Gradings and Signs on surfaces.

In order to put a $\mathbb{Z}$-grading on our morphism spaces $\hom(L_0, L_1)$, and also in order to work over a field of characteristic $\neq 2$, we need to make some other choices and restrict the class of curves that we consider.

Let S be a Riemann surface, possibly with boundary.

The first choice we must make is a nowhere vanishing C^∞ quadratic differential

$$\eta_S^2 \in \Gamma\left(S, (T^*S)^{\otimes 2}\right) \quad \text{i.e.} \quad \eta_S^2 : TS \otimes_{\mathbb{C}} TS \to \mathbb{C}$$

e.g. if z is local coordinate in $U \subseteq S$ $\quad \eta_S^2 = f\, dz^2$, $f \in C^\infty(U, \mathbb{C})$ f nowhere vanishing

η_S^2 is a complex valued quadratic function on TS

So the subset $\quad \mathcal{E}_S = \left\{ (p, v) \,\middle|\, \begin{array}{l} p \in S, \\ v \in T_pS, \end{array} \; \eta_S^2(v) \geq 0 \right\}$

is an $\mathbb{R}$-subbundle of TS (regarded as a rank 2 $\mathbb{R}$-vector bundle) called the line field.

<u>Remarks</u>: The existence of η_S^2 is equivalent to

$$2c_1(S) = 0 \text{ in } H^2(S; \mathbb{Z})$$

If S is not closed, this condition is always satisfied.
If S is closed, say genus $= g$, then this condition is satisfied iff $g = 1$. The Fukaya categories of other surfaces cannot be $\mathbb{Z}$-graded using this method.

Once we have chosen η_S^2, the definition of the Fukaya category depends on this choice in a nontrivial way. But homotopic choices (connected through space of nowhere-vanishing smooth quadratic differentials) lead to equivalent categories.

The subbundle $\xi_S \subset TS$ is an integrable distribution for dimension reasons.

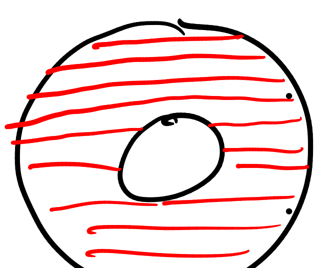
Proof: let X be a local generating vector field for ξ_S. Any two local sections are of the form fX, gX for some local functions f, g.

Then $[fX, gX] = \big(fX(g) - gX(f)\big) \cdot X \in \xi_S$

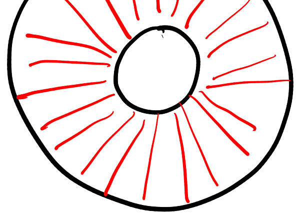
So $\Gamma(S, \xi_S)$ is closed under Lie bracket, and we may apply the Frobenius theorem.

The (unoriented) foliation that integrates ξ_S is a good way to visualize ξ_S or η_S^2.

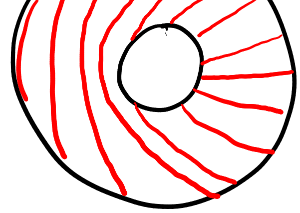
S = Annulus in $\mathbb{C}$



$\eta^2 = dz^2$



$\eta^2 = \dfrac{dz^2}{z^2} = d(\log z)^2$



$\eta^2 = \dfrac{dz^2}{z}$

(ξ not orientable)

The set of homotopy classes of line fields is a torsor for $H^1(S;\mathbb{Z})$. To see this, consider two nowhere vanishing sections $\eta_1^2, \eta_2^2 \in \Gamma(S, (T^*S)^{\otimes 2})$. The ratio $\eta_2^2/\eta_1^2 \in \Gamma(S, \underline{\mathbb{C}})$ is a nowhere vanishing section of the trivial bundle. This may be regarded as a map $S \to \mathbb{C}^\times$. Since $\mathbb{C}^\times$ is a $K(\mathbb{Z},1)$, the homotopy classes of maps are

$$[S, \mathbb{C}^\times] \simeq [S, K(\mathbb{Z},1)] \simeq H^1(S;\mathbb{Z}).$$

Now consider a curve $L \subset S$, with tangent bundle $TL \subset TS|_L$. An orientation on L amounts to a choice of (homotopy class of) nowhere vanishing section of TL. But $(TL)^{\otimes 2}$ has a __canonical__ class of nowhere vanishing sections. Thus $(T^*L)^{\otimes 2}$ does as well; let's call it ν_L^2.

Then ν_L^2 and $\eta_S^2|_L$ are two nowhere vanishing sections of $(T^*S|_L)^{\otimes 2}$, and their ratio may be regarded as a map

$$\eta_S^2/\nu_L^2 : L \to \mathbb{C}^\times$$

whose homotopy class

$$[\eta_S^2/\nu_L^2] \in [L, \mathbb{C}^\times] \simeq H^1(L;\mathbb{Z})$$

Choosing an orientation on L, we get $H^1(L;\mathbb{Z}) \simeq \mathbb{Z}$, so that this class may be interpreted as a number called the rotation number of L with respect to η_S^2 and Ξ_S. (This is an instance of the more general concept of the Maslov class of a Lagrangian submanifold.)

A curve L such that $[\eta_S^2/\nu_L^2] = 0 \in H^1(L;\mathbb{Z})$ is called gradable. Any nonclosed arc is gradable, but a null-homotopic curve is <u>never</u> gradable. Otherwise, gradability depends on the curve and the choice of line field ξ_S (η_S^2).

To say that L is gradable is to say that the subbundles TL and $\xi_S|_L$ in $TS|_L$ are homotopic. We would like to choose a specific homotopy.
One way to think of this is as a bundle over $[0,1]\times L$.
Let $\pi : [0,1]\times L \to L$ be projection.

<u>Def</u> A <u>grading</u> on L is an $\mathbb{R}$-subbundle $H \subseteq \pi^*(TS|_L)$ such that

$$H|_{\{0\}\times L} = \xi_S|_L \quad \text{and} \quad H|_{\{1\}\times L} = TL .$$

For fixed $p \in L$, $H|_{[0,1]\times\{p\}}$ is a path of $\mathbb{R}$-subspaces of T_pL. This may also be regarded as a path

$$H(t,p) : [0,1] \longrightarrow \mathbb{RP}(T_pS)$$

$$H(0,p) = \xi_S|_p \qquad H(1,p) = T_pL$$

* Let us measure rotation in $\mathbb{RP}(T_pS)$ so that closed loops are rotation through $n\pi$ for $n \in \mathbb{Z}$
* positive sense is determined by orientation of S

Let L_1 and L_2 be two curves with gradings H_1 and H_2
Let $p \in L_1 \cap L_2$ be a transverse intersection point.

We can get a path in T_pS from T_pL_1 to T_pL_2 using $H_2(t,p)$ and $H_1(1-t,p)$.

Let $\Theta \in \mathbb{R}/\pi\mathbb{Z}$ be the total angle through which this path rotates.

The absolute index of p is given by
$$i(p) = \lfloor \Theta \rfloor + 1 \in \mathbb{Z}.$$