ANNALS OF PHYSICS 134, 85–109 (1981)

The Background Field Method and the Ultraviolet Structure of the Supersymmetric Nonlinear σ-Model

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Received January 2, 1981

Calculations of the ultraviolet counterterms of the bosonic and supersymmetric nonlinear σ-models in two space-time dimensions are undertaken in order to verify conclusions of a recent argument based on differential geometry in the supersymmetric case. The background field method and the normal coordinate expansion are discussed in detail, and the generalized renormalization group pole equations applicable to the nonlinear σ-model are derived. Both component and superfield calculations of the counterterms are presented.

I. INTRODUCTION

Recently [1] the general structure of the ultraviolet divergences of supersymmetric σ-models in two space-time dimensions has been elucidated using a general argument combining the background field method and Kahler geometry. The general argument predicts that the ultraviolet divergences through two-loop order have a very specific tensor form, and that either the one-loop or two-loop divergences (or both) vanish for three interesting classes of models. The general argument can be viewed as "the theory," and in this paper we discuss the results of "the experiments," namely explicit Feynman graph calculations through two-loop order, which verify the predictions of "the theory."

The field theories we discuss are nonlinear σ-models on a general Riemannian manifold M with metric tensor $g_{ij}(\phi^k)$, where ϕ^k, for $k = 1,..., n$, are a set of coordinates on M. For the bosonic σ-model the coordinates $\phi^k(x)$ are taken to be fields over a two-dimensional Minkowski space, and the action is given by

$$I_B[\phi] = \frac{1}{2} \int d^2 x g_{ij}(\phi^k) \partial_\mu \phi^i \partial_\mu \phi^j, \tag{1.1}$$

0003-4916/81/070085-25\$05.00/0

which is invariant under general coordinate transformations on M. The supersymmetric extension is most simply formulated [2] over a superspace of two commuting coordinates x^μ and two anticommuting Majorana spinor coordinates θ_α. The supercovariant derivative is $D_\alpha = \partial/\partial\bar{\theta}_\alpha - i(\not{\partial}\theta)_\alpha$. The manifold coordinates are now taken to be superfields $\Phi^k(x, \theta)$, and the reparameterization invariant action is

$$I_S[\Phi] = \frac{1}{4i}\int d^2 x d^2\theta g_{ij}(\Phi^k)\bar{D}\Phi^i D\Phi^j \tag{1.2}$$

The component form of I_S is obtained by substituting

$$\Phi^i(x, \theta) = \phi^i(x) + \bar{\theta}\psi^i(x) + \tfrac{1}{2}\bar{\theta}\theta F^i(x) \tag{1.3}$$

(where $\psi_\alpha{}^i(x)$ are n Majorana spinor fields and $F^i(x)$ are n scalar auxiliaries) in (1.2) and performing the θ integrations. After elimination of auxiliary fields, the supersymmetric action becomes

$$\begin{aligned} I_S[\phi, \psi] = \frac{1}{2}\int d^2x\{&g_{ij}(\phi^k)\partial_\mu\phi^i\partial_\mu\phi^j + ig_{ij}(\phi^k)\bar{\psi}^i\gamma^\mu(D_\mu\psi)^j \\ &+ \tfrac{1}{6} R_{ijkl}(\bar{\psi}^i\psi^k)(\bar{\psi}^j\psi^l)\} \end{aligned} \tag{1.4}$$

The covariant derivative

$$(D_\mu\psi)^j = \partial_\mu\psi^j + \Gamma^j_{kl}\partial_\mu\phi^k\psi^l \tag{1.5}$$

and ψ^j itself transform as contravariant vectors under reparameterizations of M. The Christoffel connection and curvature tensor formed from $g_{ij}(\phi^k)$ appear in (1.4) and (1.5). Our Feynman graph calculations are performed using both superfields and the component formalism.

Power counting tells us that the counter terms which cancel the ultraviolet divergences of the theory are given by local operators of dimension 2 in the bosonic model and local superfield operators of dimension 1 in the supersymmetric case. There are two classes of counter terms, the on-shell counter terms which must be reparameterization invariant, and counter terms which vanish when the equations of motion are satisfied and which need not be invariant. Since fields (resp. superfields) are dimensionless, the counter terms necessarily involve two spatial derivatives (resp. two supercovariant derivatives).

The invariant counter terms take the form

$$\Delta I_B[\phi] = \frac{1}{2}\int d^2xT_{ij}(\phi)\partial_\mu\phi^i\partial_\mu\phi^j, \tag{1.6}$$

$$\Delta I_S[\Phi] = \frac{1}{4i}\int d^2xd^2\theta T_{ij}(\Phi)\bar{D}\Phi^i D\Phi^i \tag{1.7}$$

in the bosonic and supersymmetric theories. In each case the T_{ij} are second rank symmetric tensors which are algebraic functions of the curvature tensor of the manifold and its covariant derivatives. In the bosonic model the off-shell counter terms take the form

$$\bar{\Delta} I_B[\phi] = \int d^2x \Gamma^i(\phi) \frac{\delta I_B}{\delta \phi^i}, \tag{1.8}$$

where $\Gamma^i(\phi)$ is typically noncovariant under reparameterizations and can be constructed from the Christoffel connection on M. The Euler variation of (1.1) is

$$\frac{\delta I_B}{\delta \phi^i} = -g_{ij}(\Box \phi^j + \Gamma^j_{kl} \partial_\mu \phi^k \partial_\mu \phi^l). \tag{1.9}$$

The off-shell counter terms of the supersymmetric theory take an analogous form. We will primarily be concerned with the invariant counter terms which represent ultraviolet divergences of the "S-matrix," rather than the off-shell counter terms which can be compensated by redefinition of fields.

There is a simple argument [1] to classify the possible counter term tensors which can contribute to a given loop order in perturbation theory. Under the constant conformal scaling of the metric $g_{ij}(\phi) \to \Lambda^{-1} g_{ij}(\phi)$, Λ appears as a loop counting parameter. Hence the possible l-loop counter term tensors are algebraic functions of the curvatures of conformal weight Λ^{l-1}. Thus at one-loop order, the most general expected counter term involves the tensor

$$T^{(1)}_{ij} = a_1 R_{ij} + a_2 g_{ij} R, \tag{1.10}$$

where a_1 and a_2 are dimensionless numerical functions of the cutoff, e.g., single poles in the dimensional regularization parameter $\epsilon = d - 2$. The most general two-loop tensor is

$$\begin{aligned} T^{(2)}_{ij} = {} & b_1 R_{iklm} R_j^{\ klm} + b_2 D^k D_k R_{ij} + b_3 R_{ikjl} R^{kl} + b_4 R_{ik} R_j^{\ k} \\ & + b_5 R_{ij} R + b_6 g_{ij} R^2 + b_7 D_i D_j R, \end{aligned} \tag{1.11}$$

where the b_i can contain single and double poles in ϵ. One should note that the a_i and b_i are universal in the sense that they are completely independent of particular properties of the manifold M. These coefficients do differ, however, in the bosonic and supersymmetric theories. The background field [3, 4] method applied to (1.1) or (1.2) allows direct calculation of the coefficients by geometric and Feynman graph techniques.

In the bosonic case there are no general arguments (as opposed to explicit calculations [3, 4]) to restrict the allowed tensor forms in (1.10) and (1.11). In the supersymmetric case the allowed tensors are severely restricted [1] due to the connection [5] between models with extended supersymmetry and Kahler manifolds. The only permitted tensors are those which, after restriction from a Riemann to a Kahler manifold, become Hermitean and curl-free. This requirement allows only the one-loop tensor

$$T_{ij}^{(1)} = aR_{ij} \tag{1.12}$$

and the two-loop tensor[1]

$$T_{ij}^{(2)} = b(D^k D_k R_{ij} + 2R_{ikjl}R^{kl} + 2R_{ik}R_j{}^k), \tag{1.13}$$

which is the Lichnerowicz Laplacian acting on the Ricci tensor. Thus one predicts that supersymmetric σ-models on Ricci flat manifolds are one- and two-loop ultraviolet finite, and that models on locally symmetric manifolds ($D_m R_{ijkl} = 0$) and Einstein spaces ($R_{ij} = cg_{ij}$) are two-loop finite. By contrast, explicit calculations in the bosonic σ-model indicate the presence of a one-loop divergence [3] proportional to R_{ij} and a primitive two-loop divergence [4] proportional to $R_{iklm}R_j^{klm}$. Since the last tensor vanishes only for flat manifolds—i.e., free field theories—the bosonic σ-model always has nontrivial ultraviolet divergences. It is the specific predictions (1.12) and (1.13) which are tested by the calculations of this paper. The results confirm that the invariant ultraviolet divergences of the supersymmetric theory take the predicted tensor form, with coefficients $a = (2\pi\epsilon)^{-1}$ and $b = (4\pi\epsilon)^{-2}$.

The general metric nonlinear σ-model is not strictly renormalizable because there are infinitely many distinct counter terms. Friedan [4] has shown that the theory is renormalizable in a more general sense in which the manifold of the classical action changes due to quantum corrections. He has derived renormalization group equations which describe the change in geometry with energy scale. His considerations apply both to the bosonic and supersymmetric models. In this paper we derive the generalized renormalization grouu "pole equations" [6] which relate the coefficients of higher order poles of T_{ij} in the parameter ϵ. This investigation was suggested by the fact that the calculated coefficient b in (1.13) was a pure double pole in ϵ. Indeed the pole equations show that the specific two-loop tensor (1.13) (with the particular coefficient found) is required as a consequence of the generalized renormalizable structure of the model.

Finally we emphasize that the reason for a close check of the predictions of the geometrical argument of [1] at the two-loop level is that the argument can probably be extended to higher orders and may be powerful enough to show that all higher-loop ultraviolet divergences are absent in supersymmetric σ-models on Ricci flat or locally symmetric or Einstein spaces. Perturbation calculations provide a check that there are no hidden loopholes in the argument. The extension to higher orders is presently incomplete, but there is recent work which establishes three-loop finiteness in the supersymmetric $O(n)$ model [7] and for Ricci-flat manifolds [8]. Arguments combining background field calculations, the generalized renormalization group equations and Kahler differential geometry lead to this result. We note further that the calculations of this paper provide examples of the background field method extended beyond

[1] The tensor $D_i D_l R$, gives a counter term which vanishes on-shell [1], and is allowed in the off-shell structure of the supersymmetric model.

one-loop order,[2] of the combination of the background field method with superspace perturbation theory, and of the problem of infrared regularization in the background field method.

The background field expansion of the nonlinear σ-model action is derived in Section II using the method of Riemann normal coordinates. The diagrammatic algorithm to which this leads is presented in Section III together with one-loop calculations. Two-loop calculations for models on Ricci flat manifolds are given in Section IV for both the bosonic case and the supersymmetric model in component form. The renormalization group pole equations are derived in Section V, and the superfield calculations are presented in Section VI. Some identities used in these calculations are listed in an Appendix.

II. The Background Field Expansion

The covariant background field method for the nonlinear σ-model was first formulated [3] by Honerkamp with applications by Ecker and Honerkamp to one-loop calculation for the chiral pion Lagrangian in four dimensions. A detailed combinatoric proof that the covariant method gives the same results for the S-matrix as conventional perturbation theory has also been given [9].

Our method is computationally equivalent to that of Honerkamp and collaborators. However, it appears simplest to justify the method using the $\Omega[\phi]$ functional defined by deWitt [10] and recently discussed in a context involving nonlinear invariances [11]. For the bosonic σ-model the functional is defined by

$$\Omega_{\mathrm{B}}[\phi] = \int [d\pi^i] \exp\left(i\hbar^{-1}\left\{I_{\mathrm{B}}[\phi+\pi] - I_{\mathrm{B}}[\phi] - \frac{\delta I_{\mathrm{B}}[\phi]}{\delta\phi^i}\,\pi^i\right\}\right), \qquad (2.1)$$

where $I_{\mathrm{B}}[\phi]$ is the classical action (1.1) and the last term is expressed in a condensed notation which includes an integration over x^μ and summation over manifold indices. In the supersymmetric σ-model the action $I_{\mathrm{S}}[\Phi]$ or $I_{\mathrm{S}}[\phi, \psi]$ replaces $I_{\mathrm{B}}[\phi]$. The functional $\Omega[\phi]$ generates all diagrams containing at least one loop with external trees amputated. There is an algorithm [10] to obtain the S-matrix from $\Omega[\phi]$ by reattaching trees.

We note that $\Omega[\phi]$ is invariant under reparameterization of the manifold only if the equation of motion is satisfied, i.e., $\delta I[\phi]/\delta\phi^i = 0$, because the last term in the exponent of (2.1) is not invariant. However, $\Omega[\phi]$ contains both the on-shell and off-shell divergences of the theory. In the diagrammatic algorithm developed here and in the next section using normal coordinates as the quantum fields, the distinction between on-shell and off-shell counter terms is very simple. The former come essentially from vertices in the expansion of the invariant terms in the exponent, and the

[2] Higher loop background field calculations in gauge theories are the subject of very recent work by L. Abbott, CERN preprint, 1980.

FIG. 1. Some two-loop diagrams contained in the background field functional $\Omega[\phi]$. Diagram (a) is irreducible and contains a local divergence. Diagram (b) is one-particle-reducible and contains a nonlocal divergence. The triple line is a background field operator, and a single line denotes a quantum field.

latter from vertices in the expansion of the last term. It is not necessary to use the field equations explicitly in order to separate the invariant counter terms.

In one-loop order the divergent part of the quantity $-i \log \Omega[\phi]$ directly gives the desired counter terms of the effective action. In two-loop order $-i \log \Omega[\phi]$ contains one-particle-irreducible diagrams whose divergent parts are local expressions of the type needed, but there are nonlocal divergences corresponding to one-particle-reducible diagrams as shown in Fig. 1. The latter can be easily identified and discarded. To obtain the complete two-loop invariant divergences it is necessary to redefine $\Omega[\phi]$ to include effects of one-loop counter term insertions, and this is easily done.

To calculate the functional $\Omega[\phi]$ one can expand $I[\phi + \pi]$ in the exponential as a power series in $\pi^i(x)$ and generate a reasonably conventional diagrammatic algorithm. The problem with this approach is that the terms in the power series expansion are noncovariant since $\pi^i(x)$, which is defined as the difference between coordinate values at nearby points of the manifold, does not transform simply under reparameterization. Calculations, although correct, are not manifestly covariant and therefore awkward. This difficulty can be removed by expressing $\pi^i(x)$ as a local power series in a new field $\xi^i(x)$ which transforms as a contravariant vector. Since $I[\phi + \pi(\xi)]$ is a scalar, the coefficients of a power series expansion in $\xi^i(x)$ are covariant, and this leads to a manifestly covariant diagrammatic procedure to compute $\Omega[\phi]$.

To define the field $\xi^i(x)$, consider the two points ϕ^i and $\phi^i + \pi^i$ on the manifold. We assume that these points are close enough that there is a unique geodesic which connects them. This may be parameterized by $\lambda^i(t)$ which satisfies the differential equation

$$\ddot{\lambda}^i + \Gamma^i_{jk} \dot{\lambda}^j \dot{\lambda}^k = 0, \tag{2.2}$$

where t is a parameter proportional to arc length and dots indicate differentiation with respect to t. The solution can be chosen so that $\lambda^i(0) = \phi^i$, and $\lambda^i(1) = \phi^i + \pi^i$. The tangent vector to the geodesic at $t = 0$ is defined by $\xi^i = \dot{\lambda}^i(0)$. This vector is really a field $\xi^i(x)$ over space–time because of the implicit dependence of $\phi^i(x)$ and $\pi^i(x)$ on x^μ, and it is closely related to the quantum field of our calculations. Geometrically ξ^i is the tangent vector to the geodesic containing ϕ^i and $\phi^i + \pi^i$ whose magnitude is the square of the arc length s between these two points, i.e.,

$$g_{ij}\xi^i\xi^j = s^2 \qquad s = \int_{\phi^i}^{\phi^i+\pi^i} dt (g_{ij}\dot{\lambda}_i\dot{\lambda}_i)^{1/2}. \tag{2.3}$$

It is important to note that $\xi^i(x)$ transforms as a contravariant vector under reparameterizations of the manifold. Hence the expansion of any geometrical object on the manifold as a power series in $\xi^i(x)$ will be covariant. For a covariant tensor field $T_{k_1 \ldots k_m}(\phi)$ this expansion takes the form

$$T_{k_1 \cdots k_m}(\phi + \pi) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{\partial}{\partial \xi_{i_1}} \cdots \frac{\partial}{\partial \xi_{i_n}} \right] T_{k_1 \cdots k_m}(\phi) \xi^{i_1} \cdots \xi^{i_n} . \tag{2.4}$$

The coefficients are tensors and can be expressed in terms of covariant derivatives of T and the curvature tensor of the manifold. The easiest way to obtain the manifestly covariant form of the coefficients is to use the well-known method of normal coordinates [12, 13].

To define normal coordinates (more properly called Riemann coordinates) we return to the geodesic equation (2.2). By successive differentiation one finds that the power series solution to this equation takes the form

$$\lambda^i(t) = \phi^i + \xi^i t - \frac{1}{2} \Gamma^i_{j_1 j_2} \xi^{j_1} \xi^{j_2} t^2 - \frac{1}{3!} \Gamma^i_{j_1 j_2 j_3} \xi^{j_1} \xi^{j_2} \xi^{j_3} t^3 - \cdots, \tag{2.5}$$

where

$$\begin{aligned} \Gamma^i_{j_1 j_2 j_3} &\equiv \partial_{j_1} \Gamma^i_{j_2 j_3} - \Gamma^l_{j_1 j_2} \Gamma^i_{l j_3} - \Gamma^l_{j_1 j_2} \Gamma^i_{j_3 l} \\ &\equiv \nabla_{j_3} \Gamma^i_{j_2 j_3} , \\ \Gamma^i_{j_3 j_2 j_3 j_4} &\equiv \nabla_{j_1} \Gamma^i_{j_2 j_3 j_4} , \\ &\text{etc.}, \end{aligned} \tag{2.6}$$

and ∇_j mean "covariant differentiation on lower indices only," and all quantities are evaluated at ϕ^i.

At $t = 1$, we have $\lambda^i(1) = \phi^i(x) + \pi^i(x)$ and we can regard (2.5) as defining a coordinate transformation between a point $\phi^i + \pi^i$ in the initial coordinate chart near ϕ^i and new coordinates ξ^i. This transformation is nonsingular in a neighborhood of ϕ^i since $\partial \pi^i / \partial \xi^j = \delta^i j$ at ϕ^i. It is obvious that any two points $\phi^i + \pi^i$ and $\phi^i + \pi'^i$ on a common geodesic through ϕ^i will have normal coordinates ξ^i and $\xi^{i'}$ related by $\xi'^i = (s'/s)\, \xi^i$, where s and s' are the arc lengths from ϕ^i. Thus geodesics in normal coordinates are expressed as "straight lines" of the form $\zeta^i(t) = a^i t$. However the expansion (2.5) is valid in any coordinate system. Hence in the normal coordinate system ξ^i the expansion coefficients $\Gamma^i_{j_1 j_2 \ldots j_n}$ must vanish when symmetrized with respect to their lower indices, i.e.,

$$\Gamma^i_{(j_1 j_2 \cdots j_n)} = 0. \tag{2.7}$$

One then easily proves by induction that this condition is equivalent to

$$(\partial_{j_1} \partial_{j_2} \cdots \partial_{j_{n-2}} \Gamma^i_{j_{n-1} j_n)} = 0 \tag{2.8}$$

for all n. In this and subsequent paragraphs an upper bar indicates a relationship valid in normal coordinates.

In normal coordinates one can derive simple relations between partially symmetrized derivatives of Christoffel symbols and the curvature tensor. For example the curvature tensor

$$R^i_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^n_{jl}\Gamma^i_{nk} - \Gamma^n_{jk}\Gamma^i_{nl} \tag{2.9}$$

simplifies and becomes

$$\bar{R}^i_{jkl} = \partial_k \bar{\Gamma}^i_{jl} - \partial_l \bar{\Gamma}^i_{jk}\,. \tag{2.10}$$

By combining (2.10) with (2.8) for $n = 3$ one finds the relation

$$\partial_k \bar{\Gamma}^i_{jl} = \tfrac{1}{3}(\bar{R}^i_{jkl} + \bar{R}^i_{lkj}), \tag{2.11}$$

which will be used forthwith. Further by using (2.8) for $n = 4$ and $n = 5$ and the form of covariant derivatives of the curfature tensor in normal coordinates, one can derive

$$\partial_{(j_1}\partial_{j_2}\bar{\Gamma}^i_{j_3k)} = -\tfrac{1}{2} D_{j_1}\bar{R}^i_{j_2kj_3}\,, \tag{2.12}$$

$$\partial_{(j_1}\partial_{j_2}\partial_{j_3}\bar{\Gamma}^i_{j_4)k} = -\tfrac{3}{5}[D_{(j_1}D_{j_2}\bar{R}^i_{j_3kj_4)} + \tfrac{2}{9}\bar{R}^i_{(j_1j_2}{}^l\bar{R}_{j_3lj_4)k}], \tag{2.13}$$

where symmetrization with respect to the indices j_i (only) is indicated.

These relations may be applied to determine the first few expansion coefficients in (2.4) using the fact that the ξ^i are normal coordinates. For example,

$$\frac{\partial}{\partial \xi^i}\bar{T}_{k_1\cdots k_n}(\phi) = D_i\bar{T}_{k_1\cdots k_m}(\phi), \tag{2.14}$$

$$\frac{\partial}{\partial \xi^{i_1}}\frac{\partial}{\partial \xi^{i_2}}\bar{T}_{k_1\dots k_m}(\phi) = D_{(i_1}D_{i_2)}\bar{T}_{k_1\dots k_m}(\phi) - \frac{1}{3}\sum_{p=1}^{m}\bar{R}^j{}_{(i_1k_pi_2)}\bar{T}_{k_1\dots k_{p-1}jk_{p+1}\dots k_m}(\phi)$$

with symmeterization on i_1 and i_2 only.

Continuing in this way one can obtain the covariant expansion of any tensor through fourth order in ξ^i. The result is for a second rank tensor:

$$\begin{aligned} T_{kl}(\varphi+\pi) = {} & T_{kl}(\varphi) + D_iT_{kl}\xi^i + \frac{1}{2}\left\{D_{i_1}D_{i_2}T_{kl}(\varphi) - \frac{1}{3}R^j{}_{i_1ki_2}T_{jl} - (k\to l)\right\}\xi^{i_1}\xi^{i_2} \\ & + \frac{1}{3!}\Big[D_{i_1}D_{i_2}D_{i_3}T_{kl} - \Big(R^j{}_{i_1ki_2}D_{i_3}T_{jl}(\varphi_{i1}) \\ & + \frac{1}{2}D_{i_1}R^j{}_{i_2ki_3}T_{jl}(\varphi_{il}) + (k\to l)\Big)\Big]\xi^{i_1}\xi^{i_2}\xi^{i_3} \\ & + \frac{1}{4!}\Big[D_{i_1}D_{i_2}D_{i_3}D_{i_4}T_{kl} + \Big(\frac{3}{5}D_{i_1}D_{i_2}R^j{}_{i_3i_4k}T_{jl} \end{aligned}$$

$$+ 6R^{j}{}_{i_1 i_2 k} D_{i_3} D_{i_4} T_{jl} + \frac{1}{2} D_{i_1} R^{j}{}_{i_2 i_3 k} D_{i_4} T_{jl}$$
$$+ \frac{1}{5} R^{m}{}_{i_1 i_2 k} R^{j}{}_{i_3 i_4 m} T_{jl} + (k \to l)\Big)$$
$$+ \frac{1}{3} (R^{m}{}_{i_1 i_2 k} R^{n}{}_{i_3 i_4 l} + (k \to l))\, T_{mn} \xi^{i_1} \xi^{i_2} \xi^{i_3} \xi^{i_4}. \tag{2.15}$$

In a similar way one can obtain the expansion for a general nth rank tensor. In this case the only change needed is to change the very last term of (2.15) by a sum over pairs of indices of the arbitrary tensor $T_{k_1 k_2 \ldots k_n}$. Note that this tensorial relation is valid in a general coordinate system which is why bars have been removed. One can take the special case in which $T_{ij}(\phi)$ is the metric tensor $g_{ij}(\phi)$ of the manifold. In this case the expansion simplifies since $D_k g_{ij}$ vanishes. This expansion is one of the ingreients of the background field expansion of the σ-model action, and the explicit form is [13, 14]

$$\begin{aligned} g_{ij}(\varphi + \pi) = {} & g_{ij}(\varphi) - \frac{1}{3} R_{i k_1 j k_2}(\varphi) \xi^{k_1} \xi^{k_2} \\ & - \frac{1}{3!} D_{k_1} R_{i k_2 j k_3}(\varphi) \xi^{k_1} \xi^{k_2} \xi^{k_3} \\ & + \frac{1}{5!} \left(-6 D_{k_1} D_{k_2} R_{i k_3 j k_4}(\varphi) + \frac{16}{3} R_{k_1 j k_2}{}^{m} R_{k_3 i k_4 m} \right) \cdot \xi^{k_1} \xi^{k_2} \xi^{k_3} \xi^{k_4}. \end{aligned} \tag{2.16}$$

The other necessary ingredient of the background field method is the expansion of $\partial_\mu(\phi^i + \pi^i)$. This is a vector, and its expansion in the normal coordinate system is obtained by differentiation of (2.5) at $t = 1$.

$$\begin{aligned} \partial\mu(\varphi^i + r^i) = {} & \partial\mu\varphi^i + \partial\mu\xi^i - \Big(\frac{1}{2} \partial_j \Gamma^{i}{}_{k_1 k_2} \xi^{k_1} \xi^{k_2} \\ & + \frac{1}{3!} \partial_j \Gamma^{i}_{k_1 k_2 k_3} \xi^{k_1} \xi^{k_1} \xi^{k_3} \\ & + \frac{1}{4!} \partial_j \Gamma^{i}_{k_1 k_2 k_3 k_4} \xi^{k_1} \xi^{k_2} \xi^{k_3} \xi^{k_4}\Big) \partial_\mu \varphi^j. \end{aligned} \tag{2.17}$$

One then uses definitions (2.6) together with the normal coordinate conditions (2.7) and (2.8) and also (2.11)–(2.13) to derive the tensorial result

$$\begin{aligned} \partial_\mu(\varphi^i + \pi^i) = {} & \partial_\mu \varphi^i_{0l} + D_\mu \xi^i + \Big[\frac{1}{3} R^{i}{}_{k_1 k_2 j} \xi^{k_1} \xi^{k_2} + \frac{1}{12} D_{k_1} R^{i}{}_{k_2 k_3 j} \xi^{k_1} \xi^{k_2} \xi^{k_3} \\ & + \Big(\frac{1}{60} D_{k_1} D_{k_2} R^{i}{}_{k_3 k_4 j} - \frac{1}{45} R^{i}{}_{k_1 k_2 m} R^{m}{}_{k_3 k_4 j}\Big) \xi^{k_1} \xi^{k_2} \xi^{k_3} \xi^{k_4}\Big] \partial_\mu \varphi^i \end{aligned} \tag{2.18}$$

where $D_\mu \xi^i = \partial_\mu \xi^i + \Gamma^{i}{}_{jk} \xi^i \partial_\mu \phi^k$.

One can now combine expansion (2.16) of the metric tensor qith that of $\partial_\mu \phi^i$, namely (2.18), to obtain the background field expansion of the bosonic σ-model action

$$
\begin{aligned}
I_B[\varphi + r] = I_B[\varphi] &+ \int d^2x g_{ij} \partial_\mu \varphi^i D_\mu \xi^j \\
&+ \tfrac{1}{2} \int d\,x \Big\{ g_{ij} D_\mu \xi^i D^\mu \xi^j + R_{ik_1k_2j} \xi^{k_1} \xi^{k_2} \partial_\mu \varphi^i \partial^\mu \partial^j \varphi \\
&+ \tfrac{1}{3} D_{k_1} R_{ik_2k_3j} \xi^{k_1} \xi^{k_2} \xi^{k_3} \partial_\mu \varphi^i \partial^\mu \varphi^j \\
&+ \tfrac{4}{3} R_{ik_1k_2k_3} \xi^{k_1} \xi^{k_2} D_\mu \xi^{k_3} \partial_\mu \varphi^i \\
&+ \tfrac{1}{2} D_{k_1} R_{ik_2k_3k_4} \xi^{k_1} \xi^{k_2} \xi^{k_3} D_\mu \xi^{k_4} \partial_\mu \varphi^i \\
&+ \tfrac{1}{3} R_{k_1k_2k_3k_4} \xi^{k_2} \xi^{k_3} D_\mu \xi^{k_1} D^\mu \xi^{k_2} \\
&+ \tfrac{1}{12} (D_{k_1} D_{k_2} R_{ik_3k_4j} + 4R^m{}_{k_1k_2i} R_{mk_3k_4j})\, \xi^{k_1} \xi^{k_2} \xi^{k_3} \xi^{k_4} \partial_\mu \varphi^i \partial^\mu \varphi^j .
\end{aligned}
\tag{2.19}
$$

The linear term in $\xi^i(x)$ vanishes if the Euler–Lagrange equation is used.

One should note that the derivation of (2.20) outlined here is somewhat more complicated than necessary because it involved separate expansions for $g_{ij}(\phi + \pi)$ and $\partial_\mu(\phi^i + \pi^i)$. If one studies the whole action directly rather than separating these terms, then the combinatoric work necessary to obtain (2.19) can be simplified. (Specifically one no longer needs to use (2.8) for $n = 5$.) However(the separate expansion of a general tensor (2.15) is necessary, because the tensors occuring as lower-loop counter terms must be expanded in background field form to obtain complete higher-loop results.

The expansion of the supersymmetric action is analogous with the same tensors and numerical coefficients. However, $\Phi^i(x, \theta)$ and $\xi^i(x, \theta)$ are superfields and the spatial derivative ∂_μ is replaced by the supercovariant derivative of supersymmetry. This expansion will be)used in Section VI. In Section IV we present two-loop calculations on the supersymmetric σ-model in the component form (1.4). Since only bosonic counter terms are computed the fermion field $\psi^i(x)$ may already be considered a quantum field, and we are therefore interested only in the expansion of the Dirac terms through second order in $\xi^i(x)$. By similar techniques to those discussed in this section one finds that

$$
\begin{aligned}
g_{ij}(\varphi) \bar{\psi}^i \not{D} \psi^j = (g_{ij}(\varphi) &+ \tfrac{1}{3} R_{ik_1k_2j} \xi^{k_1} \xi^{k_2}) \bar{\psi}^i \not{D} \psi^j \\
&+ \tfrac{1}{2} R_{ijkl} \partial_\mu \varphi^l \xi^k (\bar{\psi}^i \gamma^\mu \psi^j).
\end{aligned}
\tag{2.20}
$$

(Note that the transformation of ψ^i to normal coordinates is $\psi^i = (\partial \phi^i / \partial \xi^i)\, \psi^{-j}$ and that contributions from the Jacobian occur in the expansion of $\partial_\mu \psi^i$.)

The bose and fermi fields $\xi^i(x)$ and $\psi^i(x)$ are not quite suitable as quantum fields in diagrammatic calculations. Their propagators are not standard because of the presence of $g_{ij}(\phi)$ in the kinetic terms. This problem is easily remedied by referring these Riemannian vectors to tangent frames on the manifold. Thus we introduce an n-bein $e_i{}^a(\phi)$ where n is the dimension of the manifold and define $\xi^a(x) = e_i{}^a\xi^i(x)$ and $\psi^a(x) = e_i{}^a\psi^i(x)$. After this modification the kinetic terms become $(D_\mu\xi)^a(D_\mu\xi)^a$ and $\bar{\psi}^a\gamma^\mu(D_\mu\psi)^a$, where $(D_\mu\xi)^a = \partial_\mu\xi^a + \omega_i^{ab}\partial_\mu\phi^i\xi^b$ and $D_\mu\psi^a$ is similar and where ω_i^{ab} is the spin connection on the manifold. It is easy to verify that the product $\omega_i^{ab}\partial_\mu\phi^i$ transforms as an $SO(n)$ Yang–Mills potential under local rotations of the tangent frame, and that $(D_\mu\xi)^a$ transforms as a vector under tangent frame rotations and as a scalar under coordinate reparameterizations.

III. Diagrammatic Algorithm and One-Loop Calculation

The advantage of the expansion in the variable $\xi^i(x)$ or $\xi^a(x)$ is the manifest covariance of the dependence on the background field which is evident in (2.19). Since the transformations from $\pi^i(x) \to \xi^i(x) \to \xi^a(x)$ are coordinate transformations on the manifold, the functional measure is invariant (when properly defined, see [3, 15]). Therefore the functional $\Omega[\phi^i]$ leads directly to a diagrammatic algorithm when the field variable $\xi^a(x)$ is used and the sources of invariant and off-shell counter terms effectively separate in this algorithm. For calculation of the invariant counter term of the effective action only the first two terms of the exponential in (2.1) must be considered. For calculation of the off-shell field redefinition term one must consider the additional diagrams generated by the third term with π^i expressed in terms of $\xi^i(x)$ by (2.5) (at $t = 1$) and with $\xi^i(x) = e_a{}^i(x)\,\xi^a(x)$. Note that terms linear in $\xi^a(x)$ cancel in the exponential but higher-order terms remain.

We now apply these observations to the calculation of the one-loop counter terms of the bosonic σ-model. We need the integral of the exponential in (2.1) expanded through second order in $\xi^a(x)$ including noncovariant terms

$$I_{\mathrm{B}}^{(2)}[\phi, \xi^a] = \tfrac{1}{2}\int d^2x \left\{D_\mu\xi^a D_\mu\xi^a + \left(R_{iabj}(\phi)\partial_\mu\phi^i\partial_\mu\phi^j + \frac{\partial I}{\partial\phi^i}\Gamma^i_{ab}\right)\xi^a\xi^b\right\} \tag{3.1}$$

with $\Gamma^i_{ab}(\phi) = \Gamma^i_{jk}(\phi)e_a{}^j(\phi)\,e_b{}^k(\phi)$. We found that the simplest way to generate the diagrammatic algorithm is to express $\Omega[\phi]$ using interaction picture Dyson–Wick perturbation theory. We write

$$\Omega_{\mathrm{B}}[\phi] = \langle 0 \mid \exp i\int d^2x\, L_{\mathrm{int}}(\phi, \xi) \mid 0\rangle$$

$$\int d^2x\, L_{\mathrm{int}}(\phi, \xi) \equiv I_{\mathrm{B}}^{(2)}[\phi, \xi^a] - \tfrac{1}{2}\int d^2x\partial_\mu\xi^a\partial_\mu\xi^a. \tag{3.2}$$

A similar formula is valid for higher-loop calculations. We simply replace $I^{(2)}[\phi, \xi^a]$ by the sum of all terms through sufficiently high order in $\xi^a(x)$ of the expansion of the argument of the exponential in (2.1). In the calculation of the vacuum diagrams of

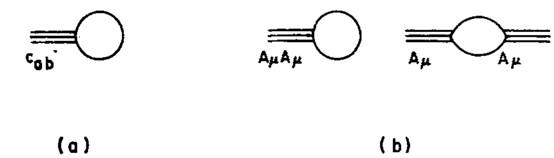

FIG. 2. Divergent one-loop diagrams whose external background field lines involve (a) the operator c_{ab} or (b) the $SO(n)$ gauge potential A_μ^{ab}.

(3.2), $\phi^i(x)$ is regarded as an external field and we simply compute all possible Wick contractions involving the quantum field $\xi^a(x)$, using

$$\xi^a(x_1)\xi^b(x_2) = i\delta^{ab}\Delta_F(x_1 - x_2). \tag{3.3}$$

There are three possibly divergent one-loop diagrams which are shown in Fig. 2. One must distinguish between graphs whose external lines involve the operator $A_\mu^{ab} = \omega_i^{ab}\partial_\mu\phi^i$, which transform as an $SO(n)$ Yang–Mills gauge potential under local tangent frame rotations in the manifold, and graphs whose external lines involve $C_{ab} = R_{iabj}\partial_\mu\phi^i\partial_\mu\phi^j + (\delta I/\delta\phi^i)\,\Gamma_{ab}^i$, which transforms as a second rank $SO(n)$ tensor. Divergences of the first set of graphs must cancel since the only possible invariant is the square of the field strength $F_{\mu\nu}^{ab}$ formed from A_μ^{ab}, namely $F_{\mu\nu}^{ab}F_{\mu\nu}^{ab}$. This is of dimension four and cannot correspond to an ultraviolet divergence of the nonlinear σ-model in two space–time dimensions (although it can and does contribute in four space–time dimensions [3]). Cancellation of the divergences can also be verified by calculation of the graphs and is a consequence of the well-known ultraviolet finiteness of a gauge theory in two dimensions.

Thus we are left with only one divergent graph involving C_{ab}, and its contribution to the effective action is

$$-i\log\Omega[\phi^1] = \tfrac{1}{2}\int d^2x\, C_{ab}(\phi(x))\xi^a(x)\xi^b(x) = \tfrac{1}{2}\int d^2x\, C_{aa}(\phi(x))i\Delta_F(0) \tag{3.4}$$

where the propagator at zero separation appears. This can be written in terms of a Euclidean momentum integral

$$\begin{aligned} I &= i\Delta_F(0) \\ &= \frac{1}{(2\pi)^n}\int d^nk_E\,\frac{1}{k_E{}^2+\mu^2} \\ &\underset{\epsilon\to 0}{\to} \frac{-1}{2\pi\epsilon} + \text{finite}, \end{aligned} \tag{3.5}$$

where we have extracted the ultraviolet divergence in the dimensional regularization parameter $\epsilon = n - 2$. Thus the counter term which must be added to the classical action $I_B[\phi]$ to cancel the one-loop divergence is

$$\Delta I_B^{(1)}[\phi] = \frac{1}{4\pi\epsilon}\int d^2x\left[R_{ij}\partial_\mu\phi^i\partial_\mu\phi^j + g^{jk}\Gamma_{jk}^i\frac{\delta I}{\delta\phi^i}\right]. \tag{3.6}$$

The covariant term involving the Ricci tensor requires a change in the classical geometry due to renormalization [4]. The noncovariant term is the effect of a field redefinition (generalized wave function renormalization) or reparameterization of the manifold of the form

$$\phi^i(x) = \phi'^i(x) + \frac{1}{4\pi\epsilon} g^{jk}(\phi')\Gamma^i{}_{jk}(\phi'). \tag{3.7}$$

Previous treatments of the ultraviolet divergences of the general bosonic σ-model were primarily concerned with the case of four space–time dimensions. However, results on the two-dimensional case are implicitly contained in previous work. The invariant one-loop counter term has been calculated [3] by a method involving Riemann normal coordinates which is very similar to ours, and both the invariant and noninvariant terms have been calculated [16] using the background field algorithm of 't Hooft [17]. The present results are also in agreement with past work on the one-loop coupling constant and field renormalization of the $O(N)$ model (for which the manifold M is the $(N-1)$-dimensional sphere) both in the $\sigma - \pi$ parameterization [18] and in conformal coordinates [19].

We now turn to the supersymmetric σ-model in component form (1.4) and consider the effect of the addition of fermions on the bosonic counter terms of the theory. We are therefore interested in graphs with external boson lines and internal boson and fermion lines. These are obtained from the functional

$$\Omega_S[\varphi] = \int [d\pi^i d\psi^i] \exp\left[i\hbar^{-1}\left(I_S(\varphi+\pi,\psi) - I_B[\varphi] - \frac{\delta I_B}{\delta\varphi^i}\pi^i\right)\right], \tag{3.8}$$

where $I_S[\phi, \psi]$ is given in (1.4). One again uses the normal coordinate expansions (2.19) and (2.20) and refers both $\xi^i(x)$ and $\psi^i(x)$ to tangent frames on the manifold. To one-loop order we require only the quadratic terms in the fields, and (3.8) factors into a product of the previous one-loop bosonic term (3.2) and the fermionic integral

$$\int [d\psi^a] \exp\left(-\frac{1}{2\hbar}\int d^2x \bar{\psi}^a \gamma^\mu(\partial_\mu\psi^a + A_\mu{}^{ab}\psi)\right). \tag{3.9}$$

Thus one-loop fermion effects are equivalent to those of an $SO(n)$ gauge theory which is ultraviolet finite in two space–time dimensions by the same argument previously applied to the external connection graphs of (3.2). Thus the supersymmetric addition of fermions does not change the one-loop bosonic counter terms of the nonlinear σ-model.

IV. Two-Loop Calculations for Ricci Flat Manifolds

In this section we present calculations of the on-shell ultraviolet behavior in two-loop order for the bosonic σ-model (1.1) and the supersymmetric σ-model in component form (1.4). In both cases we assume that the manifold is Ricci flat which simplifies

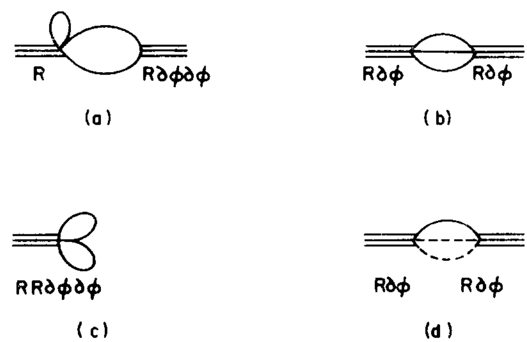

FIG. 3. The Feynman graphs discussed in Section IV with background field operators at each vertex indicated schematically. A boson quantum field ζ^a is denoted by an unbroken line, and a fermion field ψ^a by a dashed line.

the calculation, as explained below. The component calculations allow a check of our methods with previous results [4] in the bosonic σ-model and then illustrate explicitly how divergences of the additional fermion graphs cancel those of the bosonic theory leaving an ultraviolet finite result. The superfield calculations of Section VI are more general than those of this section because no restriction on the geometry is made. However, the calculations here should be useful for readers who are not familiar with superfield perturbation theory.

Another issue in the nonlinear σ-model is the infrared regularization. We follow a policy of minimal disturbance, i.e., we introduce an infrared cutoff only as necessary for unambiguous determination of the ultraviolet divergent terms. In this section (and in the previous) a simple mass cutoff in the tadpole integral I of (3.5) is sufficient. In a non-Ricci flat geometry a more elaborate infrared regularization is necessary and will be discussed in the next section.

To generate two-loop graphs we again use the Dyson–Wick algorithm expressed in (3.2) with $L_{\text{int}}(\phi, \xi, \psi)$ appropriately extended to include all terms of third and fourth order in quantum fields ξ^a and ψ^a from (2.19) and (2.20). As in the one-loop case one distinguishes between vertices involving the $SO(n)$ connection $A_i^{ab} = \omega_\mu^{ab}\partial_\mu\phi^i$ and those involving $SO(n)$ tensors. The connection vertices cannot contribute since gauge invariance requires that the field strength $F_{\mu\nu}$ appears. This is already of dimension 2 but not a Lorentz scalar, and no Lorentz scalar can ever be formed by contraction with dimensionless $SO(n)$ tensors.

For a general geometry one would have to include vertices from the one-loop counter term (3.6) with its supersymmetric completion expanded to second order in ξ^a and ψ^a. However, since the counter term vanishes in a Ricci flat geometry, this complication is avoided. There are further algebraic simplifications which make calculations easier in the Ricci flat case. For example, ultraviolet counter terms involving the tensor

$$D_k D_l R_i{}^{kl}{}_j\, \partial_\mu\phi^i\partial_\mu\phi^j \tag{4.1}$$

occur for a general geometry, but the tensor itself vanishes due to the Bianchi identity in Ricci flat manifolds. Further the graph of Fig. 3a also vanishes since the tadpole integral forces a contraction of indices giving the Ricci tensor.

For the bosonic σ-molel one finds that the only irreducible Feynman diagrams which can produce divergences are those of Fig. 3b and c. These graphs are evaluated using Wick contractions, and the ultraviolet divergences can be expressed as local integrals. After some tensor manipulations the divergent contribution to $-i$ log $\Omega[\phi]$ can be shown to take the form

$$\begin{aligned}(3\mathrm{b}) &= \frac{1}{2n} I^2 \int d^2x R_{iklm} R_j{}^{klm} \partial_\mu \phi^i \partial_\mu \phi^j, \\ (3^{\mathrm{T}}) &= \frac{1}{4} I^2 \int d^2x R_{iklm} R_j{}^{klm} \partial_\mu \phi^i \partial_\mu \phi^j,\end{aligned} \tag{4.2}$$

where I is the tadpole integral (3.5).

Note that these contributions combine to give a single pole in ϵ and are cancelled by the counter term tensor

$$T_{ij} = \frac{-1}{(2\pi)^2 4\epsilon} R_{iklm} R_j{}^{klm} \tag{4.3}$$

in agreement with [4].

We now come to the supersymmetric theory and find that there is one potentially divergent irreducible two-loop diagram involving fermions, shown in Fig. 3d, which must be added to the diagrams previously evaluated. This diagram is evaluated using supersymmetric dimensional regularization [20], i.e., Tr $\gamma^\mu \gamma^\mu = 2g^{\mu\nu}$. One then finds the local divergence

$$(3\mathrm{d}) = \frac{2-n}{4n} I^2 \int d^2x R_{iklm} R_j{}^{klm} \partial_\mu \varphi^i \partial^\mu \varphi^j. \tag{4.4}$$

This fermion contribution exactly cancels the previous boson contribution showing that the supersymmetric theory is two-loop ultraviolet finite.

V. Generalized Renormalization Group Equations

An extremely interesting interpretation of the general bosonic Riemannian metric σ-model as a renormalizable theory in which the geometry changes with renormalization mass scale has been developed by Friedan [4]. In this section we obtain his renormalization group equation by a method entirely parallel to that of 't Hooft [6] in ordinary field theory. As a bonus we derive the generalized "pole equations," which are later checked in two-loop calculations. It is interesting that there is such a natural extension from ordinary renormalizable field theories where the few coupling constants which parameterize the theory vary with mass scale to the σ-model where the manifold M which describes the theory varies with mass scale. Generalized renormalizability is a property of both the bosonic and supersymmetric σ-model, since it depends only on dimensionality and invariance considerations which are identical in both cases.

We first review 't Hooft's method in the case of a renormalizable field theory in two-dimensions with a single dimensionless coupling constant λ such as the well-known $O(N)$ models. If μ is the mass scale parameter and d is the space–time dimension then the bare coupling λ_B has dimension $[\mu^{2-n}]$. Using dimensional regularization and minimal subtraction of Feynman graphs, one determines a unique series of counter terms of the classical action such that at any mass scale μ, the bare coupling can be expressed as a function of a dimensionless renormalized coupling λ_R by a series of the form

$$\lambda_B = \mu^{2-n}\left[\lambda_R + \sum_{\nu=1}^{\infty} \frac{a_\nu(\lambda_R)}{(n-2)^\nu}\right] \tag{5.1}$$

The pole residues $a_\nu(\lambda_R)$ have no explicit dependence on μ, and the left side is entirely independent of μ. This requires that the coupling $\lambda_R(\mu)$ varies with μ and that certain relations hold among the pole residues.

The desired relations are best derived by considering two infinitesimally different mass scales μ and μ' related by $\mu = \mu'(1-\eta)$. Then (5.1) becomes

$$\lambda_B = (\mu')^{2-n}\left[\lambda_R + \eta(n-2)\lambda_R + \eta a_1(\lambda_R) + \sum_{\nu=1}^{\infty} \frac{a_\nu(\lambda_R) + \eta a_{\nu+1}(\lambda_R)}{(n-2)^\nu}\right]. \tag{5.2}$$

One must be able to interpret this as a series of the form (5.1) at the mass scale μ' and this requires the infinitessimal change in coupling

$$\lambda_R = \lambda'_R - \eta(n-2)\lambda_R - \eta\left(1 - \lambda_R \frac{\partial}{\partial\lambda_R}\right) a_1(\lambda_R). \tag{5.3}$$

If this is inserted in (5.2), one finds

$$\lambda_B = (\mu')^{2-n}\Bigg\{\lambda'_R + \sum_{\nu=1}^{\infty} \frac{a_\nu(\lambda'_R)}{(n-2)^\nu} + \eta \sum_{\nu=1}^{\infty} \frac{1}{(n-2)^\nu}\left[\left(1 - \lambda_R \frac{\partial}{\partial\lambda_R}\right) a_{\nu+1}(\lambda_R) - \left(1 - \lambda_R \frac{\partial}{\partial\lambda_R}\right) a_1(\lambda_R) \frac{\partial a_\nu(\lambda_R)}{\partial\lambda_R}\right]\Bigg\}.$$

This is of the desired form if the term linear in η cancels. Hence we find the pole equations

$$\left(1 - \lambda_R \frac{\partial}{\partial\lambda_R}\right) a_{\nu+1}(\lambda_R) = \left(1 - \lambda_R \frac{\partial}{\partial\lambda_R}\right) a_1(\lambda_R) \frac{\partial}{\partial\lambda_R} a_\nu(\lambda_R), \tag{5.5}$$

while (5.3) is equivalent to the differential equation

$$\mu \frac{\partial \lambda_R}{\partial \mu} = -\beta(\lambda_R),$$

$$\beta(\lambda_R) \equiv -(n-2)\lambda_R + \left(1 - \lambda_R \frac{\partial}{\partial \lambda_R}\right) a_1(\lambda_R), \tag{5.6}$$

where the usual β function appears.

The previous treatment can now be extended to the general metric nonlinear σ-model. We make the convention that fields $\phi^i(x)$ are dimensionless for all n (as in the case of the usual treatment of the $O(N)$ models [18]). Then the bare metric g_{ij}^B has dimension $[\mu]^{n-2}$. The bare metric is expressed as a sum of the dimensionless renormalized metric g_{ij}^R and the invariant on-shell counter term tensors $T_{ij}^{(\nu)}(g^R)$ calculated using dimensional regularization and minimal subtraction, so that

$$g_{ij}^B = \mu^{n-2}\left[g_{ij}^R + \sum_{\nu=1}^{\infty} \frac{T_{ij}^{(\nu)}(g^R)}{(n-2)^\nu}\right]. \tag{5.7}$$

The pole residue tensors have no explicit dependence on μ or n and are uniquely determined as algebraic functions of the curvature tensor of g^R and its covariant derivatives. The contribution to $T_{ij}^{(\nu)}(g^R)$ from l-loop order in the perturbation expansion is a tensor of weight Λ^{l-1} under the constant conformal transformation $g_{ij}^R \to \Lambda^{-1} g_{ij}^R$. One should also note that tensors of the form $D_i V_j + D_j V_i$ are encountered in the perturbation theory calculations where V_i is a vector field constructed from curvatures. Such tensors are excluded from (5.7) because they generate diffeomorphisms of the manifold rather than changes in the geometry and therefore give vanishing contribution to on-shell counter terms of the action [1].

The generalized renormalization group equations are obtained exactly as in ordinary field theory by requiring that a change in mass scale in (5.7) is compensated by a change in g_{ij}^R. One need only repeat the steps leading to (5.5) and (5.6) paying special attention to the variational derivations of $T_{ij}^{(\nu)}(g^R)$ with respect to changes in g^R. The results are Friedan's renormalization group equation

$$\mu \frac{\partial}{\partial \mu} g_{ij}^R = -\beta_{ij}(g^R)$$
$$\beta_{ij}(g_R) = (n-2)g_R + \left(1 + \Lambda \frac{\partial}{\partial \Lambda}\right) T^{(1)}(\Lambda^{-1} g_R)\big|_{\Lambda=1} \tag{5.8}$$

together with the new generalized pole equations

$$\left(1 + \Lambda \frac{\partial}{\partial \Lambda}\right) T^{(\nu+1)}(\Lambda^{-1} g_R) = \lim_{\substack{\eta\to 0\\ \Lambda\to 1}} \eta^{-1}\left[T^{(\nu)}\left(g_R + \eta\left(1 + \Lambda \frac{\partial}{\partial \Lambda}\right) T^{(\nu)}(\Lambda^{-1} g_R)\right) - T^{(\nu)}(g_R)\right]. \tag{5.9}$$

The chain rule has been used to relate one type of variational derivative to the derivative with respect to conformal scale or loop counting parameter. This facilitates application in perturbation theory. The other variational derivative is explicitly defined on the right side of (5.9).

The two-loop double pole tensor $T_{ij}^{(2)}$ is determined from (5.9) in terms of the one-loop single pole tensor as

$$2T_{ij}^{(2)}(g^R) = \lim \eta^{-1}[T_{ij}^{(1)}(g^R + \eta T^{(1)}) - T_{ij}^{(1)}(g^R)]. \tag{5.10}$$

If we insert the explicit one-loop counter term $T_{ij} = (2\pi)^{-1}R_{ij}$ calculated in (3.6), we see that we must calculate the first order variation of the Ricci tensor under a change in the metric proportional to the Ricci tensor. This is determined from the Palatini identity [21] as

$$2T_{ij}^{(2)}(g^R) = \frac{1}{2(2\pi)^2}[D^kD_kR_{ij} - D^kD_iR_{kj} - D^kD_jR_{ki} + D_iD_jR]. \tag{5.11}$$

This tensor coincides with (1.13), as announced in Section I. The precise coefficient will be checked in the two-loop calculations of the next section.

VI. Superfield Calculations

We begin by stating our conventions and calculating the propagator for the free massive scalar superfield. The γ-matrices used are:

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \gamma^5 = \gamma^0\gamma^1 \tag{6.1}$$

In this representation, charge conjugation is simply complex conjugation, so that a Majorana spinor satisfies $\psi_\alpha = \psi_\alpha^*$, $\alpha = 1, 2$. The Dirac adjoint is formed as usual $\bar{\psi} = \psi^*\gamma^0$. Spinor bilinears satisfy

$$\bar{\psi}\chi = \bar{\chi}\psi, \qquad \bar{\psi}\gamma^\mu\chi = -\bar{\chi}\gamma^\mu\psi, \qquad \bar{\psi}\gamma^5\chi = -\bar{\chi}\gamma^5\psi, \tag{6.2}$$

where ψ, χ are anticommuting Majorana spinors. A scalar superfield is a function of x^μ, $\mu = 0, 1$, and θ_α, $\alpha = 1, 2$, where the latter are real Grassmann numbers which form a Majorana spinor. In components a real scalar superfield can be expressed as:

$$\Phi(x, \theta) = \varphi(x) + \bar{\theta}\psi(x) + \tfrac{1}{2}\bar{\theta}\theta F(x) \tag{6.3}$$

with ϕ, F real scalar fields and ψ a Majorana spinor field. A supersymmetry transformation is given by

$$\delta\Phi = \bar{\epsilon}Q\Phi \qquad Q_\alpha = \frac{\partial}{\partial\bar{\theta}_\alpha} + i(\gamma^\mu\theta)_\alpha\partial_\mu, \tag{6.4}$$

where ϵ is a constant anticommuting Majorana spinor. The spinor derivative is

$$D_\alpha = \frac{\partial}{\partial \bar{\theta}_\alpha} - i(\gamma^\mu \theta)_\alpha \partial_\mu ,$$

$$\bar{D}_\alpha = D_\beta \gamma^0_{\beta\alpha}$$

and we have

$$\begin{aligned} \{D_\alpha , Q_\beta\} &= 0, \\ \{Q_\alpha , Q_\beta\} &= -2i(\not{\partial}\gamma^0)_{\alpha\beta} , \\ \{D_\alpha , D_\beta\} &= 2i(\not{\partial}\gamma^0)_{\alpha\beta} . \end{aligned} \tag{6.6}$$

With these conventions, the Lagrangian for the free scalar superfield of mass m is

$$\mathscr{L} = \frac{1}{2i} \int d^2x d^2\theta \left(\frac{1}{2} \bar{D}\Phi D\Phi + m\Phi\Phi\right) . \tag{6.7}$$

We will follow the method of Ref. [22] to find the Feynman rules for supergraphs with which one can obtain very simply and compactly all the results that one would get from calculating ordinary Feynman graphs in the component field formalism.

The generating functional for a self-interacting scalar superfield is defined by the path integral

$$W[J] = \int \prod_{\phi,0} d\Phi(x, \theta) \exp \int d^2x d^2\theta \left(\frac{1}{4} \bar{D}\phi D\phi + \frac{m}{2} \phi^2 + V(\phi) + J\phi\right) . \tag{6.8}$$

The gaussian integral can be done as usual

$$\begin{aligned} W_0[J] &= \int \prod_{\phi,0} d\Phi \exp \int d^2x d^2\theta \left(\frac{1}{4} \bar{D}\phi D\phi + \frac{m}{2} \phi^2 + J\phi\right) \\ &= \int \prod_{\phi,0} d\phi \exp \int d^2x d^2\theta \left(-\frac{1}{2} \phi \left(\frac{1}{2} \bar{D}D - m\right) \phi + J\phi)\right) \\ &= \exp \int d^4z_1 d^4z_2 J(z_1) G(z_1 , z_2) J(z_2), \qquad z \equiv (x^\mu, \theta^\alpha), \end{aligned}$$

where the two-point function satisfies the differential equation

$$(\tfrac{1}{2}\bar{D}_1 D_{1-m}) G(z_1 , z_2) = \delta^4(z_1 - z_2), \tag{6.9}$$

where $\delta^4(z_1 - z_2) = \delta^2(x_1 - x_2)\, \delta^2(\theta_1 - \theta_2)$ and, by definition,

$$\begin{aligned} \delta^2(\theta_1 - \theta_2) &= \frac{1}{2i} (\bar{\theta}_1 - \bar{\theta}_2)(\theta_1 - \theta_2), \\ \int d^2\theta_1 \delta^2(\theta_1 - \theta_2) F(\theta_1) &= F(\theta_2). \end{aligned} \tag{6.10}$$

In order to define $G(z_1, z_2)$ in momentum space, we notice that the operator $\frac{1}{2}\bar{D}D$ satisfies

$$(\tfrac{1}{2}\bar{D}D)^2 = -\Box \equiv -\partial_\mu\partial^\mu .$$

Then multiplying (6.9) on the right by $(\frac{1}{2}\bar{D}_1 D_1 + m)$, we find

$$(\Box_1 + m^2) G(z_1, z_2) = -(\tfrac{1}{2}\bar{D}_1 D_1 + m)\, \delta^2(x_1 - x_2)\, \delta^2(\theta_1 - \theta_2)$$

and

$$G(z_1, z_2) = \int \frac{d^2k}{(2r)^2} \frac{e^{-ik(x_1 - x_2)}}{k^2 - m^2 + i\epsilon} \left(\frac{1}{2}\,\bar{D}D(k) + m\right) \delta^2(\theta_1 - \theta_2),$$

$$D_\alpha(k) = \frac{\partial}{\partial\bar{\theta}^\alpha} - (\not{k}\theta)_\alpha . \qquad (6.11)$$

Since our calculations are given in terms of Wick contractions, we record the result:

$$\xi^a(z_1)\xi^b(z_2) = \delta^{ab} G(z_1, z_2). \qquad (6.12)$$

The Feynman rules for (6.8) can be derived in the usual way, keeping in mind that at each vertex we have to perform an integration over $d^4z_i \equiv d^2x_i d^2\theta_i$. As was shown in [22] the evaluation of a Feynman supergraph is quite easy as far as the θ-integration is concerned. The only new technique involved is partial integration of D's and $\frac{1}{2}\bar{D}D$ factors successively until all act on a single δ-function. The result can then be obtained using various identities listed in the Appendix. Essentially these identities correspond to expressions for

$$D_{\alpha_1}(k) D_{\alpha_2}(k) \cdots D_{\alpha_n}(k) \delta^2(\theta_1 - \theta_2) \,|_{\theta_1 = \theta_1} .$$

In this way all the θ-integrations can be performed, and the supergraph is then reduced to a multiloop bosonic integral.

This technique of calculation carries over to our background field method. The supersymmetric background field functional is defined as in (2.1) replacing $I_B[\phi]$ by $I_S[\Phi]$. Again we use Dyson–Wick perturbation theory as in (3.2). The interaction Lagrangian can be directly obtained from the normal coordinate expansion of the bosonic case (2.19) by replacing all field by scalar superfields, and all spatial derivatives by supercovariant derivatives, i.e.,

$$\partial_\mu \varphi^i \to D\phi^i, \qquad d^2x \to d^4z/2i ,$$

$$D_\mu \xi^i \to \hat{D}\xi^i = D\xi^i + \Gamma^i jk \xi^j D\phi^k. \qquad (6.13)$$

As before the actual quantum field is $\xi^a = e_i{}^a(\Phi)\, \xi^i$, and

$$\hat{D}\xi^a = D\xi^a + \omega_i{}^{ab}\xi^b D\phi^i. \qquad (6.14)$$

One can show that $\omega_i^{ab} D\Phi^i$ transforms as a superfield gauge connection under tangent frame rotations. Thus vertices involving this connection do not lead to ultraviolet divergences because of gauge invariance and power counting arguments similar to those of Sections III and IV.

It is necessary to introduce an infrared cutoff in order to define a meaningful perturbation theory. In order to ensure that the results of normal field theory and background field method coincide, it is safest to introduce an infrared cutoff before the breakup into background and quantum fields. On a general manifold the reparameterization invariant term

$$\frac{m}{2i} \int d^2x d^2\theta S(\Phi), \tag{6.15}$$

where $S(\Phi)$ is any scalar field on the manifold, presumably constructed from curvatures, e.g., $S = R_{ijkl}(\Phi)\, R^{ijkl}(\Phi)$ does introduce a mass scale. Depending on the geometry such terms can lead to a massive propagator in normal field theory calculations. However, such terms do not generate a well-defined massive propagator in the background field method. Therefore we introduce the noncovariant term

$$I_m = \frac{m}{2i} \int d^2x d^2\theta g_{ij}(\Phi)\Phi^i\Phi^j \tag{6.16}$$

as our infrared cutoff. We will find that the ultraviolet counterterms of dimension 1, which are of primary interest here, remain covariant, but we will obtain a counterterm structure of dimensionless operators generated by (6.16). Consistent incorporation of the one-loop counterterms of this type will be necessary to gain correct two-loop results for the invariant divergences. When (6.16) is expanded in normal coordinates, one finds the term $(m/2i)\, \delta_{ab}\xi^a\xi^b$ which combines with the kinetic term to define a massive propagator for the quantum field.

To warm up we start calculating the one-loop divergences for the supersymmetric σ-model. There is an invariant divergence coming from the background field expansion of $I_S[\Phi]$, whose value is

$$-i \log \Omega[\phi] = \frac{1}{4i} \int dz R_{iklj} g^{kl} \bar{D}\phi^i D\phi^j G(0) \tag{6.17}$$

with

$$G(0) = \int \frac{d^n k}{(2r)^k} \frac{1}{k^2 - m^2 + i\epsilon} \left(\frac{1}{2} \bar{D}D(k)\delta^2(\theta_1 - \theta_2)\right)_{\theta_1 - \theta_2}. \tag{6.18}$$

Using one of the identities listed in the Appendix one sees that the factor in parentheses is equal to i, and after Wick rotation we obtain

$$G(0) = I, \tag{6.19}$$

where I is defined in (3.5). In accord with the procedure of supersymmetric dimensional regularization all θ-integrations are performed in two dimensions, and the remaining algebra is carried out in n-dimensions with Tr $\gamma^\mu \gamma^\nu = 2g^{\mu\nu}$. The divergence above is compensated by the counterterm

$$\Delta I_0 = \frac{1}{8\pi\epsilon i} \int d^4z R_{ij}(\phi) \bar{D}\phi^i D\phi^j. \tag{6.20}$$

There is also a one-loop counterterm associated with the mass term (6.16) which is obtained from the bilinear terms in the normal coordinate expansion of (6.16). These terms are

$$I_m = \frac{m}{2!} \int d^4z \left(\frac{1}{3} R_{iklj}\xi^k\xi^l\phi^i\phi^j - \Gamma_{ikl}\xi^k\xi^l\phi^i\right) \tag{6.21}$$

The divergent parts come from a tadpole graph, and the corresponding counterterm is:

$$\Delta I_m = \frac{m}{12r\epsilon_i} \int d^4Z (R_{ij}\phi^i\phi^j - 3\Gamma_i\phi^i) \tag{6,22}$$

with $\Gamma_i \equiv \Gamma_{i,kl} g^{kl}$.

We now turn to the actual two-loop calculation. There are contributions from: (a) the invariant action expanded through fourth order in ξ, (b) the invariant counterterm (6.20) expanded to second order, (c) the one-loop mass counterterm (6.22) expanded to second order, and (d) the mass term (6.16) expanded to fourth order. The divergences due to (a) come from three graphs (see Figs. 4a, b, c) and contribute to $-i \log \Omega[\Phi]$ as follows:

$$\begin{aligned}
(4a) &= \frac{I^2}{12i} \int d^4z \left(\frac{1}{4} D^kD_kR_{ij} + \frac{1}{2} D_kD_lR_i{}^{kl}{}_j + R_i{}^kR_{k_j}\right. \\
&\quad \left. + \frac{3}{2} R_{iklm}R_j{}^{klm}\right) \bar{D}\phi^iD\phi^j, \\
(4b) &= -\frac{I^2}{8i} \int d^4z R_{iklm}R_j{}^{klm} \bar{D}\phi^i D\phi^j, \\
(4c) &= -\frac{1}{12i} \int d^4z R_{mn}R_i{}^{mn}{}_j \bar{D}\phi^i D\phi^j (I^2 - 4m^2IJ), \\
J &= \int \frac{d^nk_E}{(2r)^n} \frac{1}{(k_E{}^2 + m^2)^2}.
\end{aligned} \tag{6.23}$$

Note that J is a convergent integral at $n = 2$.

The divergent contributions due to (b) are given in Fig. 4d, e) and can be calculated from the bilinear terms of the normal coordinate expansion of (6.20), namely

$$\begin{aligned}
\Delta I_S |_{\text{bilinear}} = \frac{1}{8r\epsilon_i} \int d^4Z (&R_{ij}\hat{\bar{D}}\xi^i\hat{\bar{D}}\xi^j + 2D_kR_{ij}\bar{D}\phi^i\hat{D}\xi^j\xi^k \\
&+ \frac{1}{2} \xi^k\xi^l \bar{D}\phi^iD\phi^j(D_kD_lR_{ij} + 2R_{ikl}{}^mR_{mj}).
\end{aligned} \tag{6.24}$$

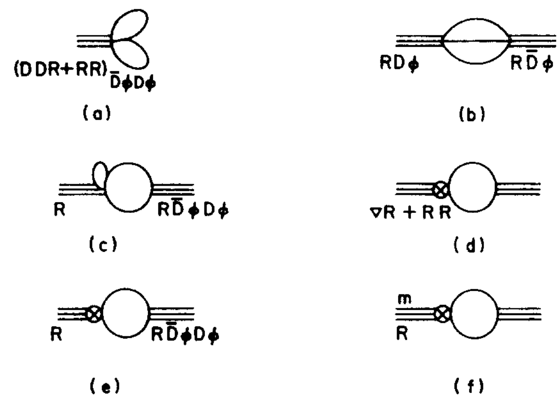

FIG. 4. Ultraviolet divergent two-loop supergraphs calculated in Section VI.

The results are given by

$$
\begin{aligned}
(4\mathrm{d}) &= \frac{1}{4\pi i\epsilon}\, I \int d^4z \left(\frac{1}{2}\, D^k D_k R_{ij} + R_{ik} R^k{}_j\right) \bar{D}\phi^i D\phi^j, \\
(4\mathrm{e}) &= -\frac{1}{8\pi i\epsilon}\,(I - 2m^2 J) \int d^4z R_k{}^l{}_j \bar{D}\phi^i D\phi^j.
\end{aligned}
\tag{6.25}
$$

The necessity for a systematic infrared cutoff can be seen in the contributions (4c–e) which formally contain mixed divergences of the form $(1/\epsilon)\log m^2$. These will eventually cancel with other contributions of the same type but a careful bookkeeping is necessary. Since these contributions vanish in the Ricci flat case a simpler infrared cutoff is adequate. Type (c) contributions lead to an ultraviolet divergence from the graph indicated in Fig. 4f. We keep only the covariant part coming from the replacement of $\phi^i\phi^j$ by $\xi^i\xi^j$. This gives the divergent contribution:

$$
(4\mathrm{f}) = -\frac{m^2}{24\pi i\epsilon}\, J \int dZ R_{iklj} R^{kl} \bar{D}\phi^i D\phi^j. \tag{6.26}
$$

We now sum all the contributions (4a–f) and obtain a pure double pole in ϵ:

$$
\sum (4\mathrm{a\text{–}f}) = -\frac{1}{4i}\left(\frac{1}{4(2r\epsilon)^2}\right) \int d^4Z (D^k D_k R_{ij} + 2[D^k, D_i]\, R_{kj})\, \bar{D}\phi^i D\phi^j. \tag{6.27}
$$

Therefore the corresponding counter term is (6.27) with opposite sign. "El momento de la verdad" has come and now we compare this with the prediction (5.11) of the renormalization group pole equations. We find complete agreement.

We now must consider possible remaining contributions of type (c) and (d). It is easy to see that none of these can lead to invariant counter terms of dimension one. Since the invariance of the leading operators of dimension one should not be altered by the addition of lower-order noncovariant operators such as the mass term, we see that noninvariant counter terms of dimension one must cancel among the various contributions. This is strictly true only on-shell, but in this section we have restricted the calculation to on-shell divergences, because we neglected vertices coming from the term proportional to the equations of motion in the definition of $\Omega[\phi]$.

Another argument that the remaining contributions of types (c) and (d) drop out is based on Hepp's theorem [23]: If we consider all the possible two-loop graphs which can be constructed with the noncovariant vertices or with the combination of covariant and noncovariant vertices, it is easy to see that the only graphs which contribute to operators of dimensions one are defined in terms of Feynman integrals whose overall degree of divergence is negative, but which contain subdivergences. Therefore by Hepp's theorem they cannot generate a two-loop counter term, and their divergences are cancelled by insertion of one-loop counter terms.

Appendix

Here we list some of the identities mentioned in Section VI, which were used in the evaluation of the two-loop graphs presented:

$$D_\alpha D_\beta \delta^2(\theta_1 - \theta_2)|_{\theta_1=\theta_2} = -i\gamma^0_{\alpha\beta}\,,$$

$$\tfrac{1}{2}\bar{D}D\delta^2(\theta_1 - \theta_2)|_{\theta_1=\theta_2} = i,$$

$$D_\alpha D_\beta \tfrac{1}{2}\bar{D}D\delta^2(\theta_1 - \theta_2)|_{\theta_1=\theta_2} = i(\not{k}\gamma^0)_{\alpha\beta}\,,$$

$$\tfrac{1}{2}\bar{D}DD_\alpha D_\beta \delta^2(\theta_{12})|_{\theta_1=\theta_2} = i(\not{k}\gamma^0)_{\alpha\beta}\,,$$

$$\tfrac{1}{2}D_\alpha \bar{D}DD_\beta \delta^2(\theta_{12})|_{\theta_1=\theta_2} = -i(\not{k}\gamma^0)_{\alpha\beta}\,,$$

$$(\tfrac{1}{2}\bar{D}D)^2\delta^2(\theta_1 - \theta_2)|_{\theta_1=\theta_2} = 0,$$

$$\tfrac{1}{2}\bar{D}DD_\alpha D_\beta \tfrac{1}{2}DD\delta^2(\theta_1 - \theta_2) = -ik^2\gamma^0_{\alpha\beta}\,,$$

$$D_\alpha(-k_1\theta_2)\delta^2(\theta_1 - \theta_2)|_{\theta_1=\theta_2} = -D_\alpha(k_1\theta_1)\delta^2(\theta_1 - \theta_2)|_{\theta_1=\theta_2}\,.$$

All these identities can be obtained by repeated use of the simple commutation relations:

$$[D_\alpha\,,\ \bar{D}D] = 4(\not{k}D)_\alpha\,,$$
$$\{D_\alpha(k),\ D_\beta(k)\} = 2(\not{k}\gamma^0)_{\alpha\beta}\,.$$

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