Cosmological heavy-neutrino problem

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A new method of deriving the Lee-Weinberg equation is presented. This method makes the approximations clear and suggest how they can be improved. A nearly exact formula for the number of heavy neutrinos left over after the annihilation process is given.

I. INTRODUCTION

An important problem that arises in determining the present particle content of the Universe is the calculation of the number of particles of each species that have survived the creation and annihilation processes that took place in the early Universe.

In 1977 Lee and Weinberg (LW) wrote a seminal paper[1] on the heavy-lepton content of the Universe. The starting point of their discussion, their Eq. (2), is an equation governing the annihilation of heavy neutral leptons $L,\bar{L}$ into light leptons ($\nu\bar{\nu},e\bar{e},\ldots$) in an expanding space-time:

$$\frac{dn}{dt}=-3\frac{\dot{R}}{R}n-\langle\sigma v\rangle(n^2-n_0{}^2)\ . \tag{1}$$

The notation in this formula, which was written down without derivation, is as follows: n is the number of heavy leptons per unit volume with n_0 the corresponding number in thermal equilibrium, R is the universal expansion factor, and $\langle\sigma v\rangle$ is a thermally averaged annihilation rate which LW do not define precisely. The heart of the LW paper is the numerical and approximate analytical solution to this formula. This analysis has the important conclusion that if a heavy lepton has a mass greater than a few MeV, then unless its mass $M\geq 2$ GeV, the present heavy-lepton number density is so high that the energy density exceeds the observed bound derived from the expansion rate of the Universe. While Lee and Weinberg do not imply that their formula (1) is *ab initio* exact, subsequently it has entered the literature[2] as the "exact" Boltzmann equation for the process. In this paper we shall show precisely how formula (1) arises from a justified approximation to the correct Boltzmann equation. We shall also present a simple and accurate approximate solution to the formula. We have chosen the LW process as a paradigm. At the end of the paper we shall describe briefly other processes to which the same methods are applicable.

II. BOLTZMANN EQUATION

To simplify our exposition, we shall explicitly work in a spatially flat universe described by the Robertson-Walker interval

$$ds^2=R(t)^2d\mathbf{x}^2-dt^2\ , \tag{2}$$

but it should be clear that our results also apply when space is curved in a homogeneous, isotropic manner. We describe the momentum of particles by the momentum vector $\mathbf{p}$ measured in a local Lorentz frame. Thus, for example, the energy and momentum of the heavy leptons are related by

$$E(p)=(\mathbf{p}^2+M^2)^{1/2}\ . \tag{3}$$

In the absence of collisions, the motion along a geodesic is described by

$$\frac{d\mathbf{p}}{d\lambda}=-\frac{\dot{R}}{R}E\mathbf{p}\ , \tag{4}$$

where the overdot denotes a derivative with respect to the comoving time t, and where λ is an affine parameter which is conveniently chosen to give

$$E=\frac{dt}{d\lambda}\ . \tag{5}$$

The Boltzmann equation describes the time evolution of the phase-space density $f(p,t)$ of the heavy leptons. (We assume, of course, that this density is homogeneous and isotropic so that it depends only upon $p=|\mathbf{p}|$ and t.) In the absence of collisions, the phase-space density is constant along a particle's world line,

$$f(p(\lambda+d\lambda),t(\lambda+d\lambda))=f(p(\lambda),t(\lambda))\ . \tag{6}$$

In view of Eqs. (4), (5), and (6) it is easy to see that, taking account of collisions, we have the well-known covariant Boltzmann equation

$$\left[E\frac{\partial}{\partial t}-\frac{\dot{R}}{R}E\mathbf{p}\cdot\frac{\partial}{\partial\mathbf{p}}\right]f(p,t)=C(p,t)\ . \tag{7}$$

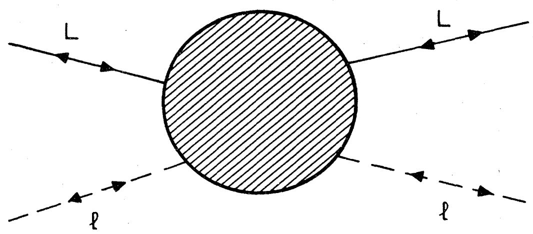

FIG. 1. Elastic collisions with light leptons (l) maintain the heavy-lepton (L) phase-space distribution in mechanical thermal equilibrium at temperature $T(t)$.

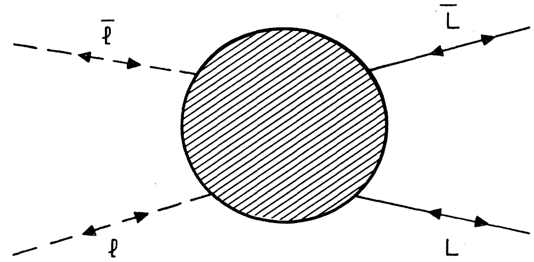

FIG. 2. Inelastic collisions maintain the chemical composition of the heavy leptons—their overall number density—at the thermal equilibrium value until the temperature $T(t)$ falls well below the heavy-lepton mass M.

The collision term $C(p,t)$ has two essentially different pieces: an elastic part $C_E(p,t)$ corresponding to the processes depicted generically in Fig. 1 and an inelastic part $C_I(p,t)$ corresponding to the processes depicted generically in Fig. 2,

$$C(p,t)=C_E(p,t)+C_I(p,t)\ . \tag{8}$$

The elastic part is a sum of terms each of the form

$$\begin{aligned}C_E(p,t)=\int\frac{d^3p'}{(2\pi)^3}\frac{1}{2E(p')}\int\frac{d^3q}{(2\pi)^3}\frac{1}{2q^0}\int\frac{d^3q'}{(2\pi)^3}\frac{1}{2q'^0}&(2\pi)^4\delta^{(4)}(p+q-p'-q')\,|\,T_E\,|^2\\&\times\{[1-f(p,t)][1-g(q,t)]f(p',t)g(q',t)\\&-[1-f(p',t)][1-g(q',t)]f(p,t)g(q,t)\}\ .\end{aligned} \tag{9}$$

Here the $g(q,t)$ is the phase-space density of a light lepton which is kept in thermal equilibrium at temperature $T(t)$ by other collision processes which are very rapid in comparison to the expansion rate $\dot{R}/R$. In the epoch which concerns us, the temperature $T(t)$ is much larger than any light-lepton mass so that we can take $q^0=|\,\mathbf{q}\,|$ and

$$g(q,t)=\frac{1}{\exp[q^0/T(t)]+1}\ . \tag{10}$$

The effect of Fermi statistics is accounted for in Eq. (9) by the Pauli blocking factor $[1-f][1-g]$. Since the density of light leptons is large, the elastic collision integrals (9) give rise to a relaxation time τ_E that is much shorter than the expansion time $R/\dot{R}$. Hence the elastic collisions keep the heavy-lepton phase-space density close to (kinetic) thermal equilibrium. To a good approximation, discussed in Appendix A, this implies that $f(p,t)$ is given by the "equilibrium" distribution

$$f(p,t)\simeq f_0(p,t)=\frac{1}{\exp\left[\alpha(t)+\dfrac{E(p)}{T(t)}\right]+1}\ . \tag{11}$$

When $f_0(p,t)$ is substituted into Eq. (9), $C_E(p,t)$ vanishes for any $\alpha(t)$, where $\alpha(t)$ is a time-dependent effective chemical potential, which must be present in $f(p,t)$ to allow the chemical composition, which is not determined by the elastic collisions, to vary with time.

The distribution $\bar{f}(p,t)$ of the antiparticles $\bar{L}$ has exactly the same form (11) but with $\alpha(t)$ replaced with $\bar{\alpha}(t)$. We assume, with LW, that in the beginning $\alpha(t)=\bar{\alpha}(t)=0$. Since the evolution equations for $f(p,t)$ and $\bar{f}(p,t)$ are identical, at later times we have $\alpha(t)=\bar{\alpha}(t)$.

Upon substituting f_0 of Eq. (11) for f in the inelastic collision integral, we obtain a sum of terms of the form

$$\begin{aligned}C_I(p,t)=\int\frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{2E(\bar{p})}\int\frac{d^3q}{(2\pi)^3}\frac{1}{2q^0}\int\frac{d^3\bar{q}}{(2\pi)^3}\frac{1}{2\bar{q}^0}&(2\pi)^4\delta^{(4)}(p+\bar{p}-q-\bar{q})\,|\,T_I\,|^2\\&\times\{[1-f_0(p,t)][1-f_0(\bar{p},t)]g(q,t)g(\bar{q},t)\\&-[1-g(q,t)][1-g(\bar{q},t)]f_0(p,t)f_0(\bar{p},t)\}\ .\end{aligned} \tag{12}$$

The time development of the number density and hence of the "chemical potential" $\alpha(t)$ is obtained by dividing the

Boltzmann equation by E and integrating over the momentum so as to obtain

$$R(t)^{-3}\frac{d}{dt}[R(t)^3 n(t)]=2\int\frac{d^3p}{(2\pi)^3}\frac{1}{E(p)}C_I(p,t)\ . \tag{13}$$

The skew symmetry of the elastic collision integral (9) shows that it vanishes upon the additional momentum integration in Eq. (13) reflecting the obvious fact that elastic collisions do not change the number density. Using Eqs. (10), (11), and (12) in Eq. (13), we now obtain

$$R(t)^{-3}\frac{d}{dt}[R(t)^3 n(t)]$$
$$=2\int\frac{d^3p}{(2\pi)^3}\frac{1}{E(p)}\int\frac{d^3\bar{p}}{(2\pi)^3}\frac{1}{2E(\bar{p})}\int\frac{d^3q}{(2\pi)^3}\frac{1}{2q^0}\int\frac{d^3\bar{q}}{(2\pi)^3}\frac{1}{2\bar{q}^0}(2\pi)^4\delta^{(4)}(p+\bar{p}-q-\bar{q})\,|T_I|^2 f_0(p,t)f_0(\bar{p},t)$$
$$\times[1-g(q,t)][1-g(\bar{q},t)]\{\exp[2\alpha(t)]-1\}\ . \tag{14}$$

Of course, this is just a generic equation whose right-hand side is understood to represent a sum of similar terms.

As it stands, Eq. (14) is very complex. As we shall see, however, the "chemical potential" $\alpha(t)$ departs from $\alpha(t)=0$ only when the temperature $T(t)$ is well below the heavy-lepton mass M. In this regime the Fermi-Dirac distribution (11) may be replaced by the Maxwell-Boltzmann form

$$f_0(p,t)\simeq\exp\left[-\alpha(t)-\frac{E(p)}{T(t)}\right]\ . \tag{15}$$

In this case

$$n(t)=e^{-\alpha(t)}n_0(t)\ , \tag{16}$$

where

$$n_0(t)=2\int\frac{d^3p}{(2\pi)^3}\exp\left[-\frac{E(p)}{T(t)}\right] \tag{17}$$

is the equilibrium density of heavy leptons at zero chemical potential. Placing these limits in Eq. (14) we obtain

$$R(t)^{-3}\frac{d}{dt}[R(t)^3 n(t)]=\langle\sigma v\rangle[n_0(t)^2-n(t)^2]\ , \tag{18}$$

which is exactly the Lee-Weinberg formula (1) but with the precise identification

$$\langle\sigma v\rangle=\int\frac{d^3p}{(2\pi)^3}\frac{2\exp[-E(p)/T(t)]}{n_0(t)}\int\frac{d^3\bar{p}}{(2\pi)^3}\frac{2\exp[-E(\bar{p})/T(t)]}{n_0(t)}$$
$$\times\int\frac{d^3q}{(2\pi)^3}\frac{1}{2q^0}[1-g(q,t)]\int\frac{d^3\bar{q}}{(2\pi)^3}\frac{1}{2\bar{q}^0}[1-g(\bar{q},t)]$$
$$\times(2\pi)^4\delta^{(4)}(p+\bar{p}-q-\bar{q})\,|T_I|^2\frac{1}{4E(p)E(\bar{p})} \tag{19}$$

for the thermally averaged cross section.

For general processes, Eq. (19) defines a temperature-dependent effective cross section. In our case, however, the exponential damping of the heavy-lepton energies constrain $|p|\lesssim\sqrt{2MT}$ and $|\bar{p}|\lesssim\sqrt{2MT}$, with $E(p)+E(\bar{p})\simeq 2M$. On the other hand, the energy-conserving δ function $\delta(2M-q^0-\bar{q}^0)$ requires that the light-lepton momentum q and $\bar{q}$ be of order M. This has two consequences. First, since q and $\bar{q}$ are much larger than p and $\bar{p}$, the momentum-conserving δ function can be replaced by $\delta^{(3)}(q+\bar{q})$. Second, since the light-lepton energies are much greater than the temperature, $g(q,t)\ll 1$, $g(\bar{q},t)\ll 1$, and the Pauli blocking factors $[1-g(q,t)]$ and $[1-g(\bar{q},t)]$ can be replaced by unity. Accordingly, Eq. (19) reduces to

$$\langle\sigma v\rangle\to(\sigma v)_0\ , \tag{20}$$

where

$$(\sigma v)_0=\int\frac{d^3q}{(2\pi)^3}\frac{1}{2q^0}\int\frac{d^3\bar{q}}{(2\pi)^3}\frac{1}{2\bar{q}^0}(2\pi)^3\delta^{(3)}(q+\bar{q})2\pi\delta(2M-q^0+\bar{q}^0)\,|T_I|^2\frac{1}{4M^2} \tag{21}$$

is the threshold limit of the total annihilation cross section for the heavy leptons, $L+\bar{L}\to$all light leptons, multiplied by the relative velocity $v\to 0$.

The heavy leptons freeze out while the Universe is radiation dominated with the dynamics governed by

$$\frac{\dot{R}}{R}=-\frac{\dot{T}}{T}=\left[\frac{8\pi G\rho}{3}\right]^{1/2}, \tag{22}$$

where

$$\rho=N_F\frac{\pi^2}{15}T^4 \tag{23}$$

is the energy for the number N_F of massless degrees of freedom with each spin state of a boson contributing $\frac{1}{2}$ to N_F and each spin state of a fermion contributing $\frac{7}{16}$ to N_F. As LW point out, for heavy-lepton masses of order 1 GeV, the bound that is obtained, the relevant temperature range is $T=10-100$ MeV, and so the massless particles are $\gamma,\nu_e\bar{\nu}_e,\nu_\mu\bar{\nu}_\mu,\nu_\tau\bar{\nu}_\tau,e^-e^+$, giving $N_F=5.4$. We follow LW in changing variables from the time to the temperature scaled by the heavy-lepton mass,

$$x=T/M\ . \tag{24}$$

It is also convenient to scale the heavy-lepton equilibrium density n_0 by the photon number density

$$n_\gamma=\frac{2\zeta(3)}{\pi^2}T^3 \tag{25}$$

and define

$$G(x)=\frac{n}{n_\gamma},\quad G_0(x)=\frac{n_0}{n_\gamma}\ . \tag{26}$$

For temperatures high in comparison with the heavy-lepton mass,

$$x\gg 1:\quad G_0(x)=\tfrac{3}{4}\ , \tag{27}$$

while for temperatures low in comparison with the mass

$$x\ll 1:\quad G_0(x)\simeq\frac{1}{2\zeta(3)}\sqrt{\pi/2}\frac{1}{x^{3/2}}e^{-1/x}\ . \tag{28}$$

Using x as the independent variable and also using Eqs. (20), (22), (23), (25), and (26), one now finds that Eq. (18) can be expressed as

$$\frac{dG(x)}{dx}=\lambda^2[G(x)^2-G_0(x)^2]\ , \tag{29}$$

where

$$\lambda=\frac{3\zeta(3)}{\pi^3}\left[\frac{5}{\pi N_F}\right]^{1/2}MM_P(\sigma v)_0$$
$$\simeq 0.15MM_P(\sigma v)_0\ , \tag{30}$$

in which

$$M_P=G^{-1/2}\simeq 1.2\times 10^{19}\ \text{GeV} \tag{31}$$

is the Planck mass.

III. ANALYTIC SOLUTION

For the LW process, the dimensionless parameter λ is very large. Following LW, the weak annihilation cross section may be estimated by

$$(\sigma v)_0=N_A\left[\frac{G_F{}^2M^2}{2\pi}\right], \tag{32}$$

where N_A is the number of open channels and the remaining factor is the cross section of the standard model with $G_F\simeq 1.2\times 10^{-5}$ GeV^{-2} the Fermi constant. Taking the open channels to be $L\bar{L}\to\nu_e\bar{\nu}_e$, $\nu_\mu\bar{\nu}_\mu$, $\nu_\tau\bar{\nu}_\tau$, e^-e^+, $\mu^-\mu^+$, $u\bar{u}$, $d\bar{d}$, and $s\bar{s}$, with quarks counted thrice to account for color, one has $N_A=14$. Inserting Eq. (32) in Eq. (30) and using various numbers that are listed above, one finds

$$\lambda\simeq 2\times 10^8M^3/\text{GeV}^3\ . \tag{33}$$

Anticipating that $M>1$ GeV, we see that λ is larger than 10^8. The large size of this dimensionless parameter makes it possible to obtain a simple approximate analytic solution to Eq. (29) which is of good accuracy. We derive this solution here for the case of interest where λ is constant. In Appendix B we present the corresponding approximate analytic solution to Eq. (29) for those cases when the general thermally averaged cross section of (19) must be used and coupling parameter λ is temperature dependent.

We first note that by a standard procedure the nonlinear, first-order Riccati differential equation (29) may be converted into a second-order, linear equation. This is accomplished by writing

$$G(x)=\frac{-1}{\lambda f(x)}\frac{df(x)}{dx}\ , \tag{34}$$

whose substitution into Eq. (29) produces

$$\left[\frac{d^2}{dx^2}-\lambda^2G_0(x)^2\right]f(x)=0\ . \tag{35}$$

Since λ is very large, a WKB solution is adequate over most of the range of x. Taking into consideration the boundary condition that $G\to G_0$ when $x\to\infty$, which is equivalent to $\alpha(0)=0$, this solution is given by

$$f(x)\cong G_0(x)^{-1/2}\exp\left[-\lambda\int^x dx'G_0(x')\right]\ . \tag{36}$$

Placing this approximate solution in Eq. (34) yields

$$G(x)\simeq G_0(x)+\frac{G_0'(x)}{2\lambda G_0(x)}\ , \tag{37}$$

where the prime denotes a derivative. We see that since λ is very large, the scaled number density $G(x)$ closely follows the equilibrium density $G_0(x)$ up to the point at which the second term on the right-hand side of Eq. (37) becomes comparable to the first term. Since this occurs when the temperature T is much less than the heavy lepton mass M, $x\ll 1$, and the limiting form (28) can be used for $G_0(x)$, giving the condition

$$\frac{\lambda}{\zeta(3)}\sqrt{\pi/2}x^{1/2}e^{-1/x}>1 \tag{38}$$

for $G(x)$ to be close to $G_0(x)$. This is satisfied if

$x \gtrsim 1/\ln\lambda$. As x decreases below $1/\ln\lambda$, $G(x)$ departs from $G_0(x)$. In the transition region the WKB approximation breaks down and some other method must be used. For $x \to 0$, the heavy-lepton number is "frozen" at a constant value, which we calculate below.

To study the transition region it is convenient to change variables from x to

$$y = \frac{1}{x} \tag{39}$$

and to write

$$f(x) = \frac{u(y)}{y} \; . \tag{40}$$

Since $G_0(x)$ can be replaced by the limiting form (28) in this region, Eq. (35) now becomes

$$\left[\frac{d^2}{dy^2} - \frac{\pi\lambda^2}{8\zeta(3)^2} \frac{e^{-2y}}{y} \right] u(y) = 0 \; . \tag{41}$$

The WKB approximation breaks down when

$$\left[\frac{\pi\lambda^2}{2\zeta(3)^2} \right]^{1/2} \frac{e^{-y}}{y^{1/2}} \lesssim 1 \; ,$$

which occurs for $y \gtrsim y_0$, with

$$\left[\frac{\pi\lambda^2}{2\zeta(3)^2} \right]^{1/2} \frac{e^{-y_0}}{{y_0}^{1/2}} \approx 1 \tag{42}$$

giving $y_0 \simeq \ln\lambda \simeq 19$. With y in the range of a few units below y_0, x is still quite large and the WKB approximation (36), which is valid, gives

$$u(y) = \text{const} \times y^{1/4} \exp\left[\frac{y}{2} - \frac{\lambda}{2\zeta(3)} \left[\frac{\pi}{2y} \right]^{1/2} e^{-y} \right] \; . \tag{43}$$

To pass from this region of y being a few units below y_0 to $y \to \infty$, we make the replacement

$$\frac{e^{-2y}}{y} \to \frac{e^{-2y}}{y_0} \tag{44}$$

in Eq. (41). This is an adequate approximation since the exponential e^{-2y} varies rapidly and effectively vanishes when y becomes much larger than y_0. With this approximation, Eq. (41) has the solution

$$u(y) = K_0 \left[\frac{\lambda}{2\zeta(3)} \left[\frac{\pi}{2y_0} \right]^{1/2} e^{-y} \right] , \tag{45}$$

where $K_0(z)$ is the modified Hankel function. Using the large argument limit of the Hankel function we have for y a few units below y_0

$$u(y) \simeq \text{const} \times {y_0}^{1/4} \exp\left[\frac{y}{2} - \frac{\lambda}{2\zeta(3)} \left[\frac{\pi}{2y_0} \right]^{1/2} e^{-y} \right] \tag{46}$$

which agrees with the limit (43) except that y is replaced by y_0 in two slowly varying factors which makes little difference since y_0 is large. We conclude that the approximate solution (45) obeys the correct initial conditions. Inserting Eqs. (45), (40), and (39) in Eq. (34) yields

$$G\left[x = \frac{1}{y} \right] = \frac{1}{\lambda} \left\{ y^2 \frac{d}{dy} \ln \left[K_0 \left[\frac{\lambda}{2\zeta(3)} \left[\frac{\pi}{2y_0} \right]^{1/2} e^{-y} \right] \right] - y \right\} . \tag{47}$$

As shown in Fig. 3, this result accurately matches on to the WKB approximation for $y < y_0$ and then accurately describes both the rapidly varying region where y passes beyond y_0 and asymptotic limit as $y \to \infty$.

The behavior of the Hankel function for small arguments now gives the desired low-temperature limit

$$G(0) = \frac{1}{\lambda} \left\{ \ln \left[\frac{\lambda}{4\zeta(3)} \left[\frac{\pi}{2y_0} \right]^{1/2} \right] + \gamma \right\} , \tag{48}$$

where $\gamma \approx 0.577$ is Euler's constant. Recalling Eq. (42), we have

$$y_0 = \tfrac{1}{2} \ln \left[\frac{\pi\lambda^2}{2\zeta(3)^2} \right] + y_1 \; , \tag{49}$$

where y_1 is a number of order unity which is not precisely determined by our fitting procedure. The ambiguity in y_1 is, however, insignificant since it gives rise to a relative error in $G(0)$ of order $y_1/2\ln^2\lambda$. Previous[1,3] rough analytic fits to Eq. (1) give results analogous to Eq. (48) but with an arbitrary constant of order unity inside the logarithm. Instead the approximation given by Eq. (48) is a leading-logarithmic approximation valid to order $1/\ln^2\lambda$. Since λ is so large, Eq. (48) is accurate to within a few percent. Using the nominal value $M = 2$ GeV or $\lambda = 1.6 \times 10^9$ inside the slowly varying logarithms in Eq. (48), we compute

$$G(0) \simeq \frac{\ln(\lambda/9)}{\lambda} \simeq \frac{19}{\lambda} \; . \tag{50}$$

The number density of heavy leptons that remain after

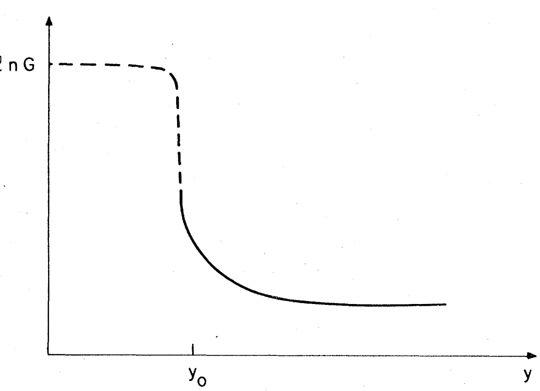

FIG. 3. Sketch of the logarithm of the scaled heavy-lepton number density G vs $y = 1/x$. The dashed line is the WKB approximation of Eq. (37). The solid line is the result given in Eq. (47). Note that the two curves join smoothly in the region of y a few units below y_0 with, in particular, the slopes of the two curves very nearly the same in this region.

the inelastic equilibrating process is no longer effective is given by

$$n(t)=n_\gamma(t)G(0) . \tag{51}$$

This result holds down to a temperature of about 1 MeV. Below this temperature, as LW point out, the photons are reheated by electron-positron annihilation,[4] increasing their number density by a factor of $\frac{11}{4}$. At yet lower temperatures both $R(t)^3n(t)$ and $R(t)^3n_\gamma(t)$ remain constant, which is to say that the number of particles within a given comoving observer volume remains fixed. Thus at our present epoch we have the number density ratio

$$\left[\frac{n}{n_\gamma}\right]_0=\frac{4}{11}G(0)\approx\frac{6.9}{\lambda} . \tag{52}$$

The present 3-K blackbody background radiation consists of about 400 photons per cm^3, and the limit on the present mass density of the Universe is about 2×10^{-29} g/cm^3. Thus if there were additional particles of mass μ with the photon number density, their mass would be bounded by $\mu\lesssim20$ eV. Applying this mass density limit to the heavy leptons and antileptons, we secure[5]

$$\frac{8}{11}G(0)M\approx\frac{14M}{\lambda}\lesssim20\text{ eV} . \tag{53}$$

Recalling Eq. (33) for λ, we see that we have the bound

$$M\gtrsim2\text{ GeV} ,$$

in agreement with the computer solution of Lee and Weinberg, as well as with their approximate analytic solution.

IV. REMARKS

We have, for simplicity, focused in this paper on the LW process. It is clear that these methods can also be applied to processes in which the initial number of particles and antiparticles is not identical as in electron-positron annihilation in the early Universe or in which the particle and antiparticle reaction rates are not identical as in processes in which baryons are produced in the very early Universe. The analysis can also be generalized to the case of particles with spin other than $\frac{1}{2}$. We will report on these matters in subsequent publications.

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APPENDIX A

As noted in the text, the distribution $f_0(p,t)$ does not exactly satisfy the Boltzmann equation, even when $\alpha(t)$ satisfies Eq. (14). However, the deviation of $f(p,t)$ from f_0 is small. We can describe this deviation by adding a momentum-dependent term $\phi(p,t)$ to $\alpha(t)$ in Eq. (11) and expanding in powers of ϕ. This gives for $f(p,t)$

$$f(p,t)\approx f_0(p,t)\{1+[1-f_0(o,t)]\phi(p,t)\}+O(\phi^2) . \tag{A1}$$

In order that this partition between α and ϕ be well defined we prescribe that to first order ϕ does not alter the number density by requiring that

$$\int d^3p\, f_0(p,t)[1-f_0(p,t)]\phi(p,t)=0 . \tag{A2}$$

The number density is then given by

$$n(t)=2\int\frac{d^3p}{(2\pi)^3}f_0(p,t) . \tag{A3}$$

When (A1) is substituted into the Boltzmann equation, two types of terms proportional to ϕ are found. One arises from the left-hand side of the equation, and involves time and momentum derivatives of ϕ. The other type arises from the collision integral $C_E(p,t)$ and is of the form

$$C_E(p,t)=\int\frac{d^3p'}{(2\pi)^3}\frac{1}{2E(p')}\int\frac{d^3q}{(2\pi)^3}\frac{1}{2q^0}\int\frac{d^3q'}{(2\pi)^3}\frac{1}{2q'^0}(2\pi)^4\delta^{(4)}(p+q-p'-q')\,|\,T_E\,|^2f_0(p,t)g(q,t)$$
$$\times f_0(p',t)g(q',t)\exp\{[q+E(p)]/T(t)+\alpha(t)\}[\phi(p',t)-\phi(p,t)] . \tag{A4}$$

The terms involving ϕ in the inelastic collision integral $C_I(p,t)$ are much smaller than these, because they are proportional to $n^2\phi$ instead of $n\phi$, and so can be neglected. However, there will remain terms independent of ϕ in the inelastic collision integral that are not compensated by the time dependence of $\alpha(t)$, because in deriving Eq. (14) for $\alpha(t)$ we have integrated over momentum p, which we are not doing here. These remaining terms in the inelastic collision integral act as an inhomogeneous term for ϕ.

The main point is now that because the collision time τ_E, defined implicitly by Eq. (A4), is much shorter than $R/(dR/dt)$, only a very small ϕ is needed to compensate for the remaining nonvanishing terms in the inelastic collision integral. We can therefore proceed by neglecting ϕ entirely in the equation that determines $\alpha(t)$, and simply replace f by f_0, as we have done in the text.

APPENDIX B

In the text we obtained an approximate but highly accurate analytic solution (47) to the rate equation (29) in which the coupling parameter λ is constant. Here we shall obtain a slightly less accurate solution for the general situation where the thermally averaged cross section $\langle \sigma v \rangle$ is temperature dependent. In this case the coupling parameter λ depends upon $x = T/M$ and the rate equation (29) becomes

$$\frac{dG(x)}{dx} = \lambda^2(x)[G(x)^2 - G_0(x)] \; . \tag{B1}$$

This generalized rate equation can again be brought into linear form. As we shall see, it proves convenient to introduce the variable

$$\xi = \xi(x) = \frac{1}{\lambda_0} \int_{x_0}^{x} dx' \lambda(x') \; , \tag{B2}$$

where λ_0 and x_0 are constants that will be specified later. In terms of this variable, the analog of Eq. (34) is to write

$$G(x) = \frac{-1}{\lambda_0 f(\xi)} \frac{df(\xi)}{d\xi} \tag{B3}$$

so that Eq. (B1) becomes

$$\left[\frac{d^2}{d\xi^2} - \lambda_0{}^2 G_0(x(\xi))^2 \right] f(\xi) = 0 \; . \tag{B4}$$

For temperatures which are higher than those of the transition region the WKB approximation again suffices, giving

$$f(\xi(x)) \simeq G_0(x)^{-1/2} \exp\left[-\int^{x} dx' \lambda(x') G_0(x') \right] \tag{B5}$$

and

$$G(x) \simeq G_0(x) + \frac{G_0'(x)}{2\lambda(x) G_0(x)} \; . \tag{B6}$$

We again assume that $\lambda(x)$ is very large in the transition region so that here $G_0(x)$ has the nonrelativistic form (28). Thus the analog of the transition point y_0 determined by Eq. (42) is the condition that

$$\frac{\pi \lambda(x_0)^2}{2\zeta(3)^2} x_0 e^{-2/x_0} = 1 \; . \tag{B7}$$

This condition determines the point x_0 that we use as the lower limit of integration in Eq. (B2). We also fix

$$\lambda_0 = \lambda(x_0) \tag{B8}$$

so that $\xi \simeq x - x_0$ for x near x_0. Since $\lambda(x_0)$ is taken to be very large, $1/x_0 \simeq \ln\lambda(x_0)$ is quite large, and the behavior across the transition region to $x = 0$ is controlled by

$$\left[\frac{d^2}{d\xi^2} - \frac{1}{x_0{}^4} \exp(2\xi/x_0{}^2) \right] f(\xi) = 0 \; . \tag{B9}$$

This has the solution

$$f(\xi) = K_0(e^{\xi/x_0{}^2}) \tag{B10}$$

which matches the WKB approximation (B5) in an interval just above the transition region. Since $1/x_0{}^2$ is very large, the argument of the modified Hankel function becomes very small as $x \to 0$, and we can use its limiting form to compute

$$f(\xi) = -\xi/x_0{}^2 + \ln 2 - \gamma \tag{B11}$$

and from Eqs. (B3) and (B2) find that the final scaled number density is given by

$$G(0)^{-1} = \int_0^{x_0} dx\, \lambda(x) + \lambda_0 x_0{}^2 (\ln 2 - \gamma) \; . \tag{B12}$$

Since x_0 is very small, we may write this result as

$$G(0)^{-1} = \int_0^{x_1} dx\, \lambda(x) \; , \tag{B13}$$

where

$$\frac{1}{x_1} = \frac{1}{x_0} - \ln 2 + \gamma \; . \tag{B14}$$

The form (B12) reduces to the result (48) of the text when λ is constant.

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[1]B. W. Lee and S. Weinberg, Phys. Rev. Lett. **39**, 165 (1977).

[2]See, for example, V. Silveira and A. Zee, University of Washington report, 1985 (unpublished), Eq. (1); L. M. Krauss, Phys. Lett. **128B**, 37 (1983).

[3]G. Steigman, Ann. Rev. Nucl. Part. Sci. **29**, 313 (1979).

[4]This assumes that the "freezing temperature" M/y_0 is less than 100 MeV. Otherwise the photons may be reheated by other annihilations. In the present case, for $M = 2$ GeV, $M/y_0 \sim 100$ MeV, but if $M \gg 2$ GeV, the condition may not be satisfied.

[5]Note that this condition involves an inverse power of M (M^{-2} for the LW process). Superficially, one might imagine that the condition will involve an exponential dependence on M, $\exp(-M/T_f)$, but actually $T_f \sim M/\ln\lambda$, where λ goes as a power of M, resulting in a power dependence of M. For other processes than the LW process, with a different dependence of cross section on M, it can happen that no lower limit on M is obtained.