q-deformation of Poincaré algebra

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Standard (Drinfeld–Jimbo) q-deformation of the Cartan–Weyl basis for $o(3, 2)$ (real form of B_2) $\simeq sp(4|\mathbb{R})$ (real form of C_2) is calculated. The limit $R\to\infty$ (R is the anti-de Sitter radius) accompanied by a particular limit of the deformation parameter q ($\mathrm{i}R \ln q\to$const.) is performed. We obtain a modified Minkowski geometry with abelian translations but the four-momentum-dependent modification of the Lorentz boost algebra. The q-covariant generalization of the Klein–Gordon equation is given.

1. Introduction

As a possible application of the notion of quantum group and quantum algebra (see e.g. refs. [1–6]) it is interesting to study the deformations of space–time symmetries and their supersymmetric extensions. At present, quantum groups appear mostly as the deformation of the internal symmetry sector (e.g. in $D=2$ conformal field theory [7], with the Virasoro algebra remaining not modified); however, it is quite plausible that the description of space–time at very short distances requires modifications which are interesting to explore. The formalism of so-called q-deformations in the framework of quasi-triangular Hopf algebras being available [3,4] it is important to derive the corresponding deformation of the Poincaré group or Poincaré algebra.

In this paper we shall discuss the quantum-algebraic aspect of the problem, however, the results could be as well obtained in the dual framework of the quantum groups [#1]. Because the Poincaré algebra is a semidirect sum of Lorentz- and translation subalgebras ($\mathscr{P}=O(3,\ 1)\oplus T_4$), the q-deformation of the Poincaré algebra is not described by the Drinfeld–Jimbo scheme [3,4] valid for the q-deformation of simple Cartan–Lie algebras. In order to obtain the q-deformation of Poincaré algebra we shall perform firstly the q-deformation of the anti-de Sitter algebra $o(3, 2)$ [$o(3, 2)\to\mathscr{U}_q(o(3, 2))$] and after introducing the de Sitter radius R we shall consider the contraction, introduced first by the Firenze group [8],

$$R\to\infty\ ,\quad \mathrm{i}R\log q\to\kappa^{-1}\quad (0<\kappa<\infty)\ . \tag{1.1}$$

We see from (1, 1) that such a prescription pro-

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[#1] Following ref. [1] we distinguish here the quantum group G_q as the q-deformation of the (algebra of functions on the) Lie group G, and the quantum algebra $\mathscr{U}_q(\hat{g})$, described by the q-deformation of the universal enveloping algebra $\mathscr{U}(\hat{g})$ of the Lie algebra $\hat{g}$.

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vides a particular way of approaching $q\to 1$, however, in such a way that a new mass-like parameter κ appears. It is the aim of this paper to show how the finite values of κ modify the Poincaré algebra.

Firstly, in section 2, we shall discuss the undeformed case ($q=1$); in particular the Cartan–Weyl basis for B_2 and C_2, the assignments of $o(5;\mathbb{C})$ generators, the real form $[o(5;\mathbb{R})$ and $o(3,2;\mathbb{R})]$ as well as the contraction $R\to\infty$. In particular, we shall consider the mapping between simple and positive roots of B_2 and C_2, described by the same two-dimensional root diagram (see e.g. ref. [9]). It should be recalled that the Drinfeld–Jimbo q-deformation distinguishes simple roots as well as the normal order in the positive root system. In section 3, we shall describe the q-deformation of the C_2 Lie algebra with suitable assignment of q-deformed $sp(4)\simeq o(5)$ generators $M_{AB}=-M_{BA}(A, B=0, 1, 2, 3, 4)$ to the Cartan–Weyl basis, and its relations with the q-deformation of B_2. Extending the involutions introducing real forms to $q\neq 1$ we shall see that the real forms $\mathscr{U}_q(o(3,2|\mathbb{R}))$ and $\mathscr{U}_q(sp(4|\mathbb{R}))$ require $|q|=1$. In section 4, we shall consider the contraction (1, 1). It appears that:

(1) O(3) space rotations are not modified.

(2) The algebra of Lorentz boosts is modified by the appearance of the rotated generators L_i by the angle $\alpha=P_0/\kappa$ [see (4.12)] and additional quadratic deformation terms in boosts and three-momentum variables.

(3) The translations commute.

(4) A second deformation (by a factor $(1/\kappa)\times\sin(P_0/\kappa)$ and the terms quadratic in three-momentum components) is obtained in the transformation law of the translations under the Lorentz boosts.

We see, therefore, that our deformation introduces a mass scale into the theory which modifies the isotropy properties of the Minkowski geometry in the cost $\mathscr{P}/E_3$, where $E_3=O(3)\oplus T_3$ denotes the euclidean group of motion.

In section 5, we introduce the q-deformed relativistic mass square operator, defined as the Casimir of the q-deformed Poincaré algebra, and the corresponding modified Klein–Gordon equation. The first correction to the Klein–Gordon equation, of order $1/\kappa^2$, contains a fourth time derivative.

It should be mentioned that the technique of contractions was recently applied to the quantum algebras describing the deformation of Lie algebras of rank one [8,10,11]. Because $D=4$ anti-de Sitter algebras are of rank two, in order to obtain the deformation of all ten generators one has to extend the Cartan–Chevalley basis of the quantized algebra to the q-deformed Cartan–Weyl basis [12–14], in consistency with the q-deformed Serre relations.

2. Anti-de Sitter algebras $o(3,2)\simeq sp(4)$ and its contraction to the Poincaré limit

We would like to recall that the anti-de Sitter algebra can be obtained as a real form of the complex Lie algebras $B_2\simeq o(5)$ and $C_2\simeq sp(4)$. The Cartan–Chevalley basis of these two algebras is described by the set of relations

$$[h_i, h_j]=0\,,\quad [e_i, e_{-j}]=\delta_{ij}h_j\,,$$
$$[h_i, e_{\pm j}]=\pm\alpha_{ij}e_{\pm j}\,, \tag{2.1}$$

where the symmetrized Cartan matrices $\alpha_{ij}=\langle\alpha_i, \alpha_j\rangle$ $(i,j=1,2)$ are

$$B_2\colon \tilde\alpha=\begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix},\quad C_2\colon \alpha=\begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix} \tag{2.2}$$

and the standard choice of simple roots in the following (see e.g. ref. [9]):

$$B_2\colon \tilde\alpha_1=(1,-1)\,,\quad \tilde\alpha_2=(0,1)\,, \tag{2.3a}$$
$$C_2\colon \alpha_1=(1/\sqrt{2},-1/\sqrt{2})\,,\quad \alpha_2=(0,\sqrt{2})\,. \tag{2.3b}$$

We choose normal orders $(\tilde\alpha_1, \tilde\alpha_3, \tilde\alpha_4, \tilde\alpha_2)$, $(-\tilde\alpha_2, -\tilde\alpha_4, -\tilde\alpha_3, -\tilde\alpha_1)$ for B_2, and $(\alpha_1, \alpha_4, \alpha_3, \alpha_2)$, $(-\alpha_2, -\alpha_3, -\alpha_4, -\alpha_1)$ for C_2, implying the following defining relations $[\alpha_3=\alpha_1+\alpha_2$, $\tilde\alpha_3=\tilde\alpha_1+\tilde\alpha_2$, $\alpha_4=2\alpha_1+\alpha_2$, $\tilde\alpha_4=\tilde\alpha_1+2\tilde\alpha_2$; $\tilde e_\rho\equiv e_{\tilde\alpha_\rho}$, $e_\rho\equiv e_{\alpha_\rho}(\rho=1,2,3,4)]$:

$$B_2\colon \tilde e_3=[\tilde e_1, \tilde e_2]\,,\quad \tilde e_{-3}=[\tilde e_{-2}, \tilde e_{-1}]\,,$$
$$\tilde e_4=[\tilde e_3, \tilde e_2]\,,\quad \tilde e_{-4}=[\tilde e_{-2}, \tilde e_{-3}]\,, \tag{2.4a}$$
$$C_2\colon e_3=[e_1, e_2]\,,\quad e_{-3}=[e_{-2}, e_{-1}]\,,$$
$$e_4=[e_1, e_3]\,,\quad e_{-4}=[e_{-3}, e_{-1}]\,. \tag{2.4b}$$

In order to relate the Cartan–Lie algebras for B_2 and C_2 we observe that the identification $\tilde\alpha_1=-\alpha_2$, $\tilde\alpha_2=-\alpha_1$ leads to the following:

(1) Identification of the Cartan matrices for B_2 and C_2.

(2) Identification of the first (second) normal order for B_2 with the second (first) normal order of C_2.

Indeed, introducing $\tilde{h}_1 = -h_2$ and

$$\tilde{e}_1 = e_{-2}, \quad \tilde{e}_2 = e_{-1}, \quad \tilde{e}_3 = e_{-3}, \quad \tilde{e}_4 = e_{-4}, \tag{2.5}$$

we see easily that the relations (2.1) and (2.4) for B_2 and C_2 are identical [#2].

The complete set of the commutation relations for the $B_2 \simeq C_2$ algebra in the Cartan–Weyl basis will be given with $q \neq 1$ in the next section. Here, we would like to write the relations expressing "physical" O(3, 2) rotation generators M_{AB}($M_{AB} = -M_{BA}$; A, $B=0$, 1, 2, 3, 4)

$$[M_{AB}, M_{CD}] = \mathrm{i}(\eta_{BC} M_{AD} - \eta_{AC} M_{BD} + \eta_{AD} M_{BC} - \eta_{BD} M_{AC}), \tag{2.6}$$

where $\eta_{AB} = (+---+)$, in terms of the Cartan–Weyl generators for C_2. We obtain

$$\begin{aligned}
&M_{12} = h_1, \quad M_{23} = (1/\sqrt{2})(e_1 + e_{-1}),\\
&M_{31} = (1/\mathrm{i}\sqrt{2})(e_1 - e_{-1}), \quad M_{04} = h_3 = h_1 + h_2,\\
&M_{34} = (-1/\sqrt{2})(e_3 - e_{-3}),\\
&M_{03} = (1/\mathrm{i}\sqrt{2})(e_3 + e_{-3}),\\
&M_{02} = \tfrac{1}{2}(e_4 - e_{-4} + e_2 - e_{-2}),\\
&M_{01} = (-1/2\mathrm{i})(e_4 + e_{-4} - e_2 - e_{-2}),\\
&M_{24} = (1/2\mathrm{i})(e_4 + e_{-4} + e_2 + e_{-2}),\\
&M_{14} = \tfrac{1}{2}(e_4 - e_{-4} - e_2 + e_{-2}).
\end{aligned} \tag{2.7}$$

In the algebra $B_2 \simeq C_2$ one can introduce two reality conditions as follows:

(1) $\mathrm{o}(3, 2; \mathbb{R}) \simeq \mathrm{sp}(4; \mathbb{R})$,

$$e_1 = e^*_{-1}, \quad e_2 = -e^*_{-2}, \tag{2.8a}$$

$$e_3 = -e^*_{-3}, \quad e_4 = -e^*_{-4},$$
$$h_1 = h^*_1, \quad h_2 = h^*_2. \tag{2.8b}$$

In such a case we obtain the algebra (2.6) with *real* generators.

(ii) $\mathrm{o}(5; \mathbb{R}) \simeq \mathrm{sp}(2, \mathbb{H}) \equiv \mathrm{sp}(4; \mathbb{C}) \cap \mathrm{u}(4)$,

$$e_1 = e^*_{-1}, \quad e_2 = e^*_{-2}, \tag{2.9a}$$

implying

$$e_3 = e^*_{-3}, \quad e_4 = e^*_{-4},$$
$$h_1 = h^*_1, \quad h_2 = h^*_2. \tag{2.9b}$$

In such a case one obtains after the replacement in (2.7) ($r, s = 1, 2, 3$)

$$M_{rs} \to M_{rs}, \quad M_{0r} \to \mathrm{i}M_{0r},$$
$$M_{04} \to M_{04}, \quad M_{4r} \to \mathrm{i}M_{4r} \tag{2.10}$$

the o(5) algebra with real generators.

In order to obtain from (2.6) the Poincaré algebra, we introduce the rescaling ($\mu, \nu = 1, 2, 3, 4$) by introducing the anti-de Sitter radius R:

$$P_\mu = \frac{1}{R} M_{0\mu} \tag{2.11}$$

and the remaining generators $M_{\mu\nu}$ describing the Lorentz group algebra are left unchanged. We obtain from (2.6) and (2.11) that

$$[M_{\mu\nu}, P_\rho] = -\mathrm{i}(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu), \tag{2.12a}$$

$$[P_\mu, P_\nu] = -\mathrm{i} \frac{M_{\mu\nu}}{R^2}. \tag{2.12b}$$

We see easily that in the limit $R \to \infty$ we obtain the Poincaré algebra.

3. Quantum algebra $U_q(\mathrm{sp}(4; \mathbb{R})) \approx U_q(\mathrm{o}(3, 2))$

In order to describe the q-deformation of $C_2 \simeq B_2$ we shall use the normal order ($\alpha_1, \alpha_4, \alpha_3, \alpha_2$) of C_2 roots in the Drinfeld–Jimbo prescription for the deformation of the Cartan–Chevalley basis (e_i, e_{-i}, h_i) and the following q-deformation of the defining relations (2.4) [#3] [12–14]:

$$e_3 \equiv e_1 e_2 - q^{-\alpha_{12}} e_2 e_1 = [e_1, e_2]_q,$$
$$e_{-3} \equiv e_{-2} e_{-1} - q^{\alpha_{12}} e_{-1} e_{-2} = [e_{-2}, e_{-1}]_{q^{-1}}, \tag{3.1}$$

[#2] The change from clockwise to anti-clockwise normal order follows from the fact that two root diagrams for C_2, one given by (2.3b) and the second given by the transformation $\tilde{\alpha}_1 \to -\alpha_2$, $\tilde{\alpha}_2 \to -\alpha_1$, are not identical, but are related by a Weyl reflection.

[#3] For simplicity we shall denote the q-deformed basis in section 3 by the same letters. We recall that in general $[e_\gamma, e_\delta]_{q^{\pm 1}} = e_\gamma e_\delta - q^{\pm\gamma\delta} e_\delta e_\gamma$ the sign $+$ ($-$) depending on the clockwise (anti-clockwise) order on the root diagram. Of course $[A, B]_q \equiv AB - qBA$.

$$e_4 \equiv e_1 e_3 - q^{-\alpha_{13}} e_3 e_1 = [e_1, e_3] ,$$

$$e_{-4} \equiv e_{-3} e_{-1} - q^{\alpha_{13}} e_{-1} e_{-3} = [e_{-3}, e_{-1}] , \quad (3.1 \text{ cont'd})$$

where we extended the scalar product defined for simple roots by the Cartan matrix to all roots ($\alpha_{\rho\tau} \equiv \langle \alpha_\rho, \alpha_\tau \rangle ; \rho, \tau = 1, 2, 3, 4$).

The complete set of commutation relations for $U_q(C_2)$ in the q-deformed Cartan–Weyl basis looks as follows ($i, j = 1, 2$):

$$\begin{aligned} &[e_i, e_{-j}] = \delta_{ij} [h_i]_q , \quad [h_i, h_j] = 0 , \\ &[h_i, e_{\pm j}] = \pm \alpha_{ij} e_{\pm j} , \\ &[e_3, e_{-3}] = [h_3]_q , \quad h_3 = h_1 + h_2 , \\ &[e_4, e_{-4}] = [h_4]_q , \quad h_4 = h_1 + h_3 , \end{aligned} \quad (3.2a)$$

$$\begin{aligned} &[e_1, e_4]_{q^{-1}} = 0 , \quad [e_{-2}, e_{-3}]_q = 0 , \\ &[e_4, e_3]_{q^{-1}} = 0 , \quad [e_{-3}, e_{-4}]_q = 0 , \\ &[e_3, e_2]_{q^{-1}} = 0 , \quad [e_{-4}, e_{-1}]_q = 0 . \end{aligned} \quad (3.2b)$$

(3.2b) is equivalent to the q-Serre relations

$$\begin{aligned} &[e_2, e_4] = (1 - q^{-1}) e_3^2 , \quad [e_{-2}, e_{-4}] = (q-1) e_{-3}^2 , \\ &[e_1, e_{-3}] = -q^{-h_1} e_{-2} , \quad [e_3, e_{-1}] = -e_2 q^{h_1} , \\ &[e_3, e_{-2}] = q^{-h_2} e_1 , \quad [e_2, e_{-3}] = e_{-1} q^{h_2} , \\ &[e_4, e_{-3}] = q^{-h_3} e_1 , \quad [e_3, e_{-4}] = e_{-1} q^{h_3} , \\ &[e_1, e_4] = -q^{h_1} e_{-3} , \quad [e_{-1}, e_4] = q^{h_1} e_3 , \\ &[e_4, e_{-2}] = -(1 - q^{-1}) q^{-h_2} e_1^2 , \\ &[e_2, e_{-4}] = -(1-q) e_{-1}^2 q^{h_2} , \end{aligned} \quad (3.2c)$$

where $[x]_q \equiv (q^x - q^{-x})/(q - q^{-1})$ and the relation $[\alpha_{12}]_q = -1$ has been extensively used.

The Drinfeld–Jimbo q-deformation $U_{\tilde q}(B_2)$ is obtained if we supplement the relations (2.5) also by the identification $\tilde q = q^{-1}$, i.e.,

$$U_{q^{-1}}(B_2) \simeq U_q(C_2) . \quad (3.3)$$

In order to obtain the real forms of (3.3), we should supplement the relations (2.8) and (2.9) by the conditions on q. One should assume [see (2.8) and (2.9)]

$$U_{q^*}(o(5)) \simeq U_q(sp(2|\mathbb{H})) , \quad (3.4a)$$

$$U_{q^*}(o(3,2)) \simeq U_q(sp(4|\mathbb{R})) . \quad (3.4b)$$

Further, we describe the relations (3.2) in the O(3, 2) basis (2.7) with the physical assignment of the generators. We introduce:

(i) Three dimensional rotations ($M_\pm, M_3$),

$$M_3 = M_{12} , \quad M_\pm = M_{23} \pm i M_{31} . \quad (3.5a)$$

We obtain

$$[M_+, M_-] = 2[M_3]_q , \quad [M_3, M_\pm] = \pm M_\pm . \quad (3.5b)$$

(ii) Lorentz boosts ($L_\pm, L_3$),

$$L_3 = M_{34} , \quad L_\pm = M_{14} \pm i M_{24} . \quad (3.6)$$

One gets

$$\begin{aligned} &[L_-, L_+] = [M_{04} + M_3]_q + [M_3 - M_{04}]_q \\ &\quad + \tfrac12 (q-1) [(K^-)^2 - q^{-1} (K^+)^2] , \\ &[L_3, L_+] = -\tfrac12 q^{-M_{04}} (1 + q^{M_3}) M_+ \\ &\quad - \tfrac14 (1-q) [K_-(N_- - L_+) - (N_- + L_+) K_+] , \\ &[L_3, L_-] = \tfrac12 M_- q^{M_{04}} (1 + q^{-M_3}) \\ &\quad + \tfrac14 (1-q) [K_+(N_+ - L_-) - (N_+ + L_-) K_-] , \end{aligned} \quad (3.7)$$

where

$$K_\pm = i M_{03} \mp M_{34} , \quad N_\pm = M_{02} \pm i M_{01} . \quad (3.8)$$

(iii) $[M, L]$ sector,

$$\begin{aligned} &[M_3, L_3] = 0 , \\ &[M_3, L_\pm] = \pm L_\pm , \\ &[M_-, L_3] = \tfrac12 (N_+ + L_-) - \tfrac12 (N_+ - L_-) q^{M_3} , \\ &[M_-, L_+] = q^{M_3} K_+ - K_- \\ &\quad + \tfrac12 (1 - q^{-1}) M_- (L_+ - N_-) , \\ &[M_-, L_-] = \tfrac12 (1-q) M_- (N_+ + L_-) , \\ &[M_+, L_3] = -\tfrac12 (N_- + L_+) + \tfrac12 q^{-M_3} (N_- - L_+) , \\ &[M_+, L_+] = \tfrac12 (1-q) M_+ (N_- + L_+) , \\ &[M_+, L_-] = q^{-M_3} K_- - K_+ \\ &\quad + \tfrac12 (1-q) (N_+ - L_-) M_+ . \end{aligned} \quad (3.9)$$

The relations (3.5), (3.7) and (3.9) describe the q-deformation of the Lorentz generators sl(2; $\mathbb{C}$) $\simeq$ o(3,1) which belong to the quantum algebra $U_q(o(3,2))$.

One can also write the q-deformation of the remaining commutators of the o(3, 2) algebra. Here we shall only write the commutators, describing the q-deformation of relation (2.12b) which provides after contraction the commutators of the translation generators P_μ.

(iv) Curved translations sector ($M_{0\mu}$),

$$[M_{04}, M_{03}] = \mathrm{i}M_{34} ,$$

$$[M_{04}, M_{02}] = \mathrm{i}M_{24} ,$$

$$[M_{04}, M_{01}] = \mathrm{i}M_{14}$$

$$\begin{aligned}[M_{03}, M_{02}] &= -\tfrac{1}{4}\mathrm{i}[M_- q^{M_{04}}(1+q^{-M_3}) + q^{-M_{04}}(1+q^{M_3})M_+] \\ &\quad + \tfrac{1}{8}\mathrm{i}(q-1)[K_+(L_- - N_+) + (N_- + L_+)K_+ \\ &\quad + (N_+ + L_-)K_- + K_-(L_+ - N_-)] ,\end{aligned}$$

$$\begin{aligned}[M_{03}, M_{01}] &= \tfrac{1}{4}[M_- q^{M_{04}}(1+q^{-M_3}) + q^{-M_{04}}(1+q^{M_3})M_+] \\ &\quad + \tfrac{1}{8}\mathrm{i}(q-1)[K_-(L_+ - N_-) + (N_- + L_+)K_+ \\ &\quad - (N_+ + L_-)K_- + K_+(N_+ - L)] ,\end{aligned}$$

$$\begin{aligned}[M_{02}, M_{01}] &= \tfrac{1}{2}\mathrm{i}\{[M_{04}+M_3]_q - [M_{04}-M_3]_q\} \\ &\quad + \tfrac{1}{4}\mathrm{i}(1-q)\{(K_-)^2 - q^{-1}(K_+)^2\} . \qquad (3.10)\end{aligned}$$

It should be stressed that the q-deformed Lorentz sector ($M_\pm$, M_3, $L_\pm$, L_3) does not form a quantum group, because it is also deformed by generators belonging before deformation to the coset O(3, 2)/O(3, 1).

4. Contraction and q-deformed Poincaré

Let us recall that $|q|=1$, i.e, $g=\mathrm{e}^{\mathrm{i}w}$. We introduce the following rescaling ($0<\kappa<\infty$) (see also ref. [8]):

$$M_{\mu\nu} = M_{\mu\nu} , \quad M_{0\mu} = R\cdot P_\mu , \quad w = \frac{1}{\kappa}\cdot\frac{1}{R} , \qquad (4.1)$$

where κ is a geometric mass parameter.

Using (4.1) for example we obtain

$$[M_3 + M_{04}]_q + [M_3 - M_{04}]_q = 2\frac{\sin(M_3/\kappa R)}{\sin(1/\kappa R)} \cos\frac{P_0}{\kappa} . \qquad (4.2)$$

Performing the limit $R\to\infty$ in (4.2) we get

$$\lim_{R\to\infty} [M_3 + M_{04}]_q + [M_3 - M_{04}]_q = 2M_3\cos(P_0/\kappa) . \qquad (4.3)$$

Using the formulae (4.1) and performing the limit $R\to\infty$ we obtain the following set of relations:

(a) Lorentz sector ($M_\pm = M_1 \pm \mathrm{i}M_2$),

$$[M_+, M_-] = 2M_3 , \quad [M_3, M_\pm] = \pm M_\pm , \qquad (4.4)$$

$$\begin{aligned}[L_-, L_+] &= 2M_3\cos\frac{P_0}{\kappa} - \frac{1}{\kappa}(P_3L_3 + L_3P_3) + \frac{1}{2\kappa^2}P_3^2 , \\ [L_3, L_\pm] &= \mp\exp\left(\mp\mathrm{i}\frac{P_0}{\kappa}\right)M_\pm \pm \frac{1}{2\kappa}L_\pm P_3 + \frac{\mathrm{i}}{2\kappa}L_3P_\mp , \qquad (4.5)\end{aligned}$$

$$\begin{aligned}&[M_3, L_3] = 0 , \\ &[M_3, L_\pm] = \pm L_\pm , \\ &[M_+, L_3] = -L_+ - \frac{\mathrm{i}}{2\kappa}M_3P_- , \\ &[M_-, L_3] = +L_- - P_+\frac{\mathrm{i}}{2\kappa}M_3 , \\ &[M_+, L_-] = +2L_3 + \frac{1}{2\mathrm{i}\kappa}P_+M_+ + \frac{1}{\kappa}P_3M_3 , \\ &[M_-, L_+] = -2L_3 + \frac{1}{2\mathrm{i}\kappa}M_-P_- - \frac{1}{\kappa}P_3M_3 , \\ &[M_\pm, L_\pm] = \frac{1}{2\mathrm{i}\kappa}M_\pm P_\mp . \qquad (4.6)\end{aligned}$$

(b) Translation sector ($P_\pm = P_2 \pm \mathrm{i}P_1$),

$$[P_\mu, P_\nu] = 0 , \qquad (4.7)$$

$$[M_i, P_0] = 0 , \quad [M_i, P_k] = \mathrm{i}\epsilon_{ikl}P_l , \qquad (4.8)$$

$$\begin{aligned}&[L_3, P_0] = \mathrm{i}P_3 , \\ &[L_3, P_3] = \mathrm{i}\kappa\sin(P_0/\kappa) , \\ &[L_3, P_2] = \frac{1}{2\mathrm{i}\kappa}P_1P_3 , \qquad (4.9)\end{aligned}$$

$$[L_3, P_1] = -\frac{1}{2i\kappa} P_2 P_3 ,$$

$$[L_\pm, P_0] = iP_1 \mp P_2 ,$$

$$[L_\pm, P_3] = \frac{1}{2i\kappa} P_\mp P_3 ,$$

$$[L_\pm, P_2] = \mp \kappa \sin\frac{P_0}{\kappa} - \frac{1}{2i\kappa} P_3^2 ,$$

$$[L_\pm, P_1] = i\kappa \sin\frac{P_0}{\kappa} \pm \frac{1}{2\kappa} P_3^2 . \qquad (4.9\text{ cont'd})$$

We see from the formulae (4.4)–(4.9) that:

(i) The three-dimensional rotations [eq. (4.4)] and the translation subgroup [eq. (4.7)] are not deformed.

(ii) The modification is obtained in the closure of the boost sector and in the transformation of four-momenta under the boost transformations.

It is easy to see that if $\kappa \to \infty$ we recover the conventional Poincaré algebra.

Let us perform the time reversal transformation T, which is an antiinvolution, changing the sign of the imaginary unit i. It is easy to check that the invariance of the relations (4.4)–(4.9) under the discrete transformation

$$T: M_i \to -M_i , \quad L_i \to L_i ,$$
$$P_i \to -P_i , \quad P_0 \to P_0 \qquad (4.10)$$

implies that

$$q \to q^* \qquad (4.11)$$

and because $|q| = 1$ the parameter κ is T-invariant. Indeed, the relations (4.5b) can be written as

$$[L_3, L_1] = i(-\cos\alpha\, M_2 + \sin\alpha\, M_1) + \frac{i}{2\kappa}(L_2 P_3 + L_3 P_2) ,$$

$$[L_3, L_2] = i(\sin\alpha\, M_2 + \cos\alpha\, M_1) - \frac{i}{2\kappa}(L_1 P_3 - L_3 P_1) ,$$

$$\alpha = (P_0/\kappa) . \qquad (4.12)$$

From the antiunitarity of T follows the the iM_i are T-invariant, and the deformation (4.5) of the boost algebra is not changed under time reversal.

In order to show that the deformed Poincaré algebra is a quantum group, one needs to define the coproduct. The following expressions were obtained as the contraction of the coproduct for $\mathcal{U}_q(o(3,2))$ and can be shown to be a homomorphism of the algebra (4.4) to (4.9):

$$\Delta M_i = M_i \otimes I + I \otimes M_i , \quad i = 1, 2, 3 ,$$

$$\Delta P_0 = P_0 \otimes I + I \otimes P_0 ,$$

$$\Delta P_i = P_i \otimes \exp\left(\frac{iP_0}{2\kappa}\right) + \exp\left(\frac{-iP_0}{2\kappa}\right) \otimes P_i , \quad i = 1, 2, 3 ,$$

$$\Delta L_3 = L_3 \otimes \exp\left(\frac{iP_0}{2\kappa}\right) + \exp\left(\frac{-iP_0}{2\kappa}\right) \otimes L_3 + \frac{i}{2\kappa} \exp\left(\frac{-iP_0}{2\kappa}\right) M_+ \otimes P_+ - \frac{i}{2\kappa} P_- \otimes M_- \exp\left(\frac{iP_0}{2\kappa}\right) ,$$

$$\Delta L_+ = L_+ \otimes \exp\left(\frac{iP_0}{2\kappa}\right) + \exp\left(\frac{-iP_0}{2\kappa}\right) \otimes L_+ + \frac{i}{2\kappa} P_- \otimes M_3 \exp\left(\frac{iP_0}{\kappa}\right) - \frac{i}{2\kappa} \exp\left(\frac{-iP_0}{2\kappa}\right) M_3 \otimes P_- + \frac{1}{\kappa} \exp\left(\frac{-iP_0}{2\kappa}\right) M_+ \otimes P_3 ,$$

$$\Delta L_- = L_- \otimes \exp\left(\frac{iP_0}{2\kappa}\right) + \exp\left(\frac{-iP_0}{2\kappa}\right) \otimes L_- + \frac{i}{2\kappa} P_+ \otimes M_3 \exp\left(\frac{iP_0}{2\kappa}\right) - \frac{i}{2\kappa} \exp\left(\frac{-iP_0}{2\kappa}\right) M_3 \otimes P_+ + \frac{1}{\kappa} P_3 \otimes M_- \exp\left(\frac{iP_0}{2\kappa}\right) .$$

The counit is defined in the usual way. The antipode S compatible with the coproduct is given by

$$S(P_\mu) = -P_\mu ,$$

$$S(M_i)=-M_i\,,$$

$$S(L_3)=-L_3-\frac{1}{2\kappa}P_3+\frac{\mathrm{i}}{2\kappa}(M_+P_+-P_-M_-)\,,$$

$$S(L_+)=-L_+-\frac{\mathrm{i}}{\kappa}P_-+\frac{1}{\kappa}M_+P_3\,,$$

$$S(L_-)=-L_-+\frac{\mathrm{i}}{\kappa}P_++\frac{1}{\kappa}P_3M_-\,.$$

This shows that the q-deformed Poincaré algebra is a quantum group.

5. First Casimir of q-deformed Poincaré and modified Klein–Gordon equation

Let us observe that all ten generators of the q-deformed Poincaré algebra commute with the following operator:

$$I_2=P_1^2+P_2^2+P_3^2+2\kappa^2\left(\cos\frac{P_0}{\kappa}-1\right)$$

$$=P_1^2+P_2^2+P_3^2-P_0^2+\frac{1}{12\kappa^2}P_0^4+\mathrm{O}(\kappa^{-4})\,, \quad (5.1)$$

which is the q-deformation of the first Casimir describing the relativistic mass squared operator. It is interesting to note that an analogous formula, obtained by putting $P_3=0$, is valid for the algebraically much simpler three-dimensional q-deformed Poincaré algebra [8]. Introducing $P_\mu=(1/\mathrm{i})(\partial/\partial x^\mu)$ one obtains the following q-generalization of the Klein–Gordon equation ($x_0\equiv t$) as the scalar realization of q-deformed Poincaré algebra:

$$\Delta\phi(\boldsymbol{x},t)+2\kappa^2[\cosh(\partial_t/\kappa)-1]\phi(\boldsymbol{x},t)$$

$$=m^2\phi(\boldsymbol{x},t)\,, \quad (5.2)$$

or in $\mathrm{O}(1/\kappa^4)$ approximation

$$(\partial_\mu\partial^\mu-m^2)\phi(\boldsymbol{x},t)$$

$$=\frac{-1}{12\kappa^2}\partial_t^4\phi(\boldsymbol{x},t)+\mathrm{O}(\kappa^{-4})\,. \quad (5.3)$$

Eq. (5.2) can be obtained from the following lagrangian density:

$$\mathscr{L}=-\tfrac{1}{2}(\nabla\phi)^2-\tfrac{1}{2}m^2\phi^2+\tfrac{1}{2}\sum_{n=1}^{\infty}\frac{\kappa^{2-2n}}{(2n)!!}(\partial_t^n\phi)^2. \quad (5.4)$$

The energy spectrum of the quanta described by eq. (5.2) (or the position of the poles of the free q-deformed propagator) is given by the formula

$$p^0=\kappa\arccos\left(1-\frac{\boldsymbol{p}^2+m^2}{2\kappa^2}\right). \quad (5.5)$$

The q-deformed mass-shell condition is invariant under the change $p_0\to p_0+2\pi n\kappa$, i.e., the energy dependence on the three-momentum $\boldsymbol{p}^2$ is a periodic function.

6. Outlook

The aim of this paper is to study the deformation of the Poincaré algebra which originates in the q-deformation of sp(4). It is not clear to us if there exists another reasonable contraction of $\mathrm{U}_q(\mathrm{sp}(4))$, in particular providing commuting translation generators.

The formulae (4.4)–(4.6) describe our proposal for the q-deformation of a Lorentz algebra, which is embedded in the q-deformation of the universal enveloping algebra of a whole Poincaré algebra (quantum Poincaré algebra). In this respect our q-deformation is different from other proposals for q-deformations of SL(2, $\mathbb{C}$) leading to the appearance of a quantum Lorentz algebra (or quantum Lorentz group as a dual object) [15–17]. It follows from our calculations that the semidirect product structure (even taking into consideration the q-deformation of commutators between Lorentz and translation generators) of the Poincaré algebra is not preserved for $q\neq 1$.

This paper contains only partial results. For completeness one should consider also the deformation of the second Casimir operator (squared of the Pauli–Lubanski four-vector), and calculate the contraction limit for the universal R-matrix. The complete description of the quantum Poincaré algebra as a quasi-triangular Hopf algebra [3] will be given elsewhere.

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References

[1] L. Faddeev, N. Reshetikhin and L. Takhtajan, Algeb. Anal. 1 (1989) 178.

[2] L. Woronowicz, Commun. Math. Phys. 111 (1987) 613; 122 (1989) 125.

[3] V.G. Drinfeld, Quantum groups, Proc. Intern. Congress of Mathematicians (Berkeley, CA, USA, 1986), p. 793.

[4] M. Jimbo, Lett. Math. Phys. 10 (1985) 63; 11 (1986) 247.

[5] Yu T. Manin, Quantum groups and non-commutative geometry (Centre de recherches Mathematique, Université de Montreal, 1989).

[6] P. Kulish, ed. Proc. First EIMI Workshop on Quantum groups (October–December 1990) (Springer, Berlin, 1991).

[7] L. Alvarez-Gaume, C. Gomez and G. Sierra, in: Physics and mathematics of strings (memorial volume for V. Knizhnik) (World Scientific, Singapore, 1990) p. 16.

[8] E. Celeghini, R. Ciachetti, E. Sorace and M. Tarlini, The three-dimensional euclidean group $E(3)_q$ and its R-matrix, J. Math. Phys., in press; Contractions on quantum groups, in: Proc. Workshop on Quantum groups (October–December 1990) (EMI, Leningrad), to be published.

[9] R. Gilmore, Lie groups, Lie algebras, and some of their applications (Wiley, New York, 1974).

[10] M. Chaichian and P. Kulish, Phys. Lett. B 234 (1990) 72.

[11] P. Kulish, Teor. Mat. Fiz. 86 (1991) 158.

[12] M. Rosso, Commun. Math. Phys. 124 (1989) 307.

[13] V.T. Tolstoj, in: Proc. Summer Workshop on Quantum groups (Clausthal, July 1989) (Springer, Berlin, 1990).

[14] H. Yamane, Publ. RIMS Kyoto Univ. 25 (1989) 503.

[15] P. Podles and S.L. Woronowicz, Commun. Math. Phys. 130 (1990) 381.

[16] U. Carow-Watamura, M. Schlieher, M. Scholl and S. Watamura, Karlsruhe preprint KA-THEP-1990-14 (May 1990).

[17] C. Gomez and G. Sierra, University of Geneva preprint UGVA-DPT 1990/08-694.