QUANTIZATION OF RELATIVISTIC SYSTEMS WITH BOSON AND FERMION FIRST- AND SECOND-CLASS CONSTRAINTS

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The general solution for the S matrix of an arbitrary Hamilton system with boson and fermion first- and second-class constraints of general form is obtained Additional diagrams arise securing unitary and gauge invariance of the theory the many-particle interaction of fermion and boson ghosts The generalized Ward identities are obtained

Introduction Quantization and construction of the S matrix for mixed systems with boson and fermion first- and second-class constraints in canonical gauges were carried out in a paper by one of the authors [1] However, in the case of gauge fields it was important to construct a generalized Hamilton formalism to obtain the S matrix also in relativistic gauges This problem of quantization in relativistic gauges of systems with boson first-class constraints (Bose gauge fields, e g gravity) was solved by Vilkovisky and one of the present authors [2] The suppression of non-physical boson degrees of freedom, which become dynamically active in relativistic gauges, was realized by the introduction of ghost fermion degrees of freedom The Hamiltonian, that governs the dynamics in this complete phase space, was uniquely determined from the requirement of unitarity in the subspace of physical states The central role in the theory was played by a principal theorem, which gave the explicit form of this generalized Hamiltonian It was discovered, that if the first-class constraints do not form a Lie algebra (which is just the case in gravity) the unitarizing Hamiltonian generally contains the four interaction of fermion ghosts The application of these results to the quantum theory of gravity was considered in ref [3] The principal theorem, obtained by Fradkin and Vilkovisky [2] has recently been extended by Batalin and Vilkovisky [4] to mixed systems with boson and fermion first-class constraints *The aim of the present paper is to extend the results of refs (1–4) to the most general case of relativistic systems, namely, the quantization in relativistic gauges of mixed systems with boson and fermion first- and second-class constraints of arbitrary rank*

The principal theorem Let us consider a dynamical system described in the phase space of boson and fermion canonical pairs (q^A, π_A) by a Hamiltonian H_0, boson and fermion second-class constraints θ_k, and functions $G_a(q, \pi)$ among which there may be boson and fermion ones and which satisfy the following (involution) relations

$$\{G_a, G_b\}_D = G_c U^c_{ab}, \qquad \{H_0, G_a\}_D = G_c V^c_a \tag{1}$$

The structure coefficients U and V are functions of canonical variables Dirac brackets $\{\ \}_D$ are defined as follows [1]

$$\{F, R\}_D = \{F, R\} - \{F, \theta_k\} Q^{-1}_{ks} \{\theta_s, R\}, \qquad Q_{ks} = \{\theta_k, \theta_s\}, \tag{2}$$

$$\{F, R\} = \frac{\partial^r F}{\partial q^A}\frac{\partial^l R}{\partial \pi_A} - (-1)^{n_R n_F}\frac{\partial^r R}{\partial q^A}\frac{\partial^l F}{\partial \pi_A}, \tag{3}$$

where r, l denote the right and left derivatives and the fermion index n_R is equal to 0 or 1 depending on whether R is a boson or a fermion Let us supplement each function G_a with two ghost degrees of freedom η^a, $\mathcal{P}_a$ of the statistics opposite to that of G_a (thus the fermion index of η, $\mathcal{P}$ is equal to $n_G + 1$ [1]) The required extension of the theorem of Vilkovisky and one of the present authors (E S) [2] to the most general case of relativistic systems is that *the following functional integral over the complete phase space does not depend on the choice of the Fermi function* $\Psi(q^A, \eta^a, \pi_A, \mathcal{P}_a)$

$$Z_\Psi = \int \mathrm{D}q^A \mathrm{D}\pi_A \mathrm{D}\eta^a \mathrm{D}\,\mathcal{P}_a M \exp \mathrm{i}\left[\int \mathrm{d}t(\pi_A \dot{q}^A + \mathcal{P}_a \dot{\eta}^a - H_\Psi)\right], \tag{4}$$

$$H_\Psi = H_1 + \{\Psi, \Omega\}_\mathrm{D}, \quad \Omega = G_a \eta^a + \sum \mathcal{P}_{a_n} \cdots \mathcal{P}_{a_1} \Omega^{a_1 \ldots a_n}, \quad H_1 = H_0 + \sum \mathcal{P}_{a_n} \cdots \mathcal{P}_{a_1} H_1^{a_1 \ldots a_n}, \tag{5, 6, 7}$$

$$M = \delta(\theta_k) \exp[\tfrac{1}{2}\, \mathrm{Tr} \pm \ln Q_{ab}], \tag{4a}$$

where $\mathrm{Tr} \pm \bar{Q}_{ab} = \Sigma_a \epsilon_a \bar{Q}_{aa}$, $Q_{ab} = \{\theta_a, \theta_b\}$ Here and below the Poisson brackets (3) are extended to the complete phase space $(q^A, \eta^a, \pi_A, \mathcal{P}_a)$

We shall define the theory to be of rank s, if it is possible to define the constraints and the structure coefficients in such a form that the Taylor expansions (6), (7) for Ω and H_1 in terms of $\mathcal{P}$ can be made as a polynomial of degree s (i e $\Omega^{a_1 \ldots a_n} \neq 0$ only for $n \leqslant s$) For example the abelian theory is of rank zero, the non-abelian theory with constant Lagrangian structure coefficients (e g Yang–Mills field, ordinary gravity) are of rank one According to our classification in refs [2–4] a relativistic S matrix for theory of rank one with first-class constraints is constructed *Our goal is to find the explicit expression for Ω and H_1 in eqs (6) and (7) for a theory of arbitrary rank with constraints of general class and to prove the gauge invariance of the S matrix in form (4) It seems that the most adequate way to do this is to generalize the method of ref [4] by taking into account also the second-class constraints and the fact that the theory is of arbitrary rank*

The two essential steps for realization of this program are as follows.

Step 1 To find the fermion generator Ω and H_1 in terms of given constraints G_a and structure constants in such a way, that the following equations hold

$$\{\Omega, \Omega\}_\mathrm{D} = 0, \qquad \{H_1, \Omega\}_\mathrm{D} = 0 \tag{8}$$

For the following consideration it is also convenient [4] to rewrite eq (1) in the form

$$\{G_a \eta^a, G_b \eta^b\}_\mathrm{D} = G_c U_b^c \eta^b = G_c U^c, \qquad \{H_0, G_a \eta^a\}_\mathrm{D} = G_c V^c, \tag{1a}$$

where $U_a^c = -(-1)^{n_a} U_{ab}^c \eta^b$, $V^c = V_a^c \eta^a$ From eqs (6), (7) for the theory of rank s we obtain the following equations

[Let us denote by $K_{\mathrm{sym}}^{\alpha_1 \ldots \alpha_n}$ the quantity

$$K_{\mathrm{sym}}^{\alpha_1 \ldots \alpha_n} = \frac{1}{n!} \sum K^{\beta_1 \ldots \beta_n} (-1)^{S\binom{\beta_1 \ldots \beta_n}{\alpha_1 \ldots \alpha_n}}.$$

Here the sum is over all the possible permutations of indices $(\beta_1 \cdots \beta_n)$ belonging to the initial set $\alpha_1 \cdots \alpha_n$ The factor $(-1)^S$ is the product of *elementary sign factors*, appearing during the permutations necessary to bring the given array $(\beta_1 \cdots \beta_n)$ to the initial one $(\alpha_1 \ldots \alpha_n)$ *The elementary sign factor* $(-1)^{(n_{\beta_i}+1)(n_{\beta_j}+1)}$ appears when the permutation of two neighboring $\mathcal{P}$ with indices β_i and β_j is made $\mathcal{P}_{\beta_i} \mathcal{P}_{\beta_j} = \mathcal{P}_{\beta_j} \mathcal{P}_{\beta_i} (-1)^{(n_{\beta_i}+1)(n_{\beta_j}+1)}$, in particular $K_{\mathrm{sym}}^{ab} = \frac{1}{2}(K^{ab} + (-1)^{(n_a+1)(n_b+1)} K^{ba})$]

$$\{\Omega, \Omega\}_\mathrm{D} = G_a(U^a + 2\Omega^a) + \sum_{n=1}^{2s} \mathcal{P}_{a_n} \cdots \mathcal{P}_{a_1} K^{a_1 \ldots a_n}, \tag{9}$$

$$\{H_1, \Omega\}_\mathrm{D} = G_a(V^a - H_1^a) + \sum_{n=1}^{2s} \mathcal{P}_{a_n} \cdots \mathcal{P}_{a_1} K_1^{a_1 \ldots a_n}, \tag{10}$$

where $\Omega^{a_1 \ldots a_n} = H_1^{a_1 \ldots a_n} = 0$, for $n > s$

$$\tfrac{1}{2} K^{a_1 \dots a_n} = ((n+1)\, \Omega^{a_1 \dots a_n a} G_a (-1)^{n_a} - B^{a_1 \dots a_n})_{\mathrm{sym}}, \tag{9a}$$

$$- K_1^{a_1 \dots a_n} = ((n+1)\, H_1^{a_1 \dots a_n a} G_a - B_1^{a_1 \dots a_n})_{\mathrm{sym}}, \tag{10a}$$

$$B^{a_1 \dots a_n} = \sum_{m=1}^{n} (n-m+1)(-1)^{(n-m+\Sigma_{l=m+1}^{n} n_{a_l})} \frac{\partial^{\mathrm{r}} \Omega^{a_1 \dots a_m}}{\partial \eta^a} \Omega^{a_{m+1} \dots a_n} - \{\Omega^{a_1 \dots a_n}, G_a \eta^a\}_{\mathrm{D}}$$
$$- \tfrac{1}{2} \sum_{m=1}^{n-1} \{\Omega^{a_1 \dots a_m}, \Omega^{a_{m+1} \dots a_n}\}_{\mathrm{D}} (-1)^{(n-m+\Sigma_{l=m+1}^{n} a_{n_l})}, \tag{9b}$$

$$B_1^{a_1 \dots a_n} = \{H_1^{a_1 \dots a_n}, G_a \eta^a\}_{\mathrm{D}} + \{H_0, \Omega^{a_1 \dots a_n}\}_{\mathrm{D}} + \sum_{m=1}^{n-1} \{H_1^{a_1 \dots a_m}, \Omega^{a_{m+1} \dots a_n}\}_{\mathrm{D}}$$
$$+ \sum_{m=1}^{n} (n-m+1) \frac{\partial^{\mathrm{r}} H_1^{a_1 \dots a_m}}{\partial \eta^a} \Omega^{a a_{m+1} \dots a_n} - \sum_{m=1}^{n-1} (m+1)\, H_1^{a_1 \dots a_n a} \frac{\partial^{\mathrm{l}} \Omega^{a_{m+1} \dots a_n}}{\partial \eta^a} \tag{10b}$$

According to eq. (8) the right-hand side of eqs. (9), (10) should vanish. We can make everyone of the first s terms in eqs. (9), (10) vanish, this gives the following s equations for determining all the coefficient functions $\Omega^{a_1 \dots a_n}$ and $H_1^{a_1 \dots a_n}$ with $n \leqslant s$:

$$\Omega^a = -\tfrac{1}{2} U^a, \qquad (n+1)\, \Omega_{\mathrm{sym}}^{a_1 \dots a_n a} G_a (-1)^{n_a} = B_{\mathrm{sym}}^{a_1 \dots a_n}, \tag{9c}$$

$$H_1^a = V^a, \qquad (n+1)\, H_{1\,\mathrm{sym}}^{a_1 \dots a_n a} G_a = B_{1\,\mathrm{sym}}^{a_1 \dots a_n} \tag{10c}$$

Each of the remaining terms in eqs. (9), (10) is equal to zero as a consequence of the higher-order Jacobi relations for Dirac brackets for a theory of rank s. In order to obtain those one can use the basic Jacobi relation

$$(-1)^{n_A n_C} \{A, \{B, C\}_{\mathrm{D}}\}_{\mathrm{D}} + (-1)^{n_C n_B} \{C, \{A, B\}_{\mathrm{D}}\}_{\mathrm{D}} + (-1)^{n_A n_B} \{B, \{C, A\}_{\mathrm{D}}\} = 0 \tag{11}$$

From eq. (11) we have ($n_\Omega = 1$, $n_{H_1} = 0$)

$$\{\Omega, \{\Omega, \Omega\}_{\mathrm{D}}\}_{\mathrm{D}} = 0, \qquad 2\{\{H_1, \Omega\}_{\mathrm{D}}, \Omega\}_{\mathrm{D}} + \{\{\Omega, \Omega\}_{\mathrm{D}}, H_1\}_{\mathrm{D}} = 0 \tag{11a}$$

For a theory of rank s eqs. (9)–(11a) give the higher order [‡1] Jacobian relations $B^{a_1 \dots a_n} = B_1^{a_1 \dots a_n} = 0$, for $n \geqslant s$.

Thus we have completed the contruction of the generator Ω and H_1, with the property (8). In particular

(1) for a theory of rank zero $\Omega = G\eta$, $H_1 = H_0$,

(2) for a theory of rank one, $\Omega = G_a \eta^a - \frac{1}{2} \mathcal{P}_a U^a$, $H_1 = H_0 + \mathcal{P}_a V_a$,

(3) for a theory of rank two

$$\Omega = G_a \eta^a - \tfrac{1}{2} \mathcal{P}_a U^a + \mathcal{P}_{a_2} \mathcal{P}_{a_1} \Omega^{a_1 a}, \qquad H_1 = H_0 + \mathcal{P}_a V^a + \mathcal{P}_{a_2} \mathcal{P}_{a_1} H_1^{a_1 a_2}, \tag{12}$$

where

$$4\Omega^{ba} G_a (-1)^{n_a} = \{U^b, G_a \eta^a\}_{\mathrm{D}} - U_a^b U^a, \quad 2H_1^{ba} G_a = \{V^b, G_a \eta^a\}_{\mathrm{D}} - \tfrac{1}{2} \{H_0, U^b\} - \tfrac{1}{2} V_a^b U^a + U_a^b V^a \tag{12a}$$

Step 2. The proof of the gauge invariance of the S matrix (4), as in ref. [4], is connected with the supersymmetry of the theory. A global abelian subalgebra first discovered in Yang–Mills theory by Becchi et al. [5] with generator Ω is realized in the theory

[‡1] At first we obtain $G_a B^{a \dots a_n}$, but if the theory is of rank s we have $B^{a_1 \dots a_n} = 0$ for all $n > s$.

Indeed, let us consider a transformation in the complete phase space $\varphi \equiv (q^A, \eta^a, \pi_A, \mathcal{P}_a)$

$$\varphi \to \tilde{\varphi} = \varphi + \{\varphi, \Omega\}_D \mu , \tag{13}$$

where μ is a fermion parameter of the type [4]

$$\mu = \int dt (\Psi - \Psi') . \tag{14}$$

From eqs (11) and (8) we have for H_Ψ ($n_\Psi = 1$)

$$\{H_\Psi, \Omega\}_D = \{\{\Psi, \Omega\}_D, \Omega\}_D + \{H_1, \Omega\}_D = -\tfrac{1}{2}\{\{\Omega, \Omega\}_D, \Psi\}_D = 0 \tag{15}$$

Therefore the Hamiltonian (5) is superinvariant and the displacement (13) leaves invariant not only the Hamiltonian (see eq. (15)) but also the action in the exponential (4) However the displacement (13) and the measure M (see eq (4a)) yields a Jacobian

$$M(\varphi)\, D\varphi = M(\tilde{\varphi})\, D\tilde{\varphi}(1 + \{\mu(\tilde{\varphi}), \Omega\}_D) \tag{16}$$

Substituting eqs (15), (16) into eq (4) we obtain for small $\Psi - \Psi'$ that $Z_\Psi = Z_{\Psi'}$, and the independence of Ψ is proved

Ward identity. Let us consider the transformation (13) of the generating functional $Z_\Psi(I)$ The generating functional $Z_\Psi(I)$ is defined by eq (4) with $\tilde{H}_\Psi = H_\Psi - I_A \varphi^A$, where I_A is the source of the field φ^A Suppose that the parameter $\mu(\varphi)$ of the transformation (13) is a constant, then from eqs (13) and (16) we obtain

$$I_A \langle \{\varphi^A, \Omega\}_D \rangle = 0 , \tag{17}$$

where $\langle B(\varphi)\rangle \equiv B\,[(\delta/i\delta I) Z_\Psi(I)]/Z_\Psi(I)$ By taking the I_A-derivative of eq (17) *we obtain the generalized Ward identities*

$$\langle \{\varphi^A, \Omega\}_D \rangle + i I_C \langle \{\varphi^C, \Omega\}_D \varphi^A \rangle (-1)^{n_{\varphi^A}} \tag{17a}$$

In particular for $\Psi = \mathcal{P}_A \chi^A$ with the help of eq (17a) we can represent the gauge variation of $Z(I)$ in the form

$$\delta_{\Delta\chi} \ln Z(I) = -i \langle \{\mathcal{P}_A \Delta\chi^A, \Omega\}_D \rangle = I_C \langle \{\varphi^C, \Omega\}_D\, \mathcal{P}_A \Delta\chi^A \rangle . \tag{17b}$$

The form (17b) also demonstrates the gauge invariance of the theory in physical subspace

Application to relativistic systems Consideration of this problem is close to ref [3] The action functional of the dynamical system with first-class ($T_\alpha(q^i, p_i)$) and second-class ($\theta_\beta(q,p)$) constraints has the form

$$S = \int dt\, [p_i \dot{q}^i - H_0(p, q) - T_\alpha \lambda^\alpha]|_{\theta_\beta = 0} . \tag{18}$$

λ^α are Lagrange multipliers, T_α is the minimal set of boson and fermion first-class constraints in involution

$$\{T_\alpha, T_\beta\}_D = T_\gamma U^\gamma_{\alpha\beta}(q, p), \qquad \{H_0, T_\alpha\}_D = T_\beta V^\beta_\alpha(q, p) \tag{19}$$

Each first-class constraint requires an introduction of two degrees of freedom of opposite statistics (η^α, $\mathcal{P}_\alpha$) and a relativistic gauge condition

$$\Phi^\alpha = -\dot{\lambda}^\alpha + \chi^\alpha(q^i, p_i, \lambda^a, \pi_a, \eta^a, \mathcal{P}_a) , \tag{20}$$

with an additional Lagrange multiplier π_2 The expression for a gauge-independent S matrix in the general class of gauge conditions (20) is given by eq (4) with the following identification of the quantities q^A, π_A, G_a introduced there

$$q^A = \begin{pmatrix} q^i \\ \lambda^\alpha \end{pmatrix}, \quad \pi_A = (p_i, \pi_\alpha), \quad G_a = (\pi_\alpha, T_\alpha), \quad \eta^a = \begin{pmatrix} \mathcal{P}^\alpha \\ C^\alpha \end{pmatrix}, \quad \bar{\mathcal{P}}_a = (\bar{C}_\alpha, \bar{\mathcal{P}}_\alpha), \quad \Psi = \bar{\mathcal{P}}_a \chi^a, \quad \chi^a = \begin{pmatrix} \chi^\alpha \\ \lambda^\alpha \end{pmatrix} \tag{21}$$

After some necessary calculations we obtain from eqs (4), (5), (6), (9), (10), (20), (21) the following expression for the generating functional when the gauge (20) does not depend on $\mathcal{P}^\alpha$ and $\bar{\mathcal{P}}_\alpha$ (here the Dirac brackets are only on q^i, p_i)

$$Z(I) = \int \mathrm{D}q^i \mathrm{D}p_i \mathrm{D}\lambda^\alpha \mathrm{D}\pi_\alpha \mathrm{D}C^\beta \mathrm{D}\bar{C}_\beta M \exp i \left[\int \mathrm{d}t (p_i \dot{q}^i - H_{\text{complete}} + I_i q^i)\right], \tag{22}$$

$$H_{\text{complete}} = H_0 + T_\alpha \lambda^\alpha + \pi_\alpha \frac{\partial^{\mathrm{r}}\Phi}{\partial \bar{C}_\alpha} - \frac{1}{2}\frac{\partial^{\mathrm{r}}\Phi}{\partial C^\alpha} U^\alpha + \{\Phi, T\}_{\mathrm{D}} - \bar{\mathcal{P}}_\alpha(\dot{C}^\alpha + U^\alpha_{\beta\gamma}\lambda^\gamma C^\beta - V^\alpha_\beta C^\beta + \tfrac{1}{2}\{U^\alpha, \Phi\}_{\mathrm{D}})$$

$$+ \sum_{n=2}^{s} \bar{\mathcal{P}}_{\alpha_n} \dots \bar{\mathcal{P}}_{\alpha_1} H^{\alpha_1 \dots \alpha_n}, \tag{23}$$

$$H^{\alpha_1 \dots \alpha_n} = \left(H_1^{\alpha_1 \dots \alpha_n} + \frac{\partial^{\mathrm{r}}\Omega^{\alpha_1 \dots \alpha_n}}{\partial C_\alpha}\lambda^\alpha\right) + \{\Omega^{\alpha_1 \dots \alpha_n}, \Phi\}_{\mathrm{D}} + (n+1)\Omega^{\alpha_1 \dots \alpha_n \alpha}(-1)^{n_\alpha}\frac{\partial^{\mathrm{r}}\Phi}{\partial C_\alpha}, \tag{24}$$

where $\Phi = \bar{C}_\alpha \Phi^\alpha$, $T = T_\alpha C^\alpha$, $\bar{\mathcal{P}}_\alpha(t) = -\int \mathrm{d}t' [\delta^{\mathrm{r}}\Phi(t')/\delta\lambda^\alpha(t)]$ and

$$H_1^{a_1 \dots a_n} + \frac{\partial^{\mathrm{r}}\Omega^{a_1 \dots a_n}}{\partial C_\alpha}\lambda^\alpha = \frac{\partial^{\mathrm{r}}\Omega^{a_1 \dots a_n}}{\partial C_\alpha}(\lambda^\alpha + \delta_{\alpha 0}),$$

T_0 is the generator of time transformation and $(\lambda_0 + 1)$ the zero–zero component of gravity, $\Omega^{a_1 \dots a_n}$ is determined by eq (9c) where $\eta_\alpha \to C_\alpha$ In particular (1) for a theory of rank one all $H^{a_1 \dots a_n} = 0$, (2) for a theory of rank two (see also [6]) only $H^{a_1 a_2} \neq 0$, and

$$H^{a_1 a_2} = \frac{\partial^{\mathrm{r}}\Omega^{a_1 a_2}}{\partial C_\alpha}(\lambda^\alpha + \delta_{\alpha 0}) + \{\Omega^{a_1 a_2}, \Phi\}_{\mathrm{D}},$$

where $\Omega^{a_1 a_2}$ is determined by eq (12a)

Unitarity of the S matrix (22) is a consequence of the principal theorem and the coincidence of eq (22) in canonical gauges with the correct unitary S matrix of ref [1]

Conclusion We obtained the general solution for the S matrix of relativistic systems of arbitrary rank with boson and fermion first- and second-class constraints A remarkable feature of the unitarizing Hamiltonian in eq (22) is the presence of many-particle interaction of ghost fields, which depend on the rank of the theory It was natural though that in real physical theories the structure of the constraints is such that the rank of the theory may not exceed unity However, in ref [6] Vasiliev and one of the authors first established that supergravity is a theory of rank two Therefore the theory of higher rank has a physical interest and this justifies the present version of our paper A remarkable feature of the theories of higher rank (this was demonstrated for supergravity in ref [6]) is the fact, that the four-particle (and possible many-particle) interaction of ghost fields survives also in the Lagrangian expression of the S matrix and therefore the usual "covariant methods" (e g. the Faddeev–Popov method) lead here to an incorrect S matrix

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