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Hands-On Fluids Challenge

Magnitude of the force that must be applied on the piston

Mass balance on the control volume (constant flow rate):

$$v_1 A_1 = v_2 A_2$$

$$v_1 \pi R^2 = v_2 \pi {R_o}^2$$

$$\Rightarrow v_2 = v_1 \left(\frac{R}{R_o}\right)^2 \tag{1}$$

Where v_1 is the velocity at which the piston is pushed and v_2 is the velocity of the jet of water as it leaves the needle tip.

Bernoulli between 1 and 2:

$$\frac{P_1}{\rho} + \alpha_1 \frac{{v_1}^2}{2} + g z_1 = \frac{P_2}{\rho} + \alpha_2 \frac{{v_2}^2}{2} + g z_2 + h_f \tag{2}$$

Where,

$$P_2 = P_{atm} + P_{capillary} = P_{atm} + \frac{\gamma}{R_o}$$

Using v_2 from (1) and assuming laminar flow ($\alpha_1 = \alpha_2 = 2$), (2) becomes:

$$\frac{P_1}{\rho} + {v_1}^2 = \frac{P_{atm} + \frac{\gamma}{R_o}}{\rho} + {v_1}^2 \left(\frac{R}{R_o}\right)^4 + g l_{needle} + h_f$$

Laminar flow can be assumed because the diameter of the needle is very small and the speed of the water is relatively low.

Solving for P_1,

$$P_1 = P_{atm} + \frac{\gamma}{R_o} + \rho {v_1}^2 \left[\left(\frac{R}{R_o}\right)^4 - 1\right] + \rho g l_{needle} + \rho h_f \tag{3}$$

All values are known in this equation, except for h_f. The losses due to friction are given as:

$$h_f = 4 f_{needle} \frac{l_{needle}}{2R_o} \frac{{v_2}^2}{2} = 4 f_{needle} \frac{l_{needle}}{2R_o} \frac{{v_1}^2}{2} \left(\frac{R}{R_o}\right)^2 \tag{4}$$

Minor losses will be neglected because they will be much smaller than the major losses, due to the ratio of the length of the needle to its diameter being very large. Since the flow was assumed to be laminar, the friction factor is:

$$f = \frac{16}{N_{Re}} \tag{5}$$

Where N_{Re} is the Reynolds number given by:

$$N_{Re} = \frac{\rho v_2 (2R_o)}{\mu} = \frac{2 \rho v_1 R^2}{\mu R_o} \tag{6}$$

Substituting (6) into (5):

$$f = \frac{8 \mu R_o}{\rho v_1 R^2} \tag{7}$$

Substituting (7) into (4):

$$h_f = \frac{8 \mu R_o}{\rho v_1 R^2} \frac{l_{needle}}{R_o} {v_1}^2 \left(\frac{R}{R_o}\right)^2$$

$$h_f = \frac{8\mu l_{needle} v_1}{\rho {R_o}^2} \tag{8}$$

Substituting (8) into (3):

$$P_1 = P_{atm} + \frac{\gamma}{R_o} + \rho {v_1}^2 \left[\left(\frac{R}{R_o} \right)^4 - 1 \right] + \rho g l_{needle} + \frac{8\mu l_{needle} v_1}{{R_o}^2} \tag{9}$$

Free body diagram of the forces acting on the piston:

Where F is the applied force to the piston, F_s is the sliding friction of the piston on the inside of the syringe and $P_1 A_1$ is the force due to the pressure of the water in the syringe. From Newton's second law,

$$\Sigma F = ma$$

For the piston to move at a constant velocity,

$$\Sigma F = 0$$

$$\therefore F = F_s + P_1 A_1 \tag{10}$$

Substituting (9) into (10),

$$F(v_1) = F_s + \left[P_{atm} + \frac{\gamma}{R_o} + \rho {v_1}^2 \left[\left(\frac{R}{R_o} \right)^4 - 1 \right] + \rho g l_{needle} + \frac{8\mu l_{needle} v_1}{{R_o}^2} \right] \pi R^2$$

As determined experimentally, $F_s = 35.6\ N$. Substituting in other known values,

$$\begin{aligned} F(v_1) = 35.6\ N + &\\ &\left[101.325 \times 10^3\ Pa + \frac{72 \times 10^{-3}\ N/m}{0.419 \times 10^{-3}\ m} + \left(1000\ kg/m^3\right) {v_1}^2 \left[\left(\frac{14.25 \times 10^{-3}\ m}{0.419 \times 10^{-3}\ m} \right)^4 - 1 \right] \right. \\ &\quad + \left(1000\ kg/m^3\right) \left(9.81\ m/s^2\right) \left(40 \times 10^{-3}\ m\right) + \\ &\quad \left. \frac{8 \left(10^{-3}\ kg/m \cdot s\right) \left(40 \times 10^{-3}\ m\right) v_1}{\left(0.419 \times 10^{-3}\ m\right)} \right] \pi \left(14.25 \times 10^{-3}\ m\right)^2 \end{aligned}$$

$$\therefore F(v_1) = 100.6 + (487.209 \times 10^{-6}) v_1 + (853.436 \times 10^3) {v_1}^2\ N$$

Where v_1 is in m/s.