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A PROPERTY OF ELECTRIC AND MAGNETIC FLUX IN NON-ABELIAN GAUGE THEORIES

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Pure non-Abelian gauge models with gauge group SU(N) are considered in a box with periodic boundary conditions at various temperatures β^{-1}. Electric and magnetic flux are defined in a gauge-invariant way. The free energy of the system satisfies an exact duality equation, following from Euclidean invariance. The equation relates properties of the electric and the magnetic fields. Conclusions that can be drawn for instance are that for $N \leqslant 3$ one cannot have both electric and magnetic confinement, and that the infrared structure of the Georgi–Glashow model is self-dual.

1. Introduction

The forces between quarks in a hadron are most likely described by a non-Abelian gauge theory without scalar fields. No precise perturbative schemes are known to compute mass spectra and scattering matrix elements in this theory. Nevertheless it is understood [1, 2] that non-Abelian local gauge symmetry can be realized in Nature in several ways: either some scalar field combination undergoes an explicit or dynamical Higgs phenomenon causing the vector bosons to become massive and quarks to become "liberated", or a disordered phase causes permanent "confinement" of quarks and absence of any reminiscence of gauge (color) symmetry. As is stressed by Mandelstam [3], local gauge invariance is not a symmetry in Hilbert space such as the usual global symmetries. Hilbert space can be set up entirely using only gauge-invariant operators acting on the vacuum. This is why local gauge invariance is sometimes obscured in the long-distance structure of a theory, and why neither of the two modes mentioned above (Higgs *versus* confinement) necessarily contains massless particles.

The confinement mode on the one hand and the complete Higgs "symmetry-breakdown" mode on the other hand are found to be dual to each other in the sense of electric-magnetic duality or Kramers–Wannier [4, 5] duality.

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However, there are also self-dual modes such as the "Georgi–Glashow mode" (to be discussed in sect. 9) and also the critical point between Higgs and confinement. As argued in ref. [1], such modes must contain massless particles.

Here we consider pure SU(N) gauge theories in four dimensions. For any closed curve C in 3-dimensional space we have operators A(C) and B(C) satisfying the commutation rule [1]

$$A(\mathrm{C})B(\mathrm{C}') = B(\mathrm{C}')A(\mathrm{C}) \exp((2\pi i n/N), \tag{1.1}$$

where n is the number of times the curve C′ winds around C in a certain direction.

The operator

$$A(\mathrm{C}) = \frac{1}{N} \mathrm{Tr}\, \mathrm{P} \exp \oint_{\mathrm{C}} igA_k(\boldsymbol{x})\, \mathrm{d}x^k \equiv \mathrm{e}^{i\Phi_\mathrm{B}}, \tag{1.2}$$

measures, in a certain sense, the total magnetic flux going through C. Here P is the symbol for path ordering along the curve, and A_k are the space components of the vector field in matrix notation. In fact we might consider Φ_B as a definition of magnetic flux*. In the Higgs mode, Φ_B tends to be quantized in units of $2\pi/N$, whereas it is only defined modulo 2π. On the other hand, A(C) can also be considered to be the creation operator of a "bare" (i.e., infinitely narrow) electric flux line along the curve C.

The operator B(C), defined in ref. [1], satisfies formally

$$(B(\mathrm{C}))^N = 1, \tag{1.3}$$

and we can set $B(\mathrm{C}) = \mathrm{e}^{i\Phi_\mathrm{E}}$ where Φ_E is the electric flux going through C. It is quantized also in units $2\pi/N$ and also defined modulo 2π. Conversely, B(C) is the creation operator of a "bare" magnetic flux line along C.

We notice a striking but not complete resemblance between the A- and B-type operators. Mandelstam [3] also attempts to write B(C) in terms of an electric non-Abelian vector potential $B^a_\mu(x)$. We will not need such a potential for our considerations.

The definition of A and B is gauge invariant so our concepts of quantized electric and magnetic flux also have a gauge-invariant interpretation. This is in contrast with the usual definition of the electromagnetic fields in terms of the covariant curls $G^a_{\mu\nu}$ which are not quantized, but not gauge-invariant either. Note however, that B(C) is only uniquely defined in theories *without* quarks or any other particles whose fields are not invariant under the center Z(N) of SU(N). This is because such particles have gauge-invariant electric charge corresponding to a total flux $\Phi_\mathrm{E} = 2\pi/N$ and thus spoil electric flux conservation.

* However, it must be borne in mind that, defined this way, flux is a not strictly additive quantity. Later, another definition will be given.

It is the purpose of this paper to study some properties of pure gauge theories before quarks are added to them. It is generally believed [6] that in such theories electric flux lines behave as unbreakable strings with universal thickness and a universal tension force (to be fitted with the experimentally measured value of 14 tons). We will investigate however the various possibilities that may arise in general in pure SU(N) gauge theories. In particular we will show that the energy of an electric flux is related to that of a magnetic flux by a dually symmetric formula. We will spell out in detail why, in the case of $N \leq 3$, only electric flux lines *or* magnetic flux lines, but not both, may behave as quantized Nambu strings [7].

One can imagine some internal parameters in a theory which we can vary. Then a transition point between the two modes discussed above might be found. Our dual equation (6.3) will then require long-range interactions associated with massless particles. But the transition can also take place *via* a mode that has the long-distance structure of the Georgi–Glashow model [8], a possibility not considered in ref. [1]. This model is characterized by ordinary photons, electrically charged bosons and magnetically charged [9] particles, and satisfies eq. (6.3) in a dually symmetric way. In that case there will be at least two critical points.

We consider a rectangular box with sides a_1, a_2, a_3 and with periodic boundary conditions. In the box is a pure SU(N) gauge system at a certain temperature. As is well-known, field theories at finite temperature $1/\beta$ can be regarded as statistical systems in a space with Euclidean metric, bounded by periodic boundary conditions in the imaginary time direction [10], with period equal to β. If we set $\beta = a_4$ then we have a box in Euclidean 4-dimensional space with sides a_μ and periodic boundary conditions in all four directions. We will then modify the boundary conditions such that we have a certain number of electric and magnetic flux quanta going in various directions through the box and consider the free energy as a_k and $\beta \to \infty$. It will turn out that the total free energy at temperature $1/\beta$ of a system with given electric and magnetic flux configuration can be expressed in terms of functional integrals over twisted bundles of gauge potentials in the box, and our relation between the electric and the magnetic energy is obtained *via* a rotation over 90° in Euclidean space. The relationship is symmetric and, as we said, rules out simultaneous electric and magnetic string formation (for $N \leq 3$). Not only will we reobtain the result of ref. [1], but also a quantitatively more precise result: the energy of magnetic flux in QCD drops exponentially with the area through which it goes, and the exponent is expressed in terms of the string constant (sect. 8). The choice of gauge in the Euclidean box must be done with some care. We elaborate on that in the appendix.

2. The twisted gauge field

If all fields are invariant under the center elements of the gauge group then we actually have an SU(N)/Z(N) theory. For such a theory, when put in a box, various different classes of periodic boundary conditions may be considered. Let us first concentrate on the 1,2 direction explicitly. The most general periodic boundary condition is

$$A_\mu(a_1, x_2) = \Omega_1(x_2) A_\mu(0, x_2)\,, \tag{2.1a}$$

$$A_\mu(x_1, a_2) = \Omega_2(x_1) A_\mu(x_1, 0)\,, \tag{2.1b}$$

where $A_\mu(x_1, x_2)$ is the vector potential in matrix notation and $\Omega_{1,2}$ are gauge rotations. Here ΩA_μ is short for $\Omega A_\mu \Omega^{-1} + (1/gi)\Omega\partial_\mu\Omega^{-1}$. How to perform the functional integrations over the values of $A_\mu(x)$ and $\Omega_{1,2}$ is explained in the appendix. To get no contradiction in the corners we must have

$$\Omega_1(a_2)\Omega_2(0) = \Omega_2(a_1)\Omega_1(0)Z\,, \tag{2.2}$$

where Z is an element of the centre Z(N) of SU(N). It is possible to find a gauge rotation $\Omega(x_1, x_2)$ such that

$$\Omega(x_1, 0) = I\,, \qquad \Omega(x_1, a_2) = \Omega_2(x_1)\,,$$

and $\Omega(x_1, x_2)$ must be continuous and differentiable on the rectangle, but not necessarily periodic. Then the transformation

$$A \to \Omega(x_1, x_2)^{-1} A\,,$$

brings the boundary conditions (2.1) into

$$A_\mu(a_1, x_2) = \Omega(x_2) A_\mu(0, x_2)\,, \tag{2.3a}$$

$$A_\mu(x_1, a_2) = A_\mu(x_1, 0)\,, \tag{2.3b}$$

where $\Omega(x_2)$ is a new gauge rotation satisfying

$$\Omega(a_2) = \Omega(0) Z\,. \tag{2.4}$$

Here Z is the same as in eq. (2.2). Further transformations on $\Omega(x_2)$ are possible, but since Z is an element of a discrete class there are no transformations that remove Z from expression (2.4). We conclude that there exist N non-gauge equivalent choices for the periodic boundary conditions on the gauge vector potential, when considered on a two-dimensional plane (torus).

This observation can be seen to hold independently for all pairs (μ, ν) of directions in (Euclidean) 4-space. Therefore, all together, there are N^6 distinct non-gauge equivalent choices for the boundary conditions [11]. We can label these by giving the six integers $n_{\mu\nu}$ for $\mu \neq \nu$; $0 \leq n_{\mu\nu} < N$. For later use, we define

$$n_{4i} = n_i, \qquad (i = 1, 2, 3)\,,$$

$$n_{ij} = m_k, \qquad (i, j, k \text{ even permutation of } 1, 2, 3)\,. \tag{2.5}$$

Functional integrals will be performed under these twisted boundary conditions:

$$W\{\boldsymbol{n}, \boldsymbol{m};\, a_\mu\} = C \int_{\{\boldsymbol{n},\boldsymbol{m}\}} \mathrm{D}A \exp S(A)\,, \tag{2.6}$$

where $\int_{\{\boldsymbol{n},\boldsymbol{m}\}} \mathrm{D}A$ stands for integration only over those fields A that are twisted according to the integers n_i, m_i. It is important that the divergent renormalization effects are independent of the choice made at the boundaries. Therefore C is a common normalization factor, independent of n_i and m_i. Clearly $W\{\boldsymbol{n}, \boldsymbol{m};\, a\}/W\{\boldsymbol{0}, \boldsymbol{0};\, a\}$ are relevant quantities, dependent on the sizes a_μ of the box. We will study these functions.

One thing will be obvious here: W must be invariant under those simultaneous permutations of $n_{\mu\nu}$ and a_μ that correspond to orthogonal rotations in Euclidean 4-space. Nevertheless, identities between W-functions thus obtained will have non-trivial consequences, as we shall see. It is important to note here that also the gauge-fixing procedure can be made Euclidean invariant (see appendix).

3. Magnetic flux in a box

Let us formally consider quantization in the $A_0 = 0$ gauge. Then the magnetic field operators commute with the vector potentials $A_i(\boldsymbol{x})$. Therefore, all classical field configurations carry a well-specified amount of magnetic flux. At first sight it might seem to be a good idea to use the operator $A(\mathrm{C})$ as defined in sect. 1 to define total magnetic flux in a certain direction in the box, by choosing C to be a loop orthogonal to that direction. However, the quantity Φ_B obtained this way is not strictly additive (except when C runs in a certain type of vacuum) and therefore the periodic boundary conditions will not guarantee its being conserved. Indeed, Φ_B would not be conserved.

A more convenient way to define magnetic flux in the 3-direction of the box can be found by first considering some curve C defined by

$$\mathrm{C} = \{\boldsymbol{x}(\sigma)\}, \qquad 0 \leq \sigma < 1\,,$$

$$\boldsymbol{x}(0) = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}, \qquad \boldsymbol{x}(1) = \begin{pmatrix} x_1 \\ x_2 \\ a_3 \end{pmatrix}. \tag{3.1}$$

The operator $B(\mathrm{C})$ as defined in ref. [1] then creates one unit of flux in the 3-direction. Nearly everywhere $B(\mathrm{C})$ is a pure gauge rotation $\Omega(\boldsymbol{x})$, singular for $\boldsymbol{x} \in \mathrm{C}$. But $\Omega(\boldsymbol{x})$ is prescribed to make a jump by a factor $\mathrm{e}^{2\pi i/N}$ at a certain angle with respect to the curve C. This cut in Ω could be located for instance at all points

$$\{\boldsymbol{x}(\sigma) + \lambda \boldsymbol{a}_1\,,\ \lambda > 0\}\,. \tag{3.2}$$

What distinguishes B(C) from a pure gauge rotation is that in the transformation law for $A_i(\boldsymbol{x})$, the (singular) derivative across this cut is replaced by zero. In other words: the "string" (3.2) is unphysical, and can be gauge transformed to other positions. The new field $A_i(\boldsymbol{x})$, after the operation B(C), can be made to satisfy the same boundary conditions (2.3) and (2.4), but Z in (2.4) has jumped by one unit. Clearly, the integer m_3, as defined in (2.5) counts how many times an operator such as B(C) has acted. In other words, m_3 counts a conserved variety of magnetic flux in the 3-direction. In general, $(m_1, m_2, m_3) = \boldsymbol{m}$ will be considered to be the (integer valued) magnetic flux vector. It is the direct analogue of the ordinary (continuously valued) magnetic flux for the Abelian case.

4. Electric flux in a box

Electric field operators do not commute with the vector potentials. Therefore we must examine the quantized theory (in the $A_0 = 0$ gauge) before establishing the concept of electric flux.

The gauge restriction $A_0 = 0$ leaves invariance under time-independent gauge rotations $\Omega(\boldsymbol{x})$. States in Hilbert space must then be representations of this invariance group. For all those $\Omega(\boldsymbol{x})$ which can be continuously and uniformly connected to the identity element we must choose the trivial representation (see, also, the appendix):

$$\{\Omega(\boldsymbol{x})\}|\psi\rangle = |\psi\rangle\,. \tag{4.1}$$

This is necessary if we want a theory that approaches the usual theory when $a_i \to \infty$.

But there are other homotopy classes of $\Omega(\boldsymbol{x})$, which cannot be deformed continuously towards the identity. First there is the familiar 2nd Chern class of mappings $\Omega(\boldsymbol{x})$ that are related to instanton effects [12] and have representations described by an arbitrary angle θ. These cannot be directly related to electric or magnetic flux. Now the gauge transformations in a box with periodic boundary conditions (in 3-space) form other homotopy classes besides those that gave the θ vacuum. Consider namely an $\Omega(\boldsymbol{x})$ with the properties

$$\Omega\begin{pmatrix}a_1\\x_2\\x_3\end{pmatrix} = \Omega\begin{pmatrix}0\\x_2\\x_3\end{pmatrix} Z_1\,, \qquad \Omega\begin{pmatrix}x_1\\a_2\\x_3\end{pmatrix} = \Omega\begin{pmatrix}x_1\\0\\x_3\end{pmatrix} Z_2\,,$$

$$\Omega\begin{pmatrix}x_1\\x_2\\a_3\end{pmatrix} = \Omega\begin{pmatrix}x_1\\x_2\\0\end{pmatrix} Z_3\,, \tag{4.2}$$

where at least one of the $Z_{1,2,3}$ is a non-trivial element of the center of SU(N). This $\Omega(\boldsymbol{x})$ is a gauge rotation that leaves the boundary conditions (2.3) and (2.4) invariant. Any choice of $Z_{1,2,3}$ forms a distinct homotopy class, characterized by three integers (k_1, k_2, k_3). Since they are an invariance of the Hamiltonian, we must have

$$\Omega|\psi\rangle = \mathrm{e}^{i\omega(k_1,k_2,k_3)}|\psi\rangle\,. \tag{4.3}$$

These angles ω need not vanish, but we do have, if we indicate elements of the homotopy class (k_1, k_2, k_3) by $\Omega[\boldsymbol{k}]$:

$$\Omega[\boldsymbol{k}_1]\Omega[\boldsymbol{k}_2] = \Omega[\boldsymbol{k}_1 + \boldsymbol{k}_2(\mathrm{mod}\,N)]\,, \tag{4.4}$$

and $(\Omega[\boldsymbol{k}])^N$ is homotopically equivalent with the identity. Therefore, ω must satisfy

$$\omega(\boldsymbol{k}) = \frac{2\pi}{N}\sum_i e_i k_i\,, \tag{4.5}$$

where $e_{1,2,3}$ are again three integers, defined modulo N. They represent new conserved quantities in the system.

Let us again consider a curve C in the 3-direction, as defined in eq. (3.1). Now we construct the operator $A(\mathrm{C})$:

$$A(\mathrm{C}) = \frac{1}{N}\,\mathrm{Tr}\,\mathrm{P}\exp ig\int_0^1 \mathrm{d}\sigma\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\sigma}\cdot\boldsymbol{A}(\boldsymbol{x}(\sigma))\,, \tag{4.6}$$

and consider

$$|\psi'\rangle = A(\mathrm{C})|\psi\rangle\,. \tag{4.7}$$

The quantity $A(\mathrm{C})$ is not invariant under $\Omega[k_1, k_2, k_3]$ if $k_3 \neq 0$:

$$A(\mathrm{C}) \to \frac{1}{N}\,\mathrm{Tr}\,\Omega(\boldsymbol{x}(0))\{\mathrm{P}\exp\ldots\}\Omega^{-1}(\boldsymbol{x}(1)) = \mathrm{e}^{-2\pi i k_3/N}A(\mathrm{C})\,. \tag{4.8}$$

Therefore,

$$A(\mathrm{C})\Omega[\boldsymbol{k}]|\psi\rangle = \Omega[\boldsymbol{k}]\,\mathrm{e}^{-2\pi i k_3/N}A(\mathrm{C})|\psi\rangle\,. \tag{4.9}$$

If $|\psi\rangle$ satisfies eq. (4.3), then

$$\Omega[\boldsymbol{k}]|\psi'\rangle = \mathrm{e}^{i\omega(\boldsymbol{k})+2\pi i k_3/N}|\psi'\rangle\,, \tag{4.10}$$

so that $|\psi'\rangle$ has e_3 replaced by e_3+1. Clearly, the integers e_i count how many times an operator of the form $A(\mathrm{C})$ acted in the various directions. Now $A(\mathrm{C})$ can be considered to be the creation operator of an electric flux line [1, 2, 6]. The conserved integers e_i therefore indicate total amount of electric flux in the three directions. Again this definition of electrix flux corresponds to the usual one in an Abelian model, apart from normalization (the total flux of a quark is normalized to be equal to one).

5. The functional integral for a fixed flux configuration

As one can conclude from the previous sections, a state $|\psi\rangle$ can be restricted to have a fixed magnetic flux (m_1, m_2, m_3) and a fixed electric flux (e_1, e_2, e_3). These six restrictions do not interfere with each other. We now wish to consider the vacuum functional integral with a given fixed $(\boldsymbol{m}, \boldsymbol{e})$ configuration.

It will be clear from the preceding that the integers m_i in eq. (2.5) must be chosen to have the values of the required magnetic flux. How do we fix the electric flux? We wish to compute the free energy F defined by

$$\mathrm{e}^{-\beta F} = \mathrm{Tr}\, P(\boldsymbol{e}, \boldsymbol{m})\, \mathrm{e}^{-\beta H}, \tag{5.1}$$

where H is the Hamiltonian of the theory and P is a projection operator that selects the required electric flux $\boldsymbol{e}$ and magnetic flux $\boldsymbol{m}$. We saw how to select the magnetic flux $\boldsymbol{m}$. The electric flux $\boldsymbol{e}$ is selected by using the operators $\Omega[\boldsymbol{k}]$ defined in sect. 4. Since $|\psi\rangle$ must satisfy eqs. (4.3) and (4.5) we may put

$$P(\boldsymbol{e}, \boldsymbol{m}) = \frac{1}{N^3} \sum_{\boldsymbol{k}} \mathrm{e}^{-2\pi i(\boldsymbol{k}\cdot\boldsymbol{e})/N}\, \Omega[\boldsymbol{k}]\,. \tag{5.2}$$

Therefore

$$\mathrm{e}^{-\beta F(\boldsymbol{e}, \boldsymbol{m}; \boldsymbol{a}, \beta)} = \frac{1}{N^3} \sum_{\boldsymbol{k}} \mathrm{e}^{-2\pi i(\boldsymbol{k}\cdot\boldsymbol{e})/N}\, \mathrm{Tr}\, \Omega[\boldsymbol{k}]\, \mathrm{e}^{-\beta H}, \tag{5.3}$$

The operator $\mathrm{e}^{-\beta H}$ is the evolution operator in Euclidean space over a distance β in the 4-direction. Taking the trace in Hilbert space implies a periodicity condition in the 4-direction [10]. But the operator $\Omega[\boldsymbol{k}]$ is also inserted. It implies a twist in the periodicity. Indeed, as can be read off immediately from eq. (4.2), we have, for each choice of $\boldsymbol{k}$, a boundary condition such as eqs. (2.3) and (2.4) both with the directions 1, 2 replaced by 1, 4 or 2, 4 or 3, 4. Thus we find

$$\mathrm{e}^{-\beta F(\boldsymbol{e}, \boldsymbol{m}; \boldsymbol{a}, \beta)} = \frac{1}{N^3} \sum_{\boldsymbol{k}} \mathrm{e}^{-2\pi i(\boldsymbol{k}\cdot\boldsymbol{e})/N}\, W\{\boldsymbol{k}, \boldsymbol{m}; a_\mu\}, \tag{5.4}$$

with $a_4 = \beta$, and W defined as in eq. (2.6).

The author has actually made an attempt to find a formulation of the Feynman rules to compute W perturbatively. But in particular $W\{0, 0; a_\mu\}$, needed for normalization, contains a quite complicated multidimensional integral, even at lowest order, that so far kept him from doing explicit numerical calculations.

In sect. 6 we will see that even without doing any numerical calculations, some conclusions can be drawn from Euclidean symmetry on W.

In the limit $\beta \to \infty$ (or $T \to 0$), F becomes the energy of the lowest state with the given flux configuration. We are interested in the behaviour of this energy as a_i become large.

6. Duality

Clearly, W will be invariant under joint rotations of a_μ and $n_{\mu\nu}$ in Euclidean space. In particular, let us perform the SO(4) rotation as given by the matrix

$$\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix} \tag{6.1}$$

(keeping in mind that $k_i = n_{4i}$ and $n_{ij} = \varepsilon_{ijk} m_k$). Let the first two components of a vector $\boldsymbol{x}$ be denoted by $\tilde{x}$, and let $\hat{\tilde{a}}$ be $\tilde{a}$ with its two components interchanged. Then we find

$$W\{\tilde{k}, k_3, \tilde{m}, m_3; \tilde{a}, a_3, \beta\} = W\{\tilde{m}, k_3, \tilde{k}, m_3; \hat{\tilde{a}}, \beta, a_3\}. \tag{6.2}$$

The consequence of this is

$$\exp\{-\beta F(\tilde{e}, e_3, \tilde{m}, m_3; \tilde{a}, a_3, \beta)\}$$
$$= N^{-2} \sum_{\tilde{k}, \tilde{l}} \exp\left\{\frac{2\pi i}{N}[-(\tilde{k}\cdot\tilde{e}) + (\tilde{l}\cdot\tilde{m})] - a_3 F(\tilde{l}, e_3, \tilde{k}, m_3; \hat{\tilde{a}}, \beta, a_3)\right\}. \tag{6.3}$$

Here N^{-2} normalizes the Fourier transforms. Notice the complete "dual" symmetry under interchange of all $\boldsymbol{e}$ with $\boldsymbol{m}$ and *vice versa* in eq. (6.3). Right- and left-hand side of eq. (6.3) differ by a Fourier transformation and dual interchange with respect to only $\tilde{e}$ and $\tilde{m}$. Later, we will frequently put e_3 and m_3 equal to zero. Eq. (6.3) will be referred to as the "duality equation". It must be stressed that so far no approximation has been made. Our duality equation (6.3) for pure non-Abelian gauge theories is exact.

7. Condensation

7.1. Light fluxes

We now may ask which asymptotic structure of F as $\boldsymbol{a}$ and β tend to infinity is compatible with duality, eq. (6.3). First let us assume that no massless physical particles occur (what may happen in the presence of massless particles, which is the most difficult case, is briefly discussed in sect. 9). The asymptotic region (for large a_μ) will then be approached exponentially. We immediately notice that not all F are allowed to tend to zero. Then, namely, βF would tend to zero too and contradiction arises with eq. (6.3). If a_3 is sufficiently large then the major contributions to the sum in eq. (6.3) come from those values of $\tilde{l}$ and $\tilde{k}$ for which F vanishes. One can then easily deduce from eq. (6.3) that of all N^4 values of $(\tilde{e}, \tilde{m})$

exactly N^2 combinations must give vanishing F. Let us call these the "light" fluxes. The others send F to infinity. These we call "heavy" fluxes. And for any pair $(\boldsymbol{e}_{(1)}, \boldsymbol{m}_{(1)}), (\boldsymbol{e}_{(2)}, \boldsymbol{m}_{(2)})$ of light fluxes with $e_3(1) = e_3(2)$ and $m_3(1) = m_3(2)$, we must have:

$$\tilde{e}_{(1)} \cdot \tilde{m}_{(2)} = \tilde{e}_{(2)} \cdot \tilde{m}_{(1)}, \qquad (\text{modulo } N) . \tag{7.1}$$

The number N^2 is necessary to cancel N^{-2} in eq. (6.3), and the condition (7.1) is necessary to cancel the imaginary part of the exponent in eq. (6.3). A further restriction follows from the requirement that W in eq. (5.4) must be positive: if $(\boldsymbol{e}, \boldsymbol{m})$ is a light flux, then $(0, \boldsymbol{m})$ must also be a light flux. This excludes some exotic solutions of eq. (7.1). In SU(2) and SU(3) it follows that either all electric or all magnetic fluxes are light. In SU(4) there is a third possibility: it could be that only the even electric and even magnetic fluxes are light, and so on.

7.2. *Heavy fluxes*

For all other values of $(\boldsymbol{e}, \boldsymbol{m})$ the free energy F must not tend to zero as $\boldsymbol{a}, \beta$ become large. This is different from the Abelian case, where the energy of any given flux, say in the x direction, behaves as

$$E \to \frac{C a_1}{a_2 a_3}, \tag{7.2}$$

where C is a constant. Thus, unlike Abelian fluxes, the heavy fluxes cannot spread out in space and produce a lower and lower energy density as the box becomes large. Physical intuition then tells us that the only possible alternative is string formation: the fluxes form narrow flux tubes with constant energy per unit of length. If the Higgs mode is realized then these heavy fluxes are the magnetic ones, known as Nielsen–Olesen flux tubes [7, 13]. But if the magnetic fluxes are the light ones then the heavy electric fluxes behave like strings as is argued in the literature [1–3, 6]. We now see how our duality equation, (6.3), forces us to accept the possibility of such an electric string mode, and tells us that the transition towards such a string mode from the Higgs mode must be through one or more phase transition(s) with massless particles at the critical point(s) [1].

From now on we will assume that the magnetic fluxes are light and the electric ones heavy (confinement mode) unless stated otherwise (the other case can always be obtained by the trivial replacement $e \leftrightarrow m$, and we will not discuss any further the exceptional situations possible in SU(4) and larger groups).

Thus we now have a clear idea about the energy of the heavy fluxes: at $\beta \to \infty$ we must have

$$F(\beta) \to E \to \rho a_1 , \tag{7.3}$$

if the flux is in the x-direction. Here E is the free energy at zero temperature and ρ is the string constant. In sect. 8 we will derive the precise asymptotic form of the energy of a (light) magnetic flux. We need, however, one more piece of information.

7.3. Factorization

We will assume absence of interference between electric and magnetic fluxes in the limit $a_\mu \to \infty$. Physically this is quite acceptable. Not only does this hold for Abelian fields; it holds as soon as we assume that strings occupy only a negligible portion of total space whereas magnetic fields fill the whole space. We will refer to this property as "factorization":

$$F(\boldsymbol{e}, \boldsymbol{m}; a_\mu) \Rightarrow F_e(\boldsymbol{e}; a_\mu) + F_m(\boldsymbol{m}; a_\mu)\,. \tag{7.4}$$

Note that $F_e(\boldsymbol{e}; a_\mu)$ will not always factorize with respect to the three components of $\boldsymbol{e}$. In the case of a square box, $\boldsymbol{e} = (1, 1, 0)$ will correspond to a string running in the diagonal direction, so for sufficiently large β:

$$\begin{aligned} F_e(1, 1, 0; a_\mu) &\to \sqrt{2} F_e(1, 0, 0; a_\mu) \\ &\to \sqrt{2} F_e(0, 1, 0; a_\mu)\,. \end{aligned} \tag{7.5}$$

Factorization can only hold for $\beta \gg a_i$ otherwise contradictions would arise between eqs. (6.3) and (7.5). If $\beta \gg a_{1,2}$ only, then factorization is still possible provided e_3 and m_3 are kept zero. This restriction is physically understandable. It means that when very long electric and magnetic flux lines are forced to stay close together then some interference will occur. If $e_3 = m_3 = 0$, eq. (6.3) implies:

$$\exp\{-\beta F_m(\tilde{m}, 0; \boldsymbol{a}, \beta)\} = N^{-1} \sum_{\tilde{l}} \exp\left\{\frac{2\pi i}{N}(\tilde{l} \cdot \tilde{m}) - a_3 F_e(\tilde{l}, 0; \hat{\tilde{a}}, \beta, a_3)\right\}, \tag{7.6}$$

$$\beta, a_3 \gg a_1, a_2\,.$$

8. Computation of the free energy

We would like to know the behavior of the energy of a magnetic field:

$$E_m(\tilde{m}, 0; \boldsymbol{a}) = \lim_{\beta \to \infty} F_m(\tilde{m}, 0; \boldsymbol{a}, \beta) \tag{8.1}$$

(where $\beta \to \infty$ first, a_i is taken large afterwards). We can use eq. (7, 6), provided we know how F_e behaves at finite β, when $a_3 \to \infty$. Expression (7.3) only holds for large β, but we can use it as a starting point.

What is F_e at finite β? Clearly this is a problem of statistical physics since β can be interpreted as an inverse temperature. As $a_3 \to \infty$ our box becomes very large and we must allow for the possibility that thermal oscillations produce more than one string, even if $\boldsymbol{e} = (1, 0, 0)$. Consider fig. 1. For the time being we ignore strings

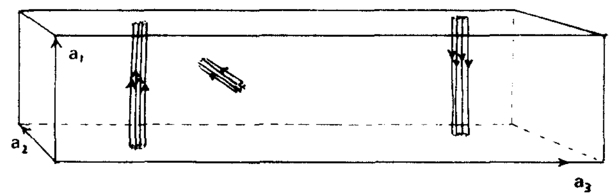

Fig. 1. Elongated box at finite temperature. Several strings may be produced by thermal oscillations. Total flux in the 1- and 2-directions is fixed (modulo N). Flux in the 3-direction is chosen to be zero.

that run diagonally. Later in this section it will be confirmed that they indeed may be neglected.

So let the total electric flux $\boldsymbol{e}$ be fixed to be $(e_1, e_2, 0)$. The Boltzmann factor for one string in the 1-direction is $\mathrm{e}^{-\beta\rho a_1}$. It may pierce the 2-3 surface at any point. Therefore, the contribution of a single 1-string to the partition sum would be

$$\lambda a_2 a_3 \,\mathrm{e}^{-\beta\rho a_1} \equiv \gamma_1 \,, \tag{8.2}$$

where λ is some elementary constant, ρ is the string constant. If two strings go in the positive 1-direction then their contribution is

$$\frac{1}{2!}\gamma_1^2 \,, \tag{8.3}$$

(a combinatorial factor for interchange symmetry is included) and so on. If β is large enough then the strings will be far apart on the average, and interactions may be neglected. If the group parameter $N > 2$ then the two string orientations (up and down) must be distinguished.

To get the total partition sum we must add all possible multi-string configurations in the 1 and 2 directions with the restriction that total flux, modulo N, is given by (e_1, e_2):

$$\begin{aligned} &\exp\left[-\beta(F_e(e_1, e_2; \boldsymbol{a}, \beta) + C(\boldsymbol{a}, \beta))\right] \\ &\quad = \sum_{n_1^\pm, n_2^\pm} \frac{1}{(n_1^+)!\,(n_1^-)!\,(n_2^+)!\,(n_2^-)!} \gamma_1^{n_1^+ + n_1^-} \gamma_2^{n_2^+ + n_2^-} \\ &\quad \times \delta_N(n_1^+ - n_1^- - e_1)\,\delta_N(n_2^+ - n_2^- - e_2) \,, \end{aligned} \tag{8.4}$$

where $C(\boldsymbol{a}, \beta)$ is an as yet to be determined normalization term. Here we took $N > 2$. If $N = 2$ we must put $n_{1,2}^- = 0$. The functions $\delta_N(x)$ are defined to be one, if x is a multiple of N, and zero otherwise. We have

$$\delta_N(x) = \frac{1}{N}\sum_{k=0}^{N-1} \mathrm{e}^{2\pi i k x/N} \,. \tag{8.5}$$

Therefore, if $N>2$:

$$e^{-\beta(F_e+C)}=N^{-2}\sum_{\tilde{k}}\exp\left(\sum_{a=1}^{2}\left(2\gamma_a\cos\frac{2\pi k_a}{N}\right)-2\pi i\tilde{k}\cdot\tilde{e}/N\right), \tag{8.6}$$

and if $N=2$:

$$e^{-\beta(F_e+C)}=\tfrac{1}{4}\sum_{\tilde{k}}(-1)^{\tilde{k}\cdot\tilde{e}}\exp\left(\sum_{a=1}^{2}\gamma_a(-1)^{k_a}\right). \tag{8.7}$$

Usually, C will be adapted so that $F(0;\boldsymbol{a},\beta)=0$. The asymptotic behavior for large a_3 can be read off from eqs. (8.2) and (8.6), (8.7). By some remarkable accident, eqs. (8.6) and (8.7) have again the form of a Fourier transform so they are easy to insert into the duality equation (7.6) in order to obtain F_m. We find:

$$\exp\left[-\beta(F_m(\tilde{m},0;\boldsymbol{a},\beta)+C(\boldsymbol{a},\beta))\right]=N^{-1}\exp\left[(2-\delta_{N2})\sum_a\hat{\gamma}_a\cos\frac{2\pi m_a}{N}\right], \tag{8.8}$$

with

$$\hat{\gamma}_1=\lambda a_1\beta\, e^{-\rho a_2 a_3}, \qquad \hat{\gamma}_2=\lambda a_2\beta\, e^{-\rho a_1 a_3}. \tag{8.9}$$

Now we can take the limit $\beta\to\infty$ to obtain the energy of a magnetic flux:

$$e^{-\beta F_m(\tilde{m},0;\boldsymbol{a},\beta)}\to e^{-\beta E_m(\tilde{m},0;\boldsymbol{a})}, \tag{8.10}$$

with

$$E_m(\tilde{m},0;\boldsymbol{a})=\sum_i E_i(m_i,\boldsymbol{a}), \tag{8.11a}$$

$$E_1(m_1,\boldsymbol{a})=R(m_1)a_1\, e^{-\rho a_2 a_3}, \tag{8.11b}$$

$$E_2(m_2,\boldsymbol{a})=R(m_2)a_2\, e^{-\rho a_1 a_3}, \tag{8.11c}$$

$$R(m)=\lambda(2-\delta_{N2})\left(1-\cos\frac{2\pi m}{N}\right). \tag{8.11d}$$

The 1 in eq. (8.11d) arises from the normalization condition. $C(\boldsymbol{a},\beta)$ in eq. (8.8) must be such that $F(0;\boldsymbol{a},\beta)\to 0$. Of course eq. (8.11) is expected to hold only for sufficiently large a_i.

In deriving the asymptotic behaviour of $E_m(\boldsymbol{m};\boldsymbol{a})$ we have neglected contributions in eq. (8.4) from strings that run diagonally, besides the ones sketched in fig. 1. As promised we can now easily justify that. They namely would have an associated Boltzmann factor

$$\gamma_{12}=\frac{\lambda a_1 a_2 a_3}{\sqrt{a_1^2+a_2^2}}\exp\{-\beta\rho\sqrt{a_1^2+a_2^2}\}, \tag{8.12}$$

etc. In eq. (8.8) this would give extra terms with

$$\hat{\gamma}_{12}=\frac{\lambda a_1 a_2 \beta}{\sqrt{a_1^2+a_2^2}}\exp\{-\rho a_3\sqrt{a_1^2+a_2^2}\}\,, \tag{8.13}$$

etc. And in eq. (8.11) we would get extra terms going like

$$\exp\{-\rho a_3\sqrt{a_1^2+a_2^2}\}\,, \tag{8.14}$$

which decrease faster than eqs. (8.11b, c) and therefore can be neglected. Eqs. (8.11) give the asymptotic behaviour of the energy of magnetic fluxes exactly. Observe that the flux energy is proportional to the length of the flux lines, and decreases exponentially with the *area* through which the flux lines go. Of course, by extrapolation, we expect also

$$E(\boldsymbol{m},\boldsymbol{a})=\sum_i E_i(m_i,\boldsymbol{a})\,, \tag{8.15}$$

for non-vanishing $m_{1,2,3}$ (if one m_i vanishes this follows directly from eq. (8.11a)). Obviously, this behaviour of the magnetic fluxes is quite opposite to that of the confining electric fluxes. The coefficient ρ in the exponent must be precisely the string constant. It multiplies an area, not a distance and therefore cannot directly be linked to the mass of a physical particle. A consequence of this rapidly decreasing energy of magnetic fields is that objects with color-magnetic charge are not confined in Quantum Chromodynamics. To the contrary, they only show short-range interactions. This was conjectured but not proved by several authors [3, 14].

9. The massless particle phase

When massless particles are present the separation between light and heavy fluxes becomes impossible. The dependence on $\boldsymbol{a}$ and β of the flux energies could be quite complicated and we have not yet succeeded in classifying the various possibilities consistent with eq. (6.3). As indicated in ref. [1], a possible phase-transition point between Higgs phase and confinement phase must show massless excitations which one could study. However, there is another realization of eq. (6.3) through massless particles that probably does not correspond to a critical transition point but may occupy a finite region in parameter space. It is when one or more unbroken U(1) groups survive the Higgs mechanism.

Let us consider the case SU(2) with an isospin-one Higgs field [8] leaving as an apparent local symmetry group the subgroup U(1). As we will see this realization which we will refer to as the "Georgi–Glashow mode" is self-dual. Indeed, Montonen and Olive [15] observed a dual resemblance between magnetic monopoles and charged vector particles in this model. Let us, by way of exercise, estimate the free energy of electric and magnetic fluxes in this case.

If all components of $\boldsymbol{a}$ and β are sufficiently large, then only the U(1) Maxwell fields determine the free energy. None of the massive particles will give a noticeable direct contribution to the free energy, because their Boltzmann factors $e^{-\beta m}$ are too small. But they give an indirect contribution, in the following way. By rare thermal fluctuations (or by quantum tunneling) a pair of oppositely charged vector bosons may be created. One member may separate, go through one of the periodic walls (i.e., wind around the torus) and meet its companion from the other side after which they annihilate. We then obtain a Maxwell field configuration in the box where the U(1) electric flux in one direction has increased by 2 units (to keep our original notation, the charged vector bosons have electric charge g = two units, and an elementary doublet would have charges $\pm\frac{1}{2}g = \pm$ one unit). Indeed, since we have $N = 2$ in our example, electric flux was only defined modulo 2. Thus, when we say that the electric flux is one unit in a certain direction, we really have to take into account all possible fluctuations that add an even number (positive or negative) to this flux, together with their Boltzmann factors. Clearly, the same must be done with the magnetic fields, because magnetic monopoles exist in this model and can be pair-created [9].

Since we are dealing with Maxwell fields, factorization (in the sense of sect. 7) holds:

$$F = F_e + F_m, \qquad F_e(\boldsymbol{e}; \boldsymbol{a}, \beta) = \sum_i F_{ei}(e_i; \boldsymbol{a}, \beta),$$
$$F_m(\boldsymbol{m}; \boldsymbol{a}, \beta) = \sum_i F_{mi}(m_i; \boldsymbol{a}, \beta). \tag{9.1}$$

Let us first compute the free energy of an electric flux. If the flux were completely fixed in the U(1) sense, we would have

$$F_{e1}(k; \boldsymbol{a}, \beta) = \frac{g^2 a_1 k^2}{8 a_2 a_3}, \tag{9.2}$$

where g is the gauge coupling constant, and k is the integer that indicates the flux in our natural units. Because of the above explained tunneling phenomenon, in our SU(2) theory we have only $k = 0$ or 1. The other values are reached by thermal excitations. If we normalize

$$F_{e1}(0; \boldsymbol{a}, \beta) = 0, \tag{9.3}$$

then

$$\exp\left[-\beta F_{e1}(1; \boldsymbol{a}, \beta)\right] = \frac{\sum_{k=-\infty}^{\infty} \exp\left[-\beta \frac{g^2 a_1}{2 a_2 a_3}(k+\frac{1}{2})^2\right]}{\sum_{k=-\infty}^{\infty} \exp\left[-\beta \frac{g^2 a_1}{2 a_2 a_3} k^2\right]}, \tag{9.4}$$

and similarly for F_{e2}, F_{e3}.

The magnetic monopoles in the theory have magnetic charge $4\pi/g$, which is two elementary flux units. Therefore,.replacing, in eq. (9.4), g by $4\pi/g$ we find the formula for the magnetic energy:

$$\exp\left[-\beta F_{\mathrm{m}1}(1;\boldsymbol{a},\beta)\right]=\frac{\sum_{k=-\infty}^{\infty}\exp\left[-\beta\frac{8\pi^2 a_1}{g^2 a_2 a_3}(k+\tfrac{1}{2})^2\right]}{\sum_{k=-\infty}^{\infty}\exp\left[-\beta\frac{8\pi^2 a_1}{g^2 a_2 a_3}k^2\right]}. \tag{9.5}$$

Using the formulas

$$\sum_{k=-\infty}^{\infty} \mathrm{e}^{-\lambda k^2}=\sqrt{\{\pi/\lambda\}}\sum_{k=-\infty}^{\infty}\mathrm{e}^{-\pi^2k^2/\lambda}, \tag{9.6a}$$

and

$$\sum_{k=-\infty}^{\infty}(-1)^k\,\mathrm{e}^{-\lambda k^2}=\sqrt{\{\pi/\lambda\}}\sum_{k=-\infty}^{\infty}\mathrm{e}^{-\pi^2(k+\frac{1}{2})^2/\lambda}, \tag{9.6b}$$

we find that eq. (7.6) is satisfied, up to an irrelevant normalization factor.

When g is varied, eqs. (9.4) and (9.5) go over into each other continuously. This is why we say that this phase is self-dual. The fact that in this "Georgi–Glashow phase" eq. (6.3) is realized in a self-dual way is in our opinion a non-trivial observation.

10. The twisted functional integral

We now know how to compute the energy of electric and magnetic fluxes, if one knows the functional integrals

$$W\{n_{\mu\nu};a_\mu\}=C\int_{\{n_{\mu\nu}\}}\mathrm{D}A\,\exp S(A) \tag{10.1}$$

in a Euclidean box with sides a_μ and twisted boundary conditions given by the integers $n_{\mu\nu}$. So we ask: what is the asymptotic form of W in the various phases (Higgs/Confinement/Georgi-Glashow)? To be specific we consider the case $N=3$ and

$$W\equiv W\{n_{12}=1,\mathrm{rest}=0;a_\mu\}/W\{0;a_\mu\}. \tag{10.2}$$

When all a_μ are larger than the "hadronic" mass scale we put

$$a_1a_2=\Sigma_1\,,\qquad a_3a_4=\Sigma_2\,, \tag{10.3}$$

We find that W essentially depends only on Σ_1 and Σ_2.

(*a*) *The Higgs phase.* When the Higgs mechanism removes the local symmetry completely in the usual way then the magnetic flux quantizes into Nielsen–Olesen tubes. Eq. (8.6) holds if F_e is replaced by F_m and $N=3$. Since factorization is assumed and all n_{4i} are kept zero, we find

$$W=e^{-\beta(F_m(m_3=1)-F_m(0))}=\frac{1-e^{-3\gamma}}{1+2e^{-3\gamma}}; \tag{10.4}$$

$$\gamma=\lambda\Sigma_1\,e^{-\rho\Sigma_2}, \tag{10.5}$$

where λ and ρ are defined in sect. 8. The latter is the string constant for the Nielsen–Olesen string.

(*b*) *The confinement phase.* The absolute confinement phase is described by eqs. (8.6)–(8.9) directly. We find

$$W=e^{-\lambda\Sigma_2}\exp[-\rho\Sigma_1] \tag{10.6}$$

(*c*) *The Georgi–Glashow phase.* We take the Georgi-Glashow phase as an example of a self-dual phase with massless particles. Eq. (9.5) applies to the case $N=2$. We can easily* extend it to $N=3$, and we find

$$W=\frac{\sum_{k=-\infty}^{\infty}\exp\{-\lambda(k+\frac{1}{3})^2\Sigma_2/\Sigma_1\}}{\sum_{k=-\infty}^{\infty}\exp\{-\lambda k^2\Sigma_2/\Sigma_1\}}=\frac{\sum_k\exp\left\{-\frac{\pi^2}{\lambda}k^2\Sigma_1/\Sigma_2+\frac{2k\pi i}{3}\right\}}{\sum_k\exp\left\{-\frac{\pi^2}{\lambda}k^2\Sigma_1/\Sigma_2\right\}}. \tag{10.7}$$

Here λ is some charge parameter.

Now let us consider the limit $\Sigma_1\to\infty$ first, then Σ_2 large, for the three cases a, b and c. We find

$$W\to 1-\text{const}\cdot e^{-\Sigma_1 f(\Sigma_2)} \tag{10.8}$$

with in case a:

$$f(\Sigma_2)=3\lambda\,e^{-\rho\Sigma_2}, \tag{10.9}$$

case b:

$$f(\Sigma_2)=\rho, \tag{10.10}$$

case c:

$$f(\Sigma_2)=\pi^2/\lambda\Sigma_2. \tag{10.11}$$

If on the other hand $\Sigma_2\to\infty$ first then

$$W\to\text{const}\cdot e^{-\Sigma_2 g(\Sigma_1)} \tag{10.12}$$

* There are various ways to describe this phase as a partial Higgs phase but the final result in terms of electrically and magnetically charged particles is independent of this description.

with in case a:

$$g(\Sigma_1) = \rho\,, \tag{10.13}$$

case b:

$$g(\Sigma_1) = \lambda\ e^{-\rho\Sigma_1}\,, \tag{10.14}$$

case c:

$$g(\Sigma_1) = \lambda/9\Sigma_1\,. \tag{10.15}$$

In all three cases the convergence towards the limit form is exponential. If W can be computed for reasonably large Σ_1, Σ_2 then it can be found out with some confidence which of the various phases is realized.

11. Conclusion

When this investigation was started it was with a view to finding a scheme for quantitative calculations for the hadron spectrum in QCD. In particular we want to express the string constant ρ in terms of the distance scales set by the renormalized coupling constant. Our starting point was eq. (5.4), in which the quantity W has a perturbation expansion that, term by term, is free of infrared divergences. Borel resummation procedures [16], corrected for instanton effects [17] could perhaps give a fairly reliable result. And then it would be easy to check which of the asymptotic forms of the previous section apply.

However, Euclidean invariance is reduced to rotations over 90° and artifacts due to this mutilation of the continuous Euclidean symmetry turned out to be formidable. (The reader is invited to compute the *zeroth* order term. He will then understand our problem, which is technical, not fundamental.)

On the other hand, qualitative study of eq. (5.4) gave us much insight in the long-distance structure of gauge theories. For instance, we found the proof of a conjecture made several times in the literature; simultaneous electric and magnetic confinement is impossible. If electric confinement is assumed, then the energy of a magnetic flux can be computed and is found to vanish exponentially as the size of the box increases. And we found that the Georgi–Glashow model, in which magnetic monopoles exist, has a long-distance structure which is self-dual. This is a statement for which we needed no such details as spin or mass spectrum of magnetic monopoles and dyons [15]. It follows solely from the gauge group structure of the model.

Finally, we hope that understanding of the asymptotic form for "twisted functional integrals" will be helpful in finding reliable calculational schemes for quantum chromodynamics.

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Appendix

We should formulate precisely the prescription for gauge fixing in the finite Euclidean box. In this appendix we will concentrate on the continuous parts of the gauge group. How to treat the various homotopy classes of gauge field configurations ("twisted fibre bundles") is explained at length in the text.

Consider the gauge condition $A_0 = 0$. We then have an Hamiltonian H depending on $\boldsymbol{A}(\boldsymbol{x})$ and $\pi(\boldsymbol{x})$, the latter being the momenta conjugate to $\boldsymbol{A}(\boldsymbol{x})$. Now H is still invariant under (time independent) gauge transformations

$$\boldsymbol{A}(\boldsymbol{x}) \to \Omega(\boldsymbol{x})\boldsymbol{A}(\boldsymbol{x}), \tag{A.1}$$

where we use the same notation as in eqs. (2.1). In a Hilbert space of all field configurations $\boldsymbol{A}(\boldsymbol{x})$, the Hamiltonian commutes with the gauge rotation Ω defined by

$$\Omega\psi\{\boldsymbol{A}(\boldsymbol{x})\} \equiv \psi\{\Omega^{-1}(\boldsymbol{x})\boldsymbol{A}(\boldsymbol{x})\}. \tag{A.2}$$

So states in Hilbert space may be chosen to be representations of Ω. If Ω is in the same homotopy class as the identity, then we must choose the trivial representation:

$$\Omega|\psi\rangle = |\psi\rangle. \tag{A.3}$$

States that satisfy eq. (A.3) form the physical subspace of the above Hilbert space. To characterize those physical states it is sufficient to specify $\psi\{\boldsymbol{A}(\boldsymbol{x})\}$ for those $\boldsymbol{A}(\boldsymbol{x})$ that satisfy a gauge condition, such as

$$A_3(\boldsymbol{x}) = A_2(x_1, x_2, 0) = A_1(x_1, 0, 0) = 0. \tag{A.4}$$

However, the condition (A.4) is only compatible with the periodic boundary conditions if the functions Ω in eqs. (2.1) are allowed to be physical degrees of freedom: in a functional integral we must integrate over the values of $\Omega_{1,2}$.

Now we wish to express $\mathrm{Tr}\, e^{-\beta H}$ in terms of functional integrals:

$$\mathrm{Tr}\, e^{-\beta H} = \sum_{\psi(t=o)} \langle\psi(t=-i\beta)|\psi(t=0)\rangle. \tag{A.5}$$

The sum is over a basis set of physical states only. We write

$$|\psi(t=0)\rangle = \int \mathrm{D}\Omega\,\Omega|\boldsymbol{A}(\boldsymbol{x})\rangle, \tag{A.6}$$

where $\boldsymbol{A}(\boldsymbol{x})$ satisfies eq. (A.4). We see that

$$\mathrm{Tr}\, e^{-\beta H} = \int \mathrm{D}A\, e^{S(A)}, \tag{A.7}$$

in Euclidean space, if: (i) the gauge is completely specified, for instance by choosing $A_4 = 0$ and eq. (A.4) at $t = 0$ and (ii) the functions Ω that describe the periodic boundary conditions in all four directions are integrated over. Comparing eq. (A.7) with eq. (5.4) we see that $\boldsymbol{e}=0$ and the values of $\boldsymbol{m}$ are summed over. The other cases can be obtained by inserting P as is done in sect. 5.

Our formulation of the periodic boundary conditions in 3-space corresponds to the requirement that only gauge-invariant expressions must be periodic. An alternative would be to require also $A_\mu(\boldsymbol{x}, t)$ to be periodic but that would exclude the possibility to define magnetic flux.

We notice that the gauge restrictions on the functional integral (A.7) imply invariance of eq. (A.7) under permutations of the Euclidean coordinates, a property we needed to derive eq. (6.3).

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