3 \times 1) + (3 \times 4i) + (-i \times 1) + (-i \times 4i) \\ &= (3 + 4) + (-1 + 12)i = 7 + 11i. \end{aligned} \tag{1.7}$$

□

Exercise 1.1.3 Let $c_1 = -3 + i$ and $c_2 = 2 - 4i$. Calculate $c_1 + c_2$ and $c_1 \times c_2$. ■

With addition and multiplication we can get all polynomials. We set out to find a solution for Equation (1.1); it turns out that complex numbers are enough to provide solutions for *all* polynomial equations.

Proposition 1.1.1 (Fundamental Theorem of Algebra). Every polynomial equation of one variable with complex coefficients has a complex solution.

Exercise 1.1.4 Verify that the complex number $-1 + i$ is a solution for the polynomial equation $x^2 + 2x + 2 = 0$. ■

This nontrivial result shows that complex numbers are well worth our attention. In the next two sections, we explore the complex kingdom a little further.

Programming Drill 1.1.1 *Write a program that accepts two complex numbers and outputs their sum and their product.*

1.2 THE ALGEBRA OF COMPLEX NUMBERS

Admittedly, the fact that we know how to handle them does not explain away the oddity of complex numbers. What *are* they? What does it mean that i squared is equal to -1?

In the next section, we see that the geometrical viewpoint greatly aids our intuition. Meanwhile, we would like to convert complex numbers into more familiar objects by carefully looking at how they are built.

Definition 1.1.1 tells us *two* real numbers correspond to each complex number: its real and imaginary parts. A complex number is thus a two-pronged entity, carrying its two components along. How about *defining* a complex number as an ordered pair of reals?

$$c \longmapsto (a, b). \tag{1.8}$$

Ordinary real numbers can be identified with pairs $(a, 0)$

$$a \longmapsto (a, 0), \tag{1.9}$$

whereas imaginary numbers will be pairs $(0, b)$. In particular,

$$i \longmapsto (0, 1). \tag{1.10}$$

Addition is rather obvious – it adds pairs componentwise:

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2). \tag{1.11}$$

Multiplication is a little trickier:

$$(a_1, b_1) \times (a_2, b_2) = (a_1, b_1)(a_2, b_2) = (a_1a_2 - b_1b_2, a_1b_2 + a_2b_1). \tag{1.12}$$

Does this work? Multiplying i by itself gives

$$i \times i = (0, 1) \times (0, 1) = (0 - 1, 0 + 0) = (-1, 0), \tag{1.13}$$

which is what we wanted.

Using addition and multiplication, we can write any complex number in the usual form:

$$c = (a, b) = (a, 0) + (0, b) = (a, 0) + (b, 0) \times (0, 1) = a + bi. \tag{1.14}$$

We have traded one oddity for another: i was previously quite mysterious, whereas now it is just $(0, 1)$. A complex number is nothing more than an ordered pair of ordinary real numbers. Multiplication, though, is rather strange: perhaps the reader would have expected a componentwise multiplication, just like addition. We shall see later that by viewing complex numbers through yet another looking glass the strangeness linked to their multiplication rule will fade away.

Example 1.2.1 Let $c_1 = (3, -2)$ and $c_2 = (1, 2)$. Let us multiply them using the aforementioned rule:

$$\begin{aligned} c_1 \times c_2 &= (3 \times 1 - (-2) \times 2, -2 \times 1 + 2 \times 3) \\ &= (3 + 4, -2 + 6) = (7, 4) = 7 + 4i. \end{aligned} \tag{1.15}$$

□

Exercise 1.2.1 Let $c_1 = (-3, -1)$ and $c_2 = (1, -2)$. Calculate their product. ■

So far, we have a set of numbers and two operations: addition and multiplication. Both operations are **commutative**, meaning that for arbitrary complex numbers c_1 and c_2,

$$c_1 + c_2 = c_2 + c_1 \tag{1.16}$$

and

$$c_1 \times c_2 = c_2 \times c_1. \tag{1.17}$$

Both operations are also **associative**:

$$(c_1 + c_2) + c_3 = c_1 + (c_2 + c_3) \tag{1.18}$$

and

$$(c_1 \times c_2) \times c_3 = c_1 \times (c_2 \times c_3). \tag{1.19}$$

Exercise 1.2.2 Verify that multiplication of complex numbers is associative. ■

Moreover, multiplication **distributes** over addition: for all c_1, c_2, c_3, we have

$$c_1 \times (c_2 + c_3) = (c_1 \times c_2) + (c_1 \times c_3). \tag{1.20}$$

Let us verify this property: first we write the complex numbers as pairs $c_1 = (a_1, b_1)$, $c_2 = (a_2, b_2)$, and $c_3 = (a_3, b_3)$. Now, let us expand the left side

$$\begin{aligned} c_1 \times (c_2 + c_3) &= (a_1, b_1) \times ((a_2, b_2) + (a_3, b_3)) \\ &= (a_1, b_1) \times (a_2 + a_3, b_2 + b_3) \\ &= (a_1 \times (a_2 + a_3) - b_1 \times (b_2 + b_3), \\ &\quad\; a_1 \times (b_2 + b_3) + b_1 \times (a_2 + a_3)) \\ &= (a_1 \times a_2 + a_1 \times a_3 - b_1 \times b_2 - b_1 \times b_3, \\ &\quad\; a_1 \times b_2 + a_1 \times b_3 + b_1 \times a_2 + b_1 \times a_3). \end{aligned} \tag{1.21}$$

Turning to the right side of Equation (1.20) one piece at a time gives

$$c_1 \times c_2 = (a_1 \times a_2 - b_1 \times b_2, a_1 \times b_2 + a_2 \times b_1) \tag{1.22}$$

$$c_1 \times c_3 = (a_1 \times a_3 - b_1 \times b_3, a_1 \times b_3 + a_3 \times b_1); \tag{1.23}$$

summing them up we obtain

$$\begin{aligned} c_1 \times c_2 + c_1 \times c_3 = (&a_1 \times a_2 - b_1 \times b_2 + a_1 \times a_3 - b_1 \times b_3, \\ &a_1 \times b_2 + a_2 \times b_1 + a_1 \times b_3 + a_3 \times b_1), \end{aligned} \tag{1.24}$$

which is precisely what we got in Equation (1.21).

Having addition and multiplication, we need their complementary operations: subtraction and division.

Subtraction is straightforward:

$$c_1 - c_2 = (a_1, b_1) - (a_2, b_2) = (a_1 - a_2, b_1 - b_2); \tag{1.25}$$

in other words, subtraction is defined componentwise, as expected.

As for division, we have to work a little: If

$$(x, y) = \frac{(a_1, b_1)}{(a_2, b_2)}, \tag{1.26}$$

then by definition of division as the inverse of multiplication

$$(a_1, b_1) = (x, y) \times (a_2, b_2) \tag{1.27}$$

or

$$(a_1, b_1) = (a_2x - b_2y, a_2y + b_2x). \tag{1.28}$$

So we end up with

$$(1) \qquad a_1 = a_2x - b_2y, \tag{1.29}$$

$$(2) \qquad b_1 = a_2y + b_2x. \tag{1.30}$$

To determine the answer, we must solve this pair of equations for x and y. Multiply both sides of (1) by a_2 and both sides of (2) by b_2. We end up with

$$(1') \qquad a_1a_2 = a_2^2x - b_2a_2y, \tag{1.31}$$

$$(2') \qquad b_1b_2 = a_2b_2y + b_2^2x. \tag{1.32}$$

Now, let us add (1′) and (2′) to get

$$a_1a_2 + b_1b_2 = (a_2^2 + b_2^2)x. \tag{1.33}$$

Solving for x gives us

$$x = \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2}. \tag{1.34}$$

We can perform the same trick for y by multiplying (1) and (2) by b_2 and $-a_2$, respectively, and then summing. We obtain

$$y = \frac{a_2b_1 - a_1b_2}{a_2^2 + b_2^2}. \tag{1.35}$$

In more compact notation, we can express this equation as

$$\frac{a_1 + b_1i}{a_2 + b_2i} = \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + \frac{a_2b_1 - a_1b_2}{a_2^2 + b_2^2}i. \tag{1.36}$$

Notice that both x and y are calculated using the same denominator, namely, $a_2^2 + b_2^2$. We are going to see what this quantity means presently. In the meantime, here is a concrete example.

Example 1.2.2 Let $c_1 = -2 + i$ and $c_2 = 1 + 2i$. We will compute $\frac{c_1}{c_2}$. In this case, $a_1 = -2,\ b_1 = 1,\ a_2 = 1$, and $b_2 = 2$. Therefore,

$$a_2^2 + b_2^2 = 1^2 + 2^2 = 5, \tag{1.37}$$

$$a_1a_2 + b_1b_2 = -2 \times 1 + 1 \times 2 = 0, \tag{1.38}$$

$$a_2b_1 - a_1b_2 = 1 \times 1 - (-2) \times 2 = 1 + 4 = 5. \tag{1.39}$$

The answer is thus $\left(\frac{0}{5}, \frac{5}{5}\right) = (0, 1) = i$. □

Exercise 1.2.3 Let $c_1 = 3i$ and $c_2 = -1 - i$. Calculate $\dfrac{c_1}{c_2}$. ■

Now, let us go back to the mysterious denominator in the quotient formula in Equation (1.36). Real numbers have a unary operation, the absolute value, given by

$$|a| = +\sqrt{a^2}. \tag{1.40}$$

We can define a generalization of this operation[2] to the complex domain by letting

$$|c| = |a + bi| = +\sqrt{a^2 + b^2}. \tag{1.41}$$

This quantity is known as the **modulus** of a complex number.

Example 1.2.3 What is the modulus of $c = 1 - i$?

$$|c| = |1 - i| = +\sqrt{1^2 + (-1)^2} = \sqrt{2}. \tag{1.42}$$

□

The geometric meaning of the modulus is discussed in the next section. For now, we remark that the quantity in the denominator of the quotient of two complex numbers is nothing more than the modulus squared of the divisor:

$$|c|^2 = a^2 + b^2. \tag{1.43}$$

This modulus must be different from zero, which always happens unless the divisor is itself zero.

Exercise 1.2.4 Calculate the modulus of $c = 4 - 3i$. ■

Exercise 1.2.5 Verify that given two arbitrary complex numbers c_1 and c_2, the following equality always holds:

$$|c_1||c_2| = |c_1c_2|. \tag{1.44}$$

■

Exercise 1.2.6 Prove that

$$|c_1 + c_2| \leq |c_1| + |c_2|. \tag{1.45}$$

When are they, in fact, equal? (Hint: Square both sides.) ■

Exercise 1.2.7 Show that for all $c \in \mathbb{C}$, we have $c + (0, 0) = (0, 0) + c = c$. That is, $(0, 0)$ is an additive identity. ■

[2] The definition given in Equation (1.40) is entirely equivalent to the more familiar one: $|a| = a$ if $a \geq 0$, and $|a| = -a$ if $a < 0$.

Exercise 1.2.8 Show that for all $c \in \mathbb{C}$ we have $c \times (1, 0) = (1, 0) \times c = c$. That is, $(1, 0)$ is a multiplicative identity. ■

In summation, we have defined a new set of numbers, $\mathbb{C}$, endowed with four operations, verifying the following properties:

(i) Addition is commutative and associative.
(ii) Multiplication is commutative and associative.
(iii) Addition has an identity: $(0, 0)$.
(iv) Multiplication has an identity: $(1, 0)$.
(v) Multiplication distributes with respect to addition.
(vi) Subtraction (i.e., the inverse of addition) is defined everywhere.
(vii) Division (i.e., the inverse of multiplication) is defined everywhere except when the divisor is zero.

A set with operations satisfying all these properties is called a **field**. $\mathbb{C}$ is a field, just like $\mathbb{R}$, the field of real numbers. In fact, via the identification that associates a real number to a complex number with 0 as the imaginary component, we can think of $\mathbb{R}$ as a subset[3] of $\mathbb{C}$. $\mathbb{R}$ sits inside $\mathbb{C}$; but $\mathbb{C}$ is a vast field, so vast, indeed, that all polynomial equations with coefficients in $\mathbb{C}$ have a solution in $\mathbb{C}$ itself. $\mathbb{R}$ is also a roomy field, but not enough to enjoy this last property (remember Equation (1.1)). A field that contains all solutions for any of its polynomial equations is said to be **algebraically complete**. $\mathbb{C}$ is an algebraically complete field, whereas $\mathbb{R}$ is not.

There is a unary operation that plays a crucial role in the complex domain. The reader is familiar with "changing signs" of real numbers. Here, however, there are *two* real numbers attached to a complex number. Therefore, there are *three* ways of changing sign: either change the sign of the real part or change the sign of the imaginary part, or both. Let us analyze them one by one.

Changing both signs of the complex number is done by multiplying by the number $-1 = (-1, 0)$.

Exercise 1.2.9 Verify that multiplication by $(-1, 0)$ changes the sign of the real and imaginary components of a complex number. ■

Changing the sign of the imaginary part only is known as **conjugation**.[4] If $c = a + bi$ is an arbitrary complex number, then the conjugate of c is $\overline{c} = a - bi$. Two numbers related by conjugation are said to be **complex conjugates** of each other.

Changing the sign of the real part ($c \longmapsto -\overline{c}$) has no particular name, at least in the algebraic context.[5]

The following exercises will guide you through conjugation's most important properties.

[3] A subset of a field that is a field in its own right is called a **subfield**: $\mathbb{R}$ is a subfield of $\mathbb{C}$.
[4] Its "geometric" name is **real-axis reflection**. The name becomes obvious in the next section.
[5] In the geometric viewpoint, it is known as **imaginary-axis reflection**. After reading Section 1.3, we invite you to investigate this operation a bit further.

Exercise 1.2.10 Show that conjugation respects addition, i.e.,

$$\overline{c_1} + \overline{c_2} = \overline{c_1 + c_2}. \tag{1.46}$$

■

Exercise 1.2.11 Show that conjugation respects multiplication, i.e.,

$$\overline{c_1} \times \overline{c_2} = \overline{c_1 \times c_2}. \tag{1.47}$$

■

Notice that the function

$$c \longmapsto \overline{c} \tag{1.48}$$

given by conjugation is **bijective**, i.e., is one-to-one and onto. Indeed, two different complex numbers are never sent to the same number by conjugation. Moreover, every number is the complex conjugate of some number. A function from a field to a field that is bijective and that respects addition and multiplication is known as a **field isomorphism**. Conjugation is thus a field isomorphism of $\mathbb{C}$ to $\mathbb{C}$.

Exercise 1.2.12 Consider the operation given by flipping the sign of the real part. Is this a field isomorphism of $\mathbb{C}$? If yes, prove it. Otherwise, show where it fails. ■

We cannot continue without mentioning another property of conjugation:

$$c \times \overline{c} = |c|^2. \tag{1.49}$$

In words, the modulus squared of a complex number is obtained by multiplying the number with its conjugate. For example,

$$(3 + 2i) \times (3 - 2i) = 3^2 + 2^2 = 13 = |3 + 2i|^2. \tag{1.50}$$

We have covered what we need from the algebraic perspective. We see in the next section that the geometric approach sheds some light on virtually all topics touched on here.

Programming Drill 1.2.1 *Take the program that you wrote in the last programming drill and make it also perform subtraction and division of complex numbers. In addition, let the user enter a complex number and have the computer return its modulus and conjugate.*

1.3 THE GEOMETRY OF COMPLEX NUMBERS

As far as algebra is concerned, complex numbers are an algebraically complete field, as we have described them in Section 1.2. That alone would render them invaluable as a mathematical tool. It turns out that their significance extends far beyond the algebraic domain and makes them equally useful in geometry and hence in physics. To see why this is so, we need to look at a complex number in yet another way. At the beginning of Section 1.2, we learned that a complex number is a pair of real

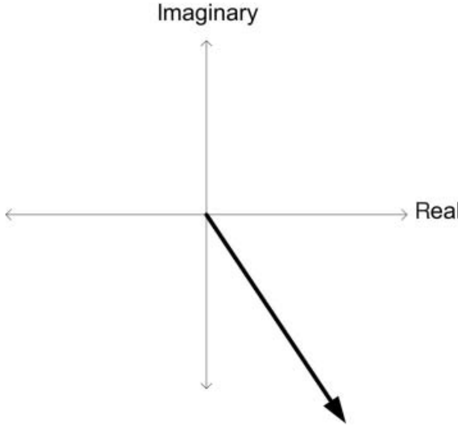

Figure 1.1. Complex plane.

numbers. This suggests a natural means of representation: real numbers are placed on the line, so pairs of reals correspond to points on the plane, or, equivalently, correspond to **vectors** starting from the origin and pointing to that point (as shown in Figure 1.1).

In this representation, real numbers (i.e., complex numbers with no imaginary part) sit on the horizontal axis and imaginary numbers sit on the vertical axis. This plane is known as the **complex plane** or the **Argand plane**.

Through this representation, the algebraic properties of the complex numbers can be seen in a new light. Let us start with the modulus: it is nothing more than the **length** of the vector. Indeed, the length of a vector, via Pythagoras' theorem, is the square root of the sum of the squares of its edges, which is precisely the modulus, as defined in the previous section.

Example 1.3.1 Consider the complex numbers $c = 3 + 4i$ depicted in Figure 1.2. The length of the vector is the hypotenuse of the right triangle whose edges have length 3 and 4, respectively. Pythagoras' theorem gives us the length as

$$length(c) = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5. \tag{1.51}$$

This is exactly the modulus of c. □

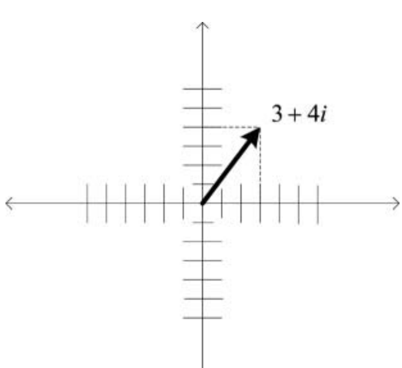

Figure 1.2. Vector $3 + 4i$.

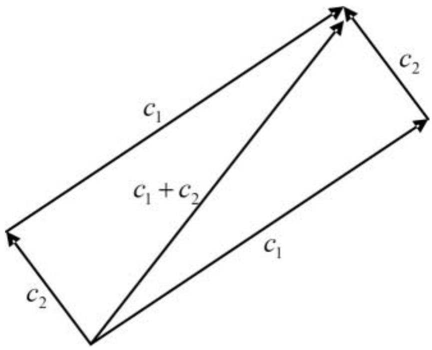

Figure 1.3. Parallelogram rule.

Next comes addition: vectors can be added using the so-called **parallelogram rule** illustrated by Figure 1.3. In words, draw the parallelogram whose parallel edges are the two vectors to be added; their sum is the diagonal.

Exercise 1.3.1 Draw the complex numbers $c_1 = 2 - i$ and $c_2 = 1 + i$ in the complex plane, and add them using the parallelogram rule. Verify that you would get the same result as adding them algebraically (the way we learned in Section 1.2). ■

Subtraction too has a clear geometric meaning: subtracting c_2 from c_1 is the same as adding the negation of c_2, i.e., $-c_2$, to c_1. But what is the negation of a vector? It is just the vector of the same length pointed in the opposite direction (see Figure 1.4).

Exercise 1.3.2 Let $c_1 = 2 - i$ and $c_2 = 1 + i$. Subtract c_2 from c_1 by first drawing $-c_2$ and then adding it to c_1 using the parallelogram rule. ■

To give a simple geometrical meaning to multiplication, we need to develop yet another characterization of complex numbers. We saw a moment ago that for every complex number we can draw a right triangle, whose edges' lengths are the real and imaginary parts of the number and whose hypotenuse's length is the modulus. Now, suppose someone tells us the modulus of the number what else do we need to know to draw the triangle? The answer is the angle at the origin.

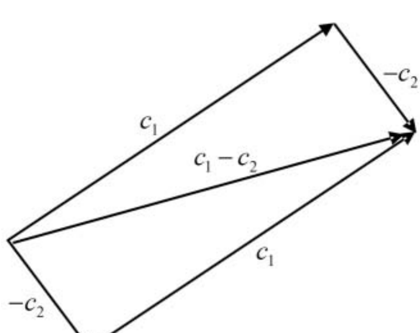

Figure 1.4. Subtraction.

The modulus ρ *and* the angle θ (notice: two real numbers, as before) are enough to uniquely determine the complex number.

$$(a, b) \longmapsto (\rho, \theta). \tag{1.52}$$

We know how to compute ρ from a, b:

$$\rho = \sqrt{(a^2 + b^2)}. \tag{1.53}$$

θ is also easy, via trigonometry:

$$\theta = \tan^{-1}\left(\frac{b}{a}\right). \tag{1.54}$$

The (a, b) representation is known as the **Cartesian representation** of a complex number, whereas (ρ, θ) is the **polar representation**.

We can go back from polar to Cartesian representation, again using trigonometry:

$$a = \rho\cos(\theta), \qquad b = \rho\sin(\theta). \tag{1.55}$$

Example 1.3.2 Let $c = 1 + i$. What is its polar representation?

$$\rho = \sqrt{1^2 + 1^2} = \sqrt{2} \tag{1.56}$$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4} \tag{1.57}$$

c is the vector of length $\sqrt{2}$ from the origin at an angle of $\frac{\pi}{4}$ radians, or 45°. □

Exercise 1.3.3 Draw the complex number given by the polar coordinates $\rho = 3$ and $\theta = \frac{\pi}{3}$. Compute its Cartesian coordinates. ■

Programming Drill 1.3.1 *Write a program that converts a complex number from its Cartesian representation to its polar representation and vice versa.*

Before moving on, let us meditate a little: what kind of insight does the polar representation give us? Instead of providing a ready-made answer, let us begin with a question: how many complex numbers share exactly the same modulus? A moment's thought will tell us that for a *fixed* modulus, say, $\rho = 1$, there is an entire circle centered at the origin (as shown in Figure 1.5).

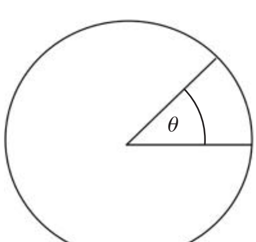

Figure 1.5. Phase θ.

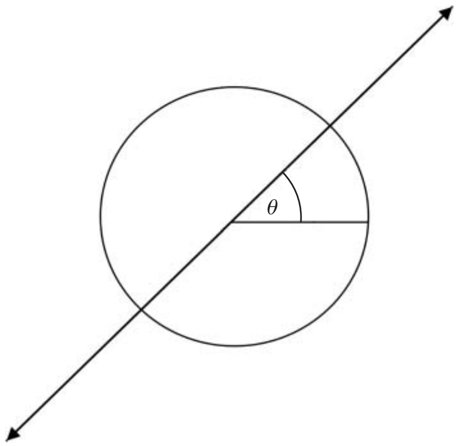

Figure 1.6. Points on a line with the same phase.

So, here comes the angle: imagine the circle as your watch, and the complex number as the needle. Angle θ tells us the "time." The "time" is known in physics and engineering as the **phase**, whereas the length of the "needle" (i.e., the modulus) is the **magnitude** of the number.

Definition 1.3.1 *A* complex number *is a magnitude and a phase.*

The ordinary positive reals are just complex numbers such that their phase is zero. The negative reals have phase π. By the same token, imaginary numbers are numbers with constant phase equal to $\frac{\pi}{2}$ (positive imaginary) or $\frac{3\pi}{2}$ (negative imaginary).

Given a constant phase, there is an entire line of complex numbers having that phase as depicted in Figure 1.6.

Observe that a complex number has a unique polar representation only if we confine the phase between 0 and 2π:

$$0 \leq \theta < 2\pi \tag{1.58}$$

(and the $\rho \geq 0$). If we restrict θ in this fashion, though, we cannot in general add angles (the sum may be bigger than 2π). A better course is to let the angle be anything and *reduce* it modulo 2π:

$$\theta_1 = \theta_2 \quad \text{if and only if } \theta_2 = \theta_1 + 2\pi k, \text{ for some integer } k. \tag{1.59}$$

Two complex numbers in polar representations will be identical if their magnitude is the same and if the angles are the same modulo 2π, as shown by the following example.

Example 1.3.3 Are the numbers $(3, -\pi)$ and $(3, \pi)$ the same? Indeed they are: their magnitude is the same and their phases differ by $(-\pi) - \pi = -2\pi = (-1)2\pi$. □

We are now ready for multiplication: given two complex numbers in polar coordinates, (ρ_1, θ_1) and (ρ_2, θ_2), their product can be obtained by simply multiplying their magnitude and *adding* their phase:

$$(\rho_1, \theta_1) \times (\rho_2, \theta_2) = (\rho_1 \rho_2, \theta_1 + \theta_2). \tag{1.60}$$

Example 1.3.4 Let $c_1 = 1 + i$ and $c_2 = -1 + i$. Their product, according to the algebraic rule, is

$$c_1 c_2 = (1 + i)(-1 + i) = -2 + 0i = -2. \tag{1.61}$$

Now, let us take their polar representation

$$c_1 = \left(\sqrt{2}, \frac{\pi}{4}\right), \qquad c_2 = \left(\sqrt{2}, \frac{3\pi}{4}\right). \tag{1.62}$$

(Carry out the calculations!) Therefore, their product using the rule described earlier is

$$c_1 c_2 = \left(\sqrt{2} \times \sqrt{2}, \frac{\pi}{4} + \frac{3\pi}{4}\right) = (2, \pi). \tag{1.63}$$

If we revert to its Cartesian coordinates, we get

$$(2 \times \cos(\pi), 2 \times \sin(\pi)) = (-2, 0), \tag{1.64}$$

which is precisely the answer we arrived at with the algebraic calculation in Equation (1.61).

Figure 1.7 is the graphical representation of the two numbers and their product. As you can see, we simply rotated the first vector by an angle equal to the phase of the second vector and multiplied its length by the length of the second vector. □

Exercise 1.3.4 Multiply $c_1 = -2 - i$ and $c_2 = -1 - 2i$ using both the algebraic and the geometric method; verify that the results are identical. ■

Reader Tip. Most of the rest of this chapter are basic ideas in complex numbers; however, they will not really be used in the text. The part on roots of unity will arise in our discussion of Shor's algorithm (Section 6.5). The rest is included for the sake

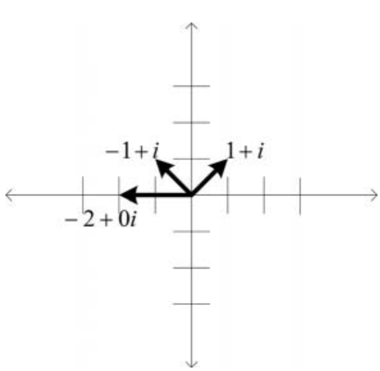

Figure 1.7. Two complex numbers and their product.

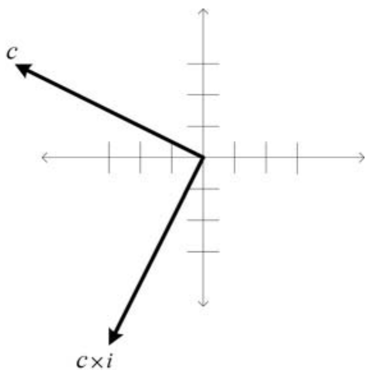

Figure 1.8. Multiplication by i.

of completeness. The restless reader can safely skim the rest of this chapter on the first reading. ♡

We have implicitly learned an important fact: multiplication in the complex domain has something to do with *rotations* of the complex plane. Indeed, observe just what happens by left or right multiplication by i:

$$c \longmapsto c \times i. \tag{1.65}$$

i has modulus 1, so the magnitude of the result is exactly equal to that of the starting point. The phase of i is $\frac{\pi}{2}$, so multiplying by i has the net result of rotating the original complex number by 90°, a right angle. The same happens when we multiply any complex number; so we can safely conclude that multiplication by i is a right-angle counterclockwise rotation of the complex plane, as shown in Figure 1.8.

Exercise 1.3.5 Describe the geometric effect on the plane obtained by multiplying by a real number, i.e., the function

$$c \longmapsto c \times r_0, \tag{1.66}$$

where r_0 is a fixed real number. ■

Exercise 1.3.6 Describe the geometric effect on the plane obtained by multiplying by a generic complex number, i.e., the function

$$c \longmapsto c \times c_0, \tag{1.67}$$

where c_0 is a fixed complex number. ■

Programming Drill 1.3.2 *If you like graphics, write a program that accepts a small drawing around the origin of the complex plane and a complex number. The program should change the drawing by multiplying every point of the diagram by a complex number.*

Now that we are armed with a geometric way of looking at multiplication, we can tackle division as well. After all, division is nothing more than the inverse operation of multiplication. Assume that

$$c_1 = (\rho_1, \theta_1) \quad \text{and} \quad c_2 = (\rho_2, \theta_2), \tag{1.68}$$

are two complex numbers in polar form; what is the polar form of $\frac{c_1}{c_2}$? A moment's thought tells us that it is the number

$$\frac{c_1}{c_2} = \left(\frac{\rho_1}{\rho_2}, \theta_1 - \theta_2\right). \tag{1.69}$$

In words, we divide the magnitudes and subtract the angles.

Example 1.3.5 Let $c_1 = -1 + 3i$ and $c_2 = -1 - 4i$. Let us calculate their polar coordinates first:

$$c_1 = \left(\sqrt{(-1)^2 + 3^2}, \tan^{-1}\left(\frac{3}{-1}\right)\right) = (\sqrt{10}, \tan^{-1}(-3)) = (3.1623, 1.8925), \tag{1.70}$$

$$c_2 = \left(\sqrt{(-1)^2 + (-4)^2}, \tan^{-1}\left(\frac{-4}{-1}\right)\right) = (\sqrt{17}, \tan^{-1}(4)) = (4.1231, -1.8158), \tag{1.71}$$

therefore, in polar coordinates the quotient is

$$\frac{c_1}{c_2} = \left(\frac{3.1623}{4.1231}, 1.8925 - (-1.8158)\right) = (0.7670, 3.7083). \tag{1.72}$$

□

Exercise 1.3.7 Divide $2 + 2i$ by $1 - i$ using both the algebraic and the geometrical method and verify that the results are the same. ■

You may have noticed that in Section 1.2, we have left out two important operations: powers and roots. The reason was that it is much easier to deal with them in the present geometric setting than from the algebraic viewpoint.

Let us begin with powers. If $c = (\rho, \theta)$ is a complex number in polar form and n a positive integer, its nth power is just

$$c^n = (\rho^n, n\theta), \tag{1.73}$$

because raising to the nth power is multiplying n times. Figure 1.9 shows a complex number and its first, second, and third powers.

Exercise 1.3.8 Let $c = 1 - i$. Convert it to polar coordinates, calculate its fifth power, and revert the answers to Cartesian coordinates. ■

What happens when the base is a number of magnitude 1? Its powers will also have magnitude 1; thus, they will stay on the same unit circle. You can think of the various powers $1, 2, \ldots$ as time units, and a needle moving counterclockwise at

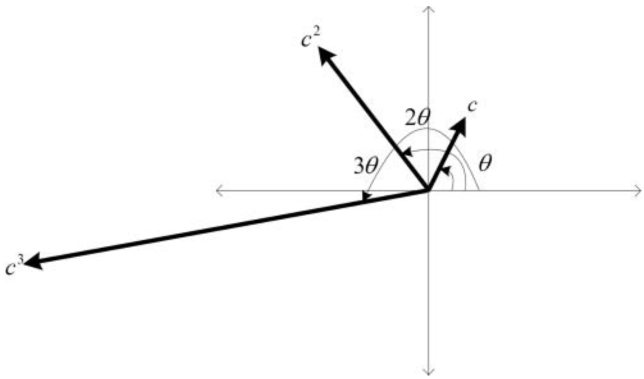

Figure 1.9. A complex number and its square and cube.

constant speed (it covers exactly θ radians per time unit, where θ is the phase of the base).

Let us move on to roots. As you know already from high-school algebra, a root is a fractional power. For instance, the square root means raising the base to the power of one-half; the cube root is raising to the power of one-third; and so forth. The same holds true here, so we may take roots of complex numbers: if $c = (\rho, \theta)$ is a complex in polar form, its nth root is

$$c^{\frac{1}{n}} = \left(\rho^{\frac{1}{n}}, \frac{1}{n}\theta\right). \tag{1.74}$$

However, things get a bit more complicated. Remember, the phase is defined only up to multiples of 2π. Therefore, we must rewrite Equation (1.74) as

$$c^{\frac{1}{n}} = \left(\sqrt[n]{\rho}, \frac{1}{n}(\theta + k2\pi)\right). \tag{1.75}$$

It appears that there are *several* roots of the same number. This fact should not surprise us: in fact, even among real numbers, roots are not always unique. Take, for instance, the number 2 and notice that there are two square roots, $\sqrt{2}$ and $-\sqrt{2}$.

How many nth roots are there? There are precisely n nth roots for a complex number. Why? Let us go back to Equation (1.75).

$$\frac{1}{n}(\theta + 2k\pi) = \frac{1}{n}\theta + \frac{k}{n}2\pi. \tag{1.76}$$

How many different solutions can we generate by varying k? Here they are:

$k = 0$	$\frac{1}{n}\theta$
$k = 1$	$\frac{1}{n}\theta + \frac{1}{n}2\pi$
$\vdots$	$\vdots$
$k = n - 1$	$\frac{1}{n}\theta + \frac{n-1}{n}2\pi$

(1.77)

That is all: when $k = n$, we obtain the first solution; when $k = n + 1$, we obtain the second solution; and so forth. (Verify this statement!)

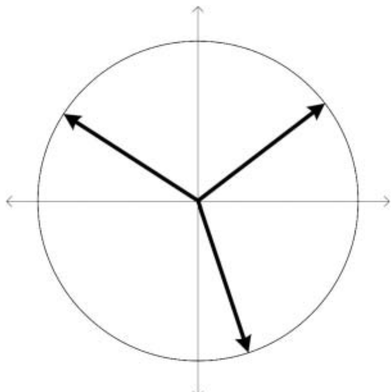

Figure 1.10. The three cube roots of unity.

To see what is happening, let us assume that $\rho = 1$; in other words, let us find nth roots of a complex number $c = (1, \theta)$ on the unit circle. The n solutions in Equation (1.77) can be interpreted in the following way: Draw the unit circle, and the vectors whose phase is $\frac{1}{n}\theta$, $\frac{1}{n}\theta$ plus an angle equal to $\frac{k}{n}$ of the entire circle, where $k = 1, \ldots, n$. We get precisely the vertices of a regular polygon with n edges. Figure 1.10 is an example when $n = 3$.

Exercise 1.3.9 Find all the cube roots of $c = 1 + i$. ■

By now we should feel pretty comfortable with the polar representation: we know that any complex number, via the polar-to-Cartesian function, can be written as

$$c = \rho(\cos(\theta) + i\ \sin(\theta)). \tag{1.78}$$

Let us introduce yet another notation that will prove to be very handy in many situations. The starting point is the following formula, known as **Euler's formula**:

$$e^{i\theta} = \cos(\theta) + i\ \sin(\theta). \tag{1.79}$$

The full justification of the remarkable formula of Euler lies outside the scope of this book.[6] However, we can at least provide some evidence that substantiates its

[6] For the calculus-savvy reader: Use the well-known Taylor expansions.

$$e^x = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots, \tag{1.80}$$

$$\sin(x) = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + \cdots, \tag{1.81}$$

$$\cos(x) = 1 - \frac{x^2}{2} + \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots. \tag{1.82}$$

Assume that they hold for complex values of x. Now, formally multiply $\sin(x)$ by i and add componentwise $\cos(x)$ to obtain Euler's formula.