Nuclear Physics B366 (1991) 74–92
North-Holland

NUCLEAR
PHYSICS B

WESS–ZUMINO EFFECTIVE ACTION FOR SUPERSYMMETRIC YANG–MILLS THEORIES

Maja MARINKOVIĆ*

Department of Physics, University of Belgrade, P.O. Box 550, 11001 Belgrade, Yugoslavia

Received 26 March 1990
(Revised 17 July 1991)

The relation between the covariant and the consistent supersymmetric gauge anomalies is analyzed. A Wess–Zumino lagrangian describing the interaction of gauge and Goldstone superfields is constructed, the result being explicit for the abelian case. The anomaly analysis also applies to the gauge vector representation of the group, which has not been done in the literature yet. The relation between the consistent gauge anomalies obtained in the gauge chiral and the gauge vector representation is discussed.

1. Introduction and notation

Along with the other anomalies, the supersymmetric non-abelian anomaly was a subject of thorough analysis in recent years. There are general results: the proof of the uniqueness of the supersymmetric gauge anomaly [1], the proof of the impossibility of expressing the anomaly in the form of finite polynomial in the gauge field e^V [2]. The anomaly was calculated in the superfield formulation by various methods [3]: perturbative, heat-kernel, differential-geometric, whereby the non-zero sums and integrals could not be expressed in closed form. It was analyzed in the Wess–Zumino gauge also, which enabled us to find the relation between the gauge and the supersymmetric anomalies and to go to higher dimensions [4].

This paper deals with a further analysis of properties of the supersymmetric gauge anomalies. The aim is to investigate the relation between the consistent and the covariant anomaly. A Wess–Zumino effective action, describing the interaction of the Goldstone bosons with the gauge fields in the supersymmetric case, is constructed. The scope is defined generally enough to allow the calculations for

* E-mail: EPMFF41@YUBGSS21

0550-3213/91/$03.50

the gauge vector representation of the gauge group as well, which has not been done in the literature yet.

The plan of the paper is the following. The details of the representations of gauge group in supersymmetry are reviewed in this section. In sect. 2 the consistent anomaly and the Wess–Zumino action are calculated for the gauge chiral representation. Analogous analysis for the gauge vector representation is given in sect. 3. For both representations the abelian case is specially treated, as there all integrals can be performed explicitly. The relation between the representations is given in sect. 4.

We start with fixing the notation.

Like in the ordinary case, the supersymmetric gauge theories can be formulated in terms of differential forms. One introduces superfields for the connection (Lie-algebra valued 1-form) and curvature F; they are related in the usual way [5,6]. The exterior derivative is defined appropriately to the superspace, $\mathrm{d} = \mathrm{d}z^A(\partial/\partial z^A), (A = a, \alpha, \dot{\alpha}; z_A = x_a, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$. But straightforward generalization is not very useful because superfields contain many component fields without direct physical meaning. In order to make minimal formulations one introduces constraints. Usually the constraints are $F_{\alpha\dot{\beta}} = 0$ ("conventional constraints"), $F_{\alpha\beta} = F_{\dot{\alpha}\dot{\beta}} = 0$ ("representation preserving constraints") and "reality conditions" which fix the $\theta = \bar{\theta} = 0$ component of the connection. As a result one gets two possibilities – gauge vector and gauge chiral representations of the gauge group.

The matter in the gauge chiral representation is given by the chiral (scalar) superfield ϕ, $\bar{D}_{\dot{\alpha}}\phi = 0$. Group transformations are $\phi \to \mathrm{e}^{-i\Lambda}\phi$, with the gauge parameter field Λ chiral, too. The transformation law of the covariant derivatives $\mathscr{D}_A$ $(A = a, \alpha, \dot{\alpha})$, $\mathscr{D}_A \to \mathrm{e}^{-i\Lambda}\,\mathscr{D}_a\,\mathrm{e}^{i\Lambda}$ gives their explicit form in terms of the gauge field V: $\mathscr{D}_\alpha = \mathrm{e}^{-V} D_\alpha\,\mathrm{e}^{V}$, $\bar{\mathscr{D}}_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}}$, $\mathscr{D}^a = -\frac{1}{4}\sigma^a_{\alpha\dot{\alpha}}\{\mathscr{D}^\alpha, \bar{\mathscr{D}}^{\dot{\alpha}}\}$. V is a real superfield, $V = V^\dagger$, and transforms under the gauge group as $\mathrm{e}^V \to \mathrm{e}^{-i\bar{\Lambda}}\,\mathrm{e}^V \mathrm{e}^{i\Lambda}$. The connections and the field strengths are given by $\Gamma_\alpha = \mathrm{e}^{-V}(D_\alpha\,\mathrm{e}^V)$, $W_\alpha = -\frac{1}{4}(\bar{D}^2\Gamma_\alpha) = -\frac{1}{4}[\bar{\mathscr{D}}_{\dot{\alpha}}, \{\bar{\mathscr{D}}^{\dot{\alpha}}, \mathscr{D}_\alpha\}]$. A more detailed list of the used quantities and their relations is given in appendix A.

The alternative to the usual gauge chiral representation – the gauge vector representation – was introduced and described in refs. [5,6]. In this representation the gauge parameter field K is a vector superfield, $K = K^\dagger$. The matter superfield φ, transforming under the group as $\varphi \to \mathrm{e}^{-iK}\varphi$, is chiral with respect to the covariant derivative $\bar{\nabla}_{\dot{\alpha}}$: $\bar{\nabla}_{\dot{\alpha}}\varphi = 0$. The group transformation law of the covariant derivatives $\nabla_A \to \mathrm{e}^{-iK}\,\nabla_A\,\mathrm{e}^{iK}$ $(A = a, \alpha, \dot{\alpha})$ can be fulfilled introducing the complex gauge superfield W, so that the covariant derivatives read: $\nabla_\alpha = \mathrm{e}^{-\bar{W}} D_\alpha\,\mathrm{e}^{\bar{W}}$, $\bar{\nabla}_{\dot{\alpha}} = \mathrm{e}^W \bar{D}_{\dot{\alpha}}\,\mathrm{e}^{-W}$, $\nabla^a = -\frac{1}{4}\sigma^a_{\alpha\dot{\alpha}}\{\nabla^\alpha, \bar{\nabla}^{\dot{\alpha}}\}$. The connections and the field strengths are $\mathscr{G}_\alpha = \mathrm{e}^{-\bar{W}}(D_\alpha\,\mathrm{e}^{\bar{W}})$, $\mathscr{W}_\alpha = -\frac{1}{4}[\bar{\nabla}_{\dot{\alpha}}, \{\bar{\nabla}^{\dot{\alpha}}, \nabla_\alpha\}]$; gauge superfield transforms under the gauge group as $\mathrm{e}^W \to \mathrm{e}^{-iK}\,\mathrm{e}^W$.

As the vector superfield has more degrees of freedom than the chiral superfield, the vector representation of the gauge group is bigger than the chiral one. The redundancy may be traced in the transformation $e^W \to e^W e^{iL}$ for chiral L, which leaves the covariant derivatives of the vector representation (and therefore connections and field strengths also) unchanged.

There is a relation between the gauge chiral and the gauge vector representation. Namely, under the transformation $e^W \to e^W e^{i\Lambda}$ the combination $e^{\overline{W}} e^W$ transforms as $e^{\overline{W}} e^W \to e^{-i\bar{\Lambda}} e^{\overline{W}} e^W e^{i\Lambda}$, so we may define the field V as $e^V = e^{\overline{W}} e^W$. So defined V is real and has transformation properties as in the gauge chiral representation. So, one may think that the basic field for both representations is complex W, with two different types of gauge transformations: vector $e^W \to e^{-iK} e^W$, $K = K^\dagger$, and chiral $e^W \to e^W e^{i\Lambda}$, $\overline{D}_{\dot\alpha}\Lambda = 0$. The corresponding covariant derivatives and field strengths are also different. But, the covariant quantities of both types are related by the non-unitary transformation: $\nabla_\alpha = e^W \mathscr{D}_\alpha e^{-W}$, $\overline{\nabla}_{\dot\alpha} = e^W \overline{\mathscr{D}}_{\dot\alpha} e^{-W}$, $\mathscr{W}_\alpha = e^W W_\alpha e^{-W}$ etc. The same transformation does not work for the connections, as they are not covariant; we have, e.g., $\mathscr{G}_\alpha = e^W \Gamma_\alpha e^{-W} + e^W(D_\alpha e^{-W})$. So an interesting question may be posed: how deep is this relation, e.g. are the anomalies in the two representations related also? We will first proceed to analyze the anomalies in each representation separately, and then discuss whether the description in terms of W is really a general one.

2. Consistent anomaly and Wess–Zumino term for gauge chiral representation

We will now give the calculation of the consistent supersymmetric non-abelian anomaly, done along the lines of ref. [8]. Apart from exploring of relation between the covariant and the consistent anomaly, the hope is also to get a more compact expression for the consistent anomaly which would be suitable for calculation of the WZ action.

The basic idea is the following. We start with the covariant anomaly ω which is a 1-form in the space U of gauge parameters. U is meant as a surface in the space of all equivalent gauge connections, which is defined by some family of gauge transformations parametrized by real u^a. The exterior derivative in this space $\delta = du^a(\partial/\partial u^a)$ is a gauge variation. The covariant anomaly is not closed, $\delta\omega \neq 0$ [8]. The closedness property is the defining property of the consistent anomaly (the "consistency conditions", or closedness of the group transformations [9]). Is it possible to construct a 1-form ω' with $\delta\omega' = 0$ starting with ω?

Why would one want to calculate the consistent from the covariant anomaly? It is because the covariant anomaly is much easier to find: the number of possible expressions is restricted by covariance. Except that, it should contain the leading term of the perturbative calculation. These two conditions are in principle enough to fix the form of the covariant anomaly. On the other hand, the covariant techniques of varying the action (like heat-kernel) also give the covariant anomaly as a result.

In order to obtain the consistent anomaly we use the standard trick of differential geometry – the homotopy operation [10]. That is to say, we introduce a family of gauge inequivalent connections $\Gamma(t)$ parametrized by t, $t \in [0,1]$. This extends the space of gauge parameters U to $\mathrm{U} \times [0,1]$, with the exterior derivative $\tilde{\delta} = \delta + \mathrm{d}t(\partial/\partial t)$ in the extended space. Let us denote the 1-form obtained from ω by replacing Γ by $\Gamma(t)$ with $\omega(t)$, the 2-form obtained from $\delta\omega$ with θ. We consider now the gauge variation of the expression $\int_0^1 \theta$, integration done over t. If θ is a closed form in $\mathrm{U} \times [0,1]$, i.e. $\tilde{\delta}\theta = 0$, we have

$$\begin{aligned}
\delta \int_0^1 \theta &= \delta \int_0^1 \tilde{\delta}\omega(t) = \int_0^1 \delta\,\tilde{\delta}\omega(t) \\
&= -\int_0^1 \mathrm{d}t \frac{\partial}{\partial t}\delta\omega(t) \quad \left(\text{as} \quad \delta\,\tilde{\delta} = \left(\tilde{\delta} - \mathrm{d}t \frac{\partial}{\partial t}\right)\tilde{\delta} = -\mathrm{d}t \frac{\partial}{\partial t}\tilde{\delta} = -\mathrm{d}t \frac{\partial}{\partial t}\delta\right) \\
&= -\int_0^1 \mathrm{d}t \frac{\partial}{\partial t}\delta\omega(t) \\
&= -\delta\omega|_{t=1} + \delta\omega|_{t=0}\,.
\end{aligned}$$

If we choose a 1-parameter family so that $\Gamma(t=1) = \Gamma$ and $\Gamma(t=0) = 0$ implying also $\delta\omega|_{t=1} = \delta\omega$ and $\delta\omega|_{t=0} = 0$, we get

$$\delta \int_0^1 \theta = -\delta\omega\,,$$

or $\delta(\omega + \int\theta) = 0$. This means that the construction gives a quantity $\omega' = \omega + \int\theta$ which is closed in U, i.e. satisfies the consistency conditions. To resume: it is possible to construct a closed 1-form ω' starting from the non-closed ω if we may generalize $\delta\omega$ into θ off the gauge orbit so that $\tilde{\delta}\theta = 0$.

We will first work out the procedure in the gauge chiral case. The covariant anomaly is [11]

$$\begin{aligned}
A &= -\frac{1}{16\pi^2}\omega \\
&= -\frac{1}{16\pi^2}\left(-\mathrm{Str}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{e}^{i\Lambda}\,\delta\,\mathrm{e}^{-i\Lambda}\,W^\alpha W_\alpha + \mathrm{Str}\int \mathrm{d}^4x\,\mathrm{d}^2\bar{\theta}\,\mathrm{e}^{i\bar{\Lambda}}\,\delta\,\mathrm{e}^{-i\bar{\Lambda}}\,\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}\right) \\
&= -\frac{1}{16\pi^2}\,\mathrm{Str}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\left(-\mathrm{e}^{i\Lambda}\,\delta\,\mathrm{e}^{-i\Lambda}\,W^\alpha\Gamma_\alpha - \mathrm{e}^{i\bar{\Lambda}}\,\delta\,\mathrm{e}^{-i\bar{\Lambda}}\,\overline{W}_{\dot\alpha}\bar{\Gamma}^{\dot\alpha}\right).
\end{aligned}$$

(Str denotes the trace symmetrized under the permutations of its arguments, e.g.

$\mathrm{Str}\,ABC = \frac{1}{2}\mathrm{Tr}(ABC + ACB)$ when A, B and C are bosonic [10].)

Let us denote the arbitrary variation of the gauge field by S, $S = \mathrm{e}^{-V}\boldsymbol{\delta}\,\mathrm{e}^{V}$, its gauge variation by S_{g}, $S_{\mathrm{g}} = \mathrm{e}^{-V}\delta\,\mathrm{e}^{V} = S_{\ell\mathrm{g}} + S_{\mathrm{rg}}$ and $S_{\ell\mathrm{g}} = i\Lambda$, $S_{\mathrm{rg}} = \mathrm{e}^{-V}(-i\bar{\Lambda})\mathrm{e}^{V}$. It holds $\bar{\mathscr{D}}_{\dot\alpha}S_{\ell\mathrm{g}} = 0$, $\mathscr{D}_\alpha S_{\mathrm{rg}} = 0$. In this notation the covariant anomaly reads

$$\omega = \mathrm{Str}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\left(S_{\ell\mathrm{g}}W^\alpha\Gamma_\alpha + S_{\mathrm{rg}}W_{\dot\alpha}\Gamma^{\dot\alpha}\right). \tag{1}$$

In order to find its gauge variation, we will first give the variations of the basic fields. They are

$$\begin{aligned}
\boldsymbol{\delta}S &= -S^2\,,\\
\boldsymbol{\delta}\Gamma_\alpha &= (\mathscr{D}_\alpha S)\,,\\
\boldsymbol{\delta}W_\alpha &= -\tfrac{1}{4}\left(\bar{\mathscr{D}}^2\mathscr{D}_\alpha S\right),\\
\boldsymbol{\delta}\Gamma_{\dot\alpha} &= \left(\bar{\mathscr{D}}_{\dot\alpha}S\right) + \Gamma_{\dot\alpha}S - S\Gamma_{\dot\alpha}\,,\\
\boldsymbol{\delta}W_{\dot\alpha} &= -\tfrac{1}{4}\left(\mathscr{D}^2\bar{\mathscr{D}}_{\dot\alpha}S\right) + W_{\dot\alpha}S - SW_{\dot\alpha}\,.
\end{aligned} \tag{2}$$

In the special case of gauge transformations

$$\begin{aligned}
\delta\Gamma_\alpha &= \left(\mathscr{D}_\alpha S_{\ell\mathrm{g}}\right),\\
\delta W_\alpha &= W_\alpha S_{\ell\mathrm{g}} - S_{\ell\mathrm{g}}W_\alpha\,,\\
\delta\Gamma_{\dot\alpha} &= \left(\bar{\mathscr{D}}_{\dot\alpha}S_{\mathrm{rg}}\right) + \Gamma_{\dot\alpha}S_{\mathrm{g}} - S_{\mathrm{g}}\Gamma_{\dot\alpha}\,,\\
\delta W_{\dot\alpha} &= W_{\dot\alpha}S_{\ell\mathrm{g}} - S_{\ell\mathrm{g}}W_{\dot\alpha}\,.
\end{aligned} \tag{3}$$

Therefore, for the gauge variation of the covariant anomaly we get

$$\begin{aligned}
\delta\omega &= \mathrm{Str}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\left(\delta S_{\ell\mathrm{g}}W^\alpha\Gamma_\alpha - S_{\ell\mathrm{g}}\delta W^\alpha\Gamma_\alpha - S_{\ell\mathrm{g}}W^\alpha\delta\Gamma_\alpha\right) + \mathrm{h.c.}\\
&= -\tfrac{1}{2}\,\mathrm{Tr}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\left(2S_{\ell\mathrm{g}}W^\alpha S_{\ell\mathrm{g}}\Gamma_\alpha + S_{\ell\mathrm{g}}W^\alpha(\mathscr{D}_\alpha S_{\ell\mathrm{g}}) - S_{\ell\mathrm{g}}(\mathscr{D}_\alpha S_{\ell\mathrm{g}})W^\alpha\right) + \mathrm{h.c.}\\
&= \mathrm{Str}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,S_{\ell\mathrm{g}}(\mathscr{D}_\alpha S_{\ell\mathrm{g}})W^\alpha + \mathrm{h.c.}\,,
\end{aligned}$$

the first term in the second line vanishes by partial integration.

If we want to define a form which is a generalization of $\delta\omega$ off the gauge orbit, we should write $\delta\omega$ in terms of S_{g} and not $S_{\ell\mathrm{g}}$ and S_{rg} separately. In order to do this, we transform further

$$\int \mathrm{d}^4x\,\mathrm{d}^4\theta\, W^\alpha(\mathscr{D}_\alpha S_{\ell\mathrm{g}})S_{\ell\mathrm{g}} = \int \mathrm{d}^4x\,\mathrm{d}^4\theta\, W^\alpha(\mathscr{D}_\alpha S_{\mathrm{g}})S_{\ell\mathrm{g}}$$

$$= \int \mathrm{d}^4x\,\mathrm{d}^4\theta \left((\mathscr{D}_\alpha W^\alpha)S_{\mathrm{g}}S_{\ell\mathrm{g}} - W^\alpha S_{\mathrm{g}}(\mathscr{D}_\alpha S_{\mathrm{g}})\right).$$

Using the Bianchi identities we get

$$\delta\omega = \mathrm{Str}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\left(W^\alpha(\mathscr{D}_\alpha S_{\mathrm{g}})S_{\mathrm{g}} + W_{\dot\alpha}\left(\bar{\mathscr{D}}^{\dot\alpha}S_{\mathrm{g}}\right)S_{\mathrm{g}}\right) + \mathrm{h.c.},$$

which can now be extended to a 2-form defined for arbitrary variations

$$\theta = \mathrm{Str}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\left(W^\alpha(\mathscr{D}_\alpha S)S + W_{\dot\alpha}(\bar{\mathscr{D}}^{\dot\alpha}S)S\right) + \mathrm{h.c.} \tag{4}$$

What is left to be done is to prove that $\hat\delta\theta = 0$ really holds. This is technically rather complicated – one uses repeatedly the algebra of covariant derivatives, partial integration, Bianchi identities. We will not present the proof here; it is given in ref. [12].

In order to find the consistent anomaly we should extend all quantities W_α, $\mathscr{D}_\alpha$, S to $\mathrm{U}\times[0,1]$ and integrate θ. A possible choice of the path of integration is $V(t) = tV$, $t\in[0,1]$, as for connections and field strengths at the end points we have $\mathrm{e}^{V(0)} = 1$, $\Gamma_\alpha(0) = 0$, $W_\alpha(0) = 0$; $\mathrm{e}^{V(1)} = \mathrm{e}^V$, $\Gamma_\alpha(1) = \Gamma_\alpha$, $W_\alpha(1) = W_\alpha$. With this choice we have

$$\hat\delta = \delta + \mathrm{d}t\,\frac{\partial}{\partial t}$$

$$S = \mathrm{e}^{-V(t)}\,\hat\delta\,\mathrm{e}^{V(t)} = \mathrm{e}^{-tV}\,\delta\,\mathrm{e}^{tV} + \mathrm{d}t\,V = S_{\mathrm{g}}(t) + \mathrm{d}t\,V$$

$$\Gamma_\alpha(t) = \mathrm{e}^{-tV}\left(\mathscr{D}_\alpha\,\mathrm{e}^{tV}\right) = \mathrm{e}^{-tV}\int_0^1 \mathrm{e}^{(1-s)tV}(\mathscr{D}_\alpha tV)\mathrm{e}^{stV}\mathrm{d}s = \int_0^t \mathrm{e}^{-sV}(\mathscr{D}_\alpha V)\mathrm{e}^{sV}\mathrm{d}s$$

$$W_\alpha(t) = -\tfrac{1}{4}\left(\bar{D}^2\Gamma_\alpha(t)\right), \quad \text{and so on}.$$

The t-integration picks out only the terms linear in $\mathrm{d}t$. Taking this into account we obtain

$$\int\theta = \mathrm{Tr}\int \mathrm{d}^4x\,\mathrm{d}^4\theta \int_0^1 S_{\mathrm{g}}\left(-V(\mathscr{D}_\alpha W^\alpha) + W^\alpha(\mathscr{D}_\alpha V) - (\mathscr{D}_\alpha V)W^\alpha\right) + \mathrm{h.c.}, \quad (5)$$

where $W_\alpha, S_{\mathrm{g}}, \mathscr{D}_\alpha$ depend on t. Written more explicitly

$$\int\theta = \mathrm{Tr}\int \mathrm{d}^4x\,\mathrm{d}^4\theta \int_0^1 \mathrm{d}t\, S_{\mathrm{g}}\Bigg(-\tfrac{1}{2}V\left(\bar{D}^{\dot\alpha}\,\mathrm{e}^{-V}\left(D^2\int_0^t \mathrm{e}^{sV}(\bar{D}_{\dot\alpha}V)\mathrm{e}^{-sV}\mathrm{d}s\right)\mathrm{e}^{V}\right)$$

$$-\tfrac{1}{4}\int_0^t \mathrm{d}s\left[(\bar{D}^2\,\mathrm{e}^{-sV}(D^\alpha V)\mathrm{e}^{sV}),(D_\alpha V)\right]$$

$$-\tfrac{1}{4}\int_0^t \mathrm{d}s\int_0^t \mathrm{d}u\left[(\bar{D}^2\,\mathrm{e}^{-sV}(D^\alpha V)\mathrm{e}^{sV}),\mathrm{e}^{-uV}[(D_\alpha V),V]\mathrm{e}^{uV}\right]$$

$$-\tfrac{1}{4}\int_0^t \mathrm{d}s\left[\mathrm{e}^{-V}\left(D^2\,\mathrm{e}^{sV}(\bar{D}_{\dot\alpha}V)\mathrm{e}^{-sV}\right)\mathrm{e}^{V},(\bar{D}^{\dot\alpha}V)\right]\Bigg) + \mathrm{h.c.}, \quad (6)$$

here, as before, $S_{\mathrm{g}} = \mathrm{e}^{-tV}\delta\,\mathrm{e}^{tV}$. This is our final result for the consistent anomaly.

Due to the non-commutativity of the factors $\mathrm{e}^V, (D_\alpha V), (\bar{D}_{\dot\alpha}V)$ etc., the integral cannot be simplified essentially, which is in accordance with the general result of ref. [2] that the consistent anomaly $\omega' = \omega + \int\theta$ cannot be written as a finite polynomial in e^V.

There are, however, two possibilities to proceed with the integration, two cases in which one gets a polynomial in tV instead of the product of exponentials. The first one is the Wess–Zumino gauge, $V^3 = 0$ there. A comment is to be made in relation with this. One may naively expect that our calculation, restricted to the WZ gauge, would give the complete gauge anomaly in the Wess–Zumino gauge as in ref. [4]. This is not true, though, because the supersymmetric and the gauge transformations are coupled in the WZ gauge. Therefore, when treating the symmetries in this gauge, one has both of the anomalies – the supersymmetric and the gauge anomaly – as a consequence of the consistency conditions. On the other hand, the superspace calculations (like ours) can never give the supersymmetric anomaly. Anyway, the gauge part of the anomaly should be, up to normal terms, the same in both cases. Considering this as an important check of the given construction, we will show it.

In the $y\theta\bar{\theta}$-basis [7], the expansions of the basic fields are

$$V = -\theta\sigma^m\bar{\theta}v_m + i\theta\theta\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\theta\lambda + \tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}(D - i\partial_m v^m),$$

$$W_\alpha = -i\lambda_\alpha + \theta_\alpha D - 2i\sigma^{mn}{}_\alpha{}^\beta\theta_\beta(\partial_m v_n + \tfrac{1}{2}iv_m v_n) + \sigma^m_{\alpha\dot{\alpha}}\theta\theta(\partial_m\bar{\lambda}^{\dot{\alpha}} + \tfrac{1}{2}i[v_m, \lambda^{\dot{\alpha}}]),$$

$$\Lambda = a,$$

$$\bar{\Lambda} = a - 2i\theta\sigma^m\bar{\theta}\partial_m a + \theta\theta\bar{\theta}\bar{\theta}\partial^2 a.$$

For the covariant anomaly, after $d^2\theta\, d^2\bar{\theta}$ integration we get

$$\omega = \operatorname{Tr}\int d^4x\, a \cdot \Big(2\varepsilon^{mnpq}(\partial_m v_n + \tfrac{1}{2}iv_m v_n)(\partial_p v_q + \tfrac{1}{2}iv_p v_q)$$
$$+\partial_m(\lambda\sigma^m\bar{\lambda} - \bar{\lambda}\bar{\sigma}^m\lambda) + \tfrac{1}{2}i\big[v_m, \lambda\sigma^m\bar{\lambda} - \bar{\lambda}\bar{\sigma}^m\lambda\big]\Big),$$

or, rescaling in the usual way $v \to -2iv$, $\lambda \to -i\lambda$, $\bar{\lambda} \to -i\bar{\lambda}$ [7]

$$\bar{\omega} = -\operatorname{Tr}\int d^4x\, a \cdot \Big(2\varepsilon^{mnpq}F_{mn}F_{pq} + \mathscr{D}_m(\lambda\sigma^m\bar{\lambda} - \bar{\lambda}\bar{\sigma}^m\lambda)\Big). \tag{7}$$

The result for the $\int\theta$ is (the calculation is outlined in appendix B)

$$\int\theta = \operatorname{Tr}\int d^4x\, a \cdot \varepsilon^{mnpq}\big(\tfrac{4}{3}\partial_m v_n\partial_p v_q - \tfrac{1}{2}v_m v_n v_p v_q + \tfrac{5}{6}i\{\partial_m v_n, v_p v_q\} + \tfrac{1}{6}iv_p(\partial_m v_n)v_q\big)$$
$$+ \operatorname{Tr}\int d^4x\, a \cdot \Big(\tfrac{2}{3}\partial_m(\lambda\sigma^m\bar{\lambda} - \bar{\lambda}\bar{\sigma}^m\lambda) + \tfrac{1}{2}i\big[v_m, \lambda\sigma^m\bar{\lambda} - \bar{\lambda}\bar{\sigma}^m\lambda\big]\Big).$$

or, rescaled

$$\int\bar{\theta} = -\operatorname{Tr}\int d^4x\, a \cdot 8\varepsilon^{mnpq}\big(\tfrac{2}{3}\partial_m v_n\partial_p v_q + v_m v_n v_p v_q + \tfrac{5}{6}\{\partial_m v_n, v_p v_q\} + \tfrac{1}{6}v_p(\partial_m v_n)v_q\big)$$
$$- \operatorname{Tr}\int d^4x\, a \cdot \Big(\tfrac{2}{3}\partial_m(\lambda\sigma^m\bar{\lambda} - \bar{\lambda}\bar{\sigma}^m\lambda) + \big[v_m, \lambda\sigma^m\bar{\lambda} - \bar{\lambda}\bar{\sigma}^m\lambda\big]\Big). \tag{8}$$

$\int\theta$, added to the covariant anomaly, gives for the consistent anomaly

$$\bar{\omega} + \int\bar{\theta} = -\operatorname{Tr}\int d^4x\, a \cdot \Big(\tfrac{8}{3}\varepsilon^{mnpq}\partial_m\big(v_n\partial_p v_q + \tfrac{1}{2}v_n v_p v_q\big) + \tfrac{1}{3}\partial_m(\lambda\sigma^m\bar{\lambda} - \bar{\lambda}\bar{\sigma}^m\lambda)\Big). \tag{9}$$

The first term on the r.h.s. of (9) is the known expression for the consistent anomaly [4, 8]. The second is a gauge variation, i.e. a normal term. That is to say, defining the gauge variations of the component fields in the WZ gauge

$$\delta v_m = (\mathscr{D}_m a) = (\partial_m a) + [v_m, a],$$

$$\delta\lambda = [\lambda, a],$$

one has

$$\delta \int d^4x \left(\lambda\sigma^m\bar{\lambda} - \bar{\lambda}\bar{\sigma}\lambda\right) v_m = \int d^4x\, \partial_m a\left(\lambda\sigma^m\bar{\lambda} - \bar{\lambda}\bar{\sigma}^m\lambda\right),$$

which confirms that the WZ-gauge-limit of (6) is really correct.

The other line of calculation is the abelian case. In the abelian case

$$S = (i\Lambda - i\bar{\Lambda})t + dt\, V, \qquad S^2 = 0$$

$$\Gamma_\alpha = (D_\alpha V), \qquad \Gamma_\alpha(t) = t\Gamma_\alpha$$

$$W_\alpha = -\tfrac{1}{4}\left(\bar{D}^2 D_\alpha V\right), \qquad W_\alpha(t) = tW_\alpha,$$

holds, so that $\int\theta$ reduces to

$$\int\theta = -\tfrac{1}{3}\int d^4x\, d^4\theta \left((D_\alpha W^\alpha)(i\Lambda - i\bar{\Lambda})V - 2W^\alpha(D_\alpha i\Lambda)V\right) + \text{h.c.}$$

$$= -\tfrac{2}{3}\int d^4x\, d^4\theta\, i\Lambda W^\alpha (D_\alpha V) + \text{h.c.} \tag{10}$$

after the t-integration. With this the result for the consistent anomaly becomes

$$\omega' = \tfrac{1}{3}\int d^4x\, d^4\theta\, i\Lambda W^\alpha \Gamma_\alpha. \tag{11}$$

The consistent anomaly is the starting point for the calculation of the Wess–Zumino effective lagrangian. According to the general prescription of ref. [9], along with the gauge fields one introduces the multiplet of the Goldstone fields $U = U_a(x)T^a$ which takes the values in the Lie algebra. U is a chiral superfield. Gauge field V, connections, etc. can be U-transformed, as well as Λ-transformed, as e^{iU} defines an element of the gauge group. U itself is not invariant under the group but transforms as $e^{iU} \to e^{i\Lambda} e^{iU}$. The WZ action is then given by

$$L_{WZ} = \int_0^1 ds\, \omega'(U, V^{sU}), \tag{12}$$

where ω' stands for the consistent anomaly in which Λ is replaced by U and all other quantities appearing in ω' are transformed by the group element e^{isU}, $s \in [0, 1]$.

In the general case, unfortunately, the expression for L_{WZ} cannot be written explicitly – not even in integral form. This is because the consistent anomaly (6) contains terms of the form e^{tV}, or $(D_\alpha V)$, the group transformation of which is impossible to write explicitly because of their non-linearity.

In the abelian case, however, we can perform the calculation of the WZ action to the end. For the gauge transformations we have

$$V \to V - i\Lambda + i\bar{\Lambda},$$

$$\Gamma_\alpha \to \Gamma_\alpha - (D_\alpha i\Lambda),$$

and similarly, for sU-transformations

$$V \to V - s(iU - i\bar{U}),$$

$$\Gamma_\alpha \to \Gamma_\alpha - s(D_\alpha iU).$$

U transforms under the gauge group as $U \to U + \Lambda$. Inserting these expressions into the WZ action (12) we get

$$\begin{aligned} L_{\mathrm{WZ}} &= \int_0^1 \mathrm{d}s \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \tfrac{1}{3} iUW^\alpha(\Gamma_\alpha - s(D_\alpha iU)) + \mathrm{h.c.} \\ &= \int \mathrm{d}^4x\, \mathrm{d}^4\theta \left(\tfrac{1}{3} iUW^\alpha\Gamma_\alpha - \tfrac{1}{12}(iU)^2(D_\alpha W^\alpha)\right) + \mathrm{h.c.} \\ &= \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \tfrac{1}{3} iUW^\alpha\Gamma_\alpha + \mathrm{h.c.} \end{aligned} \tag{13}$$

This term describes the interaction of the gauge and the Goldstone fields. It is easy to check, using (3), that $\delta L_{\mathrm{WZ}} = \omega'$ really holds.

3. Consistent anomaly and WZ term for gauge vector representation

We will now turn to the gauge vector representation. Neither consistent nor the covariant anomalies have been calculated for this case. One may try to guess the form of the covariant anomaly: it is a covariant 1-form in the gauge-parameter space containing the correct nonsupersymmetric part. We could start, e.g. with (1): $\omega = \mathrm{Str} \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, (S_{\mathrm{lg}} W^\alpha \Gamma_\alpha + S_{\mathrm{rg}} W_{\dot\alpha} \Gamma^{\dot\alpha})$, as a hint. Indeed, the correct expression for the covariant anomaly looks the same, just with all quantities written in the basis appropriate to the gauge vector representation. Let us denote $\mathscr{S}_\ell = \bar{\delta}\,\mathrm{e}^{W}\mathrm{e}^{-W}$, $\mathscr{S}_\mathrm{r} = \mathrm{e}^{-\overline{W}}\bar{\delta}\,\mathrm{e}^{\overline{W}}$, $\mathscr{S}_{\ell\mathrm{g}} = -iK$, $\mathscr{S}_{\mathrm{rg}} = iK$. The covariant anomaly in the gauge chiral representation is given by

$$A = -\frac{1}{16\pi^2}\Omega,$$

$$\Omega = \mathrm{Str}\,\tfrac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\left(\mathscr{S}_{\ell\mathrm{g}}\mathscr{W}^{\alpha}\mathscr{G}_{\alpha} + \mathscr{S}_{\mathrm{rg}}\overline{\mathscr{W}}_{\dot\alpha}\overline{\mathscr{G}}^{\dot\alpha}\right). \tag{14}$$

In order to prove this statement we have to define the Wess–Zumino gauge appropriately. The details of the definition are given in appendix C: here we just note that, in order to fix the gauge completely and get $W^3 = 0$, we use the spurious freedom of chiral transformations mentioned before. The form of the field expansions is similar to that obtained for the gauge chiral case:

$$W = -\theta\sigma^m\bar{\theta}v_m + i\theta\theta\bar{\theta}\bar{\varphi} - i\bar{\theta}\bar{\theta}\theta\varphi + \tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}(D - i\,\partial_m v^m)$$

$$-iK = -ic - \theta\sigma^m\bar{\theta}\partial_m c - \tfrac{1}{2}i\theta\theta\bar{\theta}\bar{\theta}\partial^2 c, \quad \text{etc.}$$

($y\theta\bar{\theta}$-basis). Calculating (14) we get

$$\begin{aligned}\Omega = \mathrm{Tr}\int \mathrm{d}^4x\,c\Big(&-\partial_m\{v^m, D\} - i[v_m, \{v^m, D\}]\\ &+\partial_m\left(\varphi\sigma^m\bar{\varphi} - \bar{\varphi}\bar{\sigma}^m\varphi\right) + i\left[v_m, \varphi\sigma^m\bar{\varphi} - \bar{\varphi}\bar{\sigma}^m\varphi\right]\\ &+2\varepsilon^{mnpq}\left(\partial_m v_n + iv_m v_n\right)\left(\partial_p v_q + iv_p v_q\right)\Big),\end{aligned} \tag{15}$$

the third line being exactly the covariant anomaly, the first and the second its supersymmetric completion, covariant also. We note that, checking combinations similar to (14), one sees that it is non-trivial to get all the covariant derivatives and the εFF term included in the final formula. It is also worth noting that the term under the integral and trace signs in (14) is not the $\mathrm{e}^{W}\ldots\mathrm{e}^{-W}$ transform of (1) (and therefore not equal to it) since the connections do not transform covariantly under the $\mathrm{e}^{W}\ldots\mathrm{e}^{-W}$ transformation.

In order to calculate the consistent anomaly let us find the variations first. They are

$$
\begin{aligned}
&\delta\mathscr{S}_\ell=\mathscr{S}_\ell^2\,,\\
&\delta\mathscr{S}_\mathrm{r}=-\mathscr{S}_\mathrm{r}^2\,,\\
&\delta\mathscr{W}_\alpha=\mathscr{S}_\ell\mathscr{W}_\alpha-\mathscr{W}_\alpha\mathscr{S}_\ell-\tfrac{1}{4}\left(\overline{\nabla}^2\nabla_\alpha\mathscr{S}\right),\\
&\delta\overline{\mathscr{W}}_{\dot\alpha}=-\mathscr{S}_\mathrm{r}\overline{\mathscr{W}}_{\dot\alpha}+\overline{\mathscr{W}}_{\dot\alpha}\mathscr{S}_\mathrm{r}-\tfrac{1}{4}\left(\nabla^2\overline{\nabla}_{\dot\alpha}\mathscr{S}\right),\\
&\delta\mathscr{G}_\alpha=\left(\nabla_\alpha\mathscr{S}_\mathrm{r}\right),\\
&\delta\overline{\mathscr{G}}_{\dot\alpha}=\left(\overline{\nabla}_{\dot\alpha}\mathscr{S}_\ell\right).
\end{aligned}
\tag{16}
$$

Gauge variations are in this case characterized by the condition $\mathscr{S}_{\ell\mathrm{g}}+\mathscr{S}_{\mathrm{rg}}=0$, so for them (16) reduces to:

$$
\begin{aligned}
&\delta\mathscr{W}_\alpha=\mathscr{S}_{\ell\mathrm{g}}\mathscr{W}_\alpha-\mathscr{W}_\alpha\mathscr{S}_{\ell\mathrm{g}}\,,\\
&\delta\overline{\mathscr{W}}_{\dot\alpha}=-\mathscr{S}_{\mathrm{rg}}\overline{\mathscr{W}}_{\dot\alpha}+\overline{\mathscr{W}}_{\dot\alpha}\mathscr{S}_{\mathrm{rg}}=\mathscr{S}_{\ell\mathrm{g}}\overline{\mathscr{W}}_{\dot\alpha}-\overline{\mathscr{W}}_{\dot\alpha}\mathscr{S}_{\ell\mathrm{g}}\,,\\
&\delta\mathscr{G}_\alpha=\left(\nabla_\alpha\mathscr{S}_\mathrm{r}\right),\\
&\delta\overline{\mathscr{G}}_{\dot\alpha}=\left(\overline{\nabla}_{\dot\alpha}\mathscr{S}_\ell\right).
\end{aligned}
\tag{17}
$$

We see that in this case we may calculate with the general variations from the beginning, as the generalization $\mathscr{S}_{\ell\mathrm{g}}$ and $\mathscr{S}_{\mathrm{rg}}$ to $\mathscr{S}_\ell$ and $\mathscr{S}_\mathrm{r}$ is immediately possible. The first variation of the covariant anomaly is

$$
\begin{aligned}
2\Theta=2\delta\Omega=\operatorname{Tr}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathscr{S}_\ell\mathscr{W}^\alpha\mathscr{S}_\ell\mathscr{G}_\alpha\\
+\operatorname{Str}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\left(\mathscr{S}_\ell\tfrac{1}{4}(\overline{\nabla}^2\,\nabla^\alpha\mathscr{S})\mathscr{G}_\alpha-\mathscr{S}_\ell\mathscr{W}^\alpha(\nabla_\alpha\mathscr{S}_\mathrm{r})\right)+\text{h.c.}.
\end{aligned}
\tag{18}
$$

Since Θ is now defined as $\delta\Omega$, $\delta\Theta=0$ holds trivially. This means that the condition to apply the homotopy operation is fulfilled. The path of integration can be chosen similarly as before

$$W(t) = tW,$$

$$\mathcal{S}_\ell(t) = \eth\, e^{tW} e^{-tW} = \mathrm{d}t\, W + \mathcal{S}_{\ell g}(t),$$

$$\mathcal{S}_r(t) = e^{-t\overline{W}} \eth\, e^{t\overline{W}} = \mathrm{d}t\, \overline{W} + \mathcal{S}_{rg}(t),$$

$$\mathcal{G}_\alpha(t) = e^{-t\overline{W}}\left(D_\alpha e^{t\overline{W}}\right), \quad \text{etc.}.$$

The integral which is to be performed is in this case given by

$$2\int\Theta = \mathrm{Tr}\int \mathrm{d}^4x\, \mathrm{d}^4\theta \int_0^1 t\, \mathrm{d}t\, \left(-W\mathcal{W} iK\mathcal{G}_\alpha + iK\mathcal{W}^\alpha W\mathcal{G}_\alpha\right)$$

$$+ \mathrm{Str}\int \mathrm{d}^4x\, \mathrm{d}^4\theta \int_0^1 t\, \mathrm{d}t\, \Big(iK\tfrac{1}{4}\big(\overline{\nabla}^2\, \nabla^\alpha(W+\overline{W})\big)\mathcal{G}_\alpha$$

$$- W\mathcal{W}^\alpha(\nabla_\alpha iK) - iK\mathcal{W}^\alpha\big(\nabla_\alpha\overline{W}\big)\Big) + \text{h.c.}, \qquad (19)$$

where $\mathcal{W}_\alpha, \mathcal{G}_\alpha, \nabla_\alpha$ all depend on t. As before, this integral cannot be solved because all quantities are non-commuting and t appears in exponentials. The integral (19) added to (14) gives the consistent anomaly Ω' in the gauge vector case.

We can get explicitly the final result for the consistent anomaly in the abelian case. In that case $\mathcal{S}_\ell^2 = 0$, $\mathcal{S}_r^2 = 0$, $\mathcal{G}_\alpha = (D_\alpha\overline{W})$, $\mathcal{G}_\alpha(t) = t\mathcal{G}_\alpha$, $\mathcal{W}_\alpha = W_\alpha$, $\mathcal{W}_\alpha(t) = t\mathcal{W}_\alpha$ etc., so after the t-integration we get

$$\int\Theta = \tfrac{1}{6}\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \left((D_\alpha(WW^\alpha))iK + 2iKW^\alpha\mathcal{G}_\alpha\right) + \text{h.c.} \qquad (20)$$

The consistent anomaly reads

$$\Omega' = \Omega + \int\Theta = -\tfrac{1}{6}\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, iK\left(W^\alpha\mathcal{G}_\alpha - D_\alpha(WW^\alpha)\right) + \text{h.c.} \qquad (21)$$

The situation with the WZ lagrangian is also similar to the gauge chiral case. Generally it cannot be calculated explicitly, in the abelian case it is possible. The multiplet of Goldstone fields U is now a real superfield which transforms under the gauge group as $e^{iU} \to e^{iU} e^{-iK}$, i.e. $U \to U - K$. Under the sU-transformations the fields transform as $W \to W - isU$, $\overline{W} \to \overline{W} + isU$, $\mathcal{G}_\alpha \to \mathcal{G}_\alpha + is(D_\alpha U)$ etc. Performing the integration of (14), we get for the WZ action

$$L_{\mathrm{WZ}} = \int \mathrm{d}^4x\, \mathrm{d}^4\theta \int_0^1 \mathrm{d}s \left(-\tfrac{1}{6}\right) iU\big(W^\alpha(\mathscr{G}_\alpha + is(D_\alpha U)) - D_\alpha((W - isU)W^\alpha)\big) + \text{h.c.}$$

$$= -\tfrac{1}{6}\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, iU\big(W^\alpha\big(D_\alpha(W+\overline{W})\big) - W(D_\alpha W^\alpha)\big)$$

$$-\tfrac{1}{12}\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, (iU)^2(D_\alpha W^\alpha) + \text{h.c.} \tag{22}$$

In this case also a direct check easily gives $\delta L_{\mathrm{WZ}} = \Omega'$.

4. Conclusions

The 2-forms relating the consistent and the covariant supersymmetric gauge anomalies are found for the gauge chiral and the gauge vector representations. This result is the supersymmetric analogue of the result of ref. [8]; there the corresponding form was ω^0_{2n+1} [10]. Unlike ω^0_{2n+1}, the geometric meaning of our θ and Θ is not clear. This may be due to at least two facts: First, the Chern classes for the supermanifolds are not well defined. In such a situation one may expect physics to give hints to mathematics – e.g. that the Chern classes may be something of the form (4) or (18) – in the four-dimensional case. But we have to keep in mind a second complication as well: both vector and chiral representations contain constraints, which may make it difficult to recognize the general geometric results.

The question of how the anomalies in two representations are related we cannot answer for the non-abelian case, because of the involved expressions (6) and (19). But the consideration of the abelian limit of both expressions gives a relation which fits into a general picture nicely.

We can define 1-parameter interpolation connecting the gauge vector and the gauge chiral representation using e^{tW}: $\mathrm{e}^{\overline{W}}\mathrm{e}^{tW}$, $\mathrm{e}^{-tW}\varphi$, $\mathrm{e}^{-tW}\mathscr{W}_\alpha\,\mathrm{e}^{tW}$, $\mathrm{e}^{-tW}\mathrm{e}^{-\overline{W}}(D_\alpha\,\mathrm{e}^{\overline{W}}\mathrm{e}^{tW})$ give the gauge field, matter field, field strength and connection for the values $t=1$ and $t=0$ for the gauge chiral and the gauge vector representations respectively. Denoting, in the abelian case,

$$v = \overline{W} + tW\,, \qquad \bar{v} = W + t\overline{W}\,,$$

$$g_\alpha = D_\alpha v\,, \qquad w_\alpha = W_\alpha = \mathscr{W}_\alpha\,,$$

$$\delta v = (1-t)iK - i\overline{\Lambda} + ti\Lambda\,,$$

we can try to transform the abelian consistent anomalies (11) and (21) in such a way to come to a general expression involving only v's and w_α's.

Indeed, after some work we get that they can be written in a unique way:

$$O = \tfrac{1}{6}\int d^4x\, d^4\theta \left(\delta v(D^\alpha v)w_\alpha + v(\bar{D}^{\dot\alpha}\delta\bar{v})\bar{w}_{\dot\alpha}\right) + \text{h.c.}, \tag{23}$$

where anomalies were $\omega' = O|_{t=1}$, $\Omega' = -O|_{t=0}$.

Moreover, applying the homotopy operation we can find the term relating these two anomalies, being therefore of the WZ-type. This structure is the same as in the nonsupersymmetric case, where the left–right and the vector–axial forms of anomaly are related in such a way [13]. Varying (23)

$$\delta O = \tfrac{1}{6}\int d^4x\, d^4\theta \left(-\delta v(D^\alpha \delta v)w_\alpha + \delta v(\bar{D}^{\dot\alpha}\delta\bar{v})\bar{w}_{\dot\alpha}\right) + \text{h.c.}, \tag{24}$$

and introducing generalized variations

$$\boldsymbol{\delta} v = \delta v + dt\, W = (1-t)\delta\bar{W} + t\delta V + dt\,(V - \bar{W})$$

we get a 2-form

$$T = \tfrac{1}{6}\int d^4x\, d^4\theta \left(-\boldsymbol{\delta} v(D^\alpha \boldsymbol{\delta} v)w_\alpha + \boldsymbol{\delta} v(\bar{D}^{\dot\alpha}\boldsymbol{\delta}\bar{v})\bar{w}_{\dot\alpha}\right) + \text{h.c.}$$

Integration of T over t gives

$$\tfrac{1}{12}\int d^4x\, d^4\theta \left(-V(D^\alpha(\delta\bar{W} + \delta V)) + 3\delta V(D^\alpha(V - \bar{W}))\right)W_\alpha + \text{h.c.}$$

$$= \tfrac{1}{12}\int d^4x\, d^4\theta \Big(4i\Lambda\Gamma^\alpha W_\alpha - 2iK\mathscr{G}^\alpha W_\alpha + 2iK(D_\alpha W W^\alpha)$$

$$+\delta\left\{-V^2(D^\alpha W_\alpha) - \bar{W}^2(D^\alpha W_\alpha) - V(D^\alpha V)W_\alpha - 3V(D^\alpha\bar{W})W_\alpha\right\}\Big) + \text{h.c.},$$

which, up to the gauge variation written in the last line, is equal to $\omega' + \Omega'$, and therefore interpolates between ω' and $-\Omega'$.

I want to thank J. Wess for his hospitality during my work in the Institute of Theoretical Physics in Karlsruhe; I am also grateful to I. McArthur, M. Scholl and R. Garreis for helpful discussions. Finally I would like to thank the referee for encouraging me to bring all the results to their clear closed form.

Appendix A

REPRESENTATIONS OF SUSY YM

Gauge chiral:

$$
\begin{aligned}
&e^{V} \to e^{-i\bar{\Lambda}} e^{V} e^{i\Lambda}, \\
&\mathscr{D}_\alpha = e^{-V} D_\alpha e^{V} = D_\alpha + \Gamma_\alpha, \\
&\bar{\mathscr{D}}_{\dot\alpha} = \bar{D}_{\dot\alpha}, \\
&\mathscr{D}^{a} = -\tfrac{1}{4} i \sigma^{a}_{\alpha\dot\alpha} \{ \mathscr{D}^{\alpha}, \bar{\mathscr{D}}^{\dot\alpha} \}, \\
&\Gamma_\alpha = e^{-V}\left(D_\alpha e^{V}\right), \\
&\bar{\Gamma}_{\dot\alpha} = e^{V}\left(\bar{D}_{\dot\alpha} e^{-V}\right) = -\Gamma^{\dagger}_{\alpha}, \\
&\Gamma_{\dot\alpha} = e^{-V}\left(\bar{D}_{\dot\alpha} e^{V}\right) = e^{-V} \Gamma^{\dagger}_{\alpha} e^{V}, \\
&W_\alpha = -\tfrac{1}{4}\left[\bar{\mathscr{D}}_{\dot\alpha}, \left\{ \bar{\mathscr{D}}^{\dot\alpha}, \mathscr{D}_\alpha \right\} \right] = -\tfrac{1}{4}\left(\bar{D}^{2} \Gamma_\alpha \right) \\
&\bar{W}_{\dot\alpha} = \tfrac{1}{4}\left(D^{2} \Gamma_{\dot\alpha} \right) = W^{\dagger}_{\alpha}, \\
&W_{\dot\alpha} = e^{-V} W^{\dagger}_{\alpha} e^{V} = -\tfrac{1}{4}\left(\mathscr{D}^{2} \Gamma_{\dot\alpha} \right), \\
&\mathscr{D}^{\alpha} W_\alpha = \bar{\mathscr{D}}_{\dot\alpha} W^{\dot\alpha} \quad \text{(Bianchi identities)}, \\
&\left\{ \mathscr{D}_\alpha, \bar{\mathscr{D}}_{\dot\alpha} \right\} = 2 i \sigma^{a}_{\alpha\dot\alpha} \mathscr{D}_a, \\
&\left[\mathscr{D}^{a}, \mathscr{D}_\alpha \right] = -\tfrac{1}{2} i \sigma^{a}_{\alpha\dot\alpha} W^{\dot\alpha}.
\end{aligned}
$$

Gauge vector:

$$
\begin{aligned}
&e^{W} \to e^{-iK} e^{W}, \\
&\nabla_\alpha = e^{-\bar{W}} D_\alpha e^{\bar{W}} = D_\alpha + \mathscr{G}_\alpha, \\
&\bar{\nabla}_{\dot\alpha} = e^{W} \bar{D}_{\dot\alpha} e^{-W}, \qquad (\nabla_\alpha)^{\dagger} = \bar{\nabla}_{\dot\alpha}, \\
&\mathscr{G}_\alpha = e^{-\bar{W}}\left(D_\alpha e^{\bar{W}} \right), \\
&\bar{\mathscr{G}}_{\dot\alpha} = -e^{W}\left(\bar{D}_{\dot\alpha} e^{-W} \right) = \mathscr{G}^{\dagger}_{\alpha} = -\mathscr{G}_{\dot\alpha}, \\
&\mathscr{W}_\alpha = -\tfrac{1}{4}\left[\bar{\nabla}_{\dot\alpha}, \left\{ \bar{\nabla}^{\dot\alpha}, \nabla_\alpha \right\} \right] = e^{W} W_\alpha e^{-W}, \\
&\bar{\mathscr{W}}_{\dot\alpha} = \tfrac{1}{4}\left[\nabla^{\alpha}, \left\{ \nabla_\alpha, \bar{\nabla}_{\dot\alpha} \right\} \right] = e^{-\bar{W}} \bar{W}_{\dot\alpha} e^{\bar{W}} = e^{W} W_{\dot\alpha} e^{-W} = \mathscr{W}^{\dagger}_{\alpha}, \\
&\nabla^{\alpha} \mathscr{W}_\alpha = \bar{\nabla}_{\dot\alpha} \bar{\mathscr{W}}^{\dot\alpha} \quad \text{(Bianchi identities)}.
\end{aligned}
$$

Appendix B

$\int \theta$ FOR THE GAUGE CHIRAL REPRESENTATION

We present here the details of the calculation of $\int \theta$ in the WZ gauge. The calculations are done in the $y\theta\bar{\theta}$ basis. From the expansions of the basic fields we get (keeping in mind that $V^3 = 0$)

$$\Lambda = a\,,$$

$$\bar{\Lambda} = a - 2i\theta\sigma^m\bar{\theta}\partial_m a + \theta\theta\bar{\theta}\bar{\theta}\partial^2 a\,,$$

$$V = -\theta\sigma^m\bar{\theta}v_m + i\theta\theta\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\theta\lambda + \tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}(D - i\partial_m v^m)$$

$$\begin{aligned}\delta V &= \tfrac{1}{2}i[V, \Lambda + \bar{\Lambda}] + i(\Lambda - \bar{\Lambda})\\ &= -2\theta\sigma^m\bar{\theta}\left(\partial_m a + \tfrac{1}{2}i[v^m, a]\right)) - \theta\theta\bar{\theta}_{\dot{\alpha}}\left[\bar{\lambda}^{\dot{\alpha}}, a\right] + \bar{\theta}\bar{\theta}\theta^\alpha[\lambda_\alpha, a]\\ &\quad + \tfrac{1}{2}i\theta\theta\bar{\theta}\bar{\theta}[D, a] - i\theta\theta\bar{\theta}\bar{\theta}\partial^2 a + \tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\partial_m[v^m, a]\,.\end{aligned}$$

Therefore,

$$\begin{aligned}S_g(t) &= e^{-tV}\delta\, e^{tV} = t\delta V + \tfrac{1}{2}t^2[\delta V, V]\\ &= -2t\theta\sigma^m\bar{\theta}\left(\partial_m a + \tfrac{1}{2}i[v_m, a]\right) - t\theta\theta\bar{\theta}_{\dot{\alpha}}[\bar{\lambda}^{\dot{\alpha}}, a] + t\bar{\theta}\bar{\theta}\theta^\alpha[\lambda_\alpha, a] + \tfrac{1}{2}it\theta\theta\bar{\theta}\bar{\theta}[D, a]\\ &\quad - it\theta\theta\bar{\theta}\bar{\theta}\left(\partial_m\left(\partial^m a + \tfrac{1}{2}i[v^m, a]\right) + \tfrac{1}{2}it\left[v^m, \partial_m a + \tfrac{1}{2}i[v_m, a]\right]\right),\end{aligned}$$

$$\mathscr{D}_\alpha(t) = D_\alpha + \Gamma_\alpha(t)\,,$$

$$\begin{aligned}\Gamma_\alpha(t) &= t(D_\alpha V) + \tfrac{1}{2}t^2[D_\alpha V, V]\\ &= -t\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}v_m + 2it\theta_\alpha\bar{\theta}\bar{\lambda} - it\bar{\theta}\bar{\theta}\lambda_\alpha + t\theta\theta\bar{\theta}_\alpha D\\ &\quad - 2it\sigma^{mn}{}_\alpha{}^\beta\bar{\theta}\bar{\theta}\theta_\beta\left(\partial_m v_n + \tfrac{1}{2}itv_m v_n\right) + t\sigma^m_{\alpha\dot{\alpha}}\theta\theta\bar{\theta}\bar{\theta}\left(\partial_m\bar{\lambda}^{\dot{\alpha}} + \tfrac{1}{2}it\left[v_m, \bar{\lambda}^{\dot{\alpha}}\right]\right),\end{aligned}$$

$$\begin{aligned}W_\alpha(t) &= -\tfrac{1}{4}\left(\bar{D}^2\Gamma_\alpha(t)\right)\\ &= -it\lambda_\alpha + t\theta_\alpha D - 2it\sigma^{mn}{}_\alpha{}^\beta\theta_\beta\left(\partial_m v_n + \tfrac{1}{2}itv_m v_n\right) + t\sigma^m_{\alpha\dot{\alpha}}\theta\theta\left(\partial_m\bar{\lambda}^{\dot{\alpha}} + \tfrac{1}{2}it\left[v_m, \bar{\lambda}^{\dot{\alpha}}\right]\right).\end{aligned}$$

Inserting these into (5) and integrating over $d^2\theta\, d^2\bar{\theta}$, after a lengthy calculation we get

$$\int\theta = \mathrm{Tr}\int d^4x \int_0^1 dt\Big(\tfrac{1}{2}it^2\sigma^m_{\alpha\dot\alpha}\lambda^\alpha\big\{v_m,[\bar\lambda^{\dot\alpha},a]\big\} - t^2\sigma^m_{\alpha\dot\alpha}\lambda^\alpha\big\{\bar\lambda^{\dot\alpha},\partial_m a+\tfrac{1}{2}i[v_m,a]\big\}$$

$$-t^2\eta^{pq}D\big\{v_p,\partial_q a+\tfrac{1}{2}i[v_q,a]\big\} + 2it^2(mn,pq)\big(\partial_m v_n+\tfrac{1}{2}itv_m v_n\big)\big\{v_p,\partial_q a+\tfrac{1}{2}i[v_q,a]\big\}\Big)$$

$$-\mathrm{Tr}\int d^4x\int_0^1 dt\Big(t^2\sigma^m_{\alpha\dot\alpha}\lambda^\alpha\big\{\bar\lambda^{\dot\alpha},\partial_m a+\tfrac{1}{2}i[v_m,a]\big\} - \tfrac{1}{2}it^2\sigma^m_{\alpha\dot\alpha}\lambda^\alpha\big\{v_m,[\bar\lambda^{\dot\alpha},a]\big\}$$

$$+t^2D\big\{v^m,\partial_m a+\tfrac{1}{2}i[v_m,a]\big\} + 2it^2(mn,pq)\big(\partial_m v_n+\tfrac{1}{2}itv_m v_n\big)\big\{v_q,\partial_p a+\tfrac{1}{2}i[v_p,a]\big\}\Big) + \text{h.c.},$$

where $(mn,pq) = -\frac{1}{2}(\eta^{mp}\eta^{nq}-\eta^{mq}\eta^{np}) - \frac{1}{2}i\varepsilon^{mnpq}$. Integral over t gives finally

$$2\int\theta = \mathrm{Tr}\int d^4x\, a\Big(\tfrac{4}{3}\partial_m\big(\lambda\sigma^m\bar\lambda - \bar\lambda\bar\sigma^m\lambda\big) + i\big[v_m,\lambda\sigma^m\bar\lambda-\bar\lambda\bar\sigma^m\lambda\big]$$

$$+2\varepsilon^{mnpq}\big(\tfrac{4}{3}\partial_m v_n\cdot\partial_p v_q - \tfrac{1}{2}v_m v_n v_p v_q + \tfrac{5}{6}i\{\partial_m v_n, v_p v_q\} + \tfrac{1}{6}iv_p(\partial_m v_n)v_q\big)\Big)$$

Appendix C

WZ GAUGE FOR GAUGE VECTOR REPRESENTATION

In table C.1 the sequence of chiral transformations and gauge variations from arbitrary field to nilpotent form is shown. One starts with an arbitrary field W.

TABLE C.1
The sequence of chiral transformations and gauge variations taking an arbitrary field to nilpotent form

	W	iL_1	W'	$-iK$	W_1	iL_2	W_1'
	f	$-f$	0	$-ic$	$-ic$	ic	0
θ	ϕ	$-\phi-\chi$	$-\chi$	χ	0	0	0
$\bar\theta$	$\bar\chi$	0	$\bar\chi$	$-\bar\chi$	0	0	0
$\theta\theta$	m	$-n^*-m$	$-n^*$	n^*	0	0	0
$\bar\theta\bar\theta$	n	0	n	$-n$	0	0	0
$\theta\sigma^m\bar\theta$	$-v_m+i\partial_m f$	$-i\partial_m f$	$-v_m$	$i\,\mathrm{Im}\,v_m$	$-\mathrm{Re}\,v_m$	$-\partial_m c$	$-\mathrm{Re}\,v_m-\partial_m c$
$\theta\theta\bar\theta$	$i\bar\lambda$	$\frac{1}{2}i\sigma^m\partial_m(\phi+\chi)$	$i\bar\lambda_1$	$-\bar L^*$	$i\bar\varphi^*$	0	$i\bar\varphi$
$\bar\theta\bar\theta\theta$	$-i\psi$	0	$-i\psi_1$	L	$-i\varphi$	0	$-i\varphi$
$\theta\theta\bar\theta\bar\theta$	$\frac{1}{2}(d+\frac{1}{2}\Box f)$	$-\frac{1}{4}\Box f$	$\frac{1}{2}d$	$-\frac{1}{2}i\,\mathrm{Im}\,d-\frac{1}{4}i\Box c$	$\frac{1}{2}\mathrm{Re}\,d-\frac{1}{4}i\Box c$	$\frac{1}{4}i\Box c$	$\frac{1}{2}\mathrm{Re}\,d$

The notation $L = -\frac{1}{2}i(\lambda-\psi+\frac{1}{2}\sigma^m\partial_m(\bar\phi+\bar\chi))$, $\varphi = \frac{1}{2}(\lambda+\psi+\frac{1}{2}\sigma_m\partial^m(\bar\phi+\bar\chi))$ is introduced. Field expansions are written in $x\theta\bar\theta$-basis [7].

TABLE C.2
As table C.1 but starting from fields in the WZ gauge

	W	iL_1	W'	$-iK$	W_1	iL_2	W_1'
	0	0	0	$-ic$	$-ic$	ic	0
θ	0	0	0	0	0	0	0
$\bar{\theta}$	0	0	0	0	0	0	0
$\theta\theta$	0	0	0	0	0	0	0
$\bar{\theta}\bar{\theta}$	0	0	0	0	0	0	0
$\theta\sigma^m\bar{\theta}$	$-\mathrm{Re}\, v_m$	0	$-\mathrm{Re}\, v_m$	0	$-\mathrm{Re}\, v_m$	$-\partial_m c$	$-\mathrm{Re}\, v_m - \partial_m c$
$\theta\theta\bar{\theta}$	$i\bar{\varphi}$	0	$+i\bar{\varphi}$	0	$+i\bar{\varphi}$	0	$i\bar{\varphi}$
$\bar{\theta}\bar{\theta}\theta$	$-i\varphi$	0	$-i\varphi$	0	$-i\varphi$	0	$-i\varphi$
$\theta\theta\bar{\theta}\bar{\theta}$	$\frac{1}{2}\mathrm{Re}\, d$	0	$\frac{1}{2}\mathrm{Re}\, d$	$-\frac{1}{4}i\,\Box\, c$	$\frac{1}{2}\mathrm{Re}\, d - \frac{1}{4}i\,\Box\, c$	$\frac{1}{4}i\,\Box\, c$	$\frac{1}{2}\mathrm{Re}\, d$

A chiral transformation $W + iL_1$ brings it to the form W', which is nilpotent and equivalent to W as far as covariant derivatives are concerned. Now, one fixes the gauge defining $-iK$ through W' and this gives W_1. Finally, the second chiral transformation brings W_1 to the nilpotent form W_1', actually with $W_1'^3 = 0$.

If we take that W is from the very beginning in the WZ gauge, the transformation table and the gauge variations are described in table C.2.

References

[1] O. Piguet and K. Sibold, Nucl. Phys. B247 (1984) 484

[2] S. Ferrara, L. Girardello, O. Piguet and R. Stora, Phys. Lett. B157 (1985) 179

[3] R. Garreis, M. Scholl and J. Wess, Z. Phys. C28 (1985) 623;
N.K. Nielsen, Nucl. Phys. B244 (1984) 499;
E. Gaudagnini, K. Konishi and M. Mintchev, Phys. Lett. B157 (1985) 37;
G. Girardi, R. Grimm and R. Stora, Phys. Lett. B156 (1985) 203;
L. Bonora, P. Pasti and M. Tonin, Phys. Lett. B156 (1985) 341; Nucl. Phys. B261 (1985) 249

[4] H. Itoyama, V.P. Nair and H.C. Ren, Nucl. Phys. B262 (1985) 317;
E. Guadagnini, M. Mintchev, Nucl. Phys. B269 (1986) 543

[5] J. Wess, *in* Topics in QFT and gauge theories, Proc. Salamanca 1977, Lecture Notes in Physics 77

[6] S.J. Gates, M.T. Grisaru, M. Rocek and W. Siegel, Superspace (Benjamin/Cumming, New York, 1983)

[7] J. Wess and J. Bagger, Supersymmetry and supergravity (Princeton Univ. Press, Princeton, NJ, 1983)

[8] W.A. Bardeen and B. Zumino, Nucl. Phys. B244 (1984) 421

[9] J. Wess, B. Zumino, Phys. Lett. B37 (1971) 95

[10] B. Zumino, Y.S. Wu and A. Zee, Nucl. Phys. B239 (1984) 477

[11] I.N. McArthur and H. Osborn, Nucl. Phys. B268 (1986) 573

[12] M. Marinkovic, Ph.D. Thesis, Univ. Karlsruhe (1990)

[13] L. Alvarez-Gaumé and P. Ginsparg, Ann. Phys. 161 (N.Y.) (1985) 423