TOPOLOGICAL CHARGES FOR $N = 4$ SUPERSYMMETRIC GAUGE THEORIES AND MONOPOLES OF SPIN 1

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The central charges in the supersymmetry algebra of the $N = 4$ supersymmetry gauge theory are obtained. When spontaneous symmetry breaking is imposed it is shown that the spins of the topological monopole states should be identical to those of the massive elementary particles.

An exciting recent development in field theory has been in the work of Witten and Olive [1] who showed how there can be exact mass formulae in certain supersymmetric theories which embrace both the elementary particle and topological objects such as solitons or monopoles. The mass formulae survive quantisation by requiring that the supersymmetric charge algebra be saturated in a particular fashion involving central charges which can arise as surface terms when spontaneous symmetry breakdown occurs. Witten and Olive considered mainly the $N = 2$ extended supersymmetric non abelian gauge theory which may have SSB and classical monopole solutions satisfying certain first order equations due to Bogomolny [2]. The $N = 2$ theory can be derived by dimensional reduction of the supersymmetric gauge theory in six spacetime dimensions [3] which simplifies the manipulations involved in discussing supersymmetric monopoles [4].

Our initial intention is to extend these results to the $N = 4$ extended supersymmetric theory which is the maximal theory with spins $\leqslant 1$ and has some remarkable properties [5,6]. The effect of central charges in this theory has been considered by Fayet [7], but to obtain explicit expressions we follow the route of dimensional reduction from ten spacetime dimensions [3]. The Lagrangian is, for an arbitrary gauge group [‡1]

$$\mathcal{L} = -\tfrac{1}{4}(F^{AB}, F_{AB}) + \tfrac{1}{2}\mathrm{i}(\bar{\lambda}, \Gamma^A D_A \lambda), \quad F_{AB} = \partial_A A_B - \partial_B A_A - \mathrm{i}e[A_A, A_B], \quad D_A\lambda = \partial_A\lambda - \mathrm{i}e[A_A, \lambda], \tag{1}$$

The supersymmetry transformations take the form [3]

$$\delta\lambda = \Sigma^{AB} F_{AB}\alpha, \qquad \delta A_A = \mathrm{i}\bar{\alpha}\Gamma_A\lambda = -\mathrm{i}\bar{\lambda}\Gamma_A\alpha, \tag{2}$$

with α a constant anticommuting spinor and it is essential to simultaneously impose the Majorana and Weyl conditions as allowed in 10 dimensions

$$(1 + \Gamma_{11})\lambda = 0, \qquad \bar{\lambda}(1 - \Gamma_{11}) = 0, \qquad \bar{\lambda} = \lambda^{\mathrm{T}} C. \tag{3}$$

These constraints ensure, if also applied to α, that $\mathcal{L}$ is invariant up to a total divergence. The consequential supercurrent is

$$\mathcal{J}^A = \mathrm{i}\Sigma^{BC}\Gamma^A(F_{BC}, \lambda), \qquad \bar{\mathcal{J}}^A = (\mathcal{J}^A)^{\mathrm{T}} C, \tag{4}$$

‡1 $A = 0, \ldots 9$ is a 10 dimensional index. Γ^A are 32×32 Dirac matrices $\{\Gamma^A, \Gamma^B\} = 2g^{AB} = 2\,\mathrm{diag.}\,(1, -1, \ldots -1)$. $\Gamma^{A \ldots X} = \Gamma^{[A} \ldots \Gamma^{X]}$, $\Gamma_{11} = \Gamma_0 \ldots \Gamma_9$, $(\Gamma_{11})^2 = 1$, $\Sigma^{AB} = \frac{1}{2}\Gamma^{AB} = \frac{1}{4}[\Gamma^A, \Gamma^B]$. The charge conjugation matrix C satisfies $(C\Gamma^A)^{\mathrm{T}} = C\Gamma^A$, $C^{\mathrm{T}} = -C$. A_A, λ are fields in the Lie algebra $\mathcal{L}_\mathcal{G}$ the gauge group $\mathcal{G}$ with a symmetric invariant scalar product (,). We may write any such element of $\mathcal{L}_\mathcal{G}$ as $X = X^a T_a$ with $(T_a, T_b) = \delta_{ab}$, $[T_a, T_b] = \mathrm{i}C_{abc}T_c$, C_{abc} totally antisymmetric.

which is conserved by virtue of the equations of motion. To determine the supersymmetry algebra we investigate transformations of the super-current, so from (4) and (2)

$$\delta \bar{\mathcal{G}}^A = 2\mathrm{i}\mathcal{T}^{AB}\bar{\alpha}\Gamma_B - \tfrac{1}{4}\mathrm{i}(F_{BC}, F_{DE})\bar{\alpha}\Gamma^{ABCDE} + \tfrac{1}{16}(\partial^D J^{ABC} - g^{AD}\partial_E J^{EBC})\bar{\alpha}\Gamma_{BCD}\,, \tag{5}$$

where terms vanishing under the action of the equation of motion for λ are discarded [‡2] and $\mathcal{T}^{AB}, J^{ABC}$ are the energy momentum tensor, fermion current respectively

$$\mathcal{T}^{AB} = (F^{AC}, F^B_C) + \tfrac{1}{4} g^{AB}(F^{CD}, F_{CD}) + \tfrac{1}{2}\mathrm{i}(\bar{\lambda}, \Gamma^A D^B \lambda)\,, \qquad J^{ABC} = (\bar{\lambda}, \Gamma^{ABC}\lambda)\,. \tag{6}$$

The ten dimensional theory just described attains potential realism by reduction to 4 dimensions neglecting any dependence on $x^4 \ldots x^9$. We follow ref. [3] and let [‡3]

$$\Gamma^\mu = \gamma^\mu \otimes 1 \otimes \sigma_3\,, \quad \Gamma^{3+i} = 1 \otimes \alpha^i \otimes \sigma_1\,, \quad \Gamma^{6+j} = \gamma_5 \otimes B^j \otimes \sigma_3\,, \quad \Gamma_{11} = 1 \otimes 1 \otimes \sigma_2\,, \quad C = \mathcal{C} \otimes 1 \otimes 1\,, \tag{7}$$

where $\alpha^i, \beta^j, i, j = 1,2,3$ are 4×4 real antisymmetric matrices satisfying the algebra

$$\{\alpha^i, \alpha^j\} = \{\beta^i, \beta^j\} = -2\delta^{ij}\,, \quad [\alpha^i, \alpha^j] = -2\epsilon^{ijk}\alpha^k\,, \quad [\beta^i, \beta^j] = -2\epsilon^{ijk}\beta^k\,, \quad [\alpha^i, \beta^j] = 0\,. \tag{8}$$

The gauge fields in the extra dimensions now become scalars

$$A_{3+i} \equiv A_i\,, \qquad A_{6+j} \equiv B_j \tag{9}$$

and the spinor field becomes a set of 4 Majorana fields on which α^i, β^j act

$$\lambda = \psi \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix}, \qquad \bar{\lambda} = \bar{\psi} \otimes \frac{1}{\sqrt{2}}(1, -\mathrm{i})\,, \qquad \bar{\psi} = \psi^{\mathrm{T}}\mathcal{C}\,. \tag{10}$$

The resultant Lagrangian for the 4 dimensional theory is

$$\mathcal{L} = -\tfrac{1}{4}(F^{\mu\nu}, F_{\mu\nu}) + \tfrac{1}{2}(D^\mu A_i, D_\mu A_i) + \tfrac{1}{2}(D^\mu B_j, D_\mu B_j)$$

$$+ \tfrac{1}{2}\mathrm{i}(\bar{\psi}, \gamma^\mu D_\mu \psi) - \tfrac{1}{2}e(\bar{\psi}, [\mathrm{i}\alpha^i A_i + \mathrm{i}B^j B_j \mathrm{i}\gamma_5, \psi]) - V(A, B)\,,$$

$$V(A,B) = -\tfrac{1}{4}e^2([A_i, A_j], [A_i, A_j]) - \tfrac{1}{4}e^2([B_i, B_j], [B_i, B_j]) - \tfrac{1}{2}e^2([A_i, B_j], [A_i, B_j])\,. \tag{11}$$

This theory enjoys a global O(6) symmetry which is a residuum of rotational invariance in the extra dimensions. The supersymmetry transformations are obtainable directly from (2) with $\alpha = \epsilon \otimes 2^{-1/2}\binom{1}{-\mathrm{i}}$ where ϵ is a set of 4 constant anticommuting Majorana spinors so the resulting 4 dimensional field theory is a realisation of $N = 4$ supersymmetry,

$$\delta\psi = \sigma^{\mu\nu}F_{\mu\nu}\epsilon - \gamma^\mu D_\mu \mathrm{i}(\alpha^i A_i + \beta^j B_j \mathrm{i}\gamma_5)\epsilon$$

$$+ \tfrac{1}{2}\mathrm{i}\epsilon^{ijk}\alpha^k e[A_i, A_j]\epsilon + \tfrac{1}{2}\mathrm{i}\epsilon^{ijk}\beta^k e[B_i, B_j]\epsilon + \alpha^i\beta^j e[A_i, B_j]\gamma_5\epsilon\,,$$

$$\delta A_\mu = \mathrm{i}\bar{\epsilon}\gamma_\mu\psi\,, \qquad \delta A_i = \bar{\epsilon}\alpha^i\psi\,, \qquad \delta B_j = \bar{\epsilon}B^j\gamma_5\psi\,. \tag{12}$$

[‡2] We employ the relevant Fierz identity so that if $\gamma, \bar{\delta}$ are commuting 10 dimensional Weyl spinors $(1 + \Gamma_{11})\gamma = \bar{\delta}(1 - \Gamma_{11}) = 0$, $\gamma\bar{\delta} = \frac{1}{32}\bar{\delta}\Gamma_A\gamma\Gamma^A(1 + \Gamma_{11}) - \frac{1}{32}\frac{1}{6}\bar{\delta}\Gamma_{ABC}\gamma\Gamma^{ABC}(1 + \Gamma_{11}) + \frac{1}{32}\frac{1}{5!}\bar{\delta}\Gamma_{ABCDE}\gamma\Gamma^{ABCDE}$ and it is useful to note that $\bar{\delta}\Gamma_{ABCDE}\gamma\Gamma^{ABCDE}(1 - \Gamma_{11}) = 0$.

[‡3] Our matrix conventions follow Bjorken and Drell with $\gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ but $\sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$. $\mathcal{C}$ is the usual charge conjugation matrix $\mathcal{C}^{\mathrm{T}} = -\mathcal{C}$, $(\mathcal{C}\gamma^\mu)^{\mathrm{T}} = (\mathcal{C}\gamma^\mu)$. $B = \gamma^0\mathcal{C}^{-1}$ so $B\gamma^{\mu *}B^{-1} = -\gamma^\mu$, $B^\dagger B = 1$. We let $\mathrm{i}\sigma^{ij}\epsilon_{ijk} = \sigma_k$ so that $\mathcal{C} = \gamma^0\gamma_5\boldsymbol{\sigma}$ with $\frac{1}{2}\boldsymbol{\sigma}$ being the rotation generator acting as spinors $\epsilon^{0123} = 1$.

The corresponding conserved super-current can also be similarly read off by dimensional reduction from (4) with $\mathcal{J}^\mu = \widetilde{\mathcal{J}}^\mu \otimes 2^{-1/2}\binom{1}{\mathrm{i}}$

$$\widetilde{\mathcal{J}}^\mu = \mathrm{i}\sigma^{\alpha\beta}\gamma^\mu(F_{\alpha\beta},\psi) - \gamma^\nu\gamma^\mu\alpha^i(D_\nu A_i,\psi) - \mathrm{i}\gamma^\nu\gamma^\mu\beta^j\gamma_5(D_\nu B_j,\psi)$$
$$-\tfrac{1}{2}\gamma^\mu\epsilon^{ijk}\alpha^k e([A_i,A_j],\psi) - \tfrac{1}{2}\gamma^\mu\epsilon^{ijk}\beta^k e([B_i,B_j],\psi) + \mathrm{i}\gamma^\mu\gamma_5\alpha^i\beta^j e([A_i,B_j],\psi)\,. \tag{13}$$

The vacuum for this theory with Lagrangian (11) is undetermined. For the classical ground state any solution of $F_{\mu\nu} = 0, D_\mu A_i = D_\mu B_j = 0$ and also

$$V(A,B) = 0 \Leftrightarrow [A_i,A_j] = [B_i,B_j] = [A_i,B_j] = 0 \tag{14}$$

gives zero energy density and the inherent arbitrariness in solutions of (14) is not removed by quantum corrections due to the supersymmetry of the theory. We may impose spontaneous symmetry breakdown on the theory and introduce a scale by demanding

$$(A_i,A_i) + (B_j,B_j) = v^2 \tag{15}$$

as a boundary condition at infinity. For v non zero the gauge group $\mathcal{G}$ is then broken to $\mathcal{H}$ whose generators are those which commute with the solutions of (14) and (15), $\mathring{A}_i$, $\mathring{B}_j$ defining the vacuum. Clearly, depending on how large $\mathcal{G}$ is, $\mathring{A}_i$, $\mathring{B}_j$ may, at each space-time point, span a from 1 to 6 dimensional space in $\mathcal{L}_\mathcal{G}$ corresponding to a breaking of the global O(6) to O(5) or any of its rotational subgroups. Since $\mathring{A}_i$, $\mathring{B}_j$ are necessarily among the generators of $\mathcal{H}$ and commute with all other generators in $\mathcal{L}_\mathcal{H}$ there are from 1 to 6 abelian gauge fields after SSB. It is perhaps natural to consider just the minimal scheme whereby $\mathring{A}_i$, $\mathring{B}_j$ at each point are proportional to the same element of $\mathcal{L}_\mathcal{G}$ trivially giving (14). The O(6) symmetry ensures that all directions of symmetry breakdown in the A_i, B_j space are equivalent so that, for instance, we may then take $\mathring{A}_3$ as the only non zero element, and parity is manifestly conserved. In this case $\mathcal{H}$ has the structure $\mathrm{U}(1)\otimes \mathrm{K}$, at least locally. However in our subsequent discussion we allow the symmetry breaking to be arbitrary.

From the lagrangian (11) it is easy to display the mass matrices arising from SSB and the Higgs mechanism at work. For vectors and spinors [‡4]

$$\mu_v^2 = e^2(\mathrm{ad}\,\mathring{A}_i\,\mathrm{ad}\,\mathring{A}_i + \mathrm{ad}\,\mathring{B}_j\,\mathrm{ad}\,\mathring{B}_j)\,,\qquad m_\mathrm{F} = e\mathrm{i}\alpha^i\,\mathrm{ad}\,\mathring{A}_i + e\mathrm{i}\beta^j\,\mathrm{ad}\,\mathring{B}_j\mathrm{i}\gamma_5\,, \tag{16}$$

and for scalars writing $A_i = \mathring{A}_i + A_i^{fl}$, $B_j = \mathring{B}_j + B_j^{fl}$ then acting on the 6 dimensional space $\binom{A_i^{fl}}{B_j^{fl}}$ describing fluctuations around the vacuum

$$\mu_\mathrm{s}^2 = \mu_\mathrm{v}^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - e^2\begin{pmatrix}\mathrm{ad}\,\mathring{A}\\ \mathrm{ad}\,\mathring{B}\end{pmatrix}(\mathrm{ad}\,\mathring{A}\;\;\mathrm{ad}\,\mathring{B})\,. \tag{17}$$

Manifestly the non zero eigenvalues of $\mu_\mathrm{v}^2, \overline{m}_\mathrm{F} m_\mathrm{F}, \mu_\mathrm{s}^2$ are identical and we may verify that for each massive vector there are 4 spin $\frac{1}{2}$ and 5 spin 0 states which form a representation of the $N = 4$ supersymmetry.

To obtain the central charges in the supersymmetry algebra, for this theory we consider the supersymmetry transformation of the super-current $\widetilde{\mathcal{J}}^\mu$. This can straightforwardly be obtained by dimensional reduction of (5). The extra components in the $A = 4...9$ dimensions of the stress tensor give non zero contributions

$$\mathcal{I}^{\mu 3+i} \equiv \mathcal{A}_i^\mu = -\partial_\nu(F^{\mu\nu},A_i) + (D_\nu F^{\mu\nu} - J^\mu, A_i)\,,$$
$$\mathcal{I}^{\mu 6+j} \equiv \mathcal{B}_j^\mu = -\partial_\nu(F^{\mu\nu},B_j) + (D_\nu F^{\mu\nu} - J^\mu, B_j)\,, \tag{18}$$

from (6) where $J^\mu = \mathrm{i}e[D^\mu A_i, A_i] + \mathrm{i}e[D^\mu B_j, B_j] - \frac{1}{2}e[\overline{\psi},\gamma^\mu\psi]$. Now $D_\nu F^{\mu\nu} = J^\mu$ is one of the equations of motion so that $\mathcal{A}_i^\mu, \mathcal{B}_j^\mu$ are trivially conserved. The supersymmetry transformation becomes

‡4 The adjoint representations on the Lie algebra is defined by ad $XY = [X,Y]$, $(\mathrm{ad}\,X)_{ab} \equiv (T_a, \mathrm{ad}\,XT_b)$ is the matrix representative. Thus $([A_\mu,\mathring{A}_i],[A_\nu,\mathring{A}_i]) = -(A_\mu,[\mathring{A}_i,[\mathring{A}_i,A_\nu]]) = -A_\mu^a A_\nu^b(\mathrm{ad}\,\mathring{A}_i\,\mathrm{ad}\,\mathring{A}_i)_{ab}$.

$$\delta\bar{\tilde{\mathcal{J}}}^{\mu} = 2\mathrm{i}\mathcal{G}^{\mu\nu}\bar{\epsilon}\gamma_{\nu} - 2\mathcal{A}_{i}^{\mu}\bar{\epsilon}\alpha^{i} - 2\mathcal{B}_{j}^{\mu}\bar{\epsilon}\mathrm{i}\gamma_{5}\beta^{j} + \mathrm{i}\epsilon^{\mu\alpha\beta\gamma}(F_{\alpha\beta}, D_{\gamma}A_{i})\bar{\epsilon}\gamma_{5}\alpha^{i} - \epsilon^{\mu\alpha\beta\gamma}(F_{\alpha\beta}, D_{\gamma}B_{j})\bar{\epsilon}\beta^{j} + \cdots \tag{19}$$

where

$$\mathcal{G}^{\mu\nu} = (F^{\mu\rho}, F_{\rho}^{\nu}) + (D^{\mu}A_{i}, D^{\nu}A_{i}) + (D^{\mu}B_{j}, D^{\nu}B_{j})$$
$$+ \tfrac{1}{2}\mathrm{i}(\bar{\psi}, \gamma^{\mu}D^{\nu}\psi) - g^{\mu\nu}\{-\tfrac{1}{4}(F^{\alpha\beta}, F_{\alpha\beta}) + \tfrac{1}{2}(D^{\alpha}A_{i}, D_{\alpha}A_{i}) + \tfrac{1}{2}(D^{\alpha}B_{j}, D_{\alpha}B_{j}) - V(A,B)\}, \tag{20}$$

and the discarded terms in (19) are total derivatives irrelevant subsequently. After quantisation the operator generators of symmetry transformations are of course the charges obtained from the corresponding Noether currents. Thus for a supersymmetry transformation

$$\delta O = \mathrm{i}[\bar{\epsilon}Q, O], \qquad Q = \int \mathrm{d}^3x \tilde{\mathcal{J}}^0, \tag{21}$$

so from (19) the basic algebra of Q and $\bar{Q} = Q^{\mathrm{T}}\mathcal{C}$ is

$$\{Q, \bar{Q}\} = 2\gamma P - 2\mathrm{i}\alpha^{i}Q_{i}^{A} + 2\gamma_{5}\beta^{j}Q_{j}^{B} + 2\gamma_{5}\alpha^{i}T_{i}^{A} - 2\mathrm{i}\beta^{j}T_{j}^{B}, \tag{22}$$

where $P^{\mu} = \int \mathrm{d}^3x\mathcal{G}^{0\mu}$ and Q_i^A, Q_j^B, T_i^A, T_j^B are central charges which arise from integrals over the boundary surface at infinity of three dimensional space. If $F^{0p} = E_p$, $F_{pq} = \epsilon_{pqr}B_r$ then the electric charges

$$Q_i^A = \int \mathrm{d}\boldsymbol{S}\cdot(E, A_i), \qquad Q_j^B = \int \mathrm{d}\boldsymbol{S}\cdot(E, B_j), \tag{23}$$

are effectively the momenta for the compactified space dimensions (if the gauge is not fixed and hence Gauss's law is not true then as shown by Olive and also Manton [8] the extra terms in (18) serve as generators of local gauge transformation). The magnetic charges

$$T_i^A = \int \mathrm{d}\boldsymbol{S}\cdot(B, A_i), \qquad T_j^B = \int \mathrm{d}\boldsymbol{S}\cdot(B, B_j), \tag{24}$$

are topological in character. For the $N = 4$ supersymmetry algebra (22) the basic representation for non zero mass has generally, if the central charges are not involved, dimension 32 with spins up to 2. The only exception with spins $\leqslant 1$ is the dimension 16 representation with 1 spin 1, 4 spin $\frac{1}{2}$ and 5 spin 0 states which requires the mass formula [7]

$$M^2 = Q_i^A Q_i^A + Q_j^B Q_j^B + T_i^A T_i^A + T_j^B T_j^B. \tag{25}$$

Necessarily the elementary particle masses obtained from the eigenvalues of (16) and (17) are in accord with (25).

To extend these considerations to topological objects it is convenient to restrict our interest to static, t independent, classical fields with $A_0 = 0$, and hence $\boldsymbol{E} = \boldsymbol{0}$. The energy becomes

$$P_0 = \int \mathrm{d}^3x\{\tfrac{1}{2}(B,B) + \tfrac{1}{2}(DA_i, DA_i) + \tfrac{1}{2}((DB_j, DB_j) + V(A,B)\}. \tag{26}$$

For a given topological charge the energy is minimised by embedding into the theory a solution of the Bogomolny equation (2)

$$B = DS \tag{27}$$

by taking

$$A_i = Sa_i + \hat{A}_i, \qquad B_j = Sb_j + \hat{B}_j, \qquad a_i a_i + b_j b_j = 1,$$

with $V(A,B) = 0$, $D\hat{A}_i = D\hat{B}_j = \boldsymbol{0}$ and applying the boundary condition (15). Inserting (27) and (28) into (26) and (25)

$$P_0 = \int \mathrm{d}^3x(B, DS) \equiv M, \qquad T_i^A = Ma_i, \qquad T_j^B = Mb_j, \tag{29}$$

so this realises the mass formula (25) in this classical context with $Q_i^A = Q_j^B = 0$.

The semi-classical approach to quantisation of such a topological object procedes by expanding the quantum fields about such static classical solutions, which form the dominant contribution for weak coupling. From (13) the supersymmetry charge becomes by (27) and (28)

$$Q_{\substack{\text{semi}\\ \text{classical}}} = \boldsymbol{\sigma}\cdot\int \mathrm{d}^3x\,\{\gamma^0(B,\psi) + \gamma_5\alpha^i(DA_i,\psi) + \mathrm{i}\beta^j(DB_j,\psi)\} = (1+P)\gamma^0\boldsymbol{\sigma}\cdot\int \mathrm{d}^3x(B,\psi)\,, \tag{30}$$

where

$$P = -\gamma^0\gamma_5\alpha^i a_i + \mathrm{i}\gamma^0\beta^j b_j = P^\dagger\,, \qquad P^2 = 1\,. \tag{31}$$

The degeneracy of the quantum states corresponding to this classical topological object depends crucially as the number of zero frequency solutions of the Dirac equation for ψ in the presence of the background classical fields. When the quantum field ψ is expanded in terms of the Dirac equation eigenfunctions of definite frequency the representation space for the creation and annihilation operators for the zero modes defines a set of states degenerate in energy which correspond to the monopole particles in the quantum theory possessing identical topological quantum numbers to the classical solution and having the same mass (at least in the weak coupling limit) [9]. If we assume, at least for spherically symmetric solutions of (27), that the only zero mode solutions are those obtained by supersymmetry transformation, as in (12), from the classical gauge, Higgs field solutions (27), (28) then

$$\psi_0 = \gamma^0 M^{-1/2}\boldsymbol{\sigma}\cdot Bu\,, \qquad Pu = u\,, \tag{32}$$

describes 4 independent solutions. Now we may expand ψ as

$$\psi = \psi_0 + \cdots \tag{33}$$

where the neglected terms are orthogonal to ψ_0 and the standard quantum anticommutation relations for ψ then require

$$\{u, u^\dagger\} = \tfrac{1}{2}(1+P)\,, \qquad u^\dagger B = u^{\mathrm{T}} \tag{34}$$

which, since from (30) $Q_{\text{semiclassical}} = 2M^{1/2}u$, guarantees the realisation of the supersymmetry algebra (22) on the monopole states with $P^\mu = (M, \mathbf{0})$ and T_i^A, T_j^B given in (29).

The spins of these states may be found by recognising that Q is a spin $\frac{1}{2}$ operator

$$[J, Q] = -\tfrac{1}{2}\boldsymbol{\sigma}Q\,,$$

and similarly u transforms as spin $\frac{1}{2}$ under the part of the angular momentum operator $\boldsymbol{J}$ given by

$$\boldsymbol{S} = \tfrac{1}{4}\int \mathrm{d}^3x(\psi_0^\dagger, \boldsymbol{\sigma}\psi_0) = \tfrac{1}{4}u^\dagger\boldsymbol{\sigma}u\,. \tag{35}$$

$\boldsymbol{S}$ commutes with the hamiltonian and has standard angular momentum commutation relations so it defines the semi-classical spin operator for the static monopole states. The projection operator $\frac{1}{2}(1+P)$ commutes with $\boldsymbol{\sigma}$ and $BP^*B^{-1} = P$ so we may decompose u as

$$u = \sum_{s,n}(u_s^n a_s^n + Bu_s^{n*}a_s^{n\dagger}) \tag{36}$$

where u_s^n, Bu_s^{n*}, $s = \pm 1$, $n = 1,2$ form an orthonormal set in the 8 dimensional space projected by $\frac{1}{2}(1+P)$

$$\sigma_3 u_{\pm 1} = \pm u_{\pm 1}\,, \quad \tfrac{1}{2}(\sigma_1 \pm \mathrm{i}\sigma_2)u_{\mp 1} = u_{\pm 1}\,, \quad \sum_{s,n}(u_s^n u_s^{n\dagger} + Bu_s^{n*}u_s^{n\mathrm{T}}B^\dagger) = \tfrac{1}{2}(1+P)\,.$$

The anticommutation relation (34) requires

$$\{a_s^n, a_{s'}^{n'\dagger}\} = \delta^{nn'}\delta_{ss'}\,, \qquad \{a_s^n, a_{s'}^{n'}\} = 0\,, \tag{37}$$

and as

$$S = \tfrac{1}{2} \sum_{n,s,s'} a_s^{n\dagger} (\boldsymbol{\sigma})_{ss'} a_{s'}^{n} \,, \tag{38}$$

with $(\boldsymbol{\sigma})_{ss'}$ the normal Pauli spin matrices, then a_s^n, $a_s^{n\dagger}$ are two sets of fermionic spin $\frac{1}{2}$ creation and annihilation operators for $n = 1,2$. The standard representation of (37), with (38) defining the spin, implies the unique prescription of one spin 1 state, four spin $\frac{1}{2}$ and 5 spin 0 states for the monopole, exactly as for the elementary particles.

In this light we suggest that this theory is the most probable candidate for a realisation of the intriguing conjecture of Montonen and Olive [10] of a duality between the conventional and topological sectors. In the case of the $N = 2$ theory [1,4] the same assumptions as above would give just one set of spin $\frac{1}{2}$ fermion creation and annihilation operators which would imply monopole states of spin $\frac{1}{2}$ and 2 of spin 0 (this has to be combined with angular momentum $\frac{1}{2}$ to achieve the elementary particle representation $1 \times 1, 2 \times \frac{1}{2}, 1 \times 0$). The critical assumption is the one concerning the number of fermion zero mode solutions which needs further investigation [11], especially for multiple monopole configurations.

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