DG categories (Possibly invented by M. Kelly at UIUC!)
(Ref. Bondal - Kapranov, Keller)

Abelian category: abstraction that captures structure of Ab = category of abelian groups and R-mod = cat. of modules / R, R a ring.

Triangulated category: Abstraction that captures structure of $K(\mathcal{A})$ = homotopy category of complexes in $\mathcal{A}$ or $D(\mathcal{A})$ = derived category of $\mathcal{A} = K(\mathcal{A})[qiso^{-1}]$ ($\mathcal{A}$ an abelian category)

What about the category $Ch(\mathcal{A})$ of chain complexes itself? It is most naturally a Differential Graded (DG) category.

Let k be a commutative ring (often we take k a field)

We say $\mathcal{C}$ is a k-linear category if for any objects $X, Y \in ob\, \mathcal{C}$, the hom set $hom_{\mathcal{C}}(X,Y)$ is equipped w/ struct. of k-module, and composition

$$hom_{\mathcal{C}}(Y,Z) \times hom_{\mathcal{C}}(X,Y) \to hom_{\mathcal{C}}(X,Z)$$

is k-bilinear, allowing us to write it as a map

$$hom_{\mathcal{C}}(Y,Z) \otimes_k hom_{\mathcal{C}}(X,Y) \to hom_{\mathcal{C}}(X,Z)$$

A k-linear category is called additive if it admits all finite direct sums of objects.

A <u>DG category</u> is a k-linear category $\mathcal{C}$ with the additional structures of

- a $\mathbb{Z}$-grading $\hom_{\mathcal{C}}(X,Y) = \bigoplus_{i\in\mathbb{Z}} \hom^i_{\mathcal{C}}(X,Y)$ on each hom-module
- an operator $d^i_{X,Y} : \hom^i_{\mathcal{C}}(X,Y) \longrightarrow \hom^{i+1}_{\mathcal{C}}(X,Y)$

Thus makes $(\hom^\bullet_{\mathcal{C}}(X,Y), d_{X,Y})$ into a cochain complex $d^2=0$

Such that

- composition $\circ : \hom^\bullet_{\mathcal{C}}(Y,Z) \otimes_k \hom^\bullet_{\mathcal{C}}(X,Z) \to \hom^\bullet_{\mathcal{C}}(X,Z)$ is a chain map (commutes w/ differentials)
- $d^0_{X,X}(id_X) = 0.$

The example is $Ch(R\text{-mod})$ where R is a k-algebra

Objects = cochain complexes of left R-modules $(K^\bullet, d_k)$ $\quad d_c : K^i \to K^{i+1}$ (R-module hom) $\quad d_K^2 = 0$

$$\hom((K^\bullet,d_K),(L^\bullet,d_L)) = \bigoplus \hom^p((K^\bullet,d_K),(L^\bullet,d_L))$$

$$\hom^p((K^\bullet,d_K),(L^\bullet,d_L)) = \prod_i \hom_R(K^i, L^{i+p})$$

Definition of the space $\hom^p(K,L)$ does not involve d_K or d_L. Elements of $\hom^\bullet((K^\bullet,d_K),(L^\bullet,d_L))$ are not necessarily chain maps.

Differential on $\hom^\bullet((K^\bullet,d_K),(L^\bullet,d_L))$ involves "commuting with the internal differentials"

Formally expect that if $x\in K^i$ and $f\in \hom^j(K^\bullet,L^\bullet)$ then $f(x)\in L^{i+j}$

By Leibniz, would have

$$d_L(f(x)) = d_{hom(K,L)}(f)\cdot x + (-1)^j f(d_K(x))$$

We define $d_{hom(K,L)}(f) = d_L \circ f - (-1)^{\deg f} f \circ d_K$

These definitions make $Ch(R\text{-mod})$ into a DG-category

The cohomology or homotopy category of a DG category

Given a DG category $\mathcal{C}$, we make form there is a subcategory $Z^0(\mathcal{C})$ with the same objects with morphisms the degree zero cocycles.

$$Hom_{Z^0(\mathcal{C})}(X,Y) = Z^0(hom_{\mathcal{C}}(X,Y)) = \{f \in hom^0_{\mathcal{C}}(X,Y) \mid df = 0\}$$

$Z^0(Ch(R\text{-mod}))$ is the traditional category of chain complexes and chain maps.

We can also take degree zero cohomology $H^0(\mathcal{C})$

$$Hom_{H^0(\mathcal{C})}(X,Y) = H^0(hom_{\mathcal{C}}(X,Y)) = Z^0(hom_{\mathcal{C}}(X,Y)) / d\, hom^{-1}_{\mathcal{C}}(X,Y)$$

$H^0(ch(R\text{-mod}))$ is the traditional homotopy category of chain complexes of R-modules

We can also get a $\mathbb{Z}$-graded category by considering cohomology of all degrees. $H(\mathcal{C})$

$$Hom^i_{H(\mathcal{C})}(X,Y) = H^i(hom_{\mathcal{C}}(X,Y))$$

A DG functor $F : \mathcal{A} \to \mathcal{B}$ is a functor that preserves the k-linear structure and direct sums, the grading, and such that

$$F : \hom_{\mathcal{A}}(X,Y) \to \hom_{\mathcal{B}}(FX,FY)$$

is a chain map.

Such a functor induces $H(F) : H(\mathcal{A}) \to H(\mathcal{B})$
$H^0(F) : H^0(\mathcal{A}) \to H^0(\mathcal{B})$

we say F is a <u>quasi equivalence</u> if $H(F)$ is an equivalence of ordinary categories.

That is $F : \hom_{\mathcal{A}}(X,Y) \to \hom_{\mathcal{B}}(FX,FY)$ is always a quasi-isomorphism of complexes, and $H^0(F)$ is essentially surjective

If $\mathcal{A}$ is a DG category linear over k, then an <u>$\mathcal{A}$-module</u> is a DG-functor $\mathcal{A} \to Ch(k\text{-mod})$

The collection of all DG functors $\{F : \mathcal{A} \to \mathcal{B}\}$ itself forms a DG category $Fun(\mathcal{A},\mathcal{B})$

Let $F, G : \mathcal{A} \to \mathcal{B}$ be two DG functors
Let $\hom^k_{Fun(\mathcal{A},\mathcal{B})}(F,G)$ be the set of

grading preserving natural transformations $\eta : F \Rightarrow G[k]$, forgetting the differential.
So for each $X \in Ob\,\mathcal{A}$ $\eta_X \in \hom^k_{\mathcal{B}}(FX, GX)$
The differential is just $d_{\mathcal{B}}$.

Eg $\mathcal{A}\text{-mod} = Fun(\mathcal{A}, Ch(k\text{-mod}))$ is a DG category