SUPERSYMMETRIC DIMENSIONAL REGULARIZATION VIA DIMENSIONAL REDUCTION ☆

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We introduce a modified form of dimensional regularization which manifestly preserves gauge invariance, unitarity, and global supersymmetry. The prescription is that the action which results from analytic continuation to lower dimensions is that found by dimensional reduction. We also consider its application to supergravity.

Loop calculations in gauge theories are greatly simplified by the use of regularization procedures which manifestly preserve the symmetries involved, due to the automatic satisfaction of Ward–Takahashi–Slavnov–Taylor (WTST) identities. Dimensional regularization [1,2] seems to be best suited for this purpose, and is also superior in computational convenience. Unfortunately, the standard forms of dimensional regularization do not manifestly preserve supersymmetry. This is obvious in superfield notation, since

$$\int \mathrm{d}^D x \; (D^\alpha D_\alpha)(\bar{D}^{\dot\alpha}\bar{D}_{\dot\alpha}) L(x,\theta,\bar\theta)$$

is θ-independent only for $D = 4$ [3]. On the other hand, the obvious supersymmetric generalization

$$\int \mathrm{d}^D x \, \mathrm{d}^\nu\theta \, \mathrm{d}^\nu\bar\theta \; L(x,\theta,\bar\theta) \quad (\nu = 2^{D/2-1})$$

results in propagators with momentum dependence $1/p^\nu$ [4] (as can be seen by dimensional analysis: $\mathrm{d}\theta = \partial/\partial\theta \sim m^{1/2} \to \mathrm{d}^\nu\theta\,\mathrm{d}^\nu\bar\theta \sim m^\nu$). Thus, unitarity is not manifestly preserved, and it is not clear whether the subtraction procedure of dimensional renormalization [5] can be applied without modification. (The renormalization procedure of Speer [6], which is based on propagators of the form $1/p^\nu$, but with ν unrelated to D, preserves unitarity but does not manifestly preserve gauge invariance.)

The clash between dimensional continuation and supersymmetry is due to the fact that the nature of supersymmetric multiplets differs for different dimensionalities of space time. For example, the simplest supersymmetric multiplet, the chiral (scalar-spinor) multiplet, has 1 scalar in $D = 2$ or 3, 2 scalars (scalar + pseudoscalar) in $D = 4$, 4 scalars in $D = 5$ or 6, and does not even exist in $D > 6$. This is a consequence of the dependence of the number of components of the unextended-supersymmetry generators Q_α on D. Therefore, we propose the solution that the number of components of Q_α be fixed for all D. This can be done consistently if we first define our analytic continuation from integral $D \leqslant 4$ only, which is all that is necessary for regularization of ultraviolet divergences. The result is that the O(1) generators Q_α ($\alpha = 1, ..., 4$) in $D = 4$ become O(2) generators $Q_{\alpha i}$ ($\alpha = 1,2; i = 1,2$) in $D = 2$ or 3, and O(4) generators Q_i in $D = 1$ ($i = 1, \ldots, 4$).

Explicitly, our rule for dimensional continuation is: analytically continue in the number of components D of all momenta and spacetime coordinates, but keep the number of components of all other tensors and spinors fixed. For integral $D < 4$, this is just the definition of dimensional reduction (see ref. [7] for applications to supersymmetric theories), and the analytically continued lagrangian satisfies the usual physical properties (unitarity, causality, symmetries) for $D = 1, 2, 3$. In terms of the standard dimensional regularization, a vector gauge field, for example, becomes a vector and ϵ ($D = 4 - \epsilon$) scalars. As in the standard form of dimensional regularization, WTST identities are manifestly preserved, and the usual subtraction procedure defines a correct renormalization procedure. (D-analytic

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properties are as usual, since the only modification is that Lorentz indices are treated as internal indices.) In particular, the scalar fields are treated as usual, the spinor fields are only affected by the normalization of the trace in a spinor loop (which is harmless [1]), and the vector fields introduce finite renormalizations (due to the contributions to loops of ϵ scalars). Furthermore, supersymmetry is manifestly preserved. This is particularly clear in the superfield formulation of the Feynman rules [8–10], since the algebra of θ's, of covariant derivatives D_α, and of generalized Pauli σ-matrices is identical to the usual 4D algebra (which also makes computations simpler than in the formalism of ref. [4]).

Our supersymmetric dimensional regularization also manifestly preserves chiral invariance, and so is the supersymmetric modification of the form of standard dimensional regularization which has $\{\gamma_5, \gamma_a\} = 0$ for all D [11], as opposed to the form which has $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ for all D [1]. The former form is preferred for theories in which chiral invariance is maintained by cancellation of axial anomalies: e.g., in unified theories of weak and electromagnetic interactions, the latter form gives incorrect results for 1-loop calculations of anomalous magnetic moments [11], due to violation of the axial-vector Ward identities by the subtraction procedure. Supersymmetric dimensional regularization (and the former form of standard dimensional regularization) will probably require a special prescription for calculation of axial current anomalies in theories where they do not cancel (i.e., in some of the theories where axial vectors do not couple to them).

To illustrate the supersymmetric dimensional regularization, we will discuss some examples of 1-loop supergraph calculations. 1-loop self-energy corrections have already been calculated in the literature [9,10,4] for self-interacting chiral multiplets coupled to abelian and non-abelian vector multiplets. In all these calculations θ-algebra was done in $D = 4$, as is required by our regularization prescription. (In ref. [4], where the θ-algebra was to be done for arbitrary D, arguments had to be given involving the WTST identities to show that the θ-algebra could be done in $D = 4$ and give the same result as for arbitrary D, for the case of the non-abelian vector-multiplet propagator.) The result was that all θ-dependence was contained in factors involving only external momenta, and these factors always multiplied the integrals $\int \mathrm{d}k/(k^2 + m_1^2)((k+p)^2 + m_2^2)$ and $\int \mathrm{d}k/(k^2 + m^2)$. Since these resulting integrals have no free indices, their evaluation is identical to that in the standard methods of dimensional regularization. (For the massless vector multiplet we use the Fermi–Feynman gauge $\alpha = 1$ to avoid irremovable infrared divergences due to $1/k^4$ behavior in the propagator [12]. These divergences cause problems, e.g., in evaluation of the 1-loop correction to the chiral multiplet propagator.) Supersymmetric dimensional regularization thus justifies the naive manipulations of the integrals, which lead to supersymmetric and gauge-invariant results.

Supersymmetric dimensional regularization is not fully successful when applied to supergraph calculations in supergravity [13]. The reason is that the gauge transformation law $\delta H_{\alpha\dot\beta} = D_\alpha \bar K_{\dot\beta} - \bar D_{\dot\beta} K_\alpha + \mathrm{O}(H)$ can be broken up into three parts

$$\delta H_a = \partial_a f + (D^2 + \bar D^2) g_a + \Delta_a h$$

$$(\Delta_a = -\tfrac{1}{4}\mathrm{i}\, \sigma_a^{\alpha\dot\beta} [D_\alpha, \bar D_{\dot\beta}]) , \tag{1}$$

the third of which is not preserved in the limit $D \to 4$. This can be seen by examining the transversality conditions (which occur in WTST identities corresponding to the three transformations·

$$\partial^a f_a = 0 \to f_a = (\eta_{ab} - \partial_a\partial_b/\Box) g^b , \tag{2a}$$

$$(D^2 + \bar D^2) f = 0 \to f = \hat\Pi_{1/2} g , \tag{2b}$$

$$\Delta^a f_a = 0 \to f_a = \Pi_{3/2ab} g^b + \Pi_{1ab} h^b + \Pi^{\mathrm{L}}_{1/2ab} j^b + (\hat{\hat\delta}{}_a^c - \tfrac{1}{3}\epsilon\delta_a^c)\, \Pi^{\mathrm{T}}_{1/2ab} n^b , \tag{2c}$$

where $\hat\eta_{ab} = \eta_{ab}$ restricted to the D-dimensional subspace of 4D spacetime, $\hat{\hat\eta}_{ab} = \eta_{ab} - \hat\eta_{ab}$, and $\hat\Pi_i$ and Π_{iab} are the projection operators for the irreducible representations of real scalar and vector superfields, respectively [14,13]:

$$\hat\Pi_0 = -(1/16\Box)\{D^2, \bar D^2\} , \quad \hat\Pi_{1/2} = (1/8\Box) D^\alpha \bar D^2 D_\alpha ,$$

$$\Pi_{0ab} = \hat\Pi_0 \partial_a\partial_b/\Box , \quad \Pi^{\mathrm{L}}_{1/2ab} = \hat\Pi_{1/2}\, \partial_a\partial_b/\Box ,$$

$$\Pi^{\mathrm{T}}_{1/2ab} = \tfrac{1}{3}(-\Delta_a\Delta_b/\Box + \hat\Pi_0\, \partial_a\partial_b/\Box) ,$$

$$\Pi_{1ab} = \hat\Pi_0(\eta_{ab} - \partial_a\partial_b/\Box) ,$$

$$\Pi_{3/2ab} = \tfrac{2}{3}(-(\Delta_a\Delta_b + 3\Delta_b\Delta_a)/4\Box + \hat\Pi_0\partial_a\partial_b/\Box) . \tag{3}$$

The solutions (2a) and (2b) to the transversality conditions which also occur in component-field and superfield gauge theories, respectively, have a limit $D \to 4$ which also satisfies the transversality condition, even when the functions involved have explicit D dependence (so the non-supergravity gauge invariances are maintained as $D \to 4$). However, the last term in (2c), which vanishes for $D = 4$ ($\hat{\hat{\delta}}_a^c = \epsilon = 0$), can appear in the limit $D \to 4$ if the function n^b has a $1/\epsilon$ singularity. As an example, we examine the self-energy correction for the supergravity multiplet due to a massless-chiral-multiplet loop. The lowest-order coupling of the supergravity multiplet to the chiral multiplet is given by the product of the supergravity vector superfield and the chiral-multiplet supercurrent superfield [15].

$$S_{\mathrm{INT}} = \tfrac{2}{3}\kappa \int \mathrm{d}^D x \mathrm{d}^4\theta\, H^a$$
$$\times\, (\bar{\chi}\mathrm{i}\overleftrightarrow{\partial}_a\chi + \tfrac{1}{4}\sigma_a^{\alpha\dot{\beta}}(\bar{D}_{\dot{\beta}}\bar{\chi})(D_\alpha\chi)]$$
$$= \kappa \int \mathrm{d}^D x\, \mathrm{d}^4\theta\, H^a(\bar{\chi}\mathrm{i}\overleftrightarrow{\partial}_a\chi - \tfrac{1}{3}\mathrm{i}\Delta_a\bar{\chi}\chi)\,. \tag{4}$$

The contribution of χ to the H^a self-energy is thus (with Wick-rotated momenta):

$$\kappa^2 \int \mathrm{d}^D k (2\pi)^{-D} k^{-2}(k+p)^{-2}[(2k+p)_a - \tfrac{1}{3}\mathrm{i}\Delta_a(\theta_1,p)]$$
$$\times\, [(2k+p)_b - \tfrac{1}{3}\mathrm{i}\Delta_b(\theta_2,-p)]$$
$$\times \exp\{\theta_{12}\bar{\theta}_{12}(2k+p)\}\exp\{(\theta_2\bar{\theta}_1 - \theta_1\bar{\theta}_2)p\}$$
$$= \tfrac{1}{2}(3-\epsilon)^{-1}\kappa^2 A p^4 [\Pi_{3/2\,ac}\hat{\delta}^c_b$$
$$+ (\hat{\hat{\delta}}_a^c - \tfrac{1}{3}\epsilon\delta_a^c)\Pi^{\mathrm{T}}_{1/2\,cb}](\theta_1,p)\delta^4(\theta_1-\theta_2)\,,$$
$$A = \int \mathrm{d}^{4-\epsilon}k\,(2\pi)^{\epsilon-4}k^{-2}(k+p)^{-2}$$
$$= (4\pi)^{\epsilon/2-2}\Gamma(\epsilon/2)\frac{(\Gamma(1-\epsilon/2))^2}{\Gamma(2-\epsilon)}(p^2)^{-\epsilon/2}\,, \tag{5}$$

where we have used standard tricks such as:

$$p\cdot(2k+p) = (k+p)^2 - k^2$$
$$\to \int \mathrm{d}k\,(2k+p)_a(2k+p)_b k^{-2}(k+p)^{-2}$$
$$\sim p^2\hat{\delta}_{ab} - p_a p_b\,,$$
$$\int \mathrm{d}k\,(2k+p)_a k^{-2}(k+p)^{-2} = 0\,;$$
$$(2k+p)^2 = 2(k+p)^2 + 2k^2 - p^2$$
$$\to \int \mathrm{d}k\,(2k+p)^2 k^{-2}(k+p)^{-2}$$
$$= -p^2\int \mathrm{d}k\,k^{-2}(k+p)^{-2}$$
$$\to \int \mathrm{d}k\,(2k+p)_a(2k+p)_b k^{-2}(k+p)^{-2}$$
$$= -\frac{1}{D-1}(p^2\hat{\delta}_{ab} - p_a p_b)\cdot\int \mathrm{d}k\,k^{-2}(k+p)^{-2}\,; \tag{6}$$

and supersymmetric algebraic identities such as.

$$-2\mathrm{i}\sigma_a^{\alpha\dot{\beta}}(\theta_1-\theta_2)_\alpha(\bar{\theta}_1-\bar{\theta}_2)_{\dot{\beta}}\exp\{(\theta_2\bar{\theta}_1-\theta_1\bar{\theta}_2)p\}$$
$$= \Delta_a(\theta_1,p)\delta^4(\theta_1-\theta_2)\,,$$
$$[1+\tfrac{1}{4}p^2\delta^4(\theta_1-\theta_2)]\exp\{(\theta_2\bar{\theta}_1-\theta_1\bar{\theta}_2)p\}$$
$$= \tfrac{1}{2}p^2\hat{\Pi}_{1/2}(\theta_1,p)\delta^4(\theta_1-\theta_2)\,,$$
$$\Delta_a\Delta_b = -p_a p_b + \delta_{ab}p^2\hat{\Pi}_{1/2} - \epsilon_{abcd}p^c\Delta^d\,. \tag{7}$$

Although the result (5) is gauge invariant (i.e., it satisfies $\sigma^a_{\alpha\dot{\beta}}\bar{D}^{\dot{\beta}}f_{ab} = 0$) for complex D, in the limit $D \to 4$ one gets a contribution from the second term in brackets, which violates gauge invariance (though the infinite piece is gauge invariant). This is related to the fact

$$\lim_{D\to 4}\delta^{ab}\frac{1}{\epsilon}\hat{\hat{\delta}}_{ab} = 1 \neq \delta^{ab}\lim_{D\to 4}\frac{1}{\epsilon}\hat{\hat{\delta}}_{ab} = 0$$

(since $\hat{\hat{\delta}}_{ab}$ must $\to 0$ as $D \to 4$). The problem with the third transformation in (1) might be helped by staying in its Landau gauge $\Delta\cdot H = 0$ (which is, in any case, a preferable gauge in terms of the simplifications it brings to the supergravity action [12]). Then the result (5) would be proportional to the linearized part [16] of the superconformal action $\int \mathrm{d}^D x\,\mathrm{d}^2\theta\,\phi^3 W^{\alpha\beta\gamma} \times W^{\alpha\beta\gamma}$ [17], since $\Pi_{3/2}$ projects out of the superconformal piece of H^a. It might also be possible to redefine $\Pi_{3/2}$ to absorb the second term in brackets, and correspondingly modify the renormalization subtraction procedure.

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