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USE AND MISUSE OF QCD SUM RULES, FACTORIZATION AND RELATED TOPICS

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The purpose of this paper is twofold. First, we are prompted by some recent publications to reply to the criticism of the QCD sum-rules approach contained therein. Hopefully, some of the discussion is of wider interest. In particular, we point out that the multi-gluon operators unlike the multi-quark ones, relevant to the sum rules, do not factorize at large N_c. This implies that the masterfield, even if it is found, will be of no immediate help in evaluating the quarkonium spectrum. Second, we derive new sum rules for light quarks which are sensitive to the mean intensity of the gluon field in the vacuum (the so-called gluon condensate, or $\langle \mathrm{vac}|G^2|\mathrm{vac}\rangle$). New sum rules confirm the standard value of $\langle \mathrm{vac}|G^2|\mathrm{vac}\rangle$. Some casual remarks on the π^0 transition into two virtual photons, $\pi^0 \to \gamma^* \gamma^*$, are also presented. Finally, we enumarate (in sect. 7) basic points of the sum-rule approach and discuss, in brief, the unsolved problems.

1. Introduction

The QCD sum rules [1] relate properties of low-lying resonances to vacuum expectation values of various field operators. The most frequently used vacuum expectation values are the so-called quark and gluon condensates, namely,

$$\langle \bar{q}q \rangle , \qquad \langle G^a_{\mu\nu} G^a_{\mu\nu} \rangle ,$$

where q is the quark field and $G^a_{\mu\nu}$ is the gluon field strength tensor (a is the colour index).

Strictly speaking, theoretical predictions refer to some integrals over spectral densities rather than to a particular resonance. Saturation of the sum rules by a single resonance is well justified in many cases but still introduces an uncertainty. Moreover, one usually needs some general idea on the form of the spectrum. Thus, the sum rules match fundamental QCD with phenomenology and are, so to say, semi-phenomenological.

Despite these intrinsic limitations, the sum rules have been successfully applied in many problems of hadron physics and it is difficult to find another approach which would be rooted in fundamental QCD to the same extent and could compete with the sum rules. A concise review of recent applications can be found in ref. [2]. Let us also mention the evaluation of the baryon masses [3] and, especially, the generalization of the method to the case of three-point functions [4] and to the case

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of external weak fields [5a]. The latter development resulted in computation of the baryonic magnetic moments [5a, b] (see also [5c]).

On the theoretical side, the status of the operator product expansion has become the subject of intensive investigations.[6,7] We will comment on this issue in sect. 3.

On the other hand, there appeared recently several papers [8–12] which argue that the sum rules are plagued by serious problems and should be modified in a crucial way. First, there is a series of papers [8, 10] investigating the sum rules within non-relativistic quantum mechanics. The conclusion of these papers is that original sum rules underestimate the gluon condensate considerably, by a factor of 2.5–3. Furthermore, according to ref. [11] convergence of the series of power corrections is much worse than it was expected [1] so that the original analysis makes little sense. Finally, the authors of ref. [12] claim that $\langle G^2 \rangle$ must be increased by a factor of 100!. The conclusion is drawn from an analysis of the Υ sum rules.

Of course, on general grounds alone it is quite possible that further investigation could reveal some limitations and failures of the sum rules which were originally overlooked or underestimated. However, in this particular case we believe that the criticism of refs. [8–12] is based rather on misunderstandings. In particular, the authors of refs. [8–11] ascribe to us assumptions which we have never made. Moreover, some elements of the procedure used in [11, 12] have no justifications.

In this paper we supplement these critical comments by a new determination of the gluon condensate from sum rules. The idea is that much of the controversy is centred around the numerical value of $\langle G^2 \rangle$. We will present new sum rules which refer to light quarks and result essentially in the same value of $\langle G^2 \rangle$ (the previous analyses rely primarily on heavy quarks).

Since the paper includes a few topics which are not connected directly and prompted to a great extent by other publications the presentation of the material is by necessity fragmentary.

In sect. 2 we consider sum rules in quantum mechanics, in connection with the critical remarks of refs. [8, 10]. Our conclusion is that these remarks are true but refer to the procedure invented by the authors themselves, not to the original sum rules. In sect. 3, apart from the status of OPE, we consider the factorization of matrix elements $\langle G^4 \rangle$ which reduces them to $\langle G^2 \rangle^2$. This factorization is crucial for ref. [11] to establish a discrepancy (of about a factor 2) between the expected and actual values of the higher-order corrections in the charmonium sum rules. Superficially, this factorization hypothesis looks very much the same as factorization of the four-quark operators' matrix elements exploited in ref. [1]. Our comment is that this apparent similarity is false – while factorization for the quark operators can be justified in the large-N_c limit (and then independently confirmed by the sum rules) this is not true for the gluon operators considered. Moreover, this comment refers to the so-called masterfield [19] as well. Even if this field is found explicitly this would not allow for the evaluation of the amplitudes relevant to the quarkonium spectrum.

In sect. 5 the sum-rule technique is used to determine the matrix element $\langle \mathrm{vac}|\frac{1}{2}g\bar{d}\gamma_\rho t^a \tilde{G}^a_{\rho\mu} u|\pi^+\rangle$ (where d and u are the quark fields). There are two different sets of the sum rules used to determine one and the same matrix element. One set is sensitive to the quark condensate and the other is governed by the gluon condensate. The central point is that the results agree only if $\langle \bar{q}q\rangle$ and $\langle G^2\rangle$ are close to their "standard" values. In sect. 6 we note that the matrix element $\langle \mathrm{vac}|\frac{1}{2}g\bar{d}\gamma_\rho \tilde{G}^a_{\rho\mu} t^a u|\pi^+\rangle$ is relevant to the description of the $\pi^0\gamma^*\gamma^*$ transition at intermediate photon off-shellness and fits this description well. We use the opportunity to make some further comments on $\pi^0\gamma^*\gamma^*$ transitions as well.

2. Stability plateau, predictions for the lowest-lying levels and determination of the condensate parameter

We shall make here a few comments on the criticism of the sum rules put forward in refs. [8, 10]. Since our purpose is basically illustrative let us consider the simplest possible example: the quantum mechanical, three-dimensional harmonic oscillator, which will give, however, the possibility to pose all relevant questions. The answers to these questions will, hopefully, eliminate the misunderstanding existing in the literature.

Quantum mechanics as a laboratory for probing the sum rule approach has been discussed previously in rather detailed papers [13–15, 8, 10]. We shall touch here upon one aspect, but at first, to introduce notations, we shall quote a few well-known expressions.

Consider the motion of a particle with mass m in the potential

$$V(r)=\tfrac{1}{2}m\omega^2 r^2\,, \tag{1}$$

and concentrate on the S-wave states. The following sum is the non-relativistic analogue of vacuum polarization in field theory (more exactly, of the Borel-transformed vacuum polarization):

$$S(\varepsilon)=\sum_{n=0,2,4,\ldots}|R_{n0}(0)|^2\exp(-E_n/\varepsilon)\,, \tag{2}$$

where R_{n0} is the radial wave function for zero angular momentum and E_n denotes the corresponding energy eigenvalue. Moreover, ε is an external parameter which, as usual, will regulate the energy resolution of the sum rules.

There exists a simple relation between $S(\varepsilon)$ and the time-dependent Green function, namely

$$S(\varepsilon)=4\pi G(\boldsymbol{x}_2=0,\, t_2=-i/\varepsilon|\boldsymbol{x}_1=0,\, t_1=0)\,, \tag{3}$$

which allows one to find $S(\varepsilon)$ as a series in $1/\varepsilon$ for large ε. Notice that ε in eq. (3) plays the role of inverse imaginary time. Large ε means a small euclidean time interval.

The exact answer for $S(\varepsilon)$ is also known:

$$S(\varepsilon)=\frac{2}{\sqrt{2\pi}}\left(\frac{m\omega}{\sinh(\omega/\varepsilon)}\right)^{3/2}. \tag{4}$$

Analysis of this expression in the limit $\varepsilon\to 0$ (compare with eq. (2)) gives exhaustive information on all E_n and $|R_{n0}(0)|^2$. For instance,

$$E_0=\lim_{\varepsilon\to 0}(-\mathrm{d}/\mathrm{d}(1/\varepsilon))\ln S(\varepsilon)=\tfrac{3}{2}\omega\lim_{\varepsilon\to 0}\coth\frac{\omega}{\varepsilon}=\tfrac{3}{2}\omega\,.$$

Assume, however, that, like in QCD, the small-ε asymptotics is unknown, and all we have is a few first terms of the power expansion in ω/ε. The zeroth-order corresponds to the free motion

$$S_0(\varepsilon)=\frac{2}{\sqrt{2\pi}}(m\varepsilon)^{3/2}\equiv\int_0^\infty \rho_0(E)\,\mathrm{e}^{-E/\varepsilon}\,\mathrm{d}E\,,\qquad \rho_0=\frac{\sqrt{8m^3E}}{\pi}\,; \tag{5}$$

the next term is the first Born correction and so on. In order to extract the mass it is more convenient to work with

$$(-\mathrm{d}/\mathrm{d}(1/\varepsilon))\ln S(\varepsilon)=\tfrac{3}{2}\omega\left[\frac{\varepsilon}{\omega}+\frac{\omega}{3\varepsilon}-\tfrac{1}{45}(\omega/\varepsilon)^3+\tfrac{2}{945}(\omega/\varepsilon)^5-\cdots\right]. \tag{6}$$

Indeed, if for a given value of ε the sum $S(\varepsilon)$ is saturated by the lowest-lying level, the quantity $(-\mathrm{d}/\mathrm{d}(1/\varepsilon))\ln S(\varepsilon)$ is insensitive to the residue $|R_{00}(0)|^2$, and gives directly the level energy.

Now, starting with expansion (6) one cannot tend $\varepsilon\to 0$ in the mathematical sense. Fig. 1 shows, however, that there exists a stability plateau at intermediate ε such that on the one hand the power expansion is still convergent, and on the other

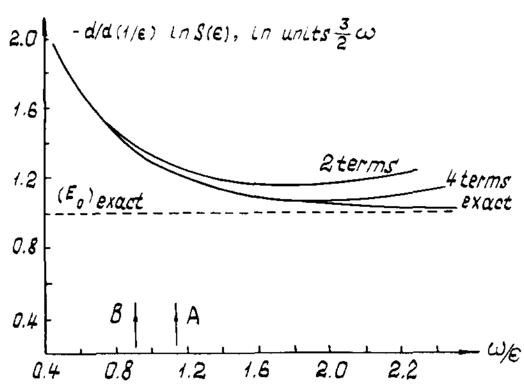


Fig. 1. Power expansion of $(-d/d(1/\varepsilon))\ln S(\varepsilon)$. The lower curve is the exact result and the other two curves correspond to the 2 and 4 terms of the expansion.

hand, the ground level *almost* saturates the sum. Analysing the curve near the plateau one obtains an approximate prediction for the ground-level energy.

The minimal value, at first sight, seems to be the best approximation. It is worth emphasizing that even at the minimum, the ground level contribution does not saturate the sum rule completely (although this contribution is dominant, of course). Thus, including the leading-power correction we get

$$(-\mathrm{d}/\mathrm{d}(1/\varepsilon)) \ln S_1(\varepsilon) = \tfrac{3}{2}\omega\left[\frac{\varepsilon}{\omega}+\frac{1}{3}\frac{\omega}{\varepsilon}\right], \tag{7}$$

and the minimal value is

$$\min_{\varepsilon}\{(-\mathrm{d}/\mathrm{d}(1/\varepsilon)) \ln S_1(\varepsilon)\} = \tfrac{3}{2}\omega\frac{2}{\sqrt{3}}, \tag{8}$$

close to the actual position of the level but still higher by a factor 1.15. Accounting for two more power corrections shifts the plateau somewhat down but nevertheless its height exceeds $\frac{3}{2}\omega$ by 6%.

In many recent publications an erroneous procedure is adopted according to which the height of the plateau in fig. 1 (and analogous plots in other problems) is taken literally as the sum rule prediction for the ground-level position (e.g. [8–10]). The recipe is supplied by a reference to [1] and then its shortcomings are criticized. We agree that *this* procedure does have shortcomings, in particular, it inevitably leads to an underestimation of the "condensate parameter", as explained below. To avoid this misfortune we always introduce a correction factor reflecting the existence of higher states.

Let us explain this remark in more detail. Assume for a moment that the value of the leading-power correction is unknown theoretically and introduce an analogue of the condensate parameter of QCD, Ω^2, phenomenologically:

$$(-\mathrm{d}/\mathrm{d}(1/\varepsilon)) \ln S_1 = \tfrac{3}{2}\varepsilon+\frac{1}{2}\frac{\Omega^2}{\varepsilon}. \tag{9}$$

(We proceed as if unaware of the fact that $\Omega=\omega$). The minimum of the curve (9) is reached at $\varepsilon=\sqrt{\frac{1}{3}}\,\Omega$ and constitutes $\sqrt{3}\,\Omega$. If one requires the height of the plateau to coincide exactly with $E_0=\frac{3}{2}\omega$, one can "fit" the parameter Ω which in this case turns out to be equal to $\frac{1}{2}\sqrt{3}\,\omega$. Correspondingly, the relative weight of the leading-power correction turns out to be essentially underestimated in comparison with its actual value. In the given example the underestimation amounts to 30%, but in similar problems it may be much stronger.

Other shortcomings mentioned in [8–10] have the same origin.

Now, a few words about our method. First of all, in real applications the exact answer is never known and we have to deal with one or at most two or three terms in the power expansion. Therefore, to ensure control over the whole series we stop at the point where the correction is still a correction, i.e. does not exceed, say, 30% of the leading term. In the given example this means that we stop at $\varepsilon=0.9\omega$ (arrow

A in fig. 1). Smaller values of ε and, in particular, the domain of the minimum are not considered at all. On the other hand, if ε is too large, higher states overshadow the ground-state contribution and a reliable calculation of the ground-state parameters becomes impossible. Thus we introduce one more boundary of the fiducial domain (arrow B in fig. 1). To this end we estimate the contribution of all higher states (let us call it the "continuum" contribution) in the approximation of free motion:

$$S_{\text{"cont"}}(\varepsilon)=\int_{E_c}^{\infty}\frac{2\sqrt{2}}{\pi}\,m^{3/2}E^{1/2}\exp\left(-E/\varepsilon\right), \tag{10}$$

(compare with eq. (5)). Here E_c is an effective "continuum" threshold to be treated as a fit parameter. Apart from E_c, $S_{\text{"cont"}}$ contains no information on the interaction. Arrow B marks the boundary where the "continuum" weight amounts to 30%. Between arrows A and B it is less than 30% and this is just our fiducial region. We write

$$S_{\text{"res"}}=r\,\mathrm{e}^{-E_0/\varepsilon},$$

with unknown constants r (residue) and E_0 (the ground-level position) and fit r, E_0, E_c by requiring coincidence between $(S_{\text{"res"}}+S_{\text{"cont"}})$ and S_{theor} inside the fiducial domain. Fig. 2 shows $S_{\text{"res"}}+S_{\text{"cont"}}$ with

$$r=\frac{4}{\sqrt{\pi}}(m\omega)^{3/2}, \qquad E_0=\tfrac{3}{2}\omega, \qquad E_c=\tfrac{5}{2}\omega, \tag{11}$$

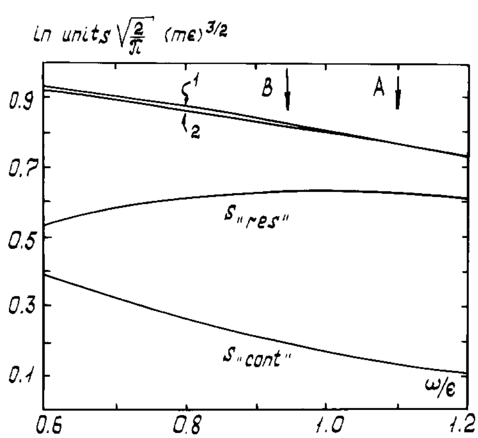


Fig. 2. Determination of E_0 and $|R_{00}(0)|^2$ for the hadronic oscillator from the sum rule. $S_{\text{"res"}}$ represents the ground state and $S_{\text{"cont"}}$ is an estimate of a higher-state contribution (eq. (10)). The double line in the upper part of the plot depicts $S_{\text{"cont"}}+S_{\text{"res"}}$ (curve 1) and $S_2=\sqrt{(2/\pi)}\,(m\varepsilon)^{3/2}(1-\frac{1}{4}(\omega/\varepsilon)^2+\frac{19}{480}(\omega/\varepsilon)^4)$ (curve 2). The meaning of arrows A and B is explained in the text.

in comparison with

$$S_2 = \sqrt{\frac{2}{\pi}}(m\varepsilon)^{3/2}\left[1 - \frac{1}{4}\left(\frac{\omega}{\varepsilon}\right)^2 + \frac{19}{480}\left(\frac{\omega}{\varepsilon}\right)^4\right]. \tag{12}$$

Near the upper boundary (A) the second-power correction amounts to ~6% and therefore it is necessary to include it into the analysis if one wishes to achieve percent accuracy. On the other hand, the "continuum" contribution here is about 20%. Even if the estimate (10) based on free motion were off by a factor of 1.2, the error would affect predictions for r and E_0 only at the one-percent level. Deviations from (11) exceeding this level would immediately destroy the beautiful agreement between "phenomenological" and theoretical parts of the sum rule.

Of course, the procedure is awkward as compared with conventional approaches to quantum mechanics. It has only one virtue: it can be generalized to the case of real hadrons. For us it is important only that a literal application of the sum-rule technique does not result in any inconsistency – contrary to the assertions of refs. [8–10].

Notice that the position of the upper boundary A depends on the number of power terms known theoretically. If one wishes to include the minimum of the stability plateau in the fiducial region one should take into account sufficiently many terms. For instance, with four or five terms of the expansion (6) one can safely shift the boundary of the fiducial region up to $\omega/\varepsilon = 2$. On the other hand, the minimum of the stability plateau is reached approximately at $\omega/\varepsilon = 1.9$ (fig. 1). It is quite clear that expanding the fiducial region results in more reliable predictions. In quantum-mechanical problems one can compute, starting from a given potential, arbitrary many power terms. However, in QCD we are limited by one or two power terms, and the length of the fiducial region and the very existence of it are not guaranteed *a priori*. There is no universal recipe. In some "lucky" channels the fiducial region is sufficiently broad, but opposite examples are also known. Each particular sum rule requires separate investigation.

3. Operator expansion and factorization of composite operators

The Wilson operator product expansion (OPE) provides a general framework for a systematic study of non-perturbative effects within the QCD sum rules. It was argued [1c, 7] that the procedure is rather simple as far as the operator dimension is not too high ($d_{cr} = 11$ in pure gluodynamics and $d_{cr} = 15$ in a theory with two color triplets of massless quarks). Namely, the coefficient functions can be calculated perturbatively while non-perturbative contributions are entirely absorbed in the vacuum matrix elements of such operators as G^2 or $\bar{q}q$. This simplified (although justified) version seems to have caused some confusion and provoked recent criticism

[6]. It might be useful to recapitulate the basic points here (we hope to present a more comprehensive discussion of the issue elsewhere).

The OPE assumes that integration over all short-wavelength fluctuations, $\rho < \mu^{-1}$, is performed explicitly and the corresponding effect is represented by coefficient functions while the effect of large-scale fluctuations, $\rho > \mu^{-1}$, is referred to the vacuum matrix elements. Then the dependence on some external parameter q, $q^2 \gg \mu^2$ can be readily traced.

It is evident that the Wilson expansion, formulated in this way, cannot be invalid in any field theory. However, the procedure necessarily requires introduction of an additional parameter (μ, normalization point) which enters both the coefficient functions and vacuum matrix elements and cancels out only in the final sum.

It is important that coefficient functions and matrix elements within the framework of the genuine OPE contain powers of μ along with conventional logarithms. Thus, for instance, the coefficient of the unit operator should look as follows

$$Q^2 \frac{\mathrm{d}}{\mathrm{d}Q^2} C_1 = [C_0 + C_1 \alpha_s(Q) + \cdots] + [C_2 + C_3 \alpha_s(\mu) + \cdots] \frac{\mu^4}{Q^4} + \cdots .$$

Because of asymptotic freedom there is a well-defined way of accounting for all short-wavelength fluctuations: one can use the standard α_s expansion (above the critical dimension one should include also small-size instantons, see ref. [7]). Of course, one assumes here that $\mu^2 \gg \Lambda^2$ where Λ is the QCD scale parameter, $\alpha_s(\mu) = 2\pi / b \ln(\mu/\Lambda)$.

The power μ dependence of C_1 is actually compensated by the corresponding power term coming from the vacuum matrix element $\langle 0|G^2|0\rangle = a\mu^4 + b\Lambda^4$, where a and b may contain a $\log \mu$ dependence. As a result, the $1/Q^4$ correction in the correlation function has the form $C_G b\Lambda / Q^4$. Just this correction is extracted from a phenomenological analysis. (Notice, that above the critical dimension, there appear power corrections of a new type associated with power terms in the coefficient functions, direct instantons.) What is essential, is that it is impossible to formulate OPE in a mathematically correct way without an explicit introduction of μ. Once this is done it becomes possible to discuss purely theoretical, not only pragmatic aspects.

Moreover, we always keep in mind the following numerical observation which is quite specific for QCD as it exists in the real world. The point is that the contribution of large-scale ($\rho \sim \Lambda^{-1}$) non-perturbatve fluctuations turns out to be enormously large. In practical applications this allows one, first, to cut short the perturbative series and, second, to choose μ in such a way that the μ dependence of the matrix elements like $\langle 0|G^2|0\rangle$ can be safely neglected. In other words, the difference between the genuine OPE and the simplified (naive) variant is not important numerically.

To be more specific, let us introduce the mass scale M^2_{crit} where the power corrections to the asymptotic freedom become sizeable [7]. Then the statement is that it is possible to choose μ^2 such that

$$M^2_{\rm crit} \gg \mu^2 \gg \Lambda^2 .$$

(Let us recall to the reader's attention that μ is defined in such a way that $\alpha_s(\mu)/\pi \ll 1$.)

To give some impression about numbers let us quote the values of $M^2_{\rm crit}$ which vary, generally speaking, from channel to channel: $M^2_{\rm crit} \approx 0.6\,\mathrm{GeV}^2$ in the $J^P = 1^-$ $\bar{\mathrm{q}}\mathrm{q}$ channel; $M^2_{\rm crit} \approx 2\,\mathrm{GeV}^2$ in the $J^P = 0^-$ $\bar{\mathrm{q}}\mathrm{q}$ channel; $M^2_{\rm crit} \approx 10\,\mathrm{GeV}^2$ in the $J^P = 0^{\pm}$ gluon channel; these are to be compared with

$$\Lambda^2 = (0.01\text{–}0.04)\,\mathrm{GeV}^2 .$$

Thus, there is enough room for a proper choice of μ.

Of course the situation (with power corrections much more important than the log ones) might not be to everybody's taste. One can ask whether there are other simple examples with the same pattern of corrections.

It can be readily shown that the two-dimensional sigma model (in the large-N limit) does share some of the features described above. In more detail, the series of log corrections in this model turns out to be finite (in the leading order of the $1/N$ expansion) and coexists with non-perturbative or power corrections. Moreover, in the leading $1/N$ approximation there is no need to be careful with the introduction of μ since the matrix elements are μ independent in this order. However, in this model $\Lambda \equiv m$ where m is the physical mass. Therefore, it is impossible to discuss next-to-leading $1/N$ corrections without explicitly introducing μ dependence.

A number of papers [6] appeared recently which doubt the validity of OPE as it is used in the sum rules. We have checked that the paradoxes mentioned in these papers basically arise due to a neglect of the normalization point μ. Another problem relevant to the Schwinger model is the following. Our construction of OPE implies the Hilbert space of states of perturbation theory, and we refer just to this operator basis, not to the exact operator solution. A more detailed discussion of these issues will be given in a separate publication.

Now we proceed to the question of convergence of the series of power corrections raised in a recent paper [11]. In this paper the next-to-leading power corrections associated with G^3 and G^4 operators have been calculated in the charmonium sum rules. Certainly, this is a very important development of the charmonium sum rules.

The results of ref. [11] are basically as follows. The coefficients in front of the G^3 operators in the ratio of moments M_n/M_{n-1} turn out to be numerically small and these can be safely neglected in the numerical analysis. On the contrary, the coefficient functions for the operators G^4 are considerably larger than one might expect starting from non-relativistic estimates. Moreover, according to ref. [11] these terms destroy the original analysis since no fiducial interval is left. In other words, there are no moments in which G^2 terms can be kept and G^4 terms can be omitted.

It is worth making two comments in connection with this. The first one is actually due to Rubinstein*. Define moments $\tilde{M}_n$ at some point $Q_0^2 \neq 0$:

$$\tilde{M}_n(Q_0^2) = \int \frac{R_c(s)\,\mathrm{d}s}{(s+Q_0^2)^n}\,.$$

Then it was found in ref. [16] that even in the G^2 approximation (the only one available at that time) the sum rules were most stable at $Q_0^2 \sim 4m_c^2$ (though they produce the same value of $\langle G^2\rangle$ as the analysis of ref. [1]). There is an important technical difference between the two analyses. At $Q_0^2 \sim 4m_c^2$ the G^4 term is relatively suppressed by a factor of ~ 4, and the results of ref. [16] stay untouched even if the recent estimate [11] of G^4 terms is taken without any criticism. Thus, the fiducial interval in the charmonium sum rules does exist.

The comment of our own is as follows. Let us still try to stick to the sum rules at $Q^2=0$ which (although successful phenomenologically) are ruined according to ref. [11]. Our observation is that the change in estimates of the G^4 matrix elements needed to reconcile them with the $Q^2=0$ sum rules is rather small. We keep here in mind the estimates based on factorization of the matrix elements $\langle G^4\rangle \sim \langle G^2\rangle^2$**. Moreover, we will show that unlike the case of the four-quark operators, $(\bar{q}q)^2$, G^4 operators do not factorize in the large-N_c limit. This might explain why the factorization hypothesis which is known to be valid to within 10% in the quark case is off by $\sim 50\%$ in the case of gluon operators.

We proceed now to a more technical elaboration of these points. There are four independent structures of order G^4***:

$$\begin{aligned}
O_4^1 &= \langle g^4 \,\mathrm{Tr}\, G_{\mu\nu}G_{\mu\nu}G_{\alpha\beta}G_{\alpha\beta}\rangle\,,\\
O_4^2 &= \langle g^4 \,\mathrm{Tr}\, G_{\mu\nu}G_{\alpha\beta}G_{\mu\nu}G_{\alpha\beta}\rangle\,,\\
O_4^3 &= \langle g^4 \,\mathrm{Tr}\, G_{\mu\nu}G_{\nu\alpha}G_{\alpha\beta}G_{\beta\mu}\rangle\,,\\
O_4^4 &= \langle g^4 \,\mathrm{Tr}\, G_{\mu\nu}G_{\alpha\beta}G_{\nu\alpha}G_{\beta\mu}\rangle\,,\\
&(G_{\mu\nu} = \tfrac{1}{2}G_{\mu\nu}^a t^a)\,.
\end{aligned} \tag{13}$$

* Two of the authors (V.Z. and V.N.) gratefully acknowledge the discussion of this point with Prof. Rubinstein in May, 1983. A similar remark has been made by the referee (September, 1983).

** Instanton based [17] estimates also discussed in ref. [11] are being thoroughly investigated by Shuryak now and are not touched upon here.

*** There are three extra operators listed in ref. [11]. Two of them ($O_4^5 = g^5 f^{abc} G_{\mu\nu}^a j_\mu^b j_\nu^c$, $O_4^7 = g^4 j_\mu^a D^2 j_\mu^a$ vanish if one neglects effects due to light quarks. Their matrix elements must be negligible anyhow. The third extra operator ($O_4^6 = g^3 f^{abc} G_{\mu\nu}^a G_{\nu\lambda}^b D^2 G_{\lambda\mu}^c$) reduces, due to the equation of motion, to those of eq. (13), $O_4^6 = 8(O_4^3 - O_4^4)$ + inessential terms with light quarks. The estimates of ref. [11] do not incorporate these implications of the equations of motion and we somewhat modify them for this reason.

The most important operator whose weight is maximal is

$$O_4^3 - O_4^4 = \tfrac{1}{4}\langle g^4 (f^{abc} G^a_{\mu\alpha} G^b_{\alpha\nu})^2 \rangle \,. \tag{14}$$

Combining the calculated coefficient functions with the factorization hypothesis,

$$\langle O_1 O_2 \rangle = \langle O_1 \rangle \langle O_2 \rangle \,, \tag{15}$$

(O_1, O_2 are some colourless operators, see also below) Nikolaev and Radyushkin get a 3.7% correction in the ratio M_n/M_{n-1} at $n=6$. Our number for the $O(m_c^{-8})$ term in M_6/M_5 within the same framework is 3.1%. The difference is due to a more consistent treatment of the matrix elements of operators $O_4^5 - O_4^7$ (see the third footnote on the previous page).

Now, the sum rules are a strikingly sensitive instrument. We can say phenomenologically that at the edge of the fiducial region, $n=6$, higher-order terms (higher than $O(G^2) \approx 10\%$ and $O(\alpha_s) \approx 10\%$) should be positive and consititute approximately 2%, not 3.1%. The conclusion of ref. [11] is that the sum rules are inconsistent because of this discrepancy. The alternative point of view, much more justified in our mind, is that the factorization does not work so well and the actual values of the matrix elements are lower by a factor of ~ 1.5.

To substantiate this standpoint we would like to demonstrate the difference between the factorization of the four-quark and four-gluon operators.

There is a single justification for the factorization (15) known in modern field theory. Namely, as was first noted by Migdal, the factorization holds in the large-N_c limit.

For example, the ρ-meson sum rules introduce the vacuum expectation value of operators like $(\bar{u}\gamma_\mu\gamma_5 t^a u)(\bar{u}\gamma_\mu\gamma_5 t^a u)$. Applying the Fierz identities we can rewrite them as follows:

$$(\bar{u}\gamma_\mu\gamma_5 t^a u)(\bar{u}\gamma_\mu\gamma_5 t^a u) = -\frac{2}{N_c}(\bar{u}\gamma_\mu\gamma_5 u)^2 + 2[(\bar{u}u)^2 - (\bar{u}\gamma_5 u)^2 + \tfrac{1}{2}(\bar{u}\gamma_\mu u)^2 + \tfrac{1}{2}(\bar{u}\gamma_\mu\gamma_5 u)^2] \,. \tag{16}$$

Now it is obvious that

$$\langle \bar{u}\gamma_\mu\gamma_5 t^a u \bar{u}\gamma_\mu\gamma_5 t^a u \rangle = 2\langle \bar{u}u \rangle^2 + O(N_c) \,. \tag{17}$$

Notice that the first term in the r.h.s. is of order N_c^2, so that the accuracy is $O(1/N_c)$. Phenomenologically it is O(5%).

Let us turn now to the four-gluon operators taking as an example

$$\Phi = f^{mab} f^{mcd} G^{[a}_{\mu_1\nu_1} G^{b]}_{\mu_2\nu_2} G^{[c}_{\mu_3\nu_3} G^{d]}_{\mu_4\nu_4} \,. \tag{18}$$

The square brackets denote antisymmetrization. The particular form of the contraction of Lorentz indices is inessential (see, e.g. (14)).

First of all we rewrite Φ identically

$$\Phi = f^{mab}f^{mcd}\left\{ G^{[a}_{\mu_1\nu_1}G^{b]}_{\mu_2\nu_2}G^{[c}_{\mu_3\nu_3}G^{d]}_{\mu_4\nu_4} - \frac{1}{N_c^2-3}(\delta^{ac}G^{[e}_{\mu_1\nu_1}G^{b]}_{\mu_2\nu_2}G^{[e}_{\mu_3\nu_3}G^{d]}_{\mu_4\nu_4} \right.$$
$$+ \delta^{ad}G^{[e}_{\mu_1\nu_1}G^{b]}_{\mu_2\nu_2}G^{[c}_{\mu_3\nu_3}G^{e]}_{\mu_4\nu_4} + \delta^{bc}G^{[a}_{\mu_1\nu_1}G^{e]}_{\mu_2\nu_2}$$
$$\times G^{[e}_{\mu_3\nu_3}G^{d]}_{\mu_4\nu_4} + \delta^{bd}G^{[a}_{\mu_1\nu_1}G^{e]}_{\mu_2\nu_2}G^{[c}_{\mu_3\nu_3}G^{e]}_{\mu_4\nu_4})$$
$$\left. + \frac{1}{(N_c^2-3)(N_c^2-2)}(\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc})G^{[e}_{\mu_1\nu_1}G^{f]}_{\mu_2\nu_2}G^{[e}_{\mu_3\nu_3}G^{f]}_{\mu_4\nu_4}\right\}$$
$$+ \frac{2N_c}{N_c^2-2}G^{[e}_{\mu_1\nu_1}G^{f]}_{\mu_2\nu_2}G^{[e}_{\mu_3\nu_3}G^{f]}_{\mu_4\nu_4}\,. \tag{19}$$

The expression in braces does not admit vacuum saturation and thus vanishes within the framework of the factorization hypothesis. On the contrary, the second term factorizes and gives

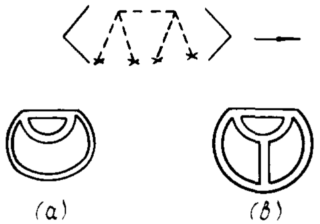
$$\frac{N_c}{(N_c^2-1)}(\langle G^e_{\mu_1\nu_1}G^e_{\mu_3\nu_3}\rangle\langle G^f_{\mu_2\nu_2}G^f_{\mu_4\mu_4}\rangle - \langle G^e_{\mu_1\nu_1}G^e_{\mu_4\nu_4}\rangle\langle G^f_{\mu_2\nu_2}G^f_{\mu_3\nu_3}\rangle)\,, \tag{20}$$

i.e. of order N_c^3. Just such a behaviour is expected since it ensures stability in N_c. Indeed, the bare quark loop $\sim N_c$. If $\langle G^4\rangle \sim N_c^3$ and $g^4 \sim N_c^{-2}$ then the $O(m_c^{-8})$ correction is also proportional to N_c.

It is important that that non-factorizable expression in braces in eq. (19) is also $O(N_c^3)$. The assertion becomes obvious in the 't Hooft diagram technique [20] (fig. 3).

What is the difference between fermionic and gluonic operators? The point is that the factorizable gluonic operator appears in the decomposition (19) with a small coefficient, $\sim 1/N_c$, while the coefficient of the non-factorizable structure is proportional to N_c^0. For four-fermion operators the weight of different terms in analogous decomposition is equal (see (16)). In other words, it is quite natural that

$$\left\{\langle\Phi\rangle \Big/ \left(\frac{N_c(N_c^2-2)}{(N_c^2-1)^2}[\langle G^e_{\mu_1\nu_1}G^e_{\mu_3\nu_3}\rangle\langle G^f_{\mu_2\nu_2}G^f_{\mu_4\nu_4}\rangle - (4\leftrightarrow 3)]\right)\right\} - 1 \sim 1\,. \tag{21}$$



(a) (б)

Fig. 3. 't Hooft diagrams for the vacuum expectation value of the four-gluon operator Φ, eq. (18). (a) Factorizable piece, three colour loops, (b) Non-factorizable piece, four colour loops times extra g^2.

At the same time the procedure of ref. [11] is actually equivalent to neglecting the r.h.s. of (21) – the prescription which has absolutely no grounds.

This simple observation may be formulated in somewhat more general terms if one invokes the notion of the masterfield [19]. Recall that the masterfield of multicolour chromodynamics is, by definition, a four-potential A_μ^{master}, essentially non-abelian and x independent in an appropriate gauge, which saturates the relations like (15). It is impossible to underestimate the simplicity and attractiveness of such an object. For us, however, only one aspect is important here. Even if the masterfield (in its original understanding [19]) exists at $N_c \to \infty$ and will be found, this will be of no help in computations of the spectrum of real hadrons, say, heavy quarkonium. Indeed, the answer for the correlation function of the quark currents in a constant non-abelian potential (even for $N_c = \infty$) has nothing to do with the genuine value of the correlation function. This is readily seen from the arguments above. In particular, the vacuum expectation value $\langle \Phi \rangle$ which basically fixes the effects $O(m_c^{-8})$ is not saturated by the masterfield since substitution of the masterfield into braces, eq. (19), would yield N_c^2 instead of N_c^3. In the non-factorizable piece of Φ, quantum fluctuations dominate, their contribution being parameterically larger than that due to the classical masterfield. One can check that this situation is general and is not specific to the operator Φ. Gluonic operators of higher dimension determining next terms of the series have the analogous colour structure and are not saturated by the masterfield because colourless factorizable suboperators can be separated only at a price of introducing extra suppression factors $(1/N_c)^k$. Our feeling is that this fact – irrelevance of the masterfield to the problem of quarkonium spectrum – is often forgotten about.

4. Bottonium sum rules

It has been claimed recently [12] that the Υ sum rules require a hundred-fold increase in the value of $\langle G^2 \rangle$ as compared to the case of the charmonium sum rules.

This conclusion, to our mind, is due to the neglect of the basic features of the Υ sum rules. The point is that the sum rules for Υ are not a mechanical repetition of the charmonium sum rules. The difference is rooted in physics, not in, say, the fit procedure.

Indeed, for the same moment-number n the role of excited states is more conspicuous in the case of Υ (see below). If one considers higher n, then the dominance of the lowest level can be made almost complete, however, the Coulomb-like corrections become far more important. This is the most difficult problem with the Υ sum rules. The physics behind it is that the Coulomb-like interaction is much more important for the heavier b-quark than for the c-quark.

The very notion of duality is somewhat changed. The J/ψ is dual to the quark continuum starting from the (fictitious) quark threshold $2m_c$ up to some energy between the J/ψ and ψ' masses. On the other hand, Υ lies below the quark threshold and it is dual (at least partly) to the "would be" Coulomb-like levels in the ($b\bar{b}$) system.

The rest of this section is devoted to a more detailed exposition of these comments (for further details see refs. [14, 21]).

Let us recall to the reader's attention that usually one considers moments of cross section which are defined as

$$M_n = \int \frac{R_b(s)\,ds}{s^{n+1}}, \tag{22}$$

Here R_b is the cross section of e^+e^- annihilation into hadrons, in units of $\sigma(e^+e^- \to \mu^+\mu^-)$, with b-quarks in the final state.

In the case of charmonium one discusses, say, $n_c \leqslant 8$. This does not suffice in the Υ case, however. The reason is that unlike the charmonium case, these moments are not saturated by the lowest state. For example, the Υ', Υ'' and Υ''' contributions to M_8 relative to that of Υ are respectively $20\pm1\%$, $9\pm0.5\%$ and $5\pm0.5\%$ which should be compared with 3% of the ψ' contribution to M_8 relative to that of J/ψ in charmonium. In other words, moments with such n for bottonium are sensitive to higher resonances and to details of the continuum threshold; the latter being poorly known experimentally at present. It is still more ambiguous to draw conclusions about the gluon condensate parameter from this kind of consideration (see e.g. ref. [12]). The reason is that the relative contribution of the term with $\langle(\alpha_s/\pi)G^2\rangle$ is $(m_b/m_c)^4 \sim 200$ smaller for bottonium sum rules than for charmonium ones at the same values of n. Therefore, for $n<9$ the power term is far smaller than the experimental uncertainty.

To suppress the contribution of higher states and to get thus "useful" sum rules one should obviously consider moments with larger n. One can readily see that the relative contribution of Υ' to the bottonium sum rules with the number n_b is the same as the ψ' relative contribution to the n_c moment if

$$\frac{n_b}{n_c} \approx \frac{m_{\psi'} - m_{J/\psi}}{m_{\Upsilon'} - m_{\Upsilon}} \frac{m_{\Upsilon}}{m_{J/4}} \approx 3.2\,. \tag{23}$$

Therefore, $n_c \leqslant 9$ corresponds to $n_b \leqslant 30$.

However, for high values of n one should take care of perturbative corrections since these are governed by the parameter $\alpha_s\sqrt{n}$ which is of order unity for such values for n. Fortunately, the large-n asymptotics of the sum rules is determined by non-relativistic dynamics of the heavy quark–antiquark pair ($b\bar{b}$ in the case considered), so that all $(\alpha_s\sqrt{n})^k$ terms can be summed up by solving the Schrödinger equation in the Coulomb potential. This has been done [21] in terms of the quark–antiquark propagator in imaginary time. The quantity $S(\varepsilon)$ considered in sect. 2 (eq. (3)) for the quark-antiquark pair is thus found in the form

$$S(\varepsilon) = \frac{1}{2\sqrt{\pi}}(m_b\varepsilon)^{3/2}\left[\Phi_s(\gamma) - \frac{\langle\pi\alpha_s G^2\rangle}{72m\varepsilon^3}X_s(\gamma)\right], \tag{24}$$

where $\gamma=\frac{2}{3}\alpha_s\sqrt{m/\varepsilon}$ is the "Coulomb" parameter and the functions $\Phi_s(\gamma)$ and $X_s(\gamma)$ describe the perturbative series in the limit of large m/ε (we shall see shortly that the ratio m/ε corresponds to n in M_n). The explicit form of $\Phi_s(\gamma)$ is

$$\Phi_s(\gamma)=1+2\sqrt{\pi}\,\gamma+\tfrac{2}{3}\pi^2\gamma^2+4\sqrt{\pi}\sum_{k=1}^{\infty}\left(\frac{\gamma}{k}\right)^3 e^{\gamma^2/k^2}\left[1+\operatorname{erf}\left(\frac{\gamma}{k}\right)\right], \quad (25)$$

($\operatorname{erf}(x)=(2/\sqrt{\pi})\int_0^x e^{-t^2}\,dt$) while the expression for $X_s(\gamma)$ is far more complicated (it can be found in ref. [21]). For values of γ relevant to the bottonium sum rules ($\gamma\leqslant 1.5$) $X_s(\gamma)$ can be approximated as

$$X_s(\gamma)\approx\Phi_s(\gamma)\exp(-0.80\gamma)\,. \quad (26)$$

Note that for $\gamma=1$ (this corresponds to $n\simeq 25$ for bottonium) $\Phi_s(\gamma)=49.12\ldots$ which shows that gluon exchange between quarks essentially determines the result instead of being a source of a moderate correction for small n.

The quantity $S(\varepsilon)$ is simply related to the integral of $R_b(s)$ with an ε-dependent exponential weight:

$$\tilde{M}\left(\frac{m_b}{\varepsilon}\right)=\int R_b(s)\exp\left[\frac{4m_b^2-s}{4m_b\varepsilon}\right]ds=\left(1-\frac{16\alpha_s(m_b)}{3\pi}\right)2\pi S(\varepsilon)\,. \quad (27)$$

On the other hand, the power weight in the moments M_n at large n can also be approximated by an exponential one

$$\begin{aligned}(4m_b^2)^{n+1}M_n&=\int ds\,R_b(s)(s/4m_b^2)^{-n-1}\\ &=\int R_b(s)\left(1+\frac{s-4m_b^2}{4m_b^2}\right)^{-n-1}ds\\ &=\int R_b(s)\exp\left[\frac{4m_b^2-s}{4m_b^2}n\right]\left(1+O\left(\frac{1}{n}\right)\right)ds\,.\end{aligned} \quad (28)$$

Thus, for large n

$$\tilde{M}\left(\frac{m}{\varepsilon}=n\right)=(4m_b^2)^{n+1}M_n\left(1+O\left(\frac{1}{n}\right)\right), \quad (29)$$

and eq. (27) can be viewed as the large-n asymptotics of the moments (22).

Note also that in eq. (27) we have included the factor $(1-16\alpha_s(m_b)/3\pi)$ which is the familiar perturbative correction to the e^+e^- decay rate of a $q\bar{q}$ resonance in terms of $|R_{n0}(0)|^2$. Thus, eqs. (24) and (27) incorporate all the terms of order $(\alpha_s\sqrt{n})^k$ and $\alpha_s(\alpha_s\sqrt{n})^k$, therefore those which are not present have relative magnitude α_s^2 or $1/n$.

The sum rules represented by eq. (27) for $m/\varepsilon\gtrsim 25$ are "useful" in the sense that they are saturated practically by the Υ resonance alone. On the other hand, the power correction is still small ($\leqslant 3\%$ for $m/\varepsilon=25$ and $\sim 10\%$ for $m/\varepsilon=45$). Thus, the region of $25\leqslant m/\varepsilon\leqslant 45$ is what we call here the fiducial region of the sum rule parameter. (It is quite possible that eq. (27) with $S(\varepsilon)$ given by eq. (24)

can be used also for $m/\varepsilon \gtrsim 45$; at least there is no deviation of the theoretical curve for $S(\varepsilon)$ from the one given by the Υ contribution up to $m/\varepsilon \approx 60$. However, the indicated region of m/ε is quite sufficient for practical purposes). The only new parameter in the bottonium sum rules is m_b, i.e. the position of the Υ resonance relative to the quark threshold $2m_b$.

An analysis along these lines [21], gives $2m_b - m_\Upsilon = 130 \pm 50$ MeV, or $m_b = 4795 \pm 25$ MeV if the b-quark mass is measured at momentum p^2 such that $m_b^2 - p^2 \approx 1\ \text{GeV}^2 \ll m_b^2$. One readily notices that Υ lies below the quark threshold in contrast to J/ψ which cites well above the bare $c\bar{c}$ threshold. This is a manifestation of the Coulomb-like attraction between quarks due to the gluon exchange. One would expect such behaviour on general grounds. In terms of the sum rules this can be explained by the fact that the J/ψ is dual to a region of the continuum cross section where characteristic values of quark velocity are not small, so that the Coulomb-like interaction described by α_s/v gives only a small correction. On the contrary, Υ is dual to the sum of the "would be" $b\bar{b}$ Coulomb bound states and the very beginning of the bare $b\bar{b}$ continuum where the cross section is strongly distorted by the Coulomb-like interaction. The center of gravity of the bare $b\bar{b}$ states to which Υ is dual is shifted to the region of the "would be" Coulomb levels which lie below the bare $b\bar{b}$ threshold. This remark explains qualitatively also why the Coulomb gluon exchange can by no means be neglected when considering the sum rules for bottonium.

As for the gluon condensate parameter, its precise value within a factor of 1.5 around the "standard" one is not crucial for an analysis of S-wave states of $b\bar{b}$. However, its enhancement, say, by a factor of 3–5 would destroy the existing agreement between eq. (27) and the data on Υ resonances.

So far we described the original analysis of the bottonium sum rules which, to our mind, constitutes the framework for any attempt to understand the bottonium spectrum within the sum rule method. The reader must keep in mind, however, that there are a few subtle points which are not fully understood yet.

First, the equations quoted above indicate that the result is extremely sensitive to the coupling constant $\alpha_s(r)$, or to a possible n dependence of its normalization point. Usually in perturbative calculations one performs computations with a fixed α_s and only at the end is it substituted for by the effective coupling constant corresponding to a characteristic off-shellness in the problem considered. In the case of the bottonium sum rules such a procedure does not seem to be adequate because of the sharp dependence on α_s. Elucidation of the effects of the "running" nature of α_s becomes one of the most urgent problems here*.

* An attempt in this direction is now being made by Shuryak (to be published). In particular, he shows that a recent analysis of Baier and Pinelis (Novosibirsk preprints, 1982) is plagued by the same fundamental shortcomings. Considering just the same problem as they did Shuryak finds the value of $\langle G^2 \rangle$ close to our original result [1] while according to the assertion of Baier and Pinelis, $\langle G^2 \rangle$ should be larger by a factor of 10.

There is another complication which emerged rather unexpectedly. The point is that the analysis sketched above is purely non-relativistic. Only in this limit does it turn out possible to sum up the whole α_s series. Naturally, predictions obtained in this way are valid up to relativistic corrections. Recent experiments have demonstrated, however, that relativistic effects on the level position can be as large as 40 or 50 MeV – this is just the measured value of fine splittings in the bottonium family (namely, $M(1^3P_2) - M(1^3P_0)$).

Literal application of the sum rules described in this section has led to a prediction of the position of P levels [21] which is about 70 MeV lower than the current experimental data. The disagreement seems to be due to either the running-α_s effects, or relativistic corrections; or both. There is a renewed interest in the problem now, and it seems premature to comment on the most recent attempts to reanalyse the bottonium sum rules.

To summarize, the bottonium case seems to be the most difficult one. The corresponding sum rules are most advanced on the technical side and still might miss some effects, such as relativistic corrections which are important numerically.

5. Matrix element $\langle 0|\tilde{a}_\mu|\pi\rangle$

In view of the preceding discussion it may be useful to have an alternative determination of $\langle G^2\rangle$ which is free of the problems related to the Coulomb-like interaction of heavy quarks. To take a fresh look at the problem we consider here new sum rules for light quarks which nevertheless are sensitive to the gluon condensate. The sum rules turn out to be powerful enough to rule out any considerable (say, by a factor 2–3) change in the "standard" fit.

Let us introduce the current

$$\tilde{a}_\mu = \tfrac{1}{2} g \bar{d}\gamma_\rho \tilde{G}^a_{\rho\mu} t^a u\,, \tag{30}$$

where $\tilde{G}_{\rho\mu} = \frac{1}{2}\epsilon_{\rho\mu\alpha\beta}G_{\alpha\beta}$, g is the strong-interaction coupling constant, t^a stand for the SU(3) Gell-Mann colour matrices, u and d are the light quark fields. The current $\tilde{a}_\mu$ has the same quantum numbers as the ordinary axial-vector current of the light quarks. Moreover, it is the only non-trivial local operator with dimension 5 with these quantum numbers. Indeed, another axial-vector of dimension 5, $\bar{d}\gamma_\rho\gamma_5 t^a G^\alpha_{\rho\mu} u$, reduces to a full derivative,

$$\bar{d}\gamma_\alpha\gamma_5(\tfrac{1}{2}g)t^a G^a_{\alpha\mu} u = i\partial_\rho(\bar{d}\gamma_\rho\gamma_5 D_\mu u) + (m_u + m_d)\bar{d}\gamma_5 D_\mu u\,, \tag{31}$$

and, as a consequence, the matrix element $\langle 0|\bar{d}\gamma_\rho\gamma_5\frac{1}{2}g t^a G^a_{\rho\mu} u|\pi^+\rangle$ vanishes in the chiral limit. This fact is often forgotten about (e.g. [22]).

Furthermore, introduce the matrix element

$$\langle 0|\tilde{a}_\mu|\pi^+\rangle = -if_\pi\delta^2 p_\mu\,, \tag{32}$$

where f_π is the $\pi\to\mu\nu$ constant, $f_\pi = 133$ MeV and the constant δ has the dimension

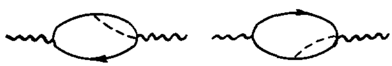


Fig. 4. Two-point function (33) in perturbation theory. Solid lines denote quarks; dashed lines, gluons; wavy lines, external currents.

of mass and is an independent dynamical characteristic of the pion. The minus sign complies with the positiveness of δ^2, see below.

We shall find δ in two different ways. First, we shall consider the non-diagonal two-point function $\langle a_\mu^+(x), \tilde{a}_\nu(0)\rangle$ and fix $f_\pi^2\delta^2$. Then we shall turn to the diagonal two-point function $\langle \tilde{a}_\mu^+(x), \tilde{a}_\nu(0)\rangle$ and fix $f_\pi^2\delta^4$.

Notice that the current quark masses are neglected. Then in momentum space

$$\pi_{\mu\nu} = i\int e^{iqx}\, dx\, \langle 0|T\{a_\mu^+(x), \tilde{a}_\nu(0)\}|0\rangle\,, \tag{33}$$

is proportional to $(q_\mu q_\nu - q^2 g_{\mu\nu})$ due to conservation of $a_\mu = \bar{d}\gamma_\mu\gamma_5 u$. Thus, there is only one independent kinematical structure.

As usual, the first step in the construction of the sum rules is the operator expansion. For $Q^2 \to \infty$ the two-point function (33) is determined by the graphs of fig. 4. Due to their two-loop nature these diagrams, however, play a minor role with respect to the condensate terms (fig. 5) in the interesting domain of $Q^2 \sim 1\ \mathrm{GeV}^2$. This fact, suppression of multiloop perturbative contributions, is common for all sum rules considered thus far. In the case considered it will be confirmed by an explicit calculation.

Apart from the numerical smallness of perturbative graphs, one should keep in mind that they are primarily dual to the continuum (not to the resonance (pion) contribution) and, therefore, the corresponding terms will partly cancel out in the right-hand and left-hand sides of the sum rules.

Diagram 5a corresponds to a gluon condensate, diagrams 5b and c give rise to the four-quark condensate. We shall limit our analysis to these leading effects, although in principle one or two next power terms are readily calculable.

Computation of graphs depicted in fig. 4 and 5c is trivial. As for the graphs of fig. 5a, b, the most economic technique of computation is that of the external field

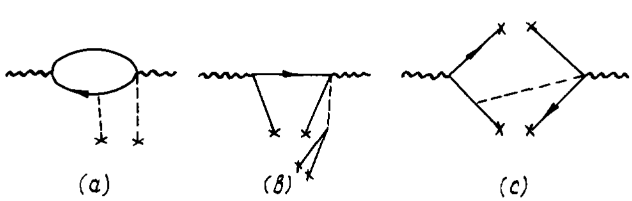


Fig. 5. Gluon and four-quark condensates in the two-point function (33). (a) The term proportional to $\langle \alpha_s G^2\rangle$; (b) the terms proportional to $\langle \bar{q}\gamma_\mu t^a q \bar{q}\gamma_\mu t^a q\rangle$.

[23]. The resulting final expression is the following:

$$\pi_{\mu\nu}=(q_\mu q_\nu-q^2 g_{\mu\nu})\left\{\frac{\alpha_s}{18\pi^3}Q^2\ln Q^2-\frac{1}{6Q^2}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\right.$$
$$\left.-\tfrac{16}{27}\pi\frac{1}{Q^4}\langle\sqrt{\alpha_s}\,\bar{q}q\rangle^2-\tfrac{16}{9}\pi\frac{1}{Q^4}\langle\sqrt{\alpha_s}\,\bar{q}q\rangle^2+\cdots\right\}, \tag{34}$$
$$Q^2\equiv -q^2\,.$$

Matrix elements $\langle\bar{q}\gamma_\mu t^a q\bar{q}\gamma_\mu t^a q\rangle$ are reduced here to $\langle\bar{q}q\rangle^2$ by means of factorization (see ref. [1] and sect. 3 above).

It is worth noting that the operator $\tilde{a}_\rho$ depends actually on the normalization point, or, in other words, has a non-vanishing anomalous dimension. The renorm-invariant combination looks as (see the paper by Shuryak and Vainshtein in ref. [23]):

$$\alpha_s^{-32/9b}\bar{\mathrm{d}}\gamma_\mu(\tfrac{1}{2}g)t^a\tilde{G}^a_{\mu\rho}u\,,$$

where $b=9$ is the first coefficient in the Gell-Mann–Low function. Strictly speaking, the sum rules should be modified for this. Choosing both the normalization point μ^2 and typical Q^2 to be of the same order, μ^2, $Q^2\sim\mathrm{GeV}^2$, we eliminate this modification. The residue found in this way will represent the pion-to-vacuum matrix element of $\tilde{a}_\rho$ normalized at $\mu=1$ GeV.

Further manoeuvres are quite standard and we omit a few steps of derivation. After the Borel transformation [1] we get

$$\frac{1}{M^2}(-f_\pi^2\delta^2)+\frac{r}{M^2}\exp(-m_{\mathrm{A}_1}^2/M^2)+\cdots$$
$$=-\left\{-\frac{\alpha_s(M)}{18\pi^3}M^2(1-\mathrm{e}^{-x_0}-x_0\,\mathrm{e}^{-x_0})+\frac{1}{6M^2}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle+\tfrac{64}{27}\pi\frac{1}{M^4}\langle\sqrt{\alpha_s}\,\bar{q}q\rangle^2\right\}.$$

Only the pion and A_1 contributions are written out explicitly on the phenomenological side of the sum rule. The A_1 residue is denoted by r:

$$r=f_{\mathrm{A}}^2\delta_{\mathrm{A}}^2\,,$$
$$\langle 0|a_\mu^+|A_1\rangle=\varepsilon_\mu m_{\mathrm{A}_1}f_{\mathrm{A}}\,,\qquad \langle A_1|\tilde{a}_\rho|0\rangle=\varepsilon_\rho m_{\mathrm{A}_1}f_{\mathrm{A}}\delta_{\mathrm{A}}^2\,, \tag{36}$$
$$m_{\mathrm{A}_1}\simeq 1.2\ \mathrm{GeV}\,,\qquad f_{\mathrm{A}}\simeq 0.17\ \mathrm{GeV}\,.$$

As usual, we accept the following model for the continuum

$$\mathrm{Im}\,\pi_{\mathrm{cont}}=\mathrm{Im}\,\pi_{\mathrm{fig.4}}\theta(s-s_0)\,, \tag{37}$$

and transfer the continuum to the r.h.s. (see the first term in braces where $x_0\equiv s_0/M^2$). The s_0 dependence is very weak. In numerical estimates we take $s_0=(1.5\text{–}1.8)\ \mathrm{GeV}^2$.

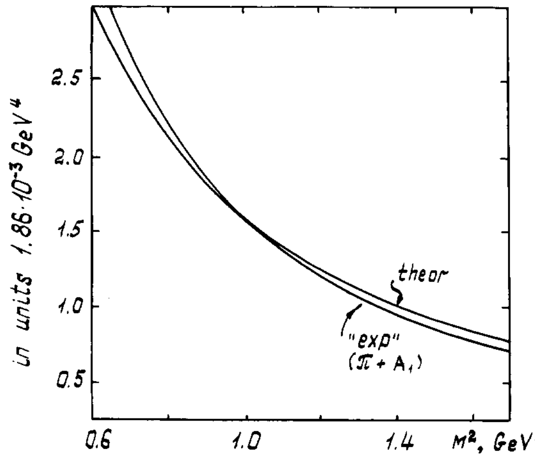


Fig. 6. Graphic representation of the sum rule (35).

Fig. 6 compares phenomenological and theoretical parts of the sum rule (35). For quark and gluon condensates we assume the "standard" values

$$\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle = 0.012\ \mathrm{GeV}^4 , \qquad \langle \sqrt{\alpha_s}\, \bar{q} q \rangle = -(0.24\ \mathrm{GeV})^3 . \tag{38}$$

Then the pion and A_1 residues turn out to be

$$\delta^2 = 0.21\ \mathrm{GeV}^2 , \qquad \delta_A^2 = 0.11\ \mathrm{GeV}^2 . \tag{39}$$

Now let us check self-consistency of this solution and consider to this end the diagonal correlation function

$$\tilde{\pi}_{\mu\nu} = i \int e^{iqx}\, d^4x \langle 0 | T\{ \tilde{a}_\mu^+(x), \tilde{a}_\nu(0)\} | 0 \rangle . \tag{40}$$

We apply the same procedure and make the same approximations as above. The only novel point is that $\tilde{\pi}_{\mu\nu}$ contains two independent structures $\sim q_\mu q_\nu$ and $\sim g_{\mu\nu}$. We write down sum rules for the structure function proportional to $q_\mu q_\nu$:

$$\frac{f_\pi^2 \delta^4}{M^2} + \frac{f_A^2 \delta_A^4}{M^2} \exp(-M_{A_1}^2/M^2) = \frac{\alpha_s(M)}{80\pi^3} M^4 (1 - e^{-x_0} - x_0\, e^{-x_0} - \tfrac{1}{2} x_0^2\, e^{-x_0})$$

$$+ \frac{1}{72} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle + \tfrac{8}{9}\pi \frac{1}{M^2} \langle \sqrt{\alpha_s}\, \bar{q} q \rangle^2 + \cdots . \tag{41}$$

Simple inspection shows that for $M^2 \sim 0.8$–$1\ \mathrm{GeV}^2$ the pion weight exceeds 90% (see eq. (39)). On the other hand, in this M^2 interval the quark condensate contribution is ~ 3 times larger than that of the gluon condensate and dominates the right-hand side. The perturbative graph of fig. 7 represented by the first term

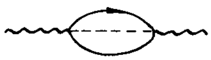


Fig. 7. Correlation function (40) in perturbation theory.

is totally negligible. Using the standard values of $\langle G^2\rangle$, $\langle\sqrt{\alpha_s}\,\bar{q}q\rangle$ we get

$$f_\pi^2\delta^4 \approx 1.2\tfrac{8}{9}\pi\langle\sqrt{\alpha_s}\,\bar{q}q\rangle^2\,, \tag{42}$$

or

$$\delta^2 \simeq 0.18\,\mathrm{GeV}^2\,. \tag{43}$$

One can expect that the accuracy of this prediction is O (10%). The estimate does not include possible uncertainties in $\langle(\alpha_s/\pi)G^2\rangle$ and $\langle\sqrt{\alpha_s}\,\bar{q}q\rangle$ and reflects only unaccounted effects due to higher-order power terms, fine structure of continuum, etc.

Now we are able to make a remark for the sake of which this whole business was initiated. Agreement between eqs. (43) and (39), within expected accuracy,

$$\delta^2 = (0.20\pm0.02)\,\mathrm{GeV}^2\,, \tag{44}$$

is extremely non-trivial. Indeed, the residue extracted from the non-diagonal correlation function (33) is very sensitive to the value of the gluon condensate. In particular, increasing $\langle(\alpha_s/\pi)G^2\rangle$ by a factor 2.5–3 automatically would entail the corresponding growth of δ^2 (roughly speaking, by 2–2.5 times). In this respect, the sum rule (35) is drastically different from the mesonic problems discussed previously [1] in which the gluon condensate represented a modest correction against the background of the dominating perturbative term.

On the other hand, such a drastic change of δ^2 would result in a sharp disagreement in the diagonal two-point function (40). Recall that the pion contribution to $\tilde{\pi}_{\mu\nu}$ is proportional to δ^4. The disagreement might be eliminated only at a price of increasing the quark condensate $\langle\sqrt{\alpha_s}\,\bar{q}q\rangle$ by a factor of 2–2.5. Needless to say, such an arbitrary change of the matrix element $\langle\sqrt{\alpha_s}\,\bar{q}q\rangle$ is impossible. Thus, by fixing the matrix element

$$\langle 0|\bar{d}\gamma_\rho(\tfrac{1}{2}g)t^a\tilde{G}^a_{\rho\mu}u|\pi\rangle\,,$$

in a consistent way from two different sum rules we, in addition, convince ourselves that the standard value of $\langle(\alpha_s/\pi)G^2\rangle$ is by no means underestimated by a factor of 2.5–3. Moreover, if one followed the claims of refs. [8–11] and made the gluon condensate larger, the result would not be self consistent.

Of course, this conclusion does not rule out the possibility of moderate (~10–40%) variations in the estimate of $\langle(\alpha_s/\pi)G^2\rangle$.

In further applications we shall use the averaged value of δ^2, see eq. (44). Note that the first attempt to find δ^2 using the PCAC technique has been undertaken in ref. [22]. Unfortunately, the theoretical consideration of this paper is erroneous in some points, although the numerical value of δ^2 obtained in [22] is close to (44). Moreover an estimate of δ^2 from a similar but not identical sum rule has been obtained in ref. [28] and the result agrees with (44).

6. $\pi^0\gamma^*\gamma^*$ vertex

In this section we shall demonstrate that the prediction for $\langle 0|\tilde{a}_\mu|\pi\rangle$ made in sect. 5 can be confirmed, at least indirectly, by considering the transition of π^0 into two virtual photons. This transition is interesting in its own right. In particular, it has been discussed in refs. [22, 24] where one can find an introduction to the problem and an exposition of the basic technique. We concentrate mostly on the points whose treatment was not quite satisfactory to our mind.

The amplitude A of the process

$$\gamma(q_1)+\gamma(q_2)\to\pi^0(p)\,, \tag{45}$$

can be written as

$$A=i\,e^2\int \mathrm{e}^{-iq_1x}\langle\pi^0|T\{j_\mu(x),j_\nu(0)\}|0\rangle\varepsilon^{(1)}_\mu\varepsilon^{(2)}_\nu \tag{46}$$

$$\equiv F(p^2,q_1^2,q_2^2)\varepsilon_{\mu\nu\alpha\beta}\varepsilon^{(1)}_\mu\varepsilon^{(2)}_\nu q_{1\alpha}q_{2\beta}\,, \tag{46}$$

where $j_\mu=\frac{2}{3}\bar{u}\gamma_\mu u-\frac{1}{3}\bar{d}\gamma_\mu d$ is the electromagnetic current. Then the triangle anomaly plus the soft-pion technique imply

$$F(0,0,0)=-\frac{\sqrt{2}\,\alpha}{\pi f_\pi}\,. \tag{47}$$

On the other hand, for a large euclidean Q^2 form factor $F(p^2=m_\pi^2=0,-Q^2,-Q^2)$ can be found by means of the operator expansion. If p^2 is put equal to zero and only two leading terms are kept, then the answer reduces to the following simple form

$$A=\frac{1}{3\sqrt{2}}8\pi\alpha\varepsilon^{(1)}_\mu\varepsilon^{(2)}_\nu\varepsilon_{\mu\nu\alpha\chi}\left\langle\pi^0\left|\frac{q_\alpha}{q^2}\,i\bar{\psi}\gamma_\chi\gamma_5\psi-\frac{8}{9}\frac{q_\alpha}{q^4}\,i\bar{\psi}\gamma_\rho(\tfrac{1}{2}g)t^a\tilde{G}^a_{\rho\chi}\psi\right|0\right\rangle\,,$$
$$\bar{\psi}\psi\to\sqrt{\tfrac{1}{2}}\,(\bar{u}u-\bar{d}d)\,, \tag{48}$$

where we account for the fact that the operator $\bar{\psi}\gamma_\rho\gamma_5G^a_{\rho\chi}t^a\psi$ can be safely omitted because of eq. (31). In the notation of eq. (32) we get

$$\frac{F(0,-Q^2,-Q^2)}{F(0,0,0)}=\tfrac{4}{3}\pi^2f_\pi^2\left\{\frac{1}{Q^2}-\frac{8}{9}\frac{\delta^2}{Q^4}-\frac{5}{6}\frac{\alpha_s(Q)}{\pi Q^2}\right\}\,, \tag{49}$$

where the perturbative correction $O(\alpha_s)$ found in [22] is also included.

Analysing the form factor $F(0,-Q^2,-Q^2)$ at intermediate Q^2 one can, in turn, obtain information on the coupling constant $g_{\rho\omega\pi}$. Although the latter has been several times discussed within the sum-rule approach [4] we would like to return to the issue and make a few critical comments about the procedure proposed in [4e].

Since the function $F(0,q_1^2,q_2^2)$ is calculated above only for a special choice of variables, $q_1^2=q_2^2=-Q^2$, there is no possibility to perform an independent borelization with respect to both q_1^2 and q_2^2 which would ensure the effective exponential suppression of "parasitic" contributions associated with the continuum. On has to limit oneself by simultaneous borelization in Q^2, as it was done in [4e]. In this case the continuum contribution is suppressed, generally speaking, only by some powers, not exponentially, and can essentially affect the result for the resonance coupling constants introducing corrections of order O (50%).

Let us explain the assertion in more detail. One can write for the function $F(0, -Q^2, -Q^2)$ the dispersion representation without subtractions [25]

$$F(0, q_1^2, q_2^2) = \frac{1}{\pi^2} \int \frac{\rho(s, s')\,\mathrm{d}s\,\mathrm{d}s'}{(s-q_1^2)(s'-q_2^2)}. \tag{50}$$

It is important that in the r.h.s. there are no terms like

$$P(q_1^2) \int \frac{\Delta(s')\,\mathrm{d}s'}{(s'-q_2^2)}, \qquad P'(q_2^2) \int \frac{\Delta(s)\,\mathrm{d}s}{s-q_1^2},$$

where P, P' denote polynomials. If such terms were present one could not apply simultaneous borelization at all, since making the simultaneous Borel transformation with respect to Q^2 would introduce non-controllable terms into the sum rules. Actually, the validity of representation (50) was accepted in [4e] without proof.

The resonance contribution we are interested in has the form (fig. 8)

$$\rho_{\text{res}} = -8\pi\alpha \frac{m^4}{3g_\rho^2} g_{\rho\omega\pi} \pi\delta(s-m^2)\pi\delta(s'-m^2), \tag{51}$$

where $g_{\rho\omega\pi}$ parametrizes the $\rho\omega\pi$ vertex

$$A(\rho\omega\pi) = -g_{\rho\omega\pi}\varepsilon_{\mu\nu\alpha\beta}\varepsilon_\mu^{(\rho)}\varepsilon_\nu^{(\omega)}p_\alpha^{(\rho)}p_\beta^{(\omega)},$$

$m^2 = m_\rho^2 = m_\omega^2$, and we have accounted for the fact that $g_\omega = 3g_\rho$. Apart from the resonance vertex of fig. 8, the three-point function (46) is contributed also by transitions like

$$\text{continuum} \to \pi + \text{continuum}, \tag{52a}$$

$$\text{continuum} \to \rho + \pi, \tag{52b}$$

etc. (fig. 9). Let us denote the spectral densities as $\rho_{\text{c-c}}$, and $\rho_{\text{c-r}}$, respectively.

Considering eq. (49) at $Q^2 \to \infty$ (the asymptotic freedom domain) one can easily convince oneself that $\rho_{\text{c-c}}$ reduces to

$$\rho_{\text{c-c}}(s, s') = \tfrac{1}{3}F(0,0,0)4\pi^2 f_\pi^2 \pi^2 \delta(s-s')\theta(s-s_0)\theta(s'-s_0), \tag{53}$$

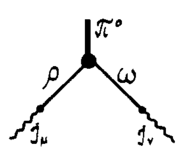


Fig. 8. Resonance contribution to the three-point function (46).

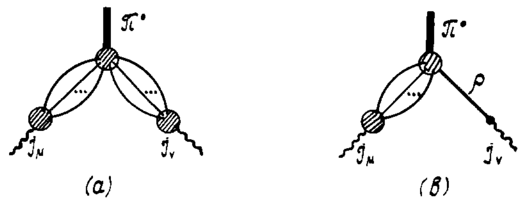


Fig. 9. Continuum contribution to the three-point function (46).

where s_0 is an effective continuum threshold. Then

$$F_{\mathrm{c-c}}(0, -Q^2, -Q^2) = \tfrac{1}{3}F(0,0,0)4\pi^2 f_\pi^2 \frac{1}{Q^2+s_0}, \tag{54}$$

(compare with (49)).

As to the spectral density $\rho_{\mathrm{c-r}}$, one can roughly estimate it starting from the following arguments. The discontinuity of the function $F(0, q_1^2, q_2^2)$ at $q_1^2 = m^2$ is actually proportional to the form factor of the process $\gamma(q_2) \to \pi\rho$. Within the framework of QCD it is well-known that asymptotics of this form factor at $q_2^2 \to \infty$ do not contain $O(q_2^{-2})$ terms, only $O(q_2^{-4})$ [26].

Therefore, the leading term in Disc $F(0, q_1^2, q_2^2)$ which is generated by $\rho_{\mathrm{res}}(s, s')$ should be cancelled by a negative contribution coming from $\rho_{\mathrm{c-r}}(s, s')$. In other words,

$$\rho_{\mathrm{res}}(s, s') + \rho_{\mathrm{c-r}}(s, s') \xrightarrow[s=m_\rho^2]{} \delta(s-m^2)\varphi(s'), \tag{55}$$

where the function φ is schematically depicted in fig. 10.

The absence of the $O(q_2^{-2})$ piece in the asymptotics of $\mathrm{Disc}_{q_1^2=m^2}F(0, q_1^2, q_2^2)$ means that the areas shaded by horizontal and vertical lines are equal to each other. The region A corresponds to ρ_{res}, the region B, to $\rho_{\mathrm{c-r}}$. One can argue that the negative "valley" is extremely stretched in energy and is not deep, so that the point s_0 lies rather high, $s_0 \gtrsim 3\ \mathrm{GeV}^2$. (Indeed, if s_0 were of order $1\ \mathrm{GeV}^2$ this would

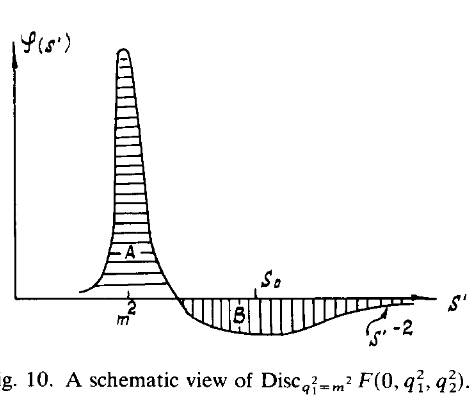


Fig. 10. A schematic view of $\mathrm{Disc}_{q_1^2=m^2}\, F(0, q_1^2, q_2^2)$.

result in a breaking of vector dominance in the $\gamma\rho\pi$ form factor at the same 1 GeV^2. However, Eletsky and Kogan (see ref. [4c]) have checked that vector dominance is valid up to $q^2 \sim 3\ \text{GeV}^2$).

For a *very rough estimate* the broad and flat minimum can be substituted by a delta function situated at s_0,

$$\varphi(s') \sim \{\delta(s'-m_\rho^2) - \delta(s'-s_0)\}\,. \tag{56}$$

Indeed, the integral characteristics of the function $\varphi(s')$ are preserved and this is basically all we need in order to estimate roughly the effects of the continuum.

Combining expressions given above we arrive at

$$F(0,-Q^2,-Q^2)_{\text{res+c-r}} = -8\pi\alpha \frac{m^4}{3g_\rho^2} g_{\rho\omega\pi}\left[\frac{1}{Q^2+m^2} - \frac{1}{Q^2+s_0}\right]^2, \tag{57}$$

(see also (54)). After simultaneous borelization in Q^2 there emerges the following sum rule

$$\begin{aligned} &-\frac{2}{3}\frac{4\pi}{g_\rho^2}\alpha m^4 g_{\rho\omega\pi}\left\{\frac{1}{M^4}\mathrm{e}^{-m^2/M^2} - \frac{2}{(s_0-m^2)M^2}(\mathrm{e}^{-m^2/M^2} - \mathrm{e}^{-s_0/M^2}) + \frac{1}{M^4}\mathrm{e}^{-s_0/M^2}\right\} \\ &\quad + \tfrac{1}{3}F(0,0,0)4\pi^2 f_\pi^2 \frac{1}{M^2}\mathrm{e}^{-s_0/M^2} \\ &= \tfrac{1}{3}F(0,0,0)4\pi^2 f_\pi^2\left\{\frac{1}{M^2} - \frac{8}{9}\frac{\delta^2}{M^4} + \cdots\right\}. \end{aligned} \tag{58}$$

We see that the piece generated by $\rho_{\text{c-c}}$ is suppressed exponentially with respect to that generated by ρ_{res}. At the same time, suppression of the term associated with $\rho_{\text{c-r}}$ is much weaker, the suppression factor is

$$2M^2/(s_0-m^2)\,, \tag{59}$$

i.e. non-exponential. It is worth emphasizing that the situation is quite general for the procedure considered. Under simultaneous borelization, transitions like (52b) are suppressed as compared to the resonance contribution only by power factors. As a consequence, there arises an essential correction.

We hope that numerous details of our model for the spectral density do not overshadow a simple and general physical picture which reduces to the following. The most interesting double-pole contribution of fig. 8 is represented in the phenomenological part of the sum rules by $M^{-4}\exp(-m^2/M^2)$. There are also uninteresting background contributions of two types. The first one (fig. 9a) which may be called "the true continuum" is suppressed by the exponential factor $\exp(-s_0/M^2)$. On the contrary, the single-pole contribution (fig. 9b) has the same exponential dependence as the double-pole piece ($M^{-2}\exp(-m^2/M^2)$) and differs from the double-pole piece only by the pre-exponential factor. In order to get a prediction for the resonance coupling constant one should either eliminate or estimate and include in analysis both background contributions. The authors of ref. [5a] have chosen the first way and have proposed a procedure of eliminating the single-pole term based on the fact that the pre-exponential factors are different. Since our aim here is mainly pedagogical, we do not follow their recipe but, instead, use a model of the spectral density allowing one to estimate all types of corrections to a purely resonance contricution of fig. 8. The main lesson is that the single-pole correction is, generally speaking, rather large. Thus, at $M^2 = m^2$ the correction due to (59) amounts to 50% even if the effective continuum threshold lies at $s_0 \simeq 5m^2$!

If one neglects the continuum entirely and keeps only the resonance contribution on the one hand, and only the leading $O(M^{-2})$ term in the right-hand side on the other, as was done in ref. [4e], then the analysis of the sum rule at $M^2 \simeq m^2$ yields

$$g_{\rho\omega\pi} = \frac{\sqrt{2}}{f_\pi} \frac{4\pi^2 f_\pi^2}{m^2} \left(\frac{g_\rho^2}{4\pi} \Big/ \frac{2\pi}{e} \right) \simeq 12\,\mathrm{GeV}^{-1}\,. \tag{60}$$

The power correction in the right-hand side first of all indicates that the point $M^2 = m^2$ lies inside the fiducial interval.

Moreover, the correction is rather important numerically. Including it in the analysis and accounting for effects due to the continuum we arrive at

$$g_{\rho\omega\pi} \simeq 17\,\mathrm{GeV}^{-1}\,, \tag{61}$$

which agrees nicely with the results of ref. [4d]. (In this reference independent borelization with respect to q_1^2, q_2^2 has been performed so that there is no ambiguity due to non-resonant contributions).

7. Conclusions

In this paper we have tried to reply to some of the criticism of the sum rules which have appered in the literature. We have also presented a new determination of the gluon condensate which confirms the old one. All this does not imply of course that we believe that the method of the sum rules is free of problems.

First of all, the method is indeed semi-phenomenological. When applied to a single resonance it is approximate. Moreover, it is easy to find examples when saturation of the sum rules by a single level is completely senseless (for quantum mechanical examples see ref. [15b]). Typically, however, the nature of the confining force is seemingly such that the saturation does work for the lowest states. There is no reliable extension to excited states. The only point we would like to insist upon is that the problem of accuracy of one or another prediction can be clarified within the sum rules themselves.

To a certain extent, the criticism of refs. [8–12] stems from the fact that the authors of these papers would like to have too much from the approximate method, namely, some kind of equation valid once and forever. Unfortunately, this is not so. Thus, one cannot assume that the height of the stability plateau (see sect. 2) is an exact answer for the lowest level. This has never been assumed by us and, indeed, such an assumption would lead to inconsistencies. Similarly, the authors of ref. [11] would like to have a theorem-like statement on the validity of factorization. Unfortunately, factorization is expected to work in some cases and to fail in others. Certainly, we have never assumed factorization to be absolute. To summarize, if the sum rules did reduce to an equation there would be much less misunderstanding and controversy in the literature.

As far as numerical fits are concerned we should admit that the determination of $\langle G^2\rangle$ from the sum rules for heavy quarks is not a simple business since it is related to a refined treatment of Coulomb-like corrections. Even in the case of charmonium the dependence on the normalization point for the effective coupling constant can affect the determination of $\langle G^2\rangle$. In the case of Y an even stronger assumption is made. Namely, it is assumed [21] that the coupling constant does not vary any longer once the normalization point becomes lower than $\mu_0^2 = 1\ \mathrm{GeV}^2$. The validity of this hypothesis is confirmed by the success of the sum rules themselves.

Keeping these problems in mind we have devoted much attention to the determination of $\langle G^2\rangle$ in this paper.

Coming to more serious and unsolved problems we remark that the sum rules predict the existence of a new mass scale in some hadronic channels [27], primarily, in gluonic channels with $J^P = 0^{\pm}$. It is expected to be large compared to the ordinary m_ρ^2, say, $20m_\rho^2$. There has been no independent confirmation of this prediction so far. Moreover, it is not clear what is the impact of this new scale on hadronic physics in general and what is the mechanism of its generation.

Finally, from the purely theoretical point of view the problems of the operator expansion within non-perturbative QCD, its validity and modification, promise to be interesting.

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