Non-commutative Euclidean and Minkowski structures

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Abstract. A noncommutative $*$-algebra that generalizes the canonical commutation relations and that is covariant under the quantum groups $SO_q(3)$ or $SO_q(1, 3)$ is introduced. The generating elements of this algebra are hermitean and can be identified with coordinates, momenta and angular momenta. In addition a unitary scaling operator is part of the algebra.

1 Introduction

Quantum spaces are non-commutative spaces. If noncommutative space-time is a realistic picture of how space-time behaves at short distances, it is natural to see what we can learn from quantum spaces. These spaces have the advantage that they are as close as possible to our ordinary space-time concept – there is a parameter q on which the structure depends and for $q = 1$ quantum spaces coincide with ordinary spaces. This is the reason why we call quantum spaces q-deformed spaces and our aim is to show that a physical system built on quantum spaces can be regarded as a q-deformation of ordinary physics.

The physical concept of space cannot be separated from the concept of symmetries. Quantum spaces are exactly those spaces that carry the symmetry structure of a quantum group. They are defined as comodules of quantum groups [1]. As such they have an additional property, the Poincaré-Birkhoff-Witt property; this means that regarded as an algebra the homogeneous polynomials have a basis of the same dimension as in the commutative case (ordinary space). For our approach this property will be essential, and thus in a physicist's language we will call it consistency.

This consistency is closely related to the existence of an R-matrix that satisfies the Quantum Yang Baxter equation. Thus our main input into the study of quantum spaces will be the R-matrix and its decomposition into projectors. Projectors decompose comodules into their irreducible parts. As physical quantities are usually associated with irreducible representations of a symmetry group, the projectors will play a very important role in developing the physical structure.

There is another essential requirement for physical objects – this is hermiticity. Thus the algebraic structure has to allow conjugation, first on a purely algebraic level. If in a physical interpretation the algebra is represented by linear operators in a Hilbert space, the conjugation operation of linear operators in the Hilbert space should be identified with the algebraic conjugation. We demand that algebraic hermiticity should correspond to essential selfadjointness of an observable.

The concept of configuration space will be generalized to the concept of phase space. There are good reasons for a physicist to do this and there might be different ways how to do it. We shall define momentum by the concept of derivatives. A differential calculus on quantum planes has been developed in [2]. It can be based on the R matrix structure as well. There is, however, no natural definition of a conjugation. The situation is as follows. The calculus can be based on $\hat{R}$ or $\hat{R}^{-1}$. Both calculi are covariant and consistent but not linearly related. Conjugation then maps one calculus to the other one. Such a conjugation would require the introduction of two types of derivatives. We could then take the real and imaginary part and thus have two types of momenta. Unfortunately it is not possible to find a consistent and covariant algebraic structure that would involve the coordinates and only one type of momenta. Commutators of the coordinates with the real part produces the imaginary part and vice versa. To enlarge phase space in this way does not seem to be the proper way out. Thus we will follow a different strategy here, one that has been developed by studying a simple one dimensional version of a quantum space [3].

The two types of derivatives can be related nonlinearly. That this is also possible for the Minkowski space has first been seen in [4]. The system has an ordered and consistent basis in terms of the coordinates and one type of derivative. Conjugation is possible but as a nonlinear operation. We take the imaginary part of the derivatives as the definition of the momenta. Then we define an algebraic structure where the commutation relations between coordinates and momenta close into the product of a unitary operator – the scaling operator – and of hermitean operators. The fact that a unitary operator has to occur was already seen in the one dimensional model – that there is only one operator of this type, the scaling operator for the higher dimensional case was first seen by W. Weich [5] when he analyzed

the three dimensional case of the Euclidean quantum space. The hermitean operators are closely related to orbital angular momentum and have a clear physical meaning. Thus we have an algebra that is generated by observables as we know them already in the undeformed system. In the deformed system, however, coordinates, momenta and orbital angular momentum are all part of one algebra, the algebra does not separate into the Heisenberg algebra and the algebra of orbital angular momenta. This new algebra can be realized in the algebra of coodinates and derivatives which have the Poincaré-Birkhoff-Witt property. Such a property is not obvious for the algebra of observables by itself.

The four dimensional case [6] has been studied in a thesis by Ocampo Durán for $SO_q(4)$ and in a thesis by Zippold for $SO_q(1,3)$.

In this paper we discuss the Euclidean version in three dimensions first. The algebra as it is represented in the coordinate-derivative basis is derived in Chap. 2. In Chap. 3 we state the algebra in terms of the observables and discuss angular momentum. The Minkowski case is studied along the same lines in Chap. 4 and Chap. 5. A notation is chosen that exhibits the $SO_q(3)$ substructure of $SO_q(1,3)$. Some useful formulas are collected in the appendices.

2 Euclidean space

The symmetry group of the three-dimensional Euclidean space is $SO(3)$. This symmetry can be generalized to the quantum group $SO_q(3)$ and the representations to the corresponding quantum spaces. This way the three-dimensional non-commutative Euclidean space is defined.

Our treatment will be based on the $\hat{R}$ matrix of $SO_q(3)$ and its projector decomposition [7]. The $\hat{R}$ matrix satisfies the Yang-Baxter equation and has the following decomposition:

$$\hat{R} = P_5 - \frac{1}{q^4}P_3 + \frac{1}{q^6}P_1 \tag{2.1}$$

$$1 = P_5 + P_3 + P_1 \tag{2.2}$$

The projectors are 9 by 9 matrices, they are explicitly given in Appendix 1.

For $SO_q(3)$, $\hat{R}$ has the eigenvalues $1, -q^{-4}, q^{-6}$ with multiplicity five, three and one respectively. The projectors project on the respective eigenspaces. They can be represented as polynomials in the $\hat{R}$ matrix. This can be seen from the characteristic equation that states that $1, -q^{-4}, q^{-6}$ are the eigenvalues of $\hat{R}$.

$$(\hat{R} - 1)(\hat{R} + \frac{1}{q^4})(\hat{R} - \frac{1}{q^6}) = 0 \tag{2.3}$$

The normalization of the projectors is easily computed, the orthogonality follows from (2.3):

$$P_5 = \frac{q^{10}}{(q^4+1)(q^6-1)}(\hat{R} + \frac{1}{q^4})(\hat{R} - \frac{1}{q^6})$$

$$P_3 = \frac{q^{10}}{(q^4+1)(q^2+1)}(\hat{R} - 1)(\hat{R} - \frac{1}{q^6}) \tag{2.4}$$

$$P_1 = \frac{q^{12}}{(q^2+1)(1-q^6)}(\hat{R} - 1)(\hat{R} + \frac{1}{q^4})$$

Note that it is only for $q \neq 1$ that the eigenvalues separate three different eigenspaces.

The quantum group symmetry is defined by the $\hat{R}TT$ relations:

$$\hat{R}^{ij}_{\ kl}T^k_{\ m}T^l_{\ n} = T^i_{\ k}T^j_{\ l}\hat{R}^{kl}_{\ mn} \tag{2.5}$$

We use index notation, repeated indices are to be summed.

Equation (2.5) defines the algebraic properties of the quantum group matrix T, in our case $SO_q(3)$.

It is obvious that in (2.5) $\hat{R}$ can be replaced by any polynomial in $\hat{R}$. The projectors are explicit examples. To make the covariance of the projector decomposition more transparent we start from the transformation law of contravariant vectors:

$$\omega(X^A) = T^A_{\ B} \otimes X^B$$
$$\omega(Y^A) = T^A_{\ B} \otimes Y^B \tag{2.6}$$

Any relation, formulated with polynomials $P[\hat{R}]$ will be covariant:

$$\omega(P[\hat{R}]XY) = P[\hat{R}]\omega(X)\omega(Y) \tag{2.7}$$

The projectors P_1, P_3 and P_5 just reflect the fact that the product of two vectors decomposes into a singlet (P_1), a triplet (P_3) and a quintuplet (P_5). The projector P_3 is a generalization of the antisymmetrizer.

We define the non-commutative Euclidean space by the relation:

$$P_3 XX = 0 \tag{2.8}$$

This generalizes the property that the coordinates of ordinary Euclidean space commute and it is covariant under $SO_q(3)$.

An explicit version of (2.8), in a bases that is adjusted to the quantum group terminology is:

$$X^3 X^+ = q^2 X^+ X^3$$
$$X^3 X^- = q^{-2} X^- X^3$$
$$X^- X^+ = X^+ X^- + \lambda X^3 X^3 \tag{2.9}$$

where q is a real deformation parameter and $\lambda = q - \frac{1}{q}$.

The relations (2.9) are compatible with the conjugation assignment:

$$\overline{X^3} = X^3$$
$$\overline{X^+} = -qX^-$$
$$\overline{X^-} = -\frac{1}{q}X^+ \tag{2.10}$$

Our basis is a generalization of $X^3, X^\pm = i(X^1 \pm iX^2)$ in the commutative space.

We now study some properties of the projectors that will be useful in the following. First we note that the $\hat{R}$ matrix (2.1) and its inverse is the unique solution of the Yang Baxter equation that can be obtained by allowing matrices that are linear combinations of the three projectors [8].

We proceed by defining a metric tensor g and an ε tensor. The one dimensional subspace, obtained by acting with P_1 on XY is invariant under $SO_q(3)$. It can be used to define an invariant metric:

$$X^3Y^3 - qX^+Y^- - \frac{1}{q}X^-Y^+ \equiv g_{AB}X^AY^B \equiv X \circ Y$$

$$g_{33} = 1, \quad g_{+-} = -q, \quad g_{-+} = -\frac{1}{q} \tag{2.11}$$

As usual, g^{AB} is defined as the inverse matrix.

If X^A transforms contravariantly, the corresponding vector:

$$X_A = g_{AB}X^B \tag{2.12}$$

transforms covariantly. From a direct calculation follows

$$X^A(X \circ X) = (X \circ X)X^A \tag{2.13}$$

The invariant length of the vector X commutes with all components of X.

The metric can be used to formulate the conjugation properties (2.10) of X:

$$\overline{X^A} = g_{AB}X^B \quad , \tag{2.14}$$

$\overline{X^A}$ transforms covariantly.

The projector P_1 can be expressed in terms of the metric tensor:

$$P_1^{AB}{}_{CD} = \frac{q^2}{1 + q^2 + q^4}g^{AB}g_{CD} \tag{2.15}$$

Another convenient concept is the ε-tensor. It combines two vectors to a third vector and thus can be obtained from P_3.

$$Z^A = X^C Y^B \varepsilon_{BC}{}^A \tag{2.16}$$

We list its components:

$$\begin{aligned}
\varepsilon_{+-}{}^3 &= q, \quad \varepsilon_{-+}{}^3 = -q, \quad \varepsilon_{33}{}^3 = 1 - q^2, \\
\varepsilon_{+3}{}^+ &= 1, \quad \varepsilon_{3+}{}^+ = -q^2, \\
\varepsilon_{-3}{}^- &= -q^2, \quad \varepsilon_{3-}{}^- = 1.
\end{aligned} \tag{2.17}$$

All other components vanish. Indices of the ε tensor will be raised and lowered with the metric g.

$$\varepsilon_{ABC} = g_{CD}\varepsilon_{AB}{}^D, \varepsilon^{FAB} = g^{FE}g^{AD}\varepsilon_{ED}{}^B \tag{2.18}$$

The relations (2.9) can be formulated with the ε-tensor:

$$X^C X^B \varepsilon_{BC}{}^A = 0 \tag{2.19}$$

The projector P_3 can be expressed with the ε-tensor:

$$P_3^{AB}{}_{CD} = \frac{1}{1 + q^4}\varepsilon^{FAB}\varepsilon_{FDC} \tag{2.20}$$

More formulas for the ε-tensor will be given in App. 1.

From (2.2) follows a simple form for P_5

$$P_5^{AB}{}_{CD} = \delta_C^A \delta_D^B - \frac{q^2}{1 + q^2 + q^4}g^{AB}g_{CD}$$
$$- \frac{1}{1 + q^4}\varepsilon^{FAB}\varepsilon_{FDC} \tag{2.21}$$

and from (2.1) another for the $\hat{R}$ matrix

$$\hat{R}_{CD}^{AB} = \delta_C^A \delta_D^B - q^{-4}\varepsilon^{FAB}\varepsilon_{FDC} - q^{-4}(q^2 - 1)g^{AB}g_{CD}. \tag{2.22}$$

The next step to a quantum mechanical system is the definition of derivatives. The formalism of derivatives in quantum spaces has been discussed in [9]. Consistent and covariant Leibniz rules have been found. In our case they are:

$$\partial_B X^A = \delta_B^A + q^4 \hat{R}^{AC}{}_{BD}X^D\partial_C \quad \text{or}$$
$$\hat{\partial}_B X^A = \delta_B^A + q^{-4}\hat{R}^{-1AC}{}_{BD}X^D\hat{\partial}_C \tag{2.23}$$

We shall work with the first definition. The derivatives ∂_A transform covariantly. The corresponding contravariant derivative is:

$$\partial^A = g^{AB}\partial_B \tag{2.24}$$

Its Leibniz rules follow from (2.23) and properties of the $\hat{R}$ matrix, such as

$$\begin{aligned}
g^{CB}\hat{R}^{AF}{}_{BD}g_{FE} &= q^{-4}\hat{R}^{-1CA}{}_{DE} \\
g^{AF}\hat{R}^{-1BE}{}_{FC}g_{ED} &= q^4\hat{R}^{AB}{}_{CD}
\end{aligned} \tag{2.25}$$

We find:

$$\partial^A X^B = g^{AB} + \hat{R}^{-1AB}{}_{CD}X^C\partial^D \tag{2.26}$$

We also know the $\partial\partial$ relations [9]. They are:

$$P_3^{AB}{}_{CD}\partial^C\partial^D = 0 \tag{2.27}$$

or, equivalently

$$\partial^C\partial^D\varepsilon_{DC}{}^A = 0. \tag{2.28}$$

A further step towards a quantum mechanical system is the introduction of a conjugation operation. For the space coordinates these are the relations (2.10) or (2.14). There is no natural way how conjugation can be defined for derivatives.

We shall first extend the algebraic system by adding conjugate derivatives:

$$\overline{\partial^A} \equiv \overline{\partial}_A \quad , \tag{2.29}$$

and then try to find consistent relations that express $\overline{\partial}_A$ in terms of ∂^A and X^A. This way we reduce the algebra again to elements generated by X and ∂. That this is possible follows from experience. To proceed we first conjugate (2.23), we invert the relation to find an expression for $\overline{\partial}X$ in terms of $X\overline{\partial}$ and raise all the indices. The result is:

$$\overline{\partial}^A X^B = -\frac{1}{q^6}g^{AB} + \hat{R}^{AB}{}_{CD}X^C\overline{\partial}^D \tag{2.30}$$

This has to be compared with (2.26). However, there is no linear relation between ∂ and $\overline{\partial}$ that would reconcile (2.30) with (2.26).

We now complete the algebra by computing the $\overline{\partial}\partial$ and $\partial\partial$ relations. There we can rely on methods developed in [4] and we find:

$$\overline{\partial}^C\overline{\partial}^D\varepsilon_{DC}{}^A = 0 \tag{2.31}$$

and

$$\partial^B\overline{\partial}^A = \frac{1}{q^4}\hat{R}^{-1BA}{}_{CD}\overline{\partial}^C\partial^D. \tag{2.32}$$

An immediate consequence of (2.32) is

$$P_3^{AB}{}_{CD}(\partial^C\overline{\partial}^D + \overline{\partial}^C\partial^D) = 0 \tag{2.33}$$

To show this we apply the projector P_3 to (2.32) and use the projector decomposition of $\hat{R}$.

We shall now show that $\overline{\partial}$ can be expressed algebraically in terms of X and ∂. There is no general formalism to be guided by; instead we follow the work done in [10]. Proving a relation means to write the relation in a fixed ordering in X and ∂, using (2.26),(2.8) and (2.28) for ordering. The ordered monominals are then a basis of the algebra. The calculations are sometimes lengthy – a more detailed description can be found in [10].

We first introduce the scaling operator Λ:

$$\Lambda = 1 + (q^4 - 1)(X \circ \partial) + q^2(q^2 - 1)^2(X \circ X)(\partial \circ \partial) \quad (2.34)$$

An explicit calculation with the strategy outlined above shows the scaling properties:

$$\Lambda X^A = q^4 X^A \Lambda$$
$$\Lambda \partial^A = q^{-4} X^A \Lambda \quad (2.35)$$

The normalization of Λ is such that it starts with 1. The inverse Λ^{-1} or the square roots $\Lambda^{-1/2}$ or $\Lambda^{1/2}$ will be defined algebraically by a power series expansion.

It turns out that

$$\overline{\partial}^A = -\frac{1}{q^6}\Lambda^{-1}[\partial^A + q^2(q^2 - 1)X^A(\partial \circ \partial)] \quad (2.36)$$

satisfies all the relations (2.30),(2.31) and (2.32). Thus we divide the $X, \partial, \overline{\partial}$ algebra by the ideal generated by (2.36) and obtain an algebra that is generated by X and ∂ only.

An immediate consequence of (2.36) is

$$\overline{\Lambda} = q^{-12}\Lambda^{-1} \quad (2.37)$$

Up to a normalization, Λ is unitary.

In a physical system, the relevant objects (observables) should be hermitean. A natural way to define hermitean momenta is

$$P^A = -\frac{i}{2}(\partial^A - \overline{\partial}^A), \quad (2.38)$$

where $\overline{\partial}^A$ is the complicated expression (2.36). P^A is a contravariant vector and from (2.28),(2.31) and (2.33) follows that it satisfies the relations:

$$P^A P^B \varepsilon_{BA}{}^C = 0 \quad (2.39)$$

Its conjugation properties, following from its definition are:

$$\overline{P^A} = P_A = g_{AB}P^B \quad (2.40)$$

Thus, momentum space is completely analogous to coordinate space. To complete the algebra, it remains to specify the XP relations. If we compute them from (2.8), (2.26) and (2.27), the right-hand-side depends on ∂ and $\overline{\partial}$ separately and not on P only. We could introduce $\partial^A + \partial^{\overline{A}}$ together with P^A and thus have an algebra in terms of hermitean objects. But this would introduce another three-vector in addition to the momentum P^A and there is no obvious interpretation of this quantity as a physical object. Our aim, however, is to define an algebra that contains only objects that could be viewed as deformations of well-known physical objects. To this end we will try to write the right-hand-side as a product of unitary operators and hermitean operators. It will turn out that this is possible and the only unitary operator will be Λ. The hermitean operators will be closely related to orbital angular momentum.

We start from (2.26) and (2.30) and compute:

$$(\partial^A - \overline{\partial}^A)X^B - \hat{R}^{-1AB}{}_{CD}X^C(\partial^D - \overline{\partial}^D) =$$
$$(1 + q^{-6})g^{AB} + (\hat{R}^{-1AB}{}_{CD} - \hat{R}^{AB}{}_{CD})X^C\overline{\partial}^D \quad (2.41)$$

From (2.1) follows that on the right-hand-side of (2.41) P_5 drops out. Only P_1 and P_3 remain, they can be expressed in terms of g^{AB} and ε^{ABF}. We find:

$$P^A X^B - \hat{R}^{-1AB}{}_{CD}X^C P^D$$
$$= -\frac{i}{2}(1 + \frac{1}{q^6})g^{AB}(1 + q^2(q^2 - 1)X \cdot \overline{\partial})$$
$$+\frac{i}{2}(1 - \frac{1}{q^4})\varepsilon^{ABF}\varepsilon_{DCF}X^C\overline{\partial}^D \quad (2.42)$$

Let us first look at the singlet term. We use (2.34) and can show that:

$$1 + q^2(q^2 - 1)X \circ \overline{\partial} = \Lambda^{-1}(1 + q^2(q^2 - 1)X \circ \partial) \quad (2.43)$$

Moreover:

$$1 + q^2(q^2 - 1)\overline{X \circ \partial} = q^{-6}\Lambda^{-1}(1 + q^2(q^2 - 1)X \circ \partial) \quad (2.44)$$

Thus there is a hermitean expression

$$W = \Lambda^{-1/2}(1 + q^2(q^2 - 1)X \circ \partial)$$
$$\overline{W} = W \quad (2.45)$$

and

$$(1 + q^2(q^2 - 1)X \cdot \overline{\partial}) = \Lambda^{-1/2}W \quad (2.46)$$

We have factorized the g^{AB} term of (2.42) into the product of the unitary operator $q^{-3}\Lambda^{-1/2}$ and a hermitean operator.

A similar calculation shows that

$$\Lambda^{1/2}X^C\overline{\partial}^D\varepsilon_{DC}{}^A = L^A \quad (2.47)$$

has the proper conjugation properties

$$\overline{L^A} = g_{AB}L^B = L_A \quad . \quad (2.48)$$

We have achieved the factorization of (2.42) into the product of the unitary operator $q^{-3}\Lambda^{-1/2}$ and hermitean operators:

$$P^A X^B - \hat{R}^{-1AB}{}_{CD}X^C P^D =$$
$$-\frac{i}{2}\Lambda^{-1/2}\{(1 + q^{-6})g^{AB}W - (1 - q^{-4})\varepsilon^{ABF}L_F\} \quad (2.49)$$

To complete the algebra we have to compute the LX, LP, WX, WP, LW and LL relations. This will be done in the X, ∂ bases as it was explained above. Though the calculations are lengthy and tedious, the result is reasonably simple. Some useful formulas for intermediate steps can be found in [10].

We find:

$$\varepsilon_{BA}{}^C L^A L^B = -\frac{1}{q^2}WL^C$$

$$L^A W = WL^A$$
$$WX^A = (1 + \lambda^2)X^A W + (q^2 - 1)^2\varepsilon^{AFC}X_C L_F$$
$$WP^A = (1 + \lambda^2)P^A W + (q^2 - 1)^2\varepsilon^{AFC}P_C L_F \quad (2.50)$$

For further comparison, it is useful to introduce a notation that exhibits the "antisymmetric" nature of L^A:

$$M^{AB} = \varepsilon^{ABF}L_F, \quad M^{AB}\varepsilon_{BAF} = (1 + q^4)L_F \quad (2.51)$$

The last two equations of (2.50) take the form:

$$WX^A = (1 + \lambda^2)X^A W + (q^2 - 1)^2 g_{DB} X^D M^{BA}$$
$$WP^A = (1 + \lambda^2)P^A W + (q^2 - 1)^2 g_{DB} P^D M^{BA} \tag{2.52}$$

Finally, the LX and LP relations:

$$L^A X^F = -\frac{1}{q^4}\varepsilon^{AFK} X_K W - \frac{1}{q^2}\varepsilon_{BC}{}^A \varepsilon^{BFD} X^C L_D$$

$$M^{AB} X^F = -q^{-4}(1 + q^4)P_3^{AB}{}_{ML} X^M (g^{LF} W + q^2 M^{LF})$$

$$L^A P^F = -\frac{1}{q^4}\varepsilon^{AFK} P_K W - \frac{1}{q^2}\varepsilon_{BC}{}^A \varepsilon^{BFD} P^C L_D$$
$$\tag{2.53}$$

The algebraic relations (2.8), (2.39), (2.49), (2.50) and (2.53) together with (2.35) and the fact that Λ commutes with L and W determine the algebra. Its "hermiticity" properties are given by: (2.14), (2.37), (2.40), (2.45) and (2.48). There is, however, one more relation that follows from the fact that all these elements have been expressed in terms of X and ∂.

We find that $L \circ L$ commutes with all the L's:

$$L^A(L \circ L) = (L \circ L)L^A \tag{2.54}$$

It is something like a Casimir operator for the L algebra. The same is true for W. We also find that the combination

$$q^4(q^2 - 1)^2 L \circ L - W^2 + 1 \tag{2.55}$$

commutes with X and ∂ and thus is central in all the algebra. Writing it in an ordered $X\partial$ basis shows that is is zero. Thus it is consistent to impose

$$W^2 - 1 = q^4(q^2 - 1)^2 L \circ L \tag{2.56}$$

on the algebra even if we do not refer to a realization of the algebra in terms of X and ∂. Such an algebraic approach will be the aim of the next chapter.

3 Euclidean phase space

In the previous chapter the Euclidean phase space algebra has been obtained by representing it in the algebra generated by X^A and ∂^A. We now consider the Euclidean phase space algebra in its own right without referring to the X^A, ∂^A algebra any more. We start with the defining relations:

$$X^C X^B \varepsilon_{BC}{}^A = 0$$
$$P^C P^B \varepsilon_{BC}{}^A = 0 \tag{3.1}$$
$$L^C L^B \varepsilon_{BC}{}^A = -\frac{1}{q^2}WL^A$$

$$L^A X^B = -\frac{1}{q^4}\varepsilon^{ABC} X_C W - \frac{1}{q^2}\varepsilon_{KC}{}^A \varepsilon^{KBD} X^C L_D$$

$$L^A P^B = -\frac{1}{q^4}\varepsilon^{ABC} P_C W - \frac{1}{q^2}\varepsilon_{KC}{}^A \varepsilon^{KBD} P^C L_D \tag{3.2}$$

$$L^A W = WL^A$$

$$WX^A = (1 + \lambda^2)X^A W + (q^2 - 1)^2 \varepsilon^{ABC} X_C L_B \tag{3.3}$$
$$WP^A = (1 + \lambda^2)P^A W + (q^2 - 1)^2 \varepsilon^{ABC} P_C L_B$$

$$P^A X^B - \hat{R}^{-1AB}{}_{CD} X^C P^D \tag{3.4}$$
$$= -\frac{i}{2}\Lambda^{-1/2}\left\{(1 + q^{-6})g^{AB} W - (1 - q^{-4})\varepsilon^{ABC} L_C\right\}$$

$$\Lambda^{1/2} X^A = q^2 X^A \Lambda^{1/2}$$
$$\Lambda^{1/2} P^A = q^{-2} P^A \Lambda^{1/2} \tag{3.5}$$
$$\Lambda^{1/2} L^A = L^A \Lambda^{1/2}$$
$$\Lambda^{1/2} W = W \Lambda^{1/2}$$

These are relations that allow us to change the order of the elements of the algebra. In addition we have to postulate:

$$q^4(q^2 - 1)^2 L \circ L = W^2 - 1 \tag{3.6}$$

Our phase space algebra is generated by the elements $X, P, L, \Lambda^{-1/2}, W$ and has to be divided by the ideal generated by the relations (3.1) to (3.6). But, different from the X, ∂ algebra where the ordered $X\partial$ monomials were a basis for the algebra, we do not have a basis of this type here. Equation (3.4) shows that an ordering of X and P involves the elements L as well and neither L nor W can be expressed in terms of ordered XP polynomials. We rather find from (3.4):

$$-\frac{i}{2}(1 - q^{-4})(1 + q^4)\Lambda^{-1/2} L^A$$
$$= (P^B X^C + q^4 X^B P^C)\varepsilon_{CB}{}^A \tag{3.7}$$

and

$$-\frac{i}{2}q^{-8}(1 + q^6)(1 + q^2 + q^4)\Lambda^{-1/2} W = P \circ X - q^6 X \circ P$$
$$\tag{3.8}$$

The conjugation properties that are consistent with the algebra and that we will impose are:

$$\overline{X^A} = X_A, \quad \overline{P^A} = P_A, \quad \overline{L^A} = L_A \tag{3.9}$$
$$\overline{W} = W, \quad \overline{\Lambda^{1/2}} = q^{-6}\Lambda^{-1/2}$$

Let us get some more insight into the physical meaning of L. Equation (3.7) suggests that L is very closely related to orbital angular momentum. Multiplying (3.2) with g_{AB} we find

$$L \circ X = -\frac{1}{q^2}(1 + q^4)X \circ L \tag{3.10}$$

We can compute $X \circ L$ and $L \circ X$ from (3.7) and find

$$L \circ X = q^2 X \circ L$$

therefore

$$X \circ L = L \circ X = 0 \tag{3.11}$$

Similarly:

$$L \circ P = P \circ L = 0 \tag{3.12}$$

These two equations have to be expected for orbital angular momentum. They are from a physical point of view very reasonable but again show that we do not have an obvious Poincaré-Birkhoff-Witt property for our algebra.

Finally, we want to study the action of the L operators on the coordinates to see in what sense the L algebra generates the $SO_q(3)$ symmetry.

To check if L commutes with $X \circ X$ we make use of (3.2) and (3.3). We find

$$L^A(X \circ X) = (X \circ X)L^A \tag{3.13}$$

We had to use (3.11) explicitly.

Thus (3.2) cannot represent the action of L on a general vector. There is, however, a way to write the action of the algebra on X that can be abstracted to an action on a general vector. To demonstrate this we start with the action of L^+ and write it in a special form using (3.11):

$$L^+X^+ = X^+L^+$$
$$L^+X^3 = X^3L^+ + q^{-2}X^+(W + q^2(1-q^2)L^3) \tag{3.14}$$
$$L^+X^- = X^-L^+ + q^{-3}X^3(W + q^2(1-q^2)L^3)$$

We abbreviate the combination of W and L^3 that occurs at the right hand side:

$$\tau^{-1/2} = W + q^2(1-q^2)L^3 \tag{3.15}$$

Using (3.11) we find that L^- reproduces $\tau^{-1/2}$ as well:

$$L^-X^+ = X^+L^- - q^{-3}X^3\tau^{-1/2}$$
$$L^-X^3 = X^3L^- - q^{-4}X^-\tau^{-1/2} \tag{3.16}$$
$$L^-X^- = X^-L^-$$

Finally:

$$\tau^{-1/2}X^+ = q^2X^+\tau^{-1/2}$$
$$\tau^{-1/2}X^3 = X^3\tau^{-1/2} \tag{3.17}$$
$$\tau^{-1/2}X^- = q^{-2}X^-\tau^{-1/2}$$

If we now assume that X and Y are two arbitrary vectors on which L^+, L^- and $\tau^{-1/2}$ act as above we find:

$$L^\pm X \circ Y = X \circ YL^\pm, \quad \tau^{-1/2}X \circ Y = X \circ Y\tau^{-1/2} \tag{3.18}$$

This suggests that $L^+, L^-, \tau^{-1/2}$ represent the algebra suitably. We are going to show that this algebra closes and that W and L^3 can be expressed in terms of L^+, L^- and $\tau^{-1/2}$, using (3.6).

We write the third component of the $L \circ L$ relation (3.1) and (3.6) explicitly:

$$qL^-L^+ - qL^+L^- = -\frac{1}{q^2}\tau^{-1/2}L^3 \tag{3.19}$$

$$q^4(1-q^2)^2(qL^+L^- + \frac{1}{q}L^-L^+)$$
$$= 1 - W^2 + q^4(q^2-1)^2L^3L^3 \tag{3.20}$$

These two equations can be combined to:

$$qL^+L^- - \frac{1}{q}L^-L^+ = \frac{1}{q^4(1-q^4)}(\tau^{-1} - 1) \tag{3.21}$$

This is the equation that shows that the algebra L^+, L^- and τ closes.

Equation (3.19) can now be used to express L^3 through elements of the algebra:

$$L^3 = q^3\tau^{1/2}(L^+L^- - L^-L^+) \tag{3.22}$$

and from (3.15) follows

$$W = \tau^{-1/2} + q^5(q^2-1)\tau^{1/2}(L^+L^- - L^-L^+) \tag{3.23}$$

The independent elements $L^+, L^-, \tau^{-1/2}$ form an algebra, closely related to $SO_q(3)$:

$$\tau^{-1/2}L^+ = q^2L^+\tau^{-1/2}$$
$$\tau^{-1/2}L^- = q^{-2}L^-\tau^{-1/2} \tag{3.24}$$
$$qL^+L^- - \frac{1}{q}L^-L^+ = \frac{1}{q^4(1-q^4)}(\tau^{-1} - 1)$$

If we define

$$T^+ = q^2\sqrt{1+q^2} \; \tau^{1/2}L^+$$
$$T^- = q^3\sqrt{1+q^2} \; \tau^{1/2}L^- \tag{3.25}$$
$$T^3 = \frac{q}{q^2-1}(1-\tau)$$

we obtain the well-known $SO_q(3)$ algebra

$$\frac{1}{q}T^+T^- - qT^-T^+ = T^3$$
$$q^2T^3T^+ - \frac{1}{q^2}T^+T^3 = (q + \frac{1}{q})T^+ \tag{3.26}$$
$$q^2T^-T^3 - \frac{1}{q^2}T^3T^- = (q + \frac{1}{q})T^-$$

The action on the coordinates is:

$$T^3X^3 = X^3T^3$$
$$T^3X^+ = q^{-4}X^+T^3 + q^{-1}(1+q^{-2})X^+$$
$$T^3X^- = q^4X^-T^3 - q(1+q^2)X^-$$
$$T^+X^3 = X^3T^+ + q^{-2}\sqrt{1+q^2}X^+$$
$$T^+X^+ = q^{-2}X^+T^+ \tag{3.27}$$
$$T^+X^- = q^2X^-T^+ + q^{-1}\sqrt{1+q^2}X^3$$
$$T^-X^3 = X^3T^- + q\sqrt{1+q^2}X^-$$
$$T^-X^+ = q^{-2}X^+T^- + \sqrt{1+q^2}X^3$$
$$T^-X^- = q^2X^-T^-$$

For physical applications [11] we have to study Hilbertspace representations of this algebra where all the hermitean elements are represented by (essentially) self-adjoint operators. Such representations are studied in [5].

4 Minkowski space

The non-commutative Minkowski space will be developed along the same line as the non-commutative Euclidean space in Chap. 2. Now the basic symmetry is the q-deformed Lorentzgroup that acts on a 4-dimensional quantum space. This quantum group has $SO_q(3)$ as a substructure and we choose a notation that makes use of this fact:

$$\{X^a\} = \{X^0, X^A\} = \{X^0, X^3, X^+, X^-\} \tag{4.1}$$

The product of two 4 vectors can be decomposed into four representations: a singlet (trace), a nonet (symmetric, traceless) and two triplets (antisymmetric, selfdual and anti-selfdual). Accordingly, there are four projectors:

$$1 = P_T + P_S + P_+ + P_- \tag{4.2}$$

These are 16 by 16 matrices, an explicit representation is given in Appendix 2.

Now there are two $\hat{R}$ matrices [9] and their inverse that can be composed from these projectors:

$$\hat{R}_I = P_S + P_T - q^2 P_+ - q^{-2} P_- \tag{4.3}$$
$$\hat{R}_{II} = q^{-2} P_S + q^2 P_T - P_+ - P_-$$

It should be noted that $\hat{R}_I$ does not separate the symmetric and $\hat{R}_{II}$ does not separate the antisymmetric eigenspaces. The corresponding RTT relations define the q-deformed Lorentzgroup $SO_q(1,3)$.

We define the non-commutative Minkowski space by the relations:

$$P_+ XX = 0, \quad P_- XX = 0 \tag{4.4}$$

From (4.2) and (4.3) follows that this is equivalent to:

$$XX = \hat{R}_I XX \tag{4.5}$$

In the basis adopted at (4.1) these relations are:

$$X^0 X^A = X^A X^0 \tag{4.6}$$

$$\varepsilon_{DC}{}^A X^C X^D = (1 - q^2) X^0 X^A$$

The time variable X^0 commutes with the space variables, the commutation relations of space variables, however, close into the time variable as well.

A consistent definition of conjugation is:

$$\overline{X^0} = X^0, \quad \overline{X^A} = g_{AB} X^B \tag{4.7}$$

The metric tensor g as well as the three-dimensional ε tensor are the same as in chapt. 2. Explicitly, the conjugation is:

$$\overline{X^0} = X^0, \overline{X^3} = X^3, \overline{X^+} = -qX^-, \overline{X^-} = -\frac{1}{q} X^+ \tag{4.8}$$

The four-dimensional metric η_{ab} can be derived from P_T, the projector on a singlet:

$$\eta_{ab} X^a Y^b = -X^0 Y^0 + X^3 Y^3 - qX^+ Y^- - \frac{1}{q} X^- Y^+$$
$$\equiv X \circ Y \tag{4.9}$$

Its non-vanishing components are:

$$\eta_{00} = -1, \eta_{33} = 1, \eta_{+-} = -q, \eta_{-+} = -\frac{1}{q} \tag{4.10}$$

Again η^{ab} is the inverse matrix. We also find that $X \circ X$ commutes with X^A.

The projector P_T can be expressed in terms of η:

$$P_T{}^{ab}{}_{cd} = \frac{1}{(q + \frac{1}{q})^2} \eta^{ab} \eta_{cd} \tag{4.11}$$

A four-dimensional ε-tensor can be defined:

$$\varepsilon^{ab}{}_{cd} = P_+{}^{ab}{}_{cd} - P_-{}^{ab}{}_{cd} \tag{4.12}$$

That this object really generalizes the ε-tensor will become clear later. We also define:

$$P_A = P_+ + P_- \tag{4.13}$$

Again it is possible to use (4.2) and write the $\hat{R}$ matrices in the form:

$$\hat{R}_I = 1 - (1 + q^2) P_+ - (1 + \frac{1}{q^2}) P_- \tag{4.14}$$

$$\hat{R}_{II} = \frac{1}{q^2} + (q^2 - \frac{1}{q^2}) P_T - (1 + \frac{1}{q^2}) P_A \tag{4.15}$$

The Leibniz rules for consistent and Lorentz-covariant derivatives that we are going to use are the same as in [4]:

$$\partial_a X^b = \delta_a{}^b + \hat{R}_{II}{}^{bc}{}_{ad} X^d \partial_c \tag{4.15}$$

and as a consequence:

$$\partial_a \partial_b = \hat{R}_I{}^{cd}{}_{ba} \partial_d \partial_c \tag{4.16}$$

We have an algebra of X and ∂ that closes and that has ordered monomials as a basis. We can raise the indices of ∂_a with η and obtain from (4.15) and (4.16):

$$\partial^a X^b = \eta^{ab} + \frac{1}{q^2} \hat{R}_{II}^{-1ab}{}_{cd} X^c \partial^d \tag{4.17}$$

and

$$\partial^a \partial^b = \hat{R}_I{}^{ab}{}_{cd} \partial^c \partial^d \tag{4.18}$$

We find that (4.18) is equivalent to

$$P_+ \partial\partial = 0, \quad P_- \partial\partial = 0 \tag{4.19}$$

Again we face the conjugation problem. We enlarge the algebra by conjugate elements. We choose a special normalization to obtain covariant relations:

$$\overline{\partial^0} = -q^4 \hat{\partial}^0 \tag{4.20}$$
$$\overline{\partial^A} = -q^4 g_{AB} \hat{\partial}^B$$

We now follow the same procedure that led from (2.23) to (2.30) and obtain, starting from (4.15):

$$\hat{\partial}^a X^b = \eta^{ab} + q^2 \hat{R}_{II}{}^{ab}{}_{cd} X^c \hat{\partial}^d \tag{4.21}$$

We had to use the identity:

$$\eta^{ab} \hat{R}_{II}{}^{cd}{}_{be} \eta_{cf} = q^{-2} \hat{R}_{II}^{-1ac}{}_{ef} \tag{4.22}$$

The $X, \partial, \hat{\partial}$ algebra closes with the relations:

$$\hat{\partial}^a \hat{\partial}^b = \hat{R}_I{}^{ab}{}_{cd} \hat{\partial}^c \hat{\partial}^d \tag{4.23}$$

and

$$\partial^a \hat{\partial}^b = \hat{R}_{II}{}^{ab}{}_{cd} \hat{\partial}^c \partial^d \tag{4.24}$$

From (4.24) follows, similar to (2.33):

$$P_+(\partial\hat{\partial} + \hat{\partial}\partial) = 0, \quad P_-(\partial\hat{\partial} + \hat{\partial}\partial) = 0 \tag{4.25}$$

Of course, (4.23) has as a consequence:

$$P_+ \hat{\partial}\hat{\partial} = 0, \quad P_- \hat{\partial}\hat{\partial} = 0 \tag{4.26}$$

Again, $\hat{\partial}$ can be expressed algebraically in X and ∂. This has been done in [4]. The scaling operator is:

$$\Lambda = 1 + \frac{1}{q^2}(1 - q^2) X \circ \partial + \frac{1}{q^2} \frac{(q^2 - 1)^2}{(q^2 + 1)^2}(X \circ X)(\partial \circ \partial) \tag{4.27}$$

with the properties:

$$\Lambda X^a = \frac{1}{q^2} X^a \Lambda$$
$$\Lambda \partial^a = q^2 \partial^a \Lambda \tag{4.28}$$
$$\Lambda \hat{\partial}^a = q^2 \hat{\partial}^a \Lambda$$

The relation between $\hat{\partial}$ and X and ∂ is:

$$\hat{\partial}_a = \Lambda^{-1}\left[\partial_a + \frac{1}{q^2}\frac{(1-q^2)}{(1+q^2)} + X_a(\partial \circ \partial)\right] \qquad (4.29)$$

It is consistent with all the relations for $\hat{\partial}$ and it generates an ideal by which the $X, \partial, \hat{\partial}$ algebra can be divided. As a consequence of (4.29) we find:

$$\overline{\Lambda} = q^8 \Lambda^{-1} \qquad (4.30)$$

We define the hermitean momentum:

$$P^a = -\frac{i}{2}(\partial^a + q^4\hat{\partial}^a) \qquad (4.31)$$

and we find

$$\overline{P^0} = P^0, \quad \overline{P^A} = g_{AB}P^B \qquad (4.32)$$

From (4.19), (4.25) and (4.26) follows:

$$P_+PP = 0, \quad P_-PP = 0 \qquad (4.33)$$

or equivalently

$$PP = \hat{R}_I PP \qquad (4.34)$$

Momentum space is completely analogous to coordinate space. We continue as in Chap. 2 and derive the relation

$$P^a X^b - \frac{1}{q^2}\hat{R}_{II}^{-1ab}{}_{cd}X^c P^d =$$

$$-\frac{i}{2}\left\{(1+q^4)\eta^{ab}\left[1+q^2\frac{q^2-1}{q^2+1}X\circ\hat{\partial}\right]\right. \qquad (4.35)$$

$$\left.+q^2(1-q^4)P_A{}^{ab}{}_{cd}X^c\hat{\partial}^d)\right\}$$

We can show that

$$U = \Lambda^{1/2}\left\{1+q^2\frac{q^2-1}{q^2+1}X\circ\hat{\partial}\right\} \qquad (4.36)$$

is a hermitean operator. The relevant formulas can be found in [4]. We shall show later that

$$V^{ij} = \Lambda^{1/2}P_A{}^{ij}{}_{kl}X^k\hat{\partial}^l \qquad (4.37)$$

has linear conjugation properties as well. Thus we can write the relation (4.35) in the form:

$$P^a X^b - \frac{1}{q^2}\hat{R}_{II}^{-1ab}{}_{cd}X^c P^d = \qquad (4.38)$$

$$-\frac{i}{2}\Lambda^{-1/2}\left\{(1+q^4)\eta^{ab}U+q^2(1-q^4)V^{ab}\right\}$$

The UX, UP, VX, VP relations are:

$$UX^a = \frac{1}{q}\frac{q^4+1}{q^2+1}X^aU - \frac{1}{2q}(q^2-1)^2\eta_{bc}X^bV^{ca} \qquad (4.39)$$

$$UP^a = \frac{1}{q}\frac{q^4+1}{q^2+1}P^aU - \frac{1}{2q}(q^2-1)^2\eta_{bc}P^bV^{ca}$$

and

$$V^{ab}X^l = P_A^{ab}{}_{cd}X^c\left\{-(q+\frac{1}{q})V^{dl}+\frac{1}{q}\eta^{dl}U\right\} \qquad (4.40)$$

$$V^{ab}P^l = P_A^{ab}{}_{cd}P^c\left\{-(q+\frac{1}{q})V^{dl}+\frac{1}{q}\eta^{dl}U\right\}$$

These relations have the same structure as (2.52) and (2.53). For UV we find

$$UV = VU \qquad (4.41)$$

and finally:

$$P_A{}^{rs}{}_{ad}g_{bc}V^{ab}V^{cd} = \frac{1}{1+q^2}UV^{rs} \qquad (4.42)$$

The object V^{ab} has - due to the projector P_A in (4.37) - six independent elements. We gain more insight if we decompose V^{ab} into its selfdual and antiselfdual components. We define:

$$R^A = P_+{}^{A0}{}_{cd}V^{cd} \qquad (4.43)$$

$$S^A = \frac{1}{q^2}P_-{}^{A0}{}_{cd}V^{cd}$$

We can invert:

$$V^{A0} = R^A + q^2 S^A$$
$$V^{0A} = -q^2 R^A - S^A \qquad (4.44)$$
$$V^{AB} = \varepsilon^{ABC}(R_C - S_C)$$
$$V^{00} = 0$$

The VV relations (4.42) now become:

$$\varepsilon_{DA}{}^K R^A R^D = \frac{1}{1+q^2}UR^K \qquad (4.45)$$

$$\varepsilon_{DA}{}^K S^A S^D = -\frac{1}{1+q^2}US^K$$

and

$$R^A S^B = q^2 \hat{R}^{AB}{}_{CD}S^C R^D \qquad (4.46)$$

The $\hat{R}$ matrix in (4.46) is the Euclidean one (2.1). The Poincaré-Birkhoff-Witt property forces it that way.

The formulas for the RX relations are lengthy. They follow from (4.40):

$$R^A X^0 = \frac{1}{q}\frac{q^4+1}{q^2+1}X^0R^A + \frac{1}{q}\frac{q^2-1}{q^2+1}\varepsilon_{LM}{}^A X^M R^L$$

$$-\frac{q}{(1+q^2)^2}X^AU$$

$$R^A X^B = \frac{1}{1+q^2}\left[q(1+q^2)X^A R^B\right.$$

$$-\frac{1}{q}(q^2-1)\varepsilon_C{}^{AB}X^0R^C$$

$$-\frac{1}{q}(q^2-1)g^{AB}g_{MC}X^M R^C$$

$$-\frac{2}{q}\varepsilon^{ABG}\varepsilon_{STG}X^T R^S$$

$$-\frac{1}{q}\frac{1}{1+q^2}g^{AB}X^0U$$

$$\left.+\frac{1}{q}\frac{1}{1+q^2}\varepsilon_{LM}{}^A g^{LB}X^M U\right] \qquad (4.47)$$

For the RP relations, X has to be replaced by P in (4.47).

The conjugation property is easily stated for R and S:

$$\overline{R^A} = -S_A \qquad (4.48)$$

This immediately leads to the SX and SP relations as we know the conjugation properties of X and P. The V^{ab} algebra has two casimirs:

$$\mathbf{R}^2 = g_{AB}R^A R^B \tag{4.49}$$

and

$$\mathbf{S}^2 = g_{AB}S^A S^B \tag{4.50}$$

U is central in the U, V^{ab} algebra. We again find an expression that is central in the X, ∂ algebra and is zero if computed in the $X\partial$ basis:

$$(q^4 - 1)^2 \frac{1}{2}(\mathbf{R}^2 + \mathbf{S}^2) - (U^2 - 1) = 0. \tag{4.51}$$

5 Minkowski phase space

We now consider the algebra generated by the elements $X^a, P^a, V^{ab}, U, \Lambda^{1/2}$ and $\Lambda^{-1/2}$. The elements of V^{ab} form a deformed antisymmetric tensor that can be decomposed into its selfdual and antiselfdual components:

$$R^A = P_+{}^{A0}{}_{cd}V^{cd} \tag{5.1}$$

$$S^A = q^{-2}P_-{}^{A0}{}_{cd}V^{cd}$$

The inverse relation is:

$$V^{A0} = R^A + q^2 S^A \tag{5.2}$$

$$V^{0A} = -q^2 R^A - S^A$$

$$V^{AB} = \varepsilon^{ABC}(R_C - S_C)$$

$$V^{00} = 0$$

Some of the relations are simpler in terms of V^{ab}, other relations in terms of R^A and S^A.

We list the relations of the algebra:

$$X^0 X^A = X^A X^0 \tag{5.3}$$

$$X^C X^D \varepsilon_{DC}{}^A = (1 - q^2)X^0 X^A$$

$$P^0 P^A = P^A P^0$$

$$P^C P^D \varepsilon_{DC}{}^A = (1 - q^2)P^0 P^A$$

$$R^C R^D \varepsilon_{DC}{}^A = (1 + q^2)^{-1}U R^A$$

$$S^C S^D \varepsilon_{DC}{}^A = -(1 + q^2)^{-1}U S^A$$

$$R^A S^B = q^2 \hat{R}^{AB}{}_{CD}S^C R^D$$

$$V^{ab}X^f = P_A{}^{ab}{}_{cd}X^c \left\{-(q + q^{-1})V^{df} + q^{-1}\eta^{df}U\right\} \tag{5.4}$$

$$V^{ab}P^f = P_A{}^{ab}{}_{cd}P^c \left\{-(q + q^{-1})V^{df} + q^{-1}\eta^{df}U\right\}$$

$$V^{ab}U = UV^{ab}$$

$$UX^a = \frac{1}{q}\frac{q^4 + 1}{q^2 + 1}X^a U - \frac{1}{2q}(q^2 - 1)^2 \eta_{bc}X^b V^{ca} \tag{5.5}$$

$$UP^a = \frac{1}{q}\frac{q^4 + 1}{q^2 + 1}P^a U - \frac{1}{2q}(q^2 - 1)^2 \eta_{bc}P^b V^{ca}$$

$$P^a X^b - q^{-2}\hat{R}_{II}^{-1\,ab}{}_{cd}X^c P^d = \tag{5.6}$$

$$= -\frac{i}{2}\Lambda^{-1/2}\left\{(1 + q^4)\eta^{ab}U + q^2(1 - q^4)V^{ab}\right\}$$

$$\Lambda^{-1/2}X^a = qX^a \Lambda^{-1/2} \tag{5.7}$$

$$\Lambda^{-1/2}P^a = q^{-1}P^a \Lambda^{-1/2}$$

$$\Lambda^{-1/2}V^{ab} = V^{ab}\Lambda^{-1/2}$$

$$\Lambda^{-1/2}U = U\Lambda^{-1/2}$$

As an additional relation we postulate

$$U^2 - 1 = \frac{1}{2}(q^4 - 1)^2(R \circ R + S \circ S) \tag{5.8}$$

This defines the algebra.

The conjugation properties are:

$$\overline{X^0} = X^0 \quad \overline{X^A} = g_{AB}X^B \tag{5.9}$$

$$\overline{P^0} = P^0 \quad \overline{P^A} = g_{AB}P^B$$

$$\overline{R^A} = -S_A, \overline{U} = U$$

$$\overline{\Lambda^{1/2}} = q^4 \Lambda^{-1/2}$$

The conjugation properties of R, S, if compared with those of L (3.9) show that R or S do not generate $SO_q(3)$. We shall see that they are related to $SO_q(1, 3)$.

As in Chap. 3, additional relations occur. They are related to the intuitive picture that V^{ab} represents orbital angular momentum. In the undeformed case ($q = 1$) this has as a consequence that $\varepsilon_{abcd}X^b V^{cd} = 0$ and $\varepsilon_{abcd}P^b V^{cd} = 0$. The analogous formulas are also true in the deformed case if we define ε as in (4.12):

$$X^a \varepsilon_{abcd}V^{cd} = 0 \tag{5.10}$$

$$P^a \varepsilon_{abcd}V^{cd} = 0$$

This again has as a consequence that:

$$\varepsilon_{abcd}V^{ba}V^{cd} = 0 \tag{5.11}$$

or

$$S \circ S = R \circ R \tag{5.12}$$

These are relations that follow from the algebra (5.3)–(5.7).

As it was the case for the Euclidean space, R^A and S^A commute with $X \circ X$ or $P \circ P$:

$$R^A X \circ X = X \circ X R^A, \quad S^A X \circ X = X \circ X S^A \tag{5.13}$$

but we have to use (5.10) to obtain this result. Again, there is a way to write the actions of the V^{ab}, U algebra on X that can be abstracted to an action on a general vector. We follow the same line of arguments as in Chap. 3. We first observe that in the R^+X relations only R^+ and

$$\rho = (q^4 - 1)R^3 + U \tag{5.14}$$

occur:

$$R^+X^+ = qX^+R^+ \tag{5.15}$$

$$R^+X^- = q^{-1}X^-R^+ + \frac{1}{(1+q^2)^2}X^0\rho - \frac{1}{(1+q^2)^2}X^3\rho$$

$$R^+X^3 = -\frac{q}{(1+q^2)^2}X^+\rho + \frac{2q}{1+q^2}X^3R^+$$
$$+ q\frac{q^2-1}{q^2+1}X^0R^+$$

$$R^+X^0 = -\frac{q}{(1+q^2)^2}X^+\rho + \frac{1}{q}\frac{(q^2-1)}{q^2+1}X^3R^+$$
$$+ \frac{1}{q}\frac{q^4+1}{q^2+1}X^0R^+$$

The same is true for the ρX relations:

$$\rho X^+ = qX^+\rho + \frac{1}{q}(q^2-1)^2(q^2+1)(X^3-X^0)R^+ \tag{5.16}$$

$$\rho X^- = q^{-1}X^-\rho$$

$$\rho X^3 = \frac{2q}{1+q^2}X^3\rho + \frac{1}{q^2}(q^2-1)^2(q^2+1)X^-R^+$$
$$- \frac{1}{q}\frac{q^2-1}{q^2+1}X^0\rho$$

$$\rho X^0 = \frac{1}{q}\frac{(q^4+1)}{q^2+1}X^0\rho - \frac{q(q^2-1)}{q^2+1}X^3\rho$$
$$+ \frac{1}{q^2}(q^2-1)^2(q^2+1)X^-R^+$$

If R^+ and ρ act on general vectors X^a and Y^a as above we find

$$R^+X \circ Y = X \circ YR^+ \tag{5.17}$$

$$\rho X \circ Y = X \circ Y\rho$$

without any additional property of the R, ρ, X, Y algebra.

From the conjugation properties of R we see that we can expect the same for S^-. The element ρ has to be replaced by $\overline{\rho} = -\sigma$

$$\sigma \equiv (q^4-1)S^3 - U \tag{5.18}$$

We find:

$$S^-X \circ Y = X \circ YS^- \tag{5.19}$$

$$\sigma X \circ Y = X \circ Y\sigma$$

The R^-X and S^+X relations are different. They depend on R^3 and U or S^3 and U respectively and it is not possible to reduce the number of independent variables. Nevertheless R^- and S^+ commute with $X \circ Y$

$$R^-X \circ Y = X \circ YR^- \tag{5.20}$$

$$S^+X \circ Y = X \circ YS^+$$

This, however, is not true for R^3, S^3 or U if we start from (5.4) and (5.5).

Still, we can proceed as in Chap. 3. We write the third components of the RR and SS relations (5.3) explicitly.

$$qR^-R^+ - qR^+R^- = \frac{1}{(1+q^2)}\rho R^3 \tag{5.21}$$

$$qS^-S^+ - qS^+S^- = -\frac{1}{(1+q^2)}\sigma S^3$$

These relations allow us to eliminate R^3 and S^3 in terms of a nonlinear relation:

$$R^3 = q(1+q^2)\rho^{-1}(R^-R^+ - R^+R^-) \tag{5.22}$$

$$S^3 = q(1+q^2)\sigma^{-1}(S^-S^+ - S^+S^-)$$

In addition we have the relation (5.8) and (5.12):

$$R^3R^3 - qR^+R^- - \frac{1}{q}R^-R^+ = \frac{1}{(q^4-1)^2}(U^2-1) \tag{5.23}$$

$$S^3S^3 - qS^+S^- - \frac{1}{q}S^-S^+ = \frac{1}{(q^4-1)^2}(U^2-1)$$

The R relations (5.22) and (5.23) can be used to express R^+R^- and independently R^-R^+ in terms of U and ρ. Now it is possible to combine R^+R^- and R^-R^+ such that the resulting expression only contains ρ

$$\frac{1}{q}R^-R^+ - qR^+R^- = \frac{1}{(q^4-1)(q^2+1)^2}(\rho^2-1) \tag{5.24}$$

$$\rho R^+ = q^2R^+\rho$$

$$\rho R^- = q^2R^-\rho$$

The $S^+S^-\sigma$ algebra follows from conjugation.

For the RS, RX and SX relations R^3 and S^3 have to be eliminated explicitly via (5.22). This leads to inhomogeneous relations as it had to be expected from the analysis of the six-generator q-deformed Lorentz algebra in [12]. A Hilbert space representation of this algebra is studied in [6].

6 Appendix 1

This appendix gives some useful formulas for the Euclidean case. The $SO_q(3)$ $\hat{R}$ matrix is block-diagonal in the basis that is labelled by $++, --; +3, 3+; 3-, -3; +-, 33, -+$. The non-zero components of $\hat{R}$ and the projectors are:

$\hat{R}^{ab}_{cd}$:

	++	--
++	1	0
--	0	1

$$\tag{6.1}$$

	+3	3+
+3	0	q^{-2}
3+	q^{-2}	$1-q^{-4}$

	3-	-3
3-	0	q^{-2}
-3	q^{-2}	$1-q^{-4}$

$$\tag{6.2}$$

	+-	33	-+
+-	0	0	q^{-4}
33	0	q^{-2}	$q^{-1}(1-q^{-4})$
-+	q^{-4}	$q^{-1}(1-q^{-4})$	$(1-q^{-2})(1-q^{-4})$

$$\tag{6.3}$$

Projectors:

P_1:

	$+-$	$3\,3$	$-+$
$+-$	q^2	$-q$	1
$3\,3$	$-q$	1	$-q^{-1}$
$-+$	1	$-q^{-1}$	q^{-2}

$$(6.4)$$

P_3:

	$+3$	$3+$
$+3$	q^4	$-q^2$
$3+$	$-q^2$	1

	$3-$	-3
$3-$	q^4	$-q^2$
-3	$-q^2$	1

$$(6.5)$$

	$+-$	$3\,3$	$-+$
$+-$	q^2	$q(q^2-1)$	$-q^2$
$3\,3$	$q(q^2-1)$	$(q^2-1)^2$	$-q(q^2-1)$
$-+$	$-q^2$	$-q(q^2-1)$	q^2

$$(6.6)$$

There are some identities:

$$\hat{R}^{AB}{}_{CD} = \hat{R}^{CD}{}_{AB} \tag{6.7}$$

$$\mathbb{1} = P_5 + P_3 + P_1$$

$$\hat{R} = P_5 - \frac{1}{q^4}P_3 + \frac{1}{q^6}P_1 = \mathbb{1} - (1+\frac{1}{q^4})P_3 + (\frac{1}{q^6}-1)P_1$$

Metric tensor:

$$g_{AB}: \quad g_{+-} = -q, \quad g_{33} = 1, \quad g_{-+} = -\frac{1}{q} \tag{6.8}$$

$$g^{AB}: \quad g^{+-} = -q, \quad g^{33} = 1, \quad g^{-+} = -\frac{1}{q}$$

$$X_A = g_{AB}X^B, X^A = g^{AB}X_B$$

$$X \circ Y = g_{AB}X^A Y^B$$

ε-tensor:

$$\varepsilon_{ABC} = g_{CD}\varepsilon_{AB}{}^D$$

$$\varepsilon_{+-3} = q, \quad \varepsilon_{-+3} = -q, \varepsilon_{333} = 1-q^2, \quad \varepsilon_{+3-} = -\frac{1}{q},$$

$$\varepsilon_{3+-} = +q, \quad \varepsilon_{-3+} = +q^3, \quad \varepsilon_{3-+} = -q \tag{6.9}$$

$$\varepsilon_{RST} = g_{RA}\,g_{SB}\,g_{TC}\,\varepsilon^{ABC} = g_{AR}\,g_{BS}\,g_{CT}\,\varepsilon^{ABC}$$

$$\varepsilon_{BA}{}^C = \varepsilon_{SBA}\,g^{SC}, \quad \varepsilon_{RBA} = \varepsilon_{BA}{}^C g_{CR}$$

$$\varepsilon^{ABF}\varepsilon_{DCF} = \varepsilon^{FAB}\varepsilon_{FDC}$$

$$\varepsilon_D{}^{FE}\varepsilon_{EFR} = (1+q^4)g_{RD},$$

$$\varepsilon^{BCF}\varepsilon_{CBA} = (1+q^4)\delta_A^F$$

$$\varepsilon_{TSE}\,\varepsilon_{DC}{}^E = q^2(g_{CT}\,g_{DS} - g_{ST}\,g_{CD})$$

$$+g_{CB}\,g_{RT}\varepsilon^{BRE}\varepsilon_{SDE} \tag{6.10}$$

$$g^{BA}\varepsilon_{ABC} = 0, \quad g^{CB}\varepsilon_{ABC} = 0$$

$$\overline{\varepsilon_{BC}{}^A} = \varepsilon_{DEA}g^{EB}g^{DC} = \varepsilon^{CBK}g_{KA}$$

$$\overline{g_{AB}} = g^{AB}$$

$$Z_C = Y^B X^A \varepsilon_{ABC}$$

Raising and lowering indices of the $\hat{R}$ matrix:

$$g^{CB}\hat{R}^{AF}{}_{BD}\,g_{FE} = q^{-4}\hat{R}^{-1CA}{}_{DE}$$

$$g^{AF}\hat{R}^{-1BE}{}_{FC}\,g_{ED} = q^{+4}\hat{R}^{AB}{}_{CD}$$

$$g^{GC}g^{ED}\hat{R}^{BA}_{DC}\,g_{AF}\,g_{BK} = \hat{R}^{GE}_{FK}$$

7 Appendix 2

This appendix gives some useful formulas for the Minkowski case. The projectors are given in a basis that is adapted to the $SO_q(3)$ subalgebra. The metric g^{AB} and the ε-tensor ε^{ABC} are the same as in Appendix 1.

The projectors are:

P_+:

	00	$C0$	$0D$	CD
00	0	0	0	0
$A0$	0	$\frac{q^2}{(1+q^2)^2}\delta_C^A$	$-\frac{1}{(1+q^2)^2}\delta_D^A$	$\frac{1}{(1+q^2)^2}\varepsilon_{DC}{}^A$
$0B$	0	$-\frac{q^4}{(1+q^2)^2}\delta_C^B$	$\frac{q^2}{(1+q^2)^2}\delta_D^B$	$-\frac{q^2}{(1+q^2)^2}\varepsilon_{DC}{}^B$
AB	0	$\frac{q^2 g^{EB}g^{FA}\varepsilon_{FEC}}{(1+q^2)^2}$	$-\frac{g^{EB}g^{FA}\varepsilon_{FED}}{(1+q^2)^2}$	$\frac{\varepsilon_{DC}{}^E g^{SB}g^{RA}\varepsilon_{RSE}}{(1+q^2)^2}$

$$(7.1)$$

P_-:

	00	$C0$	$0D$	CD
00	0	0	0	0
$A0$	0	$\frac{q^2}{(1+q^2)^2}\delta_C^A$	$-\frac{q^4}{(1+q^2)^2}\delta_D^A$	$-\frac{q^2}{(1+q^2)^2}\varepsilon_{DC}{}^A$
$0B$	0	$-\frac{1}{(1+q^2)^2}\delta_C^B$	$\frac{q^2}{(1+q^2)^2}\delta_D^B$	$\frac{1}{(1+q^2)^2}\varepsilon_{DC}{}^B$
AB	0	$-\frac{g^{EB}g^{FA}\varepsilon_{FEC}}{(1+q^2)^2}$	$\frac{q^2 g^{EB}g^{FA}\varepsilon_{FED}}{(1+q^2)^2}$	$\frac{\varepsilon_{DC}{}^E g^{SB}g^{RA}\varepsilon_{RSE}}{(1+q^2)^2}$

$$(7.2)$$

P_T:

	00	$C0$	$0D$	CD
00	$\frac{q^2}{(1+q^2)^2}$	0	0	$-\frac{q^2}{(1+q^2)^2}g_{CD}$
$A0$	0	0	0	0
$0B$	0	0	0	0
AB	$-\frac{q^2}{(1+q^2)^2}g^{AB}$	0	0	$\frac{q^2}{(1+q^2)^2}g^{AB}g_{CD}$

$$(7.3)$$

We have:

$$\mathbb{1} = P_S + P_T + P_+ + P_- \tag{7.4}$$

$$P_A = P_+ + P_- \tag{7.5}$$

$$\hat{R}_I = P_S + P_T - q^2 P_+ - q^{-2} P_- \qquad (7.6)$$

$$= \mathbb{1} - (1 + q^2)P_+ - (1 + \frac{1}{q^2})P_-$$

$$\hat{R}_{II} = q^{-2} P_S + q^2 P_T - P_+ - P_- \qquad (7.7)$$

$$= \frac{1}{q^2}\mathbb{1} + (q^2 - \frac{1}{q^2})P_T - (1 + \frac{1}{q^2})P_A$$

$$\hat{R}_I{}^{ab}{}_{cd} = \hat{R}_I{}^{cd}{}_{ab} \qquad (7.8)$$

$$\hat{R}_{IIcd}^{ab} = \hat{R}_{IIab}^{cd} \qquad (7.9)$$

Metric tensor:

$$\eta_{00} = -1, \qquad \eta_{33} = 1$$
$$\eta_{+-} = -q, \qquad \eta_{-+} = -\frac{1}{q} \qquad (7.10)$$
$$\eta^{ab} = \eta_{ab}$$

$$X \circ Y = -X^0 Y^0 + X^3 Y^3 - q X^+ Y^- - \frac{1}{q} X^- Y^+ \qquad (7.11)$$

$$= X^a \eta_{ab} Y^b$$

ε-tensor:

$$\varepsilon^{ab}{}_{cd} = P_+^{ab}{}_{cd} - P_-^{ab}{}_{cd} \qquad (7.12)$$

The $\hat{R}$ matrix and the projectors satisfy some identities:

$$\eta_{sb}\eta_{ta}P^{ab}{}_{cd} = \eta_{ud}\eta_{vc}P^{uv}{}_{st} \qquad \text{for all projectors} \qquad (7.13)$$

$$\eta_{ij}(P_A)^{jk}_{mp}\eta_{kl}\eta^{pl} = \frac{2(1 + q^2 + q^4)}{(1 + q^2)^2}\eta_{im} \qquad (7.14)$$

$$\eta^{ab}\hat{R}_{IIbe}^{cd}\eta_{cf} = q^{-2}\hat{R}_{II}^{-1ac}{}_{ef} \qquad (7.15)$$

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