CONSEQUENCES OF ANOMALOUS WARD IDENTITIES

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The anomalies of Ward identities are shown to satisfy consistency or integrability relations, which restrict their possible form. For the case of SU(3) $\times$ SU(3) we verify that the anomalies given by Bardeen satisfy the consistency relations. A solution of the anomalous Ward identities is also given which describes concisely all anomalous contributions to low energy theorems. The contributions to strong five pseudoscalar interactions, to K_{l4}, to one- and two-photon interactions with three pseudoscalars are explicitly exhibited.

It is known [1] that the naive or normal Ward identities are not always satisfied in renormalized perturbation theory. In other words, the Green's functions do not always satisfy the identities which correspond to invariance or partial invariance of the basic Lagrangian. In the case of the partial conservation equation for the axial vector current, anomalies occur if there are fundamental spinor fields in the theory. They are due to the singularities of the spinar triangle graph and, in the non-abelian case of SU(3) $\times$ SU(3), of higher polygonal graphs. It is likely that the anomalies due to these single loop graphs are not modified by higher order many loop corrections.

In this note we observe that the anomalies must satisfy consistency or integrability relations which follow from the structure of the gauge group and which are non trivial in the case of non-abelian groups. We also give explicitly a local functional satisfying the anomalous Ward identities. This functional can be interpreted as an effective action which describes all modifications to the low energy theorems due to the presence of the anomalies. We treat in detail the case of SU(3) $\times$ SU(3), but the generalization to other groups is straightforward. The consistency conditions allow one to test if an anomaly has been correctly calculated and we find that some expressions given in the literature are incorrect. Furthermore, our explicit solution of the anomalous Ward identities disagrees with partial solutions given by others who did not take into account the non-abelian structure of the group.

The strength of the axial vector anomaly depends upon the particular spinor fields entering the theory. However, as we shall see, the form of the minimal anomaly is determined, up to an overall constant, by the structure of the gauge group. In this sense one can say that is is model independent. The same is true of the effective action which is a solution of the anomalous Ward identities. The anomalous contributions to various processes it describes are related to each other in a model independent way. If the experiments should verify these relations, one would have a strong argument in favour of the point of view that the perturbation theory anomalies are actually relevant to the physical world. We make this remark because of the opinion, held by some, that nature (or the exact solution of some field theory) follows simples rules and ignores the anomalies.

Consider a hadronic system in presence of external vector and axial vector fields $V_{\mu i}$ and $A_{\mu i}$ ($i = 1, 2, \ldots 8$) and, if convenient, of other external fields. The Ward identifies for SU(3) $\times$ SU(3) can be stated by giving the variation of the connected vacuum functional W with respect to infinitesimal vector or chiral gauge transformations. They are given respectively by

$$\delta V_\mu = \beta \times V_\mu - \delta_\mu \beta, \qquad \delta A_\mu = \beta \times A_\mu$$

and

$$\delta A_\mu = \alpha \times V_\mu - \delta_\mu \alpha, \qquad \delta V_\mu = \alpha \times A_\mu ,$$

where $(\beta \times V_\mu)_i = f_{ijk} \beta_j V_{\mu k}$ etc., and similarly for the other external fields. By introducing suitable additional (scalar, pseudoscalar etc.) external fields, the normal Ward identities can be said to state the invariance of the vacuum functional, the anomalous Ward identities give a particular form to the variation.

The variation of the vacuum functional is described by infinitesimal gauge operators

$$-X = \partial_\mu \frac{\delta}{\delta V_\mu} + V_\mu \times \frac{\delta}{\delta V_\mu} + A_\mu \times \frac{\delta}{\delta A_\mu} + \dots$$

$$-Y = \partial_\mu \frac{\delta}{\delta A_\mu} + A_\mu \times \frac{\delta}{\delta V_\mu} + V_\mu \times \frac{\delta}{\delta A_\mu} + \dots = -U + \dots$$

where the dots refer to the other fields. They satisfy the commutation relations

$$[X_i(x), X_j(x')] = f_{ijk}\delta(x-x')X_k(x) ,$$

$$[X_i(x), Y_j(x')] = f_{ijk}\delta(x-x')Y_k(x) ,$$

$$[Y_i(x), Y_j(x')] = f_{ijk}\delta(x-x')X_k(x) .$$

which are the structure relations of the gauge group. The Ward identities involving the divergence of the vector current can be written as

$$X_i W = F_i = 0$$

while those involving the divergence of the axial vector current have the form

$$Y_i W = G_i = G_i[V_\mu, A_\mu]$$

where G_i is the axial vector anomaly. In writing these equations we have used the fact that W can be defined so that the vector Ward identities have no anomaly and the axial vector anomaly has a minimal form depending only upon the external vector and axial vector fields and not upon the other external fields. For SU(3) × SU(3) with quarks Bardeen [2] has given the explicit form

$$-G_i = \frac{1}{4\pi^2}\epsilon_{\mu\nu\sigma\tau}\,\mathrm{tr}\left[\frac{\lambda_i}{2}\{\tfrac{1}{4}V_{\mu\nu}V_{\sigma\tau} + \tfrac{1}{12}A_{\mu\nu}A_{\sigma\tau} + \right.$$

$$\left. + \tfrac{2}{3}\mathrm{i}(A_\mu A_\nu V_{\sigma\tau} + A_\mu V_{\nu\sigma}A_\tau + V_{\mu\nu}A_\sigma A_\tau) - \tfrac{8}{3}A_\mu A_\nu A_\sigma A_\tau\}\right]$$

in three by three matrix notation:

$$V_\mu = \tfrac{1}{2}V_{\mu i}\lambda_i , \quad A_\mu = \tfrac{1}{2}A_{\mu i}\lambda_i ,$$

$$V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - \mathrm{i}[V_\mu, V_\nu] - \mathrm{i}[A_\mu, A_\nu] ,$$

$$A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}[V_\mu, A_\nu] - \mathrm{i}[A_\mu, V_\nu] .$$

The fact that the Ward identities express the variation of a single functional W, gives immediately strong restrictions on the form of the anomalies. Combining the Ward identities with the structure relations of the gauge group one finds immediately, since the vector anomaly vanishes, the consistency conditions:

$$X_i(x)G_j(x') = f_{ijk}\delta(x-x')G_k(x) ,$$

$$Y_i(x)G_j(x') - Y_j(x')G_i(x) = 0 .$$

The first of these integrability conditions states simply that G_i belongs to an octet and is obviously satisfied by Bardeen's expression. The second is more subtle; it can be verified, by explicit calculation, that it is also satisfied. Conversely, one can show that this consistency relation can be used to determine all other terms in G_i if one knows its first term, that proportional to $\epsilon_{\mu\nu\sigma\tau}$ $\mathrm{tr}\lambda_i V_{\mu\nu}V_{\sigma\tau}$, which must occur, as we know from Adler's abelian case. In this precise sense Bardeen's expression is a minimal form for the axial vector anomaly. It should not be too surprising that it satisfies the consistency relation, since the standard procedure for calculating anomalies consists in beginning with a regularized theory, which obviously satisfies all consistency conditions, the anomalies arising from non vanishing contributions coming from the regulator terms as the regulator masses go to infinity. Nevertheless, it must be pointed out that other authors [3] have obtained forms for the axial vector anomaly different from Bardeen's. These forms do not satisfy the consistency condition and are therefore incorrect.

Having established that the anomalous Ward identities satisfy the integrability conditions, one may look for a particular functional which satisfies them exactly. The complete vacuum functional will then be given by the sum of this particular functional plus a functional satisfying the normal Ward identities and all the physical consequences of the anomalies will be deducible from the particular functional. It is easy to see that no local functional depending only upon V_μ and A_μ satisfies the anomalous Ward identities. As a minimal generalization we introduce an octet of pseudoscalar fields Π_i transforming according to a non linear realization [4] of SU(3) × SU(3), in which the usual diagonal SU(3) is represented linearly. We shall use the exponential field, which transforms by a chiral transformation as

$$\exp(-2\mathrm{i}\xi') = \exp(-\mathrm{i}\alpha)\exp(-2\mathrm{i}\xi)\exp(-\mathrm{i}\alpha) ,$$

and under an SU(3) transformation as

$$\xi' = \exp(-\mathrm{i}\beta)\,\xi\,\exp(+\mathrm{i}\beta),$$

where $\xi_i = 1/F_\pi \Pi_i$, $\xi = \frac{1}{2}\xi_i\lambda_i$, $\alpha = \frac{1}{2}\alpha_i\lambda_i$, $\beta = \frac{1}{2}\beta_i\lambda_i$, but other realizations could also be used. We seek a local functional $W[\xi, V_\mu, A_\mu]$. The anomalous Ward identity is now

$$Y_i W \equiv (Z_i + U_i)W = G_i ,$$

where Z_i operates on the ξ field in W, while

U_i is that part of Y_i, given explicitly above, which operates only on the fields V_μ and A_μ. Using the notation $\alpha \cdot U = \int \alpha_i(x) U_i(x) \mathrm{d}_4 x$ etc., we have

$$\{\alpha \cdot (Z+U)\}^n W = \{\alpha \cdot U\}^{n-1} \alpha \cdot G \ ,$$

and therefore

$$\exp\{\alpha \cdot (Z+U)\}_W = W + \{\exp(\alpha \cdot U) - 1\} \alpha \cdot G / \alpha \cdot U \ .$$

The left-hand side equals $W[\xi', V'_\mu, A'_\mu]$. Now, for each ξ, one can find an $\alpha(\xi)$ such that $\xi' = 0$: for the exponential field, for instance, $\alpha(\xi) = -\xi$. Assuming $W[0, V_\mu, A_\mu] = 0$, we find the solution

$$W[\xi, V_\mu, A_\mu] = \frac{1 - \exp(\xi \cdot U)}{\xi \cdot U} \xi \cdot G[V_\mu, A_\mu] = \int_0^1 \mathrm{d}t \exp(-t\, \xi \cdot U)\, \xi \cdot G \ .$$

We have also verified explicitly, using the infinitesimal transformation properties of the fields ξ, V_μ and A_μ, that this local functional satisfies the anomalous Ward identities. In performing this check the consistency conditions for G_i have to be used.

The above explicit solution for $W[\xi, V_\mu, A_\mu]$ can be used as an effective Lagrangian giving directly all many particle vertices which describe the low energy consequences of the anomalous terms in the Ward identities. We work out these vertices for some particular processes. First we observe that our effective action contains vertices describing purely strong interactions among pseudoscalar mesons. The simplest of these is the five pseudoscalar vertex, given by

$$\frac{1}{6\pi^2 F_\pi^5} \epsilon_{\mu\nu\sigma\tau} \mathrm{tr}(\Pi \partial_\mu \Pi \partial_\nu \Pi \partial_\sigma \Pi \partial_\tau \Pi), \quad \Pi = \tfrac{1}{2} \Pi_i \lambda_i \ .$$

Vertices with more than five pseudoscalars can be worked out just as easily. Notice that this vertex cannot be obtained as part of a chiral invariant.

Next we give vertices with three pseudoscalars and one (vector) current. For instance, the anomalous contribution to K_{l4} comes from the vector current

$$\sin\theta \frac{1}{\mathrm{i}\, 6\sqrt{2}\, \pi^2 F_\pi^3} \epsilon_{\tau\mu\nu\sigma} \partial_\mu \Pi^+ (\partial_\nu \Pi^- \partial_\sigma K^+ + \sqrt{2}\, \partial_\nu \Pi^0 \partial_\sigma K^0).$$

The one photon-three pseudoscalar interaction is given by

$$\frac{e}{\mathrm{i}\, 24\pi^2 F_\pi^3} \epsilon_{\mu\nu\sigma\tau} F_{\mu\nu} \times$$

$$\times [(\partial_\sigma \Pi^+ \partial_\tau \Pi^- + \partial_\sigma K^+ \partial_\tau K^-)(\Pi^0 + \frac{1}{\sqrt{3}}\eta) + \partial_\sigma K^0 \partial_\tau \overline{K}^0 (\Pi^0 - \sqrt{3}\,\eta)]$$

Finally, we give the two photon-three pion vertex*. it is

$$\frac{1}{F_\pi^3} \frac{e^2}{3} \frac{1}{48\pi^2} \epsilon_{\mu\nu\sigma\tau} F_{\mu\nu} \Pi^+ \Pi^- (6 A_\sigma \partial_\tau \Pi^0 - F_{\sigma\tau} \Pi^0) \ .$$

Observe that the term containing A_σ combines with the one photon-three pion vertex to give a gauge invariant expression (integrating by parts).

As explained above, a common overall multiplicative constant in all these vertices is model dependent and would be best determined from the one pion-two photon vertex

$$-\frac{e^2}{(4\pi)^2} \frac{1}{6 F_\pi} \Pi^0 \epsilon_{\mu\nu\sigma\tau} F_{\mu\nu} F_{\sigma\tau}, \quad F_\pi \approx 95 \text{ MeV} \ .$$

* In a recent paper [5] the two photon-three pion anomalous interaction was estimated. The result given there is incorrect because the authors did not take into account the non-abelian structure of the group.

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