

# A proof of Peirce's Law equivalent to Law of Excluded Middle

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January 28, 2014

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## 1 Deduction

### 1.1 Shorthands

- negation  $\neg\alpha$  is a shorthand for  $\alpha \rightarrow \perp$
- equivalent  $\alpha \Leftrightarrow \beta$  is a shorthand for  $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$

### 1.2 Rules of Inference

First, I prepare the following rules of inference, for the proof.

$$\frac{\Gamma, \alpha \vdash \beta}{\Gamma \vdash \alpha \rightarrow \beta} \quad (\text{ARR-INTRO})$$

$$\frac{\Gamma_1 \vdash \alpha \quad \Gamma_2 \vdash \alpha \rightarrow \beta}{\Gamma_1, \Gamma_2 \vdash \beta} \quad (\text{ARR-ELIM})$$

$$\begin{array}{c}
\frac{\Gamma_1 \vdash \alpha \quad \Gamma_2 \vdash \beta}{\Gamma_1, \Gamma_2 \vdash \alpha \wedge \beta} \quad (\text{CONJ-INTRO}) \\
\\
\frac{\Gamma \vdash \alpha \wedge \beta}{\Gamma \vdash \alpha} \quad (\text{CONJ-ELIM-L}) \\
\\
\frac{\Gamma \vdash \alpha \wedge \beta}{\Gamma \vdash \beta} \quad (\text{CONJ-ELIM-R}) \\
\\
\frac{\Gamma \vdash \alpha}{\Gamma \vdash \alpha \vee \beta} \quad (\text{DISJ-INTRO-L}) \\
\\
\frac{\Gamma \vdash \beta}{\Gamma \vdash \alpha \vee \beta} \quad (\text{DISJ-INTRO-R}) \\
\\
\frac{\Gamma_1 \vdash \alpha \vee \beta \quad \Gamma_2, \alpha \vdash \gamma \quad \Gamma_3, \beta \vdash \gamma}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \gamma} \quad (\text{DISJ-ELIM}) \\
\\
\frac{\Gamma, \alpha \vdash \perp}{\Gamma \vdash \neg \alpha} \quad (\text{NEG-INTRO}) \\
\\
\frac{\Gamma_1 \vdash \alpha \quad \Gamma_2 \vdash \neg \alpha}{\Gamma_1, \Gamma_2 \vdash \perp} \quad (\text{NEG-ELIM}) \\
\\
\frac{\Gamma \vdash \perp}{\Gamma \vdash \alpha} \quad (\text{BOT-ELIM})
\end{array}$$

## 2 De Morgan's Laws

### 2.1 Conjunction of Negations

*Lemma.*

$$\vdash \neg \alpha \wedge \neg \beta \rightarrow \neg(\alpha \vee \beta)$$

*Proof.*

$$\begin{array}{c}
\frac{\alpha \vee \beta \vdash \alpha \vee \beta \quad \frac{\alpha \vdash \alpha \quad \frac{\neg \alpha \wedge \neg \beta \vdash \neg \alpha \wedge \neg \beta}{\neg \alpha \wedge \neg \beta \vdash \neg \alpha}}{\alpha, \neg \alpha \wedge \neg \beta \vdash \perp} \quad \frac{\beta \vdash \beta \quad \frac{\neg \alpha \wedge \neg \beta \vdash \neg \alpha \wedge \neg \beta}{\neg \alpha \wedge \neg \beta \vdash \neg \beta}}{\beta, \neg \alpha \wedge \neg \beta \vdash \perp}}{\frac{\alpha \vee \beta, \neg \alpha \wedge \neg \beta \vdash \perp}{\neg \alpha \wedge \neg \beta \vdash \neg(\alpha \vee \beta)}} \\
\hline
\vdash \neg \alpha \wedge \neg \beta \rightarrow \neg(\alpha \vee \beta)
\end{array}$$

*Qed.*

*Lemma.*

$$\vdash \neg(\alpha \vee \beta) \rightarrow \neg\alpha \wedge \neg\beta$$

*Proof.*

$$\frac{\frac{\frac{\alpha \vdash \alpha}{\alpha \vdash \alpha \vee \beta} \quad \neg(\alpha \vee \beta) \vdash \neg(\alpha \vee \beta)}{\alpha, \neg(\alpha \vee \beta) \vdash \perp} \quad \frac{\frac{\frac{\beta \vdash \beta}{\beta \vdash \alpha \vee \beta} \quad \neg(\alpha \vee \beta) \vdash \neg(\alpha \vee \beta)}{\beta, \neg(\alpha \vee \beta) \vdash \perp}}{\neg(\alpha \vee \beta) \vdash \neg\alpha \quad \neg(\alpha \vee \beta) \vdash \neg\beta}}{\neg(\alpha \vee \beta) \vdash \neg\alpha \wedge \neg\beta} \quad \vdash \neg(\alpha \vee \beta) \rightarrow \neg\alpha \wedge \neg\beta$$

*Qed.*

*Theorem.*

$$\vdash \neg\alpha \wedge \neg\beta \Leftrightarrow \neg(\alpha \vee \beta)$$

*Proof.*

$$\frac{\frac{\text{from above}}{\vdash \neg\alpha \wedge \neg\beta \rightarrow \neg(\alpha \vee \beta)} \quad \frac{\text{from above}}{\vdash \neg(\alpha \vee \beta) \rightarrow \neg\alpha \wedge \neg\beta}}{\vdash \neg\alpha \wedge \neg\beta \Leftrightarrow \neg(\alpha \vee \beta)}$$

*Qed.*

### 3 Peirce's Law equivalent to Law of Excluded Middle

#### 3.1 Negation of Excluded Middle is False

*Lemma.*

$$\vdash \neg(\alpha \wedge \neg\alpha)$$

*Proof.*

$$\frac{\frac{\frac{\alpha \wedge \neg\alpha \vdash \alpha \wedge \neg\alpha}{\alpha \wedge \neg\alpha \vdash \alpha} \quad \frac{\alpha \wedge \neg\alpha \vdash \alpha \wedge \neg\alpha}{\alpha \wedge \neg\alpha \vdash \neg\alpha}}{\alpha \wedge \neg\alpha \vdash \perp}}{\vdash \neg(\alpha \wedge \neg\alpha)}$$

*Qed.*

*Theorem.*

$$\vdash \neg\neg(\alpha \vee \neg\alpha)$$

*Proof.*

$$\frac{\frac{\text{from above}}{\vdash \neg(\neg\alpha \wedge \neg\neg\alpha)} \quad \frac{\text{De Morgan's Law}}{\vdash \neg(\alpha \vee \neg\alpha) \rightarrow \neg\alpha \wedge \neg\neg\alpha} \quad \neg(\alpha \vee \neg\alpha) \vdash \neg(\alpha \vee \neg\alpha)}{\frac{\neg(\alpha \vee \neg\alpha) \vdash \neg\alpha \wedge \neg\neg\alpha}{\neg(\alpha \vee \neg\alpha) \vdash \perp}} \quad \vdash \neg\neg(\alpha \vee \neg\alpha)$$

*Qed.*

### 3.2 Law of Excluded Middle implies Peirce's Law

*Lemma.*

$$\vdash \neg\alpha \vee \beta \rightarrow \alpha \rightarrow \beta$$

*Proof.*

$$\frac{\frac{\frac{\alpha \vdash \alpha \quad \neg\alpha \vdash \neg\alpha}{\alpha, \neg\alpha \vdash \perp}}{\alpha, \neg\alpha \vdash \beta}}{\neg\alpha \vee \beta \vdash \neg\alpha \vee \beta} \quad \frac{\alpha \vdash \alpha \quad \beta \vdash \beta}{\beta \vdash \alpha \rightarrow \beta} \quad \frac{\neg\alpha \vee \beta \vdash \neg\alpha \vee \beta \quad \beta \vdash \alpha \rightarrow \beta}{\neg\alpha \vee \beta \vdash \alpha \rightarrow \beta} \quad \frac{}{\vdash \neg\alpha \vee \beta \rightarrow \alpha \rightarrow \beta}$$

*Qed.*

*Lemma.*

$$\vdash \alpha \rightarrow \beta \rightarrow \alpha$$

*Proof.*

$$\frac{\frac{\frac{}{\vdash \neg\beta \vee \alpha \rightarrow \beta \rightarrow \alpha} \quad \frac{\alpha \vdash \alpha}{\alpha \vdash \neg\beta \vee \alpha}}{\alpha \vdash \beta \rightarrow \alpha}}{\vdash \alpha \rightarrow \beta \rightarrow \alpha}$$

*Qed.*

*Lemma.*

$$\vdash \neg\alpha \rightarrow \alpha \rightarrow \beta$$

*Proof.*

$$\frac{\frac{\frac{}{\vdash \neg\alpha \vee \beta \rightarrow \alpha \rightarrow \beta} \quad \frac{\neg\alpha \vdash \neg\alpha}{\neg\alpha \vdash \neg\alpha \vee \beta}}{\neg\alpha \vdash \alpha \rightarrow \beta}}{\vdash \neg\alpha \rightarrow \alpha \rightarrow \beta}$$

*Qed.*

*Theorem.*

$$(\alpha \vee \neg\alpha) \vdash ((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha$$

*Proof.*

$$\frac{\frac{\frac{\frac{}{\vdash \neg\alpha \rightarrow \alpha \rightarrow \beta} \quad \frac{}{\vdash \neg\alpha \rightarrow \alpha \rightarrow \beta}}{\neg\alpha, (\alpha \rightarrow \beta) \rightarrow \alpha \vdash \alpha}}{\neg\alpha \vdash ((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha}}{\neg\alpha \vdash ((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha}} \quad \frac{\frac{\frac{}{\vdash \alpha \rightarrow ((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha} \quad \frac{\alpha \vdash \alpha}{\alpha \vdash ((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha}}{\alpha \vee \neg\alpha \vdash \alpha \vee \neg\alpha}}{\alpha \vee \neg\alpha \vdash ((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha}}$$

*Qed.*

### 3.3 Peirce's Law implies Law of Excluded Middle

*Theorem.*

$$\vdash \alpha \vee \neg \alpha$$

*Proof.*

$$\begin{array}{c}
 \text{Peirce's Law} \\
 \hline
 \vdash (((\alpha \vee \neg \alpha) \rightarrow \perp) \rightarrow \alpha \vee \neg \alpha) \rightarrow \alpha \vee \neg \alpha \\
 \hline
 \vdash \alpha \vee \neg \alpha
 \end{array}
 \qquad
 \begin{array}{c}
 \neg(\alpha \vee \neg \alpha) \vdash \neg(\alpha \vee \neg \alpha) \quad \vdash \neg \neg(\alpha \vee \neg \alpha) \\
 \hline
 \neg(\alpha \vee \neg \alpha) \vdash \perp \\
 \hline
 \neg(\alpha \vee \neg \alpha) \vdash \alpha \vee \neg \alpha \\
 \hline
 \vdash \neg(\alpha \vee \neg \alpha) \rightarrow \alpha \vee \neg \alpha
 \end{array}$$

*Qed.*

The proof ends here. Now, the law of excluded middle implies the Peirce's law and the Peirce's law implies the law of excluded middle so both proofs are equivalent.