A proof of Peirce's Law equivalent to Law of Excluded Middle

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January 28, 2014

Contents

1 Deduction

1.1 Shorthands

- negation $\neg\alpha$ is a shorthand for $\alpha \to \bot$
- equivalent $\alpha \rightleftarrows \beta$ is a shorthand for $(\alpha \to \beta) \land (\beta \to \alpha)$

1.2 Rules of Inference

First, I prepare the following rules of inference, for the proof.

$$\frac{\Gamma, \alpha \vdash \beta}{\Gamma \vdash \alpha \to \beta} \tag{Arr-Intro}$$

$$\frac{\Gamma_1 \vdash \alpha \qquad \Gamma_2 \vdash \alpha \to \beta}{\Gamma_1, \Gamma_2 \vdash \beta} \tag{Arr-Elim}$$

$$\frac{\Gamma_1 \vdash \alpha \qquad \Gamma_2 \vdash \beta}{\Gamma_1, \Gamma_2 \vdash \alpha \wedge \beta} \tag{\textsc{Conj-Intro}}$$

$$\frac{\Gamma \vdash \alpha \wedge \beta}{\Gamma \vdash \alpha} \tag{\textsc{Conj-Elim-L}}$$

$$\frac{\Gamma \vdash \alpha \wedge \beta}{\Gamma \vdash \beta} \tag{\textsc{Conj-Elim-R}}$$

$$\frac{\Gamma \vdash \alpha}{\Gamma \vdash \alpha \vee \beta} \tag{\textsc{Disj-Intro-L}}$$

$$\frac{\Gamma \vdash \beta}{\Gamma \vdash \alpha \vee \beta} \tag{\textsc{Disj-Intro-R}}$$

$$\frac{\Gamma_1 \vdash \alpha \vee \beta \qquad \Gamma_2, \alpha \vdash \gamma \qquad \Gamma_3, \beta \vdash \gamma}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \gamma} \tag{\textsc{Disj-Elim}}$$

$$\frac{\Gamma, \alpha \vdash \bot}{\Gamma \vdash \neg\alpha} \tag{\textsc{Neg-Intro}}$$

$$\frac{\Gamma_1 \vdash \alpha \qquad \Gamma_2 \vdash \neg\alpha}{\Gamma_1, \Gamma_2 \vdash \bot} \tag{\textsc{Neg-Elim}}$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \alpha} \tag{\textsc{Bot-Elim}}$$

2 De Morgan's Laws

2.1 Conjunction of Negations

Lemma.

$$\vdash \neg\alpha \wedge \neg\beta \to \neg(\alpha \vee \beta)$$

Proof.

$$\dfrac{\dfrac{\dfrac{\alpha \vee \beta \vdash \alpha \vee \beta \qquad \dfrac{\alpha \vdash \alpha \qquad \dfrac{\neg\alpha \wedge \neg\beta \vdash \neg\alpha \wedge \neg\beta}{\neg\alpha \wedge \neg\beta \vdash \neg\alpha}}{\alpha, \neg\alpha \wedge \neg\beta \vdash \bot} \qquad \dfrac{\beta \vdash \beta \qquad \dfrac{\neg\alpha \wedge \neg\beta \vdash \neg\alpha \wedge \neg\beta}{\neg\alpha \wedge \neg\beta \vdash \neg\beta}}{\beta, \neg\alpha \wedge \neg\beta \vdash \bot}}{\alpha \vee \beta, \neg\alpha \wedge \neg\beta \vdash \bot}}{\neg\alpha \wedge \neg\beta \vdash \neg(\alpha \vee \beta)}}{\vdash \neg\alpha \wedge \neg\beta \to \neg(\alpha \vee \beta)}$$

Qed.

Lemma.

$$\vdash \neg(\alpha \vee \beta) \to \neg\alpha \wedge \neg\beta$$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\alpha \vdash \alpha \vee \beta} \quad \neg(\alpha \vee \beta) \vdash \neg(\alpha \vee \beta)}{\alpha, \neg(\alpha \vee \beta) \vdash \bot}}{\neg(\alpha \vee \beta) \vdash \neg\alpha} \quad \dfrac{\dfrac{\dfrac{\beta \vdash \beta}{\beta \vdash \alpha \vee \beta} \quad \neg(\alpha \vee \beta) \vdash \neg(\alpha \vee \beta)}{\beta, \neg(\alpha \vee \beta) \vdash \bot}}{\neg(\alpha \vee \beta) \vdash \neg\beta}}{\neg(\alpha \vee \beta) \vdash \neg\alpha \wedge \neg\beta}}{\vdash \neg(\alpha \vee \beta) \to \neg\alpha \wedge \neg\beta}$$

Qed.

Theorem.

$$\vdash \neg\alpha \wedge \neg\beta \rightleftarrows \neg(\alpha \vee \beta)$$

Proof.

$$\dfrac{\dfrac{\textbf{from above}}{\vdash \neg\alpha \wedge \neg\beta \to \neg(\alpha \vee \beta)} \quad \dfrac{\textbf{from above}}{\vdash \neg(\alpha \vee \beta) \to\vdash \neg\alpha \wedge \neg\beta}}{\vdash \neg\alpha \wedge \neg\beta \rightleftarrows \neg(\alpha \vee \beta)}$$

Qed.

3 Peirce's Law equivalent to Law of Excluded Middle

3.1 Negation of Excluded Middle is False

Lemma.

$$\vdash \neg(\alpha \wedge \neg\alpha)$$

Proof.

$$\dfrac{\dfrac{\dfrac{\alpha \wedge \neg\alpha \vdash \alpha \wedge \neg\alpha}{\alpha \wedge \neg\alpha \vdash \alpha} \quad \dfrac{\alpha \wedge \neg\alpha \vdash \alpha \wedge \neg\alpha}{\alpha \wedge \neg\alpha \vdash \neg\alpha}}{\alpha \wedge \neg\alpha \vdash \bot}}{\vdash \neg(\alpha \wedge \neg\alpha)}$$

Qed.

Theorem.

$$\vdash \neg\neg(\alpha \vee \neg\alpha)$$

Proof.

$$\dfrac{\dfrac{\dfrac{\textbf{from above}}{\vdash \neg(\neg\alpha \wedge \neg\neg\alpha)} \quad \dfrac{\dfrac{\textbf{De Morgen's Law}}{\vdash \neg(\alpha \vee \neg\alpha) \to \neg\alpha \wedge \neg\neg\alpha} \quad \neg(\alpha \vee \neg\alpha) \vdash \neg(\alpha \vee \neg\alpha)}{\neg(\alpha \vee \neg\alpha) \vdash \neg\alpha \wedge \neg\neg\alpha}}{\neg(\alpha \vee \neg\alpha) \vdash \bot}}{\vdash \neg\neg(\alpha \vee \neg\alpha)}$$

Qed.

3.2 Law of Excluded Middle implies Peirce's Law

Lemma.

$$\vdash \neg\alpha \vee \beta \to \alpha \to \beta$$

Proof.

$$\dfrac{\dfrac{\neg\alpha \vee \beta \vdash \neg\alpha \vee \beta \qquad \dfrac{\dfrac{\dfrac{\alpha \vdash \alpha \quad \neg\alpha \vdash \neg\alpha}{\alpha, \neg\alpha \vdash \bot}}{\alpha, \neg\alpha \vdash \beta}}{\neg\alpha \vdash \alpha \to \beta} \qquad \dfrac{\alpha \vdash \alpha \quad \beta \vdash \beta}{\beta \vdash \alpha \to \beta}}{\neg\alpha \vee \beta \vdash \alpha \to \beta}}{\vdash \neg\alpha \vee \beta \to \alpha \to \beta}$$

Qed.

Lemma.

$$\vdash \alpha \to \beta \to \alpha$$

Proof.

$$\dfrac{\dfrac{\vdash \neg\beta \vee \alpha \to \beta \to \alpha \qquad \dfrac{\alpha \vdash \alpha}{\alpha \vdash \neg\beta \vee \alpha}}{\alpha \vdash \beta \to \alpha}}{\vdash \alpha \to \beta \to \alpha}$$

Qed.

Lemma.

$$\vdash \neg\alpha \to \alpha \to \beta$$

Proof.

$$\dfrac{\dfrac{\vdash \neg\alpha \vee \beta \to \alpha \to \beta \qquad \dfrac{\neg\alpha \vdash \neg\alpha}{\neg\alpha \vdash \neg\alpha \vee \beta}}{\neg\alpha \vdash \alpha \to \beta}}{\vdash \neg\alpha \to \alpha \to \beta}$$

Qed.

Theorem.

$$(\alpha \vee \neg\alpha) \vdash ((\alpha \to \beta) \to \alpha) \to \alpha$$

Proof.

$$\dfrac{\dfrac{(\alpha \to \beta) \to \alpha \vdash (\alpha \to \beta) \to \alpha \qquad \dfrac{\vdash \neg\alpha \to \alpha \to \beta \quad \neg\alpha \vdash \neg\alpha}{\neg\alpha \vdash \alpha \to \beta}}{\neg\alpha, (\alpha \to \beta) \to \alpha \vdash \alpha}}{\neg\alpha \vdash ((\alpha \to \beta) \to \alpha) \to \alpha}$$

$$\dfrac{\alpha \vee \neg\alpha \vdash \alpha \vee \neg\alpha \qquad \dfrac{\vdash \alpha \to ((\alpha \to \beta) \to \alpha) \to \alpha \quad \alpha \vdash \alpha}{\alpha \vdash ((\alpha \to \beta) \to \alpha) \to \alpha} \qquad \dfrac{\textbf{from above}}{\neg\alpha \vdash ((\alpha \to \beta) \to \alpha) \to \alpha}}{\alpha \vee \neg\alpha \vdash ((\alpha \to \beta) \to \alpha) \to \alpha}$$

Qed.

3.3 Peirce's Law implies Law of Excluded Middle

Theorem.

$$\vdash \alpha \vee \neg\alpha$$

Proof.

$$\dfrac{\dfrac{\textbf{Peirce's Law}}{\vdash (((\alpha \vee \neg\alpha) \to \bot) \to \alpha \vee \neg\alpha) \to \alpha \vee \neg\alpha} \qquad \dfrac{\dfrac{\dfrac{\neg(\alpha \vee \neg\alpha) \vdash \neg(\alpha \vee \neg\alpha) \quad \vdash \neg\neg(\alpha \vee \neg\alpha)}{\neg(\alpha \vee \neg\alpha) \vdash \bot}}{\neg(\alpha \vee \neg\alpha) \vdash \alpha \vee \neg\alpha}}{\vdash \neg(\alpha \vee \neg\alpha) \to \alpha \vee \neg\alpha}}{\vdash \alpha \vee \neg\alpha}$$

Qed.

The proof ends here. Now, the law of excluded middle implies the Peirce's law and the Peirce's law implies the law of excluded middle so both proofs are equivalent.