Gauge Noninvariance and Parity Nonconservation of Three-Dimensional Fermions

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The effective gauge field action due to an odd number of fermion species in three-dimensional SU(N) gauge theories is shown to change by $\pm\pi|n|$ under a homotopically nontrivial gauge transformation with winding number n. Gauge invariance can be restored by use of Pauli-Villars regularization, which, however, introduces parity nonconservation in the form of a parity-nonconserving, topological term in the effective action.

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The action, $I[A, \psi]$, for an odd number of massless fermions coupled to SU(N) gauge fields in odd dimensions is invariant under space-time reflection (which we shall call parity). I establish here, however, that the ground-state current, $\langle J_a{}^\mu\rangle$, a physical quantity, must violate this symmetry: parity is "spontaneously broken." The calculations will be limited to three dimensions, although they can be easily generalized to higher odd dimensions.

To demonstrate that parity conservation must be violated in this theory, I make use of an intriguing connection between parity nonconservation and the noninvariance of the action under homotopically nontrivial gauge transformations. The argument goes as follows: While the action, $I[A, \psi]$, is invariant under local SU(N) gauge transformations, the effective action, $I_{\text{eff}}[A]$, may not be—$I_{\text{eff}}[A]$ is obtained by integrating out fermionic degrees of freedom. In a functional integral formulation of the theory, we require $\exp iI_{\text{eff}}[A]$ to be gauge invariant. If under a gauge transformation, $I_{\text{eff}}[A]$ changes by a number, the parameters of the theory must be quantized so that number is an integral multiple of 2π.[1-4] I find that there are two ways to regulate the ultraviolet divergences in the calculation of $I_{\text{eff}}[A]$. The first way maintains parity as a good symmetry, but does not maintain gauge invariance: $I_{\text{eff}}[A]$ is found to change by an odd multiple of π under a homotopically nontrivial gauge transformation. The second way introduces a heavy Pauli-Villars regulator field and subtracts $\lim_{M\to\infty} I_{\text{eff}}[A, M]$ from $I_{\text{eff}}[A]$, thus canceling the gauge noninvariance in $I_{\text{eff}}[A]$, but introducing parity nonconservation through $I_{\text{eff}}[A, M]$—a three-dimensional mass term $M\bar{\psi}\psi$ violates parity conservation and for $M\neq 0$ the fermions have parity-nonconserving spin equal to[2] $\frac{1}{2}M/|M| = \pm\frac{1}{2}$. I find that $\lim_{M\to\infty} I_{\text{eff}}[A, M]$ contains a parity-nonconserving topological term, $\pm\pi W[A]$—$W[A]$ is the Chern-Simons secondary characteristic class[2,5]—which changes by $\pm\pi n$ under a homotopically nontrivial gauge transformation, U_n, with winding number n. Since $\langle J_a{}^\mu\rangle$ is equal to $\delta I_{\text{eff}}{}^R/\delta A_{\mu a}$ ($I_{\text{eff}}{}^R \equiv I_{\text{eff}}[A] - \lim_{M\to\infty} I_{\text{eff}}[A, M]$), $\langle J_a{}^\mu\rangle$ contains a topological term, due to $I_{\text{eff}}[A,M]$, which violates parity conservation.

As an alternative to introducing a heavy Pauli-Villars regulator to restore gauge invariance, one may simply add the topological term $\pm\pi W[A]$ to the gauge field action. Another way to restore gauge invariance is to work with an even number of fermion species, so that the effective action changes by $2\pi n$ under a large gauge transformation. In this case, parity conservation need not be violated, which is not surprising, since an even number of fermions in three dimensions can be paired to form Dirac fermions with parity-conserving mass terms.

The nonconservation of parity in odd dimensions is analogous to the nonconservation of the axial current in two and four dimensions where Pauli-Villars regularization introduces a mass which violates axial symmetry. We therefore complete the program begun by generalizing the axial anomaly to higher even dimensions[6] by establishing the existence of a similar phenomenon in odd dimensions.[7] The "anomaly" in odd dimensions appears as a parity-nonconserving topological term in the ground-state current $\langle J_a{}^\mu\rangle$, rather than as a topological term ($\sim{}^*FF$) in $\partial_\mu\langle J_a{}^\mu\rangle$ (there exists no axial current in odd dimensions, i.e., no γ_5). In both cases, the "anomaly" causes a "physical" ground-state current to violate a symmetry of the original action, $I[A, \psi]$.

To show that fermions induce a gauge-noninvariant term in the action, we begin with the functional integral

$$Z = \int d\bar{\psi}\, d\psi\, dA \exp\{i\int[\operatorname{tr}F^2/2 + i\bar{\psi}(\not{\partial} + \not{A})\psi]\, d^3x\} \quad (1)$$

and integrate over the fermion fields:

$$Z = \int dA \exp\{i(\int \mathrm{tr}(F^2/2)d^3x + I_{\mathrm{eff}}[A])\}, \quad (2)$$

where

$$I_{\mathrm{eff}}[A] = -i \ln\det(\not{\partial} + \not{A}). \quad (3)$$

I use the usual matrix notation $A = gT^a A_\mu{}^a$, where T^a are anti-Hermitian generators of the group. For definiteness, we work with SU(2) and a doublet of fermions—where $T^a = \sigma^a/2i$, and σ^a are the Pauli matrices—but the results hold in any group for which Π_3 is the additive group of integers, Z, and the fermions are in the fundamental representation. The Dirac matrices in three dimensions are Pauli matrices $(\sigma_3, i\sigma_2, i\sigma_1)$.

I now demonstrate that det $(\not{\partial} + \not{A})$ and, by (3), the effective action $I_{\mathrm{eff}}[A]$ are not gauge invariant. More precisely, I show that

$$\det(\not{\partial} + \not{A}) \to (-1)^{|n|}\det(\not{\partial} + \not{A}) \quad (4)$$

under a homotopically nontrivial gauge transformation, U_n, with winding number n. By (3) this is equivalent to $U_n : I_{\mathrm{eff}}[A] \to I_{\mathrm{eff}}[A] \pm \pi|n|$.

To prove (4), we follow closely the analogous calculations performed by Witten in four dimensions.[3] We use here a Euclidean formulation of the three-dimensional theory, and we consider gauge transformations which approach the identity at large distances; hence the base manifold is S_3 rather than R_3.

To begin, we observe that det $i\sigma_\mu(\partial_\mu + A_\mu)$, $\mu = 1, 2, 3$, may be written $\det^{1/2} i\not{D}_4$, where $\not{D}_4 = \gamma_\mu(\partial_\mu + A_\mu)$ and

$$\gamma_\mu = \begin{pmatrix} \sigma_\mu & 0 \\ 0 & -\sigma_\mu \end{pmatrix}. \quad (5)$$

Since there exists a 4×4 matrix which anticommutes with $i\not{D}_4$, the spectrum of $i\not{D}_4$ is symmetric about zero (and is real because is $i\not{D}_4$ is Hermitian). Therefore, for a particular gauge field, A_μ, we may define the square root as the product of the positive eigenvalues of $i\not{D}_4[A_\mu]$. (It is of course assumed that there is no zero eigenvalue.) Since $\det i\not{D}_4$ can be regulated by the introduction of a parity-invariant *Dirac* mass,[8] this procedure maintains parity as a good symmetry.

We now vary the gauge field along a continuous path, parametrized by τ, from $A_\mu(x^\mu, \tau) = 0$ at $\tau = -\infty$ to the pure gauge $A_\mu(x^\mu, \tau) = U_n{}^{-1}\partial_\mu U_n$ at $\tau = +\infty$, where U_n belongs to the nth homotopy class (U_n has winding number n). The spectrum of $i\not{D}_4[A]$ at $\tau = -\infty$ is identical to the spectrum at $\tau = +\infty$. However, as τ is varied from $-\infty$ to $+\infty$, the gauge field must pass through configurations in field space which are not pure gauge, because U_n is not continuously deformable to the identity. Therefore, the eigenvalues of $i\not{D}_4$ may become rearranged as τ goes from $-\infty$ to $+\infty$. In particular, one or more eigenvalues which are positive at $\tau = -\infty$ may cross zero and become negative at $\tau = +\infty$ (see Fig. 1). The square root of the determinant, $\det^{1/2} i\not{D}[A]$, defined as the product of the positive eigenvalues of $i\not{D}_4$ at $\tau = -\infty$, will therefore change sign if the number of eigenvalues which flow from positive to negative values (or vice versa) is odd.

We now recognize that the family of vector potentials, $A_\mu(x^\mu, \tau)$, is equivalent to an instanton-like four-dimensional gauge field, A^i, in the gauge $A^4 = 0$ [the space $x^i = (x^\mu, \tau)$, $i = 1, 2, 3, 4$, is the cylinder $S_3 \times R$]. The remaining components of $A^i(x^\mu, \tau)$ vary *adiabatically* as a function of $\tau = x^4$ along the path considered above.

The number of zero crossings of the eigenvalues of $i\not{D}_4[A^\mu(\tau)]$ is related to the number of normalizable zero modes of the four-dimensional operator

$$\not{D} = \gamma_i(\partial_i + A_i), \quad i = 1, 2, 3, 4, \quad (6)$$

with

$$\gamma^4 = \gamma^\tau = i\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

To see this, we write the Dirac equation $\not{D}\psi = 0$ as

$$d\psi/d\tau = -\gamma^\tau \not{D}_4 \psi. \quad (7)$$

Equation (7) is soluble in the adiabatic approximation; we choose $\psi(x^\mu, \tau) = f(\tau)\varphi^\tau(x^\mu)$, where $\varphi^\tau(x^\mu)$ satisfies the eigenvalue equation

$$\gamma^\tau \not{D}_4 \varphi^\tau(x^\mu) = \lambda(\tau)\varphi^\tau(x^\mu). \quad (8)$$

Since the spectrum of $\gamma^\tau \not{D}_4$ is equal to the spectrum of $i\not{D}_4$, the eigenvalues $\lambda(\tau)$ vary continuous-

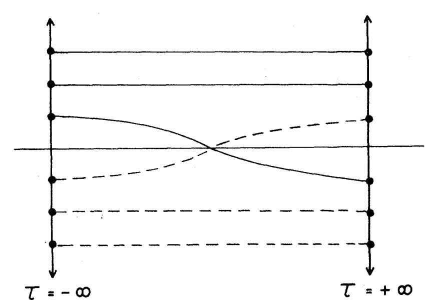

FIG. 1. The eigenvalues of $i\not{D}_4$ are plotted along the vertical axis. One initially positive eigenvalue is shown to cross zero as τ goes from $-\infty$ to $+\infty$

ly along the curves of Fig. 1. In the adiabatic approximation, Eq. (7) becomes

$$df/d\tau = -\lambda(\tau) f(\tau) \tag{9}$$

which has the solution

$$f(\tau) = f(0)\exp[-\int_0^\tau d\tau' \lambda(\tau')]. \tag{10}$$

Only if λ is positive for $\tau = +\infty$ and negative for $\tau = -\infty$ is this solution normalizable.

Therefore, there exists a one to one correspondence between the number of normalizable zero modes of $\not{D}[A^i]$ and the number of eigenvalues of $i\not{D}_4[A^\mu(\tau)]$ which pass from negative to positive (or from positive to negative) values as τ is varied from $-\infty$ to $+\infty$. The number of zero modes of $\not{D}[A^i]$ is well known from instanton studies.[9] If we choose the eigenfunctions of $\not{D}\psi_\pm = 0$ to be eigenfunctions of γ_5, $\gamma_5\psi_\pm = \pm\psi_\pm$, and denote the number of zero modes of ψ_+ (ψ_-) by n_+ (n_-), then

$$n_- - n_+ = n, \tag{11}$$

where n is the instanton number (the winding number of U_n). Therefore, the total number of zero modes, N_T, of $\not{D}$ is

$$N_T = 2n_+ + n, \tag{12}$$

which is odd if the winding number n is odd. This completes the proof of (4): The determinant changes sign under a large gauge transformation with odd winding number.

To show that gauge-invariant regularization procedures, such as Pauli-Villars regularization, restore gauge invariance at the cost of introducing parity nonconservation, I have performed two types of calculations—details will be given elsewhere.

First, one may calculate $\lim_{M\to\infty} I_{\text{eff}}[A,M]$ in perturbation theory[2,10]; I find

$$I_{\text{eff}}{}^R \equiv I_{\text{eff}}[A] - \lim_{M\to\infty} I_{\text{eff}}[A,M]$$
$$= \pm\pi W[A] + I'[A], \tag{13}$$

where $I_{\text{eff}}{}^R$ is finite and $W[A]$ is the parity-nonconserving Chern-Simons term

$$W[A] = (1/8\pi^2)\int d^3x\, \text{tr}({}^*F_\mu A^\mu - \tfrac{1}{3}A^\mu A^\nu A^\alpha \epsilon_{\mu\nu\alpha}), \tag{14}$$

with ${}^*F^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha}F_{\nu\alpha}$. Since only the vacuum polarization graph and the triangle graph are divergent, and these two graphs produce $\pm\pi W[A]$, I conclude that $I'[A]$ in (13) must be parity conserving.

Second, in the special case of gauge fields with constant field strength tensor, $F^{\mu\nu}$, the effective action (3) can be calculated exactly. The calculational procedure is the three-dimensional analog of the Euler-Heisenberg calculation of the effective action for constant field strength, as presented by Schwinger.[11] I find

$$I_{\text{eff}} = \pm W[A] + I_{\text{NA}}[A] \tag{15}$$

where $I_{\text{NA}}[A]$ is parity conserving, but is nonanalytic in the gauge field. In performing this calculation, it is necessary to introduce a parity-nonconserving fermion mass, M, to regulate divergences in a gauge-invariant manner.[2] When the mass is set equal to zero at the end of the calculation, however, the parity-nonconserving term, $\pm\pi W[A]$, does not vanish. (The sign depends upon the sign of the regulator mass.) The existence of the nonanalytic expression, $I_{\text{NA}}[A]$, is characteristic of three-dimensional gauge theories coupled to massless fermions, where such terms are known to appear in partial sums of infrared-divergent Feynman diagrams.[12] For the even more special case of an Abelian, constant field strength, the ground-state current is given exactly by

$$\langle J_a\rangle = (g^2/8\pi)\,{}^*F_a{}^\mu. \tag{16}$$

This current is identically conserved, but it violates parity conservation explicitly since ${}^*F^\mu$ is a pseudovector.

The discovery of anomalous parity nonconservation in three dimensions has wide ranging consequences. First, the violation of space-time reflection in three dimensions may have direct, measurable consequences in condensed matter physics, where models of vortex-particle interactions in superconductors are mathematically equivalent to three-dimensional QED—while the discussion here is limited to SU(N) theories, parity nonconservation occurs in $\langle J^\mu\rangle$ in QED as well. The relativistic Dirac equation used here appears also in a nonrelativistic system as is seen in the discovery of crystals which model the four-dimensional anomaly.[12] At high temperatures, four-dimensional field theories effectively reduce to three-dimensional theories. Although fermions are known to decouple at high temperatures, the effective theory in three dimensions, obtained by "integrating out" the fermions, may still be relevant in the study of high-temperature four-dimensional theories. The induced topological term $\pm W[A]$ in $I_{\text{eff}}[A]$ is known to produce a mass for the gauge fields.[2,13] Not only must parity conservation be violated in odd-dimensional theories with an odd number of fermions, but the gauge fields *must* become massive as well. Since the gauge fields acquire a mass without the need for Higgs fields, this implies that the analogous phenomenon in solid state physics, the Meissner effect, may be possible without the need for a condensate of fermion pairs. Above all, the discovery of anomalous parity nonconservation in three dimensions provides a different and simpler laboratory in which to study the subtle interplay between "anomalous" violation of symmetries and global topological properties of gauge theories.

Results similar to those presented in this paper have been obtained independently by L. Alvarez-Gaumé and E. Witten (to be published).

Recently, E. D'Hoker and E. Farhi have shown how to compensate for the gauge noninvariance of the fermion determinant in four-dimensional theories by the addition of bosonic terms to the Lagrangian.[14] Also, Niemi and Semenoff have shown how to derive the results presented here using the anomaly in two dimensions.[7]

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