Formal enlargements of A_∞ categories.

We wish to enlarge the Fukaya category for a couple of reasons.

1. To make it a <u>triangulated category</u>, so that we can do homological algebra "in it"
2. To understand in what sense this category may be <u>generated</u> by some collection of objects.

Recall that in ordinary category theory, a category $\mathcal{C}$ has a canonical enlargement

$$\hat{\mathcal{C}} := \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set}) \quad (\text{a.k.a. "Presheaves on } \mathcal{C}\text{"})$$

For any object $M \in \mathrm{ob}\,\mathcal{C}$, there is a functor

$$\mathcal{M} = \mathrm{Hom}_{\mathcal{C}}(-, M) : \mathcal{C}^{op} \to \mathrm{Set}$$

Called a <u>representable functor</u> (represented by M)

The assignment $M \mapsto \mathcal{M} = \mathrm{Hom}_{\mathcal{C}}(-, M)$ extends to a functor

$$\mathcal{Y} : \mathcal{C} \to \hat{\mathcal{C}}$$

Which is fully faithful. $\mathcal{Y}$ is the <u>Yoneda embedding</u>. It allows us to regard $\mathcal{C}$ as a subcategory of $\hat{\mathcal{C}}$.

Constructions in category theory are often formulated by saying that a certain object represents a certain functor.

Given $F : \mathcal{C}^{op} \to \mathrm{Set}$, a <u>representation of F</u> is a pair (Z, α) where $Z \in \mathrm{Ob}\,\mathcal{C}$ and
$$\alpha : \mathcal{Z} := \mathrm{Hom}_{\mathcal{C}}(-, Z) \to F$$
is a natural isomorphism, i.e., an isomorphism in $\hat{\mathcal{C}}$.

<u>Example</u>: given objects Z_0, Z_1 in $\mathcal{C}$, let $F : \mathcal{C}^{op} \to \mathrm{Set}$ be the functor
$$F(X) = \mathrm{Hom}_{\mathcal{C}}(X, Z_0) \times \mathrm{Hom}_{\mathcal{C}}(X, Z_1)$$
(with natural action on morphisms)
An object that represents F (if it exists), is called "the" product $Z_0 \times Z_1$.

In the A_∞ setting, the natural thing to do is to replace Set with the DG category Ch of cochain complexes over k, and consider the category of A_∞ functors $\hat{\mathcal{A}} = \mathrm{Fun}_{A_\infty}(\mathcal{A}, \mathrm{Ch})$ for a given A_∞ category $\mathcal{A}$.

We shall opt for an equivalent framework of A_∞-modules over $\mathcal{A}$.

<u>Def</u> Let $(\mathcal{A}, \{\mu^d_{\mathcal{A}}\}_{d \geq 1})$ be an A_∞ category. An A_∞-module $\mathcal{M}$ consists of:

- For each $X \in \mathrm{Ob}\,\mathcal{A}$, a graded k-vector space $\mathcal{M}(X)$
- structure maps

$$\mu^d_{\mathcal{M}} : \mathcal{M}(X_{d-1}) \otimes \hom_{\mathcal{A}}(X_{d-2}, X_{d-1}) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1) \to \mathcal{M}(X_0)[2-d]$$

Satisfying the following variant of the A_∞-associativity eqns.

$$\sum (-1)^{\maltese} \mu_{\mathcal{M}}^{n+1}(\mu_{\mathcal{M}}^{d-n}(b, a_{d-1}, \dots, a_{n+1}), a_n, \dots, a_1)$$
$$+ \sum (-1)^{\maltese} \mu_{\mathcal{M}}^{d-m+1}(b, a_{d-1}, \dots, \mu_{\mathcal{A}}^{m}(a_{n+m}, \dots, a_{n+1}), a_n, \dots, a_1) = 0$$
$$\maltese = \sum_{j=1}^{n} (\deg(a_j) - 1)$$

The first equation says $\mu_{\mathcal{M}}^1 \circ \mu_{\mathcal{M}}^1 = 0$, so $\mathcal{M}(X)$ is a cochain complex

For each $a \in \hom_{\mathcal{A}}^i(X_0, X_1)$, $\mu_{\mathcal{M}}^2(-, a) : \mathcal{M}(X_1) \to \mathcal{M}(X_0)$ is a cochain map (up to sign conventions)

Passing to cohomology, $X \mapsto H^\cdot(\mathcal{M}(X))$ defines a functor

$$H(\mathcal{A})^{op} \to \text{graded vector spaces.}$$

The collection of all A_∞-modules over $\mathcal{A}$, $\mathrm{mod}(\mathcal{A}) =: \mathcal{Q}$ forms an A_∞ (even DG) category.

$\hom_{\mathcal{Q}}^p(\mathcal{M}_0, \mathcal{M}_1)$ consists of collections of maps indexed by d-tuples of objects of $\mathcal{A}$, $(X_0, X_1, \dots, X_{d-1})$

$$t^d : \mathcal{M}_0(X_{d-1}) \otimes \hom_{\mathcal{A}}(X_{d-2}, X_{d-1}) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1)$$
$$\to \mathcal{M}_1(X_0)[p - d + 1]$$

$$(\mu^1_{\mathcal{Q}} t)^d (b, a_{d-1}, \dots, a_1)$$

$$= \sum (-1)^\dagger \mu^{n+1}_{\mathcal{M}_1} (t^{d-n}(b, a_{d-1}, \dots, a_{n+1}), a_n, \dots, a_1)$$

$$+ \sum (-1)^\dagger t^{n+1} (\mu^{d-n}_{\mathcal{M}_0} (b, a_{d-1}, \dots, a_{n+1}), a_n, \dots, a_1)$$

$$+ \sum (-1)^\ddagger t^{d-m+1} (b, a_{d-1}, \dots, \mu^m_{\mathcal{A}} (a_{n+m}, \dots, a_{n+1}), a_n, \dots, a_1)$$

$$(\mu^2_{\mathcal{Q}}(t_2, t_1))^d (b, a_{d-1}, \dots, a_1)$$

$$= \sum (-1)^\dagger t_2^{n+1} (t_1^{d-n}(b, a_{d-1}, \dots, a_{n+1}), a_n \dots a_1)$$

$$\dagger = \deg(a_{n+1}) + \dots + \deg(a_{d-1}) + \deg(b) - d + n + 1$$

$\mu^d_{\mathcal{Q}} = 0$ for all $d \geq 3$.

The Yoneda embedding now takes the form of an A_∞-functor

$$\mathcal{l} : \mathcal{A} \to \mathrm{mod}(\mathcal{A}) = \mathcal{Q}$$

It sets $Y \in \mathrm{Ob}\,\mathcal{A}$ to $\mathcal{l}(Y) = \mathcal{Y}$ given by

$$\left(\mathcal{Y}(X) = \hom^{\cdot}_{\mathcal{A}}(X, Y), \quad \mu^d_{\mathcal{Y}} = \mu^d_{\mathcal{A}}\right).$$

The first component $\mathcal{l}^1 : \hom_{\mathcal{A}}(Y_0, Y_1) \to \hom_{\mathcal{Q}}(\mathcal{Y}_0, \mathcal{Y}_1)$ sends c to the map

$$(b, a_{d-1}, \dots, a_1) \mapsto \mu^{d+1}_{\mathcal{A}}(c, b, a_{d-1}, \dots, a_1)$$

We can generalize $\mathcal{Y}^1$ to a map, for any A_∞-module $\mathcal{M}$:

$$\lambda : \mathcal{M}(Y) \longrightarrow hom_{\mathcal{Q}}(\mathcal{Y}, \mathcal{M})$$

$$\lambda(c)^d(b, a_{d-1}, \ldots, a_1) = \mu^{d+1}_{\mathcal{M}}(c, b, a_{d-1}, \ldots, a_1)$$

$$\left\{ \begin{array}{c} \text{Analogue in ordinary category theory} \\ Nat(Hom_{\mathcal{C}}(-,Y), F) \simeq FY \\ \alpha \longmapsto \alpha_Y(1_Y) \end{array} \right\}$$

Lemma (Seidel p. 30) λ is a quasi-isomorphism

Corollary $\mathcal{Y}$ is cohomologically full and faithful.

Thus $\mathcal{Y} : \mathcal{A} \to mod(\mathcal{A}) = \mathcal{Q}$ is a fully faithful embedding in the A_∞-sense.

Let $\mathcal{M} \in Ob\,\mathcal{Q}$ be an $\mathcal{A}$-module. A representation for $\mathcal{M}$ is a pair $(Y, [t])$ where $Y \in Ob\,\mathcal{A}$ and $[t] : \mathcal{Y} \to \mathcal{M}$ is an isomorphism in $H^0(\mathcal{Q})$.

Equivalently, there is a $c \in \mathcal{M}^0(Y)$ such that
(i) $\mu^1_{\mathcal{M}}(c) = 0$
(ii) $[t] = [\lambda(c)]$
(iii) for each $X \in Ob\,\mathcal{A}$ the map
$hom_{\mathcal{A}}(X, Y) \to \mathcal{M}(X)$, $b \mapsto (-1)^{\deg(b)} \mu^2_{\mathcal{M}}(c, b)$
is a quasi isomorphism.

<u>Direct sum</u> Given A_∞-modules $\mathcal{M}_0$ and $\mathcal{M}_1$ their direct sum has cochain complexes

$$(\mathcal{M}_0 \oplus \mathcal{M}_1)(X) = \mathcal{M}_0(X) \oplus \mathcal{M}_1(X)$$

with obvious structure maps.

If $Y_0, Y_1 \in \operatorname{Ob} \mathcal{A}$, we can ask if $\mathcal{Y}_0 \oplus \mathcal{Y}_1$ is representable by an object of $\mathcal{A}$. If it is we denote that object by $Y_0 \oplus Y_1$

<u>Tensor product by cochain complex</u> Let $(Z, d_Z) \in \operatorname{Ob} \mathrm{Ch}$ be a cochain complex and let $\mathcal{M}$ be an A_∞-module over $\mathcal{A}$

We define $Z \otimes \mathcal{M}$ by

$$(Z \otimes \mathcal{M})(X) = Z \otimes \mathcal{M}(X)$$

$$\mu^1_{Z\otimes \mathcal{M}}(z \otimes b) = (-1)^{\deg(b)-1} d_Z(z) \otimes b + z \otimes \mu^1_{\mathcal{M}}(b)$$

$$\mu^d_{Z\otimes\mathcal{M}}(z \otimes b, a_{d-1}, \dots, a_1) = z \otimes \mu^d_{\mathcal{M}}(b, a_{d-1}, \dots, a_1)$$

If $Y \in \operatorname{Ob} \mathcal{A}$, an object that represents $Z \otimes \mathcal{Y}$ is denoted $Z \otimes Y$.

<u>Shift</u> this is the special case of the above where

$$Z = k[\sigma]$$

$$Z \otimes \mathcal{M} =: \mathcal{M}[\sigma] \quad \text{and} \quad Z \otimes Y =: Y[\sigma]$$

We have $\mathrm{Hom}_{H(\mathcal{A})}(Y_0, Y_1[\sigma]) \cong \mathrm{Hom}_{H(\mathcal{A})}(Y_0, Y_1)[\sigma]$

$$\mathrm{Hom}(Y_0[\sigma], Y_1)[\sigma] \cong \mathrm{Hom}_{H(\mathcal{A})}(Y_0, Y_1)$$

<u>Cones</u> let Y_0, Y_1 be objects of $\mathcal{A}$ and $c \in hom^0_{\mathcal{A}}(Y_0, Y_1)$ be a degree zero cocycle $\mu^1_{\mathcal{A}}(c) = 0$.

the <u>abstract mapping cone</u> is $\mathcal{C} = \mathcal{C}one(c) \in Ob\, \mathcal{Q}$

$$\mathcal{C}(X) = hom_{\mathcal{A}}(X, Y_0)[1] \oplus hom_{\mathcal{A}}(X, Y_1)$$

$$\mu^d_{\mathcal{C}}((b_0, b_1), a_{d-1}, \dots, a_1)$$

$$= (\mu^d_{\mathcal{A}}(b_0, a_{d-1}, \dots, a_1), \mu^d_{\mathcal{A}}(b_1, a_{d-1}, \dots, a_1) + \mu^{d+1}_{\mathcal{A}}(c, b_0, a_{d-1}, \dots, a_1))$$

An object of $\mathcal{A}$ that represents $\mathcal{C}$ is denoted Cone(c).

* <u>Remark</u>: Cone(c), if it exists, is determined up to <u>canonical</u> isomorphism in $H^0(\mathcal{A})$.
However, if we change c to c' such that $[c] = [c']$ in $H^0(\mathcal{A})$, the objects Cone(c) and Cone(c') are <u>NOT</u> canonically isomorphic. (but are isomorphic).

This lack of canonicity for cones in the cohomology categories is one of the deficiencies of the classical theory of triangulated categories, which the theory of DG and A_∞ categories was intended to correct.