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De Sitter space as an arena for doubly special relativity

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Abstract

We show that Doubly Special Relativity (DSR) can be viewed as a theory with energy–momentum space being the four-dimensional de Sitter space. Different formulations (bases) of the DSR theory considered so far can be therefore understood as different coordinate systems on this space. The emerging geometrical picture makes it possible to understand the universality of the non-commutative structure of space–time of doubly special relativity. Moreover, it suggests how to construct the most natural DSR basis, which turns out to be the bicrossproduct basis.
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1. The DSR theory

Doubly special relativity theory is a new attempt to approach the problem of quantum gravity. This theory was proposed about a year ago by Amelino-Camelia [1] and is based on two fundamental assumptions: the principle of relativity and the postulate of existence of two observer-independent scales, of speed identified with the speed of light c,[2] and of mass κ (or length $\ell = 1/\kappa$) identified with the Planck mass. There are several theoretical indications that such a theory may replace Special Relativity as a theory of relativistic kinematics of probes whose energies are close to the Planck scale. First of all both loop quantum gravity and string theory indicate appearance of the minimal length scale. It is therefore not impossible that this scale would be present in description of ultra high energy kinematics even in the regime, in which gravitational effects are negligible. Secondly, in both inflationary cosmology [2] and in black hole physics [3] one faces the conceptual "trans-Planckian puzzle" of ordinary physical quanta being blue shifted up to the Planck energies, which as advocated by many can be solved by assuming deviation from the standard dispersion relation at high energies, and thus deviation from the standard relativistic kinematics. It should be also stressed that some DSR models might provide a resolution of observed anomalies in astrophysical data [4]. Moreover, predictions of the DSR scenario might be testable in forthcoming quantum gravity experiments [5].

Soon after appearance of the papers [1] it was realized [6,7] that the so-called κ-Poincaré algebra in

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[2] In what follows we set $c = 1$.

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the bicrossproduct basis [8] provides an example of the energy–momentum sector of DSR theory.[3] This algebra consists of undeformed Lorentz generators

$$[M_i, M_j] = i\epsilon_{ijk} M_k, \qquad [M_i, N_j] = i\epsilon_{ijk} N_k,$$
$$[N_i, N_j] = -i\epsilon_{ijk} M_k, \tag{1}$$

the standard action of rotations on momenta

$$[M_i, p_j] = i\epsilon_{ijk} p_k, \qquad [M_i, p_0] = 0, \tag{2}$$

along with the deformed action of boosts on momenta

$$[N_i, p_j] = i\,\delta_{ij}\left(\frac{\kappa}{2}\left(1 - e^{-2p_0/\kappa}\right) + \frac{1}{2\kappa}\vec{p}^{\,2}\right) - i\,\frac{1}{\kappa} p_i p_j \tag{3}$$

governed by the observer-independent mass scale κ.

The algebra (1)–(3) is, of course, not unique. The presence of the second observer-independent scale κ makes it possible to consider transformations to another DSR basis, in which (1) holds, and thus the Lorentz subalgebra is left unchanged, but one introduces new momentum variables

$$p_0' = f\left(p_0, \vec{p}^{\,2}; \kappa\right), \qquad p_i' = g\left(p_0, \vec{p}^{\,2}; \kappa\right) p_i. \tag{4}$$

By construction p_0' and p_i' transform under rotations as scalar and vector, respectively. The functions f and g are assumed to be analytical in the variables p_0 and $\vec{p}^{\,2}$, and in order to guarantee the correct low energy behavior one assumes that for $\kappa \to \infty$

$$f\left(p_0, \vec{p}^{\,2}\right) \approx p_0 + O(1/\kappa),$$
$$g\left(p_0, \vec{p}^{\,2}\right) \approx 1 + O(1/\kappa). \tag{5}$$

It can be shown [12] that also vice versa, any deformed Poincaré algebra with undeformed Lorentz sector and standard action of rotations, which has the standard Poincaré algebra as its $\kappa \to \infty$ limit can be related to the algebra (1)–(3) by transformation of the form (4). One should note in passing that this means, in particular, that any modified dispersion relation considered in the context "trans Planckian problem" can be extended to a DSR theory, and thus does not need to lead to breaking of Lorentz symmetry.

The algebra (1)–(3) does not furnish the whole physical picture of the DSR theory. To describe physical processes we need also a space–time sector of this theory. The question arises as to if it is possible to construct this sector from the energy–momentum sector. The answer turns out to be affirmative if one extends the energy–momentum DSR algebra to the quantum (Hopf) algebra. It was shown in [12] that such an extension is possible in the case of any DSR algebra, in particular, for the algebra (1)–(3) one gets the following expressions for the co-product

$$\Delta(p_i) = p_i \otimes \mathbb{1} + e^{-p_0/\kappa} \otimes p_i, \tag{6}$$
$$\Delta(p_0) = p_0 \otimes \mathbb{1} + \mathbb{1} \otimes p_0, \tag{7}$$
$$\Delta(N_i) = N_i \otimes \mathbb{1} + e^{-p_0/\kappa} \otimes N_i + \frac{1}{\kappa}\epsilon_{ijk} p_j \otimes M_k \tag{8}$$

(the co-product for rotations is trivial). Then one makes use of the so-called "Heisenberg double" prescription[4] [13] in order to get the following commutators

$$[p_0, x_0] = i, \qquad [p_i, x_j] = -i\delta_{ij},$$
$$[p_i, x_0] = -\frac{i}{\kappa} p_i. \tag{9}$$

By using the same method one finds also that the space–time of DSR theory is non-commuting

$$[x_0, x_i] = -\frac{i}{\kappa} x_i \tag{10}$$

and that position operators transform under boosts in the following way [12,13]

$$[N_i, x_j] = i\delta_{ij} x_0 - \frac{i}{\kappa}\epsilon_{ijk} M_k,$$
$$[N_i, x_0] = i x_i - \frac{i}{\kappa} N_i \tag{11}$$

(x_0 and x_i transform as scalar and vector under rotations).

It is important to note that as proved in [12] if the Heisenberg double method is used to derive the space–time sector of the DSR theory, both the space–time non-commutativity (10) and the form of the boost action on position operators (11) is universal for all the DSR theories, i.e., it is invariant of the

[3] More recently another form of the DSR theory was presented by [9]. Relations between different forms of DSR were discussed in [10,11].

[4] It should be stressed that the Heisenberg double method is not a unique way of deriving the space–time structure of the DSR theory (though appealing by its mathematical naturalness).

energy–momentum transformations (4), (5). As we will see this observation finds its natural explanation in the complementary geometrical picture of DSR, to be developed below, and this is, of course, a strong argument in favor of the Heisenberg double method.

2. DSR algebra and de Sitter space

Since the space–time algebra of Lorentz generators and positions given by (1), (10) and (11) is universal, it is worth to investigate it a bit closer. The first thing to note is that this algebra is the $SO(4,1)$ Lie algebra with Lorentz generators belonging to its $SO(3,1)$ Lie subalgebra (recall that in special relativity we have to do with the semidirect sum of $SO(3,1)$ and R^4, instead). Let us recall now that both Lorentz generators and positions can be interpreted as symmetry generators, acting on the space of momenta as "rotations" and "translations", respectively. But then it follows that the space of momenta can be identified with (a subspace of) the group quotient space $so(4,1)/so(3,1)$ which is nothing but the de Sitter space.

To see this explicitly, let us note that among infinitely many DSR bases, related to each other by transformation (4), (5) one finds the basis, in which the action of Lorentz algebra on energy–momentum sector is classical, i.e.,

$$[N_i, P_j] = i\delta_{ij} P_0, \qquad [N_i, P_0] = i P_i, \tag{12}$$

while for positions we have the universal algebra (11). Moreover, one finds that in this basis

$$\begin{aligned}
[P_i, X_j] &= -i\delta_{ij}\left(1 + \frac{1}{\kappa}P_0\right) - \frac{i}{\kappa^2}P_i P_j, \\
[X_i, P_0] &= \frac{i}{\kappa}P_i + \frac{i}{\kappa^2}P_i P_0, \\
[X_i, P_0] &= \frac{i}{\kappa}P_i + \frac{i}{\kappa^2}P_i P_0, \\
[P_0, X_0] &= i\left(1 - \frac{1}{\kappa^2}P_0^2\right),
\end{aligned} \tag{13}$$

and again the commutator of space and time is given by (10).[5]

[5] The energy–momentum sector of this basis is identical with the energy–momentum sector of Snyder's theory of non-commutative space–time [14]. The relation between this basis and Snyder's theory was discussed in [12]. Note that I take liberty to call this basis a DSR theory even though the commutators (12) are undeformed. The reason is that the deformation is still present in the action of boosts on space–time variables (cf. (11)).

As it stands, the algebra (10)–(13) looks like a particular DSR basis. The important observation is that, in agreement with general argument given at the beginning of this section, the momenta P_0 and P_i can be viewed as coordinates on de Sitter space. Indeed, let de Sitter space be defined by equation

$$-\eta_0^2 + \eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2 = \kappa^2, \tag{14}$$

and let us define the coordinates

$$P_\mu = \kappa\frac{\eta_\mu}{\eta_4}, \quad \mu = (0, \dots, 3). \tag{15}$$

It is clear that the coordinates P_μ cover only half of the whole de Sitter space (the points (η_μ, η_4) and $(-\eta_\mu, -\eta_4)$ are identified in these coordinates). If one now derives the form of generators of $SO(4,1)$ symmetry of de Sitter space in these coordinates, such that M_i, N_i belong to its $SO(3,1)$ subalgebra, while X_μ are the remaining four generators belonging to the quotient of two algebras $SO(4,1)/SO(3,1)$, one finds that they satisfy the $SO(4,1)$ relations (1), (10), (11) as well as the cross relations (13).

This simple observation clarifies the universal status of the algebra satisfied by positions and boost and rotation generators in any DSR basis (10), (11). To understand this let us look at the DSR theory from geometric perspective suggested by the construction above. From this viewpoint the space of momenta is not a flat space, as in Special Relativity, but a curved, maximally symmetric space of constant curvature. The fact that we need a maximally symmetric space is related, of course to the fact that only such space has the required number of symmetry generators, namely, six "rotations" identified with Lorentz transformations and four "translations" in the energy–momentum space, which can be identified with (non-commutative) positions. It is well known that there are only three families of maximally symmetric spaces: de Sitter, anti-de Sitter and the flat space of constant positive, negative, and zero curvature, respectively. Next, it is clear that even though on the (momentum) de Sitter space one can introduce arbitrary coordinates (each corresponding to a particular DSR basis), the form of symmetries of this space does not, of course depend on the form of the coordinate system (recall that de Sitter space is the symmetric space $so(4,1)/so(3,1)$). In other words the momentum sectors of various DSR theories are in one-to-one correspondence with differential structures that can be built on de Sitter space, while the structure of the positions/boosts/rotations, being related to the symmetries of this space is, clearly, diffeomorphism-invariant. Let us note again that the fact that Heisenberg double construction leads to algebraic structure consistent with the geometric picture of the DSR theory strongly indicates that it might be the right way of construction of the space–time sector of this theory.

At this point a question arises, namely if the coordinates (15) are the most natural ones from the geometric perspective. Indeed in these coordinates the physical meaning of positions as generators of translations in energy–momentum space is far from being manifest. It is therefore useful to try to construct a coordinate system in which the physical role played by positions exhibits itself in a more clear way. This can be done as follows. Consider the point $\mathcal{O}$ in de Sitter momentum space with coordinates $(\eta_\mu, \eta_4) = (0, \kappa)$. This point corresponds to the zero momentum in the coordinate system (15) and we assume that it corresponds to zero momentum state in any coordinates as well. Geometrically this assumption corresponds to defining the preferred point in de Sitter space, but, of course it is well motivated physically. Since de Sitter space equals $so(4,1)/so(3,1)$, the stability group of this point is just $so(3,1)$ of M_i and N_i, and the remaining four generators of $so(4,1)$, X_μ can be used to define points on de Sitter space as follows. One observes that the group elements $\exp(i\mathcal{P}_0X_0)$, $\exp(i\mathcal{P}_iX_i)$ (for fixed i) have natural interpretation of "large translations" in the momentum de Sitter space (since this space is curved the translations cannot commute, of course). Indeed

$$\exp\bigl(i\mathcal{P}_0^{(1)}X_0\bigr)\exp\bigl(i\mathcal{P}_0^{(2)}X_0\bigr) = \exp\bigl(i\bigl(\mathcal{P}_0^{(1)}+\mathcal{P}_0^{(2)}\bigr)X_0\bigr),$$
$$\exp\bigl(i\mathcal{P}_i^{(1)}X_i\bigr)\exp\bigl(i\mathcal{P}_i^{(2)}X_i\bigr) = \exp\bigl(i\bigl(\mathcal{P}_i^{(1)}+\mathcal{P}_i^{(2)}\bigr)X_i\bigr)$$

while

$$\exp(i\mathcal{P}_0X_0)\exp(i\mathcal{P}_iX_i)\exp(-i\mathcal{P}_0X_0) = \exp\bigl(ie^{-\mathcal{P}_0/\kappa}\mathcal{P}_iX_i\bigr). \tag{16}$$

Now one can define the natural coordinates on the momentum de Sitter space by labelling the point

$$\wp_\pm \equiv \mathcal{G}_\pm(\mathcal{P}_0,\mathcal{P}_i)\mathcal{O} = \exp(\pm i\mathcal{P}_iX_i)\exp(\pm i\mathcal{P}_0X_0)\mathcal{O} \tag{17}$$

with coordinates (momenta) $\mathcal{P}_\mu$.

With the help of explicit form of X_μ generators in matrix representation

$$X_0 = \frac{i}{\kappa}\begin{pmatrix} 0 & \mathbf{0} & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 1 & \mathbf{0} & 0 \end{pmatrix},$$
$$\vec{X} = \frac{i}{\kappa}\begin{pmatrix} 0 & \vec{\epsilon}^{\,T} & 0 \\ \vec{\epsilon} & \mathbf{0} & \vec{\epsilon} \\ 0 & -\vec{\epsilon}^{\,T} & 0 \end{pmatrix}, \tag{18}$$

where $\vec{\epsilon}$ is a three-vector with one non-vanishing unit component and $\vec{\epsilon}^{\,T}$ the associated transposed vector, one finds

$$\exp(\pm i\mathcal{P}_0X_0) = \begin{pmatrix} \cosh\frac{\mathcal{P}_0}{\kappa} & \mathbf{0} & \mp\sinh\frac{\mathcal{P}_0}{\kappa} \\ \mathbf{0} & \mathbb{1} & \mathbf{0} \\ \mp\sinh\frac{\mathcal{P}_0}{\kappa} & \mathbf{0} & \cosh\frac{\mathcal{P}_0}{\kappa} \end{pmatrix},$$
$$\exp(\pm i\mathcal{P}_iX_i) = \begin{pmatrix} 1+\frac{\vec{\mathcal{P}}^2}{2\kappa^2} & \mp\frac{\vec{\mathcal{P}}^T}{\kappa} & \frac{\vec{\mathcal{P}}^2}{2\kappa^2} \\ \mp\frac{\vec{\mathcal{P}}}{\kappa} & \mathbb{1} & \mp\frac{\vec{\mathcal{P}}}{\kappa} \\ -\frac{\vec{\mathcal{P}}^2}{2\kappa^2} & \pm\frac{\vec{\mathcal{P}}^T}{\kappa} & 1-\frac{\vec{\mathcal{P}}^2}{2\kappa^2} \end{pmatrix}, \tag{19}$$

and thus the coordinates $(\mathcal{P}_\mu)$ label the point $\wp_\pm = (\eta_0,\ldots,\eta_4)$ with

$$\eta_0 = -\kappa\sinh\frac{\mathcal{P}_0}{\kappa} \pm \frac{\vec{\mathcal{P}}^2}{2\kappa}e^{\mp\frac{\mathcal{P}_0}{\kappa}}, \qquad \eta_i = -\mathcal{P}_ie^{\mp\frac{\mathcal{P}_0}{\kappa}},$$
$$\eta_4 = \pm\kappa\cosh\frac{\mathcal{P}_0}{\kappa} \mp \frac{\vec{\mathcal{P}}^2}{2\kappa}e^{\mp\frac{\mathcal{P}_0}{\kappa}}. \tag{20}$$

Let us now observe that position and boost operators X_μ and N_i form the basis of the $SO(4,1)$ algebra, and so their commutators with η_A, $A = 0,\ldots,4$, are given by

$$[X_0,\eta_4] = \frac{i}{\kappa}\eta_0, \qquad [X_0,\eta_0] = \frac{i}{\kappa}\eta_4,$$
$$[X_0,\eta_i] = 0, \qquad [X_i,\eta_4] = [X_i,\eta_0] = \frac{i}{\kappa}\eta_i,$$
$$[X_i,\eta_j] = \frac{i}{\kappa}\delta_{ij}(\eta_0-\eta_4), \qquad [N_i,\eta_0] = i\eta_i,$$
$$[N_i,\eta_j] = i\delta_{ij}\eta_0, \qquad [N_i,\eta_4] = 0.$$

Using now Leibnitz identity one can read off from these equations the form of non-vanishing cross commutators

$$[\mathcal{P}_0, X_0] = i, \qquad [\mathcal{P}_i, X_0] = \pm\frac{i}{\kappa}\mathcal{P}_i,$$
$$[\mathcal{P}_i, X_j] = -i\delta_{ij}e^{\pm 2\mathcal{P}_0/\kappa} + \frac{i}{\kappa^2}\big(\vec{\mathcal{P}}^2\delta_{ij} - 2\mathcal{P}_i\mathcal{P}_j\big),$$
$$[\mathcal{P}_0, X_i] = \mp\frac{2i}{\kappa}\mathcal{P}_i. \tag{21}$$

Remarkably, the action of boosts on momenta has the form

$$[N_i, \mathcal{P}_j] = i\delta_{ij}\left(\pm\frac{\kappa}{2}\big(e^{\pm 2\mathcal{P}_0/\kappa} - 1\big) \mp \frac{1}{2\kappa}\vec{\mathcal{P}}^2\right) \pm i\frac{1}{\kappa}\mathcal{P}_i\mathcal{P}_j,$$
$$[N_i, \mathcal{P}_0] = i\mathcal{P}_i, \tag{22}$$

which is nothing but the boost action in the bicrossproduct basis (in the "$-$" case; in the "$+$" case one has to do with the bicrossproduct basis with κ replaced by $-\kappa$). Let us observe now that since η_4 commutes with boosts (and, of course, rotations as well) it must be related to the quadratic Casimir of the algebra (22). Indeed we find

$$\eta_4 = \pm\frac{1}{2\kappa}\left(\left(2\kappa\sinh\frac{\mathcal{P}_0}{2\kappa}\right)^2 - \vec{\mathcal{P}}^2 e^{\mp\mathcal{P}_0/\kappa}\right) \pm \kappa$$
$$= \pm\frac{1}{2\kappa}\big(\mathcal{C}_\pm + 2\kappa^2\big), \tag{23}$$

where $\mathcal{C}_\pm$ is the quadratic Casimir of DSR in bicrossproduct basis.

Of course, one could consider another bases, defined by the prescription

$$\tilde{\wp}_\pm \equiv \widetilde{\mathcal{G}}_\pm(\mathcal{P}_0, \mathcal{P}_i)\mathcal{O}$$
$$= \exp(\pm i\mathcal{P}_0 X_0)\exp(\pm i\mathcal{P}_i X_i)\mathcal{O}$$

which can be easily found with the help of Eq. (16) and would lead to another DSR theory. Specifically one gets (in the "$+$" case; the "$-$" one can be obtained by changing the sign of κ)

$$\eta_0 = -\kappa\sinh\frac{\mathcal{P}_0}{\kappa} + \frac{\vec{\mathcal{P}}^2}{2\kappa}e^{\mathcal{P}_0/\kappa}, \qquad \eta_i = -\mathcal{P}_i,$$
$$\eta_4 = \kappa\cosh\frac{\mathcal{P}_0}{\kappa} - \frac{\vec{\mathcal{P}}^2}{2\kappa}e^{\mathcal{P}_0/\kappa} \tag{24}$$

which leads to

$$[\mathcal{P}_0, X_0] = i, \qquad [\mathcal{P}_i, X_j] = -i\delta_{ij}\left(1 - \frac{\vec{\mathcal{P}}^2}{\kappa^2}\right)e^{\mathcal{P}_0/\kappa},$$
$$[\mathcal{P}_0, X_i] = -\frac{2i}{\kappa}\mathcal{P}_i e^{\mathcal{P}_0/\kappa}, \tag{25}$$

$$[N_i, \mathcal{P}_j] = i\delta_{ij}\left(\kappa\sinh\frac{\mathcal{P}_0}{\kappa} - \frac{\vec{\mathcal{P}}^2}{2\kappa}e^{\mathcal{P}_0/\kappa}\right),$$
$$[N_i, \mathcal{P}_0] = i\mathcal{P}_i e^{\mathcal{P}_0/\kappa}. \tag{26}$$

The quadratic Casimir of the algebra (24) is again related to η_4 and has the form

$$\mathcal{C} = -\left(2\kappa\sinh\frac{\mathcal{P}_0}{2\kappa}\right)^2 + \vec{\mathcal{P}}^2 e^{\mathcal{P}_0/\kappa}. \tag{27}$$

3. Conclusions

The main result of this Letter is that any DSR theory can be regarded as a particular coordinate system on de Sitter space of momenta. In addition, the Lorentz transformations have interpretations of stabilizers of the zero-momentum point in the de Sitter space, while positions are identified with the remaining four generators of $SO(4,1)$. Moreover, one can single out the basis, which is most natural from the geometric point of view, and this basis turns out to be the bicrossproduct one.

The observation that different DSR bases are related by diffeomorphisms in the momentum de Sitter space leads naturally to the question if this relation between different DSR theories is of any physical relevance, i.e., if the "general coordinate invariance in momentum space" can be somehow given a status of physical symmetry. To answer this question one should check if it is possible to construct a (de Sitter) momentum space theory in a manifestly coordinate independent way (which would, of course include the metric tensor on this space). This problem is currently under investigation.

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References

[1] G. Amelino-Camelia, Int. J. Mod. Phys. D 11 (2002) 35, gr-qc/0012051;
G. Amelino-Camelia, Phys. Lett. B 510 (2001) 255, hep-th/0012238;
G. Amelino-Camelia, Nature 418 (2002) 34, gr-qc/0207049.

[2] J. Martin, R.H. Brandenberger, Phys. Rev. D 63 (2001) 123501, hep-th/0005209;
R.H. Brandenberger, J. Martin, Mod. Phys. Lett. A 16 (2001) 999, astro-ph/0005432;
J.C. Niemeyer, Phys. Rev. D 63 (2001) 123502, astro-ph/0005533;
J. Kowalski-Glikman, Phys. Lett. B 499 (2001) 1, astro-ph/0006250;
A.A. Starobinsky, JETP Lett. 73 (2001) 371, astro-ph/0104043.

[3] W.G. Unruh, Phys. Rev. D 51 (1995) 2827;
S. Corley, T. Jacobson, Phys. Rev. D 54 (1996) 1568;
S. Corley, Phys. Rev. D 57 (1998) 6280;
S. Corley, T. Jacobson, Phys. Rev. D 59 (1999) 124011;
A. Blaut, J. Kowalski-Glikman, D. Nowak-Szczepaniak, Phys. Lett. B 521 (2001) 364.

[4] G. Amelino-Camelia, T. Piran, Phys. Rev. D 64 (2001) 036005, astro-ph/0008107;
S. Sarkar, Mod. Phys. Lett. A 17 (2002) 1025, gr-qc/0204092;
R. Aloisio, P. Blasi, A. Galante, P.L. Ghia, A.F. Grillo, astro-ph/0205271.

[5] J.P. Norris, J.T. Bonell, G.F. Marani, J.D. Scargle, astro-ph/9912136;
A. de Angelis, astro-ph/0009271.

[6] J. Kowalski-Glikman, Phys. Lett. A 286 (2001) 391, hep-th/0102098.

[7] N.R. Bruno, G. Amelino-Camelia, J. Kowalski-Glikman, Phys. Lett. B 522 (2001) 133, hep-th/0107039.

[8] J. Lukierski, A. Nowicki, H. Ruegg, V.N. Tolstoy, Phys. Lett. B 264 (1991) 331;
S. Majid, H. Ruegg, Phys. Lett. B 334 (1994) 348;
J. Lukierski, H. Ruegg, W.J. Zakrzewski, Ann. Phys. 243 (1995) 90.

[9] J. Magueijo, L. Smolin, Phys. Rev. Lett. 88 (2002) 190403, hep-th/0112090.

[10] J. Kowalski-Glikman, S. Nowak, Phys. Lett. B 539 (2002) 126, hep-th/0203040.

[11] J. Lukierski, A. Nowicki, hep-th/0203065.

[12] J. Kowalski-Glikman, S. Nowak, hep-th/0204245.

[13] J. Lukierski, A. Nowicki, in: H.-D. Doebner, V.K. Dobrev (Eds.), Proceedings of Quantum Group Symposium at Group 21, July 1996, Goslar, Germany, Heron Press, Sofia, 1997, p. 186;
A. Nowicki, math.QA/9803064;
J. Lukierski, hep-th/9812063.

[14] H.S. Snyder, Phys. Rev. 71 (1947) 38.