RELATIVISTIC S-MATRIX OF DYNAMICAL SYSTEMS WITH BOSON AND FERMION CONSTRAINTS

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The general solution for the S-matrix of an arbitrary Hamilton system with first-class boson and fermion constraints is obtained. No restrictions are imposed upon the structure functions of the involution of the constraints. The obtained unitarizing Hamiltonian contains a gamut of four-particle interactions of fermion and boson ghosts. The Fradkin-Vilkovisky theorem is generalized, and an extremely simple proof is given, based on global supersymmetry.

The relativistic S-matrix of a boson Hamilton system with first-class constraints was constructed in ref. [1]. The suppression of non-physical boson degrees of freedom, which become dynamically active in relativistic gauges, was realized by the introduction of auxiliary fermion degrees of freedom. The Hamiltonian that governs the dynamics in this complete phase space was uniquely determined from the requirement of unitarity in the subspace of physical states. The central role in the theory was played by a principal theorem, which gave the explicit form of this Hamiltonian. It was discovered that if the first-class constraints do not form a Lie algebra (which is just the case in the gravitational theory) the unitarizing Hamiltonian generally contains a four-fermion interaction term. The application of these results to quantum theory of gravity was considered in ref. [2].

The present paper generalizes the previous work in several respects. Firstly, the exact solution for the S-matrix was obtained in ref. [1] only under certain restrictions upon structure functions of the involution of the constraints. There was evidence [3] that in theories where theserrestrictions are not fulfilled, the six-fermion and even higher-order interactions arise in the ghost Hamiltonian. Here we shall show that in fact the mentioned restrictions are always automatically satisfied. Thus the solution for the S-matrix obtained in ref. [1], is of general validity for boson systems. The four-fermion interaction is the maximal one that can arise, if the gauge itself does not depend on fermion ghosts.

Secondly, the originally very complicated proof [4] of the Fradkin-Vilkovisky theorem gave no regular method of constructing the unitarizing Hamiltonian. Here we shall give a simple proof, based on the fact that a certain canonical transformation exists in the complete phase space, which forms a one-dimensional Abelian superalgebra. Given this superalgebra, the construction of the unitarizing Hamiltonian is straightforward.

Finally, the main purpose of the present paper is the quantization in relativistic gauges of mixed systems with first-class boson and fermion constraints. The unitarizing Hamiltonian of fermion and boson ghosts is obtained below. The results may be directly applied to the supergravity theories [5–7].

The principal theorem. We begin with the introduction of the phase space of boson and fermion (Grassman) canonical pairs: $g^A, \pi_A, A = 1, \dots (n+m)$. The definition of the Poisson brackets is [8]:

$$\{P, Q\} = \frac{\partial P}{\partial q^A}\bigg|_r \frac{\partial Q}{\partial \pi_A}\bigg|_\ell - (-1)^{n_P n_Q} \frac{\partial Q}{\partial q^A}\bigg|_r \frac{\partial P}{\partial \pi_A}\bigg|_\ell \tag{1}$$

where "r" and "ℓ" denote right and left derivatives, and n_P equals 0 or 1 depending on whether P is a boson or a fermion. A non-trivial property of (1) is that if P is a fermion, then generally:

$$\{P, P\} \neq 0, \quad \{\{P, P\}, P\} = 0. \tag{2}$$

Let us consider a dynamical system described in this phase space by a boson Hamiltonian $H_0 (g^A, \pi_A)$ and functions $G_a (q^A, \pi_A)$, $a = 1, \dots 2m$, among which there may be boson and fermion ones, and which satisfy the following (involution) relations:

$$\{G_a, G_b\} = G_c U^c_{ab}, \quad \{H_0, G_a\} = G_b V^b_a. \tag{3}$$

The structure coefficients U and V are functions of canonical variables. The order of the factors in eqs. (3) is fixed. We shall also suppose that the set of functions G_a is minimal, in a sense that $G_a = 0$ is the set of independent equations.

Let us supplement this system with $2m$ more degrees of freedom: $(\eta^a, \mathcal{P}_a)$ of the statistics opposite to that of the $2m$ functions G_a. It is convenient to attribute the statistics basically to indices $a, b, \dots$, and say that if a is a boson (fermion) label, then G_a is a boson (a fermion), while η^a and $\mathcal{P}_a$ are fermions (bosons). Let us introduce the quantity:

$$n_a = \begin{cases} 0, & \text{for boson } a \\ 1, & \text{for fermion } a. \end{cases}$$

Finally, let $\Psi (g^A, \pi_A, \eta^a, \mathcal{P}_a)$ be a completely arbitrary fermion function.

The required generalization of the Fradkin-Vilkovisky theorem [1] is that the following functional integral over the complete phase space does not depend on the choice of Ψ:

$$Z_\Psi = \int \mathrm{d}q \, \mathrm{d}\pi \, \mathrm{d}\eta \, \mathrm{d}\mathcal{P} \exp [\mathrm{i} \int \mathrm{d}t \, (\pi_A \dot{q}^A + \mathcal{P}_a \dot{\eta}^a - H_\Psi)], \tag{4}$$

$$H_\Psi = H_0 + \mathcal{P}_a V^a_b \eta^b - \{\Psi, \Omega\}, \tag{5}$$

where

$$\Omega = G_a \eta^a + \tfrac{1}{2} (-1)^{n_a} \mathcal{P}_c U^c_{ab} \eta^b \eta^a. \tag{6}$$

Here and below the Poisson brackets (1) are extended to the complete phase space, and n_a takes part in the summation.

The first step in the proof is the derivation of generalized Jacobi identities following from the involution and the minimality of the set of G_a. The lowest-order Jacobi identity is most easily derived by writing (3) in the form:

$$U^c_a \equiv -(-1)^{n_a} U^c_{ab} \eta^b, \quad \{G_a \eta^a, G_b \eta^b\} = G_c U^c_b \eta^b, \tag{7}$$

and using (2) with $P = G_a \eta^a$. The result is:

$$U^a_b U^b_c \eta^c = \{U^a_b \eta^b, G_c \eta^c\}, \tag{8}$$

where all η can be cancelled at the expense of the appearance of a cycle with sign factors. The very strong condition upon structure coefficients alone is the following higher-order Jacobi identity:

$$\{U^a_b \eta^b, U^c_f \eta^f\} = 0. \tag{9}$$

It is derived by taking the Poisson brackets of $U^a_b \eta^b$ with (7) and using (8). In the purely boson case this condition was formulated in ref. [1] as the restriction on a theory. In fact it is the exact consequence of the involution, and hence is always satisfied automatically. Similar identities hold with the quantity V.

Next let us consider a canonical transformation in the complete phase space:

$$\varphi \equiv (q, \eta; \pi, \mathcal{P}), \quad \varphi \to \tilde{\varphi} = \varphi + \{\varphi, \Omega\} \mu, \tag{10}$$

with the generator Ω and a fermion parameter μ. The main fact of the whole theory is that this transformation forms the one-dimensional Abelian superalgebra:

$$\{\Omega, \Omega\} = 0. \tag{11}$$

Since Ω is a fermion, the latter equality is highly nontrivial. It is verified on the basis of the involution and Jacobi identities (7)–(9).

The rest is based solely on the existence of a generator Ω with the property (11). Any Abelian superalgebra is nilpotent by virtue of the general identity:

$$(1 - (-1)^{n_\Omega}) \{\{\Psi, \Omega\}, \Omega\} + \{\{\Omega, \Omega\}, \Psi\} = 0, \tag{12}$$

and $n_\Omega = 1$. Therefore the construction of the Hamiltonian (5) in a form of the supertransformation of Ψ guarantees its superinvariance:

$$\{\{\Psi, \Omega\}, \Omega\} = 0, \quad \{H_\Psi, \Omega\} = 0 \tag{13}$$

at any Ψ,

The Ψ-independence of the functional integral (4) follows from the revealed supersymmetry. Indeed, let us perform the displacement of integration variables (10) with:

$$\mu = \int (\Psi' - \Psi)\mathrm{d}t. \tag{14}$$

The displacement leaves invariant the action in the exponential of (4), but yields a Jacobian:

$$\mathrm{d}\varphi = \mathrm{d}\tilde{\varphi}(1 + \int \{\Psi' - \Psi, \Omega\}\,\mathrm{d}t). \tag{15}$$

For small $\Psi' - \Psi$ this immediately gives: $Z_\Psi = Z_{\Psi'}$. The proof is completed.

In addition, the present method allows making finite changes of Ψ in the functional integral. We notice that (10) coincides with the finite transformation of the corresponding supergroup:

$$\exp(\hat{\Omega}\mu) = 1 + \hat{\Omega}\mu\,, \quad \hat{\Omega}\Psi \equiv \{\Psi, \Omega\}\,, \tag{16}$$

and (15) is the exact Jacobian of this transformation, The expression (15) is valid for any $\Psi' - \Psi$ and Ω even in the absence of the property (11). Denoting:

$$x = \int \{\Psi, \Omega\}\mathrm{d}t, \quad Z_\Psi = \langle \exp(\mathrm{i}x)\rangle\,, \tag{17}$$

and making the finite supertransformation of integration variables, we prove the general equality:

$$0 = \langle x\rangle = \langle x^2\rangle = \ldots = \langle f(x) - f(0)\rangle\,, \tag{18}$$

with an arbitrary analytic function $f(x)$. Our theorem is the particular case of this equality, corresponding to $f(x) = \exp(\mathrm{i}x)$.

Application to relativistic systems with boson and fermion constraints. The consideration here closely follows ref. [1.] The action functional of a dynamical system with first-class constraints is of the form:

$$S = \int \mathrm{d}t\,[p_i\dot{q}^i - H_0(q^i, p_i) - T_\alpha(q^i, p_i)\lambda^\alpha]\,, \tag{19}$$

$i = 1, \ldots n$; $\alpha = 1, \ldots m$, where λ^α are Lagrange multipliers, and T_α is the minimal set of boson and fermion constraints in the involution:

$$\{T_\alpha, T_\beta\} = T_\gamma U^\gamma_{\alpha\beta}(q^i, p_i)\,,$$

$$\{H_0, T_\alpha\} = T_\beta V^\beta_\alpha(q^i, p_i) \tag{20}$$

Each constraint T_α of given statistics requires the introduction of two auxiliary (ghost) degrees of freedom of the opposite statistics: $(\eta^\alpha_1, \mathcal{P}^1_\alpha)$, $(\eta^\alpha_2, \mathcal{P}^2_\alpha)$, and a relativistic gauge condition:

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda^\alpha + \chi^\alpha(g, p, \lambda, \eta, \mathcal{P}) = 0\,, \tag{21}$$

with an additional Lagrange multiplier (π_α). In eq. (21) χ^α is a completely arbitrary function.

Identifying the previously introduced quantities with:

$$q^A = \begin{pmatrix} q^i \\ \lambda^\alpha \end{pmatrix}, \pi_A = (p_i, \pi_\alpha)\,, G_a = (\pi_\alpha, T_\alpha)\,, \Psi = \mathcal{P}_a\chi^a\,,$$

$$\chi^a = \begin{pmatrix} \chi^\alpha \\ -\lambda^\alpha \end{pmatrix}, \ \mathcal{P}_a = (\mathcal{P}^1_\alpha, \mathcal{P}^2_\alpha), \ \eta^a = \begin{pmatrix} \eta^\alpha_1 \\ \eta^\alpha_2 \end{pmatrix},$$

we find that (4)–(6) is the most general expression for the gauge-independent relativistic S-matrix of the system (19). The unitarity of this S-matrix in the subspace of physical states is the corollary of the principal theorem. The proof of the unitarity is analogous to that in the boson case [1, 4]: one transforms the functional integral (4) identically to a non-relativistic gauge and verifies that the resulting expression coincides with the expression of ref. [8] for the unitary S-matrix in a non-relativistic gauge.

In the particular case, when the gauge (21) does not depend on η, $\mathcal{P}$, the explicit form of the unitarizing Hamiltonian (5) is

$$H = H_0 - G_a\chi^a - \mathcal{P}_a\{\chi^a, G_b\}\eta^b + \mathcal{P}_a U^a_{bc}\chi^c\eta^b$$
$$+ \mathcal{P}_a V^a_b\eta^b - \tfrac{1}{2}(-1)^{n_a}\mathcal{P}_s\{\chi^s, U^c_{ab}\}\eta^b\eta^a\,\mathcal{P}_c.$$

A remarkable feature of the answer is the presence of the sign factor $(-1)^{n_a}$ in the four-ghost interaction term. Due to this factor in the sum over boson and fermion labels ($a+$ and $a-$):

$$-(-1)^{n_a}U^c_{ab}\eta^b\eta^a = U^c_{a+b+}\eta^{a+}\eta^{b+}$$
$$+ 2U^c_{a-b+}\eta^{a-}\eta^{b+} + U^c_{a-b-}\eta^{a-}\eta^{b-}\,,$$

the term U_{-+} survives, i.e. the mixed commutators $\{T_-, T_+\}$ contribute to the four-ghost interaction.

There are deep physical reasons why constraints (20) do not form a Lie group ($U^\gamma_{\alpha\beta} \neq$ const) already in the boson gravity [2]. In the general Fermi-Bose case there is one more reason: no constant fermions are available. The payment for the absence of a group in involution relations is the appearance of direct interactions of four ghosts.

Conclusion. The canonical action (19) with constraints in involution is the most general form of a field theory, possessing a local Bose-Fermi invariance. The present paper gives the universal method of constructing the explicitly unitary relativistic S-matrix in any such theory. The basis of the method is global Abelian superalgebra, first discovered in the Yang-Mills theory by Becchi, Rouet and Stora [9]. In the canonical formalism the BRS-symmetry is of the universal and primary nature, being the criterion of the unitarity. The problem of quantization reduces to the solution of one equation (11) for the generator of the canonical supertransformation. The solution always exists and is given by eq. (6).

The alternative method of integration over the gauge group [10] does not work already in the boson gravity [2]. All the more it is inapplicable to supergravity theories, where the transformations of invariance do not form a group even in the Lagrange formalism [6]. The solution, obtained above, is of the general applicability.

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