Computational MR imaging
Laboratory 7: Parallel Imaging III: Non-Cartesian Imaging and Iterative Reconstruction

Code submission is due by 12:00 before the next Thursday lab section. Please upload your code to StudOn in a described format. Late submissions will not be accepted.

Learning objectives

- deal with iterative parallel imaging reconstruction for non-Cartesian sampling patterns, using gradient descent.
- explore the conjugate gradient SENSE method.

1) Derivation of gradient descent (analytical):

In gradient descent methods, we need to calculate the gradient of our objective function. In the lecture (slide 21), we said that the gradient is:

$$\frac{\partial}{\partial x}\|Ax - b\|_2^2 = 2A^T(Ax - b)$$

Show that this is indeed true. For this exercise, assume that A, x and b are real valued (the derivation for complex numbers is more involved). To simplify the notation, you can do the derivation with the assumption that A is a 2x2 matrix without loss of generality. You should only need standard linear algebra and vector analysis for this proof. Hints:

i. Remember these expressions: $\|Ax - b\|_2^2 = (Ax - b)^T(Ax - b)$

$$x^T A^T b = (b^T A x)^T$$

ii. It will be useful during the derivation to use the following substitutions to simplify the notation: $\underbrace{2b^T A}_{c^T} x$ and $x^T \underbrace{A^T A}_{B} x$

2) Iterative image reconstruction with gradient descent

a) The data file data_radial_brain_4ch.mat is loaded on these variables:
kdata (512,64,4): radial k-space data, 64 spokes, 512 readout points, 4 channels
sens_maps (256,256,4): receive coil sensitivity maps, 4 channel coil
traj (512,64): radial trajectory
cd_weights *(512,64):* density compensation
gt (256,256): Sensitivity combined fully sampled ground truth.

b) Plot the data and the sampling trajectory.

c) Single step NUFFT reconstruction.

i) Implement get_nufft_ob method.

(1) Returns a NUFFT forward operator object.
(2) The method takes *im_size* and *grid_size*.
(3) Think about Readout oversampling and its corresponding original shape.

ii) Implement get_nufft_adj_ob.
(1) Returns a NUFFT adjoint operator object.
(2) The method takes *im_size* and *grid_size*.
(3) Think about Readout oversampling and its corresponding original shape.

iii) Implement get_nufft_kdata.
(1) Convert kdata to the format that NUFFT operator can use.
(2) When dc_wieghts is given, apply it to kdata.

iv) Implement get_nufft_traj.
(1) Convert traj to the format that NUFFT operator can use.

v) Implement get_nufft_sens_maps.
(1) Convert traj to the format that NUFFT operator can use.

vi) Implement nufft_recon.
(1) Reconstruct the kdata using the NUFFT operator in one step.
(2) NUFFT reconstruction should preserve the energy between transformations. This applys to all NUFFT operations afterwards.

vii) Do a simple gridding reconstruction using density compensation.

d) Gradient descent NUFFT reconstruction.

i) Implement calc_grad
(1) Calculates the gradient of the objective function for the NUFFT operator.

$$\frac{\partial}{\partial u}||Ku - g||_2^2 = 2K^T(Ku - g)$$

(2) See the lecture note for detailed information

ii) Implement nufft_gd_recon
(1) Reconstruct the kdata by solving an objective function using the gradient descent method with the NUFFT operator.
(2) Choose a stepsize t=10-2.
(3) Initialize u with a 0s matrix.
(4) Run for e.g. 100 iterations

e) Plot results.

i) Plot the fully sampled ground truth, the single step NUFFT reconstruction, and the gradient descent reconstructed image

ii) Plot the difference images between the reconstructed images and the ground truth.

f) Plot RMSE to the ground truth and the L2 norm of the gradient over the iterations

g) Play around with hyperparameters, such as a step size and the number of iterations and find the optimal hyperparameters.

   i) Plot NMSE like the task 2.5 for different hyperparameter setups and discuss convergences of them.

3) CG-SENSE

a) Implement nufft_cg. Look at the appendix 1.
   i) Solve an objective function using the conjugate gradient method with the NUFFT operator.
   ii) Initialize x with a 0s matrix.
   iii) Think about what is b and A in the algorithm of the appendix 1.
b) Reconstruct the kdata by solving an objective function using the conjugate gradient method with the NUFFT operator. Use cg_sense method.
c) Compare the convergence behavior of CG to that of gradient descent in exercise
d) Plot the fully sampled ground truth, the single step NUFFT reconstruction, the GD reconstruction, and the CG reconstructed image and discuss their NMSE values.
e) Plot difference images of those reconstructions to the ground truth.
f) Repeat CG reconstruction with noise instead of the k-space data and plot the result in k-space. Perform at least 500 iterations. What did you just obtain?

Appendix 1

Conjugate gradient algorithm

$\mathbf{x}_0 = 0$	Initializing $\mathbf{x}_0$ to zeros
$\mathbf{r}_0 := \mathbf{b} - \mathbf{A}\mathbf{x}_0$	Initializing residual $\mathbf{r}_0$
$\mathbf{p}_0 := \mathbf{r}_0$	Initializing $\mathbf{p}_0$
$k := 0$	
repeat	
$\quad \alpha_k := \frac{\mathbf{r}_k^T \mathbf{r}_k}{\mathbf{p}_k^T \mathbf{A}\mathbf{p}_K}$	Step length
$\quad \mathbf{x}_{k+1} := \mathbf{x}_k + \alpha_k \mathbf{p}_k$	Update step
$\quad \mathbf{r}_{k+1} := \mathbf{r}_k - \alpha_k \mathbf{A}\mathbf{p}_k$	Update residual
$\quad$ if $\mathbf{r}_{k+1} < \epsilon$	
$\qquad$ break	
$\quad \beta_k := \frac{\mathbf{r}_{k+1}^T \mathbf{r}_{k+1}}{\mathbf{r}_k^T \mathbf{r}_k}$	
$\quad \mathbf{p}_{k_1} := \mathbf{r}_{k+1} + \beta_k \mathbf{p}_k$	CG Search direction
$\quad k := k + 1$	
return $\mathbf{x}_{k+1}$	