Q.1 Using Iteration method, find the roots of the following functions, corrected to four decimal places.

(a) $x^3 = x+1$

$\Rightarrow$

$f(x) = x^3 - x - 1$

Now, for initial guess

at $x=0$ $\quad f(x) = -1$

$x = 1$ $\quad f(x) = -1$

$x = 2$ $\quad f(x) = 5$ $\quad x_0 = 1.5$

Now to find $x = g(x)$

$x = (x^3 - 1)$ or $x = (x+1)^{1/3}$

Here
$g(x) = (x^3 - 1)$
$g'(x)_{x_0 = 1.5} = 3x^2$
$= 6.75$

Here,
$g(x) = (x+1)^{1/3}$
$g'(x)_{x_0=1.5} = \frac{1}{3}(x+1)^{-2/3}$
$= 0.16$

$\therefore$ $|g'(x)|$ should be minimum.
will go with $x = (x+1)^{1/3}$ and $x_0 = 1.5$

$x_1 = (1.5+1)^{1/3} = 1.35720$

$x_2 = (1.35720+1)^{1/3} = 1.33085$

$x_3 = (1.33085+1)^{1/3} = 1.32588$

$x_4 = (1.32588+1)^{1/3} = 1.32493$

$x_5 = (1.32493+1)^{1/3} = 1.32475$

$x_6 = (1.32475+1)^{1/3} = 1.32472$

$\therefore \boxed{x_r = 1.3247}$

$\therefore$ The root of given equation is 1.3247.

(b) $x = \sin x + 1/2$

$f(x) = \sin x - x + 1/2$

Let's find initial guess value

$x = 0 \quad f(x) = 0.5 \quad +ve$

$x = 1 \quad F(x) = -0.48 \quad -ve$

$\therefore x_0 = 0.5$

For $x = g(x)$ will find x such that $|g'(x)|$ should be minimum

$x = \sin x + 1/2$ OR $x = \sin^{-1}(x - 1/2)$

$g'(x) = \cos x$ $\qquad g'(x) = \dfrac{1}{\sqrt{(1-(x-1/2)^2)}}$

at $x_0 = 1.5$ $\qquad$ at $x_0 = 1.5$

$|g'(x)| = 0.999 \quad < \quad |g'(x)| = 1$

$\therefore$ we will consider $x = \sin x + 1/2$

$x_1 = \sin(0.5) + 0.5 = 0.5087$

$x_2 = \sin(0.5087) + 0.5 = 0.5088$

$x_3 = \sin(0.5088) + 0.5 = 0.5088$

$\therefore x_r = 0.5088$

The root of given equation is 0.5088

(c) $\log(x) - x + 2 = 0$.

$f(x) = \log(x) - x + 2$

$x_0 = 0$ $\quad f(x) = 2$

$x = 1$ $\quad f(x) = 2$

$x = 2$ $\quad f(x) = 0.3010$

$x = 3$ $\quad f(x) = -0.5228$ $\quad \Big] \quad x_0 = \underline{\underline{2.5}}$

$x = \log(x) + 2$ $\quad$ OR $\quad x = e^{x-2}$

$g'(x) = 1/x = 1/2.5$ $\quad \underline{\underline{<}} \quad$ $g'(x) = e^{x-2}$

$= 0.4$ $\quad\quad = 1.648$

as $g'(x)$ should be minimum at x_0

$\therefore$ we will consider $x = \log(x) + 2$ $\quad x_0 = 2.5$

$\therefore x_1 = \log(2.5) + 2 = 2.3979$

$x_2 = \log(2.3979) + 2 = 2.3798$

$x_3 = \log(2.3798) + 2 = 2.3765$

$x_4 = \log(2.3765) + 2 = 2.3759$

$x_5 = \log(2.3759) + 2 = \underline{2.3758}$

$x_6 = \log(2.3758) + 2 = \underline{\underline{2.3758}}$

$\therefore x_r = 2.3758$

$\therefore$ The root of given equation is 2.3758.

Q.2 find the root of $2x^3 = 5 + 2x$ using bisection method whose x lies between $[1, 2]$ correct it to 3 decimal place.

⇒ Using Bisection method

$f(x) = 2x^3 - 2x - 5$

$f(1) = -5$ (-ve)

$f(2) = 7$ (+ve)

The root lies between 1 & 2

1st iteration :

$\therefore f\left(\frac{1+2}{2}\right) = f(1.5) = 2(1.5)^3 - 2(1.5) - 5 = -1.25$

iteration 2 : $f(1.5) \cdot f(2) < 0$

$a = 1.5 \quad b = 2 \quad c = \frac{2+1.5}{2} = 1.75$

$\therefore f(1.75) = 2(1.75)^3 - 2(1.75) - 5 = 2.2188$

iteration 3 : $f(1.75)\, f(1.5) < 0$

$c = 1.75 + 1.5/2 = 1.625$

$f(1.625) = 2(1.625)^3 - 2(1.625) - 5 = 0.3320$

iteration 4 : $f(1.625)\, f(1.5) < 0$

$c = \frac{1.625 + 1.5}{2} = 1.5625$

$f(1.5625) = 2(1.5625)^3 - 2(1.5625) - 5 = -0.49956$

iteration 5: $f(1.5625)\,f(1.6250) < 0$

$\therefore f\left(\frac{1.5625+1.6250}{2}\right) = -0.0911$

iteration 6: $f(1.5938)\,f(1.6250) < 0$

$c = \frac{1.5938 + 1.625}{2} = 1.6094$

$f(1.6094) = 2(1.6094)^3 - 2(1.6094) - 5 = 0.1181$

iteration 7: $f(1.6094)\,f(1.5938) < 0$

$c = \frac{1.6094 + 1.5938}{2} = 1.6016$

$f(1.6016) = 2(1.6016)^3 - 2(1.6016) - 5 = 0.0129$

iteration 8: $f(1.6016)\,f(1.5938) < 0$

$c = \frac{1.6016 + 1.5938}{2} = 1.5977$

$f(1.5977) = 2(1.5977)^3 - 2(1.5977) - 5 = -0.0393$

iteration 9. $f(1.5977)\,f(1.6016) < 0$

$c = \frac{1.5977 + 1.6016}{2} = 1.5996$

$f(c) = 2(1.5996)^3 - 2(1.5996) - 5 = -0.0132$

iteration 10: $f(1.5996)\,f(1.6016) < 0$

$c = \frac{1.5996 + 1.6016}{2} = 1.6006$

$f(1.6006) = 2(1.6006)^3 - 2(1.6006) - 5 = -0.0002$

iteration 11: $f(1.6006)\, f(1.6016) < 0$

$c = \frac{1.6006 + 1.6016}{2} = 1.6011$

$f(1.6011) = 2(1.6011)^3 - 2(1.6011) - 5 = 0.0064$

iteration 12: $f(1.6011)\, f(1.6006) < 0$

$c = \frac{1.6011 + 1.6006}{2} = \underline{\underline{1.6008}}$

$f(1.6008) = 2(1.6008)^3 - 2(1.6008) - 5 = 0.0031$

iteration 13: $f(1.6008)\, f(1.6006) < 0$.

$c = \frac{1.6008 + 1.6006}{2} = \underline{\underline{1.6007}}$

$f(1.6007) = 2(1.6007)^3 - 2(1.6007) - 5 = 0.0015$

upto 3 decimal

$\therefore x_r = 1.600$

The root of given equation is approximately 1.600

Q.3 Find the root of $x^4 - x = 10$ approximately upto 6 iterations using bisection method. let $a = 1.5$ & $b = 2$.

$\Rightarrow$ Given things,

$f(x) = x^4 - x - 10$

$a = 1.5 \quad b = 2$

$f(1.5) = -6.4375$ -ve

$f(2) = 4$ +ve

$c = \frac{1.5+2}{2} = 1.75$

iteration 1 : $f(a)\,f(b) < 0$

$c = 1.75$

$f(c) = (1.75)^4 - 1.75 - 10 = -2.3710$ -ve

iteration 2 : $f(1.75)\,f(2) < 0$

$c = \frac{1.75+2}{2} = 1.875$

$f(1.875) = (1.875)^4 - 1.875 - 10 = 0.4846$

iteration 3 : $f(1.875)\,f(1.75) < 0$

$c = \frac{1.875+1.75}{2} = 1.8125$

$f(1.8125) = (1.8125)^4 - 1.8125 - 10 = -1.0202$

iteration 4 :

$f(1.8125)\,f(1.875) < 0$

$c = \frac{1.8125+1.875}{2} = 1.8437$

$f(1.8437) = (1.8437)^4 - 1.8437 - 10 = -0.2876.$

iterations 5 : $f(1.8437)\, f(1.875) < 0$

$c = \frac{1.8437 + 1.875}{2} = \frac{3.7187}{2} = 1.8593$

$f(1.8593) = (1.8593)^4 - 1.8593 - 10 = 0.0928.$

$\therefore$ The iteration 5 gives the root as approximately ~~0.0~~ $\underline{\underline{1.8593}}$.

Q.4 Find the approximated value of x till 4 iterations for $e^{-x} = 3\ln(x)$ in between 0.5 & 1.5 using Bisection method.

$\Rightarrow$ $f(x) = e^{-x} - 3\ln(x)$ till 4 iterations

$a = 0.5$ & $b = 1.5$.

iteration 1 :

$f(a) = f(0.5) = e^{-0.5} - 3\ln(0.5)$

$= 0.6065 + 2.0794$

$= 2.6859$

$f(b) = f(1.5) = e^{-1.5} - 3\ln(1.5)$

$= -0.9932$

$f(a)\, f(b) < 0$

$c = \frac{0.5 + 1.5}{2} = 1$

$f(1) = e^{-1} - 3\ln(1) = 0.3678$

2^{nd} iteration:

$$f(1)\, f(1.5) < 0$$

$$c = \frac{1+1.5}{2} = 1.25$$

$$f(1.25) = e^{-1.25} - 3\ln(1.25)$$

$$= 0.2865 - 1.2639 \quad = \cancel{-0.9298}$$

$$= -0.3829.$$

3^{rd} iteration:

$$f(1.25)\, f(1) < 0$$

$$c = \frac{1.25+1}{2} = 1.125$$

$$f(1.125) = e^{-1.125} - 3\ln(1.125)$$

$$= -0.0287$$

4^{th} iteration

$$f(1.125)\, f(1) < 0$$

$$c = \frac{1.125+1}{2} = 1.06250$$

$$f(1.0625) = e^{-1.0625} - 3\ln(1.0625) = 0.1637$$

5^{th} iteration

$$f(1.0625)\, f(1.125) < 0$$

$$c = \frac{1.0625+1.125}{2} = 1.09375$$

$$f(1.09375) = e^{-1.09375} - 3\ln(1.09375)$$

$$= 0.06612$$

$\therefore$ After 4 iteration the root is approximately $\underline{1.0625}$

~~0.1637~~

Q.5 Newton's equation $y^3 - 2y - 5 = 0$ has a root near $y = 2$, starting with $y_0 = 2$, compute $y_1, y_2, \& y_3$, the next three Newton - Raphson estimates for the root.

$\Rightarrow$ $f(y) = y^3 - 2y - 5 = 0$

$y_0 = 2$

Using Newton's Raphson equation.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

i.e $y_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}$ $\quad f'(y) = 3y^2 - 2$

$f(y_0) = (2)^3 - 2(2) - 5 = -1$

$f'(y_0) = 3(2)^2 - 2 = 10$

$\therefore y_1 = y_0 - \frac{f(y_0)}{f'(y_0)} = 2 - \frac{(-1)}{10} = \underline{\underline{2.1}}$

$y_2 = y_1 - \frac{f(y_1)}{f'(y_1)}$ $\quad f(y_1) = (2.1)^3 - 2(2.1) - 5 = 0.061$

$f'(y_1) = 3(2.1)^2 - 2 = 11.23$

$y_2 = 2.1 - \frac{0.061}{11.23} = \underline{\underline{2.09457}}$

Now

$y_3 = y_2 - \frac{f(y_2)}{f'(y_2)}$ $\quad f(y_2) = (2.094)^3 - 2(2.094) - 5$

$= 9.1818 - 9.188$

$= -0.0062$

$f'(y_2) = 3(2.094)^2 - 2$

$= 11.154$

$$y_3 = 2.094 - \frac{(-0.0062)}{11.154} = \underline{2.09455}$$

$\therefore y_1 = 2.1, \; y_2 = 2.09457, \; y_3 = 2.09455$

Q.6

Use Newton-Raphson Algorithm to find the approximate root of following equation:

$$f(x) = \sin x - \frac{(x+1)}{(x-1)}$$

with $x_0 = -0.2$ for four iterative steps [find (x_1, x_2, x_3, x_4)].

$\Rightarrow$

$$f(x) = \sin x - \frac{(x+1)}{(x-1)} \qquad x_0 = \cancel{0}\,0.2$$

$$f'(x) = \cos x - \frac{[(x-1)(1) - (x+1)(1)]}{(x-1)^2}$$

$$f'(x) = \cos x - \frac{[x-1-x-1]}{(x-1)^2} = \cos x + \frac{2}{(x-1)^2}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$f(x_0) = \sin(-0.2) - \frac{(0.2+1)}{(0.2-1)} = -0.00341 + 0.666$$

$$= \cancel{1.4965}\; 0.6701$$

$$f'(x_0) = \cos(-0.2) + \frac{2}{(0.2-1)^2} = \cancel{4.1249}\; 2.3887$$

$\therefore x_1 = -0.2 - \frac{1.4965}{4.1249} = -0.5627$

$x_1 = -0.2 - \frac{0.6201}{2.3882} = -0.4805$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$f(x_1) = f(-0.4805) = \sin(-0.4805) - \frac{(-0.4805+1)}{(-0.4805-1)}$

$= -0.008386 + \frac{0.5195}{1.4805}$

$f(x_1) = 0.3425$

$f'(x_1) = \cos(-0.4805) + \frac{2}{(-0.4805-1)^2}$

$= 1.9124.$

$x_2 = -0.4805 - \frac{0.3425}{1.9124} = -0.6595$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$f(x_2) = F(-0.6595) = \sin(-0.6595) - \frac{(-0.6595+1)}{(-0.6595-1)}$

$= 0.1936$

$f'(x_2) = \cos(-0.6595) + \frac{2}{(-0.6595-1)^2} = 1.7261$

$$x_3 = -0.6595 - \frac{0.1936}{1.7261} = \underline{\underline{-0.7716}}$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$f(x_3) = f(-0.7716) = \sin(-0.7716) - \frac{(-0.7716+1)}{(-0.7716-1)}$$

$$= 0.1156$$

0.1289

$$f'(x_3) = \cos(-0.7716) + \frac{2}{(-0.7716-1)^2} = 1.6371$$

$$x_4 = -0.7716 - \frac{0.1154}{1.6371} = \cancel{-0.8420} = \underline{\underline{-0.842}}$$

$\cancel{0.0704}$

$$f(x_4) = \sin(-0.842) - \frac{(-0.842+1)}{(-0.842-1)} = 0.071$$

$$f'(x_4) = \cos(-0.842) + \frac{2}{(-0.842-1)^2} = 1.5893$$

$$x_5 = -0.842 - \frac{0.071}{1.5893} = -0.8866$$

The required values are

$\therefore x_1 = -0.4805$, $x_2 = -0.6595$, $x_3 = -0.7716$

& $x_4 = -0.842$