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IML Assignment-1
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Q1 $A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$

for what x, scalar 'c', such that $Ax = cx$

$AX = cX$ can be re-written as $AX - cX = 0$

$\Rightarrow X(A-c) = 0$

OR

$x(A - cI) = 0$

Here, we may note that for the matrix 'A', x is an eigenvector & c is the eigenvalue.

Now, we will find eigenvalues by solving $\det(A - cI) = 0$

$$\Rightarrow \det\begin{pmatrix} 5-c & 0 & 0 \\ 1 & 5-c & 0 \\ 0 & 1 & 5-c \end{pmatrix} = 0$$

$\Rightarrow (5-c)((5-c)(5-c)) - 0 + 0 = 0$

$\Rightarrow$ Only solution is $(5-c) = 0$

$\Rightarrow \boxed{c=5}$

Now, to find the corresponding eigenvector for $c=5$ by solving $(A-5I)X = 0$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$0x_1 + 0x_2 + 0x_3 = 0$$
$$x_1 + 0x_2 + 0x_3 = 0$$
$$0x_1 + x_2 + 0x_3 = 0$$

$\Rightarrow x_1 = 0$

$\Rightarrow x_2 = 0$

so, x_3 can have any value.

so, for $\boxed{c = 5,\ x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & b & d \end{bmatrix}}$ for arbitrary a, b, d

or if $c \neq 5$, $x = 0$

Q2

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. Show that the set of all vectors $\mathbf{X}$ such that $\mathbf{AX} = \mathbf{0}$ is a subspace of $\mathbb{R}^n$. This subspace is called the **null space** of $\mathbf{A}$ and its dimension in the **nullity** of $\mathbf{A}$.

To show all vectors in set which give $AX=0$ form a subspace,

we need to show 3 properties
1. Zero vector is in subspace
2. closed under vector addition
" " " multiplication

1. Zero vector

It is evident that if $x = \vec{0}$
$AX = 0$

$\therefore$ zero vector is present

2. Closed under vector addition

Let x_1 & x_2 be 2 vectors such that $AX_1 = 0$ & $AX_2 = 0$
. To show $\rightarrow (x_1 + x_2)$ is also in the set

$$AX_1 + AX_2 = 0$$
$$A\underbrace{(X_1 + X_2)}_{\downarrow} = 0$$

same form $\rightarrow AX = 0 \longrightarrow$ Therefore $x_1 + x_2$ is also in set & subspace is closed under vector addition

3. Subspace is closed under scalar multiplication

Let x be a vector & c be scalar we need to show cX is in the set too.

$$A(cx) = c\underbrace{(Ax)}_{=0}$$
$$\therefore c(0) = 0$$

So, we may imply cX is also in the set & subspace is closed under scalar multiplication.

Hence, we may conclude that all vectors X such that $AX = 0$ form subspace $\mathbb{R}^n$

Q3

a) [redacted] Let $\mathbf{A} \in \mathbb{R}^{m\times n}$. Show that the set of all vectors $\mathbf{Y}$ such that $\mathbf{AX} = \mathbf{Y}$ has a solution for $\mathbf{X}$ is a subspace of $\mathbb{R}^m$. This subspace is called the **range space** of $\mathbf{A}$ and its dimension in the **column rank** of $\mathbf{A}$ (why?).

b) [redacted] Show that for any matrix $\mathbf{A}$

(a) **nullity** + **row rank** $= n$

(b) **nullity** + **column rank** $= n$

and conclude that
row rank of $\mathbf{A}$ = **column rank** of $\mathbf{A}$

a) To prove that all vectors Y such that $Ax = Y$ has a solution for X is a subspace of $\mathbb{R}^n$, we only need to prove 3 properties

1. Contains zero vector
2. Closed under vector addition
3. Closed under scalar multiplication

1. Contains zero vector

if we set $x = 0$, then $Ax = A(0) = 0$, so zero vector is in the subspace

2. Closed under vector addition

Let $Y1$ & $Y2$ belong to the subspace which means there exists $X1$ & $X2$ such that $AX1 = Y1$, $AX2 = Y2$. Consider $Y1 + Y2$

$$Y_1 + Y_2 = Ax_1 + Ax_2 = A(\underbrace{x_1 + x_2})$$

$A(x_1 + x_2)$ is a valid solution, $Y_1 + Y_2$ is in subspace.

3. Closed under scalar multiplication

Let Y belong to the subspace, which means there exists X such that $Ax = Y$. Consider the scalar 'c'

$$cY = c(Ax) = A(cX)$$

cX is a solution as it is a vector in $\mathbb{R}^n$, cY is in the subspace. The set of all vectors Y such that $AX=Y$ forms a subspace of $\mathbb{R}^n$.

$\searrow$ This subspace is called range space of matrix A, denoted as $R(A)$. Its dimension = column rank of A = max number of linearly independent columns in A.

6) We know that for any $A \in R^{m \times n}$, the rank of A is equal to the number of leading columns in $RREF(A) = r$.

Case 1. $\rightarrow$ if A is a non singular matrix, then $r=n$ as RREF will have no zeroes. For $AX=0$, $x=0$; so the nullity of $A=0$

$$\therefore \text{rank}(A) + \text{nullity}(A) = n+0$$
$$= n = \#\text{ columns}(A)$$

Case 2: if A is singular, $\text{Rank}(A) < \text{order}(A)$. So, there are '$r$' non-zero or leading columns of the $RREF(A)$. Hence, there will be $(n-r)$ zero rows $\longrightarrow$ solution of $AX=0$ i.e. $\rightarrow$ nullity of A is $(n-r)$

So, $\text{Rank}(A) + \text{Nullity}(A) = r+n-r = n = \#\text{ columns}(A)$

Hence, proved.

Also for column rank = row rank
Column Rank for $A \in R^{m \times n}$ is the dimension of the subspace of F^m spanned by columns of A.
& Row rank is the dimension of subspace of the space F^n of row vectors spanned by rows of A. Source — MIT Fall 2010, Algebra I

Q4

Q4. (5 Pt)

- **Weighted least-squares:** Let $\mathbf{C}$ be a positive definite matrix. Then the $\mathbf{C}$-norm is defined as $\|\mathbf{a}\|_{\mathbf{C}} = (\mathbf{a}^T\mathbf{C}\mathbf{a})^{1/2}$. The *weighted least-squares* problem is one of minimizing $\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_{\mathbf{C}}$. The most common weighting is when $\mathbf{C}$ is diagonal. Show that weigthed least-sqaures solution can be obtained by solving:

$$(\mathbf{A}^T\mathbf{C}\mathbf{A})\mathbf{x} = \mathbf{A}^T\mathbf{C}\mathbf{b}$$

Q5

Consider two subspaces U_1 and U_2, where U_1 is the solution space of the homogeneous equation system $\boldsymbol{A}_1\boldsymbol{x} = \boldsymbol{0}$ and U_2 is the solution space of the homogeneous equation system $\boldsymbol{A}_2\boldsymbol{x} = \boldsymbol{0}$ with

$$\boldsymbol{A}_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{A}_2 = \begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix}.$$

a. Determine the dimension of U_1, U_2.
b. Determine bases of U_1 and U_2.
c. Determine a basis of $U_1 \cap U_2$.

a) Dimension of U_1 & U_2

We need to find Rank(A_1) & Rank (A_2)

$$\text{RREF}(A_1) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2 pivots $\longrightarrow$ $\boxed{\text{Rank}(A_1) = 2}$

$$\text{RREF}(A_2) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Again,

$\boxed{\text{Rank}(A_2) = 2}$

So, $\boxed{\text{dimension}(U_1) = \dim(U_2) = 2}$

b)

Basis of U_1
& Basis of U_2 $\Big\rangle$ $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

c

Basis $U_1 \cap U_2$ $\longrightarrow$ $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Q 6

Which of the following sets are subspaces of $\mathbb{R}^3$?

a. $A = \{(\lambda, \lambda + \mu^3, \lambda - \mu^3) \mid \lambda, \mu \in \mathbb{R}\}$

b. $B = \{(\lambda^2, -\lambda^2, 0) \mid \lambda \in \mathbb{R}\}$

c. Let γ be in $\mathbb{R}$.
$C = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_1 - 2\xi_2 + 3\xi_3 = \gamma\}$

d. $D = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_2 \in \mathbb{Z}\}$

Q7 (10 Pt)

It is a subspace!

a. $A = \{(\lambda, \lambda+\mu^3, \lambda-\mu^3) \mid \lambda, \mu \in R\}$

1. Zero vector exists

$\lambda = 0$ & $\mu = 0$ $\Downarrow$

$(0, 0, 0)$

2. Closed under addition

$A_1 = \begin{pmatrix} \lambda_1 \\ \lambda_1 + \mu_1^3 \\ \lambda_1 - \mu_1^3 \end{pmatrix}$ $A_2 = \begin{pmatrix} \lambda_2 \\ \lambda_2 + \mu_2^3 \\ \lambda_2 - \mu_2^3 \end{pmatrix}$

$A_1 + A_2 = \begin{pmatrix} \lambda_1 + \lambda_2 \\ \lambda_1 + \mu_1^3 + \lambda_2 + \mu_2^3 \\ \lambda_1 - \mu_1^3 + \lambda_2 - \mu_2^3 \end{pmatrix}$ $\lambda_1 + \lambda_2 \in R$

$\sqrt[3]{\mu_1 + \mu_2} \in R$ & $\sqrt[3]{\mu_1 - \mu_2}$

3. Closed under scalar multiplication

$\forall c \in R$ if $A_1 = \begin{pmatrix} \lambda \\ \lambda + \mu^3 \\ \lambda - \mu^3 \end{pmatrix}$

$c \cdot A_1 = \begin{pmatrix} c \cdot \lambda \\ c\lambda + c\mu^3 \\ c\lambda - c\mu^3 \end{pmatrix}$

$c \cdot \lambda$, $\sqrt[3]{(c \cdot \mu^3)}$ both exist in R.

b. $B = (\lambda^2, -\lambda^2, 0)$

It is not closed under scalar multiplication as if $c = -1$

$c \cdot B \rightarrow (-\lambda^2, \lambda^2, 0)$

There doesn't exist $\lambda \in R$, for $-1 * \lambda^2$

c. Let $\gamma \in R$, $C = (C_1, C_2, C_3) \in R^3 \mid C_1 - 2C_2 + 3C_3 = \gamma$

1. Zero vector exists

$(0,0,0) \rceil$ $0 - 2*0 + 3*0$
$= 0 = \gamma \in R$

2. $\begin{pmatrix} C_{11} \\ C_{12} \\ C_{13} \end{pmatrix} + \begin{pmatrix} C_{21} \\ C_{22} \\ C_{23} \end{pmatrix} = \begin{pmatrix} C_{11} + C_{21} \\ C_{12} + C_{22} \\ C_{13} + C_{23} \end{pmatrix}$

$= \begin{pmatrix} C_{31} \\ C_{32} \\ C_{33} \end{pmatrix} \in R^3$

If we let $\gamma_1, \gamma_2 \in R$

so, $\gamma_1 + \gamma_2 = \gamma_3$, where
$C_{31} - 2C_{32} + 3C_{33} = C_{11} - 2C_{12} + 3C_{13} + C_{21} - 2C_{22} + 3C_{23}$

As $\gamma_1 + \gamma_2 \in R$
$\Rightarrow \gamma_3 \in R$

3. Scalar Multiplication $\rceil$

$x(C_{11} - 2C_{12} + 3C_{13})$
$= x(\gamma_1)$

$\gamma_2 = x \cdot \gamma_1$
$\Rightarrow \gamma_2 \in R$

$\boxed{\text{C is a subspace}}$

✓

d. $D = \{c_1, c_2, c_3\} \mid c_2 \in \mathbb{Z}$

D is not a subspace in $\mathbb{R}^3$ as it is not closed under scalar multiplication

$A_1 = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ taking $x = 0.5$, $A_1 = \begin{pmatrix} 0.5\,c_1 \\ 0.5\,c_2 \\ 0.5\,c_3 \end{pmatrix}$

doesn't comply with $c_2 \in \mathbb{Z}$

Q8

Q8 (15 Pt)

Consider the space of square-integrable real functions on the interval $[-\pi, \pi]$, $L_2([-\pi, \pi])$, and the associated orthonormal basis given by

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}} \right\}, \qquad n = 1, 2, \ldots$$

Consider the following two subspaces: S – space of symmetric functions, that is, $f(x) = f(-x)$, on $[-\pi, \pi]$, and A – space of antisymmetric functions, $f(x) = -f(-x)$, on $[-\pi, \pi]$.

(a) Show how any function $f(x)$ from $L_2([-\pi, \pi])$ can be written as $f(x) = f_s(x) + f_a(x)$, where $f_s(x) \in S$ and $f_a(x) \in A$.

(b) Give orthonormal bases for S and A.

(c) Verify that $L_2([-\pi, \pi]) = S \oplus A$.

a

Let $f \in L_2[-\pi, \pi]$ then f may be written as;

$$f(x) = \frac{a}{\sqrt{2\pi}} + \frac{b \cos nx}{\sqrt{\pi}} + \frac{c \sin nx}{\sqrt{\pi}} \quad \text{for some scalars } a, b, c \in N \quad m, n$$

As $\left\{ 1/\sqrt{2\pi}, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}} \right\}$ is a basis for $L_2[-\pi, \pi]$

$$\Rightarrow f(x) = f_s(x) + f_a(x)$$

$$f_s(x) = \frac{a}{\sqrt{2\pi}} + \frac{b \cos mx}{\sqrt{\pi}} \quad \& \quad f_a(x) = \frac{c \sin nx}{\sqrt{\pi}}$$

$$f_s(-x) = \frac{a}{\sqrt{2\pi}} + \frac{b \cos m(-x)}{\sqrt{\pi}} = \frac{a}{\sqrt{2\pi}} + \frac{b \cos mx}{\sqrt{\pi}}$$

$f_s(-x) = f_s(x) \Rightarrow f_s(x)$ is symmetric $\Rightarrow f_s(x) \in S$

$\&$ $f_a(-x) = \frac{c \sin(-x)}{\sqrt{\pi}} = \frac{-c \sin nx}{\sqrt{\pi}}$

$\sin(-\theta) = -\sin\theta$

$\Rightarrow f_a(x) = -f_a(-x) = f_a(x)$ is an antisymmetric function

$$\Rightarrow f_a(x) \in A$$

Hence, $f(x) = f_s(x) + f_a(x)$ where $f_s(x) \in S$
& $f_a(x) \in A$

b. As seen above $f \in \text{span}\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos nx}{\sqrt{\pi}} \right\}$ is symmetric

$$\Rightarrow \text{span}\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos nx}{\sqrt{\pi}} \right\} \subseteq S(1)$$

$$f \in S \Rightarrow f \in L_2[-\pi, \pi] \Rightarrow f \in \text{span}\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}} \right\}$$

Since $f \in S \Rightarrow f(-x) = f(x)$

$$\Rightarrow \frac{a}{\sqrt{2\pi}} + \frac{b \cos n(-x)}{\sqrt{\pi}} + \frac{c \sin(-x)n}{\sqrt{\pi}}$$

$$\Rightarrow \frac{a}{\sqrt{2\pi}} + \frac{b \cos nx}{\sqrt{\pi}} - c\frac{\sin nx}{\sqrt{\pi}} \quad \Rightarrow \frac{2c \sin nx}{\sqrt{\pi}} = 0$$

$$\Rightarrow c = 0$$

As $\sin nx$ can not be 0 $\forall x \,\&\, n$

$$f(x) = \frac{a}{\sqrt{2\pi}} + \frac{b\cos mx}{\sqrt{\pi}} \in \operatorname{span}\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos nx}{\sqrt{\pi}}\right\} \implies S \subseteq \operatorname{span}\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos nx}{\sqrt{\pi}}\right\}$$

we get

$$S = \operatorname{span}\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos nx}{\sqrt{\pi}}\right\} \longrightarrow \text{is also linearly independant}$$

Since $\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}}\right\} \longrightarrow$ basis is linearly indep.
& subset of a LI set is also LI

Therefore

$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos nx}{\sqrt{\pi}}\right\}$ is orthonormal basis for S

Similarly, we can prove $\left\{\frac{\sin nx}{\sqrt{\pi}}\right\}$ is an orthonormal basis for A

c) we know, $f(x) = f_s(x) + f_a(x)$ where $f_s(x) \in S$
& $f_a(x) \in A$

$$\implies L_2[-\pi, \pi] = S + A$$

We can say the represen. of $f \in L_2[-\pi, \pi]$ as sum of fn from S & A is unique:

let $f(x) = f_s(x) + f_a(x)$
& $f(x) = f_s'(x) + f_a'(x)$

then $f_s(x) + f_a(x) = f_s'(x) + f_a'(x)$

replacing x w/ $-x$

$f_s(x) - f_a(x) = f_s'(x) - f_a'(x)$

Adding & subtracting the equations

$2f_s(x) = 2f_s'(x)$ & $2f_a(x) = 2f_a'(x)$

$\Rightarrow f_s(x) = f_s'(x)$ $\Rightarrow f_a(x) = f_a'(x)$

Thus, repr. of $f \in L_2[-\pi, \pi]$ as sum from S & A is unique

& we can concluded that $L_2[-\pi, \pi] = S \oplus A$

Q10

[redacted] Show that $\langle \cdot, \cdot \rangle$ defined for all $\boldsymbol{x} = [x_1, x_2]^\top \in \mathbb{R}^2$ and $\boldsymbol{y} = [y_1, y_2]^\top \in \mathbb{R}^2$ by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle := x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2(x_2 y_2)$$

is an inner product.

We need to prove the following properties to prove as inner product

1. $\langle x, x \rangle \geq 0$ & $= 0$ when $x = 0$ only

$$\langle x, x \rangle = x_1^2 - (x_1 x_2 + x_2 x_1) + 2 x_2^2$$
$$= (x_1 - x_2)^2 + x_2^2 \geq 0 \quad \forall\, x_1, x_2$$

as $(x_1 - x_2)^2 \geq 0$

& $x_2^2 > 0$

& when $x_1, x_2 = 0$

then $= 0$

2. $\langle x, y \rangle = \langle y, x \rangle$

$$\langle x, y \rangle = x_1 y_1 - (x_1 y_2 + y_1 x_2) + 2(x_2 y_2)$$
$$\langle y, x \rangle = y_1 x_1 - (y_1 x_2 + x_1 y_2) + 2(y_2 x_2)$$

$$\Rightarrow \langle x, y \rangle = \langle y, x \rangle$$

3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

$$= (u_1 + v_1)w_1 - ((u_1 + v_1)w_2 + (u_2 + v_2)w_1) + 2((u_2 + v_2)w_2)$$
$$= u_1w_1 + v_1w_1 - (u_1w_2 + v_1w_2 + u_2w_1 + v_2w_1) + 2(u_2w_2 + v_2w_2)$$
$$= [u_1w_1 - (w_1w_2 + u_2w_1) + 2(u_2w_2)] + [v_1w_1 - (v_1w_2 + v_2w_1) + 2(v_2w_2)]$$
$$= \langle u, w \rangle + \langle v, w \rangle$$

$\square$ QED

4. $\langle ax, y \rangle = a\langle x, y \rangle$

$$= ax_1 - a(x_1y_2 + x_2y_1) + 2a(x_2)y_2$$
$$= a[x_1y_1 - (x_1y_2 + x_2y_1) + 2x_2y_2]$$
$$= a\langle x, y \rangle$$

$\Rightarrow \langle ax, y \rangle = a\langle x, y \rangle$

$\square$ QED

$\therefore$ it is an inner product