Two-dimensional Robotics

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Abstract

Two-dimensional robotics requires definition of essential geometric terms, their representation and essential transformations to perform a change of representation (mapping) or relate the terms of different objects (conversion).

1 Definitions

Essential terms are introduced enabling the identification, representation and manipulation of objects position and orientation as well as their linear and rotation velocity.

1.1 goniometry: point, length, angle

Classic geometry or goniometry addresses the relation between points on a plane or in space with their mutual distances and angles that exist between intersecting lines or line segments. Points are used to identify particular places or positions

symbol	name	description
P_i	point	particular place or position
l_j	length	distance between two points on a line or linesegment
$\angle P_k$ or α_k	angle	divergence between two intersecting lines

Table 1: Symbols used for goniometry

on a plane or in space. Uppercase latin letters are assigned to each distinct point. Here a (postfix) subscript i will be used to identify the point P_i in a set $\{P\}$.

The Euclidian distance between two points is the length of a line segment connecting the points. Length parameters will be identified typically using the lower case letter l and a suitable subscript.

The angle between two intersecting lines is the angle in the plane spanned by both lines. Angles are indicated by an angle symbol preceding the point of intersecting lines or linesegments. Angles are numbered explicitely in figures and concatenated numbers refer to the total angle:

$$\angle P_{12} = \angle P_1 + \angle P_2 \tag{1}$$

Angle parameters will be identified typically using the lower case letter α and a suitable subscript.

1.2 linear algebra: point, vector and frame

Only the essential concepts are introduced for describing relations between rigid bodies: Each of the symbols in table 2 can only be annotated by a trailing

symbol	name	description
P_i	point	a particular place or position
v_j	vector	a quantity specified by magnitude and a direction
Ψ_k	frame	a set of orthonormal base-vectors attached to a point

Table 2: Symbols used for basic geometry

subscript that identifies the particular instance within a larger set.

Points are used to identify particular places or positions in space. Linear algebra typically assigns a unique uppercase latin symbol to each distinct point. Here a (postfix) subscript i will be used to identify the point P_i in a set of points $\{P\}$.

Vectors (base-vector, free-vector) can be seen as a directed line-segment connecting two points A and B:

$$v_j = \vec{AB} \tag{2}$$

a (postfix) subscript j will be used to identify the vector v_j. Free vectors are characterised by magnitude and direction. Base vectors are tied to a fixed point which is the base of the vector.

Frames or co-ordinate systems are attached to points of rigid bodies and associated with orthonormal unit base-vectors to uniqely define location and attitude of the object. Here a (postfix) subscript k will be used to identify the frame Ψ_k.

1.3 robotics: geometry, kinematics and dynamics

A distinct set of latin symbols has been reserved to be used in this context: Each of the symbols in table 3 has a distinct set of valid annotations (prefix,

symbol	name	description
${}^k p_i$	pose	a combination of position and orientation
${}^k t_{i,j}$	twist	a combination of linear and rotation velocity
${}^k w_{i,j}$	wrench	a combination of force and torque

Table 3: Symbols used for robotic variables

suffix, superscript, subscript).

Pose ${}^k p_i$ is used to indicate the position and orientation of a frame Ψ_i taking another frame Ψ_k as reference. The global reference frame is identified as Ψ_0 by convention. The leading superscript defaults to zero:

$${}^0 p_i = p_i \tag{3}$$

The pose of a frame i relative to itself is zero by definition:

$${}^i p_i = 0 \tag{4}$$

Twist ${}^k t_{i,j}$ is used to indicate the linear and rotation velocity of frame Ψ_i relative to frame Ψ_j (i.e. the time-derivative of the relative pose ${}^j p_i$) expressed in frame Ψ_k.

$$ {}^k t_{i,j} = {}^k \left(\frac{d\,{}^j p_i}{dt} \right) = {}^k \left({}^j \dot{p}_i \right) \tag{5} $$

The leading superscript defaults to the global reference frame which means that the twist of a frame is expressed by default in global co-ordinates:

$$ {}^0 t_{i,j} = t_{i,j} \tag{6} $$

The twist of a frame i relative to itself is zero by definition regardless of the choice of co-ordinates:

$$ {}^j t_{i,i} = 0 \tag{7} $$

If global movement is described in global co-ordinates or the inertial reference system is used then only a single subscript will do:

$$ {}^0 t_{i,0} = {}^0 t_i = t_i \tag{8} $$

Global movement described in its own moving co-ordinate system has the next format:

$$ {}^i t_{i,0} = {}^i t_i \tag{9} $$

Relative movement obeys the next relationship:

$$ {}^k t_{i,j} = -{}^k t_{j,i} \tag{10} $$

which means that reversing the relative movement implies negation of the twist.

Wrench ${}^k w_{i,j}$ is used to indicate the force and torque excerted to frame Ψ_i from frame Ψ_j expressed in frame Ψ_k. The subscript prefix defaults to the postfix subscript which means that the wrench acting on a frame is expressed in its own coordinate system.

2 Representation

The quantities introduced in the previous section require a suitable parameterisation to be useable for elementary algebraic operations.

Three concepts need to be represented[1]:

1. description of a frame
2. transform mapping
3. transform operation

The first item will be elaborated in this section. The next two items follow in the next section.

2.1 coordinates

Pose, twist and wrench can be represented each by a row vector of suitable dimension in which the coefficients indicate the projections of a vector to the unit vectors of a given base (i.e. the used co-ordinate system of frame) extended with a combination of rotation angles and if necessary the coefficients of a rotation axis.

[1] analogous to section 2.5 of J.J. Craig, Introduction to Robotics, 1986

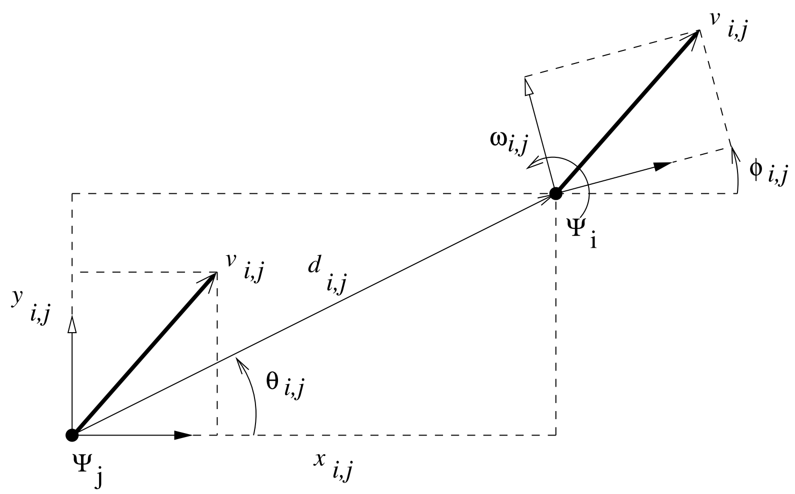

Figure 1: Represent the pose and twist of Ψ_i in Ψ_j

2.2 2D-robotics

In 2D robotics there is only planar rotation wich is positive by convention when it is rotating counter clock-wise. A 2D pose can be represented by three coefficients:

$$^{j}p_i = \begin{pmatrix} x_{i,j} \\ y_{i,j} \\ \phi_{i,j} \end{pmatrix} \tag{11}$$

where the first two coefficients represent position and the last coefficient represents the orientation. This 2D pose can be considered as a representation of a frame Ψ_i seen from frame Ψ_j having an offset position with coefficients $(x_{i,j}, y_{i.j})$ and a relative rotation with angle $\phi_{i,j}$. The offset position vector can also be represented in polar coordinates $(d_{i,j}, \theta_{i,j})$.

Relative motion in 2D expressed as twist can be represented by three coefficients:

$$^{j}t_{i,j} = \frac{d^{j}p_i}{dt} = {^{j}\dot{p}_i} = \begin{pmatrix} \dot{x}_{i,j} \\ \dot{y}_{i,j} \\ \dot{\phi}_{i,j} \end{pmatrix} \tag{12}$$

where the first two coefficients represent the linear velocity vector and the last coefficient represents the angular velocity. This 2D twist can be considered as the time-derivative of the relative position of frame Ψ_i relative to frame Ψ_j seen from frame Ψ_j having a linear velocity with coefficients $(\dot{x}_{i,j}, \dot{y}_{i,j})$ and a rotation velocity $\omega_{i,j} = \dot{\phi}_{i,j}$.

Note that the representation of twist in the objects own frame Ψ_i is typically different from the twist expressed in the reference frame Ψ_j.

Without loss of generality this is also applicable to representation of wrench.

3 Manipulation

A change of representation of points, frames and related quantities is often required for analysis and manipulation of data. Mapping is the term that is used to indicate that the reference for representation has changed and that a new representation is used for the same point or frame.

Relating different points and frames is possible using a known relation between different frames in order to compute the pose or twist of one frame from another frame. Transform is the term that is used to indicate that a representation is determined for another point of frame.

3.1 Mapping of points and frames, twist and wrench

A change from one base frame Ψ_k to another base frame Ψ_l is called a mapping of points or frames from one representation to another representation.

3.1.1 Mapping of pose

Changing the representation of a pose from one frame to another is called a transform mapping.

$$^{l}p_i = {}^{l}M_k \; {}^{k}p_i \tag{13}$$

The mapping of a pose is shown here as a premultiplication by a matrix. This mapping is completely determined the the relation between the two base frames.

Note, that the same point or frame is referred to. Its representation however has changed.

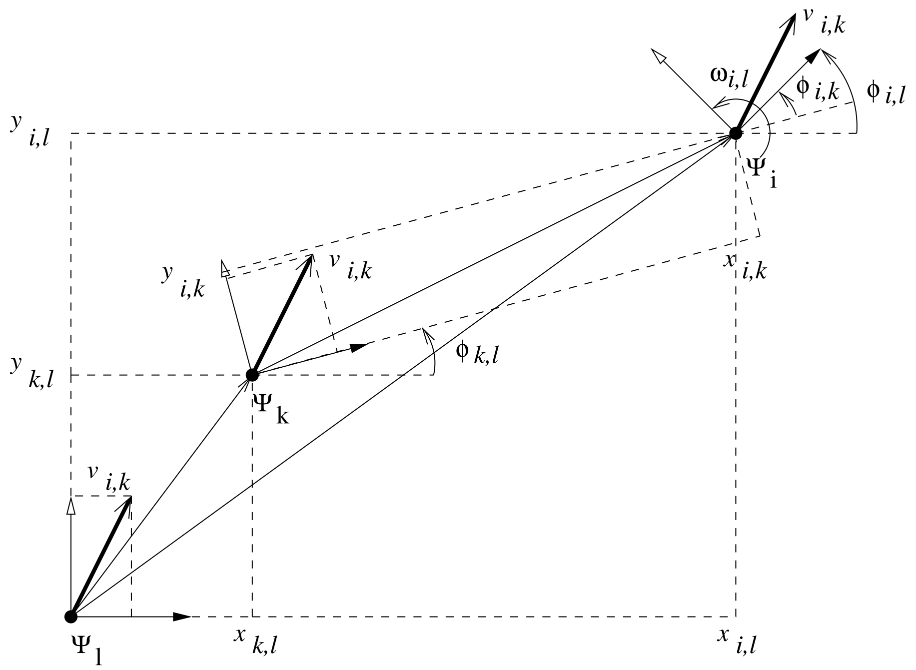

Figure 2: Mapping the pose and twist of Ψ_i from Ψ_k to Ψ_l

The mapping is completely determined by the two base frames, so we can construct the mapping ${}^lM_k$ for the pose ${}^lp_k$ of frame Ψ_k expressed in coordinates from frame Ψ_l.

$$ {}^lM_k = M({}^lp_k) \tag{14} $$

A representation of this mapping using the coefficients of ${}^lp_i$, ${}^lp_k$ and ${}^kp_i$ is:

$$ {}^lp_i = \begin{pmatrix} x_{i,l} \\ y_{i,l} \\ \phi_{i,l} \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\phi_{k,l} & -\sin\phi_{k,l} & 0 & x_{k,l} \\ \sin\phi_{k,l} & +\cos\phi_{k,l} & 0 & y_{k,l} \\ 0 & 0 & 1 & \phi_{k,l} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{i,k} \\ y_{i,k} \\ \phi_{i,k} \\ 1 \end{pmatrix} = {}^lM_k\ {}^kp_i \tag{15} $$

3.1.2 Mapping of twist and wrench

Mapping of twist and wrench from one representation into another is similar to mapping of pose. It is however not the same because linear velocity is a free vector and this vector is not changed by choosing a different base for the representation.

$$ {}^lt_{i,k} = {}^lM_k\ {}^kt_{i,k} \tag{16} $$

$$ {}^lt_{i,k} = \begin{pmatrix} {}^l\dot{x}_{i,k} \\ {}^l\dot{y}_{i,k} \\ {}^l\dot{\phi}_{i,k} \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\phi_{k,l} & -\sin\phi_{k,l} & 0 & x_{k,l} \\ \sin\phi_{k,l} & +\cos\phi_{k,l} & 0 & y_{k,l} \\ 0 & 0 & 1 & \phi_{k,l} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} {}^k\dot{x}_{i,k} \\ {}^k\dot{y}_{i,k} \\ {}^k\dot{\phi}_{i,k} \\ 0 \end{pmatrix} = {}^lM_k{}^kt_{i,k} \tag{17} $$

which uses the same matrix form of the mapping ${}^lM_k$.

3.2 Transformation of points and frames

A change of position and orientation (movement) of a point or frame is called a transformation.

$$ {}^lp_i = {}^lT_{i,k}\ {}^lp_k \tag{18} $$

The transformation of a pose is shown here as a premultiplication by a matrix. This transform operation changes pose ${}^lp_i$ to pose ${}^lp_{i'}$.

A representation of this transform operation using the coefficients of ${}^lp_i$, ${}^kp_i$ and ${}^lp_k$ is:

$$ {}^lp_i = \begin{pmatrix} x_{i,l} \\ y_{i,l} \\ \phi_{i,l} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & x_{i,k}\cos\phi_{k,l} - y_{i,k}\sin\phi_{k,l} \\ 0 & 1 & 0 & x_{i,k}\sin\phi_{k,l} + y_{i,k}\cos\phi_{k,l} \\ 0 & 0 & 1 & \phi_{i,k} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{k,l} \\ y_{k,l} \\ \phi_{k,l} \\ 1 \end{pmatrix} = {}^lT_{i,k}\ {}^lp_k \tag{19} $$

Note, movement should not be confused with motion (planar- and rotation velocity). Although the meaning of transformation is really different from that of mapping, still their representation in matrix form may have a striking appearance. The transformed point and frame are distinct from the original point and frame.

Although the tranformation is competely determined by two base frames, we still need a third base frame that is the reference for the original and transformed

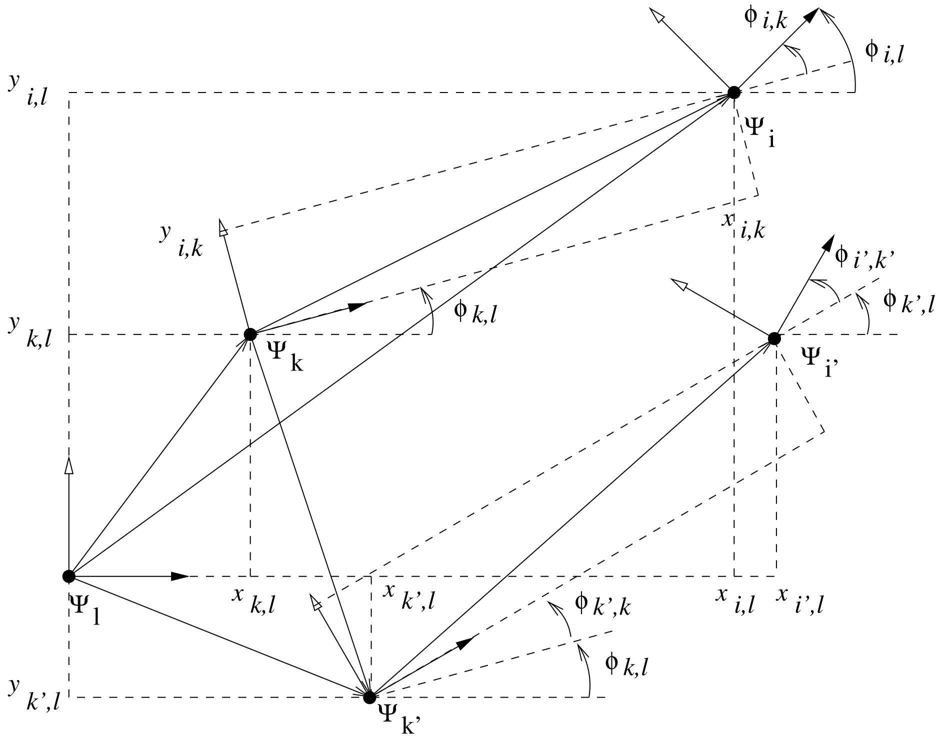

Figure 3: Transform the pose of Ψ_i to $\Psi_{i'}$

points. We can construct the transformation ${}^{j}_{k}T_i$ for the pose ${}^{j}p_i$ of frame Ψ_j expressed in coordinates from frame Ψ_k.

3.3 Conversion of twist of one frame to twist of another frame

Given two frames Ψ_i and Ψ_j and a twist t_i of the first frame, what is the twist t_j of the second frame given that both frames are attached to the same rigid body. This could be stated by the next equation:

$$ {}^{i}t_i = {}^{i,j}C_{i,j}\,{}^{j}t_j \tag{20}$$

In expanded form:

$$ {}^{i}t_i = \begin{pmatrix} {}^{i}\dot{x}_i \\ {}^{i}\dot{y}_i \\ {}^{i}\dot{\phi}_i \end{pmatrix} = \begin{pmatrix} \cos\phi_{j,i} & -\sin\phi_{j,i} & +y_{j,i} \\ \sin\phi_{j,i} & +\cos\phi_{j,i} & -x_{j,i} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} {}^{j}\dot{x}_j \\ {}^{j}\dot{y}_j \\ {}^{j}\dot{\phi}_j \end{pmatrix} = {}^{i,j}C_{i,j}\,{}^{j}t_j \tag{21}$$

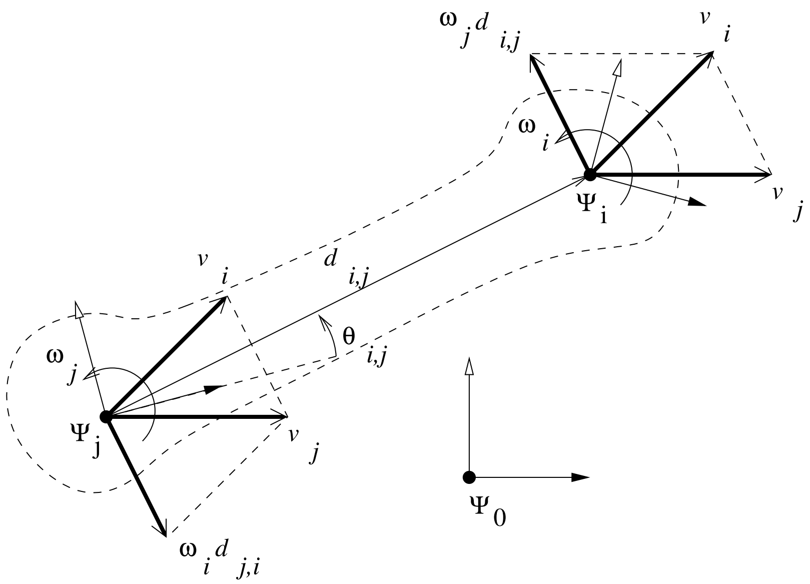

Figure 4: Conversion from twist t_i of Ψ_i to twist t_j of Ψ_j

or in the usual reversed direction:

$$ {}^{j}t_j = \begin{pmatrix} +\cos\phi_{j,i} & \sin\phi_{j,i} & x_{j,i}\sin\phi_{j,i} - y_{j,i}\cos\phi_{j,i} \\ -\sin\phi_{j,i} & \cos\phi_{j,i} & x_{j,i}\cos\phi_{j,i} + y_{j,i}\sin\phi_{j,i} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} {}^{i}\dot{x}_i \\ {}^{i}\dot{y}_i \\ {}^{i}\dot{\phi}_i \end{pmatrix} = {}^{j,i}C_{j,i}\,{}^{i}t_i \tag{22} $$

Using polar co-ordinates $(d_{j,i}, \theta_{j,i})$ instead of cartesian co-ordinates $(x_{j,i}, y_{j,i})$ gives the next more balanced expressions:

$$ {}^{i}t_i = \begin{pmatrix} {}^{i}\dot{x}_i \\ {}^{i}\dot{y}_i \\ {}^{i}\dot{\phi}_i \end{pmatrix} = \begin{pmatrix} \cos\phi_{j,i} & -\sin\phi_{j,i} & +d_{j,i}\sin\theta_{j,i} \\ \sin\phi_{j,i} & +\cos\phi_{j,i} & -d_{j,i}\cos\theta_{j,i} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} {}^{j}\dot{x}_j \\ {}^{j}\dot{y}_j \\ {}^{j}\dot{\phi}_j \end{pmatrix} = {}^{i,j}C_{i,j}\,{}^{j}t_j \tag{23} $$

and

$$ {}^{j}t_j = \begin{pmatrix} +\cos\phi_{j,i} & \sin\phi_{j,i} & d_{j,i}\sin(\phi_{j,i} - \theta_{j,i}) \\ -\sin\phi_{j,i} & \cos\phi_{j,i} & d_{j,i}\cos(\phi_{j,i} - \theta_{j,i}) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} {}^{i}\dot{x}_i \\ {}^{i}\dot{y}_i \\ {}^{i}\dot{\phi}_i \end{pmatrix} = {}^{j,i}C_{j,i}\,{}^{i}t_i \tag{24} $$