SELF-INTERACTING TENSOR MULTIPLETS IN $N = 2$ SUPERSPACE

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We give the $N = 2$ superspace action for self-interacting tensor multiplets in four dimensions and discuss the relation to the harmonic superspace recently proposed by Galperin et al.

Superspace formulations of extended theories are notoriously difficult to construct. In particular, as the number of anticommuting coordinates increases, the number of unphysical components in a superfield grows and the mass dimension of the measure increases while the dimensions of the physical fields remain unchanged. One remedy is to use constrained superfields to remove unphysical components and integrations over subspaces to decrease the dimension of the measure. The simplest example of such a subintegration is the chiral integral in $N = 1$ superspace. A chiral measure can also be used for $N = 2$ Yang–Mills theory [1] but not for the self-interacting tensor multiplet [2]. In this note we present the action for the four dimensional self-interacting tensor multiplet in $N = 2$ superspace. We use a novel subintegration which appears to break SU(2) invariance. The invariance is restored by a further integration over an internal parameter. We briefly comment on the relation to the recently introduced harmonic superspace [3].

The $N = 2$ tensor multiplet [4] is described by a real super field-strength F^a_b which satisfies Bianchi identities:

$$C_{d(a}\mathrm{D}_b{}^\alpha F_{c)}{}^d = 0, \quad C^{d(a}\bar{\mathrm{D}}^b{}_{\dot\alpha}F^{c)}{}_d = 0. \tag{1}$$

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We use the conventions of refs. [5,6]: latin letters are SU(2) isospin indices and greek letters are two-component spinor indices. We also sometimes write isospinors in boldface. F is an isovector and hence $F^1_1 = -F^2_2$. The constraints (1) are solved in terms of a chiral prepotential Φ:

$$F^a_b = \mathrm{i}(C^{bc}\mathrm{D}^2{}_{ac}\Phi - C_{ac}\bar{\mathrm{D}}^{2bc}\bar\Phi). \tag{2}$$

Since the dimension of F is 1 and the full superspace measure has dimension 0, an action with a full superspace integration must either be nonlocal or involve the prepotential Φ explicitly [4,5]. We thus look for a subintegration; however, F is not chiral and hence the obvious chiral integration cannot be used. We construct an appropriate measure by the method used in two dimensions for the $N = 4$ twisted chiral multiplet [7].

To write a lagrangian quadratic in the field-strength we need a measure with four spinor derivatives, e.g., $\tilde\nabla^2\tilde\Delta^2$. This means that the lagrangian g has to be independent of the corresponding orthogonal subspace, e.g., $\nabla_\alpha g = \Delta_{\dot\alpha} g = 0$.

We take as our starting point the explicit form of (1) (suppressing spinor indices on the derivatives):

$$\mathrm{D}_1F^2_1 = 0, \quad \mathrm{D}_2F^1_2 = 0, \quad 2\mathrm{D}_1F^1_1 - \mathrm{D}_2F^2_1 = 0,$$
$$2\mathrm{D}_2F^1_1 + \mathrm{D}_1F^1_2 = 0, \quad \bar{\mathrm{D}}^1F^1_2 = 0, \quad \bar{\mathrm{D}}^2F^2_1 = 0,$$
$$2\bar{\mathrm{D}}^1F^1_1 - \bar{\mathrm{D}}^2F^1_2 = 0, \quad 2\bar{\mathrm{D}}^2F^1_1 + \bar{\mathrm{D}}^1F^2_1 = 0. \tag{3}$$

Next we look for a function $g(F_1^1, F_2^1, F_1^2)$ annihilated by a linear combination of D_1 and D_2 and by a linear combination of $\bar{\mathrm{D}}^1$ and $\bar{\mathrm{D}}^2$. (This procedure breaks manifest SU(2) invariance.) To find these we consider:

$$(\mathrm{D}_1 + \zeta \mathrm{D}_2) g = (g_{11} + 2\zeta g_{12}) \mathrm{D}_1 F_1^1 + (\zeta g_{11} - 2g_{21}) \mathrm{D}_2 F_1^1,$$

$$(\bar{\mathrm{D}}^1 + \xi \bar{\mathrm{D}}^2) g = (g_{11} + 2\xi g_{21}) \bar{\mathrm{D}}^1 F_1^1 + (\xi g_{11} - 2g_{12}) \bar{\mathrm{D}}^2 F_1^1. \tag{4}$$

Here g_{ab} is the derivative of g with respect to F_a^b. The expressions in (4) vanish only if:

$$g_{11} + 2\zeta g_{12} = 0, \quad \zeta g_{11} - 2g_{21} = 0,$$

$$g_{11} + 2\xi g_{21} = 0, \quad \xi g_{11} - 2g_{12} = 0. \tag{5}$$

From this we see that $\zeta\xi = -1$ and that g depends on F only through a *linear* function η:

$$\eta(\zeta) \equiv (1 \quad \zeta) \begin{pmatrix} F_1^1 & F_1^2 \\ F_2^1 & -F_1^1 \end{pmatrix} \begin{pmatrix} -\zeta \\ 1 \end{pmatrix} \equiv -\boldsymbol{\zeta}^{\mathrm{t}} \mathrm{F} \mathrm{C} \boldsymbol{\zeta}$$

$$= F_1^2 - 2\zeta F_1^1 - \zeta^2 F_2^1. \tag{6}$$

($\boldsymbol{\zeta}$ is actually a *projective* isospinor, see below). The derivatives that annihilate η and hence g are:

$$\nabla \equiv \mathrm{D}_1 + \zeta \mathrm{D}_2, \quad \Delta \equiv \zeta \bar{\mathrm{D}}^1 - \bar{\mathrm{D}}^2 \tag{7}$$

(the normalizaiton of these operators is arbitrary). We can choose linearly independent operators:

$$\tilde{\nabla} \equiv \zeta \mathrm{D}_1 - \mathrm{D}_2, \quad \tilde{\Delta} \equiv \bar{\mathrm{D}}^1 + \zeta \bar{\mathrm{D}}^2. \tag{8}$$

(In general, any operators linearly independent of ∇ and Δ will do; the ones in (8) are orthogonal for real ζ, and become degenerate when $\zeta = \pm \mathrm{i}$. Combinations that are never degenerate must involve $\bar{\zeta}$, and we find analytic expressions more useful; see below.) The most general action for $i = 1, ..., n$ self-interacting tensor multiplets is

$$S = \frac{1}{2\pi \mathrm{i}} \int \mathrm{d}^4 x \oint (1 + \zeta^2)^{-4} \mathrm{d}\zeta\, \tilde{\nabla}^2 \tilde{\Delta}^2 g\{\zeta; \eta^i(\zeta)\}, \tag{9}$$

where we have chosen a convenient normalization for the measure (see below). The contour integral over ζ sums over all possible subintegrations. The possibility of such a form was suggested to us by Hitchin in connection with the $N = 1$ formulation of this theory [2] (see also refs. [7,8]). If we had not chosen $\tilde{\nabla}$ and $\tilde{\Delta}$ to be analytic in ζ, clearly, we would not have been able to use a contour integral.

We now discuss the SU(2) transformation properties of the theory. These follow from the transformation properties of the spinor derivatives: $\mathrm{D}_{a\alpha}$ transforms as an isodoublet and $\bar{\mathrm{D}}^a_{\dot{\alpha}}$ transforms in the conjugate representation. The transformation properties of the derivatives are consistent with F_b^a being an isotriplet and with formula (2). Our parametrization of an SU(2) transformation is:

$$\mathrm{U} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad a\bar{a} + b\bar{b} = 1. \tag{10}$$

Thus F and D transform as $\mathrm{F}' = \mathrm{U}\mathrm{F}\bar{\mathrm{U}}$, $\boldsymbol{D}' = \mathrm{U}\boldsymbol{D}$. To define the transformation properties of ζ and the ζ-dependent objects, we relate ζ to an isospinor $\boldsymbol{u}$:

$$\boldsymbol{u} = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} \qquad \zeta = u^2/u^1, \quad \boldsymbol{\zeta} = \boldsymbol{u}/u^1. \tag{11}$$

Thus ζ is the homogeneous coordinate on CP(1). In terms of $\boldsymbol{u}$ we have:

$$\eta = -(1/u^1)^2 \boldsymbol{u} \mathrm{F} \mathrm{C} \boldsymbol{u},$$

$$\nabla = (1/u^1) \boldsymbol{u}^{\mathrm{t}} \boldsymbol{D}, \quad \Delta = (1/u^1) \bar{\boldsymbol{D}} \mathrm{C} \boldsymbol{u},$$

$$\tilde{\nabla} = -(1/u^1) \boldsymbol{u}^{\mathrm{t}} \mathrm{C} \boldsymbol{D}, \quad \tilde{\Delta} = (1/u^1) \bar{\boldsymbol{D}} \boldsymbol{u}. \tag{12}$$

From (11) and (12) we immediately find the transformations:

$$\zeta' = (\bar{a} + \bar{b}\zeta)^{-1} (a\zeta - b), \quad \mathrm{d}\zeta' = (\bar{a} + \bar{b}\zeta)^{-2} \mathrm{d}\zeta,$$

$$\eta'(\zeta') = (\bar{a} + \bar{b}\zeta)^{-2} \eta(\zeta). \tag{13}$$

The operators ∇ and Δ also transform simply: they scale by the transformation of $(1/u^1)$. The linearly independent operators $\tilde{\nabla}$ and $\tilde{\Delta}$ transform in a more complicated way because, e.g., $\bar{\boldsymbol{D}}$ and $\boldsymbol{u}$ *both* transform contragradiently. $\tilde{\nabla}'$ and $\tilde{\Delta}'$ contain terms proportional to ∇ and Δ respectively. These we indicate by "...", as they do not contribute to the measure:

$$[(1 + \zeta^2)^{-4} \mathrm{d}\zeta\, \tilde{\nabla}^2 \tilde{\Delta}^2]'$$

$$= (\bar{a} + \bar{b}\zeta)^2 [(1 + \zeta^2)^{-4} \mathrm{d}\zeta\, \tilde{\nabla}^2 \tilde{\Delta}^2] + \dots . \tag{14}$$

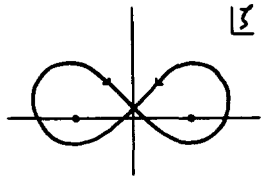

Fig. 1. The contour for $g_i = \eta \ln \eta$; the singularities enclosed by the contour are at the two roots of $\eta(\zeta) = 0$.

Then the action is invariant if g transforms as

$$g'\{\zeta';\eta'(\zeta')\} = (\bar{a} + \bar{b}\zeta)^{-2} g\{\zeta;\eta(\zeta)\}. \tag{15}$$

We now give several examples. The free tensor multiplet [4] and the improved tensor multiplet [9] are described by (this can be verified most easily by reduction to $N = 1$ superspace, see below):

$$g_f = -\eta^2/(2\zeta), \quad g_i = \eta \ln \eta. \tag{16}$$

We note the striking similarity to the $N = 1$ free and improved tensor multiplets (see, e.g., refs. [2,6]). The contour for g_f is any closed loop enclosing the origin and the contour for g_i is shown in fig. 1. The improved tensor multiplet is dual (and therefore classically equivalent) to a free theory. Interacting models can be found, e.g., by taking linear combinations of the two actions [2,8]. We further find some examples of total derivative lagrangians. Although they do not give rise to any dynamics, we include them because their forms are not obvious:

$$f_1(\zeta)\eta, \quad (A + B\zeta)f_2(\eta), \quad \zeta(A + B\zeta)f_2(\eta/\zeta^2). \tag{17}$$

Here f_1 is arbitrary, A and B are constants, and f_2 has no singularities within the contour of integration (which encloses the origin).

The superfield formulation of the tensor multiplet in six dimensions is not known to us, but we expect this construction to carry over with ∇ and Δ combined into one SU(2) Majorana–Weyl spinor [10].

The reduction to $N = 1$ superspace is straightforward [11,7]. We define $N = 1$ superfields and spinor derivatives:

$$G \equiv 2F_1^1|, \quad \chi \equiv F_2^1|, \quad \bar{\chi} \equiv F_1^2|,$$

$$\mathrm{D} \equiv \mathrm{D}_1|, \quad \bar{\mathrm{D}} \equiv \bar{\mathrm{D}}^1|, \tag{18}$$

where | indicates the $\theta^2, \bar{\theta}_2$ independent part of a superfield or an operator. The "other" derivatives D_2 and $\bar{\mathrm{D}}^2$ generate the second supersymmetry [10,7]. From (3), it follows that χ is chiral ($\bar{\mathrm{D}}\chi = 0$), and G is a real linear superfield ($\mathrm{D}^2 G = \bar{\mathrm{D}}^2 G = 0$). This is the $N = 1$ superfield content of the $N = 2$ tensor multiplet [2]. The action is obtained from (9) by writing the derivatives $\tilde{\nabla}$ and $\tilde{\Delta}$ in terms of D_1 and ∇, and $\bar{\mathrm{D}}^1$ and Δ, respectively:

$$\tilde{\nabla} = \zeta^{-1}[(1 + \zeta^2)\mathrm{D}_1 - \nabla], \quad \tilde{\Delta} = (1 + \zeta^2)\bar{\mathrm{D}}^1 - \zeta\Delta. \tag{19}$$

Because ∇ and Δ annihilate g, the measure reduces to

$$[(1 + \zeta^2)^{-4} \mathrm{d}\zeta\, \tilde{\nabla}^2 \tilde{\Delta}^2] \to \zeta^{-2} \mathrm{d}\zeta\, \mathrm{D}^2 \bar{\mathrm{D}}^2 \tag{20}$$

and $\eta(\zeta)$ reduces to:

$$\eta| = \bar{\chi} - \zeta G - \zeta^2 \chi. \tag{21}$$

To compare with ref. [2], we note that the $N = 1$ superspace lagrangian

$$f(G, \chi, \bar{\chi}) = \frac{1}{2\pi i} \oint \zeta^{-2} \mathrm{d}\zeta\, g(\zeta; \eta|) \tag{22}$$

satisfies

$$f_{GG} + f_{\chi\bar{\chi}} = 0. \tag{23}$$

The extension to more than one tensor multiplet is straightforward.

Our procedure of finding a subintegration which breaks SU(2) and then integrating over different SU(2) directions suggests a relation to the harmonic superspace recently proposed in ref. [3]. Here we briefly discuss the relation. Our ζ integration is an integration over CP(1), which is the sphere that the authors of ref. [3] describe as the coset space SU(2)/U(1). Our isospinor $\boldsymbol{u}$ corresponds to their u^{+i} and the operators $u^1\nabla$ and $u^1\Delta$ correspond to D^+ and $\bar{\mathrm{D}}^+$. Whereas we construct the integration measure from analytic operators $\tilde{\nabla}$ and $\tilde{\Delta}$, the authors of ref. [3] opt for operators orthogonal to D^+ and $\bar{\mathrm{D}}^+$, and are thus forced to introduce the complex conjugate of $\boldsymbol{u}$, (which they call u^{-i}). Consequently, they are led to a harmonic expansion on the sphere rather than to a contour integral. The resulting measures differ by terms that annihilate the integrand. The action (9) can be understood as the integral over the analytic subspace of the most general analytic superfield that can be constructed *algebraically out of an ordinary superfield* satisfying the constraints (1).

These results suggest several interesting problems: can this kind of restricted measure be generalized to other systems (higher N), and can one find a coupling to supergravity? Can the analogy to $N = 1$ theories be exploited further? Would it be useful to construct supergraph rules using the contour integration? The relation to the harmonic superspace of ref. [3] deserves further study. In particular, can the construction of analytic superfields out of ordinary superfields be extended to other cases?

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