

# Infrared and collinear divergences in gauge theories

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## Contents

1. Introduction	1099
2. Infrared divergences. The perturbation theory approach	1101
2.1 Quantum electrodynamics; 2.2 Gravitation	
3. Infrared divergences. Redefinition of the scattering operator	1107
3.1 Quantum electrodynamics; 3.2 Quantum chromodynamics. Effective Hamiltonian	
4. Collinear divergences	1112
4.1 QCD. Perturbation theory; 4.2 Massless electrodynamics. Scattering operator redefinition	
5. Conclusions	1115
6. Appendices	1116
6.1 Polarization summation formulas; 6.2 Classical emission; 6.3 Divergences in multi-vertex theories;	
6.4 Kinematics of collinear processes; 6.5 The path integral method	
References	1119

**Abstract.** The problem of infrared and collinear divergences is considered within the framework of perturbation theory and the scattering operator redefinition method. IR divergent processes in electrodynamics and gravitation (perturbation theory) are described, and for the case of electrodynamics a scattering operator free from IR divergences is constructed. For the massless electrodynamics model, a recipe for constructing a scattering operator free from both IR and collinear divergences is given. The meaning of experimental parameters entering the final formulas is discussed, and it is shown that the  $S$ -matrix factorization (i.e., the approximate independence of hard and soft processes) makes the theorem on the cancellation of divergences in observables trivial. A method for finding divergences in theories with multiparticle vertices is presented.

## 1. Introduction

Of all the nontrivial divergences encountered in quantum electrodynamics (QED), infrared ones, fully describable mathematically and easy to interpret physically, are the most harmless. Such divergences are not unique to electrodynamics, however, and are always due to the presence of massless fields, in which case the long-wave amplitudes  $A'$  (for a process with the emission of a massless particle) and  $A$  (for the main process) differ only by a factor,

$$A' \sim A \frac{g}{2pq \pm i0}, \quad (1.1)$$

where  $g$  is the coupling constant,  $p$  is the momentum of the massive particle ( $p^2 = m^2$ ), and  $q$  is that of the massless particle ( $q^2 = \omega^2 - q^2 = 0$ ; see Section 2 for more details).

From Eqn (1.1) it is seen that at low  $|q|$  the amplitude  $A'$  is by no means small, and indeed it tends to infinity as  $|q| \rightarrow 0$ . This means that the amplitude for the emission of two or more long-wave massless particles is not smaller than that for the emission of one particle, so that expanding in powers of  $g$  is meaningless. But that is not all. The calculation of the relevant cross sections requires the integration of the squares of the amplitudes  $A' \sim A/\omega$  with the measure  $d^3q/\omega \sim \omega d\omega$ ,

$$\int \frac{d^3q}{\omega} |A'|^2 \rightarrow |A|^2 \int_0 \frac{d\omega}{\omega}, \quad (1.2)$$

yielding a logarithmic divergence at the lower integration limit. It is here that one encounters a typical infrared divergence. Similar divergences are also produced by integrating over the momenta of virtual particles (inner lines of certain Feynman graphs). Although infrared divergences were first described in Refs [1-3], infinities due to the zero photon mass actually date back to total cross section calculations for Rutherford scattering in classical physics [4].

Expanding the denominator in Eqn (1.1),

$$pq = E_p \omega - \mathbf{p}\mathbf{q} = \omega |\mathbf{p}|(1 - \cos \theta) + \omega \Delta_p, \quad (1.3)$$

where  $E_p^2 = m^2 + \mathbf{p}^2$ ,  $\Delta_p = E_p - |\mathbf{p}|$ ,  $\mathbf{p}\mathbf{q} = |\mathbf{p}||\mathbf{q}| \cos \theta$ , the angle  $\theta$  being formed by the momenta of the massive particle and the emitted massless one, we see that as  $m \rightarrow 0$ , the difference  $\Delta_p \rightarrow 0$ , and as  $\theta \rightarrow 0$ , the amplitude (1.1) grows unboundedly. It follows then that if the momenta of the emitting and emitted particles are in the same direction, the corresponding amplitudes tend to infinity. In this case perturbation theory breaks down. Cross section calculations also give rise to logarithmic divergences since  $d^3q \sim \sin \theta d\theta$ ,  $1 - \cos \theta \approx \theta^2/2$ , and because the numerator of  $A'$  is propor-

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tional to  $\theta$  (see Sections 4.2 and 6.3), we have

$$\int d^3q |A'|^2 \sim |A|^2 \int_0^{\frac{d\theta}{\theta}}. \quad (1.4)$$

This is a typical collinear divergence. Letting the mass of the emitting particle tend to zero after evaluating integral (1.4) would yield a result proportional to  $\ln m$  — the reason why collinear divergences are sometimes called *mass* divergences. These divergences first emerged in [5] where the radiation correction to the muon decay process was calculated in the  $m_e \rightarrow 0$  limit ( $m_e$  being the electron mass) and where the divergences were found to cancel for observable quantities. The origin of collinear divergences is purely kinematic: a massless particle may transform into two or more real massless particles moving in the same direction. Whereas the physical origin of collinear divergences is clear, their description is not as simple as for their infrared counterparts — a consequence of the fact that the momenta of the emitted and emitting particles are not necessarily small.

To account for infrared divergences is of practical importance because radiation detectors always have a sensitivity limit and therefore miss quanta with wavelengths above a certain threshold value, i. e., with frequencies  $\omega \leq E_0$ . Such quanta are always emitted and never detected. However, in high-energy physics this threshold may be fairly high (say, hundreds of MeV [6, 7]), and including this radiation — e. g., photons — changes considerably the theoretical formulas to be compared with experiment [8]. In collider experiments, up to 10–15% of the total energy may go with undetected radiation.

The inclusion of collinear divergences is important because, first, gluons in quantum chromodynamics (QCD) are self-acting entities, i. e., behave as charged massless particles, and, second, in Eqn (1.3) the difference  $\Delta_p \sim m^2/2|p| \rightarrow 0$  not only in the limit  $m \rightarrow 0$  but also for  $|p| \rightarrow \infty$ , implying that at superhigh energies massive particles behave as massless ones (see also Ref. [9]). But these are exactly the accelerator energies which are currently available ( $E_p \gg m$ ), so while the term 'mass divergence' is good for decay processes, at superhigh energies (where  $\ln m \rightarrow \ln(m/|p|)$ ,  $|p| \rightarrow \infty$ ) the mass is irrelevant — the reason why the term 'collinear divergences' should be preferred in both cases.<sup>1</sup>

The recipe for accounting for infrared divergences was rather an exotic one [11–13]. Because the probability for the emission of a finite number of photons of infinitesimal energy is zero (the amplitudes of the corresponding processes tend to infinity whereas the total probability is normalized to unity) and because long-wave photons also generate infrared divergences, the following procedure was recommended: (1) to perform infrared regularization (e. g., by ascribing a mass to the photon); (2) to calculate the cross section by including the non-detected infrared photons; and (3) to remove the regularization. The result, it was argued, is finite and independent of the regularization parameter, and the divergences due to the emission of real non-detected photons cancel those due to virtual infrared photons.

Collinear divergences were to be treated in much the same way, [5, 14–16], by first calculating cross sections and then summing up the contributions from all massless particles

emitted into a certain solid angle; all collinear divergences canceled out in the final answer. Such a state of affairs did not look entirely satisfactory though, because in a correct theory all physical processes ought to be described in terms of probability amplitudes.

Efforts at describing infrared divergences perturbationally culminated in the work of Weinberg [17], whose concern was in fact IR divergences in gravitation — the only classical massless field theory apart from electrodynamics. The latter being used as a model theory, both theories were developed in parallel in Weinberg's study.

Preceding this work, however, was an unjustifiably unnoticed advanced-level treatment by Murota [18], who relied on the classical work of Bloch and Nordsieck [19] in his approach to the physics of the problem. The clue was the assertion by Bloch and Nordsieck that coupling the electromagnetic field and the classical current of charged particles suffices for the quantum description of processes with infrared photons. Murota found an asymptotic interaction Lagrangian (i. e., one describing the effective interaction of soft photons) and constructed an *S*-matrix free of infrared divergences. Among other things, Murota [18] employed the concept of the hierarchy of characteristic interaction times (a fast hard process and a slow soft process) to develop an innovative approach to the factorization of the *S*-matrix (i. e., to the separation of hard and soft processes in it; see Section 3.1.2 for more details). In the same work, a general formula (other than Magnus' [20]) was derived for disentangling the *T* exponent.

Later, a whole series of papers [21–28] (see also Ref. [29] for a review) contributed to the detailed analysis of the problem. As a major addendum to Murota's [18] results, the specifics of the Hilbert space of quantum mechanics as used in the context of the divergence problem was elucidated (separable and nonseparable spaces [21, 25–27]). Of major importance was Dollard's [30] suggestion that the scattering operator be redefined for slowly decreasing potentials ( $1/r$  and slower); the influence of such potentials cannot be ignored even at arbitrarily large distances. Note that an *S*-matrix representation obtained in Ref. [18] is equivalent to the representation of Ref. [30] (Section 3.1.2); in particular, it yields automatically the theorem [14–16] on the cancellation of divergences in averaged cross sections of the processes involved.

Further work along these lines was stimulated by the advent of QCD, which differs from QED in a variety of important respects.

1. The non-Abelian gauge group causes matrices  $\gamma_\mu T^a$  ( $T^a$  being group generators) to appear in diagram vertices, and whereas in QED the matrix  $\gamma_\mu$  is replaced by the number  $2p_\mu$  at low energies (see Section 2), the noncommutative nature of  $T^a$  matrices considerably complicates the perturbation theory analysis.

2. Since gluons have a charge, the problem of collinear divergences is added to the difficulties indicated above.

3. Because of the confinement phenomenon, neither perturbation theory nor the method of coherent states can be applied to gluons in the long-wave limit.

Fortunately, at small distances ( $l_0 < 10^{-14}$  cm) the running coupling constant is sufficiently small ( $\alpha_s < 0.2$ ) for perturbation theory to be applied. Furthermore, the effect of confinement is to effectively cut off low-energy quarks and gluons, so that, for practical purposes, perturbation theory and (for the structure functions) the GLAP evolution

<sup>1</sup> If phonons had a mass  $\lambda$ , then in the limit  $\lambda \rightarrow 0$  the integral (1.2) would also be proportional to  $\ln \lambda$ , therefore the terms 'mass infrared' and 'mass collinear' divergences are sometimes employed [10]. The term 'mass' will not be used in this context in what follows.

equations [31–34] prove to be sufficient. Thus, it has been shown by direct calculation [35] (see also Refs [36, 37]) that, consistent with the divergence cancellation theorem [14–16], the properly averaged cross section of the process  $e^+e^- \rightarrow \bar{q}q$  (or more precisely  $e^+e^- \rightarrow$  two jets) is finite and depends on the experimental parameters, namely  $\delta$ , the spreading angle of the quark-generated jets, and  $(1 - \epsilon)$ , the fraction of energy carried away by the jets ( $\delta, \epsilon \ll 1$ ). The analysis of various aspects of the problem continued in the 1980s. In particular, generalized coherent states with nonlinear arguments of the exponential were constructed [38] to describe the emission of soft gluons by partons.

Although the situation seems satisfactory from the practical standpoint (see Refs [10, 34, 39, 40] for review), description of the non-Abelian infrared divergences and especially the treatment of collinear divergences (even in massless electrodynamics) has not yet reached the level achieved in QED. The reason is simple: it is the assumption that in either case the field dynamics at large times and large space intervals are amenable to description. But whereas in the case of infrared divergences in QED the problem under study could be reduced to an exactly solvable problem concerning the interaction of an electromagnetic field with an external field, the analogous non-Abelian problem is not solved exactly. To fully describe processes with collinear divergences requires an ability to solve a non-trivial field-theoretical problem because emitted particles in this case are not necessarily low in energy and the only restrictions posed are those on the emission angle (see Section 4 for more details).

Section 2 of this paper describes the traditional (perturbation theory) approach to infrared divergences in QED and gravitation. In Section 3 the infrared problem is considered in the framework of the  $S$ -matrix redefinition method in both QED and QCD. Section 4, on collinear divergences, presents a recipe for constructing a scattering operator free of infrared and collinear divergences, using the massless spinor electrodynamics as an example.

Brief remarks on some problems related to the theme of the paper are discussed in Section 5. In the appendices, useful auxiliary material can be found; it is established, in particular, what types of massless particle interactions do not produce infrared and collinear divergences.

**Notation and normalization.** The metric adopted throughout is  $g_{\mu\nu} = \eta_{\mu\nu} (+---)$ ; Greek indices run from 0 to 3. Repeated indices of the same variance denote summation with an appropriate metric tensor; for example,  $q_\mu x_\mu = g^{\mu\nu} q_\mu x_\nu = q_\mu x^\mu \equiv qx$ .

We use abbreviated notation for the differentiation operator ( $\partial_\mu = \partial/\partial x^\mu$ ) and functions ( $\psi(t) \equiv \psi_t$ , etc.). The  $T$ -matrix is defined by the relations  $S_{p'p} = 1 + i(2\pi)^4 \delta^{(4)}(p' - p) T_{p'p}$  in quantum field theory and  $S_{p'p} = 1 + i2\pi\delta(E' - E) T_{p'p}$  for potential scattering. The product of functions may imply integration either over coordinates ( $JA \equiv \int dx J_\mu(x) A_\mu(x)$ , where  $dx \equiv d^4x$ ) or momenta. Throughout the paper, Heaviside's (rationalized) system of units and the convention  $\hbar = c = 1$  are used;  $e$  is the electrical charge of the electron ( $e < 0$ ). The normalization of spinors and operators are given by the following equations:

$$u_\alpha^s(p) \bar{u}_\beta^s(p) = \frac{1}{2} [(\hat{p} + m)(1 + \gamma_5 \hat{s})]_{\alpha\beta},$$

$$v_\alpha^s(p) \bar{v}_\beta^s(p) = \frac{1}{2} [(\hat{p} - m)(1 + \gamma_5 \hat{s})]_{\alpha\beta}$$

(no summation over polarizations  $\sigma = \pm 1/2$ ),  $[\gamma_\mu, \gamma_\nu]_+ = 2\eta_{\mu\nu}$ ,  $\gamma_5^2 = 1$ ,  $\hat{p} = p_\mu \gamma_\mu$  ( $s_\mu$  is the polarization vector,  $s^2 = -1$ ,  $sp = 0$ ),

$$[\hat{a}_{p,\sigma}, \hat{a}_{p',\sigma'}^\dagger]_+ = \tilde{\delta}(p', p) \delta_{\sigma\sigma'}, \quad [\hat{c}_{q,\lambda}, \hat{c}_{q',\lambda'}^\dagger]_- = \tilde{\delta}(q', q) \delta_{\lambda\lambda'},$$

where  $\tilde{\delta}(p', p) = (2\pi)^3 \cdot 2E_p \delta(p' - p)$ ,  $E_p = (m^2 + p^2)^{1/2}$ ,  $\tilde{\delta}(p', p)$  is an invariant delta function for the measure  $d\mu(p) = d^3p/[(2\pi)^3 \cdot 2E_p]$  (for the photon,  $E_q = |q| = \omega_q$ ). Field operators are defined by the expansions

$$\hat{\psi}(x) = \int d\mu(p) [\hat{a}_{p,\sigma} u^\sigma(p) \exp(-ipx) + \hat{b}_{p,\sigma}^\dagger v^\sigma(p) \exp(ipx)], \quad (1.5)$$

$$\hat{A}_\mu(x) = \int d\mu(q) [\hat{c}_{q,\lambda} \varepsilon_\mu^\lambda(q) \exp(-iqx) + \hat{c}_{q,\lambda}^\dagger \varepsilon_\mu^{\lambda*}(q) \exp(iqx)], \quad (1.6)$$

where  $\lambda = \pm 1$  is the photon polarization,  $\varepsilon_\mu \varepsilon_\mu^* = -1$ ,  $\bar{u} = u^\dagger \gamma_0$ . The symbol  $(\sim)$  designates proportionality or an asymptotic expansion.

## 2. Infrared divergences.

### The perturbation theory approach

#### 2.1 Quantum electrodynamics

Since the basic aspects of QED infrared divergences already emerge in Coulomb scattering, where modern approaches to such divergences also have their origins, a brief discussion of quantum mechanics seems to be a good starting point here.

**2.1.1 Quantum mechanics. Coulomb scattering.** Let the Hamiltonian of a particle be  $H = H_0 + V(r)$ , where in the nonrelativistic case  $H_0 = p^2/2m$ , with  $p$  the momentum and  $m$  the mass of the particle, and  $V$  — the potential energy. The evolution operator of the system,  $\hat{U}_{t,t'} = \exp[-i\hat{H}(t - t')]$ , translates the state vector of the system  $\psi_{t'}$  at time  $t'$  into the vector  $\psi_t$ . The key objective of scattering theory is to provide a recipe with which the state of the system at  $t \rightarrow \infty$  might be calculated from its state at  $t' \rightarrow -\infty$ , and it would appear at first sight that the operator  $\hat{U}_{t,t'}$  will serve well to achieve this goal.

It is known, however, that the limit  $\lim_{t \rightarrow \infty} \hat{U}_{t,-t}$  does not exist (it suffices to take  $\hat{H}$  in its own representation to prove this). As a consequence, the physically interesting matrix elements of  $\hat{U}_{\infty,-\infty}$  have an infinite phase which, first, does not provide any physical information and, second, is absent from the final probability expression. Therefore a correctly defined and physically significant operator is simply obtained by subtracting the corresponding phase (because far from the scatterer particles move as if they were free), i.e., by changing to the  $S$ -matrix,

$$\hat{U}_{t,t'} = \exp(i\hat{H}_0 t) \exp[-i\hat{H}(t - t')] \exp(-i\hat{H}_0 t'),$$

$$\hat{S} = \hat{U}_{\infty,-\infty}. \quad (2.1)$$

The operator  $\hat{S}$  is written symbolically in  $T$  exponential form as

$$\hat{S} = T \exp \left\{ -i \int_{-\infty}^{\infty} dt \hat{V}_I[r(t)] \right\}, \quad (2.2)$$

where  $\hat{V}_I[r(t)]$  is the potential energy operator in the interaction representation (hence the subscript  $I$ ). Since  $\hat{r}_I(t) = \hat{p}t/m + \hat{r}$  (i.e.,  $\hat{r}_I$  satisfies  $\dot{\hat{r}}_I = 0$ ,  $\dot{\hat{r}} \equiv \partial_t \hat{r} = \hat{p}/m$ ), the integral in Eqn (2.2) converges for potentials decreasing faster than  $1/r$  for  $|r| = r \rightarrow \infty$ . Formula (2.2) forms the basis of perturbation theory and quantum field theory.

The standard approach sketched above breaks down for potentials that decrease as  $1/r$  or slower. For example, for the Coulomb case  $\alpha/r$ , the argument of the exponential in Eqn (2.2) diverges at the upper ( $t$ ) and lower ( $-t$ ) limits, thus giving rise to an infinite phase in the matrix elements of  $S$ :  $\exp[-i(\alpha m/p) \cdot 2 \ln t] \sim \exp[-i(\alpha m/p) \cdot 2 \ln r]$ ,  $r, t \rightarrow \infty$ . It is here that the divergent 'Coulomb' phase comes in (here and for the duration of this section,  $p = |\mathbf{p}|$ ).

Fortunately, this problem is amenable to an exact solution. The asymptotic behavior of the particle wave function in a Coulomb potential is given by [41]

$$g(r, 0) \exp \left[ ipz + \frac{ixm}{p} \ln \left( 2pr \sin^2 \frac{\theta}{2} \right) \right] + \frac{f(0)}{r} \exp \left[ ipr - \frac{ixm}{p} \ln(2pr) \right], \quad r \rightarrow \infty, \quad (2.3)$$

where

$$g(r, 0) = 1 - \frac{ix^2 m^2}{2p^3 r \sin^2(\theta/2)},$$

$$f(0) = -\frac{\alpha m}{2p^2 \sin^2(\theta/2)} \exp \left[ -\frac{2ixm}{p} \ln \sin \frac{\theta}{2} + 2i\sigma_0(p) \right], \quad (2.4)$$

$\sigma_0(p)$  is the argument of the gamma function  $\Gamma(1 + ixm/p)$  (the Coulomb  $s$ -wave phase shift [42]). It is clear that the standard asymptotics [43]  $\psi \sim \exp(ipz) + [f(0)/r] \exp(ipr)$ , with a plane monochromatic wave for the large- $z$  motion of the particle, is no longer valid.

The reason why standard perturbation results like (2.2) break down is now easily understood from Eqn (2.3), which shows that even at arbitrarily large distances the motion of a particle cannot be considered free, i.e., the effect of the potential may not be neglected. Hence the logarithmic divergence occurring in Eqn (2.3). The same follows from the asymptotics  $r(t) \sim pt/m + (\alpha mp/p^3) \ln t$ ,  $t \rightarrow \infty$  of the classical solution (instead of  $pt/m + r_0$ ); it does not obey the free-motion equation  $\ddot{\mathbf{r}} = 0$ . According to Eqn (2.4), the differential scattering cross section for the exact problem is identical to the Born cross section

$$\frac{d\sigma}{d\Omega} = |f(0)|^2 = \frac{\alpha^2 m^2}{4p^4 \sin^4(\theta/2)} \quad (2.5)$$

( $d\Omega$  is the solid angle) and also to the classical Rutherford result.

The Coulomb scattering problem has also been studied for a modified potential, namely the Yukawa potential  $\exp(-\lambda r)/r$  (which is tantamount, in a sense, to ascribing a mass  $\lambda$  to the photon). According to Dalitz [44], the second-order perturbation of the scattering amplitude may be viewed as the second term in the expansion of the exponential  $\exp\{(-2ixm/p) \ln[(2p/\lambda) \sin(\theta/2)]\}$  if the factor  $2p/\lambda$  under the logarithm is identified with the divergent Coulomb phase in the second term in Eqn (2.3) (as it may according to Ref. [17]).

Thus, while the operator (2.2) is incorrectly defined for the Coulomb potential, one can 'regularize' it (changeover to the Yukawa potential) to be able to obtain differential cross sections independent of the auxiliary parameter. All this is reminiscent of the situation with the evolution operator  $\hat{U}_{t,t'}$ , where a physically irrelevant infinite phase was eliminated in order to obtain a well-defined operator.

It is appropriate that the problem we consider here should also be treated along these lines, i.e., by redefining the scattering operator so as to make it mathematically meaningful while retaining the physical information it carries. Before we do this, however (see Section 3.1.1), the situation in electrodynamics should be discussed.

**2.1.2 QED. Real infrared photons. Coulomb potential scattering.** Let us consider Coulomb potential scattering with the emission of a single photon. For this purpose we modify the Lagrangian  $\mathcal{L}_{int} = -e\bar{\psi}\gamma_\mu\psi A_\mu$  by the substitution  $A_\mu \rightarrow A_\mu + A_\mu^{ext}$ , where  $A_\mu^{ext}(x) = -g_{\mu 0}Ze/4\pi|r|$  is the external classical field,  $-Ze$  is the scatterer charge. In the lowest-order perturbation theory the  $T$ -amplitude is given by

$$T_{p'p}^{\sigma'\sigma} = -e\bar{A}_\mu^{ext}(\mathbf{p} - \mathbf{p}') \bar{u}^{\sigma'}(p') \gamma_\mu u^\sigma(p) \equiv \bar{u}_\alpha^{\sigma'}(p') M^{\alpha\beta} u_\beta^\sigma(p), \quad (2.6)$$

where  $\bar{A}_0^{ext}(\mathbf{k}) = -Ze/|\mathbf{k}|^2$ , and  $\sigma$  indicates the electronic spin state. Equation (2.6) defines the  $\bar{M}$  matrix of the problem.

A transition to processes involving photon emission by an incoming (outgoing) particle reduces to the multiplication of  $\bar{M}$  on the right (left) with the matrices

$$\frac{i}{\hat{p} - \hat{q} - m + i0} [-ie\hat{\epsilon}^{\lambda*}(q)], \quad [-ie\hat{\epsilon}^{\lambda*}(q)] \frac{i}{\hat{p}' + \hat{q} - m + i0}, \quad (2.7)$$

where  $\hat{\epsilon} \equiv \epsilon_\mu \gamma_\mu$ ,  $\epsilon_\mu^\lambda$  denotes the photon polarization,  $\epsilon\epsilon^* = -1$ , and  $qe(q) = 0$ . Since  $p^2 = m^2$  and  $q^2 = 0$ , in the limit  $q \rightarrow 0$  the matrices (2.7) are  $e(\hat{p} + m)\hat{\epsilon}^* / (-2pq + i0)$ ,  $e\hat{\epsilon}^*(\hat{p}' + m)/(2p'q + i0)$ . From Eqn (2.6), noting that  $(\hat{p} + m)\gamma_\mu u(p) = 2p_\mu u(p)$  and  $\bar{u}(p')\gamma_\mu(\hat{p}' + m) = \bar{u}(p') \cdot 2p'_\mu$ , we find the factors  $-ep\epsilon^{\lambda*}/(pq - i0)$  and  $ep'\epsilon^{\lambda*}/(p'q + i0)$  corresponding to the emission of soft photons by the incoming and outgoing particles, respectively. Thus, the emission of a soft photon is described by the amplitude

$$T_{p'q,p}^{\lambda,\sigma'\sigma} = T_{p'p}^{\sigma'\sigma} j_\mu^{\lambda*}(q), \quad j_\mu(q) = \frac{ep'_\mu}{p'q + i0} + \frac{-ep_\mu}{pq - i0}. \quad (2.8)$$

Note that Eqn (2.8) is valid for charged fields with any spin because the emission of soft photons depends only on the charge of the emitter [45] as the electric current expression  $j_\mu = ie\phi^* \partial_0 \phi$  and the commutative properties of spin matrices [17] suggest for Bose fields and Fermi fields, respectively. The identification of the sum in Eqn (2.8) with current is not fortuitous. In classical physics the 4-vector of the charged-particle current is  $j_\mu^{cl}(x, p) = ev_\mu \delta(x - \mathbf{r}t)$ ,  $v_\mu = p_\mu/E_p$ . Obviously,

$$j_\mu^{cl}(x, p) \equiv j_{\mu p}^{cl}(x) = e \int_{-\infty}^{\infty} d\tau u_\mu \delta^4(x - u\tau), \quad u_\mu = \frac{p_\mu}{m}. \quad (2.9)$$

If  $u = u(\tau)$ , then Eqn (2.9) is the transition current; the equation  $t = E_p(\tau)\tau/m$  is assumed to have only one real solution,  $\tau = tm/E_p$ .

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In our case  $u_\mu(\tau) = u_\mu$  for  $\tau < 0$  and  $u_\mu(\tau) = u'_\mu$  for  $\tau > 0$ , with

$$\int_{-\infty}^0 d\tau e u_\mu \delta(x - u\tau) = \int \frac{d^4 q}{(2\pi)^4} \exp(-iqx) \frac{-ie u_\mu}{qu - i0},$$

$$\int_0^\infty d\tau e u'_\mu \delta(x - u'\tau) = \int \frac{d^4 q}{(2\pi)^4} \exp(-iqx) \frac{ie u'_\mu}{qu' + i0}.$$

Therefore

$$j_{\mu p}^{\text{cl}}(x) = \int \frac{d^4 q}{(2\pi)^4} \exp(-iqx) j_\mu^{\text{cl}}(q)$$

$$= \int \frac{d^4 q}{(2\pi)^4} \exp(-iqx) \left[ \frac{ie u'_\mu}{qu' + i0} + \frac{-ie u_\mu}{qu - i0} \right]$$

$$\equiv i[j_\mu^+(x, p') + j_\mu^-(x, p)], \quad (2.10)$$

$$j_\mu^\pm(x, p)\theta(\mp t) = 0, \quad j_\mu^{\text{cl}}(q) = i j_\mu(q),$$

i.e.,  $j_\mu^+(x, p) = 0$  for  $t < 0$  and  $j_\mu^-(x, p) = 0$  for  $t > 0$ . Up to the factor  $i$ , the current (2.8) is identical to the current  $j_\mu^{\text{cl}}(q)$ , with  $q^2 = 0$ , of a classical charged particle moving with velocity  $v$  at  $t < 0$  and with  $v'$  at  $t > 0$ .

Thus, in complete agreement with Bloch and Nordsieck [19], the emission of soft photons is determined by the classical current of the particle. But soft photons may be emitted not only by free particles but also by virtual ones, i.e., photon lines may come not only from the external lines of charged particles but also from their inner lines. The radiation amplitudes for these latter, in contrast to Eqn (2.8), are finite for  $q \rightarrow 0$  (the propagator of a virtual charged particle does not have a  $1/q$  singularity), they produce no divergences [14, 17], and are ignored in the following analysis (the concept 'external line' is elaborated at the beginning of Section 2.1.3).

It is now straightforward to write down the probability of emitting  $N$  photons (Fig. 1). The factor associated with the emission of an infrared photon can be written in the form  $e\eta p_\mu/(pq + i\eta 0)$ , where  $\eta = -1(+1)$  for the emission by an incoming (outgoing) particle. For the emission of a succession of two photons with momenta  $q_1, q_2$  we have

$$\frac{e\eta p_\mu}{pq_1 + i\eta 0} \frac{e\eta p_\nu}{p(q_1 + q_2) + i\eta 0}. \quad (2.11)$$

Since photons may be emitted in the reverse order, the same expression with the replacement  $q_1 \leftrightarrow q_2$  should be added to Eqn (2.11), giving

$$\frac{e\eta p_\mu}{pq_1 + i\eta 0} \frac{e\eta p_\nu}{pq_2 + i\eta 0}$$

for the desired factor.

Now, using the elementary identity

$$\sum \frac{1}{a_1} \frac{1}{a_1 + a_2} \cdots \frac{1}{a_1 + \dots + a_N} = \frac{1}{a_1 a_2 \dots a_N}, \quad (2.12)$$

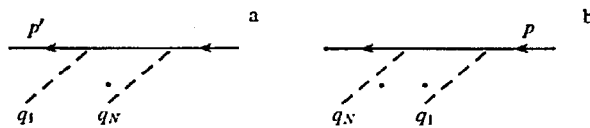


Figure 1. Emission of infrared photons by a scattered (a) and an incident (b) particle.

where the summation runs over all permutations  $a_1, \dots, a_N$ , it follows that the emission of  $N$  photons with momenta  $q_1, \dots, q_N$  by the in or out particle gives rise to the factor

$$\frac{e\eta p_{\mu_1}}{pq_1 + i\eta 0} \cdots \frac{e\eta p_{\mu_N}}{pq_N + i\eta 0}. \quad (2.13)$$

The most important non-trivial point about the emission of soft photons is the factorization of their contributions. Note that this property is valid only for  $q \rightarrow 0$ , because it is only in this limit that we are safe to make the replacement  $\gamma_\mu \rightarrow 2p_\mu$ , i.e., neglect the noncommutative nature of the matrices  $\gamma_\mu$ . Note also that a transition to non-Abelian theories (to chromodynamics, for example) gives rise to noncommutative matrices  $\hat{T}_a$ , which act as the generators of the corresponding gauge group; the simple expression (2.13) does not apply in this case.

Photons, however, are not necessarily emitted only by an in or only by an out particle: either may emit a part of the total number. For example, for  $N = 2$ , using the identity

$$\frac{1}{a'_1 a'_2} + \frac{1}{a'_1 a_2} + \frac{1}{a'_2 a_1} + \frac{1}{a_1 a_2} = \left( \frac{1}{a'_1} + \frac{1}{a_1} \right) \left( \frac{1}{a'_2} + \frac{1}{a_2} \right),$$

Eqn (2.8) becomes

$$T = T_0 j_1 \epsilon_1^* j_2 \epsilon_2^*,$$

$$j_\mu = \frac{ep'_\mu}{p'q_i + i0} - \frac{ep_\mu}{pq_i - i0} \equiv \sum_a \frac{e_a \eta_a p_\mu^a}{p^a q_i + i\eta_a 0}, \quad (2.14)$$

where  $T_0$  is the amplitude of the basic (hard) process. It is readily shown that for an arbitrary  $N$  we have

$$T = T_0 j_1 \epsilon_1^* \dots j_N \epsilon_N^*, \quad (2.15)$$

implying that photons are emitted independently.

*Generalization to an arbitrary process.* The formulas obtained for Coulomb scattering may be readily generalized to an arbitrary process. This can be seen by noting that Eqn (2.8) is true for a process with an arbitrary number of charged particles if the expression for the current is written using the sign factor  $\eta$  as in Eqn (2.14) and extending the sum to all charged particles in both initial and final states,

$$j_\mu(q, p) \equiv \sum_a \frac{e_a \eta_a p_\mu^a}{p^a q + i\eta_a 0} \equiv \sum_{a(\text{out})} \frac{e_a p_\mu^a}{p^a q + i0}$$

$$- \sum_{a(\text{in})} \frac{e_a p_\mu^a}{p^a q - i0} \equiv j_\mu^+(q, p') + j_\mu^-(q, p). \quad (2.16)$$

This expression embraces not only scattering processes but also pair creation and pair annihilation processes. Note that the currents  $j_\mu^\pm$  in Eqn (2.16) now depend on the sets of momenta  $\{p'\}, \{p\}$ . It is easy to see that the charge conservation law implies the conservation of current [17]:

$$j_\mu(q, p) q_\mu = \sum_a e_a \eta_a = \sum_{\text{out}} e_a - \sum_{\text{in}} e_a = 0. \quad (2.17)$$

This suffices to justify replacing the current (2.14) by the expression (2.16) in Eqn (2.15) (note only that  $T_0$  will now be the amplitude of the process under study rather than the function (2.6) as before). Thus, the amplitude of the emission of  $N$  infrared photons can be factored in the general case. Equation (2.15) suggests that such photons are emitted

independently both of the basic process and of one another, i.e., their emission probability must be governed by the Poisson formula — which the noncommutative nature of generators  $\hat{T}_a$  prevents in non-Abelian theories.

**Emission probabilities for real infrared photons.** According to Eqn (2.8), in the approximation we use, the probability density for a left- or a right-polarized photon is given by

$$\frac{dP_q}{d\mu(q)} = \sum_{\pm} |j\epsilon^{\pm}|^2 = -j_{\mu}(q)j_{\mu}^{*}(q)$$

[see Eqn (6.1)].

Let us now evaluate the probability density for scattering with a total energy loss  $\sum \omega_i \leq E$  due to the infrared emission. Using

$$\theta(E, \omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\tau \frac{\sin E\tau}{\tau} \exp(i\omega\tau), \quad E > 0, \quad \omega > 0$$

[this is an even function of  $\omega$  and odd function of  $E$ ; for  $E > 0, \omega > 0$ , it is identical to  $\theta(E - \omega)$ ] the desired density is

$$\begin{aligned} P_E &= |T_0|^2 \sum_{N=0}^{\infty} \frac{1}{N!} \prod_{i=1}^N \int_{\lambda_i}^E \sum_{\lambda_i} [j(q_i) \epsilon^{\lambda_i*}] [\epsilon^{\lambda_i} j^{*}(q_i)] \\ &\times \theta\left(E, \sum \omega_j\right) d\mu(q_i) \\ &= |T_0|^2 \frac{1}{\pi} \int_{-\infty}^{\infty} d\tau \frac{\sin E\tau}{\tau} \\ &\times \exp \left[ - \int_{\lambda}^E j_{\nu}(q) j_{\nu}(q)^* \exp(i\tau\omega) d\mu(q) \right]. \end{aligned} \quad (2.18)$$

The limits indicated here refer to the integrals over  $\omega_i = |q_i|$ , the integration over all  $\omega_i$  up to the upper limit  $E$  is validated by the  $\theta$  function, and the factor  $1/N!$  accounts for the phase space reduction due to photons being indistinguishable. In deriving Eqn (2.18), the polarization summation formula (6.1) and the current conservation law (2.17) ( $\lambda_{\mu} = q_{\mu}/\omega$ ) have been used.

Let us write the argument of the exponential in Eqn (2.18) in the form

$$\begin{aligned} [\dots] &\equiv N_{\tau}(E, \lambda) = A \int_{\lambda}^E \frac{d\omega}{\omega} \exp(i\omega\tau) \\ &= A \left[ \ln \frac{E}{\lambda} + \int_{\lambda}^E d\omega \frac{\exp(i\omega\tau) - 1}{\omega} \right] \\ &\equiv N_0(E, \lambda) + N'_{\tau}(E, \lambda), \end{aligned} \quad (2.19)$$

where

$$\begin{aligned} A &= - \sum_{ab} \frac{e_a \eta_a e_b \eta_b (p_a p_b)}{2(2\pi)^3 E_a E_b} \int d\mathbf{n} \frac{1}{(1 - \mathbf{v}_a \mathbf{n})(1 - \mathbf{v}_b \mathbf{n})} \\ &\equiv - \sum_{ab} e_a \eta_a e_b \eta_b \Phi(v_{ab}), \\ \Phi(v_{ab}) &= \frac{1}{8\pi^2} \frac{1}{v_{ab}} \ln \frac{1 + v_{ab}}{1 - v_{ab}}, \quad v_{ab}^2 = 1 - \frac{m_a^2 m_b^2}{(p_a p_b)^2}. \end{aligned} \quad (2.20)$$

In the rest frame of one of the particles the integration in Eqn (2.20) is elementary, and the invariant parameter  $v_{ab}$  in this case is the velocity of the second particle. From Eqns (2.18)–(2.20) (with  $\tau = 0$ ) it is easy to see that for two or more particles  $A > 0$ ; therefore  $P_E \rightarrow \infty$  as  $\lambda \rightarrow 0$ . The notation  $N_{\tau}(E, \lambda)$  appears in Eqn (2.19) for the following reason.

Dropping the  $\theta$  function from the first of Eqns (2.18) and introducing the definitions

$$N_0(E, \lambda) = \sum_{\pm} \int_{\lambda}^E |j\epsilon^{\pm}|^2 d\mu(q) \equiv \bar{N}(E, \lambda),$$

it follows that the sum in this equation contains terms of the form  $\bar{N}^N/N!$ , i.e., the probability of emission of  $N$  soft photons with energies  $\lambda < \omega < E$  is given by Poisson's formula

$$P_N = \frac{\bar{N}^N}{N!} \exp(-\bar{N}), \quad (2.21)$$

where  $\bar{N}$  is the average number of emitted photons ( $\bar{N} = \sum N P_N$ ).

That there is no normalization factor  $\exp(-\bar{N})$  in Eqn (2.18) is due to the fact that we have considered only a part of the perturbation theory diagrams: the contribution of virtual photons was left out of the calculation. Before calculating it, note that Eqns (2.19) and (2.20) reveal the main physical features of soft photon emission, namely 1) that the average number of emitted photons  $\bar{N}(E, \lambda)$  diverges logarithmically as  $\lambda \rightarrow 0$ ; and 2) that the average emitted energy

$$\bar{E} = \int_{\lambda}^E \omega \frac{dN}{d\omega} d\omega = A \int_{\lambda}^E \omega \frac{d\omega}{\omega} = \int_{\lambda}^E \frac{dE}{d\omega} d\omega$$

is finite, i.e.,  $dE/d\omega = \text{const}$  for  $\omega \rightarrow 0$ .

The fact that the total energy of emitted photons is finite while their average number is not indicates that perturbation theory does not describe this process correctly. Obviously, to describe the electromagnetic field requires a transition to collective variables, i.e., to states with an indefinite number of photons. From Eqn (2.21), the probability of emission of a finite number of infrared photons is zero because  $N_0(E, \lambda) \rightarrow \infty$  as  $\lambda \rightarrow 0$ . The distribution (2.21) is meaningful only if  $E \ll E_p$  (i.e., the emitting particle has an unlimited energy reservoir).

The following point should be noted here. While the total energy of the undetected emission  $E$  in Eqn (2.18) may seem to be identical to the sensitivity limit  $E_0$  of the photon detector, these are in fact different parameters. An experimenter not concerned with detecting soft photons may observe a loss of energy due to their emission. Obviously, the parameter  $E$  should have such a value as to make Eqns (2.8) and (2.15) valid.

**2.1.3 QED. Virtual infrared photons.** Infrared divergences are also encountered in calculating looped diagrams, i.e., in integrating over the momenta of virtual photons. However, not every virtual photon but only those linking 'outer lines' generate divergences. It is necessary to be clear about terminology at this point. We will regard as infrared photons ones with energies  $\omega \leq A$ , where  $A$  is a certain parameter securing the validity of Eqns (2.8) and (2.15) (its physical meaning will be discussed later). We will call external lines those associated with particles on the mass surface and with those that have emitted real or virtual infrared photons. To make the equations mathematically meaningful one performs an infrared regularization, whether by introducing a photon mass  $\lambda$  or by cutting off small photon-momenta in the integrals involved. We prefer here the latter approach. Clearly,  $\lambda \ll A$ .

To calculate the contribution from  $N$  virtual infrared photons, the amplitude of the main process  $T_0$  should be multiplied by the factor

$$\begin{aligned} & \frac{1}{N!2^N} \int_{\lambda}^{\Lambda} \frac{d^4 q_i}{(2\pi)^4} \prod_{i=1}^N j_{\mu}(q_i) \frac{-ig_{\mu\nu}}{q_i^2 + i0} j_{\nu}(-q_i) \\ &= \frac{1}{N!} \left[ \frac{1}{2} \int_{\lambda}^{\Lambda} d^4 q A(q) \right]^N, \end{aligned} \quad (2.22)$$

with

$$A(q) = \frac{-i}{(2\pi)^4} \frac{j_{\mu}(q) j_{\mu}(-q)}{q^2 + i0}. \quad (2.23)$$

A word now about the coefficient in front of the integrals in Eqn (2.22). By first replacing  $q_{N+i} \rightarrow -q_i$  in the amplitude for the emission of  $2N$  real infrared photons and then multiplying the result by the product of  $N$  photon propagators we obtain the integrand of (2.22). The initial amplitude (2.15) was obtained by summing over the permutations of all  $2N$  photons. Following the identification  $q_{N+i} \equiv -q_i$ , it is clear that the permutation  $q_{N+i} \leftrightarrow q_i$  is equivalent to the replacement  $q_i \rightarrow -q_i$  for the momentum of a virtual photon under the integral and that this permutation does not produce a new state — hence the factor  $2^{-N}$ . The factor  $1/N!$  accounts for the fact that photons are indistinguishable. The limits of integration indicated in (2.22) are those for  $|q|$ ; extending the  $q_0$  integration over the entire axis introduces an error which is within the accuracy of the calculation  $O(\Lambda/m)$  and at any rate does not affect the probability values.

Summing over  $N$  yields the final results:

$$\begin{aligned} \tilde{T}_0 &= T_0 \exp \left[ \frac{1}{2} \int_{\lambda}^{\Lambda} d^4 q A(q) \right], \\ \frac{|\tilde{T}_0|^2}{|T_0|^2} &= \exp \left[ \text{Re} \int_{\lambda}^{\Lambda} d^4 q A(q) \right] \\ &= \exp \left[ - \int_{\lambda}^{\Lambda} \frac{d^4 q}{(2\pi)^3} \frac{\delta(q^2)}{2} j_{\mu}(q) j_{\mu}(-q) \right] \\ &= \exp \left[ \int_{\lambda}^{\Lambda} d\mu(q) j_{\nu}(q) j_{\nu}^*(q) \right], \end{aligned} \quad (2.24)$$

where the amplitude  $\tilde{T}_0$  of the main (hard) process contains radiation corrections due to virtual infrared photons. Equation (2.24) takes into account that

$$j_{\mu}(-q) = -j_{\mu}^*(q) \quad (2.25)$$

[this equality makes the calculation of  $\text{Re } A(q)$  trivial]. We see that the argument of the exponential in Eqn (2.24) contains  $-N_0(\Lambda, \lambda) = -\tilde{N}(\Lambda, \lambda)$  [see Eqn (2.19),  $N'_0 = 0$ ].

Substituting  $|\tilde{T}_0|^2$  for  $|T_0|^2$  in Eqn (2.18) yields [17] the probability density for the hard process with radiation energy loss  $E$  [ $\lambda = 0$ ,  $\Lambda$  is given by Eqn (2.20)]:

$$\begin{aligned} \tilde{P}_E &= |T_0|^2 b(\Lambda) \exp \left[ \int_{\Lambda}^E d\mu(q) j_{\nu}(q) j_{\nu}(-q) \right] \\ &= |T_0|^2 b(\Lambda) \left( \frac{E}{\Lambda} \right)^{\Lambda}, \end{aligned} \quad (2.26)$$

where

$$\begin{aligned} b(\Lambda) &= \frac{1}{\pi} \int_{-\infty}^{\infty} d\tau \frac{\sin \tau}{\tau} \exp \left[ \Lambda \int_0^1 d\omega \frac{\exp(i\omega\tau) - 1}{\omega} \right] \\ &= 1 - \frac{\pi^2}{12} \Lambda^2 + \dots \end{aligned} \quad (2.27)$$

is a standard function [11, 17]. Equation (2.26) reproduces the Sudakov formfactor [46]; for example, for the  $e^+e^-$  scattering we have  $d\sigma = d\sigma_0 \exp[(2\alpha/\pi) \ln(s/m^2) \ln(E/\Lambda)]$ . From Eqn (2.26), it is seen that the probability of the process  $T_0$  with the emission of energy  $\sum \omega_i \leq E$  is finite and independent of the non-physical cutoff parameter  $\lambda$ . However a new parameter  $\Lambda$  has appeared. In Ref. [11] this parameter was taken to be equal to  $E$ , and in Ref. [17] it is recommended that it be the characteristic mass of the process.

The meaning of the parameter  $\Lambda$  is actually simple. Equation (2.26) in fact says that photons with energies  $\omega < \Lambda$  are not emitted and that divergences due to the emission of real soft photons exactly cancel those due to virtual infrared ones. For what values of  $\Lambda$  is this true? Recall that in reality every experiment is bounded in time, i.e., lasts a certain period  $T$ . According to the Uncertainty Principle, photons with energies  $\omega < 1/T$  are neither real nor virtual infrared photons, and their fate will only be determined after a time  $t > \omega^{-1} > T$ . Hence they cannot affect at all the result of the experiment. Thus,

$$\Lambda \sim T^{-1}. \quad (2.28)$$

For short-range forces one would reasonably assume that  $T^{-1} \sim m$ , in consistence with the recommendation of Ref. [17]. In electrodynamics, however, one would expect that  $\Lambda < m$ . It should be remembered, though, that the dependence on these parameters is logarithmic, i.e., a weak one.

Another important point is that the correction factor in Eqn (2.26) is an infinite power series in  $\alpha$ , and hence  $T_0$  in Eqn (2.26) should be calculated to the same accuracy. But then  $T_0$  should be replaced by the amplitude  $T_{0\Lambda}$  corrected for the virtual photons with energies  $\omega > \Lambda$ . In principle, we can factor out from  $T_{0\Lambda}$  the photon contribution for energies  $E_0 > \omega > \Lambda$  (or  $E > \omega > \Lambda$ ) provided that expressions like (2.15) are still valid. This will result in the replacements  $\Lambda \rightarrow E_0$ , and  $T_0 \rightarrow T_{0E_0}$  (or  $\Lambda \rightarrow E$ ,  $T_0 \rightarrow T_{0E}$ ) in Eqn (2.26).

Thus, of the four parameters  $\lambda$ ,  $\Lambda$ ,  $E_0$ ,  $E$  discussed above,  $\lambda$  is an auxiliary parameter absent from the final expression;  $\Lambda$  characterizes the experimental apparatus (photons with frequencies  $\omega \leq \Lambda$  are not emitted);  $E_0 \gg \Lambda$  determines the resolution of the experiment (photons with energies  $\omega \leq E_0$  are not detected); and  $E$  in Eqns (2.18) and (2.26) is the energy carried away by the non-detected radiation.

## 2.2 Gravitation

**2.2.1. The interaction Lagrangian and the graviton propagator.** Although the general relativity (GR) is a rather complicated theory and although the gravitational Hamiltonian differs fundamentally from its electromagnetic counterpart, the above analysis carries over almost unchanged to the theory of gravitation. The only necessary changes are purely kinematic ones and due to the fact that the spin of the graviton is 2, i.e., the vertex functions and propagators of GR diagrams must be given a different form than in QED [17]. To arrive at this conclusion, however, a certain amount

of work needs to be done. In what follows, only the most important points will be discussed.

The Lagrangian of the spinor and gravitational fields is written in the form

$$\mathcal{L} = \sqrt{-g} \left\{ \bar{\psi} [i\gamma_\mu(x) D_\mu - m] \psi + \frac{1}{\kappa^2} g^{\mu\nu} (\Gamma_{\mu\nu}^\rho \Gamma_{\rho\sigma}^\sigma - \Gamma_{\mu\sigma}^\rho \Gamma_{\rho\nu}^\sigma) \right\} \equiv \mathcal{L}_m + \mathcal{L}_{gr}, \quad (2.29)$$

where  $D_\mu = \partial_\mu - (1/4)\Gamma_{\mu\lambda}^\rho g^{\lambda\sigma} \gamma_\rho \gamma_\sigma$ ,  $[\gamma_\mu(x), \gamma_\nu(x)]_+ = 2g_{\mu\nu}(x)$ ,  $\gamma_\mu(x) = \gamma_\alpha e_\mu^\alpha(x)$ ,  $[\gamma_\alpha, \gamma_\beta]_+ = 2\eta_{\alpha\beta}$ ,  $\eta_{\alpha\beta} e_\mu^\alpha(x) e_\nu^\beta(x) = 2g_{\mu\nu}(x)$  [i.e.,  $e_\mu^\alpha(x)$  is the tetrad, and  $\eta_{\alpha\beta}$  is the metric of Minkowski space],  $\Gamma_{\mu\nu}^\rho = (1/2)g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$  are the Christoffel symbols. The graviton field  $h_{\mu\nu}$  is defined by  $g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x)$ ,  $\kappa^2 = 8\pi G$ ,  $G$  is Newton's constant ( $V = Gm_1 m_2/r$ ). Dirac's equation for the gravitational field dates back to Fock and Weyl [47, 48].

It is readily seen that in the standard gauge fixed by  $\mathcal{L}_F = (1/2)[\partial_\nu(\sqrt{-g} g^{\mu\nu})]^2$  [which is analogous to Feynman's gauge; the condition  $\partial_\nu(\sqrt{-g} g^{\mu\nu}) = 0$  specifying harmonic coordinates] the expansion of  $\mathcal{L}$  in powers of  $\kappa h$  takes the form

$$\mathcal{L} = \bar{\psi} [i\gamma_\mu \partial_\mu - i\kappa h_{\mu\nu} \gamma_\mu \partial_\nu - m + O(h^2)] \psi + \frac{1}{8} [\partial_\lambda h_{\mu\nu} \partial_\lambda h_{\rho\sigma} g_{\mu\nu\rho\sigma} + \dots] + \mathcal{L}_{gh}. \quad (2.30)$$

Here the indices of the same variance are convolved using the tensor  $\eta_{\mu\nu}$ ,  $g_{\mu\nu\rho\sigma} = \eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\nu}\eta_{\rho\sigma}$ , and  $\mathcal{L}_{gh}$  is the Lagrangian of ghost fields. From Eqn (2.30), the expressions for the three-particle vertex and the graviton propagator are

$$\kappa \gamma_\mu p_\nu, \quad \frac{1}{2} \frac{ig_{\mu\nu\rho\sigma}}{q^2 + i0} \quad (2.31)$$

[instead of  $e\gamma_\mu$  and  $-ig_{\mu\nu}/(q^2 + i0)$  as in QED]. The main points to note about the Lagrangian (2.30) are, first, the vertices of the form  $\bar{\psi}\gamma\partial\psi(\kappa h)^n$ ,  $n \geq 2$ , and, second, the vertices  $(\partial h)^2(\kappa h)^n$ ,  $n \geq 1$  for the self-interaction of spin-2 particles. In other words, the theory becomes inconceivably complicated.

It turns out, however, that the terms we have written down above suffice to describe processes with soft gravitons. Fictitious Bose fields with anomalous statistics do not lead to infrared divergences [49] because they appear only in loops (divergences are only caused by emission from outer lines, i.e., the emission of free particles). Moreover, the more gravitons there are in a vertex, the faster the decrease of the corresponding term in the interaction Hamiltonian: the theory of asymptotic expansions shows [50] that, other things being equal, the higher is the multiplicity of the Fourier integral the faster it decreases. Therefore, vertices with more than one graviton may be neglected (see also Section 6.3).

Finally, the interaction of gravitons with massless fields (and hence the self-action of gravitons) does not lead to additional collinear divergences [17], and in the present context only triple vertices need to be considered. But since in the long-wave limit these latter have a universal form independent of the spin of the 'matter field', their analysis reduces to the study of the main process with vertices (2.31) in the limit  $m \rightarrow 0$ .

**2.2.2. Infrared divergences in gravitation. Massive particles.** From what has been said in Section 2.2.1, it is clear that a transition from QED to gravitation in the theory of long-

wave radiation involves two replacements, one for the current

$$j_\mu(q, p) \rightarrow j_{\mu\nu}(q, p) = \kappa \sum_a \frac{\eta_a p_\mu^\alpha p_\nu^\alpha}{p^\alpha q + i\eta_a 0} \quad (2.32)$$

and the other for the polarization vector,  $\epsilon_\mu^\lambda(q) \rightarrow \epsilon_{\mu\nu}^\lambda(q)$ ,  $h = \pm 2$ . The soft graviton emission amplitude therefore follows from Eqn (2.8) upon replacement (2.32) to give

$$T' = T j_{\mu\nu} \epsilon_{\mu\nu}^\lambda(q). \quad (2.33)$$

The amplitude of emission of  $N$  gravitons can be written in an analogous fashion to Eqn (2.15). The expression for the contribution from  $N$  virtual gravitons is similar to Eqn (2.22) as written with the above modifications, i.e., with the replacement  $A(q) \rightarrow B(q)$ ,

$$B(q) = \frac{i}{2(2\pi)^4} \frac{j_{\mu\nu}(q) g_{\mu\nu\rho\sigma} j_{\rho\sigma}(-q)}{q^2 + i0} = \frac{i\kappa^2}{(2\pi)^4} \sum_{ab} \frac{\eta_a \eta_b [(p_a p_b)^2 - m_a^2 m_b^2 / 2]}{(p_a q + i\eta_a 0)(p_b q + i\eta_b 0)(q^2 + i0)}. \quad (2.34)$$

The real part of the integral  $\int dq B$  and that of  $\int dq A$  differ only in the coefficients of the standard function  $\Phi(v_{ab})$ , Eqn (2.20), and in Eqn (2.26) the quantity  $A$ , Eqn (2.20), is replaced by

$$B = -\kappa^2 \sum_{ab} \frac{\eta_a \eta_b [(p_a p_b)^2 - m_a^2 m_b^2 / 2]}{p_a p_b} \Phi(v_{ab}) = -\frac{\kappa^2}{2} \sum_{ab} \eta_a \eta_b m_a m_b \frac{1 + v_{ab}^2}{(1 - v_{ab}^2)^{1/2}} \Phi(v_{ab}). \quad (2.35)$$

Thus, the infrared problem in the theory of gravitation is entirely similar to that in QED: the summation over the initial and final states of infrared gravitons leads to infrared-regular expressions for the probabilities and cross sections of the processes involved.

**Massless particles.** From Eqn (2.35) one readily obtains the result [17], already mentioned in Section 2.2.1, that, unlike QED, letting the mass of hard particles tend to zero does not lead to additional collinear divergences. To see this, let  $m_a = M_a \xi^{r_a}$ , where  $M_a$  are certain masses,  $\xi$  is a parameter, and  $r_a \geq 0$ . For  $\xi \rightarrow 0$ , masses  $m_a$  tend to zero differently: taking  $r_a = 0$ , for example, leaves the mass  $m_a$  finite.

In Eqn (2.35), with  $\xi \rightarrow 0$ , the following two partial sums may be isolated:

$$1) a \neq b, \quad v_{ab} \rightarrow 1, \quad \frac{1}{v_{ab}} \ln \frac{1 + v_{ab}}{1 - v_{ab}} \rightarrow \ln \frac{4(p_a p_b)^2}{m_a^2 m_b^2},$$

$$2) a = b, \quad v_{aa} = 0, \quad \frac{1}{v_{aa}} \ln \frac{1 + v_{aa}}{1 - v_{aa}} \rightarrow 2,$$

with  $v_{ab} = [1 - m_a^2 m_b^2 / (p_a p_b)^2]^{1/2}$ .

Noting that  $\ln [4(p_a p_b)^2 / m_a^2] = \ln 4$ , we have

$$B|_{m_a \rightarrow 0} = -\frac{\kappa^2}{8\pi^2} \sum_{ab} \eta_a \eta_b (p_a p_b) \left[ \ln \frac{4(p_a p_b)^2}{m_a^2 m_b^2} + \delta^{ab} (2 - \ln 4) \right]. \quad (2.36)$$



For  $r_a > 0$ , the second term in square brackets contributes nothing, giving

$$B \xrightarrow{\xi \rightarrow 0} -\frac{\kappa^2}{8\pi^2} \sum_{ab} \eta_a \eta_b (p_a p_b) \left[ \ln \frac{4(p_a p_b)^2}{M_a^2 M_b^2} - \ln \xi^{2(r_a + r_b)} \right], \quad (2.37)$$

and the term which diverges logarithmically for  $\xi \rightarrow 0$  vanishes because of the conservation of the 4-momentum  $\sum \eta_a p_a = 0$ . [This trick fails in QED and QCD: here  $\kappa^2 p^2$  plays the role of  $e^2$  and  $g^2$ ].

Thus, letting some or all of the masses of gravitating fields to zero does not generate collinear divergences (in agreement with the results of Section 6.3). If all quanta are soft, however, then, neglecting the self-action of 'matter fields,' it follows that the long-wave theory of massless gravitating fields is free from both infrared and collinear divergences: making the replacement  $p \rightarrow \xi p$  in Eqn (2.37) we find that  $B \rightarrow 0$  as  $\xi \rightarrow 0$ . The former divergences are absent because their effective interaction with gravitons disappears (the coupling constant is proportional to  $E_p \rightarrow 0$  in this limit), and the latter, because the graviton emission amplitudes are proportional to  $\theta^2$ ,  $\theta$  being the emission angle (see Section 6.3).

### 3. Infrared divergences.

#### Redefinition of the scattering operator

The extremely simple soft-photon emission mechanism revealed in the analysis in Section 2.1 suggests that a formalism of a simpler and more general nature than perturbation theory must exist which can be employed in the long-wave region of QED. The main result of this analysis is that all one needs to describe processes with infrared photons is to include the coupling of the electromagnetic field with a classical current [19]. But QED with a classical current is an exactly solvable model (see, for example, Ref. [51]). Given the specific nature of the Coulomb potential scattering — the fact that the Coulomb potential should be taken into account at arbitrarily large distances (see Section 2.1.1) — an analogous situation is expected to persist in quantum field theory. In the present section this program is carried out for QED and QCD.

#### 3.1 Quantum electrodynamics

**3.1.1 Quantum mechanics. Coulomb scattering.** According to Eqn (2.2), the  $S$ -matrix is not defined for slowly decreasing potentials (because of the phases tending to infinity as  $t - t' \rightarrow \infty$ ). The natural way to get round this difficulty is by redefining the scattering operator [18, 30]. In a way similar to that employed in transition from the evolution operator  $\hat{U}_{t,t'}$  to the scattering operator  $\hat{U}_{t,t'}$ , Eqn (2.1), we first redefine the free Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{V} = \tilde{H}_0 + \tilde{V}, \quad (3.1)$$

by adding to it the leading order term of the asymptotics of the potential for  $r \rightarrow \infty$  (we drop the hat in tilded quantities).

For example, in the interaction representation we have

$$V_I[r(t)] = \frac{\alpha}{|\hat{\mathbf{p}}t/m + \hat{\mathbf{r}}|} \sim_{t \rightarrow \infty} \frac{\alpha m}{|t||\hat{\mathbf{p}}|} = \frac{\alpha m}{|t|(-\Delta)^{1/2}}, \quad (3.2)$$

where  $\Delta$  is the Laplace operator; we therefore postulate that

$$\tilde{H}_0 = H_0 + \frac{\alpha m}{|t|(-\Delta)^{1/2}}, \quad \tilde{V}(\mathbf{r}, t) = \frac{\alpha}{|\mathbf{r}|} - \frac{\alpha m}{|t|(-\Delta)^{1/2}}. \quad (3.3)$$

We next change the interaction representation ( $I \rightarrow \tilde{I}$ ):

$$\begin{aligned} \hat{\mathbf{r}}_{\tilde{I}} &= \exp \left[ i \int_0^t dt \tilde{H}_0 \right] \hat{\mathbf{r}} \exp \left[ -i \int_0^t dt \tilde{H}_0 \right] \\ &= \hat{\mathbf{r}}_I - \frac{\alpha m \mathbf{p}}{|\hat{\mathbf{p}}|^3} \epsilon(t) \ln |t|, \quad \hat{\mathbf{p}}_{\tilde{I}} = \hat{\mathbf{p}}_I \end{aligned} \quad (3.4)$$

[ $\epsilon(t)$  being the sign function] and replace the scattering operator (2.1) by the operator

$$\begin{aligned} \tilde{U}_{t,t'} &= \exp(i\tilde{H}_0 t) \exp[-i\tilde{H}(t-t')] \exp(-i\tilde{H}_0 t'), \\ \tilde{S} &= \tilde{U}_{\infty, -\infty}. \end{aligned} \quad (3.5)$$

Paper [30] demonstrates the existence of the strong limits  $\tilde{U}_{0,t'}$ ,  $t' \rightarrow -\infty$ ,  $\tilde{U}_{t,0}$ ,  $t \rightarrow \infty$ , thus proving the existence of the  $S$ -matrix

$$\tilde{S} = T \exp \left\{ -i \int_{-\infty}^{\infty} dt \tilde{V}_{\tilde{I}}[r(t)] \right\}. \quad (3.6)$$

The new potential decreases faster than  $1/r$ :

$$\tilde{V}_{\tilde{I}} = \frac{\alpha}{|\tilde{\mathbf{r}}_{\tilde{I}}|} - \frac{\alpha m}{|t||\hat{\mathbf{p}}|} = O\left(\frac{\ln |t|}{t^2}\right). \quad (3.7)$$

Equation (3.7) shows that the integral in Eqn (3.6) converges.

By redefining the scattering operator we have disposed of a nonsignificant, logarithmically divergent phase, thus paving the way to a mathematically correct scattering theory. It has been shown in Ref. [52] that the method described above also applies to more slowly decreasing potentials (such as  $r^{-\mu}$ ,  $\mu \geq 3/4$ ). The same paper shows that the new operator  $\tilde{S}$  is defined up to a certain unitary operator (because we actually subtracted one infinity from another).

This last circumstance is important for the interpretation of Eqns (3.4)–(3.7) because the corresponding integrals should be defined at zero — for example, by integrating over  $[t_0, t]$ ,  $t_0 > 0$ . As a result, the logarithm in Eqn (3.4) will be replaced by  $\ln |t|/t_0$ . It is here that the ambiguity discussed above manifests itself. It does not affect the physical picture, though. Note that the idea we have outlined is also workable in quantum field theory [18, 26].

**3.1.2 QED. Arbitrary process. Effective interaction Hamiltonian.** Following the discussion of Section 3.1.1, the asymptotic form of the interaction Hamiltonian for  $|t| \rightarrow \infty$  [18] is determined by substituting the Fermi operators  $\psi = \psi^{(+)} + \psi^{(-)}$ ,  $\bar{\psi} = \bar{\psi}^{(+)} + \bar{\psi}^{(-)}$ , where  $\psi^{(+)}$  ( $\bar{\psi}^{(+)}$ ) contains the electron (positron) annihilation operator, giving

$$\begin{aligned} H_{\text{int}}(t) &= e \int d^3x \bar{\psi} \gamma_\mu \psi A_\mu = e \int d^3x [\bar{\psi}^{(+)} \gamma_\mu \psi^{(+)} \\ &\quad + \bar{\psi}^{(-)} \gamma_\mu \psi^{(-)} + \bar{\psi}^{(+)} \gamma_\mu \psi^{(-)} + \bar{\psi}^{(-)} \gamma_\mu \psi^{(+)}] A_\mu. \end{aligned} \quad (3.8)$$

We will denote the integrals of the corresponding terms in Eqn (3.8) by  $H_{\text{int}}^{\pm\pm}$ .

It suffices to examine the asymptotic forms of  $H_{\text{int}}^{++}$  and  $H_{\text{int}}^{+-}$  to elucidate the essence of the problem. Substituting the expansions (1.5) and (1.6) into these operators and integrat-

ing over coordinates we find

$$H_{\text{int}}^{++} = (2\pi)^3 e \int d\mu(p', p, q) b_{\sigma'}(p') a_{\sigma}(p) \bar{v}^{\sigma'}(p') \gamma_{\mu} u^{\sigma}(p) \\ \times \{ \exp[-i(E_{p'} + E_p + \omega)t] \delta(\mathbf{p}' + \mathbf{p} + \mathbf{q}) c_{\mu}(q) \\ + \exp[-i(E_{p'} + E_p - \omega)t] \delta(\mathbf{p}' + \mathbf{p} - \mathbf{q}) c_{\mu}^{+}(q) \}, \quad (3.9)$$

$$H_{\text{int}}^{+-} = (2\pi)^3 e \int d\mu(p', p, q) b_{\sigma'}(p') b_{\sigma}^{+}(p) \bar{v}^{\sigma'}(p') \gamma_{\mu} v^{\sigma}(p) \\ \times \{ \exp[-i(E_{p'} - E_p + \omega)t] \delta(\mathbf{p}' - \mathbf{p} + \mathbf{q}) c_{\mu}(q) \\ + \exp[-i(E_{p'} - E_p - \omega)t] \delta(\mathbf{p}' - \mathbf{p} - \mathbf{q}) c_{\mu}^{+}(q) \}, \quad (3.10)$$

where  $c_{\mu} = c_{\lambda} e_{\mu}^{\lambda}$ ,  $d\mu(p', p, q)$  is the product of the corresponding one-particle measures, and  $\omega = |q|$ . The asymptotic form of such expressions for  $|t| \rightarrow \infty$  is determined by the singular points of the integrands and stationary points of the arguments of the exponentials [50]. Integrating Eqns (3.9) and (3.10) over  $\mathbf{p}'$  yields the momentum functions for the exponentials,

$$f_{++} = E_{p \pm q} + E_p \pm \omega, \quad f_{+-} = E_{p \pm q} - E_p \mp \omega \quad (3.11)$$

(with  $E_p \equiv E_p$ ).

Let us now find the asymptotics of the integrals over  $\mathbf{p}$  for  $|t| \rightarrow \infty$ . The stationary points of functions (3.11) are defined by the equations

$$\nabla_{\mathbf{p}} f_{++} = \frac{\mathbf{p} \pm \mathbf{q}}{E_{p \pm q}} + \frac{\mathbf{p}}{E_p} = 0, \quad \nabla_{\mathbf{p}} f_{+-} = \frac{\mathbf{p} \pm \mathbf{q}}{E_{p \pm q}} - \frac{\mathbf{p}}{E_p} = 0. \quad (3.12)$$

For  $m \neq 0$ , the first of these is satisfied by  $\mathbf{q} = \mp 2\mathbf{p}$ , and the second by any  $\mathbf{p}$  for  $\mathbf{q} = 0$ . According to Eqn (3.11), for  $|t| \rightarrow \infty$  the first equation yields a rapidly oscillating exponential, implying that the asymptotics of the term (3.10) dominates in this case.

Thus, of the two terms ( $H_{\text{int}}^{++}$  and  $H_{\text{int}}^{+-}$ ) describing massive electrodynamics, the main contribution to the asymptotics of  $H_{\text{int}}$ , Eqn (3.8), comes from the term (3.10). This is due to the fact that the exponentials in  $H^{+-}$  contain the difference  $E_{p'} - E_p$ , i.e., Eqn (3.10) describes the processes of emission and absorption of photons by positrons. The term (3.9), on the contrary, contains in its exponents the sum  $E_{p'} + E_p$ , which corresponds to the annihilation of electron-positron pairs. Obviously, the photon energy in this case cannot be less than  $2m$ . It is clear that of the remaining two terms,  $H^{--}$  and  $H^{+-}$ , the main contribution to the asymptotics comes from the latter one, which accounts for the emission and absorption of photons by electrons,

$$H_{\text{int}}^{--} = (2\pi)^3 e \int d\mu(p', p, q) a_{\sigma'}^{+}(p') a_{\sigma}(p) \bar{u}^{\sigma'}(p') \gamma_{\mu} u^{\sigma}(p) \\ \times \{ \exp[i(E_{p'} - E_p - \omega)t] \delta(\mathbf{p}' - \mathbf{p} - \mathbf{q}) c_{\mu}(q) \\ + \exp[i(E_{p'} - E_p + \omega)t] \delta(\mathbf{p}' - \mathbf{p} + \mathbf{q}) c_{\mu}^{+}(q) \}. \quad (3.13)$$

Thus,  $H_{\text{int}} \sim H_{\text{int}}^{--} + H_{\text{int}}^{+-}$  for  $|t| \rightarrow \infty$ . Close to critical regime ( $q \rightarrow 0$ ) we have

$$E_{p \pm q} - E_p \mp \omega \approx E_p \left( 1 \pm \frac{\mathbf{p} \cdot \mathbf{q}}{E_p^2} \right) - E_p \mp \omega = \mp \frac{\mathbf{p} \cdot \mathbf{q}}{E_p}. \quad (3.14)$$

In Eqns (3.10) and (3.13), after the integration over  $\mathbf{p}'$ , the following operations should be carried out:

1) in the coefficients before the brackets we set  $\mathbf{q} = 0$  (the integrand is set to its critical value [50]);

2) using the Gordon identities  $P_{\mu} = \pm 2m\gamma_{\mu} - i\sigma_{\mu\nu}q_{\nu} + (\hat{\mathbf{p}}' \mp m)\gamma_{\mu} + \gamma_{\mu}(\hat{\mathbf{p}} \mp m)$  (where  $P = p' + p$ ,  $q = p' - p$ ,  $\sigma_{\mu\nu} = (i/2)[\gamma_{\mu}, \gamma_{\nu}]$ ), and the orthogonality conditions  $\bar{u}^{\sigma'}(p) u^{\sigma}(p) = 2m\delta^{\sigma'\sigma}$ ,  $\bar{v}^{\sigma'}(p) v^{\sigma}(p) = -2m\delta^{\sigma'\sigma}$ , we make the replacements  $\bar{u}^{\sigma'}(p') \gamma_{\mu} u^{\sigma}(p) \rightarrow \delta^{\sigma'\sigma} \cdot 2p_{\mu}$ ,  $\bar{v}^{\sigma'}(p') \gamma_{\mu} v^{\sigma}(p) \rightarrow \delta^{\sigma'\sigma} \cdot 2p_{\mu}$ ;

3) the integration over  $\mathbf{q}$  is reduced to the region  $|q| < Q$  by substituting the function  $\theta(Q - |q|)$  into Eqns (3.10) and (3.13) (with  $Q \leq E_0$  we separate the infrared photons).

As a result we obtain the following representation for  $H'_{\text{int}} = H_{\text{int}}^{++} + H_{\text{int}}^{--}$ :

$$H'_{\text{int}} = \int d\mu(p) \frac{p_{\mu}}{E_p} \hat{p}(p) \int_{|q| < Q} d\mu(q) \left[ c_{\mu}(q) \exp\left(-\frac{itpq}{E_p}\right) \right. \\ \left. + c_{\mu}^{+}(q) \exp\left(\frac{itpq}{E_p}\right) \right] \\ = \int d^3x \int d\mu(p) \frac{p_{\mu}}{E_p} \hat{p}(p) \delta\left(x - \frac{\mathbf{p}}{E_p} t\right) \int_{|q| < Q} d\mu(q) \\ \times [c_{\mu}(q) \exp(-iqx) + c_{\mu}^{+}(q) \exp(iqx)], \quad (3.15)$$

$$\hat{p}(p) = e[a_{\sigma}^{+}(p)a_{\sigma}(p) - b_{\sigma}^{+}(p)b_{\sigma}(p)]$$

( $qx = \omega t - \mathbf{q} \cdot \mathbf{x}$ ). Recalling expressions (2.9) and (2.10) for the current of a pointlike particle and introducing the notation

$$\hat{J}_{\mu}(x, t) = \int d\mu(p) \frac{p_{\mu}}{E_p} \hat{p}(p) \delta(x - \mathbf{v}t) \equiv \int d\mu(p) \hat{N}(p) j_{\mu}(x, p), \quad (3.16)$$

where  $j_{\mu}(x, p) = e(p_{\mu}/E_p) \delta(x - \mathbf{v}t)$ ,  $e\hat{N}(p) \equiv \hat{p}(p)$ , and

$$\hat{A}_{\mu}^Q(x) = \int_{|q| < Q} d\mu(q) [\hat{c}_{\mu}(q) \exp(-iqx) + \hat{c}_{\mu}^{+}(q) \exp(iqx)], \quad (3.17)$$

Eqn (3.15) finally becomes

$$H'_{\text{int}} = \int d^3x \hat{J}_{\mu}(x) \hat{A}_{\mu}^Q(x). \quad (3.18)$$

The Hamiltonian (3.18) has a clear meaning: the emission of low-energy photons is described by the interaction with the 'classical current operator'  $\hat{J}_{\mu}(x)$ . The corresponding action is gauge invariant: since the current  $\hat{J}_{\mu}(x)$  is conserved ( $\partial_{\mu} \hat{J}_{\mu} = 0$ ), the action  $\int d^4x \hat{J}_{\mu} A_{\mu}^Q$  remains unaffected by the replacement  $A_{\mu}^Q \rightarrow A_{\mu}^Q + \partial_{\mu} \chi$ . States with a definite number of charged particles form the eigenvector of the operator  $\hat{J}_{\mu}$ :

$$\hat{J}_{\mu}(x)|p\rangle = \sum_a j_{\mu p}^{(a)}(x)|p\rangle, \\ j_{\mu p}^{(a)}(x) = e_a \frac{p_{\mu}}{E_p} \delta(x - \mathbf{v}_a t) = j_{\mu}^{\text{cl}}(x, p_a). \quad (3.19)$$

Here the summation runs over all particles that form the state  $|p\rangle$ . Noting that  $p_{\mu}/E_p = dx_{\mu}/dt$ , the one-particle action

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corresponding to Eqn (3.18) can be written as

$$e \int dt \int d^3x \hat{A}_\mu^Q(x) \frac{dx_\mu}{dt} \delta(x - vt) = e \int \hat{A}_\mu^Q(x) dx^\mu \\ = e \int \hat{A}_\mu^Q(x^{(0)} + u\tau) u_\mu d\tau, \quad (3.20)$$

where the one-dimensional integrals are along the straight line defined by  $p_\mu, u_\mu = p_\mu/m$  and  $\tau$  is the invariant time. The Hamiltonian (3.18) for the effective interaction of soft photons fails to satisfy momentum conservation — a situation which also occurs in potential scattering problems or in processes involving massive particles.

**Redefinition of the scattering operator.** We next represent the QED interaction Hamiltonian in the form (we drop hats on operators in obvious cases)

$$H_{\text{int}}(t) = H'_{\text{int}} + \int d^3x [e\bar{\psi}\gamma_\mu\psi A_\mu - J_\mu A_\mu^Q] \equiv H'_{\text{int}} + \tilde{H}_{\text{int}}, \quad (3.21)$$

and, in accord with the general ideology, change to a new scattering operator [see Eqns (3.1) and (3.5)] and the new  $S$ -matrix

$$\tilde{S} = T \exp \left[ -i \int_{-\infty}^{\infty} dt \tilde{H}_{\text{int}}^{\tilde{I}} \right], \quad (3.22)$$

where superscript  $\tilde{I}$  indicates that all the operators are taken in the new interaction representation (in what follows, this superscript will not be carried along unless necessary). These operators are expressed explicitly in terms of the initial ones, and their time dependence is now determined by the operator  $\tilde{H}_0 = H_0 + H'_{\text{int}}$ . Clearly the operator  $\tilde{S}$  is free from infrared divergences because the matrix elements of  $\tilde{H}_{\text{int}}$  vanish in the infrared limit. This result can also be obtained by direct calculation.

The equations of motion (we omit polarization indices on the operators  $\hat{a}, \hat{b}$ )<sup>2</sup>

$$\dot{a}_I^\pm(p, t) = \pm i [E_p + A_I(p, t)] a_I^\pm(p, t), \\ \dot{b}_I^\pm(p, t) = \pm i [E_p - A_I(p, t)] b_I^\pm(p, t), \quad (3.23)$$

$$\dot{c}_{\mu I}^\pm(q, t) = \pm i \left[ \omega c_{\mu I}^\pm(q, t) - \int d^3x J_{\mu I}(x, t) \exp(\pm iqx) \theta(Q - |q|) \right], \quad (3.24)$$

where

$$A(p, t) = \int d^3x A_\mu^Q(x) j_{\mu p}(x), \quad (3.25)$$

go over to the equations of motion of free fields as  $e \rightarrow 0$  and can be integrated in an elementary fashion. Recognizing that at  $t = 0$  all the representations are identical, we have ( $T$  symbolizes time ordering)

$$a_I^\pm(p, t) = T \exp \left[ \pm i \left( E_p t + \int_0^t A_I(p, t) dt \right) \right] a_I^\pm(p), \\ b_I^\pm(p, t) = T \exp \left[ \pm i \left( E_p t - \int_0^t A_I(p, t) dt \right) \right] b_I^\pm(p), \quad (3.26)$$

<sup>2</sup> It is sometimes convenient to attach the '+' sign to operators  $a, b$ , and  $c$ ;  $\hat{a}^-(p)$  etc.

$$c_{\mu I}^\pm(q, t) = \exp(\pm i\omega t) \left[ c_{\mu}^\pm(q) \mp i \int_0^t dt \int d^3x \hat{J}_\mu(x, t) \right. \\ \left. \times \exp(\mp iqx) \theta(Q - |q|) \right]. \quad (3.27)$$

The equations and solutions (3.24) and (3.27) are written only for fields  $A_\mu^Q$ . Those for fields with  $|q| > Q$  are identical to expressions for the standard interaction representation. In obtaining Eqn (3.27) we have made use of the equality  $\hat{J}_\mu^{\tilde{I}} = \hat{J}_\mu$  which follows from Eqns (3.15), (3.16), and (3.23).

Let us discuss the results obtained. From the representation (3.20) and Eqns (3.25)–(3.27) we obtain the following expression for the Fermi field:

$$\psi_I(x) = \int d\mu(p) T \exp \left[ -ie \int^x A_\mu^Q(y) dy^\mu \right] \\ \times [\hat{a}_{p,\sigma} u^\sigma(p) \exp(-ipx) + \hat{b}_{p,\sigma}^\dagger v^\sigma(p) \exp(ipx)], \\ y^\mu = u^\mu \tau, \quad dy^\mu = u^\mu d\tau.$$

Similar to the selection of a gauge [53], the appearance of the phase  $e \int A_\mu^Q dy^\mu$  in charged fields ('exponentiation') has a simple meaning: the corresponding degrees of freedom of the field  $A_\mu$  are withdrawn from dynamics. According to the analysis at the end of this section, quanta with energies  $\omega < \Lambda$  are not actually emitted in the course of an experiment and do not affect its results [see Eqn (2.26)] but it is these photons which give rise to the infrared problem. The time evolution of such concomitant quanta is completely determined by equations of motion for charged fields (particles). For a particle moving in the direction  $\mathbf{n}$ , in the gauge  $A_0 = 0$  the field  $A_\mu^Q$  is concentrated outside a cylinder with an axis  $\mathbf{n}$  and radius  $Q^{-1}$ ,  $Q \sim \Lambda$  (see the discussion at the end of this section).

The creation and annihilation operators in the new representation have the same commutation relations as in the old. It is these operators which produce in and out states in the new scattering theory. As compared to the previous interaction representation, charged particles acquire phases

$$\chi_{\text{out}} = \int_0^\infty dt \int d^3x A_\mu^Q(x) j_\mu^{(+)}(x, p), \\ \chi_{\text{in}} = \int_{-\infty}^0 dt \int d^3x A_\mu^Q(x) j_\mu^{(-)}(x, p), \quad (3.28)$$

where  $j_\mu^{(\pm)} = i j_\mu^\pm$  and the currents are defined by Eqn (2.10). Using the properties of these currents the time integration in Eqn (3.28) will be extended over the entire axis.

Expanding the exponentials in power series in currents we see that they describe the processes of emission of infrared photons by in and out particles [cf. Eqns (2.8) and (2.15)] — a consequence of the fact that in the  $\tilde{I}$  representation charged particles are considered together with the concomitant infrared photons. Further,  $j_{\mu p}^{(\pm)}(x)$  are classical real currents, so that the transition from  $a^\pm, b^\pm$  to  $\tilde{a}^\pm, \tilde{b}^\pm$  is performed via a unitary operator (hence the same commutation relations). Note, however, that we obtain zero when calculating the projection of the old one-particle state onto the new one.

Now let  $\chi^{(\pm)}$  contain photon creation (+) or annihilation (−) operators, i.e.,  $\chi = \chi^{(+)} + \chi^{(-)}$  (for  $\chi_{\text{in}}$  and  $\chi_{\text{out}}$ ). Then for  $U_\chi \equiv T \exp(i\chi)$  we have

$$U_\chi = \exp[i\chi^{(+)}] \exp[i\chi^{(-)}] \exp \left[ -\frac{1}{2} \int dt_1 dt_2 \right. \\ \left. \times \int d^3x_1 d^3x_2 j_\mu^{(\pm)}(x_1, p) (-g_{\mu\nu}) \Delta_c^Q(x_1 - x_2) j_\nu^{(\pm)}(x_2, p) \right]. \quad (3.29)$$

Here  $-g_{\mu\nu}\Delta_c^Q$  is the propagator of the field  $A_\mu^Q$  and we have made use of Hori's [54] formula for disentangling the  $T$  exponent

$$T \exp(ij\varphi) = \exp\left[-\frac{1}{2}j\Delta_c j\right] : \exp(ij\varphi) :, \quad (3.30)$$

where  $\Delta_c$  is the causal propagator of the free scalar field  $\varphi$  and where the contracted notation  $j\varphi \equiv \int dx j(x)\varphi(x)$ ,  $dx = d^4x$ , etc., has been introduced. From the definitions (2.8) and (2.10) we obtain the argument of the last exponential in Eqn (3.29):

$$-\frac{1}{2} \int_Q \frac{dq}{(2\pi)^4} j_\mu^{(\pm)}(-q) \frac{-i}{q^2 + i0} j_\mu^{(\pm)}(q) \equiv -\frac{1}{2} \int_Q d^4q A^{(\pm)}(q) \quad (3.31)$$

[see Eqn (2.23)], where the domain of integration is  $|q| < Q$ .

Twice the real part of Eqn (3.31), with  $(\pm)$  omitted, is identical to the argument of the exponential in Eqn (2.24). It is readily seen that it is negative and divergent [see Eqns (2.19) and (2.20)]:

$$\begin{aligned} & \frac{i}{2} \int_Q \frac{dq}{(2\pi)^4} j_\mu^{(\pm)}(-q) (-i \cdot 2\pi) \frac{\delta(q^2)}{2} j_\mu^{(\pm)}(q) \\ &= -\frac{1}{2} \sum_{a,b} e_a e_b \int_Q d\mu(q) \frac{p^a p^b}{(p^a q)(p^b q)} \\ &= -\ln \frac{Q}{\lambda} \sum_{a,b} \frac{e_a e_b}{2} \Phi(v_{ab}). \end{aligned} \quad (3.32)$$

For one particle ( $v_{aa} = 0$ ,  $\Phi(0) = 1/4\pi^2$ ) we have  $-(\alpha/2\pi) \ln(Q/\lambda) \rightarrow -\infty$ ,  $\lambda \rightarrow 0$ . Hence, dropping the polarization indices,

$$\begin{aligned} & \int d\mu(p_1, p_2) \psi_1^*(p_1) \psi_2(p_2) \langle 0 | a(p_1) \tilde{a}^+(p_2) | 0 \rangle \\ &= \int d\mu(p) \psi_1^*(p) \psi_2(p) \left(\frac{\lambda}{Q}\right)^{\alpha/2\pi} \rightarrow 0, \quad \lambda \rightarrow 0 \end{aligned} \quad (3.33)$$

i.e., the states of charged particles in the  $\tilde{I}$  representation are orthogonal to those in the conventional interaction representation.

The meaning of this result is straightforward. As is known [55], the Hilbert space of quantum field theory is nonseparable (i.e., the set of basis vectors is uncountable). The space of standard perturbation theory is a separable Hilbert space (Fock space). When applied to a Fock space vector, the operator  $U_\chi$ , generating the cloud of soft photons, removes the vector from this space — hence the orthogonality property (3.33). This seems to contradict the formal unitary property of the operator  $U_\chi$ :  $U_\chi^\dagger U_\chi = 1$ . The explanation is, however, that  $U_\chi$  is unitary in a nonseparable Hilbert space (von Neumann space) of which the Fock space is a subspace [21, 26, 56, 57]. This example illustrates von Neumann's theorem [58] concerning the possibility of unitary non-equivalent representations of canonical commutation relations (commutators of the operators  $a$  and  $\tilde{a}$  are identical) in quantum field theory.

At the same time, the meaning of the cancellation of infrared divergences in perturbation theory becomes clear: including the contribution of virtual infrared photons gives rise to a factor which cancels out the corresponding infinite factor resulting from the summation over the states of real

infrared photons — much in the way such factors cancel out when the norm of the state  $U_\chi|0\rangle$  [cf. Eqns (3.32) and (2.24)] is calculated. Also the meaning of the  $S$ -matrix is cleared up: infrared photons are withdrawn from the dynamics in this case.

**Factorization of the  $S$ -matrix.** With the time hierarchy concept of Ref. [18] (see Section 1), one easily obtains a theorem on the factorization of the  $S$ -matrix. Since the time of a hard collision is much less than that of the emission of a long-wave quantum, the  $S$ -matrix is obtained by sandwiching an infrared-regular matrix  $S^h$  (for hard processes) by operators for the motion of charged particles before and after such a collision, i.e., for scattering in which soft photons take place [18]:

$$S^M = U_{\infty,0} S^h U_{0,-\infty}. \quad (3.34)$$

To see this, note that by definition,

$$S = T \exp \left[ -i \int_{-\infty}^{\infty} (H'_{\text{int}} + \tilde{H}_{\text{int}}) dt \right] \quad (3.35)$$

[see Eqns (3.18) and (3.21)]. The term  $H'_{\text{int}}$  accounts for soft quantum processes, in which only small energy changes occur in the course of the interaction. Hence, the exponential in Eqn (3.35) is dominated by the integral of  $H'_{\text{int}}$  over a large time interval; small changes in the range of integration have little effect on the result.  $\tilde{H}_{\text{int}}$ , on the contrary, describes hard processes with a large energy (momentum) transfer, i.e., the integration of  $\tilde{H}_{\text{int}}$  over time in Eqn (3.35) may be performed over a finite interval. Integrating over the entire time axis introduces no appreciable error because at large times particles are so far apart that their interaction may be neglected (the slowly decreasing part of  $\tilde{H}_{\text{int}}$  being removed).

It follows from the above argument that the operator (3.35) for finite time may be represented in the form

$$S^M_{TT'} = U_{Tt} S^h_{tt'} U_{t'T'}, \quad T \gg t, \quad |T'| \gg |t'|, \quad (3.36)$$

$$S^h_{tt'} = T \exp \left( -i \int_{t'}^t \tilde{H}_{\text{int}} dt \right),$$

$$U_{Tt} = T \exp \left( -i \int_t^T H'_{\text{int}} dt \right), \quad (3.37)$$

where the operators are taken in the standard interaction ( $I$ ) representation. We now may, as shown above, take the double limit  $t \rightarrow \infty$ ,  $t' \rightarrow -\infty$  in the first exponential in Eqn (3.37) and  $T \rightarrow \infty$ ,  $t \rightarrow 0$  (i.e.,  $T' \rightarrow -\infty$ ,  $t' \rightarrow 0$  in  $U_{t'T'}$ ) in the second, thus arriving at representation (3.34).

It is readily seen that representation (3.36) is equivalent to the corresponding expression for the  $S$ -matrix free of infrared divergence in the approach of Ref. [30]. This is seen by writing down the operators

$$S^h_{tt'} = \exp(iH_0 t) \exp[-i(H_0 + \tilde{H}_{\text{int}})(t - t')] \exp(-iH_0 t'), \quad (3.38)$$

$$U_{Tt} = \exp(iH_0 T) \exp[-i(H_0 + H'_{\text{int}})(T - t)] \exp(-iH_0 t), \quad (3.39)$$

which when substituted into Eqn (3.36) together with a similar representation for  $U_{t'T'}$  yield

$$\begin{aligned} S^M_{TT'} &= \exp(iH_0 T) \exp(-i\tilde{H}_0 T) \tilde{S}_{tt'} \exp(i\tilde{H}_0 T') \\ &\times \exp(-iH_0 T') \equiv U_T \tilde{S}_{tt'} U_{T'}^+, \end{aligned} \quad (3.40)$$

where  $\tilde{S}_{\mu'}$  is the scattering operator in the  $\tilde{T}$  representation [see Eqn (3.22)]. The operator  $U_T$  relates the  $I$  and  $\tilde{T}$  representations,  $A_I = U_T A_{\tilde{T}} U_T^\dagger$ . In Eqn (3.38) we may, as we did in Eqn (3.36), change to infinite times for  $\tilde{S}_{\mu'}$ , i.e., go over to the  $\tilde{S}$ -matrix (3.22).

**Kinoshita–Lee–Nauenberg theorem** [15, 16]. Representation (3.34) clearly demonstrates that infrared divergences cancel after averaging over the ensemble of photons in the initial and final states [16] (in view of  $UU^\dagger = 1$ ); and that the  $S$ -matrix (3.22) or the matrix  $S^h$  (3.34), (3.37) are infrared regular. This is seen by noting that infrared fields are in fact removed from the interaction operator  $\tilde{H}_{\text{int}}$  (3.21) and that the photon emission amplitude tends to zero if the photon momentum  $q \rightarrow 0$ .

Note here that the magnitude of the parameter  $Q$  in the definition of  $H'_{\text{int}}$  determines the meaning of the  $\tilde{S}$ -matrix. For  $Q \sim E$  (see at the end of Section 2.1.3),  $\tilde{S}$  can be identified with  $S^h$ , i.e., with the scattering operator for hard processes. If  $Q \sim \Lambda$ ,  $\tilde{S}$  is again free from infrared divergences but also describes processes in which infrared photons are involved ( $\omega > \Lambda$ ).

### 3.2 Quantum chromodynamics. Effective Hamiltonian

For the purpose of definiteness, the discussion in this section is limited to quantum chromodynamics, i.e., to the gauge theory of group  $SU(3)$ ; all formulas are generalized automatically to any semi-simple group. As mentioned in the introduction, a transition to non-Abelian gauge theories gives rise to a whole series of problems. The most obvious of these is due to the noncommutative nature of the generators of group  $T_a$ , i.e., the absence of factorization (2.15) in the infrared limit (gluons possess color and cannot therefore be emitted independently). The most challenging problem is due to the appearance of charged massless fields (gluons) because difficulties arising from the noncommutative nature of  $T_a$  matrices are compounded by the appearance of collinear divergences. Finally, the phenomenon of confinement makes the situation apparently hopeless because the problem of the dynamics at large distances (confinement problem) is not solved yet [59–61].

In a sense, however, the above difficulties actually simplify the problem. First, confinement leads to an effective small-momentum cutoff for quarks and gluons:  $|\mathbf{p}| > r_h^{-1}$  (where  $r_h$  is the hadron size), i.e., collinear divergences appear only for  $|\mathbf{p}| \rightarrow \infty$  (see Section 1), when all particles behave as massless. But then — and this is a second point — the asymptotic freedom of QCD enables perturbation theory to be applied. Because present day experiments are mostly high-energy (aimed, in particular, at the study of inclusive processes), this circumstance actually dictates what lines of research are to be pursued in the field, emphasizing such things as the asymptotic behavior of hard process amplitudes (e.g., the pion formfactor [40, 62]) and of the kernels of GLAP evolution equations (see Refs [10, 40, 63] and references therein). In this section the  $S$ -matrix redefinition method outlined above is applied to QCD. We will start with analyzing the noncommutative nature of group generators in order to introduce the reader into the specifics of the problem.

At first sight, generalizing the method of Section 3.1 to QCD seems absurd because free quarks do not exist and so there is no point in considering their interaction with low-energy gluons (there are no excitations for fields with wavelengths longer than the confinement radius). The

problem does make sense, however, when viewed from the standpoint of the hierarchy of characteristic interaction times. In this approach, only relative space-time scales of the processes of interest are important. In QED, such scales were determined by the size of the laboratory ( $L \sim T$ ) and the characteristic time of the hard process [ $1/Q_h$ , where  $Q_h$  is the momentum transfer,  $L \gg 1/Q_h$ ]. In QCD, the correspondence of  $L \leftrightarrow r_h$  (with  $r_h$  the hadron size, of order  $10^{-13}$  cm) is admissible, i.e., the scattering problems within hadrons are studied. For energies currently available (of order 1 TeV) we have  $Q_h r_h \sim 10^3 - 10^4$ , so that the problem of scattering seems quite reasonable to address. The gauge invariance of the  $S$ -matrix for quarks and gluons secures this property of the states in the collision, so that the analysis of Section 3.1 may be applied quite meaningfully to QCD problems. Note that in this context it is gluons with energies  $\omega \sim r_h^{-1}$  which are considered soft.

The analysis, which parallels almost completely that presented in Section 3.1, starts from the standard QCD Lagrangian

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}^a)^2 + \sum_f \bar{\psi}_f(i\hat{D} - m_f)\psi_f \quad (3.41)$$

in which we keep only one flavor  $f$  in the second term. In Eqn (3.41)  $F_{\mu\nu} = i[D_\mu, D_\nu]/g$ ,  $D_\mu = \partial_\mu - ig\hat{A}_\mu$ ,  $\hat{D} = D_\mu\gamma_\mu$ ,  $\hat{A}_\mu = A_\mu^a T_a$ ,  $T_a = (1/2)\lambda_a$  are  $SU(3)$  group generators and  $\lambda_a$  are the Gell-Mann matrices<sup>3</sup>. The arguments leading to Eqn (2.8) remain unchanged except for the current and polarization vector expressions, which become

$$j_\mu(q) \rightarrow j_\mu^c(q) = j_\mu(q)T^c, \quad \varepsilon^\lambda(q) \rightarrow \varepsilon^\lambda(q) = \varepsilon^\lambda(q)\chi^c, \quad (3.42)$$

where  $\chi^c$  denotes the gluon color state,  $c = 1, \dots, 8$ .

Due to the noncommutative nature of the  $T_c$  operators, Eqn (2.14) is incorrect (it is necessary that the currents be ordered). While the analysis in Section 2.1.2 could in principle be extended to QCD, the method of Section 3.1 is simpler to apply. Because all the essential features of Eqns (3.8)–(3.15) are retained in QCD, the required Hamiltonian [the analogue of Eqns (3.15)–(3.18)] is immediately written as

$$H'_{\text{int}} = \int d\mu(p) \frac{p_\mu}{E_p} \hat{\rho}^a(p) \int_Q d\mu(q) \left[ c_\mu^a(q) \exp\left(-\frac{itpq}{E_p}\right) + c_\mu^{a+}(q) \exp\left(\frac{itpq}{E_p}\right) \right] = \int d^3x \hat{\mathcal{T}}_\mu^a(x) \hat{A}_\mu^a(x), \quad (3.43)$$

$$\hat{\rho}^a = -g[a_\sigma^+(p)T^a a_\sigma(p) - b_\sigma^+(p)T^a b_\sigma(p)],$$

$$\hat{\mathcal{T}}_\mu^a(x) = \int d\mu(p) \frac{p_\mu}{E_p} \hat{\rho}^a(p) \delta(x - vt) \equiv \int d\mu(p) \hat{N}^a(p) j_\mu(x, p). \quad (3.44)$$

Note that all the equations of Section 3.1 — i.e., Eqns (3.26), (3.36), and (3.37) — retain their form after an obvious modification (the inclusion of the  $T_a$  matrices and their ordering in the  $T$  exponentials).

Since the gluon sector requires accounting for collinear divergences, we present here the effective Hamiltonian for the

<sup>3</sup> The hat is for the convolution with the  $\gamma_\mu$  and  $\lambda_a$  matrices, either separately or in concert. This does not introduce any confusion provided a non-convoluted index is used.

interaction of hard ( $|p| \gg Q$ ) and soft ( $|q| < Q$ ) gluons only [64]. The three-gluon interaction operator  $H_{\text{int}}^{(3)}$  follows from Eqns (3.43) and (3.44) by making the replacement

$$\hat{J}_\mu^c \rightarrow -ig \int_{|p|>\Lambda} d\mu(p) \frac{p_\mu}{E_p} \delta(x - vt) f^{abc} \hat{c}_\sigma^{a+}(p) \hat{c}_\sigma^b(p),$$

$$\Lambda \gg Q, \quad (3.45)$$

and the biquadratic operator  $H_{\text{int}}^{(4)}$  is

$$H_{\text{int}}^{(4)} = \frac{g^2}{2} \int_{|p|>\Lambda} \frac{d^3x d\mu(p)}{E_p} \delta(x - vt) f^{abc} f^{ade} \\ \times [\hat{c}_{p,\mu}^{b+} \hat{c}_{p,\nu}^c \hat{A}_{x,\mu}^{dQ} \hat{A}_{x,\nu}^{eQ} + (e \leftrightarrow c) + \hat{c}_{p,\mu}^{b+} \hat{c}_{p,\mu}^d \hat{A}_{x,\nu}^{cQ} \hat{A}_{x,\nu}^{eQ}], \quad (3.46)$$

where  $f^{abc}$  are the structural constants of the group. Equations (3.45) and (3.46) enable the equations of motion for hard gluons to be constructed and their asymptotic states to be obtained in symbolic form. The present method is also applicable to gravitation theory.

## 4. Collinear divergences

### 4.1 QCD. Perturbation theory

*Sterman-Weinberg formula.* The theory of collinear divergences has not been developed as well as the theory of infrared divergences, and there are serious reasons for that. First, this problem has not been that topical. Such divergences may occur in theories with massless charged particles such as the gravitational or gluon fields. As regards the former, theoretical efforts have mostly concentrated on the renormalization problem because at the energies currently available graviton interaction studies are beyond the reach of present-day experimentation. The gluon field is not observed free (confinement!) and it is only at energies much above the inverse hadron radius that its behavior in a closed volume is one of a massless field. It is from the study of the unreal case of a zero-mass electron [5] that the problem arose.

Second, collinear divergences are far more difficult to treat than infrared ones. While the latter problem does not take any more than a correct treatment of the field and a classical source coupling, in the former an exact solution of a two-dimensional field model is needed. The problem became particularly topical in the transition to superhigh energies, where leptons and quarks behave as massless particles. A practical way out of this was found in the framework of standard perturbation theory, in which the asymptotic freedom of QCD provides a small coupling constant and where the Kinoshita-Lee-Nauenberg theorem [15, 16] secures that divergences cancel out to any order of approximation.

The analysis of the processes  $e^+e^- \rightarrow \bar{q}q$ ,  $\bar{q}qg$  ( $q, g$  designating the quark and the gluon, respectively) to lowest nontrivial order of perturbation theory has confirmed the general conclusions of Refs [14-16]. In Ref. [35] the partial cross section of the process  $e^+e^- \rightarrow$  two jets has been calculated. The particular cross sections computed include:  $\sigma_a(e^+e^- \rightarrow \bar{q}(qg), (\bar{q}g)q)$ , where a hard gluon together with a quark  $q$  (or  $\bar{q}$ ) form one of the jets (the quark and the gluon being collinear);  $\sigma_b(e^+e^- \rightarrow \bar{q}qg)$ , where an infrared gluon may or may not belong to one of the jets;  $\sigma_c(e^+e^- \rightarrow \bar{q}q)$ , where, unlike the first two cases, the one-loop diagram is included in addition to the tree diagram. The term jet is here

understood to mean a quark (antiquark) or alternatively a quark (antiquark) plus a gluon emitted into a cone of angle  $\delta \ll 1$  (the cone cuts a circumference of radius  $\delta$  on the unit sphere). It is assumed that the two jets take essentially all of the energy  $E$  of the system  $e^+e^-$ , i.e.,  $(1-\epsilon)E$ , where  $\epsilon \ll 1$ ,  $E \rightarrow \infty$ .

It turns out that all three cross sections diverge logarithmically upon removal of the infrared regularization (i.e., for  $\lambda \rightarrow 0$ , where  $\lambda$  is the gluon mass). The sum  $\sigma$ , however, is finite (see a note in Section 6.5):

$$\sigma = \sigma_a + \sigma_b + \sigma_c \\ = \left( \frac{d\sigma}{d\Omega} \right)_0 \Omega \left[ 1 - \frac{g_E^2}{3\pi^2} \left( 3 \ln \delta + 4 \ln \delta \ln 2\epsilon + \frac{\pi^2}{3} - \frac{5}{2} \right) \right] \quad (4.1)$$

(Sterman-Weinberg formula [35]), where

$$\left( \frac{d\sigma}{d\Omega} \right)_0 = \frac{\alpha^2}{4E^2} (1 + \cos^2 \vartheta) \sum 3 \left( \frac{e_q}{e} \right)^2 \quad (4.2)$$

is the lowest-order differential cross section of the  $e^+e^- \rightarrow \bar{q}q$  process ( $\alpha = e^2/4\pi$ ,  $e_q$  are the electric charges of the quarks,  $g_E$  is the running coupling constant,  $\vartheta$  is the angle between the jet and electron-beam directions; and the summation runs over the quark aromas). Note the parameter  $\Omega$  in Eqn (4.1). The role of the parameters  $\delta$  and  $\Omega$  here is much the same as that of  $\Lambda$  and  $E \sim E_0$  in Eqn (2.26); the small value of the first parameter in the pair secures the validity of the expressions used, whereas the second parameter ( $\pi\delta^2 \ll \Omega \ll 1$ ) characterizes the angular resolution of the experiment (the accuracy of the jet direction determination). Clearly, the method is applicable to any process to any order of perturbation theory.

The cancellation of the divergences in Eqn (4.1) suggests the existence of a common approach to such processes. It turns out that time hierarchy (see the end of Section 3.1.2) and  $S$ -matrix factorization also take place in the collinear case, although the corresponding transitions are not necessarily associated with small energy changes in this case. Formally, the reason is that propagator poles always determine the time-asymptotic behavior, and physically, that forward scattering (at small angles  $\theta \ll 1$ ) implies the formation of a state with a small transverse momentum  $k_\perp$  ( $k_\perp/k \approx 0$ ), i.e., with a transverse dimension of order  $k_\perp^{-1}$  — a process which takes a time of order  $k_\perp^{-1} = t_{\text{col}}$  to occur. If the characteristic hard-process time is  $t_h \sim 1/k$ , then  $t_h/t_{\text{col}} \sim \theta \ll 1$ . Hence we may expect a factorization of the type (3.34), with new bordering matrices  $U$  now describing small-transverse-momentum processes.

Since the general picture of infrared and collinear divergences is mainly determined by classifying processes into fast and slow ones — something which does not depend on the internal quantum numbers of the particles — it remains unchanged when we ascribe quantum numbers to photons (i.e., go over to non-Abelian theories). The main difference here is that it is impossible to disentangle the  $T$  exponents corresponding to divergence-generating processes. While in the case of infrared QCD divergences operators  $U$  were rather simple to construct, the construction of  $U_{\text{col}}$  actually requires a knowledge of the exact solution of the corresponding two-dimensional field theory (a problem which has not yet been addressed in a general form). The factorization of the  $S$ -matrix leads to the Kinoshita-Lee-Nauenberg theorem, which ensures that probabilities of

interest are finite following the ensemble summation over the initial and final states of massless particles. In the next section a method of constructing a scattering operator free of collinear divergences is described using massless electrodynamics as an example.

#### 4.2 Massless electrodynamics. Scattering operator redefinition

**Effective Hamiltonian.** Thus, the general strategy for dealing with collinear divergences is identical to that developed for the infrared case (Section 3): first to find the effective interaction Hamiltonian accounting for collinear divergences and then to subtract it from the interaction Hamiltonian [as in Eqn (3.21)]. The scattering operator with the new interaction is free of collinear divergences.

Before calculating the effective interaction, we first transform the Hamiltonian (3.8). Substituting the expansions (1.5) and (1.6) for massless fields and integrating over  $\mathbf{x}$ , we see that there are only two types of time dependent exponentials: in two cases, particle energies in the argument of the exponential have the same sign [these are the coefficients of the operators  $\hat{a}^+\hat{b}^+\hat{c}^+$  and  $\hat{a}\hat{b}\hat{c}$  as exemplified by the first term in Eqn (3.9)]; in the remaining six cases, two terms in the exponentials always have the same sign [e.g., the second term in Eqn (3.9) and both terms in Eqn (3.10)]. Terms of the former type contain oscillating exponentials (the trivial case  $\mathbf{p}' = \mathbf{p} = \mathbf{q} = 0$  describes vacuum transitions) and cannot describe the asymptotic dynamics of massless particles.

The remaining six terms will be transformed as follows. Momenta  $\mathbf{p}'$ ,  $\mathbf{q}$  are usually associated with fields  $\psi$ ,  $A_\mu$ . Let us redefine the integration variables  $\mathbf{p}'$ ,  $\mathbf{p}$ ,  $\mathbf{q}$  such that the momentum  $\mathbf{p}'$  is associated with the field which differs in the sign of the energy in the exponential from the other two. In Eqn (3.9) we interchange  $\mathbf{q}$  and  $\mathbf{p}'$  in the second term, and in Eqn (3.10) we interchange  $\mathbf{p}$  and  $\mathbf{p}'$  in the first term while leaving the notation in the second term unchanged. Then after the integration over  $\mathbf{p}'$  all the exponential arguments have the same form  $\pm i f(\mathbf{p}, \mathbf{q}) t$ , where

$$f(\mathbf{p}, \mathbf{q}) = E_{\mathbf{p}+\mathbf{q}} - E_{\mathbf{p}} - E_{\mathbf{q}} = |\mathbf{p} + \mathbf{q}| - |\mathbf{p}| - |\mathbf{q}|. \quad (4.3)$$

Since the integration is over  $\mathbf{p}$  and  $\mathbf{q}$  only, the critical points of the function  $f(\mathbf{p}, \mathbf{q})$  are found from the equations  $\nabla_{\mathbf{p}} f = 0$ ,  $\nabla_{\mathbf{q}} f = 0$ . By the symmetry of  $f$  with respect to the interchange  $\mathbf{p} \leftrightarrow \mathbf{q}$ , it suffices to employ only one of them,

$$\nabla_{\mathbf{q}} f(\mathbf{p}, \mathbf{q}) = \frac{\mathbf{p} + \mathbf{q}}{|\mathbf{p} + \mathbf{q}|} - \frac{\mathbf{q}}{|\mathbf{q}|} = 0. \quad (4.4)$$

This is satisfied by  $\mathbf{q} = a\mathbf{p}$ , where  $a$  obeys  $(1+a)/|1+a| = a/|a|$ , i.e.,  $a > 0$  or  $a < -1$ .

Substituting the obtained solution into  $f$ , it is found that  $f(\mathbf{p}, \mathbf{q}) = E_p \varphi(a)$ , where  $\varphi(a) = |1+a| - 1 - |a| = \{0 \text{ for } a > 0; 2a \text{ for } -1 \leq a \leq 0; -2 \text{ for } a < -1\}$ . It is clear that an oscillating exponential will not appear for  $a > 0$ , i.e., the set of critical points is defined by the condition  $\mathbf{q} = a\mathbf{p}$ ,  $a \geq 0$ . In the neighborhood of a critical point we have

$$\begin{aligned} f(\mathbf{p}, \mathbf{q}) &= \sqrt{(E_p + E_q)^2 - 2pq} - E_p - E_q \\ &\approx \frac{-pq}{E_p + E_q} = \frac{-(p+q)q}{E_p + E_q}, \quad pq \rightarrow 0. \end{aligned} \quad (4.5)$$

Next, the functions  $f(\mathbf{p}, \mathbf{q})$  in the exponentials should be replaced by an approximate expression (4.5) and the pre-

factors be set to their critical values. Actually the prefactors vanish at these points (see Section 6.4), so that one should add small transverse terms to the momenta [e.g.,  $\mathbf{q} = a\mathbf{p} + \mathbf{q}_\perp$ ,  $\mathbf{q}_\perp = (0; \mathbf{q}_\perp, 0)$ ; see Eqn (6.22)], in which only terms linear in  $|\mathbf{q}_\perp|$  must be kept when taking the limit  $\mathbf{q}_\perp \rightarrow 0$ .

The resulting asymptotic Hamiltonian

$$\begin{aligned} H'_{\text{int}} &= \frac{e}{2} \int \frac{d\mu(\mathbf{p}, \mathbf{q})}{E_p + E_q} \left\{ [\hat{a}_\sigma^+(\mathbf{p} + \mathbf{q}) \hat{a}_\sigma(\mathbf{p}) \bar{u}^{\sigma'}(\mathbf{p} + \mathbf{q}) \gamma_\mu u^\sigma(\mathbf{p}) \right. \\ &\quad + \hat{b}_{\sigma'}(\mathbf{p}) \hat{b}_\sigma^+(\mathbf{p} + \mathbf{q}) \bar{v}^{\sigma'}(\mathbf{p}) \gamma_\mu v^\sigma(\mathbf{p} + \mathbf{q})] \hat{c}_\lambda(\mathbf{q}) \varepsilon_\mu^\lambda(\mathbf{q}) \\ &\quad \left. + \hat{b}_{\sigma'}(\mathbf{q}) \hat{a}_\sigma(\mathbf{p}) \bar{v}^{\sigma'}(\mathbf{q}) \gamma_\mu u^\sigma(\mathbf{p}) \hat{c}_\lambda^*(\mathbf{p} + \mathbf{q}) \varepsilon_\mu^{\lambda*}(\mathbf{p} + \mathbf{q}) \right\}_{q_\perp \rightarrow 0} \\ &\quad \times \exp\left(\frac{-itpq}{E_p + E_q}\right) + \text{H.c.} \end{aligned} \quad (4.6)$$

is further transformed as follows.

1. We define the vicinity  $K^+$  of the critical points  $\mathbf{q} \rightarrow a\mathbf{p} + \mathbf{q}_\perp$ ,  $\mathbf{p} \rightarrow \mathbf{p} - \mathbf{q}_\perp$  to be the set

$$\{\mathbf{q} \in K^+ : |\mathbf{q}_\perp| \leq K, \quad a \geq 0\}, \quad (4.7)$$

where  $K$  is a certain positive constant ( $K \ll E$  if  $E$  is the characteristic energy of the process). In Eqn (4.6), an integration over the region  $K^+$ ,

$$\int_{K^+} d\mu(\mathbf{q}) \equiv \int_0^\infty \frac{da}{4\pi a} \int_{|\mathbf{q}_\perp| \leq K} \frac{d^2 \mathbf{q}_\perp}{4\pi^2}, \quad (4.8)$$

is performed and in some of the terms the substitution  $\mathbf{q}_\perp \rightarrow -\mathbf{q}_\perp$  is made to uniform the expressions.

2. In view of the vanishing of the prefactors at  $\mathbf{q}_\perp \rightarrow 0$ , the approximation

$$\begin{aligned} &\bar{u}^{\sigma'}(\mathbf{p} + \mathbf{q}) \gamma_\mu u^\sigma(\mathbf{p} - \mathbf{q}_\perp) \varepsilon_\mu^\lambda(\mathbf{q} + \mathbf{q}_\perp) \\ &\approx \bar{u}^{\sigma'}(\mathbf{p} + \mathbf{q}) \gamma_\mu u^\sigma(\mathbf{p}) \varepsilon_\mu^\lambda(\mathbf{q} + \mathbf{q}_\perp) \\ &+ \bar{u}^{\sigma'}(\mathbf{p} + \mathbf{q}) \gamma_\mu u^\sigma(\mathbf{p} - \mathbf{q}_\perp) \varepsilon_\mu^\lambda(\mathbf{q}) \end{aligned} \quad (4.9)$$

is used in the first term in Eqn (4.6) and similar relations in the remaining terms. The terms  $O(q_\perp^2)$  are neglected in the prefactors.

3. By the symmetry of the integrand in Eqn (4.6) with respect to the interchange  $\mathbf{p} \leftrightarrow \mathbf{q}$ , all functions containing  $\mathbf{q}_\perp$  are considered to be dependent on  $\mathbf{q}$ . For example, the second term in Eqn (4.9) is rewritten as  $\bar{u}(\mathbf{p} + \mathbf{q}) \gamma_\mu u(\mathbf{q} - \mathbf{q}_\perp) \varepsilon_\mu(\mathbf{p})$ , after which we set  $\mathbf{q} = a\mathbf{p}$ ,  $a \geq 0$  everywhere.

4. As in Eqn (3.15), we rewrite the exponential in the form

$$\begin{aligned} \exp\left(\frac{-it(p+q)q}{E_p + E_q}\right) &= \int d^3x \delta(\mathbf{x} - \mathbf{v}_{\mathbf{p}+\mathbf{q}} t) \exp(-iqx), \\ v_{\mathbf{p}+\mathbf{q}}^\mu &= \frac{p^\mu + q^\mu}{E_p + E_q} \end{aligned} \quad (4.10)$$

( $q^2 = 0$ ). For  $\mathbf{q} = a\mathbf{p}$ , it is obvious that  $v_{\mathbf{p}+\mathbf{q}}^\mu = v_p^\mu$ .

5. Finally, infrared divergences should be taken care of because the collinear regular region  $\{\bar{K}^+ : \mathbf{q} \in K^+\}$  complementary to  $K^+$  contains points that produce such divergences. This region is one of small  $|\mathbf{q}_\perp|$  and small  $|a|$ , i.e., it is defined by  $q^2 \leq Q^2$ , including  $qp/E_p < 0$ . The corresponding term should also be present in the asymptotic form of the Hamiltonian, which is achieved by modifying the region of integration in Eqn (4.8) by the replacement  $K^+ \rightarrow K_Q^+$  (Fig. 2),

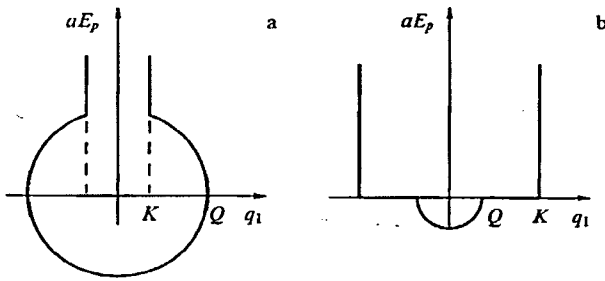


Figure 2. Domains of integration  $K_Q^+$ :  $Q > K$  (a),  $Q < K$  (b);  $E_p = |p|$ ,  $q_\perp = (q_1, q_2)$ . The sections  $q_2 = 0$  of the corresponding regions are shown.

where

$$\{q \in K_Q^+ : |q_\perp| \leq K, a \geq 0; |q| \leq Q, a < 0\}, \quad Q < K, \quad (4.11)$$

$$\{q \in K_Q^+ : |q_\perp| \leq K, aE_p \geq Q; |q| \leq Q, aE_p < Q\}, \quad Q > K. \quad (4.12)$$

As a result, the following representation of the effective Hamiltonian is obtained (Fig. 3; we omit hats on current and field operators  $A_\mu, \bar{\psi}, \psi; b = 1 + a$ ):

$$H'_{\text{int}}(t) = \int d^3x [\mathcal{J}_\mu^{(+)} A_\mu^{(+)\mathcal{K}} + \bar{\psi}^{(+)\mathcal{K}} \mathcal{J}^{(+)} + \bar{\mathcal{J}}^{(+)} \psi^{(+)\mathcal{K}} + \text{H.c.}], \quad (4.13)$$

where, for example,

$$\mathcal{J}_\mu^{(+)} A_\mu^{(+)\mathcal{K}} = \int_{|q_\perp| \leq K} \frac{da}{4\pi a} \int d\mu(p) \mathcal{J}_\mu^{(+)}(x, p, a) A_\mu^{(+)\mathcal{K}}(x, p, a), \quad \epsilon_1 E_p \sim Q, \quad (4.14)$$

$$A_\mu^{(+)\mathcal{K}}(x, p, a) = \int_{|q_\perp| \leq K} \frac{d^2 q_\perp}{4\pi^2} \hat{e}_\lambda(ap + q_\perp) \times \epsilon_\mu^\lambda(ap + q_\perp) \exp(-iqx), \quad (4.15)$$

$$\mathcal{J}_\mu^{(+)}(x, p, a) = eV_\mu^{\sigma'\sigma}(p, a) \delta(x - v_p t) \times [\hat{a}_\sigma^+(bp) \hat{a}_\sigma(p) - \hat{b}_\sigma^+(bp) \hat{b}_\sigma(p)], \quad (4.16)$$

$$V_\mu^{\sigma'\sigma}(p, a) = \frac{\bar{u}^{\sigma'}(p+q) \gamma_\mu u^\sigma(p)}{2E_p + 2E_q} = \frac{\bar{v}^{\sigma'}(p) \gamma_\mu v^\sigma(p+q)}{2E_p + 2E_q}, \quad q = ap.$$

Using (6.27), the current  $V_\mu$  takes a simpler form  $V_\mu^{\sigma'\sigma} \sim p_\mu \delta^{\sigma'\sigma}$ . The remaining two terms in (4.13) are also given by

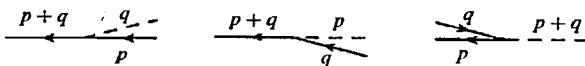


Figure 3. Diagrams associated with certain characteristic terms in the Hamiltonian (4.13). Added to these must be diagrams with reversed arrows and another six similar diagrams with photons in final states.

integrals of the type (4.14), with fields and currents

$$\psi^{(+)\mathcal{K}}(x, p, a) = \int_{|q_\perp| \leq K} \frac{d^2 q_\perp}{4\pi^2} \hat{a}_\sigma(ap + q_\perp) u^\sigma(ap + q_\perp) \times \exp(-iqx), \quad (4.17)$$

$$\bar{\mathcal{J}}^{(+)}(x, p, a) = \frac{e\delta(x - v_p t)}{2(E_p + E_g)} [\bar{u}^{\sigma'}(bp) \hat{e}_\lambda(p) \hat{a}_\sigma^+(bp) \hat{c}_\lambda(p) + \bar{v}^{\sigma'}(p) \hat{e}_\lambda^*(bp) \hat{b}_\sigma^-(p) \hat{c}_\lambda^*(bp)] \quad (4.18)$$

for  $\bar{\mathcal{J}}^{(+)}\psi^{(+)\mathcal{K}}$  and similarly for  $\bar{\psi}^{(+)\mathcal{K}}\mathcal{J}^{(+)}$ , with obvious alterations. Specifically,  $\bar{\psi}^{(+)\mathcal{K}}$  is obtained from Eqn (4.17) by making the replacements  $\hat{a} \rightarrow \hat{b}$ ,  $u \rightarrow \bar{v}$ , and  $\mathcal{J}^{(+)}$  follows from Eqn (4.18) upon the replacements  $\hat{a}^+ \rightarrow \hat{b}^+$ ,  $\hat{b} \rightarrow \hat{a}$ ,  $\bar{u} \rightarrow \bar{v}$ ,  $\bar{v} \rightarrow u$  with appropriate permutations. The current (4.16) is conserved, and the Hamiltonian (4.13) is invariant with respect to the gauge transformations  $\epsilon_\mu(q) \rightarrow \epsilon_\mu(q) + q_\mu \Lambda(q)$ . Incidentally, Eqn (4.13) implies that, as in the infrared case, collinear divergences are generated only by free particles. Note that representation (4.14) allows one to interpret Eqn (4.13) as a set of one-dimensional theories dependent on the parameter  $p$ .

**Redefinition of the scattering operator.** An  $S$ -matrix free of infrared and collinear divergences is constructed in a standard way (see Section 3). In accord with Eqns (3.1) and (3.21), we define the new interaction operator  $\tilde{H}_{\text{int}} = H_{\text{int}} - H'_{\text{int}}$ , where  $H'_{\text{int}}$  is specified by Eqn (4.13). The desired  $S$ -matrix is given by Eqn (3.22), in which the interaction representation  $\tilde{I}$  is determined by the 'free' Hamiltonian  $\tilde{H}_0 = H_0 + H'_{\text{int}}$  [similar to Eqn (3.4)]. That the  $S$ -matrix is free from infrared and collinear divergences follows from the way it is constructed and can also be verified directly. For this it suffices to show that the operators  $\int_{-\infty}^0 H'_{\text{int}} dt$  and  $\int_0^\infty H'_{\text{int}} dt$  reproduce the amplitudes for the emission of infrared and collinear quanta by in and out particles, respectively.

To this end we set  $m = 0$  in Eqns (2.7), giving

$$\frac{\hat{p} - \hat{q}}{-2pq + i0} e\hat{\epsilon}^*(q), \quad e\hat{\epsilon}^*(q) \frac{\hat{p}' + \hat{q}}{2p'q + i0}, \quad p^2 = q^2 = p'^2 = 0. \quad (4.19)$$

Matrices (4.19) multiply the spinors  $u(p)$  and  $\bar{u}(p')$  on the left and on the right, respectively. Noting that  $\hat{p}u(p) = \bar{u}(p')\hat{p}' = q\epsilon(q) = 0$ , we find that for  $q = ap$ ,  $q = ap'$ ,  $a > 0$  the numerators in Eqn (4.19) vanish. To separate out the leading terms of the expansions we employ the representation (6.22) and similar expressions for  $p'$ . For  $q_\perp = 0$ , the momenta  $p$  and  $q$  are collinear. Since  $pq \sim q_\perp^2$  [see Eqn (6.23)], it is necessary to keep the terms  $O(|q_\perp|)$  in the numerators of the amplitudes (4.19). Using the representation  $q = q_\perp + (q\bar{p})p/p\bar{p} + O(q_\perp^2)$ , the desired amplitudes are found to be

$$e \frac{-2(p\bar{p}/q\bar{p}) q_\perp \epsilon^*(q) + \hat{\epsilon}^*(q) \hat{q}_\perp}{-2pq + i0} u(p), \quad \bar{u}(p') \frac{-2(p'\bar{p}'/q'\bar{p}') q_\perp \epsilon^*(q) - \hat{q}_\perp \hat{\epsilon}^*(q)}{2p'q + i0} e : \quad (4.20)$$

(where the replacement  $\epsilon(q) \rightarrow \epsilon(p)$  can be made).

At this point, expressions analogous to (4.19) can be written down for the pair creation amplitudes in the initial (virtual positron) and final states. For typical processes (with



the replacement  $p' \rightarrow q$  we have instead of Eqn (4.19)

$$e\bar{u}(q)\hat{\varepsilon}(p) \frac{1}{\hat{p}-\hat{q}+i0}, \quad e\bar{u}(p)\gamma_\mu v(q) \frac{-g_{\mu\nu}}{(p+q)^2+i0}. \quad (4.21)$$

Using the momentum representation (6.22), (6.23), the amplitudes (4.21) are calculated to be

$$e\bar{u}(q) \frac{(p\hat{p}/q\hat{p})\hat{q}_\perp\hat{\varepsilon}(p) - 2q_\perp\varepsilon(p)}{-2pq+i0}, \quad -e \frac{\bar{u}(p)\gamma_\nu v(q+q_\perp)}{2pq+i0} \quad (4.22)$$

in the limit  $q_\perp \rightarrow 0$ .

The amplitudes (4.20), (4.22) must be reproduced by the asymptotic interaction Hamiltonian. We omit the detailed proof of this and note only that the integration over  $t$  from  $(-\infty, 0)$  to  $(0, \infty)$  yields correct denominators in Eqns (4.20) and (4.22) and that the matrix elements of  $H_{\text{int}}$  and  $H'_{\text{int}}$  between the states in the critical region are identical to terms  $O(q_\perp^2)$  by construction. Note that, for example, the term (4.14) reproduces the first term in the numerator of the first fraction in Eqn (4.20), whereas the term  $(\bar{\psi}^{(+)\kappa}\mathcal{J}^{(+)})^+$  reproduces the second term of the numerator [see Eqn (6.26)]. The Hamiltonian  $\tilde{H}_0 = H_0 + H'_{\text{int}}$  reproduces correctly the emission probabilities for quanta in the critical region because the auxiliary vector  $h$  in Eqns (6.26) and (6.27) determines only the phases of transition currents. The spreading angle of the cone (jet) is clearly determined by the parameter  $K$  ( $\delta \sim K/E_p$ ), and the energy  $E$  carried away by the undetected emission may be related to the parameter  $\epsilon \sim E/E_p$  [similar to Eqn (4.14)]. The role of the parameter  $Q$  in Eqns (4.11) and (4.12) is similar to that of  $A$  (see the end of Section 2.1.3).

## 5. Conclusions

In conclusion, some problems related to the subject of this paper will be discussed in brief.

1. *Eikonal*. The approximation used in Section 2 is identical to the so-called *eikonal approximation* of quantum field theory [65, 66], which essentially represents the following modification of high-energy particle propagators:  $(2pk + k^2)^{-1} \rightarrow (2pk)^{-1}$  ( $|\mathbf{p}| \rightarrow \infty$ ,  $\mathbf{k}$  being the momentum or the sum of the momenta of virtual quanta). The fact that it is equivalent to the approximation (2.11), (2.13) is obvious. The differences between the two theories (in their traditional interpretation) are as follows:

(1) Whereas the eikonal approach always treats soft quanta as virtual, in infrared theory their emission is also considered.

(2) Infrared theory always involves a process which fixes a 'point' (zero) on the time axis (and which takes much less time than the emission of soft quanta); this is why it is sensible to speak of the long-wave emission of in and out particles. This feature is absent from eikonal theories, in which all emission (exchange) processes stand on an equal footing and in which incident and scattered particles neither emit nor absorb virtual quanta.

We now consider Coulomb scattering to illustrate the argument above. Referring to the hard process with the emission of  $N$  soft photons (see Section 2), suppose we wish to find the amplitude of small-angle particle scattering. To this end one must modify the amplitude (2.15) by replacing  $\varepsilon_\mu^*(q_i)$  by the amplitude  $\tilde{A}_\mu^{\text{ext}}(q_i)$  of exchange by a photon with

an external current. Since all the photons are then virtual and the main process is as good as any other ( $p' - p \sim q_i$ ), what we must do next is to integrate over the photon momenta  $q_i$  ( $i=1, \dots, N$ ), to take into account the fact that photons are indistinguishable [factor  $1/(N+1)!$ ], and to sum over all  $N$ . Then from Eqns (2.6), (2.15)

$$\begin{aligned} T_{p'p}^{\sigma'\sigma} &= \sum_{N=0}^{\infty} \frac{1}{(N+1)!} [-e\bar{u}^{\sigma'}(p')\gamma_\mu \tilde{A}_\mu^{\text{ext}}(p-p')u^\sigma(p)] \\ &\times \int \prod_{i=1}^N \frac{d^4 q_i}{(2\pi)^4} j_\mu(q_i, p) \tilde{A}_\mu^{\text{ext}}(q_i) \\ &= -e \int d^3 x \exp[i(\mathbf{p}-\mathbf{p}')\mathbf{x}] \bar{u}^{\sigma'}(p') \\ &\times \gamma_\mu A_\mu^{\text{ext}}(\mathbf{x}) u^\sigma(p) \frac{\exp(-i j^{\text{cl}} \tilde{A}^{\text{ext}}) - 1}{-i j^{\text{cl}} \tilde{A}^{\text{ext}}}, \end{aligned} \quad (5.1)$$

where  $\tilde{A}_\mu^{\text{ext}}(q) = 2\pi\delta(q_0)\tilde{A}_\mu^{\text{ext}}(\mathbf{q})$  and  $j_\mu^{\text{cl}}(q, p) = i j_\mu(q, p)$  [see Eqn (2.10)].

In the standard eikonal formulation [65] one needs to evaluate the sum of the form

$$-e \sum_{N=0}^{\infty} \frac{1}{N+1} \sum_{k=0}^N \frac{1}{k!(N-k)!} (j_{\text{out}} A)^k A (j_{\text{in}} A)^{N-k} \quad (5.2)$$

which follows from Eqn (5.1) by using the binomial formula  $(jA)^N = (j_{\text{in}}A + j_{\text{out}}A)^N$  [note that the incoming (outgoing) currents were labeled by the  $+$  ( $-$ ) signs in Eqn (2.10)]. Following the ideology of Sections 2 and 3, the integration in Eqn (5.1) should be performed over the region  $|q| < Q$  [as, e.g., in Eqn (3.15)]. In eikonal theories, however, it is customary to integrate over all the momenta of virtual particles without estimating the error so introduced.

2. *Bloch–Nordsieck model and the continuum integration method*. As discussed in Sections 2 and 3, the dynamics of long-wave photons are described by the exactly solvable Bloch–Nordsieck model [19]. An especially elegant formulation of this problem (and many others, in fact) has been obtained using the method of continuum integration [66–68] (see Section 6.5). The Bloch–Nordsieck model with propagator  $1/pq$  and a modified vertex ( $\gamma_\mu \rightarrow v_\mu$ ) has recently been applied to the problem of infrared divergences at finite temperatures [69–71].

3. *Unitarity*. This property imposes nontrivial restrictions on the interaction of massless fields: in the local limit, effective interactions of the renormalizable type must have complex coupling constants [72]. In a recent paper [73], the amplitudes of the 'decays' of massless particles were studied.

4. Some time ago, the gravitational scattering of a trial particle with Planck energies  $E$  ( $GE^2 \approx 1$ ) came under study [74–78]. In Ref. [74], the amplitude of particle scattering by the background metric of the scatterer was calculated. The authors of Ref. [75] obtained an equivalent result by using the eikonal approximation for a Reggised graviton, whereas in the string approach of Refs [76, 77] an expansion in the number of loops at high energies was resummed. In the decomposition of the metric tensor into 2D longitudinal components and 2D transverse (relative to the incident momentum) components in Ref. [78], the 2D theory of longitudinal high-energy processes is treated exactly and the exchange by transverse quanta perturbationally. Note that for currently available energies the effects of the long-wave gravitational emission are negligibly small [45] and that at

Planck energies significant changes in our views on space and time are to be expected.

5. A highly topical problem is one of high-energy scattering in the QCD framework. The current state of this field is summarized by Lipatov [79], who uses a theory with an exactly solvable Hamiltonian to treat processes with a small momentum transfer.

6. *Summary.* The phenomenon of infrared and collinear divergences consists essentially in the unbounded growth of the emission amplitudes of massless particles when either their energy or the emission angle tend to zero. This also results in the divergent cross sections and other physical quantities when these latter are evaluated by perturbation theory. Success in describing such processes depends on the concept of interaction time hierarchy: hard processes (large momentum-transfer) are rapid and occur in small volumes, whereas soft ones (small momentum- or transverse-momentum-transfer) are slow. A consequence of the time hierarchy is the factorization of the  $S$ -matrix [see Eqns (3.34) and (3.40)]; the operators that describe soft processes and generate divergences appear as sandwiching operators for the hard scattering operator. This immediately yields the theorem that the divergences in observable quantities cancel out, the reason being the isometry property of the sandwiching operators which manifests itself when the magnitudes of the amplitudes are squared and quantum states in the critical region (for example, the states of undetected soft photons) summed over. The matrix elements of the products of such operators by their conjugates do not any longer produce divergences, which is precisely the reason why theoretical formulas contain only experiment-specific parameters — the duration of the experiment and the sensitivity of the detectors used.

## 6. Appendices

### 6.1 Polarization summation formulas

If particle spins are not detected, they are summed over in the final state and averaged over in the initial one. The relevant formulas are as follows.

*Photons, Gluons* ( $J = 1$ ). If we choose a basis of the form  $\varepsilon_\mu^\pm(q) = (0; 1, \pm i, 0)/\sqrt{2}$ ,  $\lambda_\mu = (1; 0, 0, 1)$ ,  $\bar{\lambda}_\mu = (1; 0, 0, -1)$ , then  $q_\mu = \omega\lambda_\mu$  is a standard vector. In Euclidean space, a basis with the isotropic unit vectors  $\lambda^2 = \bar{\lambda}^2 = 0$  is not orthogonal,  $\lambda\bar{\lambda} = 2$ . For the orthogonal basis  $\varepsilon_\mu^\pm$ ,  $\varepsilon_\mu^3 = (\lambda_\mu - \bar{\lambda}_\mu)/2$ ,  $\varepsilon_\mu^0 = (\lambda_\mu + \bar{\lambda}_\mu)/2$ , we have the completeness condition

$$\sum_{\pm} \varepsilon_\mu^\pm \varepsilon_\nu^{\pm*} + \varepsilon_\mu^3 \varepsilon_\nu^{3*} - \varepsilon_\mu^0 \varepsilon_\nu^{0*} = -g_{\mu\nu},$$

which yields the spin summation formula

$$\sum_{\pm} \varepsilon_\mu^\pm(q) \varepsilon_\nu^{\pm*}(q) = -g_{\mu\nu} + \frac{1}{2}(\lambda_\mu \bar{\lambda}_\nu + \bar{\lambda}_\mu \lambda_\nu) \equiv \Pi_{\mu\nu}. \quad (6.1)$$

*Gravitons* ( $J = 2$ ). The polarization tensor for gravitons must possess the properties  $\varepsilon_{\mu\nu} = \varepsilon_{\nu\mu}$ ,  $\varepsilon_{\mu\mu} = 0$ ,  $q_\mu \varepsilon_{\mu\nu}(q) = 0$ . In addition, one must secure the normalization condition  $\varepsilon_{\mu\nu}^\pm \varepsilon_{\mu\nu}^{\pm*} = 1$ , and the  $S$ -matrix should be invariant with respect to the transformation  $\varepsilon_{\mu\nu}^\pm(q) \rightarrow \varepsilon_{\mu\nu}^\pm(q) + q_\mu f_\nu(q) + q_\nu f_\mu(q)$ ,  $q_\mu f_\mu = 0$ . The above requirements are satisfied by the tensor [45]

$$\varepsilon_{\mu\nu}^\pm(q) = \varepsilon_\mu^\pm(q) \varepsilon_\nu^{\pm*}(q). \quad (6.2)$$

By Eqn (6.2), the required sum over polarizations is expressed as follows in terms of the tensor  $\Pi_{\mu\nu}$  defined by Eqn (6.1):

$$\begin{aligned} \Pi_{\mu\nu\rho\sigma}(q) &= \sum_{\pm} \varepsilon_{\mu\nu}^\pm(q) \varepsilon_{\rho\sigma}^{\pm*}(q) \\ &= a \Pi_{\mu\nu} \Pi_{\rho\sigma} + b(\Pi_{\mu\rho} \Pi_{\nu\sigma} + \Pi_{\mu\sigma} \Pi_{\nu\rho}). \end{aligned} \quad (6.3)$$

From the fact that  $\varepsilon_{\mu\mu} = 0$  and using the equalities  $\Pi_{\mu\mu} = -2$ ,  $\Pi_{\mu\rho} \Pi_{\mu\sigma} = -\Pi_{\rho\sigma}$  we have  $\Pi_{\mu\mu\rho\sigma} = -2(a+b)\Pi_{\rho\sigma} = 0$ , i.e.,  $a = -b$ . Further, since the graviton has only two polarization states, the normalization factor  $b$  is found to be given by  $\sum_{\mu\nu} \Pi_{\mu\nu\mu\nu} = 4b = 2$ . Thus, the tensor (6.3) in the numerator of the graviton propagator is [45]

$$\Pi_{\mu\nu\rho\sigma} = \frac{1}{2}(\Pi_{\mu\rho} \Pi_{\nu\sigma} + \Pi_{\mu\sigma} \Pi_{\nu\rho} - \Pi_{\mu\nu} \Pi_{\rho\sigma}). \quad (6.4)$$

Note that  $\varepsilon_\mu^{\pm*} = \varepsilon_\mu^\mp$  when checking Eqn (6.4) directly.

### 6.2 Classical emission

The amplitude of the emission of a long-wave photon, Eqn (2.8), is related directly to the emission of a classical charged particle. If the retarded solution of the field equations,

$$\begin{aligned} A_\mu(\mathbf{x}, t) &= - \int d^4y D_{\text{ret}}(x-y) j_\mu^{\text{cl}}(y) \\ &= \frac{1}{4\pi} \int d^3y \frac{j_\mu^{\text{cl}}(\mathbf{y}, t-R)}{|\mathbf{x}-\mathbf{y}|}, \quad R = |\mathbf{R}| = |\mathbf{x}-\mathbf{y}|, \end{aligned} \quad (6.5)$$

where  $D_{\text{ret}}(x) = -\theta(t)[\delta(t-r) - \delta(t+r)]/4\pi r$ , is rewritten in Fourier component form, we have

$$A_\mu(\mathbf{x}, \omega) = \frac{1}{4\pi} \int \frac{d^3y}{|\mathbf{x}-\mathbf{y}|} \int \frac{d^3q}{(2\pi)^3} \exp(i\omega R + i\mathbf{q}\mathbf{y}) j_\mu^{\text{cl}}(\mathbf{q}, \omega). \quad (6.6)$$

Assume now that the charges are distributed in a bounded region in space. Since  $\max |\mathbf{y}|/r \rightarrow 0$  for  $|\mathbf{x}| = r \rightarrow \infty$  and since  $R \approx r - \mathbf{y}\mathbf{n}$  ( $\mathbf{n} = \mathbf{x}/r$ ), the asymptotic behavior is then

$$A_\mu(\mathbf{x}, \omega) \sim \frac{1}{4\pi r} \exp(i\omega r) j_\mu^{\text{cl}}(\omega \mathbf{n}, \omega), \quad \mathbf{q} = \omega \mathbf{n}. \quad (6.7)$$

Since the Poynting vector  $\mathbf{S} = \mathbf{n}(\mathbf{E}^2 + \mathbf{H}^2)/2$ , the energy  $d\mathcal{E}$  emitted in the frequency range  $d\omega$  during a collision is given by  $d\mathcal{E} = 2 \int dS \mathbf{H}^2 d\omega/2\pi$ , where  $\mathbf{H} = \text{rot } \mathbf{A}_\omega$ ,  $dS$  is an area element of a sphere of radius  $r$ , the factor 2 in front of the integral accounts for the system's negative time emission, and the integration region is assumed to be  $\omega > 0$ . Thus,

$$\begin{aligned} \frac{d\mathcal{E}}{d\omega} &= \frac{1}{\pi} \frac{1}{(4\pi)^2} \int d\mathbf{n} |\mathbf{q} \times \mathbf{j}^{\text{cl}}|^2 = \frac{\omega^2}{16\pi^3} \int d\mathbf{n} (|\mathbf{j}|^2 - |\mathbf{n}\mathbf{j}|^2), \\ j_\mu^{\text{cl}} &= i j_\mu. \end{aligned} \quad (6.8)$$

From current conservation  $[q_\mu j_\mu(q) = 0]$ , the expression in brackets in Eqn (6.8) is  $-j_\mu(q) j_\mu^*(q)/\omega^2$ , so that

$$d\mathcal{E} = \int d\mathbf{n} \left[ (-j_\mu j_\mu^*) \omega \frac{d\mu(q)}{d\mathbf{n}} \right] \quad (6.9)$$

(which agrees with the formula for the photon emission probability density; see Section 2.1.2). In the non-relativistic

limit the transition current of a point-like particle is

$$j_\mu(q, p) = e \left( \frac{p'_\mu}{p'q} - \frac{p_\mu}{pq} \right) \sim e \frac{v'_\mu - v_\mu}{\omega}, \quad v_\mu = \frac{p_\mu}{E_p}; \quad (6.10)$$

and hence [80]

$$\frac{d\mathcal{E}}{d\omega} = \frac{e^2}{16\pi^3} \frac{8\pi}{3} (v' - v)^2 = \frac{2\alpha}{3\pi} (v' - v)^2. \quad (6.11)$$

The difference  $e(v' - v)/2\pi = \ddot{\mathbf{d}}_\omega$  is the second time derivative of the Fourier component of the system's dipole moment  $\mathbf{d}$  for  $\omega \rightarrow 0$ . The expression so obtained is the spectral emission density of a pointlike particle for  $\omega \rightarrow 0$ .

Clearly, expression (6.7) can also be obtained from the asymptotic behavior of the Lenard-Wiechert potential for  $r \rightarrow \infty$

$$A_\mu = \frac{e}{4\pi} \frac{1}{R - \mathbf{v}R} \frac{p_\mu}{E_p} \approx \frac{e}{4\pi r} \frac{p_\mu}{E_p - \mathbf{p}\mathbf{n}} = \frac{e\omega}{4\pi r} \frac{p_\mu}{pq}, \quad \mathbf{v} = \frac{\mathbf{p}}{E_p}, \quad (6.12)$$

assuming that  $A_\mu(\mathbf{x}, \omega)$  is calculated in the limit  $\omega \rightarrow 0$  and that  $p_\mu(t) = p_\mu(p'_\mu)$  as  $t \rightarrow -\infty (\infty)$ .

### 6.3 Divergences in multi-vertex theories

If a charged particle can emit two or more massless particles locally, the situation with infrared and collinear divergences is different. In this case divergences are produced by the amplitudes of the form  $A_n \sim TR_n$  [cf. Eqns (2.7) and (2.8)] with

$$R_n^\pm = \frac{1}{(p \pm \sum q_i)^2 - m^2} = \frac{1}{\pm 2p \sum q_i + (\sum q_i)^2}. \quad (6.13)$$

Let us now consider the following possibilities: 1) infrared divergences ( $m \neq 0$ ,  $\omega_i = |\mathbf{q}_i| \rightarrow 0$ ); 2) infrared and collinear divergences ( $m = 0$ ,  $\omega_i \rightarrow 0$ ,  $E_p = |\mathbf{p}| \gg \omega_i$ ); 3) collinear divergences ( $m = 0$ ;  $\mathbf{q}$  and  $\mathbf{p}$  are arbitrary). Case 2) is added for methodological reasons.

1) The probability for the emission of  $n$  photons with energies  $0 < \omega_i < \Lambda$  is

$$P_n \propto \int_0^\Lambda |R_n|^2 \prod_i^n d\mu(q_i) \propto \int_0^\Lambda \frac{\prod \omega_i d\omega_i d\mathbf{n}_i}{|E_p \sum \omega_i - \mathbf{p} \sum \mathbf{q}_i|^2} \propto \int_0^\Lambda \frac{\prod \omega_i d\omega_i}{|\sum \omega_i|^2} \equiv I_n \quad (6.14)$$

[we omit the obviously convergent integrals and terms of  $O(\omega^2)$ ,  $\mathbf{n}_i = \mathbf{q}_i/\omega_i$ ]. That the integral in Eqn (6.14) converges for  $n > 1$  can be seen by recalling the well-known inequality [81]

$$\left( \frac{\sum p_i \omega_i}{\sum p_i} \right)^{\sum p_i} \geq \prod \omega_i^{p_i}, \quad \omega_i \geq 0, \quad p_i > 0, \quad (6.15)$$

and setting  $p_i = 1$  in Eqn (6.15), i.e., using the inequality  $\sum \omega_i/n \geq \prod \omega_i^{1/n}$ . This gives

$$I_n \leq \int_0^\Lambda \frac{\prod \omega_i d\omega_i}{n^2 \prod \omega_i^{2/n}} < \infty, \quad n > 1. \quad (6.16)$$

2) The second case, in addition to infrared divergences, also includes collinear ( $m = 0$ ) ones, provided one may neglect the second term in the denominator of the second fraction in Eqn (6.13). The answer depends on the spin of the particles. If the emitted particle has a spin  $J = 0, 1, 2, \dots$ , then an additional factor  $(pe(q_i))^J$  usually appears in the amplitude. The vectors  $\epsilon^\pm(q)$  lie in a plane orthogonal to the vector  $\mathbf{q}$  (see Section 6.1). If  $\theta$  is the angle between the vectors  $\mathbf{p}$  and  $\mathbf{q}$ , the angle formed by  $\mathbf{p}$  and  $\epsilon^\pm(q)$  lies in the interval  $[\pi/2 + \theta, \pi/2 - \theta]$ , giving

$$|pe(q_i)|^J \leq |E_p \sin \theta_i|^J.$$

Using the equalities  $p \sum q_i = E_p (\sum \omega_i - \sum \mathbf{n}_i \omega_i)$  and  $d\mathbf{n}_i = \sin \theta_i d\theta_i d\varphi_i$ , where  $\mathbf{n} = \mathbf{p}/E_p$ ,  $\mathbf{n}^2 = 1$ ,  $\mathbf{n}\mathbf{n}_i = \cos \theta_i$ , we find

$$P_n \propto \int_0^\Lambda \frac{\prod \omega_i d\omega_i |pe(q_i)|^{2J} d\mathbf{n}_i}{|\sum \omega_i - \sum \omega_i \cos \theta_i|^2} \propto \int_0^\Lambda \frac{\prod \omega_i d\omega_i \theta_i^{2J+1} d\theta_i}{|\sum \omega_i \theta_i^2|^2} \equiv K_n \quad (6.17)$$

for which, again using Eqn (6.15), we find the estimate

$$K_n \leq \frac{1}{n^2} \int_0^\Lambda \prod \omega_i^{1-2/n} \theta_i^{2J+1-4/n} d\omega_i d\theta_i, \quad (6.18)$$

implying that the integral (6.17) converges if the inequalities

$$n > 1, \quad n > \frac{2}{J+1} \quad (6.19)$$

are satisfied.

If the first (second) of these does not hold, infrared (collinear) divergences appear. Thus, according to Eqn (6.19), in the  $n = 1$ ,  $J = 0, 1$  case we have infrared and collinear divergences, whereas in the  $n = 2$ ,  $J = 0$  case collinear divergences occur. For  $n \geq 3$ , both types are absent. For  $J = 2$  collinear divergences are absent in theories with the factor  $[pe(q_i)]^2$  in emission amplitudes, as they are in gravitation.

3) In the general case in which  $m = 0$  and  $E_p$  and  $\omega_i$  are arbitrary, we take advantage of the fact that in the vicinity of the singularity (6.13), i.e. for  $\sum \theta_{ij}^2 > 0$ , when  $\cos \theta_{ij} = (\mathbf{n}_i \mathbf{n}_j)$ , we have the inequality

$$\delta = p \sum q_i \pm \left( \sum q_i \right)^2 > 0, \quad (6.20)$$

because in this case one can always choose a reference frame in which  $\sum \mathbf{q}_i = 0$  and  $\delta = E_p \sum \omega_i \pm (\sum \omega_i)^2 > 0$  if the inequality  $E_p > \sum \omega_i$  holds (the lower sign is for the emission of an incident particle). Thus,

$$|R_n| = \frac{1}{|p \sum q_i + \delta|} \leq \frac{2}{E_p \sum \omega_i \theta_i^2}, \quad (6.21)$$

i.e., we arrive at integrals of the type (6.17) and inequalities (6.19). For  $J > 0$ , the additional integration over  $p$  in Eqn (6.14) does not lead to new divergences.

Thus, divergences appear in the following theories with  $n + 2$  particles in vertices and a factor  $(pe)^J$  in amplitudes:

- 1) Infrared Divergences (ID):  $J = 0, 1, 2, \dots$  for  $n = 1$ ;  
 2) Collinear Divergences (CD):  $J = 0, 1$  for  $n = 1$  and  $J = 0$  for  $n = 2$ .

In  $D + 1$  space-time instead of the inequalities (6.19) we have

$$n > \frac{2}{D-1}, \quad n > \frac{4}{2J+D-1},$$

from which it follows that at  $D = 2$  divergences appear in the following cases:

- 1) ID:  $J = 0, 1, 2, \dots$  for  $n = 1, 2$ ;  
 2) CD:  $J = 0$  for  $n = 1, 2, 3, 4$  and  $J = 1$  for  $n = 1$ .

For  $D = 4$ , only collinear divergences with  $J = 0$  exist. The above formulas fail for  $D = 1$ . If the factor  $(p\varepsilon)^J$  is absent, then the theory is equivalent to one with  $J = 0$  [as, e.g., in the case  $\mathcal{L}_{\text{int}} \sim (A_\mu^2)^2$ ]. The obtained estimates are useful in the study of emission by virtual particles, and in particular in dealing with effective Lagrangians.

#### 6.4 Kinematics of collinear processes

**Vectors.** In massless QED, the first of the three isotropic collinear vectors  $p', p, q$  is expressed in terms of the other two in view of the conservation of the 4-momentum in the vertex, and Eqn (4.4) gives  $q = ap$ ,  $a > 0$ . In prefactors (4.6), it is necessary to take account of 'weakly non-collinear processes', i.e., to include terms  $O(|q_\perp|)$ , where  $q_\perp p = 0$ . We therefore use the following standard representation for the vectors:

$$p = E_p(1; 0, 0, 1), \quad \tilde{p} = E_p(1; 0, 0, -1), \\ q = (\omega; q_\perp, |q_\parallel|), \quad \omega^2 = q_\parallel^2 + q_\perp^2. \quad (6.22)$$

The vector  $q_\perp = (0; q_\perp, 0)$  is expressed in terms of  $p, \tilde{p}$  and  $q$  as follows:  $q_\perp = q - (q \cdot \tilde{p} p + q \cdot p \tilde{p}) / (p \cdot \tilde{p}) \equiv q - q_\parallel$ . Here and hereafter we use the dot notation for the scalar product of 4-vectors. We have ( $q_\perp \rightarrow 0$ )

$$q_\perp \cdot p = q_\perp \cdot \tilde{p} = 0, \quad p \cdot \tilde{p} = 2E_p^2, \\ q \cdot p = E_p(\omega - |q_\parallel|) \approx E_p \frac{q_\perp^2}{2\omega}, \quad q \cdot \tilde{p} \approx 2E_p \omega. \quad (6.23)$$

**Polarization.** Since  $q \cdot \varepsilon^\lambda(q) = 0$  for the photon, expressions of the type

$$p \cdot \varepsilon(a\bar{p} + q_\perp) = -q_\perp \cdot \frac{\varepsilon(a\bar{p} + q_\perp)}{a} \\ = -q_\perp \cdot \frac{\varepsilon(ap)}{a} + O(q_\perp^2), \quad \varepsilon(aq) = \varepsilon(q), \quad a > 0 \quad (6.24)$$

in Eqns (4.6) and (4.9) are  $O(|q_\perp|)$ . In Eqn (6.24)  $\bar{p}$  is the vector with the components  $(E_{\bar{p}}; \mathbf{p})$  and  $E_{\bar{p}}^2 = \mathbf{p}^2 + q_\perp^2/a^2$  since  $(a\bar{p} + q_\perp)^2 = 0$ .

Similar formulas for fermions are not so simple. A typical expression for Eqn (4.6) is  $\bar{u}(p)\gamma_\mu u(a\bar{p} + q_\perp)\varepsilon_\mu(ap)$ , whose proportionality to  $|q_\perp|$  is demonstrated using the relations (see a very useful discussion in Ref. [82])

$$u_\pm(p') = C\tilde{p}'\hat{h}u_\pm(p), \quad C^{-2} = 4p' \cdot hp \cdot h - 2p' \cdot ph^2, \\ u_{+/-} \equiv u_{L/R}, \quad u_\pm = \frac{1 \pm \gamma_5}{2} u. \quad (6.25)$$

Here  $h$  is a certain 4-vector and  $p' \cdot h, p \cdot h \neq 0$ . Setting  $h \cdot q_\perp = 0$  shows that

$$u(a\bar{p} + q_\perp) = C(a\bar{p} + \hat{q}_\perp)\hat{h}u(ap) + O(q_\perp^2), \\ C^{-1} \approx 2\sqrt{ap \cdot h} + O(q_\perp^2),$$

i.e., to terms  $O(q_\perp^2)$  we have

$$\bar{u}(p)\hat{\varepsilon}^\lambda(ap)u(a\bar{p} + q_\perp) \approx \frac{1}{2\sqrt{ap \cdot h}} \\ \times [\bar{u}(p)\gamma_\mu(2ap \cdot h + \hat{q}_\perp \hat{h})u(ap)]\varepsilon_\mu^\lambda(p) \\ \approx \frac{1}{2\sqrt{ap \cdot h}} \bar{u}(p)\hat{\varepsilon}^\lambda(ap)\hat{q}_\perp \hat{h}u(ap), \quad (6.26)$$

since  $\bar{u}(p)\gamma_\mu u(ap) \sim p_\mu$  and  $p \cdot \varepsilon(p) = 0$ . This is seen by noting that, according to Ref. [82]

$$\bar{u}_\pm(p')\gamma_\mu u_\pm(p) = \frac{2(p'_\mu p \cdot h + p_\mu p' \cdot h - h_\mu p' \cdot p \pm i\varepsilon_{\mu\nu\rho\sigma}p'_\nu p_\rho h_\sigma)}{(4p' \cdot hp \cdot h - 2p' \cdot ph^2)^{1/2}}, \quad (6.27)$$

where  $\varepsilon_{0123} = 1$ , and the proportionality of  $\bar{u}(p)\gamma_\mu u(ap)$  to the vector  $p_\mu$  follows from Eqn (6.27) and the expansion  $\bar{u}\gamma_\mu u = \bar{u}_+\gamma_\mu u_+ + \bar{u}_-\gamma_\mu u_-$ . From Eqns (6.26) and (6.27) it follows that the prefactors in Eqn (4.6) vanish as  $q_\perp \rightarrow 0$ . The vector  $h$  is absent from the current products  $j_\mu^* j_\nu$  [82] and hence from the probabilities of interest.

#### 6.5 The path integral method

Widely used in modern quantum field theory, path integration also offers a good method for treating processes with infrared divergences [67; 68, p. 368] or performing eikonal calculations [66]. Note, however, that whereas in full-scale quantum field theory this method leads to a fundamentally new calculational scheme, i.e., the quasiclassical approximation, eikonal theory and the treatment of long-wave quanta depend on the simplification of the physical picture for their success. Thus, the Bloch–Nordsieck model [19] replaces Dirac's  $\gamma$  matrices by  $c$  numbers, and the more comprehensive model of Section 3 replaces current by the 'classical current operator,' Eqn (3.16), to make the problem exactly solvable. Nevertheless, the path integral method may be useful in these cases as well.

Let us first find a continuum integral representation for the  $S$ -matrix. The scattering operator is defined by the  $T$  exponent (2.2) [or (3.22)], and the main technical challenge here is to change over to a normally ordered operator. The starting point is Hori's formula (3.30) which allows any causally ordered operator  $TF[\hat{\phi}]$  to be disentangled — i.e., normally ordered. To this end we use the functional Fourier transform [68, Ch. 8]

$$F[\phi] = \int d[\chi] \exp(i\chi\phi) \tilde{F}[\chi] \quad (6.28)$$

[in the notation of Eqn (3.30)]; the necessary constant factors have been incorporated into the measure  $d[\chi]$  of the functional integration. Then, using Eqn (3.30) one arrives at

$$TF[\hat{\phi}] = \int d[\chi] \tilde{F}[\chi] \exp\left(-\frac{1}{2}\chi\Delta_c\chi\right) : \exp(i\chi\hat{\phi}) :. \quad (6.29)$$

Substituting this into the inverse Fourier transform

$$\tilde{F}[\chi] = \int d[\eta] \exp(-i\chi\eta) F[\eta],$$

making the shift  $\eta \rightarrow \eta + \hat{\phi}$  and integrating over  $\chi$  we find

$$\begin{aligned} TF[\hat{\phi}] &= \int d[\eta\chi] : \exp \left( i\chi\hat{\phi} - i\chi\eta - \frac{1}{2} \chi\Delta_c\chi \right) : F[\eta] \\ &= \int d[\eta] \exp \left( -\frac{1}{2} \eta\Delta_c^{-1}\eta \right) : F[\eta + \hat{\phi}] : \end{aligned}$$

(with the integration measure incorporating all the constant factors that appear). Further, recalling the definition of the causal operator,  $\Delta_c = i/(p^2 - m^2 + i\epsilon)$ , and noting that  $\eta\Delta_c^{-1}\eta = -i \int d^4x \eta(-\partial_\mu^2 - m^2 + i\epsilon)\eta = -2iS_0[\eta]$ , where  $S_0$  is the free field action, the desired representation is

$$TF[\hat{\phi}] = \int d[\eta] \exp(iS_0[\eta]) : F[\eta + \hat{\phi}] : \quad (6.30)$$

For the  $S$ -matrix we rewrite Eqn (6.30) in the form

$$\hat{S} = \int d[\eta] : \exp(iS_0[\eta] + iS_{\text{int}}[\eta + \hat{\phi}]) : , \quad (6.31)$$

where  $\hat{\phi}$  are free field operators. The contribution of the vacuum diagrams is eliminated by dividing the operator (6.31) by the vacuum average of  $\hat{S}$ . Representation (6.31) is also valid for Fermi fields provided the specifics of integration over such fields [61; 68, p. 354] is taken into account. From here on  $d[\eta]$  denotes the measure of integration over all quantum fields (e.g., in spinor electrodynamics  $d[\eta] \rightarrow d[\bar{\psi}\psi A]$ ).

In the framework of representation (6.31) the emission of real infrared photons is described as follows. Using Eqns (3.18), (3.19), and (3.36), and the current properties (2.10), (2.16) it is found that the emission of real infrared photons in the hard process  $\alpha \rightarrow \beta$  is given by the operator

$$\hat{S}_{\beta\alpha}^{\text{IR}} = \int d[\eta_\mu] : \exp(iS_0[\eta_\mu] + iS'_{\beta\alpha}[\eta_\mu + \hat{A}_\mu^Q]) : , \quad (6.32)$$

where

$$S'_{\beta\alpha}[\eta_\mu] = - \int d^4x j_{\mu\beta\alpha}(x) \eta_\mu(x) ,$$

$$\begin{aligned} j_{\mu\beta\alpha}(x) &= i \int \frac{d^4q}{(2\pi)^4} \exp(-iqx) j_{\mu\beta\alpha}(q, p) \\ &= \int \frac{d^4q}{(2\pi)^4} \exp(-iqx) \left[ \sum_\beta \frac{ie_\beta p_\mu^{\beta}}{p'^\beta q + i0} - \sum_\alpha \frac{ie_\alpha p_\mu^\alpha}{p^\alpha q - i0} \right] . \end{aligned}$$

It is assumed that a gauge-specifying term [53] is incorporated into  $S_0$  (i.e., that Feynman's gauge is used).

Integrating now over  $\eta_\mu$ , Eqn (6.32) becomes

$$\begin{aligned} \hat{S}_{\beta\alpha}^{\text{IR}} &= \exp \left[ -\frac{1}{2} j_{\mu\beta\alpha} \Delta_{Qc}^{\mu\nu} j_{\nu\beta\alpha} \right] : \exp \left[ -i \int d^4x j_{\mu\beta\alpha}(x) \hat{A}_\mu^Q(x) \right] : \\ &\equiv \exp \left( -\frac{1}{2} N \right) : \exp(-ij_{\beta\alpha} A^Q) : . \end{aligned} \quad (6.33)$$

(following the infrared regularization procedure), where  $\Delta_{Qc}^{\mu\nu}$  denotes the propagator of the field  $A_\mu^Q$ , Eqn (3.17), and  $\text{Re } N = \bar{N} = N_0(Q, \lambda)$  is the average number of emitted photons [see Eqns (2.23) and (2.24)]. It is readily seen that

$$\langle q_1 \epsilon_1, \dots, q_N \epsilon_N | \hat{S}_{\beta\alpha}^{\text{IR}} | 0 \rangle = j_1 \epsilon_1^* \dots j_N \epsilon_N^* \exp \left( -\frac{1}{2} N \right) \quad (6.34)$$

reproduces the product of the factors in front of the amplitude of the main process  $T_0 (\equiv T_{\beta\alpha})$  which describe the creation of real (2.15) and virtual (2.24) photons.

The probability density of a hard process  $\alpha \rightarrow \beta$  with the total energy loss due to infrared emission  $\sum \omega_i \leq E$  gives [according to Eqns (2.18)–(2.20), (2.27)] a factor  $b(A) (E/\lambda)^A$ , whereas  $\text{Re } N = A \ln(Q/\lambda)$  from Eqn (2.19). As a result, we obtain expression (2.26) (with the replacement  $A \rightarrow Q$ ),

$$\tilde{P}_E = |T_{\beta\alpha}|^2 b(A) \left( \frac{E}{\lambda} \right)^A \left( \frac{\lambda}{Q} \right)^A = |T_{\beta\alpha}|^2 b(A) \left( \frac{E}{Q} \right)^A , \quad (6.35)$$

which is regular for  $\lambda \rightarrow 0$ .

The by far more primitive (no-pair-creation) Bloch–Nordsieck model [19] is obtained by the replacement  $\gamma_\mu \rightarrow u_\mu$  in spinor electrodynamics ( $u_\mu$  being a constant unit vector,  $u^2 = 1$ ) — which is equivalent to the replacement  $p_\mu \rightarrow u_\mu$  in, for example, Eqns (2.13) or (6.12). The continuum integration calculation of the propagator of a charged fermion within this model is given in Refs [67; 68, p. 368]. Clearly, for  $u = p/m$  the Bloch–Nordsieck model accounts well for infrared emission by incident or scattered particles.

The Hamiltonian (4.13) for collinear processes is also represented by the continuum integral (6.31), but whether this latter can be calculated explicitly is still an open question.

*Note.* Since in perturbation theory the multiplicative cancellation of divergences in Eqn (6.35) becomes additive [e.g.,  $(1 + A \ln(E/\lambda))(1 + A \ln(\lambda/Q)) \approx 1 + A \ln(E/Q)$ ,  $A = O(\alpha)$ ], a question may occur to a nonspecialist, how is it that three positive quantities (4.1) divergent at  $\lambda \rightarrow 0$  sum to a finite quantity? Let us write down those terms in the relations  $d_{a,b,c} = \sigma_{a,b,c}/[\Omega(d\sigma/d\Omega)_0]$  [35] which diverge as  $\lambda \rightarrow 0$ :

$$d_a = g_R^2 (-4 \ln 2\epsilon - 3) \ln \frac{E}{\lambda} > 0 \quad (\epsilon \ll 1) ,$$

$$d_b = g_R^2 \left( 2 \ln^2 \frac{E}{\lambda} + 4 \ln 2\epsilon \ln \frac{E}{\lambda} \right) > 0 ,$$

$$d_c = 1 + g_R^2 \left( -2 \ln^2 \frac{E}{\lambda} + 3 \ln \frac{E}{\lambda} \right)$$

( $g_R^2 = g_E^2/3\pi^2$ ,  $g_E$  is the renormalized coupling constant). We see that the sum of the above terms is unity. The last expression is positive if  $2g_R^2 \ln^2(E/\lambda) < 1$ , i.e., only if the condition for the applicability of perturbation theory is satisfied. That the sum of positive cross sections (4.1) for  $\lambda \rightarrow 0$  is finite comes as no surprise because each cross section is positive only for a finite (large enough) value of  $\lambda$ , and the gluon mass  $\lambda$  does not enter the sum.

*Note added in proof.* As was discovered in the preparation of materials for the 100th anniversary of V A Fock's birth (22 December 1998), the Coulomb scattering problem (see Section 2.1.1) was also solved by V A Fock. Obtained in February of 1928, the solution was published by Fock in his *Principles of Quantum Mechanics* (Leningrad: Kubuch, 1932) and by W Gordon [*Zs. Phys.* 48 180 (1928)]. For details see L V Prokhorov *Fiz. Elem. Chostits At. Yadra* 31 47 (2000).

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