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DECAY OF THE FALSE VACUUM AT FINITE TEMPERATURE

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In this paper we study kinetics of the first-order phase transitions in gauge theories with spontaneous symmetry breaking. A general theory of the decay of the metastable phase (false vacuum) at a finite temperature is developed. A number of concrete examples are considered, which will make it possible to study kinetics of phase transitions in a wide class of theories without complicated computer calculations.

1. Introduction

In recent years there has been a considerable interest in cosmological consequences of grand unified theories [1]. There are two main reasons for that. Firstly, these theories may provide a natural solution of the problem of baryon asymmetry of the universe [2], and thus the study of these theories may be of great importance for cosmology. Secondly, it has become clear that in the near future an investigation of particle properties in the energy range $E \sim 10^{15}$ GeV, necessary for a thorough study of grand unified theories, can be performed neither with the help of cosmic ray experiments, nor by constructing new accelerators. The only laboratory wherein elementary particles of such energies once existed was our universe at the very early stages of its evolution. Therefore, one can expect that after carrying out certain experiments on proton decay, one of the most important sources of experimental information concerning grand unified theories will be an investigation of cosmological consequences of these theories.

A necessary stage in the investigation of the physical processes in the early universe is a development of the theory of phase transitions with symmetry breaking, which, according to grand unified theories, must have proceeded in the course of the cooling of the universe at the very early stages of its evolution [3–5]. The theory of the above-mentioned phase transitions turns out to be of principle importance for the theory of cosmological baryon production [6], for solving the primordial monopole problem [7] and for the development of a number of interesting and non-standard scenarios of the universe evolution (cold universe [5], inflationary universe [8, 9], etc.). Particularly important in this connection is an investigation of the kinetics of phase transitions in grand unified theories. In recent years there appeared many papers on this subject [8–15], but being complicated, the question has not been completely clarified.

The present paper is the first one in the series of articles devoted to the kinetics of phase transitions in grand unified theories. Before describing the content of the present paper and elucidating the subjects of the other papers it would be relevant to recall some basic points of the theory of phase transitions in gauge theories [3–5].

Symmetry breaking in gauge theories is due to the appearance of a non-zero Higgs classical scalar field φ*. The field φ is usually time independent (at least in the unitary-type "physical" gauges) and determined by minimization of the so-called effective potential $\mathcal{V}(\varphi(\boldsymbol{x}), T)$ [3–5], which at the points of its extrema $\varphi(\boldsymbol{x})$ coincides simply with the free energy density F of matter at a temperature T in the presence of the time-independent field $\varphi(\boldsymbol{x})$**. The minimum of $\mathcal{V}(\varphi(\boldsymbol{x}), T)$ corresponds, as a rule, to the fields $\varphi(\boldsymbol{x})$, which are constant not only in time, but also in space, $\varphi(x) = \varphi_0 = \text{const.}$ (although there are some exceptions from this rule, such as, for example, π condensation). For this reason, in most parts of this paper we will consider the function $\mathcal{V}(\varphi, T)$ disregarding the terms that depend on the spatial derivatives of the field φ and will denote it by $V(\varphi, T)$. It is this quantity that is usually referred to as an effective potential, but we would like to emphasize that it is not obligatory, and all the formalism, connected with the definition of the function $V(\varphi, T)$ is extended directly to the case of the fields depending on the spatial coordinates x***, see, for example, ref. [16].

The typical form of the function $V(\varphi, T)$ at zero temperature is presented in fig. 1, curve A, although in some theories $V(\varphi, 0)$ may take a more complicated form (see, for example, fig. 1, curve B). In both cases the absolute minimum of $V(\varphi, 0)$, shown in fig. 1, is realized at $\varphi_0 = 0$.

At a sufficiently high temperature T, which exceeds all the characteristic mass parameters in the theory, the only minimum of $V(\varphi, T)$ is the state $\varphi = 0$ [3–5]‡. This means that symmetry breaking in field theory (at zero temperature) occurs as a result of some phase transition from the state $\varphi = 0$ to the state $\varphi_0 \neq 0$, as temperature lowers. If the magnitude of the field φ changes smoothly from 0 to φ_0 as T decreases, we are dealing with the second-order phase transition. It is just such transitions that occur in the Weinberg–Salam type theories at temperatures $T \sim 10^2$ GeV. However, in grand unified theories during phase transitions at $T \sim 10^{14} - 10^{15}$ GeV, a non-zero scalar field arises discontinuously, the amplitude of this discontinuity being of the order of φ_0 [18] (see also refs. [29, 20]). Such a phase transition (first-order one) occurs if in some temperature interval the quantity

* Throughout the paper we will restrict ourselves, for the sake of simplicity, to the consideration of a one-component Higgs field; extension of our results to the case of many-component fields is rather trivial.

** More precisely, $\mathcal{V}(\varphi(x), T)$ coincides with the density of the thermodynamic potential Ω [3–5], but in what follows we will consider the situation when the chemical potentials of particles are zero so that $F = \Omega$.

*** This remark is due to D.A. Kirzhnits.

‡ In the case of many-component fields φ there exist some exceptions from this rule [3, 17, 6].

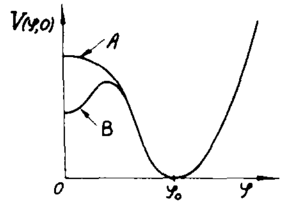

Fig. 1. (A) Typical shape of the effective potential $V(\varphi, 0)$ in theories with spontaneous symmetry breaking. (B) The effective potential $V(\varphi, 0)$ with two minima. Potentials of this type occur in gauge theories at sufficiently small mass of the Higgs meson φ.

$V(\varphi, T)$ has two minima: at $\varphi = 0$ and at $\varphi \neq 0$, see fig. 2. In thise case the phase transition starts at a temperature T_0, at which the value of $V(\varphi, T)$ in the minimum at $\varphi \neq 0$ becomes less than $V(0, T)$, i.e. when the state $\varphi = 0$ becomes metastable. However, at temperatures close to T_0 the phase transition proceeds very slowly. Therefore the phase transition actually occurs at lower temperatures, roughly speaking, when the time necessary for completing the phase transition becomes less than the age of the universe by this moment. Thus, to study kinetics of the first-order phase transitions and, in particular, to estimate the degree of supercooling of the phase $\varphi = 0$, which in some cases turns out to be very large, it is necessary to develop a theory of a decay of the metastable state $\varphi = 0$.

The phase transition from the phase $\varphi = 0$ to the phase $\varphi \neq 0$ proceeds by formation and a subsequent expansion of bubbles of a new, energetically favourable phase $\varphi \neq 0$ inside the phase $\varphi = 0$. Note that the potential energy of the bubble is proportional to $-r^3$, where r is the bubble radius, and the surface energy is proportional to $+r^2$. Therefore only bubbles of a sufficiently large radius are produced, i.e. the phase transition proceeds due to a barrier tunneling. A theory of bubble production at zero temperature in models where the potential $V(\varphi, 0)$ has two minima was first suggested by Kobsarev, Okun' and Voloshin [21], and then developed by Coleman and Callan [22]. An extension of this theory to the case of non-zero temperatures can be found in our papers [10].

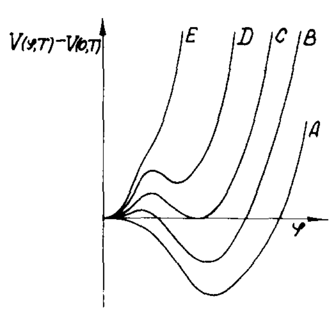

Fig. 2. The behaviour of the effective potential $V(\varphi, T)$ corresponding to first-order phase transition. (A) $0 < T < T_c$ (at the point T_c there appears the minimum $V(\varphi, T)$ at $\varphi = 0$); (B) $T_c < T < T_0$; (C) $T = T_0$; (D) $T_0 < T < T_{c_1}$ (at the point T_{c_1} the minimum at $\varphi \neq 0$ disappears); (E) $T > T_{c_1}$.

The present paper is devoted to a more detailed discussion of the theory of decay of a metastable phase (false vacuum, in the terminology of Coleman [22]) at a non-zero temperature and also to an investigation of some particular models, the analysis of which will considerably simplify the study of kinetics of phase transitions in various versions of grand unified theories.

The paper is organized as follows. In sect. 2 we discuss a general theory of the formation of bubbles of a new phase both at zero and at a non-zero temperature. In sect. 3 the so-called thin-wall approximation is developed, in the framework of which one can obtain a simple description of bubble formation for the cases when supercooling of the metastable phase is sufficiently small. In sect. 4 the bubble formation in a strongly supercooled phase is studied. The results obtained in sects. 3 and 4 will make it possible to analyze the bubble production in a wide class of theories without complicated computer calculations, like was done, for example, by Witten [11]. In sect. 5 we analyse the process of expansion of the bubbles produced. In the conclusion the main results of the work are briefly summarized.

In other papers of this series we analyse the kinetics of phase transitions in grand unified theories [23], the primordial monopole problem [24], the phase transition in the Coleman–Weinberg theory and the new inflationary universe scenario [9, 25, 41].

2. Bubble formation

Let us consider the theory of a scalar field with a lagrangian

$$L(\varphi)=\tfrac{1}{2}(\partial_\mu\varphi)^2-V(\varphi,0)\,. \tag{2.1}$$

Following ref. [22], we will discuss the process of barrier tunneling, that leads to the appearance of bubbles of a new phase with $\varphi\neq 0$, as a classical motion in imaginary time, i.e. in euclidean space. To calculate the probability of such a process in quantum field theory (i.e. at zero temperature T), one should first solve the euclidean equation of motion

$$\Box\varphi=\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2}+\Delta\varphi=\frac{\mathrm{d}V(\varphi,0)}{\mathrm{d}\varphi}\equiv V'(\varphi,0) \tag{2.2}$$

with the boundary condition $\varphi\to 0$ at $\boldsymbol{x}^2+t^2\to\infty$. If we now normalize the quantity $V(\varphi)$ so that $V(0)=0$ (i.e. redefine $V(\varphi)\to V(\varphi)-V(0)$), the probability of tunneling per unit time per unit volume will be given by

$$\Gamma=A\,\mathrm{e}^{-S_4(\varphi)}\,, \tag{2.3}$$

where $S_4(\varphi)$ is the euclidean action corresponding to the solution φ of eq. (2.2),

$$S_4(\varphi)=\int \mathrm{d}^4x\left[\frac{1}{2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2+\frac{1}{2}(\nabla\varphi)^2+V(\varphi,0)\right], \tag{2.4}$$

and the pre-exponential factor A is given by

$$A = \left(\frac{S_4(\varphi)}{2\pi}\right)^2 \left(\frac{\det'[-\Box + V''(\varphi)]}{\det[-\Box + V''(0)]}\right)^{-1/2} . \tag{2.5}$$

Here det′ implies that the zero eigenvalues of the operator $-\Box + V''(\varphi)$ are to be omitted when computing the determinant. This operator has 4 zero modes, which correspond to the possibility of translation of the solution $\varphi(x)$ along any of the 4 axes in euclidean space. The product of the contributions $(S_4(\varphi)/2\pi)^{1/2}$ from each of the zero modes gives the factor $(S_4(\varphi)/2\pi)^2$ in (2.5).

The derivation of eqs. (2.4), (2.5) is presented in ref. [22] and is based on the calculation of the imaginary part of the potential $V(\varphi)$ in the false (metastable) vacuum $\varphi = 0$. Before passing over to the extension of these results to the case $T \neq 0$, we would like to make some remarks.

First of all one should note that for obtaining a complete answer for Γ, it is necessary to perform a summation of contributions to Γ from all possible solutions $\varphi(x)$ of eq. (2.2). Fortunately, however, it is sufficient in most cases to restrict ourselves to the simplest O(4)-symmetric solution $\varphi(\boldsymbol{x}^2 + t^2)$, since it is this solution that provides the minimum of action $S_4(\varphi)$ [23]. In this case eq. (2.2) takes a somewhat simpler form

$$\frac{d^2\varphi}{dr^2} + \frac{3}{r}\frac{d\varphi}{dr} = V'(\varphi, 0) , \tag{2.6}$$

where $r = \sqrt{\boldsymbol{x}^2 + t^2}$ with the boundary conditions $\varphi \to 0$ at $r \to \infty$, $d\varphi/dr = 0$ at $r = 0$.

Unfortunately, usually it is impossible to solve analytically even this equation, and one can obtain the solution and the corresponding value of $S_4(\varphi)$ only by means of computer calculations. The calculation of determinants is an even more complicated problem. However, usually it is sufficient to have a rough estimate of the pre-exponential factor A. Such an estimate can be obtained if one takes into account that the factor A has the dimension m^4, and its value is determined by three different dimensional quantities: $\varphi(0)$, $\sqrt{V''(\varphi)}$ and r, where r is the characteristic bubble size. In most of the theories to be investigated all these quantities differ by no greater than an order of magnitude, so that to have a rough estimate one may assume that

$$\frac{\det'[-\Box + V''(\varphi)]}{\det[-\Box + V(0)]} = O(r^{-4}, \varphi^4(0), (V''(\varphi))^2) , \tag{2.7}$$

where the quantities r and $V''(\varphi)$ should be understood as some typical mean values of these parameters for the solution $\varphi(r)$ of eq. (2.6).

Now let us consider the case $T \neq 0$ which is most important for us. In order to extend the above-mentioned results to this case [10], it is sufficient to remember that quantum statistics of bosons (fermions) at $T \neq 0$ is formally equivalent to quantum field theory in the euclidean space–time, periodic (anti-periodic) in the

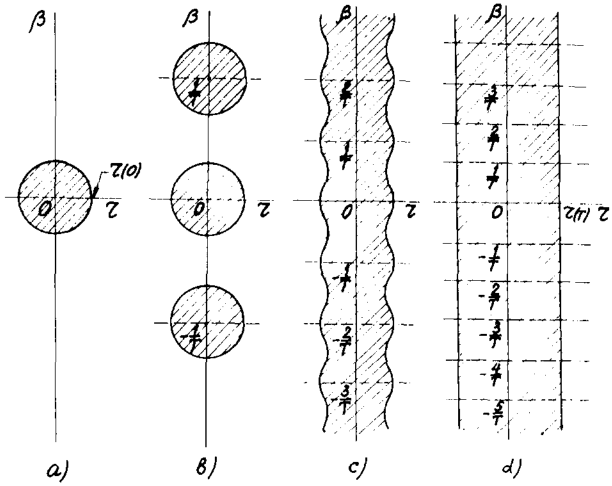

Fig. 3. Solution of (2.2) at different values of temperature. (a) $T=0$; (b) $T \ll r^{-1}(0)$; (c) $T \sim r^{-1}(0)$; (d) $T \gg r^{-1}(0)$. The dashed regions contain the classical field $\varphi \neq 0$. For simplicity we have shown the bubbles for the case, when their wall thickness is less than the bubble radius.

"time" direction β with period T^{-1} [27]. As has been mentioned in the introduction, at a finite temperature one should use the effective potential $V(\varphi, T)$ instead of the zero-temperature potential $V(\varphi) = V(\varphi, 0)$, see refs. [3–5]. Calculation of the imaginary part of $V(\varphi, T)$ in an unstable vacuum can be performed in complete analogy with what has been done in ref. [22] for the case $T=0$. In fact, the only modification is that instead of the O(4)-symmetric solution of eq. (2.2) one should look for the O(3)-symmetric (with respect to spatial coordinates) solution, periodic in the "time" direction β with period T^{-1}. At $T=0$ the solution of eq. (2.2) corresponding to the minimal value of the action $S_4(\varphi)$ is the O(4)-symmetric bubble with a certain typical radius $r(0)$, fig. 3a. At $T \ll r_{(0)}^{-1}$ the solution will be a series of such bubbles placed at a distance T^{-1} from one another in the "time" direction β, fig. 3b. At $T \sim r^{-1}(0)$ the bubbles become overlapping, fig. 3c. At $T > r^{-1}(0)$ (and this case is of most interest and importance for us) the solution is a cylinder, whose spatial cross section is the O(3)-symmetric bubble of some new radius $r(T)$, fig. 3d. In this case, in the calculation of the action $S_4(\varphi)$ the integration over β is reduced simply to multiplication by T^{-1}, i.e. $S_4(\varphi) = T^{-1} S_3(\varphi)$, where $S_3(\varphi)$ is a three-dimensional action corresponding to the O(3)-symmetric bubble [10]:

$$S_3(\varphi) = \int \mathrm{d}^3x\, \mathcal{V}(\varphi(x), T) = \int \mathrm{d}^3x\, [\tfrac{1}{2}(\nabla\varphi)^3 + V(\varphi, T)]\,. \tag{2.8}$$

To calculate $S_3(\varphi)$ it is necessary to solve the equation

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\varphi}{\mathrm{d}r} = V'(\varphi, T)\,, \tag{2.9}$$

with the boundary conditions $\varphi \to 0$ at $r \to \infty$, $\mathrm{d}\varphi/\mathrm{d}r = 0$ at $r = 0$, where $r^2 = x^2$. A complete expression for the probability of tunneling per unit time per unit volume in the high-temperature limit ($T \gg r_{(0)}^{-1}$) is obtained in the way completely analogous to the one used in ref. [22] for the derivation of relations (2.4), (2.5) and is as follows [10]:

$$\Gamma(T) = T\left(\frac{S_3(\varphi, T)}{2\pi T}\right)^{3/2} \left(\frac{\det'[-\Delta + V''(\varphi, T)]}{\det[-\Delta + V''(0, T)]}\right)^{-1/2} \mathrm{e}^{-S_3(\varphi, T)/T}. \quad (2.10)$$

Here, as before, det′ implies that three zero eigenvalues of the operator $-\Delta + V''(0, T)$ are to be omitted when computing the determinant. The contribution of three zero modes of this operator, which correspond to the translations of the solution $\varphi(\boldsymbol{x})$ in three spatial directions, give the factor $(S_3(\varphi, T)/2\pi T)^{3/2}$ in (2.10), and the factor T arises when the periodicity with period T^{-1} of euclidean space in the "time" direction β is taken into account.

Eqs. (2.4), (2.5) and particularly (2.10) will be the basis of our further investigations, and therefore they should be discussed thoroughly.

(1) In the derivation of eq. (2.10) the methods of equilibrium quantum statistics were used. In particular, it was implicitly assumed that the mean free path $l(T)$ of particles in the hot dense gas with temperature T is much less than the bubble radius $r(T)$, and that there is enough time for thermal equilibrium between the matter inside and outside the bubble to be established. Unfortunately, in many interesting cases these conditions do not hold. For example, the typical value of the mean free path of particles in grand unified theories at temperature T varies from $\mathrm{O}(10^{-1})T^{-1}$ to $\mathrm{O}(10)T^{-1}$, whereas the typical value of $r(T)$ at the point of the phase transition varies from T^{-1} to $\mathrm{O}(10)T^{-1}$ [23]. In the case $r(T) \lesssim l(T)$ eq. (2.10) is not directly applicable, but may still be useful. Indeed a preliminary analysis of small bubble formation with $r(T) \lesssim l(T)$ indicates that in this case the probability of the bubble formation is smaller than $\Gamma(T)$ (2.10). Therefore (2.10) may serve as an upper limit for the rate of small bubble formation. We hope to return to the problem of small bubble formation in a separate publication, and in the present paper we shall restrict ourselves to the investigation of the case $l(T) \ll r(T)$.

(2) The criterion $T \gg r^{-1}(0)$ of the applicability of expression (2.10) for $\Gamma(T)$ is not quite exact. First of all, notations $r(0)$ and $r(T)$ have an exact meaning only if the bubble wall thickness Δr is much less than the bubble radius. This condition is far from always being fulfilled; see, for example, fig. 7 in sect. 4. Second, cylindric O(3)-symmetric (with respect to $\boldsymbol{x}$) solutions exist, as a rule, not only in the limit $T \gg r^{-1}(0)$, but at all temperatures T. The only problem is to find such a solution among many possible solutions $\varphi(\boldsymbol{x}, \beta)$ of eq. (2.2), which corresponds to the minimum of action $S_4(\varphi, T)$. A more exact criterion of the applicability of eq. (2.10) for $\Gamma(T)$ is the following.

At $T \gg r^{-1}(0)$ one should use eq. (2.10) for $\Gamma(T)$. At $T \lesssim r^{-1}(0)$ one should compare the action S_3/T for the O(3)-symmetric solution and the action $S_4(\varphi)$ for the O(4)-symmetric solution for the theory with effective potential $V(\varphi, T)$. If $S_3(\varphi)/T < S_4(\varphi)$, one should use eq. (2.10) for $\Gamma(T)$. If $S_3(\varphi)/T > S_4(\varphi)$ and the radius of the O(4)-symmetric bubble is less than $(2T)^{-1}$, then one should use eqs. (2.4), (2.5). We will return to the discussion of this point in sect. 4.

(3) Now let us try to understand the physical meaning of the quantity $S_3(\varphi, T)$. To this end we will take into account the fact that the euclidean action for a time-independent field $\varphi(r)$ is equal to the three-dimensional integral of the effective potential $\mathcal{V}(\varphi(r), T)$, which includes both the "surface" energy of the bubble, proportional to $(\nabla\varphi)^2$, and the "volume" energy $V(\varphi, T)$, see (2.8). If we now take into account that our equations include, in fact, not $\mathcal{V}(\varphi(r), T)$ but $\mathcal{V}(\varphi(r), T) - \mathcal{V}(0, T)$ [we have normalized the effective potential $V(\varphi, T)$ to be zero at $\varphi = 0$], and also that on the time-independent solutions $\varphi(r)$ the potential $\mathcal{V}(\varphi(r), T)$ is equal to the free energy $F(\varphi(r), T)$ (see the introduction), it becomes clear that the factor $e^{-S_3/T}$ in eq. (2.10) is simply the known factor $e^{-\Delta F/T}$ in the theory of boiling, where ΔF is the change of the free energy of matter due to bubble formation [28]. Thus, the formalism developed here makes it possible to describe in a unique way the formation of bubbles both due to quantum fluctuations (2.4), (2.5) and due to thermodynamic fluctuations (2.10). This fact will become particularly clear in the limiting case, when the bubble wall thickness is much less than the bubble radius. As will be seen later on (see sect. 3), the expression $e^{-S_3/T}$ in this limit coincides exactly with the well-known expression for the bubble production probability, entering textbooks on statistical mechanics [28].

(4) It is usually extremely difficult to compute the determinants in (2.10), just as in the case $T = 0$. However, the dimensional estimates may prove useful here too. The expression $(\det'[-\Delta + V''(\varphi,T)]/\det[-\Delta + V''(0,T)])^{-1/2}$ at $T = 0$ has the dimension m^3 (which corresponds to the three zero modes of the operator $-\Delta + V''(\varphi,T)$). Therefore, this expression may be of the order of φ^3, $(V''(\varphi,T))^{3/2}$, $r^{-3}(T)$ or T^3. Typically, the quantities φ, $(V'')^{1/2}$ and $r^{-1}(T)$ turn out to be of the same order of magnitude, and then in the most interesting case $T \gtrsim r^{-1}(T)$ (fig. 3) the expression $(\det'[-\Delta + V''(\varphi,T)]/\det[-\Delta + V''(0,T)])^{-1/2}$ is expected to be of the order of T^3, i.e.

$$\Gamma(T) \sim T^4 \left(\frac{S_3(\varphi, T)}{2\pi T}\right)^{3/2} e^{-S_3(\varphi,T)/T}. \tag{2.11}$$

(5) It should be noted that the computation and renormalization of the determinants requires some care in our case. The point is that the phase transitions occur at finite temperature due to the fact that the form of $V(\varphi, T)$ is temperature-dependent when quantum corrections are taken into account. In particular, the quantity $S_3(\varphi, T)$ in (2.10) is not a classical action, but is an effective action that takes into account quantum effects in some approximation. Therefore to derive eq. (2.10) one should use not the usual quasiclassical approximation, but some more elaborate approximation, which takes into account quantum (temperature)

corrections in a self-consistent way. Such an approximation, which in a self-consistent way takes into account leading T corrections to masses in the propagators of particles, was developed in [4], see also [29]. This approximation makes possible a kind of effectively quasiclassical treatment of bubble formation, and is well suited for the derivation of eq. (2.10). However, renormalization of the quantities calculated in this approximation is not quite trivial [4]. We will not dwell on this question here, referring the reader to papers [4] and [29], since we will not actually compute the determinants in (2.10) in the present paper, and the results of computation of the effective potential by this method coincide with the corresponding results of the one-loop approximation up to higher order corrections in the coupling constants [4, 29].

An important question, which arises nevertheless in this connection, is that once we have started modifying the equations of motion using quantum corrections and have used the effective action instead of the classical one, we should also modify those terms of eq. (2.2), which contain derivatives of the field φ with respect to x and t, and not the effective potential alone. Fortunately, in the lowest order of the self-consistent approximation mentioned above [4], the terms with derivatives are not modified at all, and it is only the potential $V(\varphi, T)$ that is temperature dependent. This is connected with the fact that the leading corrections in T to the polarization operator do not depend on the momentum of the incoming particle [4]. As a result, the corrections to the kinetic terms are usually inessential and have little effect upon the results of the calculation of the quantity $\Gamma(T)$ (2.10)*, although some exceptions from this rule are possible.

As is seen from (2.10), (2.11), the main problem to be solved in the calculation of the probability of bubble production is a computation of the value of $S_3(\varphi, T)$ [or $S_4(\varphi)$]. Furthermore, to obtain a reasonable estimate for the determinants and also to study the kinetics of bubble expansion, one should know the form of the function $\varphi(r)$ and the typical radius of the bubble. Appropriate results are usually obtained by means of computer calculations, which seriously complicates the investigation. Therefore it is very instructive and useful to study those rare cases in which all the necessary results can be obtained analytically. One such example is considered in the next section. In what follows we will analyse not only the case $T \gg r^{-1}(0)$, which is of primary interest for us, but also the case $T = 0$, since it gives us information on the probability of the phase transition from the strongly supercooled metastable phase, when $T \ll r^{-1}(0)$.

(6) Euclidean solutions of eq. (2.2) in the theory of bubble production play a role similar to that of instantons in Yang–Mills theory. Just as in Yang–Mills theory, for our semiclassical approximation to be reliable the action $S_4(\varphi, T)$ corresponding

* The opposite statement contained in ref. [30] is based on the analysis of the infrared-divergent theory with $V''(0,T) = 0$. This is just the reason of the pathologic behaviour of $S_3(\varphi, T)$ noticed in ref. [30].

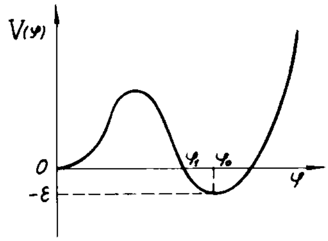

Fig. 4. The effective potential in the case of weak supercooling of the phase $\varphi = 0$ [the quantity $\varepsilon = V(0, T) - V(\varphi_0, T)$ is small].

to the above-mentioned solutions should be large, $S_4(\varphi, T) \gg 1$. In what follows we assume this condition to be satisfied.

3. Thin wall approximation

In the theory of bubble formation there exist two limiting cases, when the solution of the problem is essentially simplified. One of them is realized in the situation when the difference between the values of $V(\varphi, T)$ at its minima at $\varphi = 0$ and at $\varphi_0(T) \neq 0$ is much greater than the height of the barrier $V(\varphi)$ between $\varphi = 0$ and $\varphi = \varphi_0(T)$. This case is discussed in the next section. Here we shall consider another limiting case, when $|V(\varphi_0(T))| = \varepsilon$ is much smaller than the height of the barrier $V(\varphi)$, see fig. 4. It is easily understood that as ε decreases, the gain in the volume energy due to bubble production $\sim \varepsilon r^3$ becomes sufficiently large as compared with the surface energy $\sim r^2$ only at very large values of r. When the size of the bubble becomes much greater than the wall thickness (the wall is the region where $d\varphi/dr$ is large), it is possible to neglect the second term in (2.6), (2.9) as compared with the first term, i.e. these equations reduce to the equation which describes tunneling in one-dimensional space:

$$\frac{d^2\varphi}{dr^2} = V'(\varphi, T) . \tag{3.1}$$

The solution of this equation in the limit $\varepsilon \to 0$ is

$$r = \int_{\varphi}^{\varphi_0} \frac{d\varphi}{\sqrt{V(\varphi, T)}} ; \tag{3.2}$$

the form of the curve $\varphi(r)$ is shown in fig. 5. Let us first consider tunneling in quantum field theory ($T = 0$). In this case the action S_4 on the O(4)-symmetric bubble is equal to

$$\begin{aligned} S_4 &= 2\pi^2 \int_0^\infty r^3 \, dr \left[\frac{1}{2} \left(\frac{d\varphi}{dr} \right)^2 + V(\varphi) \right] \\ &= -\tfrac{1}{2}\varepsilon \pi^2 r^4 + 2\pi^2 r^3 S_1 , \end{aligned} \tag{3.3}$$

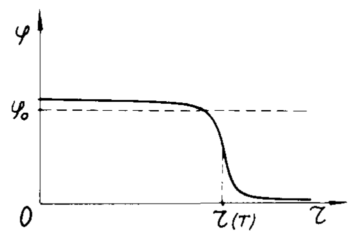

Fig. 5. Typical form of the solution of eqs. (2.6), (2.9) at $\varepsilon \to 0$.

where S_1 is the bubble wall surface energy (surface tension), equal to the action, corresponding to the one-dimensional theory (3.1),

$$S_1 = \int_0^\infty \mathrm{d}r \left[\frac{1}{2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}r}\right)^2 + V(\varphi)\right] = \int_0^{\varphi_0} \mathrm{d}\varphi\, \sqrt{2V(\varphi)}\,, \tag{3.4}$$

and the integral in (3.4) should be calculated in the limit $\varepsilon \to 0$.

The bubble radius $r(0)$ is calculated by minimization of S_4 (3.3) with respect to r:

$$r(0) = \frac{3S_1}{\varepsilon}\,, \tag{3.5}$$

whence it follows that

$$S_4 = \frac{27\pi^2 S_1^4}{2\varepsilon^2}\,. \tag{3.6}$$

Note that the order of magnitude of the bubble wall thickness is equal simply to $(V''(0))^{-1/2}$. Therefore, taking account of (3.5), the condition of applicability of this approximation, called the thin wall approximation, is as follows:

$$\frac{3S_1}{\varepsilon} \gg (V''(0))^{-1/2}\,. \tag{3.7}$$

The results presented above were obtained by Coleman [22].

These results can be easily extended to the case $T \gg r^{-1}(0)$. To this end it is sufficient to take into account that

$$S_3 = 4\pi \int_0^\infty r^2\, \mathrm{d}r \left[\frac{1}{2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}r}\right)^2 + V(\varphi,T)\right]$$
$$= -\tfrac{4}{3}\pi r^3 \varepsilon + 4\pi r^2 S_1(T)\,, \tag{3.8}$$

whence

$$r(T) = \frac{2S_1(T)}{\varepsilon}\,, \tag{3.9}$$

$$S_3 = \frac{16\pi S_1^3(T)}{3\varepsilon^2}. \tag{3.10}$$

The expression for the rate of the bubble formation thus obtained,

$$\Gamma(T) \sim e^{-16\pi S_1^3/3\varepsilon^2 T}, \tag{3.11}$$

coincides with the well-known expression from textbooks on statistical mechanics [28]. The only (but a very important) difference is that we can *calculate* the surface tension S_1 (3.4), which enters eq. (3.11) [$V(\varphi)$ in (3.4) should be understood as $V(\varphi, T)$].

Now let us take into account that in many interesting cases the function $V(\varphi, T)$, presented in fig. 4, can be approximated by the expression

$$V(\varphi) = \tfrac{1}{2}M^2(T)\varphi^2 - \tfrac{1}{3}\delta(T)\varphi^3 + \tfrac{1}{4}\lambda\varphi^4. \tag{3.12}$$

Let us investigate bubble formation in this theory in more detail, since for the potential (3.12) the integral in (3.4) can be taken exactly, and thus it becomes possible to obtain an analytic expression for S_1, S_3, S_4 and $r(T)$.

Indeed, it is readily seen that at the values of the parameters M, δ and λ, at which the depths of the minima at $\varphi = 0$ and at $\varphi = \varphi_0(T)$ become equal to each other ($\varepsilon \to 0$), eq. (3.12) transforms into

$$V(\varphi) = \tfrac{1}{4}\lambda\varphi^2(\varphi_0 - \varphi)^2, \tag{3.13}$$

the value of φ_0 in this case being equal to

$$\varphi_0 = \frac{2\delta}{3\lambda}, \tag{3.14}$$

and the parameters M, λ and δ are related as follows:

$$2\delta^2 = gM^2\lambda. \tag{3.15}$$

From (3.4), (3.12) and (3.14) it follows that

$$S_1 = \frac{\sqrt{\lambda}\varphi_0^3}{6\sqrt{2}} = \frac{2\sqrt{2}\delta^3}{3^4\lambda^{5/2}}, \tag{3.16}$$

whence for the case $T = 0$ we obtain

$$S_4 = \frac{\pi^2 2^5 \delta^{12}}{3^{13}\lambda^{10}\varepsilon^2}, \qquad r(0) = \frac{2^{3/2}\delta^3}{3^3\lambda^{5/2}\varepsilon}, \tag{3.17}$$

and for the case $T \gg r^{-1}(0)$

$$S_3 = \frac{\pi\lambda^{3/2}\varphi_0^9}{2^{1/2}3^4\varepsilon^2} = \frac{2^{17/2}\pi\delta^9}{3^{13}\lambda^{15/2}\varepsilon^2}, \qquad r(T) = \frac{2^{5/2}\delta^3}{3^4\lambda^{5/2}\varepsilon}. \tag{3.18}$$

Now we will turn concretely to the study of phase transitions in gauge theories at high temperatures. In this case the typical expression for $V(\varphi, T)$ is

$$V(\varphi, T)=\tfrac{1}{2}\gamma(T^2-T_c^2)\varphi^2-\tfrac{1}{3}\alpha T\varphi^3+\tfrac{1}{4}\lambda\varphi^4\,, \qquad (3.19)$$

where T_c is the temperature above which the symmetric phase $\varphi=0$ is metastable, γ and α are some numerical coefficients [3–5]. The temperature T_0, at which the quantities $V(\varphi, T)$ for the phases with $\varphi=0$ and $\varphi=\varphi_0(T)$ become equal to each other, is determined by the relation

$$T_0^2\left(1-\frac{2\alpha^2}{9\gamma\lambda}\right)=T_c^2\,. \qquad (3.20)$$

It is also easy to determine the quantity ε as a function of deviation of the temperature T from T_0:

$$\varepsilon=\frac{4T_0T_c^2\alpha^2\gamma}{9\lambda^2}\Delta T, \qquad (3.21)$$

where $\Delta T=T_0-T$. With the aid of relations (3.14)–(3.21) one can easily obtain the expressions for the quantities S_1, S_3 and $r(T)$, which we are interested in. We will write the corresponding expressions for the most important case $(T_0-T_c)/T_c\ll 1$:

$$S_1=\frac{2^{3/2}\alpha^3}{3^4\lambda^{5/2}}T_c^3\,, \qquad (3.22)$$

$$\frac{S_3}{T}=\frac{2^{9/2}\pi\alpha^5}{3^9\gamma^2\lambda^{7/2}}\frac{1}{x^2}\,, \qquad (3.23)$$

$$r(T)=\sqrt{\frac{2}{\lambda}}\,\frac{\alpha}{9\gamma T_c x}\,, \qquad (3.24)$$

where $x=\Delta T/T_c\ll 1$.

Thus, the thin wall approximation makes it possible to obtain analytical expressions for S_1, S_3, S_4 and $r(T)$ in a very important class of theories (3.12), (3.19). For the example of the theories (3.12), (3.19) we would like to specify, however, the question of the applicability limits of this method. For this case the integral in (3.2) can be calculated exactly, and one can easily find the bubble wall thickness Δr, which turns out to be equal to

$$\Delta r\approx\frac{2}{\varphi_0}\sqrt{\frac{2}{\lambda}}\approx\frac{3\sqrt{\lambda}}{\sqrt{2}\alpha T_c}\,. \qquad (3.25)$$

Now let us write the conditions of applicability of the thin wall approximation in the form

$$r>N\Delta r, \qquad N\gg 1\,. \qquad (3.26)$$

In this case from (3.23), (3.24) it follows that

$$x < \frac{\alpha^2}{3^3 N\gamma\lambda}, \tag{3.27}$$

$$\frac{S_3}{T} > N^2 \frac{2^{9/2}\pi\alpha}{3^3\lambda^{3/2}}. \tag{3.28}$$

From this already for $N = 2$ it follows that

$$x = \frac{\Delta T}{T_c} < 2\cdot 10^{-2}\frac{\alpha^2}{\gamma\lambda}, \tag{3.27'}$$

$$\frac{S_3}{T} > 10\frac{\alpha}{\lambda^{3/2}}. \tag{3.28'}$$

Another series of inequalities can be obtained from the condition $T \gg r^{-1}(T)$:

$$x \ll \sqrt{\frac{2}{\lambda}}\frac{\alpha}{9\gamma}, \tag{3.29}$$

$$\frac{S_3}{T} \gg \frac{2^{7/2}\pi\alpha^3}{3^5\lambda^{5/2}} \tag{3.30}$$

[as a rule, these constraints are less restrictive than the constraints (3.27′), (3.28′)]. From the constraints (3.27)–(3.30) it follows that the thin wall approximation is applicable only to processes proceeding with a comparatively small supercooling x, and the probability of such processes must be sufficiently strongly suppressed by the factor $e^{-S_3/T}$. These conditions are consistent with each other only for phase transitions proceeding at a sufficiently low rate. In many cases the rate of the phase transitions in realistic theories of weak, strong and electromagnetic interactions in the domain of validity of the thin wall approximation turns out to be much less than the rate of expansion of the universe, and as a result, the phase transition in these theories usually starts only at those values of the parameters at which the thin wall approximation becomes inapplicable (though it is not a general rule). Nevertheless, the method itself and its results sometimes turn out to be useful, and, besides, they may play an important heuristic role for the development of the theory of the metastable phase decay, see, for example, ref. [32].

4. Outside the limits of the thin wall approximation

As has already been mentioned, there exists another case when the theory of bubble formation is greatly simplified. Namely, if the depth of the minimum $V(\varphi)$ at the point φ_0 is sufficiently large, the maximum value of the field $\varphi(r)$ on solution of eqs. (2.6), (2.9) usually becomes of the order of φ_1, where $V(\varphi_1) = V(0)$, $\varphi_1 \ll \varphi_0$. In this case, when solving eqs. (2.6), (2.9), one can disregard the details of the

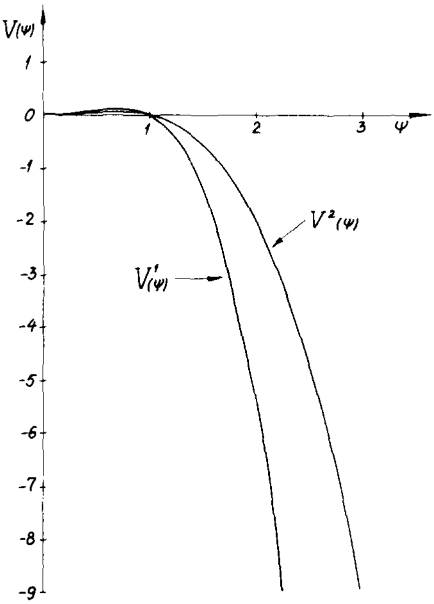

Fig. 6. Potentials $V^1(\psi)$ (4.9) and $V^2(\psi)$ (4.10).

behaviour of $V(\varphi)$ at $\varphi \gg \varphi_1$, and at $\varphi \lesssim \varphi_1$ it is often possible to approximate the potential $V(\varphi)$ by a function of the two main types, see fig. 6,

$$V^1(\varphi) = \tfrac{1}{2}M^2\varphi^2 - \tfrac{1}{4}\lambda\varphi^4 , \tag{4.1}$$

$$V^2(\varphi) = \tfrac{1}{2}M^2\varphi^2 - \tfrac{1}{3}\delta\varphi^3 \tag{4.2}$$

[compare the latter expression with (3.12)].

At zero temperature and $M = 0$, eq. (2.6) for the theory (4.1) is solved exactly [33]:

$$\varphi(r) = \sqrt{\frac{8}{\lambda}}\frac{\rho}{r^2+\rho^2}, \tag{4.3}$$

where ρ is an arbitrary parameter of dimension of length (arbitrariness in the choice of the parameter ρ is a consequence of the absence of any mass parameter in the theory (4.1) at $M = 0$). The action on the solutions (4.3) at all ρ is equal to

$$S_4 = \frac{8\pi^2}{3\lambda}. \tag{4.4}$$

The solution (4.3) is widely used in the theory of Yang–Mills instantons [34], and also in the analysis of higher orders of perturbation theory [35].

To find the total probability of bubble formation, one should integrate, with a certain weight, the contributions from the solutions (scalar instantons) with all values of ρ, as is done in the theory of Yang–Mills instantons [36].

At $T=0$ and arbitrary $M \neq 0$ eq. (2.6) in the theory (4.1) has no exact solutions of the instanton type [37] for the same reason for which instantons are absent from the theory of massive Yang–Mills fields. However, the solution (4.3) at $\rho \ll M^{-1}$ is practically not affected by the presence of the mass M in the theory (4.1). Thus, in the theory (4.1) at $T=0$, $M \neq 0$ there exist "almost exact solutions" of eq. (2.6), which coincide with (4.3) in the limit $\rho \to 0$. This means that there exists the whole class of trajectories (4.3) in euclidean space that lead to the formation of bubbles of the field φ. The action on these trajectories in the theory (4.1) almost coincides with $8\pi^2/3\lambda$ at $\rho^{-1} \ll M$, and becomes equal to $8\pi^2/\lambda$ in the limit $\rho \to 0$. As a result, tunneling in the theory (4.1) at $T=0$ does exist, and to describe this tunneling one should integrate the contributions to $\Gamma(0)$ from all the "solutions" (4.3) with the action (4.4) at $\rho^{-1} \ll M$, as is done in the theory of Yang–Mills instantons, when the Yang–Mills fields acquire mass [36, 38]. It is in this not quite exact sense, that we shall speak later about the solutions of eq. (2.6) in the theory (4.1) at $M \neq 0$. Note also that under an arbitrarily small modification of the theory (4.1) (for example, when $\lambda\varphi^4$ is replaced by $\lambda\varphi^4 \ln(\varphi/\varphi_0)$) there may appear exact solutions of eq. (2.6) with the action approximately equal to $8\pi^2/3\lambda$ (4.4).

In other cases of interest for us [in the theory (4.1) at a high temperature and in the theory (4.2) at high and low temperatures] there exist exact solutions of the corresponding equations. One has not succeeded in obtaining these solutions analytically, but in the above-mentioned cases computer calculations prove to be very informative. Indeed, after the change of variables $r = M^{-1}R$ and also

$$\varphi = \sqrt{\frac{2}{\lambda}}\, M\psi \tag{4.5}$$

for the theory (4.1) and

$$\varphi = \frac{3}{2}\frac{M^2}{\delta}\psi \tag{4.6}$$

for the theory (4.2), eq. (2.2) for these theories is reduced, respectively, to the equations

$$\Box\psi = \psi - 2\psi^3, \tag{4.7}$$

$$\Box\psi = \psi - \tfrac{3}{2}\psi^2, \tag{4.8}$$

which correspond to the potentials

$$V^1(\psi) = \tfrac{1}{2}\psi^2(1-\psi^2), \tag{4.9}$$

$$V^2(\psi) = \tfrac{1}{2}\psi^2(1-\psi), \tag{4.10}$$

see fig. 6 [the derivatives in (4.7), (4.8) should be taken not with respect to r, but with respect to $R = Mr$]. Note that the transformations (4.5), (4.6) are so chosen

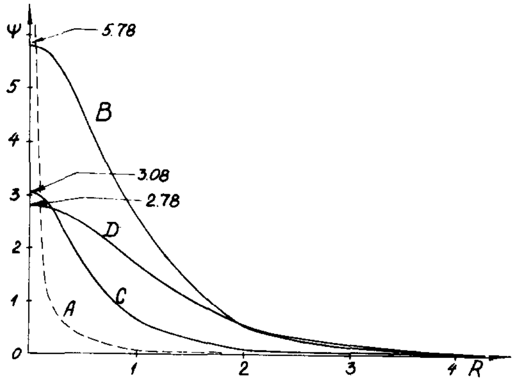

Fig. 7.

that the point φ_1, at which the potential $V(\varphi)$ vanishes, becomes $\psi=1$ for both theories.

The meaning of these manipulations is that now, having solved eqs. (4.7) and (4.8) by means of computer calculations (fig. 7), we can obtain the solution of eqs. (2.6), (2.9) for arbitrary M, λ, and δ just by the field rescaling (4.5), (4.6). The quantities S_4 at $T=0$ and S_3 at $T \gg r^{-1}(0)$ in the theories (4.1), (4.2) are calculated in a similar way (the value of S_3 in the theory (4.1) was first calculated in refs. [39, 11]). Namely,

$$S_4^1(\varphi)=\frac{2}{\lambda}S_4^1(\psi)\,, \qquad S_4^1(\psi)\approx\tfrac{4}{3}\pi^2\,, \tag{4.11}$$

$$S_4^2(\psi)=\frac{9}{4}\frac{M^2}{\delta^2}S_4^2(\psi)\,, \qquad S_4^2(\psi)\approx 91\,, \tag{4.12}$$

$$\frac{S_3^1(\varphi)}{T}=\frac{2M}{\lambda T}S_3^1(\psi)\,, \qquad S_3^1(\psi)\approx 9.5\,, \tag{4.13}$$

$$\frac{S_3^2(\varphi)}{T}=\frac{9M^3}{4\delta^2 T}S_3^2(\psi)\,, \qquad S_3^2(\psi)\approx 19.4\,. \tag{4.14}$$

Here $S_4^i(\psi)$, $S_3^i(\psi)$ are the quantities S_4 and S_3 in the theories (4.9), (4.10), respectively.

Now let us discuss the most important case, the solutions of eqs. (4.7), (4.8) at superhigh temperatures, fig. 7, curves C and D.

An important feature of all these solutions is the fact that both the maximum field amplitude $\psi(0)\approx 3$ and the typical bubble radius, at which $\psi(R)$ decreases strongly as compared with $\psi(0)$, turn out to be of the order of unity. This leads to some important consequences to be discussed below.

(1) In order to make an estimate of the probability of bubble formation in the theory with some potential $V(\varphi)$ using eqs. (4.13), (4.14), it is sufficient that $V(\varphi)$ coincide to a sufficient accuracy with one of the potentials (4.1), (4.2) at $\varphi \lesssim 3\varphi_1$.

(2) The behaviour of $\varphi(r)$ does not depend on the behaviour of $V(\varphi)$ at $\varphi \gtrsim 3\varphi_1$. Therefore, the behaviour of the determinants in (2.10) depends only on the behaviour of $\varphi(r)$ and $V(\varphi)$ at $\varphi \lesssim 3\varphi_1$, but not, say, on the value of φ_0 at the minimum of $V(\varphi)$. Since the typical value of r for these solutions is of the order of M^{-1}, and T in (2.10) is assumed to be much larger than r^{-1}, in this case the maximum quantity of dimension or mass, which the determinants in (2.10) may depend on, is temperature, and we are thus again led to the estimate (2.11) for the probability of bubble production $\Gamma(T)$. In most cases the quantities $r^{-1}(T)$, $\varphi(0)$, $\sqrt{V''(0)}$ differ from T by no more than an order of magnitude, since at superhigh temperatures $V''(0) = M^2(T) \sim aT^2$, where a is some combination of coupling constants in the theory, the typical value of which lies in the interval from 10^{-1} to 1. Therefore, irrespective of a concrete dependence of the determinants on T, $r^{-1}(T)$, $\varphi(0)$ etc., the estimate (2.11) for $\Gamma(T)$ is expected to be valid at least with an accuracy of 2–3 orders of magnitude. Such an accuracy is usually quite sufficient.

We should emphasize that the temperature T in eq. (2.11) is the temperature at the moment of bubble formation, but not, say, the temperature T_0 at which $V(0, T_0) = V(\varphi_0(T_0), T_0)$. The disregard of this circumstance has led, for example, the authors of ref. [14] to an overestimation of the probability of bubble formation in the Coleman–Weinberg model [31] by more than 20 orders of magnitude.

(3) The main difference of curves C and D from those obtained in the thin wall approximation consists in the fact that the value of $\psi(R)$ for these solutions can be much less than the equilibrium value of ψ_0, corresponding to the minimum of the potential $V(\psi)$. This circumstance will be of ultimate importance for the analysis of phase transitions in the Coleman–Weinberg theory [9, 25].

(4) The potentials of the type (4.1) often occur in the investigation of the problems, in which supercooling at a phase transition point is extremely large, see, for example, refs. [11, 12, 15]. The potentials of the type (4.2) occur in the cases when supercooling is not too large but the thin wall approximation is still inapplicable [23]. In a general case the potential $V(\varphi)$ coincides neither with (4.1) nor with (4.2). The information obtained above is, however, sufficient to make a rough estimate of the bubble formation rate for a wide class of potentials $V(\varphi)$, which can be approximated by functions (4.1) or (4.2) at $\varphi \lesssim 3\varphi_1$. To estimate the accuracy of the results obtained by this method let us see what would happen if we tried to approximate the potential $V^1(\psi)$ (4.9) by the potential $V^2(\psi)$ (4.10). In spite of the fact that in the region $\psi \lesssim 3$ these functions differ considerably, see fig. 6, the value of the action (4.13) and (4.14) differ only by a factor of 2.

To obtain a better approximation one should bear in mind that the main contribution to $S_3(\psi)$ for curves C and D, fig. 7, is given by the integration in the region $0.5 < R < 2.5$. In this region the potential $V^1(\psi)$ is better approximated not by the potential $V^2(\psi)$, but by the potential $cV^2(\psi)$, where c is some numerical coefficient of the order of 3. The value of the action in the theory with the potential $cV^2(\psi)$

is obtained from the action in the theory with the potential $V^2(\psi)$ by the division of $S_3^2(\psi)$ by $\sqrt{c}$, see (4.14). In this case at $c \sim 3$ the action S_3 in the theory with the potential $V^1(\psi)$ would be approximated by the action in the theory with the potential $cV^2(\psi)$ with an accuracy of 15%. In most cases, which will be considered in subsequent publications, such an accuracy is quite sufficient for a qualitatively correct description of the kinetics of the phase transition.

(5) Let us now try to understand in which situations one should use the low-temperature approximation (2.4), (2.5), (4.11), (4.12) or the high-temperature approximation (2.10), (4.13), (4.14) for the calculation of $\Gamma(T)$. We shall suppose the values of the action to be large, so that there will be no need to compare the pre-exponential factors in (2.4), (2.10).

First of all we are going to compare the quantities S_4^1 and S_3^1/T. From (4.11) and (4.13) it follows that $S_3^1/T < S_4^1$ at

$$T > 0.72M\,. \tag{4.15}$$

Note that the condition $T \ll r^{-1}(0)$ is fulfilled for sufficiently small ρ (4.3) at all temperatures. Therefore, it can be expected that at least at $T < 0.72M$ the minimum action is achieved at $\rho \to 0$ on the O(4)-symmetric solution (4.3), which at small ρ is not affected by the periodicity of the "time" β with the period T^{-1}. This means that at $T < 0.72M$ for the calculation of $\Gamma(T)$ one should use the low-temperature equation (4.11), and at $T > 0.72M$ one should use eq. (4.13), which determines the minimum of action S_3^1/T in the high-temperature range. The transition between the O(4)-symmetric and O(3)-symmetric cylindric solutions proceeds discontinuously at the point $T = 0.72M$ (first-order phase transition between the trajectories in the functional space, which give the maximum contribution to $\Gamma(T)$).

Now let us discuss the solutions of eq. (4.8). From (4.12) and (4.14) it follows that $S_3^2/T < S_4^2$ at

$$T > 0.21M\,. \tag{4.16}$$

In the region $T < 0.21M$ the O(4)-symmetric solution of eq. (2.6) is almost not affected by the periodicity of the "time" β with period T^{-1}, and therefore it minimizes the action almost exactly. Thus, in the theory (4.2) the transition from the O(4)-symmetric solution (figs. 3a, b) to the O(3)-symmetric solution (fig. 3d) proceeds discontinuously, so that at $T < 0.21M$ one should use the low-temperature equation (4.12), and at $T > 0.21M$ one should use the high-temperature equation (4.14).

Due to the exponential dependence of $\Gamma(T)$ (2.4), (2.10) on the value of the action $S_4(\varphi, T)$ the transition region from the low-temperature regime to the high-temperature one usually turns out to be very narrow, such that the inequalities (4.15), (4.16) can be understood literally as the conditions of applicability of the high-temperature approximation; the sign $\gg$ would be unnecessary in both cases.

(6) In conclusion we will briefly discuss the most typical case, when the potentials V^1, V^2 have the form

$$V^1(\varphi, T) = \tfrac{1}{2}\gamma(T^2 - T_c^2)\varphi^2 - \tfrac{1}{4}\lambda\varphi^4 , \tag{4.1'}$$

$$V^2(\varphi, T) = \tfrac{1}{2}\gamma(T^2 - T_c^2)\varphi^2 - \tfrac{1}{3}\alpha T\varphi^3 . \tag{4.2'}$$

At a sufficiently high temperature from (4.1), (4.13), (4.14) it follows that

$$\frac{S_3^1}{T} \approx \frac{19[\gamma(T^2 - T_c^2)]^{1/2}}{\lambda T} , \tag{4.13'}$$

$$\frac{S_3^2}{T} \approx \frac{44[\gamma(T^2 - T_c^2)]^{3/2}}{\alpha^2 T^3} , \tag{4.14'}$$

whence at $y = (T - T_c)/T_c \ll 1$ we obtain

$$\frac{S_3^1}{T} \approx 27\lambda^{-1}(\gamma y)^{1/2} , \tag{4.13''}$$

$$\frac{S_3^2}{T} \approx \frac{124(\gamma y)^{3/2}}{\alpha^2} . \tag{4.14''}$$

From (4.15) it follows that relations (4.13′), (4.13″) are applicable at

$$\frac{T^2 - T_c^2}{T^2} \lesssim \frac{2}{\gamma} , \tag{4.17}$$

and relations (4.14′), (4.14″) hold at

$$\frac{T^2 - T_c^2}{T^2} \lesssim \frac{25}{\gamma} . \tag{4.18}$$

In realistic theories the parameter γ as a rule turns out to be less than unity, so that eqs. (4.13′)–(4.14″) will be valid in most cases of interest, see e.g. [14, 15, 23].

Note finally that eqs. (4.13′), (4.13″) hold for any theory in which the potential $V(\varphi, T)$ may be approximated by (4.1′) at

$$\varphi \lesssim 3\sqrt{\frac{2\gamma}{\lambda}(T^2 - T_c^2)} \approx 6T\sqrt{\frac{\gamma y}{\lambda}} , \tag{4.19}$$

and eqs. (4.14′), (4.14″) are valid for any theory, wherein $V(\varphi, T)$ may be approximated by expression (4.2′) at

$$\varphi \lesssim \frac{9}{2}\,\frac{\gamma(T^2 - T_c^2)}{\alpha T} \approx \frac{9\gamma y}{\alpha} T . \tag{4.20}$$

5. Expansion of bubbles

In the previous sections we have studied the process of bubble formation. Now we are in a position to investigate how the bubbles expand and fill the entire space.

At zero temperature an expanding bubble is described by the function $\varphi(\boldsymbol{x}^2 - t^2)$, which is an analytic continuation of the euclidean solution $\varphi(\boldsymbol{x}^2 + t^2)$ [22]. Therefore the amplitude of the field φ is constant on the hyperboloid $\boldsymbol{x}^2 - t^2 = \text{const}$. This means that the bubble wall, which was at rest at the moment of the bubble formation $t = 0$, at $t \gg r(0)$ moves with a velocity v almost equal to the velocity of light $c = 1$.

The theory of bubble expansion at $T \neq 0$ is much more complicated. Here we will give only a simplest estimate for velocity of the bubble wall in the non-relativistic limit $v \ll 1$. In this limit the temperature inside the bubble cannot differ from the temperature outside the bubble, i.e. the bubble wall expansion is isothermal due to the large heat conductivity of the ultrarelativistic gas. When the bubble radius r becomes sufficiently large, $r \gg r(T)$, the influence of the surface tension on the motion of the wall can be neglected. We shall also assume that the field φ inside the bubble at that time is equal to its equilibrium value $\varphi_0(T)$. Therefore the pressure inside the bubble is $p_1 = -F_1 = -V(\varphi_0, T)$, and outside the bubble $\varphi = 0$, $p_2 = -V(0, T)$. The bubble wall velocity v increases until the extra pressure of the particles reflected and accelerated by the bubble wall $\Delta p(v)$ compensates the difference between p_1 and p_2:

$$\Delta p(v) = p_1 - p_2 = V(0, T) - V(\varphi_0, T) = \varepsilon \,. \tag{5.1}$$

Let us assume for simplicity that at $\varphi = 0$ all particles are massless, and that at $\varphi = \varphi_0(T)$ N scalar particles aquire masses $m_i \gg T$ and Q vector particles aquire masses $M_i \gg T$. These particles cannot penetrate into the bubble. Since the bubble wall moves with some velocity $v \neq 0$, it not only reflects such particles backwards to the space with $\varphi = 0$, but also increases their energy. This leads to the extra pressure of these particles onto the moving wall

$$\Delta p(v) = \sigma v \,, \tag{5.2}$$

where

$$\sigma = \tfrac{1}{30}\pi^2 T^4 (N + 3Q) \,. \tag{5.3}$$

The bubble wall velocity in this case is determined by eqs. (5.1)–(5.3):

$$v = \frac{\varepsilon}{\sigma} = \frac{30\varepsilon}{\pi^2 T^4 (N + 3Q)} \,. \tag{5.4}$$

Note, that $v \to 0$ in the limit $\varepsilon \to 0$.

At large ε the bubble wall velocity becomes large (though not at the initial stages of the bubble expansion), and the process may become not isothermal, but adiabatic. A possible description of the bubble expansion in this case was suggested in [40].

If the duration of the phase transition is comparable with the age of the Universe by the moment of the phase transition, then while describing bubble expansion one should take into account the expansion of the Universe. Let us consider for simplicity a flat Friedman universe with the metric

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2) \tag{5.5}$$

(in the early universe the effects connected with spatial curvature were insignificant). Time evolution of the scale factor a is described by the equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\rho}{3M_p^2}, \tag{5.6}$$

where $M_p = 1.2 \cdot 10^{19}$ GeV is the Planck mass and ρ is the matter density. The bubble wall velocity (5.2) in the space (5.3) means now not the rate of change of the wall coordinates x, y, z with time, but the rate of change in the interval ds, i.e. of the physical distance.

If the rate of universe expansion is not too large, i.e. if $\dot{a}/a \ll r^{-1}(T)$, where $r(T)$ is a typical bubble radius, then the theory of bubble formation remains practically unaffected by the universe expansion [32]. Usually this condition actually holds (see, however, the theory of the phase transition in the Coleman–Weinberg model [9, 14, 15, 25]). However, even in this case the universe expansion should be taken into account to describe the process of occupation of the universe by the bubbles of the new phase. The description of this process is particularly simple at $T = 0$ [or in the situation when at the moment of phase transition the particle energy density T^4 proves to be much smaller than $\varepsilon = V(0) - V(\varphi_0)$]. In this case, as is shown in ref. [13], the part of the coordinate volume of the universe, which remains in the old phase $\varphi = 0$ by the moment t after the beginning of phase transition (which starts at the moment t_0) is equal to

$$P(t) = \exp\left\{-\int_{t_0}^{t} dt_1\, \Gamma(t_1) a^3(t_1) V(t_1, t)\right\}. \tag{5.7}$$

Here $\Gamma(t_1)$ is the probability of bubble formation per unit physical volume per unit time, which was calculated in the previous sections. The quantity $\Gamma(t_1)a^3(t_1)$ is equal to the probability of bubble formation per unit coordinate volume per unit time, and $V(t_1, t)$ is a coordinate volume occupied at the moment t by a bubble produced at the moment t_1:

$$V(t_1, t) = \frac{4\pi}{3}\left(\frac{r(t_1)}{a(t_1)} + \int_{t_1}^{t} \frac{dt_2}{a(t_2)}\right)^3. \tag{5.8}$$

In the derivation of these equations in [13], however, it was essentially used that the velocity of bubble wall expansion in vacuum is almost exactly equal to $c = 1$. Therefore eqs. (5.5), (5.6) are valid only for the phase transitions from a strongly supercooled metastable phase, such that the energy of particles in this state $\sim T^4$

is much less than ε. In a general case a modification of these relations at $T \neq 0$ is rather complicated, since the metric of space between the bubbles at $T \neq 0$ differs from (5.3). However, if the rate of the phase transition is much greater than the rate of Universe expansion, the analog of relations (5.5), (5.6) can be easily written as follows:

$$P(t) = \exp\left\{-\tfrac{4}{3}\pi \int_{t_0}^{t} dt_1\, \Gamma(t_1) \left[r(t_1) + \int_{t_1}^{t} v(t_1, t_2)\, dt_2\right]^3\right\}, \tag{5.9}$$

where $r(t_1)$ is the radius of the bubbles produced at the moment t_1, and $v(t_1, t_2)$ is the velocity at which the walls of the bubble, produced at the time t_1, move at the time t_2, the velocity v at large $t_2 - t_1$ being estimated by (5.3), (5.4).

At the final stage of the phase transition the free energy difference $\varepsilon = V(0) - V(\varphi_0)$ between symmetric and asymmetric states $\varphi = 0$ and $\varphi = \varphi_0(T)$ transforms into thermal energy of the new phase. A theory of this stage of the phase transition is extremely complicated and will not be developed here; however, we would like to discuss some features of this process.

Usually it is supposed that all the energy of the symmetric state $\varphi = 0$ transforms first into kinetic energy of the bubble walls, and thermalization occurs due to the wall collisions. However, at finite temperature the transformation of the free energy of the supercooled state $\varphi = 0$ to thermal energy proceeds not only due to the bubble wall collisions, but also due to the motion of the walls with some speed v inside dense hot matter. Moreover, in some cases thermalization may proceed in a completely different way. For example, in some versions of the Coleman–Weinberg theory the time necessary for the field φ inside the bubbles to grow up to its equilibrium value $\varphi_0(T)$ is much greater than the time which is necessary for all space to be filled with the bubbles of the field $\varphi \ll \varphi_0(T)$. In this case at the moment when all space becomes filled by the bubbles of the field φ, the field φ everywhere remains much less than $\varphi_0(T)$, the energy of the field φ remains almost equal to $V(0)$, and the energy of the bubble walls is much smaller than the potential energy of the field φ. In such theories thermalization occurs not due to the wall collisions, but due to the interactions of particles created by the classical almost homogeneous field φ, convergently oscillating near its equilibrium value $\varphi_0(T)$ [9, 25, 41].

6. Conclusion

In the present paper we have developed a formalism, which makes possible a comparatively simple investigation of the kinetics of phase transitions in a wide class of theories with spontaneous symmetry breaking.

In particular, the results obtained here make it possible to obtain analytical expressions for the probability of bubble production, which are valid both in the case of weak supercooling (sect. 3), and in the case of strong supercooling of the metastable phase (sect. 4). Applications of these results and their generalization taking account of various effects connected with universe expansion will be contained in a number of publications devoted to the study of phase transitions in grand unified theories [9, 23–25].

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