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EXACT GELL-MANN–LOW FUNCTION OF SUPERSYMMETRIC KÄHLER SIGMA MODELS

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We consider a broad class of Kähler supersymmetric sigma models in two-dimensional space-time. The exact Gell-Mann–Low function is found within the framework of the method proposed earlier [1,2] and based on analysis of classical solutions. It is shown that the exact beta function accounting for all orders in the coupling constant actually coincides with the one-loop result.

1. Introduction

In this work we shall consider the simplest kählerian sigma models in two-dimensional space-time. Our purpose is to find exact Gell-Mann–Low functions for the supersymmetric models using the method proposed within the framework of gluodynamics in [1] and used in [2] for the O(3) sigma model.

We shall dwell on four classes of models in which the fields are defined on the following Kähler manifolds*:

$$
\begin{aligned}
X_{\mathrm{I}} &= \mathrm{SU}(n+m)/\mathrm{SU}(n)\otimes\mathrm{SU}(m)\otimes\mathrm{U}(1)\,,\\
X_{\mathrm{II}} &= \mathrm{Sp}(n)/\mathrm{SU}(n)\otimes\mathrm{U}(1)\,,\\
X_{\mathrm{III}} &= \mathrm{SO}(2n)/\mathrm{SU}(n)\otimes\mathrm{SO}(2)\,,\\
X_{\mathrm{IV}} &= \mathrm{SO}(n+2)/\mathrm{SO}(n)\otimes\mathrm{SO}(2) \qquad (n\geqslant 2)\,.
\end{aligned} \tag{1}
$$

In all the cases we shall find *the β-function to all orders in the coupling constant and show that it coincides with the result of one-loop calculation*:

$$\beta = (\mathrm{d}/\mathrm{d}\ln M_0)\frac{g^2}{4\pi} = -b\left(\frac{g^2}{4\pi}\right)^2, \tag{2}$$

* Besides the four series of compact homogeneous symmetric Kähler manifolds listed above there exist also two exceptional ones [3]: $X_{\mathrm{V}} = E_6/\mathrm{spin}(10)\otimes\mathrm{SO}(2)$ and $X_{\mathrm{VI}} = E_7/E_6\otimes\mathrm{SO}(2)$, which will not be discussed in this paper (see, however, eq. (42)).

where M_0 is the mass of an ultraviolet regulator and

$$b_{\mathrm{I}}=m+n\,,\qquad b_{\mathrm{II}}=n+1\,,\qquad b_{\mathrm{III}}=n-1\,,\qquad b_{\mathrm{IV}}=n\,. \tag{3}$$

Notice that a very popular CP(n) model is a special case in the first series ($m=1$).

Vanishing of all coefficients of the β-function except the first one has been recently demonstrated [2] in the supersymmetric O(3) model. Eq. (2) extends this result to a very broad class of sigma models on the Kähler manifolds. Thus we see that the situation is general.

At present it is hardly necessary to prove that investigation of sigma models is of great interest. They find applications everywhere, from hadronic phenomenology [4] to pure mathematics [5]. There is a striking parallel between 2d sigma models and 4d gauge theories – first noted by Polyakov [6]. Suffice it to recall that both theories are asymptotically free [6, 7]. Moreover, instantons, non-trivial solutions of classical equations of motion, were found practically simultaneously in the O(3) model and QCD [8, 9].

In sigma models (just like in gauge theories) one can increase the "number of colors" and study the $1/n$ expansion. To this end, instead of the simplest models, O(3) or SU(2), one considers O(n) and SU(n). Unfortunately, the price one has to pay is rather high – instantons present in the O(3) model disappear, and the analogy with QCD becomes much poorer.

Also well-developed are alternative schemes that possess instantons for any n. One of the most popular known examples is the CP(n) model. The latter enter in a more general class of the so-called Grassmann models in which the fields are defined on the manifold $\mathrm{X_I}=\mathrm{SU}(n+m)/\mathrm{SU}(n)\otimes\mathrm{SU}(m)\otimes\mathrm{U}(1)$. One can show that for all manifolds listed in eq. (1)

$$\pi_2(\mathrm{X})=\mathbb{Z}$$

(since these are compact Kähler manifolds), which in turn guarantees the existence of topologically non-trivial solutions of the classical equations of motion.

Instantons in CP(n) and Grassmann models were discussed previously in refs. [10–15].

If in the first stages theorists concentrated on purely bosonic models, in recent years their attention has switched to models with fermions, in particular, to supersymmetric schemes. Supersymmetrization of the simplest sigma models was first carried out in [16, 17] (see also [18]). In constructing a supersymmetric version the $N=1$ supersymmetry is built in automatically. However, as was noted in ref. [16] for the O(3) model one certainly gets an extended supersymmetry ($N=2$). Zumino seems to be the first to have established the connection between this fact and the Kähler structure of manifolds on which σ-fields live [19]. He has also given [19] a general formulation of all Kähler sigma models, explicitly realizing the full $N=2$ supersymmetry. Definitions and conventions referring to the Kähler manifolds used throughout the paper are borrowed from textbooks [20–22].

Finishing our minireview of the history of the sigma models we would like to add a few words about the method which will be applied for the determination of the exact β-function. In ref. [1] it was shown that the differential instanton contribution to the vacuum energy in certain SUSY theories can be *fixed exactly* (i.e. to all orders in the coupling constant). The corresponding calculation bears in essence a purely classical character; its results depend only on the number of zero modes (bosonic and fermionic), the bare coupling constant g_0 and the ultraviolet cutoff M_0. Moreover, since we are dealing with the renormalizable theories, the explicit M_0-dependence must be compensated by an implicit one coming from $g(M_0)$. In this way we fix the exact M_0-dependence of the bare coupling constant and determine the Gell-Mann–Low function to all orders in g^2. The coefficients of the β-function are expressed in terms of the number of zero modes and have a geometrical meaning – they are related to the number of symmetries of the classical action realized on instantons in a non-trivial way.

We would like to emphasize once more that the analysis resulting in eq. (2) touches only upon classical aspects of the model. Actually the supersymmetry allows one to substitute a field-theoretical system by a classical one with a finite number of degrees of freedom. This is the main advantage of the method [1], demonstrating that the lion's share of the standard perturbative calculation of the β-function consists of superfluous operations cancelling each other in the final answer.

In four-dimensional gauge theories there exists a general theorem [23a], stating that for extended supersymmetries ($N \geqslant 2$) all coefficients of the beta function – with the possible exception of the first one – vanish. However, as far as we can judge, the proof of the theorem given in [23a] is inapplicable literally to two-dimensional sigma models. Nevertheless, eq. (2) obtained by a straightforward computation clearly indicates that such a proof must also exist for the Kähler sigma models at hand. Some arguments pointing in this direction are presented in ref. [23b].

This paper is organized as follows. In sects. 2 and 3 we discuss in some detail the calculation of the beta function in the Grassmann models. Sect. 4 is devoted to three other classes. In sect. 5 basic results are summarized and related issues that might be of interest for further analysis are listed.

2. Grassmann models – preliminaries

2.1. BOSON AND FERMION ZERO MODES

The Grassmann models have naturally grown from the well-known CP(n) models. The σ-fields are defined on X_I, the manifold of m-dim complex planes $\mathbb{C}^m$ in $(n+m)$-dim complex space $\mathbb{C}^{n+m}$ passing through the origin. If $m=1$ we come back to CP(n):

$$\mathrm{CP}(n) = X_I(m=1, n) = \mathrm{SU}(n+1)/\mathrm{U}(n).$$

As has been already mentioned, our derivation is based on the analysis of non-trivial solutions of the duality equations in euclidian space-time, instantons. First of all, we shall describe the instanton structure in the bosonic Grassmann models and determine the number n_B of collective coordinates in the sector with the unit topological charge*. Then we shall consider the anomaly in the divergency of the axial current and fix the number of fermion zero modes n_F.

Dimensionality of the manifold X_I is easily calculable: $\dim(X_I) = (m+n)^2 - m^2 - n^2 = 2mn$. Independent variables of the bosonic Grassmann model (coordinates on the manifold) are $m \times n$ complex fields. Sometimes we shall use a combined notation:

$$z = x_1 + ix_2, \qquad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right) \equiv \partial .$$

Thus, W_i^α is an m by n matrix containing $2mn$ real variables. Deleting the superscript α and thus putting $m=1$ we arrive at the CP(n) model, parametrized by a complex n-vector. Note that we use bars and crosses to denote complex and hermitian conjugation of matrices W_i^α respectively.

If the manifold under consideration has the structure $X = G/H$ (see eq. (1)), then the isotropy subgroup $H = H_1 \times H_2 \times U(1)$ is realized on W_i^α linearly (it corresponds to a symmetry in the tangent space at $W=0$):

$$W_i^\alpha \to e^{i\delta} H_1^{\alpha\beta} W_j^\beta H_2^{ij} . \tag{4}$$

Other generators from G act non-linearly**:

$$W_i^\alpha \to W_i^\alpha + \varepsilon_i^\alpha + W_i^\beta \bar{\varepsilon}_j^\beta W_j^\alpha . \tag{5}$$

Certainly the action of the model must be invariant under these transformations. Usually it is written in the following form:

$$\begin{aligned} S &= \frac{2}{g^2}\int d^2x\, h^{ij}_{\alpha\bar\beta}\,\partial_\mu W^i_\alpha\,\partial_\mu \overline{W^j_\beta} \\ &= \frac{4}{g^2}\int d^2x\, h^{ij}_{\alpha\bar\beta}\left(\partial W^i_\alpha\,\overline{\partial W^j_\beta} + \bar\partial W^i_\alpha\,\partial\,\overline{W^j_\beta}\right). \end{aligned} \tag{6}$$

* The instanton solutions and their properties have been studied previously in refs. [14,24]. Unfortunately we disagree with the expressions for the number of zero modes quoted in these papers.

** Literally speaking, the transformations (4) and (5) are valid for the Grassmann models. For the other three series there are some differences, to be discussed in sect. 4. They do not touch, however, at principle points.

The corresponding expression for the topological charge is

$$Q = \frac{\varepsilon_{\mu\nu}}{2\pi} \int d^2x\, h^{i\bar{j}}_{\alpha\bar{\beta}}\, \partial_\mu W^i_\alpha\, \partial_\mu \overline{W^j_\beta}$$

$$= \frac{1}{\pi} \int d^2x\, h^{i\bar{j}}_{\alpha\bar{\beta}} \left(\partial W^i_\alpha\, \overline{\partial W^j_\beta} - \bar{\partial} W^i_\alpha\, \partial \overline{W^j_\beta} \right). \tag{7}$$

Here $h^{i\bar{j}}_{\alpha\bar{\beta}}$ is the metric on the manifold. By definition, for the Kähler spaces

$$\partial h^{i\bar{j}}_{\alpha\bar{\beta}} / \partial W^k_\gamma = \partial h^{k\bar{j}}_{\gamma\bar{\beta}} / \partial W^i_\alpha, \tag{8}$$

and thus locally

$$h^{i\bar{j}}_{\alpha\bar{\beta}} = \partial^2 F / \partial W^i_\alpha \partial \overline{W^j_\beta}\,, \tag{9}$$

where F is some function of W^i_α and $\overline{W^j_\beta}$, sometimes called the Kähler potential.

Distinct Kähler manifolds differ by the choice of the potential F. For Grassmann spaces

$$F = \ln\det(I + W^+ W) = \operatorname{tr}\ln(I + W^+ W)$$

$$= \ln\det(I + WW^+) = \operatorname{tr}\ln(I + WW^+), \tag{10}$$

where

$$(WW^+)_{i\bar{j}} = \sum_{\alpha=1}^{m} W^\alpha_i \overline{W^\alpha_j}\,,$$

$$(W^+ W)_{\bar{\alpha}\beta} = \sum_{i=1}^{n} \overline{W^\alpha_i}\, W^\beta_i. \tag{11}$$

In eq. (10) the argument of the logarithm in the first line represents a square $n \times n$ matrix, while in the second line it represents a square $m \times m$ matrix.

Combining eqs. (6) and (7) we readily get

$$S \pm \frac{4\pi}{g^2} Q = \frac{1}{g^2} \left[\int d^2x\, h^{i\bar{j}}_{\alpha\bar{\beta}} \left(\partial_\mu W^i_\alpha \pm i\varepsilon_{\mu\nu}\, \partial_\nu W^i_\alpha \right) \left(\partial_\mu \overline{W^i_\alpha} \mp i\varepsilon_{\mu\nu}\, \partial_\nu \overline{W^i_\alpha} \right) \right]. \tag{12}$$

Thus in each class of configurations with a given topological charge the minimum of the action is achieved if and only if the field W^i_α satisfies the duality equations,

$$\partial W^i_\alpha = 0 \qquad \left(\partial = \tfrac{1}{2}(\partial_1 - i\partial_2) \right), \tag{13}$$

or

$$\bar{\partial} W_\alpha^i = 0 \qquad \left(\bar{\partial} = \tfrac{1}{2}(\partial_1 + i\partial_2)\right). \tag{14}$$

For such fields

$$S = \frac{4\pi}{g^2}|Q|, \tag{15}$$

and for $Q > 0$ we are dealing with a Q-instanton configuration (antiinstantons if $Q < 0$).

The solution of the duality equation (14) is an arbitrary analytic (meromorphic) function $W_\alpha^i(z)$*.

Moreover, it is necessary to choose a trivial vacuum configuration, $W_\alpha^i(z) = \text{const.}$ The point is that in computing the instanton contribution we always normalize it to the vacuum one. The constant representing the vacuum field can be chosen arbitrarily. However, once chosen, it immediately fixes the asymptotic behaviour of $W_\alpha^i(z)_{\text{inst}}$ at $|z| \to \infty$ (for details see ref. [2]). For definiteness let us stick to the condition $\lim_{|z| \to \infty} W_\alpha^i(z) = 0$. Then the general instanton solution reduces to

$$W_\alpha^i(z) = \sum_{l=1}^{k_{i,\alpha}} \frac{\left(a_\alpha^i\right)_l}{z - \left(b_\alpha^i\right)_l}, \tag{16}$$

where $(a_\alpha^i)_l$ and $(b_\alpha^i)_l$ are some complex matrices. Among all the solutions of type (16) we must select only functions that correspond to the unit topological charge, $Q = 1$. Fortunately this is a relatively simple task. Notice that the density of the topological charge figuring in eq. (7) reduces to the laplacian of the Kähler potential,

$$Q = \frac{1}{\pi} \int d^2x \, \Delta F. \tag{17}$$

Since the integrand is a full derivative of a function falling off rapidly enough at infinity, the integral is saturated by the domain near the poles of the matrix $W^+ W$. Each pole $[(z-b)(\bar{z}-\bar{b})]^{-k}$ evidently adds k units to the topological charge. It is clear that if at least two constants $(b_\alpha^i)_l$ were different, Q would be larger than 1. On the other hand, we are interested in the simplest non-trivial solution with $Q = 1$, and hence we choose all $(b_\alpha^i)_l$ to be equal:

$$W_\alpha^i(z) = \frac{a_\alpha^i}{z - b}. \tag{18}$$

* The so-called toron configurations [25] are not discussed here. For these configurations the finiteness of the action is achieved due to the introduction of a large box of size L (plus certain periodic boundary conditions). The field itself is of order $1/L$ (in 2d). Then both the action and topological charge are finite in the limit $L \to \infty$.

For the case of the CP(n) model ($m = 1$, α is absent) this condition is sufficient – the configuration (18) possesses the unit topological charge.

Interestingly enough, for the Grassmann model ($m > 1$) the parametrization (18) does not guarantee $Q = 1$: for some sets of a^i_α the topological charge turns out to be $Q \geqslant 2$. Indeed, substituting eq. (18) into eq. (10) we readily convince ourselves that

$$F = \ln P_r\left[|z-b|^{-2}\right], \tag{19}$$

where P_r is a polynomial in $|z-b|^{-2}$ of power r, and

$$r = \operatorname{rank}\left(a^i_\alpha\right). \tag{20}$$

Combining now eqs. (19) and (17) we conclude that

$$Q = r. \tag{21}$$

In other words, the topological charge of the configuration (18) is determined by the rank of the $m \times n$ matrix a^i_α. To single out the one-instanton solution we must parametrize the most general matrix of the unit rank. It is rather obvious that any such matrix reduces to

$$a^i_\alpha = \lambda^i \mu_\alpha, \tag{22}$$

where λ^i and μ_α are complex vectors. Notice that the transformation

$$\lambda^i \to C\lambda^i, \qquad \mu_\alpha \to C^{-1}\mu_\alpha,$$

with C being an arbitrary complex number, leaves the matrix a^i_α (22) intact. If so, we may always assume that, say, $\lambda^1 = 1$, and hence the matrix (22) is given by $(n + m - 1)$ complex parameters. The complex number b figuring in eq. (18) is an extra one. As a result, the one-instanton solution in the Grassmann model depends on $2(m + n)$ real collective coordinates.

Taking the derivative of the instanton solution over every collective coordinate we get zero modes. Thus the number of bosonic zero modes for the X_{I} model is equal to

$$n_{\mathrm{B}} = 2(m+n). \tag{23}$$

We proceed now to the discussion of the fermionic zero modes' number. First of all, let us recall the basic points of the supersymmetrization. The recipe given by Zumino and corresponding to the explicit realization of $N = 2$ supersymmetry is very simple. If the Kähler potential F of the bosonic model is known, then

$$S_{\mathrm{SUSY}} = \frac{1}{2g^2}\int \mathrm{d}^2x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,F(W, \overline{W}). \tag{24}$$

Now W becomes a chiral superfield,

$$\begin{aligned} W_\alpha^i(x_{\mathrm{ch}},\theta) &= W_\alpha^i(x_{\mathrm{ch}}) + \sqrt{2}\,\theta\chi_\alpha^i(x_{\mathrm{ch}}) + \theta^2 F_\alpha^i(x_{\mathrm{ch}}) \\ &= W_\alpha^i(x) + \sqrt{2}\,\theta\chi_\alpha^i(x) + \theta^2 F_\alpha^i(x) + i\theta\gamma_\mu\bar\theta\partial_\mu W_\alpha^i(x) \\ &\quad + \sqrt{\tfrac{1}{2}}\, i\theta^2\partial_\mu\chi_\alpha^i\gamma_\mu\bar\theta + \tfrac{1}{4}\theta^2\bar\theta^2\,\partial^2 W_\alpha^i(x)\,, \end{aligned} \tag{25}$$

where x_{ch} stands for the chiral argument

$$(x_{\mathrm{ch}})_\mu = x_\mu + i\theta\gamma_\mu\bar\theta\,, \tag{26}$$

χ_α^i is the Majorana spinor superpartner of W_α^i and F_α^i is an auxiliary boson field entering the lagrangian with no kinetic term.

Substituting the explicit expression for the superfield W_α^i in eq. (24), integrating over θ and $\bar\theta$, and eliminating the auxiliary field F_α^i we arrive at [5]

$$\begin{aligned} \mathcal{L}_{\mathrm{SUSY}} = \frac{2}{g^2}\Big[&-h^{ij}_{\alpha\bar\beta}\partial_\mu W_i^\alpha\,\partial_\mu\overline{W_j^\beta} - \tfrac{1}{2} i h^{ij}_{\alpha\bar\beta}\chi_i^\alpha\gamma_\mu\,\nabla_\mu\overline{\chi_j^\beta} \\ &- \tfrac{1}{2} i h^{ij}_{\alpha\bar\beta}\overline{\chi_j^\beta}\,\gamma_\mu\overline{\nabla}_\mu\chi_i^\alpha + \tfrac{1}{4}R^{ijkl}_{\alpha\bar\beta\gamma\bar\delta}(\chi_i^\alpha\chi_k^\gamma)\big(\overline{\chi_j^\beta}\,\overline{\chi_l^\delta}\big)\Big]\,. \end{aligned} \tag{27}$$

Here

$$(\nabla_\mu\chi)_i^\alpha = \partial_\mu\chi_i^\alpha + \Gamma^{\alpha jk}_{i\beta\gamma}\big(\partial_\mu W_j^\beta\big)\chi_k^\gamma\,,$$

where the Γ's denote the Christoffel symbols on the manifold, $R^{ijkl}_{\alpha\bar\beta\gamma\bar\delta}$ is the curvature tensor, and we have used the definition (9) of the metric. Recall that for the Kähler manifold

$$\Gamma^a_{bc} = h^{a\bar d}\,\partial_b h_{c\bar d}\,, \qquad \Gamma^{\bar a}_{\bar b\bar c} = \overline{\Gamma^a_{bc}}\,,$$

while all other Christoffel symbols vanish.

Performing the chiral transformation over fermions we convince ourselves that at the classical level the lagrangian (27) implies conservation of the axial current

$$j_\mu^5 = \frac{2}{g^2}h^{ij}_{\alpha\bar\beta}\overline{\chi_j^\beta}\,\gamma_\mu\gamma_5\chi_i^\alpha\,. \tag{28}$$

As is well known, at the quantum level (taking into account the ultraviolet regularization), this current is no longer conserved – there emerges the anomaly in the divergence of j_μ^5. The value of $\partial_\mu j_\mu^5$ is easily calculable starting from the Schwinger diangle anomaly; there is no need for perform an explicit calculation anew (this is so simple, though, that it can be immediately repeated directly in the sigma models).

If the fermion Green function satisfies the equation

$$\frac{2}{g^2}\left[i\gamma_\mu\left(\partial_\mu\delta^\alpha_\beta\delta^i_j+\left(A_\mu(x)\right)^{\alpha i}_{j\beta}\right)\right]G^{j\beta}_{\gamma k}(x,y)=\delta(x-y)\delta^i_k\delta^\alpha_\gamma\,, \tag{29}$$

then, according to Schwinger [26]

$$\partial_\mu j^5_\mu=\frac{1}{\pi}\varepsilon_{\mu\nu}\partial_\mu(A_\nu)^{\alpha i}_{i\alpha}. \tag{30}$$

Comparing eqs. (29) and (27) we convince ourselves that in our case

$$\begin{aligned}\partial_\mu j^5_\mu&=\frac{1}{2\pi}\varepsilon_{\mu\nu}\partial_\mu\left[\Gamma^{\alpha ji}_{i\beta\alpha}\partial_\nu W^\beta_j-\overline{\Gamma^{\alpha ji}_{i\beta\alpha}}\,\partial_\nu\overline{W^\beta_j}\right]\\&=\frac{1}{2\pi}\varepsilon_{\mu\nu}\left[\frac{\partial\Gamma^{\alpha ji}_{i\beta\alpha}}{\partial\overline{W^\gamma_k}}\partial_\mu\overline{W^\gamma_k}\,\partial_\nu W^\beta_j-\frac{\partial\overline{\Gamma^{\alpha ji}_{i\beta\alpha}}}{\partial W^\gamma_k}\partial_\mu W^\gamma_k\,\partial_\nu\overline{W^\beta_j}\right]\\&=-\frac{1}{2\pi}\left[R^{jk}_{\beta\bar\gamma}\varepsilon_{\mu\nu}\,\partial_\mu\overline{W^\gamma_k}\,\partial_\nu W^\beta_j-R^{jk}_{\bar\beta\gamma}\varepsilon_{\mu\nu}\,\partial_\mu W^\gamma_k\,\partial_\beta\overline{W^\beta_j}\right]\\&=-\frac{2}{\pi}R^{ij}_{\alpha\bar\beta}\left(\bar\partial\,\overline{W^\beta_j}\,\partial W^\alpha_i-\partial\overline{W^\beta_j}\,\bar\partial W^\alpha_i\right).\end{aligned} \tag{31}$$

Here we have used the fact that in Kähler spaces the Ricci tensor is related to the Christoffel symbols by the following relation [20]:

$$R_{\alpha\bar\beta}=-\partial\Gamma^\gamma_{\alpha\gamma}/\partial\overline{W^\beta}\,. \tag{32}$$

Moreover, in any symmetric Kähler space the Ricci tensor is proportional to the metric [20]

$$R_{\alpha\bar\beta}=\kappa h_{\alpha\bar\beta}. \tag{33}$$

The proportionality coefficient κ depends on the particular manifold under consideration. If one uses this relation in eq. (31) and integrates both sides over the whole space, one arrives at the index theorem:

$$\Delta q_5=2\kappa Q=2\kappa\qquad\text{for } Q=1\,. \tag{34}$$

(Compare eqs. (31) and (33) with the definition of the topological charge (7).) Here q_5 stands for the axial charge. For the Grassmann model the theorem (34) has been already discussed in the literature [15].

As usual, the index theorem allows one to establish the number of zero modes provided that all the modes possess definite chirality. This is just the case in the sector with the unit topological charge [15]. Then

$$n_{\mathrm{F}} = 2\kappa .$$

Now the final effort: let us find κ. The simplest method seems to exploit eq. (33) that allows us to calculate the curvature at a single point $W_\alpha^i = 0$. Notice that for small W_α^i

$$F = h^{i\bar{j}}_{\alpha\bar{\beta}}(0) W_i^\alpha \overline{W}_j^\beta - \tfrac{1}{4} R^{i\bar{j}k\bar{l}}_{\alpha\bar{\beta}\gamma\bar{\delta}}(0) W_i^\alpha \overline{W_j^\beta} W_k^\gamma \overline{W_l^\delta} + \mathrm{O}(W^6) . \tag{35}$$

Expanding the explicit expression for the Kähler potential (10) we observe that for weak external fields (small W_α^i) the metric $h^{i\bar{j}}_{\alpha\bar{\beta}}$ coincides with the Minkowski one:

$$h^{i\bar{j}}_{\alpha\bar{\beta}} = \delta_{\alpha\beta}\delta_{ij} ,$$

while

$$R^{i\bar{j}k\bar{l}}_{\alpha\bar{\beta}\gamma\bar{\delta}} = \delta_{\alpha\beta}\delta_{jk}\delta_{\gamma\delta}\delta_{il} + \delta_{\alpha\delta}\delta_{kl}\delta_{\beta\gamma}\delta_{ij} . \tag{36}$$

Contracting subscripts and superscripts (γ, δ), (k, l) in eq. (36) yields $\kappa = (m+n)$, and consequently in the Grassmann models

$$(n_{\mathrm{F}})_{\mathrm{I}} = 2(m+n) . \tag{37}$$

Thus the numbers of boson and fermion zero modes are equal to each other. Moreover, in this model the functional dependence of the modes is identical – each boson zero mode $(W_\alpha^i)_0$ corresponds to the fermion one:

$$\left(\chi_\alpha^i\right)^{(1)} = \left(W_\alpha^i\right)_0 \xi , \qquad \left(\chi_\alpha^i\right)^{(2)} = 0 ,$$

where $^{(1),(2)}$ are the Lorentz (spinor) indices and ξ is a complex Grassmann parameter. This fact will be exploited later. We would like to emphasize that this situation – coincidence of the numbers of zero modes – is quite general and takes place in other classes of sigma models, to be discussed in sect. 4. (Note that this statement is also valid in the $N=2$ supersymmetric Yang-Mills theory in 4d [1].)

3. One-instanton contribution to the vacuum-to-vacuum transition

3.1. THE EXACT GELL-MANN–LOW FUNCTION

Now, when the numbers of boson and fermion zero modes are found we can fix (up to an overall numerical constant) the instanton measure I. The basic principles of the calculation are perfectly analogous to those discussed in refs. [1,2], and we refer the reader to these papers for details. Here we only sketch basic points of the derivation.

(i) To the leading approximation I is determined by the instanton action. Under our definition of the coupling constant the action is $4\pi/g_0^2$ times the topological charge and

$$I \sim e^{-4\pi/g_0^2}.$$

(ii) In order to find the one-loop correction we represent W_α^i in the form

$$W_\alpha^i = \left(W_\alpha^i\right)_{\text{inst}} + \delta W_\alpha^i,$$

expand the action up to bilinear terms in δW_α^i, and doing the corresponding Gauss integrals arrive at standard determinants – one in the numerator for the fermions and another in the denominator for the bosons. The ultraviolet regularization is realized according to Pauli-Villars (see [27] or the review paper [28]). Generally speaking, in the case at hand one needs an infrared regularization as well since most of the zero modes are non-normalizable (the normalization integral diverges at large distances). However, once an infrared cutoff is introduced (see, for instance, [29]) normalization factors (proportional to $\ln R$) appearing in the numerator and denominator cancel each other. The reason is simple – the boson and fermion modes form pairs characterized by an identical functional dependence on z (see above). The final answer for I is infrared stable.

(iii) Non-zero modes, boson and fermion, cancel each other leading to the unit contribution to I. This fact, a direct consequence of the supersymmetry, was first noted in ref. [30]. We shall try to explain it at a qualitative level. (A more technical proof is given, for example, in the review paper [31].)

The ordinary supersymmetry implies degeneracy of boson and fermion excitations over the genuine "empty" vacuum (more exactly, the vacuum state of the perturbation theory). The energy of fluctuations in a non-vanishing external field is different for bosons and fermions, generally speaking. Certainly, each non-zero boson mode is accompanied, as previously, by two fermion ones. The latter, however, represent excitations over "another" vacuum in another external field, resulting from the original field by a supersymmetry transformation.

Assume, however, that we have at our disposal several supersymmetry generators. Then the cancellation of non-zero modes can be achieved provided that the external field is chosen in such a way that it is annihilated by one of these generators. The instantons, being self-dual field configurations, just possess the property. For instance, in QCD the instanton field stays intact under the supersymmetry transformation of a "wrong" chirality [1]. As for the Kähler sigma models, the place of the QCD chiral transformations is occupied now by the $N = 2$ supersymmetry. Self-dual instantons transform non-trivially under the action of only one of two independent linear combinations of generators [2]. Thus the method of ref. [1] seems to be applicable only to the *Kähler* sigma models.

(iv) Following the above line of reasoning we arrive at the conclusion: the instanton measure is determined exclusively by the zero modes. Accounting for them results in the occurrence of an integral over collective coordinates (to be denoted generically by $\mathrm{d}\mu$). Besides that, in passing to the collective coordinates there emerges a jacobian depending on the collective coordinates, coupling constant and regulator mass M_0. The latter comes from the regularizing determinants and is due to the fact that the number of boson and fermion zero modes is unbalanced (contrary to the situation of the non-zero modes). Each real boson zero mode yields a factor of $(M_0^2/g_0^2)^{1/2}$, while the fermion mode produces $(M_0/g_0^2)^{-1/2}$.

Assembling all factors together we obtain

$$\mathrm{d}I \sim M_0^{n_\mathrm{B}-(n_\mathrm{F}/2)} g_0^{n_\mathrm{F}-n_\mathrm{B}} \exp\left[-4\pi/g_0^2\right] \mathrm{d}\mu\,, \tag{38}$$

where n_B and n_F are the numbers of (real) zero modes.

(v) Expression (38) is valid not only at the one-loop level, but to all orders in g^2. Higher loops vanish due to the definite duality of the instanton field. The corresponding non-renormalization theorem has been proven in ref. [1] (see also ref. [31]).

The transition amplitude (38) can be considered as an effective fermion interaction with n_F fermion legs. Therefore, apart from proper vertex corrections – they, as has been already noted, are absent – one should take into account insertions to external lines, i.e. renormalization of the fermion wave function. Fortunately, under our convention for the lagrangian, the Z-factor renormalizing the wave functions reduces to unity.

In the literature one may sometimes encounter the assertion [32] according to which there are two distinct renormalization counterterms in sigma models – the first one corresponds to renormalization of the coupling constant while the second one renormalizes the wave function. Indeed, if perturbation theory is constructed in such a way that the full symmetry of the model is violated at intermediate stages one may obtain two counterterms. No surprise, however, that if one uses the background field method then the explicit global G-invariance of the whole expression referring to the Kähler homogeneous space $\mathrm{M} = \mathrm{G}/\mathrm{H}$ of a *definite size* is guaranteed. This fact immediately implies, in turn, that $Z_{\text{wave function}} = 1$. (In this connection see also appendix C in ref. [31].) It is worth adding that in calculating the instanton measure we actually use just the background field method. Therefore, within the framework of our approach we are dealing with the single-charge theory. The presence of only one renormalization constant will be exploited below.

(vi) Thus, the M_0 and g_0 dependence of the one-instanton contribution to the amplitude of the vacuum-to-vacuum transition is fixed exactly to all orders in g_0^2. (Non-perturbative corrections might, generally speaking, appear in eq. (38). The existence of such corrections would lead to extra terms like $\exp(-c/g_0^2)$ in the

β-function (2). We do not discuss them here.) Recall now that the theories at hand are renormalizable, and hence the physical quantities cannot depend on the regulator mass. This means that explicit dependence on M_0 is exactly compensated by implicit dependence, arising due to $g_0 = g(M_0)$. (There are no other parameters to participate in the renormalization.) Taking into account the equality $n_B = n_F$ valid for $N = 2$ supersymmetry we obtain

$$\frac{g_0^2}{4\pi} = \frac{4}{n_B} \frac{1}{\ln\left(M_0^2/m^2\right)} . \tag{39}$$

The value of n_B is already known, see eq. (23). This immediately results in eq. (2) with $b_I = m + n$, our final answer for the Grassmann models. For $m = 1$ it reduces to the β-function of the supersymmetric CP(n) model. Furthermore, substituting $m = n = 1$ we find for the CP(1) (O(3) or, more exactly, O(3)/O(2)) model that $b = 2$, in accordance with the result of ref. [2].

Note that the one-loop coefficient of the β-function is rather easily calculable in the ordinary perturbation theory. At the one-loop level fermions do not contribute to the Gell-Mann–Low function [16], and the latter coincides with that of the purely bosonic theory. The β-function of the bosonic version is obviously related to the curvature of the manifold. A simple way to fix the boson loop (up to an overall normalization factor) is described in the appendix. The first coefficient b obtained in this way always coincided with eq. (3).

4. Other models

Consider now other Kähler manifolds enumerated in eq. (1). It has been already mentioned that these form a complete set of compact, homogeneous, symmetric Kähler spaces. We need almost all of these properties in our analysis. First of all non-compact manifolds usually have $\pi_2(X) = 0$, and thus no two-dimensional instantons*. Second, the Kähler structure of the sigma models is actually equivalent to the occurrence of $N = 2$ supersymmetry, which is crucial for cancellation of non-zero modes in the instanton background. Moreover, to fix the β-function unambiguously we must ensure that there is a single Z-factor in the theory

* It goes without saying that using additional information, namely the obvious isomorphism of non-compact and compact manifolds, one readily obtains β-functions for the non-compact case by merely changing the sign of g^2. Thus

$$\beta[\mathrm{SU}(m,n)/\mathrm{SU}(m)\otimes \mathrm{SU}(n)\otimes \mathrm{U}(1)] = -\beta[\mathrm{SU}(m+n)/\mathrm{SU}(m)\otimes \mathrm{SU}(n)\otimes \mathrm{U}(1)]$$
$$= -(m+n)$$

(no asymptotic freedom), etc.

considered. In this way we naturally come to homogeneity of the space. (Note, however, that the *one-loop* β-function is not sensitive to extra Z-factors.) As to the last requirement, of the space to be symmetric, we believe it is not essential: we could safely discuss non-symmetric spaces. In fact nothing new will arise for any Einstein space*. (Recall that we needed symmetrical manifolds only to ensure eq. (33) – the proportionality between the Ricci and metric tensors.) Surely the analysis may be extended to an even wider class of models. However, we would like to emphasize that beyond the set of Einstein spaces $R_{\alpha\bar{\beta}}$ is no longer proportional to $g_{\alpha\bar{\beta}}$, and the axial anomaly connects the number of the fermion zero modes n_F with another topological charge:

$$Q_1 = \frac{\varepsilon_{\mu\nu}}{2\pi} \int R_{\alpha\bar{\beta}}\, \partial_\mu W_\alpha\, \partial_\nu \overline{W}_\beta ,$$

which generally speaking differs from

$$Q = \frac{\varepsilon_{\mu\nu}}{2\pi} \int h_{\alpha\bar{\beta}}\, \partial_\mu W_\alpha\, \partial_\nu \overline{W}_\beta .$$

Finally, the σ-models in more than two dimensions are non-renormalizable and it is unclear whether one should (or can) use the method in this case (see point (vi) in sect. 3).

What are the changes that it is necessary to introduce in the discussion of sects. 2, 3? They all are of a technical character, not of principal nature. First of all, different Kähler manifolds are populated by different fields. If, say, in the case of X_I the fields W_α^i are arbitrary complex $m \times n$ matrices ($\dim_R X_I = 2mn$), for X_{II} they are complex symmetrical $n \times n$ square matrices. It is obvious that the real dimension of X_{II} is $d_{II} = n(n+1)$. Moreover, the fields defined on X_{III} are complex antisymmetrical $n \times n$ matrices ($d_{III} = n(n-1)$). One easily concludes from this fact that the analogue of the transformation (4) generated by the isotropy group for the latter two series is

$$W_\alpha^i \to U_j^i W_\beta^j U_\alpha^\beta \qquad (W \to UWU^T, \qquad U \in SU(n)) .$$

Finally, in the fourth case (X_{IV}) the superscript α is absent and the field W_i is a complex n-vector. The dimensions of all these spaces are presented in table 1. Furthermore, the Kähler potential is chosen to ensure invariance of the metric

$$ds^2 = h_{ij}^{\alpha\bar{\beta}}\, dW_\alpha^i\, \overline{dW_\beta^j}$$

* Note, however, that one should be careful in choosing the Kähler potential in a non-symmetric case.

TABLE 1

Manifold	G	H	Fields	D_X(real)	n_B	$2\kappa = n_F$
X_I	$SU(m+n)$	$SU(m)\otimes SU(n)\otimes U(1)$	complex $m\times n$ matrix	$2mn$	$2(m+n)$	$2(m+n)$
X_{II}	$Sp(n)$	$SU(n)\otimes U(1)$	complex symmetric $n\times n$ matrix	$n(n+1)$	$2(n+1)$	$2(n+1)$
X_{III}	$SO(2n)$	$SU(n)\otimes U(1)$	complex antisymmetric $n\times n$ matrix	$n(n-1)$	$4(n-1)$*	$4(n-1)$*
X_{IV}	$SO(n+2)$ $n\geqslant 2$	$SO(n)\otimes SO(2)$	complex n-vector	$2n$	$2n$	$2n$

*For the manifold X_{III} the minimal non-trivial topological charge is $Q_{\min}=2$. For this reason $n_F = 4\kappa$ in this case.

on the manifold under the transformations (4), (5).

$$F = \ln\det\left(\delta_{ij} + W^i_\alpha \overline{W}^j_\alpha\right) = \ln\det\left(\delta_{\alpha\beta} + \overline{W^i_\alpha}\, W^i_\beta\right)$$

is clearly invariant under the linear transformation (4), while under (5)

$$\delta F = \ln(I + \varepsilon^+ W + \varepsilon W^+) = \mathrm{tr}(\varepsilon^+ W + \varepsilon W^+).$$

The second derivative of this expression vanishes.

Thus, for the three series X_I, X_{II} and X_{III} we use the expression (10) and the subsequent line of reasoning to determine n_B. (We shall return to the space X_{IV} a bit later.) Any symmetrical $n\times n$ matrix of unit rank is parametrized by a single complex vector λ_α: $a^i_\alpha = \lambda^i\lambda_\alpha$. Accounting for the parameter b (see eq. (18)) we obtain $(n+1)$ complex collective coordinates:

$$(n_B)_{II} = 2(n+1).$$

There is a subtle point in the case of the manifold X_{III}. The point is that our definition of the topological charge (7) leads in this case to the minimal non-zero value of topological charge $Q_{\min}=2$ rather than 1. More generally, Q can take only even values. In principle, one may change normalization of Q in eq. (7) for X_{III}, but we shall not do this in order to conserve the notational uniformity.

Thus, the one-instanton solution for X_{III} has $Q=2$ and the action $S = 4\pi Q/g_0^2 = 8\pi/g_0^2$. In accordance with this fact is the observation that there are no antisymmetric matrices of unit rank. The lowest non-zero rank of such a matrix is two. What is the number of complex parameters that specify antisymmetric square matrices of rank two? First of all we should choose a complex 2-hyperplane in $\mathbb{C}^n$. There are $2(n-2)$ ways to do this. (Note that the set of these 2-hyperplanes is in fact the Grassmann space $X_I(m=2, n-2)$, and its dimension is already known.) In the reference frame connected with the chosen hyperplane the antisymmetric rank-2 matrix is parametrized by a single complex number:

$$\left(\begin{array}{cc|c} 0 & a & \multirow{2}{*}{0} \\ -a & 0 & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right).$$

As a result we get $(2n-3)$ complex parameters. Adding b to them we arrive at

$$(n_B)_{III} = 4(n-1).$$

The number of the fermionic zero modes is fixed in a manner similar to that used for the Grassmann models. At first, the proportionality coefficient κ in relation (33) is determined. To this end we write down the Ricci and metric tensors for weak fields, expanding the Kähler potential F in the vicinity of $W_\alpha^i = 0$. The result can be read off eq. (36) (we should (anti)symmetrize within pairs $\alpha i, \beta j, \gamma k, \delta l$). Omitting trivial algebra we obtain directly the answer

$$\kappa_{II} = n+1, \qquad \kappa_{III} = n-1.$$

It is worth noting that while

$$(n_F)_{II} = 2\kappa_{II} = 2(n+1),$$

for the third series

$$(n_F)_{III} = 4\kappa_{III} = 4(n-1),$$

since, as has been already mentioned, in this case the instanton topological charge is $Q=2$.

The numbers of fermionic zero modes are presented in table 1.

Let us return now to the discussion of the Kähler space X_{IV}. The transformation law (5) should now be substituted by

$$W_i \to W_i + \varepsilon_i + 2(\bar{\varepsilon}_j W_j) W_i - \bar{\varepsilon}_i (W_j W_j), \tag{40}$$

and the corresponding covariant Kähler potential takes the form [33]

$$F = \ln(1 + 2\overline{W}_i W_i + |W_i W_i|^2). \tag{41}$$

Any other (more simple) choice of the potential could lead to a wider isotropy group than is necessary. The variation of the potential (41) under transformation (40) is given by the same formula as previously:

$$\delta F = 2\,\mathrm{Tr}(\varepsilon_i \overline{W}_i + \bar{\varepsilon}_i W_i).$$

Proceeding to the calculation of n_{B} and n_{F} we observe that any vector treated as a one-column matrix has unit rank. However, since the potential (41) contains, besides the bilinear combination $\overline{W}_i W_i$, a quadrilinear term $|W_i W_i|^2$, a solution of the type

$$W_i = \frac{a_i}{z-b},$$

generally speaking will not have the unit topological charge. One can easily convince oneself that for the non-vanishing quadrilinear term $Q=2$. Therefore, in order to single out the one-instanton configurations we should impose an additional condition:

$$\sum_i a_i^2 = 0.$$

It is worth emphasizing that here $a_i^2 = 0$, not $|a_i|^2 = 0$, and this constraint for complex numbers is easily satisfied. Thus the vector a_i yields $(n-1)$ complex parameters. Together with b this implies that

$$(n_{\mathrm{B}})_{\mathrm{IV}} = 2n.$$

In calculating κ_{IV} it is convenient to rescale the fields W_i and write F as

$$F = \ln\left(1 + \overline{W}_i W_i + \tfrac{1}{4}(W_i W_i)^2\right).$$

Doing so we guarantee that the metric for weak fields coincides with the flat one, $h_{i\bar{j}} = \delta_{i\bar{j}}$. Furthermore,

$$R_{i\bar{j}k\bar{l}} = \left(-\delta_{ik}\delta_{\bar{j}\bar{l}} + \delta_{i\bar{j}}\delta_{k\bar{l}} + \delta_{i\bar{l}}\delta_{\bar{j}k}\right) + \mathrm{O}(W^2).$$

Contracting the Riemann tensor over (kl) results in the Ricci tensor

$$R_{i\bar{j}} = n\delta_{i\bar{j}}.$$

The latter equality implies in turn that

$$\kappa_{\mathrm{IV}} = n,$$

and consequently

$$(n_{\mathrm{F}})_{\mathrm{IV}} = 2n.$$

The values of n_{B} and n_{F} are given in table 1, while the corresponding coefficients of the Gell-Mann–Low function are presented in eq. (3). We quote in this section (with no derivation) the coefficients for exceptional spaces:

$$b_{\mathrm{V}}=12, \qquad d_{\mathrm{V}}=32, \qquad (n_{\mathrm{B}})_{\mathrm{V}}=(n_{\mathrm{F}})_{\mathrm{V}}=24;$$
$$b_{\mathrm{VI}}=18, \qquad d_{\mathrm{VI}}=54, \qquad (n_{\mathrm{B}})_{\mathrm{VI}}=(n_{\mathrm{F}})_{\mathrm{VI}}=36. \tag{42}$$

Note also that dimensions and quadratic Casimir coefficients (cf. the appendix for normalization) for relevant exceptional groups are equal to

$$b_{\mathrm{E}_6}=12, \qquad d_{\mathrm{E}_6}=78;$$
$$b_{\mathrm{E}_7}=18, \qquad d_{\mathrm{E}_7}=133.$$

5. Conclusions

In this paper we have found the exact β-function for a broad class of the supersymmetric Kähler sigma models, analysing the theory at the classical level without invoking the standard perturbation theory and performing no computation of Feynman graphs. All terms in the expansion in the coupling constant, except the first one, vanish. Since the manifolds discussed are compact, the first coefficient is always negative, thus implying asymptotic freedom.

We have not used all the opportunities provided by the instanton calculus in supersymmetric theories. In particular it would be interesting to calculate a correlation function like

$$\langle 0|T\left\{\prod_{k=1}^{m+n} h^{i\bar{j}}_{\alpha\bar{\beta}}(x_k)\chi^{\alpha}_{i}(x_k)\bar{\chi}^{\beta}_{j}(x_k)\right\}|0\rangle, \tag{43}$$

arising due to fermionic zero modes in the instanton field. Analogously to the CP($N-1$) sigma model [29] the correlator (43) cannot depend on coordinates (due to the supersymmetry) and should reduce to a non-zero constant. This fact – the breakdown of the cluster decomposition – naturally suggests the occurrence of the condensate $\langle 0|h^{i\bar{j}}_{\alpha\bar{\beta}}\chi^{\alpha}_{i}\bar{\chi}^{\beta}_{j}|0\rangle$. In this way one can determine the value of the fermionic condensate and compare it with analogous result of the $1/n$ expansion.

It would be also instructive to consider contributions to (43) associated with toron configurations [25] that are likely to exist in σ-models (a similar discussion for gluodynamics has been recently given in ref. [34]).

Finally, the question of how the supersymmetry manages to coexist with the condensate

$$\langle 0|h^{i\bar{j}}_{\alpha\bar{\beta}}\chi^{\alpha}_{i}\overline{\chi^{\beta}_{j}}|0\rangle$$

is not absolutely clear. Indeed the operator $h^{i\bar{j}}_{\alpha\bar{\beta}}\chi^{\alpha}_{i}\overline{\chi^{\beta}_{j}}$ is not the lowest component of a superfield, rather it occupies a middle position. From this point of view the appearance of the vacuum expectation value $\langle 0|h^{i\bar{j}}_{\alpha\bar{\beta}}\chi^{\alpha}_{i}\overline{\chi^{\beta}_{j}}|0\rangle$ would normally mean that the supersymmetry is broken. On the other hand, it is apparently *un*broken. What is the solution of the puzzle? Observe that the lower components of the superfield incorporating $h^{i\bar{j}}_{\alpha\bar{\beta}}\chi^{\alpha}_{i}\overline{\chi^{\beta}_{j}}$ are not invariant with respect to the action of the group G. In other words, there is no G-invariant operator whose supertransformation reduces to $h^{i\bar{j}}_{\alpha\bar{\beta}}\chi^{\alpha}_{i}\overline{\chi^{\beta}_{j}}$: any possible candidate has a lower symmetry. The solution is quite similar to the problem of the gluino condensate $\bar{\lambda}\lambda$ in gluodynamics. Although $\bar{\lambda}^a\lambda^a = \{Q_{\alpha},(A^a_{\mu}\gamma_{\mu}\lambda^a)_{\alpha}\}(4i)^{-1}$ (Q is the supercharge) the vacuum expectation value $\langle\bar{\lambda}\lambda\rangle \neq 0$ does not seem to result in the supersymmetry breaking in the gauge-invariant sector.

Note added in proof

Recently two of us (A.M. and A.P.) succeeded in applying the method of this paper to the calculation of the beta function in $N=4$ supersymmetric 2-dim sigma models, see preprint ITEP-131, 1984 and ZhETF Pis'ma, 40 (1984) 38.

Appendix

In this appendix we show how one can find the one-loop β-function of the sigma models I–IV in standard perturbation theory. Recall that at the one-loop level fermions give no contribution to the β-function [16] so that one can consider purely bosonic theories.

Given any homogeneous space G/H, the Lie algebra of the group G is naturally decomposed into the sum

$$\mathrm{L_G} = \mathrm{L_H} + \mathrm{L_{G/H}}$$

of the Lie algebra $\mathrm{L_H}$ of H and the space (not algebra) $\mathrm{L_{G/H}}$. The latter represents the tangent space of the manifold G/H. Correspondingly, some generators T^a of the fundamental representation of the $\mathrm{L_G}$ algebra belong to $\mathrm{L_H}$ (we denote them by $T^{a'}$), while the remainder belong to $\mathrm{L_{G/H}}$ ($T^{a''}$).

Let us use the original formulation of the σ-models, which is based on exponential parametrization, with the following lagrangian:

$$\mathcal{L} = \frac{1}{g^2}\int \mathrm{Tr}\left(A^+\partial_{\mu}A\right)^2 \mathrm{d}^2x,$$

where

$$A = \exp\left(i\omega^{a''}T^{a''}\right).$$

Then it seems quite obvious that up to an overall numerical (non-parametric) factor the one-loop coefficient of the β-function coincides with the eigenvalue of the quadratic Casimir operator [35]*:

$$f^{a''cd}f^{b''cd} = b_{G/H}\delta^{a''b''}. \tag{A.1}$$

If G/H is a symmetric space then [22]

$$[L_H L_{G/H}] \subset L_{G/H},$$
$$[L_{G/H} L_{G/H}] \subset L_H.$$

In other words, among the structure constants only $f^{a'b'c'}$ and $f^{a''b''c'}$ are non-vanishing, while $f^{a''b'c'} = f^{a''b''c''} = 0$.

But if it is so, the following relations are valid:

$$2f^{a''c''d'}f^{b''c''d'} = b_{G/H}\delta^{a''b''}$$
$$\Rightarrow 2f^{a''c''d'}f^{a''c''d'} = d_{G/H}b_{G/H},$$

where $d_{G/H}$ stands for the real dimension of the manifold G/H, $d_{G/H} = d_G - d_H$. Moreover,

$$f^{a'c'd'}f^{b'c'd'} = b_H\delta^{a'b'}$$
$$\Rightarrow f^{a'c'd'}f^{a'c'd'} = d_H b_H,$$
$$f^{acd}f^{bcd} = b_G\delta^{ab}$$
$$\Rightarrow 3f^{a''c''d'}f^{a''c''d'} + f^{a'c'd'}f^{a'c'd'} = d_G b_G.$$

These three equalities immediately lead to

$$b_{G/H} = \frac{2}{3d_{G/H}}(d_G b_G - d_H b_H). \tag{A.2}$$

Of course, the groups G and H have been assumed simple. The generalization of eq. (A.2) for $H = H_1 \otimes \cdots \otimes H_l$ is obvious:

$$b_{G/H} = \frac{2}{3d_{G/H}}\left(d_G b_G - \sum_{\alpha=1}^{l} d_{H_\alpha} b_{H_\alpha}\right). \tag{A.3}$$

* Note also that the curvature tensor of any symmetric space (cf., e.g. [22]) $R_{a''b''c''d''}$ is proportional to $f^{a''b''e'}f^{c''d''e'}$; this assertion must be compared with the analysis of sect. 2 and with eq. (A.1).

For the Grassmann models

$$\mathrm{X_I} = \mathrm{SU}(m+n)/\mathrm{SU}(m)\otimes\mathrm{SU}(n)\otimes\mathrm{U}(1),$$

$$\mathrm{G} = \mathrm{SU}(m+n), \qquad \mathrm{H} = \mathrm{SU}(m)\otimes\mathrm{SU}(n)\otimes\mathrm{U}(1).$$

Since

$$b_{\mathrm{SU}(n)} = n, \qquad d_{\mathrm{SU}(n)} = n^2 - 1,$$

$$b_{\mathrm{U}(1)} = 0, \qquad d_{\mathrm{U}(1)} = 1,$$

the one-loop β-function of the $\mathrm{X_I}$ model is proportional to $(m+n)$

$$b_{\mathrm{I}} \sim (m+n). \tag{A.4}$$

Likewise, eq. (A.2) can be applied to the other spaces:

$$\mathrm{X_{II}} = \mathrm{Sp}(n)/\mathrm{SU}(n)\otimes\mathrm{U}(1),$$

$$b_{\mathrm{Sp}(n)} = 2(n+1), \qquad d_{\mathrm{Sp}(n)} = n(2n+1),$$

$$b_{\mathrm{II}} \sim 2(n+1); \tag{A.5}$$

$$\mathrm{X_{III}} = \mathrm{SO}(2n)/\mathrm{SU}(n)\times\mathrm{SO}(2),$$

$$b_{\mathrm{SO}(n)} = n-2, \qquad d_{\mathrm{SO}(n)} = \tfrac{1}{2}n(n-1),$$

$$b_{\mathrm{III}} \sim 2(n-1); \tag{A.6}$$

$$\mathrm{X_{IV}} = \mathrm{SO}(n+2)/\mathrm{SO}(n)\otimes\mathrm{SO}(2),$$

$$b_{\mathrm{IV}} \sim n. \tag{A.7}$$

It is worth noting that complete calculation of the one-loop coefficients for symmetric spaces is known in the literature (see, e.g. [36]). In all cases the results quoted in eqs. (A.4)–(A.7) coincide with those of ref. [36].

In conclusion, it may be in order to recall why the fermions do not contribute to the one-loop β-function. To understand this fact it suffices to look at the lagrangian

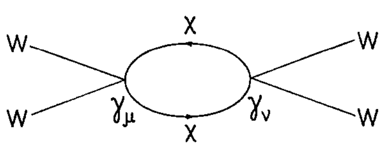

Fig. 1.

(27). The fermion tadpole is obviously proportional to $\mathrm{Tr}(G(x,x)\gamma_\mu)$, G being the fermion Green function. Hence, we get zero. As far as the graph in fig. 1 is concerned, it has no logarithm in the two-dimensional theory because of the transversality of the fermion vertex (conservation of the vector current).

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