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Physical aspects of the space–time torsion

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Abstract

We review many quantum aspects of torsion theory and discuss the possibility of the space–time torsion to exist and to be detected. The paper starts, in Section 2, with a pedagogical introduction to the classical gravity with torsion, that includes also interaction of torsion with matter fields. Special attention is paid to the conformal properties of the theory. In Section 3, the renormalization of quantum theory of matter fields and related topics, like renormalization group, effective potential and anomalies, are considered. Section 4 is devoted to the action of spinning and spinless particles in a space–time with torsion, and to the discussion of possible physical effects generated by the background torsion. In particular, we review the upper bounds for the magnitude of the background torsion which are known from the literature. In Section 5, the comprehensive study of the possibility of a theory for the propagating completely antisymmetric torsion field is presented. It is supposed that the propagating field should be quantized, and that its quantum effects must be described by, at least, some effective low-energy quantum field theory. We show, that the propagating torsion may be consistent with the principles of quantum theory only in the case when the torsion mass is much greater than the mass of the heaviest fermion coupled to torsion. Then, universality of the fermion–torsion interaction implies that torsion itself has a huge mass, and cannot be observed in realistic experiments. Thus, the theory of quantum matter fields on the classical torsion background can be formulated in a consistent way, while the theory of dynamical torsion meets serious obstacles. In Section 6, we briefly discuss the string-induced torsion and the possibility to induce torsion action and torsion itself through the quantum effects of matter fields. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The development of physics, until recent times, went from experiment to theory. New theories were created when the previous ones did not fit with some existing phenomena, or when the theories describing different classes of phenomena have shown some mutual contradictions. At some point this process almost stopped. However, recent data on the neutrino oscillations should be, perhaps, interpreted in such a way that the Minimal Standard Model of particle physics does not describe the full spectrum of existing particles. On the other hand, one can mention the supernova observational evidence for a positive cosmological constant and the lack of the natural explanation for the inflation. This probably indicates that the extension of the Standard Model must also include gravity. The desired fundamental theory is expected to provide the solution to the quantum gravity problem, hopefully explain the observable value of the cosmological constant and maybe even predict the low-energy observable particle spectrum. The construction of such a fundamental theory meets obvious difficulties: besides purely theoretical ones, there is an extremely small link with the experiments or observations. Nowadays, the number of theoretical models or ideas has overwhelming majority over the number of their possible verifications. In this sense, today the theory is very far ahead of experiment.

In such a situation, when a fundamental theory is unknown or it cannot be verified, one might apply some effective approach and ask what could be the traces of such a theory at low energies. In principle, there can be two kinds of evidences: new fields or new low-energy symmetries. The Standard Model is composed by three types of fields: spinors, vectors and scalars. On the other hand, General Relativity yields one more field—metric, which describes the properties of the space–time. Now, if there is some low-energy manifestation of the fundamental theory, it could be some additional characteristics of the space–time, different from the fields included into the Standard Model. One of the candidates for this role could be the space–time torsion, which we are going to discuss in this paper. Torsion is some independent characteristic of the space–time, which has a very long history of study (see [99] for the extensive review and references, mainly on various aspects of classical gravity theory with torsion). In this paper we shall concentrate on the quantum aspects of the theory, and will look at the problem, mainly, from quantum point of view. Our purpose will be to apply the approach which is standard in the high-energy physics when things concern the search for some new particle or interaction. One has to formulate the corresponding theory in a consistent way, first at the level of lower complexity, and then investigate the possibility of experimental manifestations. After that, it is possible to study more complicated models. Indeed, for the case of torsion, which has not been ever observed, the study of experimental manifestations reduces to the upper bounds on the torsion parameters from various experiments. Besides this principal line, the extensive introduction to the gravity with torsion will be given in Section 2.

For us, the simplest level of the torsion theory will be the classical background for quantum matter fields. As we shall see, such a theory can be formulated in a consistent way. The next level is, naturally, the theory of dynamical (propagating) torsion, which should be considered in the same manner as metric or as the constituents of the Standard Model. We shall present the general review of the original publications [18,22] discussing the restrictions in implementing torsion into a gauge theory such as the Standard Model. Towards the end of the paper, we

discuss the possibility of torsion induced in string theory and through the quantum effects of matter fields.

2. Classical torsion

This section mainly contains introductory material, which is necessary for the next sections, where the quantum aspects of torsion will be discussed.

2.1. Definitions, notations and basic concepts

Let us start with the basic notions of gravity with torsion. In general, our notations correspond to those in [165,34]. The metric $g_{\mu\nu}$ and torsion $T_{\beta\gamma}^\alpha$ are independent characteristics of the space–time. In order to understand better, how the introduction of torsion becomes possible, let us briefly review the construction of covariant derivative in General Relativity. We shall mainly consider the algebraic aspect of the covariant derivative. For the geometric aspects, related to the notion of parallel transport, the reader is referred, for example, to [99].

The partial derivative of a scalar field is a covariant vector (one-form). However, the partial derivative of any other tensor field does not form a tensor. But, one can add to the partial derivative some additional term such that the sum is a tensor. The sum of partial derivative and this additional term is called covariant derivative. For instance, in the case of the (contravariant) vector A^α the covariant derivative looks like

$$\nabla_\beta A^\alpha = \partial_\beta A^\alpha + \Gamma_{\beta\gamma}^\alpha A^\gamma, \quad (2.1)$$

where the last term is a necessary addition. The covariant derivative (2.1) is a tensor if and only if the affine connection $\Gamma_{\beta\gamma}^\alpha$ transforms in a special non-tensor way. The rule for constructing the covariant derivatives of other tensors immediately follows from the following facts:

- (i) The product of co- and contravariant vectors A^α and B_α should be a scalar. Then

$$\nabla_\beta (A^\alpha B_\alpha) = \partial_\beta (A^\alpha B_\alpha)$$

and therefore

$$\nabla_\beta B_\alpha = \partial_\beta B_\alpha - \Gamma_{\beta\alpha}^\gamma B_\gamma. \quad (2.2)$$

(ii) In the same manner, one can note that the contraction of any tensor with some appropriate set of vectors is a scalar, and arrive at the standard expression for the covariant derivative of an arbitrary tensor

$$\nabla_\beta T_{\gamma_1 \dots}^{\alpha_1 \dots} = \partial_\beta T_{\gamma_1 \dots}^{\alpha_1 \dots} + \Gamma_{\beta\lambda}^{\alpha_1} T_{\gamma_1 \dots}^{\lambda \dots} + \dots - \Gamma_{\beta\gamma_1}^\tau T_{\tau \dots}^{\alpha_1 \dots} - \dots. \quad (2.3)$$

Now, (2.1) and (2.2) become particular cases of (2.3). At this point, it becomes clear that the definition of $\Gamma_{\beta\gamma}^\alpha$ contains, from the very beginning, some ambiguity. Indeed, (2.3) remains a tensor if one adds to $\Gamma_{\beta\gamma}^\alpha$ any tensor $C_{\beta\gamma}^\alpha$:

$$\Gamma_{\beta\gamma}^\alpha \rightarrow \Gamma_{\beta\gamma}^\alpha + C_{\beta\gamma}^\alpha. \quad (2.4)$$

A very special choice of $\Gamma_{\beta\gamma}^\alpha$, which is used in General Relativity, appears as a consequence of two requirements:

- (i) symmetry $\Gamma_{\beta\gamma}^\alpha = \Gamma_{\gamma\beta}^\alpha$ and
- (ii) metricity of the covariant derivative $\nabla_\alpha g_{\mu\nu} = 0$.

If these conditions are satisfied, one can apply (2.3) and obtain the unique solution for $\Gamma_{\beta\gamma}^\alpha$:

$$\Gamma_{\beta\gamma}^\alpha = \left\{ \begin{array}{c} \alpha \\ \beta\gamma \end{array} \right\} = \frac{1}{2} g^{\alpha\lambda} (\partial_\beta g_{\lambda\gamma} + \partial_\gamma g_{\lambda\beta} - \partial_\lambda g_{\beta\gamma}) . \quad (2.5)$$

Expression (2.5) is called Christoffel symbol, it is a particular case of the affine connection. Indeed, (2.5) is a very important object, because it depends on the metric only. Eq. (2.5) is the simplest one among all possible affine connections. It is very useful to consider (2.5) as some “reference point” for all the connections. Other connections can be considered as (2.5) plus some additional tensor as in (2.4). It is easy to prove that the difference between any two connections is a tensor.

When the space–time is flat, the metric and expression (2.5) depend just on the choice of the coordinates, and one can choose them in such a way that $\left\{ \begin{array}{c} \alpha \\ \beta\gamma \end{array} \right\}$ vanishes everywhere. On the contrary, if we consider, as in (2.4),

$$\hat{\Gamma}_{\beta\gamma}^\alpha = \left\{ \begin{array}{c} \alpha \\ \beta\gamma \end{array} \right\} + C_{\beta\gamma}^\alpha , \quad (2.6)$$

then the tensor $C_{\beta\gamma}^\alpha$ (and, consequently, the whole connection $\hat{\Gamma}_{\beta\gamma}^\alpha$) cannot be eliminated by a choice of the coordinates. Even if one takes the flat metric, the covariant derivative based on $\hat{\Gamma}_{\beta\gamma}^\alpha$ does not reduce to the coordinate transform of partial derivative. Thus, the introduction of an affine connection different from Christoffel symbol means that the geometry is not completely described by the metric, but has another, absolutely independent characteristic-tensor $C_{\beta\gamma}^\alpha$. The ambiguity in the definition of $\Gamma_{\beta\gamma}^\alpha$ is very important, for it enables one to introduce gauge fields different from gravity, and thus describe various interactions.

In this paper we shall consider the particular choice of the tensor $C_{\beta\gamma}^\alpha$. Namely, we suppose that the affine connection $\tilde{\Gamma}_{\beta\gamma}^\alpha$ is not symmetric:

$$\tilde{\Gamma}_{\beta\gamma}^\alpha - \tilde{\Gamma}_{\gamma\beta}^\alpha = T_{\beta\gamma}^\alpha \neq 0 . \quad (2.7)$$

At the same time, we postulate that the corresponding covariant derivative satisfies the metricity condition $\tilde{\nabla}_\mu g_{\alpha\beta} = 0$.² The tensor $T_{\beta\gamma}^\alpha$ is called torsion.

Below, we use notation (2.5) for the Christoffel symbol, and the notation with tilde for the connection with torsion and for the corresponding covariant derivative. The metricity condition enables one to express the connection through the metric and torsion in a unique way as

$$\tilde{\Gamma}_{\beta\gamma}^\alpha = \Gamma_{\beta\gamma}^\alpha + K_{\beta\gamma}^\alpha , \quad (2.8)$$

² The breaking of this condition means that one adds one more tensor to the affine connection. This term is called non-metricity, and it may be important, for example, in the consideration of the first order formalism for General Relativity. However, we will not consider the theories with non-metricity here.

where

$$K_{\cdot\beta\gamma}^\alpha = \frac{1}{2}(T_{\cdot\beta\gamma}^\alpha - T_{\beta\cdot\gamma}^\alpha - T_{\gamma\cdot\beta}^\alpha) \quad (2.9)$$

is called the contorsion tensor. The indices are raised and lowered by means of the metric. It is worthwhile noting that the contorsion is antisymmetric in the first two indices: $K_{\alpha\beta\gamma} = -K_{\beta\alpha\gamma}$, while torsion $T_{\cdot\beta\gamma}^\alpha$ itself is antisymmetric in the last two indices.

The commutator of covariant derivatives in the space with torsion depends on the torsion and on the curvature tensor. First of all, consider the commutator acting on the scalar field φ . We obtain

$$[\tilde{\nabla}_\alpha, \tilde{\nabla}_\beta]\varphi = T_{\cdot\alpha\beta}^\lambda \partial_\lambda \varphi, \quad (2.10)$$

that indicates a difference with respect to the commutator of the covariant derivatives ∇_α based on the Christoffel symbol (2.5). In the case of a vector, after some simple algebra we arrive at the expression

$$[\tilde{\nabla}_\alpha, \tilde{\nabla}_\beta]P^\lambda = T_{\cdot\alpha\beta}^\tau \tilde{\nabla}_\tau P^\lambda + \tilde{R}_{\cdot\tau\alpha\beta}^\lambda P^\tau, \quad (2.11)$$

where $\tilde{R}_{\cdot\tau\alpha\beta}^\lambda$ is the curvature tensor in the space with torsion:

$$\tilde{R}_{\cdot\tau\alpha\beta}^\lambda = \partial_\alpha \tilde{\Gamma}_{\cdot\tau\beta}^\lambda - \partial_\beta \tilde{\Gamma}_{\cdot\tau\alpha}^\lambda + \tilde{\Gamma}_{\cdot\gamma\alpha}^\lambda \tilde{\Gamma}_{\cdot\tau\beta}^\gamma - \tilde{\Gamma}_{\cdot\gamma\beta}^\lambda \tilde{\Gamma}_{\cdot\tau\alpha}^\gamma. \quad (2.12)$$

Using (2.10), (2.11) and that the product $P^\lambda B_\lambda$ is a scalar, one can easily derive the commutator of covariant derivatives acting on a one-form B_λ and then calculate such a commutator acting on any tensor. In all cases the commutator is the linear combination of curvature (2.12) and torsion.

Curvature (2.12) can be easily expressed through the Riemann tensor (curvature tensor depending only on the metric), covariant derivative ∇_α (torsionless) and contorsion as

$$\tilde{R}_{\cdot\tau\alpha\beta}^\lambda = R_{\cdot\tau\alpha\beta}^\lambda + \nabla_\alpha K_{\cdot\tau\beta}^\lambda - \nabla_\beta K_{\cdot\tau\alpha}^\lambda + K_{\cdot\gamma\alpha}^\lambda K_{\cdot\tau\beta}^\gamma - K_{\cdot\gamma\beta}^\lambda K_{\cdot\tau\alpha}^\gamma. \quad (2.13)$$

Similar formulas can be written for the Ricci tensor and for the scalar curvature with torsion:

$$\tilde{R}_{\tau\beta} = \tilde{R}_{\cdot\tau\alpha\beta}^\alpha = R_{\tau\beta} + \nabla_\lambda K_{\cdot\tau\beta}^\lambda - \nabla_\beta K_{\cdot\tau\lambda}^\lambda + K_{\cdot\gamma\lambda}^\lambda K_{\cdot\tau\beta}^\gamma - K_{\cdot\tau\gamma}^\lambda K_{\cdot\lambda\beta}^\gamma \quad (2.14)$$

(note that it is not symmetric) and

$$\tilde{R} = g^{\tau\beta} \tilde{R}_{\tau\beta} = R + 2\nabla^\lambda K_{\cdot\lambda\tau}^\tau - K_{\cdot\tau\lambda}^\lambda K_{\cdot\tau}^{\tau\gamma} + K_{\tau\gamma\lambda} K^{\tau\lambda\gamma}. \quad (2.15)$$

It proves useful to divide torsion into three irreducible components:

- (i) the trace vector $T_\beta = T_{\cdot\beta\alpha}^\alpha$,
- (ii) the (sometimes, it is called pseudotrace) axial vector $S^\nu = \epsilon^{\alpha\beta\mu\nu} T_{\alpha\beta\mu}$ and
- (iii) the tensor $q_{\cdot\beta\alpha}^\alpha$, which satisfies two conditions $q_{\cdot\beta\alpha}^\alpha = 0$ and $\epsilon^{\alpha\beta\mu\nu} q_{\alpha\beta\mu} = 0$.

Then, the torsion field can be expressed through these new fields as³

$$T_{\alpha\beta\mu} = \frac{1}{3}(T_{\beta}g_{\alpha\mu} - T_{\mu}g_{\alpha\beta}) - \frac{1}{6}\varepsilon_{\alpha\beta\mu\nu}S^{\nu} + q_{\alpha\beta\mu}. \quad (2.16)$$

Using the above formulas, it is not difficult to express curvatures (2.13)–(2.15) through these irreducible components. We shall write only the expression for scalar curvature, which will be useful in what follows:

$$\tilde{R} = R - 2\nabla_{\alpha}T^{\alpha} - \frac{4}{3}T_{\alpha}T^{\alpha} + \frac{1}{2}q_{\alpha\beta\gamma}q^{\alpha\beta\gamma} + \frac{1}{24}S^{\alpha}S_{\alpha}. \quad (2.17)$$

2.2. Einstein–Cartan theory and non-dynamical torsion

In order to start the discussion of gravity with torsion, we first consider a direct generalization of General Relativity, which is usually called Einstein–Cartan theory. Indeed, our consideration will be very brief. For further information one is recommended to look at the review [99]. Our first aim is to generalize the Einstein–Hilbert action

$$S_{\text{EH}} = -\frac{1}{\kappa^2} \int d^4x \sqrt{-g} R$$

for the space with torsion. It is natural to substitute scalar curvature R by (2.17), despite the change of the coefficients for the torsion terms in (2.17) cannot be viewed as something wrong. As we shall see later, in quantum theory the action of gravity with torsion is induced with the coefficients which are, generally, different from the ones in (2.18). The choice of the volume element dV_4 should be done in such a manner that it transforms like a scalar and also reduces to the usual d^4x for the case of a flat space–time and global orthonormal coordinates. Since we have two independent tensors: metric and torsion, the correct transformation property could be, in principle, satisfied in infinitely many ways. For example, one can take $dV_4 = d^4x \sqrt{-g}$ as in General Relativity, or $dV_4 = d^4x \sqrt{\det(S_{\mu}T_{\nu} - S_{\nu}T_{\mu})}$, or choose some other form. However, if we request that the determinant becomes d^4x in a flat space–time limit with zero torsion, all the expressions similar to the last one are excluded. In this paper we postulate, as usual, that the volume element in the space with torsion depends only on the metric and hence it has the form $dV_4 = d^4x \sqrt{-g}$. Then, according to (2.17), the most natural expression for the action of gravity with torsion will be

$$S_{\text{EC}} = -\frac{1}{\kappa^2} \int d^4x \sqrt{-g} \tilde{R} = -\frac{1}{\kappa^2} \int d^4x \sqrt{-g} \left(R - 2\nabla_{\alpha}T^{\alpha} - \frac{4}{3}T_{\alpha}T^{\alpha} + \frac{1}{2}q_{\alpha\beta\gamma}q^{\alpha\beta\gamma} + \frac{1}{24}S^{\alpha}S_{\alpha} \right). \quad (2.18)$$

The second term in the last integrand is a total derivative, so it does not affect the equations of motion, which have the non-dynamical form $T^{\alpha}_{\beta\gamma} = 0$. Therefore, on the mass shell the theory (2.18) is completely equivalent to General Relativity. The difference appears when we add the

³ In the most of this paper, we consider the four-dimensional space–time. More general, n -dimensional formulas concerning classical gravity with torsion can be found in Ref. [102]. More detailed classification of the torsion components can be found in Ref. [44].

external source for torsion. Imagine that torsion is coupled to some matter fields, and that the action of these fields depends on torsion in such a way that it contains the term

$$S_m = \int d^4x \sqrt{-g} K_{\beta\gamma}^\alpha \Sigma_\alpha^{\beta\gamma}, \quad (2.19)$$

where the tensor $\Sigma_\alpha^{\beta\gamma}$ is constructed from the matter fields (it is similar to the dynamically defined energy–momentum tensor) but may also depend on metric and torsion. One can check that, for the Dirac fermion minimally coupled to torsion, the $\Sigma_\alpha^{\beta\gamma}$ is nothing but the expression for the spin tensor of this field. One can use this as a hint and choose

$$\Sigma_\alpha^{\beta\gamma} = \frac{1}{\sqrt{-g}} \frac{\delta S_m}{\delta K_{\beta\gamma}^\alpha} \quad (2.20)$$

as the dynamical definition of the spin tensor for the theory with the classical action S_m . Unfortunately, in some theories this formula gives the result different from the one coming from the Noether theorem, and only for the minimally coupled Dirac spinor (see the next section) the result is the same. Next, since there is no experimental evidence for torsion, we can safely suppose it to be very weak. Then, as an approximation, $\Sigma_\alpha^{\beta\gamma}$ can be considered independent of torsion. In this case, the equations following from the action $S_{EC} + S_m$ have the structure

$$K \sim \kappa^2 \Sigma \sim \frac{1}{M_P^2} \cdot \Sigma, \quad (2.21)$$

where $M_P = 1/\kappa$ is the Planck mass. Then, torsion leads to the contact spin–spin interaction with the classical potential

$$V(\Sigma) \sim \frac{1}{M_P^2} \cdot \Sigma^2. \quad (2.22)$$

Some discussion of this contact interaction can be found in [99]. In Section 2.6 we shall provide an example, illustrating the possible importance of this interaction in the early Universe. Since the last expression (2.22) contains a $1/M_P^2 \approx 10^{-38} \text{ GeV}^{-2}$ factor, it can only lead to some extremely weak effects at low energies. Therefore, the effects of torsion, in the Einstein–Cartan theory, are suppressed by the torsion mass which is of the Planck order M_P . Even if one introduces kinetic terms for the torsion components, the situation would remain essentially the same, as far as we consider low-energy effects.

An alternative possibility is to suppose that torsion is light or even massless. In this case torsion can propagate, and there would be a chance to meet really independent torsion field. The review of theoretical limitations on this kind of theory [18,22] is one of the main subject of the present review (see Section 4). These limitations come from the consistency requirement for the effective quantum field theory for such a “light” torsion.

2.3. Interaction of torsion with matter fields

In order to construct the actions of matter fields in an external gravitational field with torsion we impose the principles of locality and general covariance. Furthermore, in order to preserve

the fundamental features of the original flat-space theory, one has to require the symmetries of a given theory (gauge invariance) in flat space–time to hold for the theory in curved space–time with torsion. It is also natural to forbid the introduction of new parameters with the dimension of inverse mass. This set of conditions enables one to construct the consistent quantum theory of matter fields on the classical gravitational background with torsion. The form of the action of a matter field is fixed except the values of some new parameters (non-minimal and vacuum ones) which remain arbitrary. This procedure which we have described above, leads to the so-called non-minimal actions.

Along with the non-minimal scheme, there is a (more traditional) minimal one. According to it the partial derivatives ∂_μ are substituted by the covariant ones $\tilde{\nabla}_\mu$, the flat metric $\eta_{\mu\nu}$ by $g_{\mu\nu}$ and the volume element d^4x by the covariant expression $d^4x\sqrt{-g}$. We remark, that the minimal scheme gives, for the case of the Einstein–Cartan theory, action (2.18), while the non-minimal scheme would make the coefficients at the torsion terms arbitrary.

We define the minimal generalization of the action for the scalar field as

$$S_0 = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right\}. \quad (2.23)$$

Obviously, the last action does not contain torsion. One has to note some peculiar property of the last statement. If one starts from the equivalent flat-space expression

$$S_0 = \int d^4x \left\{ -\frac{1}{2} \varphi \partial^2 \varphi - V(\varphi) \right\},$$

then the generalized action does contain torsion. This can be easily seen from the following simple calculation:

$$\tilde{\nabla}^2 \varphi = g^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \varphi = \square \varphi + T^\mu \partial_\mu \varphi. \quad (2.24)$$

Thus, at this point, the minimal scheme contains a small ambiguity which can be cured through the introduction of the non-minimal interaction. For one real scalar, one meets five possible non-minimal structures (cf. to expression (2.18)) $\varphi^2 P_i$, where [39]

$$P_1 = R, \quad P_2 = \nabla_\alpha T^\alpha, \quad P_3 = T_\alpha T^\alpha, \quad P_4 = S_\alpha S^\alpha, \quad P_5 = q_{\alpha\beta\gamma} q^{\alpha\beta\gamma}. \quad (2.25)$$

Correspondingly, there are five non-minimal parameters $\xi_1 \dots \xi_5$. The general non-minimal free field action has the form

$$S_0 = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi + \frac{1}{2} m^2 \varphi^2 + \frac{1}{2} \sum_{i=1}^5 \xi_i P_i \varphi^2 \right\}. \quad (2.26)$$

A more complicated scalar content gives rise to more non-minimal terms [36]. In particular, for the complex scalar field ϕ one can introduce, into the covariant Lagrangian, the following additional term:

$$\Delta L(\phi, \phi^\dagger) = i \xi_0 T^\mu (\phi^\dagger \cdot \partial_\mu \phi - \partial_\mu \phi^\dagger \cdot \phi).$$

In the case of a scalar φ coupled to a pseudoscalar χ , there are other possible non-minimal terms:

$$\Delta L(\varphi, \chi) = \frac{1}{2} \xi'_0 S^\mu (\varphi \partial_\mu \chi - \chi \partial_\mu \varphi) + \varphi \chi \sum_{j=6}^{10} \xi_j D_j,$$

where

$$D_6 = \nabla_\mu S^\mu, \quad D_7 = T_\mu S^\mu, \quad D_8 = \epsilon_{\mu\nu\alpha\beta} S^\mu q^{\nu\alpha\beta},$$

$$D_9 = \epsilon_{\mu\nu\alpha\beta} q_\lambda^{\mu\nu} q^{\lambda\alpha\beta} \quad D_{10} = \epsilon_{\mu\nu\alpha\beta} q_{\cdot\lambda}^{\mu\nu} q^{\alpha\lambda\beta}.$$

More general scalar models can be treated in a similar way.

For the Dirac spinor,⁴ the minimal procedure leads to the expression for the hermitian action

$$S_{1/2,\min} = \frac{i}{2} \int d^4x \sqrt{g} (\bar{\psi} \gamma^\alpha \tilde{\nabla}_\alpha \psi - \tilde{\nabla}_\alpha \bar{\psi} \gamma^\alpha \psi - 2im \bar{\psi} \psi). \quad (2.27)$$

Here $\gamma^\mu = e_a^\mu \gamma^a$, where γ^a is usual (flat-space) γ -matrix, and e_a^μ is tetrad (vierbein) defined through the standard relations

$$e_a^\mu \cdot e_{\mu b} = \eta_{ab}, \quad e_a^\mu \cdot e^{\nu a} = g^{\mu\nu}, \quad e_\mu^a \cdot e_{\nu a} = g_{\mu\nu}, \quad e_\mu^a \cdot e^{\mu b} = \eta^{ab}.$$

The covariant derivative of a Dirac spinor $\tilde{\nabla}_\alpha \psi$ should be defined to be consistent with the covariant derivative of tensors. We suppose that

$$\tilde{\nabla}_\mu \psi = \partial_\mu \psi + \frac{i}{2} \tilde{w}_\mu^{ab} \sigma_{ab} \psi, \quad (2.28)$$

where \tilde{w}_μ^{ab} is a new object which is usually called spinor connection, and

$$\sigma_{ab} = \frac{i}{2} (\gamma_a \gamma_b - \gamma_b \gamma_a).$$

The conjugated expression is

$$\tilde{\nabla}_\mu \bar{\psi} = \partial_\mu \bar{\psi} - \frac{i}{2} \bar{\psi} \tilde{w}_\mu^{ab} \sigma_{ab}. \quad (2.29)$$

Now, we consider how the covariant derivative acts on the vector $\bar{\psi} \gamma^\alpha \psi$. As we already learned in Section 2.1, if the connection provides the proper transformation law for the vector, it does so with any tensor. Therefore, the only one equation for the spinor connection \tilde{w}_μ^{ab} is

$$\tilde{\nabla}_\mu (\bar{\psi} \gamma^\alpha \psi) = \partial_\mu (\bar{\psi} \gamma^\alpha \psi) + \tilde{\Gamma}_{\cdot\lambda\mu}^\alpha (\bar{\psi} \gamma^\lambda \psi) = \nabla_\mu (\bar{\psi} \gamma^\alpha \psi) + K_{\cdot\lambda\mu}^\alpha (\bar{\psi} \gamma^\lambda \psi). \quad (2.30)$$

Substituting (2.28) and (2.29) into (2.30), after some algebra, we arrive at the formula for the spinor connection

$$\tilde{w}_{\mu ab} = w_{\mu ab} + \frac{1}{4} K_{\cdot\lambda\mu}^\alpha (e_a^\lambda e_{b\alpha} - e_b^\lambda e_{a\alpha}), \quad (2.31)$$

where

$$w_{\mu ab} = \frac{1}{4} (e_{b\alpha} \partial_\mu e_a^\alpha - e_{a\alpha} \partial_\mu e_b^\alpha) + \frac{1}{4} \Gamma_{\lambda\mu}^\alpha (e_{b\alpha} e_a^\lambda - e_{a\alpha} e_b^\lambda) \quad (2.32)$$

is spinor connection in the space–time without torsion.

⁴ Various aspects of the minimal fermion–torsion interaction have been considered in many papers, e.g. [57,12,99].

Substituting (2.31) into (2.37), and performing integration by parts, we arrive at two equivalent forms for the spinor action

$$\begin{aligned} S_{1/2,\min} &= i \int d^4x \sqrt{g} \bar{\psi} \left(\gamma^\alpha \tilde{\nabla}_\alpha - \frac{1}{2} \gamma^\alpha T_\alpha - im \right) \psi \\ &= i \int d^4x \sqrt{g} \bar{\psi} \left(\gamma^\alpha \nabla_\alpha - \frac{i}{8} \gamma^5 \gamma^\alpha S_\alpha - im \right) \psi, \end{aligned} \quad (2.33)$$

where we use standard representation for the Dirac matrices, such that $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$ and $(\gamma^5)^2 = 1$. Also, $\gamma^5 = \gamma_5$. The first integral is written in terms of the covariant derivative $\tilde{\nabla}_\alpha$ with torsion, while the last is expressed through the torsionless covariant derivative ∇_α . Indeed the last form is more informative, for it tells us that only the axial vector S_μ couples to fermion, and the other two components: T_μ and $q_{\beta\gamma}^\alpha$ completely decouple. It is important to note, that in the first of integrals (2.33) the $\gamma^\alpha T_\alpha$ -term has an extra factor of i as compared to the electromagnetic (or any vector) field. This indicates, that if being taken separately from the first term $\gamma^\alpha \tilde{\nabla}_\alpha$, the $\gamma^\alpha T_\alpha$ -term would introduce an imaginary part into the spinor action, and therefore it has no sense. Of course, the same concerns the term $i \int \bar{\psi} \gamma^\alpha \tilde{\nabla}_\alpha \psi$. The imaginary terms, coming from the two parts, cancel, and give rise to a hermitian action for the spinor, equivalent to (2.27) or to the second integral in (2.33).

The non-minimal interaction is a bit more complicated. Using covariance, locality, dimension, and requesting that the action does not break parity one can construct only two non-minimal (real, of course) terms with the structures already known from (2.33):

$$S_{1/2,\text{non-min}} = i \int d^4x \sqrt{g} \bar{\psi} \left(\gamma^\alpha \nabla_\alpha + \sum_{j=1,2} \eta_j Q_j - im \right) \psi \quad (2.34)$$

with

$$Q_1 = i\gamma^5 \gamma^\mu S_\mu, \quad Q_2 = i\gamma^\mu T_\mu$$

and two arbitrary non-minimal parameters η_1, η_2 . The minimal theory corresponds to $\eta_1 = -\frac{1}{8}$, $\eta_2 = 0$. Let us again note that the T_μ -dependent term in (2.34) is different from the last term in the first representation in (2.33). In (2.34) all the terms are real. We observe, that the interaction of the torsion trace T_μ with fermion is identical to the one of the electromagnetic field. Therefore, in some situations when torsion is considered simultaneously with the external electromagnetic field A_μ , one can simply redefine A_μ such that the torsion trace T_μ disappears.

Consider the symmetries of the Dirac spinor action non-minimally coupled to the vector and torsion fields

$$S_{1/2} = i \int d^4x \bar{\psi} [\gamma^\mu (\partial_\mu + ieA_\mu + i\eta\gamma_5 S_\mu) - im] \psi. \quad (2.35)$$

As compared to (2.34), here we have changed the notation for the non-minimal parameter of interaction between spinor fields and the axial part S_μ of torsion: $\eta_1 \rightarrow \eta$. Furthermore, we used the possibility to redefine the external electromagnetic potential A_μ in such a way that it absorbs the torsion trace T_μ .

The new interaction with torsion does not spoil the invariance of the action under usual gauge transformation:

$$\psi' = \psi e^{\alpha(x)}, \quad \bar{\psi}' = \bar{\psi} e^{-\alpha(x)}, \quad A'_\mu = A_\mu - e^{-1} \partial_\mu \alpha(x). \quad (2.36)$$

Furthermore, the massless part of action (2.35) is invariant under the transformation in which the axial vector S_μ plays the role of the gauge field

$$\psi' = \psi e^{\gamma_5 \beta(x)}, \quad \bar{\psi}' = \bar{\psi} e^{\gamma_5 \beta(x)}, \quad S'_\mu = S_\mu - \eta^{-1} \partial_\mu \beta(x). \quad (2.37)$$

Thus, in the massless sector of the theory one faces generalized gauge symmetry depending on the scalar, $\alpha(x)$, and pseudoscalar, $\beta(x)$, parameters of the transformations, while the massive term is not invariant under the last transformation.

Consider whether the massless vector field might couple to torsion. Here, one has to use the principle of preserving the symmetry. The minimal interaction with torsion breaks the gauge invariance for the vector field, since

$$\tilde{F}_{\mu\nu} = \tilde{\nabla}_\mu A_\nu - \tilde{\nabla}_\nu A_\mu = F_{\mu\nu} + 2A_\lambda K^\lambda_{[\mu\nu]}$$

is not invariant. The possibility to modify the gauge transformation in the theory with torsion has been studied in [139,104]. In this paper we opt to keep the form of the gauge transformation unaltered, and postulate that the gauge vector does not couple to torsion. The reasons for this choice is the following. First of all, when one is investigating the quantum field theory in external torsion field, it is natural to separate the effects of external field from the purely matter sector. Thus, the modification of the gauge transformation does not fit with our approach. Furthermore, the most important part S_μ of the torsion tensor does not admit the fine-tuning of the gauge transformation. In other words, for the most interesting case of purely antisymmetric torsion it is not possible to save gauge invariance for the vector coupled to torsion in a minimal way.

Let us consider the non-minimal interaction for the special case of an abelian gauge vector. One can introduce several non-minimal terms which do not break the gauge invariance: $\int d^4x \sqrt{-g} F_{\mu\nu} K^{\mu\nu}$. The most general form of $K^{\mu\nu}$ is

$$\begin{aligned} K^{\mu\nu} = & \theta_1 \epsilon^{\mu\nu\alpha\beta} T_\alpha S_\beta + \theta_2 \epsilon^{\mu\nu\alpha\beta} \partial_\alpha S_\beta + \theta_3 \epsilon^{\mu\nu\alpha\beta} q^\lambda_{\alpha\beta} S_\lambda \\ & + \theta_4 q^\lambda_{\mu\nu} T_\lambda + \theta_5 (\partial_\mu T_\nu - \partial_\nu T_\mu) + \theta_6 \partial_\lambda q^\lambda_{\mu\nu}. \end{aligned} \quad (2.38)$$

For the non-abelian vector, the non-minimal structures like (2.38) are algebraically impossible. In the Standard Model, where all vectors are non-abelian, the non-minimal terms (2.38) do not exist.

And so, we have constructed the actions of free scalar, spinor and vector fields coupled to torsion. In general, there are two types of actions: minimal and non-minimal. As we shall see in the next sections, the non-minimal interactions with spinors and scalars provide certain advantages at the quantum level, for they give the possibility to construct renormalizable theory [36].

2.4. Conformal properties of torsion

Conformal symmetry with torsion has been studied in many papers (see, for example, [141,36, 102] and references therein). Here, we shall summarize the results obtained in [36,102].

For the torsionless theory the conformal transformation of the metric, scalar, spinor and vector fields take the form

$$g_{\mu\nu} \rightarrow g'_{\mu\nu} = g_{\mu\nu} e^{2\sigma}, \quad \varphi \rightarrow \varphi' = \varphi e^{-\sigma}, \quad \psi \rightarrow \psi' = \psi e^{-3/2\sigma}, \quad A_\mu \rightarrow A'_\mu = A_\mu, \quad (2.39)$$

where $\sigma = \sigma(x)$. In the absence of torsion, the actions of free fields are invariant if they are massless, besides in the scalar sector one has to put $\xi = \frac{1}{6}$. Indeed, the interaction terms of the gauge theory (gauge, Yukawa and 4-scalar terms) are conformally invariant.

The problem is to define the conformal transformation for torsion, such that the free actions formulated in the previous section would be invariant for these or that values of the non-minimal parameters. It turns out that there are three different ways to choose the conformal transformation for torsion:

- (i) *Weak conformal symmetry* [36]: Torsion does not transform at all, $T^\lambda_{\cdot\mu\nu} \rightarrow T'^\lambda_{\cdot\mu\nu} = T^\lambda_{\cdot\mu\nu}$. The conditions of conformal symmetry are absolutely the same as in the torsionless theory.
- (ii) *Strong conformal symmetry* [36]: In this version, torsion transforms as

$$T^\lambda_{\cdot\mu\nu} \rightarrow T'^\lambda_{\cdot\mu\nu} = T^\lambda_{\cdot\mu\nu} + \omega(\delta^\lambda_\nu \partial_\mu - \delta^\lambda_\mu \partial_\nu) \sigma(x). \quad (2.40)$$

This transformation includes an arbitrary parameter,⁵ ω . Indeed the above transformation means that only the torsion trace transforms

$$T_\mu \rightarrow T'_\mu = T_\mu + 3\omega \partial_\mu \sigma(x).$$

Other components of torsion remain inert under (2.40). For any value of ω , the free actions (2.26) and (2.34) are invariant if they depend only on the axial vector S_μ and tensor $q^\lambda_{\cdot\mu\nu}$, but not on the trace T_μ . Of course, this is quite natural, because only T_μ transforms. The restrictions imposed by the symmetry are $\xi_2 = \xi_3 = \eta_2 = 0$. The immediate result of the strong conformal symmetry is the modified Noether identity, which now reads

$$\frac{\delta S}{\delta g_{\mu\nu}} \delta g_{\mu\nu} + \sum \frac{\delta S}{\delta \Phi} \delta \Phi + \frac{\delta S}{\delta T^\alpha_{\cdot\beta\gamma}} \delta T^\alpha_{\cdot\beta\gamma} = 0, \quad (2.41)$$

where Φ is the full set of matter fields. Using the equations of motion, the definition of the energy–momentum tensor $T_{\mu\nu} = -(2/\sqrt{-g})\delta S/\delta g^{\mu\nu}$, and the definition of the spin tensor given in (2.20), we get

$$T^{\mu\nu} \delta g_{\mu\nu} + (\Sigma^{\mu\nu}_\lambda - \frac{1}{2} \Sigma^{\mu\nu}_{\cdot\lambda}) \delta T^\lambda_{\cdot\mu\nu} = 0,$$

that gives

$$2T^\mu_\mu - 3\omega \nabla_\mu \Sigma^{\mu\nu}_\nu = 0. \quad (2.42)$$

Along with the standard relation for the trace of the energy–momentum tensor $T^\mu_\mu = 0$, here we meet an additional identity $\nabla_\mu \Sigma^{\mu\nu}_\nu = 0$. It is interesting that this identity is exact, for it is

⁵ Instead of introducing an arbitrary numerical parameter ω , one could replace, in the transformation rule for torsion (2.40), the parameter σ for some other, independent parameter. This observation has been done in 1985 by A.O. Barvinsky and V.N. Ponomaryev in the report on my Ph.D. Thesis [165].

not violated by the anomaly on quantum level. In order to understand this, we remember that the T_μ -dependence is purely non-minimal, and if it does not exist in classical theory, it cannot appear at the quantum level.

(iii) *Compensating conformal symmetry* [141,102]: This is the most interesting and complicated version of the conformal transformation.

Let us consider the massless scalar, non-minimally coupled to metric and torsion (2.26). One can add to it the $\lambda\varphi^4$ -term, without great changes in the results. Then

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + \frac{1}{2} \sum_{i=1}^5 \xi_i P_i \varphi^2 - \frac{\lambda}{4!} \varphi^4 \right\}, \quad (2.43)$$

where P_i were defined at (2.25).

The equations of motion for the torsion tensor can be split into three independent equations written for the components T_α , S_α , $q_{\alpha\beta\gamma}$; they yield

$$T_\alpha = \frac{\xi_2}{\xi_3} \cdot \frac{\nabla_\alpha \varphi}{\varphi}, \quad S_\mu = q_{\alpha\beta\gamma} = 0. \quad (2.44)$$

Replacing these expressions back into action (2.43), we obtain the on-shell action

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} \left(1 - \frac{\xi_2^2}{\xi_3} \right) g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + \frac{1}{2} \xi_1 \varphi^2 R - \frac{\lambda}{4!} \varphi^4 \right\}, \quad (2.45)$$

that can be immediately reduced to the torsionless conformal action

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + \frac{1}{12} \varphi^2 R - \frac{\lambda'}{4!} \varphi^4 \right\}, \quad (2.46)$$

by an obvious change of variables, whenever the non-minimal parameters satisfy the condition

$$\xi_1 = \frac{1}{6} \left(1 - \frac{\xi_2^2}{\xi_3} \right) \quad (2.47)$$

and some relation between λ and λ' . Some important observation is in order. It is well known, that the conformal action (2.46) is classically equivalent to the Einstein–Hilbert action of General Relativity, but with the opposite sign [59,168]. In order to check this, we take such a wrong-sign action:

$$S_{\text{EH}}[g_{\mu\nu}] = \int d^4x \sqrt{-\hat{g}} \left\{ \frac{1}{\kappa^2} \hat{R} + \Lambda \right\}. \quad (2.48)$$

This action depends on the metric $\hat{g}_{\mu\nu}$. Performing conformal transformation $\hat{g}_{\mu\nu} = g_{\mu\nu} \cdot e^{2\sigma(x)}$, we use the standard relations between geometric quantities of the original and transformed metrics:

$$\sqrt{-\hat{g}} = \sqrt{-g} e^{4\sigma}, \quad \hat{R} = e^{-2\sigma} [R - 6\Box\sigma - 6(\nabla\sigma)^2]. \quad (2.49)$$

Substituting (2.49) into (2.48), after integration by parts, we arrive at

$$S_{\text{EH}}[g_{\mu\nu}] = \int d^4x \sqrt{-g} \left\{ \frac{6}{\kappa^2} e^{2\sigma} (\nabla\sigma)^2 + \frac{e^{2\sigma}}{\kappa^2} R + \Lambda e^{4\sigma} \right\},$$

where $(\nabla\sigma)^2 = g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma$. If one denotes

$$\varphi = e^\sigma \cdot \sqrt{\frac{12}{\kappa^2}},$$

action (2.48) becomes

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{12}R\varphi^2 + \Lambda \left(\frac{\kappa^2}{12} \right)^2 \cdot \varphi^4 \right\}, \quad (2.50)$$

that is nothing but (2.46). And so, the metric–scalar theory described by the action of Eq. (2.50) and the metric–torsion–scalar theory (2.43) with constraint (2.47) are equivalent to the General Relativity with cosmological constant.

One has to notice that the first two theories exhibit an extra local conformal symmetry, which compensates an extra (with respect to (2.48)) scalar degree of freedom. Moreover, (2.50) is a particular case of a family of similar actions, linked to each other by the reparametrization of the scalar or (and) the conformal transformation of the metric [168]. The symmetry transformation which leaves action (2.50) stable is

$$g'_{\mu\nu} = g_{\mu\nu} \cdot e^{2\rho(x)}, \quad \varphi' = \varphi \cdot e^{-\rho(x)}. \quad (2.51)$$

The version of the Brans–Dicke theory with torsion (2.43) is conformally equivalent to General Relativity (2.48) provided that the new condition (2.47) is satisfied and there are only external conformally covariant sources for S_μ , $q^\alpha_{\cdot\beta\gamma}$, and for the transverse component of T_α . Such a sources do not spoil the conformal symmetry.

Now, we can see that the introduction of torsion provides some theoretical advantage. If we start from the positively defined gravitational action (2.48), the sign of the scalar action (2.50) should be negative, indicating to the well-known instability of the conformal mode of General Relativity (see, for example, [97] and also [189,130] for the recent account of this problem and further references). It is easy to see that the metric–torsion–scalar theory may be free of this problem, if we choose $\xi_2^2/\xi_3 - 1 > 0$. In this case one meets the equivalence of the positively defined scalar action (2.43) to action (2.48) with the negative sign. The negative sign in (2.48) signifies, in turn, the positively defined gravitational action. Without torsion one can achieve positivity in the gravitational action only by the expense of taking the negative kinetic energy for the scalar action in (2.50).

The equation of motion (2.44) for T_α may be regarded as a constraint that fixes the conformal transformation for this vector to be consistent with the one for the metric and scalar. Then, instead of (2.51), one has

$$g'_{\mu\nu} = g_{\mu\nu} \cdot e^{2\rho(x)}, \quad \varphi' = \varphi \cdot e^{-\rho(x)}, \quad T'_\alpha = T_\alpha - \frac{\xi_2}{\xi_3} \cdot \partial_\alpha \rho(x). \quad (2.52)$$

The on-shell equivalence can be also verified using the equations of motion [102]. It is easy to check, by direct inspection, that even off-shell, the theory with torsion (2.43), satisfying relation (2.47), may be conformally invariant whenever we define the transformation law for the torsion trace according to (2.52): and also postulate that the other pieces of torsion: S_μ and $q^\alpha_{\cdot\beta\gamma}$, do not transform. The quantities $\sqrt{-g}$ and R transform as in (2.49). One may introduce into the action other conformal invariant terms depending on torsion. For instance,

$$S = -\frac{1}{4} \int d^4x \sqrt{-g} T_{\alpha\beta} T^{\alpha\beta},$$

where $T_{\alpha\beta} = \partial_\alpha T_\beta - \partial_\beta T_\alpha$. In the spinor sector one has to request $\eta_2 = 0$ in (2.34), as it was for the strong conformal symmetry. It is important to notice, for the future, that on the quantum level this condition does not break renormalizability, even if $\xi_{2,3}$ are non-zero.

One can better understand the equivalence between General Relativity and conformal metric–scalar–torsion theory (2.43), (2.47) after presenting an alternative form for the symmetric action. All torsion-dependent terms in (2.43) may be unified in the expression

$$\mathcal{P} = -\frac{1}{6} \frac{\xi_2^2}{\xi_3} R + \xi_2 (\nabla_\mu T^\mu) + \xi_3 T_\mu T^\mu + \xi_4 S_\mu^2 + \xi_5 q_{\mu\nu\lambda}^2. \quad (2.53)$$

It is not difficult to check that the transformation law for this new quantity is especially simple: $\mathcal{P}' = e^{-2\rho} \mathcal{P}$. Using new quantity, the conformal invariance of the action becomes obvious:

$$S_{\text{inv}} = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + \frac{1}{12} R \varphi^2 + \frac{1}{2} \mathcal{P} \varphi^2 \right\}. \quad (2.54)$$

In order to clarify the role of torsion in our conformal model, we construct one more representation for the metric–scalar–torsion action with local conformal symmetry. Let us start, once again, from action (2.43), (2.47) and perform only part of transformations (2.52):

$$\varphi \rightarrow \varphi' = \varphi \cdot e^{-\rho(x)}, \quad T_\alpha \rightarrow T'_\alpha = T_\alpha - \frac{\xi_2}{\xi_3} \cdot \partial_\alpha \rho(x). \quad (2.55)$$

Of course, if we supplement (2.55) by the transformation of the metric, we arrive at (2.52) and the action does not change. On the other hand, (2.55) alone might lead to an alternative conformally equivalent description of the theory. Taking ρ such that

$$\varphi \cdot e^{-\rho(x)} = \frac{12}{\kappa^2} \left(1 - \frac{\xi_2}{\xi_3} \right) = \text{const.},$$

we obtain, after some algebra, the following action:

$$S = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} \left\{ R + \frac{3}{\kappa^2 (1 - \xi_2/\xi_3)} \left[\xi_4 S_\mu^2 + \xi_5 q_{\mu\nu\lambda}^2 + \xi_3 \left(T_\alpha - \frac{\xi_2}{\xi_3} \nabla_\alpha \ln \varphi \right)^2 \right] \right\}. \quad (2.56)$$

This form of the action does not contain interaction between curvature and the scalar field. At the same time, the latter is present until we use the equations of motion (2.44) for torsion. Torsion trace looks here like a Lagrange multiplier, and only using torsion equations of motion, one can obtain the action of GR. It is clear that one can arrive at the same action (2.56), making the transformation of the metric as in (2.52) instead of (2.55).

To complete this part of our consideration, we mention that the direct generalization of the Einstein–Cartan theory including an extra scalar may be conformally equivalent to General Relativity, provided that the non-minimal parameter takes an appropriate value. To see this, one uses relation (2.17) and replace it into the “minimal” action

$$S_{\text{ECBD}} = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + \frac{1}{2} \xi \tilde{R} \varphi^2 \right\}. \quad (2.57)$$

It is easy to see that condition (2.47) is satisfied for the special value $\xi = \frac{1}{3}$, contrary to the famous $\xi = \frac{1}{6}$ in the torsionless case. The effect of changing conformal value of ξ due to the

non-trivial transformation of torsion has been discussed in [148,102] (see also further references there).

2.5. Gauge approach to gravity: higher derivative gravity theories with torsion

There are many good reviews on the gauge approach to gravity (see, for example, [99]), and since the aim of the present paper is to treat torsion from the field-theoretical point of view, we shall restrict ourselves to a brief account of the results and some observations.

In Section 2.1 we have introduced covariant derivative and found, that this can be done in different ways, because one can add to the affine connection any tensor $C^\alpha_{\beta\gamma}$ (see Eq. (2.6)). Any such extension of the affine connection is related to some additional physical field, exactly because $C^\alpha_{\beta\gamma}$ is a tensor and cannot be removed by a coordinate transformation. It was already mentioned in Section 2.1, that the introduction of covariant derivative is related to the general coordinate transformations. The aim of the gauge approach to gravity is to show that the same construction, including the metric and the covariant derivative based on an arbitrary connection, can be achieved through the local version of the Lorentz–Poincare symmetry. If one requests the theory to be invariant with respect to the Poincare group with the infinitesimal parameters depending on the space–time point, one has to introduce two compensating fields: vierbein e^a_α and some independent spinor connection W^{bc}_μ [179,115,99,100] (see also [112,34] and [176,136,110,151,89] for alternative considerations). In this way, one naturally arrives at the gauge approach to gravity. One of the important applications of this approach is the natural and compact formulation of simple supergravity [63], where the supersymmetric generalization of the Einstein–Cartan theory emerges.

The gauge approach, exactly as the one of Section 2.1, does not provide any reasonable restrictions on W^{bc}_μ , and one has to introduce (or not) these restrictions additionally. In this paper we suppose that the covariant derivative possess metricity $\nabla_\alpha e^\mu_a = 0$. Then W^{bc}_μ becomes \tilde{w}^{bc}_μ -spinor connection with torsion. The interesting question to answer is whether the description of the gravity with torsion in terms of the variables $(e^a_\alpha, \tilde{w}^{bc}_\mu)$ is equivalent to the description in terms of the variables $(g_{\mu\nu}, T^\alpha_{\beta\gamma})$.

One can make the following observation. The first set $(e^a_\alpha, \tilde{w}^{bc}_\mu)$ corresponds to the first order formalism, while the second set $(g_{\mu\nu}, T^\alpha_{\beta\gamma})$ to the second order formalism. The origin of this is that in the last case the non-torsional part of the affine connection is a function of the metric, while, within the gauge approach, the variables $(e^a_\alpha, \tilde{w}^{bc}_\mu)$ are mutually independent *completely*. One has to note that, in gravity, the equivalence between the first and second order formalism is a subtle matter. The simplest situation is the following. One takes the action

$$S_1 = \int d^4x \sqrt{-g} g^{\mu\nu} R_{\mu\nu}(\Gamma), \quad (2.58)$$

where $R_{\mu\nu}(\Gamma) = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\mu \Gamma^\lambda_{\nu\lambda} + \Gamma^\lambda_{\mu\nu} \Gamma^\tau_{\lambda\tau} - \Gamma^\lambda_{\mu\tau} \Gamma^\tau_{\lambda\nu}$ depends on the connection $\Gamma^\lambda_{\mu\nu}$, which is independent on the metric $g_{\mu\nu}$. Then the equations for these two fields

$$\frac{\delta S_1}{\delta g_{\mu\nu}} = 0, \quad \frac{\delta S_1}{\delta \Gamma^\lambda_{\mu\nu}} = 0$$

lead to the conventional Einstein equations and also to the standard expression for the affine connection (2.5). In this case the first order formalism is equivalent to the usual second order formalism. However, this is not true if one chooses some other action for gravity. For instance, introducing higher derivative terms or adding to (2.58) additional terms depending on the non-metricity, one can indeed lose classical equivalence between two formalisms.⁶ For the general action, the only possibility to link the connection with the metric is to impose the metricity condition.

Let us come back to our case of gravity with torsion. From the consideration above it is clear that the descriptions in terms of the variables $(e^a_\alpha, \tilde{w}^{bc}_\mu)$ and $(g_{\mu\nu}, T^\alpha_{\beta\gamma})$ cannot be equivalent unlike we work with the Einstein–Cartan action (2.58). But, the non-equivalence comes only from the usual difference between first and second order formalisms, and has nothing to do with torsion. In order to have comparable situations, we have to replace the first set by $(e^a_\alpha, \Delta w^{bc}_\mu)$, where $\Delta w^{bc}_\mu = \tilde{w}^{bc}_\mu - w^{bc}_\mu$ and w^{bc}_μ depends on the vierbein through (2.32). The equivalence between the sets $(g_{\mu\nu}, T^\alpha_{\beta\gamma})$ and $(e^a_\alpha, \Delta w^{bc}_\mu)$ really takes place and this can be checked explicitly. First of all, one has to establish the equivalence (invertible relation) between metric and vierbein. This can be achieved by deriving

$$\frac{\delta g_{\mu\nu}}{\delta e^a_\alpha} = 2 \delta^\alpha_{(\mu} e_{\nu)a} \quad \text{and} \quad \frac{\delta e^b_\beta}{\delta g_{\mu\nu}} = \frac{1}{2} e^{b(\mu} \delta^\nu_{\beta)} . \quad (2.59)$$

In a similar fashion, one can calculate the derivatives

$$\frac{\delta \Delta w^{ab}_\mu}{\delta T_{\lambda\rho\sigma}} = \frac{1}{2} e^{\lambda[b} e^{a][\rho} e^{\sigma]}_\mu + \frac{1}{4} \delta^\lambda_\mu e^{a[\rho} e^{\sigma]b} \quad \text{and} \quad \frac{\delta T_{\lambda\rho\sigma}}{\delta \Delta w^{ab}_\mu} = 4 e_{\lambda[a} e_{b][\sigma} \delta^\mu_{\rho]} . \quad (2.60)$$

Thus, the transformation from one set of variables $(g_{\mu\nu}, T^\alpha_{\beta\gamma})$ to another one $(e^a_\alpha, \Delta w^{bc}_\mu)$ is non-degenerate and two (second order in the torsion-independent part) descriptions are equivalent. If some statement about torsion is true for the $(g_{\mu\nu}, T^\alpha_{\beta\gamma})$ variables, it is also true for the $(e^a_\alpha, \Delta w^{bc}_\mu)$ variables, and *v.v.*

Now, since we established the equivalence between different variables, we can try to formulate some torsion theories more general than the one based on the Einstein–Cartan action. On the classical level the consideration can be based only on the general covariance and other symmetries. Let us restrict ourselves to the local actions only.

As we have already learned, there are three possible descriptions of gravity with torsion:

(a) In terms of the metric $g_{\mu\nu}$, torsion $T^\alpha_{\beta\gamma}$ and Riemann curvature $R^\lambda_{\alpha\beta\gamma}$. When useful, torsion tensor can be replaced by its irreducible components $T_\alpha, S_\beta, q^\lambda_{\gamma\tau}$.

(b) In terms of the metric $g_{\mu\nu}$, torsion $T^\alpha_{\beta\gamma}$ and curvature (2.12) with torsion $\tilde{R}^\lambda_{\alpha\beta\gamma}$. The relation between two curvature tensors, with and without torsion, is given by (2.13).

(c) In terms of the variables $(e^c_\alpha, \tilde{w}^{ab}_\mu)$ and the corresponding curvature

$$R^{ab}_{\mu\nu} = \partial_\mu \tilde{w}^{ab}_\nu - \partial_\nu \tilde{w}^{ab}_\mu + \tilde{w}^{ac}_\mu \tilde{w}^b_{c\nu} - \tilde{w}^{ac}_\nu \tilde{w}^b_{c\mu} .$$

We consider possibility (a) as the most useful one and will follow it in this paper, in particular for the construction of the new actions.

⁶ We remark that on the quantum level there is no equivalence even for the (2.58) action [35].

The invariant local action can be always expanded into the power series in derivatives of the metric and torsion. It is natural to consider torsion to be of the same order as the affine connection, that is $T \sim \partial g$. Then, in the second order (in the metric derivatives) we find just those terms which were already included into the Einstein–Cartan action (2.18), but possibly with other coefficients. In the next order one meets numerous possible structures of the mass dimension 4, which were analyzed in Ref. [48]. Indeed, this action, which includes more than 100 dynamical terms, and many surface terms, does not look attractive for deriving physical predictions of the theory. It is important that this general action, and many its particular cases, describe torsion dynamics. In fact, all interesting and physically important cases, like the action of vacuum for the quantized matter fields, second order (in α') string effective action, possible candidates for the torsion action [18] are nothing but the particular cases of the bulky action of [48]. In what follows, we consider some of the mentioned particular cases of [48]. Some other torsion actions which will not be presented here: the general fourth derivative actions with absolutely antisymmetric torsion, with and without local conformal symmetry, were described in [34].

2.6. An example of the possible effect of classical torsion

There is an extensive bibliography on different aspects of classical gravity with torsion. We are not going to review these publications here, just because our main target is the quantum theory. However, it is worthwhile to present a short general remark and consider a simple but interesting example.

The expression “classical action of torsion” can be used only in some special sense. In the Einstein–Cartan theory, with or without matter, torsion does not have dynamics and therefore can only lead to the contact interaction between spins. On the other hand, the spin of the particle is essentially quantum characteristic. Therefore, the classical torsion can be understood only as the result of a semi-classical approximation in some quantum theory. Now, without going into the details, let us suppose that such an approximation can be done and consider its possible effects. The most natural possibility is the application to early cosmology, which has been studied long ago (see, for example, discussion in [99]). Here, we are going to consider this issue in a very simple manner.

One can suppose that in the early Universe, due to quantum effects of matter, the average spin (axial) current is non-zero. Let us demonstrate that this might lead to a non-singular cosmological solution. For simplicity, we suppose that torsion is completely antisymmetric and that there is a conformally constant spinor current

$$J^\mu = \langle \bar{\psi} \gamma^5 \gamma^\mu \psi \rangle . \quad (2.61)$$

The Einstein–Cartan action (2.18), with this additional current is [38]⁷

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{16\pi G} (R + \theta S_\mu S^\mu) + S_\mu J^\mu \right] . \quad (2.62)$$

⁷ The one-loop quantum calculations in this model, and the construction of the on-shell renormalization group, has been performed in [38].

We have included an arbitrary coefficient θ into the Einstein–Cartan action, but it could be equally well included into the definition of the global current (2.61). It is worth mentioning that at the quantum level the introduction of such coefficient is justified. Since torsion does not have its own dynamics, on shell it is simply expressed through the current

$$S^\mu = \frac{8\pi G}{\theta} J^\mu. \quad (2.63)$$

Replacing (2.63) back into action (2.62) we arrive at the expression

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{16\pi G} R + \frac{4\pi G}{\theta} J_\mu J^\mu \right], \quad (2.64)$$

which resembles the Einstein–Hilbert action with the cosmological constant. However, the analogy is incomplete, because the square of the current J^μ has conformal properties different from the ones of the cosmological constant. Consider, for the sake of simplicity, the conformally flat metric

$$g_{\mu\nu} = \eta_{\mu\nu} \cdot a^2(\eta),$$

where η is the conformal time. According to (2.61), the current J^μ has to be replaced by $J^\mu = a^{-4}(\eta) \cdot \bar{J}^\mu$, where \bar{J}^μ is constant. Denoting

$$\frac{32}{3} \frac{\pi G^2}{\theta} \eta_{\mu\nu} \bar{J}^\mu \bar{J}^\nu = K = \text{const.},$$

we arrive at the action and the corresponding equation of motion for $a(\eta)$:

$$S = -\frac{3}{8\pi G} \int d\eta \int d^3x \left[(\nabla a)^2 - \frac{K}{a^2} \right], \quad \frac{d^2 a}{d\eta^2} = \frac{K}{a^3}. \quad (2.65)$$

This equation can be rewritten in terms of physical time t (where, as usual, $a(\eta) d\eta = dt$) as

$$a^2 \ddot{a} + a \dot{a}^2 = K a^{-3}.$$

After the standard reduction of order, the integral solving this equation is written in the form

$$\int \frac{a^2 da}{\sqrt{C a^2 - K}} = t - t_0, \quad (2.66)$$

where C is the integration constant. The last integral has different solutions depending on the signs of K and C . Consider all the possibilities:

(1) The global spinor current is time-like and $K > 0$. Then, Eq. (2.66) shows that: (i) C is positive, and (ii) $a(t)$ has minimal value $a_0 = \sqrt{K/C} > 0$. Thus, the presence of the global time-like spinor current, in the Einstein–Cartan theory, prevents the singularity. Indeed, since such a global spinor current can appear only as a result of some quantum effects, one can consider this as an example of quantum elimination of the Big Bang singularity. The singularity is prevented, in this example, at the scale comparable to the Planck one. This is indeed natural, since the dimensional unity in the theory is the Newton constant. The dimensional considerations [99] show that in the Einstein–Cartan theory the effects of torsion become relevant only at the Planck scale. Finally, the explicit solution of Eq. (2.66) has the form

$$\text{arccosh} \left(\sqrt{\frac{C}{K}} a \right) + a \sqrt{a^2 - \frac{K}{C}} = \frac{2C^{3/2}}{K} (t - t_0), \quad (2.67)$$

where C is an integration constant. The value of C can be easily related to the minimal possible value of a . In the long-time limit we meet the asymptotic behavior $a \sim t^{2/3}$. The importance of torsion, in the Einstein–Cartan theory, is seen only at small distances and times and for the scale factor comparable to $a_0 = \sqrt{K/C}$. At this scale torsion prevents singularity and provides the cosmological solution with bounce.

(2) The spinor current is space-like and $K < 0$. Then, for any value of C , there are singularities. In the case of positive C the solution is

$$\frac{a}{C} \sqrt{1 + \frac{C}{|K|} a^2} - \frac{|K|^{1/2}}{C^{3/2}} \ln \left[\sqrt{\frac{C}{|K|}} a + \sqrt{1 + \frac{C}{|K|} a^2} \right] = 2(t - t_0), \quad (2.68)$$

while in case of negative C the solution is

$$-\frac{a}{|C|} \sqrt{1 - \left| \frac{C}{K} \right| a^2} + \left| \frac{K}{C^3} \right|^{1/2} \arcsin \left(\left| \frac{C}{K} \right|^{1/2} a \right) = 2(t - t_0) \quad (2.69)$$

and for $C = 0$ it is the simplest one

$$a(t) = [3|K|(t - t_0)]^{1/3} \sim t^{1/3}.$$

(3) The last case is when the spinor current is light-like and $K = 0$. Then, $C > 0$ and the solution is

$$a(t) = [2\sqrt{C}(t - t_0)]^{1/2} \sim t^{1/2}. \quad (2.70)$$

This is, of course, exactly the same solution as one meets in the theory without torsion. Light-like spin vector decouples from the conformal factor of the metric.

The above solutions are, up to our knowledge, new (see, however, Refs. [160,190,8] where other, similar, non-singular solutions were obtained) and may have some interest for cosmology. We notice that the second-derivative inflationary models with torsion have attracted some interest recently, in particular they were used for the analysis of the cosmic perturbations [147,81].

3. Renormalization and anomalies in curved space–time with torsion

The classical theory of torsion, which has been reviewed in the previous section, is not really consistent, unless quantum corrections are taken into account. The consistency of a quantum theory usually includes such requirements as unitarity, renormalizability and the conservation of fundamental symmetries on the quantum level. In many cases, these requirements help to restrict the form of the classical theories, and thus improve their predictive power.

The condition of unitarity is relevant for the propagating torsion. But, for the study of the quantum theory of matter on classical curved background with torsion, it is useless. Therefore, we have to start by formulating the renormalizable quantum field theory of the matter fields on curved background with torsion and related issues like anomalies.

There is an extensive literature devoted to the quantum field theory in curved space–time (see, for example, books [23,90,80,34] and references therein). In the book [34] the quantum theory

on curved background with torsion has been also considered. We shall rely on the formalism developed in [36,165,37,4,33,32,34,102], and consider some additional applications.⁸

3.1. General description of renormalizable theory

Let us start out with some gauge theory (some version of SM or GUT) which is renormalizable in flat space–time, and describe its generalization for the curved background with torsion. The theory includes spinor, vector and scalar fields linked by gauge, Yukawa and 4-scalar interactions, and is characterized by gauge invariance and may be by some other symmetries. It is useful to introduce, from the very beginning, the non-minimal interactions between matter fields and torsion. One can note, that the terms describing the matter self-interaction have dimensionless couplings and hence they cannot (according to our intention not to introduce the inverse-mass dimension parameters), be affected by torsion. Thus, the general action can be presented in the form [36]

$$S = \int d^4x \sqrt{g} \left\{ -\frac{1}{4}(G_{\mu\nu}^a)^2 + \frac{1}{2}g^{\mu\nu}\mathcal{D}_\mu\phi\mathcal{D}_\nu\phi + \frac{1}{2}\left(\sum \xi_i P_i + M^2\right)\phi^2 - V_{\text{int}}(\phi) \right. \\ \left. + i\bar{\psi}\left(\gamma^\alpha\mathcal{D}_\alpha + \sum \eta_j Q_j - im + h\phi\right)\psi\right\} + S_{\text{vac}}, \quad (3.1)$$

where \mathcal{D} denotes derivatives which are covariant with respect to both gravitational and gauge fields but do not contain torsion. $\xi_i P_i$ and $\eta_j Q_j$ are non-minimal terms described in Section 2.3. The last term in (3.1) represents the vacuum action, which is a necessary element of the renormalizable theory. We shall discuss it below, especially in the next section.

The action of a renormalizable theory must include all the terms that can show up as counterterms. So, let us investigate which kind of counterterms one can meet in the matter fields sector of the theory with torsion. We shall consider both general non-minimal theory (3.1) and its particular minimal version. Some remark is in order. The general consideration of renormalization in curved space–time, based on the BRST symmetry, has been performed in [30,34]. The generalization to the theory with torsion is straightforward and it is not worth to present it here. Instead, we are going to discuss the renormalization in a more simple form, using the language of Feynman diagrams, and also will refer to the general statements about the renormalization of the gauge theories in presence of the background fields [64,7,113].

The generating functional of Green's functions, in the curved space–time with torsion, can be postulated in the form

$$Z[J, g_{\mu\nu}, T_{\beta\gamma}^\alpha] = \mathcal{N} \int d\Phi \exp\{iS[\Phi, g, T] + i\phi J\}, \quad (3.2)$$

where Φ denotes all the matter (non-gravitational) fields ϕ (with spins 0, 1/2, 1) and the Faddeev–Popov ghosts c, \bar{c} . J are the external sources for the matter fields ϕ . In the last term, in the exponential, we are using condensed (DeWitt) notations. $\mathcal{N}=Z^{-1}[J=0]$ is the normalization factor.

⁸ In part, we repeat here the content of Section 4 of [34], but some essential portion of information was not known at the time when [34] was written, or has not been included into that edition.

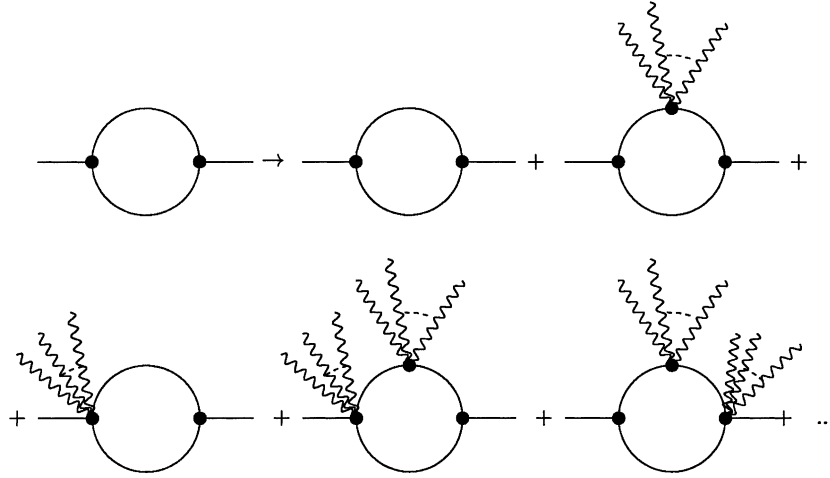


Fig. 1. The straight lines correspond to the matter (in this case scalar with $\lambda\varphi^3$ interaction) field, and wavy lines to the external field (in this case metric). A single diagram in flat space–time generates an infinite set of families of diagrams in curved space–time. The first of these generated diagrams is exactly the one in the flat space–time, and the rest have external gravity lines.

Besides the source term, (3.2) depends on the external fields $g_{\mu\nu}$ and $T^\alpha_{\beta\gamma}$. One has to define how to modify the perturbation theory in flat space–time so that it incorporates the external fields. The corresponding procedure is similar to that for the purely metric background. One has to consider the metric as a sum of $\eta_{\mu\nu}$ and of the perturbation $h_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}.$$

Then, we expand the action $S[\Phi, g, T]$ such that the propagators and vertices of all the fields (quantum and background) are the usual ones in the flat space–time. The internal lines of all the diagrams are only those of the matter fields, while external lines are both of matter and background gravitational fields (metric $h_{\mu\nu}$ and torsion). As a result, any flat-space diagram gives rise to the infinite set of diagrams, with increasing number of the background fields tails. An example of such set is depicted at Fig. 1.

Let us now remind three relevant facts. *First*, when the number of vertices increases, the superficial degree of divergence for the given diagram may only decrease. Therefore, the insertion of new vertices of interaction with the background fields $g_{\mu\nu}$, $T^\alpha_{\beta\gamma}$ cannot increase the degree of divergence. In other words, for any flat-space diagram, all generated diagrams with gravitational external tails have the same or smaller index of divergence than the original diagram. *Second*, since we are working with the renormalizable theory, the number of the divergent n -loop diagrams, in flat space–time, is finite. As a result, after generating the diagrams with external gravity (metric and torsion) tails, we meet a finite number of *families* of divergent diagrams at any loop order. Furthermore, including an extra vertex of interaction with external field one can convert the quadratically divergent diagram into a logarithmically divergent one. For example, the quadratically divergent diagram of Fig. 2a generates the logarithmically divergent ones

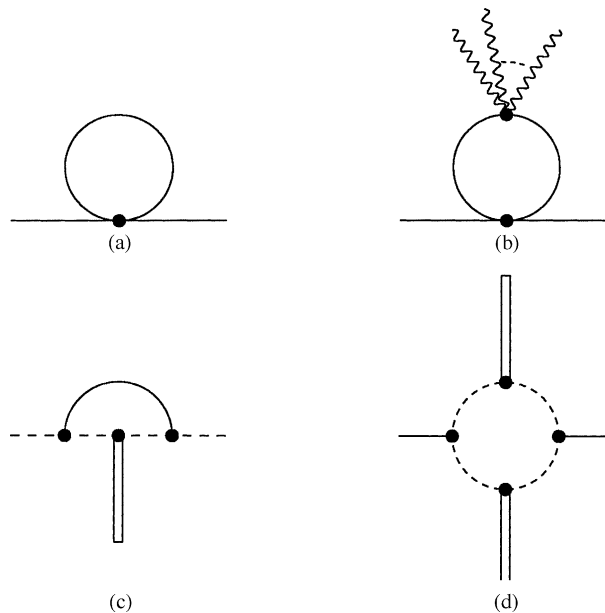


Fig. 2. (a) Quadratically divergent graph for the $\lambda\varphi^4$ -theory. (b) The example of logarithmically divergent graph generated by the graph at Fig. 2(a) and the procedure presented at Fig. 1. This diagram contributes to the $R\varphi^2$ -type counterterm. (c) Dashed lines represent spinor, continuous lines represent scalar and double line—external torsion S_μ . This diagram gives rise to the $\bar{\psi}\gamma^5\gamma^\mu S_\mu\psi$ -type counterterm. (d) This diagram produces $\varphi^2 S^2$ -type counterterm.

of Fig. 2b. The diagrams from Fig. 2b give rise to the $R\varphi^2$ -type counterterm. Similarly, the diagrams of Figs. 2c and d produce $\bar{\psi}\gamma^5\gamma^\mu S_\mu\psi$ and $\varphi^2 S^2$ -type counterterms.

Third, there are general proofs [182] that the divergences of a gauge invariant theory can be removed, at any loop, by the gauge invariant and local counterterms.⁹ Indeed these theorems apply only in the situation when there is no anomaly. In the present case we have regularizations (say, dimensional [106,123], or properly used higher derivative [171,10]) which preserve, on the quantum level, both general covariance and gauge invariance of the model. Thus, we are in a position to use general covariance and gauge invariance for the analysis of the counterterms. Anomalies do not threaten these symmetries, for in the four-dimensional space–time there are no gravitational anomalies.

Taking all three points into account, we arrive at the following conclusion. The counterterms of the theory in an external gravitational background with torsion have the same dimension as the counterterms for the corresponding theory in flat space–time. These counterterms possess general covariance and gauge invariance, which are the most important symmetries of the classical action.

⁹ In case of the diffeomorphism invariance, this can be also proved using the existence of the explicitly covariant perturbation technique based on the local momentum representation and Riemann normal coordinates. For example, in [149] this technique has been described in details and applied to the extensive one-loop calculations. In the case of gravity with torsion, the local momentum representation have been used in Ref. [53].

At this stage one can explain why the introduction of the non-minimal interaction between torsion and matter (spin-1/2 and spin-0 fields) is so important. The reason is that the appearance of the non-minimal counterterms is possible, for they have proper symmetries and proper dimensions. Let us imagine that we have started from the minimal theory, that is take $\eta_1 = \frac{1}{8}$, $\eta_2 = 0$ and $\xi_{1,2,3,4,5} = 0$. Then, the classical action depends on the metric $g_{\mu\nu}$ and on the axial vector component S_μ of torsion. Thus, the vertices of interaction with these two fields will modify the diagrams and one can expect that the counterterms depending on $g_{\mu\nu}$ and S_μ will appear. According to our analysis these counterterms should be of three possible forms (see Figs. 2b–d):

$$\int d^4x \sqrt{-g} S_\alpha S^\alpha \varphi^2, \quad \int d^4x \sqrt{-g} R \varphi^2, \quad \int d^4x \sqrt{-g} \bar{\psi} \gamma^5 \gamma^\alpha S_\alpha \psi$$

and therefore these three structures should be included into the classical action in order to provide renormalizability. Therefore, the essential non-minimal interactions with torsion are the ones which contain the torsion pseudotrace S_μ . If the space–time possesses torsion, the non-minimal parameters η_1 and ξ_4 have the same status as the ξ_1 parameter has for the torsionless theory. Of course, ξ_1 remains to be essential—independent of whether torsion is present.

The special role of the two parameters η_1 , ξ_4 , as compared to others: η_2 and $\xi_{2,3,5}$, is due to the fact that minimally only S_μ -component of torsion interacts with matter fields. It is remarkable that not only spinors but also scalars have to interact with torsion if we are going to have a renormalizable theory.

The terms which describe interaction of matter fields with T_μ and $q^\alpha_{\beta\gamma}$ components of torsion, can be characterized as purely non-minimal. One can put parameters $\xi_{2,3,4}$, η_2 to be zero simultaneously without jeopardizing the renormalizability. Indeed, if the η_2 -term is included, it is necessary to introduce also the $\xi_{2,3}$ -type terms. In the case of abelian gauge theory with complex scalars one may need to introduce some extra non-minimal terms [39] (see also Sections 2.3 and 3.3).

Besides the non-minimal terms, one can meet the vacuum structures which satisfy the conditions of dimension and general covariance. The action of vacuum depends exclusively on the gravitational fields $g_{\mu\nu}$ and $T^\alpha_{\beta\gamma}$. Hence, the corresponding counterterms result from the diagrams which have only the external tails of these fields. The most general form of the vacuum action for gravity with torsion has been constructed in [48]. This action satisfies the conditions of covariance and dimension, but it is very bulky for it contains 168 terms constructed from curvature, torsion and their derivatives. Using the torsionless curvature, one can distinguish the terms of the types

$$R^2, \quad R_{\dots} T^2, \quad R_{\dots} \nabla T, \quad T^2 \nabla T, \quad T^4$$

plus total derivatives.

It turns out, that the number of necessary terms can be essentially reduced without giving up the renormalizability. At the one-loop level, we meet just an algebraic sum of the closed loops of free vectors, fermions and scalars, and only the last two kind of fields contribute to the torsion-dependent vacuum sector. Therefore, calculating closed scalar and spinor loops one can fix the necessary form of the classical action of vacuum, such that this action is sufficient for renormalizability but does not contain any unnecessary terms. Since we are

considering the renormalizable theory, the divergent vertices in the matter field sector are local and have the same algebraic structure as the classical action. For this reason, the structures which do not emerge as the one-loop vacuum counterterms, will not show up at higher loops too. Hence, one can restrict the minimal necessary form of the vacuum action, using the one-loop calculations.

3.2. One-loop calculations in the vacuum sector

In this section we derive the one-loop divergences for the free matter fields in an external gravitational field with torsion. As we already learned in the previous sections, only scalar and spinor fields couple to torsion, so we restrict the consideration by these fields.

For the purpose of one-loop calculations we shall consistently use the Schwinger–DeWitt technique. One can find the review of this method, its generalizations and developments and the list of many relevant references in [64,16,180,11]. Also, in Section 5, some new application of this technique will be given. Now, we need just a simplest version of the Schwinger–DeWitt technique. The one-loop contribution to the effective action $\bar{\Gamma}^{(1)} = (i/2) \text{Tr} \ln \hat{H}$ has the following integral representation:

$$\bar{\Gamma}^{(1)} = -\frac{i}{2} \text{Tr} \int_0^\infty \frac{ds}{s} \frac{i \mathcal{D}^{1/2}(x, x')}{(4\pi i s)^{n/2}} \exp \left\{ -ism^2 + \frac{i}{2s} \sigma(x, x') \right\} \sum_{k=0}^\infty (is) k \hat{a}_k(x, x'), \quad (3.3)$$

where $\sigma(x, x')$ is the world function (geodesic distance between two close points, $\sigma = \frac{1}{2} \nabla_\mu \sigma \nabla^\mu \sigma$) and $\mathcal{D}^{1/2}(x, x')$ is the Van Vleck–Morette determinant

$$\mathcal{D}^{1/2}(x, x') = \left| \det \left(-\frac{\partial^2 \sigma}{\partial x^\mu \partial x^\nu} \right) \right|,$$

n is the parameter of the dimensional regularization. The details about the dimensional regularization in the Schwinger–DeWitt technique can be found in [16].

For the minimal differential operator

$$\hat{H} = \hat{1} \square + 2\hat{h}^\lambda \nabla_\lambda + \hat{\Pi} \quad (3.4)$$

acting on the fields of even Grassmann parity, the divergent part of the functional trace (3.3) is a factor of the coincidence limit of the trace

$$\text{tr} \lim_{x' \rightarrow x} \hat{a}_2(x, x')$$

of the second coefficient of the Schwinger–DeWitt expansion. Direct calculation yields [64]

$$\begin{aligned} \Gamma_{\text{div}}^{(1)}(\hat{H}) &= \frac{i}{2} \text{Tr} \ln \left(-\frac{\hat{H}}{\mu^2} \right) \Big|_{\text{div}} = -\frac{\mu^{n-4}}{\varepsilon} \int d^n x \sqrt{-g} \text{tr} \left[\frac{\hat{1}}{180} (R_{\mu\nu\alpha\beta}^2 - R_{\mu\nu}^2 + \square R) \right. \\ &\quad \left. + \frac{1}{6} \square \hat{P} + \frac{1}{2} \hat{P} \cdot \hat{P} + \frac{1}{12} \hat{S}_{\mu\nu} \cdot \hat{S}^{\mu\nu} \right], \end{aligned} \quad (3.5)$$

where $\varepsilon = (4\pi)^2(n-4)$ is the parameter of dimensional regularization, μ is the dimensional parameter, $\hat{1}$ is the identity matrix in the space of the given fields,

$$\hat{P} = \hat{\Pi} + \frac{\hat{1}}{6}R - \nabla_\alpha \hat{h}^\alpha - \hat{h}^\alpha \hat{h}_\alpha$$

and

$$\hat{S}_{\mu\nu} = (\nabla_\nu \nabla_\mu - \nabla_\mu \nabla_\nu) \hat{1} + \nabla_\nu \hat{h}_\mu - \nabla_\mu \hat{h}_\nu + \hat{h}_\nu \hat{h}_\mu - \hat{h}_\mu \hat{h}_\nu .$$

One has to note that the last formula is nothing but the commutator of the covariant derivatives

$$D_\alpha = \nabla_\alpha + \hat{h}_\alpha .$$

Of course, expression (3.5) can be written in terms of the covariant derivative with torsion $\tilde{\nabla}_\alpha$, but it is useful to separate the torsion-dependent terms. The last observation is that for operator (3.4) acting on the fields of odd Grassmann parity, expression (3.5) changes its sign. In a complicated situations with the operators of mixed Grassmann parity (like that we shall meet in Section 5) it is useful to introduce special notation *Str* for the supertrace.

Let us first consider the calculation of divergences for the especially simple case of free scalar field. The one-loop divergences are given by Eq. (3.5), where

$$\hat{H}_{sc} = -\frac{1}{2} \frac{\delta^2 S_0}{\delta^2 \varphi} = \square - m^2 - \sum \xi_i P_i .$$

Here we use notation (2.26) of the previous section. Applying (3.5), one immediately obtains

$$\begin{aligned} \Gamma_{\text{div}}^{(1)}(\text{scalar}) &= -\frac{i}{2} \text{Tr} \ln \left(\frac{\hat{H}_{sc}}{\mu^2} \right) \Big|_{\text{div}} \\ &= -\frac{\mu^{n-4}}{\varepsilon} \int d^n x \sqrt{-g} \left[\frac{1}{180} (R_{\mu\nu\alpha\beta}^2 - R_{\mu\nu}^2 + \square R) + \frac{1}{6} \square \hat{P} + \frac{1}{2} \hat{P}^2 \right] , \end{aligned} \quad (3.6)$$

where

$$\hat{P} = \frac{1}{6}R - \sum_i \xi_i P_i - m^2 .$$

As it was already mentioned above, in order to provide renormalizability one has to include into the classical action of vacuum all the structures that can appear as counterterms. For the scalar field on the external background of gravity with torsion, the list of the integrands of the vacuum action consists of $R_{\mu\nu\alpha\beta}^2$ and $R_{\mu\nu}^2$, five total derivatives $\square P_i$, 10 products $P_i P_j$ and six mass dependent terms: m^4 and $m^2 P_i$. The total number of necessary vacuum structures is 23, and 7 of them are total derivatives. This number of 23 can be compared, from one side, with the 6 terms

$$R_{\mu\nu\alpha\beta}^2, R_{\mu\nu}^2, R^2, \square R, m^4, m^2 R,$$

which emerge in the torsionless theory, and from the other side, with the 168 algebraically possible covariant terms constructed from curvature, torsion and their derivatives [48].

It is sometimes useful to have another basis for the torsionless fourth derivative terms. We shall use the following notations:

$$C^2 = C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 2 R_{\alpha\beta} R^{\alpha\beta} + \frac{1}{3} R^2$$

for the square of the Weyl tensor, which is conformal invariant at four dimensions and

$$E = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 R_{\alpha\beta} R^{\alpha\beta} + R^2$$

for the integrand of the Gauss–Bonnet topological term. The inverse relations have the form

$$R_{\mu\nu\alpha\beta}^2 = 2C^2 - E + \frac{1}{3} R^2 \quad \text{and} \quad R_{\mu\nu}^2 = \frac{1}{2} C^2 - \frac{1}{2} E + \frac{1}{3} R^2. \quad (3.7)$$

Let us now consider the fermionic determinant, which has been studied by many authors (see, for example, [87,116,142,32,53,93]). One can perform the calculation by writing the action through the covariant derivative without torsion [142,32]. So, we start from the general non-minimal action (2.35). The divergent contribution from the single fermion loop is given by the expression

$$\Gamma_{\text{div}}[g, A, S] = -i \text{Tr} \ln \hat{H}|_{\text{div}}, \quad (3.8)$$

where

$$\hat{H} = i\gamma^\alpha (\mathcal{D}_\alpha - im) \quad \text{and} \quad \mathcal{D}_\alpha = \nabla_\alpha + ieA_\alpha + i\eta\gamma^5 S_\alpha$$

is generalized covariant derivative. For the massless theory this covariant derivative respects general covariance, the abelian gauge symmetry (2.36) and the additional gauge symmetry (2.37). In the massive case this last symmetry is softly broken, but the above notation is still useful.

In order to calculate functional determinant (3.8), we perform the transformation which preserves covariance with respect to the derivative \mathcal{D}_α . First observation is that, by dimensional reasons, the (3.8) is even in the mass m . In other words, (3.8) does not change if we replace m by $-m$. It proves useful to introduce the conjugate derivative, $D_\mu^* = \partial_\mu + ieA_\mu - i\eta\gamma^5 S_\mu$ and the conjugated operator $\hat{H}^* = i\gamma^\alpha (\mathcal{D}_\alpha + im)$. Then, we can perform the transformation:

$$\begin{aligned} \Gamma_{\text{fermion}} &= -\frac{i}{2} \text{Tr} \ln \hat{H} \cdot \hat{H}^* = -\frac{i}{2} \text{Tr} \ln \{-\gamma^\mu D_\mu \gamma^\nu D_\nu - m^2\} \\ &= -\frac{i}{2} \text{Tr} \ln \{-(\gamma^\mu \gamma^\nu D_\mu^* D_\nu + m^2)\}. \end{aligned} \quad (3.9)$$

After a simple algebra, one can write two useful forms for the last operator: the non-covariant (with respect to \mathcal{D}_α):

$$-\hat{H} \cdot \hat{H}^* = \nabla^2 + R^\mu \nabla_\mu + \Pi,$$

with (here $\sigma^{\mu\nu} = e_a^\mu e_b^\nu \sigma^{ab} = (i/2)[\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu]$)

$$R^\mu = 2ieA_\mu + 2\eta \sigma^{\mu\nu} S_\nu \gamma_5,$$

$$\begin{aligned} \Pi &= ie\nabla^\mu A_\mu + i\eta\gamma^5 \partial_\mu S^\mu - e^2 A^\mu A_\mu + \frac{ie}{2} \gamma^\mu \gamma^\nu F_{\mu\nu} - \frac{1}{4} R + m^2 \\ &+ \frac{i}{2} \eta \gamma^\mu \gamma^\nu \gamma_5 S_{\mu\nu} + \eta^2 S_\mu S^\mu + 2ie\eta \sigma^{\mu\nu} \gamma^5 A_\mu S_\nu, \end{aligned} \quad (3.10)$$

where $S_{\mu\nu} = \partial_\mu S_\nu - \partial_\nu S_\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and covariant

$$-\hat{H} \cdot \hat{H}^* = D^2 + E^\mu D_\mu + F, \quad (3.11)$$

with

$$E^\mu = -2i \eta \gamma_\nu \gamma^\mu \gamma^5 S^\nu, \quad F = m^2 - \frac{1}{4}R + \frac{ie}{2} \gamma^\mu \gamma^\nu F_{\mu\nu} + \frac{i}{2} \eta \gamma^5 \gamma^\mu \gamma^\nu S_{\mu\nu}. \quad (3.12)$$

The intermediate expressions, for the covariant version, are

$$\hat{P} = i\eta \gamma^5 \nabla_\mu S^\mu - 2\eta^2 S_\mu S^\mu + \frac{ie}{2} \gamma^\mu \gamma^\nu F_{\mu\nu} - \frac{1}{12}R + m^2 \quad (3.13)$$

and

$$\begin{aligned} \hat{S}_{\alpha\beta} = & \frac{1}{4} \gamma^\rho \gamma^\lambda R_{\beta\alpha\rho\lambda} - ie F_{\alpha\beta} + \eta \sigma_{\alpha\nu} \gamma^5 \nabla_\beta S^\nu - \eta \sigma_{\beta\nu} \gamma^5 \nabla_\alpha S^\nu \\ & + (\gamma_\alpha \gamma_\mu \gamma_\beta \gamma_\nu - \gamma_\beta \gamma_\mu \gamma_\alpha \gamma_\nu) \eta^2 S^\mu S^\nu. \end{aligned} \quad (3.14)$$

The divergent part of (3.9) can be easily evaluated using the general formula (3.5). After some algebra we arrive at the following divergences:

$$\begin{aligned} \bar{\Gamma}_{\text{div}}^{(1)}(\text{Dirac spinor}) = & \frac{\mu^{n-4}}{\varepsilon} \int d^n x \sqrt{-g} \left\{ \frac{2}{3} e^2 F_{\mu\nu}^2 + \frac{2}{3} \eta^2 S_{\mu\nu}^2 - 8 m^2 \eta^2 S_\mu S^\mu - \frac{1}{3} m^2 R + 2m^4 \right. \\ & + \frac{1}{72} R^2 - \frac{1}{45} R_{\mu\nu}^2 - \frac{7}{360} R_{\mu\nu\rho\lambda}^2 - \frac{4}{3} \eta^2 \square (S^\mu S_\mu) \\ & \left. + \frac{4}{3} \eta^2 \nabla_\mu (S^\nu \nabla_\nu S^\mu - S^\mu \nabla_\nu S^\nu) - \frac{1}{30} \square R \right\}. \end{aligned} \quad (3.15)$$

The form of divergences (3.15) is related to the symmetry transformation (2.37). For instance, diffeomorphism and gauge invariance (2.36) are preserved. The one-loop divergences contain the $S_{\mu\nu}^2$ -term, that indicates that in the massless theory symmetry (2.37) is also preserved. And the appearance of the massive divergent $m^2 S^2$ term reveals that the symmetry under transformation (2.37) is softly broken by the fermion mass. Symmetry (2.37) and the form of divergences (3.15) will be extensively used later on, in Section 5, when we try to formulate the consistent theory for the propagating torsion. Indeed, it is very important that the longitudinal $(\partial_\nu S_\nu)^2$ -term is absent in (3.15), for it would break symmetry (2.37).

The one-loop divergences coming from the fermion sector, add some new necessary terms to the vacuum action. The terms which were not necessary for the scalar case, but appear from spinor loop are (remind that T_μ is hidden inside A_μ):

$$S_{\mu\nu} S^{\mu\nu}, \quad T_{\mu\nu} T^{\mu\nu}, \quad \nabla_\mu (S_\nu \nabla^\nu S^\mu - S^\mu \nabla^\nu S_\nu), \quad (3.16)$$

where we denoted, as in Section 2, $T_{\mu\nu} = \partial_\mu T_\nu - \partial_\nu T_\mu$. Finally, for the theory including scalars, gauge vectors and fermions, the total number of necessary vacuum structures is 26, and 7 of them are surface or topological terms. One can see, that in such theory 142 of the algebraically possible counterterms [48] never show up, most of them depend on the $q_{\beta\gamma}^\alpha$ -component of torsion.

3.3. One-loop calculations in the matter fields sector

The one-loop calculation in the matter fields sector in curved space–time with torsion is very important and it has been carried out for various gauge models [36,37,39,164,4].

The first such calculation was done in [36,37] for several $SU(2)$ models. The renormalization of the same models in flat space–time has been studied earlier in [183], where they were taken as examples of theories with the asymptotic freedom in all (gauge, Yukawa and scalar) couplings taking place on the special solutions of the renormalization group equations. Later on, the same models have been used in [31] for the first calculation of the one-loop divergences and the study of the renormalization group for the complete gauge theory in an external gravitational field. The one-loop renormalization in the torsion-dependent non-minimal sector [36] possesses some universality, and this motivated consequent calculations performed in [36,37,39,4] for the abelian, $O(N)$ and $SU(N)$ models with various field contents. Here, we present only a brief account of these calculations, so that the origin of the mentioned universality becomes clear.

As we have already learned, there is some difference in the interaction with torsion for abelian and non-abelian gauge models. In the last case, the massless vector field does not interact with torsion at all, while in the first the non-minimal interaction (2.38) is yet possible. Therefore, besides being technically simpler, the abelian gauge theory is somehow more general and we shall use it to discuss the details of the one-loop renormalization. We mention, that the one-loop calculation for the abelian model was first performed in [140], but has been correctly explained only in [39].

Let us consider the abelian gauge model including gauge, Yukawa and four-scalar interactions. The classical action is some particular case of (3.1), it is given by the sum of non-minimal matter actions (2.26), (2.34) and (2.38), interaction terms and the action of vacuum with all 26 terms described in the previous section:

$$\begin{aligned}
 S = \int d^4x \sqrt{-g} \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + K^{\mu\nu} F_{\mu\nu} \right. \\
 + \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + \frac{1}{2} m^2 \varphi^2 + \frac{1}{2} \sum_{i=1}^5 \xi_i P_i \varphi^2 - \frac{1}{24} f \varphi^4 \\
 \left. + i \bar{\psi} \left(\gamma^\alpha \nabla_\alpha + i e \gamma^\alpha A_\alpha - i m - i h \varphi + \sum_{j=1,2} \eta_j Q_j \right) \psi \right\} + S_{\text{vac}} .
 \end{aligned} \tag{3.17}$$

For the sake of the one-loop calculations one can use the same formula (3.5). However, since we are going to consider, simultaneously, the fields with different Grassmann parity, this formula must be modified by replacing usual traces, Tr and tr , by the supertraces, Str and str .

The decomposition of the fields into classical background A_μ , φ , $\bar{\psi}$, ψ and quantum ones a_μ , σ , $\bar{\chi}$, χ is performed in the following way:

$$A_\mu \rightarrow A_\mu + a_\mu, \quad \varphi \rightarrow \varphi + i\sigma, \quad \bar{\psi} = \bar{\psi} + \bar{\chi}, \quad \psi = \psi + \chi, \quad \chi = -\frac{i}{2} \gamma^\lambda \nabla_\lambda \eta . \tag{3.18}$$

In the abelian theory, we can fix the gauge freedom by simple condition $\nabla_\mu a^\mu = 0$. The Faddeev–Popov ghosts decouple from other quantum fields. They contribute only to the vacuum sector, which we already studied in the previous section. The resulting operator (3.4) has the block structure which emerges from the bilinear expansion of the classical action [39]

$$S^{(2)} = \frac{1}{2} \int d^4x \sqrt{-g} (a^\mu | \sigma | \bar{\chi}) \left(\hat{H} \right) \begin{pmatrix} a^\nu \\ \sigma \\ \eta \end{pmatrix}. \quad (3.19)$$

The details of the calculation can be found in [140,36]. The matrix structure of the operators \hat{P} and $\hat{S}_{\lambda\tau}$ is the following:

$$\hat{P} = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} \quad \text{and} \quad \hat{S}_{\lambda\tau} = \begin{pmatrix} S_{11,\lambda\tau} & S_{12,\lambda\tau} & S_{13,\lambda\tau} \\ S_{21,\lambda\tau} & S_{22,\lambda\tau} & S_{23,\lambda\tau} \\ S_{31,\lambda\tau} & S_{32,\lambda\tau} & S_{33,\lambda\tau} \end{pmatrix},$$

where the terms essential for our consideration are [39]

$$\begin{aligned} S_{31} &= S_{32} = 0, & P_{32} &= 2h\psi, & P_{31} &= -2e\gamma_\nu\psi, \\ P_{13} &= -\frac{ie}{2}\nabla_\rho\bar{\psi}\gamma_\mu\gamma^\rho + ieh\bar{\psi}\gamma_\mu\varphi - \frac{1}{2}e^2\bar{\psi}\gamma_\mu A_\rho\gamma^\rho - \frac{i}{4}e\bar{\psi}\gamma_\mu\gamma^\rho \sum \eta_j Q_j\gamma_\rho, \\ P_{23} &= \frac{i\hbar}{2}\nabla_\rho\bar{\psi}\gamma^\rho - i\hbar^2\bar{\psi}\varphi + \frac{1}{2}eh\bar{\psi}A_\rho\gamma^\rho + \frac{i}{4}h\bar{\psi}\gamma^\rho \sum \eta_j Q_j\gamma_\rho. \end{aligned} \quad (3.20)$$

Now, substituting \hat{P} and $\hat{S}_{\lambda\tau}$ into (3.5), after some algebra we arrive at the divergent part of the one-loop effective action

$$\begin{aligned} \bar{I}_{\text{div}}^{(1)} &= -\frac{\mu^{n-4}}{\varepsilon} \int d^n x \sqrt{-g} \left\{ -\frac{2e^2}{3} F_{\mu\nu} F^{\mu\nu} + \frac{8e\eta_2}{3} F_{\mu\nu} \nabla^\mu T^\nu + 2h^2 g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \right. \\ &\quad + \left(\frac{f^2}{8} - 2h^4 \right) \varphi^4 + \frac{1}{2} \varphi^2 \left[- \left(\sum_i \xi_i P_i - \frac{1}{6} R \right) f + \frac{2}{3} h^2 R - 16h^2 \eta_1^2 S_\mu S^\mu \right] \\ &\quad + i(2e^2 + h^2) \bar{\psi} \gamma^\mu (\nabla_\mu + ieA_\mu - i\eta_2 T_\mu) \psi + (2e^2 - h^2) \eta_1 \bar{\psi} \gamma^5 \gamma^\mu S_\mu \psi \\ &\quad \left. + (8e^2 - 2h^2) h \bar{\psi} \varphi \psi \right\} + \text{vacuum divergent terms}. \end{aligned} \quad (3.21)$$

The above expression is in perfect agreement with the general considerations of Section 3.1. Let us comment on some particular points.

1. One can see that the non-minimal divergences of the $\bar{\psi} \gamma^5 \gamma^\mu S_\mu \psi$ and $\varphi^2 S_\mu S^\mu$ -types really emerge, even if the starting action includes only minimal interaction with $\eta_2 = \frac{1}{8}$, $\eta_2 = 0$ and $\xi_{2,3,4,5} = 0$. Thus, exactly as we have supposed, the non-minimal interaction of the S_μ -component of torsion with spinor and scalar is necessary for the renormalizability. The list of essential parameters includes ξ_1, η_1, ξ_4 parameters of the action which are not related to T_μ or $q_{\beta\gamma}^\alpha$.

Another potentially important parameter is θ_2 , because the corresponding topological counterterm (2.38) can emerge at higher loops.

2. All terms with T_μ and $q_{\beta\gamma}^\alpha$ components of torsion are purely non-minimal. Furthermore, the substitution $eA_\mu + \eta_2 T_\mu \rightarrow eA_\mu$ explains all details of the renormalization of the parameter η_2 . In particular, this concerns the non-minimal interaction of the gauge vector A_μ with torsion trace T_μ , which leads to the renormalization of the non-minimal parameter θ_5 in (2.38). It is worth mentioning that such a mixing is not possible for the non-abelian case [36].

3. One can observe some simple hierarchy of the parameters. The non-minimal parameters ξ_i, η_j, θ_k do not affect the renormalization of the coupling constants e, h, f and masses. One can simply look at Fig. 1 to understand why this is so. In turn, vacuum parameters do not affect the renormalization of neither coupling constants nor the non-minimal parameters ξ_i, η_j, θ_k . One important consequence of this is that the renormalization group equations in the minimal matter sector are independent on external fields so that the renormalization of the couplings and masses is exactly the same as in the flat space–time. The renormalization group in the mixed non-minimal sector depends on the matter couplings, but does not depend on the vacuum parameters. Finally, the renormalization in the vacuum sector depends, in general, on the non-minimal parameters.

4. The last observation is the most complicated one. The contributions to the spinor sector may come only from the mixed sector of the products of operators (3.20). Now, since $S_{31} = S_{32} = 0$, all the fermion renormalization comes from two traces:

$$\text{tr}(P_{13} \cdot P_{31}) \quad \text{and} \quad \text{tr}(P_{23} \cdot P_{32}).$$

It is easy to see, that the arrangement of the γ -matrices in the expressions for

$$P_{13}, P_{31}, P_{23}, P_{32}$$

is universal in the sense that it does not depend on the gauge group. For any non-abelian theory this arrangement is the same as for the simple abelian model under discussion. Therefore, in the fermionic sector, the signs of the counterterms will always be equal to the ones we meet in the abelian model. The renormalization of the essential parameter η_1 has the form

$$\eta_1^0 = \eta_1 \left(1 - \frac{C}{\varepsilon} h^2 \right), \quad (3.22)$$

with $C = 2$ in the abelian case. Using the above consideration we can conclude that the renormalization of this parameter in an arbitrary gauge model will have the very same form (3.22) with positive coefficient C . Of course, the value of this coefficient may depend on the gauge group. In the theory with several fermion fields the story becomes more complicated, because there may be more than one non-minimal parameters η_1 , and in general they may be different for different fields. The renormalization can mix these parameters, but the consideration presented above is a useful hint to the sign universality of the β -function for η_1 , which really takes place [36,39,37,4].

The details of the calculations of the one-loop divergences in the variety of $SU(2)$, $SU(N)$, $O(N)$ models including the finite and supersymmetric models can be found in [36,39,37,4], part of them was also presented in [34]. Qualitatively, all these calculations resemble the sample

we have just considered, so that all the complications come from the cumbersome group relations and especially from the necessity to work with many-fermion models. The results are in complete agreement with our analysis, in particular this concerns the universal sign of C in (3.22).

Let us now discuss another, slightly different, example. Consider, following [164], the Nambu–Jona–Lasinio (NJL) model in curved space–time with torsion. This model is regarded as an effective theory of the SM which is valid at some low-energy scale. If we are interested in the renormalization of the theory in an external gravitational field, then the NJL model may be regarded as the special case of the theory with the Higgs scalars [103,131]. Our purpose is to study the impact of torsion for this effective theory.

Consider the theory of N -component spin $\frac{1}{2}$ field with four-fermion interaction in an external gravitational field with torsion. The action is

$$S_{njl} = \int d^4x \sqrt{-g} \{ L_{gb} + i\bar{\psi}\gamma^\mu (D_\mu - i\eta_1 \gamma^5 S_\mu) \psi + G(\bar{\psi}_L \psi_R)^2 \} . \quad (3.23)$$

Here, L_{gb} is the Lagrangian of the gauge boson field, D_μ is the covariant derivative with respect to both general covariance and gauge invariance, G is the dimensional coupling constant. The above Lagrangian (103) is direct generalization of the one of the paper [103] for the case of gravity with torsion. The introduction of the non-minimal interaction with torsion reflects the relevance of such an interaction at high energies.

Introducing the auxiliary Higgs field H , one can cast the Lagrangian in the form

$$S_{njl} = \int d^4x \sqrt{-g} \{ L_{gb} + i\bar{\psi}\gamma^\mu (D_\mu - i\eta_1 \gamma^5 S_\mu) \psi + (\bar{\psi}_L \psi_R H + \bar{\psi}_R \psi_L H^\dagger) - m^2 H^\dagger H \} . \quad (3.24)$$

It is easy to see that, in this form, theory (3.24) is not renormalizable due to the divergences in the scalar and gravitational sectors. However, our previous analysis can be successfully applied here if we add to (3.24) an appropriate action of the external fields. First, we note that the scalar field is non-dynamical, so that both kind of divergences are similar to the vacuum ones in ordinary gauge theories in curved space–time. Then, one can provide the renormalizability by introducing the Lagrangian L_{ext} of external fields $H, H^\dagger, g_{\mu\nu}, S_\mu$ into action (3.24).

If we do not consider surface terms, the general form of L_{ext} is

$$L_{\text{ext}} = g^{\alpha\beta} \mathcal{D}_\alpha H \mathcal{D}_\beta H^\dagger + \xi_1 R H^\dagger H + \xi_4 H^\dagger H S_\mu S^\mu - \frac{\lambda}{2} (H^\dagger H)^2 + L_{\text{vac}} , \quad (3.25)$$

where the form of L_{vac} corresponds to the possible counterterms (3.15), as it has been discussed in the previous section. Some terms in (3.24) were transferred to L_{ext} . This reflects their role in the renormalization.

When analyzing possible divergences of theories (3.23) and (3.24), one meets a serious difference. Theory (3.23) is not renormalizable, while theory (3.24), (3.25) is. The reason is the use of the fermion-bubble approximation. Then, (3.24) becomes the theory of the free spinor field in the external background composed by gauge boson, scalar, metric and torsion fields. All possible divergences in this theory emerge at the one-loop level, and only in the external field sector. In this sense, one can formulate the renormalizable NJL model in the external gravitational field with torsion.

The direct calculations give the following result for the divergent part of the effective action (we omit the gauge boson and surface terms):

$$\begin{aligned} \bar{\Gamma}_{\text{div}}^{(1)} = & -\frac{2N\mu^{n-4}}{\varepsilon} \int d^n x \sqrt{-g} \left\{ g^{\alpha\beta} \mathcal{D}_\alpha H \mathcal{D}_\beta H^\dagger + \frac{1}{6} R H^\dagger H \right. \\ & \left. + 4\eta_1^2 H^+ H S_\mu S^\mu - (H^+ H)^2 - \frac{1}{3} S_{\mu\nu} S^{\mu\nu} + \frac{1}{20} C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} \right\} + \dots \end{aligned} \quad (3.26)$$

The first terms in (3.26) are the same as in a purely metric theory [130,131], while others are typical for the theory with external torsion.

3.4. Renormalization group and universality in the non-minimal sector

The renormalizability of the Quantum Field Theory in curved background enables one to formulate the renormalization group equation for the effective action and parameters of the theory. The derivation of these equations in the space-time with torsion is essentially the same as for the purely metric background. Let us outline the formulation of the renormalization group [30,34]. The renormalized effective action Γ depends on the matter fields Φ (as before, we denote *all* kind of non-gravitational fields in this way), parameters P (they include all couplings, masses, non-minimal parameters and the parameters of the vacuum action), external fields $g_{\mu\nu}$, $T_{\beta\gamma}^\alpha$, dimensional parameter μ and the parameter of the dimensional regularization n . The renormalized effective action is equal to the bare one:

$$\Gamma[g_{\mu\nu}, T_{\beta\gamma}^\alpha, \Phi, P, \mu, n] = \Gamma_0[g_{\mu\nu}, T_{\beta\gamma}^\alpha, \Phi_0, P_0, n] . \quad (3.27)$$

Taking derivative with respect to μ we arrive at the equation

$$\left[\mu \frac{\partial}{\partial \mu} + \beta_P \frac{\partial}{\partial P} + \int d^n x \sqrt{-g} \gamma \Phi \frac{\delta}{\delta \Phi} \right] \Gamma[g_{\mu\nu}, T_{\beta\gamma}^\alpha, \Phi, P, \mu, n] = 0 . \quad (3.28)$$

Here β and γ functions are defined in a usual way:

$$\beta_P(n) = \mu \frac{\partial P}{\partial \mu} \quad \text{and} \quad \gamma(n) \Phi = \mu \frac{\partial \Phi}{\partial \mu} . \quad (3.29)$$

The conventional $n=4$ beta- and gamma functions are defined through the limit $n \rightarrow 4$. Using the dimensional homogeneity of the effective action we get, in addition to (3.28), the equation

$$\left[\frac{\partial}{\partial t} + \mu \frac{\partial}{\partial \mu} + d_P \frac{\partial}{\partial P} + d_\Phi \int d^n x \sqrt{-g} \Phi \frac{\delta}{\delta \Phi} \right] \Gamma[g_{\mu\nu}, T_{\beta\gamma}^\alpha, \Phi, P, \mu, n] = 0 , \quad (3.30)$$

where d_P and d_Φ are classical dimensions of the parameters and fields. Combining (3.28) and (3.30), setting $t=0$, replacing the operator $2g_{\mu\nu}(\delta/\delta g_{\mu\nu})\Gamma[g_{\mu\nu}, \dots]$ by $(\partial/\partial t)\Gamma[e^{-2t}g_{\mu\nu}, \dots]$, and taking the limit $n \rightarrow 4$, we arrive at the final form of the renormalization group equation appropriate for the study of the short-distance limit:

$$\left[\frac{\partial}{\partial t} - (\beta_P - d_P) \frac{\partial}{\partial P} - (\gamma - d_\Phi) \int d^n x \sqrt{-g} \Phi \frac{\delta}{\delta \Phi} \right] \Gamma[g_{\mu\nu} e^{-2t}, T_{\beta\gamma}^\alpha, \Phi, P, \mu] = 0 . \quad (3.31)$$

The general solution of this equation is

$$\Gamma[g_{\mu\nu} e^{-2t}, T_{\beta\gamma}^\alpha, \Phi, P, \mu] = \Gamma[g_{\mu\nu}, T_{\beta\gamma}^\alpha, \Phi(t), P(t), \mu] , \quad (3.32)$$

where fields and parameters satisfy the equations

$$\begin{aligned}\frac{d\Phi}{dt} &= [\gamma(t) - d_\Phi]\Phi, & \Phi(0) &= \Phi, \\ \frac{dP}{dt} &= \beta_P(t) - d_P, & P(0) &= P.\end{aligned}\quad (3.33)$$

In fact, torsion does not play much role in the above derivation, mainly because it does not transform under scaling. In fact, this is natural, because the physical interpretation of the UV limit in curved space–time is the limit of short distances. But, geometrically, the distance between two points does not depend on torsion, so it is not a big surprise that torsion is less important than metric here.

From Eq. (3.32) follows that the investigation of the short-distance limit reduces to the analysis of Eqs. (3.33). Our main interest will be the behavior of the non-minimal and vacuum parameters related to torsion, but we shall consider the renormalization of other parameters when necessary.

First of all, according to our previous discussion of the general features of the renormalization, the matter couplings and masses obey the same renormalization group equations as in the flat space–time. Furthermore, the running of ξ_1 and those vacuum parameters, which are not related to torsion, satisfy the same equations as in the torsionless theory. Let us concentrate our attention on the parameters related to torsion. Consider the most important equation for the non-minimal parameter η_1 . Using the universality of its one-loop renormalization (3.22), and the classical dimension (in $n \neq 4$) for the Yukawa coupling

$$(2\pi)^2 \frac{dh}{dt} = \frac{n-4}{2}h + \beta_h(4),$$

we can derive the universal form of the renormalization group equation at $n = 4$

$$(4\pi)^2 \frac{d\eta_1}{dt} = C\eta_1 h^2. \quad (3.34)$$

Here, according to (3.22), the constant C is positive, but its magnitude depends on the gauge group. For instance, in the case of abelian theory $C = 2$, and for the adjoint representation of the $SU(N)$ group the value is $C = 1$ [4]. The physical interpretation of the universal running (3.34) is obvious: the interaction of fermions with torsion becomes stronger in the UV limit (short-distance limit in curved space–time). This provides, at the first sight, an attractive opportunity to explain very weak (if any) interaction between torsion and matter fields. Unfortunately, the numerical effect of this running is not sufficient. Let us consider the simplest case of the constant Yukawa coupling $h = h_0$. Then, replacing, as usual, d/dt for $\mu d/d\mu$, (3.34) gives

$$\frac{\eta_1(\mu)}{\eta_1(\mu_{UV})} = \left(\frac{\mu}{\mu_{UV}} \right)^{Ch_0^2/(4\pi)^2}. \quad (3.35)$$

It is easy to see, that even the 50-order change of the scale μ changes η_1 for less than one order. The effect of (3.35) is stronger for the heavy fermions with larger magnitude of the Yukawa coupling. If we suppose, that all the fermions emerge after some superstring phase transition, and that the high-energy values of the non-minimal parameters are equal for all the spinors, the low-energy values of these parameters will not differ for more than 2–3 times. Indeed, there

may be some risk in the above statement, related to the non-perturbative low-energy effects of QCD. However, since the quarks are confined inside the nucleus, and there is no chance to observe their interaction with an extremely weak background torsion (see the next section for the modern upper bounds for the background torsion), the effect of the $\eta_1(t)$ running does not have much physical importance.

In the non-minimal scalar sector we meet standard equation for the ξ_1 parameter, which does not depend on torsion [36,34]

$$(4\pi)^2 \frac{d\xi_1}{dt} = \left(\xi_1 - \frac{1}{6} \right) [k_1 g^2 + k_2 h^2 + k_3 f], \quad (3.36)$$

where the magnitudes of k_1, k_2, k_3 depend on the gauge group, but always $k_1 < 1$ and $k_{2,3} > 0$. The equation for the scalar–torsion interaction parameter ξ_4 has the form [36]

$$(4\pi)^2 \frac{d\xi_4}{dt} = [k_1 g^2 + k_2 h^2 + k_3 f] \xi_4 - k_4 h^2 \eta_1^2 \quad (3.37)$$

with $k_4 > 0$. The asymptotic behavior of ξ_1 depends on the gauge group and on the multiplet composition of the model. For some models $\xi_1 \rightarrow 1/6$ in the UV limit $t \rightarrow \infty$ [31,40], and one meets the asymptotic conformal invariance. For other theories, including the minimal $SU(5)$ GUT [149], the asymptotic behavior is the opposite: $|\xi_1| \rightarrow \infty$ at UV. It is remarkable, that due to the non-homogeneous term in the beta-function (3.37) for ξ_4 , this parameter has an universal behavior which does not depend on the gauge group. It is easy to see, that in all cases

$$|\xi_4(t)| \rightarrow \infty \quad \text{at } t \rightarrow \infty. \quad (3.38)$$

Consequently, the interaction of scalar with torsion also gets stronger at shorter distances, and weaker at long distances. Qualitatively, the result is the same as for the parameter η_1 .

Consider, for example, the $SU(2)$ gauge model with one charged scalar multiplet and two sets of fermion families in the fundamental representation of the gauge group. In flat space–time, the β -functions for this model (which is quite similar to the SM) have been derived in Ref. [183]. The Lagrangian of the renormalizable theory in curved space–time with torsion has the form [36]

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu} + g^{\mu\nu} \left(\partial_\mu \phi^\dagger + \frac{ig}{2} \tau^a A_\mu^a \phi^\dagger \right) \left(\partial_\nu \phi - \frac{ig}{2} \tau^a A_\nu^a \phi \right) - \frac{f}{8} (\phi^\dagger \phi)^2 \\ & + \sum_{i=1}^5 \xi_i P_i \phi^\dagger \phi + i\xi_0 T^\alpha (\phi^\dagger \partial_\alpha \phi - \partial_\alpha \phi^\dagger \cdot \phi) + \sum_{k=1}^m i \bar{\chi}^{(k)} \left(\gamma^\mu \nabla_\mu + \sum_{j=1,2} \delta_j Q_j \right) \chi^{(k)} \\ & + \sum_{k=1}^{m+n} i \bar{\psi}^{(k)} \left(\gamma^\mu \nabla_\mu - \frac{ig}{2} \tau^a \gamma^\mu A_\mu^a + \sum_{j=1,2} \eta_j Q_j \right) \psi^{(k)} - h \sum_{k=1}^m \left(\bar{\psi}^{(k)} \chi^{(k)} \phi + \phi^\dagger \bar{\chi}^{(k)} \psi^{(k)} \right). \end{aligned} \quad (3.39)$$

Let us write, for completeness, the full set of the β -functions for the coupling constants and non-minimal parameters.

(i) *Couplings* [138]:

$$\begin{aligned}
 (4\pi)^2 \frac{dg^2}{dt} &= -b^2 g^2, \quad b^2 = -\frac{43 - 4(m+n)}{3} g^4, \\
 (4\pi)^2 \frac{dh^2}{dt} &= (3 + 4m)h^4 - \frac{9}{2} g^2 h^2, \\
 (4\pi)^2 \frac{df}{dt} &= 3f^2 - 32mh^4 + 9g^4 + 8mfh^2 - 9fg^2.
 \end{aligned} \tag{3.40}$$

Likewise all three $SU(2)$ models of Voronov and Tyutin [183], this one is asymptotically free in all effective couplings, but only in the special regime, when they are proportional to each other. The necessary condition of the asymptotic freedom in g^2 is $m+n \leq 10$ and for h^2 it is $m+n \geq 8$. The asymptotic freedom in $f(t)$ happens only for $m+n=10$ and $m=1$. On the special asymptotically free solutions of the renormalization group equations one meets the following behavior:

$$g^2(t) = \frac{g^2}{1 + b^2 g^2 t / (4\pi)^2}, \quad h^2(t) = \frac{1}{2} g^2, \quad 0 < f \leq g^2. \tag{3.41}$$

(ii) *Non-minimal parameters* [31,36]:

$$\begin{aligned}
 (4\pi)^2 \frac{d\eta_1}{dt} &= h^2(\eta_1 + \delta_1), \quad (4\pi)^2 \frac{d\delta_1}{dt} = 2h^2\eta_1, \\
 (4\pi)^2 \frac{d\eta_2}{dt} &= h^2 \left(\eta_2 - \delta_2 - \frac{1}{2}\xi_0 \right), \quad (4\pi)^2 \frac{d\delta_2}{dt} = h^2(-2\eta_2 + \xi_0), \\
 (4\pi)^2 \frac{d\xi_0}{dt} &= -4m^2 h^2(\delta_2 - \eta_2), \\
 (4\pi)^2 \frac{d\xi_1}{dt} &= \left(\xi_1 - \frac{1}{6} \right) \cdot A, \quad (4\pi)^2 \frac{d\xi_{2,5}}{dt} = A\xi_{2,5}, \quad \text{where } A = \frac{3}{2}f - \frac{9}{2}g^2 + 4mh^2, \\
 (4\pi)^2 \frac{d\xi_3}{dt} &= A\xi_3 + 4mh^2(\eta_2 - \delta_2)^2, \quad (4\pi)^2 \frac{d\xi_4}{dt} = A\xi_4 - mh^2(\eta_1 + \delta_1)^2.
 \end{aligned} \tag{3.42}$$

The solutions of these equations are rather cumbersome [165] and we will not write them here, but only mention that they completely agree with the general analysis given above (the same is true for all other known examples). Finally, the asymptotic behavior of the non-minimal parameters is the following [165,36]:

$$\xi_1 - \frac{1}{6}, \xi_0, \eta_2, \delta_2 \rightarrow 0, \quad \eta_1, \delta_1 \rightarrow \infty \operatorname{sgn}(\delta_1 + 2\eta_1), \quad \xi_4 \rightarrow +\infty.$$

A special case is the renormalization group equations for the NJL model (3.24). Since this theory can be viewed as the theory of free spinor fields in an external scalar and gravitational fields, the exact β -functions coincide with the one-loop ones. The renormalization group equations for

the essential effective couplings ξ_4 , η_1 have the form

$$\begin{aligned} (4\pi)^2 \frac{d\eta_1^2}{dt} &= \frac{8N}{3} \eta_1^4, \\ (4\pi)^2 \frac{d\xi_4}{dt} &= 2N \left(\xi_4 - \frac{8}{3} \eta_1^2 \right). \end{aligned} \quad (3.43)$$

The analysis of these equations shows that the strength of the interaction of spinor and scalar fields with torsion increases at short distances.

Let us now consider the renormalization group for the parameters of the vacuum energy. If we write the vacuum action as

$$S_{\text{vac}} = \int d^4x \sqrt{-g} \sum_{k=1}^{26} p_k J_k, \quad (3.44)$$

where p_k are parameters and J_k are vacuum terms (for instance, $J_1 = C^2$, $J_2 = E, \dots$), the one-loop divergences will have the form

$$\Gamma_{\text{div}} = -\frac{\mu^{n-4}}{\epsilon} \int d^n x \sqrt{-g} \sum_{k=1}^{26} \Delta_k J_k. \quad (3.45)$$

Here Δ_k are the sums of the contributions from the free scalar, vector and spinor fields. The relations between renormalized and bare parameters have the form

$$p_k^0 = \mu^{n-4} \left[p_k + \frac{\Delta_k}{\epsilon} \right].$$

The β -functions are derived according to the standard rule $\beta_k = \mu dp_k/d\mu$. It is important, from the technical point of view, that the one-loop vacuum divergences depend only on the non-minimal parameters ξ , η , but not on the couplings g, h, f . The renormalization of the non-minimal parameters does not include the μ^{n-4} factor, and that is why we can derive universal expressions for the vacuum β -functions and renormalization group equations

$$\frac{dp_k}{dt} = \beta_k = (4-n)p_k - \frac{\Delta_k}{(4\pi)^2}, \quad p_k(0) = p_{k0}. \quad (3.46)$$

Standard $n=4$ beta-functions can be obtained through the limit $n \rightarrow 4$. Since the total number of vacuum terms is 26, it does not have much sense to study the details of scaling behavior for all of them. We shall just indicate some general properties. One can distinguish the Δ_k coefficients which are parameter independent (like the Δ_1, Δ_2), the ones which are proportional to the squares of the masses of scalars or spinors m^2 , the ones which are proportional to the squares of the non-minimal parameters of the torsion–matter interaction η_j^2 or ξ_i^2 . In general, all these types of terms will have distinct asymptotic behavior. Let us consider, for simplicity, only those parameters which are related to the completely antisymmetric torsion and correspond to the massless theory. For the same reason we can take some finite model, in which the beta-functions for masses equal zero (this can be also considered as an approximation, because typically η_1 runs stronger than the mass). We can notice that if the scalar mass does not run, the

behavior of ξ_4 is $\xi_4(t) \sim \eta_2^2(t)$. Finally, for the torsion-independent terms we get the asymptotic behavior

$$p_k(t) - p_{k0} \sim -\frac{\Delta_k}{(4\pi)^2} t ,$$

while for the η_1^2 -type terms the behavior is

$$p_k - p_{k0} \sim e^{2Ch^2 t/(4\pi)^2} \sim \eta_1^2(t)$$

and for the ξ_4^2 -type term

$$p_k - p_{k0} \sim e^{4Ch^2 t/(4\pi)^2} \sim \eta_1^4(t) .$$

Thus, the running of the torsion-dependent vacuum terms is really different from the running of the torsion-independent ones. For the last ones, we meet the usual power-like scaling, which may signify asymptotic freedom in UV or (for the massless theories) in IR—this depends on the relative sign of p_{k0} and Δ_k . For the torsion-dependent terms the behavior is approximately exponential, exactly as for the non-minimal parameter η_1 . Indeed, the numerical range of the running is not too large, because of the $1/(4\pi)^2$ -factor.

3.5. *Effective potential of scalar field in the space–time with torsion: spontaneous symmetry breaking and phase transitions induced by curvature and torsion*

Let us investigate further the impact of the renormalization and renormalization group in the matter field sector of the effective action. We shall follow [33] and consider the effective potential of the massless scalar field in the curved space–time with torsion. The effective potential V is defined as a zero-order term in the derivative expansion of the effective action of the scalar field φ :

$$\Gamma[\varphi] = \Gamma_0 + \int d^4x \sqrt{-g} \left\{ -V(\varphi) + \frac{1}{2} Z(\varphi) g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + \dots \right\} , \quad (3.47)$$

where the dots stand for higher derivative terms, and Γ_0 is the vacuum effective action. We shall discuss the derivation of Γ_0 in the next sections.

The classical potential of the scalar field has the form

$$V_{\text{cl}} = af\varphi^4 - \sum_{i=1}^5 b_i \xi_i P_i \varphi^2 , \quad (3.48)$$

where we have used notations (2.25). If the space–time metric is non-flat and the scalar field couples to spinors through the Yukawa interaction, two of the non-minimal parameters ξ_1 and ξ_4 are necessary non-zero. Therefore, even for the flat space–time metric the potential feels torsion through the parameter ξ_4 . Here we consider the general case of the curved metric and take all parameters ξ_i arbitrary for the sake of generality.

The quantum corrections to the classical potential (3.48) can be obtained using the renormalization group method [54,30,34,33]. The renormalization group equation for the effective potential follows from the renormalization group equation (3.28) for the whole effective action. Since (3.28) is linear, all terms in expansion (3.47) satisfy this equation independently. It is

supposed that the divergences were already removed by the renormalization of the parameters, and therefore in this case one can put $n=4$ from the very beginning. Thus, we get

$$\left[\mu \frac{\partial}{\partial \mu} + \delta \frac{\partial}{\partial \alpha} + \beta_P \frac{\partial}{\partial P} + \int d^4x \sqrt{-g} \gamma \varphi \frac{\delta}{\delta \varphi} \right] V(g_{\mu\nu}, T_{\beta\gamma}^\alpha, \varphi, P, \mu) = 0. \quad (3.49)$$

Here P stands, as before, for all the parameters of the theory: gauge, scalar and Yukawa couplings and non-minimal parameters ξ_i . α is the gauge fixing parameter corresponding to the term $\mathcal{L}_{gf} = (1/2\alpha)(\nabla_\mu A^\mu)^2$ and δ is the renormalization group function corresponding to α .

We shall solve (3.49) in the approximation $\varphi^2 \gg |P_i|$ for all P_i , and neglect higher order terms. Physically, this approximation corresponds to the weakly oscillating metric and weak external torsion. In the gravitational field without torsion this approximation has been used in [108]. Similar method can be applied to the derivation of other terms in the effective action, including higher order terms [41] (see also [34]). The initial step is to write the effective potential in the form $V = V_1 + V_2$, where V_1 does not depend on the external fields $g_{\mu\nu}$, $T_{\beta\gamma}^\alpha$, and $V_2 = \sum_{i=1}^5 V_{2i} P_i$. Since all P_i are linear independent, V_{2i} must satisfy Eq. (3.49) independently. Then this equation is divided into the following set of equations for V_1, V_{2i} :

$$(\mathcal{D} - 4\gamma) \frac{d^4 V_1}{d\varphi^4} = 0, \quad (\mathcal{D} - 2\gamma) \frac{d^2 V_{2i}}{d\varphi^2} = 0, \quad (3.50)$$

where

$$\mathcal{D} = -(1 + \gamma) \frac{\partial}{\partial t} + \delta \frac{\partial}{\partial \alpha} + \beta_P \frac{\partial}{\partial P} \quad (3.51)$$

and $t = (1/2) \ln \varphi^2 / \mu^2$. If we use the standard initial conditions [34]

$$\left. \frac{d^4 V_1}{d\varphi^4} \right|_{t=0} = 4af, \quad \left. \frac{d^2 V_{2i}}{d\varphi^2} \right|_{t=0} = -2b_i \xi_i P_i,$$

the solution of Eq. (3.50) can be easily found in the form

$$\frac{d^4 V_1}{d\varphi^4} = 4af(t)\sigma^4(t), \quad \frac{d^2 V_{2i}}{d\varphi^2} = -2b_i \xi_i(t)\sigma^2(t)P_i, \quad (3.52)$$

where

$$\sigma(t) = \exp \left\{ - \int_0^t \bar{\gamma}[P(t')] dt' \right\}.$$

The effective charges $P(t) \equiv (f(t), \xi_i(t), \dots)$ satisfy the renormalization group equations [34]

$$\dot{P}(t) = \bar{\beta}_P(t), \quad P(0) = P, \quad \text{where } \bar{\beta}_P = \frac{\beta_P}{1 + \gamma} \quad \text{and} \quad \bar{\gamma} = \frac{\gamma}{1 + \gamma}. \quad (3.53)$$

In the one-loop approximation one has to take the linear dependence on t , so that $P(t) = P + \beta_P \cdot t$. Then Eqs. (3.52) become

$$\begin{aligned} \frac{d^4 V_1}{d\varphi^4} &= 4a \left[f + \frac{1}{2}(\beta_f - 4f\gamma) \ln \frac{\varphi^2}{\mu^2} \right], \\ \frac{d^2 V_{2i}}{d\varphi^2} &= -2b_i \left[\xi_i + \frac{1}{2}(\beta_{\xi_i} - 2\xi_i\gamma) \ln \frac{\varphi^2}{\mu^2} \right]. \end{aligned} \quad (3.54)$$

Integrating these equations with the renormalization (boundary) conditions

$$\left. \frac{d^2 V_1}{d\varphi^2} \right|_{\varphi=0} = 0, \quad \left. \frac{d^4 V_1}{d\varphi^4} \right|_{\varphi=\mu} = 4af, \quad \left. \frac{d^2 V_{2i}}{d\varphi^2} \right|_{\varphi=\mu} = -2b_i \xi_i P_i,$$

we arrive at the final expression for the one-loop effective potential

$$V = af\varphi^4 + A \left(\ln \frac{\varphi^2}{\mu^2} - \frac{25}{6} \right) \varphi^4 - \sum_{i=1}^5 \left[b_i \xi_i + B_i \left(\ln \frac{\varphi^2}{\mu^2} - 3 \right) \right] P_i \varphi^2 \quad (3.55)$$

with the coefficients

$$A = \frac{a}{2}(\beta_f - 4f\gamma) \quad \text{and} \quad B_i = \frac{1}{2}b_i(\beta_{\xi_i} - 2\xi_i\gamma). \quad (3.56)$$

Eq. (3.55) is the general expression for the one-loop effective potential in the linear in P_i approximation. One can substitute the values of a and b from some classical theory, together with the corresponding β and γ functions, and derive the quantum corrections using (3.55) and (3.56). The gauge fixing dependence enters the effective potential through the γ -function of the scalar field. We remark that, in general, the problem of gauge dependence of the effective potential is not simple to solve. It was discussed, for instance, in [144]. In the case of $f \sim g^4$ and $\xi_i \sim g^2$ the gauge fixing dependence goes beyond the one-loop approximation and the corresponding ambiguity disappears. These relations are direct analogs of the one which has been used in [54] for the similar effective potential on flat background without torsion.

Let us present, as an example, the explicit expression for the effective potential for theory (3.39) with $m+n=10$ and $m=1$. The necessary β -functions are given in (3.42). The γ -function can be easily calculated to be

$$\gamma(\alpha) = \frac{3}{4}\alpha g^2 - \frac{9}{4}g^2 + 2h^2.$$

Then, using the general formulas (3.55), (3.56) one can easily derive the effective potential for the theory (3.39).¹⁰

$$\begin{aligned} V = & \frac{f}{8}(\varphi^\dagger \varphi)^2 - \sum_{i=1}^5 \xi_i P_i \varphi^\dagger \varphi + \frac{3f^2 + 9g^4 - 32h^4 - 3\alpha f g^2}{(4\pi)^2} \varphi^\dagger \varphi \left(\ln \frac{|\varphi|^2}{\mu^2} - \frac{25}{6} \right) \\ & - \frac{1}{2(4\pi)^2} \varphi^\dagger \varphi \left(\ln \frac{|\varphi|^2}{\mu^2} - 3 \right) \left[\frac{3}{2}(f - \alpha g^2) \sum_{i=1}^5 \xi_i P_i - \frac{1}{6} \left(\frac{3}{2}f + 4h^2 - \frac{9}{2}g^2 \right) P_1 \right. \\ & \left. - h^2(\delta_2 - \eta_2)^2 P_3 - h^2(\delta_1 + \eta_1)^2 P_4 \right]. \end{aligned} \quad (3.57)$$

Consider, using the general expressions (3.55) and (3.56) the spontaneous symmetry breaking. It is easy to see that, for the $\sum_{i=1}^5 b_i \xi_i P_i > 0$ case, the classical potential (3.48) has a minimum at

$$\varphi_0^2 = \frac{1}{2af} \sum_{i=1}^5 b_i \xi_i P_i, \quad (3.58)$$

¹⁰ Some small misprints of [33] are corrected here.

so that the spontaneous symmetry breaking might occur, at the tree level, without having negative mass square. It is not difficult to take into account the quantum corrections to φ_0^2 , just solving the equations $\partial V / \partial \varphi = 0$ iteratively. For instance, after the first step one obtains

$$\varphi_1^2 = \varphi_0^2 - \frac{A}{af} \varphi_0^2 \left(\ln \frac{\varphi_0^2}{\mu^2} - \frac{11}{3} \right) + \frac{1}{2af} \sum_{i=1}^5 B_i P_i \left(\ln \frac{\varphi_0^2}{\mu^2} - 2 \right) .$$

One may consider a marginal case, when $\sum b_i \xi_i P_i < 0$, so that there is no spontaneous symmetry breaking at the tree level. Suppose also that the $\xi_i \approx 0$, so that the absolute value of the sum $\sum b_i \xi_i P_i$ is very small, and that $\sum b_i \beta_{\xi_i} P_i > 0$, such that the sign of $\sum b_i \xi_i P_i$ changes under the quantum corrections. Approximately, $B_i = \frac{1}{2} b_i \beta_{\xi_i}$. Then, from the equation $\partial V / \partial \varphi = 0$ one gets

$$\varphi^2 - \sum_{i=1}^5 \frac{b_i \beta_{\xi_i} P_i}{2af} - \frac{11}{3} \frac{A}{af} \varphi^2 + \ln \frac{\varphi^2}{\mu^2} \left(\frac{A}{af} \varphi^2 - \frac{1}{4af} \sum_{i=1}^5 b_i \beta_{\xi_i} P_i \right) . \quad (3.59)$$

One can denote the positive expression

$$\varphi_0^2 = \frac{1}{2af} \sum_{i=1}^5 b_i \beta_{\xi_i} P_i .$$

Then, after the first iteration Eq. (3.59) gives

$$\varphi_1^2 = \varphi_0^2 + \frac{11}{3} \frac{A}{af} \varphi_0^2 - \ln \frac{\varphi_0^2}{\mu^2} \left(\frac{A}{af} \varphi_0^2 - \frac{1}{4af} \sum_{i=1}^5 b_i \beta_{\xi_i} P_i \right) . \quad (3.60)$$

It is easy to see, that in this case the spontaneous symmetry breaking emerges only due to the quantum effects. In all the cases: classical or quantum, the effect of spontaneous symmetry breaking is produced by expressions like $b_i \xi_i P_i$ or $b_i \beta_{\xi_i} P_i$. In particular, the effect can be achieved only due to the torsion, without the Ricci curvature scalar $P_1 = R$.

Let us now investigate the possibility of phase transitions induced by curvature and torsion. We shall be interested in the first order phase transitions, when the order parameter $\langle \varphi \rangle$ changes by jump. It proves useful to introduce the dimensionless parameters $x = \varphi^2 / \mu^2$ and $y_i = P_i / \mu^2$. The equations for the critical parameters x_c, y_{ic} corresponding to the first order phase transition, are [117]:

$$V(x_c, y_{ic}) = 0, \quad \left. \frac{\partial V}{\partial x} \right|_{x_c, y_{ic}} = 0, \quad \left. \frac{\partial^2 V}{\partial x^2} \right|_{x_c, y_{ic}} > 0 . \quad (3.61)$$

These equations lead to the following conditions:

$$2A^2 = - \sum_{i=1}^5 q_i \varepsilon_i D_i \pm \left[\sum_{i,j} (D_i D_j - 4A^2 B_i B_j) \varepsilon_i \varepsilon_j q_i q_j \right]^{1/2} ,$$

$$af - \frac{8}{3} A + A \ln x - \frac{1}{2} \sum_{i=1}^5 B_i \varepsilon_i q_i > 0 , \quad (3.62)$$

where

$$D_i = Ab_i \xi_i - afB_i - \frac{5}{6}Ab_i, \quad \varepsilon_i = \text{sign } P_i, \quad q_i = \frac{y_{ic}}{x}. \quad (3.63)$$

Besides (3.62), the quantities q_i have to satisfy the conditions $0 < q_i \ll 1$. The last inequality means that our approximation $\varphi^2 \gg |P_i|$ is valid.

In order to analyze the above conditions one has to implement such relations for the parameters that the result would be gauge fixing independent. Let us, for this end, take the relation $af = \frac{11}{3}A$, as it has been done in [54]. Then $f \sim g^4$. Consider the special case $|\xi_i| \ll g^2$. Then from (3.62) follows [33] $D_i \approx -\frac{9}{2}AB_i$, $q_i \approx A/4.3B_i\varepsilon_i$. We notice that in the first of (3.62) one has to take positive sign, otherwise the $q_i \ll 1$ condition does not hold. In this approximation $A > 0$, $B_i > 0$, therefore one has to take all $\varepsilon_i > 0$. As we see, the theory admits the first order phase transition, which may be induced by curvature and (or) torsion. In fact, there are other possibilities, for instance where all or part of the non-minimal parameters satisfy opposite relations $|\xi_i| \gg g^2$ [33]. We will not present the discussion of these possibilities here. An important observation is that, in the point of minimum, the effective potential generates the induced gravity with torsion:

$$S_{\text{ind}} = - \int d^4x \sqrt{-g} V(\varphi_c) = - \int d^4x \sqrt{-g} \left\{ A_{\text{ind}} - \frac{1}{16\pi G_{\text{ind}}} \sum_i \theta_i^{(\text{ind})} P_i \right\} \quad (3.64)$$

with

$$A_{\text{ind}} = af\varphi_c^4 + A \left[\ln \frac{\varphi_c^2}{\mu^2} - \frac{25}{6} \right] \varphi_c^4 \quad (3.65)$$

and

$$-\frac{1}{16\pi G_{\text{ind}}} \theta_i^{(\text{ind})} = \sum_{i=1}^5 \left[b_i \xi_i + B_i \left(\ln \frac{\varphi_c^2}{\mu^2} - 3 \right) \right] \varphi_c^2. \quad (3.66)$$

It is reasonable to take $\theta_i^{(\text{ind})} = 1$, and then other $\theta_i^{(\text{ind})}$ will give the coefficients in the induced analog of the Einstein–Cartan action (2.18). Since these coefficients depend on the non-minimal parameters ξ_i , and these parameters have different scale dependence (see, for example, (3.42)), the coefficients of the induced action are, in general, different from the ones in (2.18), which correspond to the $\int \sqrt{-g} \tilde{R}$ -type action of the Einstein–Cartan theory.

3.6. Conformal anomaly in the spaces with torsion: trace anomaly and modified trace anomaly

As we have already learned in Section 2.4, three different types of local conformal symmetry are possible in the theory of gravity with torsion. Consequently, one meets different versions of conformal anomaly, which violates these symmetries at the quantum level. In this section, we shall consider only the vacuum sector, and just note that the trace anomaly in the matter field sector with torsion (which requires the renormalization of composite operators similar to the one performed in [28]) has not been performed yet. On the other hand, this anomaly would not

be very different from the one in the purely metric theory, and the vacuum effects look much more interesting.

Let us start from the anomaly corresponding to the *weak conformal symmetry* [32]. In this case, torsion does not transform and the Noether identity, in the vacuum sector, is just the same as in the purely metric gravity:

$$T^\mu_\mu = -\frac{2}{\sqrt{-g}} g_{\mu\nu} \frac{\delta S}{\delta g_{\mu\nu}} = 0. \quad (3.67)$$

The last identity indicates that the vacuum action can be chosen to be conformal invariant.¹¹ The vacuum action may include the non-conformal terms, but their renormalization is not necessary in the case of conformal invariant free massless fields. But these terms may be, indeed, important from other points of view. In particular, one can include the Einstein–Cartan action into S_{vacuum} , and treat the anomaly-induced effective action as quantum correction to the classical action of gravity with torsion at the very high energy, when the particle masses are negligible.

Consider the anomaly in identity (3.67). We shall use the dimensional regularization, which is the most useful for this purpose [60,70]. The divergent part of the one-loop effective action that emerges after integrating over one real scalar and one Dirac spinor, have the form (3.6) and (3.15). If we restrict consideration by the case of a purely antisymmetric torsion, the result for the N_0 real scalars, $N_{1/2}$ Dirac spinors and N_1 gauge bosons will be

$$\begin{aligned} \bar{\Gamma}_{\text{div}}(N_0, N_{1/2}, N_1) = & -\frac{\mu^{n-4}}{\varepsilon} \int d^n x \sqrt{-g} \left\{ \left(\frac{N_0}{120} + \frac{N_{1/2}}{20} + \frac{N_1}{10} \right) C^2 \right. \\ & - \left(\frac{N_0}{360} + \frac{11 N_{1/2}}{360} + \frac{31 N_1}{180} \right) E + \left(\frac{N_0}{180} + \frac{N_{1/2}}{30} - \frac{N_1}{10} \right) \square R \\ & - \frac{2\eta^2 N_{1/2}}{3} S_{\mu\nu} S^{\mu\nu} + \frac{N_0}{2} \xi^2 (S^\mu S_\mu)^2 + \left(\frac{4N_{1/2}}{3} \eta^2 - \frac{N_0}{6} \xi \right) \square (S^\mu S_\mu) \\ & \left. - \frac{4N_{1/2}}{3} \eta^2 \nabla_\mu (S_\nu \nabla^\nu S^\mu - S^\mu \nabla_\nu S^\nu) \right\} \\ = & -\frac{\mu^{n-4}}{\varepsilon} \int d^n x \sqrt{-g} a C^2 + b E + c \square R + d S_{\mu\nu}^2 + e S^4 + f \square S^2 \\ & + g \nabla_\mu (S_\nu \nabla^\nu S^\mu - S^\mu \nabla_\nu S^\nu). \end{aligned} \quad (3.68)$$

The standard arguments show that the one-loop effective action of vacuum is conformal invariant before the local counterterm ΔS is introduced [70]. Consider the general expression

¹¹ In fact, this is true (exactly as in the purely metric theory) only in the one-loop approximation. At higher loops, the non-minimal parameter ξ_1 of the scalar–curvature interactions departs from the one-loop conformal fixed point [167]. As a result, in order to preserve the renormalizability, one needs, strictly speaking, a non-conformal vacuum action. But, as it was noticed in [167], the coefficients in front of the non-conformal terms can be safely kept very small, and one can always consider the conformal invariant vacuum action as a very good approximation. In order to avoid the discussion of this issue, we consider, in this section, the vacuum effects of the free matter fields.

for the one-loop effective action

$$\Gamma = S + \bar{\Gamma} + \Delta S, \quad (3.69)$$

where $\bar{\Gamma}$ is the quantum correction to the classical action and ΔS is an infinite local counterterm which is called to cancel the divergent part of $\bar{\Gamma}^{(1)}$. Then, the anomalous trace is

$$T = \langle T_\mu^\mu \rangle = -\frac{2}{\sqrt{-g}} g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} \bar{\Gamma} \Big|_{n=4} = -\frac{2}{\sqrt{-g}} g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} \Delta S \Big|_{n=4}. \quad (3.70)$$

The most simple way of calculating this expression is to perform the local conformal transformation

$$g_{\mu\nu} = \bar{g}_{\mu\nu} \cdot e^{2\sigma}, \quad \sigma = \sigma(x), \quad \det(\bar{g}_{\mu\nu}) = \text{const}. \quad (3.71)$$

and use the identity

$$-\frac{2}{\sqrt{-g}} g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} A[g_{\mu\nu}] = -\frac{1}{\sqrt{-\bar{g}}} e^{-4\sigma} \frac{\delta}{\delta \sigma} A[\bar{g}_{\mu\nu} e^{2\sigma}] \Big|_{\bar{g}_{\mu\nu} \rightarrow g_{\mu\nu}, \sigma \rightarrow 0}. \quad (3.72)$$

When this operator acts on

$$\Delta S = \frac{\mu^{n-4}}{\epsilon} \int d^n x \sqrt{-\bar{g}} e^{(n-4)\sigma} (a \bar{C}^2 + \dots),$$

the $1/(n-4)$ -factor cancels and we immediately arrive at the expression

$$T = -\frac{1}{(4\pi)^2} [a C^2 + b E + c \square R + d S_{\mu\nu}^2 + e S^4 + f \square S^2 + g \nabla_\mu (S_\nu \nabla^\nu S^\mu - S^\mu \nabla_\nu S^\nu)] \quad (3.73)$$

with the same coefficients a, b, \dots, g as in (3.68). The derivation of the anomaly for the general torsion case can be done in the same way [165]. The important thing is that for the case of the *weak conformal symmetry* all components of torsion S_μ, T_μ and $q_{\beta\gamma}^\alpha$ do not transform.

The calculation of the anomaly for the case of *strong conformal symmetry* always reduces to the one for the *weak conformal symmetry*. As it was already noted in Section 2.4, the Noether identity (2.42) separates into two independent identities: one of them is (3.67), and second simply requests that the actions does not depend on the torsion trace T_μ . As it was mentioned in Section 2.4, the second identity cannot be violated by the anomaly, and we are left with (3.67) and with the corresponding anomaly (3.73).

Let us now consider the most interesting case of the *compensating conformal symmetry*, which will lead us to the modified trace anomaly. We shall follow Ref. [102]. This version of conformal symmetry depends on the torsion trace T_μ and on the non-minimal interaction of this trace with scalar. In the spinor sector the symmetry requires that there is no any interaction with T_μ , so that $\eta_2 = 0$. Therefore, we can restrict our consideration to the case of a single scalar field.

It is easy to see that the Noether identity corresponding to symmetry (2.52) looks as follows:

$$2g_{\mu\nu} \frac{\delta S_t}{\delta g_{\mu\nu}} + \frac{\xi_2}{\xi_3} \partial_\mu \frac{\delta S_t}{\delta T_\mu} - \varphi \frac{\delta S_t}{\delta \varphi} = 0.$$

Then, due to the conformal invariance of the vacuum divergences (coefficient at the $n=4$ pole), the vacuum action may be chosen in such a way that

$$-\sqrt{-g}\mathcal{T} = 2g_{\mu\nu}\frac{\delta S_{\text{vac}}}{\delta g_{\mu\nu}} + \frac{\xi_2}{\xi_3}\partial_\mu\frac{\delta S_{\text{vac}}}{\delta T_\mu} = 0. \quad (3.74)$$

The new form (3.74) of the conformal Noether identity indicates the modification of the conformal anomaly. In the theory under discussion, the anomaly would mean $\langle\mathcal{T}\rangle \neq 0$ instead of usual $\langle T_\mu^\mu \rangle \neq 0$. Therefore, we have a special case here and one cannot directly use the relation between the one-loop counterterms and the conformal anomaly derived above, just because this relation does not take into account the non-trivial transformation law for the torsion field.

One can derive this new anomaly directly, using the same method as before. However, it is possible to find $\langle\mathcal{T}\rangle$ in a more economic way, after performing a special decomposition of the background fields. Let us try to change the background variables in such a way that the transformation of torsion is absorbed by that of the metric. The crucial observation is that \mathcal{P} , from (2.53), transforms, under (2.52),¹² as $\mathcal{P}' = \mathcal{P} \cdot e^{-2\sigma(x)}$. The non-trivial transformation of torsion is completely absorbed by \mathcal{P} . Since \mathcal{P} only depends on the background fields, we can present it in any useful form. One can imagine, for instance, \mathcal{P} to be of the form $\mathcal{P} = g^{\mu\nu}\Pi_\mu\Pi_\nu$ where the vector Π_μ does not transform, exactly as the axial vector S_μ . After that, the calculation readily reduces to the case of an antisymmetric torsion (3.73), described above.

For the single scalar, the one-loop divergences have the form

$$\Gamma_{\text{div}}^{(1)} = -\frac{\mu^{n-4}}{(4\pi)^2(n-4)} \int d^n x \sqrt{-g} \left\{ \frac{1}{120}C^2 - \frac{1}{360}E + \frac{1}{180}\square R + \frac{1}{6}\square\mathcal{P} + \frac{1}{2}\mathcal{P}^2 \right\}. \quad (3.75)$$

Taking into account the arguments presented above, one can immediately cast the anomaly under the form

$$\langle\mathcal{T}\rangle = -\frac{1}{(4\pi)^2} \left[\frac{1}{120}C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} - \frac{1}{360}E + \frac{1}{180}\square R + \frac{1}{6}\square\mathcal{P} + \frac{1}{2}\mathcal{P}^2 \right]. \quad (3.76)$$

3.7. Integration of conformal anomaly and anomaly-induced effective actions of vacuum: application to inflationary cosmology

One can use the conformal anomaly to restore the induced effective action of vacuum. This action can be regarded as a quantum correction to the classical gravitational action. We notice, that the induced action proved to be the best tool in the analysis of anomaly, see e.g. [14,73], including the theory with torsion [32].

The equation for the finite part of the one-loop correction $\bar{\Gamma}$ to the effective action can be obtained from anomaly. Let us consider, first, the weak conformal symmetry. Then

$$-\frac{2}{\sqrt{-g}}g_{\mu\nu}\frac{\delta\bar{\Gamma}}{\delta g_{\mu\nu}} = T. \quad (3.77)$$

In the case of purely metric gravity this equation has been solved in [153,77]. For the torsion theory with the weak conformal symmetry the solution has been found in [32] (see also [34]).

¹² As a consequence, the action $\int \sqrt{-g}\mathcal{P}\varphi^2$ is conformal invariant. This fact has been originally discovered in [141].

Finally, for the most complicated case of the compensating conformal symmetry, the problem has been solved in [102].

We start from the case of purely antisymmetric torsion, corresponding to the strong conformal symmetry. The simplest possibility in solving (3.77) is to divide the metric in two parts: the conformal factor $\sigma(x)$ and the fiducial metric $\bar{g}_{\mu\nu}(x)$ with fixed determinant (3.71), and write the (3.80) via (3.72). Since torsion does not transform, we put $S_\mu = \bar{S}_\mu$. Then we get [32,34]

$$\begin{aligned} \bar{I} = & S_c[\bar{g}_{\mu\nu}, \bar{S}_\mu] + \frac{1}{(4\pi)^2} \int d^4x \sqrt{-\bar{g}} \left\{ a\sigma\bar{C}^2 + b\sigma \left(\bar{E} - \frac{2}{3}\bar{\nabla}^2\bar{R} \right) + 2b\sigma\bar{\Delta}_4\sigma + d\sigma\bar{S}_{\mu\nu}^2 \right. \\ & + e\sigma(\bar{S}_\mu\bar{S}^\mu)^2 + (f + g/2)\bar{S}^2(\bar{\nabla}\sigma)^2 + g(\bar{S}^\mu\bar{\nabla}_\mu\sigma)^2 - g\bar{\nabla}_\mu\sigma(\bar{S}_\nu\bar{\nabla}^\nu\bar{S}^\mu - \bar{S}^\mu\bar{\nabla}_\nu\bar{S}^\nu) \\ & \left. - f\bar{\nabla}_\mu\sigma\bar{\nabla}^\mu\bar{S}^2 - \frac{1}{12} \left(c + \frac{2}{3}b \right) [\bar{R} - 6(\bar{\nabla}\sigma)^2 - (\bar{\square}\sigma)]^2 \right\} + S_c[\bar{g}_{\mu\nu}, \bar{S}_\mu], \end{aligned} \quad (3.78)$$

where $S_c[\bar{g}_{\mu\nu}, \bar{S}_\mu]$ is an unknown functional of the metric and torsion which serves as an integration constant for any solution of (3.77). Indeed, if one succeeds to rewrite (3.78) in terms of the original variables $g_{\mu\nu}, S_\mu$, the action $S_c[\bar{g}_{\mu\nu}, \bar{S}_\mu]$ must be replaced by an arbitrary conformal-invariant functional of these variables. It is, in principle, possible to proceed and, following [153], derive the covariant form of the induced action (3.78). This action contains, exactly as in the torsionless case, the local and non-local pieces [60,153].

Action (3.78), being the quantum correction to the Einstein–Cartan theory, can serve as a basis for the non-singular cosmological model with torsion. This model has been constructed in [32] (see also [34]). Without going into technical details, we just summarize that, for the conformally flat metric $g_{\mu\nu} = \eta_{\mu\nu}a^2$ and isotropic torsion axial vector $S_\mu = (T, 0, 0, 0)$, the dynamical equations have approximate classical solution of the form (in physical time)

$$a(t) = a(0)e^{Ht}, \quad T(t) = T(0)e^{-2Ht}. \quad (3.79)$$

This solution has an obvious physical interpretation: torsion exponentially decreases during inflation, and that is why it is so weak today. Of course, this concerns only specific background torsion, but the result is indeed relevant for the cosmological applications of torsion.

Let us now derive the conformal non-invariant part of the effective action of vacuum, which is responsible for the modified conformal anomaly (3.76). Taking into account our previous treatment of the compensating conformal transformation of torsion, we consider it hidden inside the quantity \mathcal{P} of Eq. (2.53), and again imagine \mathcal{P} to be of the form $\mathcal{P} = g^{\mu\nu}\Pi_\mu\Pi_\nu$. Then, the equation for the effective action $\Gamma[g_{\mu\nu}, \Pi_\alpha]$ is¹³

$$-\frac{2}{\sqrt{-g}}g_{\mu\nu}\frac{\delta\Gamma}{\delta g_{\mu\nu}} = \langle \mathcal{T} \rangle. \quad (3.80)$$

In order to find the solution for Γ , we can factor out the conformal piece of the metric $g_{\mu\nu} = \bar{g}_{\mu\nu} \cdot e^{2\sigma}$, where $\bar{g}_{\mu\nu}$ has fixed determinant and put $\mathcal{P} = \bar{\mathcal{P}} \cdot e^{-2\sigma(x)}$, that corresponds to $\bar{\Pi}_\alpha = \Pi_\alpha$.

¹³ We remark that this equation is valid only for the “artificial” effective action $\Gamma[g_{\mu\nu}, \Pi_\alpha]$, while the effective action in original variables $g_{\mu\nu}, T_{\beta\gamma}^\alpha$ would satisfy the modified equation (3.74). The standard form of Eq. (3.80) for the effective action is achieved only through the special decomposition of the external fields.

The result can be obtained directly from the effective action derived in [32], and we get

$$\Gamma = S_c[\bar{g}_{\mu\nu}; \bar{P}] - \frac{1}{12} \frac{1}{270(4\pi)^2} \int d^4x \sqrt{-g(x)} R^2(x) + \frac{1}{(4\pi)^2} \int d^4x \sqrt{-\bar{g}} \left\{ \sigma \left[\frac{1}{120} \bar{C}^2 - \frac{1}{360} \left(\bar{E} - \frac{2}{3} \bar{\square} \bar{R} \right) + \frac{1}{2} \bar{\mathcal{P}}^2 \right] + \frac{1}{180} \sigma \bar{\Delta} \sigma - \frac{1}{6} (\bar{\nabla}_\mu \sigma) \bar{\nabla}^\mu \bar{\mathcal{P}} + \frac{1}{6} \bar{\mathcal{P}} (\bar{\nabla}_\mu \sigma)^2 \right\}, \quad (3.81)$$

where $S_c[\bar{g}_{\mu\nu}; \bar{\mathcal{P}}]$ is an unknown functional of the metric $\bar{g}_{\mu\nu}(x)$ and $\bar{\mathcal{P}}$, which acts as an integration constant for any solution of (3.80).

Now, one has to rewrite (3.81) in terms of the original field variables, $g_{\mu\nu}, T^\alpha_{\beta\gamma}$. Here, we meet a small problem, because we only have, for the moment, the definition $\Pi_\alpha = \bar{\Pi}_\alpha$ for the artificial variable Π_α , but not for the torsion. Using the previous result (2.52), we can define

$$T^\alpha_{\beta\gamma} = \bar{T}^\alpha_{\beta\gamma} - \frac{1}{3} [\delta^\alpha_\gamma \partial_\beta \sigma - \delta^\alpha_\beta \partial_\gamma \sigma],$$

where $\bar{T}^\alpha_{\beta\gamma}$ is an arbitrary tensor. Also, we call $\bar{T}^\alpha = \bar{g}^{\alpha\beta} \bar{T}_\beta$, etc. Now, we can rewrite (3.81) in terms of metric and torsion components

$$\begin{aligned} \Gamma = & S_c[\bar{g}_{\mu\nu}; \bar{T}^\alpha_{\beta\gamma}] - \frac{1}{12} \frac{1}{270(4\pi)^2} \int d^4x \sqrt{-g(x)} R^2(x) \\ & + \frac{1}{(4\pi)^2} \int d^4x \sqrt{-\bar{g}} \left\{ + \frac{1}{180} \sigma \bar{\Delta} \sigma + \frac{1}{120} \bar{C}^2 \sigma - \frac{1}{360} \left(\bar{E} - \frac{2}{3} \bar{\square} \bar{R} \right) \sigma \right. \\ & + \frac{1}{72} \sigma \left[-\frac{\xi_2^2}{\xi_3} \bar{R} + 6\xi_2 (\bar{\nabla}_\mu \bar{T}^\mu) + 6\xi_3 \bar{T}_\mu \bar{T}^\mu + 6\xi_4 \bar{S}_\mu \bar{S}^\mu + 6\xi_5 \bar{q}_{\mu\nu\lambda} \bar{q}^{\mu\nu\lambda} \right]^2 \\ & \left. + \frac{1}{6} [(\bar{\square}^2 \sigma + (\bar{\nabla}_\mu \sigma)^2)] \left[-\frac{\xi_2^2}{\xi_3} \bar{R} + 6\xi_2 (\bar{\nabla}_\mu \bar{T}^\mu) + 6\xi_3 \bar{T}_\mu \bar{T}^\mu + 6\xi_4 \bar{S}_\mu \bar{S}^\mu + 6\xi_5 \bar{q}_{\mu\nu\lambda} \bar{q}^{\mu\nu\lambda} \right] \right\}. \end{aligned} \quad (3.82)$$

This effective action is nothing but the generalization of expression (3.78) for the case of general metric–torsion background and compensating conformal symmetry. The curvature dependence in the last two terms appears due to the non-trivial transformation law for torsion. The physical interpretation of action (3.82) coincides with the one of (3.78) in case $T_\mu = q^\alpha_{\beta\gamma} = 0$.

From a technical point of view, Eq. (3.82) is a very interesting example of an exact derivation of the anomaly-induced effective action for the case when the background includes, in addition to metric, another field with the non-trivial conformal transformation.

3.8. Chiral anomaly in the spaces with torsion: cancellation of anomalies

Besides the conformal trace anomaly, in the theory with torsion one can meet anomalies of other Noether identities. In particular, the systematic study of chiral anomalies has been performed in Refs. [17,1,3]. In many cases, due to the special content of the gauge theory, the anomaly cancel. The most important particular example is the Standard Model of particle physics, which is a chiral theory where left and right components of the spinors emerge in

a different way. The violation of the corresponding symmetries could, in principle, lead to the inconsistency of the theory [92]. However, it does not happen in the SM, because the dangerous anomalies cancel.

The history of chiral anomaly in curved space–time with torsion started simultaneously with the derivation of divergences (always related to the $a_2(x, x')$ -coefficient) for the corresponding fermion operator [87]. In this paper, however, the explicit result has not been achieved because of the cumbersome way of calculation. Let us indicate only some of the subsequent calculations [116,142,135,192,52,51,66] and other papers devoted to the closely related issues like index theorems, topological structures and Wess–Zumino [188] conditions [143,46,128,191,150]. We shall not go into details of these works but only present the most important and simple expression for the anomaly and give a brief review of other results.

For the massless Dirac fermion, there is an exact classical symmetry (2.37), and the corresponding Noether identity is $\nabla_\mu J_5^\mu = 0$, where (2.61)

$$J_5^\mu = \bar{\psi} \gamma^5 \gamma^\mu \psi .$$

The anomaly appears due to the divergences coming from the fermion loop. The mechanism of the violation of the Noether identity can be found in many books (for instance, in [55,109]). However, the standard methods of calculating anomaly using Feynman diagrams are not very useful for the case of gravity with torsion. In principle, one can perform such calculations using the methods mentioned at the beginning of this section: either introducing external lines of the background fields, or using the local momentum representation. The most popular are indeed the functional methods (see e.g. [79]), which provide the covariance of the divergences automatically.

The investigation of the anomaly in curved space–time using the so-called analytic regularization based on the Schwinger–DeWitt (Seeley–Minakshisundaram) expansion for the elliptic operator on the compact manifolds with positive-defined metric has been performed in [155]. The anomaly can be calculated on the Riemann or Riemann–Cartan [142]) manifold with the Euclidean signature. The Euclidean rotation can be done in a usual way, and the result can be analytically continued to the pseudo-Euclidean signature. Then the vacuum average of the axial (spinor) current is

$$\langle J_5^\mu \rangle = \frac{\int d\bar{\psi} d\psi J_5^\mu \exp\{\int d^4x \sqrt{g} \bar{\psi} \gamma^\mu D_\mu \psi\}}{\int d\bar{\psi} d\psi \exp\{\int d^4x \sqrt{g} \bar{\psi} \gamma^\mu D_\mu \psi\}} ,$$

where $D_\alpha = \nabla_\alpha + i\eta\gamma_5 S_\alpha$ is covariant derivative with the antisymmetric torsion. It can be easily reduced to the minimal covariant derivative $\tilde{\nabla}_\alpha = \nabla_\alpha - (i/8)\gamma_5 S_\alpha$, but we will not do so, and keep η arbitrary in order to have correspondence with other sections of this section.

The vacuum average of the axial current divergence can be presented as [155,142]

$$\langle \nabla_\mu J_5^\mu \rangle = \frac{\delta}{\delta\alpha(x)} \frac{\int d\bar{\psi} d\psi \exp\{\int d^4x \sqrt{g} (\bar{\psi} \gamma^\mu D_\mu \psi - J_5^\mu \partial_\mu \alpha)\}}{\int d\bar{\psi} d\psi \exp\{\int d^4x \sqrt{g} \bar{\psi} \gamma^\mu D_\mu \psi\}} \Big|_{\alpha=0} .$$

The analysis of [155,142] shows that this expression is nothing but

$$\mathcal{A} = \langle \nabla_\mu J_5^\mu \rangle = 2 \lim_{x \rightarrow x'} \text{tr} \gamma^5 a_2(x, x') , \quad (3.83)$$

where $a_2(x, x')$ is the second coefficient of the Schwinger–DeWitt expansion (3.3). Applying this formula to the theory with torsion, that is using expressions (3.5), (2.53), and (3.14), one can obtain the expression for the anomaly in the external gravitational field with torsion [142]

$$\mathcal{A} = \frac{2}{(4\pi)^2} \left[\nabla_\mu \mathcal{K}^\mu + \frac{1}{48} \varepsilon_{\rho\sigma\alpha\beta} R^{\alpha\beta}{}_{\cdot\mu\nu} R^{\mu\nu\rho\sigma} + \frac{1}{6} \eta^2 \varepsilon^{\mu\nu\alpha\beta} S_{\mu\nu} S_{\alpha\beta} \right], \quad (3.84)$$

where

$$\mathcal{K}^\mu = -\frac{2}{3} \eta (\square + 4\eta^2 S_\lambda S^\lambda - \frac{1}{2} R) S^\mu.$$

One can consider the anomalies of other Noether currents [51] and obtain more general expressions.

From the physical point of view, the most important question is whether the presence of torsion preserves the cancellation of anomalies in the matter sector. If this would not be so, the introduction of torsion could face serious difficulty. This problem has been investigated in [51] and especially in [66]. The result is that the presence of an external torsion does not modify the anomalous divergence of the baryonic current $J_B^\mu = (1/N_c) Q^\dagger \gamma^\mu Q$ while it changes the leptonic current $J_L^\mu = L^\dagger \gamma^\mu L$ by expression (3.84). Independent on whether neutrinos are massless or massive, the cancellation of anomalies holds in the presence of external torsion. In particular, weak external torsion does not affect the quantization of the SM charges [66]. Therefore, the existence of the *background* torsion does not lead to any inconsistency. On the other hand, the leptonic current gains some additional contributions, and this could, in principle, lead to some effects like anisotropy of polarization of light coming from distant galaxies [65]. However, taking the current upper bound on the background torsion from various experiments (see Section 4.6) one can see that the allowed magnitude of the background torsion is insufficient to explain the experimental data which have been discussed in the literature [138].

4. Spinning and spinless particles and the possible effects on the classical background of torsion

The purpose of this section is to construct the non-relativistic approximation for the quantum field theory on torsion background and also develop the consistent formalism for the spinning and spinless particles on the torsion background. We shall follow the original papers [13,158,84]. First, we construct the non-relativistic approximation for the spinor field and particle, then use the path integral method to construct the action of a relativistic particle. The action of a spin 1/2 massless particle with torsion has been first established in [156] on the basis of global supersymmetry. The same action has been rediscovered in [154], where it was also checked using the index theorems. In [150] the action of a massive particle has been obtained through the squaring of the Dirac operator. This action does not possess supersymmetry, and does not have explicit link to the supersymmetric action of [156,154]. In [84] the action of a spinning particle on the background of torsion and electromagnetic field has been derived in the framework of the Berezin–Marinov path integral approach. This action possesses local supersymmetry in the standard approximation of weak torsion field and includes the previous actions of [156,154,150] as a limiting case.

In the last part of this section we present a brief review of the possible physical effects and of the existing upper bounds on the magnitude of the background torsion from some experiments. Since our purpose is to investigate the effects of torsion, it is better not to consider the effects of the metric. For this reason, in this and next sections we shall consider the flat metric $g_{\mu\nu} = \eta_{\mu\nu}$.

4.1. Generalized Pauli equation with torsion

The Pauli-like equation with torsion has been (up to our knowledge) first discussed in [159], and derived, in a proper way, in [13]. After that, the same equation has been obtained in [95,169,120], and the derivation of higher order corrections through the Foldy–Wouthuysen transformation with torsion has been done in [158]. In [120] the Foldy–Wouthuysen transformation has been applied to derive Pauli equation.

The starting point is action (2.35) of Dirac fermion in external electromagnetic and torsion fields:

$$S = \int d^4x \{ i\bar{\psi} \gamma^\mu (\partial_\mu + i\eta \gamma_5 S_\mu + ieA_\mu) \psi + m\bar{\psi} \psi \} .$$

The equation of motion for the fermion ψ can be rewritten using the standard representation of the Dirac matrices (see, for example, [19])

$$\beta = \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \vec{\alpha} = \gamma^0 \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} .$$

Here, as before, $\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$. The equation has the form

$$i\hbar \frac{\partial \psi}{\partial t} = [c\vec{\alpha} \vec{p} - e\vec{\alpha} \vec{A} - \eta \vec{\alpha} \vec{S} \gamma_5 + e\Phi + \eta \gamma_5 S_0 + mc^2 \beta] \psi . \quad (4.1)$$

Here, the dimensional constants \hbar and c were taken into account, and we denoted

$$A_\mu = (\Phi, \vec{A}), \quad S_\mu = (S_0, \vec{S}) .$$

Following the simplest procedure of deriving the non-relativistic approximation [19] we write

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} e^{imc^2 t/\hbar} . \quad (4.2)$$

Within the non-relativistic approximation $\chi \ll \varphi$. From Eq. (4.1) it follows that

$$\left(i\hbar \frac{\partial}{\partial t} - \eta_1 \vec{\sigma} \cdot \vec{S} - e\Phi \right) \varphi = (c\vec{\sigma} \cdot \vec{p} - e\vec{\sigma} \cdot \vec{A} - \eta S_0) \chi \quad (4.3)$$

and

$$\left(i\hbar \frac{\partial}{\partial t} - \eta \vec{\sigma} \cdot \vec{S} - e\Phi + 2mc^2 \right) \chi = (c\vec{\sigma} \cdot \vec{p} - e\vec{\sigma} \cdot \vec{A} - \eta S_0) \varphi = 0 . \quad (4.4)$$

At low energies, the term $2mc^2 \chi$ in the l.h.s. of (4.4) is dominating. Thus, one can disregard other terms and express χ from (4.4). Then, in the leading order in $1/c$ we meet the equation

$$i\hbar \frac{\partial \varphi}{\partial t} = \left[\eta \vec{\sigma} \cdot \vec{S} + e\Phi + \frac{1}{2mc^2 \chi} (c\vec{\sigma} \cdot \vec{p} - e\vec{\sigma} \cdot \vec{A} - \eta S_0)^2 \right] \varphi . \quad (4.5)$$

The last equation can be easily written in the Schrödinger form

$$i\hbar \frac{\partial \varphi}{\partial t} = \hat{H} \varphi \quad (4.6)$$

with the Hamiltonian

$$\hat{H} = \frac{1}{2m} \vec{\pi}^2 + B_0 + \vec{\sigma} \cdot \vec{Q}, \quad (4.7)$$

where

$$\begin{aligned} \vec{\pi} &= \vec{p} - \frac{e}{c} \vec{A} - \frac{\eta_1}{c} \vec{\sigma} S_0, \\ B_0 &= e\Phi - \frac{1}{mc^2} \eta^2 S_0^2, \\ \vec{Q} &= \eta \vec{S} + \frac{\hbar e}{2mc} \vec{H}. \end{aligned} \quad (4.8)$$

Here, $\vec{H} = \text{rot } \vec{A}$ is the magnetic field strength. This equation is the analog of the Pauli equation for the general case of external torsion and electromagnetic fields.

The above expression for the Hamiltonian can be compared to the standard one, which contains only the electromagnetic terms. Some torsion-dependent terms resemble the ones with the magnetic field. At the same time, the term $-(\eta_1 S_0/mc) \vec{p} \cdot \vec{\sigma}$ does not have analogies in quantum electrodynamics.

4.2. Foldy–Wouthuysen transformation with torsion

One can derive the next to the leading order corrections to the non-relativistic approximation (4.6) in the framework of the Foldy–Wouthuysen transformation with torsion [158]. The initial Hamiltonian of the Dirac spinor in external electromagnetic and torsion fields can be presented in the form

$$H = \beta m + \mathcal{E} + \mathcal{G}, \quad (4.9)$$

where

$$\mathcal{E} = e\Phi + \eta \gamma_5 \vec{\alpha} \vec{S} \quad \text{and} \quad \mathcal{G} = \vec{\alpha} (\vec{p} - e\vec{A}) - \eta \gamma_5 S_0 \quad (4.10)$$

are even and odd parts of the expression. From this instant, if this is not indicated explicitly, we shall use the conventional units $c = \hbar = 1$.

Our purpose is to find a unitary transformation which separates “small” and “large” components of the Dirac spinor. In other words, we need to find a Hamiltonian which is block diagonal in the new representation. We use a conventional prescription (see, for example, [24]):

$$H' = e^{i\mathcal{S}} (H - i\partial_t) e^{-i\mathcal{S}}, \quad (4.11)$$

where \mathcal{S} has to be chosen in an appropriate way. We shall try to find \mathcal{S} and H' in a form of the weak-relativistic expansion, and thus start by taking \mathcal{S} to be of order $1/m$ (with $\hbar = c = 1$).

Then, to the usual accuracy, we arrive at the standard result

$$H' = H + i[\mathcal{S}, H] - \frac{1}{2}[\mathcal{S}, [\mathcal{S}, H]] - \frac{i}{6}[\mathcal{S}, [\mathcal{S}, [\mathcal{S}, H]]] + \frac{1}{24}[\mathcal{S}, [\mathcal{S}, [\mathcal{S}, [\mathcal{S}, H]]]] \\ - \dot{\mathcal{S}} - \frac{i}{2}[\mathcal{S}, \dot{\mathcal{S}}] + \frac{1}{6}[\mathcal{S}, [\mathcal{S}, \dot{\mathcal{S}}]] + \dots \quad (4.12)$$

One can easily see that \mathcal{E} and \mathcal{G} given above (anti)commute with β in a usual way

$$\mathcal{E}\beta = \beta\mathcal{E}, \quad \mathcal{G}\beta = -\beta\mathcal{G} \quad (4.13)$$

and therefore one can safely use the standard prescription for the lowest order approximation:

$$\mathcal{S} = -\frac{i}{2m}\beta\mathcal{G}. \quad (4.14)$$

This gives

$$H' = \beta m + \mathcal{E}' + \mathcal{G}', \quad (4.15)$$

where \mathcal{G}' is of order $1/m$. Now one has to perform second Foldy–Wouthuysen transformation with $\mathcal{S}' = -(i/2m)\beta\mathcal{G}'$. This leads to the

$$H'' = \beta m + \mathcal{E}' + \mathcal{G}'' \quad (4.16)$$

with $\mathcal{G}'' \approx 1/m^2$; and then a third Foldy–Wouthuysen transformation with $\mathcal{G}'' = -(i/2m)\beta\mathcal{G}''$ removes odd operators in the given order of the non-relativistic expansion, so that we finally obtain the usual result

$$H''' = \beta \left(m + \frac{1}{2m}\mathcal{G}^2 - \frac{1}{8m^3}\mathcal{G}^4 \right) + \mathcal{E} - \frac{1}{8m^2}[\mathcal{G}, ([\mathcal{G}, \mathcal{E}] + i\dot{\mathcal{G}})]. \quad (4.17)$$

Substituting our \mathcal{E} and \mathcal{G} from (4.10), after some algebra we arrive at the final form of the Hamiltonian

$$H''' = \beta \left[m + \frac{1}{2m}(\vec{p} - e\vec{A} - \eta S_0 \vec{\sigma})^2 - \frac{1}{8m^3}\vec{p}^4 \right] + e\Phi - \eta(\vec{\sigma} \cdot \vec{S}) - \frac{e}{2m}\vec{\sigma} \cdot \vec{H} \\ - \frac{e}{8m^2} \left[\text{div } \vec{E} + i\vec{\sigma} \cdot \text{rot } \vec{E} + 2\vec{\sigma} \cdot [\vec{E} \times \vec{p}] \right] + \frac{\eta}{8m^2} \{ \vec{\sigma} \cdot \nabla \dot{S}_0 - [p_i, [p^i, (\vec{\sigma} \cdot \vec{S})]_+]_+ \\ + 2 \text{rot } \vec{S} \cdot \vec{p} - 2i(\vec{\sigma} \cdot \nabla)(\vec{S} \cdot \vec{p}) - 2i(\nabla \vec{S})(\vec{\sigma} \cdot \vec{p}) \}, \quad (4.18)$$

where we have used standard notation for the anticommutator $[A, B]_+ = AB + BA$. As usual, \vec{E} denotes the strength of the external electric field $\vec{E} = -(1/c)\partial\vec{A}/\partial t - \text{grad } \Phi$. One can proceed in the same way and get separated Hamiltonian with any given accuracy in $1/m$.

The first five terms of (4.18) reproduce the Pauli-like equation with torsion (4.7). Other terms are the next-to-the-leading order weak-relativistic corrections and torsion-dependent corrections to the Pauli-like equation (4.6). In those terms we follow the system of approximation which is standard for the electromagnetic case [24]; that is we keep the terms linear in interactions. One can note that for the case of the constant torsion and electromagnetic fields one can achieve the exact Foldy–Wouthuysen transformation [137]. We do not reproduce this result here, because

torsion (if it exists) is definitely weak and the the leading order approximation [158] is certainly the most important one.

Further simplifications of (4.18) are possible if we are interested in constant torsion. This version of torsion can be some kind of relic cosmological field or it can be generated by the vacuum quantum effects. In this case we have to keep only the constant components of the pseudovector S_μ . Then the effects of torsion will be: (i) a small correction to the potential energy of the spinor field, which sometimes looks just like a correction to the mass, and (ii) the appearance of a new gauge-invariant spin–momentum interaction term in the Hamiltonian.

4.3. Non-relativistic particle in the external torsion field

In this section we start from the simple derivation of the action for the non-relativistic particle. This action will be used later on to test the more general relativistic expression.

If we consider (4.7) as the Hamiltonian operator of some quantum particle, then the corresponding classical energy has the form

$$H = \frac{1}{2m} \vec{\pi}^2 + B_0 + \vec{\sigma} \cdot \vec{Q}, \quad (4.19)$$

where $\vec{\pi}, B_0, \vec{Q}$ are defined by (16) and $\vec{\pi} = m\vec{v}$. Here $\vec{v} = \dot{\vec{x}}$ is the velocity of the particle. The expression for the canonically conjugated momenta \vec{p} follows from (4.19).

$$\vec{p} = m\vec{v} + \frac{e}{c} \vec{A} + \frac{\eta}{c} \vec{\sigma} S_0. \quad (4.20)$$

One can consider the components of the vector $\vec{\sigma}$ as internal degrees of freedom, corresponding to spin.

Let us perform the canonical quantization of the theory. For this, we introduce the operators of coordinate \hat{x}_i , momenta \hat{p}_i and spin $\hat{\sigma}_i$ and demand that they satisfy the equal-time commutation relations of the following form:

$$[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}, \quad [\hat{x}_i, \hat{\sigma}_j] = [\hat{p}_i, \hat{\sigma}_j] = 0, \quad [\hat{\sigma}_i, \hat{\sigma}_j] = 2i\epsilon_{ijk} \hat{\sigma}_k. \quad (4.21)$$

The Hamiltonian operator \hat{H} which corresponds to energy (4.19) can be easily constructed in terms of the operators $\hat{x}_i, \hat{p}_i, \hat{\sigma}_i$. From it we may write the equations of motion

$$\begin{aligned} i\hbar \frac{d\hat{x}_i}{dt} &= [\hat{x}_i, \hat{H}], \\ i\hbar \frac{d\hat{p}_i}{dt} &= [\hat{p}_i, \hat{H}], \\ i\hbar \frac{d\hat{\sigma}_i}{dt} &= [\hat{\sigma}_i, \hat{H}]. \end{aligned} \quad (4.22)$$

After the computation of the commutators in (4.22) we arrive at the explicit form of the operator equations of motion. Now, we can omit all terms which vanish when $\hbar \rightarrow 0$. Thus, we obtain the non-relativistic, quasi-classical equations of motion for the spinning particle in the external

torsion and electromagnetic fields. Note that the operator ordering problem is irrelevant because of the $\hbar \rightarrow 0$ limit. The straightforward calculations lead to the equations [13]:

$$\frac{d\vec{x}}{dt} = \frac{1}{m} \left(\vec{p} - \frac{e}{c} \vec{A} - \frac{\eta}{c} \vec{\sigma} S_0 \right) = \vec{v}, \quad (4.23)$$

$$\begin{aligned} \frac{d\vec{v}}{dt} = & e\vec{E} + \frac{e}{c} [\vec{v} \times \vec{H}] - \eta \nabla (\vec{\sigma} \cdot \vec{S}) \\ & + \frac{\eta}{c} \left[(\vec{v} \cdot \vec{\sigma}) \nabla S_0 - (\vec{v} \cdot \nabla S_0) \vec{\sigma} - \frac{dS_0}{dt} \vec{\sigma} \right] + \frac{\eta^2}{mc^2} \nabla (S_0^2) + \frac{\eta}{c} S_0 [\vec{\sigma} \times \vec{R}] \end{aligned} \quad (4.24)$$

and

$$\frac{d\vec{\sigma}}{dt} = [\vec{R} \times \vec{\sigma}], \quad \text{where } \vec{R} = \frac{2\eta}{\hbar} \left[\vec{S} - \frac{1}{c} \vec{v} S_0 \right] + \frac{e}{mc} \vec{H}. \quad (4.25)$$

Eqs. (4.23)–(4.25) contain the torsion-dependent terms which are similar to the magnetic terms, and also some terms which have a qualitatively new form.

4.4. Path-integral approach for the relativistic particle with torsion

In this section, we are going to construct a path-integral representation for a propagator of a massive spinning particle in external electromagnetic and torsion fields. The consistent method of constructing the path-integral representation has been developed by Berezin and Marinov [21]. Various aspects of the Berezin–Marinov approach were investigated in the consequent publications (see, for example, [29]). In this section we shall follow Ref. [84] where the path-integral representation has been generalized for the background with torsion. It was demonstrated in [75,85], that a special kind of path-integral representations for propagators of relativistic particles allow one to derive gauge-invariant pseudoclassical actions for the corresponding particles. Let us remark that in [76] some path-integral representation for massive spinning particle in the presence of the torsion was derived using the perturbative approach to path integrals.

First, we consider the path-integral representation of the scalar field propagator in external torsion S_μ and electromagnetic A_μ fields. As we already know, the scalar field interacts with torsion non-minimally, and this interaction is necessary for the renormalizability of scalar coupled to the fermions. Therefore, the Klein–Gordon equation in external electromagnetic and torsion fields has the form

$$[\hat{\mathcal{P}}^2 + m^2 + \xi S^2] \varphi(x) = 0, \quad (4.26)$$

where $\mathcal{P}_\mu = i\partial_\mu - eA_\mu$, $S^2 = S_\mu S^\mu$ and ξ is an arbitrary non-minimal parameter. This is the very same parameter which was called ξ_4 in Section 2. Since in this section there are no other ξ , it is reasonable to omit index 4, exactly as we omitted the index in η_1 .

Our consideration is very similar to the one presented in [85] for the torsionless case. The propagator obeys the equation

$$[\hat{\mathcal{P}}^2 + m^2 + \xi S^2] D^c(x, y) = -\delta(x, y). \quad (4.27)$$

The Schwinger representation for the propagator is

$$D^c(x, y) = \langle x | \hat{D}^c | y \rangle .$$

Here $|x\rangle$ are eigenvectors for some Hermitian operators of coordinates X^μ and the corresponding canonically conjugated momenta operators are P_μ . Then, the following relations hold:

$$\begin{aligned} X^\mu |x\rangle &= x^\mu |x\rangle, & \langle x | y \rangle &= \delta^4(x - y), & \int |x\rangle \langle x| dx &= I, \\ [P_\mu, X^\nu]_- &= -i\delta_\mu^\nu, & P_\mu |p\rangle &= p_\mu |p\rangle, & \langle p | p' \rangle &= \delta^4(p - p'), \\ \int |p\rangle \langle p| dp &= I, & \langle x | P_\mu | y \rangle &= -i\partial_\mu \delta^4(x - y), & \langle x | p \rangle &= \frac{1}{(2\pi)^2} e^{ipx}, \\ [\Pi_\mu, \Pi_\nu]_- &= -ieF_{\mu\nu}(X), & \Pi_\mu &= -P_\mu - eA_\mu(X), & F_{\mu\nu}(X) &= \partial_\mu A_\nu - \partial_\nu A_\mu . \end{aligned} \quad (4.28)$$

We can write (4.27) in an operator way

$$\hat{F}\hat{D}^c = \hat{1}, \quad \text{where } \hat{F} = m^2 + \xi S^2 - \Pi^2 ,$$

and use the Schwinger proper-time representation:

$$\hat{D}^c = \hat{F}^{-1} = i \int_0^\infty e^{-i\lambda(\hat{F} - i\epsilon)} d\lambda , \quad (4.29)$$

where $\epsilon \rightarrow 0$ at the end of calculations. For the massive theory ϵ -term can be included into the mass, and that is why we do not write it in what follows. Indeed, for the massless case ϵ is important for it stabilizes the theory in the IR domain. In principle, for the constant torsion and $\xi S^2 > 0$, torsion term can stabilize the proper time integral even in the massless case. It proves useful to denote $\hat{\mathcal{H}} = \hat{F} \cdot \lambda$, and rewrite the previous expression for the Green's function (4.29)

$$\begin{aligned} \hat{D}^c &= \hat{D}^c(x_{\text{out}}, x_{\text{in}}) = i \int_0^\infty \langle x_{\text{out}} | e^{-i\hat{\mathcal{H}}(\lambda)} | x_{\text{in}} \rangle d\lambda \\ &= i \lim_{N \rightarrow \infty} \int_0^\infty d\lambda_0 \int_{-\infty}^{+\infty} dx_1 \dots dx_N d\lambda_1 \dots d\lambda_N \prod_{k=1}^N \langle x_k | e^{-i\hat{\mathcal{H}}(\lambda_k)/N} | x_{k-1} \rangle \delta(\lambda_k - \lambda_{k-1}) , \end{aligned} \quad (4.30)$$

where $x_0 = x_{\text{in}}$, $x_N = x_{\text{out}}$. For large enough N one can approximate

$$\langle x_k | e^{-i\hat{\mathcal{H}}(\lambda_k)/N} | x_{k-1} \rangle \approx \left\langle x_k \left| 1 - \frac{i}{N} \hat{\mathcal{H}}(\lambda_k) \right| x_{k-1} \right\rangle . \quad (4.31)$$

Introducing the Weyl symbol $\mathcal{H}(\lambda, x, p)$ of the operator $\hat{\mathcal{H}}(\lambda)$ as [20]

$$\mathcal{H}(\lambda, x, p) = \lambda(m^2 + \xi S^2 - \mathcal{P}^2), \quad \mathcal{P}_\mu = -p_\mu - eA_\mu ,$$

we can express each of (4.31) in terms of the Weyl symbols in the middle points $\bar{x}_k = (x_k + x_{k-1})/2$:

$$\langle x_k | e^{-i\hat{\mathcal{H}}(\lambda_k)/N} | x_{k-1} \rangle \approx \int \frac{dp_k}{(2\pi)^4} \exp \left\{ \left[p_k(x_k - x_{k-1}) - \frac{\mathcal{H}(\lambda_k, \bar{x}_k, p_k)}{N} \right] \right\} . \quad (4.32)$$

Then the expression for the propagator becomes

$$\hat{D}^c = i \lim_{N \rightarrow \infty} \int_0^\infty d\lambda_0 \int_{-\infty}^{+\infty} \prod_{k=1}^N dx_k d\lambda_k \frac{dp_k}{(2\pi)^4} \frac{d\pi_k}{2\pi} \times \exp \left\{ \left[p_k(x_k - x_{k-1}) - \frac{\mathcal{H}(\lambda_k, \bar{x}_k, p_k)}{N} + \pi_k(\lambda_k - \lambda_{k-1}) \right] \right\}, \quad (4.33)$$

where we have also used the Fourier representation for the delta-functions $\delta(\lambda_k - \lambda_{k-1})$ of (4.30). Eq. (4.33) is the definition of the Hamiltonian path integral for the propagator of scalar particle

$$\hat{D}^c = i \int_0^\infty d\lambda_0 \int_{x_{\text{in}}}^{x_{\text{out}}} Dx \int D\lambda \int Dp D\pi \exp \left\{ i \int_0^1 d\tau [\lambda(\mathcal{P}^2 - m^2 - \xi S^2) + p\dot{x} + \pi\dot{\lambda}] \right\}. \quad (4.34)$$

The integral is taken over the path $x^\mu(\tau), p_\mu(\tau), \lambda(\tau), \pi(\tau)$ with fixed ends $x(0) = x_{\text{in}}, x(1) = x_{\text{out}}, \lambda(0) = \lambda_0$. Integrating over the momenta p_μ , one arrives at the Lagrangian form of the path-integral representation (where we substituted $\lambda = \theta/2$ in order to achieve the conventional form)

$$\hat{D}^c = \frac{i}{2} \int_0^\infty d\theta_0 \int_{x_{\text{in}}}^{x_{\text{out}}} Dx \int_{\theta_0} D\theta M(\theta) \int D\pi e^{i \int_0^1 L d\tau}, \quad (4.35)$$

where the Lagrangian of the scalar particle has the form

$$L = -\frac{\dot{x}^2}{2\theta} - \frac{\theta}{2}(m^2 + \xi S^2) - ex_\mu A^\mu + \pi\dot{\theta}. \quad (4.36)$$

It is easy to see that the presence of torsion does not make any essential changes in the derivation of the particle action. The result may be obtained by simple replacement $m^2 \rightarrow m^2 + \xi S^2$.

Let us now consider the path-integral representation for the propagator of spinning particle. We shall follow the original paper [84], where the technique of Ref. [75] was applied to the case of the spinning particle on the torsion and electromagnetic background. One can also consult [84] for the further list of references on the subject.

Consider the causal Green's function $\Delta^c(x, y)$ of the equation of motion corresponding to action (2.35). $\Delta^c(x, y)$ is the propagator of the spinor particle.

$$[\gamma^\mu(\hat{\mathcal{P}}_\mu + \eta\gamma^5 S_\mu) - m]\Delta^c(x, y) = -\delta^4(x - y). \quad (4.37)$$

It proves useful to introduce, along with $\gamma^0, \gamma^1, \gamma^2, \gamma^3, \gamma^5$, another set of the Dirac matrices $\Gamma^0, \Gamma^1, \Gamma^2, \Gamma^3, \Gamma^4$. These matrices are defined through the relations

$$\Gamma^4 = i\gamma^5, \quad \Gamma^\mu = \Gamma^4\gamma^\mu. \quad (4.38)$$

It is easy to check that the matrices $\Gamma^n, n = 0, 1, \dots, 4$, form a representation of the Clifford algebra in five dimensions:

$$[\Gamma^n, \Gamma^m]_+ = 2\eta^{nm}, \quad \eta_{nm} = \text{diag}(1, -1, -1, -1, -1). \quad (4.39)$$

In order to proceed, we need a homogeneous form of the operator $(\Delta^c)^{-1}$. Therefore, we perform the Γ^4 -transformation for $\Delta^c(x, y)$.

$$\tilde{\Delta}^c(x, y) = \Delta^c(x, y)\Gamma^4.$$

The new propagator $\tilde{\Delta}^c(x, y)$ obeys the equation

$$[\Gamma^\mu(\hat{\mathcal{P}}_\mu - i\eta\Gamma^4 S_\mu) - m\Gamma^4]\tilde{\Delta}^c(x, y) = \delta^4(x - y). \quad (4.40)$$

Similar to the scalar case, we present $\tilde{\Delta}^c(x, y)$ as a matrix element of an operator $\tilde{\Delta}^c$.

$$\tilde{\Delta}^c(x, y) = \langle x | \tilde{\Delta}^c | y \rangle.$$

All further notations are those of (4.28). The formal solution for the operator $\tilde{\Delta}^c$ is

$$\tilde{\Delta}^c = \hat{\mathcal{F}}^{-1}, \quad \hat{\mathcal{F}} = \Pi_\mu \Gamma^\mu - m\Gamma^4 - i\eta\Gamma^\mu \Gamma^4 S_\mu.$$

The operator $\hat{\mathcal{F}}$ may be written in an equivalent form

$$\hat{\mathcal{F}} = \Pi_\mu \Gamma^\mu - m\Gamma^4 - \frac{i}{6}\eta\epsilon_{\mu\nu\alpha\beta}S^\mu\Gamma^\nu\Gamma^\alpha\Gamma^\beta, \quad (4.41)$$

using the following formula:

$$\Gamma_\mu\Gamma^4 = \frac{1}{6}\epsilon_{\mu\nu\alpha\beta}\Gamma^\nu\Gamma^\alpha\Gamma^\beta, \quad \epsilon_{0123} = 1.$$

The last relation is important, for it replaces the product of an even number of Γ 's for the product of an odd number of Γ 's. Together with the Γ^4 -transformation of the propagator, this helps us to provide the homogeneity of the equation, so that $\hat{\mathcal{F}}$ becomes purely fermionic operator. Now, $\hat{\mathcal{F}}^{-1}$ can be presented by means of an integral [75]:

$$\hat{\mathcal{F}}^{-1} = \int_0^\infty d\lambda \int d\chi e^{i[\lambda(\hat{\mathcal{F}}^2 + i\epsilon) + \chi\hat{\mathcal{F}}]} d\chi. \quad (4.42)$$

Here λ, χ are the parameters of even and odd Grassmann parity. Taken together, they can be considered as a superproper time [75]. Indeed, λ commutes and χ anticommutes with $\hat{\mathcal{F}}$:

$$[\lambda, \hat{\mathcal{F}}] = 0, \quad [\chi, \hat{\mathcal{F}}]_+ = 0.$$

Calculating $\hat{\mathcal{F}}^2$ we find

$$\hat{\mathcal{F}}^2 = \Pi^2 - m^2 - \eta^2 S^2 - \frac{ie}{2}F_{\mu\nu}\Gamma^\mu\Gamma^\nu + \hat{K}_{\mu\nu}\Gamma^\mu\Gamma^\nu + \eta\partial_\mu S^\mu\Gamma^0\Gamma^1\Gamma^2\Gamma^3, \quad (4.43)$$

where

$$\hat{K}_{\mu\nu} = \frac{i\eta}{2}[\Pi^\alpha, S^\beta]_+ \epsilon_{\alpha\beta\mu\nu}, \quad \Pi^2 = P^2 + e^2 A^2 + e[P_\mu, A^\mu]_+. \quad (4.44)$$

Thus, we get the integral representation for the propagator:

$$\tilde{\Delta}^c = \int_0^\infty d\lambda \int d\chi \exp[-i\hat{\mathcal{H}}(\lambda, \chi)],$$

where

$$\begin{aligned} \hat{\mathcal{H}}(\lambda, \chi) = & \lambda \left(m^2 + \eta^2 S^2 - \Pi^2 + \frac{ie}{2} F_{\mu\nu} \Gamma^\mu \Gamma^\nu - \hat{K}_{\mu\nu} \Gamma^\mu \Gamma^\nu \right. \\ & \left. - \eta \partial_\mu S^\mu \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \right) - \chi \left(\Pi_\mu \Gamma^\mu - \frac{im\eta}{6} \epsilon_{\kappa\mu\nu\alpha} S^\kappa \Gamma^4 \Gamma^\mu \Gamma^\nu \Gamma^\alpha \right). \end{aligned}$$

The Green's function $\tilde{\Delta}^c(x_{\text{out}}, x_{\text{in}})$ has the form

$$\tilde{\Delta}^c(x_{\text{out}}, x_{\text{in}}) = \int_0^\infty d\lambda \int \langle x_{\text{out}} | e^{-i\hat{\mathcal{H}}(\lambda, \chi)} | x_{\text{in}} \rangle d\chi. \quad (4.45)$$

Now, we are going to represent the matrix element entering in expression (4.45) by means of a path integral [75,84]. The calculation goes very similar to the scalar case. We write, as usual, $e^{-i\hat{\mathcal{H}}} = (e^{-i\hat{\mathcal{H}}/N})^N$, then insert $(N-1)$ identities $\int |x\rangle \langle x| dx = I$ and introduce N additional integrations over λ and χ

$$\begin{aligned} \tilde{\Delta}^c(x_{\text{out}}, x_{\text{in}}) = & \lim_{N \rightarrow \infty} \int_0^\infty d\lambda_0 \int \prod_{k=1}^N \langle x_k | e^{(-i/N)\hat{\mathcal{H}}(\lambda_k, \chi_k)A} | x_{k-1} \rangle \\ & \times \delta(\lambda_k - \lambda_{k-1}) \delta(\chi_k - \chi_{k-1}) d\chi_0 dx_1 \dots dx_{N-1} d\lambda_1 \dots d\lambda_N d\chi_1 \dots d\chi_N, \end{aligned} \quad (4.46)$$

where $x_0 = x_{\text{in}}$, $x_N = x_{\text{out}}$. Using approximation (4.31), and introducing the Weyl symbol $\mathcal{H}(\lambda, \chi, x, p)$ of the symmetric operator $\hat{\mathcal{H}}$

$$\begin{aligned} \mathcal{H}(\lambda, \chi, x, p) = & \lambda \left(m^2 + \eta^2 S^2 - \mathcal{P}^2 + \frac{ie}{2} F_{\mu\nu} \Gamma^\mu \Gamma^\nu - K_{\mu\nu} \Gamma^\mu \Gamma^\nu \right. \\ & \left. - \eta \partial_\mu S^\mu \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \right) - \chi \left(\mathcal{P}_\mu \Gamma^\mu - \frac{im\eta}{6} \epsilon_{\kappa\mu\nu\alpha} S^\kappa \Gamma^4 \Gamma^\mu \Gamma^\nu \Gamma^\alpha \right) \end{aligned}$$

with $K_{\mu\nu} = -\eta \mathcal{P}^\alpha S^\beta \epsilon_{\alpha\beta\mu\nu}$, we can express the matrix elements (4.31) in terms of the Weyl symbols at the middle point \bar{x}_k . Then (4.31) can be replaced by the expressions

$$\int \frac{dp_k}{(2\pi)^4} \exp i \left[p_k \frac{x_k - x_{k-1}}{\Delta\tau} - \mathcal{H}(\lambda_k, \chi_k, \bar{x}_k, p_k) \right] \Delta\tau, \quad (4.47)$$

where $\Delta\tau = 1/N$. Such expressions with different values of k do not commute due to the Γ -matrix structure and, therefore, have to be replaced into (4.46) in such a way that the numbers k increase from the right to the left. For the two δ -functions, accompanying each matrix element in expression (4.46), we use the integral representations

$$\delta(\lambda_k - \lambda_{k-1}) \delta(\chi_k - \chi_{k-1}) = \frac{i}{2\pi} \int d\pi_k dv_k \exp \{ i [\pi_k (\lambda_k - \lambda_{k-1}) + v_k (\chi_k - \chi_{k-1})] \},$$

where v_k are odd variables. Then we attribute to the Γ -matrices in (4.47) an index k . At the same time we attribute to all quantities the “time” τ_k according to the index k they have, $\tau_k = k\Delta\tau$. Then, $\tau \in [0, 1]$. Introducing the T-product, which acts on Γ -matrices, it is possible to gather all the expressions, entering in (4.46), in one exponent and postulate that at equal

times the Γ -matrices anticommute. Finally, we arrive at the propagator

$$\begin{aligned} \tilde{A}^c(x_{\text{out}}, x_{\text{in}}) = & \text{T} \int_0^\infty d\lambda_0 \int d\chi_0 \int_{x_{\text{in}}}^{x_{\text{out}}} Dx \int Dp \int_{\lambda_0} D\lambda \int_{\chi_0} D\chi \int D\pi \int Dv \\ & \times \exp \left\{ i \int_0^1 \left[\lambda \left(\mathcal{P}^2 - m^2 - \eta^2 S^2 - \frac{ie}{2} F_{\mu\nu} \Gamma^\mu \Gamma^\nu + K_{\mu\nu} \Gamma^\mu \Gamma^\nu \right. \right. \right. \\ & \left. \left. + \eta \partial_\mu S^\mu \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \right) + \chi \left(\mathcal{P}_\mu \Gamma^\mu - \frac{im\eta}{6} \epsilon_{\kappa\mu\nu\alpha} S^\kappa \Gamma^4 \Gamma^\mu \Gamma^\nu \Gamma^\alpha \right) \right. \\ & \left. \left. + p\dot{x} + \pi\dot{\lambda} + v\dot{\chi} \right] d\tau \right\}, \end{aligned} \quad (4.48)$$

where $x(\tau)$, $p(\tau)$, $\lambda(\tau)$, $\pi(\tau)$, are even and $\chi(\tau)$, $v(\tau)$ are odd functions. The boundary conditions are $x(0) = x_{\text{in}}$, $x(1) = x_{\text{out}}$, $\lambda(0) = \lambda_0$, $\chi(0) = \chi_0$. The operation of T-ordering acts on the Γ -matrices which formally depend on the time τ . Expression (4.48) can be transformed as follows:

$$\begin{aligned} \tilde{A}^c(x_{\text{out}}, x_{\text{in}}) = & \int_0^\infty d\lambda_0 \int d\chi_0 \int_{\lambda_0} D\lambda \int_{\chi_0} D\chi \int_{x_{\text{in}}}^{x_{\text{out}}} Dx \int Dp \int D\pi \int Dv \\ & \times \exp \left\{ i \int_0^1 \left[\lambda \left(\mathcal{P}^2 - m^2 - \eta^2 S^2 - \frac{ie}{2} F_{\mu\nu} \frac{\delta_l}{\delta\rho_\mu} \frac{\delta_l}{\delta\rho_\nu} + K_{\mu\nu} \frac{\delta_l}{\delta\rho_\mu} \frac{\delta_l}{\delta\rho_\nu} \right. \right. \right. \\ & \left. \left. + \eta \partial_\mu S^\mu \frac{\delta_l}{\delta\rho_0} \frac{\delta_l}{\delta\rho_1} \frac{\delta_l}{\delta\rho_2} \frac{\delta_l}{\delta\rho_3} \right) + \chi \left(\mathcal{P}_\mu \frac{\delta_l}{\delta\rho_\mu} - m \frac{\delta_l}{\delta\rho_4} - \frac{i\eta}{6} \epsilon_{\kappa\mu\nu\alpha} S^\kappa \frac{\delta_l}{\delta\rho_\mu} \frac{\delta_l}{\delta\rho_\nu} \frac{\delta_l}{\delta\rho_\alpha} \right) p\dot{x} \right. \\ & \left. \left. + \pi\dot{\lambda} + v\dot{\chi} \right] d\tau \right\} \text{T exp} \int_0^1 \rho_n(\tau) \Gamma^n d\tau \Big|_{\rho=0}, \end{aligned}$$

where five odd sources $\rho_n(\tau)$ are introduced. They anticommute with the Γ -matrices by definition. One can represent the quantity $\text{T exp} \int_0^1 \rho_n(\tau) \Gamma^n d\tau$ via a path integral over odd trajectories [75],

$$\begin{aligned} \text{T exp} \int_0^1 \rho_n(\tau) \Gamma^n d\tau = & \exp \left(i \Gamma^n \frac{\partial_l}{\partial \Theta^n} \right) \int_{\psi(0)+\psi(1)=\Theta} \exp \left[\int_0^1 (\psi_n \dot{\psi}^n - 2i\rho_n \psi^n) d\tau \right. \\ & \left. + \psi_n(1) \psi^n(0) \right] \mathcal{D}\psi \Big|_{\Theta=0} \end{aligned} \quad (4.49)$$

with the modified integration measure

$$\mathcal{D}\psi = D\psi \left[\int_{\psi(0)+\psi(1)=0} D\psi \exp \left\{ \int_0^1 \psi_n \dot{\psi}^n d\tau \right\} \right]^{-1}.$$

Here Θ^n are odd variables, anticommuting with the Γ -matrices, and $\psi^n(\tau)$ are odd trajectories of integration. These trajectories satisfy the boundary conditions indicated below the signs of

integration. Using (4.49) we get the Hamiltonian path-integral representation for the propagator:

$$\begin{aligned} \tilde{\Delta}^c(x_{\text{out}}, x_{\text{in}}) = & \exp\left(i\Gamma^n \frac{\partial_l}{\partial \Theta^n}\right) \int_0^\infty d\lambda_0 \int d\chi_0 \int_{\lambda_0} D\lambda \int_{\chi_0} D\chi \int_{x_{\text{in}}}^{x_{\text{out}}} Dx \int Dp \int D\pi \int Dv \\ & \times \int_{\psi(0)+\psi(1)=\Theta} \mathcal{D}\psi \exp \left\{ i \int_0^1 \left[\lambda \left(\mathcal{P}_\mu + \frac{i}{\lambda} \psi_\mu \chi + d_\mu \right)^2 - \lambda(m^2 + \eta^2 S^2) \right. \right. \\ & + 2i\lambda e F_{\mu\nu} \psi^\mu \psi^\nu + 16\lambda \eta \partial_\mu S^\mu \psi^0 \psi^1 \psi^2 \psi^3 + 2i\chi \left(m\psi^4 + \frac{2}{3} \psi^\mu d_\mu \right) \\ & \left. \left. - i\psi_n \dot{\psi}^n + p\dot{x} + \pi\dot{\lambda} + v\dot{\chi} \right] d\tau + \psi_n(1)\psi^n(0) \right\} \Big|_{\Theta=0}, \end{aligned} \quad (4.50)$$

where

$$d_\mu = -2i\eta \epsilon_{\mu\nu\alpha\beta} S^\nu \psi^\alpha \psi^\beta. \quad (4.51)$$

Integrating over the momenta, we get the Lagrangian path-integral representation for the propagator,

$$\begin{aligned} \tilde{\Delta}^c(x_{\text{out}}, x_{\text{in}}) = & \exp\left(i\Gamma^n \frac{\partial_\ell}{\partial \Theta^n}\right) \int_0^\infty d\theta_0 \int d\chi_0 \int_{\theta_0} \mathcal{M}(\theta) D\theta \int_{\chi_0} D\chi \int_{x_{\text{in}}}^{x_{\text{out}}} Dx \int D\pi \int Dv \\ & \times \int_{\psi(0)+\psi(1)=\Theta} \mathcal{D}\psi \exp \left\{ i \int_0^1 \left[-\frac{z^2}{2\theta} - \frac{\theta}{2} M^2 - \dot{x}_\mu (eA^\mu - d^\mu) + i\theta e F_{\mu\nu} \psi^\mu \psi^\nu \right. \right. \\ & \left. \left. + i\chi \left(m\psi^4 + \frac{2}{3} \psi^\mu d_\mu \right) - i\psi_n \dot{\psi}^n + \pi\dot{\theta} + v\dot{\chi} \right] d\tau + \psi_n(1)\psi^n(0) \right\} \Big|_{\Theta=0}, \end{aligned} \quad (4.52)$$

where $\theta = 2\lambda$ and the measure $\mathcal{M}(\theta)$ has the form:

$$\mathcal{M}(\theta) = \int \mathcal{D}p \exp \left[\frac{i}{2} \int_0^1 \theta p^2 d\tau \right] \quad (4.53)$$

and

$$M^2 = m^2 + \eta^2 S^2 - 16\eta \partial_\mu S^\mu \psi^0 \psi^1 \psi^2 \psi^3, \quad z^\mu = \dot{x}^\mu + i\chi \psi^\mu. \quad (4.54)$$

The discussion of the role of measure (4.53) can be found in [75].

The exponential in integrand (4.52) can be considered as an effective non-degenerate Lagrangian action of a spinning particle in electromagnetic and torsion fields. It consists of

two principal parts. The term

$$S_{\text{GF}} = \int_0^1 (\pi \dot{\theta} + v \dot{\chi}) d\tau ,$$

can be treated as a gauge fixing term corresponding to the gauge conditions $\dot{\theta} = \dot{\chi} = 0$. The other terms can be treated as a gauge-invariant action of a spinning particle. It has the form

$$S = \int_0^1 \left[-\frac{z^2}{2\theta} - \frac{\theta}{2} M^2 - \dot{x}_\mu (e A^\mu - d^\mu) + i\theta e F_{\mu\nu} \psi^\mu \psi^\nu \right. \\ \left. + i\chi \left(m\psi^4 + \frac{2}{3} \psi^\mu d_\mu \right) - i\psi_n \dot{\psi}^n \right] d\tau , \quad (4.55)$$

where z^μ , M^2 and d_μ have been defined in (4.54) and (4.51). Action (4.55) is a generalization of the Berezin–Marinov action [21,29] to the background with torsion. One can easily verify that (4.55) is invariant under reparametrizations:

$$\delta x = \dot{x} \xi, \quad \delta \theta = \frac{d(\theta \xi)}{dt}, \quad \delta \psi^n = \dot{\psi}^n \xi \quad (n = 0, 1, 2, 3, 4), \quad \delta \chi = \frac{d(\chi \xi)}{dt} .$$

I could establish the explicit form of the local supersymmetry transformations, which generalize the ones for the Berezin–Marinov action, only in the linear in torsion approximation. These transformations have exactly the same form as in the case without torsion (see, for example, [86]):

$$\delta x^\alpha = i\psi^\alpha \epsilon, \quad \delta \theta = i\chi \epsilon, \quad \delta \psi^\alpha = \frac{1}{2\theta} (\dot{x}^\alpha + i\chi \psi^\alpha) \epsilon \quad (\alpha = 0, 1, 2, 3), \\ \delta \chi = \dot{\epsilon}, \quad \delta \psi^5 = \left[\frac{m}{2} - \frac{i}{m\theta} \psi^5 \left(\dot{\psi}^5 - \frac{m}{2} \chi \right) \right] \epsilon$$

with $\epsilon = \epsilon(\tau)$. The demonstration of supersymmetry is technically non-trivial, and in particular one needs the identity:

$$\psi^\alpha \psi^\beta \psi^\mu \psi^\nu \varepsilon_{\alpha\beta\mu\lambda} \partial_\nu S^\lambda = \frac{1}{4} \psi^\alpha \psi^\beta \psi^\mu \psi^\nu \varepsilon_{\alpha\beta\mu\nu} \partial_\lambda S^\lambda .$$

In the general case one can establish the supersymmetry of the action through the structure of the Hamiltonian constraints in the course of quantization [84].

Let us analyze the equations of motion for the theory with action (4.55). These equations contain some unphysical variables, that are related to the reparametrization and supersymmetry invariance. One can choose the gauge conditions $\chi = 0$ and $\theta = 1/m$ to simplify the analysis. Then we need only two equations

$$\frac{\delta_r S}{\delta \psi^\alpha} = 2i\dot{\psi}_\alpha - 2i\theta e F_{\alpha\beta} \psi^\beta - \frac{i}{e} \dot{x}_\alpha \chi + \frac{2i}{3} \chi d_\alpha + 4i\eta \varepsilon_{\mu\nu\alpha\beta} \dot{x}^\mu S^\nu \psi^\beta \\ - \frac{8\eta}{3} \chi \varepsilon_{\mu\nu\alpha\beta} \psi^\mu S^\nu \psi^\beta - \frac{4\eta\theta}{3} \partial_\lambda S^\lambda \varepsilon_{\mu\nu\alpha\beta} \psi^\mu \psi^\nu \psi^\beta , \quad (4.56)$$

$$\frac{\delta S}{\delta x^\alpha} = \frac{d}{d\tau} \left(\frac{\dot{x}_\alpha}{\theta} \right) + e \dot{x}^\beta F_{\beta\alpha} + i\theta e F_{\mu\nu, \alpha} \psi^\mu \psi^\nu + e \dot{x}_\mu \partial_\alpha A^\mu + \frac{d}{d\tau} \left(\frac{i}{\theta} \psi_\alpha \chi - e A_\alpha + d_\alpha \right) \\ + \eta \theta S^\mu \partial_\alpha S_\mu - 8\eta \theta (\partial_\alpha \partial_\mu S^\mu) \psi^0 \psi^1 \psi^2 \psi^3 - \dot{x}_\mu (\partial_\alpha d^\mu) - \frac{2i}{3} \chi \psi_\mu (\partial_\alpha d^\mu) = 0 . \quad (4.57)$$

Now, in order to perform the non-relativistic limit we define the three-dimensional spin vector $\vec{\sigma}$ as [21]

$$\sigma_k = 2i\epsilon_{kjl}\psi^l\dot{\psi}^j, \quad \psi^j\psi^l = \frac{i}{4}\epsilon^{kjl}\sigma_k, \quad \dot{\psi}^j\psi^l = \frac{i}{4}\epsilon^{kjl}\dot{\sigma}_k \quad (4.58)$$

and consider

$$\psi^0 \approx 0, \quad x^0 \approx 1, \quad \dot{x}^i \approx v^i = \frac{dx^i}{dx^0} \quad (4.59)$$

as a part of the non-relativistic approximation. Furthermore, we use standard relations for the components of the stress tensor:

$$F_{0i} = -E_i = \partial_0 A_i - \partial_i A_0 \quad \text{and} \quad F_{ij} = \epsilon_{ijk} H^k.$$

Substituting these formulas into (4.57) and (4.56), and disregarding the terms of higher orders in the external fields, we arrive at the equations:

$$\begin{aligned} m\dot{\vec{v}} &= e\vec{E} + \frac{e}{c}[\vec{v} \times \vec{H}] - \eta\nabla(\vec{\sigma} \cdot \vec{S}) - \frac{\eta}{c} \frac{dS_0}{dt} \vec{\sigma} + \frac{\eta}{c}(\vec{v} \cdot \vec{\sigma})\nabla S_0 + \dots, \\ \dot{\vec{\sigma}} &= \left[\left(\frac{e}{mc} \vec{H} + \frac{2\eta}{\hbar} \vec{S} - \frac{2\eta S_0}{c\hbar} \vec{v} \right) \times \vec{\sigma} \right]. \end{aligned} \quad (4.60)$$

They coincide perfectly with the classical equations of motion (4.24), (4.25) obtained from the generalized Pauli equation. This correspondence confirms our interpretation of action (4.55). Additional arguments in favor of this interpretation were obtained in the framework of canonical quantization [84]. The quantization leads us back to the Dirac equation on torsion and electromagnetic background. Therefore, we have all reasons to consider (4.55) as the correct expression for the action of spin-1/2 particle in the external background of A_μ and S_μ fields.

4.5. Space-time trajectories for the spinning and spinless particles in an external torsion field

In this section, we shall consider several particular examples of motion of spinless and spinning particles in an external torsion field. Let us start from the scalar particle with action (4.36). For the sake of simplicity we do not consider electromagnetic field. Using the gauge condition $\pi = 0$, and replacing the solution for the auxiliary field θ back into the action, we cast it in the form

$$S = - \int_0^1 d\tau \sqrt{(m^2 + \xi S_\mu S^\mu) \dot{x}^2}. \quad (4.61)$$

It is obvious, already from action (4.36), that the role of the constant torsion axial vector $S_\mu = \text{const.}$ is just to change the value of the mass of the scalar particle $m^2 \rightarrow m^2 + \xi S^2$. Indeed, this is true only until the metric is flat. As far as we take a curved metric, the square $S_\mu S^\mu$ depends on it, even for a constant torsion. Let us suppose that $S_\mu S^\mu$ is coordinate dependent. Performing the variation over the coordinate x^α , after some algebra we arrive at the equation of motion

$$\ddot{x}_\alpha = \frac{\dot{x}^2}{2} \frac{\xi \partial_\alpha (S^\mu S_\mu)}{m^2 + \xi S^2}.$$

Taking into account that the square of the 4-velocity is constant, $\dot{x}^2 = 1$, we obtain

$$\ddot{x}_\alpha = \frac{1}{2} \partial_\alpha \ln \left| 1 + \frac{\xi S^2}{m^2} \right|. \quad (4.62)$$

Thus, in case of the non-constant $S_\mu S^\mu$ the motion of particle corresponds to the additional four-acceleration (4.62), and the torsion leads to the potential of the form

$$V_S = -\ln \sqrt{|m^2 + \xi S^2|/m^2}.$$

Indeed, this potential is constant for the scalar minimally coupled to torsion $\xi = 0$. However, as we have seen in the previous section, for any scalar which interacts with spinors, the non-minimal interaction $\xi \neq 0$ is nothing but a consistency condition. So, if torsion would really exist, and if its square is not constant, the force producing (4.62) should exist too. At the same time, the only scalar which is supposed to couple to the fermions through Yukawa interaction, is the Higgs. Since the Higgs mass is supposed to be, at least, of the order of 100 GeV, all experimental manifestations of the Higgs particle are possible only at the high-energy domain. Since torsion (if exists) is a very weak field, there are extremely small chances to observe torsion through acceleration (4.62).

Let us now discuss the spin 1/2 case. Here we follow, mainly, Ref. [158]. The physical degrees of freedom of the particle are its coordinate \vec{x} and its spin $\vec{\sigma}$. For the sake of simplicity, we shall concentrate on the non-relativistic case and consider the motion of a spinning particle in a space with constant axial torsion $S_\mu = (S_0, \vec{S})$, but without electromagnetic field. In this case the equations of motion (4.24), (4.25) have the form

$$\begin{aligned} \frac{d\vec{v}}{dt} &= -\eta \vec{S}(\vec{v} \cdot \vec{\sigma}) - \frac{\eta S_0}{c} \frac{d\vec{\sigma}}{dt}, \\ \frac{d\vec{\sigma}}{dt} &= +\frac{2\eta}{\hbar} [\vec{S} \times \vec{\sigma}] - \frac{2\eta S_0}{\hbar c} [\vec{v} \times \vec{\sigma}]. \end{aligned} \quad (4.63)$$

Consider first the case when $S_0 = 0$ so that only \vec{S} is present. Since $\vec{S} = \text{const.}$, we can safely put $S_{1,2} = 0$. The solution for the spin can be easily found to be

$$\sigma_3 = \sigma_{30} = \text{const.}, \quad \sigma_1 = \rho \cos\left(\frac{2\eta S_3 t}{\hbar}\right), \quad \sigma_2 = \rho \sin\left(\frac{2\eta S_3 t}{\hbar}\right), \quad (4.64)$$

where $\rho = \sqrt{\sigma_{10}^2 + \sigma_{20}^2}$. For the first two components of the velocity we have $v_1 = v_{10} = \text{const.}$, $v_2 = v_{20} = \text{const.}$, but the solution for v_3 turns out to be complicated. For $\sigma_3 \neq 0$ the solution is

$$\begin{aligned} v_3(t) &= \left[v_{30} + \frac{\rho \hbar (\sigma_3 v_{10} \hbar - 2m v_{20})}{4m^2 + \hbar^2 \sigma_3^2} \right] e^{-(\eta S_3 \sigma_3 / m)t} - \frac{\rho \hbar}{4m^2 + \hbar^2 \sigma_3^2} \\ &\quad \times \left[(\sigma_3 v_{10} \hbar - 2m v_{20}) \cos\left(\frac{2\eta S_3 t}{\hbar}\right) + (\sigma_3 v_{20} \hbar + 2m v_{10}) \sin\left(\frac{2\eta S_3 t}{\hbar}\right) \right]. \end{aligned} \quad (4.65)$$

Physically, such a solution means (i) precession of the spin around the direction of \vec{S} and (ii) oscillation of the particle velocity in this same direction accompanied (for $\sigma_3 \neq 0$) by the exponential damping of the initial velocity in this direction. We remark that the value of the

relic torsion field should be very weak so that very precise experiments will be necessary to measure these (probably extremely slow) precession, oscillation and damping.

Consider another special case $\vec{S} = 0$ and $S_0 \neq 0$, which is the form of the torsion field motivated by the isotropic cosmological model (3.79). The equations of motion have a form:¹⁴

$$\begin{aligned}\frac{d\vec{v}}{dt} &= \frac{2\eta^2 S_0^2}{c\hbar} [\vec{v} \times \vec{\sigma}], \\ \frac{d\vec{\sigma}}{dt} &= -\frac{2\eta S_0}{\hbar} [\vec{v} \times \vec{\sigma}].\end{aligned}\quad (4.66)$$

In order to analyze these equations we note, that the squares \vec{v}^2 , $\vec{\sigma}^2$ and the product $(\vec{v} \cdot \vec{\sigma})$ are integrals of motion. Consequently, the magnitudes of the two vectors, and the angle between them do not change during the motion. Another obvious integral of motion is the linear combination

$$\vec{w} = \frac{2\eta S_0}{\hbar} \left(\vec{v} + \frac{\eta S_0}{c} \vec{\sigma} \right). \quad (4.67)$$

Therefore, the evolution of \vec{v} and $\vec{\sigma}$ performs such that the plane of two vectors is rotating around the constant vector \vec{w} . By elementary means one can check that the frequency of this rotation is exactly $w = |\vec{w}|$, so that the period is $T = 2\pi/w$. In this case, the magnitude and direction of the precession of spin and acceleration of the particle depends on the magnitude and mutual orientation of its spin and velocity. It is interesting that in the case $[\vec{\sigma}, \vec{v}] = 0$ both spin and velocity are constants, but as far as $\vec{\sigma}$ and \vec{v} are not exactly parallel, the period of precession depends only on the magnitude of the vector \vec{w} . In other words, both vectors may be infinitesimally non-parallel, and the frequency of the precession will not be infinitesimal (but the amplitude of the precession will). The last observation is that, for the gas of particles with random orientation of velocities, their precession in the S_0 field would be also random. We note that the possibility of the accelerating motion of the spinning particles in an external torsion field has been discussed also in [193,169,99].

The torsion field is supposed to act on the spin of particles but not on their angular momentum [170,193]. Therefore, a motion like the one described above will occur for individual electrons or other particles with spin as well as for macroscopic bodies with fixed spin orientation but it does not occur for the (charged or neutral) bodies with random orientations of spins.

4.6. Experimental constraints for the constant background torsion

In the previous and present sections we are considering the approach in which torsion is purely background field. Thus, we shall describe only those possible effects which do not need propagating torsion. A brief discussion of the possible experimental manifestations of the propagating torsion will be given in the next section. In principle, the background torsion may produce two kinds of effects: the change of the trajectory for the particles with (or even without—for the case of Higgs particle) spin, or the change of the spectrum due to the torsion-dependent terms

¹⁴ The analysis of Eq. (4.66) in our paper [158] was wrong. I am very grateful to Luiz Garcia de Andrade who found this mistake and noticed me about it.

in the Dirac equation. The possible experiments with the motion of particles are quite obvious. Let the electromagnetic field to be absent. Then, according to the results of the previous section, the interaction with torsion twists the particle trajectory. Then, any charged particles may be the source of electromagnetic radiation. The structure of the radiation provides the opportunity to look for torsion effects. However, for a very feeble torsion, the electromagnetic radiation (as a second order effect) will be very weak and this way of detecting torsion is not really promising.

Let us now comment on the spectroscopy part, using the non-relativistic approximation. We shall follow the consideration of [13]. Consider the Schroedinger equation (4.6) with the Hamiltonian operator (4.7) without electromagnetic field. It is evident that the effect of a torsion field can modify the particles spectrum. Some modifications are similar to the ones which arise in the electromagnetic field. At the same time, another modifications are possible due to the qualitatively new term $(\eta/mc)S_0 \vec{p} \cdot \vec{\sigma}$ in (4.7).

It is natural to suppose that torsion is feeble enough and therefore one can consider it as a perturbation. This perturbation might lead to the splitting of the known spectral lines and hence one can, in principle, derive an upper bound for the background torsion using the spectral analysis experiments.¹⁵

One can expect the splitting of the spectral lines for the hydrogen atom (such a splitting has been also discussed in [82] for torsion coupled to massive electrodynamics). Consider the constant torsion $S_\mu = \text{const.}$ and estimate possible modifications of the spectrum. In this particular case the Hamiltonian operator is

$$\hat{H} = \frac{1}{2m} \hat{\pi}^2 + \eta \hat{S} \cdot \hat{\sigma} - \frac{\eta}{2mc} (\hat{S}_0 \hat{p} + \hat{p} \hat{S}_0) \cdot \hat{\sigma},$$

where

$$\vec{\pi} = \vec{p} - \frac{e}{c} \vec{A}.$$

In the framework of the non-relativistic approximation $|\vec{p}| \ll mc$ and hence the second S_0 dependent term in the brackets can be omitted. The remaining term $\eta \vec{S} \cdot \vec{\sigma}$ admits the standard interpretation and gives the contribution $\pm \eta S_3$ into the spectrum. Thus, if the S_3 component of the torsion tensor is not equal to zero, the energy level is splitted into two sub-levels with the difference $2\eta S_3$. If now, the weak transversal magnetic field is switched on then the cross between the new levels will arise and the energy absorption takes place at the magnetic field frequency $\omega = 2\eta/S_3$. Note that the situation is typical for the magnetic resonance experiments, however in the present case the effect arises due to the torsion, but not to the magnetic field effects.

There were several attempts to draw numerical bounds on the background torsion using known experiments and the torsion corrections to the Schrödinger or Dirac equation. One can distinguish two approaches. One of them is more traditional, it does not really distinguish between purely background and propagating torsion. It is sufficient to suppose that the torsion mass is dominating over the possible kinetic terms. In this case the effect of torsion is to

¹⁵ Indeed, torsion effects will compete with the relativistic corrections and with the fine structure effects coming from QED. Therefore, this our consideration has mainly pedagogical purpose. At the end of the section we shall briefly present the modern limits on torsion coming from the complete studies.

provide the contact spin–spin interactions. As an examples of works done in this direction one can mention [99] (one can find there more references) and [95], where the Pauli-like equation (4.6), (4.8) has been applied together with the Einstein–Cartan action for torsion. The most recent and complete upper bound for torsion parameters from the contact interaction have been obtained in [18]. We shall present the corresponding results in the next section after discussing the problem of torsion mass and propagating torsion.

An alternative way is to suppose the existence, in our part of the Universe, of some constant torsion axial vector S_μ , and to look for its possible manifestations. The most recent publication with the analysis of this possibility and the derivation of the corresponding upper bound for torsion is [120], where the constraints on the space–time torsion were obtained from the data on the Hughes–Drever experiments on the basis of the Pauli-like equation. The limit on the magnitude of the space component of the antisymmetric contorsion has been obtained in [120] using the experimental data concerning violation of Lorentz invariance [50].

Besides these papers devoted to the search of the torsion effects, there were, in the last decades, numerous publications on the same subject, but without explicit mentioning the word “torsion”. These works were devoted to the search for the Lorentz and CPT violations coming from various odd terms in the Dirac equation. The most popular form (there are others) of such an insertion is the b_μ axial vector. The modified form of the Dirac equation is

$$(i\gamma^\mu D_\mu - \gamma_5 \gamma^\mu b_\mu - m)\psi = 0, \quad (4.68)$$

where $D_\mu = \partial_\mu - iA_\mu$. It is easy to see that the b_μ is nothing but the normalized axial vector of torsion $b_\mu = \eta S_\mu$ and, fortunately, we have the possibility to use the corresponding data on the limits coming from the CPT and Lorentz anomalies. There is no reason to repeat the details of the existing numerous reviews on the subject (see, for example, [25,118] and references therein), so we shall just present the main results. The violation of the Lorentz and CPT symmetries occurs because b_μ is a constant vector with the fixed space component. Consequently, any Lorentz boost breaks the form of the Dirac equation. The limits on the magnitude of the b_μ fields come from the studies of neutral-meson oscillations in the kaon system, experimental test with leptons and barions using Penning traps, comparative spectroscopy of the hydrogen and antihydrogen atoms, measurements of muon properties, clock-comparison Hughes–Drever type experiments, observation of the anomaly in the behavior of the spin-polarized torsion pendulum and tests with the spin-polarized solids [26]. The overall limits on $|b|$ differ and depend on the type of experiment. In particular, these limits are different for different fermions. These limits are typically from 10^{-25} to 10^{-30} GeV, so that the universal phenomenological bound, valid for all fermion species, is between 10^{-27} and 10^{-30} GeV. If we really associate b_μ vector with torsion, and remember the renormalization-group based arguments (see Section 3.4) about the universality of the fermion–torsion interaction, the estimates for different fermions can be put together and we arrive at the total universal limit $|b| < 10^{-30}$ GeV. Thus, the limits derived from numerous laboratory experiments, are very small. They leave no real chance to use torsion for the explanation of physical phenomena [65] like the anomaly in the polarization of light coming from distant galaxies [138]. The same concerns the creation of particles by external torsion field [157] and the helicity flip for the solar neutrino which could be, in principle, induced by torsion [96].

5. The effective quantum field theory approach for the dynamical torsion

The theoretical description of any new field, including torsion, must have two important elements: the interaction of this field with the well established matter fields and the proper dynamics of the new field. From Quantum Field Theory point of view, any classical description may be considered as an approximation to some complete theory including quantum effects. Following this line, one has to construct the theory of torsion in such a way that it would pass the necessary test of consistency as a quantum theory. As we have seen in the previous two sections, the interaction of torsion with matter does not lead to any difficulty, until we consider torsion as a purely background field. However, this semi-classical theory is definitely incomplete if we do not attempt to formulate torsion dynamics. The simplest approach is just to postulate the Einstein–Cartan theory (2.18) as a torsion action. As we have already learned in Section 2, in this case torsion does not propagate and leads only to the contact spin–spin interactions. Furthermore, since the torsion mass is of the Planck order of magnitude, such a contact interaction is suppressed, at low energies, by the Planck mass. As a result there are very small chances to observe torsion at low energies. The main purpose of the present section is to follow the recent papers [18,22] where we have discussed an alternative possibility for a smaller torsion mass. We start the section by making a short account of the previous works on the dynamical torsion, and then proceed by applying the ideas of effective quantum field theory to the formulation of torsion dynamics. We shall mainly concentrate on the theoretical aspects, and provide only a short review of the phenomenological bounds on the torsion mass and couplings. The interested reader is referred to the second paper in Ref. [18] for further phenomenological details.

5.1. Early works on the quantum gravity with torsion

Since the early days (see [99] for a review) torsion has been considered as an object related to quantum theory. Thus, it is natural to discuss propagating torsion in terms of Feynman diagrams, Green's functions and S -matrix instead of using the dynamical equations.

As any other propagating field, torsion must satisfy the condition of unitarity. So, it is natural that the attempts to construct the theory of the propagating torsion started from the study of the constraints imposed by the unitarity [132,133,161,162,134]. The initial motivation of [132,133,161,162] was to construct the theory of quantum gravity which would be both unitary and renormalizable. So, let us briefly describe the general situation in Quantum Gravity (see [163] for more extensive review). It is well known that the program of quantizing General Relativity met serious difficulty, because this quantum theory is non-renormalizable by power counting. If taking only the superficial logarithmic divergences of the diagrams into account, the dimension (number of derivatives of the metric) of the n -loop counterterms is $d = 2 + 2n$. Then, with every new order of the loop expansion the dimension of the counterterms grows up and hence one needs an infinite number of the renormalization conditions to extract a single prediction of the theory in the high-energy region. The non-renormalizability of quantum General Relativity becomes apparent already at the one-loop level for the case of gravity coupled to matter [107,61] and at two-loop level for the pure gravity [88]. The situation in supergravity (see e.g. [83]) is better in the sense that the on-shell divergences do not show up at second

($N = 1$ case) or even higher (perhaps seventh for $N = 8$ supergravity) loops. However, the supersymmetry does not solve the principal problem, and all known versions of supergravity generalizations of General Relativity are expected to be non-renormalizable.

At the same time, it is fairly simple to construct renormalizable theory of the gravitational field by adding the fourth-derivative terms [175]

$$S_{\text{HD}} = \int d^4x \sqrt{-g} (\alpha R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \beta R_{\mu\nu} R^{\mu\nu} + \gamma R^2 + \delta \square R) \quad (5.1)$$

into the classical action. It is better to write the above expression in another basis, so that the algebraic properties of the terms become more explicit. Let us use (3.7), so that Eq. (5.1) can be written as

$$S_{\text{HD}} = \int d^4x \sqrt{-g} (a_1 C^2 + a_2 E + a_3 R^2 + a_4 \square R). \quad (5.2)$$

It is well known (see e.g. [175,34]), that the contributions of the terms of Eq. (5.2) to the propagator of the gravitational perturbations are very different. This propagator can be divided into irreducible parts through introducing the projectors to spin-2, spin-1 and spin-0 states. It turns out that spin-1 states can be completely removed by the gauge fixing (of course, the Faddeev–Popov ghosts must be taken into account). Furthermore, the C^2 term contributes to the gauge fixing independent spin-2 part of the propagator, while the R^2 -term contributes only to the spin-0 part and does not contribute to the spin-2 part. The term $\int E$ does not contribute to the propagator at all, even if we change the dimension of the space–time from 4 to n . In $n = 4$ this term is topological, so it is supposed not to affect the vertices either (algebraically, the situation is not so simple [43], but there are no indications of the non-trivial quantum effect of this term). At the same time, for $n \neq 4$ all three terms contribute to all vertices: to the interactions of the metric components of all spins.

Now, suppose we take a theory with the action

$$S_t = \int d^4x \sqrt{-g} \left[-\frac{1}{\kappa^2} R + a_1 C^2 + a_2 E + a_3 R^2 \right] \quad (5.3)$$

with $a_1 \neq 0$ and $a_3 \neq 0$. In this case the propagators of all components, including ghosts [34], behave like k^{-4} in the high-energy domain. Similarly, there are vertices proportional to the fourth, second and zero power of the momenta. In this situation, the power counting shows that the superficial degree of divergence of the IPI diagrams does not depend on the loop order, and the maximal possible dimension of the logarithmic counterterms is 4. Taking the locality and general covariance of the counterterms [175,184] into account, we see that they have the same form as the classical action (5.3). Thus, the theory is renormalizable.

Still, it is not perfect. Spin-2 sector of the propagator has the form

$$G^{(2)}(k) \sim \frac{1}{k^2(k^2 + m_2^2)} = \frac{1}{m_2^2} \left[\frac{1}{k^2} - \frac{1}{k^2 + m_2^2} \right], \quad (5.4)$$

where $m_2 \sim 1/\kappa^2$ is the mass of the non-physical ghost. Another massive pole exists in spin-0 sector, but its position depends on the gauge fixing while spin-2 part (5.4) is gauge independent. Excluding the non-physical particles from the physical spectrum, one breaks unitarity [175]. The situation described above is quite general, because further modifications of the action

do not change the situation. For instance, introducing more higher derivative terms into the action cannot provide unitarity [9]. It is quite obvious that the situation cannot be improved by changing the dynamical variables, because this operation cannot change the position of the massive gauge independent pole of the propagator.

Some observation is in order. If we put, in action (5.3), the coefficient $a_1 = 0$, the theory will not be renormalizable (despite it still possesses higher derivatives). The point is that such a theory contains high derivative vertices of the interaction of spin-2 states with themselves and with other states, but spin-2 part of the propagator behaves like $G^{(2)}(k) \sim k^{-2}$. It is easy to see that the power counting in this theory would be even worse than in quantum General Relativity. Thus, the renormalizability of quantum gravity theory with high derivatives crucially depends on the k^{-4} UV behavior of spin-2 propagator. In turn, since spin-2 sector always contains a k^{-2} part due to the Einstein term, the (5.4) structure of the propagator looks as an unavoidable consequence of the renormalizability. Of course, these considerations do not have absolute sense, one can try to look for an action which could solve the problem of quantum gravity.

The original idea of [132] was to use the first order formalism and to establish the combination of the R^2 -terms which could preserve the renormalizability and, simultaneously, provide the absence of the massive unphysical pole. As it was already explained in Section 2.5, the first order formalism treats the affine connection as an object independent on the metric. Anyhow, the connection can always be presented as a sum of the Christoffel symbol and some additional tensor which includes torsion and non-metricity. Furthermore, if one is interested in the propagator of perturbations on the flat background, it is possible to classify all the fields by the representations of the Lorentz group. Indeed, the non-metricity tensor and $q^\alpha_{\beta\gamma}$ -component of torsion both contain spin-2 states. At least one of these states should be massless in order to provide the correct Newtonian limit. Then, if there are other, massive spin-2 states, the general structure of the complete spin-2 propagator should be equivalent to (5.4). Of course, the word “equivalent” in the last statement cannot be understood as an identity. For instance, in the first order formalism the propagator may be free of high derivatives at all. The equivalence signifies that the r.h.s. of (5.4) will be restored, in the linearized theory, after elimination of an extra (one can call them auxiliary) fields.

The first attempt to find the renormalizable and ghost-free theory of Ref. [132] did not include torsion, which was implemented later in Refs. [133,161,162,134]. In all these papers, the gravitational field has been parametrized by the vierbein e^a_μ and spinor connection w^μ_{ab} , but as we have discussed in Section 2, these variables are equivalent to another ones— $(g_{\mu\nu}, T^\alpha_{\beta\gamma})$: if one does not introduce a non-metricity. In principle, one could include, as we mentioned above, all curvature and torsion depending terms (there are 168 of them [48]) plus all possible terms depending on the non-metricity. The technical part of the original papers [132,133,161,162] is quite cumbersome and we will not reproduce it here. The final result has been achieved for the theories with torsion but without non-metricity. The number of ghost-free R^2 -type actions has been formulated, but all of them are non-renormalizable. Obviously, these actions resemble (5.3) with $a_1 = 0$, for they possess high derivatives in the vertices but not in the propagators. If one performs the loop calculation in such theory, the ghost-free structure of the propagator will be immediately destroyed. Therefore, at the quantum level the unitarity of the classical S -matrix can be spoiled, if the theory is not renormalizable. The conclusion is that, for the consistent quantum theory, one needs both unitarity and renormalizability.

In the next sections we will not focus on the problem of quantum gravity. Instead, we shall concentrate on the possibility to have a theory of the propagating torsion, which should be consistent at the quantum level. The application of the ideas of the effective field theories enables one to weaken the requirement of renormalizability (see e.g. [187,69]), but even in this case we shall meet serious problems and limitations in constructing the theory of the propagating torsion. As to the formulation of consistent quantum gravity, the string theory is, nowadays, the only one visible candidate.

5.2. General note about the effective approach to torsion

In order to construct the action of a propagating torsion, we have to apply two requirements: unitarity and renormalizability. The problem of renormalizability is casted in another form if we consider it in the content of effective field theory [187,69]. In the framework of this approach one has to start with the action which includes all possible terms satisfying the symmetries of the theory. Usually, such an action contains higher derivatives at least in a vertices. However, as far as one is interested in the lower energy effects, those high derivative vertices are suppressed by the great massive parameter which should be introduced for this purpose. Then, those vertices and their renormalization are not visible and effectively at low energies one meets renormalizable and unitary theory. The gauge invariance of all the divergences is guaranteed by the corresponding theorems [182] and thus this scheme may be applied to the gauge theories including even gravity [186,68]. Within this approach, it is important that the lower-derivative counterterms have the same form as the terms included into the action. This condition, together with the symmetries and the requirement of unitarity, may help to construct the effective field theories for the new interactions such as torsion.

If one starts to formulate the dynamical theory for torsion in this framework, the sequence of steps is quite definite. First, one has to establish the field content of the dynamical torsion theory and the form of the interactions between torsion and other fields. Then, it is necessary to take into account the symmetries and formulate the action in such a way that the resulting theory is unitary and renormalizable as an effective field theory. Indeed, there is no guarantee that all these requirements are consistent with each other, but the inconsistency might indicate that some symmetries are lost or that the theory with the given particle content is impossible. In the next sections we consider how this scheme works for torsion, and then compare the situation with that of effective low-energy quantum gravity (see e.g. [68]).

Among the torsion components (2.16) S_μ , T_μ , $q^\alpha_{\beta\gamma}$, only S_μ is really important, because only this ingredient of torsion couples to spinors in a minimal way. Therefore, in what follows, we shall restrict the consideration to the axial vector S_μ which parameterizes the completely antisymmetric torsion.

5.3. Torsion–fermion interaction again: softly broken symmetry associated with torsion and the unique possibility for the low-energy torsion action

In this section, we consider the torsion–spinor system without scalar fields. Thus, we start from the action (2.35) of the Dirac spinor non-minimally coupled to the vector and

torsion fields

$$S_{1/2} = i \int d^4x \bar{\psi} [\gamma^\alpha (\partial_\alpha + ieA_\alpha + i\eta\gamma_5 S_\alpha) - im] \psi.$$

First one has to establish its symmetries. At this stage we consider the vector field A_μ as an abelian one but later we will focus on the vector fields of the SM which are non-abelian. The symmetries include the usual gauge transformation (2.36), and also softly broken symmetry (2.37):

$$\psi' = \psi e^{i\gamma_5 \beta(x)}, \quad \bar{\psi}' = \bar{\psi} e^{i\gamma_5 \beta(x)}, \quad S'_\mu = S_\mu - \eta^{-1} \partial_\mu \beta(x).$$

The massive term is not invariant under the last transformation.

The symmetries of the theory have serious impact on the renormalization structure. In particular, since the symmetry under (2.37) is softly broken, it does not forbid massive counterterms in the torsion sector and hence S_μ has to be a massive field. Below we consider the torsion mass as a free parameter which should be defined on a theoretical basis, and may be also subject of experimental restrictions.

As far as torsion is taken as a dynamical field, one has to incorporate it into the SM along with other vector fields. Let us discuss the form of the torsion action in the framework of the effective approach—that is focusing on the low-energy effects. The higher derivative terms may be included into the action, but they are not seen at low energies. Thus, we restrict the torsion action by the lower-derivative terms and arrive at the expression:

$$S_{\text{tor}} = \int d^4x \{ -a S_{\mu\nu} S^{\mu\nu} + b (\partial_\mu S^\mu)^2 + M_{ts}^2 S_\mu S^\mu \}, \quad (5.5)$$

where $S_{\mu\nu} = \partial_\mu S_\nu - \partial_\nu S_\mu$ and a, b are some positive parameters. Action (5.5) contains both transverse vector mode and the longitudinal vector mode where the last one is equivalent to the scalar [45]. In particular, in the $a = 0$ case only the scalar mode, and for $b = 0$ only the vector mode propagates. It is well known [74] (see also [45] for the discussion of the theory (5.5)) that in the unitary theory of the vector field the longitudinal and transverse modes cannot propagate simultaneously,¹⁶ and therefore one has to choose either a or b to be zero.

In fact the only correct choice is $b = 0$. In order to see this one has to reveal that symmetry (2.37), which is spoiled by the massive terms only, is preserved in the renormalization of the dimensionless coupling constants of the theory (at least at the one-loop level). In other words, the divergences and corresponding local counterterms, which produce the dimensionless renormalization constants, do not depend on the dimensional parameters such as the masses of the fields. This structure of renormalization resembles the one in the Yang–Mills theories with spontaneous symmetry breaking. As we shall see later, even the $b = 0$ choice is not free of problems, but at least they are not related to the leading one-loop effects, as in the opposite $a = 0$ choice. Thus, the only one possible torsion action is given by Eq. (5.26) with $b = 0$. In order to illustrate this, we remind that the divergences coming from fermion loop are given

¹⁶ One can easily check this without even looking into the textbooks: the kinetic terms for the transverse vector and for the “longitudinal” scalar, coming from the $a(\partial_\mu S^\mu)^2$ -term, have opposite signs, while both fields share massive term. As a result one of these fields must have, depending on the signs of a, b, M_{ts}^2 , either negative mass or negative kinetic energy. Hence, the theory with both components always includes either ghost or tachyon.

by expression (3.15), which is in a perfect agreement with transformation (2.37). Namely, the one-loop divergences contain $S_{\mu\nu}^2$ and the massive term while the $(\partial_\nu S^\nu)^2$ term is absent.

It is well known that the fermion loop gives rise, in theory (2.35), to axial anomaly. But, as we have discussed in Section 3.8, the problem of anomaly does not spoil our attempts to implement torsion into the fermion sector of the SM. And so, the only possible form of the torsion action which can be coupled to the spinor field (2.35) is

$$S_{\text{tor}} = \int d^4x \left\{ -\frac{1}{4} S_{\mu\nu} S^{\mu\nu} + M_{ts}^2 S_\mu S^\mu \right\}. \quad (5.6)$$

In the last expression we put the conventional coefficient $-1/4$ in front of the kinetic term. With respect to the renormalization this means that we (in a direct analogy with QED) can remove the kinetic counterterm by the renormalization of the field S_μ and then renormalize the parameter η in action (2.35) such that the combination ηS_μ is the same for the bare and renormalized quantities. Instead one can include $1/\eta^2$ into the kinetic term of (5.6), that should lead to the direct renormalization of this parameter while the interaction of torsion with spinor has minimal form (2.33) and S_μ is not renormalized. Therefore, in the case of a propagating torsion the difference between minimal and non-minimal types of interactions is only a question of notations on both classical and quantum levels.

5.4. Brief review of the possible torsion effects in high-energy physics

As we have already seen, spinor–torsion interactions enter the Standard Model as interactions of fermions with a new axial vector field S_μ . Such an interaction is characterized by the new dimensionless parameter—coupling constant η . Furthermore, the mass of the torsion field M_{ts} is unknown, and its value is of crucial importance for the possible experimental manifestations of the propagating torsion and finally for the existence of torsion at all (see the discussion in the last part of this section and in Section 6). In the present section we consider η and M_{ts} as arbitrary parameters and review the limits on their values from the known experiments [18]. Later on we shall see that the consistency of the fermion–torsion system can be achieved for the heavy torsion only, such that $M_{ts} \gg M_{\text{fermion}}$. However, we shall follow [18] where one can find the discussion of the “light” torsion with the mass of the order of 1 GeV.

The strategy of [18] was to use known experiments directed to the search of the new interactions. One can regard torsion as one of those interactions and obtain the limits for the torsion parameters from the data which already fit with the phenomenology.

Torsion, being a pseudovector particle interacting with fermions, might change different physical observables. For instance, this specific type of interaction might lead to the forward–backward asymmetry. The last has been precisely measured at the LEP e^+e^- collider, so the upper bounds for torsion parameters may be set from those measurements. One can consider two different cases: (i) torsion is much heavier than other particles of the SM and (ii) torsion has a mass comparable to that of other particles. In the last case one meets a propagating particle which must be treated on an equal footing with other constituents of the SM. Contrary to that, the very heavy torsion leads to the effective contact four-fermion interactions. Let us briefly review all mentioned possibilities.

(i) *Forward–backward asymmetry*: Any parity violating interactions eventually give rise to the space asymmetry and could be revealed in the forward–backward asymmetry of the particle scattering. Axial-vector type interactions of torsion with matter fields is this case of interactions. But the source of asymmetry also exists in the SM electroweak interactions because of the presence of the $\gamma_\mu \gamma_5$ structure in the interactions of Z - and W -bosons with fermions. The interactions between Z -boson and fermions can be written in general form as

$$L_{Zff} = -\frac{g}{2 \cos \theta_W} \sum_i \bar{\psi}_i \gamma^\mu (g_V^i - g_A^i \gamma^5) \psi Z_\mu, \quad (5.7)$$

where, θ_W is Weinberg angle, $g = e/\sin \theta_W$ (e —positron charge); and the vector and axial couplings are:

$$g_V^i \equiv t_{3L}(i) - 2q_i \sin^2 \theta_W, \quad (5.8)$$

$$g_A^i \equiv t_{3L}(i). \quad (5.9)$$

Here t_{3L} is the weak isospin of the fermion and i has the values $+1/2$ for u_i and v_i while it is $-1/2$ for d_i and e_i . Here $i = 1, 2, 3$ is the index of the fermion generation and q_i is the charge of the ψ_i in units of charge of positron.

The forward–backward asymmetry for $e^+e^- \rightarrow l^+l^-$ is defined as

$$A_{FB} \equiv \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B}, \quad (5.10)$$

where $\sigma_F(\sigma_B)$ is the cross section for l^- to travel forward(backward) with respect to electron direction. Such an asymmetries are measured at LEP [125].

In the SM, from asymmetries one derives the ratio g_V/g_A of vector and axial-vector couplings, but the presence of torsion could change the result. In fact, the measured electroweak parameters are in a good agreement with the theoretical predictions and hence one can establish the limits on the torsion parameters based on the experimental errors. The contribution from torsion exchange diagrams has been calculated in [18]. From those calculations one can establish the limits on η and M_{ts} taking into account the mentioned error of the experimental measurements. Deviations of the asymmetry from SM predictions would be an indication of the presence of the additional torsion-like type axial-vector interactions. The analysis of [18] shows that the electron A_{FB}^e asymmetry is the best observable among others asymmetries to look for torsion. The details of the corresponding analysis can be easily found in [18] and we will not reproduce them here.

(ii) *Contact interactions*: Since the torsion mass comparable to the mass of the fermions leads to problems (which will be explained in the following sections), it is especially important for us to consider the case of heavy torsion. Since the massive term dominates over the covariant kinetic part of the action, the last can be disregarded. Then the total action leads to the algebraic equation of motion for S_μ . The solution of this equation can be substituted back into the action and thus produce the contact four-fermion interaction term

$$\mathcal{L}_{\text{int}} = -\frac{\eta^2}{M_{ts}^2} (\bar{\psi} \gamma_5 \gamma^\mu \psi) (\bar{\psi} \gamma_5 \gamma_\mu \psi). \quad (5.11)$$

As one can see the only quantity which appears in this approach is the ratio M_{ts}/η and therefore for the very heavy torsion field the phenomenological consequences depend only on

this single parameter. As it was mentioned above, the axial-vector type interactions would give rise to the forward–backward asymmetry which have been precisely measured in the $e^+e^- \rightarrow l^+l^-(q\bar{q})$ scattering (here $l = (\tau, \mu, e)$ stands for the leptons and q for quarks) at LEP collider with the center-mass energy near the Z-pole. Due to the resonance production of Z-bosons the statistics is good (several million events) and it allowed to measure electroweak (EW) parameters with high precision. There are several experiments from which the constraints on the contact four-fermion interactions come [18]:

- (1) Experiments on polarized electron–nucleus scattering: SLAC e-D scattering experiment [152], Mainz e-Be scattering experiment [98] and bates e-C scattering experiment [173].
- (2) Atomic physics parity violations measures [122] electron–quark coupling that are different from those tested at high-energy experiment provides alternative constraints on new physics (see also Section 4.6).
- (3) e^+e^- experiments—SLD, LEP1, LEP1.5 and LEP2 (see for example [124,145,126,5,121]).
- (4) Neutrino–Nucleon DIS experiments—CCFR collaboration obtained a model independent constraint on the effective $\nu\nu qq$ coupling [129].

Consider the limits on the contact interactions induced by the torsion. The contact four-fermion interaction may be described by the Lagrangian [72] of the most general form:

$$L_{\psi'\psi\psi\psi} = g^2 \sum_{i,j=L,R} \sum_{q=u,d} \frac{\epsilon_{ij}}{(A_{ij}^\epsilon)^2} (\bar{\psi}'_i \gamma_\mu \psi'_i) (\bar{\psi}_j \gamma^\mu \psi_j) . \quad (5.12)$$

Subscripts i, j refer to different fermion helicities: $\psi_i^{(\prime)} = \psi_{R,L}^{(\prime)} = (1 \pm \gamma_5)/2 \cdot \psi^{(\prime)}$; where $\psi^{(\prime)}$ could be quark or lepton; A_{ij} represents the mass scale of the exchanged new particle; coupling strength is fixed by the relation: $g^2/4\pi = 1$, the sign factor $\epsilon_{ij} = \pm 1$ allows for either constructive or destructive interference with the SM γ and Z-boson exchange amplitudes. Formula (5.12) can be successfully used for the study of the torsion-induced contact interactions because it includes an axial–axial current interaction (5.11) as a particular case.

Recently, the global study of the electron–electron–quark–quark ($eeqq$) interaction sector of the SM have been done using data from all mentioned experiments [15]. For the axial–axial $eeqq$ interactions (5.12) takes the form (we put $g^2 = 4\pi$):

$$L_{eeqq} = -\frac{4\pi}{(A_{AA}^\epsilon)^2} (\bar{e} \gamma_\mu \gamma_5 e) (\bar{q} \gamma^\mu \gamma_5 q) . \quad (5.13)$$

The limit for the contact axial–axial $eeqq$ interactions comes from the global analysis of Ref. [15]:

$$\frac{4\pi}{A_{AA}^2} < 0.36 \text{ TeV}^{-2} . \quad (5.14)$$

Comparing the parameters of the effective contact four-fermion interactions of general form (5.13) and contact four-fermion interactions induced by torsion (5.11) we arrive at the following relations:

$$\frac{\eta^2}{M_{ts}^2} = \frac{4\pi}{A_{AA}^2} . \quad (5.15)$$

From (5.14) and (5.15) one gets the following limit on torsion parameters:

$$\frac{\eta}{M_{ts}} < 0.6 \text{ TeV}^{-1} \Rightarrow M_{ts} > 1.7 \text{ TeV} \cdot \eta. \quad (5.16)$$

The last relation puts rigid phenomenological constraints on the torsion parameters η, M_{ts} , and one can also take into account that the modern scattering-based analysis cannot be relevant for the masses beyond the 3 TeV. It is easy to see that bound (5.16) is incompatible with the one established in [95], because in this paper the torsion mass has been taken to be of the Planck order of magnitude.

An additional restriction can be obtained from the analysis of the TEVATRON data but since they concern mainly the light torsion, we will not give the details here. In [18], one can find the total limits on torsion from an extensive variety of experiments (see also [47] for the consequent analysis concerning the torsion coming from small extra dimensions).

In conclusion, we learned that torsion, if exists, may produce some visible effects, while the phenomenological analysis put some limits on the torsion parameters η, M_{ts} . Of course, these limits will improve if the experimental data and(or) the theoretical derivation of observables in the SM become more precise. In case one detects some violation of the phenomenological bounds, one can suppose that this is a manifestation of some new physics. This new physics can be torsion or something else (GUT, supersymmetry, higher dimensions and so on). In each case one has to provide the consistent quantum theory for the corresponding phenomena. Therefore, the relevance of phenomenological considerations always depends on the formal field-theoretical investigation. As we shall see in the next sections, in the case of torsion the demands of the theory are more restrictive than the phenomenological limits.

5.5. First test of consistency: loops in the fermion–scalar systems break unitarity

To this point, we have considered only interaction between torsion and spinors. Now, in order to implement torsion into the SM, one has to include scalar and Yukawa interactions. When introducing scalar field we shall follow the same line as in the previous section and try first to construct the renormalizable theory. Hence, the first thing to do is to analyze the structure of the possible divergences. The divergent diagrams in the theory with a dynamical torsion include, in particular, all such diagrams with external lines of torsion and internal lines of other fields. Those grafts are indeed the same one meets in quantum field theory on a classical torsion background. Therefore, one has to include into the action all terms which were necessary for the renormalizability when torsion was a purely external field. All such terms are already known from our investigation of quantum field theory on an external torsion background. Besides the non-minimal interaction with spinors, one has to introduce the non-minimal interaction $\varphi^2 S^2$ between scalar field and torsion as in (2.26) and also the terms which played the role of the action of vacuum (see, e.g., (3.6) or (3.68)) in the form

$$S_{\text{tor}} = \int d^4x \left\{ -\frac{1}{4} S_{\mu\nu} S^{\mu\nu} + M_{ts}^2 S_\mu S^\mu - \frac{1}{24} \zeta (S_\mu S^\mu)^2 \right\} + \text{surface terms}. \quad (5.17)$$



Fig. 3. Two-loop diagrams corresponding to torsion self-interaction and torsion–scalar interaction.

Here ζ is some new parameter, and the coefficient $1/24$ stands for the sake of convenience only. The necessity of the $(S_\mu S^\mu)^2$ term in the classical action follows from the fact that such a term emerges from the scalar loop with a divergent coefficient. Then, if not included into the classical action, it will appear with infinite coefficient as a quantum correction. On the other hand, by introducing this term into the classical action we gain the possibility to remove the corresponding divergence renormalizing the coupling ζ .

So, if one implements torsion into the complete SM including the scalar field, the total action includes the following new terms: torsion action (5.17) with the self-interacting term, and non-minimal interactions between torsion and spinors (2.35) and scalars (2.26). However, at the quantum level, such a theory suffers from a serious difficulty.

The root of the problem is that the Yukawa interaction term $h\phi\bar{\psi}\psi$ is not invariant under transformation (2.37). Unlike the spinor mass, the Yukawa constant h is massless, and this non-invariance may affect the renormalization in the massless sector of the theory. In particular, the non-invariance of the Yukawa interaction causes the necessity of the non-minimal scalar–torsion interaction in (2.26) which, in turn, requires an introduction of the self-interaction term in (5.17). Those terms do not pose any problem at the one-loop level, but already at the second loop one meets two dangerous diagrams presented at Fig. 3.

These diagrams are divergent and they can lead to the appearance of the $(\partial_\mu S^\mu)^2$ -type counterterm. No symmetry is seen which forbids these divergences. Let us consider, following [18], the diagrams presented at Fig. 3 in more details. Using the actions of the scalar field coupled to torsion (2.26) and torsion self-interaction (5.17), we arrive at the following Feynman rules:

- (i) scalar propagator: $G(k) = i/(k^2 + M^2)$ where $M^2 = 2M_{\text{ls}}^2$,
- (ii) torsion propagator: $D_\mu^\nu(k) = [i/(k^2 + M^2)](\delta_\mu^\nu + k_\mu k^\nu/M^2)$,
- (iii) the $\phi^2 S^2$ —vertex: $V^{\mu\nu}(k, p, q, r) = -2i\zeta\eta^{\mu\nu}$,
- (iv) vertex of torsion self-interaction: $V^{\mu\nu\alpha\beta}(k, p, q, r) = (i\zeta/3)g_{(4)}^{\mu\nu\alpha\beta}$,

where $g_{(4)}^{\mu\nu\alpha\beta} = g^{\mu\nu}g^{\alpha\beta} + g^{\mu\beta}g^{\alpha\nu} + g^{\mu\alpha}g^{\nu\beta}$ and k, p, q, r denote the outgoing momenta.

The only one thing that we would like to check is the violation of the transversality in the kinetic two-loop counterterms. We shall present the calculation in some details because it is quite instructive. To analyze the loop integrals we have used dimensional regularization and in particular the formulas from [123,101]. It turns out that it is sufficient to trace the $1/\epsilon^2$ -pole, because even this leading divergence requires the longitudinal counterterm. The contribution to the mass-operator of torsion from the second diagram from Fig. 3 is given by the following integral:

$$\Pi_{\alpha\beta}^{(2)}(q) = -2\zeta^2 \int \frac{d^n k}{(2\pi)^4} \frac{d^n p}{(2\pi)^4} \frac{\eta_{\alpha\beta} + M^{-2}(k-q)_\alpha(k-q)_\beta}{(p^2 + M^2)[(k-q)^2 + M^2][(p+k)^2 + M^2]} . \quad (5.18)$$

First, one has to note that (as in any local quantum field theory) the counterterms needed to subtract the divergences of the above integrals are local expressions, hence the divergent part of the above integral is finite polynomial in the external momenta q^μ . Therefore, in order to extract these divergences one can expand the factor in the integrand into the power series in q^μ :

$$\frac{1}{(k-q)^2 + M^2} = \frac{1}{k^2 + M^2} \left[1 + \frac{-2k \cdot q + q^2}{k^2 + M^2} \right]^{-1} = \frac{1}{k^2 + M^2} \sum_{l=1}^{\infty} (-1)^l \left(\frac{-2k \cdot q + q^2}{k^2 + M^2} \right)^l \quad (5.19)$$

and substitute this expansion into (5.18). It is easy to see that the divergences hold in this expansion till the order $l=8$. On the other hand, each order brings some powers of q^μ . The divergences of the above integral may be cancelled only by adding the counterterms which include high derivatives.¹⁷

To achieve the renormalizability one has to include these high-derivative terms into action (5.17). However, since we are aiming to construct the effective (low-energy) field theory of torsion, the effects of the higher derivative terms are not seen and their renormalization is not interesting for us. All we need are the second derivative counterterms. Hence, for our purposes expansion (5.19) can be cut at $l=2$ rather than at $l=8$ and moreover only $O(q^2)$ terms should be kept. Thus, one arrives at the known integral [101]

$$\begin{aligned} \Pi_{\alpha\beta}^{(2)}(q) &= -6\xi^2 q^2 \eta_{\alpha\beta} \int \frac{d^n k}{(2\pi)^4} \frac{d^n p}{(2\pi)^4} \frac{1}{p^2 + M^2} \frac{1}{(k^2 + M^2)^2} \frac{1}{(p+k)^2 + M^2} + \dots \\ &= -\frac{12\xi^2}{(4\pi)^4(n-4)^2} q^2 \eta_{\alpha\beta} + (\text{lower poles}) + (\text{higher derivative terms}). \end{aligned} \quad (5.20)$$

Another integral looks a bit more complicated, but its derivation can be done in a similar way. The contribution to the mass-operator of torsion from the first diagram from Fig. 3 is given by the integral

$$\begin{aligned} \Pi_{\alpha\lambda}^{(1)}(q) &= -\frac{\xi^2}{108} g^{(2)\alpha\rho\sigma\beta} g^{(2)\lambda\tau\mu\nu} \int \frac{d^n k}{(2\pi)^4} \int \frac{d^n p}{(2\pi)^4} \frac{1}{k^2 + M^2} \left(\eta_{\tau\beta} + \frac{k_\tau k_\beta}{M^2} \right) \\ &\quad \times \left(\eta_{\rho\mu} + \frac{(p-q)_\rho (p-q)_\mu}{M^2} \right) \frac{1}{p^2 + M^2} \\ &\quad \times \left(\eta_{\sigma\nu} + \frac{(p+k)_\sigma (p+k)_\nu}{M^2} \right) \frac{1}{(k+q)^2 + M^2}. \end{aligned} \quad (5.21)$$

Now, we perform the same expansion (5.19) and, disregarding lower poles, finite contributions and higher derivative divergences arrive at the following leading divergence:

$$\Pi_{\alpha\lambda}^{(1)}(q) = -\frac{\xi^2}{(4\pi)^4(n-4)^2} q^2 \eta_{\alpha\lambda} + \dots \quad (5.22)$$

¹⁷ This is a consequence of the power-counting non-renormalizability of the theory with massive vector fields.

Thus, we see that both diagrams from Fig. 3 really give rise to the longitudinal kinetic counterterm and no any simple cancellation of these divergences is seen.

In order to understand the situation better let us compare it with the one that takes place for the usual abelian gauge transformation (2.36). In this case, the symmetry is not violated by the Yukawa coupling, and (in the abelian case) the $A^2\varphi^2$ counterterm is impossible because it violates gauge invariance. The same concerns also the self-interacting A^4 counterterm. The gauge invariance of the theory on quantum level is controlled by the Ward identities. In principle, the non-covariant counterterms can show up, but they can be always removed, even in the non-abelian case, by the renormalization of the gauge parameter and in some special class of (background) gauges they are completely forbidden. Generally, the renormalization can be always performed in a covariant way [182].

In the case of transformation (2.37) if the Yukawa coupling is inserted there are no reasonable gauge identities at all. Therefore, in the theory of torsion field coupled to the SM with scalar field there is a conflict between renormalizability and unitarity. The action of the renormalizable theory has to include the $(\partial_\mu S^\mu)^2$ term, but this term leads to the massive ghost. This conflict between unitarity and renormalizability reminds another one which is well known—the problem of massive unphysical ghosts in the high-derivative gravity [175]. The difference is that in our case, unlike higher-derivative gravity, the problem appears due to the non-invariance with respect to transformation (2.37). We shall proceed with the discussion of this analogy in Section 5.8.

5.6. Second test: problems with the quantized fermion–torsion systems

The problem of consistency of the fermion–torsion system has been studied in [22]. Since the result of this study had crucial importance for the theoretical possibility of torsion, we shall present many details of the investigation of [22] here.

In order to understand the source of the problems, let us first write the fermion action with torsion using the Boulware-like parametrization. For pedagogical reasons, we first consider the usual vector case, that is, repeat the transformation of [27]. The action of the massive vector field V_μ , in original variables, has the form

$$S_{m\text{-vec}} = \int d^4x \left\{ -\frac{1}{4} V_{\mu\nu} V^{\mu\nu} + \frac{1}{2} M^2 V_\mu V^\mu + i\bar{\psi}[\gamma^\alpha(\partial_\alpha - igV_\alpha) - im]\psi \right\}, \quad (5.23)$$

where $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ and, after the change of the field variables [27]:

$$\psi = \exp \left\{ \frac{ig}{M} \cdot \varphi \right\} \cdot \chi, \quad \bar{\psi} = \bar{\chi} \cdot \exp \left\{ -\frac{ig}{M} \cdot \varphi \right\}, \quad v_\mu = V_\mu^\perp - \frac{1}{M} \partial_\mu \varphi, \quad (5.24)$$

the new scalar, φ , is completely factored out:

$$S_{m\text{-vec}} = \int d^4x \left\{ -\frac{1}{4} (V_{\mu\nu}^\perp)^2 + \frac{1}{2} M^2 V_\mu^\perp V^{\perp\mu} + \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi + i\bar{\chi}[\gamma^\alpha(\partial_\alpha + igV_\alpha^\perp)]\chi \right\}. \quad (5.25)$$

Let us now consider the fermion–torsion system given by the action¹⁸

$$S_{\text{tor-fer}} = \int d^4x \left\{ -\frac{1}{4} S_{\mu\nu} S^{\mu\nu} + \frac{1}{2} M^2 S_\mu S^\mu + i\bar{\psi} [\gamma^\alpha (\partial_\alpha + i\eta\gamma_5 S_\alpha) - im] \psi \right\}. \quad (5.26)$$

The change of variables, similar to the one in (5.24), has the form

$$\psi = \exp \left\{ \frac{i\eta}{M} \gamma^5 \varphi \right\} \chi, \quad \bar{\psi} = \bar{\chi} \exp \left\{ \frac{i\eta}{M} \gamma^5 \varphi \right\}, \quad S_\mu = S_\mu^\perp - \frac{1}{M} \partial_\mu \varphi, \quad (5.27)$$

where S_μ^\perp and $S_\mu^\parallel = \partial_\mu \varphi$ are the transverse and longitudinal parts of the axial vector, respectively, the latter being equivalent to the pseudoscalar φ . One has to notice that, contrary to (5.24), but in full accordance with (2.37), the signs of both the exponentials in (5.27) are the same. In terms of the new variables, the action becomes

$$S_{\text{tor-fer}} = \int d^4x \left\{ -\frac{1}{4} S_{\mu\nu}^\perp S^{\perp\mu\nu} + \frac{1}{2} M^2 S_\mu^\perp S^{\perp\mu} + i\bar{\chi} [\gamma^\alpha (\partial_\alpha + i\eta\gamma_5 S_\alpha^\perp) - im \cdot e^{(2i\eta/M)\gamma^5 \varphi}] \chi + \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi \right\}, \quad (5.28)$$

where $S_{\mu\nu}^\perp = \partial_\mu S_\nu^\perp - \partial_\nu S_\mu^\perp = S_{\mu\nu}$. Contrary to the vector case (5.24), for the torsion axial vector (5.28) the scalar mode does not decouple. Moreover, the interaction has the following unpleasant features:

- (i) it is Yukawa-type, resembling the problems with the ordinary scalar,
- (ii) besides, the interaction is exponential, which makes almost certain that the model is not power-counting renormalizable.

On quantum level, symmetry (2.37) manifests itself through the Ward identities for the Green's functions. The analysis of the Ward identities arising in the fermion–torsion system confirms that the new scalar mode does not decouple and that one cannot control the φ -dependence on quantum level [22]. However, at first sight, there is a hope that the above properties would not be fatal for the theory. With respect to point (i), one can guess that the only result of the non-factorization, which can be dangerous for the consistency of the effective quantum theory, would be the propagation of the longitudinal mode of the torsion, and this does not directly follow from the non-factorization of the scalar degree of freedom in the classical action. On the other hand, (ii) indicates the non-renormalizability, which might mean just the appearance of higher-derivative divergences. But, this does not matter within the effective approach. Thus, a more detailed analysis is necessary. In particular, the one-loop and especially two-loop calculations in theory (5.26) may be especially helpful.

The *one-loop calculation* in theory (5.26) can be performed using the generalized Schwinger–DeWitt technique developed by Barvinsky and Vilkovisky [16], but the application of this technique here is highly non-trivial. The problem has been solved in [22]. First of all, we note that the derivation of divergences in the purely torsion sector is not necessary, since the result is (3.15), that is the same as for the spinor loop on torsion background.

¹⁸ For the sake of simplicity we consider a single spinor only. Since torsion is an abelian massive vector, the results cannot depend on the gauge group.

Of course, one has to disregard all curvature-dependent terms in (3.15), since now we work on the flat metric background.

Now, we are in a position to start the complete calculation of divergences. The use of the background field method supposes the shift of the field variables into a background and a quantum part. However, in the case of the (axial)vector–fermion system, the simple shift of the fields leads to an enormous volume of calculations, even for a massive vector. Such a calculation becomes extremely difficult for the axial massive vector (5.26). That is why, in [22], we have invented a simple trick combining the background field method with the Boulware transformation (5.27) for the quantum fields.

According to the method of [22], one has to divide the fields into background $(S_\mu, \psi, \bar{\psi})$ and quantum $(t_\mu^\perp, \varphi, \chi, \bar{\chi})$ parts, according to

$$\begin{aligned}\psi &\rightarrow \psi' = e^{i(\eta/M)\gamma_5\varphi} \cdot (\psi + \chi), \\ \bar{\psi} &\rightarrow \bar{\psi}' = (\bar{\psi} + \bar{\chi}) \cdot e^{i(\eta/M)\gamma_5\varphi}, \\ S_\mu &\rightarrow S'_\mu = S_\mu + t_\mu^\perp - \frac{1}{M}\partial_\mu\varphi.\end{aligned}$$

The one-loop effective action depends on the quadratic (in quantum fields) part of the total action:

$$\begin{aligned}S^{(2)} = \frac{1}{2} \int d^4x \left\{ t_\mu^\perp (\square + M^2) t^{\perp\mu} + \varphi (-\square) \varphi + t_\mu^\perp (-2\eta\bar{\psi}\gamma^\mu\gamma_5)\chi + \varphi \left(-\frac{4m\eta^2}{M^2} \bar{\psi}\psi \right) \varphi \right. \\ \left. + \bar{\chi} (-2\eta\gamma^\nu\gamma_5\psi) t_\nu^\perp + \bar{\chi} \left(\frac{4im\eta}{M} \gamma^5\psi \right) \varphi + \varphi \left(\frac{4im\eta}{M} \bar{\psi}\gamma^5 \right) \chi + \bar{\chi} (2i\gamma^\mu D_\mu + 2m)\chi \right\}.\end{aligned}\quad (5.29)$$

Making the usual change of the fermionic variables $\chi = -(i/2)(\gamma^\mu D_\mu + im)\tau$, and substituting $\varphi \rightarrow i\varphi$, we arrive at the following expression:

$$S^{(2)} = \frac{1}{2} \int d^4x (t_\mu^\perp \varphi \bar{\chi}) \cdot \hat{\mathbf{H}} \cdot \begin{pmatrix} t_\nu^\perp \\ \varphi \\ \tau \end{pmatrix},$$

where the Hermitian bilinear form $\hat{\mathbf{H}}$ has the form

$$\hat{\mathbf{H}} = \begin{pmatrix} \theta^{\mu\nu}(\square + M^2) & 0 & \theta_\beta^\mu (L^{\beta\alpha}\partial_\alpha + M^\beta) \\ 0 & \square + N & A^\alpha\partial_\alpha + B \\ P_\beta\theta^{\beta\nu} & Q & \hat{1}\square + R^\lambda\partial_\lambda + \Pi \end{pmatrix}, \quad (5.30)$$

$\theta^{\mu\nu} = \delta^\mu_\nu - \partial^\mu_{\square} \partial_\nu$, being the projector on the transverse vector states. The elements of the matrix operator (5.30) are defined according to (5.29):

$$\begin{aligned}R^\mu &= 2\eta\sigma^{\mu\nu}\gamma_5 S_\nu, \quad \Pi = i\eta\gamma_5(\partial_\mu S^\mu) + \frac{i}{2}\eta\gamma^\mu\gamma^\nu\gamma_5 S_{\mu\nu} + \eta^2 S_\mu S^\mu + m^2, \\ L^{\alpha\beta} &= -i\eta\bar{\psi}\gamma_5\gamma^\alpha\gamma^\beta, \quad M^\beta = \eta^2\bar{\psi}\gamma^\beta\gamma^\alpha S_\alpha + \eta m\bar{\psi}\gamma_5\gamma^\beta,\end{aligned}$$

$$\begin{aligned}
A^\alpha &= 2i\eta \frac{m}{M} \bar{\psi} \gamma_5 \gamma^\alpha, \quad B = 2\eta^2 \frac{m}{M} \bar{\psi} \gamma^\beta S_\beta - 2\eta \frac{m^2}{M} \bar{\psi} \gamma_5, \\
N &= 4\eta^2 \frac{m}{M^2} \bar{\psi} \psi, \quad P^\beta = -2\eta \gamma^\beta \gamma_5 \psi, \quad Q = -4\eta \frac{m}{M} \gamma_5 \psi.
\end{aligned} \tag{5.31}$$

The operator $\hat{\mathbf{H}}$ given above might look like the minimal second order operator $(\square + 2h^\lambda \nabla_\lambda + \Pi)$; but, in fact, it is not minimal because of the projectors $\theta^{\mu\nu}$ in the axial vector–axial vector $t_\mu^\perp - t_\nu^\perp$ sector. That is why one cannot directly apply the standard Schwinger–DeWitt expansion to derive the divergent contributions to the one-loop effective action, and some more sophisticated method is needed. Such a method, which can be called the generalized Schwinger–DeWitt technique in the transverse space, has been developed in [22], where the reader can find the complete details concerning the one-loop calculations in both theories (5.25) and (5.26). Let us, for the sake of brevity, present only the final result for the one-loop divergences in the theory (5.26):

$$\begin{aligned}
\Gamma_{\text{div}}^{(1)} &= -\frac{\mu^{n-4}}{\varepsilon} \int d^n x \left\{ -\frac{2\eta^2}{3} S_{\mu\nu} S^{\mu\nu} + 8m^2 \eta^2 S^\mu S_\mu - 2m^4 + \frac{3}{2} M^4 + \right. \\
&\quad \left. + \left(\frac{8\eta^2 m^3}{M^2} - 6\eta^2 m \right) \bar{\psi} \psi + 8 \frac{\eta^4 m^2}{M^4} (\bar{\psi} \psi)^2 + 4i \frac{\eta^2 m^2}{M^2} \bar{\psi} \gamma^\mu D_\mu^* \psi \right\}.
\end{aligned} \tag{5.32}$$

It is interesting to note that the above expression (5.32) is not gauge invariant. One can trace the calculations back in order to see that the non-invariant terms come as a contribution of the scalar φ (see, e.g., Eq. (5.28)). Thus, the non-invariant divergences emerge because there are variables, in which the violation of symmetry (2.37) is not soft.

Consider the one-loop renormalization and the corresponding renormalization group. Expression (5.32) shows that theory (5.26) is not renormalizable, but the new type of divergences are suppressed if we suppose that the torsion has very big mass $m \ll M_{ts}$ (remember our notation $M_{ts}^2 = 2M^2$). Let us, for a while, take this relation as a working hypothesis. Then, the relations between bare and renormalized fields and the coupling η follow from (5.32):

$$\begin{aligned}
S_\mu^{(0)} &= \mu^{(n-4)/2} S_\mu \left(1 + \frac{1}{\varepsilon} \cdot \frac{8\eta^2}{3} \right), \quad \psi^{(0)} = \mu^{(n-4)/2} \psi \left(1 + \frac{1}{\varepsilon} \cdot \frac{2\eta^2 m^2}{M^2} \right), \\
\eta^{(0)} &= \mu^{(4-n)/2} \left(\eta - \frac{1}{\varepsilon} \cdot \frac{8\eta^3}{3} \cdot \left[1 + \frac{m^2}{M^2} \right] \right).
\end{aligned} \tag{5.33}$$

Similar relations for the parameter $\tilde{\lambda} = (M^4/m^2)\lambda$ of the $\lambda(\bar{\psi}\psi)^2$ -interaction, have the form

$$\tilde{\lambda}^{(0)} = \mu^{4-n} \left[\tilde{\lambda} + \frac{16\eta^4}{\varepsilon} - \frac{8\tilde{\lambda}\eta^2 m^2}{M^2 \varepsilon} \right]. \tag{5.34}$$

These relations lead to a renormalization group equation for η , which contains a new term proportional to $(m/M)^2$:

$$(4\pi)^2 \frac{d\eta^2}{dt} = \frac{8}{3} \left[1 + \frac{m^2}{M^2} \right] \eta^4, \quad \eta(0) = \eta_0. \tag{5.35}$$

Indeed, for the case $m \ll M$ and in the low-energy region, this equation reduces to the one presented in [18] (that is identical to the similar equation of QED). In any other case, the theory of torsion coupled to the massive spinors is inconsistent, and Eq. (5.35) is meaningless.

One can also write down the renormalization group equation for the parameter $\tilde{\lambda}$ defined above. Using (5.34), we arrive at the following equation:

$$(4\pi)^2 \frac{d\tilde{\lambda}}{dt} = 16\eta^4. \quad (5.36)$$

This equation confirms the lack of a very fast run for this parameter. Indeed, all this renormalization group consideration has meaning only under the assumption that $m \ll M$. In order to provide the one-loop renormalizability, it is necessary to add the $(\bar{\psi}\psi)^2$ -term to the classical action. However, as we have learned, in Section 3, on the example of the Nambu–Jona–Lasinio model, such a term does not affect the one-loop renormalization of other sectors of the action. However, it becomes extremely important for the two-loop contributions.

The *two-loop calculation* is the crucial test for the consistency of the fermion–torsion effective theory. The point is that there are no one-loop diagrams which include any non-symmetric vertices and can contribute to the dangerous longitudinal term in the torsion action. However, there are two-loop diagrams contributing to the propagator of the axial vector, S_μ . The question is whether there are longitudinal divergences $(\partial_\mu S^\mu)^2$ -type at the two-loop level. Then, it is reasonable to start from the diagrams which can exhibit $1/\epsilon^2$ -divergences.

The leading $1/\epsilon^2$ -two-loop divergences of the mass operator for the axial vector S_μ come from two distinct types of diagrams: the ones with the $(\bar{\psi}\psi)^2$ -vertex and the ones without this vertex. As we shall see, the most dangerous diagrams are those with four-fermion interaction. As we have seen above, this kind of interaction is a remarkable feature of the axial vector theory, which is absent in a massive vector theory. Now, we shall calculate divergent $1/\epsilon^2$ -contributions from two diagrams with the $(\bar{\psi}\psi)^2$ -vertex, using expansion (5.19); in Appendix B of [22] this calculation has been checked using Feynman parameters.

Consider the first diagram of Fig. 4. It can be expressed, after making some commutations of the γ -matrices, as

$$\Pi_{\mu\nu}^1 = -\lambda\eta^2 \text{tr}\{I_\nu \cdot I_\mu\}, \quad (5.37)$$

where $\lambda \sim m^2/M^4$ is the coupling of the four-fermion vertex, the trace is taken over the Dirac spinor space and

$$I_\nu(p) = \int \frac{d^n p}{(2\pi)^n} \frac{\not{p} - m}{p^2 - m^2} \gamma_\nu \gamma_5 \frac{\not{p} - \not{q} - m}{(p - q)^2 - m^2}. \quad (5.38)$$

Following [18], we can perform the expansion

$$\frac{1}{(p - q)^2 - m^2} = \frac{1}{p^2 - m^2} \sum_{l=0}^{\infty} (-1)^l \left(\frac{-2p \cdot q + q^2}{p^2 - m^2} \right)^l. \quad (5.39)$$

Now, as in the previous section, one can omit the powers of q higher than 2. Besides, when performing the integrations, we trace just the divergent parts, thus arriving (using the integrals

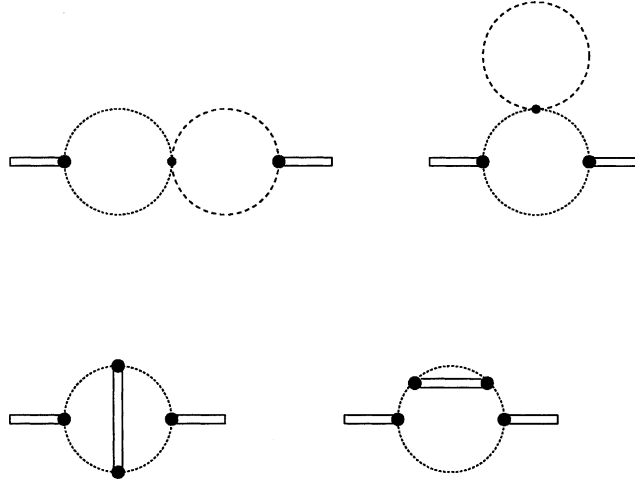


Fig. 4. Two-loop “dangerous” diagrams, which can contribute to the $(\partial_\mu S^\mu)^2$ -counterterm. The first two give non-zero contributions and the second two cannot cancel them.

from [101]) at the expressions:

$$I_v = \frac{i}{\epsilon} \left\{ -\frac{1}{6} q^2 \gamma_v - 2m^2 \gamma_v - \frac{1}{6} \gamma_\alpha \gamma_v \gamma_\beta q^\alpha q^\beta + m q_v \right\} + \dots,$$

where the dots stand for the finite and higher-derivative divergent terms. Substituting this into (5.37), we obtain the leading divergences of the diagram:

$$\Pi_{\mu\nu}^{1,\text{div}} = -\frac{\lambda \eta^2}{\epsilon^2} \left\{ +16m^4 \eta_{\mu\nu} + \frac{28}{3} m^2 q_\mu q_\nu - \frac{16}{3} m^2 q^2 \eta_{\mu\nu} \right\} + \dots. \quad (5.40)$$

This result shows that the construction of the first diagram contains an $1/\epsilon^2$ -longitudinal counterterm.

The contribution of the second two-loop diagram of Fig. 4 to the polarization operator, $\Pi_{\mu\nu}^2$, is written, after certain transformations, in the following way:

$$\Pi_{\mu\nu}^2 = -\lambda \eta^2 \text{tr} \{ I_{\nu\mu} \cdot J \}, \quad (5.41)$$

where

$$I_{\nu\mu} = \int \frac{d^n p}{(2\pi)^n} \frac{\not{p} - m}{p^2 - m^2} \gamma_\nu \frac{\not{p} - \not{q} + m}{(p - q)^2 - m^2} \gamma_\mu \frac{\not{p} - m}{p^2 - m^2} \quad (5.42)$$

and

$$J = \int \frac{d^n p}{(2\pi)^n} \frac{\not{k} - m}{k^2 - m^2}. \quad (5.43)$$

It proves useful to introduce the following definitions:

$$A_{\alpha\nu\beta\mu\rho} = \gamma_\alpha \gamma_\nu \gamma_\beta \gamma_\mu \gamma_\rho,$$

$$B_{\alpha\nu\mu\beta} = -q^\rho \gamma_\alpha \gamma_\nu \gamma_\rho \gamma_\mu \gamma_\beta + m(\gamma_\alpha \gamma_\nu \gamma_\mu \gamma_\beta - \gamma_\nu \gamma_\alpha \gamma_\mu \gamma_\beta - \gamma_\alpha \gamma_\nu \gamma_\beta \gamma_\mu),$$

$$C_{\alpha\nu\mu} = m^2(\gamma_\nu\gamma_\alpha\gamma_\mu - \gamma_\nu\gamma_\mu\gamma_\alpha - \gamma_\nu\gamma_\alpha\gamma_\mu) + mq^\beta(\gamma_\nu\gamma_\beta\gamma_\mu\gamma_\alpha + \gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\mu),$$

$$D_{\nu\mu} = -m^2q^\beta\gamma_\nu\gamma_\beta\gamma_\mu + m^3\gamma_\nu\gamma_\mu.$$

Then, the first integral can be written as

$$I_{\nu\mu} = \int \frac{d^n p}{(2\pi)^n} \frac{A_{\alpha\nu\beta\mu\rho} \cdot p^\alpha p^\beta p^\rho + B_{\alpha\nu\mu\beta} \cdot p^\alpha p^\beta + C_{\alpha\nu\mu} p^\alpha + D_{\nu\mu}}{(p^2 - m^2)^2((p - q)^2 - m^2)}. \quad (5.44)$$

Using expansion (5.39), and disregarding higher powers of q , as well as odd powers of p in the numerator of the resulting integral, one obtains, after using the integrals of [101]:

$$I_{\nu\mu} = \frac{i}{\epsilon} \left\{ \frac{1}{4} B_{\nu\mu\alpha}^\alpha + \frac{1}{12} (A_{\nu\alpha\mu\rho}^\alpha + A_{\nu\rho\mu\alpha}^\alpha + A_{\rho\nu\alpha\mu}^\alpha) q^\rho \right\} + \dots,$$

which gives, after some algebra,

$$I_{\nu\mu} = \frac{i}{\epsilon} \left\{ m\gamma_\mu\gamma_\nu + 3m\eta_{\mu\nu} - \frac{2}{3}\gamma_\rho q^\rho \eta_{\mu\nu} + \frac{1}{3}\gamma_\mu q_\nu + \frac{1}{3}\gamma_\nu q_\mu \right\} + \dots. \quad (5.45)$$

The divergent contribution to J is

$$J = -\frac{i}{\epsilon} m^3 + \dots.$$

Now, the calculation of (5.41) is straightforward:

$$\Pi_{\mu\nu}^{2,\text{div}} = \frac{\lambda\eta^2}{\epsilon^2} 8m^4\eta_{\mu\nu} + \dots. \quad (5.46)$$

As we see, this diagram does not contribute to the kinetic counterterm (with accuracy of the higher-derivative terms), and hence the cancellation of the contributions to the longitudinal counterterm coming from $\Pi_{\mu\nu}^1$ do not take place.

One has to note that other two-loop diagrams do not include the $(\bar{\psi}\psi)^2$ -vertex. Thus, even if those diagrams contribute to the longitudinal counterterm, the cancellation with $\Pi_{\mu\nu}^1$ should require some special fine-tuning between λ and η . In fact, one can prove, without explicit calculation, that the remaining two-loop diagrams of Fig. 4 do not contribute to the longitudinal $1/\epsilon^2$ -pole. In order to see this, let us note that the leading (in our case $1/\epsilon^2$) divergence may be obtained by consequent substitution of the contributions from the subdiagrams by their local divergent components. Since the local counterterms produced by the subdiagrams of the last two graphs of Fig. 4 are minus the one-loop expression (5.32), the corresponding divergent vertices are $1/\epsilon$ factor classical vertices. Hence, in the leading $1/\epsilon^2$ -divergences of the last two diagrams of Fig. 4, one meets again the same expressions as in (5.32). The result of our consideration is, therefore, the non-cancellation of the $1/\epsilon^2$ -longitudinal divergence (5.40). This means that theory (5.26), without additional restrictions on the torsion mass, like $m \ll M$, is inconsistent at the quantum level.

5.7. Interpretation of the results: do we have a chance to meet propagating torsion?

Obviously, we have found a very pessimistic answer about the possibility of a propagating torsion. Torsion is not only helpless in constructing the renormalizable quantum gravity, as people thought [132,161], but it cannot be renormalizable by itself. Moreover, even if we give up

the requirement of power counting renormalizability and turn to the effective approach, the theory remains inconsistent. In this situation one can try the following: (i) Invent, if possible, some restriction on the parameters of the theory, for which the problems disappear. (ii) Analyze the approach from the very beginning in order to look for the possible holes in the analysis. Let us start with the first option, and postpone the second one for the next section.

The problems with the non-renormalizability and with the violation of unitarity become weakened if one input severe restrictions on the torsion mass, which has to be much greater than the mass of the heaviest fermion (say, t -quark, with a mass about 175 GeV), and continue to use an effective quantum field theory approach, investigating the low-energy observables only. This approach implies the existence of a fundamental theory which is valid at higher energies.

Hence, in order to have a propagating torsion, one has to satisfy a double inequality:

$$m_{\text{fermion}} \ll M_{\text{torsion}} \ll M_{\text{fundamental}} . \quad (5.47)$$

Usually, the fundamental scale is associated with the Planck mass, $M_P \approx 10^{19}$ GeV. Indeed, the identification of the torsion mass with the Planck scale means, for $M_{\text{torsion}} \approx M_P$, that at the low (much less than M_P) energies, when the kinetic and other terms are negligible, we come back to the Einstein–Cartan theory (2.18). As we have readily noticed, in this theory torsion is not propagating, but it can mediate contact interactions. These interactions are too weak for the high-energy experiments described in Section 5.4, but maybe, in future, they can be detected in very precise experiments like the ones in the atomic systems.

It is important to remark that, in principle, the effective quantum field theory approach may be used only at the energies essentially smaller than the typical mass scale of the fundamental theory. If the mass of torsion is comparable to the fundamental scale M_P , all the torsion degrees of freedom should be described directly in the framework of the fundamental theory. For instance, in the low-energy effective actions of the available versions of string theory torsion enters with the mass $1/\alpha'$, which is conventionally taken to be of the Planck order. But, other degrees of freedom associated to torsion also have a mass of the same order. Thus, it is unclear why one can take only this “static” mode and neglect infinite amount of others with the same huge mass. Later on, in Section 6, we shall discuss torsion coming from the string theory, and see that the standard approach to string forbids propagating torsion at all. Then, the Einstein–Cartan theory (2.18) becomes some kind of universal torsion theory for the low-energy domain.

On the other hand, relation (5.47) still leaves a huge gap in the energy spectrum, which is not completely covered by the present theoretical consideration. In other words, (5.47) might be inconsistent with the string theory, but it does not contradict the effective approach. Of course, this gap cannot be closed by any experiment, because the mass of torsion is too big. Even the restrictions coming from the contact experiments [18] achieve only the region $M < 3$ TeV. And that is not really enough to satisfy (5.47) for all the fermions of the Standard Model. It is clear that the existence of fermions with masses many orders of magnitude larger than m_t (like the ones which are expected in many GUTs or SSM) can close the gap on the particle spectrum and “forbid” propagating torsion.

5.8. What is the difference with metric?

The situation with torsion is similar to the one with quantum gravity. In both cases, there is a conflict between renormalizability and unitarity. In the case of quantum gravity, there are models which are unitary (General Relativity and its supersymmetric generalizations) but non-renormalizable, and other models, with higher derivatives, which are renormalizable but not unitary.

For the case of torsion, there are models which are unitary at the tree level (take, for instance, all versions described in [162], or simply our action (5.6)), but non-renormalizable. At the same time, it is not difficult to formulate renormalizable theory of torsion. For that one has to take, say, the general metric–torsion action of [48], which will provide renormalizable metric–torsion theory. The same action of [48], but with the flat metric $g_{\mu\nu} = \eta_{\mu\nu}$ will give a renormalizable theory of the torsion alone. However, such a theory will not be unitary, because both transverse and longitudinal components of the axial vector S_μ will propagate.

In some sense, this analogy is natural, because both metric and torsion are geometric characteristics of the space–time manifold rather than usual fields. Therefore, one of the options is to give up the quantization of these two fields and consider them only as a classical background. This option cures all the problems at the same time, and leaves one a great freedom in choosing the model for metric, torsion and other gravity components: there are no constraints imposed by quantum theory anymore. Indeed, there are very small number of the shortcomings in this point of view [163]. The most important of them is the quantum-mechanical inconsistency of the systems composed by quantum and classical constituents (see, for example, [185]).

If one does not accept this option, it is only possible to consider both metric and torsion as effective low-energy interactions resulting from a more fundamental theory like string. There is, however, a great difference between metric and torsion. The effective field theory permits the long-distance propagation of the metric waves, because metric has massless degrees of freedom. Indeed, there may be other degrees of freedom, with the mass of the Planck order, which are non visible at low energies. But, the massless degrees of freedom “work” at very long distances, and provide the long-range gravitational force. Then, the study of an effective quantum field theory for the metric does not meet major difficulties [68,187]. In case of torsion, the massless degrees of freedom are forbidden, because symmetry (2.37) is violated by the spinor mass.

It might happen, that some new symmetries will be discovered, which make the consistent quantum theory of the propagating torsion possible. However, in the framework of the well-established results, the remaining possibilities are that torsion does not exist as an independent field, or it has a huge mass, or it is a purely classical field which should not be quantized. Indeed, in the first two cases there are no chances of detecting torsion experimentally.

6. Alternative approaches: induced torsion

As we have seen in the previous section, the formulation of the theory of propagating torsion meets serious difficulties. In this situation it is natural to remind that some analog of the space–time torsion is generated in string theory, and then try to check whether this is consistent with the situation in the effective field theory. On the other hand, we know that usual metric gravity

can be induced not only in string theory, but also through the quantum effects of the matter fields (see, for example, [2] for a review and further references). This short section is devoted to a brief description of these two approaches: torsion induced in string theory and torsion induced by the quantum effects of matter fields.

6.1. Is that torsion induced in string theory?

The covariant (super)string action has a geometric interpretation as a two-dimensional non-linear sigma model. Then, the completeness of the spectrum (it can be seen as the requirement of the correspondence between string and the sigma-model) requires an additional Wess–Zumino–Witten term to be included. The action of a bosonic string has the form

$$S_{\text{str}} = \int d^2z \sqrt{|h|} \left\{ \frac{1}{2\alpha'} h^{ab} g_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu + \frac{1}{\alpha'} \frac{\epsilon^{ab}}{\sqrt{|h|}} b_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu + \Phi(X)^{(2)}R + T(X) \right\}. \quad (6.1)$$

Here $\mu, \nu = 1, 2, \dots, D$; and $a, b = 1, 2$. $g_{\mu\nu}(X)$ is the background metric, $b_{\mu\nu}(X)$ is the antisymmetric tensor background field, $\Phi(X)$ is the dilaton field, $T(X)$ is the tachyon background field, which we do not consider in what follows.¹⁹ $^{(2)}R$ is the two-dimensional curvature, $X^\mu = X^\mu(z)$ are string coordinates. The parameter α' may be considered as the parameter of the string loop expansion. Since the dimension of α' is the inverse of the mass, usually α' is associated with the inverse square of the Planck mass $1/M_{\text{P}}^2$. Indeed, this choice is strongly motivated by the common belief that the (super)string theory is the theory which should unify all the interactions including gravitational, in one quantum theory.

The X^μ are the dynamical fields defined on the world sheet. At the same time they are coordinates of the D -dimensional space with the metric $g_{\mu\nu}$. Initially, the geometry of this D -dimensional space is not known, it is generated by the quantum effects of the two-dimensional theory. In the sigma-model approach (see, for example, [94,178] for the general review and [114] for the technical introduction and complete list of results of the higher-loop calculations of the string effective action) the effective D -dimensional action of the background fields $g_{\mu\nu}(X), b_{\mu\nu}(X), \Phi(X)$ appears as a result of the imposition of the Weyl invariance principle at each order of the perturbative expansion of the two-dimensional effective action. For example, the dilaton term in (6.1) is not Weyl invariant, but it has an extra factor of α' . Therefore, after integration over the fields X^μ we find that this classical term contributes to the trace of the energy–momentum tensor and that this contribution is of the first order in α' . Other similar contribution comes from the one-loop effects, including the renormalization of the composite operators in the $\langle T_\mu^\mu \rangle$ expression [78,42,146,114]. Requesting the cancellation of two contributions we get a set of the one-loop effective equations: conditions on the background fields $g_{\mu\nu}(X), b_{\mu\nu}(X), \Phi(X)$. The corresponding action is interpreted as a low-energy effective action of string.

¹⁹ Tachyon does not pose a problem for the superstring.

The one-loop effective equations do not directly depend on the antisymmetric field $b_{\mu\nu}(X)$, but rather on the stress tensor

$$H_{\mu\nu\lambda} = \partial_{[\mu} b_{\nu\lambda]} . \quad (6.2)$$

The covariant calculation includes the expansion in Riemann normal coordinates [105,106,114]), and the coefficients of the expansion for $b_{\mu\nu}$ depend only on $H_{\mu\nu\lambda}$ and its derivatives, but not directly on the $b_{\mu\nu}$ components. After all, the geometry of the D -dimensional space is characterized by three kinds of fields: metric $g_{\mu\nu}$, dilaton Φ and completely antisymmetric field $H_{\mu\nu\lambda}$, which satisfies the constraint $\varepsilon^{\mu\nu\alpha\beta} \partial_\alpha H_{\nu\alpha\beta} = 0$ coming from (6.2). In principle, the field $H_{\mu\nu\lambda}$ can be interpreted as a space–time torsion. In order to understand better the correspondence with our previous treatment of torsion, let us parametrize the $H_{\mu\nu\lambda}$ field by the axial vector $S^\mu = \varepsilon^{\mu\nu\alpha\beta} H_{\nu\alpha\beta}$. Then the constraint $\varepsilon^{\mu\nu\alpha\beta} \partial_\alpha H_{\nu\alpha\beta}$ gives $\partial_\alpha S^\alpha = 0$, and we arrive at the conclusion that the string-induced torsion has only transverse axial vector component—exactly the same answer which is dictated by the effective field theory approach (see the beginning of Section 5.3).

In fact, on the way to this interpretation $H_{\mu\nu\lambda} \sim T_{\mu\nu\lambda}$ one has to check only one thing: how this field interacts to the fermions. Indeed, we are interested in the answer after the compactification into the four-dimensional space–time. However, compactification cannot change the general form of the interaction. If we suppose that the fermions interact to the $H_{\mu\nu\lambda}$ field through the axial current, then $H_{\mu\nu\lambda}$ can be identified as the space–time torsion $T_{\mu\nu\lambda}$. One can note that this is the form of the interaction which looks quite natural from the point of view of dimension and covariance. This is not the unique possible choice, indeed. If we consider the tensor $b_{\mu\nu}$ as an independent field (see e.g. [172]), the situation becomes quite different. In four dimensions, the antisymmetric field is dual to the axion-specific kind of scalar field (see e.g. [34]). In a conventional supergravity theory, which might be taken as the low-energy limit of the superstring, the axion field is present and the formulation of its quantum theory does not pose any problem (in a sharp contrast to torsion!). Of course, the interaction of axion with fermions is quite different from the one of (2.34), so in this case the antisymmetric field cannot be associated to the space–time torsion.

On the other hand, the renormalizable and local supergravity theory is not the unique possible choice. There are some arguments [119] that the non-local effects can play a very important role in the string effective actions, and one of the popular phenomenological models for these effects are related to the b_μ field, which we already mentioned by the end of Section 4.6. The b_μ field is a close analog of torsion, and therefore the available predictions of the string theory are not completely definite.

Let us now discuss the form of the string-induced action for torsion. At the one-loop level, this is some kind of the metric–dilaton–torsion action considered in Section 2.4. As an example of the string-induced action we reproduce the one for the bosonic string

$$S_{\text{eff}} \sim \int d^D x \sqrt{|g|} e^{-2\Phi} \left[-R - 2\mathcal{D}^2 \Phi + \frac{1}{3} H_{\lambda\alpha\beta} H^{\lambda\alpha\beta} \right] , \quad (6.3)$$

where one can put $\Phi = \text{const.}$ for the analysis of the torsion dynamics. Since the torsion square enters the expression in the linear combination with the Ricci scalar, the torsion mass equals to

the square of the Planck mass. This is, again, in a perfect agreement with our results about the effective approach to the propagating torsion. The expressions similar to (6.3) show up for all known versions of string theory. For superstring the one-loop result is exactly the same [56]. For the heterotic string the torsion term is the same as in (6.3), the difference concerns only the dilaton. No version of string is known, which would produce the zero torsion mass in the low-energy effective action. Perhaps, this is more than a simple coincidence, for the massless torsion should lead to problems in the effective framework.

The propagation of torsion can be caused by the higher loop string corrections to the low-energy effective action. It was noticed by Zweibach in 1985, that the definition of the higher order corrections includes an ambiguity related to the choice of the background fields [194,62,177,111]. In particular, one can perform such a reparametrization, that the propagators of all three fields: metric, torsion and dilaton do not depend on higher derivatives. For the theory with torsion the corresponding analysis has been done in Ref. [111]. The general form of the transformation is

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}, \quad b_{\mu\nu} \rightarrow b_{\mu\nu} + \delta b_{\mu\nu}, \quad \Phi \rightarrow \Phi + \delta\Phi, \quad (6.4)$$

where

$$\begin{aligned} \delta g_{\mu\nu} = & x_1 R_{\mu\nu} + x_2 R g_{\mu\nu} + x_3 H_{\mu\nu}^2 + x_4 \nabla_\mu \nabla_\nu \Phi + x_5 \nabla_\mu \Phi \nabla_\nu \Phi \\ & + x_6 H^2 g_{\mu\nu} + x_7 g_{\mu\nu} \nabla^2 \Phi + x_8 (\nabla \Phi)^2 + \dots, \\ \delta b_{\mu\nu} = & x_9 \nabla_\lambda H_{\mu\nu}^\lambda + x_{10} H_{\mu\nu}^\lambda \nabla_\lambda \Phi + \dots, \\ \delta \Phi = & x_{11} R + x_{12} H^2 + x_{13} \nabla^2 \Phi + x_{14} (\nabla \Phi)^2 + \dots. \end{aligned} \quad (6.5)$$

In these transformations, the coefficients $x_1, x_2, \dots, x_{14}, \dots$ are chosen in such a way that the high-derivative terms contributing to the propagators disappear. The transformation can be continued (in Ref. [111] the proof is presented until the third order in α' , but the statement is likely to hold at any order, see for example, [9]).

The motivation to make a transformation (6.4) is to construct the low-energy theory of gravity free of the high-derivative ghosts. At the same time, one can make several simple observations:

(i) At higher orders the transformation is not uniquely defined, at least if we request just a ghost-free propagator. For instance, let us take a $R_{\mu\nu} R^\mu_\alpha R^{\alpha\nu}$ -term. This term can be easily removed by some transformation similar to (6.4), despite it is innocent—it does not contribute to the propagator of the metric perturbations. Usually, all such terms are removed in order to simplify the practical calculations, but the validity of this operation is not obvious.

(ii) If we are not going to quantize the effective theory, the necessity of whole (6.4) is not clear. It is well known, that in many cases the physically important classical solutions are due to higher derivatives (for example, inflation may be achieved in this way [174,127]).

(iii) If we are going to quantize the effective theory (see corresponding examples in [68] for metric and [18,22] for torsion), one has to repeat reparametrization (6.4) after calculating

any loop in the effective quantum field theory. This operation becomes a necessary component of the whole effective approach, and it substitutes the standard consideration of Ref. [186,68]. Of course, since the effective approach cares about lower derivatives only, the physical consequences of two definitions must be the same.

Indeed, transformation (6.4) kills the torsion kinetic term, so that the torsion action consists of the mass and interaction terms. In the second loop, for the bosonic and heterotic strings, the interaction terms of the H^4 and RH^2 -types emerge (see, for example, [114]). These terms cannot be removed by (6.4). For the superstring, the two- and three-loop contributions cancel, and at the four-loop level the torsion terms can be “hidden” inside the R^4 -type terms. It is unlikely that this can be done at higher loops but, since we are interested in the torsion dynamics, it has no real importance. The conclusion is that string torsion does not propagate, and that in this respect the effective low-energy action of string agrees with the results of the effective quantum field theory.

6.2. Gravity with torsion induced by quantum effects of matter

Despite induced torsion looks as an interesting possibility, it seems there are not many publications on this issue (except [58,32,102] devoted to the anomaly-induced action with torsion). One can note, however, that the action of gravity with torsion may be induced in the very same manner as the action for gravity without torsion. Let us repeat the consideration typical for the standard approach (see, for example, [2]), but with torsion.

One has to suppose that there are gravity fields: metric and torsion, which couple to quantized matter fields. For instance, it can be the non-minimal interaction described in Section 2.3. The non-minimal interaction is vitally important, for otherwise the theory is non-renormalizable. The geometry of the space–time is described by some vacuum action, depending on the metric and torsion, but this action is, initially, not defined completely. The exact sense of the last statement will be explained in what follows. Now, the action of gravity with torsion $S_{\text{grav}}[g_{\mu\nu}, T^\alpha_{\beta\gamma}]$ is a result of the quantum effects of matter on an arbitrary curved background. This means the following representation for this action (cf. to (3.2)):

$$e^{iS_{\text{grav}}[g_{\mu\nu}, T^\alpha_{\beta\gamma}]} = \int \mathcal{D}\Phi e^{iS_{\text{matter}}[\Phi, g_{\mu\nu}, T^\alpha_{\beta\gamma}]} . \quad (6.6)$$

Here, Φ means the whole set of non-gravitational fields: quarks, gluons, leptons, scalar and vector bosons and their GUT analogs, gauge ghosts, etc. Consequently, $S_{\text{matter}} = S_{\text{matter}}[\Phi, g_{\mu\nu}, T^\alpha_{\beta\gamma}]$ is the action of the fields Φ . After these non-gravitational fields are integrated out, the remaining action can be interpreted as an action of gravity and the solutions of the dynamical equations following from this action will define the space–time geometry. Then, the geometry depends on the quantum effects of matter, including spontaneous symmetry breaking which occurs at different scales, phase transitions, loop corrections, non-perturbative effects and scale dependence governed by the renormalization group.

An important observation is in order. As we have already indicated in Section 3, the action must also include a vacuum part. For the massive theory one has to introduce not only the $R^2, R \cdot T^2, R \nabla T, T^2 \nabla T$ and T^4 -terms, but also R and T^2 -type terms, similar to the ones in the Einstein–Cartan action. Then, the lower derivative action of gravity cannot be completely

induced. The induced part will always sum up with the initial vacuum part which is subject of an independent renormalization. Of course, for the massless theory, one can choose the R^2 -type vacuum action, and then the low-energy term will be completely induced through the dimensional transmutation mechanism (see Section 3.5).

Let us now comment on the induced part of the S_{grav} . One can introduce various reasonable approximations and evaluate action (6.6) in the corresponding framework. We shall list the following approaches in deriving the one-loop effective vacuum action:

(i) The anomaly-induced action (see [102] and Section 3.6 of this report for the best available result in gravity with torsion). In some known cases this looks as the best approximation [174,32,73,14]. Indeed, in all cases, except [32,102], the theory without torsion has been considered. Usually, the anomaly-induced action has been treated as a quantum correction to the Einstein action. The advantage of the anomaly-induced action is that it includes the non-local pieces. This is especially important for the cosmological applications (see, e.g. solution (3.79) as an example of such application).

(ii) The induced action of gravity with torsion which emerges as a result of the phase transition induced by torsion or curvature (see Section 2.5 for the case with torsion). We remark that the derivation of the effective action can be continued beyond the effective potential, so that the next terms in the derivative expansion could be taken into account. For gravity without torsion this has been done in [41] (see also [34]), and it is technically possible to realize similar calculus for the theory with torsion. Then, after the phase transition, in the critical point one meets not only the non-minimal version (3.66) of the Einstein–Cartan theory, but also the next order corrections, including high derivative terms, terms of higher order in torsion and curvature, the ones described in [48], etc. The common property of all these terms is locality. The expansion parameter will be the inverse square of the Planck mass, therefore these terms will be negligible at low energies.

(iii) Alternatively, one can simply take the contributions of the free fields and take into account the higher orders of the Schwinger–DeWitt expansion (3.3). Starting from $\text{tr } \hat{a}_3(x, x)$, all these terms will be finite and they are indeed contributions to S_{grav} . In general, these terms are not very much different from the ones described in (ii). There is a possibility to sum the Schwinger–DeWitt expansion, but this has been achieved [11] only for the especially simple backgrounds without torsion.

(iv) The non-local terms can be taken into account using the generalized Schwinger–DeWitt technique. Such calculations have been performed by Vilkovisky et al. (see [180] for the review and further references). This method seems more appropriate for the low-energy region, for the non-localities are not directly related to the high-energy behavior of the massive fields (as it is in the (i) case). The calculations for torsion gravity has not been performed yet. It may happen, that the non-local effects are relevant for torsion (as they are, perhaps, for gravity), and then the effective field theory approach does not give full information. In the content of string theory similar possibility led to the consideration of the (already mentioned) b_μ -field, which strongly resembles torsion.

(v) Another way of deriving the non-local piece is to take the renormalization-group improved action [67]. The corresponding corrections are obtained through the replacement of $1/\epsilon$, in the expression for vacuum divergences, by the $\ln(\square/\mu^2)$. Indeed, these terms can be relevant only in the high-energy region.

All mentioned possibilities concern the definition of the torsion action, but one can make a stronger question about inducing the torsion itself. In principle, the axial current and the interactions similar to the torsion–fermion one, can be induced in a different ways. One can mention, in this respect, the old works on the anomalous magnetic field [137], recent works on the anomalous effects of the spinning fluid [181] and the torsion-like contact interactions coming from extra dimensions [47]. The phenomenological bounds on the contact interaction derived in the last work are similar to the ones of [18].

7. Conclusions

We have considered various aspects of the space–time torsion. There is no definite indications, from experiments, whether torsion exists or not, but it is remarkable that purely theoretical studies can put severe limits on this hypothetical field which would be an important geometrical characteristic of the space–time. The main point is that the consistency of the propagating torsion is extremely restricted. Indeed, there is no problem in writing the classical action for torsion, and this can be done in many ways. Also, the quantum field theory on classical torsion background can be successfully formulated, and we presented many results in this area of research. However, without the dynamical theory for the torsion itself the description of this phenomena is incomplete, and one can only draw phenomenological upper bound for the background torsion from known experiments.

The serious problems show up when one demands the consistency of the propagating torsion theory at the quantum level [18,22]. Then, we have found that there is no theory of torsion which could be simultaneously unitary and renormalizable. Moreover, there is no theory which can be consistent even as an effective theory, when we give up the requirement of the power counting renormalizability. The only one possibility is to suppose that torsion has a huge mass—much greater than the mass of the heaviest fermion. Then, if we assume that torsion couples to *all* fermions, its mass has to greatly exceed the TeV level. Hence, torsion cannot propagate long distances and can only produce contact spin–spin interactions. The necessity of a huge torsion mass can explain the weakness of torsion and difficulties in its observation. Up to our knowledge, this is the first example of rigid restrictions on the geometry of the space–time, derived from the quantum field theory principles.

In this review, we avoided to discuss some, technically obvious, possibilities—like a spontaneous symmetry breaking which would provide, simultaneously, the mass to all the fermions and to the torsion. The reason is that such an approach would require torsion to be treated as a matter field, and in particular to be related to some internal symmetry group—like $SU(2)$, for example. Besides possible problems with anomalies, this would mean that we do not consider, anymore, torsion as part of gravity, but instead take it as a matter field. And that certainly goes beyond the scope of the present review, devoted to the field-theoretical investigation of the space–time torsion.

Thus, the only solution is to take a string- or matter-induced torsion. As we have seen in the previous section, both approaches do not give any definite information. First of all, the torsion induced in string theory has a mass of the Planck order. Thus, it cannot be really treated as an independent field, but rather as a string degree of freedom, integrated into the complete

string spectrum. Furthermore, if we take supergravity as a string effective theory, there is no torsion, but rather there is an axion-scalar field with quite different interaction to fermions. Equally, the investigation of the matter-induced torsion offers some possibilities concerning torsion action, but in all available cases the torsion mass is of the Planck order. Anyhow, the possibility of induced torsion remains, and it is still interesting to look for more precise bounds on a background torsion and to develop the formal aspects of quantum field theory on curved background with torsion.

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