GAUGE ALGEBRA AND QUANTIZATION

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In respectful memory of Professor Berezin

Quantization of a general gauge theory in the lagrangian approach is accomplished in closed form. The generating equation is found, containing all the relations of the open gauge algebra. A new class of diagrams is revealed, required by BRS-symmetry, but completely definable only from the requirement of unitarity.

1. Introduction. The canonical quantization of a general gauge theory in arbitrary gauges including relativistic ones was carried out in refs. [1–3]. However, there remained the problem of transition to configuration space. For particular cases: –gravity and simple supergravity theories – such a transition was accomplished in refs. [4,5]. As an intermediate step to the solution of this problem in the general case one may try to construct Feynman rules directly in the lagrangian formalism proceeding from some requirements which replace the requirement of unitarity. Such an approach was developed in refs. [6,7]. The authors of ref. [7] revealed the structure of the general gauge algebra which is very similar (and in fact must be equivalent) to the algebra of first-class constraints, previously derived in ref. [3].

In the present paper we find the closed equation generating the gauge algebra and use it to construct the functional integral in the configuration space of a gauge field. The Feynman rules thus obtained generalize and supplement the results of ref. [7]. It must be realized that the principles of quantization used in the lagrangian approach are weaker than the condition of unitarity, and as a result a certain class of diagrams remains undefined. Therefore the problem of comparison with the canonical formalism still persists.

We begin with our definition of the gauge field. The field [‡1] ϕ^i ($i = 1, ..., n = n_+ + n_-$), $\epsilon(\phi^i) \equiv \epsilon_i$, is a gauge field, if its action $\mathcal{S}(\phi)$ is a boson satisfying the following postulates. There exists at least one stationary point ϕ_0, and in its neighbourhood the Noether identities hold:

$$(\partial_r \mathcal{S}/\partial\phi^i) R^i_\alpha \equiv 0\ , \quad \alpha = 1, ..., m = m_+ + m_-\ ,$$

$$m_\pm < n_\pm\ , \tag{1}$$

$\mathcal{S}(\phi)$ and $R^i_\alpha(\phi)$ are infinitely differentiable functions, and

$$\operatorname{rank}(R^i_\alpha)|_{\phi=\phi_0} = m_+ + m_-\ , \quad \epsilon(R^i_\alpha) = \epsilon_i + \epsilon_\alpha\ , \tag{2}$$

$$\operatorname{rank}(\partial^2_r \mathcal{S}/\partial\phi^i\partial\phi^k)|_{\phi=\phi_0} = (n_+ - m_+) + (n_- - m_-)\ , \tag{3}$$

ϵ_α is the Grassman parity of the parameters of gauge transformations. These are the usual conditions of

‡1 n_+ (n_-) denotes the number of bosonic (fermionic) components of the field, $\epsilon(A)$ denotes the Grassman parity of a quantity A, right and left derivatives are ∂_r and ∂_l. For the theory of manifolds and such notions as rank and determinant in the general Bose–Fermi case, see the reviews [8].

gauge invariance [‡2] plus the requirement of existence of the quasiclassical expansion.

2. Antifields, antibrackets and the master equation. Given any set of boson and fermion fields $\Phi^A, A = 1, ..., N$, $\epsilon(\Phi^A) \equiv \epsilon_A$, we introduce new variables, Φ^*_A, of the opposite statistics: $\epsilon(\Phi^*_A) = \epsilon_A + 1$, and call them antifields. For functions on the "phase space" of Φ and Φ^* we define the following operation

$$(F, H) = (\partial_r F/\partial\Phi^A)\, \partial_l H/\partial\Phi^*_A - (\partial_r F/\partial\Phi^*_A)\, \partial_l H/\partial\Phi^A \tag{4}$$

and call it antibrackets. The main properties of antibrackets are

$$\epsilon[(F,H)] = \epsilon(F) + \epsilon(H) + 1\ , \tag{5}$$

$$(F,H) = -(-1)^{(\epsilon_F+1)(\epsilon_H+1)}(H,F)\ , \tag{6}$$

$$(-1)^{(\epsilon_F+1)(\epsilon_H+1)}(F,(G,H)) + \text{cycl. perm.}\, F, G, H = 0\ . \tag{7}$$

For any fermion

$$(F,F) \equiv 0\ , \tag{8}$$

while for a boson generally

$$(B,B) = 2(\partial_r B/\partial\Phi^A)\, \partial_l B/\partial\Phi^*_A \not\equiv 0\ . \tag{9}$$

For any G:

$$((G,G),G) = (G,(G,G)) = 0\ .$$

The properties (8), (9) are opposite to that of the usual Poisson brackets [9,2]. One more unusual property is that the infinitesimal canonical transformation

$$\bar{\Phi}^A = \Phi^A + (\Phi^A, H)\ , \quad \bar{\Phi}^*_A = \Phi^*_A + (\Phi^*_A, H)\ ,$$

$$\epsilon(H) = 1\ ,$$

does not preserve the volume element of the phase space:

$$\partial(\bar{\Phi}, \bar{\Phi}^*)/\partial(\Phi, \Phi^*) = 1 + 2(-1)^{\epsilon_A}(\partial_r/\partial\Phi^A)\, \partial_l H/\partial\Phi^*_A\ . \tag{10}$$

The technique of antibrackets will be used to quantize the gauge field, and property (10) will prove to be responsible for the incompleteness of the lagrangian quantization.

The master equation is

$$(S,S) = 0\ , \tag{11}$$

for a boson S. It will be shown that under certain boundary conditions its solution serves as the action generating Feynman rules in any field theory. The important property of the master equation is that any of its solutions is gauge-invariant. Indeed, the differentiation of (11) gives Noether identities:

$$(S, \partial_r S/\partial Z) = 0\ , \tag{12}$$

where Z is any of the variables Φ^A, Φ^*_A. As seen from (12), the second derivatives of S serve as generators of gauge transformations, and the matrix $\partial^2 S$ at a stationary point of S is nilpotent. The $2N$ Noether identities (12) are linearly dependent. Let r be the number of independent identities among (12). Then r is also the rank of the matrix $\partial^2 S$ at a stationary point, and always $r \leqslant N$ We shall say, that S is the proper solution of the master equation if $r = N$. The solution S is proper if and only if the matrix $\partial^2 S$ at a stationary point has no other zero-eigenvalue eigenvectors except those contained in itself. The requirement that the solution be proper will arise when quantizing the gauge field and will lead to the necessity of introduction of ghosts.

3. Quantization. We shall construct the S-matrix as the functional integral Z in the configuration space of a gauge field proceeding from three requirements: gauge-independence, nondegeneracy and correctness of the classical limit.

Let Φ^A be any set of fields, which includes the given gauge field. As shown in ref. [2], the most general gauge arbitrariness contained in any gauge theory is one fermionic function of all fields. Let $\Psi(\Phi)$ be this arbitrary fermionic function. Let us introduce antifields Φ^*_A and look for such an action $W(\Phi, \Phi^*)$ that its restriction to the surface

$$\Sigma: \quad \Phi^*_A = \partial_r \Psi(\Phi)/\partial\Phi^A\ , \tag{13}$$

generates the correct Feynman rules:

$$Z_\Psi = \int \exp\left\{\frac{i}{\hbar} W\left(\Phi, \frac{\partial_r \Psi}{\partial\Phi}\right)\right\} \prod_A d\Phi^A\ . \tag{14}$$

‡2 The postulates of the gauge theory can in fact be formulated in terms of equations of motion only, and the generators R^i_α can be expressed through the action $\mathcal{S}(\phi)$.

The action $W(\Phi, \Phi^*)$ must be found from the three requirements listed above.

Let us define the following BRS-transformation of the integration variables in (14):

$$\delta\Phi = (\Phi, W)|_\Sigma \cdot \mu , \tag{15}$$

where μ is a fermionic parameter. One may verify that the functional integral (14) is BRS-invariant if $W(\Phi, \Phi^*)$ satisfies the equation

$$\tfrac{1}{2}(W, W) = \mathrm{i}\hbar\Delta W , \quad \Delta \equiv (\partial_\mathrm{r}/\partial\Phi^A)\, \partial_1/\partial\Phi^*_A , \tag{16}$$

which is equivalent to the linear equation

$$\Delta \exp[(\mathrm{i}/\hbar)\, W] = 0 . \tag{17}$$

The gauge-dependence of the S-matrix is a consequence of its BRS-invariance. The proof of gauge-independence is analogous to that of ref. [2]: one makes the BRS-transformation of the integration variables in (14) with parameter $\mu = (\mathrm{i}/\hbar)\,\delta\Psi$ and shows that $Z_\Psi = Z_{\Psi+\delta\Psi}$.

The right-hand side of eq. (16) is the contribution of the jacobian of the BRS-transformation. The non-triviality of this jacobian [see eq. (10)] is responsible for the dependence of W on h. We may represent W as

$$W = S + \sum_{p=1}^{\infty} \hbar^p M_p , \tag{18}$$

where S is the classical part of the action, and the remainder is the contribution of the quantum integration measure, which secures the invariance of the functional integral. We learn that the measure generally depends on the gauge (through Φ^*) and is not purely one-loop, but acquires new contributions at each order in h. One finds from (16):

$$(S,S) = 0 , \quad (M_1, S) = \mathrm{i}\Delta S ,$$

$$(M_p,S) = \mathrm{i}\Delta M_{p-1} - \frac{1}{2}\sum_{q=1}^{p-1} (M_q, M_{p-q}) , \quad p \geqslant 2 . \tag{19}$$

The equation for the classical part of W is just the master equation considered above.

Thus the requirement of gauge-independence led to a differential equation for $W(\Phi, \Phi^*)$, but the contents of the field Φ and the boundary conditions remained arbitrary.

The requirement of the correctness of the classical limit imposes the following boundary condition on the classical part of W:

$$\phi^i \subset \Phi , \quad S(\Phi, \Phi^*)|_{\Phi^*=0} = \mathcal{S}(\phi) , \tag{20}$$

where ϕ^i is the given gauge field, and $\mathcal{S}(\phi)$ is its action introduced above. Further, the integral (14) virtually realizes quantization of the gauge action $S(\Phi, \Phi^*)$ in the gauge (13). For non-degeneracy of the integral it is necessary that S be the proper solution of the master equation, because otherwise N gauge conditions (13) will be insufficient to remove the invariance of S. However, if the boundary value (20) of S is already the gauge action, then there are initially m zero-eigenvalue eigenvectors R^i_α not included into the matrix of second derivatives. In order to include R^i_α in $\partial^2 S$ we introduce m auxiliary fields C^α, $\epsilon(C^\alpha) = \epsilon_\alpha + 1$, and require that

$$C^\alpha \subset \Phi , \quad \partial_\mathrm{r}\partial_1 S(\Phi, \Phi^*)/\partial C^\alpha \partial\phi^*_i|_{\Phi^*=0} = R^i_\alpha(\phi) . \tag{21}$$

Thus the minimal content of Φ is

$$\Phi_{\min} = \{\phi^i, C^\alpha\} \tag{22}$$

and (21) is the second boundary condition for the classical part of W.

The requirement of nondegeneracy imposes also restrictions on the gauge fermion Ψ. In order to represent the gauge in conventional form, we shall introduce $2m$ additional fields

$$\bar{C}_\alpha , \quad \pi_\alpha \subset \Phi , \quad \epsilon(\bar{C}_\alpha) = \epsilon_\alpha + 1 , \quad \epsilon(\pi_\alpha) = \epsilon_\alpha , \tag{23}$$

and require that

$$\frac{\partial_1\Psi}{\partial\bar{C}_\alpha} \equiv \Psi^\alpha , \quad \frac{\partial_\mathrm{r}\Psi^\alpha}{\partial\phi^i} R^i_\beta \equiv D^\alpha_\beta , \quad \det D^\alpha_\beta \neq 0 . \tag{24}$$

Then Ψ^α will play the role of conventional gauge conditions [‡3] removing the initial invariance, π_α will be the Lagrange multipliers for these gauge conditions, and $\bar{C}_\alpha$, C^α will be the Faddeev–Popov ghosts. Besides (23) Φ may contain an arbitrary number of pairs of fields

$$\Lambda_{\dots} , \Pi_{\dots} \subset \Phi , \quad \epsilon(\Lambda_{\dots}) = \epsilon(\Pi_{\dots}) + 1 . \tag{25}$$

Their permissibility is a part of the gauge freedom.

The dependence of W on the additional fields (23) and (25) must be trivial:

$$W(\Phi, \Phi^*) = W(\Phi_{\min}, \Phi^*_{\min}) + \bar{C}^{*\alpha}\pi_\alpha + \Lambda^{*\cdots}\Pi_{\dots} .$$

Then $W(\Phi_{\min}, \Phi^*_{\min})$ satisfies eqs. (18), (19) in the

‡3 The existence of such gauge conditions in a neighbourhood of the stationary point follows from the postulate (2).

minimal sector (22). The solution of the master equation with boundary conditions (20), (21) will be considered below. The boundary conditions to the equations for the quantum measure remain undefined. Eqs. (19) define only the transformation properties of the measure and admit a considerable arbitrariness. The measure can be completely determined only from comparison with the canonical formalism [1–3] in which the condition of unitarity fixes everything.

It is interesting to note, that the proper solution of the master equation is the exceptional particular case of a gauge theory when het generators of the gauge transformations are initially included into the matrix of second derivatives of the gauge action. This is the reason why the quantization of the action $S(\Phi, \Phi^*)$ in the gauge (13) does not require any ghosts as seen from eq. (14).

The master equation governs also the behaviour of the effective action. If we define the following source-dependent functional

$$Z[J, \langle\Phi^*\rangle] = \int \exp\{(i/\hbar)[W(\Phi, \langle\Phi^*\rangle + \partial_r \Psi/\partial\Phi) + J_A \Phi^A]\} \prod_A \mathrm{d}\Phi^A ,$$

and construct the effective action $\Gamma[\langle\Phi\rangle, \langle\Phi^*\rangle]$ as the Legendre transform of $(\hbar/i)\ln Z$ with respect to the variable J, then

$$(\Gamma, \Gamma) = 0 , \tag{26}$$

in consequence of eq. (16). In the case of the Yang–Mills theory eq. (26) was used to prove the gauge-invariant renormalizability [10, 11].

4. The gauge algebra. The main theorem is that the solution of the master equation with boundary conditions (20), (21) always exists as a power series in antifields ($\Phi = \Phi_{\min}$):

$$S(\Phi, \Phi^*) = \sum_{n=0}^{\infty} \Phi^*_{A_n} \ldots \Phi^*_{A_1} S^{A_1 \ldots A_n}(\phi, C) . \tag{27}$$

The coefficients are polynomials in C and are infinitely differentiable in ϕ in a neighbourhood of the stationary point. The proof is based on the properties of the gauge theory, postulated above, and will be given in the extended version of the present paper.

The coefficients $S^{A_1 \ldots A_n}(\phi, C)$ (with C differentiated away) are the structure functions of the gauge algebra. For the sake of clearity we shall write down several lowest-order equations for these coefficients. On account of the boundary conditions the structure functions at $n = 0$ and $n = 1$ are

$$S(\phi, C) = \mathcal{S}(\phi) , \quad S^i(\phi, C) = R^i_\alpha(\phi) C^\alpha . \tag{28}$$

With these identifications the master equation gives:

$$n = 0: \quad \frac{\partial_r \mathcal{S}}{\partial\phi^i} R^i_\alpha C^\alpha = 0 , \tag{29}$$

$n = 1$:

$$\frac{\partial_r R^i_\alpha C^\alpha}{\partial\phi^k} R^k_\beta C^\beta + R^i_\gamma S^\gamma(\phi, C) + 2S^{ik}(\phi, C) \frac{\partial_r \mathcal{S}}{\partial\phi^k} = 0 , \tag{30}$$

$$\frac{\partial_r S^\gamma(\phi, C)}{\partial\phi^i} R^i_\alpha C^\alpha + \frac{\partial_r S^\gamma(\phi, C)}{\partial C^\alpha} S^\alpha(\phi, C) + 2S^{\gamma i}(\phi, C) \frac{\partial_r \mathcal{S}}{\partial\phi^i} = 0 , \tag{31}$$

$n = 2$:

$$\frac{\partial_r S^{ik}(\phi, C)}{\partial\phi^j} R^j_\alpha C^\alpha + \frac{\partial_r S^{ik}(\phi, C)}{\partial C^\gamma} S^\gamma(\phi, C) + 3S^{ikj}(\phi, C) \frac{\partial_r \mathcal{S}}{\partial\phi^j} - (-1)^{\epsilon_k} \left[\frac{\partial_r R^i_\alpha C^\alpha}{\partial\phi^j} S^{jk}(\phi, C) + R^i_\mu S^{\mu k}(\phi, C)\right] + (-1)^{\epsilon_k(\epsilon_i + 1)} \left[\frac{\partial_r R^k_\alpha C^\alpha}{\partial\phi^j} S^{ji}(\phi, C) + R^k_\mu S^{\mu i}(\phi, C)\right] = 0 , \tag{32}$$

and so on. Eq. (29) is the Noether identity. Eq. (30) gives the general form of the commutator of gauge transformations. The new structure functions $S^\gamma(\phi, C)$ and $S^{ik}(\phi, C)$, entering this equation, are quadratic in C. Eq. (31) gives the general form of the Jacobi identity. The new structure function $S^{\gamma i}(\phi, C)$, entering the Jacobi identity, is cubic in C. Eq. (32) and all higher-order equations are the new ones.

The master equation contains all the relations of the gauge algebra, and its solution $S(\Phi, \Phi^*)$ is the generating expression for all structure functions.

In the particular case of a gauge theory when the coefficient $S^{ik}(\phi, C)$ in eq. (30) turns out to be zero, the algebra is said to be closed [‡4]. Then the exact solution of the master equation is linear in antifields:

$$S(\Phi, \Phi^*) = \mathcal{S}(\phi) + \phi_i^* R_\alpha^i(\phi)\, C^\alpha - \tfrac{1}{2} C_\gamma^* t_{\alpha\beta}^\gamma(\phi)\, C^\beta C^\alpha (-1)^{\epsilon_\alpha} , \tag{33}$$

where

$$t_{\alpha\beta}^\gamma(\phi) = -(-1)^{\epsilon_\alpha} \frac{\partial_r}{\partial C^\beta} \frac{\partial_r}{\partial C^\alpha} S^\gamma(\phi, C) .$$

In this case eqs. (19) for the quantum measure also admit the closed solution

$$M_p = 0 , \quad p > 1 ; \quad M_1 = M_1(\phi) ,$$

and for $M_1(\phi)$ one finds:

$$\frac{\partial_r M_1}{\partial \phi^k} R_\alpha^k = \mathrm{i} A_\alpha(\phi) ,$$

$$A_\alpha(\phi) \equiv \frac{\partial_r R_\alpha^k}{\partial \phi^k} (-1)^{\epsilon_k(\epsilon_\alpha + 1)} - t_{\beta\alpha}^\beta (-1)^{\epsilon_\beta} . \tag{34}$$

To solve eq. (34) we introduce arbitrary gauge conditions (24) and consider the following initial-value problem for the ordinary differential equation

$$\mathrm{d}g^i/\mathrm{d}x = R_\alpha^i(g)\, D_\beta^{-1\alpha}(g)\, \Psi^\beta(\phi) , \quad g^i|_{x=1} = \phi^i . \tag{35}$$

The general solution of eq. (34) is:

$$M_1 = M^{\mathrm{inv}} + \mathrm{i} \int_0^1 \mathrm{d}x\, A_\alpha(g)\, D_\beta^{-1\alpha}(g)\, \Psi^\beta(\phi) ,$$

where g is the solution of eqs. (35), and M^{inv} is an arbitrary gauge-invariant functional. It can be shown, that the transition to another gauge Ψ in the above solution is equivalent to the change of M^{inv}. The arbitrariness in M^{inv} is the result of the incompleteness of the lagrangian quantization.

[‡4] In this case the finite gauge transformations generally form the quasigroup considered in ref. [12].

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