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FERMIONIC ZERO MODES FOR COSMIC STRINGS

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In the presence of a cosmic string, we determine separately the number of right- and left-moving transverse zero modes of the Dirac equation for $2n$ chiral fermions which get their masses through couplings to various scalar fields with arbitrary phase variations around the string. This result provides us with an unambiguous criterion for string superconductivity in "realistic" grand unified and superstring inspired models. As an application, we determine the zero modes of a superheavy L-R string of a specific superstring inspired model. We find that this string is superconducting, through fermionic charge carriers, provided that electric charge and colour is conserved on the string core.

1. Introduction

The existence of transverse zero energy modes of the Dirac equation in the presence of a string can make the string superconducting [1]. This effect opened up the possibility of a variety of new phenomena. In particular, it has been suggested that cosmic strings in some grand unified models behave as superconducting loops of wire. They can have electromagnetic interactions with observable effects as sources of cosmic rays or synchrotron radiation [1,2]. What has attracted even more attention is a proposal put forward by Ostriker, Thompson and Witten [3]. They suggested that such superconducting current carrying loops, which lost their energy electromagnetically at an early epoch, may in fact explain the big voids observed today in the distribution of galaxies as well as the formation of galaxies themselves. The properties of superconducting cosmic strings are still under investigation [4] and no doubt there will be more suggestions about their role in cosmology. Many interesting models are known which have superconducting strings with fermionic charge carriers. These include SO(10) and E_6 GUT models [1], axionic string models [5] and, more recently, superstring inspired models [6]. What do those theories have in common? There are one or more fermions which get their mass through couplings to scalar Higgs fields which vary in phase as one moves around the string. The simplest system of this kind has been studied by Jackiw and Rossi [7]. They considered a single massive fermion acquiring its mass through a Yukawa interaction with a single scalar field which varies in phase by $2\pi n$ (n is an integer) around

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the string. They found $|n|$ zero modes. Their result is in agreement with a related index theorem [8]. So, in this simple case, the string is superconducting if $n \neq 0$. Unfortunately, the above analysis leaves out most cases of physical interest where we have several scalar Higgs fields with varying phases around the string, coupled to several fermion fields. This situation arises in realistic GUTS as well as superstring inspired models. The purpose of this article is to extend the analysis of ref. [7] in order to include these very important cases.

The problem we are dealing with can be simply stated: Given the fermionic mass matrix in the presence of a string, we would like to determine all the fermion zero modes in order to study the superconductivity of the string. In a previous paper [9] we proved an index theorem which sets a lower bound to the number of zero modes. This provides us with an unambiguous criterion for superconductivity of strings, if the index $I \neq 0$. In the present article we have succeeded in determining separately the number of right- and left-moving transverse fermionic zero modes (see ref. [1]). The result is consistent with the index theorem and goes beyond that. It is especially useful in the case of zero index where string superconductivity was previously ambiguous. In deriving our results, as we shall explain shortly, we have made only very simple assumptions about the fermion mass matrix and, thus, we believe that all cases of physical interest are included. As an interesting application we will study the zero modes in the stable string that arises in a superstring inspired model [10, 11] when the extra $U(1)_{L-R}$ gauge symmetry is broken. Provided that electric charge and colour is conserved on the string, it is found that this string is superconducting. In sect. 2 we describe the system and derive the equations for transverse fermionic zero modes. In sect. 3 the equations for right-movers are solved, after satisfying regularity conditions at the origin and at infinity; this determines the number of right-movers. The number of left-movers is similarly determined in sect. 4, and agreement with the index theorem is established. Finally, in sect. 5 we briefly describe a superstring inspired model and apply our results.

2. Description of the system

The Dirac equations for a system of fermions coupled to a string through their interactions with a number of scalar fields was derived in ref. [9]. We will repeat here some of the steps, in order to make our exposition self contained. Consider a theory with n left-handed fermion fields $\Psi_\alpha(= -\gamma_5\Psi_\alpha)$, n right-handed fermion fields X_α and the following fermionic part of the lagrangian

$$L = \bar{\Psi}_\alpha i \not\partial \Psi_\alpha + \bar{X}_\alpha i \not\partial X_\alpha - \bar{X}_\alpha M_{\alpha\beta}\Psi_\beta - \bar{\Psi}_\alpha M^{*}_{\alpha\beta} X_\beta, \quad \alpha, \beta = 1, 2, \dots, n. \tag{1}$$

We do not include gauge interactions, although present in general, since they are not expected to affect the calculation of zero modes. The $n \times n$ mass matrix $M_{\alpha\beta}$ contains the scalar fields together with the corresponding coupling constants.

In the background of a string (or vortex line) along the x^3-axis, we will assume that

$$M_{\alpha\beta} = S_{\alpha\beta}(r)\mathrm{e}^{iq_{\alpha\beta}\varphi}, \tag{2}$$

with r, φ the polar coordinates on the transverse x^1-x^2 plane. The integers $q_{\alpha\beta}$ are not arbitrary. As it is explained in ref. [9]

$$q_{\alpha\beta} = -\hat{q}_\alpha - q_\beta, \tag{3}$$

where $\hat{q}_\alpha$ and q_β are integer charges of the fermion fields $\overline{X}_\alpha$ and Ψ_β with respect to the generator of the vortex appropriately normalized. It is also assumed that the radial functions $S_{\alpha\beta}(r)$ are everywhere finite and become constant exponentially fast outside the core of the vortex. If the size of the core is L then, for large r,

$$S_{\alpha\beta}(r) \sim S^{\infty}_{\alpha\beta} + \mathrm{O}(\mathrm{e}^{-r/L}). \tag{4}$$

It is further assumed that all the fermions acquire non-zero masses in vacuo (away from the string).

Due to the cylindrical symmetry of the string, one can separate the transverse from the longitudinal degrees of freedom. The Dirac equations which follow from eq. (1) are

$$i\,\partial\!\!\!/\,\Psi_\alpha - M^{*}_{\alpha\beta}X_\beta = 0, \qquad i\,\partial\!\!\!/\,X_\alpha - M_{\alpha\beta}\Psi_\beta = 0. \tag{5}$$

Write

$$\Psi_\alpha(x^0, x^1, x^2, x^3) = f(x^0, x^3)\psi_\alpha(x^1, x^2)n_-,$$

$$X_\alpha(x^0, x^1, x^2, x^3) = f(x^0, x^3)\chi_\alpha(x^1, x^2)n_+. \tag{6}$$

Here Ψ_α, χ_α and f are ordinary functions and the constant spinors n_+ and n_- satisfy $\gamma^5 n_\pm = \pm n_\pm$ since Ψ_α and X_α are left- and right-handed fermions respectively. The last condition does not fix n_+ completely. There are two possibilities

(a) $\gamma^0 n_\pm = -\gamma^3 n_\pm$. From the longitudinal part of eqs. (5), we get

$$(\partial_0 - \partial_3)f = 0. \tag{7}$$

Thus, this mode will propagate along the string only in one direction (to the left) and is called left-mover (L-mover). The transverse zero-mode equations for L-movers are

$$(\partial_1 + i\,\partial_2)\psi_\alpha - iM^{*}_{\alpha\beta}\chi_\beta = 0, \qquad (\partial_1 - i\,\partial_2)\chi_\alpha + iM_{\alpha\beta}\psi_\beta = 0. \tag{8}$$

(b) The other possibility is $\gamma^0 n_\mp = \gamma^3 n_\pm$. With this ansatz we find the transverse zero-mode equations for right-movers (R-movers)

$$(\partial_1 - i\,\partial_2)\psi_\alpha - iM^{*}_{\alpha\beta}\chi_\beta = 0, \qquad (\partial_1 + i\,\partial_2)\chi_\alpha + iM_{\alpha\beta}\psi_\beta = 0. \tag{9}$$

Our task is, therefore, to solve eqs. (8) and (9). More precisely, for given $M_{\alpha\beta}$ we want to determine the number of normalizable solutions of the above equations. It is useful to recall here the result of ref. [9], that the difference between the number of R-movers and L-movers (the index) is given by

$$I = \text{no. R-movers} - \text{no. L-movers} = \sum_{\alpha=1}^{n} q_{\alpha\alpha}. \tag{10}$$

3. Constructing solutions for R-movers

Our first task is to try and separate the angular dependence of zero-modes. Starting with the transverse zero mode equations for R-movers (eq. (9)), we will use the following ansatz

$$\psi_\alpha(r,\varphi) = \mathrm{e}^{-i\varphi Q_\alpha}\psi_\alpha(r), \qquad \chi_\alpha(r,\varphi) = \mathrm{e}^{i\varphi \hat{Q}_\alpha}\chi_\alpha(r). \tag{11}$$

Here Q_α and $\hat{Q}_\alpha$ are integers which will have to be chosen appropriately so that the φ-dependence of eq. (9) cancels. When we substitute in eq. (9) and use eq. (2), we find that the necessary condition for this cancellation is that, for all α and β,

$$S_{\alpha\beta}(r)\left(q_{\alpha\beta} - \hat{Q}_\alpha - Q_\beta - 1\right) = 0. \tag{12}$$

The resulting equations are

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{Q_\alpha}{r}\right)\psi_\alpha - iS^{*}_{\alpha\beta}\chi_\beta = 0, \qquad \left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\hat{Q}_\alpha}{r}\right)\chi_\alpha + iS_{\alpha\beta}\psi_\beta = 0. \tag{13}$$

Without loss of generality we can assume that the matrix $S_{\alpha\beta}$ is irreducible. Then, from eqs. (3) and (12), we obtain

$$Q_\alpha = l - q_\alpha, \qquad \hat{Q}_\alpha = -l - \hat{q}_\alpha - 1, \quad \alpha = 1,\dots,n, \tag{14}$$

where l is an arbitrary integer which will be fixed later by regularity conditions. The $2n$ eqs. (13) constitute a first order linear system. We can suppress the indices if we introduce the notation

$$X(r)=\begin{bmatrix}\psi_1\\ \vdots\\ \psi_n\\ \chi_1\\ \vdots\\ \chi_n\end{bmatrix},\qquad Q=\operatorname{diag}(Q_1,\dots,Q_n,\hat{Q}_1,\dots,\hat{Q}_n),$$

$$A(r)=\left[\begin{array}{c|c}0 & iS^{\dagger}(r)\\ \hline -iS(r) & 0\end{array}\right]_{2n\times 2n}. \tag{15}$$

Then eq. (13) can be written as

$$(\mathrm{d}/\mathrm{d}r)X=(Q/r+A(r))X. \tag{16}$$

Our problem is, therefore, reduced to the determination of normalizable solutions of eq. (16). This system has $2n$ linearly independent solutions X_j, $j=1,2,\dots,2n$, but some X_j may in fact diverge. One has to examine the behaviour of the solutions near the origin and at infinity. In physical terms this means looking at distances inside the string core where the fermion fields must remain finite and at distances much larger than any inverse mass where the fields must decay sufficiently fast.

(a) At the origin eq. (16) has a singular point at $r=0$. We will assume that $A(r)$, which is essentially the radial part of the mass matrix, converges for $r<L$ (L could be the size of the string core) and can be written as

$$A(r)=\sum_{k=0}^{\infty}A_k r^k, \tag{17}$$

with A_k being constant matrices. Then one can use an asymptotic expansion to determine the solution of eq. (16). According to a general theorem due to Kneser (see ref. [12]) the $2n$ linearly independent solutions of eq. (16) are

$$X_j(r)=\sum_{k=0}^{\infty}r^{k+Q_j}p_{jk}(\ln r),\quad j=1,\dots,2n. \tag{18}$$

Here $Q_{n+\alpha}=\hat{Q}_\alpha$ ($\alpha=1,\dots,n$) and $p_{jk}(\ln r)$ are vector polynomials in $\ln r$ of degree smaller than $2n$. The polynomials p_{jk} can be determined recursively in terms of the constants A_k, and the series in eq. (18) converges for $0<r<L$. One can further show that when the matrix Q is diagonal, as here, p_{j0} are just the constant eigenvectors of Q, that is, $Qp_{j0}=Q_jp_{j0}$, for $j=1,\dots,2n$. From the above theorem

we conclude that the leading term of each solution X_j of eq. (16) is $r^{Q_j}p_{j0}$ and therefore only solutions with $Q_j \geq 0$ remain finite as $r \to 0$. If we denote by $m(l)$ the number of solutions which are finite at the origin for a given l, it follows from the above and from eq. (14) that

$$m(l) = \sum_{j=1}^{2n} \theta(Q_j) = \sum_{\alpha=1}^{n} \left(\theta(Q_\alpha) + \theta(\hat{Q}_\alpha)\right) = \sum_{\alpha=1}^{n} \left(\theta(l - q_\alpha) + \theta(-l - \hat{q}_\alpha - 1)\right), \tag{19}$$

with $\theta(x)$ the "step" function defined for any integer x,

$$\theta(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases} \tag{20}$$

It is also easy to see that

$$\theta(x) + \theta(-1 - x) = 1, \tag{21}$$

and therefore eq. (19) can also be written as

$$m(l) = n + \sum_{\alpha=1}^{n} \left[\theta(l - q_\alpha) - \theta(l + \hat{q}_\alpha)\right]. \tag{22}$$

(b) At infinity eq. (16) has asymptotically constant coefficients, since Q/r vanishes and $A(r)$ becomes constant (see eq. (4)). Denote

$$A_\infty \equiv \lim_{r \to \infty} A(r) = \left[\begin{array}{c|c} 0 & iS_\infty^\dagger \\ \hline -iS_\infty & 0 \end{array}\right]. \tag{23}$$

As we shall see the matrix A_∞ essentially determines the behaviour of solutions of eq. (16) as $r \to \infty$ and we will need to know its eigenvalues. The masses of the fermions in vacuo $\mu_1, \mu_2, \ldots, \mu_n$ are determined by the constants S_∞. Note that $\mathrm{diag}(\mu_1, \ldots, \mu_n) = U_R^+ S_\infty U_L$, with U_L and U_R two unitary transformations that act on left- and right-handed fields. Then one can easily see that A_∞ is unitary equivalent to

$$A'_\infty = \left[\begin{array}{ccc|ccc} & & & \mu_1 & & \\ & 0 & & & \ddots & \\ & & & & & \mu_n \\ \hline \mu_1 & & & & & \\ & \ddots & & & 0 & \\ & & \mu_n & & & \end{array}\right]. \tag{24}$$

It, therefore, follows that the eigenvalues of A'_∞, and those of A_∞ too, are $\pm\mu_1, \pm\mu_2, \ldots, \pm\mu_n$.

Can one replace $Q/r + A(r)$ by its asymptotic value A_∞ in eq. (16) in order to determine the large r behaviour of solutions? This may look reasonable, but it is not always correct. Fortunately, in our case, we can apply Levinson's theorem [13]. It is essential that the eigenvalues of A_∞ are all distinct (this is a physically acceptable assumption for the masses of the fermions). Then, for large r, eq. (16) has $2n$ linearly independent solutions

$$X(r) \sim \boldsymbol{\alpha}_{\pm i} \exp\left[\pm\mu_i r + \int^r f_{\pm i}(\tau)\,\mathrm{d}\tau\right], \quad i = 1, \ldots, n, \tag{25}$$

with $\boldsymbol{\alpha}_{\pm i}$ the eigenvectors of A_∞ which correspond to the eigenvalues $\pm\mu_i$ respectively, and $f_{\pm i}(r)$ $2n$ functions which vanish as $r \to \infty$ like $1/r$ or faster. We will not give here any more details about the above result or its derivation; the interested reader should consult ref. [13]. For our purpose it is sufficient to note, from eq. (25), that the solutions with $+\mu_i$ diverge, whereas those with $-\mu_i$ converge, as $r \to \infty$.

The solutions of eq. (16) form a $2n$-dimensional linear space. The two sets of solutions which we have just described, at the origin and at infinity, are two different bases of this space. We will now need to connect the two sets, in order to determine the number of solutions which are everywhere finite. Suppose that $X_j(r)$, $j = 1, \ldots, 2n$ is the set described by eq. (18) and, for simplicity, assume that these solutions are rearranged so that, for $j = 1, \ldots, m \equiv m(l)$, the X_j's are finite as $r \to 0$. Also denote by $y_j(r)$ the set of solutions (25), and suppose that, for $j = 1, \ldots, n$, y_j converges as $r \to \infty$. The two sets must be related by

$$X_j(r) = \sum_{k=1}^{2n} b_{jk}\, y_k(r), \quad j = 1, 2, \ldots, 2n. \tag{26}$$

The constants b_{jk} cannot be explicitly determined and, in general, depend on the mass matrix in a complicated way. We will assume that, for a general irreducible mass matrix, one gets a constant matrix b without any singular minors. Clearly, the most general solution finite at the origin, depends on $m \equiv m(l)$ arbitrary constants c_j

$$X(r) = \sum_{j=1}^{m} c_j X_j(r). \tag{27}$$

These constants c_j must be chosen so that $X(r)$ does not diverge at infinity. Using eq. (26), the condition is

$$\sum_{j=1}^{m} c_j b_{jk} = 0, \quad \text{for} \quad k = n+1, \ldots, 2n. \tag{28}$$

To determine the number of independent solutions of the above equation, one, of course, needs to know the rank of the $m \times n$ truncated matrix b_{jk} ($j = 1, \ldots, m$; $k = n+1, \ldots, 2n$). From what was said before about this matrix, it is natural to assume that its rank is equal to its smallest dimension. It then follows that the number of finite solutions of eq. (16) can be written as

$$\text{no. of solutions} = \tfrac{1}{2}(m - n + |m - n|). \tag{29}$$

The above expression corresponds to a given choice for Q_α and $\hat{Q}_\alpha$ in the ansatz of eq. (11). When we vary the parameter l in eq. (14) we get inequivalent solutions and this brings us to our main result. We conclude that the total number of R-moving zero modes is

$$\begin{aligned}\text{no. R-movers} &= \tfrac{1}{2}\sum_l \left[(m(l) - n) + |m(l) - n|\right] \\ &= \tfrac{1}{2}\sum_l \left[\left(\sum_{\alpha=1}^{n} \theta(l - q_\alpha) - \theta(l + \hat{q}_\alpha)\right) + \left|\sum_\alpha \theta(l - q_\alpha) - \theta(l + \hat{q}_\alpha)\right|\right].\end{aligned} \tag{30}$$

The sum over l extends over all integers. (But note that there are contributions only for finite range of values of l.) Once we know the fermion charges $q_\alpha, \hat{q}_\alpha$ with respect to the generator of the string, the above expression can be easily computed. A simple example will illustrate the use of the above result. Suppose that there are four chiral fermions $\bar{\chi}_1$, $\bar{\chi}_2$, ψ_1, ψ_2 and their charges $\hat{q}_1, \hat{q}_2, q_1, q_2$ with respect to the broken U(1) generator are $0, 1, -1, 0$ respectively. Then, in the presence of a string, we may have the following mass terms

$$\Delta L = \bar{\chi}_1 \psi_1 M e^{i\varphi} + \bar{\chi}_2 \psi_2 \mu\, e^{-i\varphi} + \bar{\chi}_1 \psi_2 \lambda + \bar{\chi}_2 \psi_1 \lambda' + \text{h.c.}. \tag{31}$$

Here M, μ, λ and λ' are v.e.v.'s of Higgs fields (including Yukawa coupling constants), and φ is the azimuthal angle around the string. If $\lambda = \lambda' = 0$, the mass matrix is diagonal and it is well known that the first two terms will give us one R-mover and one L-mover. When λ, λ' are non-zero, we can apply our results. We find, for Q in eq. (14),

$$\hat{Q}_1 = -l - 1, \qquad \hat{Q}_2 = -l - 2, \qquad Q_1 = l + 1, \qquad Q_2 = l. \tag{32}$$

At least two of the above eigenvalues are negative and, consequently, $m(l) \leq 2$. We can, thus, see from eq. (30) that there are no R-movers. Since the index of the above mass matrix is zero, the number of L-movers will also be zero. We conclude that even a small φ-independent mixing forbids the existence of zero modes. Alternatively, if the charges of the fermions are $0, 0, -1, 1$ the following couplings are allowed

$$\Delta L' = \bar{\chi}_1 \psi_1 M e^{i\varphi} + \bar{\chi}_2 \psi_2 \mu\, e^{-i\varphi} + \bar{\chi}_1 \psi_2 \lambda\, e^{-i\varphi} + \bar{\chi}_2 \psi_1 \lambda' e^{i\varphi} + \text{h.c.}. \tag{33}$$

In this case one has

$$\hat{Q}_1 = -l-1, \qquad \hat{Q}_2 = -l-1, \qquad Q_1 = l+1, \qquad Q_2 = l-1, \tag{34}$$

and, from eq. (30), we find that there is one R-mover (for the value $l=1$) and consequently, one L-mover.

The previous analysis illustrates the effect of the φ-independent matrix elements of the mass matrix on the number of R-movers. This can also be understood in terms of a more general theorem which follows directly from eq. (30): The number of R-movers determined by the $2n$ charges $(\hat{q}_1, \dots, \hat{q}_n, q_1, \dots, q_n)$ is equal to the number of R-movers determined by the $2n-2$ charges $(\hat{q}_1, \dots, \hat{q}_{n-1}, q_1, \dots, q_{n-1})$, if $\hat{q}_n + q_n = 0$. Consequently, if a particular matrix element M_{ij} of the mass matrix remains constant around the string $(q_{ij} = 0)$, then, when calculating the number of R-movers, we can get a simplification by removing the ith row and jth column of $q_{\alpha\beta}$. If there are several constant matrix elements we remove all the corresponding rows and columns; this can lead to considerable simplification in applications.

4. The number of L-movers and the index

One can go through exactly the same steps in order to solve eq. (8) for L-movers. Use the ansatz

$$\psi_\alpha = e^{iQ_\alpha\varphi}\psi_\alpha(r), \qquad \chi_\alpha = e^{-i\hat{Q}_\alpha\varphi}\chi_\alpha(r). \tag{35}$$

The resulting condition for the integers Q_α and $\hat{Q}_\alpha$ is

$$q_{\alpha\beta} + \hat{Q}_\alpha + Q_\beta + 1 = 0. \tag{36}$$

It, then, follows that the fields ψ_α, χ_α satisfy the same eq. (13), which was solved previously. Eq. (36) implies

$$Q_\alpha = -l + q_\alpha - 1, \qquad \hat{Q}_\alpha = l + \hat{q}_\alpha, \quad \alpha = 1, \dots, n. \tag{37}$$

Consequently, the number of L-moving zero modes is given by the expression

$$\text{no. L-movers} = \tfrac{1}{2}\sum_l \left[\left(\sum_{\alpha=1}^{m} \theta(l+\hat{q}_\alpha) - \theta(l-q_\alpha)\right) + \left|\sum_\alpha \theta(l+\hat{q}_\alpha) - \theta(l-q_\alpha)\right|\right]. \tag{38}$$

If we now compare the two expressions in eqs. (30) and (38), we can rederive the index [9]

$$I = \text{no. R-movers} - \text{no. L-movers} = \sum_{l} \sum_{\alpha=1}^{n} \left(\theta(l - q_\alpha) - \theta(l + \hat{q}_\alpha)\right)$$

$$= \sum_{\alpha} \left(\sum_{l'} \theta(l') - \theta(l' + q_\alpha + \hat{q}_\alpha) \right) = \sum_{\alpha=1}^{n} (-q_\alpha - \hat{q}_\alpha) = \sum_{\alpha} q_{\alpha\alpha} = q \,. \tag{39}$$

5. An application: Superconductivity of the L–R string

We will now apply our results to a particular "realistic" model inspired by superstring theory. As it is well known, there is a whole class of such models which are based on rank five or six subgroups of E_6. We will concentrate on the model introduced in ref. [10] and further studied in ref. [11]. It is based on a subgroup G of E_6 which includes one extra U(1) factor apart from the standard model G = $SU(2)_c \times SU(2)_L \times U(1)_Y \times U(1)_{L-R}$. The breaking of the extra $U(1)_{L-R}$ gives rise to cosmic strings. The field content of the model is determined by various phenomenological and cosmological considerations and consists of three families of chiral fields which transform as 27's of E_6. There are also additional incomplete 27, $\overline{27}$'s. These include $SU(3)_c \times SU(2)_L \times U(1)_Y$ singlet fields S_i, $i = 1, 2, 3$ (and their mirrors $\tilde{S}_i$), which are responsible for the breaking of $U(1)_{L-R}$ at an intermediate scale. The complete list of the fields and their quantum numbers is given in table 1 of ref. [10]; the couplings of the fields are also listed there and will be used in what follows in order to determine some of the fermion mass matrices in the presence of a string.

By definition, around an L–R string one performs a non-trivial L–R transformation. Consequently, all the scalar fields which have non-zero $U(1)_{L-R}$ charge and are singlets with respect to $SU(3)_c \times SU(2)_L \times U(1)_Y$ acquire a phase around the string. One can then write, for these fields (N belongs to the complete 27's and has no mirrors) [11]

$$S_i = \langle S_i \rangle e^{i\varphi}, \qquad \tilde{S}_i = \langle S_i \rangle e^{-i\varphi} \quad (i = 1, 2, 3), \qquad N = \langle N \rangle e^{i\varphi}. \tag{40}$$

Here $\langle S_i \rangle$ and $\langle N \rangle$ are vacuum expectation values of the scalar singlets ($\langle N \rangle$ is of order M_W) and φ the azimuthal angle around the string. For the Weinberg–Salam Higgs doublets H and $\overline{H}$, one can determine only the relative phase and we may write

$$H = \langle H \rangle^{in\varphi}, \qquad \overline{H} = \langle \overline{H} \rangle e^{-i(n-1)\varphi}, \quad n = \text{integer}. \tag{41}$$

We will examine the down quark mass matrix by first ignoring fermion states from incomplete multiplets. The terms $gg_c S$, $gD_c N$ and $Q_D D_c H$ in the superpotential give

$$
M = \begin{array}{c} \\ g_c \\ D_c \end{array}
\begin{array}{c} \begin{array}{cc} \quad g \quad & \quad Q_D \quad \end{array} \\
\left(\begin{array}{c|c} \langle S_1\rangle e^{i\varphi} & 0 \\ \hline \langle N\rangle e^{i\varphi} & \langle H\rangle e^{in\varphi} \end{array}\right)_{18\times 18} \end{array}
\tag{42}
$$

Here, the heavy quark states g, g_c mix with the "ordinary" d-quark states D_c, Q_D. Since the fermion fields come in three families and three colours each, the blocks are of dimension 9×9. When $n \neq -1$, the index $I \neq 0$ and the existence of zero modes and, consequently, string superconductivity is guaranteed. The case $n=-1$ is of interest because it gives $I=0$ and it was previously unknown whether the L-R string is still superconducting or not. However, our results as summarized in eqs. (30) and (38) imply that, in this case, there are nine R-movers and nine L-movers (one for each family and each colour), and the cosmic string is superconducting. There may be of course additional zero modes coming from the lepton and down quark sectors. This result also holds when we introduce extra fermion fields from the incomplete 27 and $\overline{27}$'s. Then enlarged down quark mass matrix takes the form

$$
M = \begin{array}{c} \\ \tilde{g} \\ g_c' \\ g_c \\ D_c \end{array}
\begin{array}{c} \begin{array}{cccc} \quad g' \quad & \quad \tilde{g}_c \quad & \quad g \quad & \quad Q_D \quad \end{array} \\
\left[\begin{array}{c|c|c|c} \dfrac{\langle S_3\rangle^2}{M_c} & \langle S_3\rangle e^{-i\varphi} & \dfrac{\langle S_3\rangle^2}{M_c} & 0 \\ \hdashline \langle S_1\rangle e^{i\varphi} & 0 & \langle S_1\rangle e^{i\varphi} & 0 \\ \hdashline \langle S_1\rangle e^{i\varphi} & 0 & \langle S_1\rangle e^{i\varphi} & 0 \\ \hdashline \langle N\rangle e^{i\varphi} & 0 & \langle N\rangle e^{i\varphi} & \langle H\rangle e^{-i\varphi} \end{array}\right] \end{array} .
\tag{43}
$$

The first two diagonal blocks are φ-independent. According to the theorem of the last section the new rows and columns will not change the number of zero-modes and therefore superconductivity persists. The above results imply that the L-R string is superconducting. But there is a caveat. The mass terms which were written are dictated by conservation of electric charge and colour, which of course is true outside the core of the string. We do not know if this is also true on the string. Since the exact dynamics of the breaking of $U(1)_{L-R}$ is unknown, one may also include fermion bilinears which do not conserve electric charge and/or colour on the string

[14]. For those more exotic cases the mass matrix that one can write on the string for one family of fermions in the 27 of E_6 is

$$\bar{\psi}_i M_{ij} \psi_j^c, \quad i, j = 1, \ldots, 27.$$

In order to apply the analysis of the previous sections one needs to know the phases of M_{ij} around the L–R string, and for this one needs to know the charges C_i of the fermions ψ_i with respect to the generator C of the string. We have examined the various possibilities and concluded

(1) The index for the large matrix M_{ij} is always zero. This follows directly from

$$\sum_i C_i = 0.$$

(2) If the charges C_i are zero or come in positive–negative pairs, then there are no zero modes. This is possible, for example, when the 27 of E_6 take the following C_i's

ψ_i	Q	U_c	D_c	g	g_c	L	E_c	H	$\bar{H}$	N
C_i	0	2	−1	0	−1	1	−2	1	−2	1

In those cases only, fermionic zero modes no longer exist and superconductivity due to fermionic carriers disappears. (However there may be bosonic superconductivity [14].)

(3) Introducing more families or extra fermion fields from incomplete 27 and $\overline{27}$ will not change these results.

Other conductivity phenomena in the absence of zero modes have been studied in refs. [14, 15].

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