

Monopoles, affine algebras and the gluino condensate

N. Michael Davies

Department of Physics, University of Durham, Durham, DH1 3LE, United Kingdom

Timothy J. Hollowood

Department of Physics, University of Wales Swansea, Swansea, SA2 8PP, United Kingdom

Valentin V. Khoze

Department of Physics, University of Durham, Durham, DH1 3LE, United Kingdom

(Received 28 November 2002; accepted 9 April 2003)

We examine the low-energy dynamics of four-dimensional supersymmetric gauge theories and calculate the values of the gluino condensate for all simple gauge groups. By initially compactifying the theory on a cylinder we are able to perform calculations in a controlled weakly coupled way for a small radius. The dominant contributions to the path integral on the cylinder arise from magnetic monopoles which play the role of instanton constituents. We find that the semi-classically generated superpotential of the theory is the affine Toda potential for an associated twisted affine algebra. We determine the supersymmetric vacua and calculate the values of the gluino condensate. The number of supersymmetric vacua is equal to c_2 , the dual Coxeter number, and in each vacuum the monopoles carry a fraction $1/c_2$ of topological charge. As the results are independent of the radius of the circle, they are also valid in the strong coupling regime where the theory becomes decompactified. For gauge groups $SU(N)$, $SO(N)$ and $USp(2N)$ our results for the gluino condensate are in precise agreement with the “weak coupling instanton” expressions (and not with the “strong coupling instanton” calculations). For the exceptional gauge groups we calculate the values of the gluino condensate for the first time. © 2003 American Institute of Physics. [DOI: 10.1063/1.1586477]

I. INTRODUCTION AND SUMMARY OF RESULTS

The goal of this article is to provide new calculations of the values of the gluino condensate $\langle \text{tr} \lambda^2 \rangle$ in four-dimensional $\mathcal{N}=1$ supersymmetric Yang-Mills theory for all the simple gauge groups. Our approach is based on the idea¹ that when the space-time is partially compactified on the cylinder $\mathbb{R}^3 \times S^1$, magnetic monopoles play the role of instanton constituents, directly contribute to the path integral and completely determine the values of the gluino condensate. Our results for the gluino condensate are summarized in Table I and a universal formula in terms of Lie algebra data is given in (56). For classical gauge groups our results are in precise agreement with the known expressions derived in the “weak-coupling instanton” approach in Refs. 2–5. For the exceptional gauge groups the condensates are calculated for the first time.

It is somewhat of a miracle that some features of gauge theories which have supersymmetry can be understood exactly. Sometimes this success arises from viewing these theories as being embedded in string theory, a classical example being the duality of Maldacena.⁶ Generally, however, we can make use of the fact that supersymmetry leads to very restrictive Ward identities, giving rise to powerful holomorphy properties (see the review in Ref. 7). Regarding this later point, the full functional form of certain correlators is fixed up to an overall constant. Sometimes these correlators have a dependence on the couplings which can be identified with specific gauge theory configurations, in particular with instantons, but in other cases this is not so.⁸ In the former case, it is tempting to suppose that a semi-classical instanton calculation will yield the value of the correlator. In particular, we have in mind multi-point functions of the gluino operator $\text{tr} \lambda^2$ in $\mathcal{N}=1$ supersymmetric gauge theory. It is our thesis that one must be very careful in applying a semi-classical analysis to a strongly-coupled theory and such calculations will only be correct if

TABLE I. The values of the gluino condensate in the Pauli-Villars scheme (2).

Gauge group	$\langle \text{tr } \lambda^2 \rangle / (\Lambda^3 16\pi^2)$
$SU(N)$	1
$SO(N)$	$2^{4/(N-2)-1}$
$USp(2N)$	$2^{1-2/(N+1)}$
G_2	$2^{-1/2} 3^{1/4}$
F_4	$2^{-1/9} 3^{-1/3}$
E_6	$2^{-1/2} 3^{-1/4}$
E_7	$2^{-7/9} 3^{-1/3}$
E_8	$2^{-13/15} 3^{-2/5} 5^{-1/6}$

they are performed in a weakly-coupled phase, where semi-classical methods are rigorously justified. It is then possible to infer the value of the correlator in a strongly-coupled phase, if that phase is continuously connected to the weakly-coupled phase by using holomorphicity. It was the misuse of a semi-classical analysis directly in a strongly-coupled phase that led to the gluino condensate puzzle.

This famous puzzle is the inconsistency between two conceptually different approaches followed in the early literature of calculating the gluino condensate in pure $\mathcal{N}=1$ supersymmetric gauge theory. In the first methodology^{9–11}—and in the present context the “suspect” method, because it involves a semi-classical analysis directly in the strongly-coupled confining phase of the gauge theory—the so-called strong-coupling instanton (SCI) approach, the gluino condensate $\langle \text{tr } \lambda^2 \rangle$ is determined via an explicit one-instanton calculation of a certain multi-point function of $\text{tr } \lambda^2$. Cluster decomposition arguments are then invoked in order to extract the one-point function $\langle \text{tr } \lambda^2 \rangle$. In the second methodology,² the so-called weak-coupling instanton (WCI) approach—and for us the safe method—the calculation is performed with additional matter fields whose presence ensures that the non-Abelian gauge group is broken and the theory is in a weakly-coupled Higgs phase and a “constrained instanton” calculation is justified.¹² Holomorphicity is then used to decouple the matter fields and to flow continuously to the confining phase of the original gauge theory. As is well known, these two methods give two different values for the gluino condensate:^{2,13,11,5}

$$\left\langle \frac{\text{tr } \lambda^2}{16\pi^2} \right\rangle_{SU(N)} = \begin{cases} \frac{2}{[(N-1)!(3N-1)]^{1/N}} \Lambda^3 & \text{SCI,} \\ \Lambda^3 & \text{WCI.} \end{cases} \quad (1)$$

[These results are quoted for an $SU(N)$ gauge theory in the Pauli-Villars scheme with Λ being the corresponding dimensional transmutation scale of the theory defined in Eq. (2) below.] The reason for the discrepancy between the SCI and WCI calculations, as well as the question as to which is correct, has been a long-standing controversy.^{2,11,14,15} This controversy was reexamined in Ref. 16 using recently developed multi-instanton methods,^{17,18} for a comprehensive review see Ref. 19. By evaluating the k -instanton contribution to gluino correlation functions in the large number of colors limit it was shown that an essential step in the SCI calculation of the gluino condensate, namely the use of cluster decomposition in the instanton sector, is actually invalid. The central idea pursued in Ref. 1 and in the present paper is that there are additional configurations which contribute to the gluino condensate implying that the SCI calculation only gives part of the answer. The existence of other contributions to multi-point correlators of $\text{tr } \lambda^2$, which are non-instantonic invalidates the application of cluster decomposition to a purely instantonic contribution.

In Ref. 1 we provided an alternative way to deform the theory in order to connect the confining phase continuously to a weak-coupled phase: in this case a Coulomb rather than a Higgs phase. The idea is to consider the theory partially compactified on the cylinder $R^3 \times S^1$. In this scenario, the gauge field can have a non-trivial Wilson loop around the circle which acts like an

adjoint-valued Higgs field breaking the gauge group to its maximal Abelian subgroup and so the theory is in a Coulomb phase. For small enough radius, the resulting theory is arbitrarily weakly coupled and the gluino condensate can be reliably calculated. It is then argued, based on the usual argument of holomorphy, that the result is actually independent of the radius and is therefore easily extrapolated to the confining phase where the radius goes to infinity and the theory becomes decompactified.

However, there is a significant bonus in this scenario: the additional configurations missing in the instanton calculation can explicitly be identified at small radius with BPS monopoles in the gauge theory with the component of the gauge field around the circle playing the role of a Higgs field. However, we should point out that this in no way means that BPS monopoles quantitatively describe the physics in the decompactification limit. In this scenario, the one-point function $\langle \text{tr } \lambda^2 \rangle$ directly receives a semi-classical contribution from single monopoles, unlike the SCI situation in \mathbb{R}^4 , where, as described above, only multi-functions receive contributions. (There are precisely 2 gluino zero modes in the single-monopole background. These are supersymmetric fermion zero modes of the monopole. In distinction with instanton configurations on \mathbb{R}^4 , there are no, e.g., superconformal gluino zero modes in the monopole background. In fact, classical conformal invariance of the theory is explicitly broken by the compactification on the cylinder.) The monopoles consequently carry fractional topological charge. Hence, the theory on the cylinder uncovers a very pleasing picture of instanton constituents, or instanton partons, that were argued to play an important role in confinement of ordinary QCD.^{20–24} The fact that an instanton configuration on the cylinder is actually a composite of fundamental monopoles has been the subject of number of earlier works.^{25–30} These generalize the notion of a periodic instanton, or “caloron,”^{31–33} to the case when the gauge field has a non-trivial Wilson line around the circle. It is only in this case that the instanton constituents can be pulled apart and identified with monopoles. It turns out that in ordinary QCD the Wilson line of the gauge field around the circle is energetically favored to vanish: in this case the monopoles have no role to play in the physics. On the contrary, as we shall explicitly show, in $\mathcal{N}=1$ supersymmetric gauge theories, a non-trivial superpotential is generated by the monopoles whose supersymmetric vacua have a non-trivial value for the Wilson line and hence monopole effects are important. Other recent references which consider supersymmetric gauge theories on a cylinder and monopole effects are to be found in Refs. 34–36.

In $\mathcal{N}=1$ supersymmetric gauge theories, the first coefficient of the β -function is $b_0 = 3c_2$, where c_2 is the dual Coxeter number of the gauge group listed in the table in Appendix A. We will use a definition of the dynamical scale Λ in the Pauli–Villars renormalization scheme via the RG-invariant exact relation

$$\Lambda^3 = \mu^3 \frac{1}{g^2(\mu)} \exp \frac{2\pi i \tau(\mu)}{c_2}. \quad (2)$$

[If one chooses to use instead another exact definition of Λ ,¹⁵ more in line with standard QCD conventions, $\tilde{\Lambda}^3 = (\mu^3 16\pi^2 / 3c_2 g^2) \exp(-8\pi^2 / c_2 g^2)$, then the values of the gluino condensate in Table I have to be adjusted accordingly.] Here μ is the Pauli–Villars regulator mass and τ is the usual complexified coupling incorporating both the gauge coupling constant g and the theta angle ϑ :

$$\tau = \frac{\vartheta}{2\pi} + \frac{4\pi i}{g^2}. \quad (3)$$

The article is organized in the following way. In Sec. II we consider, in general terms, the effect of compactifying the $\mathcal{N}=1$ gauge theory on $\mathbb{R}^3 \times S^1$. In Sec. III we discuss the various semi-classical configurations that can contribute to the functional integral and explain the relation between monopoles and instantons on the cylinder. Regarding this point, our considerations in this article are purely field-theoretical; for an elegant D-brane discussion of the $SU(N)$ dynamics on $\mathbb{R}^3 \times S^1$ see Refs. 1 and 25. In Sec. IV we derive the form of the superpotential in the low energy

TABLE II. The associated affine algebra.

Gauge group	Lie algebra	Affine Toda potential
$G(\text{Simply-laced})$	g	$g^{(1)}$
$SO(2r+1)$	b_r	$a_{2r-1}^{(2)}$
$USp(2r)$	c_r	$d_{r+1}^{(2)}$
G_2	g_2	$d_4^{(3)}$
F_4	f_4	$e_6^{(2)}$

effective three-dimensional theory generated by monopoles (46). It turns out that this potential is precisely the affine Toda potential for a specific affine algebra. For the simply-laced cases the affine algebra is the untwisted affinization of the Lie algebra of the gauge group while for the non-simply-laced cases it is affine algebra whose Kac–Dynkin diagram is obtained from the untwisted affine diagram with long roots changed to short roots, and vice versa. The affine algebras (in Kac’s notation³⁷) are listed in Table II. The Toda potential is in complete agreement with M(F)-theory considerations,^{38–40} although we will find some additional prefactors that feed into the calculation of the gluino condensate in an essential way. From the superpotential, we find that the number of supersymmetric vacua is equal to dual Coxeter number of the gauge group in complete agreement with the Witten index.⁴¹ The values of the gluino condensate in each vacuum are then found and the results are summarized in Table I. For all classical groups these are in agreement with the WCI results of Refs. 2–5. In Appendix A we summarize our Lie algebra conventions and Appendix B contains a brief discussion of the measure needed for integrating over the collective coordinate space of fundamental monopoles.

II. $\mathcal{N}=1$ GAUGE THEORY ON THE CYLINDER

In this section, we consider the effect of compactifying the pure $\mathcal{N}=1$ gauge theory on a cylinder of radius R . To this end, let us take x_0 to be periodic with period $2\pi R$. We then impose periodic boundary conditions on the gauge field and gluino:

$$v_m(x_\mu, x_0) = v_m(x_\mu, x_0 + 2\pi R), \quad \lambda(x_\mu, x_0) = \lambda(x_\mu, x_0 + 2\pi R). \quad (4)$$

(In our notation the four-dimensional indices run over $m, n, \dots = 0, 1, 2, 3$ while our three-dimensional indices run over $\mu, \nu, \dots = 1, 2, 3$.) Notice that the periodicity of the fermions preserves supersymmetry.

Smooth finite-action gauge fields on the cylinder were classified in Ref. 33. In particular, at finite radius instanton configurations do not exhaust the set of semi-classical contributions. The complete set of semi-classical configurations is characterized by three pieces of data. First, there is a topological or instanton charge (or second Chern class) generalized from \mathbb{R}^4 to the cylinder:

$$k = \frac{1}{16\pi^2} \int_{\mathbb{R}^3 \times S^1} d^4x \, \text{tr} v_{mn} * v^{mn}. \quad (5)$$

An important feature of the cylinder is that k is not quantized in integer units. However, when k is an integer there are solutions with action $S = 8\pi^2 k / g^2 - ik \vartheta$ that, for scale size much smaller than R , are identifiable as instantons of the uncompactified theory. The second piece of data involves the Wilson loop of the gauge field around the circle:

$$\oint_{S^1} dx_m v_m = \int_0^{2\pi R} dx_0 v_0 \equiv \varphi. \quad (6)$$

We will then define the VEV of φ as the asymptotic value at spatial infinity in \mathbb{R}^3 :

$$\langle \varphi \rangle = \lim_{|x_\mu| \rightarrow \infty} \varphi \cdot \mathbf{H}, \quad (7)$$

where we have fixed a portion of the global gauge symmetry by choosing the Wilson loop (7) to lie within the Cartan subalgebra of the Lie algebra g associated to the gauge group G . (Our Lie algebra conventions are summarized in Appendix A. We will denote $r = \text{rank } G$ vectors in bold-face.) This still leaves the freedom to perform global gauge transformations from the Weyl group W_g of G .

A nonzero value for $\langle \varphi \rangle$ acts as an adjoint-valued Higgs field that generically breaks the gauge group to its maximal Abelian subalgebra $U(1)^r$. The classical moduli space \mathcal{M}_{cl} , parametrized by the vector $\langle \varphi \rangle$, is the quotient

$$\mathcal{M}_{\text{cl}} = \frac{\mathbb{R}^r}{2\pi \cdot \Lambda_W^* \rtimes W_g}, \quad (8)$$

where Λ_W^* is the co-weight lattice. The form of the quotient is explained in the following way: we have already noted that fixing $\langle \varphi \rangle$ to be in the Cartan subalgebra leaves the freedom to perform global gauge transformations in the Weyl group. On top of this, theories with $\langle \varphi \rangle$ differing by 2π times any co-weight vector are equivalent. To see this last point, consider the following topologically nontrivial gauge transformation

$$U(x_0) = \exp\left(\frac{ix_0}{R} \boldsymbol{\omega}^* \cdot \mathbf{H}\right), \quad (9)$$

for any co-weight $\boldsymbol{\omega}^* \in \Lambda_W^*$. This transforms the component of the gauge field around the circle as $v_0 \rightarrow v_0 + \boldsymbol{\omega}^* \cdot \mathbf{H}/R$, and hence $\langle \varphi \rangle \rightarrow \langle \varphi \rangle + 2\pi \boldsymbol{\omega}^*$. The transformation (9) is periodic in the adjoint representation of the gauge group and consequently in the pure gauge theory where all fields are adjoint-valued $\langle \varphi \rangle$ is identified with $\langle \varphi \rangle + 2\pi \boldsymbol{\omega}^*$. (Because $\boldsymbol{\alpha} \cdot \boldsymbol{\omega}^* \in \mathbb{Z}$ for any root $\boldsymbol{\alpha}$ and co-weight $\boldsymbol{\omega}^*$.)

We will find it convenient to choose the VEV $\langle \varphi \rangle$ to lie in a “fundamental cell”

$$0 \leq \langle \varphi \rangle \cdot \boldsymbol{\alpha}_i < 2\pi, \quad i = 1, \dots, r, \quad (10)$$

where $\boldsymbol{\alpha}_i$ are the simple roots of g . [Notice that this region is still an over-parametrization of the quotient (8).] The regions where $\langle \varphi \rangle \cdot \boldsymbol{\alpha}_i = 0$, for some set of i 's, correspond to submanifolds of \mathcal{M}_{cl} where a non-Abelian subgroup of the gauge symmetry is restored.

The final piece of data arises from the fact that finite action configurations can also carry three-dimensional magnetic charge. This is an r -vector-valued quantity \mathbf{g} in the charge space of the unbroken $U(1)^r$ Abelian symmetry that can be defined via a surface integral over the two-sphere at spatial infinity in \mathbb{R}^3 of the magnetic field $B_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho} v_{\nu\rho}$:

$$-\frac{1}{2\pi} \int_{S^2} dS_\mu B_\mu \equiv \mathbf{g} \cdot \mathbf{H}. \quad (11)$$

The magnetic charge is subject to the usual generalization of the Dirac quantization rule^{42,43} which requires that

$$\mathbf{g} \in \Lambda_R^*, \quad (12)$$

the co-root lattice of g .

Classically, the Wilson loop $\langle \varphi \rangle$ is not determined and, so, as we have explained, there is a moduli space \mathcal{M}_{cl} of inequivalent theories. An important question is whether this classical degeneracy persists in the quantum theory. At this point, the behavior depends crucially on whether one has periodic or anti-periodic boundary conditions on the fermions. In the latter—thermal—case,

Ref. 33 argued that nontrivial values of the asymptotic Wilson loop (7) are suppressed in the infinite volume limit. Consequently, the classically flat directions are lifted by thermal quantum corrections and the true vacuum of the theory is $\langle \varphi \rangle = 0$. In this case, the configurations with magnetic charges are not relevant, since they require nonvanishing VEV, and the semi-classical physics is described by instantons only. Remarkably, for the theory on the cylinder, with periodic boundary conditions on the fermions, the argument of Ref. 33 does not apply and, as we shall see in the following sections, the opposite scenario ensues; namely, the following.

- (i) The semi-classical physics of the theory on the cylinder is described by configurations of BPS monopoles. There are $r+1$ types of “fundamental” monopole which carry only four bosonic and two (adjoint) fermionic zero modes. To those who are sufficiently initiated into monopole calculus in gauge theories with arbitrary gauge group, this will be a surprise: one would expect to have only r such monopoles (each with a magnetic charge equal to one of the r simple roots). The additional monopole, needed to make up the full complement of $r+1$ types, is specific to the compactification on the cylinder since, unlike the other, it is a nontrivial function of “time” x_0 .^{25,26,29,30} The magnetic charge of the new monopole is such that when all $r+1$ types of monopoles are present with a specific degeneracy, the magnetic charges cancel and the resulting configuration carries only a unit of instanton charge. Hence, remarkably, instantons on the cylinder can be understood as composite configurations of monopoles.^{25–30}
- (ii) The classical moduli space of the gauge theory on the cylinder (6) is lifted in the quantum theory in a nontrivial way. The quantum vacua correspond to a single point in \mathcal{M}_{cl} cell along with an additional c_2 -fold degeneracy, which has no counterpart in the classical theory, and corresponds precisely to the expectations based on a refined Witten index⁴¹ and the WCI counting.^{44,4}

III. SEMI-CLASSICAL CONFIGURATIONS

In the weak-coupling limit, the path integral is dominated by field configurations which are of minimal action in each topological sector. These configurations satisfy the four-dimensional self-dual, or anti-self-dual, equations $v_{mn} = \pm *v_{mn}$. As we have explained there are two quantum numbers carried by semi-classical configurations: the topological charge and the magnetic charge.

First of all, let us consider solutions which are independent of the coordinate around the circle x_0 . These are simply BPS monopoles in the three-dimensional theory^{45–48} with the time direction taken to be along x_0 . Monopole solutions in a gauge theory with a simple gauge group G can in turn be constructed out of the $SU(2)$ BPS monopole in the following way.⁴³ The idea is to take a regular embedding $SU(2) \subset G$, associated to a positive root α of G :

$$\tau^1 = \frac{1}{2}(E_\alpha + E_{-\alpha}), \quad \tau^2 = \frac{1}{2i}(E_\alpha - E_{-\alpha}), \quad \tau^3 = \frac{1}{2}\alpha^* \cdot \mathbf{H}, \quad (13)$$

which obey the $SU(2)$ algebra

$$[\tau^a, \tau^b] = i\epsilon_{abc}\tau^c. \quad (14)$$

(Here, $\alpha^* = 2\alpha/\alpha^2$ is the co-root associated to α .) The monopole solution is then

$$v_0(x_\nu) = \Phi^c(v; x_\nu)\tau^c + \frac{1}{2\pi R} \left(\langle \varphi \rangle - \frac{1}{2}(\langle \varphi \rangle \cdot \alpha)\alpha^* \right) \cdot \mathbf{H}, \quad v_\mu(x_\nu) = v_\mu^c(v; x_\nu)\tau^c, \quad (15)$$

where $\Phi^c(v; x_\nu)$ is the Higgs field and $v_\mu^c(v; x_\nu)$ are the spatial components of the gauge field (in the gauge $v_0=0$) of the $SU(2)$ BPS monopole. The long distance behavior of this solution is

$$\lim_{|x_\mu| \rightarrow \infty} \Phi^c(v; x_\nu)\tau^c = \frac{v}{2}\alpha^* \cdot \mathbf{H}, \quad (16)$$

where

$$v = \frac{\alpha \cdot \langle \varphi \rangle}{2\pi R}. \quad (17)$$

For this solution to be well defined, we must have $v > 0$, which is automatic if α is a positive root and $\langle \varphi \rangle$ lies in the fundamental cell (10), in which case it has magnetic charge, topological charge and action given by

$$\mathbf{g} = \alpha^*, \quad k = \alpha^* \cdot \frac{\langle \varphi \rangle}{2\pi}, \quad S = \frac{4\pi}{g^2} \alpha^* \cdot \langle \varphi \rangle. \quad (18)$$

For completeness, we give the explicit solution for the SU(2) BPS monopole in “hedgehog” gauge,

$$v_\mu^c(v; x_\nu) = \epsilon_{\mu\nu c} \frac{x_\nu}{|x|^2} \left(1 - \frac{v|x|}{\sinh v|x|} \right), \quad (19a)$$

$$\Phi^c(v; x_\nu) = \frac{x_c}{|x|^2} (v|x| \coth v|x| - 1). \quad (19b)$$

The asymptotic value of the magnetic field of the hedgehog solution, as $|x| \rightarrow \infty$, is

$$B_\mu^c \rightarrow -\frac{x_\mu x^c}{|x|^4}, \quad (20)$$

while in unitary gauge

$$B_\mu^c \rightarrow -\frac{x_\mu}{|x|^3} \delta^{c3}, \quad B_\mu \equiv B_\mu^c \tau^c \rightarrow -\frac{x_\mu}{2|x|^3} \alpha^* \cdot H. \quad (21)$$

However, these x_0 -independent solutions do not exhaust the set of solutions with a given magnetic charge α^* .²⁵ A whole tower of other solutions which are x_0 dependent can be generated in the following way. First of all, we start with the solution (15) with $\langle \varphi \rangle$ lying in the fundamental cell (10). We then write down the same solution with a shifted VEV $\langle \varphi' \rangle = \langle \varphi \rangle + \pi n \alpha^*$, where $n \in \mathbb{Z}$. For this solution to be well defined we must have

$$v' = \frac{\alpha \cdot \langle \varphi' \rangle}{2\pi R} = \frac{\alpha \cdot \langle \varphi \rangle}{2\pi R} + \frac{n}{R} > 0. \quad (22)$$

For $\alpha = \alpha_i$, a simple root, (10) implies that $n \geq 0$. Acting on the solution with the (nonperiodic) gauge transformation,

$$V_n(x_0) = \exp\left(\frac{inx_0}{2R} \alpha^* \cdot \mathbf{H}\right), \quad (23)$$

has the effect of restoring the VEV $\langle \varphi \rangle$ to its original value. The new solution is then given by

$$v_0(x_\nu) = \Phi^c(v + n/R; x_\nu) \tilde{\tau}^c + \frac{1}{2\pi R} \left(\langle \varphi \rangle - \frac{1}{2} (\langle \varphi \rangle \cdot \alpha + 2\pi n) \alpha^* \right) \cdot \mathbf{H}, \quad (24)$$

$$v_\mu(x_\nu) = v_\mu^c(v + n/R; x_\nu) \tilde{\tau}^c,$$

where v is given as in (17) and the SU(2) generators are conjugated with $V_n(x_0)$:

$$\tilde{\tau}^c = V_n(x_0) \tau^c V_n(x_0)^{-1}. \quad (25)$$

Notice although $V_n(x_0)$ is not a periodic gauge transformation the generators $\tilde{\tau}^c$ are periodic functions of x_0 . The solution (24) has the same magnetic charge as (15), but the topological charge is $k = \alpha^* \cdot \langle \varphi \rangle / 2\pi + n$. This solution can be interpreted as a composite configuration of the original monopole plus an instanton of charge n .

However, there are also towers of solutions of the self-dual equations that have a magnetic charge equal to some *negative* root.²⁵ We should emphasize that these solutions are *not* anti-monopoles which would satisfy the *anti*-self-dual equations. To construct these solutions we can start with our solution (15) with $\langle \varphi \rangle$ lying in the fundamental cell. We now define a new solution with a VEV $\langle \varphi' \rangle = \kappa_\alpha(\langle \varphi \rangle) + \pi n \alpha^*$, where κ_α is the Weyl reflection in α . For the solution to be well defined we must have

$$v' = \frac{\alpha \cdot \langle \varphi' \rangle}{2\pi R} = -\frac{\alpha \cdot \langle \varphi \rangle}{2\pi R} + \frac{n}{R} > 0. \quad (26)$$

For $\alpha = \alpha_i$, a simple root, this means $n > 0$. To reinstall the original VEV, we then perform a Weyl reflection in α and the gauge transformation (23). The resulting solution is

$$v_0(x_\nu) = \Phi^c(n/R - v; x_\nu) \tilde{\tau}^c + \frac{1}{2\pi R} \left(\langle \varphi \rangle - \frac{1}{2} (-\langle \varphi \rangle \cdot \alpha + 2\pi n) \alpha^* \right) \cdot \mathbf{H},$$

$$v_\mu(x_\nu) = v_\mu^c(n/R - v; x_\nu) \tilde{\tau}^c, \quad (27)$$

where v is given in (17) and the SU(2) generators are now conjugated with $V_n(x_0) \kappa_\alpha$:

$$\tilde{\tau}^c = V_n(x_0) \kappa_\alpha \tau^c \kappa_\alpha V_n(x_0)^{-1}. \quad (28)$$

It can be easily verified that this solution is again periodic in x_0 . The resulting solution has magnetic charge $-\alpha^*$ and topological charge $k = -\alpha^* \cdot \langle \varphi \rangle / 2\pi + n$.

It will be important for later to determine the number of adjoint fermion, or gluino, zero modes of these monopole solutions. Each classical solution has at least two adjoint fermion zero modes protected by supersymmetry. These modes can be generated from the purely bosonic solution by acting with the generators of supersymmetry that do not leave the configuration invariant. This gives the universal expression for these supersymmetric modes

$$\lambda_\alpha = \sigma_\alpha^{mn\beta} \xi_\beta v_{mn}, \quad (29)$$

where v_{mn} is the field strength. For future reference we give the long-distance behavior of the supersymmetric fermion zero modes (29) of our fundamental monopole solutions (15) and (27) with $n = 1$:

$$\lambda_\alpha = \sigma_\alpha^{mn\beta} \xi_\beta v_{mn} = -2(\sigma^\nu \xi)_\alpha B_\nu \rightarrow 4\pi(\mathcal{S}_F \xi)_\alpha \alpha^* \cdot \mathbf{H}, \quad (30)$$

where $\mathcal{S}_F(x) = \sigma_\mu x_\mu / (4\pi |x_\mu|^3)$ is the massless fermion propagator in three dimensions.

Solutions with only the supersymmetric zero modes have four associated bosonic zero modes which correspond to moving the center-of-mass of the monopole in \mathbb{R}^3 as well as performing global gauge rotations by $\exp((i/2) \Omega \alpha^* \cdot \mathbf{H})$. Hence these solutions are special in that they are elementary or “fundamental”: the other solutions have additional moduli that correspond to pulling the configuration apart into their fundamental constituents.

As might have been expected there are r solutions of the form (15) where α is a simple root α_i lying at the bottom of the more general tower of solutions (24). This gives us $i = 1, \dots, r$ fundamental monopole solutions with two adjoint-valued fermion zero modes, magnetic charge α_i^* , and the topological charge $k = \alpha_i^* \cdot \langle \varphi \rangle / 2\pi$. Solutions higher in the tower, with $n > 0$, have

$2(1 + nc_2)$ fermion zero modes,^{25,26} as we expect for a configuration of a fundamental monopole and n instantons. In addition to these r fundamental monopoles, there is one other solution that is fundamental.^{25,26} This is a solution which has a negative magnetic charge equal to the lowest root $\alpha_0^* \equiv \alpha_0$ (although the solution, as we explained above is *not* an anti-monopole) lying in the second tower (27) with $n = 1$ and hence with topological charge $k = -\alpha_0^* \cdot \langle \varphi \rangle / 2\pi + 1$.

Since $\sum_{i=0}^r k_i^* \alpha_i^* = 0$, the quantum numbers of the solutions suggest that a pure instanton solution, carrying zero magnetic charge and unit topological charge, is a composite configuration with k_i^* fundamental α_i monopoles, for each $i = 0, \dots, r$. (Here k_i^* are the dual “Kac labels,” or co-marks, defined in Appendix A.) This turns out to be the case^{25,26} and the resulting configuration has exactly $2c_2$ ($4c_2$) exact fermionic (bosonic) zero modes as expected for a singly-charged instanton with gauge group G .

IV. MONOPOLE CONTRIBUTIONS TO THE SUPERPOTENTIAL

In this section, we will explain how the fundamental monopoles described in the last section lift the classical degeneracy of the theory parametrized by the asymptotic value of the Wilson loop $\langle \varphi \rangle$ (7). The idea is to consider the low energy three-dimensional effective theory corresponding to the massless Abelian components of the fields formed by integrating out all the massive fields.

For this analysis to hold we must first assume there is no root α such that $\langle \varphi \rangle \cdot \alpha = 0$, so that the unbroken gauge group is maximally Abelian $U(1)^r$. We will also assume that the Wilson line VEV $\langle \varphi \rangle$ lies in the fundamental region (10). After that we can integrate out (1) all non-Abelian fields on $\mathbb{R}^3 \times S^1$, and (2) all the massive Kaluza–Klein modes on S^1 , i.e., the modes with nonzero Matsubara frequency $\omega_m = m/R$, to flow to the Abelian theory on \mathbb{R}^3 . We emphasize that the periodicity in $\langle \varphi \rangle \sim \langle \varphi \rangle + 2\pi \omega^*$, $\omega^* \in \Lambda_W^*$, is a property of the full microscopic theory but not of the low-energy theory on \mathbb{R}^3 . Indeed, the large gauge transformation (9) is x_0 -dependent and has the effect of mixing up the massless and massive Kaluza–Klein modes.

The fields of the low energy theory consist of the Wilson loop φ , i.e., the component v_0 of the gauge field averaged over the circle, along with r massless photons corresponding to the components of v_μ in the Cartan subalgebra of the gauge group. Along with these bosonic fields there are superpartners corresponding to the Abelian components of the gluino.

It turns out to be convenient to use the fact that massless Abelian gauge fields in three dimensions can be eliminated in favor of scalar fields by a duality transformation. To construct the classical effective action, we start with the action of the pure gauge theory in four dimensions and dimensionally reduce to three dimensions keeping only the Abelian components of the fields. From (6), the component v_0 of the four-dimensional gauge field is replaced by $\varphi \cdot \mathbf{H} / (2\pi R)$ and the resulting three-dimensional effective action is

$$S_{\text{cl}} = \frac{2\pi R}{g^2} \int d^3x \left\{ \frac{1}{4\pi^2 R^2} (\partial_\mu \varphi)^2 - \frac{1}{2} (\mathbf{v}_{\mu\nu})^2 + 2i\bar{\lambda} \cdot \bar{\sigma}_\mu \mathcal{D}_\mu \lambda \right\} - \frac{i\vartheta}{8\pi^2} \int d^3x \epsilon_{\mu\nu\rho} \partial_\mu \varphi \cdot \mathbf{v}_{\nu\rho}. \quad (31)$$

[It is useful to notice that in our normalization $\text{tr}(\mathbf{a} \cdot \mathbf{H} \mathbf{b} \cdot \mathbf{H}) = \mathbf{a} \cdot \mathbf{b}$.] In order to construct the dual description of the three-dimensional gauge field one adds a new term to the action involving a field σ which serves as a Lagrange multiplier for the Bianchi identity constraint:

$$S_{\text{cst}} = -\frac{i}{4\pi} \int d^3x \epsilon_{\mu\nu\rho} \partial_\mu \sigma \cdot \mathbf{v}_{\nu\rho} = -\frac{i}{2\pi} \int_{S^2} dx_\mu \sigma \cdot \mathbf{B}_\mu. \quad (32)$$

The Abelian field strength $\mathbf{v}_{\mu\nu}$ can now be integrated-out of the path integral as a Gaussian field to obtain the classical effective action, whose bosonic part is

$$S_{\text{cl}}^{\text{bos}} = \frac{1}{2\pi R} \int d^3x \left\{ \frac{1}{g^2} (\partial_\mu \varphi)^2 + \frac{g^2}{16\pi^2} \left(\partial_\mu \sigma + \frac{\vartheta}{2\pi} \partial_\mu \varphi \right)^2 \right\}. \quad (33)$$

This can be written compactly in terms of the single complex field

$$\mathbf{z} = i(\tau\boldsymbol{\varphi} + \boldsymbol{\sigma}), \quad (34)$$

as

$$S_{\text{cl}}^{\text{bos}} = \frac{1}{8\pi^2 R} \int d^3x \frac{1}{\text{Im } \tau} \partial^\mu \mathbf{z}^+ \cdot \partial_\mu \mathbf{z}. \quad (35)$$

We have eliminated the r massless photons in favor of an r -vector scalar field $\boldsymbol{\sigma}$. Notice that since the magnetic charge \mathbf{g} is quantized in the co-root lattice it follows from (32) that $\boldsymbol{\sigma}$ is physically equivalent to $\boldsymbol{\sigma} + 2\pi\boldsymbol{\omega}$ for any weight $\boldsymbol{\omega} \in \Lambda_W$. Once again, we also have the freedom to perform Weyl reflections and so $\boldsymbol{\sigma}$ is valued in the quotient

$$\frac{\mathbb{R}^r}{2\pi \cdot \Lambda_W \rtimes W_g}, \quad (36)$$

to compare with $\boldsymbol{\varphi}$ which is valued in the slightly different quotient (8). Obviously these spaces are the same for the simply-laced groups.

The fact that both (real) scalar fields $\boldsymbol{\varphi}$ and $\boldsymbol{\sigma}$ can be amalgamated into a single complex field \mathbf{z} is no coincidence. Since the original four-dimensional theory was $\mathcal{N}=1$ supersymmetric, the effective theory written in terms of the bosonic fields $\boldsymbol{\varphi}$ and $\boldsymbol{\sigma}$, along with the Abelian components of the gluino $\boldsymbol{\lambda}_\alpha$, must form a representation of four-dimensional $\mathcal{N}=1$ supersymmetry (corresponding to $\mathcal{N}=2$ in three dimensions) which must be a chiral superfield since we have taken the dual of all the vector fields. In particular the bosonic fields must be expressible in terms of a single complex field as we have found in (34).

The $\mathcal{N}=2$ (in 3D) supersymmetric completion of the classical action (35) can be most succinctly expressed in terms of a four-dimensional $\mathcal{N}=1$ chiral superfield \mathbf{X} with scalar component \mathbf{z} and fermionic component $\boldsymbol{\lambda}_\alpha$, the Abelian component of the gluino. The supersymmetric version of (35) written in superspace is then

$$S_{\text{cl}} = \frac{1}{8\pi^2 R} \int d^3x \frac{1}{\text{Im } \tau} \mathbf{X}^+ \cdot \mathbf{X} \big|_{\theta\theta\bar{\theta}\bar{\theta}}. \quad (37)$$

Quantum effects can modify the classical expression (37). However, modifications must preserve supersymmetry. As long as we are at a generic point in the classical moduli space, we expect to be able to integrate out all the massive fields to be left with an effective theory in terms of the superfield \mathbf{X} . The most general possible low energy effective action, i.e., involving at most two derivatives or four fermions, and preserving $\mathcal{N}=2$ supersymmetry in three dimensions, is

$$S_{\text{eff}} = \int d^3x \{ \mathcal{K}(\mathbf{X}, \mathbf{X}^+) \big|_{\theta\theta\bar{\theta}\bar{\theta}} + \mathcal{W}(\mathbf{X}) \big|_{\theta\theta} + \bar{\mathcal{W}}(\mathbf{X}^+) \big|_{\bar{\theta}\bar{\theta}} \}, \quad (38)$$

which involves an arbitrary D -term $\mathcal{K}(\mathbf{X}, \mathbf{X}^+)$ as well as a superpotential $\mathcal{W}(\mathbf{X})$. It is the superpotential that is responsible for lifting the classical degeneracy and which we must determine.

In the classical theory (37) the superpotential vanishes identically. Quantum corrections will modify the theory in a complicated way depending on the couplings. However, the superpotential, by the standard arguments,^{49,7,34} must be holomorphic in the fields \mathbf{X} and the complexified coupling τ . In particular, up to the overall factor, the superpotential can only depend on R through the running of τ via the dimensionless quantity $R|\Lambda|$, where Λ is the usual Pauli–Villars scale of strong coupling effects in the pure gauge theory in \mathbb{R}^4 . [This appears when the fields are not canonically normalized. In our case scalar fields arise from the Wilson line and the dual photon, which are dimensionless. This leads to an overall factor of R/g^2 in Eq. (46) below.] We intend to compute the superpotential at weak-coupling, for which $R \ll |\Lambda|^{-1}$ and the VEV of the effective

Higgs field (17) is large and a semi-classical analysis should be reliable. In this regime the superpotential will receive contributions from the minimal action configurations in each topological sector which have exactly two gluino zero modes; in other words, from the $r+1$ fundamental monopoles described in the last section. As usual holomorphy then forbids any perturbative corrections to the semi-classical contributions and, as a consequence, fixes the R dependence, a fact that ultimately will allow us to take R to be large.

In the presence of the dual photon field σ , the action of the fundamental monopole associated to the root α_j , $j=0, \dots, r$, is given in terms of the VEV of the scalar field z by

$$S_j = -2\pi i \tau \delta_{j0} - i \tau \alpha_j^* \cdot \langle \varphi \rangle - i \alpha_j^* \cdot \langle \sigma \rangle \equiv -2\pi i \tau \delta_{j0} - \alpha_j^* \cdot \langle \mathbf{z} \rangle. \quad (39)$$

Here τ is the complexified coupling (3).

We determine the form of the superpotential by calculating the monopole contribution to the large distance behavior of the correlator of two components of the massless gluino field

$$\langle \lambda_\alpha(x) \otimes \lambda_\beta(0) \rangle. \quad (40)$$

In the background of the α_j monopole, only the component $\lambda_\alpha \propto \alpha_j$ is nontrivial; in fact from (30) one finds the long-distance behavior to be

$$\lambda_\alpha^{LD}(x) = 4\pi \alpha_j^* S_F(x-a) \gamma_\alpha \xi_\gamma, \quad (41)$$

where $S_F(x) = \sigma_\mu x_\mu / (4\pi |x_\mu|^3)$ is the massless fermion propagator in three dimensions, a_μ is the position of the monopole in \mathbb{R}^3 and ξ_α are the Grassmann collective coordinates corresponding to the two supersymmetric zero modes.

In order to evaluate the contribution to the superpotential from the monopole, we need the measure for integrating over the moduli space of the monopole derived in Appendix B. A fundamental monopole has a moduli space that is parametrized by a_μ , the position in \mathbb{R}^3 and by the $U(1)$ phase angle $0 \leq \Omega \leq 2\pi$. Along with this, there are two Grassmann collective coordinates ξ_α , corresponding to the two supersymmetric zero modes. From Eq. (79) the measure is

$$\int d\mu_{\text{mon}}^{(j)} = \frac{2}{\alpha_j^2} \frac{\mu^3 R}{g^2} e^{-S_j} \int d^3 a d\Omega d^2 \xi. \quad (42)$$

Performing the integrals over the phase angle and the Grassmann collective coordinates, we find that

$$\langle \lambda_\alpha(x) \otimes \lambda_\beta(0) \rangle = \frac{2^6 \pi^3 \mu^3 R}{g^2 \alpha_j^2} \alpha_j^* \otimes \alpha_j^* e^{2\pi i \tau \delta_{j0} + \alpha_j^* \cdot \langle \mathbf{z} \rangle} \int d^3 a S_F(x-a) \gamma_\alpha S_F(a) \beta_\gamma. \quad (43)$$

Amputating this correlator we find the associated vertex in the effective action:

$$\left(\frac{2\pi R}{g^2} \right)^2 \frac{2^5 \pi^3 \mu^3 R}{g^2 \alpha_j^2} e^{2\pi i \tau \delta_{j0} + \alpha_j^* \cdot \langle \mathbf{z} \rangle} (\alpha_j^* \cdot \lambda)^2. \quad (44)$$

In the above, the numerical factor in the bracket reflects our normalization for the kinetic term of $\alpha \cdot \lambda$ which follows from (31). The vertex (44) is generated by a term in the effective potential of the form

$$\frac{4\pi \mu^3 R}{g^2 \alpha_j^2} e^{2\pi i \tau \delta_{j0} + \alpha_j^* \cdot \mathbf{X}}. \quad (45)$$

[In order to get the correct numerical factor, notice that the fermionic component of \mathbf{z} and the gluino λ are related via $\psi = 2^{5/2} \pi^2 g^{-2} R \lambda$. This follows from the fact⁵⁰ that the superpartner of \mathbf{v}_0 is $(\lambda + \bar{\lambda})/\sqrt{2}$.] Hence, summing over the effects of all $r+1$ fundamental monopoles we deduce that the monopole-generated superpotential of the theory is

$$\mathcal{W}_{\text{mono}}(\mathbf{X}) = \frac{2\pi\mu^3 R}{g^2} \left(\sum_{j=1}^r \frac{2}{\alpha_j^2} e^{\alpha_j^* \cdot \mathbf{X}} + \frac{2}{\alpha_0^2} e^{2\pi i \tau + \alpha_0^* \cdot \mathbf{X}} \right). \quad (46)$$

This is an affine Toda potential for an associated affine algebra. Notice that to give the usual expression for a Toda potential one can remove the pre-factors $2/\alpha_j^2$ by a shift in the field:

$$\mathbf{X} \rightarrow \mathbf{X} + \sum_{j=1}^r \ln(\alpha_j^2/2) \boldsymbol{\omega}_j + \frac{\boldsymbol{\rho}}{c_2} \left(2\pi i \tau - \sum_{j=0}^r \ln(\alpha_j^2/2) \right), \quad (47)$$

where $\boldsymbol{\rho} = \sum_{j=1}^r \boldsymbol{\omega}_j$ is the Weyl vector. For the simply-laced groups, the associated affine algebra is the untwisted affinization of the original Lie algebra, $g^{(1)}$ in Kac's notation,³⁷ while for the non-simply-laced groups the corresponding affine algebra is twisted in the way described in Table II. In these cases, the Kac–Dynkin diagram of the affine algebra is obtained from the Kac–Dynkin diagram of the untwisted affinization $g^{(1)}$ by changing long roots into short roots, and vice versa. In Kac's notation³⁷ this leads to the twisted affinization of a different algebra. The same superpotential has been deduced from entirely different considerations involving M theory compactified on certain eight-dimensional manifolds,^{38–40,51,52} although the $2/\alpha_j^2$ prefactors, that we shall find crucial in order to get results for the gluino condensate that agree with other calculations, are absent. It is also interesting that the integrable systems related to the Toda potentials that we have found above are precisely those that appear in the “Seiberg–Witten theory” of the $\mathcal{N}=2$ gauge theory with the same gauge group in four dimensions.^{53,54} Naturally this is no accident since the $\mathcal{N}=1$ theory can be obtained from the $\mathcal{N}=2$ theory by soft breaking mass terms.

Importantly, although we have calculated the superpotential in the limit $R \ll |\Lambda^{-1}|$, at weak coupling, there can be no additional dependence on R and the result can be continued to any R , and in particular to the decompactification limit.^{34,35}

One may wonder how the superpotential relates to that calculated in Ref. 55 for the three-dimensional $\mathcal{N}=2$ supersymmetric gauge theory. The way that this superpotential arises from the $R \rightarrow 0$ limit of our superpotential is explained in a slightly different context in Ref. 34. The point is that to take the three-dimensional limit, one should take it in such a way that the three-dimensional gauge coupling, which is classically given by $g_3^2 = g^2/(2\pi R)$, is fixed. In other words, as $R \rightarrow 0$, we should simultaneously take the limit $g \rightarrow 0$ in the superpotential (46). In this limit, the additional term corresponding to the affine root is removed to give

$$\mathcal{W}_{3-d} = \frac{\mu^3}{g_3^2} \sum_{j=1}^r \frac{2}{\alpha_j^2} e^{\alpha_j^* \cdot \mathbf{X}}. \quad (48)$$

In other words, in this limit, the affine Toda potential becomes the Toda potential for a nonaffine algebra. This is what one expects because the affine term in the superpotential on the cylinder is generated by the additional monopole solution that only exists on the cylinder and not in \mathbb{R}^3 . The genuinely three-dimensional superpotential (48) is the generalization of that of Ref. 55 from $SU(2)$ to arbitrary gauge group. Just as in the $SU(2)$ case, it does not have a stationary point and therefore the theory does not have a vacuum state.

V. VACUUM STRUCTURE AND THE GLUINO CONDENSATE

The superpotential (46) gives rise to a number of supersymmetric vacua which satisfy

$$\frac{2}{\alpha_j^2} e^{\alpha_j^* \cdot \mathbf{X}} = \frac{2}{\alpha_0^2} k_j^* e^{2\pi i \tau} e^{\alpha_0^* \cdot \mathbf{X}}, \quad (49)$$

for $j = 1, \dots, r$. Writing $\mathbf{X} = \sum_{j=1}^r a_j \omega_j$ we have

$$e^{a_j} = \frac{k_j^* \alpha_j^2 e^{2\pi i \tau}}{2\kappa}, \quad (50)$$

where $\kappa = e^{\sum_{j=1}^r a_j k_j^*}$ is determined self-consistently as the solution of the equation

$$\kappa^{c_2} = e^{2\pi i (c_2 - 1) \tau} \prod_{j=0}^r \left(\frac{k_j^* \alpha_j^2}{2} \right)^{k_j^*}. \quad (51)$$

There are consequently c_2 supersymmetric ground states given by the c_2 roots of (51) which are related by $\mathbf{X} \rightarrow \mathbf{X} + 2\pi i \rho / c_2$. (The vector $\rho = \sum_{j=1}^r \omega_j$ is the Weyl vector and recall that \mathbf{X} is identified with $\mathbf{X} + 2\pi i \rho$ as a consequence of the fact that σ is identified with $\sigma + 2\pi \rho$, since $\rho \in \Lambda_W$.) These vacua correspond to a fixed value of φ :

$$\varphi = \left(\frac{2\pi}{c_2} + \frac{g^2}{4\pi} \ln |\kappa| \right) \rho - \frac{g^2}{4\pi} \sum_{j=1}^r \ln(k_j^* \alpha_j^2 / 2) \omega_j, \quad (52)$$

and c_2 values of σ given by

$$\sigma = -\frac{\vartheta}{2\pi} \varphi + \frac{\vartheta + 2\pi u}{c_2} \rho, \quad (53)$$

where $u = 1, 2, \dots, c_2$. Notice that as expected the c_2 vacua are related by $\vartheta \rightarrow \vartheta + 2\pi$.

The value of the superpotential in one of the vacua is

$$\langle \mathcal{W}_{\text{mono}} \rangle = \frac{2\pi \mu^3 R}{g^2} \cdot \frac{e^{2\pi i \tau} c_2}{\kappa} = \frac{2\pi \mu^3 R}{g^2} \cdot \frac{c_2 e^{2\pi i \tau / c_2 + 2\pi i u / c_2}}{\prod_{j=0}^r (k_j^* \alpha_j^2 / 2)^{k_j^* / c_2}} = 2\pi R \Lambda^3 \cdot \frac{c_2 e^{2\pi i u / c_2}}{\prod_{j=0}^r (k_j^* \alpha_j^2 / 2)^{k_j^* / c_2}}, \quad (54)$$

where in the final expression we have eliminated the Pauli–Villars mass scale μ in favor of the Lambda parameter using the exact relation (2). The value of the gluino condensate in each vacuum can be extracted by using the general relation

$$\left\langle \frac{\text{tr } \lambda^2}{16\pi^2} \right\rangle = b_0^{-1} \Lambda \frac{\partial}{\partial \Lambda} \left\langle \frac{1}{2\pi R} \mathcal{W}_{\text{mono}} \right\rangle, \quad (55)$$

adapted to the three-dimensional superpotential. The first coefficient of the beta-function is $b_0 = 3c_2$ giving

$$\left\langle \frac{\text{tr } \lambda^2}{16\pi^2} \right\rangle = \frac{\Lambda^3 e^{2\pi i u / c_2}}{\prod_{j=0}^r (k_j^* \alpha_j^2 / 2)^{k_j^* / c_2}}. \quad (56)$$

The gluino condensate can also be evaluated directly without having to rely on the identity (55). The idea is to consider the fundamental monopole contributions to the one-point function $\langle \text{tr } \lambda^2 / 16\pi^2 \rangle$ in a given vacuum, say the u^{th} . The contribution of the α_j monopole to the condensate in this vacuum is

$$\left\langle \frac{\text{tr } \lambda^2}{16\pi^2} \right\rangle_{j\text{-mono}} = \int d\mu_{\text{mon}}^{(j)} \frac{\text{tr } \lambda^2}{16\pi^2} \Big|_{j\text{-mono}}. \quad (57)$$

To evaluate (57), we can use the normalization of the adjoint fermion zero modes from Ref. 50,

$$\int d^3a d^2\xi \frac{\text{tr } \lambda^2}{16\pi^2} \Big|_{j\text{-mono}} = \frac{g^2 \text{Re } S_j}{8\alpha_j^2 \pi^3 R}. \quad (58)$$

Computing the remaining integral over the phase angle gives

$$\left\langle \frac{\text{tr } \lambda^2}{16\pi^2} \right\rangle_{j\text{-mono}} = \frac{\mu^3 \text{Re } S_j}{4\pi^2 \alpha_j^2} e^{-S_j}. \quad (59)$$

In the supersymmetric vacua

$$S_j = -2\pi i \tau - \ln \left(\frac{k_j^* \alpha_j^2}{2\kappa} \right) \quad (60)$$

and so inserting the value for κ in (51) we have

$$\left\langle \frac{\text{tr } \lambda^2}{16\pi^2} \right\rangle_{j\text{-mono}} = \frac{k_j^* \Lambda^3 e^{2\pi i u/c_2}}{c_2} \cdot \frac{1}{\prod_{j=0}^r (\alpha_j^2 k_j^*/2)^{k_j^*/c_2}}. \quad (61)$$

Summing over the contributions from the $r+1$ fundamental monopoles gives (56).

We conclude the section with the observation that in the supersymmetric vacua the $r+1$ fundamental monopoles have equal topological charge

$$\alpha_j \cdot \frac{\langle \varphi \rangle}{2\pi} = 1 - \alpha_0 \cdot \frac{\langle \varphi \rangle}{2\pi} = \frac{1}{c_2} \quad (62)$$

(for $j=1, \dots, r$) independent of j . In addition, as we have discussed in Sec. III, the configuration which becomes the singly-charged instanton in the uncompactified theory is obtained by considering a multi-monopole solution which consists of k_j^* of the j th fundamental monopole. In this very precise sense they realize the old dream of thinking of the instanton in terms of a set of constituents, or instanton quarks.²⁰⁻²⁴ It was anticipated that the instanton quarks would cause, or at least play a major role in, confinement. In the theory on the cylinder this old idea again receives confirmation. Notice that in the quantum vacuum states the dual photon becomes massive which is equivalent to the confinement of the original Abelian electric photons.

APPENDIX A: SOME LIE ALGEBRA CONVENTIONS

In this appendix we give a brief review of particular details of Lie algebras that we will need. For more details on Lie algebras the reader may consult Refs. 56.

Let $\{H^i\}$ be a maximal set of simultaneously diagonalizable, mutually commuting generators, $[H^i, H^j] = 0$. The indices i, j run from 1 to r , the *rank* of the Lie algebra. We normalize the Cartan generators to one,

$$\text{tr}(H^i H^j) = \delta^{ij}, \quad (A1)$$

and often think of the r -vector \mathbf{H} . The remainder of the generators are the step operators E_α with

$$[\mathbf{H}, E_\alpha] = \alpha E_\alpha. \quad (A2)$$

The normalization condition (A1) makes the length squared of any long root to be equal to 2.

TABLE III. Lie algebra data.

G	g	c_2	$\{k_i^*\}$	$\{\alpha_i^2\}$
$SU(r+1)$	a_r	$r+1$	$\{1,1,\dots,1\}$	$\{2,\dots,2\}$
$SO(2r+1)$	b_r	$2r-1$	$\{1,1,2,\dots,2,1\}$	$\{2,\dots,2,1\}$
$USp(2r)$	c_r	$r+1$	$\{1,\dots,1\}$	$\{2,1,\dots,1,2\}$
$SO(2r)$	d_r	$2r-2$	$\{1,1,2,\dots,2,1,1\}$	$\{2,\dots,2\}$
G_2	g_2	4	$\{1,2,1\}$	$\{2,2,2/3\}$
F_4	f_4	9	$\{1,2,3,2,1\}$	$\{2,2,2,1,1\}$
E_6	e_6	12	$\{1,1,1,2,2,2,3\}$	$\{2,\dots,2\}$
E_7	e_7	18	$\{1,1,2,2,2,3,3,4\}$	$\{2,\dots,2\}$
E_8	e_8	30	$\{1,2,2,3,3,4,4,5,6\}$	$\{2,\dots,2\}$

We will denote a set of simple roots as α_j , $j=1,\dots,r$. These span the root lattice Λ_R . The lowest root is then denoted as α_0 . The co-roots are defined via

$$\alpha^* \equiv \frac{2}{\alpha^2} \alpha \quad (\text{A3})$$

and these span the co-root lattice Λ_R^* . The weight lattice Λ_W is dual to the co-root lattice and is spanned by the fundamental weights ω_j where

$$\omega_i \cdot \alpha_j^* = \delta_{ij}. \quad (\text{A4})$$

Similarly one can define the co-weight lattice Λ_W^* which is dual to the root lattice and is spanned by the co-weights ω_i^* where

$$\omega_i^* \equiv \frac{2}{\alpha_i^2} \omega_i. \quad (\text{A5})$$

We will also need to define the dual Kac labels, or co-marks, k_i^* . By definition $k_0^* = 1$ and the remaining co-marks are given by the expansion of the lowest co-root in terms of the co-simple-roots:

$$\alpha_0^* = - \sum_{i=1}^r k_i^* \alpha_i^*. \quad (\text{A6})$$

Finally

$$c_2 \equiv \sum_{i=0}^r k_i^* \quad (\text{A7})$$

is the *dual Coxeter number*. (The Kac labels, or marks, and Coxeter number are similarly defined but will not be needed here.)

In Table III we summarize all the Lie algebra data that we need. As well as listing the dual Kac labels and dual Coxeter number we also list the root lengths α_j^2 for $j=0,\dots,r$. (Note that the set of dual Kac labels and root lengths are ordered in the same way.)

APPENDIX B: THE MONOPOLE COLLECTIVE COORDINATE MEASURE

In this appendix we briefly discuss the measure for integrating over the collective coordinates of a fundamental monopole. A fundamental monopole has a moduli space that is identical to the BPS monopole in $SU(2)$. Therefore, it is parametrized by a_μ , the position in \mathbb{R}^3 , and by the $U(1)$ phase angle $0 \leq \Omega \leq 2\pi$. Along with this, there are two Grassmann collective coordinates ξ_α ,

corresponding to the two supersymmetric zero modes. The measure for integrating over the monopole moduli space is obtained in the standard way by changing variables in the path integral from field fluctuations around the monopole to the monopole's collective coordinates:

$$\int d\mu_{\text{mon}} = \mu^3 e^{-S} \int \frac{d^3 a}{(2\pi)^{3/2}} J_a \int_0^{2\pi} \frac{d\Omega}{(2\pi)^{1/2}} J_\Omega \int \frac{d^2 \xi}{J_F}, \quad (\text{B1})$$

where S is the monopole action (39) and μ is the Pauli–Villars mass scale. The Jacobian factors J_a and J_F were calculated in Ref. 50:

$$J_a = (\text{Re } S)^{3/2}, \quad J_F = 2 \text{Re } S, \quad (\text{B2})$$

and S is the monopole action. The remaining Jacobian J_Ω is given by

$$J_\Omega = \frac{2\pi R (\text{Re } S)^{1/2}}{\alpha \cdot \langle \varphi \rangle}. \quad (\text{B3})$$

To derive this, we start with the general expression for the bosonic zero mode,

$$Z_m = \frac{\partial v_m^{(\Omega)}}{\partial \Omega} + D_m \Lambda, \quad (\text{B4})$$

where $v_m^{(\Omega)}$ is the Ω -rotated monopole solution in the singular gauge,

$$v_m^{(\Omega)} = e^{i\Omega \tau^3} v_m e^{-i\Omega \tau^3}, \quad (\text{B5})$$

and $D_m \Lambda$ is added to keep the zero mode in the covariant background gauge. Since

$$\frac{\partial v_m^{(\Omega)}}{\partial \Omega} = i \left[\frac{1}{2} \alpha^* \cdot \mathbf{H}, v_m \right], \quad (\text{B6})$$

the choice of Λ is obvious [recall (16)]:

$$\Lambda = \frac{2\pi R}{\alpha \cdot \langle \varphi \rangle} \Phi^c \tau^c - \frac{1}{2} \alpha^* \cdot \mathbf{H}. \quad (\text{B7})$$

This gives

$$Z_m = \frac{2\pi R}{\alpha \cdot \langle \varphi \rangle} D_m(\Phi^c \tau^c) = \frac{2\pi R}{\alpha \cdot \langle \varphi \rangle} v_{m0}, \quad (\text{B8})$$

and

$$J_\Omega \equiv \sqrt{\langle Z_m | Z_m \rangle} = \frac{2\pi R (\text{Re } S)^{1/2}}{\alpha \cdot \langle \varphi \rangle}. \quad (\text{B9})$$

Gathering all factors together, we find that the measure is

$$\frac{\mu^3 R}{g^2} \cdot \frac{2}{\alpha^2} \cdot e^{-S} \int d^3 a \, d\Omega \, d^2 \xi. \quad (\text{B10})$$

In contradistinction with the three-dimensional calculation of Ref. 50, our present calculation is locally four-dimensional, i.e., in the path integral we have integrated over the fluctuations around the monopole configuration in $\mathbb{R}^3 \times S^1$. Thus, the UV-regularized determinants over nonzero eigenvalues of the quadratic fluctuation operators cancel between fermions and bosons due to supersymmetry as in Ref. 57. The ultra-violet divergences are regularized in the Pauli–Villars scheme, which explains the appearance of the Pauli–Villars mass scale μ .

- ¹N. M. Davies, T. J. Hollowood, V. V. Khoze, and M. P. Mattis, Nucl. Phys. B **559**, 123 (1999).
- ²V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B **260**, 157 (1985).
- ³M. A. Shifman and A. I. Vainshtein, Nucl. Phys. B **296**, 445 (1988).
- ⁴A. Morozov, M. Ol'shanetsky, and M. Shifman, Nucl. Phys. B **304**, 291 (1988).
- ⁵D. Finnell and P. Pouliot, Nucl. Phys. B **453**, 225 (1995).
- ⁶O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, Phys. Rep. **323**, 183 (2000).
- ⁷K. Intriligator and N. Seiberg, Nucl. Phys. Proc. Suppl. **45BC**, 1 (1996).
- ⁸N. M. Davies and V. V. Khoze, J. High Energy Phys. **0001**, 015 (2000).
- ⁹V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B **229**, 394 (1983); **229**, 407 (1983).
- ¹⁰G. C. Rossi and G. Veneziano, Phys. Lett. B **138**, 195 (1984); D. Amati, G. C. Rossi, and G. Veneziano, Nucl. Phys. B **249**, 1 (1985).
- ¹¹D. Amati, K. Konishi, Y. Meurice, G. C. Rossi, and G. Veneziano, Phys. Rep. **162**, 169 (1988).
- ¹²I. Affleck, M. Dine, and N. Seiberg, Nucl. Phys. B **241**, 493 (1984).
- ¹³J. Fuchs and M. G. Schmidt, Z. Phys. C **30**, 161 (1986).
- ¹⁴A. Kovner and M. Shifman, Phys. Rev. D **56**, 2396 (1997).
- ¹⁵M. Shifman and A. Vainshtein, hep-th/9902018.
- ¹⁶T. J. Hollowood, V. V. Khoze, W. Lee, and M. P. Mattis, Nucl. Phys. B **570**, 241 (2000).
- ¹⁷N. Dorey, V. V. Khoze, and M. P. Mattis, Phys. Rev. D **54**, 2921 (1996); **54**, 7832 (1996).
- ¹⁸N. Dorey, T. J. Hollowood, V. V. Khoze, M. P. Mattis, and S. Vandoren, Nucl. Phys. B **552**, 88 (1999).
- ¹⁹N. Dorey, T. J. Hollowood, V. V. Khoze, and M. P. Mattis, Phys. Rep. **371**, 231 (2002).
- ²⁰A. A. Belavin, V. A. Fateev, A. S. Schwarz, and Y. S. Tyupkin, Phys. Lett. B **83**, 317 (1979).
- ²¹V. A. Fateev, I. V. Frolov, and A. S. Shvarts, Nucl. Phys. B **154**, 1 (1979).
- ²²B. Berg and M. Luscher, Commun. Math. Phys. **69**, 57 (1979).
- ²³C. G. Callan, R. Dashen, and D. J. Gross, Phys. Rev. D **17**, 2717 (1978).
- ²⁴H. Osborn, Ann. Phys. **135**, 373 (1981).
- ²⁵K. Lee and P. Yi, Phys. Rev. D **56**, 3711 (1997).
- ²⁶K. Lee, Phys. Lett. B **426**, 323 (1988).
- ²⁷W. Nahm, "Self-dual Monopoles and Calorons," in *Lecture Notes in Physics Vol. 201*, edited by G. Denado *et al.* (Springer, New York, 1984), p. 189.
- ²⁸H. Garland and M. K. Murray, Commun. Math. Phys. **120**, 335 (1988).
- ²⁹K. Lee and C. Lu, Phys. Rev. D **58**, 025011 (1998).
- ³⁰T. C. Kraan and P. van Baal, Phys. Lett. B **428**, 268 (1998); Nucl. Phys. B **533**, 627 (1998); Phys. Lett. B **435**, 389 (1998).
- ³¹B. J. Harrington and H. K. Shepard, Phys. Rev. D **17**, 2122 (1978).
- ³²P. Rossi, Nucl. Phys. B **149**, 170 (1979).
- ³³D. J. Gross, R. D. Pisarski, and L. G. Yaffe, Rev. Mod. Phys. **53**, 43 (1981).
- ³⁴N. Seiberg and E. Witten, hep-th/9607163.
- ³⁵O. Aharony, A. Hanany, K. Intriligator, N. Seiberg, and M. J. Strassler, Nucl. Phys. B **499**, 67 (1997).
- ³⁶N. Dorey, J. High Energy Phys. **9907**, 021 (1999).
- ³⁷V. G. Kac, *Infinite Lie Algebras*, Progress in Mathematics, Vol. 44 (Birkhäuser, Boston, 1983).
- ³⁸E. Witten, Nucl. Phys. B **474**, 343 (1996).
- ³⁹S. Katz and C. Vafa, Nucl. Phys. B **497**, 196 (1997).
- ⁴⁰C. Vafa, Adv. Theor. Math. Phys. **2**, 497 (1998).
- ⁴¹E. Witten, Nucl. Phys. B **202**, 253 (1982); J. High Energy Phys. **9802**, 006 (1998); A. Keurentjes, A. Rosly, and A. V. Smilga, Phys. Rev. D **58**, 081701 (1998).
- ⁴²P. A. M. Dirac, Proc. R. Soc. London, Ser. A **133**, 60 (1931); F. Englert and P. Windey, Phys. Rev. D **14**, 2728 (1976); P. Goddard, J. Nuyts, and D. Olive, Nucl. Phys. B **125**, 1 (1977).
- ⁴³E. J. Weinberg, Nucl. Phys. B **167**, 500 (1980); **203**, 445 (1982).
- ⁴⁴S. F. Cordes and M. Dine, Nucl. Phys. B **273**, 581 (1986).
- ⁴⁵G. 't Hooft, Nucl. Phys. B **79**, 276 (1974).
- ⁴⁶A. M. Polyakov, JETP Lett. **20**, 194 (1974).
- ⁴⁷E. B. Bogomol'nyi, Sov. J. Phys. **24**, 97 (1977).
- ⁴⁸M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. **35**, 760 (1975).
- ⁴⁹N. Seiberg, Phys. Lett. B **318**, 469 (1993).
- ⁵⁰N. Dorey, V. V. Khoze, M. P. Mattis, D. Tong, and S. Vandoren, Nucl. Phys. B **502**, 94 (1997).
- ⁵¹C. Gomez, hep-th/9706131.
- ⁵²C. Gomez and R. Hernandez, Nucl. Phys. (Proc. Suppl.) **61A**, 172 (1998); Int. J. Mod. Phys. A **12**, 5141 (1997).
- ⁵³E. Martinec and N. Warner, Nucl. Phys. B **459**, 97 (1996).
- ⁵⁴T. J. Hollowood, Adv. Theor. Math. Phys. **2**, 335 (1998).
- ⁵⁵I. Affleck, J. A. Harvey, and E. Witten, Nucl. Phys. B **206**, 413 (1982).
- ⁵⁶R. Gilmore, *Lie groups, Lie algebras and some of their representations* (Wiley, New York, 1974); H. Samelson, *Notes on Lie algebras* (Springer, New York, 1990); R. N. Cahn, *Semisimple Lie algebras and their representations*, J. Adams, *Lectures on exceptional Lie groups*, Chicago, 1996, edited by Z. Mahmud and M. Mimura (Benjamin/Cummings, 1984).
- ⁵⁷A. D'Adda and P. Di Vecchia, Phys. Lett. B **73**, 162 (1978).