Computational MRI

Compressed Sensing

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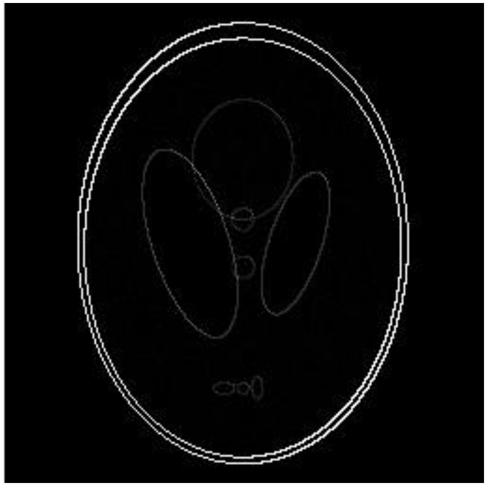
Compressed sensing: The big picture

- Exploit image sparsity/compressibility to reconstruct undersampled data

Original (non-sparse)

Gradient (sparse)



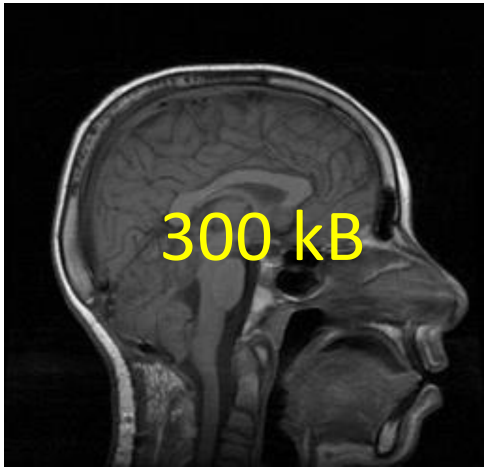
Which image would require fewer samples?

  – Nyquist: same FOV, same number of samples
  – Common sense: fewer non-zero pixels, fewer samples

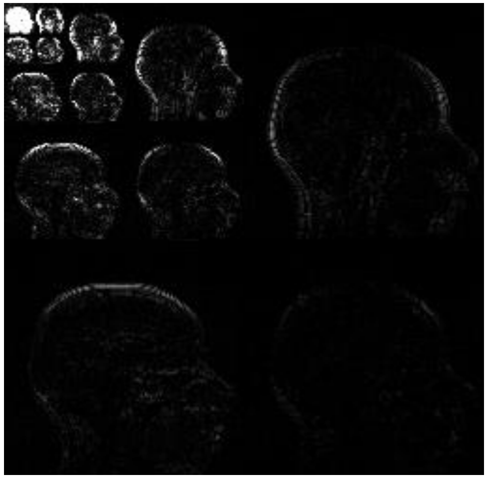
Image compression

- Essential tool for modern data storage and transmission
- Exploit pixel correlations to reduce number of bits
- First reconstruct, then compress

Fully-sampled acquisition
(Nyquist rate)

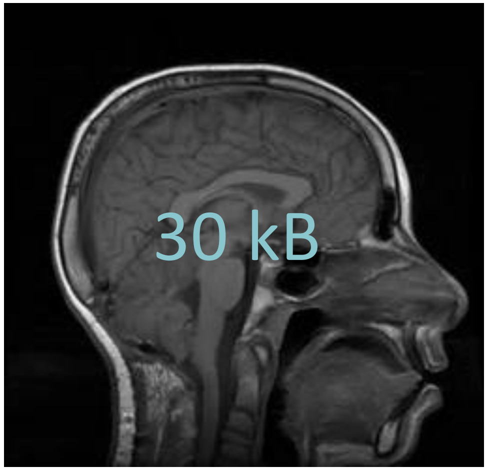


Sparsifying transform
(e.g. wavelets)



Store or transmit
non-zero coefficients only

Recover image from
sparse coefficients



10-fold compression

Nyquist sampling is inefficient

- Question
  - Why do we need to acquire samples at the Nyquist rate if we are going to throw away most of them?

- Answer
  - We don’t. Do **compressed sensing** instead
  - Build data compression in the acquisition
  - First compress, then reconstruct

Candès E,Romberg J,Tao T. IEEE Trans Inf Theory 2006; 52(2): 489-509

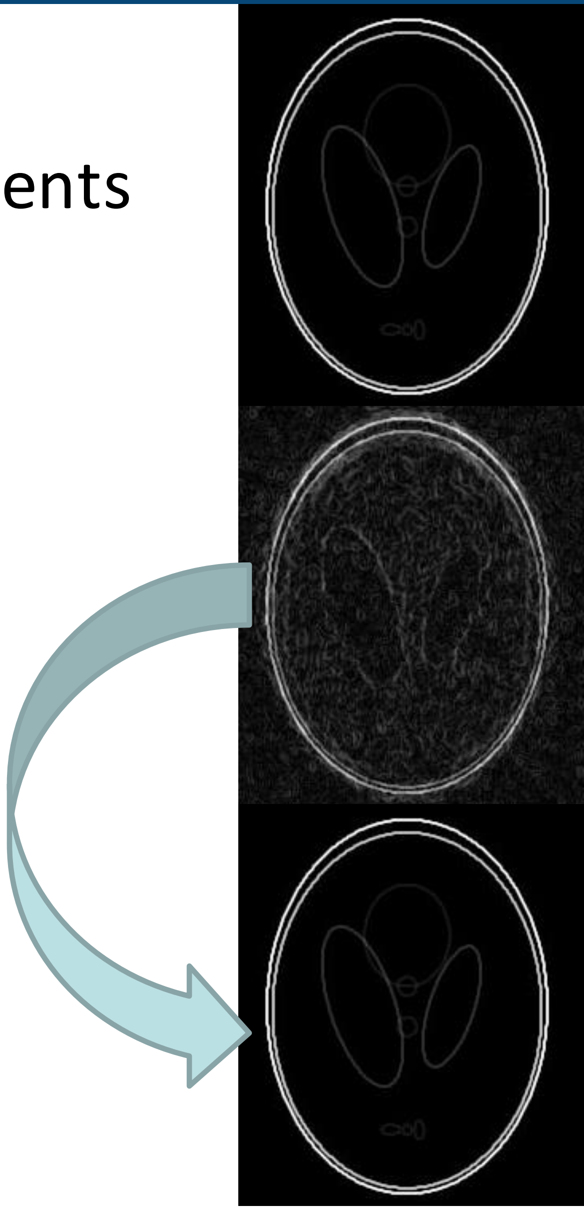
Donoho D. IEEE Trans Inf Theory 2006; 52(4): 1289-1306.

Compressed sensing components

- Sparsity
  - Represent images with a few coefficients
  - Transform: wavelets, gradient, etc.

- Incoherence
  - Noise-like aliasing artifacts

- Non-linear reconstruction
  - Remove aliasing artifacts

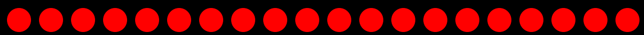


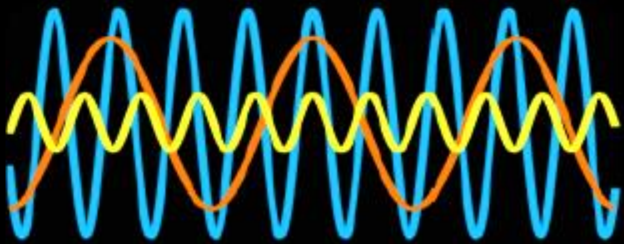
Simple compressed sensing example

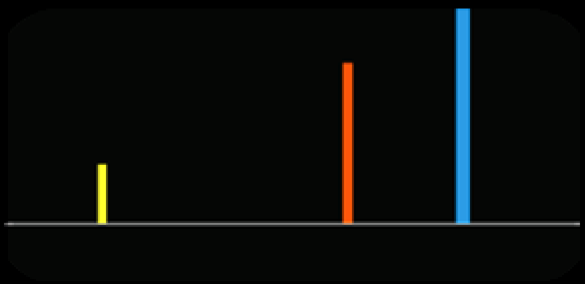
k-space

Image space

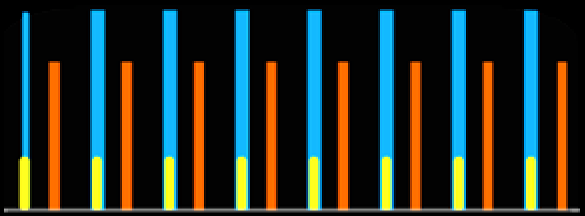
No undersampling





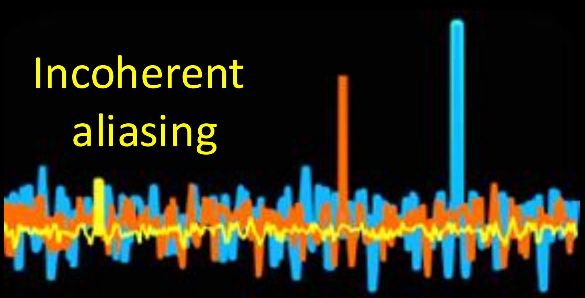


Regular undersampling



Coherent aliasing

Random undersampling



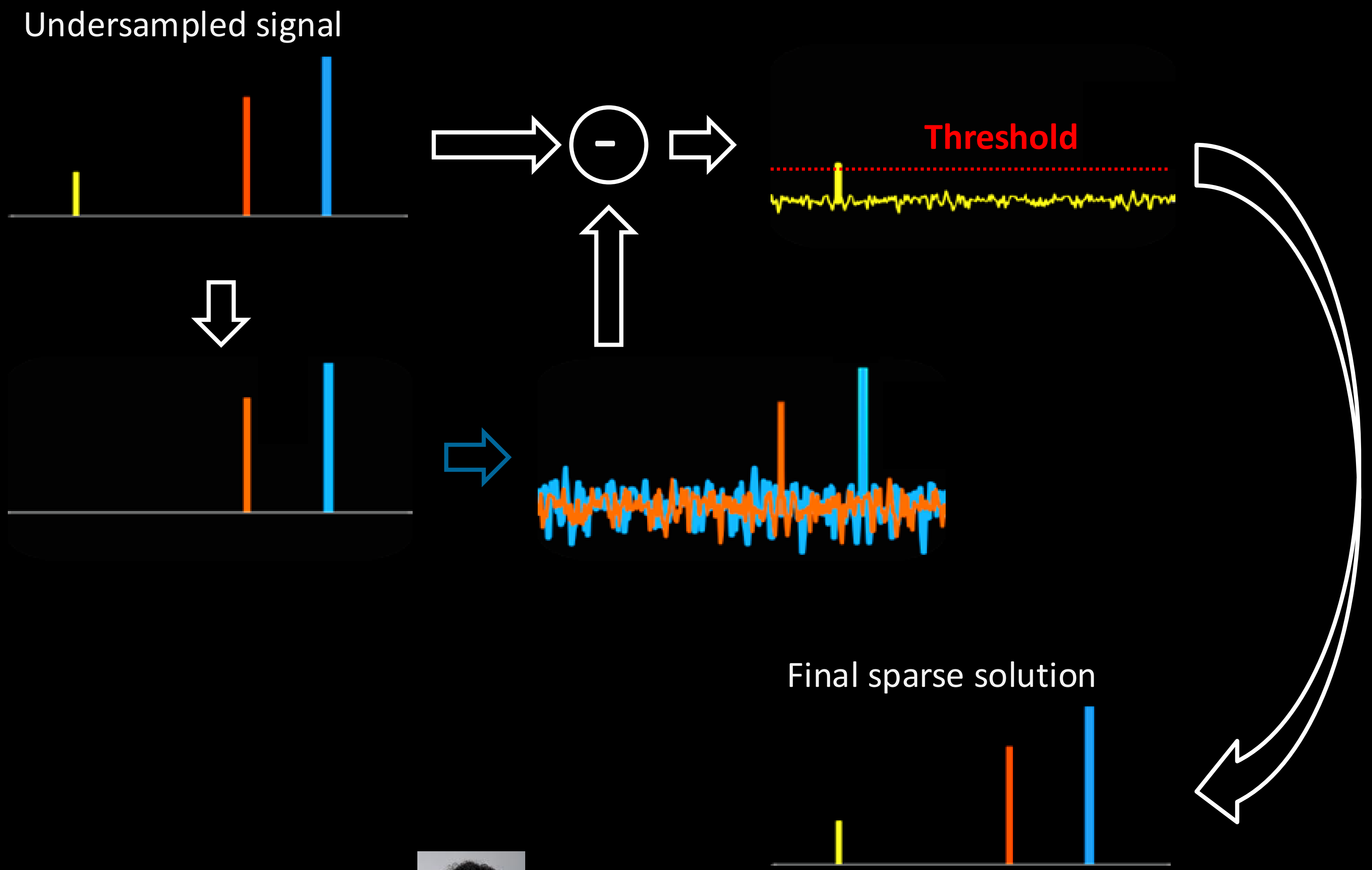
Courtesy of Miki Lustig

Simple compressed sensing example

Find sparse solution: Simple iterative algorithm

- Threshold so that 2 largest impulses are preserved
- Apply undersampling
- Subtract from previous signal
- Threshold again

Simple compressed sensing example



Courtesy of Miki Lustig

Sparsity

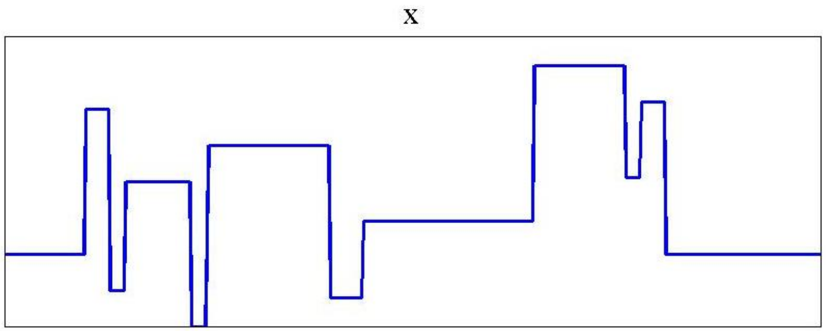
Sparsifying transforms

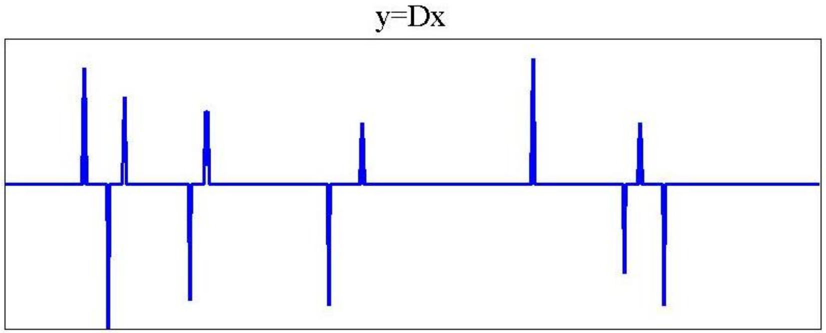
- Finite differences

$$y(n) = x(n) - x(n-1)$$

In matrix form : $y = Dx$

$$D = \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \end{bmatrix}$$





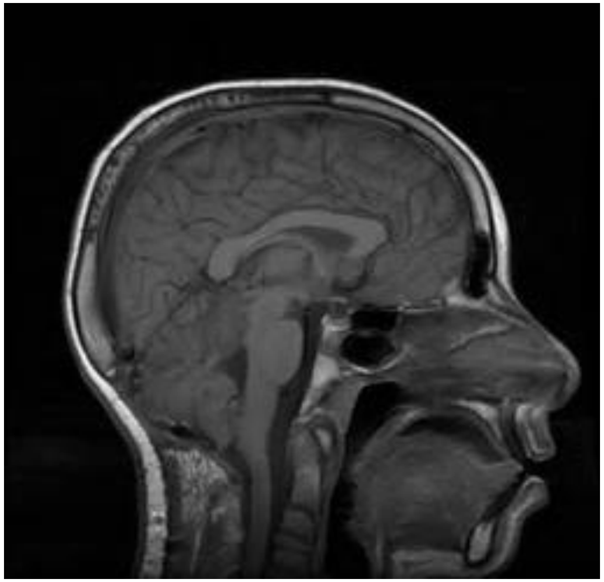
- Total variation

$$TV(x) = \sum_{n=2}^{N} |x(n) - x(n-1)| \qquad \min TV(x) = \min \|Dx\|_1$$

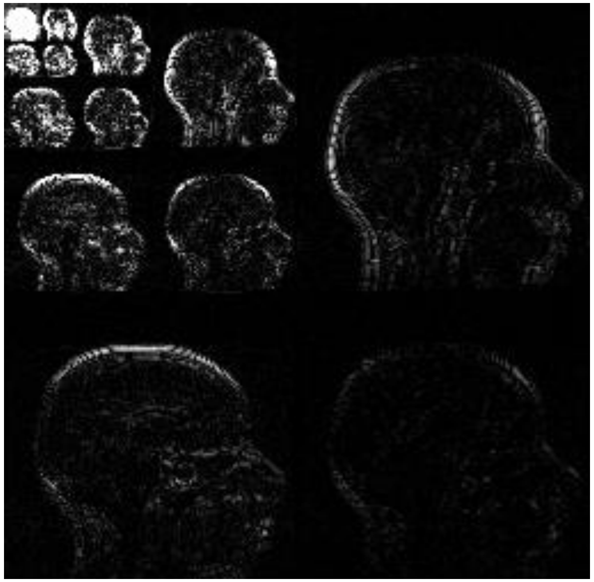
Sparsifying transforms

- Wavelets
  - Multi-resolution image representation
  - Recursive application of the wavelet function modified by the scaling function (each resolution is twice of that of the previous scale)

Brain image



Daubechies 4-tap wavelet transform



Incoherence

Transform Point Spread Function (TPSF)

- Encoding model: $\mathbf{s} = \mathbf{E}\mathbf{m}$
- Representation model: $\mathbf{p} = \mathbf{W}\mathbf{m}$ (**p** is sparse)

W is orthogonal

$$\mathbf{s} = \mathbf{E}\mathbf{W}^{H}\mathbf{p}$$

- Transform point spread function for position *r*

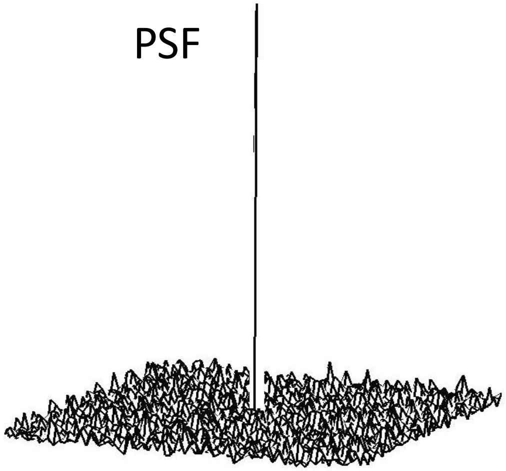
$$TPSF(r) = \mathbf{W}\mathbf{E}^{H}\mathbf{E}\mathbf{W}^{H}{}_{(r)}$$

Transform Point Spread Function (TPSF)

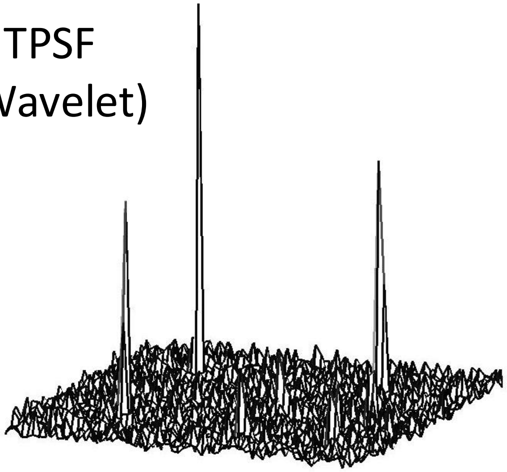
- Tool to check incoherence in the sparse domain
- Computation
  - Apply inverse FFT to sampling mask (1=sampled, 0=otherwise)
  - Apply sparsifying transform
- Incoherence = ratio of the main peak to the std of the pseudo-noise (incoherent artifacts)

Sampling mask

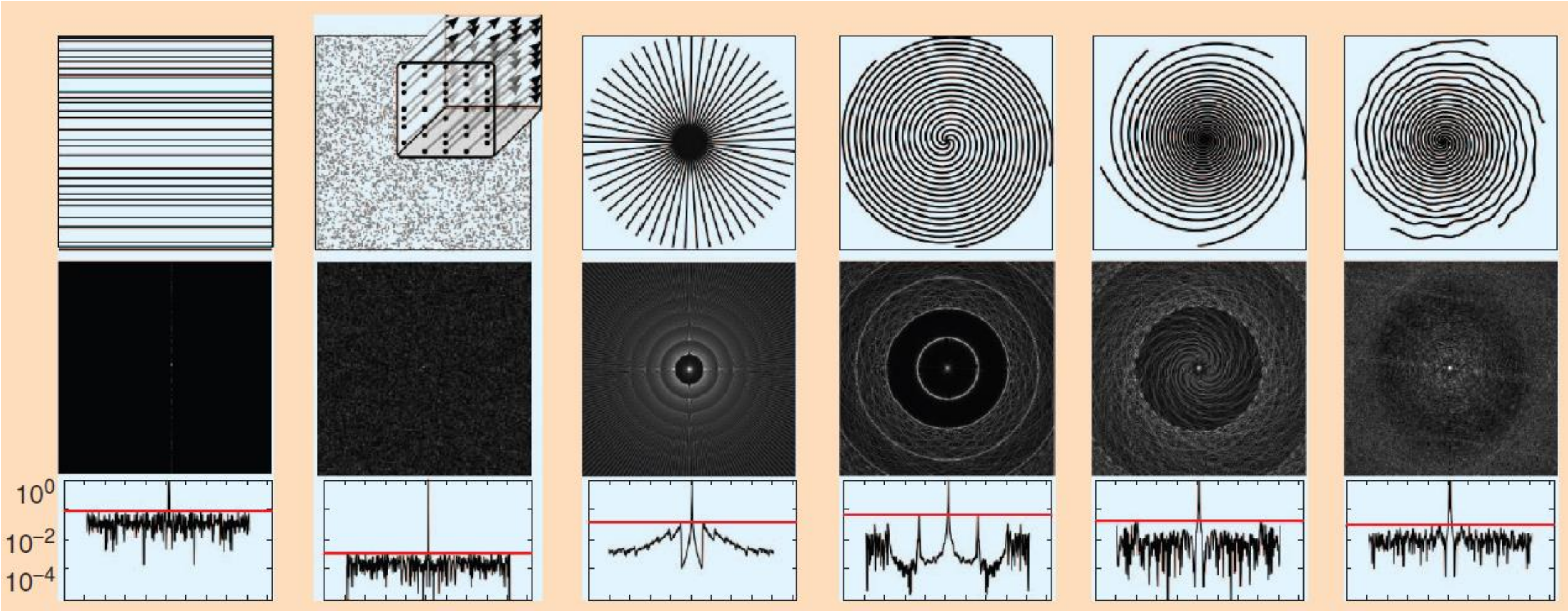
PSF



TPSF
(Wavelet)



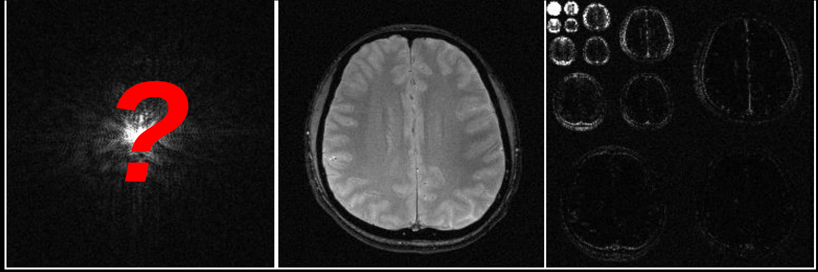
Incoherence of k-space trajectories

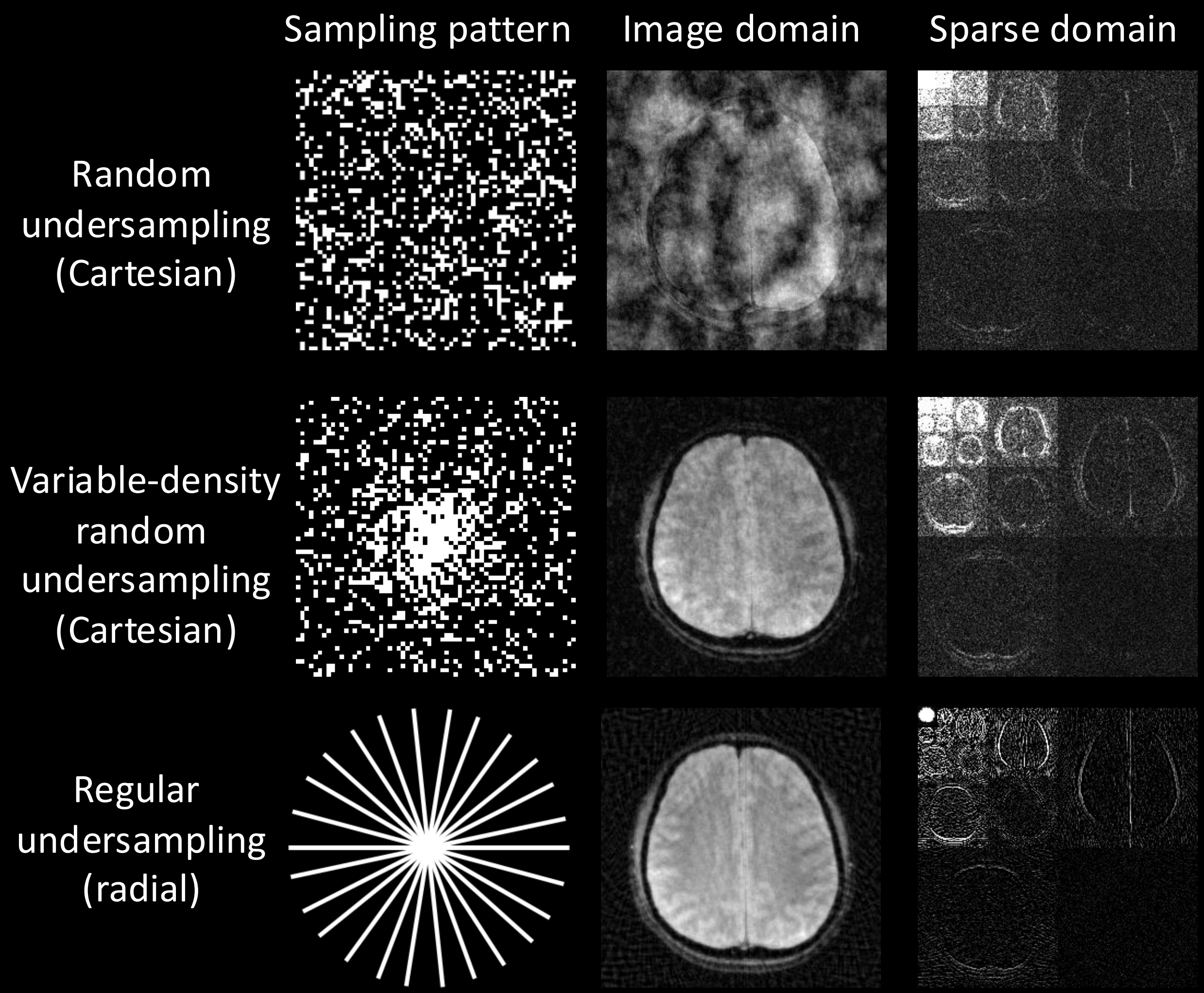


Lustig M, Donoho DL, Santos JM, Pauly JM. Compressed Sensing. MRI, IEEE Signal Processing Magazine, 2008; 25(2): 72-82

Incoherent sampling patterns

- What are the best samples?



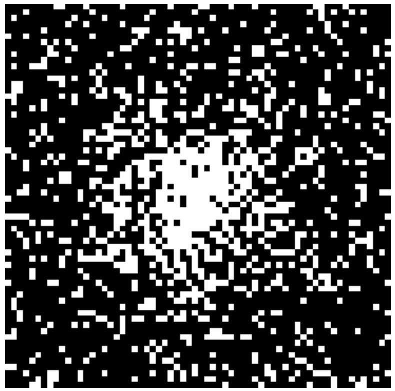


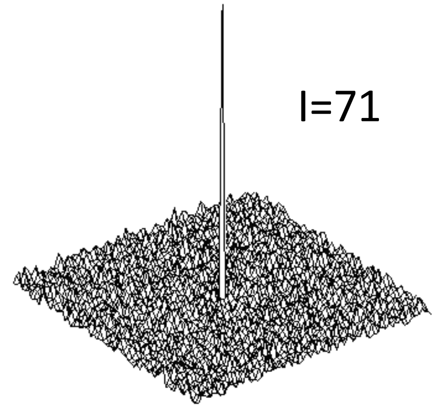
Dimensionality and incoherence

Sampling pattern
(R=4)

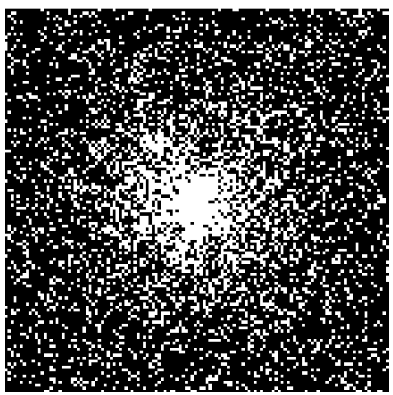
PSF

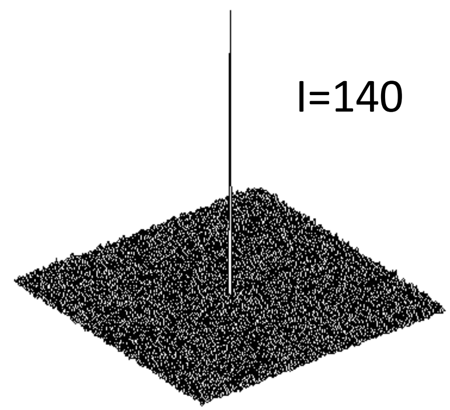
64x64
space





128x128
space





Sparse reconstruction

Compressed sensing reconstruction

- Acquisition model: $\mathbf{d} = \mathbf{Em}$

d : acquired data
E : undersampled Fourier transform
m : image to reconstruct

- Sparsifying transform: $\mathbf{T}$

$$\min_{\mathbf{m}} \|\mathbf{Tm}\|_1 \text{ subject to } \|\mathbf{Em} - \mathbf{d}\|_2 < \varepsilon$$

Sparsity

Data consistency

$$\|\mathbf{x}\|_1 = \sum_{n=1}^{N} |x_i| \quad (l_1\text{- norm of } \mathbf{x})$$

Compressed sensing reconstruction

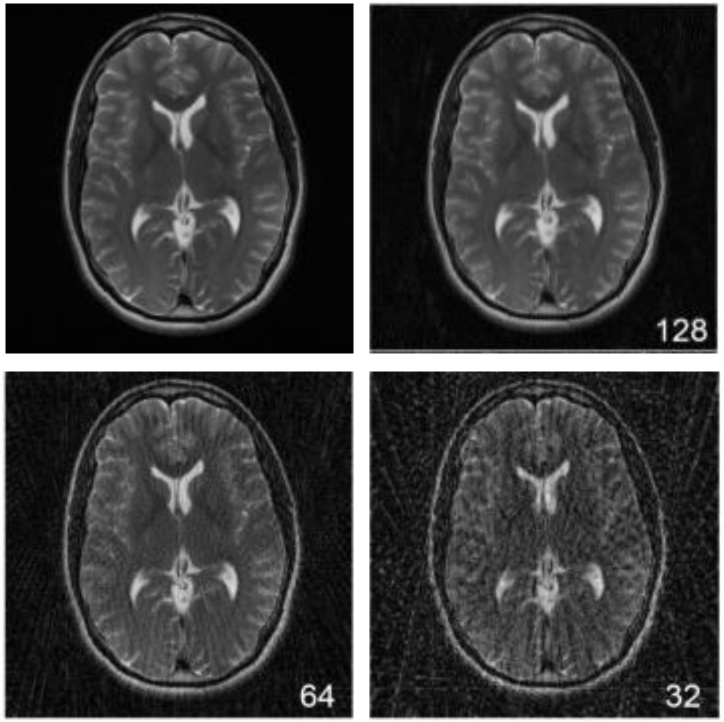
- Unconstrained optimization (in practice)

$$\min_{\mathbf{m}} \left\| \mathbf{Em} - \mathbf{d} \right\|_2^2 + \lambda \left\| \mathbf{Tm} \right\|_1$$

- Regularization parameter λ
  - Trade-off between data fidelity and removal of aliasing artifacts
  - High λ: artifact removal and denoising at the expense of image corruption (blurring, ringing, blocking, etc)
  - Low λ : no image corruption, but residual aliasing

Example: TV Norm, radial subsampling

Data set	Total Variation (a.u.)
Original (256×256)	2801



Rudin et al., Phys. D 60: 259-268 (1992)
Block et al., MRM 57: 1086-1098 (2007)
Knoll et al., MRM 65: 480-491 (2011)
Knoll et al., MRM 67: 43-41 (2012)

Compressed sensing reconstruction

- Gradient descent
  - Cost function: $C(\mathbf{m}) = \|\mathbf{Em} - \mathbf{d}\|_2^2 + \lambda \|\mathbf{Tm}\|_1$
  - Iterations: $\mathbf{m}_{n+1} = \mathbf{m}_n - \alpha \nabla C(\mathbf{m}_n)$

$$\nabla C(\mathbf{m}_n) = 2\mathbf{E}^H\left(\mathbf{Em}_n - \mathbf{d}\right) + \lambda \mathbf{T}^H \mathbf{M}^{-1} \mathbf{Tm}_n$$

M is a diagonal matrix: $M_{ii} = \sqrt{(\mathbf{Tm})_i^* (\mathbf{Tm})_i + \mu}$

Approximate assumption: Regularized l1 norm

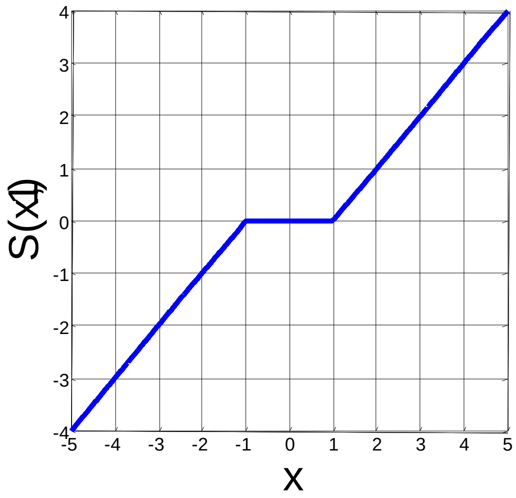
$$|x| = \sqrt{x^* x + \mu}$$

See lab exercise 7 for proof

Compressed sensing reconstruction

- Proximal gradient descent (iterative soft-thresholding)
  - Soft-thresholding operation

$$S(x,\lambda)=\begin{cases}0,\ \text{if } |x|\le\lambda \\ \dfrac{x}{|x|}\left(|x|-\lambda\right),\ \text{if } |x|>\lambda\end{cases}$$

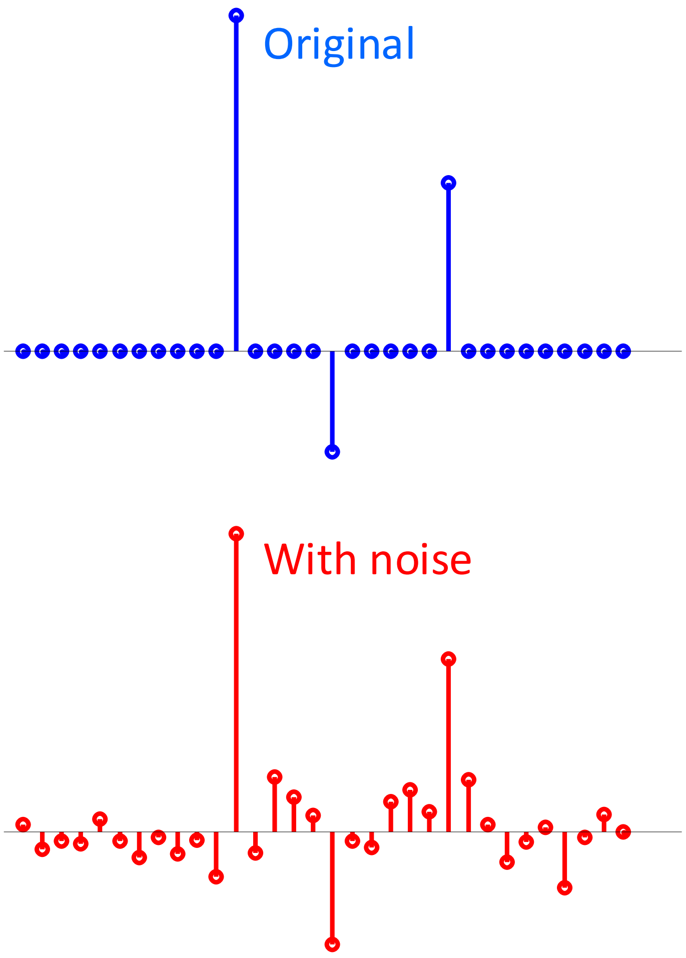


  - Our gradient gradient-descent algorithm becomes

$$\mathbf{m}_{n+1}=\mathbf{T}^{-1}\left[S\left(\mathbf{T}\left[\mathbf{m}_n-\mathbf{E}^H\left(\mathbf{E}\mathbf{m}_n-\mathbf{d}\right)\right],\lambda\right)\right]$$

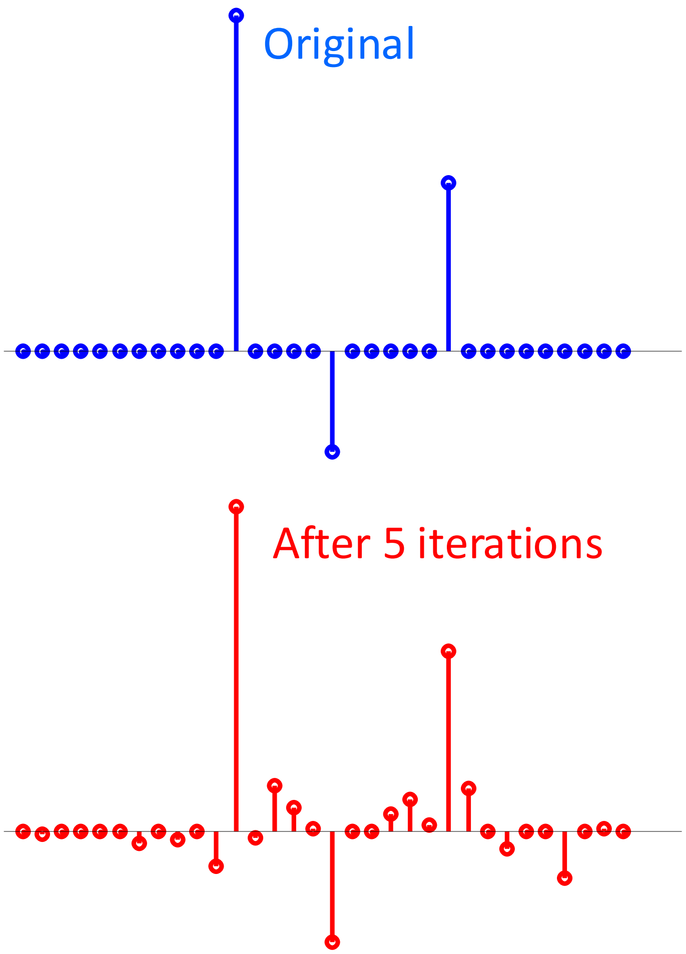
Iterative soft-thresholding

- Sparse signal denoising



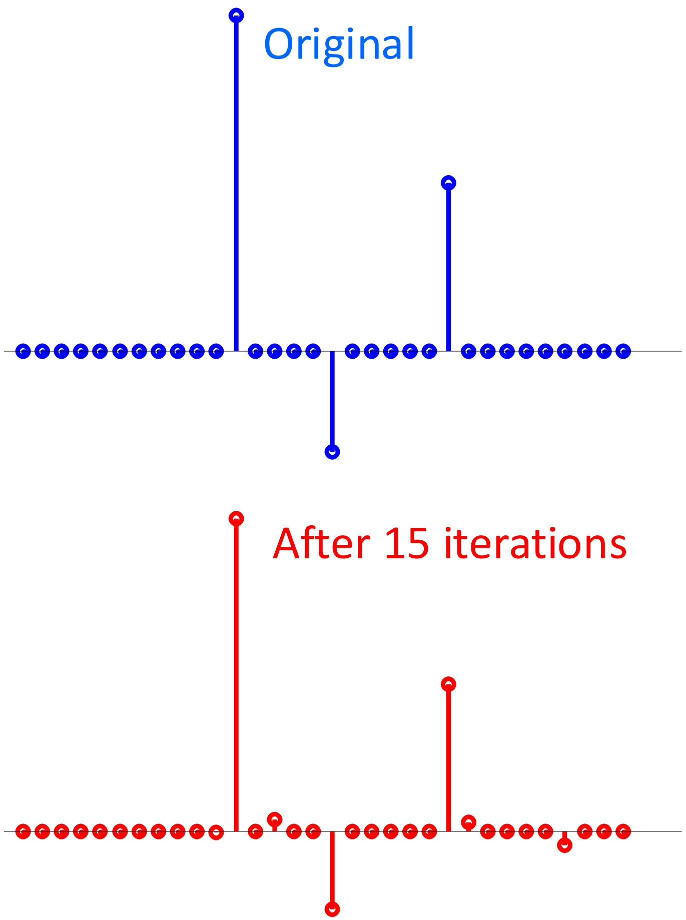
Iterative soft-thresholding

- Sparse signal denoising

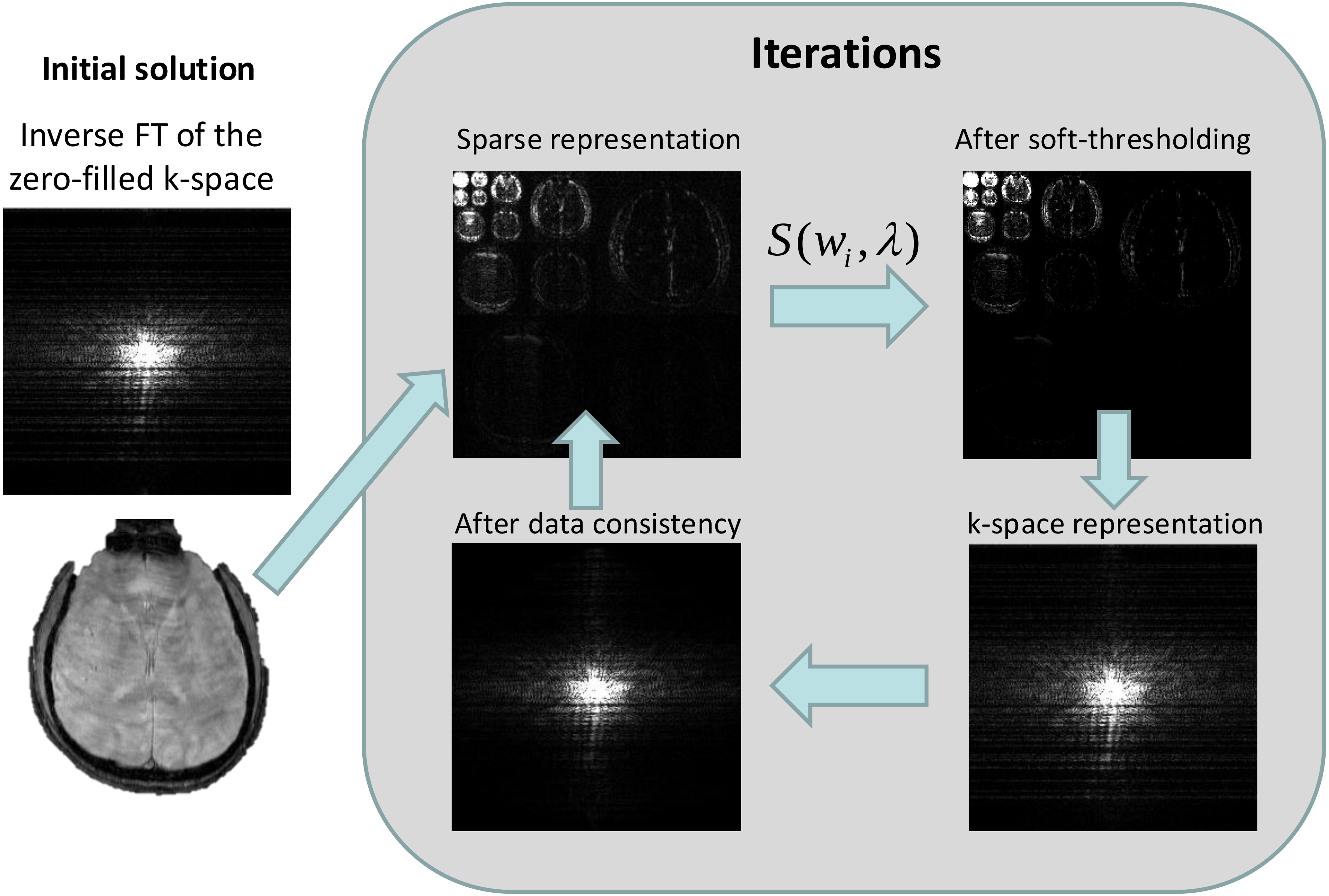


Iterative soft-thresholding

- Sparse signal denoising



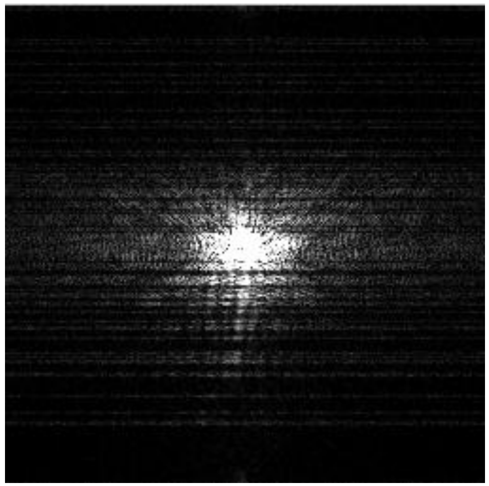
Iterative soft-thresholding



Iterative soft-thresholding (R=3)

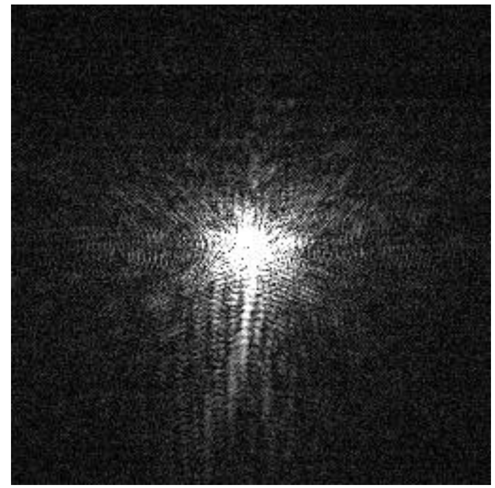
Initial solution

Inverse FT of the zero-filled k-space



After 30 iterations

Iterative soft-thresholding



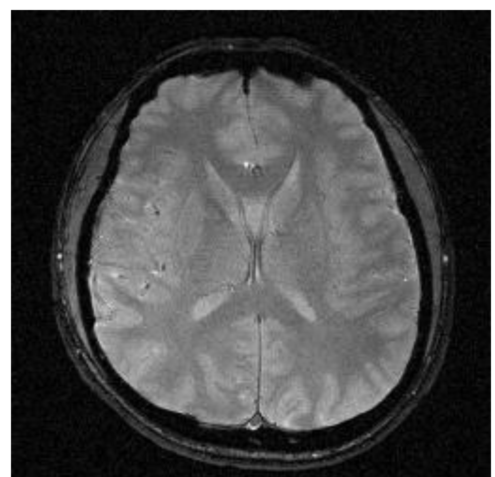
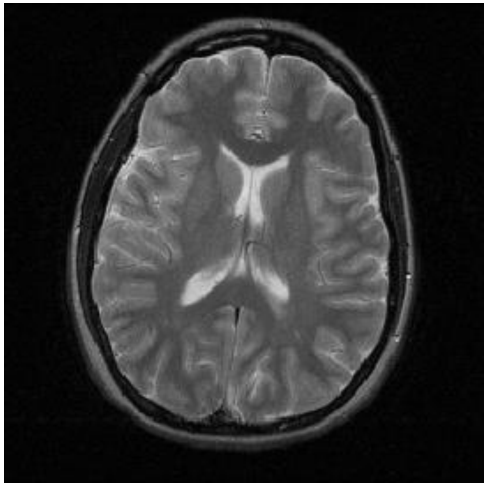


Image quality in compressed sensing

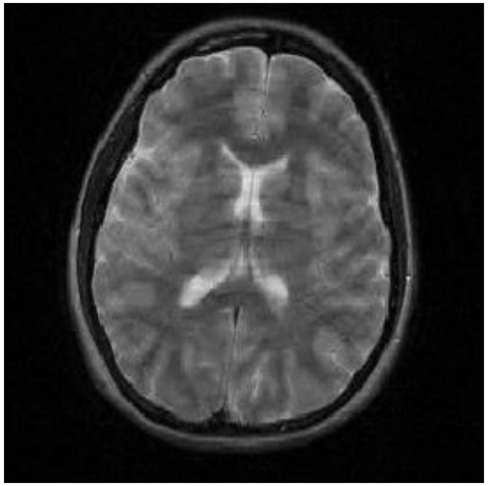
- SNR is not a good metric

- Loss of small coefficients in the sparse domain
  - Loss of contrast
  - Blurring
  - Blockiness
  - Ringing
  - Images look more synthetic

R=2



R=4



Combination of compressed sensing and parallel imaging

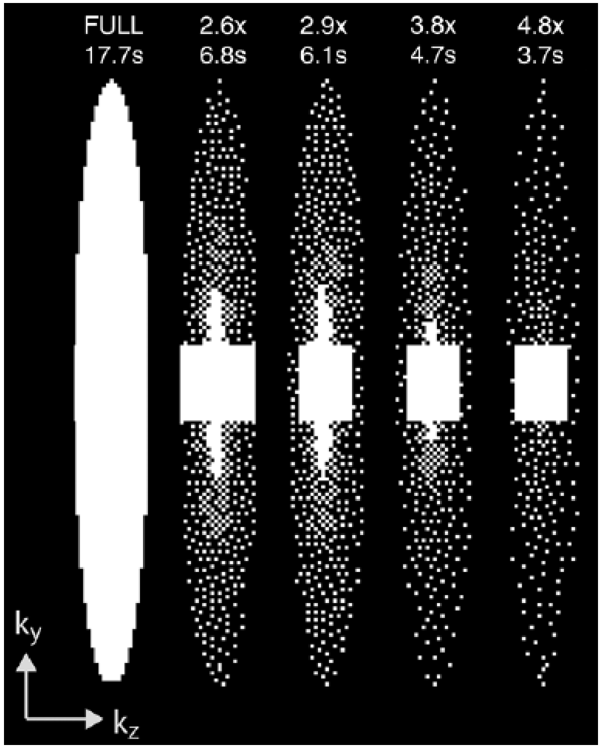
Why would CS & PI make sense?

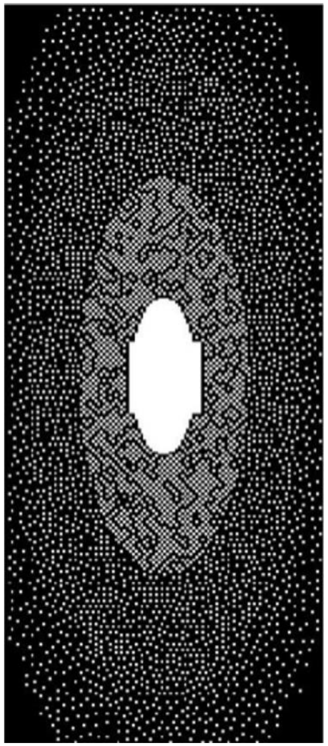
- Image sparsity and coil-sensitivity encoding are complementary sources of information
- Compressed sensing can regularize the inverse problem in parallel imaging
- Parallel imaging can reduce the incoherent aliasing artifacts

Challenges of CS & PI?

- CS requires irregular k-space sampling while PI requires regular k-space sampling

Poisson disk sampling





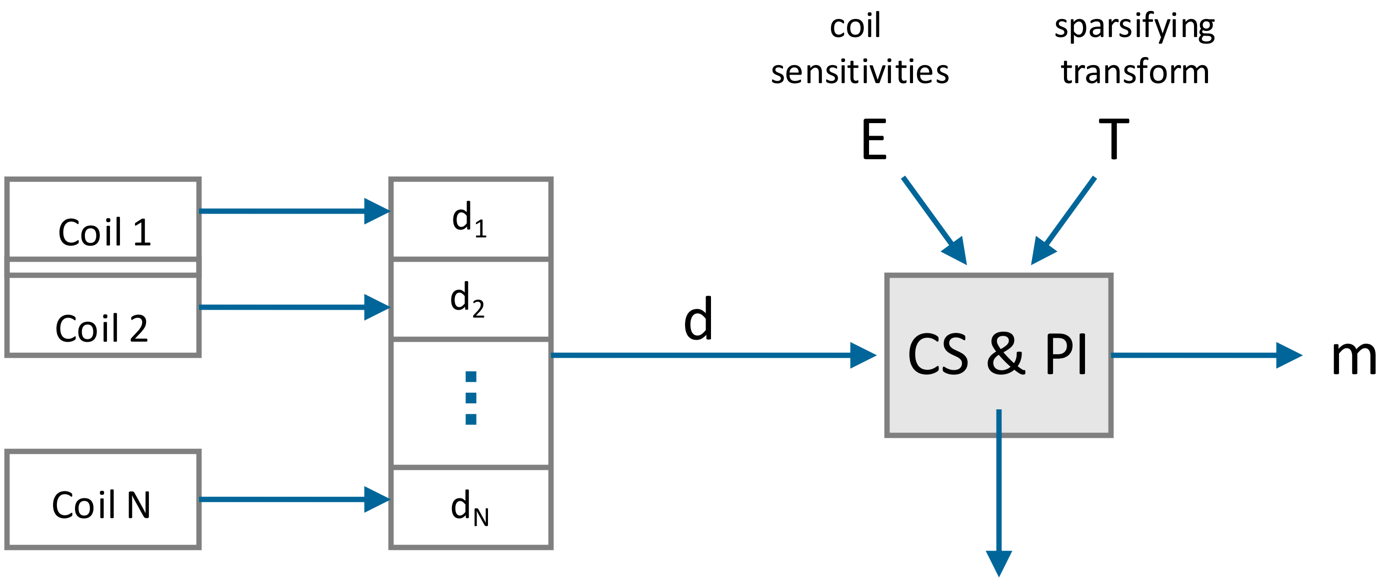
Approaches for CS&PI

- CS with SENSE parallel imaging model
  - Multicoil imaging with variational regularization
- CS with GRAPPA parallel imaging model
  - l_1-SPIRiT

Block et al. MRM 2007
Knoll et al. MRM 2011
Lustig M et al. MRM 2010

CS with SENSE parallel imaging model

- Include parallel imaging in data consistency term



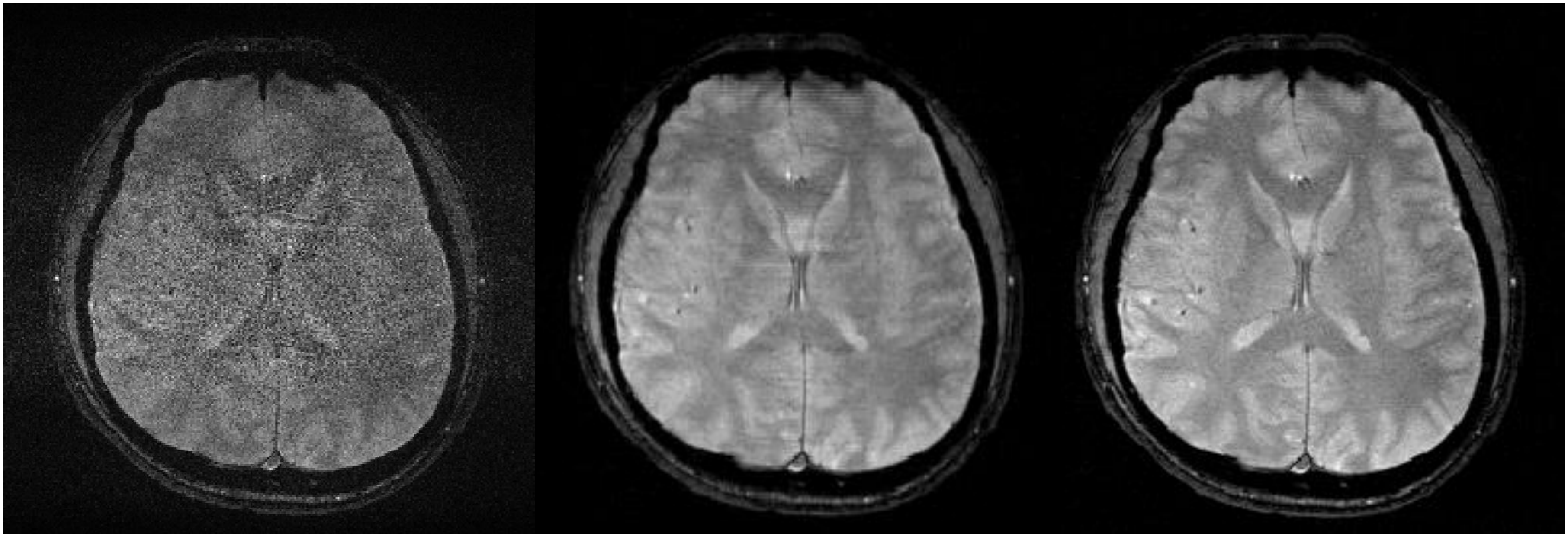
$$\min \|\mathbf{Em} - \mathbf{d}\|_2^2 + \lambda \|\mathbf{Tm}\|_1$$

SENSE data consistency

CS & PI for 2D imaging

- Siemens 3T Tim Trio
- 12-channel matrix coil array
- 4-fold acceleration

GRAPPA | Coil-by-coil CS | Joint CS & PI



Summary

- Compressed sensing
  - New sampling theorem
    - Information rate rather than pixel rate
  - Ingredients
    - Sparsity
    - Incoherence
    - Non-linear reconstruction

- Fast imaging tool for MRI
  - MR images are naturally compressible
  - Data acquisition in k-space facilitates incoherence
  - Can be combined with parallel imaging