CHIRAL FERMION DETERMINANTS AND THEIR ANOMALIES ☆

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Although the eigenvalues of the operator $\sigma_\mu(\partial_\mu + A_\mu)$ are not meaningful, the corresponding determinant does make sense. The change of the determinant generated by an infinitesimal gauge transformation is shown to be given by the explicit expression for the anomaly due to Zumino, Wu Yong-Shi and Zee. The ambiguities in the definition of the current (covariant versus consistent anomalies) find a simple interpretation in terms of the short distance properties of the propagator.

In the functional integral representation of the generating functional, the chiral anomalies are generated by the determinant of the Dirac operator. This property was exploited by Fujikawa [1], who has shown that the anomaly may be extracted directly from the local properties of the Dirac operator. On the other hand, Wess and Zumino [2] have shown that in four dimensions the structure of the anomaly is fixed by the integrability condition alone. Zumino, Wu Yong-Shi and Zee [3] have extended the analysis of the integrability condition to an arbitrary number of space–time dimensions and have explicitly determined the chiral anomaly in the general case. The purpose of the present note is to show that one can give unambiguous meaning to the determinant of the Weyl operator, to exhibit its transformation properties under gauge transformations and to show how the ambiguities in the definition of the current (consistent versus covariant anomalies [4]) manifest themselves in the properties of the determinant. We comment on related work towards the end of the paper. The differential operator

$$\mathrm{D} = \mathrm{i}\sigma_\mu(\partial_\mu + A_\mu) \tag{1}$$

describes the propagation of chiral fermions in d-dimensional euclidean space. The matrices σ_μ obey the Weyl algebra

$$\sigma_\mu\tilde{\sigma}_\nu + \sigma_\nu\tilde{\sigma}_\mu = 2\delta_{\mu\nu} . \tag{2}$$

The Lie-algebra valued external vector field $A_\mu(x)$ is represented by an antihermitean matrix which commutes with σ_ν.

If d is odd, all irreducible representations of the Weyl algebra are equivalent. They act on a vector space of dimension $2^{(d-1)/2}$ and, in a suitable basis, satisfy $\sigma_\mu = \tilde{\sigma}_\mu = \sigma_\mu^\dagger$. In this basis, the Weyl matrices obey the Dirac algebra; D is hermitean and coincides with $\not{D}$.

If d is even, there are two inequivalent irreducible representations of the Weyl algebra, which act on a vector space of dimension $2^{(d-2)/2}$ and can be distinguished by the matrix

$$\Gamma = (-\mathrm{i})^{d/2}\sigma_1\tilde{\sigma}_2\sigma_3 \ldots \tilde{\sigma}_d , \quad \Gamma^2 = 1 , \tag{3}$$

which is either equal to 1 or equal to -1, depending on the representation. We use the notation $\Gamma = \mathrm{sign}(\sigma)\,1$ The representations $\sigma'_\mu = \tilde{\sigma}_\mu$ and $\sigma''_\mu = \sigma_\mu^\dagger$ are inequivalent to σ_μ. In a suitable basis we have $\tilde{\sigma}_\mu = \sigma_\mu^\dagger$. The operator D is not hermitean; it coincides with the component $\frac{1}{2}\not{D}(1 + \gamma_{d+1})$ of the Dirac operator.

As it is well known from the perturbative analysis of the corresponding one-loop graphs, ln det D is unique only up to a local polynomial in the field A_μ and its derivatives. The ambiguity stems from the fact that the operator D possesses arbitrarily large eigenvalues. The standard procedure to arrive at a well-defined renormalized determinant is to cut the product $\Pi_n\lambda_n$ of eigenvalues off at the upper end, $|\lambda_n| < \Lambda$, to divide by the exponential of a suitable local polynomial

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(whose coefficients grow with Λ) and to pass to the limit $\Lambda \to \infty$. If d is even, the eigenvalues of D are however not euclidean invariant; although the renormalization procedure just described is legitimate, it unnecessarily spoils one of the symmetries of the problem [5]. We instead pin the determinant down in an euclidean invariant manner by specifying how it changes as the field $A_\mu(x)$ is deformed. Formally, the change in the determinant generated by a variation in A_μ is given by

$$\delta \ln \det \mathrm{D} = \mathrm{Tr}(\delta \mathrm{D} \mathrm{D}^{-1}) . \tag{4}$$

Since the propagator $(x | \mathrm{D}^{-1} | y)$ is however singular at $x = y$, we need to regularize this expression. We choose the regularization

$$L_\epsilon(\delta A, A) = \mathrm{Tr}(\delta \mathrm{D} \mathrm{D}^{-1}\, \mathrm{e}^{-\epsilon \mathrm{D}\tilde{\mathrm{D}}})$$

$$= \int_\epsilon^\infty \mathrm{d}\lambda\, \mathrm{Tr}(\delta \mathrm{D} \tilde{\mathrm{D}}\, \mathrm{e}^{-\lambda \mathrm{D} \tilde{\mathrm{D}}}) , \tag{5}$$

where $\tilde{\mathrm{D}} = \mathrm{i}\tilde{\sigma}_\mu(\partial_\mu + A_\mu)$. Note that, in the basis $\tilde{\sigma}_\mu = \sigma_\mu^\dagger$, the operator $\tilde{\mathrm{D}}$ coincides with $\mathrm{D}^\dagger$; the eigenvalues of $\mathrm{D}\tilde{\mathrm{D}}$ are therefore positive, such that the exponential occurring in (5) makes sense. (In general, the small eigenvalues of $\mathrm{D}\tilde{\mathrm{D}}$ may give rise to an infrared divergence of the integral (5) at $\lambda \to \infty$. In the perturbative expansion of ln det D in powers of the field $A_\mu(x)$, there are however no infrared divergences, provided $A_\mu(x)$ vanishes sufficiently rapidly at $|x| \to \infty$. We restrict ourselves to this perturbative framework, thereby excluding nontrivial winding number, normalizable zero modes, etc.) In the limit $\epsilon \to 0$ the functional L_ϵ behaves like

$$L_\epsilon = \epsilon^{-(N-2)}\, l_{N-2} + \ldots + \epsilon^{-1}\, l_1 - \ln(\epsilon\mu)\, l_0$$

$$+ L + \mathrm{O}(\epsilon) , \tag{6}$$

where the coefficients l_i of the singular terms are local polynomials in the fields $\delta A_\mu, A_\mu$ and their derivatives (the explicit form of these polynomials may be extracted from the heat kernel expansion to be discussed below). We renormalize the functional L_ϵ by subtracting the singular contributions, which play the role of counter terms. The renormalized functional is identified with the finite part L.

The main point in the following analysis is that if d is even, the quantity $L(\delta A, A)$ cannot be written as the variation of some functional of A only (which could then be identified with the renormalized determinant): the commutator of two variations

$$\delta_2 L_\epsilon(\delta_1 A, A) - \delta_1 L_\epsilon(\delta_2 A, A) = C_\epsilon(\delta_1 A, \delta_2 A, A) \tag{7}$$

does not vanish as $\epsilon \to 0$. Using the properties

$$\delta(\mathrm{e}^{-\lambda A}) = -\int_0^\lambda \mathrm{d}\nu\, \mathrm{e}^{-(\lambda-\nu)A}\, \delta A\, \mathrm{e}^{-\nu A} ,$$

$$\tilde{\mathrm{D}}\, \mathrm{e}^{-\lambda \mathrm{D}\tilde{\mathrm{D}}}\, \mathrm{D} = -\mathrm{d}(\mathrm{e}^{-\lambda \tilde{\mathrm{D}}\mathrm{D}})/\mathrm{d}\lambda , \tag{8}$$

C_ϵ may be rewritten in the form

$$C_\epsilon = \int_0^\epsilon \mathrm{d}\lambda\, \mathrm{Tr}\{\delta_1 \mathrm{D}\, \mathrm{e}^{-(\epsilon-\lambda)\tilde{\mathrm{D}}\mathrm{D}}\, \delta_2 \tilde{\mathrm{D}}\, \mathrm{e}^{-\lambda \mathrm{D}\tilde{\mathrm{D}}}$$

$$- (\sigma_\mu \leftrightarrow \tilde{\sigma}_\mu)\} . \tag{9}$$

If d is even, this expression tends to a finite limit C as $\epsilon \to 0$, given by the local polynomial ($N = d/2$):

$$C = \kappa N \int \mathrm{str}\{\delta_1 A, \delta_2 A, F^{(N-1)}\} ,$$

$$\kappa = (\mathrm{i}/2\pi)^N\, \mathrm{sign}(\sigma)/N! . \tag{10}$$

If d is odd, there is no anomaly: C_ϵ vanishes and L_ϵ is therefore integrable to a gauge invariant regularized determinant.

In (10) we have used the notation introduced by Zumino, Wu Yong-Shi and Zee [3]. The objects involved are Lie-algebra valued differential forms, constructed as follows. One introduces a set of d anticommuting variables $\mathrm{d}x_1, \mathrm{d}x_2, \ldots, \mathrm{d}x_d$. In terms of these, the one-form A and the two-form F are defined by

$$A = \mathrm{d}x_\mu A_\mu , \quad F = \tfrac{1}{2} \mathrm{d}x_\mu\, \mathrm{d}x_\nu\, F_{\mu\nu} , \tag{11}$$

where $F_{\mu\nu}$ is the field strength tensor associated with A_μ. The quantity $\mathrm{str}\{\delta_1 A, \delta_2 A, F^{(N-1)}\}$ stands for the symmetric trace of $\delta_1 A, \delta_2 A$ and $(N-1)$ factors of F. (Up to signs, the symmetric trace of a string of Lie-algebra valued differential forms is the trace of the product of these forms, averaged over all permutations of the factors. For a definition of the symmetric trace and for a discussion of the properties of this notion the reader is referred to ref. [3].) Since $\mathrm{str}\{\delta_1 A, \delta_2 A, F^{(N-1)}\}$ is a d-form, it is proportional to the product $\mathrm{d}x_1 \cdot \mathrm{d}x_2 \cdot \ldots \cdot \mathrm{d}x_d$ which is identified with the volume element dx. More explicitly, the functional C is the volume integral

$$C = 2^{(1-N)} \kappa N \int dx\, \epsilon_{\mu_1 \ldots \mu_d}$$

$$\times \operatorname{str}\{\delta_1 A_{\mu_1}, \delta_2 A_{\mu_2}, F_{\mu_3\mu_4}, \ldots, F_{\mu_{d-1}\mu_d}\}\,, \qquad (12)$$

where the symmetric trace occurring here is the ordinary trace of the product of the matrices involved, averaged over all permutations of the factors.

To verify the claim (10), we make use of the heat kernel expansion

$$(x|e^{-\lambda D\tilde{D}}|y)$$

$$= (4\pi\lambda)^{-N} \exp(-\tfrac{1}{4}\lambda^{-1}z^2) \sum_{n=0}^{\infty} \lambda^n H_n(x|y)\,,$$

where $z = x - y$ and $N = d/2$. The coefficients H_n in this expansion may be obtained from the recursion relation

$$(z_\mu D_\mu + n)\, H_n = -D\tilde{D} H_{n-1}\,,$$

$$z_\mu D_\mu H_0 = 0\,, \quad H_0(x|x) = 1\,, \qquad (13)$$

with $D_\mu = \partial_\mu + A_\mu$. The analogous expansion coefficients $\tilde{H}_n(x|y)$ of the operator $\exp(-\lambda\tilde{D}D)$ are obtained from $H_n(x|y)$ with the substitution $\sigma_\mu \leftrightarrow \tilde{\sigma}_\mu$.

The recursion (13) implies that $H_n(x|y)$ is a polynomial in the matrix $\sigma_\mu\tilde{\sigma}_\nu$ of degree n (H_0 is a scalar, H_1 contains a scalar piece and a term linear in $\sigma_\mu\tilde{\sigma}_\nu$, H_2 contains up to two powers of $\sigma_\mu\tilde{\sigma}_\nu$, etc.). The trace over the σ-matrices called for in (9) is therefore a sum of contributions proportional to

$$\Delta_{\mu_1 \ldots \mu_{2k}} = \operatorname{tr}(\sigma_{\mu_1}\tilde{\sigma}_{\mu_2}\sigma_{\mu_3} \ldots \tilde{\sigma}_{\mu_{2k}})$$

$$- \operatorname{tr}(\tilde{\sigma}_{\mu_1}\sigma_{\mu_2} \ldots \sigma_{\mu_{2k}})\,. \qquad (14)$$

As a consequence of the Weyl algebra, Δ vanishes if one or several of the indices $1, \ldots, d$ are absent in the set $\mu_1, \ldots, \mu_{2k}$. In particular, Δ vanishes if $k < N$. For $k = N\ (= d/2)$ we have

$$\Delta_{\mu_1 \ldots \mu_d} = (2i)^{d/2} \operatorname{sign}(\sigma)\, \epsilon_{\mu_1 \ldots \mu_d}\,. \qquad (15)$$

Hence the term $\tilde{H}_m \otimes H_n$ in the heat kernel expansion of the two exponentials in (9) only contributes if it involves a sufficient number of σ-matrices; this is the case only if $m + n \geqslant N - 1$. On the other hand, counting powers of ϵ, one easily checks that the contribution from $\tilde{H}_m \otimes H_n$ is of order $\epsilon^{m+n+1-N}\{1 + O(\epsilon)\}$. Hence we obtain a nonvanishing contribution to C_ϵ if and only if $m + n = N - 1$. Furthermore, (i) only the values of $\tilde{H}_m(x|y)$ and $H_n(x|y)$ at $x = y$ matter and (ii) in the expansion of $\tilde{H}_m$ and H_n in powers of σ-matrices only the term with the largest possible number of σ-matrices counts. For these contributions the recursion relation (13) implies

$$H_n(x|x) \triangleq (1/n!)(\tfrac{1}{2}\sigma_\mu\tilde{\sigma}_\nu F_{\mu\nu})^n\,, \qquad (16)$$

where the symbol $\triangleq$ indicates that the equality only holds up to a polynomial in $\sigma_\mu\tilde{\sigma}_\nu$ of degree $n - 1$. The trace over the σ-matrices is now easily evaluated with (14), (15); the result is the expression (12) given above

Since C_ϵ does not contain any singularities at $\epsilon = 0$, the counter terms $l_{N-2}, \ldots, l_0$ occurring in (6) are integrable, but the finite part L is not. To arrive at an integrable functional, we look for a local polynomial $l(\delta A, A)$ with the property

$$\delta_2 l(\delta_1 A, A) - \delta_1 l(\delta_2 A, A) = -C(\delta_1 A, \delta_2 A, A)\,. \quad (17)$$

Using the technique of Zumino et al. [3,4] it is not difficult to solve this equation. The solution takes the form

$$l(\delta A, A) = -\kappa N \int_0^1 dt\, \operatorname{str}\{\delta A, A_t, F_t^{(N-1)}\} \qquad (18)$$

where $A_t = tA$ and where F_t is the field strength associated with A_t. The renormalized determinant may therefore be defined by integrating the relation

$$\delta \ln \det D = L(\delta A, A) + l(\delta A, A) \qquad (19)$$

along the path A_t from $t = 0$ to $t = 1$:

$$\ln \det D = \int_0^1 dt [L(A, A_t) + l(A, A_t)]\,. \qquad (20)$$

This completes our construction of the determinant of D.

It is now a straightforward matter to calculate the change in the determinant generated by an infinitesimal gauge transformation: it suffices to evaluate the functionals L and l for the particular case

$$\delta A_\mu = -iD_\mu\alpha = -i(\partial_\mu\alpha + [A_\mu, \alpha])\,. \qquad (21)$$

To obtain the value of L, we insert $i\delta D = [D, \alpha]$ in the regularized expression (5) and get

$$L_\epsilon = \mathrm{i}\,\mathrm{Tr}\{\alpha(\mathrm{e}^{-\epsilon \mathrm{D}\tilde{\mathrm{D}}} - \mathrm{e}^{-\epsilon \tilde{\mathrm{D}}\mathrm{D}})\}\,. \tag{22}$$

With the heat kernel expansion and the trace properties of the σ-matrices discussed above, one easily verifies that there are no singularities in this expression at $\epsilon = 0$. In the limit $\epsilon = 0$ we find

$$L(-\mathrm{i}\mathrm{D}\alpha, A) = \mathrm{i}\kappa \int \mathrm{tr}(\alpha F^N)\,. \tag{23}$$

Note the close correspondence with Fujikawa's analysis [1]: the right-hand side of (22) coincides with $\mathrm{tr}\{\alpha\gamma_{d+1}\exp(-\epsilon \not{D}^2)\}$, where $\not{D}$ is the Dirac operator. The contribution from L is however not the whole story. To evaluate the additional contribution from the local polynomial $l(-\mathrm{i}\mathrm{D}\alpha, A)$, we use the explicit representation (18), integrate by parts and exploit $\mathrm{D}F = 0$. Adding the result to (23), we obtain

$$\delta \ln \det \mathrm{D} = \mathrm{i}\kappa \int_0^1 \mathrm{d}t \int [\mathrm{tr}(\alpha F_t^N)$$

$$- Nt(1-t)\,\mathrm{str}\{[\alpha, A], A, F_t^{(N-1)}\}]\,. \tag{24}$$

This coincides with the expression for the anomaly constructed by Zumino, Wu Yong-Shi and Zee [3].

The above decomposition of the determinant shows that the current $J_\mu(x, A)$, defined as the response of the generating functional to a deformation of the gauge field

$$\delta \ln \det \mathrm{D} = \int \mathrm{d}x\, \mathrm{tr}\{\delta A_\mu(x) J_\mu(x, A)\} \tag{25}$$

consists of two parts, $J_\mu = L_\mu + l_\mu$. By construction, the current L_μ is gauge covariant: it is the finite part of the gauge covariant expression

$$L_\mu(x, A)_\epsilon = \mathrm{i}\,\underset{\sigma}{\mathrm{tr}}\{\sigma_\mu(x|\mathrm{D}^{-1}\mathrm{e}^{-\epsilon \mathrm{D}\tilde{\mathrm{D}}}|x)\}\,. \tag{26}$$

The remainder, l_μ, is a gauge-noncovariant local polynomial. One is not forced to define the current as the response of the generating functional to deformations of the gauge field [4]. Any current, which differs from J_μ only by a local polynomial in the field A_μ and its derivatives, is acceptable. In particular, one may identify the current with the covariant quantity L_μ, in which case the anomaly equation takes the covariant form (23), or, equivalently,

$$\mathrm{D}_\mu L_\mu = (\mathrm{i}/4\pi)^N (1/N!)$$

$$\times\, \epsilon_{\mu_1 \ldots \mu_d} F_{\mu_1\mu_2} \ldots F_{\mu_{d-1}\mu_d}\,. \tag{27}$$

Finally, we comment on an alternative approach to the problem [6,7]. The Dirac operator

$$\not{D} = \mathrm{i}\gamma_\mu(\partial_\mu + v_\mu + a_\mu\gamma_{d+1}) = \begin{pmatrix} 0 & \tilde{\mathrm{D}}_- \\ \mathrm{D}_+ & 0 \end{pmatrix} \tag{28}$$

decomposes into the two Weyl operators D_+, $\tilde{\mathrm{D}}_-$ belonging to the fields $A_\mu = v_\mu \pm a_\mu$. In contrast to the eigenvalues of the Weyl operator, the eigenvalues of the Dirac operator are euclidean-invariant. If the product of the eigenvalues of $\not{D}$ is renormalized either with the ϵ- regularization or with the ζ-function technique, one therefore arrives at an euclidean-invariant determinant of the Dirac operator. Formally, $\det \not{D}$ is equal to the product $\det \mathrm{D}_+ \cdot \det \tilde{\mathrm{D}}_-$. This allows one to define the determinant of the Weyl operator in terms of $\det \not{D}$, choosing $v_\mu = a_\mu = \frac{1}{2}A_\mu$ (such that $\det \tilde{\mathrm{D}}_- =$ constant). Normalizing the determinant of the free Weyl operator $\mathrm{D}_0 = \mathrm{i}\sigma_\mu\partial_\mu$ to 1, this amounts to the prescription [7]

$$\ln \det \mathrm{D}_\epsilon = -\frac{1}{2}\int_\epsilon^\infty \frac{\mathrm{d}\lambda}{\lambda}\mathrm{Tr}(\mathrm{e}^{-\lambda \not{D}^2} - \mathrm{e}^{-\lambda \not{D}_0^2})$$

$$= -\int_\epsilon^\infty \frac{\mathrm{d}\lambda}{\lambda}\mathrm{Tr}(\mathrm{e}^{-\lambda \mathrm{D}\tilde{\mathrm{D}}_0} - \mathrm{e}^{-\lambda \mathrm{D}_0\tilde{\mathrm{D}}_0})\,. \tag{29}$$

(The logarithm of the renormalized Weyl determinant is what remains of $\ln \det \mathrm{D}_\epsilon$ when the singularities at $\epsilon = 0$ are removed – compare eq. (6) above.) In four dimensions, the transformation properties of the quantity defined by (29) were worked out in ref. [6]. It turns out that, with this definition, even the modulus of $\det \mathrm{D}$ fails to be gauge invariant ($\ln \det \mathrm{D}$ contains a gauge-dependent contribution which does not involve the ϵ-tensor; the contribution is an irrelevant, euclidean-invariant, local polynomial, which one is free to remove). Quite apart from this unnecessary complication, the spin structure of the noncovariant quantity $\exp(-\lambda \mathrm{D}\tilde{\mathrm{D}}_0)$ is much more complicated than the spin structure of the covariant operator $\exp(-\lambda \mathrm{D}\tilde{\mathrm{D}}$ which occurs in the method used above. It does not appear to be possible to extend the analysis of refs. [6,7] to an arbitrary number of dimensions.

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