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TWO-LOOP BACKGROUND FIELD CALCULATIONS FOR ARBITRARY BACKGROUND FIELDS

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The ultraviolet divergences of two-loop vacuum graphs in the presence of an arbitrary background field are determined for four-dimensional ϕ^4 and gauge theories on flat space. Dimensional regularisation is employed and the heat kernel is used to analyse the short-distance singularities in the products of propagators, in the presence of the background field, that occur for two-loop graphs. The single and double poles in $\varepsilon = 4 - d$ are determined in a concise fashion, giving known results for the β function. A procedure for determining the remaining finite parts in terms of explicit convergent integrals in four dimensions is described.

1. Introduction

Calculation of the functional integrals defining a quantum field theory by perturbative expansions about some background field, in many cases a solution of the classical equations for which the classical action is stationary, occur in many situations when non-perturbative effects in the overall quantum theory are being investigated. Such calculations further can define an effective quantum action incorporating quantum corrections to the classical result, as in the case of the effective potential for constant background scalar fields whose minimum determines the vacuum ground state of the quantum theory [1,2]. Background field methods also enjoy an especial attraction in the discussion of gauge theories, since, with an appropriate gauge choice, manifest covariance is retained with respect to gauge transformations on the background field [4–11].

Thus, when applying the method to a non-abelian gauge theory, the quantum gauge field is expanded as

$$A^q_\mu = A_\mu + g'_0 a_\mu , \tag{1.1}$$

where A_μ is the fixed background field, g'_0 an essentially arbitrary constant and a_μ the fluctuation field which is integrated over in the functional integral according to the usual effective measure on gauge equivalence classes produced by the Faddeev-

Popov determinant in conjunction with a gauge condition imposed on a_μ. If, for a gauge transformation $g(x) \in \mathcal{G}$ the gauge group,

$$A_\mu \to g^{-1} A_\mu g + g^{-1} \partial_\mu g, \tag{1.2}$$

then letting

$$a_\mu \to g^{-1} a_\mu g \tag{1.3}$$

as a change of variables in the functional integral ensures that the partition function $Z[A] = \exp(W[A])$ is invariant, with respect to the transformations (1.2), for gauge conditions on the quantum fluctuations a_μ involving only $\mathcal{G}$ invariant functions of $\mathcal{D} \cdot a = \mathcal{D}_\mu a_\mu$, where

$$\mathcal{D}_\mu X = \partial_\mu X + \left[A_\mu, X \right], \tag{1.4}$$

and also if any external source is coupled only to a_μ and transforms according to (1.3).

For field theories in four dimensions the standard perturbative renormalisation procedures are still valid when expanding about a background field but, as a result of invariance under gauge transformations (1.2), the renormalisation analysis for $W[A]$ is significantly simplified [7, 12], and verified by one-loop calculations [4, 5, 13, 14]. Abbott [15], and others [16], were subsequently able to straightforwardly recover the two-loop results for the β function for gauge theories [17] using background field methods. Recently [18] this calculation has been extended to general background gauges for a_μ, as compared with Abbott's use of a Feynman type gauge, with of course the same results. For this work the vacuum functional $W[A]$ is regarded as expanded to quadratic order in A_μ so that the calculation for β was ultimately concerned with Feynman diagrams with two external legs and internal lines represented by conventional free propagators for the a_μ and ghost fields. However, the treatment differs from standard perturbation theory [and is sufficient to determine $\beta(g)$ from the two-point amplitude alone] in that the vertices depend on the choice of the background gauge and there is a crucial distinction between the external A_μ lines and internal a_μ lines in the rules for constructing the amplitude [15].

In the present paper we discuss analogous calculations for the euclidean gauge theory where the background gauge field A_μ is regarded as O(1) and is not expanded. Thus to two-loop order there are essentially only 3 new graphs to be considered for $W[A]$ but the propagators associated with internal lines are required to be exact to all orders in A_μ. As is natural for gauge theories where gauge invariance is crucial and calculational simplicity almost as important we employ dimensional regularisation so that the amplitudes associated with the vacuum Feynman graphs are defined to be unique analytic functions of $\varepsilon = 4 - d$ and physical results are given in terms of the poles and finite parts as $\varepsilon \to 0$. It is necessary to suppose that the background field is formally extended to d dimensions but the precise prescription is later shown to be unimportant in the limit $\varepsilon \to 0$ when $W[A]$ is expressed in terms of renormalised quantities using standard minimal subtraction counterterms.

The d-dimensional Feynman amplitudes that are then required for perturbative background field calculations are here discussed directly in terms of the configuration space integrals representing the associated Feynman diagram with no external legs. There is no requirement for introducing subsidiary parameter integrals. The integrands involve for each vertex some function of the background field at that point and for each internal line the appropriate propagator for the fluctuating quantum field such as a_μ. The propagators are given by the Green functions for the differential operators that act on the fluctuation fields in the quadratic piece when the action for the full quantum field, such as in (1.1), is expanded about its value for the background field. We assume that these Green functions are well defined, as integrable distributions analytic in d, in d dimensions for suitably smooth (analytic in d) background fields. A complete mathematical definition of the d-dimensional amplitude, providing the required regularisation, requires a unique prescription for the configuration space integrals for d in some interval that allows analytic continuation to $d = 4$. For the potentially singular contributions that contain short-distance divergences and lead to poles as $\varepsilon \to 0$ the analytic behavior for $\varepsilon \sim 0$ is obtained unambiguously by relating these singular integrals to momentum-space integrals through Fourier transforms and then employing methods which agree identically with the standard dimensional regularisation evaluation. The details to two-loop order are contained in this paper; to higher order similar procedures should be feasible although further analysis is necessary. Since the Green functions are supposed exact in d dimensions, and the usual formal manipulations of the d-dimensional configuration space integrals are valid, any symmetries, such as gauge invariance, which may be defined in the presence of the background field for arbitrary d are preserved, in accord with the usual virtues of dimensional regularisation.

For sufficiently small background fields the Green functions and hence the background field Feynman amplitudes can be expanded in the background field. Our procedures guarantee that the result, to arbitrary order n in the expansion in the external background field, coincides with that obtained by conventional dimensionally regularised calculations in momentum space with conventional free propagators and n external legs. Thus for gauge theories to $O(A^2)$ our calculations would become identical to Abbott, although the number of separate diagrams necessary to consider before the expansion is significantly less.

To analyse the poles in ε, representing ultraviolet divergences, it is important to have explicit forms for the short-distance behaviour of the exact propagators. For propagators which are Green functions for elliptic differential operators on flat euclidean space of the form

$$\Delta_x = -1D_x^2 + B(x), \qquad D_\mu = \partial_\mu + A_\mu(x) \tag{1.5}$$

(here $A_\mu(x)$ and $B(x)$ can be taken to be general matrices), this can be achieved using now well-known heat kernel techniques [3, 19]. The heat kernel for Δ satisfies

$$\left(\frac{\partial}{\partial t} + \Delta_x\right)\mathcal{G}_\Delta(x, y; t) = 0,$$

$$\mathcal{G}_\Delta(x, y; 0) = \delta^d(x-y), \tag{1.6}$$

and the corresponding Green function is

$$G_\Delta(x, y) = \int_0^\infty \mathrm{d}t\, \mathcal{G}_\Delta(x, y; t),$$

$$\Delta_x G_\Delta(x, y) = \delta^d(x-y). \tag{1.7}$$

The precise details of the extension to d dimensions and the exact definition of (1.6) and (1.7) is unimportant here and may be considered in the usual formal manner. Since we require the results to be related to the normal discussion in momentum space through Fourier transforms we have

$$\delta^d(x) = \frac{1}{(2\pi)^d}\int \mathrm{d}^d k\, \mathrm{e}^{ik\cdot x}. \tag{1.8}$$

For $t \to 0+$, $\mathcal{G}_\Delta$ has the asymptotic expansion [3, 20, 21], on flat d-dimensional euclidean space

$$\mathcal{G}_\Delta(x, y; t) = \frac{1}{(4\pi t)^{d/2}}\mathrm{e}^{-|x-y|^2/4t}\sum_{n=0} a_n^\Delta(x, y)t^n, \qquad a_0^\Delta(x, x) = 1, \tag{1.9}$$

where by (1.6)

$$n a_n^\Delta(x, y) + (x-y)\cdot D_x a_n^\Delta(x, y) = -\Delta_x a_{n-1}^\Delta(x, y),$$

$$(x-y)\cdot D_x a_0^\Delta(x, y) = 0. \tag{1.10}$$

The iterative solution of (1.10) is routine, with, for Δ as in (1.5), the results of subsequent importance being

$$
\begin{aligned}
a_1^\Delta(x,x) &= -B(x), \\
a_2^\Delta(x,x) &= \tfrac{1}{12}F_{\alpha\beta}(x)F_{\alpha\beta}(x)1 + \tfrac{1}{2}B(x)^2 - \tfrac{1}{6}\mathcal{D}_x^2 B(x), \\
F_{\alpha\beta} &= \partial_\alpha A_\beta - \partial_\beta A_\alpha + \left[A_\alpha, A_\beta\right],
\end{aligned} \tag{1.11}
$$

where the covariant derivative $\mathcal{D}$ is as in (1.4).

Corresponding to (1.9) there is an associated expansion for the Green function G_Δ exhibiting, since $a_n^\Delta(x, y)$ are regular functions, the leading singular behavior as $x \to y$. For our purposes it is useful to write

$$
\begin{aligned}
G_\Delta(x,y) = {}& G_0(x-y)a_0^\Delta(x,y) + R_1(x-y)a_1^\Delta(x,y) \\
&+ R_2(x-y)a_2^\Delta(x,y) + \bar{G}_\Delta(x,y),
\end{aligned} \tag{1.12}
$$

where

$$
\begin{aligned}
G_0(x) &= \frac{\Gamma(\frac{1}{2}d-1)}{4\pi^{d/2}}|x|^{2-d}, \\
R_1(x) &= \frac{1}{16\pi^2}\left\{\frac{2}{\varepsilon}\mu^{-\varepsilon} + \frac{\Gamma(\frac{1}{2}d-2)}{\pi^{d/2-2}}|x|^{4-d}\right\}, \\
R_2(x) &= \frac{1}{32\pi^2}\left\{-\frac{|x|^2}{\varepsilon}\mu^{-\varepsilon} + \frac{\Gamma(\frac{1}{2}d-3)}{2\pi^{d/2-2}}|x|^{6-d}\right\}.
\end{aligned} \tag{1.13}
$$

R_1 and R_2 are so defined that they are regular as $\varepsilon \to 0$, with limits given in (A.5) so for $d = 4$ (1.12) reproduces the standard short-distance expansion of G_Δ [3]. Together with G_0 they satisfy

$$
\begin{gathered}
-\partial^2 G_0(x) = \delta^d(x), \qquad \partial_\mu R_1(x) = -\tfrac{1}{2}x_\mu G_0(x), \qquad -\partial^2 R_1(x) = G_0(x), \\
\partial_\mu R_2(x) = -\tfrac{1}{2}x_\mu R_1(x), \qquad -\partial^2 R_2(x) = 2R_1(x) - \frac{\mu^{-\varepsilon}}{16\pi^2}.
\end{gathered} \tag{1.14}
$$

By its definition in (1.12) $\bar{G}_\Delta(x, y)$ has thus been arranged to have no poles in ε and also to be regular for $x \sim y$, even for two derivatives, when $\varepsilon \to 0$. In (1.12) we have introduced a mass scale μ to ensure dimensional homogeneity. G_Δ is independent of μ and hence any arbitrariness is compensated by $\bar{G}_\Delta$ in (1.12) but it is convenient to identify μ with standard dimensional regularisation scale mass so as to consistently keep track of dimensions. The expansion (1.12) is an exact analog in configuration space for arbitrary background fields to that used by 't Hooft [13].

From (1.12) by analytic continuation in d from $d<2$ it follows that

$$G_\Delta(x,x)=\frac{2}{\varepsilon}\frac{\mu^{-\varepsilon}}{16\pi^2}a_1^\Delta(x,x)+D_\Delta(x), \tag{1.15}$$

with

$$D_\Delta(x)=\bar{G}_\Delta(x,x) \tag{1.16}$$

finite as $\varepsilon\to 0$. The pole term in (1.15) can also be obtained directly with the representation (1.7) using (1.9). Further from (1.12), (1.14) and (1.10)

$$\Delta_x\bar{G}_\Delta(x,y)=-R_2(x-y)\Delta_x a_2^\Delta(x,y)+\frac{\mu^{-\varepsilon}}{16\pi^2}a_2^\Delta(x,y),$$

giving the relation, used later and which has a well-behaved limit for $\varepsilon\to 0$,

$$\Delta_x\bar{G}_\Delta(x,y)|_{y=x}=\frac{\mu^{-\varepsilon}}{16\pi^2}a_2^\Delta(x,x). \tag{1.17}$$

In background field calculations the one-loop contributions are given by determinants of operators such as Δ. The one-loop divergences are then easily obtained using the same asymptotic expansion (1.9) of $\mathcal{G}_\Delta$, which is the kernel for $e^{-t\Delta}$, as above. If for $|x|\to\infty$, $A_\mu(x)\to 0$, $B(x)\to m^2$ so that $\Delta\to\Delta_0=1(-\partial^2+m^2)$ then

$$K_\Delta(t,d)=\operatorname{Tr}(e^{-t\Delta}-e^{-t\Delta_0}) \tag{1.18}$$

should be infrared finite, although the functional trace contains an integral over all d-dimensional euclidean space. In terms of the corresponding zeta function

$$\zeta_\Delta(s,d)=\frac{1}{\Gamma(s)}\int_0^\infty dt\,t^{s-1}K_\Delta(t,d), \tag{1.19}$$

the functional determinant of Δ may then be defined by

$$-\ln\det\Delta/\det\Delta_0=\zeta'_\Delta(0,d)=\int_0^\infty\frac{dt}{t}K_\Delta(t,d), \tag{1.20}$$

analytically continued in d from $d<2$. Since $\zeta_\Delta(0,d)=0$ for $d\neq 4$ the result (1.20) is unchanged under rescaling of Δ, $\Delta\to\lambda\Delta$. The conventional zeta function regularised definition for the functional determinants [20] as expressed in (1.20) is $F_\Delta=\zeta'_\Delta(0,4)$ where $\zeta_\Delta(s,4)$ is obtained for $s\sim 0$ by analytic continuation in s from $\mathrm{Re}(s)>2$. This may be related to the dimensionally regularised result in (1.20) by decomposing

ζ_Δ as

$$\zeta_\Delta(s,d) = \zeta_\Delta(s,d)_{\mathrm{F}} + \frac{1}{\Gamma(s)} \sum_{n=1}^{2} \frac{b_n^\Delta(d)}{s+n-\frac{1}{2}d},$$

$$b_n^\Delta(d) = \frac{1}{(4\pi)^{d/2}} \int \mathrm{d}^d x \, \mathrm{tr}\big(a_n^\Delta(x,x) - a_n^{\Delta_0}(x,x)\big), \tag{1.21}$$

where $\zeta_\Delta(s,d)_{\mathrm{F}}$ is obtained by taking $K_\Delta(t,d) \to K_\Delta(t,d) - \theta(1-t)\Sigma_{n=1}^2 b_n^\Delta(d) t^{n-d/2}$ in (1.19) and, from (1.9), is regular in d, s for ε, $s \sim 0$. Hence

$$-\ln\det\Delta/\det\Delta_0 = F_\Delta - \gamma b_2^\Delta(4) - 2b_2^{\Delta\prime}(4) + \frac{2b_2^\Delta(4)}{\varepsilon} + \mathrm{O}(\varepsilon), \tag{1.22}$$

displaying explicitly the pole at $\varepsilon = 0$. Zeta function regularisation is essentially equivalent to the Pauli-Villars method which would involve introducing a regularising factor $(1 - \Sigma_j e_i \mathrm{e}^{-tM_i^2})$ in (1.19) and (1.20) and letting $M_i^2 \to \infty$ with $\Sigma_i e_i = 1$, $\Sigma_i e_i M_i^2 = 0$. Then, with $d = 4$, the Pauli-Villars regularised result for the determinants in (1.20) is [23]

$$F_\Delta^{\mathrm{PV}} = F_\Delta - b_1^\Delta(4) \sum_i e_i M_i^2 \ln M_i^2 + b_2^\Delta(4) \sum_i e_i \ln M_i^2. \tag{1.23}$$

For appropriate A_μ, B in (1.5) the operator Δ may have normalisable zero modes. In this case, such as arises in semiclassical calculations, G_Δ has to be defined on the space of functions orthogonal to the zero modes and there are corresponding modifications in the treatment of determinants from (1.18) and (1.20). For simplicity we neglect the consideration of zero modes here.

To two-loop order poles in ε arise from short-distance singularities of products of two or more propagators. The products of the singular functions G_0, R_1, R_2 in (1.13) are well defined without ambiguity for general d and have poles as $\varepsilon \to 0$, with residues which are δ-functions or derivatives of δ-functions, when their dimension is equal to or greater than four. Thus for G_0^2 the leading singular behaviour as $\varepsilon \to 0$ is

$$G_0(x)^2 \sim \frac{1}{8\pi^2} \frac{1}{\varepsilon} \delta^d(x). \tag{1.24}$$

In the appendix a complete list of such singular contributions as $\varepsilon \to 0$ is given for products of two or three combinations of G_0, R_1, R_2 with up to two overall derivatives. For products involving R_1, without any derivative, or R_2 where there are logarithms in the four-dimensional limit, double and single poles in ε occur. These results allow a complete determination of the potential poles in two-loop vacuum graphs in the background field method.

In the next section we illustrate these techniques for scalar ϕ^4 theory where, lacking derivatives, group and Lorentz indices, two-loop calculations become almost trivial. A similar calculation for a general curved spatial background has been described by Toms [24] but with, from our point of view, a less complete use of heat kernel techniques. Sect. 3 contains our main results determining the two-loop divergences for general background fields in gauge theories and recovering the known two-loop β function. We use a Feynman type background gauge so that the a_μ propagator is given by a Green function for an elliptic operator of the form (1.5). In sect. 4 we discuss how the finite part of background field amplitudes, after minimal subtraction of divergences, can be obtained. A few remarks are offered by way of a conclusion concerning the extension of these results to conformally flat spaces. In the appendix there is a list of relevant formulae such as (1.24).

2. Application to ϕ^4 theory

For the simple renormalisable ϕ^4 theory then if

$$\phi = \phi_c + f, \tag{2.1}$$

for ϕ_c a fixed background field, we are concerned with the partition function Z defined by a functional integral over f,

$$Z[\phi_c] = e^{(1/\hbar)W[\phi_c]} = \int d[f]\, e^{-(1/\hbar)S[\phi]+(1/\hbar)(J,f)}, \tag{2.2}$$

with normalisation $W[0] = 0$. S is the euclidean action

$$S[\phi] = \int d^dx \left\{ \tfrac{1}{2}Z_\phi\, \partial\phi\cdot\partial\phi + \tfrac{1}{2}Z_m m^2\phi^2 + \frac{1}{4!}Z_\lambda \lambda\mu^\varepsilon\phi^4 \right\}, \tag{2.3}$$

with m, λ finite mass, coupling parameters and μ is now the dimensional regularisation mass scale. In (2.3) the Z's, and correspondingly also S, W, are regarded as represented as a power series in $\hbar$, $Z_{\phi,m,\lambda} = 1 + \Sigma_{n=1} Z^{(n)}_{\phi,m,\lambda}\hbar^n$, or equivalently as an expansion in the number of loops. Henceforth $\hbar = 1$. With the usual minimal subtraction scheme Z_ϕ, Z_m, Z_λ are power series in λ containing just poles in ε, chosen so as to ensure W is finite for $\varepsilon \to 0$ order by order in the perturbation expansion. In (2.2)

$$(J, f) = \int d^dx\, J(x) f(x), \tag{2.4}$$

and initially we take

$$J = j(\phi_c) = \frac{\delta S^{(0)}}{\delta\phi}[\phi]\bigg|_{\phi=\phi_c}. \tag{2.5}$$

$S^{(0)}$ is the classical action so that if ϕ_c is a classical solution in 4 dimensions, depending on λ, m^2, then with an appropriate, essentially arbitrary, extension to d dimensions we expect $j(\phi_c) = O(\varepsilon)$. Such instanton solutions with finite action exist for $m^2 = 0$, $\lambda < 0$ and are relevant for considering the large-order behaviour of the perturbation expansion [25]. One-loop calculations with dimensional regularisation have been carried out for this case using conformal compactification to S^4 or S^d [26].

From (2.2) to zeroth order of course,

$$W^{(0)}[\phi_c] = -S^{(0)}[\phi_c]. \tag{2.6}$$

For perturbative calculations the action is expanded as

$$S[\phi] - (j, f) = S[\phi_c] + \tfrac{1}{2}(f, Mf) + S_I[f, \phi_c], \tag{2.7}$$

where

$$M = -\partial^2 + m^2 + \tfrac{1}{2}\lambda\mu^\varepsilon\phi_c^2, \tag{2.8}$$

whose Green function G_M gives the propagator for the quantum field f. S_I is the interaction which is treated perturbatively with

$$S_I^{(0)}[f, \phi_c] = \lambda\mu^\varepsilon \int d^dx \left(\tfrac{1}{6}\phi_c f^3 + \frac{1}{4!}f^4\right). \tag{2.9}$$

To one-loop order S_I is unimportant and then

$$W^{(1)}[\phi_c] = -S^{(1)}[\phi_c] - \tfrac{1}{2}\ln\det M/\det M_0. \tag{2.10}$$

With M prescribed by (2.8), (1.11) implies in this case [27]

$$a_0^M(x, y) = 1, \qquad a_1^M(x, x) = -\left(m^2 + \tfrac{1}{2}\lambda\mu^\varepsilon\phi_c(x)^2\right),$$

$$a_2^M(x, x) = \tfrac{1}{2}\left(m^2 + \tfrac{1}{2}\lambda\mu^\varepsilon\phi_c(x)^2\right)^2 - \tfrac{1}{6}\lambda\mu^\varepsilon\,\partial^2\phi_c(x)^2. \tag{2.11}$$

so that, since $M|_{\phi_c=0} = M_0$, from (1.21)

$$b_2^M(d) = \frac{\lambda\mu^\varepsilon}{(4\pi)^{d/2}}U[\phi_c], \qquad U[\phi] = \int d^dx\left(\tfrac{1}{2}m^2\phi^2 + \tfrac{1}{8}\lambda\mu^\varepsilon\phi^4\right). \tag{2.12}$$

Hence (1.22) implies that it is sufficient to take

$$S^{(1)}[\phi] = \frac{\hat{\lambda}}{\varepsilon}U[\phi], \qquad \hat{\lambda} = \frac{\lambda}{16\pi^2}, \tag{2.13}$$

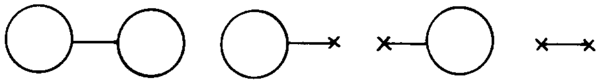


Fig. 1. Two-loop one-particle reducible background field vacuum graphs for ϕ^4 theory.

and in the limit $\varepsilon \to 0$ (2.10) gives the finite result

$$W^{(1)}[\phi_c] = \tfrac{1}{2}F_M + (X + \ln\mu)\hat{\lambda}U[\phi_c],$$
$$X = \tfrac{1}{2}(\ln 4\pi - \gamma). \tag{2.14}$$

X is the standard factor occurring with minimal subtraction, the possible d dependence of $U[\phi_c]$ away from $d=4$ disappears in the physical limit (2.14).

To two-loop order the relevant graphs are given by the interaction $S_I^{(0)}$ in (2.9) together with the one-loop counterterms from (2.13)

$$S_I^{(1)}[f,\phi_c] = \frac{\hat{\lambda}}{\varepsilon}\int d^dx\,\{\phi_c(m^2 + \tfrac{1}{2}\lambda\mu^\varepsilon\phi_c^2)f + \tfrac{1}{2}(m^2 + \tfrac{3}{2}\lambda\mu^\varepsilon\phi_c^2)f^2 + O(f^3)\}. \tag{2.15}$$

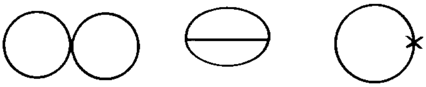
From (1.15), with (2.11) in this case,

$$G_M(x,x) = -\frac{2}{\varepsilon}\frac{\mu^{-\varepsilon}}{16\pi^2}\left(m^2 + \tfrac{1}{2}\lambda\mu^\varepsilon\phi_c(x)^2\right) + D_M(x), \tag{2.16}$$

so that for the one particle reducible graphs in fig. 1, with vertices determined by the interactions (2.9) and (2.15), the associated contribution to $W^{(2)}$ is

$$W_{\rm red}^{(2)}[\phi_c] = \tfrac{1}{8}\lambda^2\mu^{2\varepsilon}\int d^dx\,d^dy\,\phi_c(x)D_M(x)G_M(x,y)\phi_c(y)D_M(y), \tag{2.17}$$

which is finite, as expected, when $\varepsilon \to 0$. For the one-particle irreducible graphs shown in figs. 2a–c the corresponding relevant contributions, after subtractions to



(a) (b) (c)

Fig. 2. Two-loop one-particle irreducible background field graphs for ϕ^4 theory.

ensure $W^{(2)}[0]=0$, are

$$
\begin{aligned}
W^{(2)}_{1\mathrm{PI}}[\phi_c]_a &= -\tfrac{1}{8}\lambda\mu^\varepsilon \int \mathrm{d}^d x \left\{ G_M(x,x)^2 - G_{M_0}(x,x)^2 \right\}, \\
W^{(2)}_{1\mathrm{PI}}[\phi_c]_b &= \tfrac{1}{12}\lambda^2\mu^{2\varepsilon} \int \mathrm{d}^d x\, \mathrm{d}^d y\, \phi_c(x) G_M(x,y)^3 \phi_c(y), \\
W^{(2)}_{1\mathrm{PI}}[\phi_c]_c &= -\frac{\hat{\lambda}}{\varepsilon}\frac{1}{2} \int \mathrm{d}^d x \left\{ \left(m^2 + \tfrac{3}{2}\lambda\mu^\varepsilon \phi_c(x)^2\right) G_M(x,x) - m^2 G_{M_0}(x,x) \right\}.
\end{aligned} \tag{2.18}
$$

Using (2.16) the pole terms for a and c can be identified:

$$
\begin{aligned}
W^{(2)}_{1\mathrm{PI}}[\phi_c]_{a+c} = &-\tfrac{1}{8}\lambda\mu^\varepsilon \int \mathrm{d}^d x \left\{ D_M(x)^2 - D_{M_0}(x)^2 \right\} \\
&- \frac{\hat{\lambda}}{\varepsilon}\frac{1}{2}\lambda\mu^\varepsilon \int \mathrm{d}^d x\, \phi_c(x)^2 D_M(x) \\
&+ \frac{\hat{\lambda}^2}{\varepsilon^2}\frac{1}{2} \int \mathrm{d}^d x\, \phi_c(x)^2 \left(3m^2 + \tfrac{5}{4}\lambda\mu^\varepsilon \phi_c(x)^2\right).
\end{aligned} \tag{2.19}
$$

For b it is sufficient to recognize that if

$$
\begin{aligned}
S_M(x,y) &= R_1(x-y) a_1^M(x,x) + D_M(x), \\
I(x,y) &= \phi_c(x) \left\{ G_0(x-y)^3 + 3G_0(x-y)^2 S_M(x,y) \right\} \phi_c(y),
\end{aligned} \tag{2.20}
$$

then using the expansions (1.12) for G_M it is easy to see that

$$
\phi_c(x) G_M(x,y)^3 \phi_c(y) - I(x,y)
$$

contains no terms as singular as or more singular than $O((x-y)^{-4})$ in the limit $\varepsilon=0$ and so is integrable without divergences. Using (A.4) to give the poles in ε from $G_0^3, G_0^2 R_1, G_0^2$,

$$
\begin{aligned}
\tfrac{1}{12}\lambda^2\mu^{2\varepsilon} \int \mathrm{d}^d x\, \mathrm{d}^d y\, I(x,y) = &-\frac{\hat{\lambda}^2}{\varepsilon^2}\frac{1}{2} \int \mathrm{d}^d x\, \phi_c(x)^2 \left(m^2 + \tfrac{1}{2}\lambda\mu^\varepsilon \phi_c(x)^2\right) \\
&- \frac{\hat{\lambda}^2}{\varepsilon}\frac{1}{12} \int \mathrm{d}^d x\, \phi_c(x) \left(-\tfrac{1}{2}\partial^2 + 3m^2 + \tfrac{3}{2}\lambda\mu^\varepsilon \phi_c(x)^2\right) \phi_c(x) \\
&+ \frac{\hat{\lambda}}{\varepsilon}\frac{1}{2}\lambda\mu^\varepsilon \int \mathrm{d}^d x\, \phi_c(x)^2 D_M(x) + F[\phi_c],
\end{aligned} \tag{2.21}
$$

with F finite as $\varepsilon \to 0$. Hence it is necessary, combining (2.19) and (2.21), to choose

$$S^{(2)}[\phi] = \frac{\hat{\lambda}^2}{\varepsilon^2}\frac{1}{2}\int \mathrm{d}^d x\left(2m^2\phi^2 + \tfrac{3}{4}\lambda\mu^\varepsilon\phi^4\right)$$
$$-\frac{\hat{\lambda}^2}{\varepsilon}\frac{1}{4}\int \mathrm{d}^d x\left(\tfrac{1}{6}\partial\phi\cdot\partial\phi + m^2\phi^2 + \tfrac{1}{2}\lambda\mu^\varepsilon\phi^4\right). \qquad (2.22)$$

Then the total two-loop one-particle irreducible vacuum amplitude becomes

$$W_{1\mathrm{PI}}^{(2)}[\phi_c] = -\tfrac{1}{8}\lambda\mu^\varepsilon\int \mathrm{d}^d x\left\{D_M(x)^2 - D_{M_0}(x)^2\right\}$$
$$+\tfrac{1}{12}\lambda^2\mu^{2\varepsilon}\int \mathrm{d}^d x\,\mathrm{d}^d y\left\{\phi_c(x)G_M(x,y)^3\phi_c(y) - I(x,y)\right\} + F[\phi_c],$$
$$(2.23)$$

which, by its construction, is finite for $\varepsilon = 0$. An explicit expression depends on determining F from (2.21) after subtracting the double and single poles in ε from the integral over $I(x, y)$. A formula for F which achieves this is obtained in sect. 4. Manifestly any potential arbitrariness resulting from possible variations in the choice of I in (2.20) cancels in (2.23).

To two-loop order now calculated from (2.3), (2.13) and (2.22) we obtain

$$Z_\phi = 1 - \frac{\hat{\lambda}^2}{12\varepsilon}, \qquad Z_m = 1 + \frac{\hat{\lambda}}{\varepsilon} + 2\frac{\hat{\lambda}^2}{\varepsilon^2} - \frac{1}{2}\frac{\hat{\lambda}^2}{\varepsilon},$$
$$Z_\lambda = 1 + 3\frac{\hat{\lambda}}{\varepsilon} + 9\frac{\hat{\lambda}^2}{\varepsilon^2} - 3\frac{\hat{\lambda}^2}{\varepsilon}, \qquad (2.24)$$

regaining the standard results [28]

$$\beta(\hat{\lambda}) = -\varepsilon\left[\frac{\mathrm{d}}{\mathrm{d}\hat{\lambda}}\ln\left(\frac{Z_\lambda}{Z_\phi^2}\lambda\right)\right]^{-1} = -\varepsilon\hat{\lambda} + 3\hat{\lambda}^2 - \tfrac{17}{3}\hat{\lambda}^3 + \mathrm{O}(\hat{\lambda}^4),$$
$$\gamma_m(\hat{\lambda}) = -\beta(\hat{\lambda})\frac{\mathrm{d}}{\mathrm{d}\hat{\lambda}}\ln\frac{Z_m}{Z_\phi} = \hat{\lambda} - \tfrac{5}{6}\hat{\lambda}^2 + \mathrm{O}(\hat{\lambda}^3),$$
$$\gamma_\phi(\hat{\lambda}) = \beta(\hat{\lambda})\frac{\mathrm{d}}{\mathrm{d}\hat{\lambda}}\ln Z_\phi = \tfrac{1}{6}\hat{\lambda}^2 + \mathrm{O}(\hat{\lambda}^3). \qquad (2.25)$$

If the background field ϕ_c is not constrained in terms of J by (2.5), and can then be regarded as independent, there are additional contributions which, letting

$$\bar{J} = J - j(\phi_c), \qquad (2.26)$$

Fig. 3. Additional tree graph $O(\bar{J}^2)$ for a source $\bar{J}$.

give a connected vacuum amplitude $W[\phi_c, \bar{J}]$. This may be expanded in powers of $\bar{J}$ about $W[\phi_c]$. Thus from the tree graph of fig. 3

$$W^{(0)}[\phi_c, \bar{J}] = -S^{(0)}[\phi_c] + \tfrac{1}{2}\int d^dx\, d^dy\, \bar{J}(x) G_M(x, y) \bar{J}(y) + O(\bar{J}^3), \tag{2.27}$$

and at the one-loop level, corresponding to the graphs in fig. 4,

$$W^{(1)}[\phi_c, \bar{J}] = W^{(1)}[\phi_c] - \tfrac{1}{2}\lambda\mu^{\varepsilon}\int d^dx\, d^dy\, \bar{J}(x) G_M(x, y) \phi_c(y) D_M(y) + O(\bar{J}^2), \tag{2.28}$$

where $W^{(1)}$ for $\bar{J}=0$ is given in (2.10). The extra terms resulting from the power expansion in $\bar{J}$ are automatically finite as $\varepsilon \to 0$ given the renormalisation counter-terms determined for $\bar{J}=0$. In general $W[\phi_c, \bar{J}]$ is finite once it is renormalised appropriately for one particular choice of $\bar{J}$. The standard generating functional for one-particle irreducible graphs $\Gamma[\phi_c]$ corresponds to a particular choice of $\bar{J}$, as in

$$\Gamma[\phi_c] = W[\phi_c, \bar{J}]\big|_{\delta W/\delta \bar{J} = 0}. \tag{2.29}$$

It is easy to see from (2.27) and (2.28) that, with $\bar{J}$ so determined in terms of ϕ_c by (2.29) neglecting corrections of higher order in $\bar{J}$, the extra terms in (2.27) and (2.28) then just cancel the two-loop one-particle reducible contribution $W^{(2)}_{\text{red}}$ in (2.17).

If ϕ_c is a classical solution and $J=0$ then $\bar{J} = -j(\phi_c) = O(\varepsilon)$. Since they are finite in terms of $\bar{J}$ the additional contributions containing $\bar{J}$ then vanish as $\varepsilon \to 0$. This demonstrates that any arbitrariness in extending the classical solution to d dimensions is irrelevant in the physical limit.

3. Application to gauge theories

As related in the introduction our main endeavour is to extend the two-loop calculations of the previous section to gauge theories. Thus we now consider an

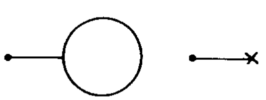


Fig. 4. Additional one-loop graph $O(\bar{J})$ for a source $\bar{J}$.

arbitrary classical background gauge field A_μ, extended to d dimensions, expanded in terms of components A_μ^a over a set of antihermitian generators for the fundamental representation of the gauge group $\mathcal{G}$ so that,

$$A_\mu = A_\mu^a t_a, \qquad [t_a, t_b] = c_{abc} t_c, \quad \mathrm{tr}(t_a t_b) = -\tfrac{1}{2}\delta_{ab}. \tag{3.1}$$

In terms of the field strength $F_{\alpha\beta}$, defined in (1.11), the standard euclidean action which determines the quantum theory functional integral is

$$S[A] = \tfrac{1}{4}\int \mathrm{d}^d x\,(F_{\alpha\beta} F_{\alpha\beta}), \tag{3.2}$$

where we introduce the convenient notation $(t_a t_b) = \delta_{ab}$. Using a covariant background gauge and expanding the quantum field A_μ^{q} about the background A_μ as in (1.1) the partition function, with a source J_μ, is represented by

$$\begin{aligned} Z[A, J] &= \mathrm{e}^{W[A,J]} \\ &= \int \mathrm{d}[a] \det\left(-\frac{1}{\xi^{1/2}}\mathcal{D}\cdot\mathcal{D}^{\mathrm{q}}\right)\exp\left\{-\frac{1}{g_0^2}S[A^{\mathrm{q}}] - \frac{1}{2\xi_0}(\mathcal{D}\cdot a, \mathcal{D}\cdot a)\right. \\ &\qquad\qquad \left. + \frac{1}{Z_a^{1/2}}(J, a)\right\}, \end{aligned}$$

$$g_0^2 = Z_g g^2 \mu^\varepsilon, \qquad \xi_0 = Z_\xi \xi. \tag{3.3}$$

$\mathcal{D}_\mu^{\mathrm{q}}$ is the covariant derivative, in the adjoint representation, involving the full quantum field A_μ^{q}, as for $\mathcal{D}_\mu$ in terms of A_μ in (1.4), and

$$(J, a) = \int \mathrm{d}^d x\,(J_\mu(x) a_\mu(x)). \tag{3.4}$$

The renormalisation constants Z_g, Z_ξ, $Z_a = 1 + \mathrm{O}(g^2)$ are determined so that Z, W are finite order by order in perturbation theory. The choice of g_0' in (1.1), which fixes the scale for a_μ, is arbitrary but it is convenient here to choose $g_0' = g_0$. This corresponds to the quantum fluctuations a_μ being unrenormalised, in the conventional sense, and in this case

$$Z_a = Z_\xi. \tag{3.5}$$

Generally J_μ is supposed to be determined in terms of the background A_μ. For A_μ a classical solution so that if

$$j_\mu(A) = \frac{\delta S[A]}{\delta A_\mu} = \mathcal{D}_\nu F_{\mu\nu}, \tag{3.6}$$

we have $j_\mu = O(\varepsilon)$, it is of interest to consider just the physical vacuum amplitude W for $J_\mu = 0$. According to 't Hooft [7], if J_μ is determined by

$$\frac{\delta}{\delta A_\mu} W[A, J(A)] = -J_\mu(A), \tag{3.7}$$

then

$$W[A, J(A)] = \Gamma[A],$$

with Γ the standard one-particle irreducible generating functional in an unconventional gauge. In our case we set, analogous to (2.5),

$$J_\mu = \frac{1}{g\mu^{\varepsilon/2}} j_\mu(A), \tag{3.8}$$

which coincides with (3.7) to zeroth order in the loop expansion.

The quantum action in (3.3) can be expanded in terms of the fluctuations

$$S[A^{\mathrm{q}}] = S[A] + g_0(j, a) + \tfrac{1}{2} g_0^2 (a, \Delta a) + S_{\mathrm{I}}[a], \tag{3.9}$$

where

$$\Delta_{\mu\nu} = -\delta_{\mu\nu}\mathcal{D}^2 + \mathcal{D}_\mu \mathcal{D}_\nu - 2F_{\mu\nu}^{\mathrm{ad}} \tag{3.10}$$

(for X in the Lie algebra of $\mathcal{G}$, X^{ad} denotes the corresponding matrix acting on the adjoint representation) and

$$S_{\mathrm{I}}[a] = \int \mathrm{d}^d x \left\{ g_0 \left(\mathcal{D}_\alpha a_\beta [a_\alpha, a_\beta] \right) + \tfrac{1}{4} g_0^2 \left([a_\alpha, a_\beta][a_\alpha, a_\beta] \right) \right\}. \tag{3.11}$$

In the usual fashion the leading terms in the perturbative evaluation of (3.3) involve functional determinants, in this case of the ghost operator $-\mathcal{D}^2$ and the effective operator on the quadratic fluctuations

$$\Delta^\xi_{\mu\nu} = \Delta_{\mu\nu} - \frac{1}{\xi} \mathcal{D}_\mu \mathcal{D}_\nu. \tag{3.12}$$

Normalising so that $W[0] = 1$ it is possible in general to rewrite (3.3), with J_μ fixed by (3.8), in the form

$$\mathrm{e}^{W[A]} = \mathrm{e}^{W_1[A]} \langle \exp(H[A, a]) \rangle / \langle \exp(H[0, a]) \rangle,$$

$$W_1[A] = -\frac{1}{Z_g g^2 \mu^\varepsilon} S[A] - \tfrac{1}{2} \ln\det \Delta^\xi / \det \Delta^\xi_0 + \ln\det\left(-\frac{1}{\xi^{1/2}} \mathcal{D}^2 \right) \Big/ \det\left(-\frac{1}{\xi^{1/2}} \mathcal{D}_0^2 \right),$$

$$H[A,a] = -S_{\mathrm{I}}[a] + \mathrm{Tr}\ln\left(1 - g_0 G(\mathscr{D}\cdot a)^{\mathrm{ad}}\right)$$
$$+\frac{1}{2\xi}\left(Z_\xi^{-1}-1\right)(a,\mathscr{D}\mathscr{D}\cdot a) + \left(Z_a^{-1/2} - Z_g^{-1/2}\right)(J,a). \quad (3.13)$$

In (3.13) $\langle\,\cdot\,\rangle$ denotes the expectation value in terms of the free gaussian measure on a_μ so that

$$\left\langle a_\mu^a(x) a_\nu^b(y)\right\rangle = G_{\mu\nu}^{\xi ab}(x,y) = G_{\nu\mu}^{\xi ba}(y,x), \quad (3.14)$$

which is the Green function for Δ^ξ, and correspondingly $G^{ab}(x,y) = G^{ba}(y,x)$ is the Green function for $-\mathscr{D}^2$, where Δ^ξ and $-\mathscr{D}^2$ both act on the adjoint representation.

As in (2.6) the zero-loop vacuum functional is just the classical action

$$W^{(0)}[A] = -\frac{1}{g^2\mu^\varepsilon}S[A], \quad (3.15)$$

with the extraneous mass scale μ disappearing for $\varepsilon = 0$. If $Z_g^{-1} \to (Z_g^{-1})^{(1)}$ in (3.13) then W_1 becomes just the one-loop contribution. When $j_\mu = 0$ it is easy to see that

$$\det\Delta^\xi = \det\Delta\det\left(-\frac{1}{\xi}\mathscr{D}^2\right),$$

with Δ restricted to fields satisfying $\mathscr{D}\cdot a = 0$. In this physical case the ξ dependence in W_1 given by (3.13) manifestly cancels.

The determination of the divergences of $\det\Delta^\xi$ for general ξ can be achieved by heat kernel methods [29] but it is certainly simpler to use $\xi = 1$. This Feynman type gauge is imposed from now on, since two-loop calculations become unduly complicated otherwise. The virtue of $\xi = 1$ is that Δ^1 is at once an elliptic operator of the form (1.5), acting on d-dimensional vector fields in the adjoint representation, with $B \to -2F^{\mathrm{ad}}$. The analysis of divergences of the determinants in (3.13) then follows straightforwardly since, from (1.11),

$$b_1^{\Delta^1}(d) = b_1^{-\mathscr{D}^2}(d) = 0,$$
$$b_2^{\Delta^1}(d) = \left(8 - \tfrac{1}{3}d\right)\frac{C_2}{(4\pi)^{d/2}}S[A],$$
$$b_2^{-\mathscr{D}^2}(d) = -\frac{1}{3}\frac{C_2}{(4\pi)^{d/2}}S[A]. \quad (3.16)$$

C_2 is the usual quadratic Casimir, if $t_a^{\text{ad}} = T_a = -T_a^{\text{T}}$,

$$(T_a)_{bc} = -c_{abc}, \qquad \operatorname{tr}(T_a T_b) = -C_2\,\delta_{ab}. \tag{3.17}$$

With minimal subtraction of the ε pole in $W^{(1)}[A]$, (1.22) thereby requires

$$\left(Z_g^{-1}\right)^{(1)} = \frac{g^2}{\varepsilon}\left(b_2^{\Delta^1}(4) - 2b_2^{-\mathscr{D}^2}(4)\right) = \frac{1}{\varepsilon}\frac{22}{3}\frac{g^2 C_2}{16\pi^2}. \tag{3.18}$$

Then as $\varepsilon \to 0$

$$W^{(1)}[A] = \tfrac{1}{2}\left(F_{\Delta^1} - 2F_{-\mathscr{D}^2}\right) + \left\{X + \ln\mu + \frac{1}{22}\right\}\frac{22}{3}\frac{C_2}{16\pi^2}S[A] \tag{3.19}$$

is finite, with X given in (2.14). The extra $\frac{1}{22}$ in (3.19) comes from the overall d-dependent factor in $b_2^{\Delta^1}(d)$ shown in (3.16). In the usual way the arbitrariness in μ can be compensated for by a redefinition of g, and can be eliminated in terms of an invariant renormalisation group mass scale Λ. With the minimal subtraction scheme then to one-loop order the standard prescription defining Λ_{MS} is

$$\frac{1}{g^2} = \frac{22}{3}\frac{C_2}{(4\pi)^2}\ln\frac{\mu}{\Lambda_{\text{MS}}}, \tag{3.20}$$

so that $W^{(0)} + W^{(1)}$ is just represented by the expression in (3.19) but with $\ln\mu \to \ln\Lambda_{\text{MS}}$. Alternatively using the Pauli-Villars method of regularisation it is only necessary, from (1.23), to replace the { } factor in (3.19) by $\ln\mu' = \sum_i e_i \ln M_i$. As discussed above μ' is eliminated and the zero loop term (3.15) is absorbed by letting $\ln\mu' \to \ln\Lambda_{\text{PV}}$. For equivalence between the two schemes $W^{(0)} + W^{(1)}$ must be unaffected by these arbitrary choices of scale. Hence

$$\frac{\Lambda_{\text{PV}}}{\Lambda_{\text{MS}}} = \mathrm{e}^{X + 1/22}, \tag{3.21}$$

in accord with the most recent evolution of this number [30].

The two-loop vacuum graphs for the gauge theory have the same structure as those for the model scalar theory discussed in the previous section. The one-particle reducible graphs from (3.13) give rise to

$$W_{\text{red}}^{(2)}[A] = \tfrac{1}{2}\int \mathrm{d}^d x\,\mathrm{d}^d y\, V_{\mu,a}^{(1)}(x) G_{\mu\nu}^{1ab}(x,y) V_{\nu,b}^{(1)}(y), \tag{3.22}$$

where $V_{\mu,a}$ represents the a_μ to vacuum amplitude. By virtue of the choice (3.8) $V_{\mu,a}^{(0)} = 0$ and the leading contribution is produced by the one-loop graphs of fig. 5. If

$$\mathscr{D}_\alpha G_{\mu\nu}^1(x,y)|_{y=x} = H_{\alpha\mu\nu}(x), \qquad \mathscr{D}_\alpha G(x,y)|_{y=x} = H_\alpha(x), \tag{3.23}$$

then

$$V^{(1)}_{\mu,a} = g\mu^{\varepsilon/2}\mathrm{tr}\big(T_a\{H_{\alpha\alpha\mu} + H_{\mu\alpha\alpha} - 2H_{\alpha\mu\alpha} - H_\mu\}\big)$$
$$+ \frac{1}{g\mu^{\varepsilon/2}}\big(Z_a^{-1/2} - Z_g^{-1/2}\big)^{(1)} j_{\mu,a}. \tag{3.24}$$

From the general analysis of the short-distance behaviour of Green functions in sect. 1 it follows that as $\varepsilon \to 0$

$$H_{\alpha\mu\nu} \sim \frac{1}{\varepsilon}\frac{1}{16\pi^2}\big(2\mathfrak{D}_\alpha F_{\mu\nu} + \tfrac{1}{3}\delta_{\mu\nu} j_\alpha\big)^{\mathrm{ad}},$$
$$H_\alpha \sim \frac{1}{\varepsilon}\frac{1}{16\pi^2}\frac{1}{3} j_\alpha^{\mathrm{ad}}. \tag{3.25}$$

Using these pole terms in (3.24), in conjunction with (3.18), shows that a finite limit as $\varepsilon \to 0$ is obtained if

$$\big(Z_a^{-1}\big)^{(1)} = -\frac{1}{\varepsilon}\frac{10}{3}\frac{g^2 C_2}{16\pi^2}. \tag{3.26}$$

If $J_\mu = 0$, so that effectively $Z_a^{-1/2} \to 0$, (3.24) contains a potential pole term $O(j_{\mu,a}/\varepsilon)$. For A_μ a solution of the classical equations in 4 dimensions so that $j_\mu = O(\varepsilon)$ there would be a finite non-zero contribution in (3.24) as $\varepsilon \to 0$ depending on the method of continuation of the classical solution to d dimensions. However in this case there are additional two-loop graphs, since now $V^{(0)}_\mu = -j_\mu/g\mu^{\varepsilon/2}$ is non-zero, and these cancel the apparent j_μ/ε poles in (3.22). Hence $W^{(2)}$ is then uniquely defined by (3.22) for $\varepsilon = 0$ where $V^{(1)}_\mu$ is given by (3.24) but with the j_μ/ε terms subtracted off.

The one-particle irreducible two-loop graphs are those in fig. 6, for simplicity the space arguments x, y of the propagators $G^1_{\alpha\beta}$, G in the corresponding integrals are henceforth suppressed. For fig. 6a the four-gluon vertex gives

$$W^{(2)}_{1\mathrm{PI}}[A]_{\mathrm{a}} = -\tfrac{1}{4}g^2\mu^\varepsilon \int \mathrm{d}^d x \big\{\mathrm{tr}\big(T_a G^1_{\alpha\beta}\big)\mathrm{tr}\big(T_a G^1_{\alpha\beta}\big)$$
$$+ \mathrm{tr}\big(T_a G^1_{\alpha\beta} T_a G^1_{\alpha\beta} - T_a G^1_{\alpha\alpha} T_a G^1_{\beta\beta}\big)\big\}\big|_{y=x}. \tag{3.27}$$

Fig. 5. One-loop a_μ vacuum graphs.

(a) (b) (c) (d)

Fig. 6. Two-loop one-particle irreducible background field vacuum graphs for gauge theory.

From (1.15)

$$G^{1}_{\alpha\beta}(x,x)=\frac{1}{\varepsilon}\frac{4\mu^{-\varepsilon}}{16\pi^{2}}F^{\mathrm{ad}}_{\alpha\beta}(x)+D_{\alpha\beta}(x), \tag{3.28}$$

and, as a consequence of (3.17),

$$T_a^{2}=-C_2 1, \qquad T_aT_bT_a=-\tfrac{1}{2}C_2T_b. \tag{3.29}$$

(3.27) can be written as a part involving poles in ε and a remainder which is finite as $\varepsilon\to 0$,

$$W^{(2)}_{1\mathrm{PI}}[A]^{\mathrm{pole}}_{\mathrm{a}}=-\frac{1}{\varepsilon^{2}}24\frac{g^{2}}{\mu^{\varepsilon}}\frac{C_2^{2}}{(16\pi^{2})^{2}}S[A]-\frac{1}{\varepsilon}3g^{2}\frac{C_2}{16\pi^{2}}\int \mathrm{d}^{d}x\,\mathrm{tr}\big(F^{\mathrm{ad}}_{\alpha\beta}D_{\beta\alpha}\big),$$

$$\begin{aligned}W^{(2)}_{1\mathrm{PI}}[A]^{\mathrm{finite}}_{\mathrm{a}}=-\tfrac{1}{4}g^{2}\mu^{\varepsilon}\int \mathrm{d}^{d}x\,\big\{&\mathrm{tr}(T_aD_{\alpha\beta})\mathrm{tr}(T_aD_{\alpha\beta})\\&+\mathrm{tr}\big(T_aD_{\alpha\beta}T_aD_{\alpha\beta}-T_aD_{\alpha\alpha}T_aD_{\beta\beta}\big)\big\}.\end{aligned} \tag{3.30}$$

The expressions corresponding to figs. 6b, c are more complicated. The explicit appearance of group indices can be avoided by introducing the notation

$$(X,Y,Z)=c_{abc}c_{a'b'c'}X^{aa'}Y^{bb'}Z^{cc'}, \tag{3.31}$$

so that if

$$W^{(2)}_{1\mathrm{PI}}[A]_{\mathrm{b,c}}=\tfrac{1}{2}g^{2}\mu^{\varepsilon}\int \mathrm{d}^{d}x\,\mathrm{d}^{d}y\,I_{\mathrm{b,c}}(x,y), \tag{3.32}$$

then

$$\begin{aligned}I_{\mathrm{b}}&=\big(\mathcal{D}_\alpha G, G\overleftarrow{\mathcal{D}}_\beta, G^{1}_{\alpha\beta}\big)+\partial G_0\cdot\partial G_0G_0C_2N,\\
I_{\mathrm{c}}&=\big(\mathcal{D}_\alpha G^{1}_{\beta\gamma}\overleftarrow{\mathcal{D}}_\delta, G^{1}_{\alpha\delta}, G^{1}_{\beta\gamma}\big)-\big(\mathcal{D}_\alpha G^{1}_{\beta\gamma}\overleftarrow{\mathcal{D}}_\delta, G^{1}_{\alpha\gamma}, G^{1}_{\beta\delta}\big)\\
&\quad-\big(\mathcal{D}_\alpha G^{1}_{\beta\gamma}, G^{1}_{\alpha\delta}\overleftarrow{\mathcal{D}}_\gamma, G^{1}_{\beta\delta}\big)+\big(\mathcal{D}_\alpha G^{1}_{\beta\gamma}, G^{1}_{\alpha\gamma}\overleftarrow{\mathcal{D}}_\delta, G^{1}_{\beta\delta}\big)\\
&\quad+\big(\mathcal{D}_\alpha G^{1}_{\beta\gamma}, G^{1}_{\beta\delta}\overleftarrow{\mathcal{D}}_\gamma, G^{1}_{\alpha\delta}\big)-\big(\mathcal{D}_\alpha G^{1}_{\beta\gamma}, G^{1}_{\beta\gamma}\overleftarrow{\mathcal{D}}_\delta, G^{1}_{\alpha\delta}\big)\\
&\quad-(d-1)\,\partial G_0\cdot\partial G_0G_0C_2N.\end{aligned} \tag{3.33}$$

In (3.33) the corresponding contribution for $A_\mu = 0$ has been subtracted (N is the dimension of $\mathcal{G}$). This removes the leading divergences of $I_{\mathrm{b,c}}$ as $s = x - y \to 0$. For both expressions $I_{\mathrm{b,c}}(x, y) = I_{\mathrm{b,c}}(y, x)$ which constrains the nonleading divergences depending on the background field.

To determine only the pole terms in (3.32) at $\varepsilon \to 0$ it is sufficient to expand all the Green functions appearing in (3.3) as in (1.12) and consistently use the results in (A.4). If it is ultimately desired to decompose (3.32) into pieces containing poles in ε and finite remainders, as in (3.30), it is more useful to isolate first the singular parts $I^{\mathrm{S}}_{\mathrm{b,c}}(x, y)$ of (3.33). These are required to contain all the dependence on $s = x - y$ as, or more singular than, s^{-4} as $s \to 0$ in the limit $\varepsilon = 0$. Such terms may be isolated using the expansion (1.12) for each Green function. The final result involves the group element $P(x, y)$ giving parallel transport along the straight line from y to x for the connection A_μ, belonging to the adjoint representation of $\mathcal{G}$. This occurs since from (1.10)

$$a_0^{\Delta^1}(x, y) = \delta_{\mu\nu} P(x, y), \qquad a_0^{-\mathcal{D}^2}(x, y) = P(x, y). \tag{3.34}$$

For $x \sim y$

$$\mathcal{D}_\mu P(x, y)\big|_{y=x} = 0, \qquad \mathcal{D}_\mu \mathcal{D}_\nu P(x, y)\big|_{y=x} = \tfrac{1}{2} F^{\mathrm{ad}}_{\mu\nu}(x), \tag{3.35}$$

and with consequential results in (A.7) we find after some simplification, for $\mathcal{D}_\alpha \mathcal{D}_\beta \bar{G}_{\mathrm{diag.}}(x) = \mathcal{D}_\alpha \mathcal{D}_\beta \bar{G}(x, y)|_{y=x}$,

$$\begin{aligned} I^{\mathrm{S}}_{\mathrm{b}}(x, y) = {} & C_2^2\big\{ \tfrac{1}{24}(2d-1) G_0(s)^3 s_\alpha s_\beta - 2\, \partial_\alpha G_0(s)\, \partial_\beta G_0(s) R_2(s) \\ & - (3 - \tfrac{1}{6}d) G_0(s)\, \partial_\alpha G_0(s)\, \partial_\beta R_2(s) \big\} (F_{\alpha\mu} F_{\beta\mu}) \\ & + C_2^2 \tfrac{1}{12} \big\{ (d-2) G_0(s)^2 R_1(s) + \partial G_0(s) \cdot \partial G_0(s) R_2(s) \big\} (F_{\mu\nu} F_{\mu\nu}) \\ & + C_2\, \partial_\alpha G_0(s) G_0(s) s_\beta \\ & \times \big\{ \tfrac{1}{4} \mathrm{tr}\big(D_{\alpha\mu} F^{\mathrm{ad}}_{\mu\beta} + F^{\mathrm{ad}}_{\beta\mu} D_{\mu\alpha} \big) - 2\, \mathrm{tr}\big(\mathcal{D}_\alpha \mathcal{D}_\beta \bar{G}_{\mathrm{diag.}} \big) \big\} \\ & - C_2\, \partial_\alpha G_0(s)\, \partial_\beta G_0(s) \mathrm{tr}\big(P(y, x) \bar{G}^1_{\alpha\beta}(x, y) \big). \end{aligned} \tag{3.36}$$

Only the last term is more singular than s^{-4} when $\varepsilon = 0$. For those terms which are local in A_μ this is forced by the requirement of invariance under background field gauge transformations (1.2) so that such terms must involve $(F_{\alpha\beta} F_{\gamma\delta})$ as the lowest dimension local gauge invariant constructed out of A_μ. For the terms of $\mathrm{O}(s^{-4})$ when $\varepsilon = 0$ the regular coefficient functions may be evaluated at x, or any other convenient point differing by $\mathrm{O}(x - y)$.

The poles in $W^{(2)}_{1\mathrm{PI}}[A]_{\mathrm{b}}$ as $\varepsilon \to 0$ which arise from the singular behaviour of $I_{\mathrm{b}}(x, y)$ as for $x-y \to 0$ may now be straightforwardly obtained by letting $I_{\mathrm{b}} \to I^{\mathrm{S}}_{\mathrm{b}}$ and using the results in (A.4). With also, from (1.17),

$$\mathrm{tr}\left(\mathfrak{D}^2 \bar{G}_{\mathrm{diag}}\right) = \frac{C_2}{16\pi^2}\frac{\mu^{-\varepsilon}}{12}\left(F_{\mu\nu}F_{\mu\nu}\right), \tag{3.37}$$

we get

$$\begin{aligned} W^{(2)}_{1\mathrm{PI}}[A]^{\mathrm{pole}}_{\mathrm{b}} &= \frac{1}{\varepsilon}\frac{2}{3}\frac{g^2}{\mu^{\varepsilon}}\frac{C_2^2}{(16\pi^2)^2} S[A] \\ &\quad + \frac{1}{\varepsilon}\frac{1}{2}g^2\frac{C_2}{16\pi^2}\int \mathrm{d}^d x\, \mathrm{tr}\left(\left(-\tfrac{1}{6}\mathfrak{D}^2\,\delta_{\beta\alpha} - \tfrac{1}{3}\mathfrak{D}_\beta \mathfrak{D}_\alpha - \tfrac{1}{3}F^{\mathrm{ad}}_{\beta\alpha}\right)\bar{G}^1_{\alpha\beta\,\mathrm{diag.}}\right). \end{aligned} \tag{3.38}$$

Similarly

$$\begin{aligned} I^{\mathrm{S}}_{\mathrm{c}}(x, y) = {} & C_2^2\left\{\tfrac{1}{24}(23 + 3d - 2d^2)G_0(s)^3 s_\alpha s_\beta + 2(d-1)\,\partial_\alpha G_0(s)\,\partial_\beta G_0(s) R_2(s)\right. \\ & \left. + \tfrac{1}{6}(6 + 7d - d^2)G_0(s)\,\partial_\alpha G_0(s)\,\partial_\beta R_2(s)\right\}\left(F_{\alpha\mu}F_{\beta\mu}\right) \\ & - C_2^2\left\{\tfrac{1}{48}(25 - d)G_0(s)^3 s^2 + \tfrac{1}{12}(d-1)\,\partial G_0(s)\cdot\partial G_0(s) R_2(s)\right. \\ & \left. + \tfrac{1}{24}(176 - 95d + 3d^2)G_0(s)^2 R_1(s)\right\}\left(F_{\mu\nu}F_{\mu\nu}\right) \\ & - C_2\left\{\tfrac{3}{4}G_0(s)\,\partial_\alpha G_0(s) s_\beta + \partial_\alpha\,\partial_\beta G_0(s) R_1(s)\right\}\mathrm{tr}\left(D_{\alpha\mu}F^{\mathrm{ad}}_{\mu\beta} + F^{\mathrm{ad}}_{\alpha\mu}D_{\mu\beta}\right) \\ & + C_2 G_0(s)^2\left\{\mathrm{tr}\left(\mathfrak{D}_\mu \bar{G}^1_{\nu\nu}\overleftarrow{\mathfrak{D}}_{\mu\,\mathrm{diag}} - \mathfrak{D}_\mu \bar{G}^1_{\nu\mu}\overleftarrow{\mathfrak{D}}_{\nu\,\mathrm{diag}}\right) + \tfrac{1}{4}(d-2)\mathrm{tr}\left(F^{\mathrm{ad}}_{\mu\nu}D_{\nu\mu}\right)\right\} \\ & + 2C_2 G_0(s)\,\partial_\alpha G_0(s) s_\beta \mathrm{tr}\left(\mathfrak{D}_\beta \mathfrak{D}_\alpha \bar{G}^1_{\mu\mu\,\mathrm{diag}} - \mathfrak{D}_\beta \mathfrak{D}_\mu \bar{G}^1_{\alpha\mu\,\mathrm{diag}}\right) \\ & - C_2\,\partial_\alpha G_0(s)\,\partial_\beta G_0(s)\mathrm{tr}\left(P(y, x)\bar{G}^1_{\alpha\beta}(x, y)\right) \\ & + C_2\,\partial G_0(s)\cdot\partial G_0(s)\mathrm{tr}\left(P(y, x)\bar{G}^1_{\alpha\alpha}(x, y)\right), \end{aligned} \tag{3.39}$$

and hence

$$W^{(2)}_{1\mathrm{PI}}[A]^{\mathrm{pole}}_{\mathrm{c}} = \left(\frac{24}{\varepsilon^2} - \frac{1}{\varepsilon}\right)\frac{g^2}{\mu^\varepsilon}\frac{C_2^2}{(16\pi^2)^2}S[A]$$

$$+\frac{1}{\varepsilon}\frac{1}{2}g^2\frac{C_2}{16\pi^2}\int \mathrm{d}^d x\,\mathrm{tr}\Big(\big(-\tfrac{19}{6}\mathcal{D}^2\delta_{\beta\alpha} + \tfrac{11}{3}\mathcal{D}_\beta\mathcal{D}_\alpha - \tfrac{1}{3}F^{\mathrm{ad}}_{\beta\alpha}\big)\overline{G}^1_{\alpha\beta\,\mathrm{diag}}\Big). \tag{3.40}$$

Adding then the two-loop pole contributions from (3.30), (3.38) and (3.40)

$$W^{(2)}_{1\mathrm{PI}}[A]^{\mathrm{pole}}_{\mathrm{a+b+c}} = -\frac{1}{\varepsilon}\frac{1}{3}\frac{g^2}{\mu^\varepsilon}\frac{C_2^2}{(16\pi^2)^2}S[A]$$

$$+\frac{1}{\varepsilon}\frac{5}{3}g^2\frac{C_2}{16\pi^2}\int \mathrm{d}^d x\,\mathrm{tr}\big(\Delta_{\beta\alpha}\overline{G}^1_{\alpha\beta\,\mathrm{diag}}\big), \tag{3.41}$$

with Δ defined in (3.10). It remains to consider the contribution resulting from the gauge fixing counterterms, corresponding to fig. 6d. Using (3.5) and (3.26), for our choice $\xi = 1$, this is

$$W^{(2)}_{1\mathrm{PI}}[A]_{\mathrm{d}} = -\frac{1}{\varepsilon}\frac{5}{3}g^2\frac{C_2}{16\pi^2}\int \mathrm{d}^d x\,\mathrm{tr}\big(\mathcal{D}_\beta\mathcal{D}_\alpha G^1_{\alpha\beta\,\mathrm{diag}}\big). \tag{3.42}$$

From the expansion (1.12)

$$\mathcal{D}_\beta\mathcal{D}_\alpha G^1_{\alpha\beta\,\mathrm{diag}} - \mathcal{D}_\beta\mathcal{D}_\alpha\overline{G}^1_{\alpha\beta\,\mathrm{diag}} = \frac{\mu^{-\varepsilon}}{16\pi^2}\frac{1}{\varepsilon}\big(2\mathcal{D}_\beta\mathcal{D}_\alpha a^{\Delta^1}_{1\alpha\beta} - a^{\Delta^1}_{2\alpha\alpha}\big)_{\mathrm{diag}}$$

$$= \frac{\mu^{-\varepsilon}}{16\pi^2}\frac{1}{12}F^{\mathrm{ad}}_{\alpha\beta}F^{\mathrm{ad}}_{\alpha\beta}, \tag{3.43}$$

so that adding (3.42) to (3.41) just ensures that $-\frac{1}{3} \to \frac{2}{9}$ in the first term and $\Delta \to \Delta^1$ in the second. The pole terms non-local in A_μ finally disappear by using, from (1.17),

$$\mathrm{tr}\big(\Delta^1_{\beta\alpha}\overline{G}^1_{\alpha\beta\,\mathrm{diag}}\big) = C_2\frac{\mu^{-\varepsilon}}{16\pi^2}\big(2 - \tfrac{1}{12}d\big)\big(F_{\alpha\beta}F_{\alpha\beta}\big). \tag{3.44}$$

Ultimately from all these contributions for arbitrary A_μ there remains a single pole

$$W^{(2)}_{1\mathrm{PI}}[A]_{\mathrm{a+b+c+d}} \sim \frac{1}{\varepsilon}\frac{34}{3}\frac{g^2}{\mu^\varepsilon}\frac{C_2^2}{(16\pi^2)^2}S[A], \tag{3.45}$$

which can now be cancelled by the correct choice of $(Z_g^{-1})^{(2)}$. Hence we reproduce the known [17] two-loop results,

$$Z_g^{-1} = 1 + \frac{1}{\varepsilon}\frac{22}{3}\frac{g^2 C_2}{16\pi^2} + \frac{1}{\varepsilon}\frac{34}{3}\left(\frac{g^2 C_2}{16\pi^2}\right)^2,$$

$$\beta(g) = -\varepsilon\left[\frac{\mathrm{d}}{\mathrm{d}g}\ln Z_g g^2\right]^{-1}$$

$$= -\tfrac{1}{2}\varepsilon g - \tfrac{11}{3} g \frac{g^2 C_2}{16\pi^2} - \tfrac{34}{3} g\left(\frac{g^2 C_2}{16\pi^2}\right)^2. \tag{3.46}$$

The remainder after subtraction of the poles in ε and which are finite when $\varepsilon \to 0$ may now be obtained more explicitly by writing

$$W_{1\mathrm{PI}}^{(2)}[A]_{\mathrm{b,c}}^{\mathrm{finite}} = \tfrac{1}{2} g^2 \mu^{\varepsilon} \int \mathrm{d}^d x\, \mathrm{d}^d y \{I_{\mathrm{b,c}}(x, y) - \bar{I}_{\mathrm{b,c}}(x, y)\} + F_{\mathrm{b,c}}[A],$$

$$\tfrac{1}{2} g^2 \mu^{\varepsilon} \int \mathrm{d}^d x\, \mathrm{d}^d y\, \bar{I}_{\mathrm{b,c}}(x, y) = W_{1\mathrm{PI}}^{(2)}[A]_{\mathrm{b,c}}^{\mathrm{pole}} + F_{\mathrm{b,c}}[A], \tag{3.47}$$

where $\bar{I}_{\mathrm{b,c}}(x, y)$ is chosen so that it agrees with $I_{\mathrm{b,c}}^{\mathrm{S}}(x, y)$, as in (3.36) and (3.39), as far as the singular behaviour when $s \to 0$ which generates poles in ε but also so that no spurious infrared divergences resulting from the integrations for large $x, y \to \infty$ are introduced. A method for obtaining a finite integral expression for $F_{\mathrm{b,c}}[A]$ in 4 dimensions is outlined in the next section. The final result is then, for the minimal subtraction scheme of renormalization

$$W_{1\mathrm{PI}}^{(2)}[A] = W_{1\mathrm{PI}}^{(2)}[A]_{\mathrm{a+b+c}}^{\mathrm{finite}} + \frac{5}{9}\frac{g^2}{\mu^{\varepsilon}}\left(\frac{C_2}{16\pi^2}\right)^2 S[A], \tag{3.48}$$

with the last term coming from the d-dependent factor in (3.44).

4. Finite parts

As we have shown, the determination of the poles in ε for the two-loop dimensionally regularised vacuum amplitude for arbitrary background fields is essentially an algebraic exercise. It is also possible, using our methods, to obtain explicit well-defined integral expressions for the finite parts, as $\varepsilon \to 0$, after the pole terms have been subtracted.

To this end we consider integrals of the form

$$S = \int \mathrm{d}^d x \, \mathrm{d}^d y \, \alpha(x) H(x-y) \beta(y), \tag{4.1}$$

where α, β are smooth integrable functions (within some suitable class of test functions) and $H(s)$ is one of the singular functions of s, listed in appendix A which, as a distribution, has poles when $\varepsilon \to 0$. For an explicit example we use

$$H(s) = G_0(s)^2, \tag{4.2}$$

but other cases can be discussed in an exactly similar fashion. With (4.2) then from (1.24) there is a pole as $\varepsilon \to 0$,

$$S^{\text{pole}} = \frac{1}{\varepsilon} \frac{\mu^{-\varepsilon}}{8\pi^2} \int \mathrm{d}^d x \, \alpha(x) \beta(x). \tag{4.3}$$

As before the appropriate power of the scale μ is inserted in preserve dimensions, any arbitrariness will be compensated by the remaining finite part.

The pole in $G_0(s)^2$, which gives (4.3), is not, of course, directly apparent from its functional form but it can be revealed by Fourier transforms. Let

$$\mathcal{F}(\alpha)(k) = \frac{1}{(2\pi)^{d/2}} \int \mathrm{d}^d x \, \mathrm{e}^{ik\cdot x} \alpha(x), \tag{4.4}$$

and further

$$\tilde{H}(k) = \int \mathrm{d}^d s \, \mathrm{e}^{ik\cdot s} H(s). \tag{4.5}$$

$\tilde{H}$ can be evaluated using (A.1) and the result contains an explicit pole in ε arising from the singular behaviour of $H(s)$ as $s \to 0$. This can then be subtracted to give $\tilde{H}_{\text{reg}}$, which is a finite distribution as $\varepsilon \to 0$, and hence

$$S^{\text{finite}} = S - S^{\text{pole}} = \int \mathrm{d}^d k \, \mathcal{F}(\alpha)(-k) \tilde{H}_{\text{reg}}(k) \mathcal{F}(\beta)(k). \tag{4.6}$$

This expression for S^{finite} should, for suitable α, β, then be finite and well behaved as $\varepsilon \to 0$. For H given by (4.2)

$$\begin{aligned} \tilde{H}_{\text{reg}}(k) &= \tilde{H}(k) - \frac{1}{\varepsilon} \frac{\mu^{-\varepsilon}}{8\pi^2} \\ &\underset{\varepsilon \to 0}{=} \frac{1}{16\pi^2} \left(\ln 4\pi - \gamma + 2 - \ln \frac{k^2}{\mu^2} \right). \end{aligned} \tag{4.7}$$

For the ϕ^4 theory, discussed in sect 2, the crucial requirement to obtain an expression for the finite part is to determine F defined by (2.21), with I given by (2.20). The integral over I in (2.21) is clearly a sum of three terms of the form (4.1). Thus the above method can be applied with H replaced by G_0^3, $G_0^2 R_1$ and G_0^2 giving, with a_1^M from (2.11),

$$\begin{aligned} F[\phi_c]\big|_{\varepsilon=0} = {} & \tfrac{1}{24}\hat{\lambda}^2 \int \mathrm{d}^4k\, \mathcal{F}(\phi_c)(-k) k^2 \left(\ln \frac{k^2}{\mu^2} - 2X - \tfrac{13}{4} \right) \mathcal{F}(\phi_c)(k) \\ & + \tfrac{1}{8}\hat{\lambda}^2 \int \mathrm{d}^4k\, \mathcal{F}\left(m^2\phi_c + \tfrac{1}{2}\lambda\phi_c^3\right)(-k) \left(\ln \frac{k^2}{\mu^2} - 2X - 1 \right)^2 \mathcal{F}(\phi_c)(k) \\ & - \tfrac{1}{4}\hat{\lambda}\lambda \int \mathrm{d}^4k\, \mathcal{F}(\phi_c D_m)(-k) \left(\ln \frac{k^2}{\mu^2} - 2X - 2 \right) \mathcal{F}(\phi_c)(k), \end{aligned} \tag{4.8}$$

with $2X = \ln 4\pi - \gamma$. This result in (2.23) with $\varepsilon = 0$ achieves the desired finite expression for the two-loop amplitude. Actual evaluation requires only knowledge of the Green function G_M for the given background field ϕ_c.

A similar result can be obtained for gauge theories if $\bar{I}_{b,c}$ can be chosen in (3.47) so that it is a sum of terms of the form (4.1). $I^S_{b,c}$, given by (3.36) and (3.39), have to be modified, while retaining the leading short-distance behaviour, to give $\bar{I}_{b,c}$ the required property. If I^S contains a term $H(s)f(x)$, with $H(s) = O(s^{-4})$ when $\varepsilon = 0$, it would be sufficient to replace it in $\bar{I}$ by $H(s)(f(x)f(y))^{1/2}$.

5. Conclusion

Apart from the usual tedious complication associated with evaluating multiple contractions over group and Lorentz vector indices we have described a method of calculating two-loop vacuum amplitudes for arbitrary background fields, with dimensional regularisation, that preserves many of the simplifications present at one loop. Any arbitrariness in the dimensional continuation can ultimately be absorbed in the renormalisation constants containing the poles in ε and the minimal subtraction scheme provides a unique prescription for this choice. To ensure exact correspondence with conventional momentum-space calculations we subtract the poles in ε for our background field calculations with the residue some d-dimensional x integral over some function of the background fields, but without any explicit d-dependent factors. When the poles are finally cancelled by the minimal subtraction choice of counterterms there is then no dependence at all on the continuation to d dimensions in the limit $\varepsilon = 0$.

The calculations described in this paper have all been made on flat euclidean space although it should be possible to extend them to general curved backgrounds.

For conformally invariant theories it is possible in a much more simple fashion to discuss conformally flat spaces with a metric of the form

$$g_{\mu\nu} = \Omega^2 \delta_{\mu\nu} . \tag{5.1}$$

For S^d, of radius a, we can take

$$\Omega(x) = 2\left(1 + \frac{x^2}{a^2}\right)^{-1} . \tag{5.2}$$

Considering just gauge theories the d-dimensional action is simply modified, for metric (5.1), from (3.2) to

$$S_\Omega[A] = \tfrac{1}{4}\int \mathrm{d}^d x\, \Omega^{-\varepsilon}\left(F_{\alpha\beta}F_{\alpha\beta}\right). \tag{5.3}$$

It is then possible to treat the dependence on the conformal factor Ω perturbatively by expanding

$$\Omega^{-\varepsilon} = 1 - \varepsilon \ln \Omega + \cdots , \tag{5.4}$$

and regarding the terms of order ε and higher as part of the interaction. The gauge-fixing terms can be left unchanged since, for $J_\mu = 0$ and A_μ a background classical solution, the partition function or $W[A]$ is then independent of the particular prescription for gauge fixing. To one loop the Ω dependence is just determined by the pole in Z_g^{-1} and the first term in the expansion (5.4) which together give a finite result as $\varepsilon \to 0$. From (3.16) then

$$W_\Omega^{(1)}[A] = W^{(1)}[A] + \frac{11}{6}\frac{C_2}{16\pi^2}\int \mathrm{d}^4 x \ln \Omega\left(F_{\alpha\beta}F_{\alpha\beta}\right). \tag{5.5}$$

A possible one-loop contribution

$$\varepsilon \tfrac{1}{2}\int \mathrm{d}^d x \ln \Omega(x) \operatorname{tr}\left(\Delta_{\mu\nu} G^1_{\nu\mu}(x, y)|_{y=x}\right) \tag{5.6}$$

can be shown to vanish as $\varepsilon \to 0$. The result (5.5) is exactly equivalent to the results for the dependence of functional determinants on the conformal scale that can be derived by zeta function techniques [21, 31].

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Note added

While this paper was being written we received a preprint [33]. The basic aims are similar to our work but the methods of calculation and of implementing dimensional regularisation differ considerably.

Appendix

To discuss the singular behaviour of $|x|^{-2\alpha}$ we use d-dimensional Fourier transforms which are defined in a distribution theoretic sense [32]:

$$\int \mathrm{d}^d x \, \frac{1}{|x|^{2\alpha}} \mathrm{e}^{ik\cdot x} = \pi^{d/2} \frac{\Gamma\left(\frac{1}{2}d-\alpha\right)}{\Gamma(\alpha)} \left(\tfrac{1}{4}k^2\right)^{\alpha-d/2}. \tag{A.1}$$

The right-hand side of (A.1) has poles when $\alpha - \frac{1}{2}d = 0, 1, \ldots$, with residues which are integral powers of k^2. Hence

$$\frac{1}{|x|^{2\alpha}} \sim \frac{1}{\frac{1}{2}d-\alpha+n} \frac{\pi^{d/2}}{\Gamma\left(\frac{1}{2}d+n\right)} \frac{1}{2^{2n}n!} \left(\partial^2\right)^n \delta^d(x), \qquad n=0,1,\ldots. \tag{A.2}$$

Similarly

$$\frac{x_\mu x_\nu}{|x|^{2\alpha+2}} \sim \frac{1}{\frac{1}{2}d-\alpha+n} \frac{\pi^{d/2}}{\Gamma\left(\frac{1}{2}d+n+1\right)} \frac{1}{2^{2n}n!} \left(\tfrac{1}{2}\delta_{\mu\nu}\partial^2 + n\,\partial_\mu\partial_\nu\right)\left(\partial^2\right)^{n-1} \delta^d(x). \tag{A.3}$$

Using these results we find

$$G_0(x)^2 \sim \frac{1}{8\pi^2}\frac{1}{\varepsilon}\delta^d(x), \tag{A.4a}$$

$$G_0(x)\,\partial_\mu G_0(x) \sim \frac{1}{16\pi^2}\frac{1}{\varepsilon}\partial_\mu \delta^d(x), \tag{A.4b}$$

$$\partial_\mu G_0(x)\,\partial_\nu G_0(x) \sim \frac{1}{16\pi^2}\frac{1}{\varepsilon}\frac{1}{6}\left(\partial^2\delta_{\mu\nu} + 2\partial_\mu\partial_\nu\right)\delta^d(x), \tag{A.4c}$$

$$\partial_\mu G_0(x)\,\partial_\nu R_1(x) \sim \frac{1}{16\pi^2}\frac{1}{\varepsilon}\frac{1}{2}\delta_{\mu\nu}\delta^d(x), \tag{A.4d}$$

$$G_0(x)^3 \sim \frac{1}{(16\pi^2)^2}\frac{1}{\varepsilon}\frac{1}{2}\partial^2\delta^d(x), \tag{A.4e}$$

$$G_0(x)^2 R_1(x) \sim \frac{\mu^{-2\varepsilon}}{(16\pi^2)^2}\left(\frac{2}{\varepsilon^2}+\frac{1}{\varepsilon}\right)\delta^d(x), \tag{A.4f}$$

$$G_0(x)^2\,\partial_\mu G_0(x) \sim \frac{1}{(16\pi^2)^2}\frac{1}{6}\frac{1}{\varepsilon}\partial_\mu\partial^2\delta^d(x), \tag{A.4g}$$

$$G_0(x)\,\partial_\mu G_0(x) R_1(x) \sim \frac{\mu^{-2\varepsilon}}{(16\pi^2)^2}\left(\frac{1}{\varepsilon^2}+\frac{1}{4}\frac{1}{\varepsilon}\right)\partial_\mu \delta^d(x), \tag{A.4h}$$

$$G_0(x)^2\,\partial_\mu R_1(x) \sim \frac{1}{(16\pi^2)^2}\frac{1}{\varepsilon}\frac{1}{2}\partial_\mu \delta^d(x), \tag{A.4i}$$

$$\partial_\mu G_0(x)\,\partial_\nu G_0(x) G_0(x) \sim \frac{1}{(16\pi^2)^2}\frac{1}{\varepsilon}\frac{1}{48}\left(\tfrac{1}{2}\delta_{\mu\nu}\partial^2+2\partial_\mu\partial_\nu\right)\partial^2\delta^d(x), \tag{A.4j}$$

$$\partial_\mu G_0(x)\,\partial_\nu G_0(x) R_1(x) \sim \frac{\mu^{-2\varepsilon}}{(16\pi^2)^2}\frac{1}{6}\left(\frac{1}{\varepsilon^2}-\frac{1}{12}\frac{1}{\varepsilon}\right)\left(\delta_{\mu\nu}\partial^2+2\partial_\mu\partial_\nu\right)\delta^d(x), \tag{A.4k}$$

$$G_0(x)\,\partial_\mu G_0(x)\,\partial_\nu R_1(x) \sim \frac{1}{(16\pi^2)^2}\frac{1}{\varepsilon}\frac{1}{12}\left(\delta_{\mu\nu}\partial^2+2\partial_\mu\partial_\nu\right)\delta^d(x), \tag{A.4l}$$

$$G_0(x)\,\partial_\mu R_1(x)\,\partial_\nu R_1(x) \sim \frac{1}{(16\pi^2)^2}\frac{1}{\varepsilon}\frac{1}{4}\delta_{\mu\nu}\delta^d(x), \tag{A.4m}$$

$$\partial_\mu G_0(x)\,\partial_\nu R_1(x) R_1(x) \sim \frac{\mu^{-2\varepsilon}}{(16\pi^2)^2}\frac{1}{2}\left(\frac{1}{\varepsilon^2}+\frac{1}{4}\frac{1}{\varepsilon}\right)\delta_{\mu\nu}\delta^d(x), \tag{A.4n}$$

$$\partial_\mu G_0(x)\,\partial_\nu G_0(x) R_2(x) \sim -\frac{\mu^{-2\varepsilon}}{(16\pi^2)^2}\frac{1}{2}\left(\frac{1}{\varepsilon^2}+\frac{1}{4}\frac{1}{\varepsilon}\right)\delta_{\mu\nu}\delta^d(x), \tag{A.4o}$$

$$G_0(x)\,\partial_\mu G_0(x)\,\partial_\nu R_2(x) \sim \frac{\mu^{-2\varepsilon}}{(16\pi^2)^2}\frac{1}{2}\left(\frac{1}{\varepsilon^2}+\frac{1}{4}\frac{1}{\varepsilon}\right)\delta_{\mu\nu}\delta^d(x), \tag{A.4p}$$

$$\partial_\mu\partial_\nu G_0(x) G_0(x) \sim \frac{1}{16\pi^2}\frac{1}{\varepsilon}\frac{2}{3}\left(\partial_\mu\partial_\nu-\tfrac{1}{4}\delta_{\mu\nu}\partial^2\right)\delta^d(x), \tag{A.4q}$$

$$\partial_\mu\partial_\nu G_0(x) R_1(x) \sim -\frac{1}{16\pi^2}\frac{1}{\varepsilon}\frac{1}{2}\delta_{\mu\nu}\delta^d(x), \tag{A.4r}$$

$$G_0(x)\,\partial_\mu\partial_\nu R_1(x) \sim -\frac{1}{16\pi^2}\frac{1}{\varepsilon}\frac{1}{2}\delta_{\mu\nu}\delta^d(x), \tag{A.4s}$$

$$\partial_\mu\partial_\nu G_0(x) G_0(x)^2 \sim \frac{1}{(16\pi^2)^2}\frac{1}{\varepsilon}\frac{1}{12}\left(\partial_\mu\partial_\nu-\tfrac{1}{4}\delta_{\mu\nu}\partial^2\right)\partial^2\delta^d(x), \tag{A.4t}$$

$$\partial_\mu \partial_\nu G_0(x) G_0(x) R_1(x) \sim \frac{\mu^{-2\varepsilon}}{(16\pi^2)^2} \frac{1}{9}\left(\frac{6}{\varepsilon^2} + \frac{1}{\varepsilon}\right)\left(\partial_\mu \partial_\nu - \frac{1}{d}\delta_{\mu\nu}\partial^2\right)\delta^d(x), \tag{A.4u}$$

$$\partial_\mu \partial_\nu G_0(x) R_1(x)^2 \sim -\frac{\mu^{-2\varepsilon}}{(16\pi^2)^2}\left(\frac{1}{\varepsilon^2} + \frac{1}{4}\frac{1}{\varepsilon}\right)\delta_{\mu\nu}\delta^d(x), \tag{A.4v}$$

$$G_0(x)^2 \partial_\mu \partial_\nu R_1(x) \sim \frac{1}{(16\pi^2)^2}\frac{1}{\varepsilon}\frac{1}{6}\left(\partial_\mu \partial_\nu - \delta_{\mu\nu}\partial^2\right)\delta^d(x), \tag{A.4w}$$

$$G_0(x)\partial_\mu \partial_\nu R_1(x) R_1(x) \sim -\frac{\mu^{-2\varepsilon}}{(16\pi^2)^2}\frac{1}{2}\left(\frac{1}{\varepsilon^2} + \frac{3}{4}\frac{1}{\varepsilon}\right)\delta_{\mu\nu}\delta^d(x), \tag{A.4x}$$

$$\partial_\mu \partial_\nu G_0(x) G_0(x) R_2(x) \sim 0, \tag{A.4y}$$

$$G_0(x)^2 \partial_\mu \partial_\nu R_2(x) \sim -\frac{\mu^{-2\varepsilon}}{(16\pi^2)^2}\left(\frac{1}{\varepsilon^2} + \frac{1}{4}\frac{1}{\varepsilon}\right)\delta_{\mu\nu}\delta^d(x). \tag{A.4z}$$

Those results with double poles in ε require careful expansion of (A.1) to obtain the correct residue for the single pole. They arise when for $\varepsilon \to 0$ the products of singular functions contain logarithms. In this limit

$$\begin{aligned} G_0(x) &= \frac{1}{4\pi^2|x|^2}, \\ R_1(x) &= -\frac{1}{16\pi^2}\left(\gamma + \ln\pi + \ln\mu^2|x|^2\right), \\ R_2(x) &= \frac{1}{64\pi^2}|x|^2\left(\gamma + \ln\pi - 1 + \ln\mu^2|x|^2\right). \end{aligned} \tag{A.5}$$

The calculation of the singular parts of the two-loop integrands displayed in (3.33) requires a further collection of relations which can be listed separately. These involve the short-distance expansion of various triple products, according to the definition (3.31), of the heat kernel coefficients a_0, a_1, a_2, for the operators $\Delta^1_{\mu\nu}$ and $-\mathcal{D}^2$, and also with one or two covariant derivatives acting on them. Using, from (3.31)

$$\begin{aligned} &(X,1,1) = C_2\mathrm{tr}(X), \qquad (X,T_a,1) = -\tfrac{1}{2}C_2\mathrm{tr}(T_a X), \\ &(X,P,P) = C_2\mathrm{tr}(P^{-1}X), \end{aligned} \tag{A.6}$$

and (3.35) together with other consequences of the recurrence relations (1.10) the relevant relations used to derive (3.36) and (3.39) are, with $s = x - y \to 0$,

$$
\begin{aligned}
&\left(P, P, P\right) = C_2 N, \\
&\left(\mathcal{D}_\mu P, P, P\right) = 0(s^4), \\
&\left(\mathcal{D}_\mu P, P\overleftarrow{\mathcal{D}}_\nu, P\right) \sim \tfrac{1}{8} s_\beta s_\alpha C_2^2 \left(F_{\alpha\mu} F_{\beta\nu}\right), \\
&\left(\mathcal{D}_\mu P\overleftarrow{\mathcal{D}}_\nu, P, P\right) \sim -\tfrac{1}{4} s_\beta s_\alpha C_2^2 \left(F_{\alpha\mu} F_{\beta\nu}\right), \\
&\left(a_1^{-\mathcal{D}^2}, P, P\right) \sim -\tfrac{1}{12} s_\beta s_\alpha C_2^2 \left(F_{\alpha\gamma} F_{\beta\gamma}\right), \\
&\left(a_{1\mu\nu}^{\Delta^1}, P, P\right) \sim -\tfrac{1}{12} s_\beta s_\alpha C_2^2 \left(F_{\alpha\gamma} F_{\beta\gamma}\right)\delta_{\mu\nu}, \\
&\left(a_{1\mu\nu}^{\Delta^1}, \mathcal{D}_\rho P, P\right) \sim \tfrac{1}{2} s_\alpha C_2^2 \left(F_{\mu\nu} F_{\alpha\rho}\right), \\
&\left(\mathcal{D}_\rho a_1^{-\mathcal{D}^2}, P, P\right) \sim -\tfrac{1}{6} s_\alpha C_2^2 \left(F_{\alpha\gamma} F_{\rho\gamma}\right), \\
&\left(\mathcal{D}_\rho a_{1\mu\nu}^{\Delta^1}, P, P\right) \sim -s_\alpha C_2^2 \left\{\tfrac{1}{6}\left(F_{\alpha\gamma} F_{\rho\gamma}\right)\delta_{\mu\nu} + \left(F_{\alpha\rho} F_{\mu\nu}\right)\right\}, \\
&\left(a_{1\mu\nu}^{\Delta^1}, \mathcal{D}_\alpha P\overleftarrow{\mathcal{D}}_\beta, P\right) \sim \tfrac{1}{2} C_2^2 \left(F_{\mu\nu} F_{\alpha\beta}\right), \\
&\left(\mathcal{D}_\rho a_{1\mu\nu}^{\Delta^1}\overleftarrow{\mathcal{D}}_\lambda, P, P\right) \sim C_2^2 \left\{\tfrac{1}{6}\left(F_{\rho\gamma} F_{\lambda\gamma}\right)\delta_{\mu\nu} - \left(F_{\mu\nu} F_{\rho\lambda}\right)\right\}, \\
&\left(a_{1\mu\nu}^{\Delta^1}, a_{1\lambda\rho}^{\Delta^1}, P\right) \sim 2 C_2^2 \left(F_{\mu\nu} F_{\lambda\rho}\right), \\
&\left(a_2^{-\mathcal{D}^2}, P, P\right) \sim -\tfrac{1}{12} C_2^2 \left(F_{\alpha\beta} F_{\alpha\beta}\right), \\
&\left(a_{2\mu\nu}^{\Delta^1}, P, P\right) \sim -\tfrac{1}{12} C_2^2 \left(F_{\alpha\beta} F_{\alpha\beta}\right)\delta_{\mu\nu} + 2 C_2^2 \left(F_{\mu\beta} F_{\nu\beta}\right).
\end{aligned}
\tag{A.7}
$$

Other possible combinations are either zero or cannot contribute to the order required.

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