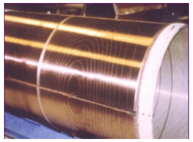
Computational MRI

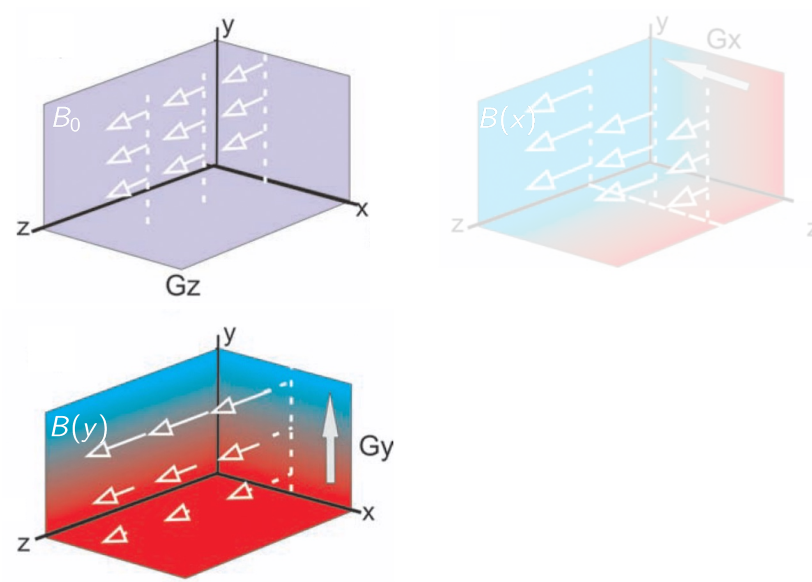
Fourier Image Reconstruction Basics

Based on a lecture by Ricardo Otazo

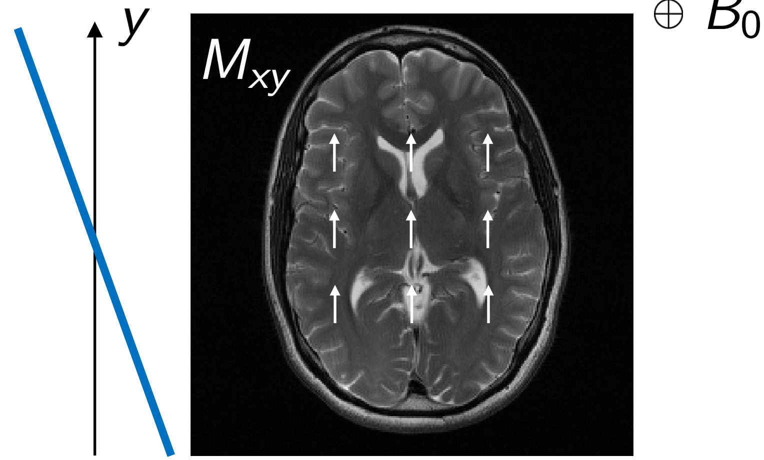
Friedrich-Alexander-Universität
Computational Imaging Lab

Interaction with gradient fields G





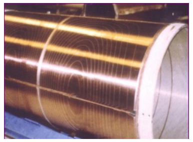
$$\omega(y) = \omega_0 + \gamma G_y y$$

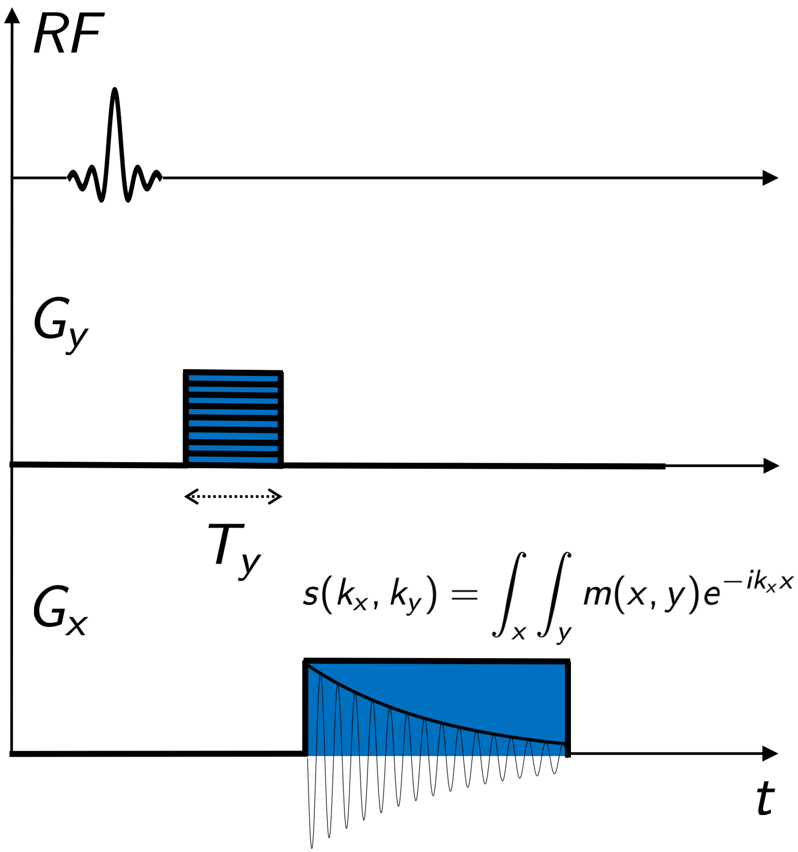


$$B(y) = B_0 + G_y y$$

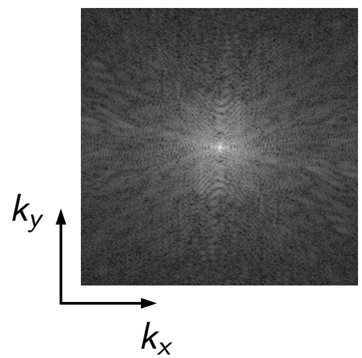
Plewes 2012

Interaction with gradient fields G

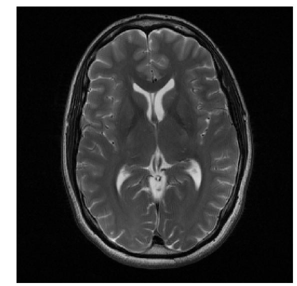




$$s(k_x, k_y) = \int_x \int_y m(x, y) e^{-ik_x x} e^{-ik_y y} dxdy$$



$$\xrightarrow{\mathcal{F}\mathcal{T}^{-1}}$$



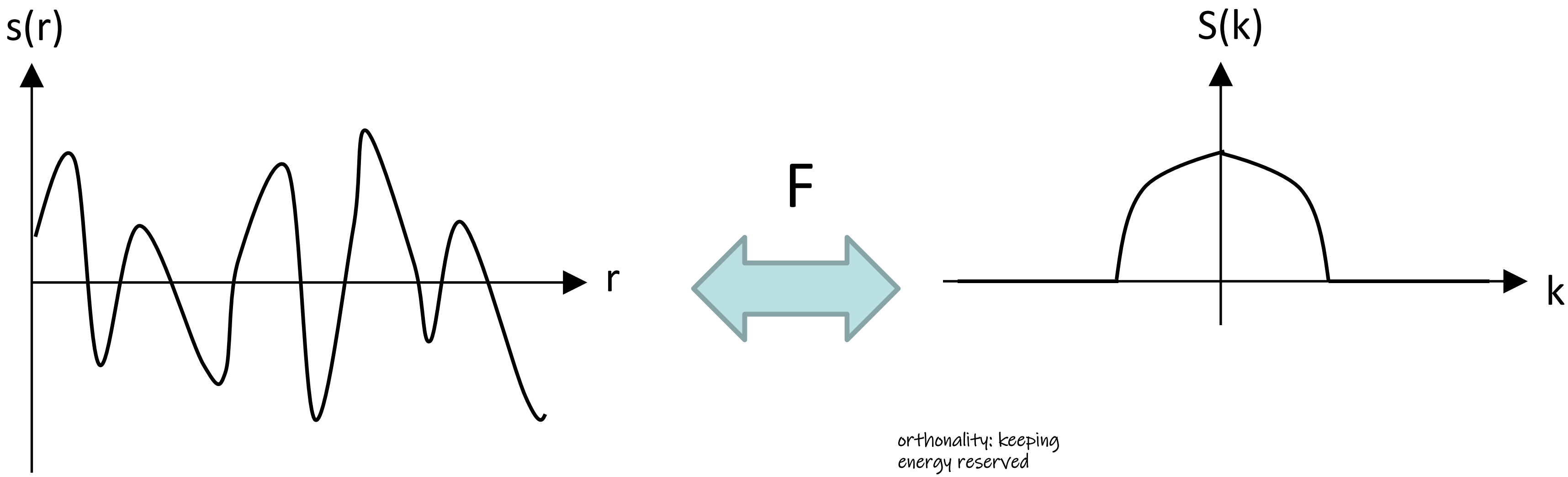
$$\phi(x, t) = \gamma x \int_0^t G_x d\tau \equiv k_x x$$

$$\phi(y, G_y) = \gamma y \int_0^{T_y} G_y d\tau \equiv k_y y$$

Fourier transform

$$S(k) = \int_{-\infty}^{\infty} s(r)\, e^{-i2\pi kr}\, dr \qquad \text{(forward)}$$

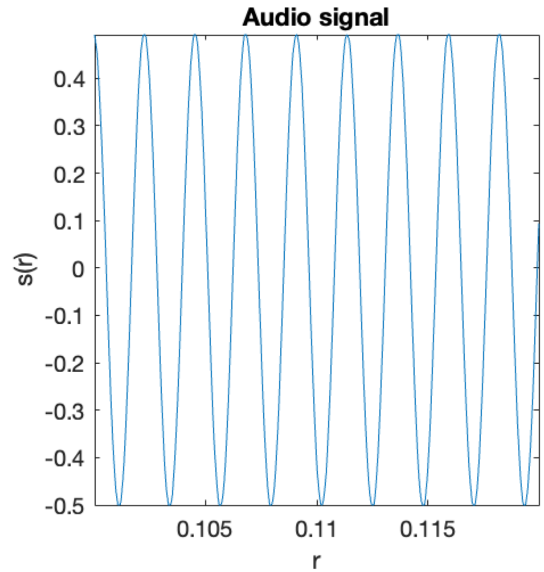
$$s(r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(k)\, e^{i2\pi kr}\, dk \qquad \text{(inverse)}$$

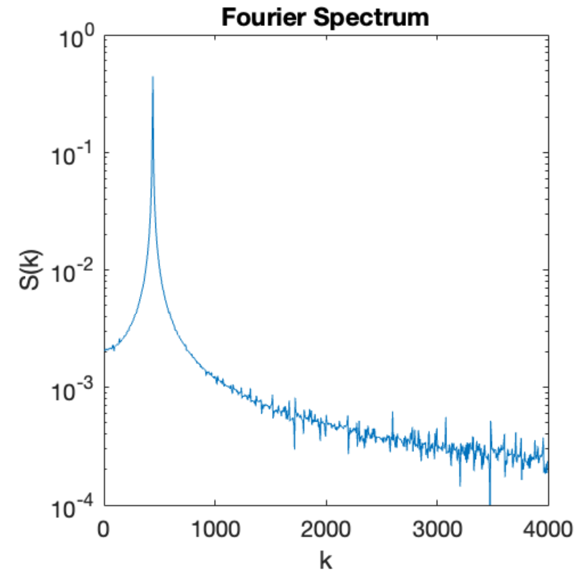


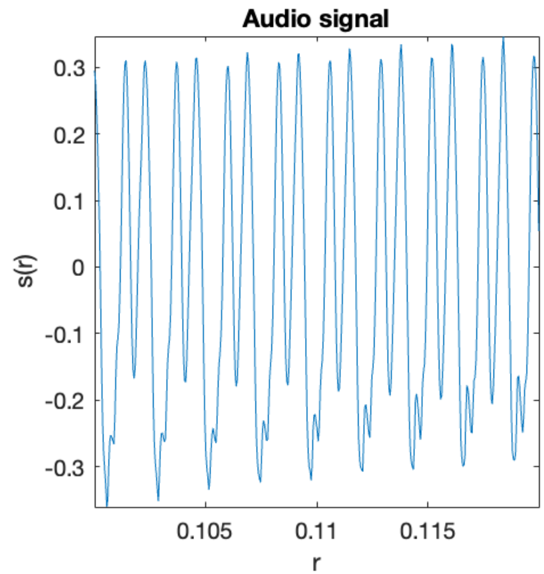
orthonality: keeping
energy reserved

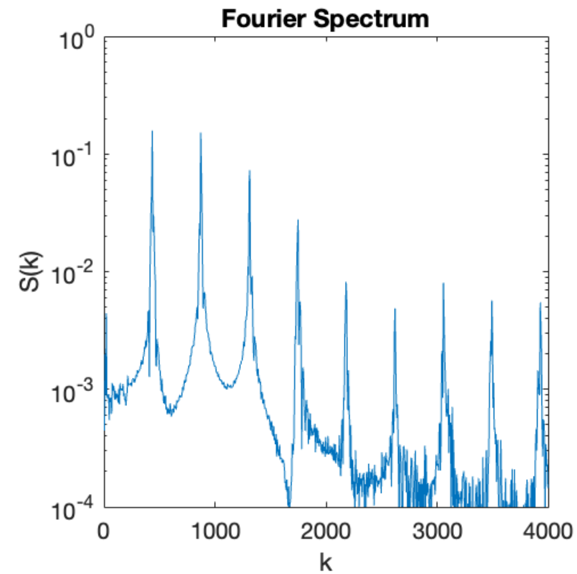
1D example

A4 (440Hz)









Fourier transform properties

Linearity: $$F\{as_1(r)+bs_2(r)\}=aS_1(k)+bS_2(k)$$

Shifting: $$F\{s(r-r_0)\}= e^{-i2\pi kr_0}S(k)$$

Modulation: $$F\{e^{i2\pi k_0 r}s(r)\}=S(k-k_0)$$

Conjugate symmetry: $$s(r) real \Rightarrow S(-k)=S^*(k)$$

Scaling: $$F\{s(ar)\}=\frac{1}{|a|}S\left(\frac{k}{a}\right)$$

Fourier transform properties

Parseval's formula: $$\int s_1(r)s_2(r)\,dr = \int S_1(k)S_2(k)\,dk$$

assumption:
orthonormality

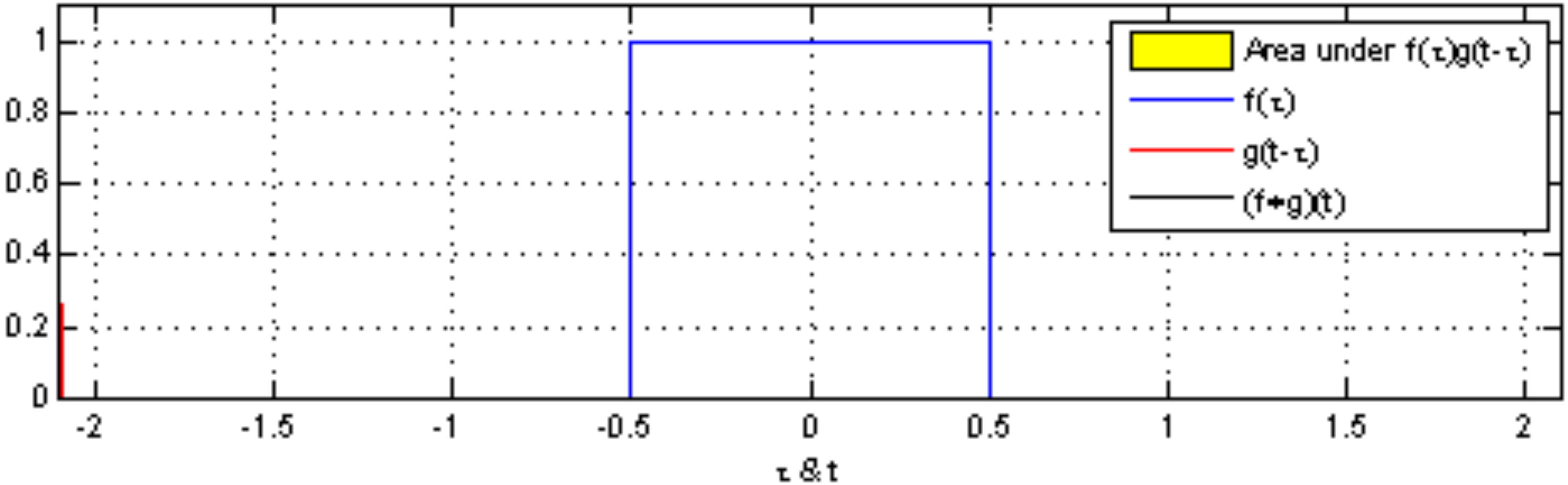
Convolution & multiplication

$$F\{s_1(r) * s_2(r)\} = S_1(k)S_2(k)$$

$$F\{s_1(r)s_2(r)\} = S_1(k) * S_2(k)$$

Convolution

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$



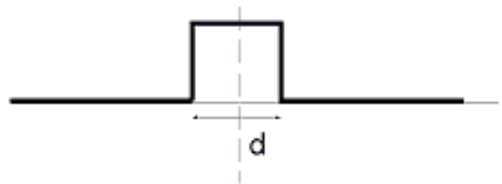
Source: Wikipedia

Fourier transform of basic functions

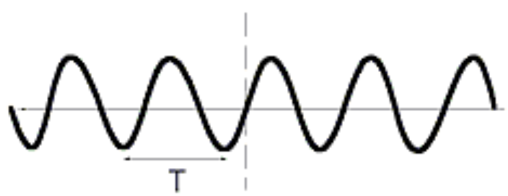
	s(r)	*S(k)*

Impulse
$\delta(r)$

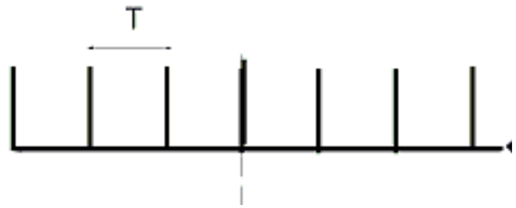
Rectangle



Sine



Comb
$\sum_{n=-\infty}^{\infty}\delta(r-nT)$



What is the Fourier representation of a triangle?

Multidimensional Fourier transform

$$S(\mathbf{k}) = \int_{-\infty}^{\infty} s(\mathbf{r})\, e^{-i2\pi \mathbf{k}\cdot\mathbf{r}} d\mathbf{r} \qquad \text{(forward)}$$

$$s(\mathbf{r}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\mathbf{k})\, e^{i2\pi \mathbf{k}\cdot\mathbf{r}} d\mathbf{k} \qquad \text{(inverse)}$$

2D $\Rightarrow$

$$\mathbf{r} = (x, y)$$

$$\mathbf{k} = (k_x, k_y)$$

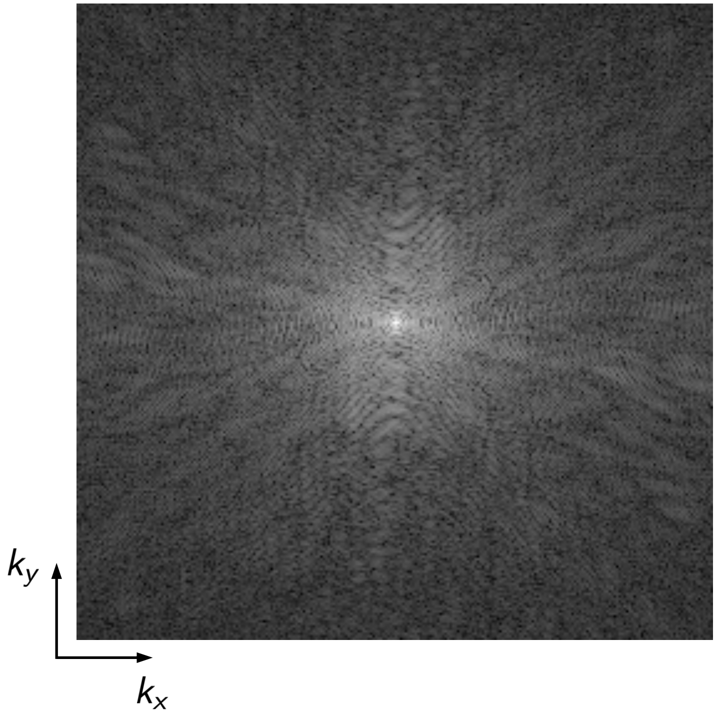
$\Rightarrow$

$$S(k_x . k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(x, y)\, e^{-i2\pi(k_x x + k_y y)} dx dy$$

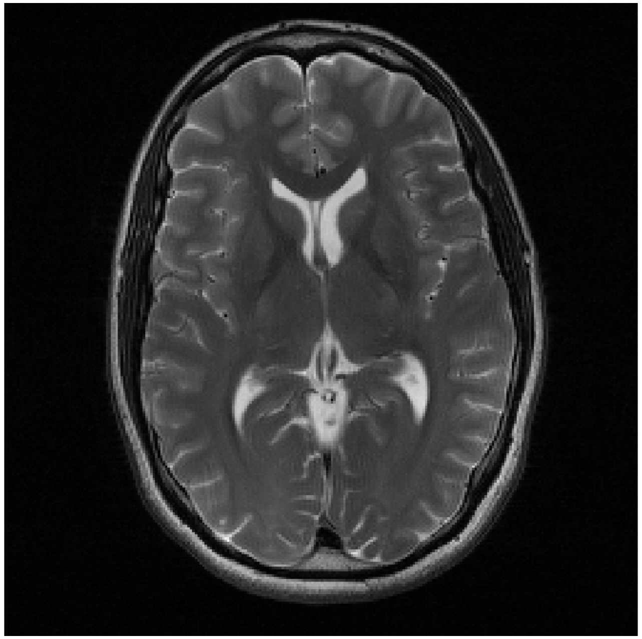
$$s(x, y) = \frac{4}{\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(k_x, k_y)\, e^{i2\pi(k_x x + k_y y)} dk_x dk_y$$

The multidimensional Fourier transform is separable

K-space

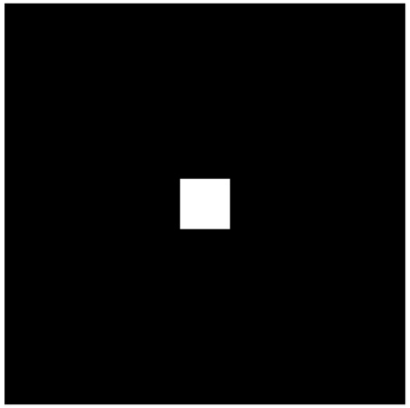


$$\xrightarrow{\mathcal{FT}^{-1}}$$

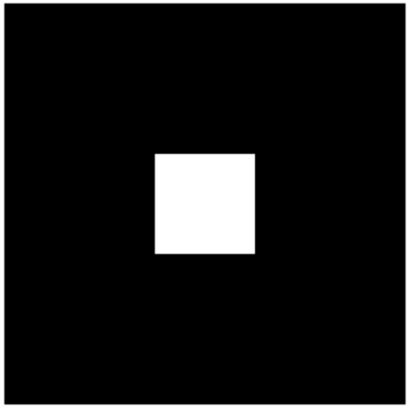


k-space and spatial frequencies

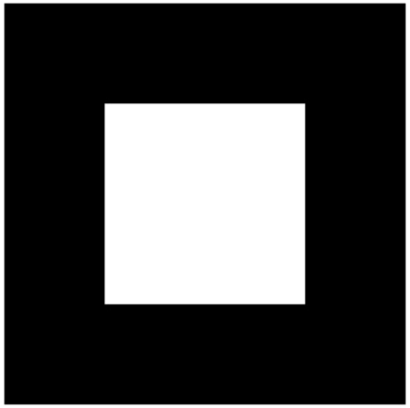
64x64



128x128



256x256



k-space and spatial frequencies

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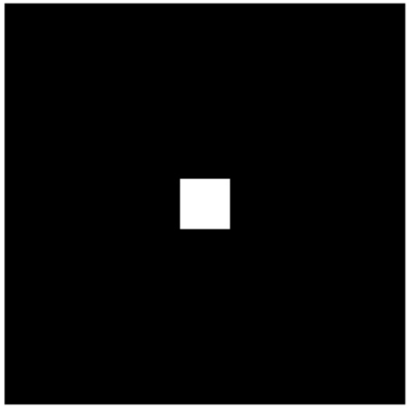

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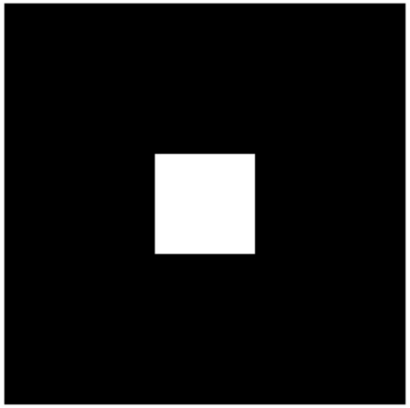

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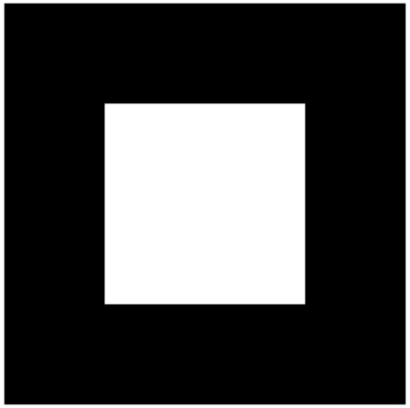
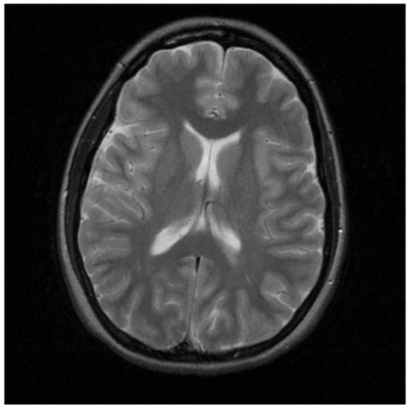


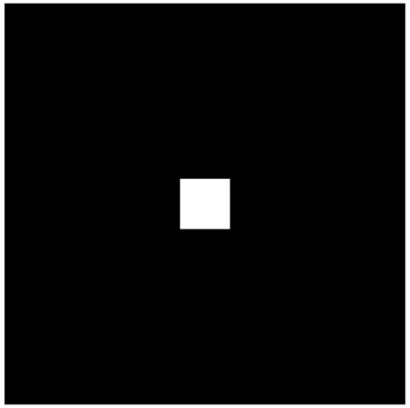


k-space and spatial frequencies

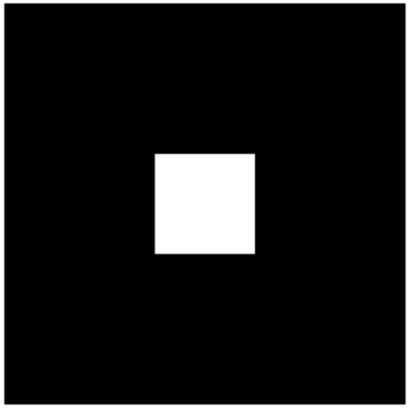
64x64 128x128 256x256

k-space and spatial frequencies

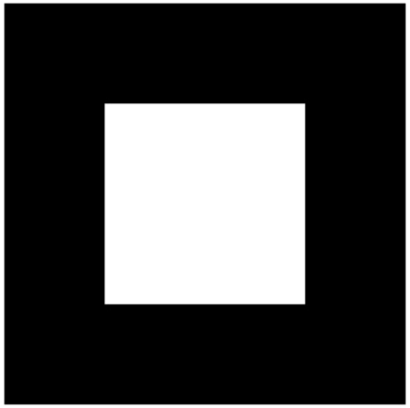
64x64



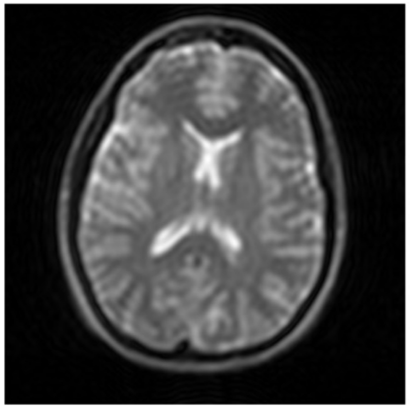
128x128

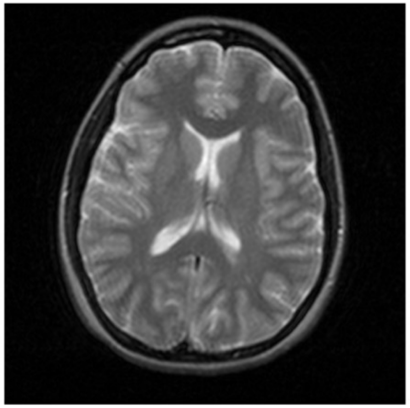


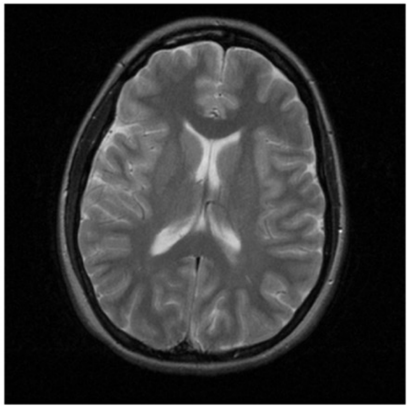
256x256



gibbs ringing
artifacts: generated
from side lobs







k-space and spatial frequencies

64x64 128x128 256x256

k-space and spatial frequencies

64x64

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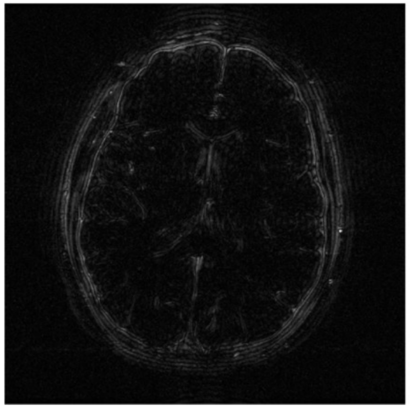

k-space and spatial frequencies

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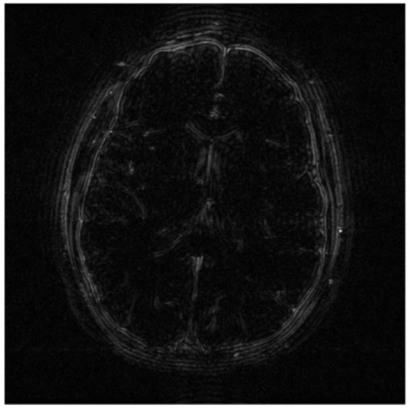
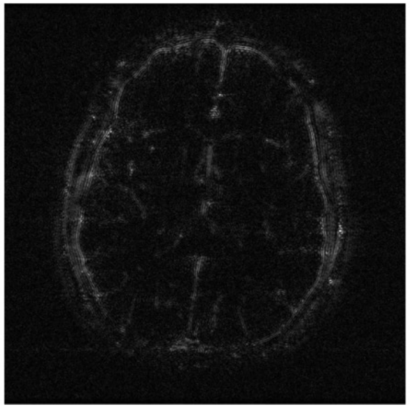


128x128



256x256

k-space and spatial frequencies

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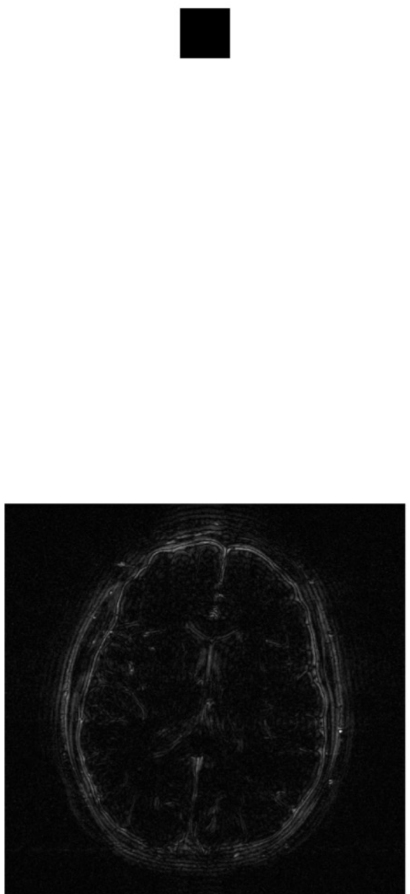

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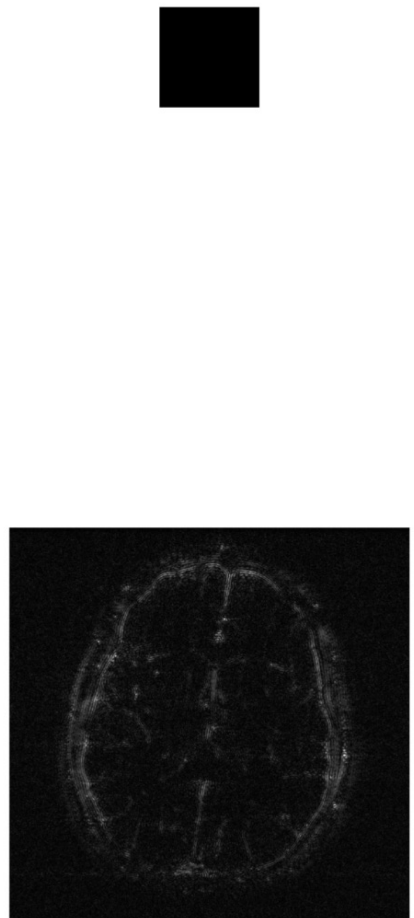

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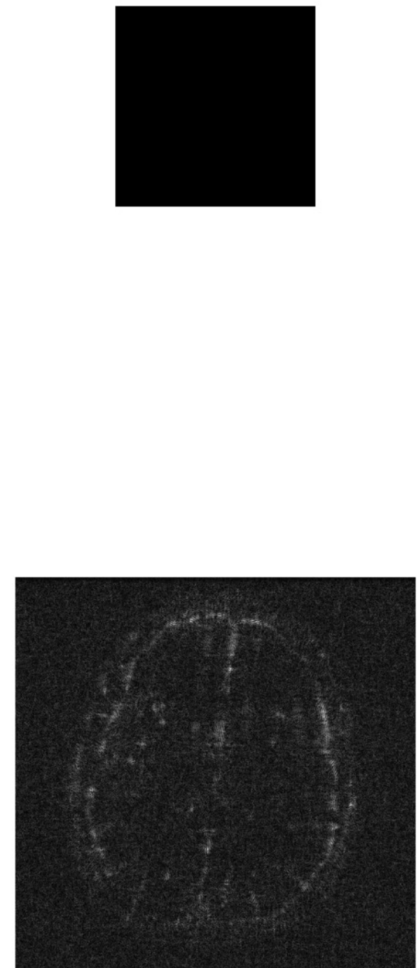
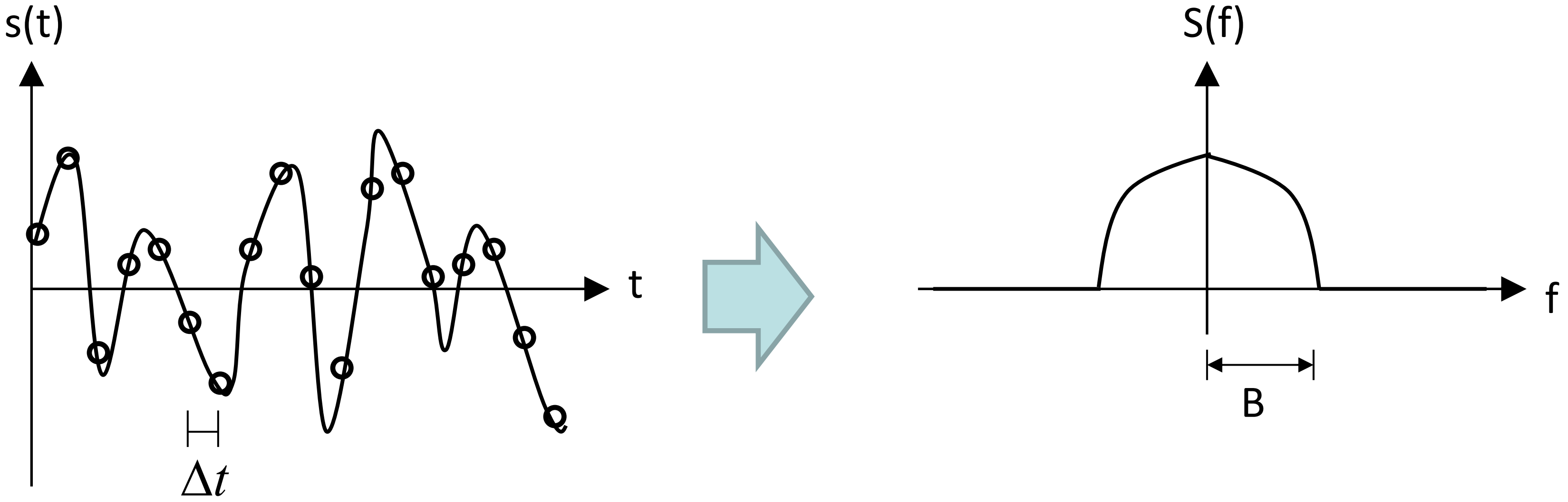


Sampling of continuous signals

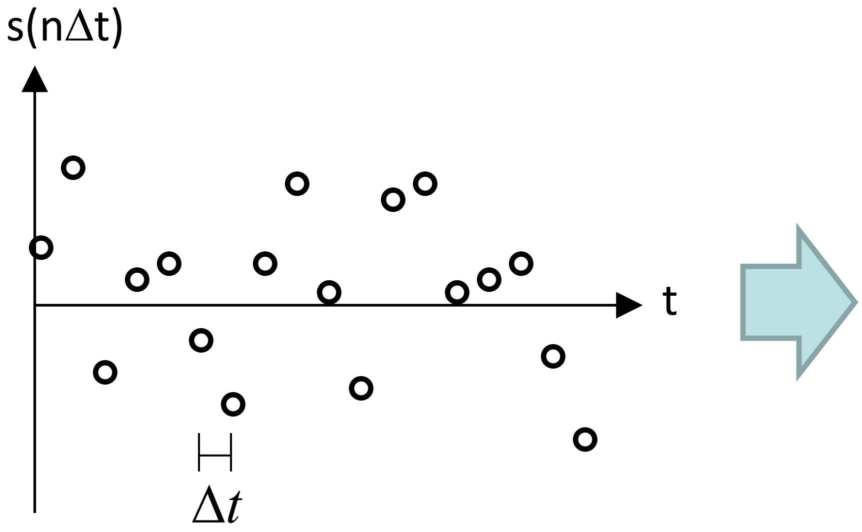
- Nyquist/Shannon theorem
  - A signal with bandwidth B can be reconstructed from its samples if they are taken regularly with a period no larger than 1/2B

$$\text{Nyquist rate}: \Delta t = \frac{1}{2B}$$

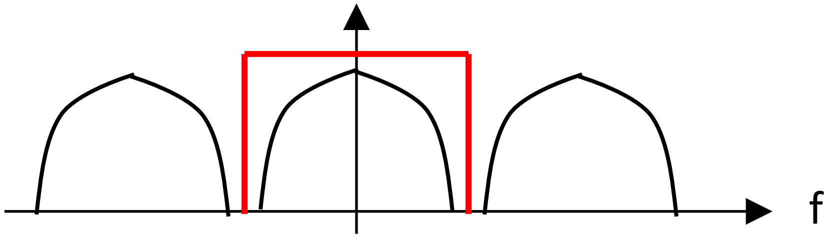


Sampling of continuous signals

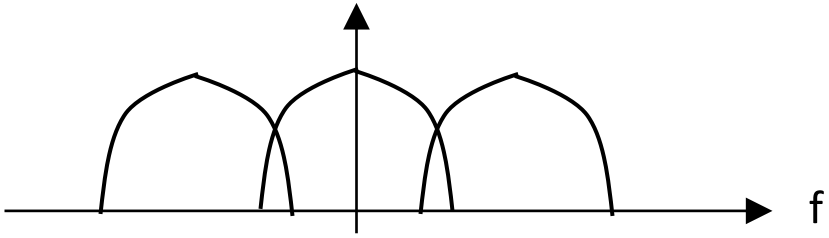
- Nyquist/Shannon theorem



$$\Delta t \le \frac{1}{2B} \quad \Rightarrow \text{ no aliasing}$$



$$\Delta t > \frac{1}{2B} \quad \Rightarrow \text{ aliasing}$$



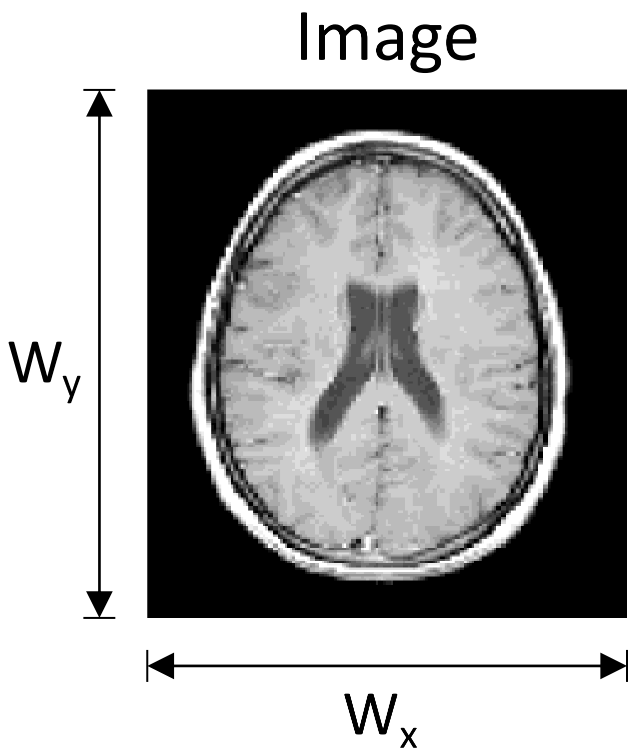
Sampling of MRI signals

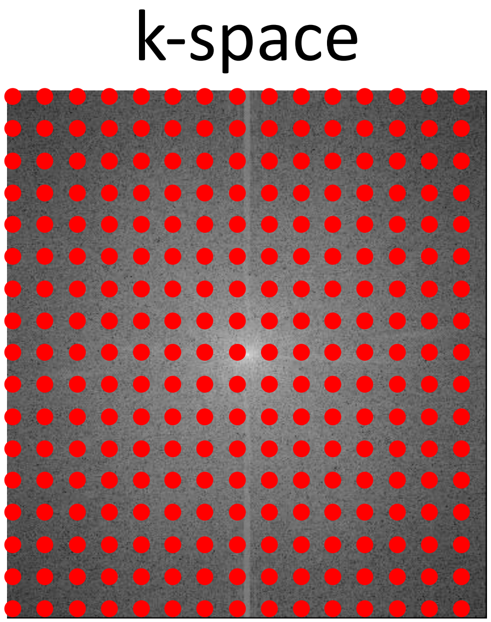
- Where do we sample?

time $\Rightarrow$ k-space image

- How do we apply the sampling theorem?
  - bandwidth: image
  - sampling rate: k-space

Cartesian sampling of k-space

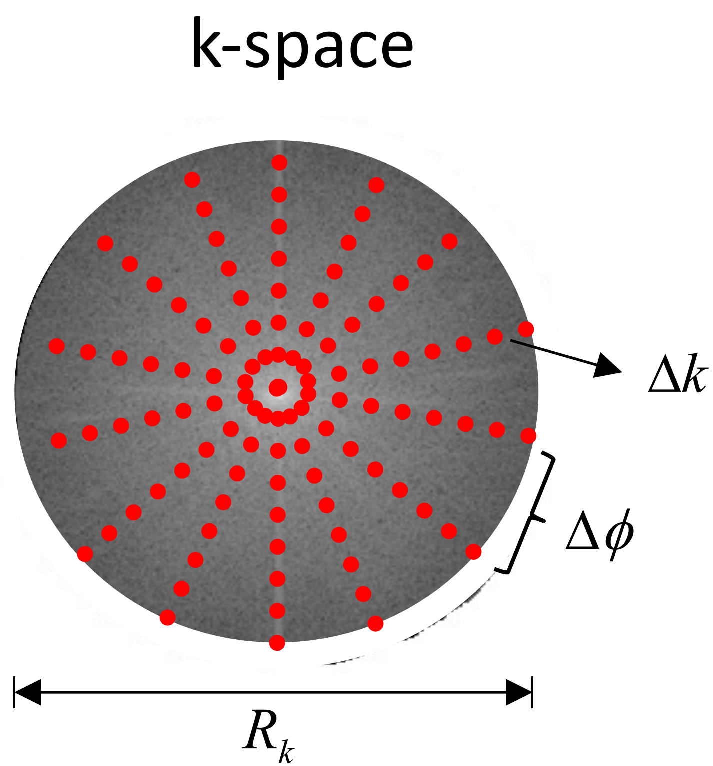




Nyquist rate:

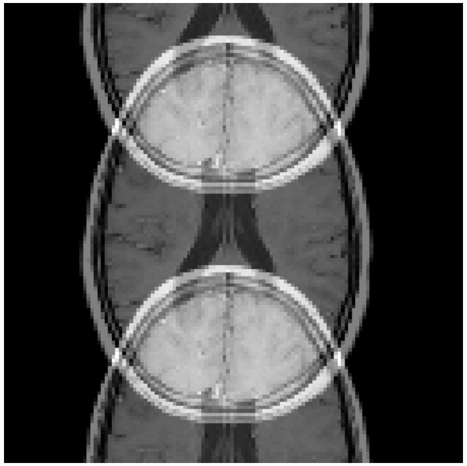
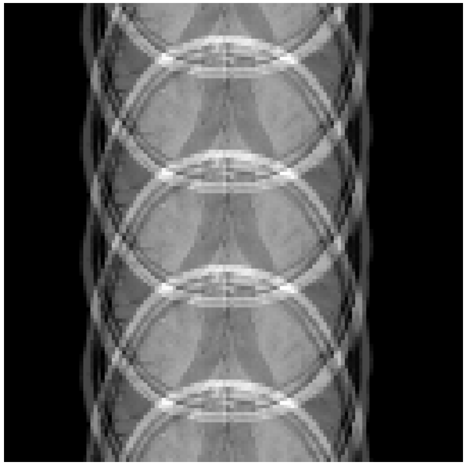
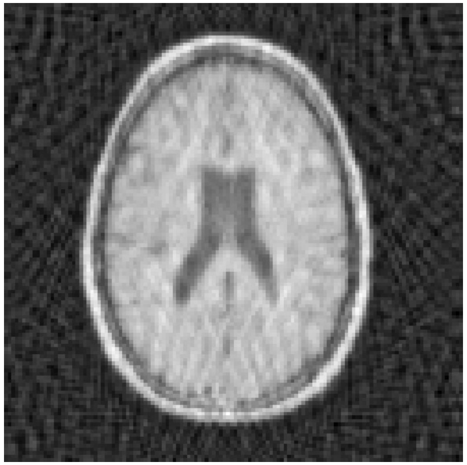
$$\Delta k_x = \frac{1}{W_x}; \quad \Delta k_y = \frac{1}{W_y}$$

Radial sampling of k-space



Nyquist rate (approximate): $N_{radial} = \dfrac{\pi}{2} N_{Cartesian}$

Aliasing examples

	Cartesian		Radial
$\Delta k_y = \dfrac{2}{W_y}$		$\Delta\phi = 2\Delta\phi_{Nyquist}$	
$\Delta k_y = \dfrac{4}{W_y}$		$\Delta\phi = 4\Delta\phi_{Nyquist}$	

Discrete Fourier transform (DFT)

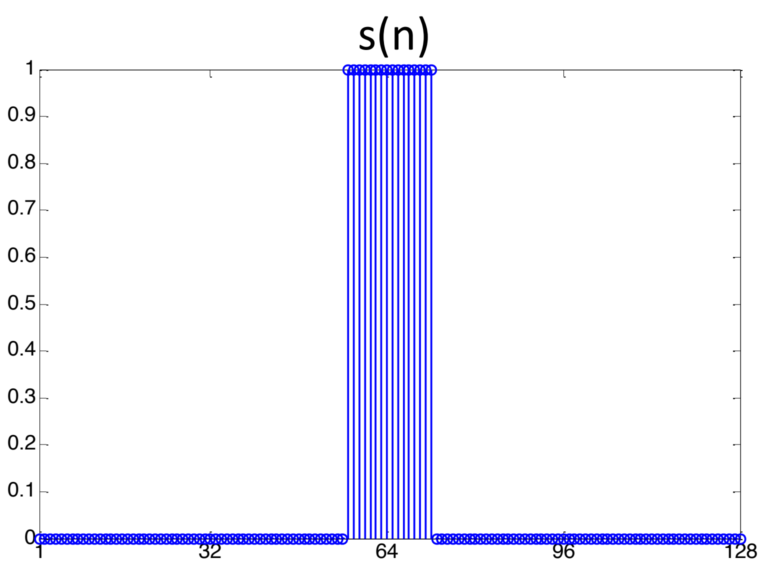
- Discrete signals (sequence of numbers)
- Fast implementation: FFT

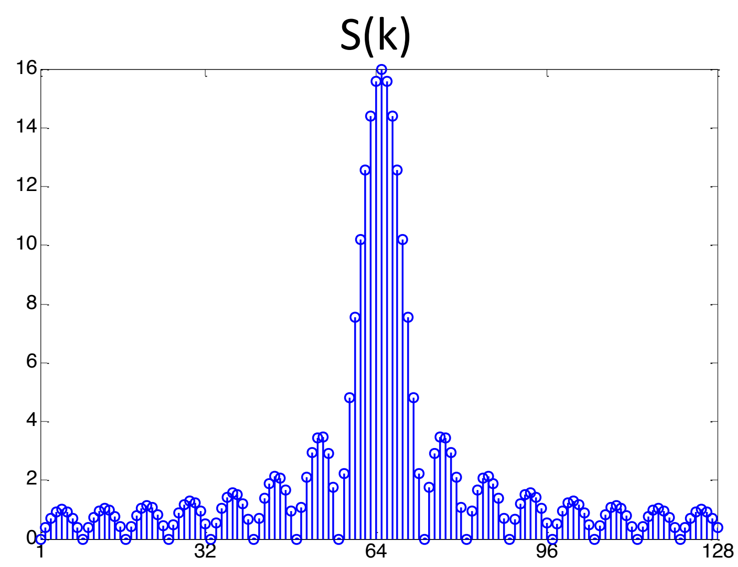
$$S(k) = \sum_{n=0}^{N-1} s(n) e^{-i\frac{2\pi}{N}nk} \qquad \text{(forward)}$$

$$s(n) = \frac{1}{N} \sum_{k=0}^{N-1} S(k) e^{i\frac{2\pi}{N}nk} \qquad \text{(inverse)}$$

Python (NumPy)

```
S=np.fft(s)
s=np.ifft(S)
```

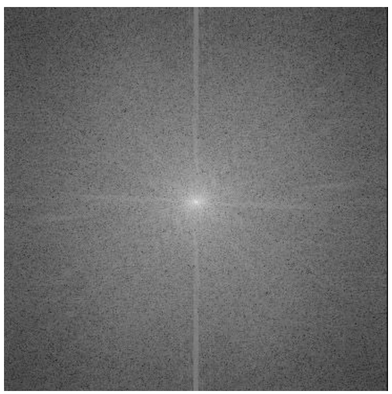




DFT reconstruction of Cartesian k-space data

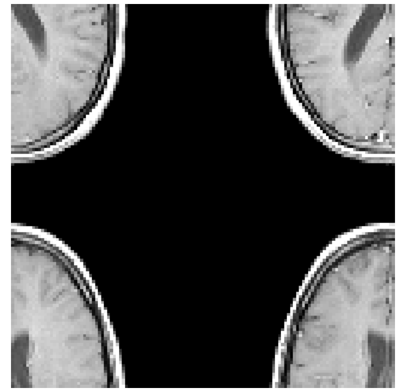
- S(k) is known at $k=n\Delta k$ $\left(-\frac{N}{2} \leq n \leq \frac{N}{2}\right)$

NxN k-space data



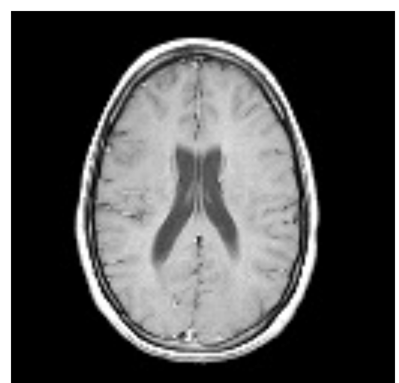
$-\frac{N}{2}$ $\frac{N}{2}$

`s=np.ifft2(S)`



Solution: r and k from 0 to N-1

```
s=np.sqrt(np.size(S))*np.fft.fftshift
(np.fft.ifft2(np.fft.ifftshift(S)))
```



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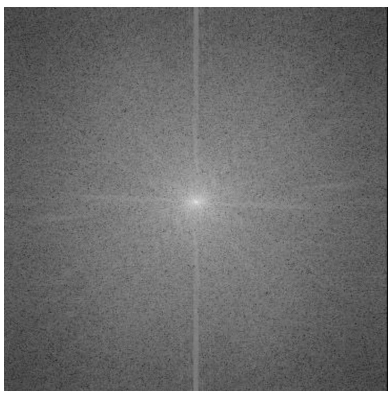
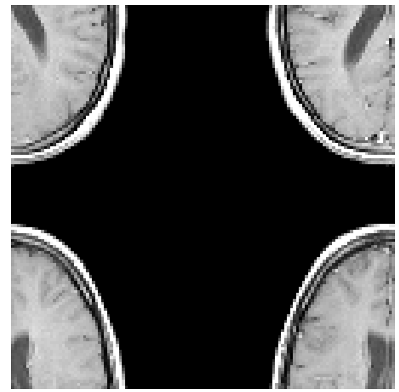
NxN k-space data



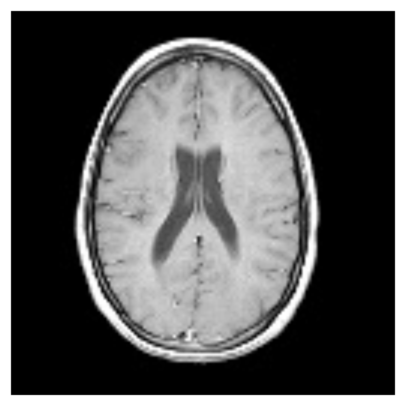
$-\frac{N}{2}$ $\frac{N}{2}$

```
S=np.fft2(s)
```



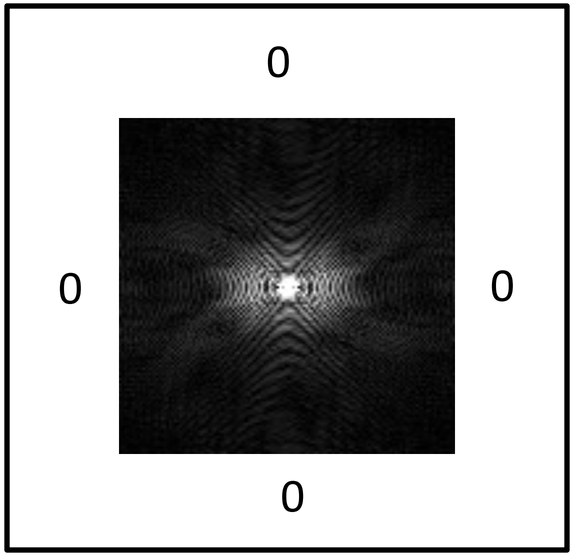
Solution: r and k from 0 to N-1

```
S=1/np.sqrt(np.size(s))*np.fft.fftshift
(np.fft.fft2(np.fft.ifftshift(s)))
```

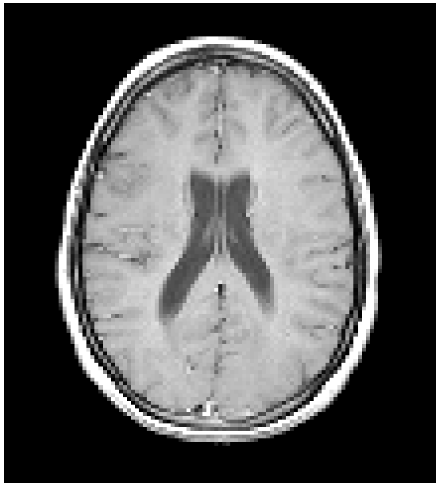


DFT reconstruction of Cartesian k-space data

- Zero-padding in k-space (Fourier interpolation)
  - Decreases the pixel size but does not increase resolution



Original 128x128

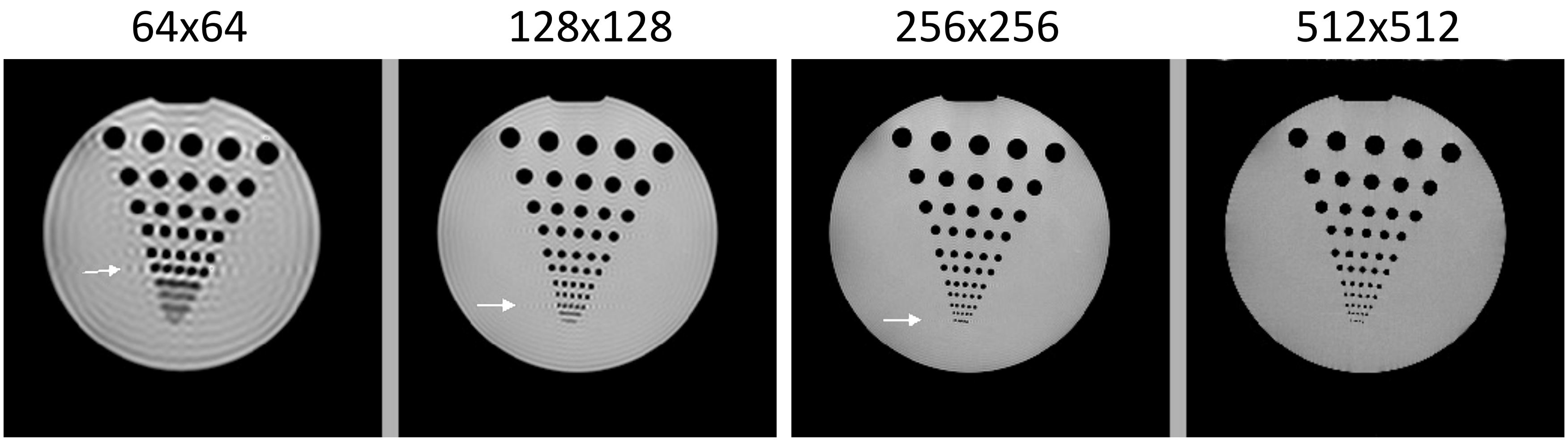


Zero-padded 256x256

$$p_x = \frac{W_x}{N_{x,padded}}; p_y = \frac{W_y}{N_{y,padded}}$$

DFT reconstruction of Cartesian k-space data

- Gibbs ringing
  - Spurious ringing around sharp edges
  - Caused by k-space truncation
  - Gets stronger for decreasing N)



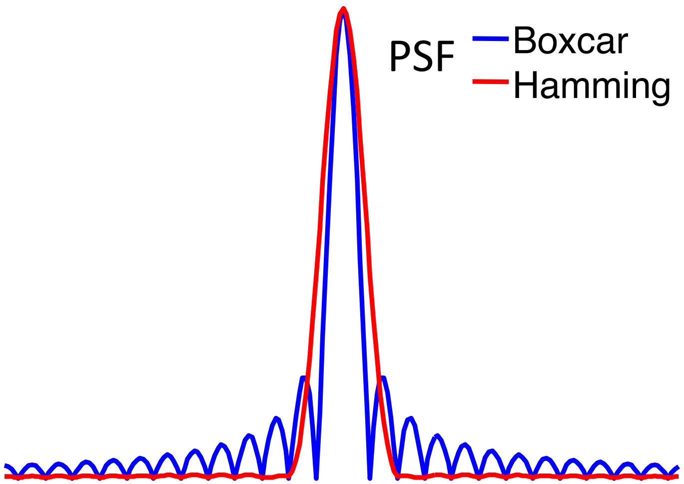
DFT reconstruction of Cartesian k-space data

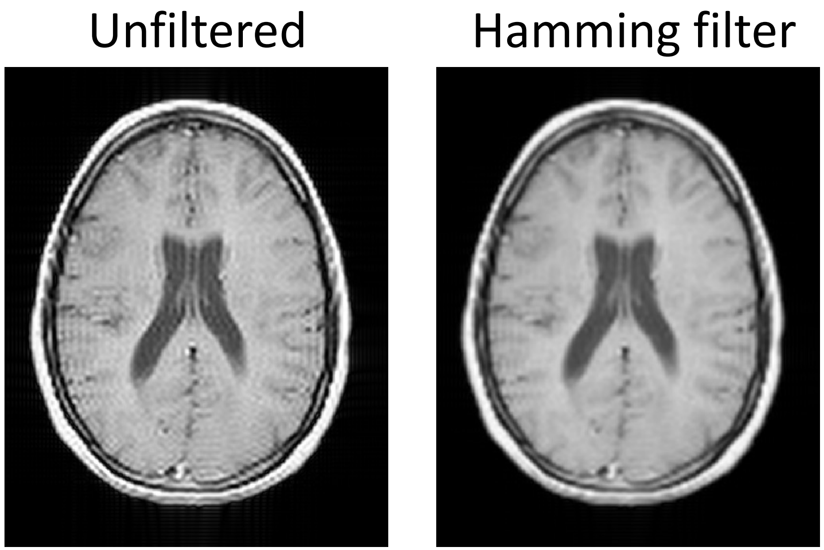
- k-space filtering or windowing
  - Reduce Gibbs ringing at the expense of resolution loss

$$S_W(k) = S(k)W(k)$$

- Hamming filter

$$W(k) = 0.54 + 0.46\cos\left(\frac{2\pi n}{N}\right)$$





DFT reconstruction of Cartesian k-space data

- Signal-to-noise ratio (SNR) (simplified)

$$SNR \propto V\sqrt{T}$$

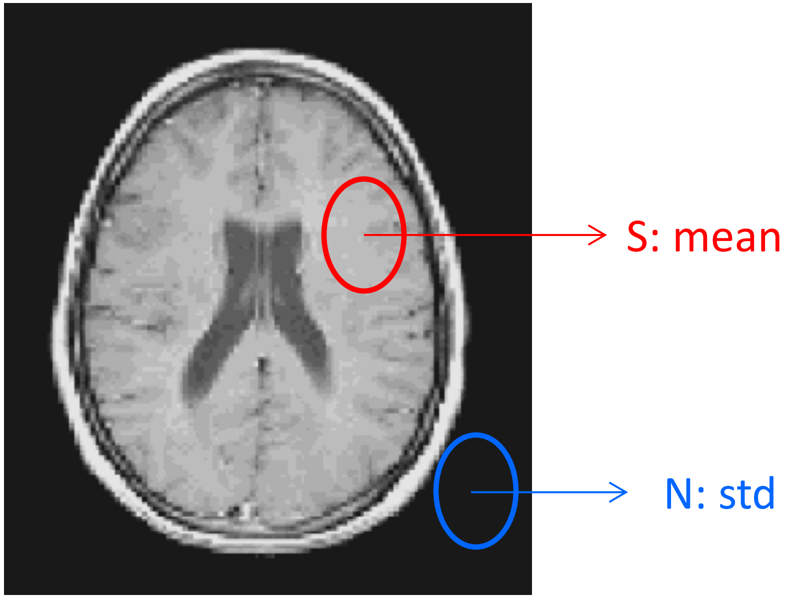
V: voxel volume

T: cumulated readout duration

DFT reconstruction of Cartesian k-space data

- Signal-to-noise ratio (SNR)

$$SNR = \frac{S}{N} = \frac{\text{Pixel signal amplitude}}{\text{Standard deviation of background}}$$



Simplification: Not entirely correct for many real-world measurements!

Dietrich JMRI 2007