Using the left end points of the subintervals, $c_j = x_{j-1}$, gives the left-hand Riemann sum

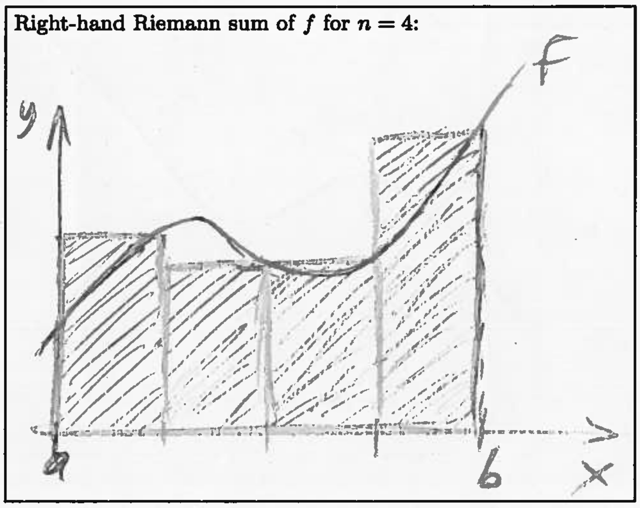
$$\sum_{j=1}^{n} f(x_{j-1})\Delta x = \sum_{j=0}^{n-1} f(x_j)\Delta x.$$

Remark 3.2. (i) Equation (3.1) introduces a new quantity – the definite integral of f over $[a, b]$ – and *defines* (that is what the ":=" means) it to be equal to the limit for $n \to \infty$ of the Riemann sums on the right. Note that on the right-hand side of (3.1), different choices could be made for the points c_j. One should therefore show now that all these possible different choices lead to the same limit. That is, one should show that $\int_a^b f(x)\,\mathrm{d}x$ as defined above is *well-defined*! This will be given as an exercise at the end of this section.

(ii) The following figure illustrates the right-hand Riemann sum for $n = 4$,

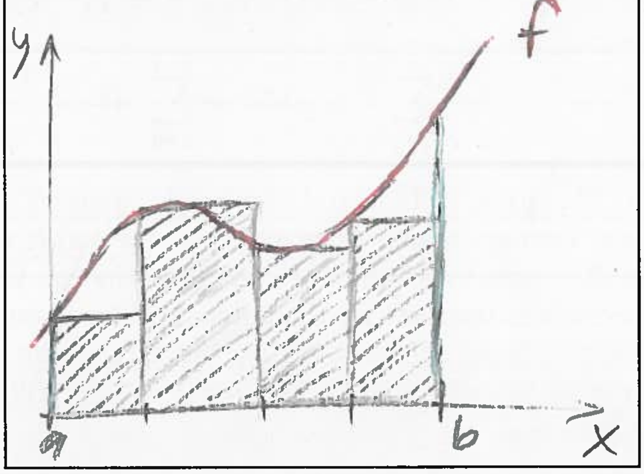
$$\sum_{j=1}^{4} f(x_j)\Delta x,$$

of some function f.



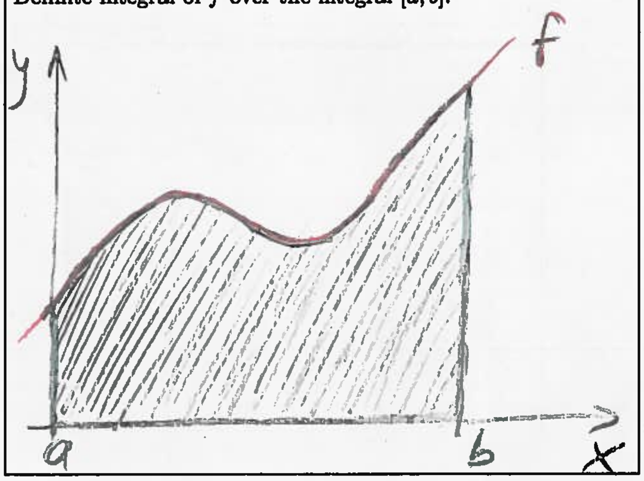
The rectangles in this figure have areas "height" · "width" $= f(x_j) \cdot \Delta x$, and they are added up to obtain the Riemann sum. For comparison, the left-hand Riemann sum is

Left-hand Riemann sum of f for $n = 4$:



We see that in both cases, the Riemann sums approximate the area under the curve $y = f(x)$, and as in the example of the oddly-shaped room in the introduction, those approximations will get better if we choose n to be larger. We can therefore interpret their limit, i.e., $\int_a^b f(x)\,\mathrm{d}x$, as the signed area under the curve $y = f(x)$:

Definite integral of f over the integral $[a, b]$:



Here, the word "signed" was added since areas below the x-axis contribute negatively to the definite integral.

Example 3.3. Consider $f(x) = {}^3\!/_5\, x$, $a = 0$, $b = 5$. First, let us find the left-hand Riemann sum for $n = 5$ of f. The partition of $[a, b] = [0, 5]$ is

$$\Delta x = \frac{b-a}{n} = \frac{5-0}{5} = 1,$$
$$x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ \ldots, x_5 = 5,$$

Now suppose that $f(x)$ is continuous on the interval $[-a, a]$, where $a > 0$. Then we have

$$\text{(7)} \qquad \text{If } f(x) \text{ is even, i.e. } f(-x) = f(x), \text{ then } \int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_{0}^{a} f(x)\,\mathrm{d}x,$$

$$\text{(8)} \qquad \text{If } f(x) \text{ is odd, i.e. } f(-x) = -f(x), \text{ then } \int_{-a}^{a} f(x)\,\mathrm{d}x = 0.$$

Remark 3.5. (i) All those properties are proven starting from the definition 3.1 of the definite integral. (There is nothing else to start from, is there?) Only the proof of the last property is written out below.

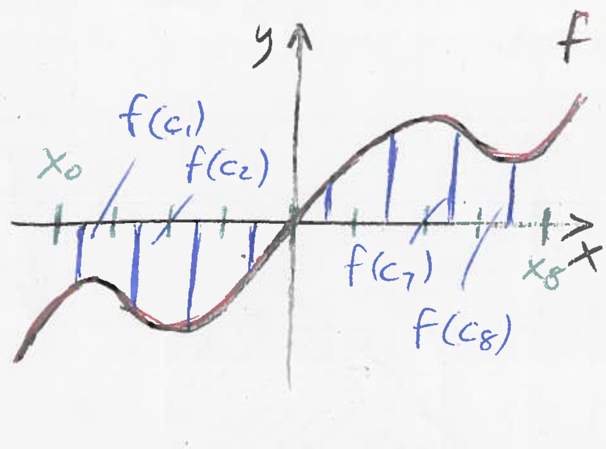
(ii) Properties (3) and (4) state that integration is *linear*, and they can by proven by using the corresponding laws for sums and limits (e.g., pull the constant α out of the sum and then out of the limit in definition 3.1).

Proof. For the proof of (8), we consider an odd function and we use only even n for the limit $n \to \infty$. Since all choices for the points c_j lead to the same limit, we make a choice that suits our purpose well: let the c_j be the midpoints of their subintervals. The Riemann sums are

$$\sum_{j=1}^{n} f(c_j)\Delta x = \Delta x \cdot [f(c_1) + f(c_2) + \ldots f(c_{n-1}) + f(c_n)],$$

and comparing to

An odd function and its Riemann sum, using midpoints:



we see that this Riemann sum is zero since $f(c_1) + f(c_n) = 0$, $f(c_2) + f(c_{n-1}) = 0, \ldots$ This gives

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = \lim_{n\to\infty} 0 = 0.$$

Now, the above assertion that $f(c_j) + f(c_{n-j+1}) = 0$ seems to be confirmed by the sketch, but we should prove it properly: After working out formulas for the partition points x_j, we find

$$c_j = -a + \Delta x\left(j - \frac{1}{2}\right) = -a + \frac{2a}{n}\left(j - \frac{1}{2}\right)$$

for the midpoints c_j of the subintervals. This gives

$$\begin{aligned} c_{n-j+1} &= -a + \frac{2a}{n}\left(n - j + 1 - \frac{1}{2}\right) \\ &= -a + 2a + \frac{2a}{n}\left(-j + \frac{1}{2}\right) = -c_j, \end{aligned}$$

and therefore, since f is odd,

$$f(c_j) + f(c_{n-j+1}) = f(c_j) + f(-c_j) = f(c_j) - f(c_j) = 0.$$

□

Definition 3.6 (Mean). For continuous $f : [a, b] \to \mathbb{R}$, the real number

$$\bar{f} := \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

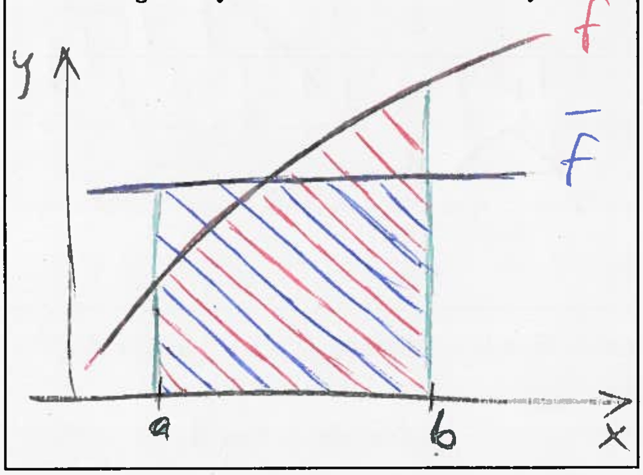
is called the *mean* or *average* of f over the interval $[a, b]$.

Remark 3.7. We have

$$\bar{f} \cdot (b - a) = \int_a^b f(x)\,\mathrm{d}x,$$

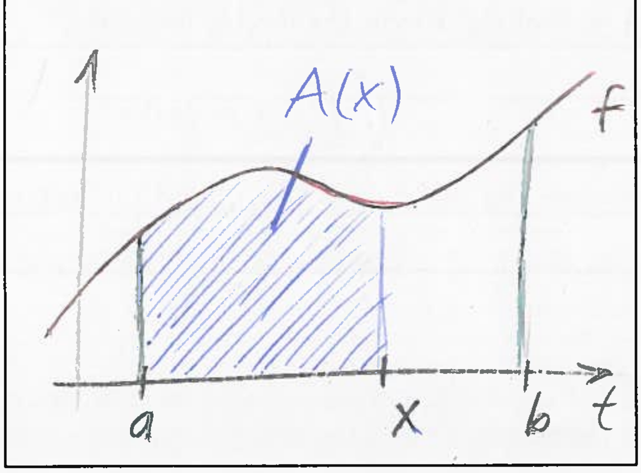
that is, the area under the constant function $\bar{f}$ over $[a, b]$ is the same as the area under the graph of f:

Definite integrals of f and of the constant function $\bar{f}$:



Note that this is similar for the average of numbers – if Alice has an average of $\bar{m}$ on all the tests in a module, and Bob scored exactly $\bar{m}$ each time, then they have the same total mark.

The area function for a continuous function f:



Theorem 3.11 (Fundamental Theorem of Calculus (FTC)). Let $f : [a, b] \to \mathbb{R}$ be continuous.

(i) The derivative of the area function is

$$A'(x) = \frac{\mathrm{d}}{\mathrm{d}x} A(x) = f(x).$$

That is,

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\,\mathrm{d}t = f(x).$$

(ii) If F is any *antiderivative* of f, i.e. a function with $F'(x) = f(x)$, then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

Example 3.12. We solve the following definite integral by guessing a function whose derivative is the integrand. Once this guessed antiderivative is written down, one should always check whether it really differentiates to the original integrand.

$$\int_1^3 (x^3 - 6x)\,\mathrm{d}x = \left(\frac{x^4}{4} - 3x^2 \Big|_1^3 \right) \qquad \left[\text{check} : \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{x^4}{4} - 3x^2 \right) = x^3 - 6x \quad \checkmark \right]$$

$$= \left(\frac{3^4}{4} - 3 \cdot 3^2 \right) - \left(\frac{1^4}{4} - 3 \cdot 1^2 \right) = \frac{81}{4} - 27 - \frac{1}{4} + 3 = -4$$

The vertical line in the second expression stands for "evaluate at $x = 3$ and then subtract the evaluation at $x = 1$" – as on the right-hand side of FTC (ii). To foreshadow the proof of the FTC, note that the choice of antiderivative is not unique, but different choices seem to be leading to the same result; e.g.,

$$\int_1^3 x^3 - 6x\,\mathrm{d}x = \left(\frac{x^4}{4} - 3x^2 + 42 \Big|_1^3 \right) = \frac{81}{4} - 27 + 42 - \frac{1}{4} + 3 - 42 = -4.$$

Remark 3.42. (i) Just as the integral $\int_a^b f(x)\,\mathrm{d}x$ is the area between the curve $y = f(x)$ and the interval $[a, b]$ of the x-axis, the double integral

$$\int_a^b \int_c^d f(x, y)\,\mathrm{d}y\,\mathrm{d}x$$

is the volume between the surface $z = f(x, y)$ and the rectangle

$$[a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ c \leq y \leq d\}$$

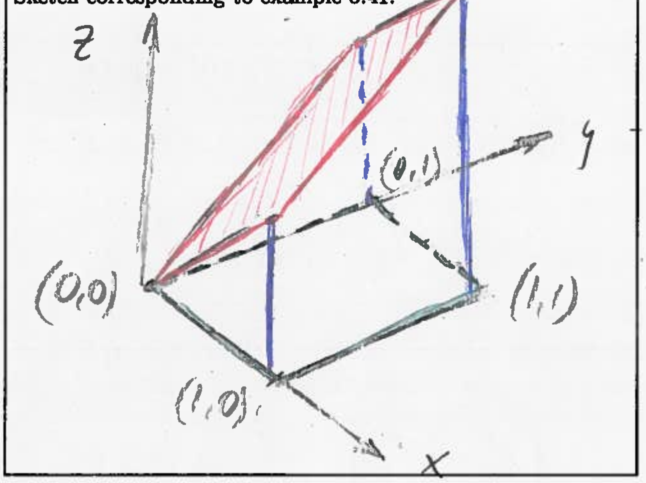
of the xy-plane.

(ii) To check whether this interpretation agrees with the result $I = 1$ we have found above, note that the graph of the integrand $f(x, y) = x + y$ is a plane and has heights

$$f(0,0) = 0, \quad f(0,1) = 1, \quad f(1,0) = 1, \quad f(1,1) = 2,$$

over the corner points of the domain of integration, $D = [0, 1] \times [0, 1]$. This means that the volume that is to be found is that of a cuboid of size $1 \times 1 \times 2$ that is diagonally cut in half:

Sketch corresponding to example 3.41:



This solid has the volume

$$V = \frac{1 \cdot 1 \cdot 2}{2} = 1,$$

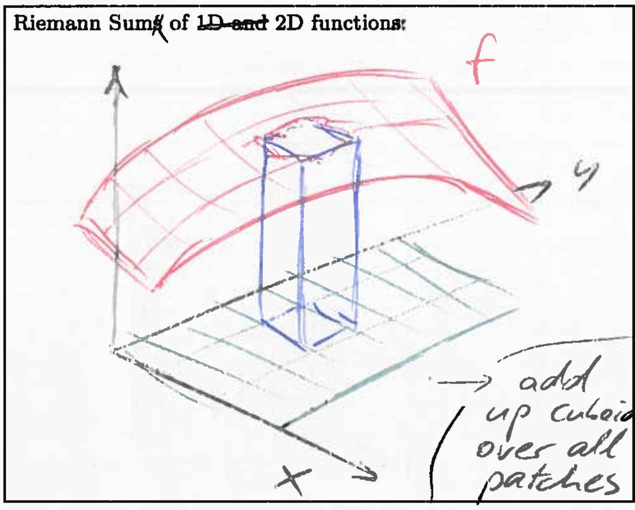
which we had also obtained with the integration above.

(iii) We write $\mathrm{d}A$ for $\mathrm{d}x\mathrm{d}y$,

$$\mathrm{d}A = \mathrm{d}x\mathrm{d}y,$$

meaning, roughly, that the change of area is equal to the change of x times the change of y. $\mathrm{d}A$ is called the *area element*. The order of the integrations $\mathrm{d}x$ and $\mathrm{d}y$ can be swapped, but one needs to be careful about the boundaries, cf. later examples.

(iv) There also is a formulation of Riemann sums for functions of several variables. We will not address this further in MTH1002, but the idea is as follows:



Example 3.43. Integrate the function $f(x,y) = {}^{y}\!/_{x}$ over the domain $D = [3,6] \times [1,2]$.
Sol.:

$$I = \iint_D f \,\mathrm{d}A = \int_1^2 \int_3^6 \frac{y}{x} \,\mathrm{d}x \,\mathrm{d}y = \int_1^2 \left[\int_3^6 \frac{y}{x} \,\mathrm{d}x \right] \mathrm{d}y.$$

The inside integral is with respect to x, and it therefore treats y like a constant – it is therefore permissible to pull out y,

$$I = \int_1^2 y \left[\int_3^6 \frac{1}{x} \,\mathrm{d}x \right] \mathrm{d}y = \int_1^2 y \left[\ln x|_3^6 \right] \mathrm{d}y = \ln 2 \int_1^2 y \,\mathrm{d}y = \ln \sqrt{8}.$$

Remark 3.44. All domains of integration so far have been rectangles. In this case, the x and y boundaries are constant, and the order of integration can be swapped easily – convince yourself of that by re-doing one of the problems above integrating with respect to y and then w.r.t. x. Next we study non-rectangular domains of integration, for which the boundaries of the inner integral depend on the outer variable.

Example 3.45.

$$\begin{aligned} I &= \int_0^1 \int_0^{x^2} 1 \,\mathrm{d}y \,\mathrm{d}x = \int_0^1 \left[\int_0^{x^2} 1 \,\mathrm{d}y \right] \mathrm{d}x \\ &= \int_0^1 \left(y|_0^{x^2} \right) \mathrm{d}x = \int_0^1 (x^2 - 0) \,\mathrm{d}x = \int_0^1 x^2 \,\mathrm{d}x = \frac{1}{3}. \end{aligned}$$

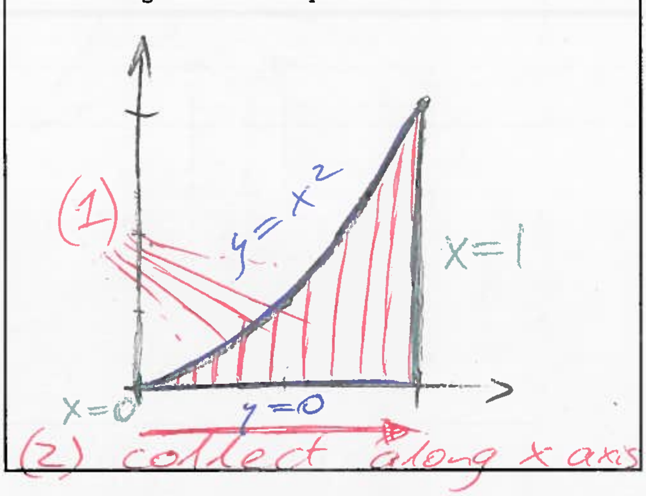
Remark 3.46. (i) In the previous example, the domain of integration was

$$D = \{(x,y) \,|\, 0 \le x \le 1, \, 0 \le y \le x^2\}.$$

Integrating the function $f(x,y) = 1$ over a subset of $\mathbb{R}^2$ gives the area of that set. This is similar to 1D integration: integrating $f(x) = 1$ over an interval gives the length of that interval.

(ii) Our choice to integrate first w.r.t. y and then w.r.t. x in the previous example corresponds to the following steps: (1) for each $x \in [0, 1]$, integrate over each vertical line in the sketch, then (2) collect those values in the x direction.

Order of integration for example 3.45:

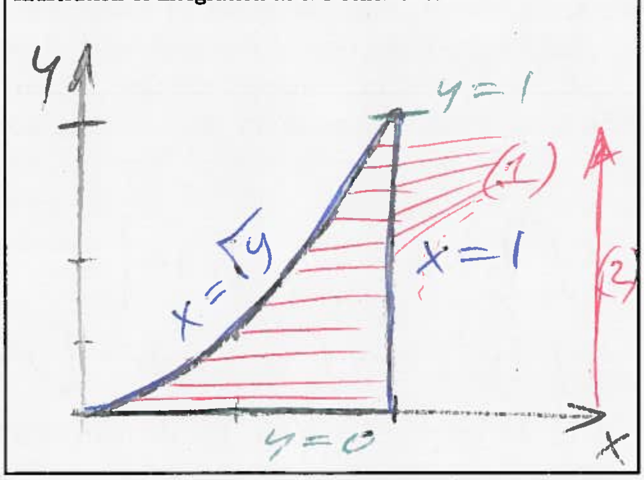


We could also find $I = \iint_D 1 \, \mathrm{d}A$ by integrating the other way around. For this, one has to solve the equations that define the boundaries for the other variable:

$$I = \int_0^1 \int_{\sqrt{y}}^1 1 \, \mathrm{d}x \, \mathrm{d}y$$

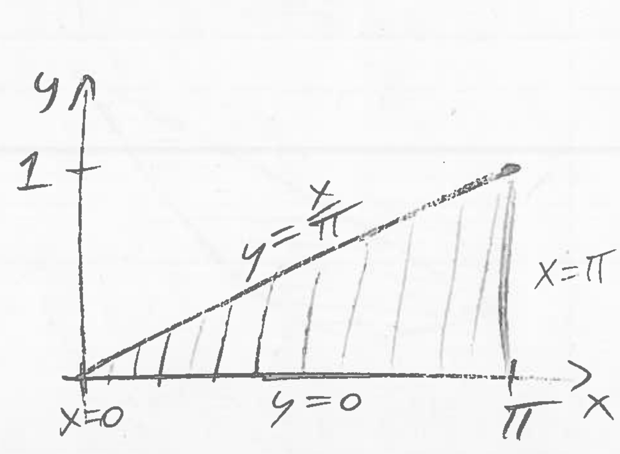
– check that this gives the same result.

Illustration of integration in the other order:



Example 3.47. Integrate $f(x, y) = y \sin x$ over the triangle D with corner points $(0, 0)$, $(\pi, 0)$, and $(\pi, 1)$.
Sol.:

Sketch of the domain of integration:



$$
\begin{aligned}
\iint f \,\mathrm{d}A &= \int_0^{\pi} \int_0^{x/\pi} y \sin x \,\mathrm{d}y \,\mathrm{d}x = \int_0^{\pi} \sin x \left[\int_0^{x/\pi} y \,\mathrm{d}y \right] \mathrm{d}x \\
&= \int_0^{\pi} \sin x \left(\frac{y^2}{2} \Big|_0^{x/\pi} \right) \mathrm{d}x = \int_0^{\pi} \sin x \frac{x^2}{2\pi^2} \,\mathrm{d}x \\
&= \frac{1}{2\pi^2} \int_0^{\pi} x^2 \cdot \sin x \,\mathrm{d}x = \cdots = \frac{\pi^2 - 4}{2\pi^2}.
\end{aligned}
$$

Remark 3.48. It is helpful to always start a computation of a 2D integral with a sketch of the domain of integration. Labelling its boundaries with the formulas that describe them helps to correctly write out $\iint_D f(x,y)\,\mathrm{d}A$ as $\int_{\cdot}^{\cdot} \int_{\cdot}^{\cdot} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$. The integration can be carried out in either order, but the next example and the corresponding exercise below show that for some domains, one order of integration is easier than the other.

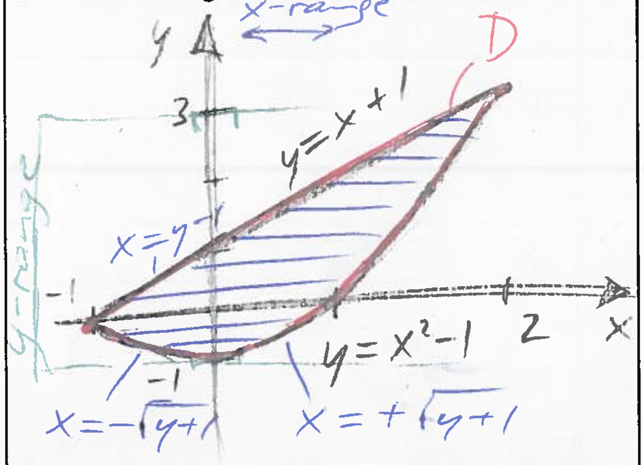
Example 3.49. (i) Let D be the region bounded by the line $y = x + 1$ and by the parabola $y = x^2 - 1$. Find

$$I = \iint_D xy + 2 \,\mathrm{d}A.$$

Sol.: The domain of integration is the region between the line and the parabola below. We find the intersection points as follows:

$$
\begin{aligned}
x + 1 &\overset{!}{=} x^2 - 1 \\
\rightarrow \quad 0 &= x^2 - x - 2 \\
&\Rightarrow \quad x_1 = -1, \quad x_2 = 2.
\end{aligned}
$$

The domain of integration D:



Let us integrate with respect to x first and then w.r.t. y; that is, the outside integral is w.r.t. y. The sketch shows that y ranges from $y = -1$ to $y = 3$. Note that the right x boundary is $x = +\sqrt{y+1}$ for any $y \in [-1, 3]$. However, for the left x boundary, we have to distinguish two cases: $x = -\sqrt{y+1}$ for any $y \in [-1, 0]$, and $y = x + 1 \leftrightarrow x = y - 1$ for any $y \in [0, 3]$. We therefore split D along $y = 0$ to obtain

$$
\begin{aligned}
I &= \iint_{D_-} xy + 2\,\mathrm{d}A + \iint_{D_+} xy + 2\,\mathrm{d}A \\
&= \int_{-1}^{0} \int_{-\sqrt{y+1}}^{+\sqrt{y+1}} xy + 2\,\mathrm{d}x\,\mathrm{d}y + \int_{0}^{3} \int_{y-1}^{+\sqrt{y+1}} xy + 2\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{-1}^{0} \left(\frac{x^2 y}{2} + 2x \Big|_{-\sqrt{y+1}}^{+\sqrt{y+1}} \right) \mathrm{d}y + \int_{0}^{3} \left(\frac{x^2 y}{2} + 2x \Big|_{y-1}^{\sqrt{y+1}} \right) \mathrm{d}y \\
&= \int_{-1}^{0} \frac{(y+1)y}{2} + 2\sqrt{y+1} - \frac{(y+1)y}{2} + 2\sqrt{y+1}\,\mathrm{d}y \\
&\qquad + \int_{0}^{3} \frac{(y+1)y}{2} + 2\sqrt{y+1} - \frac{(y-1)^2 y}{2} - 2y + 2\,\mathrm{d}y \\
&= \int_{-1}^{0} 4\sqrt{y+1}\,\mathrm{d}y + \int_{0}^{3} \frac{-y^3 + 3y^2 - 4y + 4}{2} + 2\sqrt{y+1}\,\mathrm{d}y \\
&= \frac{8}{3}\left((y+1)^{3/2} \Big|_{-1}^{0} \right) + \frac{1}{2}\left(-\frac{y^4}{4} + \frac{3y^3}{3} - \frac{4y^2}{2} + 4y \Big|_{0}^{3} \right) + \frac{4}{3}\left((y+1)^{3/2} \Big|_{0}^{3} \right) \\
&= \left(\frac{8}{3} - 0 \right) + \frac{1}{2}\left(-\frac{3^4}{4} + 3^3 - 2 \cdot 3^2 + 4 \cdot 3 - 0 \right) + \frac{4}{3}\left(4^{3/2} - 1 \right) = \frac{8}{3} + \frac{3}{8} + \frac{28}{3} = \frac{99}{8}.
\end{aligned}
$$

(ii) For $D = [0, 2] \times [0, 1]$, find

$$I = \iint_D \left| (x+y)^2 - 1 \right| \,\mathrm{d}A.$$

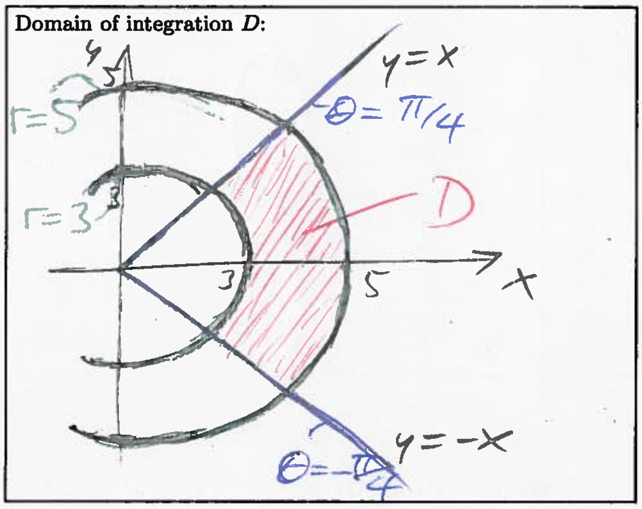
You could now also carry out that integration in Cartesian coordinates x, y – as an additional exercise for the previous section and to convince yourself of the benefits of integrating in polar coordinates. Of course, not all 2D integrals are easier to solve in polar coordinates.

(ii) Let

$$D = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0,\ |y| \leq x,\ 9 \leq x^2 + y^2 \leq 25\},$$

and find $\iint_D x \, \mathrm{d}A$.

Sol.: The domain of integration is a segment of the ring with outer radius $r = 5$ and inner radius $r = 3$:



We see that r ranges from 3 to 5 and θ from $-\pi/4$ to $\pi/4$ – again, a rectangle! That is, for

$$U = [3, 5] \times \left[-\frac{\pi}{4}, \frac{\pi}{4}\right],$$

we have $D = \Phi(U)$. This gives

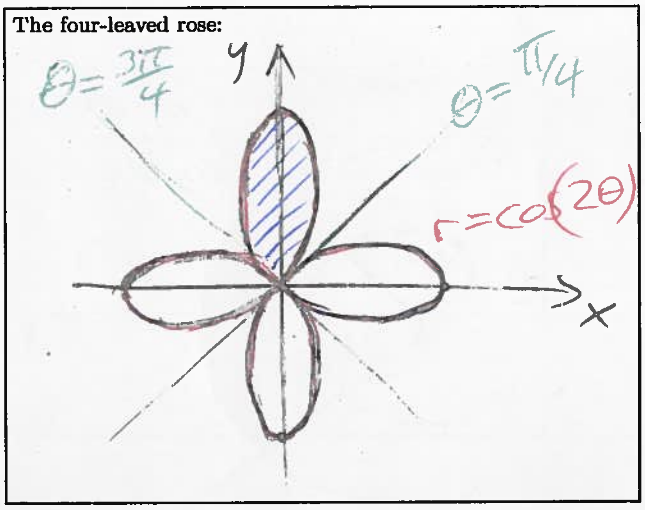
$$\begin{aligned}\iint_D x \, \mathrm{d}A &= \int_3^5 \int_{-\pi/4}^{\pi/4} r\cos\theta \, r \, \mathrm{d}\theta \, \mathrm{d}r = \int_3^5 r^2 \int_{-\pi/4}^{\pi/4} \cos\theta \, \mathrm{d}\theta \, \mathrm{d}r \\ &= \int_3^5 r^2 \left(\sin\theta\big|_{-\pi/4}^{\pi/4}\right) \mathrm{d}r = \sqrt{2}\int_3^5 r^2 \, \mathrm{d}r = \sqrt{2} \cdot \frac{98}{3} \approx 46.198.\end{aligned}$$

This result, 46.2, has a physical meaning – compare to the application in the introduction to this chapter to see what it is[13]. Note that integration in Cartesian coordinates would have been more laborious, as one would have to split up D into at least three pieces[14].

[13] Hint: If you divide that number by $\iint_D 1 \, \mathrm{d}A$, the result will help you balance a cut-out of D on a single point.

[14] Hint: E.g., letting y be the outer integral: there are three different sets of formulas for the x boundaries.

(iii) Find the area A enclosed by one loop of the four-leaved rose $r = \cos(2\theta)$.
Sol.: Here, a curve in the xy-plane is not given by an explicit (e.g. $y = x^2$) or implicit (e.g. $x^2 + y^2 = 1$) formula in x and y, but via an equation in polar coordinates. Those are *polar curves*. For example, $r = 1$ is the unit circle centred at the origin, and $\theta = {}^{\pi}/_{3}$ describes the ray that leaves the origin at an angle of 60°. Examples of polar curves – including the four-leaved rose – can be viewed in the interactive MTH1000 worksheet "Exercise 5: Polar coordinates". You can also enter commands like `polar plot r=cos(2t)` into WolframAlpha. To find the area within one loop of the four-leaved rose,



we integrate the function $f = 1$ over $\theta \in [{}^{\pi}/_{4}, {}^{3\pi}/_{4}]$ and $r \in [0, \cos 2\theta]$:

$$A = \iint_D 1 \, \mathrm{d}A = \int_{\pi/4}^{3\pi/4} \int_0^{\cos 2\theta} r \, \mathrm{d}r \, \mathrm{d}\theta = \frac{1}{2} \int_{\pi/4}^{3\pi/4} \cos^2(2\theta) \, \mathrm{d}\theta$$

$$= \frac{1}{4} \int_{\pi/4}^{3\pi/4} 1 + \cos(4\theta) \, \mathrm{d}\theta = \frac{\pi}{8} + \frac{1}{16} \left(\sin(4\theta)\Big|_{\pi/4}^{3\pi/4} \right) = \frac{\pi}{8}.$$

Example 3.55. Find the volume within the ball of radius R in $\mathbb{R}^3$ (centred at the origin).
Sol.: Denote that ball by B_R,

$$B_R = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq R^2\} \subseteq \mathbb{R}^3.$$

Besides using a basic geometry formula, there are two approaches to finding the volume of B_R: (1) compute the volume of the upper hemisphere by integrating the 2D function whose graph forms the surface of the upper hemisphere over the the disk D_R of radius R, and then multiply by 2; or (2) find the integral of the 3D function $f(x, y, z) = 1$ over B_R. In maths terms:

$$(1) \qquad V = 2 \iint_{D_R} \sqrt{R^2 - x^2 - y^2} \, \mathrm{d}A,$$

$$(2) \qquad V = \iiint_{B_R} 1 \, \mathrm{d}V.$$

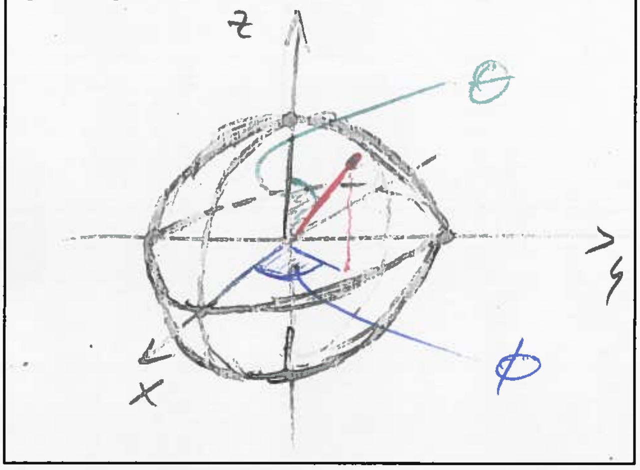
(1) can be carried out nicely in polar coordinates, but we will take the second approach now, as it is a good opportunity to apply the general form of theorem 3.52.

For this, we need *spherical polar coordinates*, i.e. a form of polar coordinates for $\mathbb{R}^3$. You do not need to know this transformation for the MTH1002 exam, but it is good to preview it for your second year in the programme:

$$\begin{array}{rcccl} \Phi & : & \mathbb{R}_0^+ \times [0, 2\pi] \times [0, \pi] & \to & \mathbb{R}^3 \\ & & \begin{pmatrix} r \\ \phi \\ \theta \end{pmatrix} & \mapsto & \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\phi\sin\theta \\ r\sin\phi\sin\theta \\ r\cos\theta \end{pmatrix} \end{array}$$

For example, for fixed $r = r_0$, the points (x, y, z) will traverse the surface of the ball of radius r_0, with $\theta = 0$ corresponding to the north pole, $\theta = \pi$ to the south pole, and $\theta = {}^{\pi}/_{2}$ to the equator.

Spherical polar coordinates in $\mathbb{R}^3$:



The Jacobian for this transformation is

$$J_\Phi = \begin{bmatrix} \cos\phi\sin\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\theta & 0 & -r\sin\theta \end{bmatrix},$$

whose determinant has an absolute value of $|\det J_\Phi| = r^2 \sin\theta$. Hence the *volume element* (the 3D version of the area element) is

$$\mathrm{d}V = r^2 \sin\theta \, \mathrm{d}r\mathrm{d}\phi\mathrm{d}\theta.$$