Physics Letters B 481 (2000) 397–407

Holomorphy minimal homotopy and the 4D, $N = 1$ supersymmetric Bardeen–Gross–Jackiw anomaly term [1]

S. James Gates Jr. [a,2], Marcus T. Grisaru [b,4], Silvia Penati [c,5]

[a] *Department of Physics, University of Maryland, College Park, MD 20742-4111, USA*
[b] *Physics Department, McGill University, Montreal, QC H3A 2T8, Canada*
[c] *Dipartmento di Fisica dell'Universitá di Milano–Bicocca and INFN, Sezione di Milano, via Celoria 16, I-20133 Milano, Italy*

Received 7 February 2000; accepted 5 April 2000
Editor: M. Cvetič

Abstract

By use of a special homotopy operator, we present an explicit, closed-form and *simple* expression for the left-right Bardeen–Gross–Jackiw anomalies described as the proper superspace integral of a superfunction. © 2000 Elsevier Science B.V. All rights reserved.

PACS: 03.70.+k; 11.15.-q; 11.10.-z; 11.30.Pb; 11.30.Rd
Keywords: Gauge theories; Chiral anomaly; Supersymmetry

1. Introduction

Years ago, Bardeen [1] as well as Gross and Jackiw [2] (BGJ) considered the question of the simultaneous quantum consistency of conservation laws involving vector and axial vector non-Abelian currents. The results of these studies are now widely appreciated. While classically both types of currents can be simultaneously conserved, when the effects of relativistic quantum theory are taken into account both currents *cannot* be simultaneously conserved. One of the conservation laws must be broken due to an anomaly in the corresponding Ward identities. The implications of the anomaly are, in the words of Zumino, Wu and Zee, ''ubiquitous'' [3]. A topic closely related to this is the form of the WZNW term [4] whose variation produces the appropriate anomalies, and its extension in the presence of 4D, $N=1$ supersymmetry.

Earlier work on the latter subject has framed the issue by formulating a superspace WZNW action described solely in terms of chiral superfields. How-

[1] Supported in part by National Science Foundation Grants PHY-98-02551, PHY-9604587 and by MURST Grant 12/2-11/480.
[2] E-mail: gatess@wam.umd.edu
[3] E-mail: grisaru@brandeis.edu
[4] On leave from Brandeis University.
[5] E-mail: silvia.penati@mi.infn.it

PII: S0370-2693(00)00464-0

ever, in Ref. [5] a surprising alternative proposal has been made. Namely, if the spin-$1/2$ fields (''pionini'') accompanying the usual pions are Dirac fields, then a supersymmetric WZNW term may exist wherein some spin-0 degrees of freedom are assigned to supersymmetric representations *other* than the chiral multiplet! The nonminimal scalar superfield, a variant representation that is dual to the chiral superfield, has been proposed. We call such models that use both chiral and nonminimal superfields, chiral-nonminimal (CNM) models. In the second work of Ref. [5], initial steps began toward the description of a 4D, $N = 1$ supersymmetric extension of the *gauged* WZNW term.

In working toward this last goal we have made a search of the literature [6]. Although there have appeared many prior discussions of 4D, $N = 1$ supersymmetric gauge theories and non-Abelian anomalies (see also [4]), to the best of our knowledge, there has been no discussion of the 4D, $N = 1$ superfield action for the BGJ anomalies that pays *special* attention to the simplest expression for these anomalies. Many of the results given so far have explicit dependence on either the prepotential V or $\delta_G V$. Since the gauge variation $\delta_G V$ is a transcendental expression, the clarity of such formulations is lessened. As the anomaly is always defined up to cohomologically trivial terms, we were led to the belief that the prepotential should *only* enter via its natural superspace appearance as e^V along with superconnections and the fermionic field strength.

It is thus a purpose of this work to offer an ''improved'' description of the 4D, $N = 1$ supersymmetric BGJ anomaly. We begin with the integrated form of the left-right anomalies that appear in the (would-be) current conservation laws for the non-supersymmetric theory and reformulate the Wess–Zumino consistency condition in terms of two operators, Δ and δ_R. We also observe that the BGJ anomaly term, defined in terms of geometrical ''monomials of the anomaly'' possesses interesting properties with respect to these operators. The monomials as well as the Δ and δ_R operators all have 4D, $N = 1$ extensions which we use to reconceptualize the supersymmetric problem. In the course of our analysis, we also show how to use the results of Ref. [7] combined with a special choice of homotopy to reach our goals.

2. Algebraic realization of the WZ consistency condition on the left–right BGJ anomaly term

Our discussion starts by considering the integrated form of the left-right BGJ anomalies. For this purpose, we use the results of Refs. [3,8] with a set of gauge fields, $A_{\underline{a}}{}^{(L)}$ ($\underline{a} \equiv \alpha\dot{\alpha}$), minimally coupled to purely left handed spinors, $\bar{\zeta}^{\dot{\alpha}}$ and a set of gauge fields, $A_{\underline{a}}{}^{(R)}$, minimally coupled to purely right handed spinors, ψ^{α}

$$
\begin{aligned}
&\mathscr{S}(J_L + J_R) = \mathscr{S}(J_L) + \mathscr{S}(J_R)\,,\\
&\mathscr{S}(J_L) = \int d^4x \left[-i\,\zeta^{\alpha} \nabla_{\underline{a}}{}^{(L)} \bar{\zeta}^{\dot{\alpha}} \right],\\
&\nabla_{\underline{a}}{}^{(L)} \equiv \partial_{\underline{a}} - iA_{\underline{a}}{}^{(L)\mathrm{I}} t_{\mathrm{I}}\,,\\
&\mathscr{S}(J_R) = \int d^4x \left[-i\,\bar{\psi}^{\dot{\alpha}} \nabla_{\underline{a}}{}^{(R)} \psi^{\alpha} \right],\\
&\nabla_{\underline{a}}{}^{(R)} \equiv \partial_{\underline{a}} - iA_{\underline{a}}{}^{(R)\mathrm{I}} t_{\mathrm{I}}\,.
\end{aligned} \tag{2.1}
$$

Here t_{I} denotes a hermitian matrix representation of the group generators. We note that in supersymmetric theories, the left-right split is most natural since the superfields that contain the fermions are already formulated in terms of chiral spinors. In the subsequent discussion we will make use of the following notational devices,

$$
\begin{aligned}
&\delta_G(\lambda) A_{\underline{a}} = \partial_{\underline{a}}\lambda + i\left[\lambda, A_{\underline{a}}\right] \equiv \partial_{\underline{a}}\lambda + i L_{\lambda} A_{\underline{a}}\,,\\
&\lambda = \lambda^I t_I\,,\\
&\delta_G(\lambda) F_{\underline{ab}} = i L_{\lambda} F_{\underline{ab}}\,,\\
&F_{\underline{ab}} \equiv \partial_{\underline{a}} A_{\underline{b}} - \partial_{\underline{b}} A_{\underline{a}} - i\left[A_{\underline{a}}, A_{\underline{b}}\right].
\end{aligned} \tag{2.2}
$$

We emphasize that throughout the following discussion the λ's are functions of x.

Due to the gauge transformation properties above, it becomes possible to define new operators that we denote by Δ and δ_R via the equation

$$\delta_G(\lambda) = \Delta(\lambda) + \delta_R(\lambda)\,, \tag{2.3}$$

where we choose

$$\delta_R(\lambda) A_{\underline{a}} = i L_{\lambda} A_{\underline{a}}\,, \quad \delta_R(\lambda) F_{\underline{ab}} = i L_{\lambda} F_{\underline{ab}}\,. \tag{2.4}$$

These obviously imply that

$$\Delta(\lambda) A_{\underline{a}} = \partial_{\underline{a}}\lambda\,, \quad \Delta(\lambda) F_{\underline{ab}} = 0\,, \tag{2.5}$$

and show that under the action of Δ, the connection and field strength transform as in an abelian gauge theory. We also note that the operator Δ satisfies $\Delta^2 = 0$, i.e. nilpotency. Although we will not exploit this property in the present work, we believe this is significant since the Δ operator satisfies a Poincaré lemma, suggesting an immediate relation to exact short sequences and topology. As we shall see below, the decomposition in (2.3) is also significant for the BGJ anomaly since actually the Δ-operator, not the δ_R-operator, determines it. We find it very satisfying that the nilpotent operator, reminiscent of the exterior derivative, plays the more fundamental role.

We may write the left BGJ non-Abelian gauge anomaly as

$$\mathscr{S}^{(L)}_{\rm BGJ}(\lambda) = \left(\frac{1}{48\pi^2}\right)\int d^4x\, \lambda^{\rm I}\, {\rm G}^{\rm I}_{(L)}(A^{(L)}) \equiv \left(\frac{1}{48\pi^2}\right)\int d^4x\, \omega^1_4(\lambda, A^{(L)}, F^{(L)}) . \tag{2.6}$$

We have written this last function with its arguments to emphasize that *only* connections and field strengths are allowed to enter in the above construction. The function ${\rm G}^{\rm I}_{(L)}(A^{(L)})$ can be explicitly written in the form,

$$\begin{aligned}\lambda^{\rm I}{\rm G}^{\rm I}_{(L)}(A^{(L)}) &= {\rm Tr}\Big\{\lambda\Big[F_{\underline{a}\underline{b}}{}^{(L)}\tilde{F}^{\underline{a}\underline{b}(L)} + i\tilde{F}^{\underline{a}\underline{b}(L)}A_{\underline{a}}{}^{(L)}A_{\underline{b}}{}^{(L)} \\ &\quad + iA_{\underline{a}}{}^{(L)}\tilde{F}^{\underline{a}\underline{b}(L)}A_{\underline{b}}{}^{(L)} + iA_{\underline{a}}{}^{(L)}A_{\underline{b}}{}^{(L)}\tilde{F}^{\underline{a}\underline{b}(L)} \\ &\quad - \epsilon^{\underline{a}\underline{b}\underline{c}\underline{d}}A_{\underline{a}}{}^{(L)}A_{\underline{b}}{}^{(L)}A_{\underline{c}}{}^{(L)}A_{\underline{d}}{}^{(L)}\Big]\Big\} .\end{aligned} \tag{2.7}$$

In writing this, we have expressed the answer in terms of the field strength $F_{\underline{a}\underline{b}}$ and the dual field strength $\tilde{F}_{\underline{a}\underline{b}} = \frac{1}{2}\epsilon_{\underline{a}\underline{b}\underline{c}\underline{d}}F^{\underline{c}\underline{d}}$. The right integrated BGJ non-Abelian anomaly can be obtained from the left one by the replacements ${\rm G}_{(L)} \to -{\rm G}_{(R)}$, $A_{\underline{a}}{}^{(L)} \to A_{\underline{a}}{}^{(R)}$ and $F_{\underline{b}\underline{c}}{}^{(L)} \to F_{\underline{b}\underline{c}}{}^{(R)}$. The leading term in (2.6), (2.7) has exactly the same form as that for the Abelian anomaly with the difference that an extra factor of the group generator $t^{\rm I}$ (contained in λ) is present under the trace operation.

The BGJ non-Abelian gauge anomaly term (2.6), (2.7) is by definition a solution of the Wess–Zumino consistency condition,

$$\Delta(\lambda_1)\,\mathscr{S}^{(L)}_{\rm BGJ}(\lambda_2) - \Delta(\lambda_2)\,\mathscr{S}^{(L)}_{\rm BGJ}(\lambda_1) = i\,\mathscr{S}^{(L)}_{\rm BGJ}([\lambda_1,\lambda_2]), \tag{2.8}$$

where we have used the $\Delta(\lambda)$ operator in place of the more traditional $\delta_G(\lambda)$ operator. We note that using the δ_G operator leads to the same expression, but with a minus sign on the RHS of (2.8). It is also of interest to note that

$$\delta_R(\lambda_1)\,\mathscr{S}^{(L)}_{\rm BGJ}(\lambda_2) - \delta_R(\lambda_2)\,\mathscr{S}^{(L)}_{\rm BGJ}(\lambda_1) = -i2\,\mathscr{S}^{(L)}_{\rm BGJ}([\lambda_1,\lambda_2]), \tag{2.9}$$

which differs from the WZ consistency condition in (2.8) by a factor of (-2) on the right hand side. Analogous identities hold for the right BGJ anomaly $S^{(R)}_{\rm BGJ}$.

The function ω^1_4 in (2.6) can be organized according to the powers of the connections that enter its different terms. So that we have

$$\omega^1_4(\lambda, A^{(L)}, F^{(L)}) \equiv \mathscr{A}_0(\lambda) + i\,\mathscr{A}_2(\lambda) - \mathscr{A}_4(\lambda) = \sum_{\ell=0}^{2}(i)^{\ell}\mathscr{A}_{2\ell}(\lambda), \tag{2.10}$$

where the subscripts of $\mathscr{A}_{2l}$ denotes the powers of the connection that enter, i.e.

$$\begin{aligned}\mathscr{A}_0(\lambda) &= {\rm Tr}\Big\{\lambda\, F_{\underline{a}\underline{b}}\tilde{F}^{\underline{a}\underline{b}}\Big\}, \\ \mathscr{A}_2(\lambda) &= {\rm Tr}\Big\{\lambda\Big[A_{\underline{a}}A_{\underline{b}}\tilde{F}^{\underline{a}\underline{b}} + A_{\underline{a}}\tilde{F}^{\underline{a}\underline{b}}A_{\underline{b}} + \tilde{F}^{\underline{a}\underline{b}}A_{\underline{a}}A_{\underline{b}}\Big]\Big\}, \\ \mathscr{A}_4(\lambda) &= \epsilon^{\underline{a}\underline{b}\underline{c}\underline{d}}{\rm Tr}\{\lambda\, A_{\underline{a}}A_{\underline{b}}A_{\underline{c}}A_{\underline{d}}\},\end{aligned} \tag{2.11}$$

and we dropped the L superscript for notational convenience. For reasons that will become clear later, we call the $\mathscr{A}_i$'s, ''the basis monomials of the anomaly.''

It is instructive (especially in view of our goal to treat the 4D, $N = 1$ supersymmetric case) to ask precisely how the WZ consistency condition gets satisfied using the monomials. In order to do this, it is useful to introduce a further notational device. Let the symbol $\{\}^{\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}}$ be defined by

$$\{\}^{\underline{a}\underline{b}\underline{c}\underline{d}} \equiv \epsilon^{\underline{a}\underline{b}\underline{c}\underline{d}} \widetilde{\mathrm{Tr}}\{\} \,, \tag{2.12}$$

where, given any matrices $\mathscr{X}$, $\mathscr{Y}$ and $\mathscr{Z}$, we define

$$\widetilde{\mathrm{Tr}}\{\mathscr{X}\lambda_0\mathscr{Y}\lambda_1\mathscr{Z}\} \equiv \mathrm{Tr}\{\mathscr{X}\lambda_0\mathscr{Y}\lambda_1\mathscr{Z}\} - \mathrm{Tr}\{\mathscr{X}\lambda_1\mathscr{Y}\lambda_0\mathscr{Z}\} \,. \tag{2.13}$$

A simple set of calculations reveals

$$\begin{aligned}
&\Delta(\lambda_1)\,\mathscr{A}_0(\lambda_2) - \Delta(\lambda_2)\,\mathscr{A}_0(\lambda_1) = 0\,,\\
&\Delta(\lambda_1)\mathscr{A}_2(\lambda_2) - \Delta(\lambda_2)\mathscr{A}_2(\lambda_1)\\
&\quad = \mathscr{A}_0([\lambda_1,\lambda_2]) + i\Big\{\lambda_1\lambda_2\, A_{\underline{a}}F_{\underline{bc}}A_{\underline{d}}\\
&\qquad + \tfrac{1}{2}\big[A_{\underline{a}}\lambda_1 A_{\underline{b}}\lambda_2 + \lambda_1 A_{\underline{a}}\lambda_2 A_{\underline{b}}\\
&\qquad - \lambda_1 A_{\underline{a}}A_{\underline{b}}\lambda_2\big]F_{\underline{cd}}\Big\}^{\underline{a}\underline{b}\underline{c}\underline{d}},\\
&\Delta(\lambda_1)\mathscr{A}_4(\lambda_2) - \Delta(\lambda_2)\mathscr{A}_4(\lambda_1)\\
&\quad = i\,\mathscr{A}_4([\lambda_1,\lambda_2]) + \mathscr{A}_2([\lambda_1,\lambda_2])\\
&\qquad - \Big\{\lambda_1\lambda_2\, A_{\underline{a}}F_{\underline{bc}}A_{\underline{d}}\\
&\qquad + \tfrac{1}{2}\big[A_{\underline{a}}\lambda_1 A_{\underline{b}}\lambda_2 + \lambda_1 A_{\underline{a}}\lambda_2 A_{\underline{b}}\\
&\qquad - \lambda_1 A_{\underline{a}}A_{\underline{b}}\lambda_2\big]F_{\underline{cd}}\Big\}^{\underline{a}\underline{b}\underline{c}\underline{d}}.
\end{aligned} \tag{2.14}$$

In writing these results, we have neglected total divergences. It is also useful to note that we have repeatedly used the identities

$$\partial_{\underline{a}}A_{\underline{b}} = \tfrac{1}{2}F_{\underline{a}\underline{b}} + iA_{\underline{a}}A_{\underline{b}}\,, \quad \partial_{\underline{a}}F_{\underline{b}\underline{c}} = -iF_{\underline{a}\underline{b}}A_{\underline{c}} + iA_{\underline{a}}F_{\underline{b}\underline{c}}\,. \tag{2.15}$$

which are valid under the $\{\}^{\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}}$ symbol. It is seen that upon introducing constants a_0, a_2 and a_4, we obtain

$$\begin{aligned}
&\Delta(\lambda_1)\left[\sum_{\ell=0}^{2} a_{2\ell}\mathscr{A}_{2\ell}(\lambda_2)\right]\\
&\quad - \Delta(\lambda_2)\left[\sum_{\ell=0}^{2} a_{2\ell}\mathscr{A}_{2\ell}(\lambda_1)\right]\\
&\quad = a_2\mathscr{A}_0([\lambda_1,\lambda_2]) + a_4\mathscr{A}_2([\lambda_1,\lambda_2])\\
&\qquad + i\,a_4\,\mathscr{A}_4([\lambda_1,\lambda_2]) - (a_4 - i\,a_2)\\
&\qquad \times\Big\{\big[-2\,A_{\underline{a}}\lambda_1\lambda_2\,A_{\underline{b}} + \lambda_1 A_{\underline{a}}\lambda_2 A_{\underline{b}}\\
&\qquad - \lambda_1 A_{\underline{a}}A_{\underline{b}}\lambda_2\big]F_{\underline{cd}}\Big\}^{\underline{a}\underline{b}\underline{c}\underline{d}}.
\end{aligned} \tag{2.16}$$

Imposing the WZ consistency result (2.8) upon this last equation, we find that the constants must satisfy $a_0 = -ia_2 = -a_4$ which up to an overall normalization reproduces (2.6), (2.7). It is also of note that each of the monomials *separately* and *exactly* (i.e. no total divergences are dropped) satisfies the equation

$$\begin{aligned}
&\delta_R(\lambda_1)\,\mathscr{A}_{2\ell}(\lambda_2) - \delta_R(\lambda_2)\,\mathscr{A}_{2\ell}(\lambda_1)\\
&\quad = -i2\,\mathscr{A}_{2\ell}([\lambda_1,\lambda_2])\,.
\end{aligned} \tag{2.17}$$

This approach also emphasizes that it is the abelian part of the gauge field transformation law that determines the form of the BGJ anomaly when working in the basis defined by the anomaly monomials in (2.11). The condition in (2.17) does not lead to algebraic relations among the $a_{2\ell}$ coefficients.

It is also interesting to investigate how the WZ consistency condition is satisfied when the anomaly is given as

$$\begin{aligned}
&\lambda^I \mathrm{G}^I_{(L)}(A^{(L)})\\
&\quad = \epsilon^{\underline{a}\underline{b}\underline{c}\underline{d}}\,\mathrm{Tr}\Big\{\lambda\big[\partial_{\underline{a}}(A_{\underline{b}}F_{\underline{c}\underline{d}} + iA_{\underline{b}}A_{\underline{c}}A_{\underline{d}})\big]\Big\}\\
&\quad \equiv \mathscr{B}_1(\lambda) + i\,\mathscr{B}_3(\lambda)\,.
\end{aligned} \tag{2.18}$$

Here $\mathscr{B}_1$ denotes the term linear in the gauge field and $\mathscr{B}_3$ denotes the term cubic in the gauge field. Once again the leading term in (2.18) is of the form of the Abelian anomaly except for the presence of λ under the trace operation. In this way of writing the anomaly we find that the WZ consistency condition leads to

$$\delta_R(\lambda_1)\,\mathscr{S}^{(L)}_{\rm BGJ}(\lambda_2) - \delta_R(\lambda_2)\,\mathscr{S}^{(L)}_{\rm BGJ}(\lambda_1) = -i\,\mathscr{S}^{(L)}_{\rm BGJ}([\lambda_1,\lambda_2])\,, \quad (2.19)$$

as a trivial consequence of the rigid transformations (2.4), and

$$\Delta(\lambda_1)\,\mathscr{S}^{(L)}_{\rm BGJ}(\lambda_2) - \Delta(\lambda_2)\,\mathscr{S}^{(L)}_{\rm BGJ}(\lambda_1) = 0\,, \quad (2.20)$$

as the only non–trivial condition.

The condition (2.20) can be used to determine algebraically the non-abelian anomaly in the form (2.18). In fact, we can start with the definition

$$\omega^1_4 = b_1\,\mathscr{B}_1 + b_3\,\mathscr{B}_3 \quad (2.21)$$

where b_1 and b_3 are generic constants. To show that solutions to Eq. (2.20) exist in this form we note that up to total derivatives

$$\Delta(\lambda_1)\mathscr{B}_1(\lambda_2) - \Delta(\lambda_2)\mathscr{B}_1(\lambda_1) = \epsilon^{\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}}\widetilde{\rm Tr}\{(\partial_{\underline{a}}\lambda_1)(\partial_{\underline{b}}\lambda_2)\,F_{\underline{c}\,\underline{d}}\}\,,$$

$$\Delta(\lambda_1)\mathscr{B}_3(\lambda_2) - \Delta(\lambda_2)\mathscr{B}_3(\lambda_1) = i\,\epsilon^{\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}}\widetilde{\rm Tr}\{(\partial_{\underline{a}}\lambda_1)(\partial_{\underline{b}}\lambda_2)\,F_{\underline{c}\,\underline{d}})\}\,, \quad (2.22)$$

where we have used the Bianchi identities (2.15). As long as the condition $b_3 = ib_1$ is valid, we see that (2.20) is satisfied.

We note that the non-Abelian consistency condition as encoded in the δ_R-equation which does not lead to any algebraic condition on the b_i coefficients, whereas the Δ-equation simply yields an Abelian-like condition which fixes the constants b_1 and b_3 up to an overall normalization factor. Again, the structure of the anomaly is completely determined by the nilpotent Δ-operator.

Since it is our goal to study the possibility of a supersymmetric generalization of (2.7), it behooves us to make one final set of notational changes to facilitate the use of *Superspace* [9] conventions. We note

$$A_{\underline{a}} = A_{\alpha\dot\alpha}\,,$$

$$\epsilon^{\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}} = i\left[C^{\alpha\delta}C^{\beta\gamma}C^{\dot\alpha\dot\beta}C^{\dot\gamma\dot\delta} - C^{\alpha\beta}C^{\gamma\delta}C^{\dot\alpha\dot\delta}C^{\dot\beta\dot\gamma}\right], \quad (2.23)$$

$$F_{\underline{a}\,\underline{b}} = \left[C_{\dot\alpha\dot\beta}\,f_{\alpha\beta} + C_{\alpha\beta}\,\bar f_{\dot\alpha\dot\beta}\right],$$

$$\tilde F_{\underline{a}\,\underline{b}} = i\left[C_{\dot\alpha\dot\beta}\,f_{\alpha\beta} - C_{\alpha\beta}\,\bar f_{\dot\alpha\dot\beta}\right], \quad (2.24)$$

so that the final form in which we write (2.6, 2.7) is

$$\begin{aligned}\mathscr{S}^{(L)}_{\rm BGJ}(\lambda) = {} & \left(\frac{1}{24\pi^2}\right)\int d^4x\,{\rm Tr}\Big\{\lambda\Big[\,i f^{\alpha\beta}f_{\alpha\beta} - i\bar f^{\dot\alpha\dot\beta}\bar f_{\dot\alpha\dot\beta} \\ & + \tfrac12 f^{\alpha\beta}\,A_\alpha{}^{\dot\beta}\,A_{\beta\dot\beta} + \tfrac12 A_\alpha{}^{\dot\beta}\,f^{\alpha\beta}\,A_{\beta\dot\beta} \\ & + \tfrac12 A_\alpha{}^{\dot\beta}\,A_{\beta\dot\beta}\,f^{\alpha\beta} - \tfrac12\bar f^{\dot\alpha\dot\beta}\,A^\alpha{}_{\dot\alpha}\,A_{\alpha\dot\beta} \\ & - \tfrac12 A^\alpha{}_{\dot\alpha}\,\bar f^{\dot\alpha\dot\beta}\,A_{\alpha\dot\beta} - \tfrac12 A^\alpha{}_{\dot\alpha}\,A_{\alpha\dot\beta}\,\bar f^{\dot\alpha\dot\beta} \\ & - \frac{i}{2}A^{\alpha\dot\alpha}\,A_{\alpha\dot\beta}\,A^{\beta\dot\beta}\,A_{\beta\dot\alpha} \\ & + \frac{i}{2}A^{\alpha\dot\alpha}\,A_{\beta\dot\alpha}\,A^{\beta\dot\beta}\,A_{\alpha\dot\beta}\Big]\Big\}\,, \end{aligned} \quad (2.25)$$

whereas Eq. (2.18) takes the form

$$\begin{aligned}\mathscr{S}^{(L)}_{\rm BGJ}(\lambda) = \left(\frac{1}{24\pi^2}\right)\int d^4x\,{\rm Tr}\Big\{\lambda\,\partial^{\underline{a}}\Big(& i A_\alpha{}^{\dot\beta}\bar f_{\dot\alpha\dot\beta} \\ & - iA^\beta{}_{\dot\alpha}\,f_{\alpha\beta} + \tfrac12 A_\alpha{}^{\dot\beta}\,A^\beta{}_{\dot\beta}\,A_{\beta\dot\alpha} \\ & - \tfrac12 A^\beta{}_{\dot\alpha}\,A_\beta{}^{\dot\beta}\,A_{\alpha\dot\beta}\Big)\Big\}\,. \end{aligned} \quad (2.26)$$

A superfield action for the anomaly should contain the terms in either (2.25) or (2.26) at a minimum.

3. Preliminaries for 4D, $N = 1$ supersymmetric BGJ anomaly term

Having completed the discussion of relevant structures in the non-supersymmetric case, the next obvious step is to consider analogous structures in the supersymmetric extensions. The superspace Yang–Mills covariant superderivative $\nabla_{\underline{A}} = D_{\underline{A}}$

$-i\Gamma_{\underline{A}}$ (where $\Gamma_{\underline{A}}$ is a matrix in the Lie algebra of the gauge group and $\nabla_{\underline{A}} \equiv (\nabla_\alpha, \nabla_{\dot\alpha}, \nabla_{\underline{a}})$) is totally expressed in the chiral representation in terms of a pseudoscalar hermitian-matrix general superfield V as

$$\nabla_\alpha \equiv \mathrm{e}^{-V} D_\alpha \mathrm{e}^V , \quad \nabla_{\dot\alpha} \equiv \overline{D}_{\dot\alpha} ,$$
$$\nabla_{\underline{a}} \equiv -i\{ \nabla_\alpha , \nabla_{\dot\alpha} \} , \tag{3.1}$$

and the spinorial field strengths are given by ($\overline{\Gamma}_{\dot\alpha} = -(\Gamma_\alpha)^*$)

$$W_\alpha = \overline{D}^{\,2}\Gamma_\alpha = i\overline{D}^{\,2}\left(\mathrm{e}^{-V} D_\alpha \mathrm{e}^V\right),$$
$$\overline{W}_{\dot\alpha} = D^{\,2}\overline{\Gamma}_{\dot\alpha} = iD^{\,2}\left(\mathrm{e}^V \overline{D}_{\dot\alpha} \mathrm{e}^{-V}\right). \tag{3.2}$$

Here we use *Superspace* [9] notations and conventions supplemented by superspace conjugation rules on the derivatives, $(D_\alpha)^* = -\overline{D}_{\dot\alpha}$, $(D^{\alpha})^* = \overline{D}^{\dot\alpha}$ and $(\partial_{\underline{a}})^* = \partial_{\underline{a}}$.

From the results in (3.1) and (3.2) we see that

$$\overline{D}_{\dot\alpha}\Gamma_\alpha = i\,\Gamma_{\underline{a}}, \quad \overline{D}^{\dot\alpha}\Gamma_{\underline{a}} = -i2\,W_\alpha , \tag{3.3}$$

$$\overline{D}_{\dot\alpha}W_\alpha = 0, \quad \nabla^\alpha W_\alpha + \overline{D}^{\dot\alpha}\left(\mathrm{e}^{-V}\,\overline{W}_{\dot\alpha}\,\mathrm{e}^V\right) = 0. \tag{3.4}$$

where the covariant derivative acting on superfields in the adjoint representation of the gauge group is defined as $\nabla^\alpha W_\alpha \equiv D^\alpha W_\alpha - i\{\Gamma^\alpha, W_\alpha\}$. Most of the following equations are familiar from the literature,

$$D_\alpha \mathrm{e}^V = -i\,\mathrm{e}^V\,\Gamma_\alpha , \quad D_\alpha \mathrm{e}^{-V} = i\,\Gamma_\alpha\,\mathrm{e}^{-V},$$
$$\overline{D}_{\dot\alpha}\mathrm{e}^V = i\,\overline{\Gamma}_{\dot\alpha}\,\mathrm{e}^V, \quad \overline{D}_{\dot\alpha}\mathrm{e}^{-V} = -i\,\mathrm{e}^{-V}\,\overline{\Gamma}_{\dot\alpha},$$
$$\partial_{\underline{a}}\mathrm{e}^V = -i\,\mathrm{e}^V\,\Gamma_{\underline{a}} + i\,\overline{\Gamma}_{\underline{a}}\,\mathrm{e}^V,$$
$$\partial_{\underline{a}}\mathrm{e}^{-V} = -i\,\mathrm{e}^{-V}\,\overline{\Gamma}_{\underline{a}} + i\,\Gamma_{\underline{a}}\,\mathrm{e}^{-V}, \tag{3.5}$$

but to our knowledge the last two have not appeared before.

The results of (3.3) and (3.5) are the supersymmetric analogs of the first result given in (2.15) for the non-supersymmetric theory. In both cases, the equations tell us how to calculate the first derivatives of gauge-variant objects in terms of other geometrical quantities, the superconnections. Similarly, the results in (3.4) are known to be the supersymmetric analogs of the second result given in (2.15) for the non-supersymmetric theory. In both cases these are the Bianchi identities.

The infinitesimal gauge transformations of e^V, Γ_α and W_α are given by

$$\delta_G(\Lambda)\,\mathrm{e}^V = i\,[\,\overline{\Lambda}\,\mathrm{e}^V - \mathrm{e}^V\,\Lambda\,],$$
$$\delta_G(\Lambda)\,\mathrm{e}^{-V} = i\,[\,\Lambda\,\mathrm{e}^{-V} - \mathrm{e}^{-V}\,\overline{\Lambda}\,],$$
$$\delta_G(\Lambda)\,\Gamma_{\underline{A}} = D_{\underline{A}}\Lambda + i\,L_\Lambda\,\Gamma_{\underline{A}},$$
$$\delta_G(\Lambda)\,W_\alpha = i\,L_\Lambda\,W_\alpha , \tag{3.6}$$

where the gauge parameter superfields satisfy $\overline{D}_{\dot\alpha}\Lambda = D_\alpha\,\overline{\Lambda} = 0$.

In the bosonic case we defined the operator Δ and saw that the WZ consistency condition can be entirely reformulated in terms of this operator. The Δ-operator has a natural extension to a supersymmetric YM theory. In this case it can be split into a sum of a holomorphic operator Δ_1 and its antiholomorphic conjugate $\overline{\Delta}_1$ according to

$$\Delta = \Delta_1 + \overline{\Delta}_1, \tag{3.7}$$

where both Δ_1 and $\overline{\Delta}_1$ annihilate all field strengths and factors e^V and e^{-V}. However, they act on superconnections as

$$\Delta_1\,\Gamma_\alpha = D_\alpha\,\Lambda, \quad \overline{\Delta}_1\,\Gamma_\alpha = 0,$$
$$\Delta_1\,\Gamma_{\underline{a}} = \partial_{\underline{a}}\Lambda, \quad \overline{\Delta}_1\,\Gamma_{\underline{a}} = 0. \tag{3.8}$$

In common with its non-supersymmetric precursor, the holomorphic supersymmetric operator Δ_1 and its antiholomorphic conjugate $\overline{\Delta}_1$ satisfy, $(\Delta_1)^2 = (\overline{\Delta}_1)^2 = \Delta_1\overline{\Delta}_1 = \overline{\Delta}_1\Delta_1 = 0$.

In a similar manner, the supersymmetric operator δ_R can also be split into the sum of a holomorphic operator δ^1_R and anti-holomorphic operator $\overline{\delta}^1_R$,

$$\delta_R = \delta^1_R + \overline{\delta}^1_R , \tag{3.9}$$

where

$$\delta^1_R\,\mathrm{e}^V = -i\,\mathrm{e}^V\,\Lambda, \quad \delta^1_R\,\mathrm{e}^{-V} = i\,\Lambda\,\mathrm{e}^{-V},$$
$$\delta^1_R\,\Gamma_{\underline{A}} = i\,L_\Lambda\,\Gamma_{\underline{A}},\ \delta^1_R\,W_\alpha = i\,L_\Lambda\,W_\alpha ,$$
$$\delta^1_R\,\overline{\Gamma}_{\underline{A}} = 0 \quad , \quad \delta^1_R\,\overline{W}_{\dot\alpha} = 0. \tag{3.10}$$

We also note that ''tilde-variables'' defined by

$$\tilde{\Gamma}_{\dot\alpha} \equiv \mathrm{e}^{-V}\,\overline{\Gamma}_{\dot\alpha}\,\mathrm{e}^V, \quad \tilde{\Gamma}_{\underline{a}} \equiv \mathrm{e}^{-V}\,\overline{\Gamma}_{\underline{a}}\,\mathrm{e}^V,$$
$$\tilde{W}_{\dot\alpha} \equiv \mathrm{e}^{-V}\,\overline{W}_{\dot\alpha}\,\mathrm{e}^V, \tag{3.11}$$

only transform under the action of δ^1_R according to Eq. (3.10). We will call ''holomorphic'' any quantity which manifests this behavior under the δ^1_R transformation.

The superfield form of $\mathscr{S}(J_L+J_R)$ is usually assumed to be of the form

$$\mathscr{S}_{C^2}(J_L+J_R) = \int d^8Z\left[\overline{\Phi}_+ \mathrm{e}^{V^{(R)}}\Phi_+ + \Phi_- \mathrm{e}^{-V^{(L)}}\overline{\Phi}_-\right], \tag{3.12}$$

so that the spinor ζ_α are contained in Φ_- and the spinors ψ_α are contained Φ_+. We use the notation $d^8Z \equiv d^4x d^2\theta d^2\overline{\theta}$. For a finite gauge transformation, the matter superfields Φ_- and Φ_+ and the Yang–Mills gauge superfields $V^{(L)}$and $V^{(R)}$ transform as

$$\begin{aligned}
(\Phi_+)' &= \exp\left[i\Lambda^{(R)I}t_I\right]\Phi_+ ,\\
\left(\overline{\Phi}_+\right)' &= \overline{\Phi}_+\exp\left[-i\overline{\Lambda}^{(R)I}t_I\right],\\
\left(\overline{\Phi}_-\right)' &= \exp\left[i\overline{\Lambda}^{(L)I}t_I\right]\overline{\Phi}_- ,\\
(\Phi_-)' &= \Phi_-\exp\left[-i\Lambda^{(L)I}t_I\right],\\
\left(\mathrm{e}^{-V^{(L)}}\right)' &= \mathrm{e}^{i\Lambda^{(L)}}\mathrm{e}^{-V^{(L)}}\mathrm{e}^{-i\overline{\Lambda}^{(L)}},\\
\left(\mathrm{e}^{V^{(R)}}\right)' &= \mathrm{e}^{i\overline{\Lambda}^{(R)}}\mathrm{e}^{V^{(R)}}\mathrm{e}^{-i\Lambda^{(R)}}.
\end{aligned} \tag{3.13}$$

We shall construct the anomaly associated with either Left or Right gauge group.

4. The 4D, $N = 1$ supersymmetric BGJ anomaly term in the Wess–Zumino gauge

The BGJ anomaly as written in Eq. (2.18) strongly suggests that any supersymmetric extension can be written proportional to $D_{\underline{A}}\Lambda$. On the basis of dimensional analysis (in mass units $dim[V]=0$, $dim[\Gamma_\alpha]=\frac{1}{2}$, $dim[\Gamma_{\underline{a}}]=1$ and $dim[W_\alpha]=\frac{3}{2}$) one can then write all possible monomials proportional to $D_{\underline{A}}\Lambda$ times connections $\Gamma_{\underline{A}}$, field strengths W_α, $\overline{W}_{\dot\alpha}$ or their derivatives and generic functions of e^V. The physical bosonic content of any single monomial can be easily found by linearizing in V and reducing in components in the Wess–Zumino gauge, defined by the following conditions

$$V| = 0, \quad D_\alpha V| = 0, \quad D^2 V| = 0. \tag{4.1}$$

In this gauge, the reality condition $\Lambda| = \overline{\Lambda}| \equiv \lambda$ holds for the gauge parameters. Moreover, due to the chirality constraint, they satisfy $\overline{D}_{\dot\alpha}\, D_\alpha\, \Lambda| = D_\alpha\, \overline{D}_{\dot\alpha}\, \overline{\Lambda}| = i\,\partial_{\underline{a}}\lambda$. Performing the reduction for all the geometrical objects and keeping only the physical bosonic components we have found that the only monomial structures proportional to $D_{\underline{A}}\Lambda$ and linear in V, which give contributions to the bosonic physical sector are

$$\begin{aligned}
\int d^8Z\,\mathscr{C}_1 &\equiv \int d^8Z\,\mathrm{Tr}\{D^\alpha\Lambda\{W_\alpha, V\}\} + \text{h.c.}\\
&\to -\int d^4x\,\mathscr{B}_1,\\
\int d^8Z\,\mathscr{C}_2 &\equiv \int d^8Z\,\mathrm{Tr}\left\{\partial_{\underline{a}}\Lambda\left[\Gamma^\alpha, \left[\overline{\Gamma}^{\dot\alpha}, V\right]\right]\right\} + \text{h.c.}\\
&\to 2\,i\int d^4x\,\mathscr{B}_3,\\
\int d^8Z\,\mathscr{C}_3 &\equiv \int d^8Z\,\mathrm{Tr}\left\{D^\alpha\Lambda\left\{\Gamma_{\underline{a}}, \left[\overline{\Gamma}^{\dot\alpha}, V\right]\right\}\right\} + \text{h.c.}\\
&\to \int d^4x\,[\mathscr{B}_1 + 2\,i\,\mathscr{B}_3],\\
\int d^8Z\,\mathscr{C}_4 &\equiv \int d^8Z\,\mathrm{Tr}\left\{D^\alpha\Lambda\left\{\overline{\Gamma}_{\underline{a}}, \left[\overline{\Gamma}^{\dot\alpha}, V\right]\right\}\right\} + \text{h.c.}\\
&\to -4\,i\int d^4x\,\mathscr{B}_3,\\
\int d^8Z\,\mathscr{C}_5 &\equiv i\int d^8Z\,\mathrm{Tr}\left\{D^\alpha\Lambda\left\{\partial_{\underline{a}}\overline{\Gamma}^{\dot\alpha}, V\right\}\right\} + \text{h.c.}\\
&\to \int d^4x\,[\mathscr{B}_1 + 2\,i\,\mathscr{B}_3],\\
\int d^8Z\,\mathscr{C}_6 &\equiv i\int d^8Z\,\mathrm{Tr}\left\{D^\alpha\Lambda\left\{\overline{D}^{\dot\alpha}\overline{\Gamma}_{\underline{a}}, V\right\}\right\} + \text{h.c.}\\
&\to \int d^4x\,[\mathscr{B}_1 + 2\,i\,\mathscr{B}_3],\\
\int d^8Z\,\mathscr{C}_7 &\equiv \int d^8Z\,\mathrm{Tr}\left\{D^\alpha\Lambda\left(\left\{\left\{\Gamma_{\underline{a}}, \overline{\Gamma}^{\dot\alpha}\right\}, V\right\}\right.\right.\\
&\qquad \left.\left. -\left\{\left\{\overline{\Gamma}_{\underline{a}}, \overline{\Gamma}^{\dot\alpha}\right\}, V\right\}\right)\right\} + \text{h.c.}\\
&\to \int d^4x\,[-3\mathscr{B}_1 + 2\,i\,\mathscr{B}_3].
\end{aligned} \tag{4.2}$$

It is easily seen that one can find many suitable linear combinations of the superfield monomials (the $\mathscr{C}$'s) whose component reduction gives the correct bosonic combination $(-\mathscr{B}_1 - i\mathscr{B}_3)$. For example we conclude that, within the WZ gauge,

$$\mathscr{S}_{\mathrm{BGJ}}(\lambda) = \left(\frac{1}{48\pi^2}\right) \int d^8 Z \left[\mathscr{C}_1 - \tfrac{1}{2}\mathscr{C}_2 + \mathrm{h.c.} \right], \tag{4.3}$$

contains the proper expression for the anomaly. Therefore, the expression for the supersymmetric BGJ anomaly necessarily contains this particular linear combination up to extra terms which could be required in order to satisfy the supersymmetric WZ consistency condition but whose physical bosonic sector is cohomologically trivial.

As shown in Eq. (4.2), the reduction to the WZ gauge allows only for the determination of the linear V dependence of the supersymmetric anomaly on the prepotential. In the next section, by exploiting the general approach of Ref. [7], we determine the simplest structure of the supersymmetric monomials as functions of the geometric objects of the theory.

5. The holomorphic 4D, $N = 1$ SUSY BGJ anomaly term, the MAO formalism and the minimal homotopy

Outside of the Wess–Zumino gauge it is reasonable to seek a supersymmetric extension of the BGJ anomaly in the form,

$$\mathscr{S}_{\mathrm{BGJ}}(\Lambda, \bar{\Lambda}) = \left[\tilde{S}_{\mathrm{BGJ}}(\Lambda) + \mathrm{h.c.} \right]. \tag{5.1}$$

We shall call the action $\tilde{S}_{\mathrm{BGJ}}(\Lambda)$ the ''holomorphic BGJ anomaly action'' and its hermitian conjugate the ''anti-holomorphic BGJ anomaly action.'' This action is subject to two conditions: (a.) it must satisfy the superfield WZ consistency condition and (b.) it must not be cohomologically trivial. In terms of the Δ_1 and δ_R^1 operators, the superfield WZ consistency condition for the purely holomorphic BGJ anomaly action (and equivalently, for the purely anti-holomorphic BGJ anomaly action) is

$$\begin{aligned} &\left[\delta_R^1(\Lambda_1) + \Delta_1(\Lambda_1) \right] \tilde{\mathscr{S}}_{\mathrm{BGJ}}(\Lambda_2) \\ &\quad - \left[\delta_R^1(\Lambda_2) + \Delta_1(\Lambda_2) \right] \tilde{\mathscr{S}}_{\mathrm{BGJ}}(\Lambda_1) \\ &\quad = -i\, \tilde{\mathscr{S}}_{\mathrm{BGJ}}([\Lambda_1, \Lambda_2]), \end{aligned} \tag{5.2}$$

and also a second *independent* condition given by

$$\begin{aligned} Re\Big\{ &\left[\bar{\delta}_R^1(\Lambda_1) + \bar{\Delta}_1(\Lambda_1) \right] \tilde{\mathscr{S}}_{\mathrm{BGJ}}(\Lambda_2) \\ &- \left[\bar{\delta}_R^1(\Lambda_2) + \bar{\Delta}_1(\Lambda_2) \right] \tilde{\mathscr{S}}_{\mathrm{BGJ}}(\Lambda_1) \Big\} = 0. \end{aligned} \tag{5.3}$$

The supersymmetric BGJ anomaly is expressible in terms of a real super 4-form F_{ABCD}. The superspace geometry of all 4D, $N = 1$ irreducible super p-forms was established many years ago [10]. Components of the real 4-form satisfy the following constraints,

$$\begin{aligned} &F_{\alpha\beta\gamma\underline{D}} = F_{\dot{\alpha}\beta\gamma\underline{D}} = F_{\dot{\alpha}\beta\underline{c}\,\underline{d}} = 0, \\ &F_{\alpha\beta\underline{c}\,\underline{d}} = C_{\dot{\gamma}\dot{\delta}} C_{\alpha(\gamma} C_{\delta)\beta} \bar{\mathscr{F}}, \end{aligned} \tag{5.4}$$

where $\bar{D}_{\dot{\alpha}} \mathscr{F} = 0$. The remaining non-vanishing field strength superfields take the forms

$$\begin{aligned} &F_{\alpha\underline{b}\,\underline{c}\,\underline{d}} = -\epsilon_{\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}} \bar{D}^{\dot{\alpha}} \bar{\mathscr{F}}, \\ &F_{\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}} = i\epsilon_{\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}} \left[D^2 \mathscr{F} - \bar{D}^2 \bar{\mathscr{F}} \right]. \end{aligned} \tag{5.5}$$

As was given in the first considerations of irreducible super p-forms [10], the super 4-form defined by (5.4) and (5.5) is super-closed ($(dF)_{\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}\,\underline{E}} = 0$).

The definition of the anomaly in terms of the 4-forms (5.5) is

$$\begin{aligned} S_{\mathrm{BGJ}} &\equiv \frac{1}{4!} \int_{R^4} \epsilon^{\underline{a}} \epsilon^{\underline{b}} \epsilon^{\underline{c}} \epsilon^{\underline{d}} F_{\underline{abcd}} \\ &= \frac{i}{4} \int d^4 x \left[D^2 \mathscr{F} - \bar{D}^2 \bar{\mathscr{F}} \right] \end{aligned} \tag{5.6}$$

The essential problem is then reduced to one of specifying the form of the chiral superfield $\mathscr{F}$ in terms of a gauge parameter chiral superfield Λ and the YM gauge superfield V.

As in the bosonic case, one could in principle use the conditions in (5.2) and (5.3) to determine algebraically the supersymmetric BGJ anomaly. This would amount to consider the most general non-trivial linear combination of ''monomials'', expressed in terms of e^V, superconnections, field strengths and possibly their derivatives, linear in Λ, and impose the conditions (5.2) and (5.3) to determine the coefficients of the linear combination. Here we prefer to determine an explicit expression for the supersymmetric anomaly by the use of a special homotopy operator along the lines of the work by McArthur and Osborn (MAO) [7] who gave the clearest and most succinct discussion of the issues involved with the solution of the supersymmetric Wess–Zumino consistency conditions.

In this reference, the anomaly is written as the sum of two terms L and $\int_0^1 dy X(y)$ where L is the covariant anomaly obtained from a regularized form of the one-loop effective action, and X is a local functional added in order to satisfy the WZ consistency conditions. It is constructed by using a homotopy, described by the parameter y, which is a class of maps denoted by g_y satisfying the boundary conditions $g_{y=0}=\boldsymbol{I}$ and $g_{y=1}=\mathrm{e}^V$ (for our purpose, in a K-gauge where $\bar{g}=1$). In the class of maps g_y satisfying the previous boundary conditions we choose the *minimal homotopy* defined as

$$g_y \equiv \boldsymbol{I}+y(\mathrm{e}^V-\boldsymbol{I}) . \tag{5.7}$$

Its main merit is that it is *linear* in e^V. Let us denote its inverse by $\mathscr{G}$

$$\mathscr{G}\equiv g_y^{-1}=\frac{\boldsymbol{I}}{\boldsymbol{I}+y(\mathrm{e}^V-\boldsymbol{I})} \rightarrow \mathscr{G}|_{y=0}=\boldsymbol{I},$$
$$\mathscr{G}|_{y=1}=\mathrm{e}^{-V}, \tag{5.8}$$

which satisfies

$$y\,\mathscr{G}\,\mathrm{e}^V=\boldsymbol{I}-(1-y)\,\mathscr{G},$$
$$y(\mathrm{e}^V-1)\,\mathscr{G}=\boldsymbol{I}-\mathscr{G}. \tag{5.9}$$

$$\frac{\partial^n}{\partial y^n}\mathscr{G}=(-1)^n\, n!\,(\mathrm{e}^V-1)^n\,\mathscr{G}^{n+1}. \tag{5.10}$$

The minimal homotopy is uniquely characterized by the simplicity that is evident in the equations

$$\delta_G g_y=-iy(\mathrm{e}^V\Lambda-\bar{\Lambda}\mathrm{e}^V),$$
$$\delta_G \mathscr{G}=iy\,\mathscr{G}(\mathrm{e}^V\Lambda-\bar{\Lambda}\mathrm{e}^V)\,\mathscr{G},$$
$$dg_y=-iy\,\mathrm{e}^V \times\Big[d\omega^\alpha\,\Gamma_\alpha-d\omega^{\dot\alpha}\,\tilde{\Gamma}_{\dot\alpha}+d\omega^{\underline{a}}\big(\Gamma_{\underline{a}}-\tilde{\Gamma}_{\underline{a}}\big)\Big] +dy(\mathrm{e}^V-\boldsymbol{I}),$$
$$d\mathscr{G}=iy\,\mathrm{e}^V \times\mathscr{G}\Big[d\omega^\alpha\,\Gamma_\alpha-d\omega^{\dot\alpha}\,\tilde{\Gamma}_{\dot\alpha}+d\omega^{\underline{a}}\big(\Gamma_{\underline{a}}-\tilde{\Gamma}_{\underline{a}}\big)\Big]\mathscr{G} -dy(\mathrm{e}^V-1)\,\mathscr{G}^2, \tag{5.11}$$

where $(d\omega^\alpha, d\omega^{\dot\alpha}, d\omega^{\underline{a}})$ are the basis one-forms canonically dual to $(D_\alpha,\bar{D}_{\dot\alpha},\partial_{\underline{a}})$. Due to the form of (5.7), the connections in (5.11) are independent of the y-variable. The second line in (5.11) allows the definition of a homotopically extended superconnection form via $\hat{\gamma}=i\mathscr{G}dg_y$.

The gauge superfield appears solely via exponential dependence. This is the hallmark of the class of homotopy operators of the form $g_y=\boldsymbol{I}+f(y)(\mathrm{e}^V-\boldsymbol{I})$. For the minimal one in (5.7), simple and explicit calculations show

$$[\delta_G(\Lambda^1),\delta_G(\Lambda^2)]\begin{pmatrix} g_y\\ \mathscr{G}\end{pmatrix}=\delta_G(\Lambda^3)\begin{pmatrix} g_y\\ \mathscr{G}\end{pmatrix},$$
$$\Lambda^3\equiv -i[\Lambda^1,\Lambda^2],$$
$$(d\,\delta_G-\delta_G\,d)\,g_y=-\big[\Delta_1+\bar{\Delta}_1\big]\,dg_y,$$
$$(d\,\delta_G-\delta_G\,d)\,\mathscr{G}=-\mathscr{G}\big(\big[\Delta_1+\bar{\Delta}_1\big]\,dg_y\big)\mathscr{G}. \tag{5.12}$$

We can now state the relation of the minimal homotopy to the irreducible super 4-form field strength $\mathscr{F}$ in (5.5). The minimal homotopy allows explicit evaluation of all quantities defined in Ref. [7]. The ‘‘X’’-function in our conventions is given by

$$X(f_1,f_2) =\tfrac{1}{3}\int d^8Z\,\mathrm{Tr}_s\Big[f_1\cdot\big(\bar{D}^{\dot\alpha}f_2\big)\cdot\tilde{\mathscr{W}}_{\dot\alpha} +f_2\cdot\big[\mathscr{G}D^\alpha(g_y f_1\mathscr{G})g_y\big]\cdot\mathscr{W}_\alpha\Big]+\mathrm{h.c.}, \tag{5.13}$$

where the functions f_1 and f_2 depend (again in our conventions) on the choice of the homotopy according to the following definitions

$$f_1=g_y^{-1}\left(\frac{\partial g_y}{\partial y}\right)dy \quad , \quad f_2=\mathscr{G}\,\delta_R^1 g_y. \tag{5.14}$$

The Tr_s operation is defined by

$$\mathrm{Tr}_s[\mathscr{A}\cdot\mathscr{B}\cdot\mathscr{C}]$$
$$=\mathrm{Tr}[\mathscr{A}(\mathscr{B}\mathscr{C}+\mathscr{C}\mathscr{B})+\mathscr{B}(\mathscr{C}\mathscr{A}+\mathscr{A}\mathscr{C})$$
$$+\mathscr{C}(\mathscr{A}\mathscr{B}+\mathscr{B}\mathscr{A})],$$
$$\mathrm{Tr}_s[\mathscr{A}\cdot\mathscr{B}^\alpha\cdot\mathscr{C}_\alpha]$$
$$=\mathrm{Tr}[\mathscr{A}(\mathscr{B}^\alpha\mathscr{C}_\alpha-\mathscr{C}_\alpha\mathscr{B}^\alpha)+\mathscr{B}^\alpha(\mathscr{C}_\alpha\mathscr{A}+\mathscr{A}\mathscr{C}_\alpha)$$
$$-\mathscr{C}_\alpha(\mathscr{A}\mathscr{B}^\alpha+\mathscr{B}^\alpha\mathscr{A})]. \tag{5.15}$$

In the case of minimal homotopy one obtains

$$f_1\equiv\mathscr{G}(\mathrm{e}^V-1),\quad f_2\equiv-iy\mathscr{G}\,\mathrm{e}^V\Lambda,$$
$$\mathscr{G}\big[D_\alpha(g_y f_1\mathscr{G})\big]g_y\equiv-i\mathrm{e}^V\mathscr{G}^2\Gamma_\alpha,$$
$$\overline{D}_{\dot\alpha}f_2\equiv y(1-y)\,\mathrm{e}^V\mathscr{G}\tilde{\Gamma}_{\dot\alpha}\mathscr{G}\Lambda, \tag{5.16}$$

whereas the homotopically extended field strength $\mathscr{W}_\alpha$ and its ''tilde'' conjugate $\tilde{\mathscr{W}}_{\dot\alpha}$ are defined as follows,

$$\mathscr{W}_\alpha\equiv y\,\mathrm{e}^V\mathscr{G}\,w_\alpha=\{\boldsymbol{I}-(1-y)\mathscr{G}\}\,w_\alpha,$$
$$w_\alpha\equiv W_\alpha-(1-y)\Big[\tilde{\Gamma}^{\dot\alpha}\mathscr{G}\Gamma_{\underline{a}}$$
$$+(1-y)\,\tilde{\Gamma}^{\dot\alpha}\mathscr{G}\tilde{\Gamma}_{\dot\alpha}\mathscr{G}\Gamma_\alpha$$
$$-i\tfrac{1}{2}\Big(\overline{D}^{\dot\alpha}\tilde{\Gamma}_{\dot\alpha}-i\,\tilde{\Gamma}^{\dot\alpha}\tilde{\Gamma}_{\dot\alpha}\Big)\mathscr{G}\Gamma_\alpha\Big],$$
$$\tilde{\mathscr{W}}_{\dot\alpha}\equiv y\,\mathrm{e}^V\mathscr{G}\,\tilde{w}_{\dot\alpha}=\{\boldsymbol{I}-(1-y)\mathscr{G}\}\,\tilde{w}_{\dot\alpha},$$
$$\tilde{w}_{\dot\alpha}\equiv\tilde{W}_{\dot\alpha}+(1-y)\Big[\tilde{\Gamma}_{\underline{a}}\mathscr{G}\Gamma^\alpha$$
$$-(1-y)\,\tilde{\Gamma}_{\dot\alpha}\mathscr{G}\Gamma^\alpha\mathscr{G}\Gamma_\alpha$$
$$-i\tfrac{1}{2}\tilde{\Gamma}_{\dot\alpha}\mathscr{G}(D^\alpha\Gamma_\alpha+i\,\Gamma^\alpha\Gamma_\alpha)\Big]. \tag{5.17}$$

Above, W_α is the standard field strength defined in (3.2), while the tilde quantities [6] appearing on the RHS of Eq. (5.17) are defined in (3.11).

We also work in chiral representation (with holomorphy manifest) to find

$$\mathscr{F}=-\frac{1}{2\pi^2}\overline{D}^2\Big\{\mathrm{Tr}(\Lambda\Gamma^\alpha W_\alpha)$$
$$-\tfrac{1}{3}\int_0^1 dy\,y\,\mathrm{Tr}_s\Big(\mathrm{e}^V\mathscr{G}\Lambda\cdot\mathrm{e}^V\mathscr{G}^2\Gamma^\alpha\cdot\mathscr{W}_\alpha$$
$$+(\mathrm{I}-\mathrm{e}^V\mathscr{G})\cdot\mathrm{e}^V\mathscr{G}\tilde{\Gamma}^{\dot\alpha}\mathscr{G}\Lambda\cdot\tilde{\mathscr{W}}_{\dot\alpha}\Big)\Big\}$$
$$\equiv\overline{D}^2\mathscr{P}(\Lambda;\mathrm{e}^V). \tag{5.18}$$

Written in this form, only the inverse minimal homotopy $\mathscr{G}$ appears. Making the symmetrization of the trace explicit we eventually find

$$\mathscr{P}=-\frac{1}{2\pi^2}\Big\{\mathrm{Tr}\Big[\Lambda\Big(\Gamma^\alpha W_\alpha-\int_0^1 dy\,y\Big(2\tilde{\mathscr{W}}^{\dot\alpha}\tilde{\pi}_{\dot\alpha}$$
$$+[\mathscr{W}^\alpha,\pi_\alpha]\mathscr{G}\,\mathrm{e}^V$$
$$-\{\tilde{\mathscr{W}}^{\dot\alpha},\mathscr{G}\,\mathrm{e}^V\}\tilde{\pi}_{\dot\alpha}\Big)\Big)\Big]\Big\}, \tag{5.19}$$

where we have defined

$$\pi_\alpha\equiv\mathrm{e}^V\mathscr{G}^2\Gamma_\alpha\quad,$$
$$\tilde{\pi}_{\dot\alpha}\equiv\mathscr{G}(-\pi_\alpha)^\dagger g_y=\mathrm{e}^V\mathscr{G}\tilde{\Gamma}_{\dot\alpha}\mathscr{G}. \tag{5.20}$$

Therefore, upon defining $d^6Z\equiv d^4xd^2\theta$ we find

$$\tilde{S}_{\mathrm{BGJ}}(\Lambda)\equiv i\tfrac{1}{4}\int d^6Z\mathscr{F}=i\tfrac{1}{4}\int d^8Z\mathscr{P} \tag{5.21}$$

for the holomorphic BGJ anomaly action.

The superfield action given by (5.1) and (5.21) contains the component action defined by (2.6) and (2.7). Furthermore, using standard arguments, the WZNW term can be obtained from the replacement

$$\mathscr{P}(\Lambda;\mathrm{e}^V)\to\int_0^1 dw\,\mathscr{P}(\Lambda;\mathscr{U}^\dagger\mathrm{e}^V\mathscr{U}),$$
$$\mathscr{U}\equiv\mathrm{e}^{-iw\Lambda}. \tag{5.22}$$

(details concerning these results will be reported in an extended version of this paper [11]).

[6] Care should be taken to note that W_α and its ''tilde-conjugate'' are defined by (3.2) and (3.11), respectively. On the other hand, $\mathscr{W}_\alpha$ and its ''tilde-conjugate'' are defined by (5.17).

One reason for the comparative simplicity of our result contrasted with those in Refs. [6] and [7], is precisely the use of the minimal homotopy. A non-minimal choice of the homotopy that has been widely discussed previously is defined by $\tilde{g}_y \equiv \exp[\, yV\,]$. The gauge variation of this expression is vastly more complicated than the first result given in (5.11). Many other previous expressions for the anomaly are non-minimal (i.e. contain cohomologically trivial terms) as can be seen explicitly in many places (e.g. the work by Guadagnini, Konishi and Minchev [6]). Cohomological and topological non-minimality appear to be the source of much of the opacity of the literature on the topic of supersymmetric BGJ anomalies.

''*The labour we delight in physics pain.*'' – W. Shakespeare

Acknowledgements

The authors wish to acknowledge discussions with I. McArthur and W. Siegel. S.P. thanks the Physics Department of University of Maryland for hospitality during the period when part of this work was performed. S.J.G. likewise acknowledges the hospitality of Dipartimento di Fisica dell'Universitá di Milano–Bicocca and INFN, Sezione di Milano.

References

[1] W.A. Bardeen, Phys. Rev. D 184 (1969) 1848.

[2] D.J. Gross, R. Jackiw, Phys. Rev. D 2 (1972) 477.

[3] B. Zumino, Y.-S. Wu, A. Zee, Nucl. Phys. B 239 (1984) 477.

[4] J. Wess, B. Zumino, Phys. Lett. B 37 (1971) 95; A.P. Balachandran et al., Phys. Rev. D 27 (1983) 1369; S. Novikov, Sov. Math. Dokl. 24 (1981) 222; E. Witten, Nucl. Phys. B 223 (1983) 422, 433; Ö. Kaymakcalan, S. Rajeev, J. Schecter, Phys. Rev. D 30 (1984) 594; K.C. Chou, H.Y. Guo, K. Wu, Phys. Lett. B 134 (1984) 67; H. Kawai, S.-H.-H. Tye, Phys. Lett. B 140 (1984) 403; J. Mañes, Nucl. Phys. B 250 (1985) 369; N.K. Pak, P. Rossi, Nucl. Phys. B 250 (1985) 279.

[5] S.J. Gates Jr., Phys. Lett. B 365 (1996) 132; Nucl. Phys. B 485 (1997) 145; S.J. Gates Jr., M.T. Grisaru, M.E. Knutt-Wehlau, M. Roček, O.A. Soloviev, Phys. Lett. B 396 (1997) 167.

[6] N.K. Nielsen, Nucl. Phys. B 244 (1984) 499; T.E. Clark, S. Love, Phys. Lett. B 138 (1984) 289; L. Bonora, P. Pasti, M. Tonin, Phys. Lett. B 156 (1985) 341; G. Girardi, R. Grimm, S. Stora, Phys. Lett. B 156 (1985) 203; E. Guadagnini, K. Konishi, M. Mintchev, Phys. Lett. B 157 (1985) 37; R. Garreis, M. Scholl, J. Wess, Zeit. Phys. C 28 (1985) 623; L. Bonora, P. Pasti, M. Tonin, Nucl. Phys. B 261 (1985) 249; M. Pernici, F. Riva, Nucl. Phys. B 267 (1986) 61; O. Piguet, K. Sibold, Renormalized Supersymmetry: The Perturbation-Theory of $N = 1$ Supersymmetric Gauge Theories in Flat Space-time Birkháuser Verlag, 1985, Berlin, Germany; H. Itoyama, V.P. Nair, H.-C. Ren, Nucl. Phys. B 262 (1985) 317; Phys. Lett. B 168 (1986) 78; F. Gieres, Geometry of Supersymmetric Gauge Theories, Springer, Berlin, 1988; M. Marinković, Nucl. Phys. B 366 (1991) 74; Y. Ohshima, K. Okuyama, H. Suzuki, H. Yasuta, Phys. Lett. B 457 (1999) 291.

[7] I.N. McArthur, H. Osborne, Nucl. Phys. B 268 (1986) 573.

[8] B. Zumino, Nucl. Phys. B 253 (1985) 477, and the long list of references therein.

[9] S.J. Gates Jr., M.T. Grisaru, M. Roček, W. Siegel, Superspace, Benjamin Cummings, Reading, MA, 1983.

[10] S.J. Gates Jr., Nucl. Phys. B 184 (1981) 381; S.J. Gates Jr., M.T. Grisaru, M. Roček, W. Siegel, Superspace, Benjamin Cummings, Reading, MA, 1983, pp. 181–197.

[11] S.J. Gates Jr., M.T. Grisaru, S. Penati, in preparation.