Limitations on the uses of Euclidean semiclassical methods

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(Received 16 February 1979)

The breakdown of semiclassical methods based on solutions of the Euclidean equations of motion is studied for the case of the quartic potential. The model is nontrivial, yet sufficiently tractable that techniques for careful treatment of non-Gaussian fluctuation modes can be illustrated clearly. It is shown how such fluctuations prohibit one from employing the semiclassical method once the dilute-gas approximation becomes invalid. Speculations are advanced regarding the relevance of these results to quantum chromodynamics.

I. INTRODUCTION

Euclidean classical solutions have been used to compute tunneling effects via the path integral in a number of physical problems.[1] Such semiclassical techniques can be expected to be accurate when tunneling is an adequate representation of the dominant physical effects. Conversely, when qualitative effects of a different character dominate, one should expect to find strong signals for the inapplicability of the method of approximation.

Consider the quartic potential of Fig. 1,

$$V(x) = V_0 - \tfrac{1}{2}\mu^2 x^2 + \frac{1}{4!} g^2 x^4 , \tag{1.1}$$

where $V_0 = 3\mu^4/2g^2$ has been adjusted so that $V = 0$ at the minima $x_\pm = \pm\sqrt{6}\mu/g$. For fixed μ and sufficiently small g, the lowest-energy wave functions are symmetric and antisymmetric linear combinations of functions peaked sharply about the classical minima of V. The corresponding energies are well approximated by harmonic levels about these minima, split by a small amount attributable to tunneling.

As g is increased for fixed μ, the outer walls of the potential become steeper and squeeze the classically allowed region. There is a corresponding increase in the zero-point energy in the classical minima. Eventually, the energy of the lowest state becomes larger than the classical barrier height, and the ground-state wave function becomes broadly spread about $x = 0$. At the same time, the energy separating the first excited state from the ground state grows comparable in magnitude to the ground-state energy,[2] behaving characteristically as $g^{2/3}$.

The aim of this paper is to elucidate what happens to the Euclidean semiclassical approximation as the variable parameter g is increased for fixed μ. From Coleman's review we already know that the approximation to the path integral corresponding to well-separated nodes of the approximate classical solutions—the dilute-gas approximation (DGA)—breaks down for large g because dominant configurations in the path integral are not truly dilute.

Callan, Dashen, and Gross have argued that one can go beyond the DGA by using a broader class of constrained Euclidean classical solutions.[3] A method for constructing constrained solutions of the desired class has been described by Rajaraman.[4] This method is developed for our present applications in Sec. II.

Careful treatment of the fluctuations about a given constrained classical solution is of vital importance to assessing the validity of the corresponding steepest-descent estimate of the path integral. A general formalism for handling this problem has been delineated by Gervais, Neveu, and Virasoro.[5] Unfortunately, their formalism is not well suited for dealing with linear constraints of the form introduced by Rajaraman. Consequently the fluctuation problem for linearly constrained solutions is tackled afresh in Sec. III from a ped-

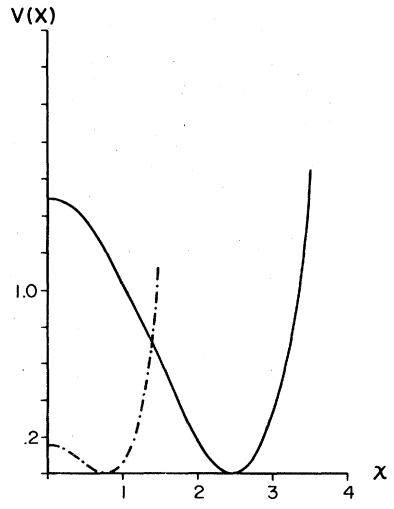

FIG. 1. The potential $V(x) = V_0 - \frac{1}{2}\mu^2 x^2 + \frac{1}{4} g^2 x^4$. The solid curve is for $\mu^2 = 1$, $g^2 = 1$. The dashed curve is for $\mu^2 = 1$, $g^2 = 10$.

estrian point of view, with first principles kept clearly in sight. In Sec. IV we show how the fluctuations destroy the semiclassical approximation once the conditions for the validity of the DGA are violated.

A remark is in order regarding a certain unevenness in the technical aspects of the paper. Sections II and III and the Appendices contain considerable technical detail because it is essential that the key mathematical aspects of the approximation method be understood clearly. Indeed, certain unexpected features which may be of general interest have arisen in our study of the collective coordinates. The upshot of the formulational stage of the problem is, as usual, a transformation of the Euclidean path integral into a classical statistical-mechanical problem. The statistical-mechanical problem so obtained [Eq. (3.21)] is quite complicated. At this stage, the level of technical analysis is reduced considerably. Following other workers in this field, we merely advance plausibility arguments regarding the behavior of the new problem. The breakdown of the approximation scheme that is discovered occurs within this context. Nevertheless, the result is natural and credible from a physical point of view.

The subject of this paper is interesting for its own sake. However, there also exists a surreptitious aim of the investigation. For the quartic potential Eq. (1.1), the "problem" is to find how the Euclidean path-integral method behaves in strong coupling. But if we were ignorant of the results of the Schrödinger approach to the problem, and knew only that the Euclidean method was failing, what would we conclude about the physics? Not ungenerously, this appears to be the present situation in quantum chromodynamics (QCD). The paper ends on an optimistic note in Sec. V.

II. CONTRAINED CLASSICAL SOLUTIONS

A. Generalities

We wish to compute the transition function

$$K_E(x_+, T/2;\, x_-, -T/2) = \mathfrak{N}_0 \int_{x_-}^{x_+} \mathfrak{D}x \exp\left[\left(-\frac{1}{\hbar}\right)\int_{-T/2}^{T/2} dt\left(\frac{\dot{x}^2}{2}+V(x)\right)\right]. \tag{2.1}$$

It is useful to introduce scaled variables $\phi_0 \equiv gx/\sqrt{6}\mu$, $\tau \equiv \mu t/2$, so that

$$S_{+-} \equiv \frac{g^2}{2\mu^3}\int_{-T/2}^{T/2} dt\left[\frac{\dot{x}^2}{2}+V(x)\right] = \int_{-\mu T/4}^{\mu T/4} d\tau\ \left[\frac{1}{2}\left(\frac{\partial\phi_0}{\partial\tau}\right)^2+(\phi_0{}^2-1)^2\right]. \tag{2.2}$$

Introduce the shift

$$\phi_0(\tau) = \phi(\tau)+\chi(\tau), \tag{2.3a}$$

$$\phi_0(\pm\mu T/4) = \phi(\pm\mu T/4) = \pm 1. \tag{2.3b}$$

Then, with $\tau_\pm \equiv \pm(\mu T/4)$, $\hbar^* \equiv g^2\hbar/2\mu^3$,

$$K_E(1, \tau_+, -1, \tau_-) = \mathfrak{N}\int_0^0 \mathfrak{D}\chi \exp\left(-\frac{1}{\hbar^*}S_{+-}[\phi+\chi]\right). \tag{2.1'}$$

Dropping the subscripts on S_{+-},

$$S[\phi+\chi] = \sum_{n=0}\int\cdots\int \frac{d\tau_1\cdots d\tau_n}{n!}\chi(\tau_1)\cdots\chi(\tau_n)S^{[n]}\Big|_\phi. \tag{2.4}$$

The goal is to saturate the integral (2.1') using choices for ϕ for which a truncation of Eq. (2.4) is a good approximation. It is known that in general it is necessary to make use of functions ϕ which pass through $\phi=0$ more than once. Such functions are not solutions of the classical Euclidean equations of motion with boundary conditions (2.3b).

One method for obtaining functions with the desired properties is to introduce impulses at intermediate times τ_j. To this end, add and subtract constraints $[\sum_j \lambda_j \int d\tau\,\phi(\tau)\,\delta(\tau-\tau_j)]$ onto S:

$$S[\phi+\chi] = \left\{S[\phi]+\sum_j \lambda_j\phi_{(j)}\right\} + \chi\left\{S'[\phi]+\sum_j\lambda_j\delta_j\right\} + \frac{\chi^2}{2}S'' - \sum_j\lambda_j[\phi_{(j)}+\chi_{(j)}] + \cdots. \tag{2.5}$$

Here we denote $\phi_{(j)} = \phi(\tau=\tau_j)$, and $\chi\delta_j \equiv \int d\tau\chi(\tau)\,\delta(\tau-\tau_j) = \chi_{(j)}$.

B. The solutions

The term in Eq. (2.5) linear in the fluctuations χ is eliminated by choosing ϕ such that

$$\left(S'[\phi]+\sum_j\lambda_j\delta_j\right)\Big|_{\phi=\phi^*} = 0, \tag{2.6a}$$

$$\phi^*_{(j)} = 0. \tag{2.6b}$$

We write the solution ϕ^* in the form

$$\phi^*(\tau;\, \tau_1\cdots\tau_n) = \sum_{j=1}^{n+1}\phi_j(\tau)\,\theta(\tau-\tau_{j-1})\,\theta(\tau_j-\tau), \tag{2.7}$$

where $\tau_0 \equiv \tau_-$, $\tau_{n+1} \equiv \tau_+$. Thus,

$$\frac{\partial^2\phi_j}{\partial\tau^2} = V'(\phi_j), \tag{2.8a}$$

$$\phi_j(\tau_{j-1}) = \phi_j(\tau_j) = 0, \tag{2.8b}$$

$$\dot{\phi}_1(\tau_0) = \dot{\phi}_{n+1}(\tau_{n+1}), \tag{2.8c}$$

$$\lambda_j = (\dot{\phi}_{j+1} - \dot{\phi}_j)|_{\tau=\tau_j}. \tag{2.8d}$$

As $\tau_0 \to -\infty$, $\tau_{n+1} \to +\infty$, $\dot{\phi}_1(\tau_0)$, and $\dot{\phi}_{n+1}(\tau_{n+1})$ vanish for finite $(\tau_1, \ldots, \tau_n)$.

For the potential of interest, Eqs. (2.8) are conveniently integrated using the conserved Euclidean energies E_j,

$$\dot{\phi}_j(\tau) = \pm\{2[E_j + (\phi_j^2 - 1)^2]\}^{1/2}. \tag{2.9}$$

The E_j, in turn, are determined by the initially specified times (τ_{j-1}, τ_j). If the solution ϕ_j has no nodes in $[\tau_{j-1}, \tau_j]$,

$$(\tau_j - \tau_{j-1}) = \left(\frac{2}{1+\epsilon_j}\right)^{1/2} K\left(\frac{1-\epsilon_j}{1+\epsilon_j}\right). \tag{2.10}$$

In this equation, $E_j \equiv -\epsilon_j^2$, and the elliptic function K is indexed as in Ref. 6.

Figure 2 graphs Eq. (2.10). The instanton solution centered at the origin

$$\phi^* = \tanh\sqrt{2}\,\tau \tag{2.11}$$

corresponds to $|\tau_j - \tau_{j-1}| \to \infty$, $\epsilon_j \to 0$. Another interesting limiting solution occurs for the case $\epsilon_j \approx 1$, $\tau_j - \tau_{j-1} \approx \pi/2$,

$$\phi^*_{\rm cp} \approx (1-\epsilon_j)^{1/2} \sin 2(\tau - \tau_{j-1}). \tag{2.12}$$

This latter solution corresponds to harmonic oscillations about the minimum of $(-V)$ at $\phi = 0$. These oscillations have very small amplitudes fixed by the nonlinear terms in the equation of motion.

One must bear in mind that the E_j are not *uniquely* determined by the impulse times (τ_{j-1}, τ_j) because solutions to Eq. (2.8) with nodes in $[\tau_{j-1}, \tau_j]$ are not excluded. For completeness, a brief discussion of the general situation is presented in Appendix A. There the assignment of Lagrange multipliers λ_j is also discussed. However, for most of our work, consideration of solutions ϕ_j nodeless in $[\tau_{j-1}, \tau_j]$ is sufficiently general.

In brief, solutions which will prove to be useful in the saturation of the path integral (2.1) have the following structure. In $[\tau_0, \tau_1]$, ϕ^* is basically half an instanton. An impulse acts at τ_1 with strength λ_1 so that ϕ^* will reach a maximum amplitude $(1-\epsilon_1)^{1/2}$ and then pass through zero at τ_2. The effective energy ϵ_1 and the impulse λ_1 are calculable in terms of $(\tau_2 - \tau_1)$. This process continues until τ_n, at which time the final impulse acts in such a manner that ϕ^* is again half an instanton.

C. The classical action

Using Eq. (2.7) we find

$$S[\phi^*] = \sum_j \int_{\tau_{j-1}}^{\tau_j} d\tau \left[\frac{\dot{\phi}_j^2}{2} + V(\phi_j)\right] = \sum_j S_j. \tag{2.13}$$

It is important to observe that there is no explicit dependence on the parameters λ_j in this expression. Analytically,[6] for ϕ_j nodeless in $[\tau_{j-1}, \tau_j]$,

$$S_j = \frac{[2(1+\epsilon_j)]^{1/2}}{3}\left[2\mathcal{E}\left(\frac{1-\epsilon_j}{1+\epsilon_j}\right) - \frac{\epsilon_j(2-\epsilon_j)}{1+\epsilon_j} K\left(\frac{1-\epsilon_j}{1+\epsilon_j}\right)\right]. \tag{2.14}$$

Of course, $\epsilon_j = \epsilon_j(\tau_j - \tau_{j-1})$ through Eq. (2.10).

Figure 3 graphs $S_j = S_j(\tau_j - \tau_{j-1})$. It is evident from this graph that S_j changes very little as a function of $\Delta\tau_j = \tau_{j-1}$ from $\Delta\tau = \infty$ down to $\Delta\tau \approx 3\pi/2$ (which is a numerical accident as far as one can tell). Over the range $[3\pi/2, \infty]$ it is sometimes convenient to describe the exponentially

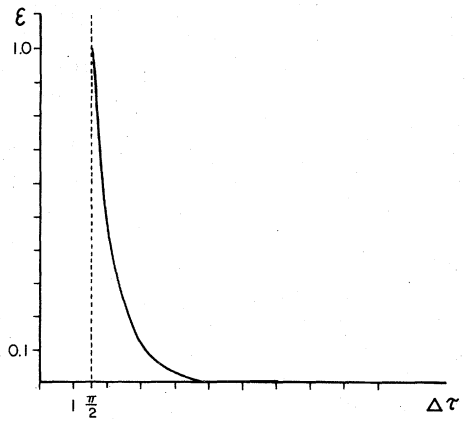

FIG. 2. A graph of the effective energy parameter ϵ versus the time interval between nodes in dimensionless units.

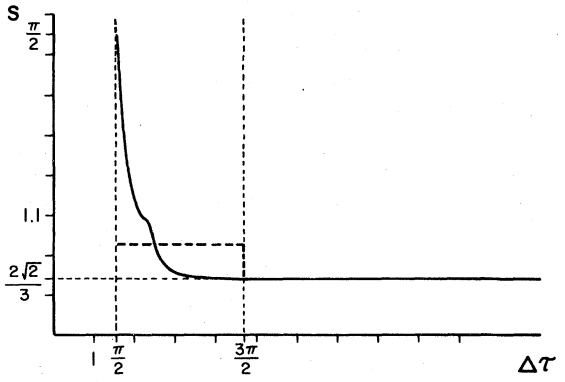

FIG. 3. A graph of the classical Euclidean action in units of $g^2/2\mu^3$ versus the time interval between nodes in dimensionless units.

small difference in S from its asymptotic ($\Delta\tau=\infty$) value as attributable to an "interaction."

With the full picture in view, however, it is more useful for our present purposes to treat the action approximately as a constant in $[3\pi/2,\infty]$. The numerically significant variations in the action occur within the small interval $[\pi/2, 3\pi/2]$. In this view, $\Delta\tau=\pi/2$ is an impenetrable hard core for the constrained classical solutions, r_h, while $r_s \approx 3\pi/2$ is an outer soft core.

Thus, suppose that all intervals $\Delta\tau_j \gg r_s$. Then $S[\phi^*]$ is minimal if all the ϕ_j are nodeless solutions (DGA), for if any one interval contains a solution with an extra node, S_j for that interval is at least twice what is would be without the extra node.

Next consider the effect of changing some pair of times so that their difference becomes smaller than r_s. Clearly, $S[\phi^*]$ will experience a net increase because the increase in the chosen interval will not be compensated by a decrease in S in the neighboring intervals in which $\Delta\tau$ is increasing—S is nearly a constant in those neighboring intervals.

Alternatively, if τ_0 and τ_{n+1} are finite and the number of constraints "n" is chosen so large that at least some $\Delta\tau_j < r_s$, $S(\phi^*)$ will be larger than it would have been for some smaller "n." In fact, there is a maximal value of n, $N_{\rm cp} \approx (\tau_{n+1}-\tau_0)/r_h$, at which $S(\phi^*_{\rm cp}) \approx N_{\rm cp} r_h \approx (\tau_{n+1}-\tau_0)$ is as large as it can be. (Here cp refers to close-packed.) This configuration gives a nonvanishing contribution to the energy splitting

$$\lim_{T\to\infty}\left[-T^{-1}\ln K_E(x_+, T/2; x_-, -T/2)\right].$$

Unfortunately, this configuration is unstable. We shall see why in the following sections.

III. COLLECTIVE COORDINATES

When ϕ^* satisfying Eq. (2.8) is substituted into Eq. (2.1′), we obtain

$$K_E(1,\tau_{n+1};-1,\tau_0)=\mathfrak{N}\exp\left(-\frac{1}{\hbar^*}\sum_j S_j\right)\int_0^0 \mathfrak{D}\chi\, e^{-\sigma(\chi)/\hbar^*}, \tag{3.1}$$

$$\sigma(\chi)=\int_{\tau_0}^{\tau_{n+1}} d\tau\left[\frac{\dot\chi^2}{2}+\frac{\chi^2}{2}V''(\phi^*)+\frac{\chi^3}{3!}V'''(\phi^*)+\frac{\chi^4}{4!}-\sum_j \lambda_j\chi(\tau)\,\delta(\tau-\tau_j)\right]. \tag{3.2}$$

These expressions are exact for any times τ_j and any consistent parameters λ_j that we choose. Being exact, the expressions are useless. The art of the approximation method is to choose the τ_j and λ_j so that $\sigma(\chi)$ may be truncated at order χ^2, as in Eq. (2.5), without losing any significant part of the integral over χ.

The principal complication in achieving this saturation is that given some $\phi^*=\phi^*(\tau;\{\tau_n,\lambda_n\})$, there may exist certain fluctuations $\tilde\chi$ such that $\phi^*+\tilde\chi$ is identically a path $\tilde\phi^*=\tilde\phi^*(\tau;\{\tilde\tau_n,\tilde\lambda_n\})$. That is, a given classical path ϕ^* plus its fluctuations may be identical to a different classical path, obtained under a different set of constraints. Which, then, is the "best" ϕ^* with which to begin?

When the relevant fluctuations $\tilde\chi$ are large on the scale set by the Gaussian approximation, one should sum over the various ϕ^*, each with its own set of small fluctuations. This is standard in the theory of saddle-point integration.

However, it may also happen that certain sets of classical solutions are close together on the scale described above. The extremum of the classical action in such cases may be a very broad minimum. It becomes necessary to separate fluctuations which fall rapidly away from the saddle point from those along which the action is slowly varying. We turn to defining these notions in a precise manner.

A. Symmetry modes

If τ_0, τ_{n+1} in K_E tend to $\mp\infty$, the theory is time translationally invariant. In particular, if we denote $\phi(t+\delta t)\equiv\phi^\delta\approx(1+L)\phi$, where $L=\delta t\, d/dt$, then $S[\phi^\delta]=S[\phi]$. But

$$S[\phi^\delta]\approx S[\phi]+(L\phi)S'+\tfrac{1}{2}(L\phi)S''(L\phi). \tag{3.3}$$

Thus if $\phi=\phi^*$ such that $S'[\phi^*]=0$, $(L\phi^*)$ is an eigenmode of S'' with eigenvalue zero. This is the most well-known source of non-Gaussian components in the small-fluctuation integral about a classical solution. The zero modes present by virtue of a symmetry of S must be eliminated in favor of collective coordinates.[7]

It is crucial to observe at the outset that the argument based on symmetry reviewed above is not sufficiently general to deal with the problem of non-Gaussian fluctuations for constrained solutions of the type Eq. (2.8). Let us see why not.

B. More general non-Gaussian fluctuations

Consider two configurations which give the same value to $S[\phi_0]$. There may be functions ϕ_0 not related by any obvious symmetry for which S is degenerate, but it will actually be sufficient to consider only ϕ_0 related by symmetry. We will use the following notation:

$$\phi_0 = \phi^* + \chi, \tag{3.4a}$$

$$\phi_0^g \approx (1+L)\,\phi_0 \equiv \tilde{\phi}^* + \tilde{\chi}. \tag{3.4b}$$

It is possible to specialize to $\tilde{\phi}^* = (1+L)\phi^*$. This was the case in the preceding subsection. More generally, however, it is possible to let

$$\delta\phi \equiv \tilde{\phi}^* - \phi^* \neq L\phi^*, \tag{3.5a}$$

$$\delta\chi \equiv \tilde{\chi} - \chi = L(\phi^* + \chi) - \delta\phi. \tag{3.5b}$$

From the discussion which introduced this subsection, we seek to ascertain whether a fluctuation taking a solution ϕ_1^* into a "nearby" solution ϕ_2^* and/or its translational image is non-Gaussian. A precise construction of nearby solutions is presented in Appendix A. All we need here is the definition $\phi_1^* = \phi^*(\{\tau_j\})$, $\phi_2^* = \phi^*(\{\tau_j + \delta\tau_j\})$ where all $\delta\tau_j$ are infinitesimal and no interval between nodes is less than r_h. That is,

$$S'(\phi_1^*) + \sum_n \alpha_n \delta(\tau - \tau_n) = 0, \tag{3.6a}$$

$$S'(\phi_2^*) + \sum_n \beta_n \delta(\tau - \tau_n - \delta\tau_n) = 0. \tag{3.6b}$$

The parameters α_j (β_j) depend upon the τ_j ($\tau_j + \delta\tau_j$) as has been discussed previously.

Taking first the case that all the $\delta\tau_j$ are different, we have, for $\chi^* \equiv \phi_2^* - \phi_1^*$,

$$S(\phi_2^*) \cong S(\phi_1^*) + \frac{\chi^{*2}}{2} S''(\phi_1^*) - \sum_n \alpha_n \chi^*_{(n)}. \tag{3.7}$$

But

$$S'(\phi_2^*) \cong S'(\phi_1^*) + \chi^* S''(\phi_1^*). \tag{3.8}$$

It follows that

$$S(\phi_1^* + \chi^*) = S(\phi_1^*) - \tfrac{1}{2}\int d\tau\, \chi^*(\tau) \sum_n [\alpha_n \delta(\tau - \tau_n) + \beta_n \delta(\tau - \tau_n - \delta\tau_n)]. \tag{3.9}$$

So the fluctuation χ^* moves us from ϕ_1^* to the nearby solution ϕ_2^*. Equation (3.9) confirms that the action is linear in this fluctuation and not quadratic. Consequently, the χ integral about ϕ_1^* is not Gaussian damped in the direction χ^*. This non-Gaussian behavior is not a symmetry mode of the system since the $\delta\tau_j$ may all be different. Rather, it is a consequence of working with constrained solutions.

Of course, for a fixed set of values $\{\delta\tau_j\}$, χ^* as defined above is a rather discrete fluctuation. But as the $\delta\tau_j$ are varied, a small $(n-1)$-dimensional surface will be swept out in function space near ϕ_1^*. The fluctuation integral is not Gaussian damped on this surface.

C. Translational invariance

If $\tau_\pm \to \pm\infty$, $\delta\tau_j = \delta\tau$ for all j, and $\beta_j = \alpha_j$ for all j, then the shift of the constrained solution corresponds to a uniform displacement. Since the constraints were both added and subtracted in Eq. (2.5), K_E should be invariant under this transformation. We now check that the formalism respects this invariance.

In Appendix A, the change of the solution ϕ^* over each interval between constraints is calculated as the constraint times change infinitesimally. From (A3) we find that if all the times change by the same amount,

$$\chi^*(\delta\tau) = -\delta\tau\dot{\phi}_1^*(\tau). \tag{3.10}$$

Thus, $\phi_2^*(\tau) \cong \phi_1^*(\tau - \delta\tau) = (\phi_1^*)^{g^{-1}}$. It follows that Eq. (3.9) becomes

$$\sum_j \alpha_j \int d\tau\, \dot{\phi}_1^*(\tau)\left[\delta(\tau - \tau_j) + \delta\left(\tau - \tau_j - \delta\tau\right)\right] = 0. \tag{3.11}$$

Formally, this equation is ill-defined because ϕ_1^* experiences jumps precisely at the τ_j. Nevertheless, if the formal expression is treated carefully as discussed in Appendix A, Eq. (3.11) becomes a rather obvious condition on the Lagrange multipliers α_j. Since specific boundary conditions are imposed on the entire solution ϕ_1^*, only $(n-1)$ parameters α_j can be independent. Stated differently, by displacement invariance only $(n-1)$ relative time differences are required to specify the constrained solution.

D. Small eigenvalues

In the preceding subsections, we have seen that there exist $(n-1)$ non-Gaussian fluctuation modes corresponding to $(n-1)$ relative shifts in the constraint times τ_j. There exists a further non-Gaussian mode corresponding to a uniform translation of the constrained solution. However, the argument of Sec. II A does not imply that even this latter fluctuation is an eigenstate of S'' with eigenvalue zero. The reason is that $S'(\phi^*) = \sum_j \lambda_j \delta(\tau - \tau_j) \neq 0$. Consequently, the usual method for introducing collective coordinates to eliminate the zero eigenmodes of S'' is not strictly applicable in the presence of constraints.

Nevertheless, there is a close connection between the non-Gaussian modes we have described and the low-lying eigenfunctions of S'' which we

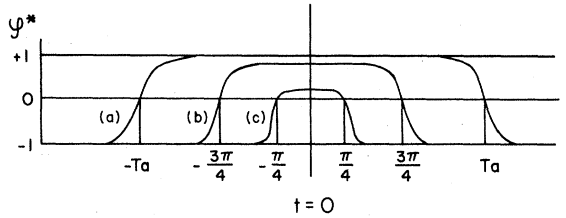

FIG. 4. Sketches of constrained classical solutions for (a) node separation times much larger than r_s, (b) node separation times of order r_s, (c) node separation times equal to $r_h + \delta$.

shall now discuss. Let

$$\chi(\tau) = \sum_m c_m \chi_m(\tau), \tag{3.12a}$$

$$S''(\phi^*)\chi_m = \left[-\frac{\partial^2}{\partial\tau^2} + 4(3\phi^{*2} - 1)\right]\chi_m = \nu_m \chi_m . \tag{3.12b}$$

The eigenstates χ_m are orthonormal and complete in $[\tau_-, \tau_+]$. The potential $V'' = 4(3\phi^{*2} - 1)$ is only piecewise continuous for constrained solutions ϕ^*, but this presents no difficulty for the eigenvalue problem.

Let us first deal with a simplified situation illustrated in Fig. 4. What happens at distant early or late times does not bear on the present limited considerations. Figure 5 displays the potential V'' in which the small fluctuations move for the solutions of Fig. 4.

It is clear that when the constraint times are very far apart, as in Fig. 4(a), in lowest approximation there are eigenvalues $\nu_0 = 0$ corresponding to translational invariance. For any finite time separation ($2\tau_a$), however, there will be exponentially small corrections to these levels due to tunneling. As τ_a decreases, the lowest corrected eigenvalue becomes increasingly negative, while the second would-be translational level grows eventually to order unity.

Referring to Fig. 6, we see that if $\delta\tau_1 = \delta\tau_2$, so that the solution ϕ_1^* is being translated, the new curve ϕ_2^* intersects the old curve ϕ_1^* at one time. On the other hand, if $\delta\tau_1 = -\delta\tau_2$, ϕ_2^* and ϕ_1^* never intersect. Consequently, the fluctuation $\chi^* = \phi_2^* - \phi_1^*$ closely resembles the nodeless eigenfunction χ_0 if $\delta\tau_1 = -\delta\tau_2$, or the eigenfunction with one node χ_1 if $\delta\tau_1 = +\delta\tau_2$.

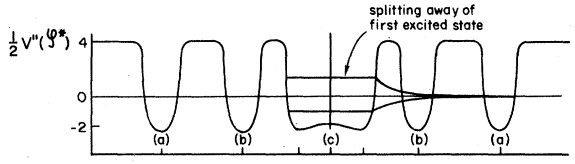

FIG. 5. The potential governing the small fluctuations $V''(\phi^*)$ for the solutions of Fig. 4.

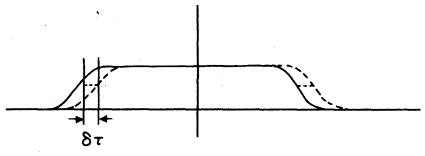

FIG. 6. A small displacement of the constraint times corresponding to a bodily shift of the constrained solution.

It is essential to bear in mind that the fluctuations $\chi_\pm^* \equiv \chi^*(\delta\tau_1 = \pm\delta\tau_2)$ are not *identical* to χ_0 or χ_1. They cannot be identical because it is required that the $\dot{\chi}_k$ be smooth at τ_1 and τ_2, while $\dot{\chi}^*$ jumps at precisely those times.

Thus, there are two distinct sources of non-Gaussian behavior in the fluctuation integral about constrained solutions ϕ^*. In addition to the χ^* discussed in Sec. III B, there are eigenfunctions of S'' with small and even negative eigenvalues. We have just argued that these effects bear a certain resemblance to one another, but they are strictly distinct effects.

In the case that τ_a is very large, the difference between $\chi_\pm^*$ and χ_1 becomes exponentially small. More precisely, the discontinuity in $\dot{\chi}_\pm^*$ tends to zero. Thus in the DGA limit, it is an excellent approximation to introduce collective coordinates for the near-symmetry modes and to neglect the linear terms in $\sigma(\chi)$ appearing in Eq. (3.8).

We shall follow this known example as a guide. Collective coordinates will be introduced to replace all those low-lying eigenfunctions of S'' which closely resemble the fluctuations χ^* corresponding to shifts in the τ_j.

Before implementing this procedure and assessing its effect, let us briefly discuss the case of n times. For very large relative times, there will be an n-fold degenerate zero mode at lowest approximation. Tunneling will broaden these levels into a band of n levels with both positive and negative magnitudes. As the relative times are brought closer together, these levels will split by non-negligible amounts.

For the same reason as was discussed for $n = 2$, the eigenfunction corresponding to a uniform shift of the n times has $(n-1)$ nodes, and so it is the highest-lying state of the set. The lower-lying eigenfunctions with fewer nodes correspond to certain "collective," or "normal-mode"-like, changes in the $(n-1)$ relative times. The lowest-energy, nodeless, eigenfunction corresponds to alternating compressions and rarifications of the nodes of the classical solution.

E. The Jacobian

The general paths ϕ_0 in K_E do not depend upon the special $\{\tau_n\}$ at which constraints were introduced. Thus we write

$$\phi_0(\tau) = \phi^*(\tau;\{\tau_n\}) + \chi(\tau;\{\tau_n\}). \tag{3.13}$$

To ensure that the fluctuations χ retained for Gaussian integration are orthogonal to the n lowest eigenmodes of S'', we impose the n conditions

$$\int d\tau \chi(\tau;\{\tau_n\})\chi_j = 0, \quad j = 0, \ldots, n-1. \tag{3.14}$$

But since the χ_j are complete, Eq. (3.14) amounts to $C_j = 0$, $j = 0, \ldots, n-1$.

Thus, to impose the desired conditions, we write

$$\prod_{j=1}^{n} dC_{j-1} = \left(\prod_{j=1}^{n} d\tau_j\right)\left|\frac{\partial(C_0, \ldots, C_{n-1})}{\partial(\tau_1, \ldots, \tau_n)}\right| \equiv \left(\prod_{j=1}^{n} d\tau_j\right)\mathcal{J}. \tag{3.15}$$

Consequently,

$$\prod_{j=1}^{n}\int d\tau_j \mathcal{J}\left[\prod_{j=1}^{n-1}\delta(C_{j-1})\right] = 1 \tag{3.16}$$

and

$$\prod_{j=1}^{n}\int dC_{j-1} F[\{C_k\}] = \prod_{j=1}^{n}\int d\tau_j \mathcal{J}_0 F[\{C_{k<n}=0; C_{k\geq n}\}], \tag{3.17}$$

where $\mathcal{J}_0$ is $\mathcal{J}$ evaluated at $C_{j-1} = 0, j = 1, \ldots, n$.

Now calculation of $\mathcal{J}$ involves computing the n^2 expressions

$$\frac{\partial C_i}{\partial \tau_j} = -\int d\tau \frac{\partial \phi^*}{\partial \tau_j}\chi_i + \int d\tau \chi(\tau;\{\tau_n\})\frac{\partial \chi_i}{\partial \tau_j}. \tag{3.18}$$

[We have used Eq. (3.13) to obtain this result.] In this equation, the derivatives $\partial\chi_i/\partial\tau_j \neq 0$ because if the τ_j change, V'' changes:

$$V''[\phi^*(\tau;\{\tau_n+\delta\tau_n\})] \cong V''[\phi^*(\tau;\{\tau_n\})] + \sum_j \delta\tau_j \frac{\partial\phi^*}{\partial\tau_j} V'''[\phi^*(\tau;\{\tau_n\})]. \tag{3.19}$$

The change in V'' is proportional to $\delta\tau$ and leads to a change in the eigenfunctions χ_i which can be computed in ordinary Rayleigh-Schrödinger perturbation theory. For our purposes it is sufficient to neglect this small change, as is conventionally done.[7] (In a theory requiring renormalization, contributions such as this one are required for replacing the bare coupling by a running coupling in the instanton Jacobian factors.)

Notice that in both (3.18) and (3.19) we have used the formal expression $\partial\phi^*/\partial\tau_j$. This is evaluated in the standard manner,

$$\frac{\partial\phi^*}{\partial\tau_j} = \lim_{\delta\tau_j\to 0}\frac{\phi^*(\tau;\ldots\tau_j+\delta\tau_j\ldots) - \phi^*(\tau;\ldots\tau_j\ldots)}{\delta\tau_j}, \tag{3.20}$$

where the difference in the numerator on the right-hand side is given by Eq. (A3).

F. Assembling the pieces

Initially, arbitrarily chosen constrained solutions ϕ^* were introduced into the path integral. The arbitrariness of these solutions resides partly in the arbitrariness of the choice of times τ_j at which ϕ^* is required to pass through zero. A second element of arbitrariness is that ϕ^* may pass through zero at other times as well. To avoid double counting, it is sensible to choose ϕ^* to have no nodes other than its constrained nodes, and to include solutions with extra nodes as contributions within a $(\sum_n)$ over all possible numbers of constraints. This point will be discussed further in the next section. At present, we confine our attention to fixed "n" and to ϕ^* with nodes at $\{\tau_n\}$ and no further nodes.

Consideration of the small fluctuations about a particular constrained solution leads to a picture in which the constraint times are not rigidly fixed after all. Small variations in the constraint times lead to new classical solutions which are well represented as the old solution plus a small fluctuation. Based on our knowledge of the action, summarized in Fig. 3, we expect the DGA approach to dealing with these fluctuations to be adequate for relative times larger than r_s. When any relative times are smaller than r_s, the action varies significantly. This is accompanied by significant variation in the Jacobian of the transformation between the non-Gaussian fluctuations and the collective coordinates.

A calculation which attempted to be as accurate as possible would resolve the fluctuations χ^* corresponding to relative displacements of the constraint times into all its components along the eigenfunctions χ_k of S''. In the space of the coefficients C_k, one should construct the n-dimensional hypersurface onto which these fluctuations project. The full fluctuation integral would then contain Gaussian integrals in the directions orthogonal to this hypersurface, and a complicated non-Gaussian integral on the hypersurface.[1]

However, we have argued that the χ^* of interest have especially large components along those eigenfunctions χ_k with small eigenvalues. Thus, an ap-

proximate claculation may be carried out by identifying the above-mentioned hypersurface as the cube with axes C_j, $j=0,1,\ldots,n-1$. This set of coefficients is replaced by collective coordinates $\{\tau_n\}$. The calculation is not identical to that of the DGA because the Jacobian of the transformation between $(\prod dc)$ and $(\prod d\tau)$ is a complicated function of the $\{\tau_n\}$.

In this manner, one obtains the approximate expression

$$K_E(1,\tau_+;-1,\tau_-)\approx\sum_n \mathfrak{N}'_{(n)}\exp\left[\left(-\frac{1}{\hbar^*}\right)\sum_{j=1}^{n}S_j\right]$$
$$\times\int\cdots\int d\tau_1\cdots d\tau_n\,\mathcal{J}_0(\{\tau_n\})\left\{\prod_{k=n}^{\infty}\left(\frac{2\pi\hbar^*}{\nu_k}\right)^{1/2}\exp\left[\left(\frac{1}{2\hbar^*}\right)\frac{[\sum_{j=1}^{n}\lambda_j\chi_k(\tau_j)]^2}{\nu_k}\right]\right\}. \quad (3.21)$$

The normalization factor $\mathfrak{N}'_{(n)}$ may be chosen in a convenient manner as discussed by Coleman.[1]

G. Two examples

Consider first a pair of times initially separated by an interval much larger than r_s. All other time intervals in the problem are also assumed large, so we neglect them. The DGA Jacobian will then emerge.

Explicitly, for large $\Delta\tau_j=\tau_j-\tau_{j-1}$, we find $\epsilon(\Delta\tau_j)\approx 8e^{-\sqrt{2}\Delta\tau_j}$. Thus if (for all j) $\tau_j\to\tau_j+\delta\tau_j$, ϵ changes by an amount $\delta\epsilon\approx-\sqrt{2}\epsilon(\Delta\tau_j)(\delta\tau_j-\delta\tau_{j-1})$. If ϕ_j is the solution in $[\tau_{j-1},\tau_j]$, and $\phi_j+\delta\phi_j$ is the solution in $[\tau_{j-1}+\delta\tau_{j-1},\tau_j+\delta\tau_j]$, we find $\delta\phi_j\approx-\delta\tau_{j-1}\dot\phi_j(\tau)+O(e^{-\sqrt{2}\Delta\tau_j})$ for $\tau<\tau_j^*\equiv\frac{1}{2}(\tau_{j-1}+\tau_j)$, with a similar result for $\tau>\tau_j^*$.

In addition, ϕ_j is constructed so that $\dot\phi_j$ is only exponentially different from the slope of the true instanton at the nodes [see Eq. (B1d)]. Since this jth interval occurs within other large intervals by hypothesis, the discontinuity in ϕ_j at the nodes is exponentially small (of order $e^{-\sqrt{2}\Delta\tau_j}$). Thus the Lagrange multipliers are zero to an excellent degree of approximation.

We are justified in applying the symmetry argument of Sec. IIIA to the extent that it is legitimate to neglect the Lagrange multipliers. Thus, restricting our attention to the interval $[\tau_{j-1},\tau_j]$, the lowest-energy wave functions in the two identical wells centered at τ_{j-1} and τ_j are approximately

$$\chi_0=\frac{\theta_L\dot\phi_L-\theta_R\dot\phi_R}{\{\frac{1}{2}[(\dot\phi_L,\dot\phi_L)+(\dot\phi_R,\dot\phi_R)]\}^{1/2}}, \quad (3.22a)$$

$$\chi_1=\frac{\theta_L\dot\phi_L+\theta_R\dot\phi_R}{\{\frac{1}{2}[(\dot\phi_L,\dot\phi_L)+(\dot\phi_R,\dot\phi_R)]\}^{1/2}}. \quad (3.22b)$$

The notation is taken from Appendix B. Subscripts L (R) refer to the left (right) of τ_j^*.

Making use of Eqs. (B1e) and (B1f), we have for example

$$\frac{\partial C_0}{\partial\tau_j^*}\approx\frac{1}{[(\dot\phi_L,\dot\phi_L)]^{1/2}}\int d\tau(\dot\phi_L{}^2\theta_L-\dot\phi_R{}^2\theta_R)$$
$$=0. \quad (3.23)$$

This reflects our earlier statement that the bodily displacement of the solution has a node. This displacement is therefore orthogonal to the lowest eigenfunction, which is nodeless.

Continuing in this manner, one easily verifies that there will be nonzero entries in the Jacobian matrix only on the "diagonal," i.e., where χ_k and $\partial\phi^*/\partial\tau_k$ have the same node structure. Each such term contributes a factor $[(\dot\phi_L,\dot\phi_L)]^{1/2}=[(\dot\phi_R,\dot\phi_R)]^{1/2}$. These norms equal those of the single-instanton solution. Extension of the argument to reproduce the DGA result for all the times is straightforward.

Much more interesting is the case in which $r_s>\Delta\tau_j>r_h$. Again, we consider this interval to be immersed in a dilute instanton gas. In the limit that $\epsilon(\Delta\tau_j)=1-\delta$, $\delta\ll1$, the solution in $[\tau_{j-1},\tau_j]$ is characterized as follows:

$$\Delta\tau_j\approx\frac{\pi}{2}(1+3\delta/8), \quad (3.24a)$$

$$\phi_j\approx\sqrt{\delta}(1+3\delta/8)\sin2(1-3\delta/8)(\tau-\tau_{j-1})+2^{-5}\delta^{3/2}\sin6(1-3\delta/8)(\tau-\tau_{j-1}), \quad (3.24b)$$

$$\dot\epsilon(\Delta\tau_j)\simeq-16/3\pi, \quad (3.24c)$$

$$\phi_j[\tau_{j-1}+\delta\tau_{j-1},\tau_j+\delta\tau_j]\approx\phi_j-\delta\tau_{j-1}\dot\phi_j$$
$$+\frac{(\delta\tau_j-\delta\tau_{j-1})}{\sqrt{\delta}}\frac{8}{3\pi}\sin2(1-3\delta/8)(\tau-\tau_{j-1})+O(\sqrt{\delta}\,[\delta\tau_j-\delta\tau_{j-1}]). \quad (3.24d)$$

Under the stated conditions, for $\tau>\tau_j$ and for $\tau<\tau_{j-1}$, the potential for the small fluctuations $V''(\phi^*)$

will be very close to its maximum value (+8) and constant. In the special interval, the amplitude of ϕ^* is at most $\sqrt{\delta}$, so $V''(\phi^*)$ will be very close to its minimum value (−4). It is therefore an excellent approximation to treat the potential as a rectangular well.

For notational ease, let $j=2$. The lowest eigenfunctions for the potential are then approximately

$$\chi_0 = A_0\left[e^{\mu_0(\tau-\tau_1)}\cos\frac{\omega_0\tau_R}{2}\theta(\tau_1-\tau)+\cos\omega_0(\tau-\tau_c)\,\theta(\tau-\tau_1)\,\theta(\tau_2-\tau)+e^{\mu_0(\tau_2-\tau)}\cos\frac{\omega_0\tau_R}{2}\theta(\tau-\tau_2)\right], \tag{3.25a}$$

$$\chi_1 = A_1\left[-e^{\mu_1(\tau-\tau_1)}\sin\frac{\omega_1\tau_R}{2}\theta(\tau_1-\tau)\right.$$
$$\left.+\sin\omega_1(\tau-\tau_c)\,\theta(\tau-\tau_1)\,\theta(\tau_2-\tau)+e^{\mu_1(\tau_2-\tau)}\sin\frac{\omega_1\tau_R}{2}\theta(\tau-\tau_2)\right]. \tag{3.25b}$$

In these equations, $\tau_R=\tau_2-\tau_1$, $\tau_c=\frac{1}{2}(\tau_1+\tau_2)$, and $\mu_n=(8-\nu_n)^{1/2}$, $\omega_n=(4+\nu_n)^{1/2}$.

The eigenvalues ν_0 and ν_1 are easily computed. They are recorded in Appendix B. There we also record A_0 and A_1, and present the details of the calculation of the Jacobian. Retaining only a term with a strong dependence on δ, we have

$$\mathcal{J}_0 \approx \frac{1.42}{\sqrt{\delta}}+O(\delta^0). \tag{3.26}$$

Qualitatively, $\mathcal{J}_0$ diverges as $\delta\to 0$ because there are an infinitely large number of small oscillatory solutions in $[\tau_{j-1},\tau_j]$. That is, there is a large entropy $\sim\ln\delta^{-1}$. Nevertheless, this divergence is integrable. If we allow $\Delta\tau_j$ to range from $r_h=\pi/2$ to $r_s=3\pi/2$, we obtain

$$\left|\int d\delta/\sqrt{\delta}\right| \simeq \frac{8\sqrt{3}}{3}.$$

In later work we shall refer to $\mathcal{J}_0$ of Eq. (3.26) as $\mathcal{J}(\mathrm{cp})$, cp = close-packed, for obvious reasons.

The two limits studied above have a clear physical interpretation. Configurations with n constraint times such that $(\tau_+-\tau_-)/n \gg r_s$ supply a factor $(1/n!)(\tau_+-\tau_-)^n$ to the path integral because each constraint time can be translated independently over the entire interval $[\tau_-,\tau_+]$. However, when pairs of constraint times are close together, the effect is to reduce a factor $(\tau_+-\tau_-)^2$ for independent translations to a factor $(\tau_+-\tau_-)$ for the center-of-mass translation times a small finite factor for the allowed relative motions.

We have not studied the details of the transition from $\Delta\tau=r_s$ down to $\Delta\tau=r_h$. The mathematics is very messy in this region, and does not add significantly to a qualitative understanding of the system. Having exposed the "kinematics" of the constrained solutions and their fluctuations in the DGA and cp limits, we are ready to study the "dynamics" of the saturation scheme.

IV. BEYOND DGA?

A. Breakdown of DGA

Equation (3.21) poses a difficult statistical-mechanical problem for which the constraint times τ_j are coordinates of the "particles" of the system. When the "temperature" $\hbar^*$ varies, the summand integrand in that formula maximizes in different regions as a function of the arguments n and $\{\tau_n\}$. Coleman's argument for the breakdown of the DGA may be phrased as follows.[1] Let all the Lagrange multipliers be set equal to zero, and let the Jacobian be of the standard DGA independent-particle form. We write the sum $\sum$ in terms of a "free-energy"

$$\sum \equiv \sum_n (T^n/n!)(Ke^{-S})^n$$
$$= \sum_n e^{-F_n}, \tag{4.1}$$

where $F_n \approx n(S-\ln TK)+n(\ln n-1)$ using Sterling's formula. Now $\partial F_n/\partial n=0$ for $n^*=TKe^{-S}$, and $\sum \approx e^{-F_{n^*}}$. But the form (4.1) for $\sum$ is only valid when $n^* r_h/T \ll 1$. This implies that $h^* \ll 2S_0$, where $S_0=2\sqrt{2}/3$ is the action of a single instanton.

If $\hbar^*$ is so large that the DGA breaks down, i.e., is internally inconsistent, it is possible that Eq. (3.21) is still a good approximation—there may be other regions of the summand integrand which dominate the sum and integral for large $\hbar^*$, and which approximate the full K_E very well. The task is to identify those other regions, if they exist.

As a simple illustration, let us discuss what happens when Coleman's estimate for n^* is such that the instantons are occupying a non-negligible fraction of the available "volume" $(\tau_+-\tau_-)$. Clearly, in such circumstances it is erroneous to assign a statistical weight $(\tau_+-\tau_-)^n/n!$ to the configuration. What then?

For definiteness, consider the case of two constraints in isolation. The DGA statistical weight is

$$I_2 = \frac{(\tau_+ - \tau_- - 2r_s)(\tau_+ - \tau_- - r_s)}{2} . \quad (4.2)$$

On the other hand, if the constraints were treated as close-packed, we saw in Sec. III that we should assign the configuration a statistical weight

$$I_2' = \kappa \int_{\tau_- + r_h/2}^{\tau_+ - r_h/2} d\tau_c = \kappa(\tau_+ - \tau_- - r_h) . \quad (4.3)$$

Here $\kappa = (1.42)8\sqrt{3}/3 = 6.56$. Taking $r_h = \pi/2$, $r_s = 3\pi/2$, we find that $I_2' > I_2$ if $(\tau_+ - \tau_-) < 24.62$. Of course, for I_2 to be applicable at all, it is necessary that $(\tau_+ - \tau_-) > 2r_s = 9.42$. We find then a range of $(\tau_+ - \tau_-)$ in which the statistical weight for a close-packed configuration is larger than that for a dilute configuration.

This calculation does not prove that the cp configuration dominates the path integral. Other factors enter, especially the action, which depends strongly on $\hbar^*$. Nevertheless, a suggestive picture emerges from such considerations. As the temperature of the gas increases, pairs of particles are produced, so the gas becomes dense. Energetically (the action plays the role of energy) a close-packing of the gas is disfavored. But it might become favorable for condensation of close-packed pairs to occur nevertheless because of the phase space liberated by this process. We will now develop these ideas.

B. Dressed Instantons

The reduction of the full problem into one utilizing only the DGA and cp limits which we have been discussing can be formalized by approximating the action by the dashed curve in Fig. 3. The full contribution to K_E of a pair of constraint times separated by large times from any other constraint times is then

$$z_{12} = \mathfrak{N}_{12} \iint d\tau_1 d\tau_2 \mathcal{J}_0(\tau_1, \tau_2; \tau_{j\neq 1,2}) \exp\left[-\frac{1}{\hbar^*} S(12)\right] \left\{ \prod_{k=n}^{\infty} \left(\frac{2\pi\hbar^*}{\nu_k}\right)^{1/2} \exp\left[\frac{\lambda_1^2 \chi_k^2(\tau_1)}{2\hbar^* \nu_k}\right] \exp\left[\frac{\lambda_2^2 \chi_k^2(\tau_2)}{2\hbar^* \nu_k}\right] \right\} . \quad (4.4)$$

For the given conditions, the only terms in $\mathcal{J}_0$ that depend explicitly on τ_1 and τ_2 are contained in a 2×2 subdeterminant evaluated as in Sec. IId. Changing coordinates to $\tau_R = \tau_2 - \tau_1$, $\tau_c = \frac{1}{2}(\tau_1 + \tau_2)$, the truncated model amounts to the following:

$$z_{12} \approx \int_{r_h}^{r_s} d\tau_R \int_{\tau_0}^{\tau_3} d\tau_c \mathcal{J}(\mathrm{cp}) \exp\left[\frac{-S(\mathrm{cp})}{\hbar^*}\right] \left\{ \prod_k (\cdots) \right\} + \int_0^{\tau_3 - \tau_0} d\tau_R \int_{\tau_0}^{\tau_3} d\tau_c J(\mathrm{DG}) \exp\left(-\frac{S_0}{\hbar^*}\right) \left\{ \prod_k (\cdots) \right\} \Big|_{\lambda_1 = \lambda_2 = 0} . \quad (4.5)$$

It should be clear that Eq. (4.5) embodies a well-defined approximation to the two-body forces between the particles of the analog gas. Any approximation of this type neglects many-body forces which are also present in the full problem. The nontrivial nature of the many-body interaction can be illustrated by examining three particles.

Consider a configuration in which nodes (123) are all close together. We analyze the "kinematics" of these three particles following the lines of Sec. III.

There exist three low-lying eigenfunctions of S'' which correspond to (1) total displacement, (2) a breathing mode of [1] and [3] relative to [2] subject to zero total displacement, and (3) a similarly constrained bound motion of [13] relative to [2].

As for the case of two particles [cf. Eq. (B6)], the various $\partial\phi_j^*/\partial\tau_k$ can be organized into collective modes with the same node structure as the low-lying eigenstates described above. We obtain significant contributions to the Jacobian (of the form $\delta^{-1/2}$) from those $\partial C_j/\partial\tau_k$ which contain projections of the collective $\partial\phi_j^*/\partial\tau_k$ into the corresponding eigenfunctions. Thus, there will be one factor of $\delta^{-1/2}$ for each relative time.

However, a glimpse at the formulas in Appendix B should convince the reader that the numerical coefficient multiplying $\delta^{-1/2}$ is a complicated function of the eigenvalues in the well $V''(\phi^*)$. As we pass from two particles at separation $\approx r_h$, to three particles pairwise separated by $\approx r_h$, V'' broadens by an amount $\approx r_h$. The low-lying eigenvalues in the potential change nontrivially under such a broadening. (Of course, there is a similar change in the eigenvalues depending on whether our cp approximation is taken at separation r_h or r_s or somewhere in between. This can be fixed once and for all. The present problem is that for large numbers of particles closely packed with respect to each other, the well breadth increases with the number of particles.)

The remarks above may be summarized in the formula for n close-packed relative time differences:

$$\mathcal{J}_0(n\ \mathrm{cp}) \cong \frac{\rho_n}{(\delta^n/2)} . \quad (4.6)$$

It is *not* the case that $\rho_n = (\rho_1)^n$. Rather, one might consider $\ln[\rho_n/(\rho_1)^n]$ to be a many-body potential.

Technically, it is not difficult to use a computer

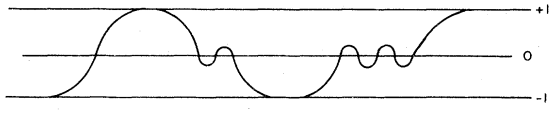

FIG. 7. An example of a constrained solution containing close-packed configurations interspersed among dilute-gas solutions.

to obtain the function ρ_n. However, this would be a wasted effort for our present purposes. We shall proceed to introduce a resummation of Eq. (3.21) making use of DG and cp configurations without worrying about the detailed form of ρ_n.

The elementary units of the proposed resummation are constrained solutions with $\Delta\tau \gg r_s$ and constrained solutions with $\Delta\tau \approx r_h$. A particular constrained solution built from these units is illustrated in Fig. 7. The idea underlying this resummation is that although we have seen that specified constraint times are meaningless once fluctuations are taken into account, the collective-coordinate integrals break up into two pieces as in Eq. (4.5). Rather than attempting to examine whether condensation occurs from the dilute-gas into close-packed configurations as $\hbar^*$ increases, we introduce both configurations on an equal footing in the sum Eq. (3.21). The utility of this procedure will become evident momentarily.

We see in Fig. 7 that clusters of close-packed solutions are interspersed with dilute-gas solutions. The close-packed clusters are of two kinds, even and odd. The even (odd) clusters contain an even (odd) number of constrained nodes. Minimal changes in the derivatives $\dot{\phi}_j(\tau_j)$ occur if the slopes of the dilute-gas solutions incoming and outgoing from a given cluster are of opposite (equal) sign for even (odd) clusters as in Fig. 8(a). We shall limit our considerations to configurations of this kind.

Figure 8(b) indicates that each even and odd cluster can contain an infinite number of terms in the limit that $(\tau_+ - \tau_-) \to \infty$. Evidently, an expansion of any graph of Fig. 8 corresponds to an infinite sum of the graphs of Fig. 7. A sum over all possible locations of even (E) and odd (O) blobs includes all possible mixtures of dilute-gas and cp contributions to Eq. (3.21).

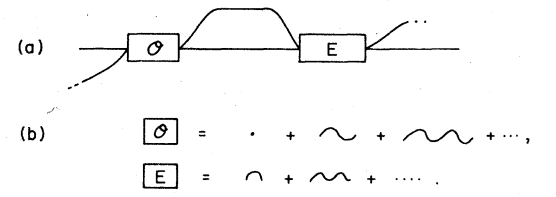

FIG. 8. A segment of a resummation of the series of possible close-packed and dilute configurations.

Graphically, we may interpret an O blob as a dressing of a primitive instanton. Similarly, an E blob corresponds to annihilation or creation of half-instanton pairs. The entire gas thus consists of dilute gas of "nonlocal" dressed instantons which interact through pair-number-changing vertices.

Let us momentarily assume that $\rho_n = (\rho_1)^n$, and set all constraint parameters equal to zero. Then, e.g.,

$$O \approx \sum_{m=0}^{\infty} \left(\frac{K\rho_1}{(2\pi\hbar^*)^{1/2}} \right)^{2m} \exp(-2mS_{\rm cp}/\hbar^*). \quad (4.7)$$

This insertion "renormalizes" the "bare" action of the primitive instanton. There will be no real conflict with a DGA estimate built from such dressed instantons, with the centers of the dressing blobs playing the role of instanton node times, provided the insertion O itself maximizes for a value of "m" in the sum (4.7) such that O occupies a small fraction of the available volume and has a reasonably defined center. For example, for very small $\hbar^*$ the usual DGA estimate corresponds to dominance of the term $m = 0$ in Eq. (4.7).

The true problem is more complicated because $\rho_n \neq (\rho_1)^n$ and because in general the constraint parameters λ_n cannot be ignored. Nevertheless, the method would be tractable with the aid of a computer if numerical accuracy was a goal. Qualitatively, however, the picture of dressed instantons developed from the cp truncation resembles closely the "meron" picture which has been proposed.[3] In the next subsection, we offer a brief critique of the meron picture as it applies to the quartic potential.

C. Merons

All of the classical solutions employed in the analysis of the preceding sections have had an energy parameter ϵ strictly less than 1. The limiting period $r_h = \pi/2$ is the period of small harmonic oscillations about the minimum of $-V(\phi)$ at $\phi = 0$. However, a solution of the classical equations with $\epsilon = 1$ also exists, although it is disjoint from all the other solutions because it is time independent: $\phi(\tau) = 0$. This solution is perfectly acceptable over arbitrary finite time intervals between impulses, and plays the role of a meron pair for this problem.

In some ways the zero solution resembles the cp solutions. The amplitudes of the latter are nearly zero, and a graph of a string of oscillations with fixed $\epsilon \approx 1$ is barely distinguishable from that of the zero solution. Further, the action of such a string of n oscillations is approximately $nr_h = n\pi/2$ while the action of a $\phi = 0$ solution over a time interval nr_h is also nr_h. A string of oscillations with the

same ϵ parameter does not require impulses between periods, so the λ_j are all zero over the interval of the string, as they are for $\phi=0$ in the same interval. Finally, $V''(\phi^*)$ has been consistently approximated with $\phi=0$ in the cp regions.

There are also important differences between the cp solutions and $\phi=0$. The cp solutions come in approximately quantized lengths. That is, a continuous string of n oscillations with $\epsilon\approx 1$ is almost exactly nr_h units long. Now if very small impulses are imparted at the nodes along the string, some of the oscillations may broaden slightly. Thus a series of such mildly constrained solutions exists over an interval $(nr_h+\Delta)$, where strictly Δ is a continuously variable amount. But sensibly if $\Delta > r_s$, the impulse contributions to the action are minimized by breaking the interval into (say) an $(n+1)r_h$-length continuous string with a smaller variable remainder Δ'. (Action arguments will always favor going to DG configurations. The point here is to stay within the cp class of configurations.) The solution $\phi=0$, on the other hand, is always strictly of continuously variable duration.

In addition, the small fluctuations about the two types of classical solution have different structures. A first this is mystifying because we have used the same V'' for the two solutions. However, the point is that projections of the fluctuations onto the eigenfunctions with small eigenvalues were suppressed in the cp case by making use of nearby classical solutions. This method is not available in the case of the $\phi=0$ solution because there are not enough nearby solutions. The only variables describing $\phi=0$ in the interval $[\tau_{j-1},\tau_j]$ are τ_{j-1} and τ_j. Thus, there are only two variables available for use as collective coordinates, while as $\Delta\tau_j$ increases there is an ever increasing number of small eigenvalues. Thus, non-Gaussian fluctuations overwhelm the estimate based on the solution $\phi=0$.

Clearly, the same flaw invalidates utilization of long strings of oscillations with no intermediate constraints. If one "long" solution is used with no available freedom to vary the intermediate time intervals, there will exist more small eigenvalues than available parameters to serve as collective coordinates. Again, non-Gaussian fluctuations will overwhelm the estimate.

D. Breakdown of the semiclassical approximation

The arguments of the preceding subsection enable us to recognize, finally, symptoms for the complete breakdown of the semiclassical approximation method for the quartic potential. Only a brief reflection is needed to understand that insertion of small impulses cannot ultimately rescue the scheme from the instabilities uncovered above.

That is, the presence of n constraints enables us to introduce collective coordinates formally to eliminate n low-lying eigenmodes of S''. But there may be more than n low-lying modes. We have only to count the number of bound states in the potential V'' to see that if the cp E or O blobs grow indefinitely, there will be more low eigenmodes than there are available collective-coordinate parameters.

Specifically, consider the rectangular well with variable trough length w. The even-parity bound states in this potential have energies ν_n determined from the equation

$$(\nu_n+4)^{1/2}\tan\left(\frac{w(\nu_n+4)^{1/2}}{2}\right)=(8-\nu_n)^{1/2}. \tag{4.8}$$

Let $x_n=\nu_n+4$, $\theta_n=w\sqrt{x_n}/2$, so that (4.8) becomes

$$\tan\theta_n=\frac{(3w^2-\theta_n^2)^{1/2}}{\theta_n}. \tag{4.8'}$$

The right-hand side vanishes for $\theta_n=\sqrt{3}w$. Now the function $\tan\theta$ has exactly m positive branches in the interval $m\pi>\theta>0$. Thus, Eq. (4.8') will have m roots if $\sqrt{3}w=m\pi$. If $(m+1)\pi>\sqrt{3}w>m\pi$, there will be $(m+1)$ roots, etc. Since $\nu_n=(2\theta_n^2/w^2)-4$, the root corresponding to the mth branch has an eigenvalue $\nu_{(m)}\lesssim+2$ for large m; the root corresponding to the first branch is $\nu_{(1)}\sim O(1/m^2)-4$.

Similar arguments apply for the odd-parity solutions. The odd eigenvalues are determined by the equation

$$\cot\theta_n=-\frac{(3w^2-\theta_n^2)^{1/2}}{\theta_n}. \tag{4.9}$$

There are m negative branches of $\cot\theta$ in the interval $m\pi>\theta>0$, so if $\sqrt{3}w=m\pi$, there will be m odd-parity eigenvalues.

In all then if $w=m\pi/\sqrt{3}$, there will be $2m$ alternating even and odd bound states. Now a cp configuration with n relative times occupies a maximal interval $\Delta\tau=nr_s\approx 3n\pi/2$. Thus $w=3n\pi/2$ is the width of the rectangular well for such a configuration. There are, excluding the center-of-mass motion, n collective coordinates at our disposal. These can be identified with all of the bound states of the potential if the potential has width $\tilde{w}=n\pi\sqrt{3}/6$. But $\tilde{w}<3n\pi/2$. Thus, the well required for the cp configuration under consideration has more bound states than the number of available collective coordinates.

A wary reader might question whether the criterion being imposed for the breakdown of the semiclassical method is rigorously necessary. What if the bound states unaccounted for by collective coordinates have sufficiently large eigenvalues that a Gaussian approximation is quantitatively valid? For some of the eigenvalues near

numerical value of order 1, this criticism is valid. But as w increases, there is an increasing number of eigenvalues which are numerically near zero. Thus, a mild dressing of the bare instantons can be expected to extend the range of the DGA into a slightly larger $\hbar^*$ regime, but the picture collapses entirely once large-scale dressing is required.

Instabilities occur in Eq. (3.21). The full path integral (2.1′) does not suffer from these problems because the χ^4 terms neglected in (3.21) provide the required damping on the integral regardless of the coefficient of the χ^2 term. This is as it should be. When the fluctuations are sufficiently violent, the full nonlinear structure of the theory must be invoked to ensure that the theory remains well defined.

V. SUMMARY AND CONCLUSIONS

A. Results for the quartic potential

It is entirely obvious *a priori* that violent quantum fluctuations destroy a tunneling description of the level splittings for a quartic potential in strong coupling. But the breakdown of the tunneling description within the Euclidean path-integral method can only be seen if one treats the fluctuations carefully. From the point of view of methodology, the treatment of collective coordinates for constrained solutions described in this paper may be of general applicability.

A discussion of an interacting gas of dressed instantons introduced in Sec. IV was cut short because detailed study of that version of the model was not warranted for the principal aims of this investigation. The purpose of introducing those notions was to motivate the meron description of the interacting gas in a natural manner. The action of the theory behaves in such a way that introduction of a two-component description (DG and cp) suggests itself quite naturally. Thus, the statistical-mechanics problem is one of coexistence. But the cp "phase" is mathematically almost indistinguishable from the meron approximation. The one place in which the cp and meron descriptions differ is in the treatment of collective coordinates. The meron is obviously unstable; the meron picture immediately suggests why the cp description is also unstable. Consequently, when the conditions for the DGA fail, the entire method fails.

Let us now consider the above method in reverse order. If all that we knew was that the DGA was showing symptoms of breaking down in strong coupling, and that we were being driven to consider Euclidean solutions which corresponded to *maxima* of the Minkowski potential energy, what should we conclude?

The simple case of the quartic potential behaves in the following way. In weak coupling, the wave function is sharply peaked at the minima of the Minkowski, or "true," potential $V(x)$. Likewise the Euclidean classical solutions which dominate the path integral spend most of their "time" at the classical minima of V. In strong coupling, the wave function is broadly peaked about the classical *maximim* of the true potential V. Evidently the Euclidean path integral mimics this situation as best it can by driving one to solutions which spend most of their "time" at this same maximum.

We have seen that the Euclidean method fails in strong coupling because the small fluctuations about the $\phi(\tau)=0$ solution are not legitimately small. These must be stabilized by the quartic terms in the fluctuations, which means that one is back to solving the original problem. Faced with this purely mathematical problem, one is initially tempted to seek saddle points in the complex ϕ domain. For example, if $\phi \to i\phi$, the Lagrangian of the Euclidean problem becomes $L'_E = -[\frac{1}{2}\dot{\phi}^2 - (\phi^2+1)^2]$. Evidently solutions of the classical Euler-Lagrange equations which follow from this Lagrangian are exactly those of a Minkowski version of the theory. However, the contours of the path integral cannot be rotated to attain this result because the integrals have essential singularities in the directions

$$\phi = |\phi| e^{i\pi/4}.$$

So for the quartic potential, the only possible rotation available is back from Euclidean time to Minkowski time.

Of course, for this potential different methods of approximation are available. For example, there exists a WKB scheme which can be implemented using the Minkowski classical solutions.[8] However, from Ref. 8 one learns that the Minkowski classical path $\phi=0$ for this potential is insignificant in the strong-coupling regime. (The well of Ref. 8 has the opposite sign for the quadratic term in the potential. For strong coupling this is irrelevant.) Rather, one should use time-dependent solutions with classical energies larger than or equal to the barrier height.

B. Bridges to QCD

The present discussion bears loosely on the gauge theory in the following manner. Following Callan, Dashen, and Gross,[3] it is instructive to consider the single Belavin-Polyakov-Schwartz-Tyupkin (BPST) instanton[9] as a specific solution within the spherical ansatz

$$A^a_\mu[\phi] = \frac{\eta^a_{\mu\nu} x^\nu}{R^2}[1+\phi(\lambda)]\,. \tag{5.1}$$

Here $R=(x_\mu x_\mu)^{1/2}$ and $\lambda = \ln R$. The Euclidean action

of the Yang-Mills theory becomes

$$S_E[A[\phi]]=\frac{3\pi^2}{g^2}\int_{-\infty}^{\infty}d\lambda\left[\left(\frac{\partial\phi}{\partial\lambda}\right)^2+(\phi^2-1)^2\right]. \quad (5.2)$$

This is the same as Eq. (2.2) with $\lambda=\sqrt{2}\tau$, $\mu^3=\sqrt{2}\pi^2$. The instanton solution (2.11) centered at $\tau=\tau_0$ is precisely the BPST instanton solution for $A_\mu^a[x;\rho=\exp(\lambda_0)]$. In addition, the meron solution $\phi=0$ reproduces the Euclidean meron solution of QCD using (5.1).[10]

Gaussian fluctuations for the QCD theory result in passage to a running coupling constant $g^2 \to g^2(\mu\rho)$, where $8\pi^2/g^2(\mu\rho)\approx -11\ln\rho\mu+\cdots$. Further, an integral over scale sizes must be evaluated. For our present purposes, however, consider the shift $A_\mu^a \to A_\mu^a[\phi]+a_\mu^a$ in the functional integral prior to selection of a particular solution ϕ. Ignoring complications in the measure and initially neglecting the effects of the linear coupling to the fluctuations, one would find that an integral over scale sizes still appeared, and that the coupling became the running coupling. Thus, one would be integrating over a sequence of double-well potentials, ranging from those for which the DGA is valid through those (for large ρ) for which the DGA breaks down.

Within the above framework, the integral over ρ would drive us to solutions with $\phi\approx 0$. At this point, however, a reexamination of the full structure of the fluctuations, including the neglected linear coupling, would reveal a breakdown in the approximation scheme of the type described earlier.

Any specific ansatz such as (5.1) is much too restrictive to allow extraction of general features of the theory. There is even a misleading element present in that $S_E(\phi)$ does not possess the periodicity of the classical gauge field vacuum. That this essential feature of the theory may have useful consequences for the approximation method can be illustrated with the example of the pendulum

$$L=\tfrac{1}{2}\dot{\theta}^2-g(1-\cos\theta). \quad (5.3)$$

For this pendulum, the classical energy is given by

$$E=\tfrac{1}{2}\dot{\theta}^2+g(1-\cos\theta). \quad (5.4)$$

Upon rotation to the Euclidean domain,

$$\epsilon=\tfrac{1}{2}\dot{\theta}^2+g(\cos\theta-1) \quad (5.5)$$

is a conserved quantity. But if

$$\theta_M=\theta_E\pm\pi, \quad (5.6a)$$

$$\epsilon+g=E-g, \quad (5.6b)$$

the Euclidean classical solutions and the Minkowski classical solutions are the same. There exists an intimate relationship between the classical Euclidean solutions and the classical Minkowski solutions which is absent in the case of the quartic potential.

The relevance of this intimacy for the evaluation of the path integral can be elucidated straightforwardly for the case of the pendulum. First, $\theta=(2n+1)\pi$ are solutions of the Minkowski equations of motion with $E=2g$. These sit at the peaks of the classical potential. Then $\theta=2n\pi$ are Euclidean solutions with $\epsilon=0$. These Euclidean solutions are at the maxima of the Euclidean potential. More generally, unbounded Minkowski solutions exist for $E>2g$. Shifting θ by π, these are also unbounded solutions of the Euclidean theory with $\epsilon>0$.

Now the DGA for the pendulum utilizes very special solutions with $\epsilon=0$ which move from one minimum of $g(\cos\theta-1)$ to another.[11] The fluctuations about these solutions become large as g becomes very small. But for this theory, solutions are available to meet the desired boundary conditions without additional constraints. So mathematically, use of the Euclidean path integral for $\epsilon\geq 0$ will be indistinguishable from use of a Minkowski path integral (rendered proper by giving t a small imaginary part) which is saturated by solutions with $E\geq 2g$.

Generically, let us say that merons are classical solutions with energies equal to or greater than the barrier heights of the problem. Technically, this is not the same use of the term as in Ref. 3, but there is no real cause for confusion. Merons then are "extended" solutions rather than solutions localized about the minima of the classical potential. We have encountered two possibilities in our discussion. For the quartic well, Minkowski merons do not continue analytically into Euclidean merons. For the pendulum, they do, provided we add a trivial shift.

It may happen within any given theory that for some limit of the coupling, quantum fluctuations lead to ground states which are extended in the sense defined above: The ground-state energy may lie above the classical barrier heights. In such cases, surely Minkowski merons are the appropriate objects on which to base a semiclassical approximation. But it may be that Euclidean merons exist with isomorphic mathematics in the path integral. In such cases, the Euclidean calculations may be extended into the quantum domain. We now pass to speculations regarding the situation in QCD.

C. Speculations

(1) It is known that the Euclidean QCD meron corresponding to $\phi=0$ in (5.1) solves the classical equations of motion almost everywhere. However, this solution is a maximum of the action rather

than a minimum, and once constraints are introduced to meet typical boundary conditions, S'' has at least one negative eigenvalue. Moreover, the Euclidean meron is the analytic continuation of a finite-energy, finite-action Minkowski solution.[10]

Thus, consider the following conjecture. Attempts to saturate the Euclidean functional integral with merons correspond to approaching the strong-coupling vacuum from below. That is, strong fluctuations are pushing one away from classical solutions localized near the minima of the classical potential and toward solutions which are extended, i.e., merons. Alternatively, it should be possible to attempt a Minkowski-space (WKB) calculation utilizing merons of the de Alfaro-Fubini-Furlan[10] type with appropriate constraints. The Minkowski merons have finite energy to be sure, but they are not necessarily particle-like solutions. The zero-point energy of the strong-coupling vacuum can lie a finite amount above the zero-point energy of the perturbative vacuum.

The point is that since QCD is nonlinear but has no scale, the concept of a vacuum is ill defined. There exists no criterion for whether quantum fluctuations are large or small without a scale relative to which the effective magnitude of these fluctuations may be measured. Presumably, the running coupling constant organizes these fluctuations, and one can speak of strong or weak coupling within the context of responses to external probes or in scattering processes. If one constructs the quantum vacuum as the classical vacuum plus a specific set of quantum fluctuations about the classical vacuum, it is evident that the quantum vacuum must be defined differently at different distance scales.[12]

Figure 9 illustrates the point of view that is being advocated. Our current understanding of the $A_0=0$ gauge[13] leads us to introduce cylindrical coordinates in the space of vector potentials. If gauge transformations which do not change winding number, assigned to the $\hat{z}$ direction in this space, are suppressed from the figure, one expects the general structure which is displayed for the potential

$$V[A]=\tfrac{1}{2}\int d\vec{x}\, B_i^a(\vec{x})\, B_i^a(\vec{x})\,. \tag{5.7}$$

The $\hat{\theta}$ axis schematically labels directions in A space in which $V[A]$ is periodic with respect to winding-number-changing gauge transformations, with its minima corresponding to n vacuums. In gauge-independent directions in A space, denoted $\hat{r}$, V grows (symbolically) as A^4. After subtraction of the usual infinities due to the infinite number of degrees of freedom of the field, the zero-point

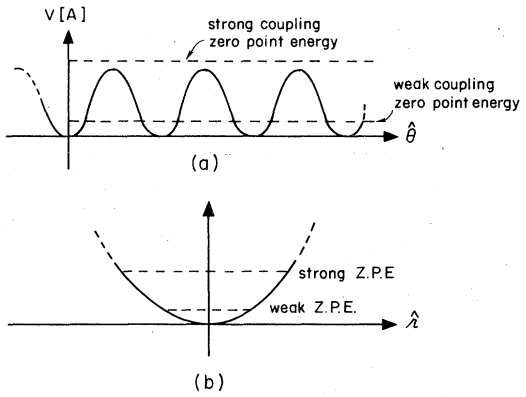

FIG. 9. An idealization of the classical potential energy for Yang-Mills theory in the $A_0=0$ gauge.

energy increases with the scale size of observations over the top of the hills in the $\hat{\theta}$ direction: The effective potential senses the periodicity of the classical potential only weakly.

Within this picture, Minkowski merons in a general sense are solutions of the classical theory with classical energies above the maxima of the periodic potential. Euclidean merons exist much as in the case of the pendulum. In particular, it is readily verified that the merons $A_\mu^a[\phi=0]$ have zero Euclidean energy, as do the instantons.[14] For QCD, unlike the case of the pendulum, there are differences between the Minkowski and Euclidean merons, e.g., finite versus singular action. So it may actually prove more convenient to compute in Minkowski space, although entropy is technically difficult to compute in that formalism as well as in the Euclidean formalism. But this is largely irrelevant. The significant aspect of our conjecture is that QCD merons have enough analyticity that their use in the Euclidean saturation scheme corresponds physically to Minkowski solutions with classical energies near the true strong-coupling ground-state energy. Of course, stability questions of the kind posed in this paper must be answered quantitatively for any such scheme of approximation to be valid.

More generally, however, whether a QCD meron saturation scheme works or not is not our concern in this paper. Rather, the point is that our arguments suggest a new picture for the quantum domain in QCD. The available evidence suggests that quantum fluctuations drive the zero-point energy above the maxima of the classical potential. Thus, the vacuum wave functional is not strongly peaked at the classical minima, but rather resembles the wave function of a conduction electron in a metal. This may be a useful picture outside the semiclassical framework from several points of view.

(2) Ordinary perturbation theory splits the total Hamiltonian into H_0, which contains all terms quadratic in the canonical coordinates and canonical momenta, and H_{int} which contains the remainder of the potential. But if the *effective* potential for QCD is roughly a trough as suggested in Fig. 9, with the periodic classical potential a minor ripple on the bottom of the trough, it makes sense to diagonalize the canonical momenta in the direction of the trough. This is formally the situation in Hamiltonian lattice gauge theory.[15]

(3) If the minimum of the effective potential is treated momentarily as though it were itself a classical potential energy value, one would say that the true classical minimum of

$$V[A]=\tfrac{1}{2}\int d\vec{\mathbf{x}}\, B_i^a(x)\, B_i^a(x)$$

occurred for $B_i^a(x)=0$, or $A=g^{-1}\partial g$, pure gauge, while the effective potential's minimal energy would have to correspond to the density $B^2(x)$ being nonzero in some regions of space. We should associate the quantum ground-state energy with the presence of a smooth, pseudoclassical nonzero density of $B^2(x)$. But this can only occur if the vacuum is characterized by average vector potentials $A_i^a(x)$ far from pure gauges. The standard lore holds that this is a necessary condition for confinement, based on the properties of the Wilson loop.

(4) Nambu has recently proposed a direct connection between $\langle 0|F^a_{\mu\nu}F^{\mu\nu a}|0\rangle$ and the magnitude of the slope of Regge trajectories.[16] Thus if the QCD flux line between quarks is an adequate mathematical representation of the mesons with large angular momenta, a good calculation of $\langle 0|B^2|0\rangle$ is as important for understanding the spectrum of the light quarks as a good calculation of the Wilson loop is for understanding the spectrum of the very massive quarks. As was mentioned in the preceding paragraph, a nonsingular value for this expectation value should be obtained when the level of the effective potential is much higher than the level of the classical potential.

Instantons in QCD are mere crutches which enable one to begin to explore the physical consequences of the periodic classical potential.[12] They demonstrate the richness of the gauge-field world. In the absence of quarks, instantons should not be expected to fare better than ordinary perturbation theory—it is likely that quantum fluctuations wipe out the relevance of the classical potential. But *both* perturbation theory and instanton approximation methods presage their fates. Ultimately the relevance of instantons for pure gauge-field theory might be simply that their use is more efficient than perturbation theory in signaling the nature of the true quantum ground state.[17]

ACKNOWLEDGMENTS

This investigation was inspired by stimulating lectures and discussion sessions on QCD instanton methods led by Callan, Dashen, and Gross at the 1978 La Jolla Summer Workshop. A paper with conclusions in the same spirit as those presented here has appeared[18]—Witten has explored the role of instantons in nonlinear σ models with large N. I do not see any contradictions between Witten's qualitative findings and those presented here, but then I do not see any real contradictions between his findings and those of Callan, Dashen, and Gross, either. In conducting this investigation, I have benefited from Wiegel's paper (Ref. 1) and extended his arguments; I have also benefited from conversations with colleagues at UCSC, especially J. Greensite and M. Nauenberg. This work was supported by a grant from the National Science Foundation.

APPENDIX A

1. Setting up the constrained solutions

Equation (2.9) is readily integrated with boundary conditions (2.8b) to give

$$\tau-\tau_{j-1}=\frac{(\pm)}{[2(1+\epsilon_j)]^{1/2}}\int_0^{\phi_j(\tau)/(1-\epsilon_j)^{1/2}} dx\left[(1-x^2)\left(1-\frac{1-\epsilon_j}{1+\epsilon_j}x^2\right)\right]^{-1/2}. \tag{A1}$$

The solution $\phi_j(\tau)$ corresponding to + (−) in (A1) rises (falls) from zero to its max (min) value + (−) $(1-\epsilon_j)^{1/2}$ at $\tau^*=\frac{1}{2}(\tau_j+\tau_{j-1})$. For $\tau>\tau^*$, $\phi_j(\tau)$ is easily constructed by reflection about τ^* of the form given in (A1).

There exist other solutions to the equations of motion in the interval $[\tau_{j-1},\tau_j]$. From Fig. 2 we see that given $\Delta\tau_j=(\tau_j-\tau_{j-1})>\pi/2$, there exists some $\epsilon_j^{(0)}$ corresponding to the nodeless solution (A1). But if $\Delta\tau_j/2>\pi/2$, there will be another solution with a node at $[\tau_{j-1}+\frac{1}{2}\Delta\tau_j]$. This solution has an energy $\epsilon_j^{(1)}=\epsilon(\Delta\tau_j/2)$. There will be solutions of this type up to $N_{\max}$ nodes, where $N_{\max}\pi/2<\Delta\tau_j$ but $(N_{\max}+1)\pi/2>\Delta\tau_j$.

Correspondingly, there are $N_{\max}$ possible solutions for the Lagrange multipliers λ_j. For exam-

ple, if the solution ϕ_1 in $[\tau_0, \tau_1]$ is nodeless, then as $\tau_0 \to -\infty$, $\phi_1(\tau_1) \to \sqrt{2}$. By Eq. (2.8d), $\phi_2(\tau_1) = \sqrt{2} + \lambda_1$. Consequently $E_2 = \frac{1}{2}\phi_2^2(\tau_1) - 1$ implies that $\epsilon_2 = [1 - \frac{1}{2}(\lambda_1 + \sqrt{2})^2]^{1/2}$. But since we consider τ_2 to be the given parameter for the problem, ϵ_2 can only take on certain values

$$\epsilon_2^{(k)} = \epsilon\left(\frac{\tau_2 - \tau_1}{k}\right),$$

where $k = 1, \ldots, N_{\max} + 1$. This implies that λ_1 is to be solved for in terms of $\epsilon_2^{(k)}$, where $(k-1)$ is the number of nodes of the solution ϕ_2 in $[\tau_1, \tau_2]$.

Next, given a particular $\epsilon_2^{(k)}$, we compute

$$\dot{\phi}_2^{(k)}(\tau_2) = \pm[2(E_2^{(k)} + 1)]^{1/2}.$$

Again, Eq. (2.8d) gives us $\dot{\phi}_3^{(k)}(\tau_2) = \lambda_2 + \dot{\phi}_2^{(k)}(\tau_2)$, and we repeat the argument: $\epsilon_3 = \epsilon(\tau_3 - \tau_2)$, therefore $\dot{\phi}_3 = \dot{\phi}_3(\tau_3 - \tau_2)$ and $\lambda_2 = \lambda_3(\tau_3 - \tau_2)$. It is obvious that the λ_j simply make the initial velocity $\dot{\phi}_{j+1}(\tau_j)$ take such values that the solution can pass through zero at τ_{j+1}, with or without extra nodes in between.

2. Nearby functions

Consider two nodeless solutions of the equation of motion (2.8a) denoted ϕ_α, $\alpha = 1, 2$. The function ϕ_1 is a solution in the interval characterized by an energy $\epsilon_1 = \epsilon(\tau_2 - \tau_1)$. Similarly, ϕ_2 is a solution in $[\tau_1 + \delta\tau_1,\ \tau_2 + \delta\tau_2]$ with energy parameter $\epsilon_2 = \epsilon(\tau_2 + \delta\tau_2 - \tau_1 - \delta\tau_1)$.

We shall say that ϕ_1 and ϕ_2 are "nearby functions" if $\phi_2 = \phi_1 + \delta\phi$, $\epsilon_2 = \epsilon_1 + \delta\epsilon$ with $\delta\phi$ and $\delta\epsilon$ much smaller than unity for infinitesimal $\delta\tau_\alpha$ except possibly near the nodes. Thus, retaining only terms first order in small quantities, we have from Eq. (2.10)

$$\frac{\delta\tau_2 - \delta\tau_1}{\delta\epsilon} = (3 + \epsilon_1)(\tau_2 - \tau_1)/2(1 - \epsilon_1^2) - \frac{\sqrt{2}}{\epsilon_1(1 - \epsilon_1)(1 + \epsilon_1)^{1/2}} E\left(\frac{1 - \epsilon_1}{1 + \epsilon_1}\right). \tag{A2}$$

It is convenient to denote $\delta\epsilon/(\delta\tau_2 - \delta\tau_1) \equiv \dot{\epsilon}$. This is the change in the energy parameter of the solution produced by infinitesimal changes in the times at which the solution is constrained to pass through zero.

The change in the form of the solution itself, $\delta\phi$, can be computed from Eq. (A1). Choosing the (+) sign in that equation for illustration, we find

$$\begin{aligned}\delta\phi(\tau) = {} & -\delta\tau_1\,\dot{\phi}_1(\tau) + \dot{\epsilon}(\delta\tau_2 - \delta\tau_1)\left(\frac{(\tau - \tau_1)\dot{\phi}_1}{2(1+\epsilon_1)} - \frac{\phi_1}{2(1-\epsilon_1^2)} + \frac{\dot{\phi}_1}{2\sqrt{2}\,\epsilon_1(1+\epsilon_1)(1-\epsilon_1)^{1/2}}\right. \\ & \times\left\{\left(\frac{1+\epsilon_1}{1-\epsilon_1}\right)^{1/2}\left[E\left(\psi_1 \frac{1-\epsilon_1}{1+\epsilon_1}\right) - \frac{2\epsilon_1}{1+\epsilon_1} F\left(\psi_1 \frac{1-\epsilon_1}{1+\epsilon_1}\right)\right]\right. \\ & \left.\left. - \frac{1}{2}\sin 2\psi\left[\frac{1-\epsilon_1}{\cos^2\psi + \epsilon_1(1+\sin^2\psi)}\right]^{1/2}\right\}_{\psi \equiv \phi_1/(1-\epsilon_1)^{1/2}}\right) \\ \equiv {} & -\delta\tau_1\,\dot{\phi}_1(\tau) + \delta\epsilon\,\phi_1'(\tau).\end{aligned} \tag{A3}$$

Some discussion of this equation is in order. Suppose that $\delta\tau_1 > 0$. Then $\phi_2 = \phi_1 + \delta\phi$ should be defined only for $\tau \geq \tau_1 + \delta\tau_1$. In particular, $\phi_2(\tau_1 + \delta\tau_1) = 0 = \phi_1(\tau_1 + \delta\tau_1) + \delta\phi(\tau_1 + \delta\tau_1)$ must be satisfied. It is easy to verify that (A3) does satisfy this requirement. In addition, ϕ_1 is maximal at $\tau_1^* = \frac{1}{2}(\tau_1 + \tau_2)$, where its value is $\phi_1(\tau_1^*) = (1 - \epsilon_1)^{1/2}$. From (A3) we readily find that $\phi_2(\tau_2^*) = (1 - \epsilon_1 - \delta\epsilon)^{1/2}$, where $\tau_2^* = \frac{1}{2}(\tau_1 + \tau_2 + \delta\tau_1 + \delta\tau_2)$, and that this is its maximal value. Then the shifted solution ϕ_2 can be constructed for $\tau > \tau_2^*$ by reflection about τ_2^*. An analogous set of arguments applies when the (−) sign in (A1) is chosen.

APPENDIX B

Consider first the solution graphed in Fig. 4(a). Denote the solution in the first interval, which we take to be $[\tau_0, \tau_1]$, by $\phi^* = \phi_{\mathrm{I}}$. Then

$$\epsilon_{\mathrm{I}} \approx 8e^{-2\sqrt{2}(\tau_1 - \tau_0)}, \tag{B1a}$$

$$\varphi_{\mathrm{I}} \approx \tanh\sqrt{2}\,(\tau - \tau_1). \tag{B1b}$$

Similarly, if ϕ_{III} is the solution in the final interval $[\tau_2, \tau_3]$,

$$\varphi_{\mathrm{III}} \approx \tanh\sqrt{2}\,(\tau_2 - \tau). \tag{B1c}$$

But in the intermediate interval $[\tau_1, \tau_2]$, ϕ_{I} can be extended up to $\tau_c = (\tau_1 + \tau_2)/2$, and ϕ_{III} can be extended down to τ_c with errors of order $e^{-(\tau_2 - \tau_1)}$ with respect to the true constrained solution in $[\tau_1, \tau_2]$. Thus, it is an excellent approximation to write the entire solution in the form

$$\begin{aligned}\varphi_{(a)}^* = {} & \tanh\sqrt{2}\,(\tau - \tau_1)\theta(\tau_c - \tau) \\ & + \tanh\sqrt{2}\,(\tau_2 - \tau)\theta(\tau - \tau_c) \\ \equiv {} & \theta_L\varphi_L + \theta_R\varphi_R.\end{aligned} \tag{B1d}$$

Introducing $\tau_R = \tau_2 - \tau_1$, we have at the same level

of approximation

$$\frac{\partial \varphi^*_{(a)}}{\partial \tau_c} = -\frac{\partial \varphi_L}{\partial \tau}\theta_L - \frac{\partial \varphi_R}{\partial \tau}\theta_R , \tag{B1e}$$

$$\frac{\partial \varphi^*_{(a)}}{\partial \tau_R} = \frac{1}{2}\left(\frac{\partial \varphi_L}{\partial \tau}\theta_L - \frac{\partial \varphi_R}{\partial \tau}\theta_R\right). \tag{B1f}$$

On observing that $\partial\phi_L/\partial\tau > 0$ up to τ_c, while $\partial\phi_R/\partial\tau < 0$ for $\tau > \tau_c$, these equations form the basis for Eq. (3.22).

Next consider the solution of Fig. 4(c):

$$\varphi^*_{(c)} \equiv \varphi_{\rm I}\theta(\tau_1 - \tau) + \varphi_{\rm II}\theta(\tau - \tau_1)\theta(\tau_2 - \tau) + \varphi_{\rm III}\theta(\tau - \tau_2) . \tag{B2a}$$

In this equation, $\phi_{\rm I}$ and $\phi_{\rm III}$ are the same as in Eq. (B1), and we recall [Eq. (3.24b)] that

$$\varphi_{\rm II} \approx \sqrt{\delta}\left(1+\frac{3\delta}{8}\right)\sin 2\left(1-\frac{3\delta}{8}\right)(\tau - \tau_1) .$$

Then

$$\frac{\partial \varphi^*_{(c)}}{\partial \tau_c} = -\frac{\partial \tau_{\rm I}}{\partial \tau}\theta(\tau_1 - \tau) - \frac{\partial \varphi_{\rm III}}{\partial \tau}\theta(\tau - \tau_2) - 2\sqrt{2}\cos 2\left(1-\frac{3\delta}{8}\right)(\tau - \tau_1)\theta(\tau - \tau_1)\theta(\tau_2 - \tau) , \tag{B2b}$$

$$2\frac{\partial \varphi^*_{(c)}}{\partial \tau_R} = \frac{\partial \varphi_{\rm I}}{\partial \tau}\theta(\tau_1 - \tau) - \frac{\partial \varphi_{\rm III}}{\partial \tau}\theta(\tau - \tau_2) + 2\sqrt{\delta}\cos 2\left(1-\frac{3\delta}{8}\right)(\tau - \tau_1) + \frac{16}{3\pi\sqrt{\delta}}\left(1+\frac{3\delta}{8}\right)\sin 2\left(1-\frac{3\delta}{8}\right)(\tau - \tau_1)\ \theta(\tau - \tau_1)\theta(\tau_2 - \tau) . \tag{B2c}$$

We have taken $\delta = \delta(\tau_2 - \tau_1)$ into account in obtaining these equations, as was explained in Appendix A.

The potential in which the fluctuations move is graphed in Fig. 5(c). The lowest eigenfunctions χ_0 and χ_1 of this potential were written in terms of the eigenvalues through ω_n and μ_n in Eq. (3.25a). For a well width $(\pi/2)(1+3\delta/8)$ we find

$$\begin{aligned}
&\lambda_0(\delta) \approx -1.90(1+0.59\delta) ,\\
&\omega_0(\delta) \approx 2.90(1-0.278\delta) ,\\
&\mu_0(\delta) \approx 3.14(1+0.068\delta) ,\\
&\lambda_1(\delta) \approx 3.84(1-3.45\delta) ,\\
&\omega_1(\delta) \approx 2.80(1-0.84\delta) ,\\
&\mu_1(\delta) \approx 2.04(1+1.608\delta) .
\end{aligned} \tag{B3}$$

In addition, the normalization constants are

$$\begin{aligned}
A_0^{-2} &= \frac{\tau_R}{2} + \frac{\sin\omega_0\tau_R}{2\omega_0} + \frac{1}{\mu_0}\cos^2\frac{\omega_0\tau_R}{2} ,\\
A_1^{-2} &= \frac{\tau_R}{2} - \frac{\sin\omega_1\tau_R}{2\omega_1} + \frac{1}{\mu_1}\sin^2\frac{\omega_1\tau_R}{2} .
\end{aligned} \tag{B4}$$

According to Eq. (3.18), we require, e.g.,

$$\begin{aligned}
\partial C_0/\partial\tau_R &\cong -\int d\tau \frac{\partial \varphi^*_{(c)}}{\partial \tau_R}\chi_0\\
&= \left(\frac{-A_0}{2}\right)\left(\int_{-\infty}^{\tau_1} d\tau \frac{\partial \varphi_{\rm I}}{\partial \tau} e^{\mu_0(\tau-\tau_1)}\cos\frac{\omega_0\tau_R}{2} - \int_{\tau_2}^{\infty}\frac{\partial \varphi_{\rm III}}{\partial \tau} e^{\mu_0(\tau_2-\tau)}\cos\frac{\omega_0\tau_R}{2}\right.\\
&\qquad \left. + \int_{\tau_1}^{\tau_2} d\tau \cos\omega_0(\tau-\tau_1)[2\sqrt{\delta}\cdots \text{ (as in (B2c))}]\right).
\end{aligned} \tag{B5}$$

All the terms are known on the right-hand side of this equation. A calculation shows that there is only a weak dependence on δ through $\mu(\delta)$, $\omega(\delta)$, etc. Keeping only a strong dependence due to the change in δ as the constraint times change, we obtain

$$\frac{\partial C_0}{\partial \tau_R} \approx (-A_0)\left[\frac{\sqrt{2}}{2}\mu_0\gamma_0\cos\frac{\omega_0\pi}{4} + \frac{32}{3\pi\sqrt{\delta}}\frac{\cos(\omega_0\pi/4)}{4-\omega_0^2}\right]. \tag{B6a}$$

In this equation, $\gamma_0 = \beta(\sqrt{\delta}\,\mu_0/4) - \sqrt{\delta}/\mu_0$, where β is an incomplete β function.

Proceeding in the same manner, we find

$$\frac{\partial C_0}{\partial \tau_c} \approx \theta(\sqrt{\delta}) , \tag{B6b}$$

$$\frac{\partial C_1}{\partial \tau_R} \approx \theta(\sqrt{\delta}) , \tag{B6c}$$

$$\frac{\partial C_1}{\partial \tau_c} \approx -\sqrt{2}\, A_1 \mu_1 \gamma_1 \sin\frac{\omega_1 \pi}{4} . \tag{B6d}$$

Consequently,

$$\mathcal{J}_{(cp)} \approx \sqrt{2}\, A_0 A_1 \mu_1 \left| \gamma_1 \sin\frac{\omega_1 \pi}{4} \right| \left| \frac{\sqrt{2}}{2} \mu_0 \gamma_0 \cos\frac{\omega_0 \pi}{4} + \frac{32}{3\pi\sqrt{\delta}} \frac{\cos(\omega_0 \pi/4)}{4 - \omega_0^2} \right| . \tag{B7}$$

[1] F. W. Wiegel, Phys. Rep. 16C, 57 (1975); S. Coleman, lectures delivered at the 1977 International School of Subnuclear Physics "Ettore Majorana" (unpublished). A. M. Polyakov, Nucl. Phys. B121, 429 (1977) also contains a treatment of the double well.

[2] J. L. Richardson and R. Blankenbecler, Phys. Rev. D 19, 496 (1979).

[3] C. Callan, R. Dashen, and D. Gross, Phys. Rev. D 17, 2717 (1978); 19, 1826 (1979); and lectures at the La Jolla Workshop, 1978 (unpublished).

[4] R. Rajaraman, Phys. Rev. D 15, 2866 (1977).

[5] J. L. Gervais, A. Neveu, and M. A. Virasoro, Nucl. Phys. B138, 45 (1978).

[6] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1965).

[7] J. L. Gervais, A. Jevicki, and B. Sakita, Phys. Rep. 23C, 281 (1976).

[8] R. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D 10, 4114 (1974).

[9] A. Belavin, A. Polyakov, P. Schwartz, and Y. Tyupkin, Phys. Lett. 59B, 851 (1975).

[10] V. De Alfaro, S. Fubini, and G. Furlan, Phys. Lett. 65B, 163 (1976); 72B, 203 (1977).

[11] J. B. Marion, *Classical Dynamics of Particles and Systems*, second edition (Academic, New York, 1970).

[12] See also H. Pagels and E. Tomboulis, Princeton report, 1978 (unpublished).

[13] M. Creutz, I. J. Muzinich, and T. N. Tudron, Phys. Rev. D 19, 531 (1979).

[14] R. Jackiw, Rev. Mod. Phys. 49, 681 (1977).

[15] K. Wilson, Phys. Rev. D 10, 2445 (1974); J. Kogut and L. Susskind, *ibid.* 11, 395 (1975).

[16] Y. Nambu, in *Proceedings of the 19th International Conference on High Energy Physics*, edited by S. Homma, M. Kawaguchi, and H. Miyasawa (Phys. Soc. of Japan, Tokyo, 1979).

[17] J. Greensite has recently proposed an explicit vacuum wave functional for QCD in the $A_0 = 0$ gauge [Nucl. Phys. B (to be published)].

[18] E. Witten, Nucl. Phys. B149, 285 (1979).