# PART II FLUCTUATIONS IN AN ISOTROPIC UNIVERSE



...ical Idea:

- Structure in the universe is the result of the growth, driven by gravity, of primordial fluctuations in the density of the universe.

Note:

- At high redshift universe is indistinguishable from Einstein-de Sitter ($\Omega = 1$), so that small overdensities present then would have local values of $\Omega > 1$, would eventually stop expanding and collapse to form virialized structures.

- Galaxy systems have formed ("collapsed") relatively late (recently) in the history of the Universe. For example, the mean density of a galaxy like the Milky Way is

$$\langle \rho_{gal} \rangle \approx \frac{M}{r^3} \approx \frac{V^2 r}{G r^3} \approx \frac{(200\ \mathrm{km/s})^2\ 10\ \mathrm{kpc}}{G\ 10^3\ \mathrm{kpc}^3} \approx 10^{+8}\ \frac{M_\odot}{\mathrm{kpc}^3}$$

$$\rho_{crit} = \frac{3H_0^2}{8\pi G} \approx 277.7\ h^2\ \frac{M_\odot}{\mathrm{kpc}^3}$$

$$\Rightarrow \text{For } h \approx 0.6 \implies \frac{\rho_{gal}}{\rho_{crit}} \approx 10^6$$

Similarly, for a galaxy cluster $\dfrac{\rho_{cluster}}{\rho_{crit}} \approx 200 \quad 1000$

But, since $\langle \rho \rangle \propto (1+z)^3$ $\Rightarrow$ the Universe itself was denser than a typical galaxy at $z \sim 100$

cluster at $z \sim 10$

Galaxy systems then must have formed at $z < 100$.

This is long after recombination, and well into the matter-dominated era

More importantly, it implies that

"galaxy and cluster formation epochs are accessible to observation."

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## Growth of Density Fluctuations – Linear theory – Non relativistic case.

Recall the standard equations of hydrodynamics.

– Continuity: $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

– Euler's eq.: $$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\vec{\nabla} p - \vec{\nabla}\phi$$

– Poisson's eq.: $$\nabla^2 \phi = 4\pi G \rho$$

which, in "Lagrangian" form, i.e., in a reference frame that accompanies the fluid in its mean motion, can be written as

$$(1)\quad \begin{cases} \dfrac{d\rho}{dt} = -\rho\,\nabla\cdot\vec{v} \\ \dfrac{d\vec{v}}{dt} = -\dfrac{1}{\rho}\vec{\nabla}p - \vec{\nabla}\phi \\ \nabla^2\phi = 4\pi G\rho \end{cases}$$

using the "total" or "convective" derivative operator

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{v}\cdot\nabla)$$

For the problem of interest to us, the "fluid element rest frame" is that given by comoving coordinates. (i.e. the frame "expands" with the fluid rather than moving with it). For example, if one writes Hubble's law as

$$\vec{v} = H\,\vec{r}$$

we have

$$\frac{d\rho}{dt} = -\rho\,\nabla\cdot\vec{v} = -3H\rho \quad\Rightarrow\quad \frac{d\rho}{\rho} = -3\,H\,dt = -3\,\frac{dR}{R}$$

$$\Rightarrow\quad \rho = \rho_0\,R^{-3} = \rho_0\,(1+z)^3$$

Typically one searches for a solution to the set of eq. (1) above and then perturbs them. Let us call $(\rho_0, \vec{v}_0, p_0, \phi_0)$ this solution.
To first order, we have

$$\vec{v} = \vec{v}_0 + \delta\vec{v}\,,\quad \rho = \rho_0 + \delta\rho\,,\quad p = p_0 + \delta p \quad \phi = \phi_0 + \delta\phi$$

Now

$$\frac{d(\rho_0+\delta\rho)}{dt} = -(\rho_0+\delta\rho)\ \nabla\cdot(\vec{v}_0+\delta\vec{v})$$

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$$\frac{d\rho_0}{dt} + \frac{d\,\delta\rho}{dt} \quad - \rho_0\,\nabla\cdot\vec{v}_0 \quad \rho_0\,\nabla\cdot\delta\vec{v} - \delta\rho\,\nabla\cdot\vec{v}_0 - \underbrace{\delta\rho\,\nabla\cdot\delta\vec{v}}_{\simeq 0\ \text{(second order)}}$$

$$\Rightarrow \frac{d}{dt}\delta\rho + \frac{\delta\rho}{\rho_0}\frac{d\rho_0}{dt} = -\rho_0\,\nabla\cdot\delta\vec{v} \quad \longleftarrow \left(\text{using } \frac{d\rho_0}{dt} \quad -\rho_0\,\nabla\cdot\vec{v}_0\right)$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\delta\rho}{\rho_0}\right) = -\nabla\cdot\delta\vec{v}}$$

$\frac{\delta\rho}{\rho_0} \equiv \Delta$ = density contrast or overdensity

Similarly, one [illegible] Euler's eq that

$$\boxed{\frac{d(\delta\vec{v})}{dt} + (\delta\vec{v}\cdot\nabla)\,\vec{v}_0 = -\frac{1}{\rho_0}\nabla\delta p - \nabla\delta\phi}$$

and

$$\boxed{\nabla^2\delta\phi = 4\pi G\,\delta\rho}$$

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It is convenient to write these eqs in comoving coordinates, so

$$\vec{x} = R(t)\,\vec{r}$$

$$\Rightarrow \vec{v} = \frac{d\vec{x}}{dt} = \frac{dR}{dt}\vec{r} + R(t)\frac{d\vec{r}}{dt} = \vec{v}_0 + \delta\vec{v}$$

$\vec{v}_0 \equiv$ "Hubble flow term

$\delta\vec{v} = R(t)\frac{d\vec{r}}{dt} = R(t)\,\vec{u} =$ velocity perturbation" ; $\vec{u} \equiv$ perturbed comoving velocity

We have now, by substituting above

$$\frac{d\vec{u}}{dt} + 2\left(\frac{\dot{R}}{R}\right)\vec{u} = -\frac{1}{\rho_0 R^2}\nabla_c \delta p - \frac{1}{R^2}\nabla_c \phi$$

( $\nabla_c$ is gradient relative to comoving coordinates)

Consider now "adiabatic" perturbations, i.e. those for which

$$\delta p = c_s^2 \, \delta\rho$$

$c_s$: sound speed

Combining the eqs. above we have

$$\frac{d^2\Delta}{dt^2} + 2\frac{\dot{R}}{R}\frac{d\Delta}{dt} = \frac{c_s^2}{\rho_0 R^2}\nabla_c^2 \delta\rho + 4\pi G \delta\rho$$

Proposing a wave-like solution, $\Delta = \frac{\delta\rho}{\rho_0} = e^{i(\vec{k}_c\cdot\vec{r} - \omega t)}$

we have

$$\boxed{\frac{d^2\Delta}{dt^2} + 2\frac{\dot{R}}{R}\frac{d\Delta}{dt} = \Delta\left(4\pi G\rho_0 - c_s^2 k^2\right)}$$

where $k = |\vec{k}| = |\vec{k}_c|/R$ = "proper" wave vector

## Jeans' Instability



Assume a non-expanding medium, $\dot{z} = 0$. Then we have

$$\frac{d^2\Delta}{dt^2} = \Delta\left(4\pi G\rho_0 - c_s^2 k^2\right)$$

$\Rightarrow \quad \omega^2 = c_s^2 k^2 - 4\pi G\rho_0$ this is the Jean's dispersion relation.

As usual,

**CASE 1** $\omega^2 > 0 \Rightarrow k_J > \sqrt{\frac{4\pi G\rho_0}{c_s^2}} \Rightarrow$ solutions are oscillatory.

$\Rightarrow$ perturbations on scales smaller than

$$\lambda_J = \frac{2\pi}{k_J} = \left(\frac{\pi}{G\rho_0}\right)^{1/2} c_s \quad \text{are stable}$$

**CASE 2** $\omega^2 < 0$

$$\Rightarrow \quad \Delta = \Delta_0 \, e^{(\Gamma t + i\vec{k}\cdot\vec{r})}$$

where $\Gamma = \pm\left[4\pi G\rho_0\left(1 - \frac{\lambda_J^2}{\lambda^2}\right)\right]^{1/2}$

$+ \Rightarrow$ "growing" mode

$- \Rightarrow$ "decaying" mode

( For $\lambda \gg \lambda_J$ the growth rate is

$$\Gamma \approx (4\pi G\rho_0)^{1/2}$$

so that perturbations grow exponentially on a time scale

$$\tau = \Gamma^{-1} = (4\pi G\rho_0)^{-1/2} \sim \frac{1}{\sqrt{G\rho_0}} \qquad (\lambda >> \lambda_J)$$

## Jeans' instability in an expanding medium -

When the term $2\frac{\dot{R}}{R}\frac{d\Delta}{dt}$ is non zero the stability criterion

is the same, but the growth rates change -

Look, for example, at the case $\lambda >> \lambda_J$

We have in this case that the pressure term is unimportant,

$$\frac{d^2\Delta}{dt^2} + 2\left(\frac{\dot{R}}{R}\right)\frac{d\Delta}{dt} = 4\pi G\rho_0 \Delta$$

$\boxed{\Omega_0 = 1}$ $\qquad \frac{\dot{R}}{R} = \frac{2}{3t} \qquad 4\pi G\rho = \frac{2}{3t^2}$

so $$\frac{d^2\Delta}{dt^2} + \frac{4}{3t}\frac{d\Delta}{dt} - \frac{2}{3t^2}\Delta = 0$$

Proposing $\Delta = a t^n$ we find $n(n-1) + \frac{4}{3}n - \frac{2}{3} = 0$

so

$$\Delta \propto \begin{cases} t^{2/3} \propto 1/(1+z) & \text{"growing" solution} \\ t^{-1} & \text{"decaying" solution} \end{cases}$$

growth therefore algebraic and not exponential if $\Omega = 1$.

Another important case is

$\boxed{\Omega = 0}$ Solutions are

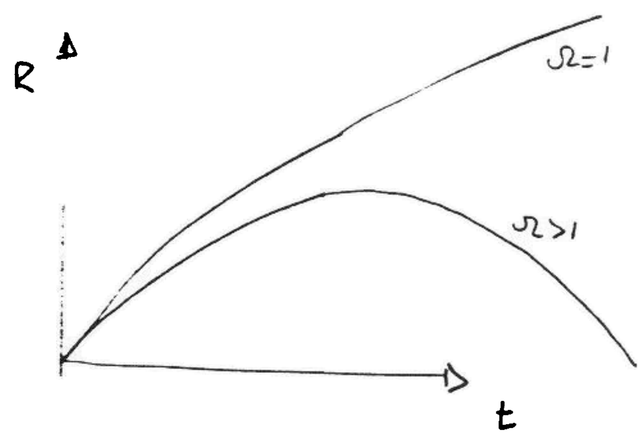
$$\Delta \propto \begin{cases} t^0 & \to \text{constant amplitude} \\ t^{-1} & \to \text{decaying solution} \end{cases}$$

Linear perturbations do not grow if $\Omega \ll 1$.

Another important result can be obtained if the evolution of the perturbation is deduced by comparing Friedman solutions for universes with different values of $\Omega$.

For $\frac{\delta\rho}{\rho} \ll 1$ one finds that

$$\Delta = \frac{3}{5}\,\frac{\Omega - 1}{\Omega}\,R$$




The merit of thinking it in this way is that in no place one assumes that the perturbation is all in causal contact, i.e. within the particle horizon. A perturbation on a scale larger than the horizon still grows linearly with R at early times.

For the general case the solution can be found numerically.

A few things to note

$\boxed{\Lambda = 0}$ $\Rightarrow$ $\Delta \underset{\sim}{\propto} R$ for $\Omega_0 z \gg 1$ but growth essentially stops at $z \sim \frac{1}{\Omega_0}$ if $\Omega_0 < 1$ and $\Omega_\Lambda = 0$; i.e., growth stops as the universe becomes non decelerated.

$\Rightarrow$ Since recombination ($z \sim 1000$) fluctuations have grown by a factor $\sim 1000$ ($\Omega_0 = 1$)

$\sim 100$ ($\Omega_0 = 0.1$).

These are quite modest numbers $\Rightarrow$ • CMB fluctuations must be large

• " " are larger if $\Omega_0 < 1$.

(this is actually one traditional way to argue for the existence of non baryonic DM).

$\boxed{\Lambda \neq 0,\ \Omega_0 + \Omega_\Lambda = 1}$ In this case growth continues to lower redshift. Why?

Recall that

$$\dot{R} = H_0 \left( \frac{\Omega_0}{R} + (1 - \Omega_0) R^2 \right)^{1/2}$$

in this case.

When the first term dominates we are in the "linear with R" growth case. When the second term dominates we switch to the "non decelerated" universe case, when growth is stopped. ($\dot{R} \approx$ constant)

This happens when

$$R \approx \left(\frac{\Omega_0}{1-\Omega_0}\right)^{1/3} \quad \text{or} \quad (1+z) \approx \Omega_0^{-1/3} \quad (\Omega_0 \ll 1)$$

## The RELATIVISTIC CASE

At early times, in the radiation dominated era, the perturbations are in a radiation dominated plasma, so the relativistic hydrodynamical equations are needed. Using these, as well as the eq. of state

$$P = \frac{1}{3}\mathcal{E} = \frac{1}{3}\rho c^2$$

one finds a "wave" equation similar to the one found above.

$$\frac{d^2\Delta}{dt^2} + 2\left(\frac{\dot{R}}{R}\right)\frac{d\Delta}{dt} = \Delta\left(\frac{32\pi G\rho}{3} - k^2 c_s^2\right)$$

and Jeans' critical length reads $\lambda_J = c_s\left(\frac{3\pi}{8G\rho}\right)^{1/2}$

with the important condition $c_s = \frac{c}{\sqrt{3}}$, as appropriate for a fully relativistic plasma.

Solutions are found as in the previous analysis. Growing mode is given by

$$\Delta \propto t \propto R^2 \propto (1+z)^{-2}.$$

Again, this is algebraic in cosmic time, and nowhere the scale of the perturbation relati[illegible] because [illegible] linear

# GALAXY FORMATION IN A BARYONIC UNIVERSE

Here we assume that all of the mass of the universe is in baryonic form and discuss the problems associated with galaxy formation in such a scenario.

We will use the following facts

- Jeans' length is maximum scale for stable fluctuations.

$$\lambda_J = \frac{2\pi}{k_J} = c_s \left(\frac{\pi}{G\rho}\right)^{1/2}$$

Jean's mass is defined as

$$M_J = \frac{\pi}{6} \lambda_J^3 \rho$$

- Perturbations on scales much smaller than $\lambda_J$ propagate as sound waves with angular frequency

$$\omega = \left[ 4\pi G\rho \left( \frac{\lambda_J^2}{\lambda^2} - 1 \right) \right]^{1/2}$$

- Perturbations on scales larger than $\lambda_J$ are unstable, and grow as

$$\Delta \propto R \quad \text{(matter-dominated era)}$$
$$\propto R^2 \quad \text{(radiation-dominated era)}$$

provided $\Omega_0 \sim 1$

Growth is suppressed in non decelerated phases of expansion.

- For an Einstein de Sitter universe the particle horizon at early time is

$$r_H(t) = R(t) \int_0^t \frac{c\,dt}{R(t)} \quad \frac{2c}{H_0} R^{3/2}(t) \quad \frac{2c}{H_0} \frac{t}{t_0} = 3ct \quad \text{if matter dominated}$$

$$= 2ct \quad \text{if radiation dominated}$$

Case 1 – Radiation dominated era

$$\lambda_J = \frac{c}{\sqrt{3}}\left(\frac{3\pi}{8G\rho}\right)^{1/2}$$

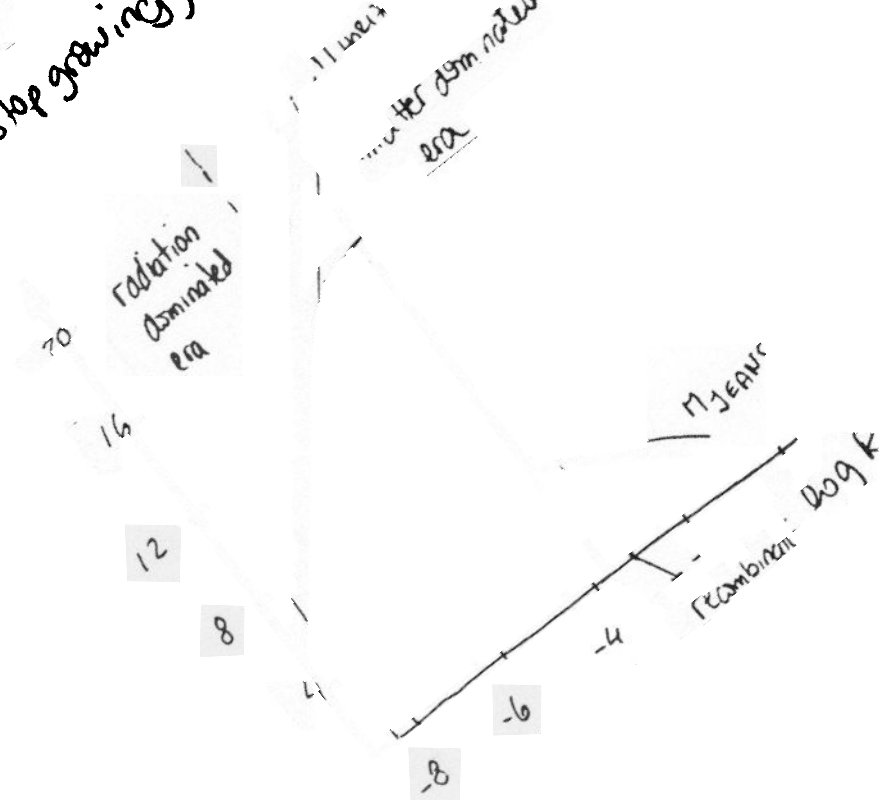
$\rho \approx \rho_{rad} \propto R^{-4}$ since $R \propto t^{1/2} \Rightarrow \rho \propto t^{-2}$

$\Rightarrow \lambda_J \propto t$ Recall that the horizon scale is $r_H = 2ct$

$\Rightarrow \frac{\lambda_J}{r_H} \approx$ constant " " " in other words the Jeans mass is comparable the mass inside the horizon.

This is important, since it implies that perturbations on a given scale are "d" (stop growing) once they become smaller than the horizon scale

see Figure



## CASE 2 - Matter dominated era

Above results are OK as long as $z > z_{eq} = 2.4\,10^4\, \Omega_0 h^2$ and the dynamics of the universe is dominated by radiation. The Jeans mass changes dramatically as the universe becomes matter dominated (but baryons and radiation are still coupled because atoms are fully ionized) and as, finally, atoms recombine.

The baryon mass inside the horizon is

$$M_{H,b} = \rho_b \frac{\pi r_H^3}{6} \simeq \frac{3\,10^{22}}{(\Omega_0 h^2)^{1/2}} R^{3/2} M_\odot \qquad \left( \begin{array}{l} \text{recall that} \\ r_H \propto t \\ R \propto t^{2/3} \\ \rho_b \propto R^{-3} \end{array} \right)$$

$$\Rightarrow \quad M_{H,b} \propto R^{3/2} \propto t$$

For the Jeans' mass we need to evaluate how the sound speed evolves once matter becomes important as a source of gravity (but not of pressure, continuing with our assumption of matter as pressureless "dust")

Since $\quad p = p_r + p_m = p_r$

$$\rho = \rho_r + \rho_m \quad \text{, where } \rho_r = \frac{\rho_{eq}}{2\,(R/R_{eq})^4} \text{ and } \rho_m = \frac{\rho_{eq}}{2\,(R/R_{eq})^3}$$

Setting $\quad x = R/R_{eq} \quad \Rightarrow \quad \frac{\delta\rho_r}{\rho_{eq}} = -\frac{2}{x^5} \quad ; \quad \frac{\delta\rho_m}{\rho_{eq}} = \frac{-3}{2x^4}$

$$\delta p_r = -\frac{2}{3}\frac{c^2}{x^5}$$

which implies that

$$c_s^2 = \frac{\delta P}{\delta \rho} = \frac{c^2}{3}\left(\frac{-2/x^5}{-2/x^5 - 3/2x^4}\right) = \frac{c^2}{3}\left(\frac{1}{1+\frac{3}{4}x}\right)$$

or

$$c_s^2 = \frac{c^2}{3}\,\frac{4\rho_r}{4\rho_r + 3\rho_m}$$

so at late times, when $\rho_m \gg \rho_r$ the sound speed goes from

$$c_s^2 \approx \frac{1}{3}c^2 \quad \text{(radiation dominated era)}$$

to

$$c_s^2 \approx \frac{4}{9}c^2\,\frac{\rho_r}{\rho_m} \;\propto 1/R \;\ll c \quad \text{(matter dominated era)}$$

Using the latter, we have

$$c_s = c\left(\frac{4}{9}\frac{\rho_r}{\rho_m}\right)^{1/2} = \frac{10^6(1+z)^{1/2}}{(\Omega_b h^2)^{1/2}}\ \mathrm{m\,s^{-1}}$$

Going back now to the expression for Jeans' mass,

$$M_J \propto \lambda_J^3 \rho \propto \frac{c_s^3}{\rho^{3/2}}\,\rho \propto \frac{(1+z)^{3/2}\,(1+z)^3}{(1+z)^{9/2}} \simeq \text{constant !}$$

$$\boxed{M_J \approx \frac{3.75\,10^{15}}{(\Omega_b h^2)^2}\ M_\odot}$$

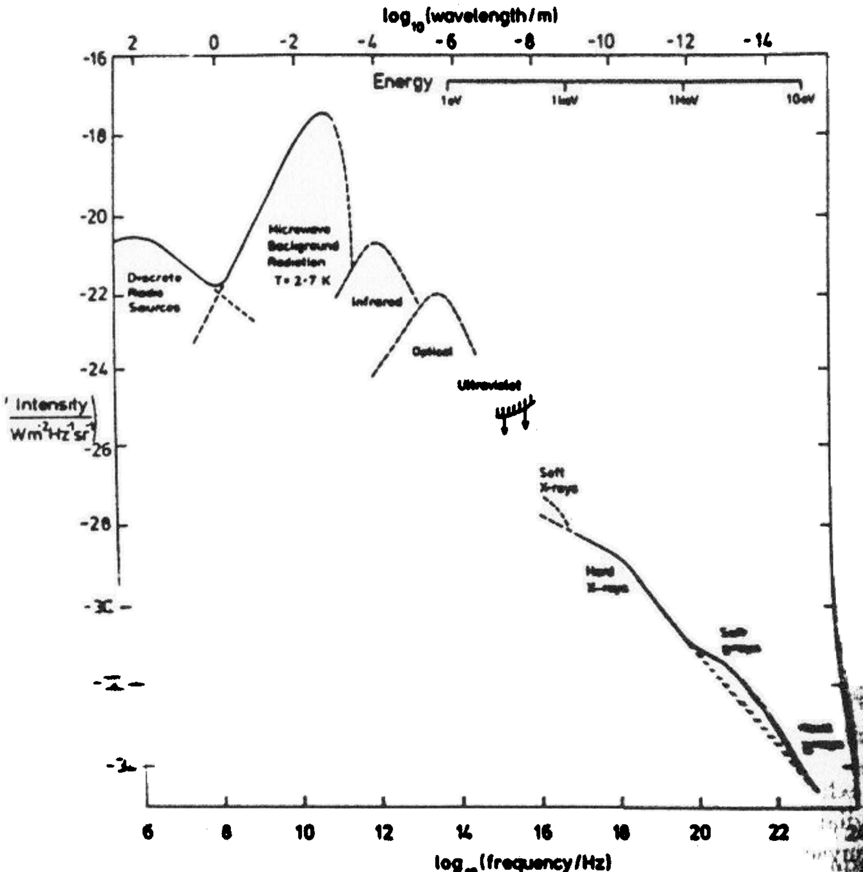



**Fig. 9.1.** (a) The spectrum of the extragalactic background radiation as it was known in 1969 (Longair and Sunyaev 1971). This figure still provides a good indication of the overall spectral energy distribution of the background radiation. The solid lines indicate regions of the spectrum in which extragalactic background radiation has been detected. The dashed lines were theoretical estimates of the background intensity due to discrete sources and should not be taken too seriously. (b) The spectrum of the extragalactic background radiation plotted as $I = \nu I_\nu = \lambda I_\lambda$ (Longair 1995, courtesy of Dr. Andrew Blain). This presentation shows the amount of energy $\varepsilon = 4\pi I/c$ present per unit volume throughout the Universe at the present epoch. The solid lines in the radio, millimetre, X- and $\gamma$-ray wavebands show the observed background intensities. The circles, crosses and squares correspond to upper limits to the background intensity in the far infrared to ultraviolet wavebands and are usually conservative upper limits. We will return to discuss the significance of these limits in Sect. 18.2



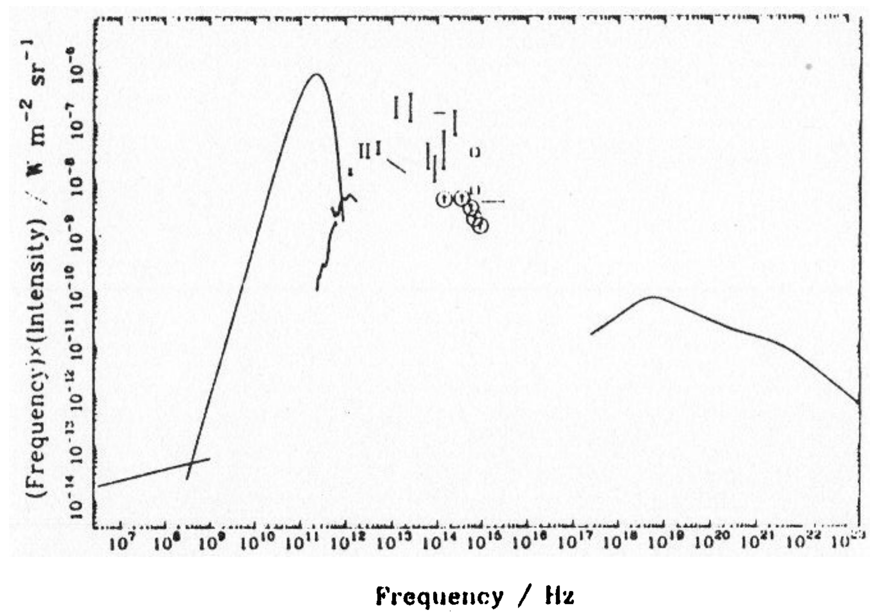



**Fig. 9.1.** continued

**Table 9.1.** The energy densities and photon number densities of the extragalactic background radiation in different regions of the electromagnetic spectrum. Note that these are usually rough estimates which are only useful for making order of magnitude calculations.

| Waveband | Energy density of radiation eV m$^{-3}$ | Number density of photons m$^{-3}$ |
|---|---|---|
| Radio | $\sim 5 \times 10^{-2}$ | $\sim 10^6$ |
| Microwave | $3 \times 10^5$ | $10^8$ |
| Infrared | ? | ? |
| Optical | $\sim 2 \times 10^3$ | $\sim 10^3$ |
| Ultraviolet | ? | ? |
| X-ray | 75 | $3 \times 10^{-3}$ |
| $\gamma$-ray | 25 | $3 \times 10^{-6}$ |

various sky surveys. In Table 9.1, typical energy densities and number densities of the photons in each of the wavebands in which a positive detection of the extragalactic background radiation has been made are listed. It must be emphasised that these are rough estimates and, for precise work, integrations should be taken over the appropriate regions of the spectrum. These figures are, however, often useful for making order of magnitude estimates.

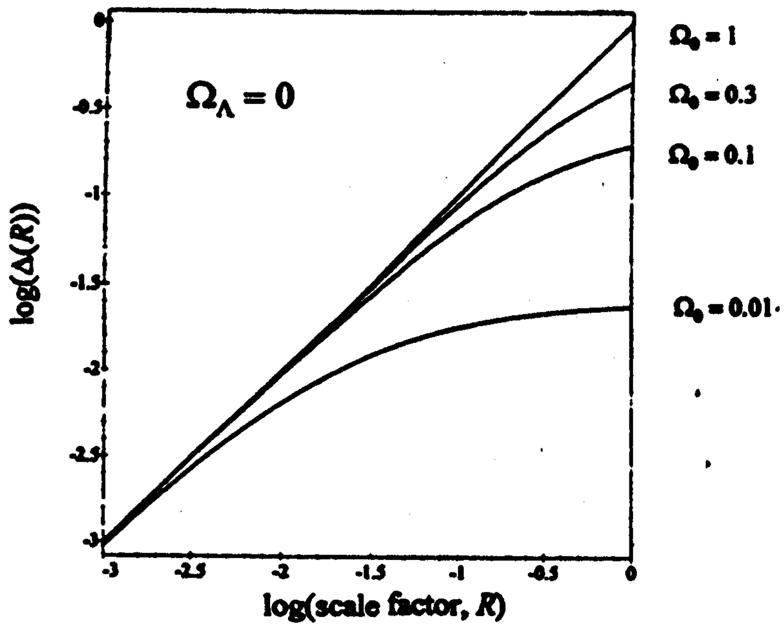



**Fig. 11.4.** The growth of density perturbations over the range of scale-factors $R = 10^{-3}$ to 1 for world models with $\Omega_\Lambda = 0$ and density parameters $\Omega_0 = 0.01$, 0.1, 0.3 and 1.

only modestly from $R = 10^{-1}$ to 1. In this case, which is of considerable astrophysical interest, the growth of the density contrast is only by a factor of 190 from the epoch of recombination to the present epoch. If $\Omega_0$ were 0.01, the growth of the density contrast would be even smaller, a factor of only 24.

Similar calculations can be carried out for the cases in which $\Omega_\Lambda \neq 0$. Those which are of the greatest interest are the flat models for which $(\Omega_0 + \Omega_\Lambda) = 1$. Fig. 11.5 shows the development of the fluctuations over the range of scale-factors from $R = 1/30$ to the present epoch $R = 1$, in all cases, the fluctuations having amplitude $\Delta = 10^{-3}$ at $R = 10^{-3}$. The growth of the density contrast is somewhat greater in the cases $\Omega_0 = 0.1$ and 0.3 as compared with the corresponding cases with $\Omega_\Lambda = 0$. For example, in the case $\Omega_0 = 0.1$, the growth of the fluctuation from $R = 1/1000$ to 1 is 610. Inspection of Fig. 11.5 shows that the fluctuations continue to grow to greater values of the scale-factor $R$, corresponding to smaller redshifts, as compared with the models with $\Omega_\Lambda = 0$. The redshift at which the growth rate of the instability slows down can be deduced from (11.46). If $\Omega_0 + \Omega_\Lambda = 1$, the expression for $\dot{R}$ becomes

$$\dot{R} = H_0 \left(\frac{\Omega_0}{R} + \Omega_\Lambda R^2\right)^{1/2} = H_0 \left[\frac{\Omega_0}{R} + (1 - \Omega_0)R^2\right]^{1/2} . \qquad (11.48)$$



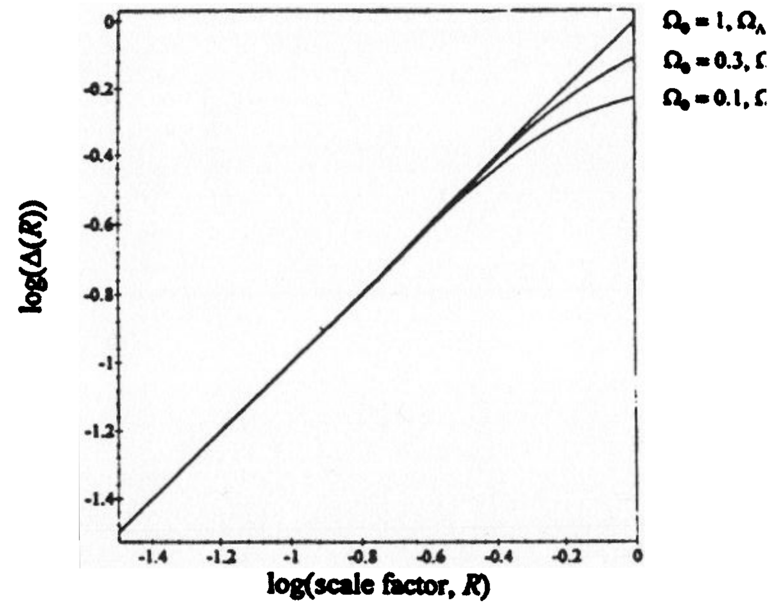



**Fig. 11.5.** The growth of density perturbations over the range of scale-fa 1/30 to 1 for world models with $\Omega_0 + \Omega_\Lambda = 0$ and density parameters 0.3 and 1.

Thus, when the first term in square brackets is greater than th the same dynamics as in the case of the critical model is found, corre to growth as $\Delta \propto R$. When the second term dominates, the universe accelerate under the influence of the vacuum fields, or the repulsive the $\Lambda$-term, and then growth is suppressed – recall that even if the e is unaccelerated, there is no growth of the perturbations. Thus, the i can grow to scale factors such that $\Omega_0/R = (1 - \Omega_0)R^2$, that is,

$$R \approx \left(\frac{\Omega_0}{1 - \Omega_0}\right)^{1/3} \quad \text{or} \quad (1 + z) \approx \Omega_0^{-1/3} \quad \text{if } \Omega_0 \ll 1 .$$

Notice that, expressing the development of the fluctuations in term factor rather than cosmic time disguises the fact that the time depe the models with and without $\Omega_\Lambda$ are significantly different. Fig. 1 the cosmic time-scale factor relation for three of the world models above. It can be seen that the effect of the cosmological constant is out the cosmic time-scale allowing more time for the density pertur grow.

We have considered only the cases of Friedman models with $\Omega_\Lambda$ flat cosmological models with $\Omega_0 + \Omega_\Lambda = 1$, but the space of co world models is much greater than these cases. Generally, the re

## Jeans' mass after recombination

After atoms recombine at $z \sim 1{,}000$ photons start behaving like free particles and their contribution to the pressure support of a perturbation ceases. The sound speed then drops dramatically to that of a gas with the same temperature as the radiation at recombination ($T_{gas} = T_{rad}$ until recombination because of the close coupling between atoms + photons in a fully ionized universe).

So

$$c_s = \left(\frac{5kT}{3m_p}\right)^{1/2} \quad \text{(adiabatic sound speed of gas at temperature } T\text{)}$$

and

$$M_J = \left(\frac{\pi \lambda_J^3}{6}\right)\rho \simeq 1.6\, 10^5 \,(\Omega_0 h^2)^{-1/2}\, M_\odot \quad \left(\begin{array}{l} T \approx 3{,}000\,K \\ z \approx 1{,}000 \end{array}\right)$$

Note: → this is the mass typical of globular clusters.

→ after recombination perturbations on all mass scales $\gtrsim 10^6\, M_\odot$ are free to grow again.

→ $T_{gas} \propto R^{-2}$

$T_{rad} \propto R^{-1}$

$$\left(\begin{array}{l} \text{from } PV^\gamma = \text{const} \\ \Rightarrow T \propto R^{-3(\gamma-1)} \end{array} ; \quad \begin{array}{l} \gamma = 4/3 \text{ for rad.} \\ \gamma = 5/3 \text{ for gas} \end{array}\right)$$

⇒ today $T_{gas} \ll T_{rad}$. This is one reason why we have always neglected the pressure of ordinary matter (i.e. "dust" approx.)

## Silk damping

Fluctuations that enter the horizon before $t_{eq}$ then oscillate until recombination, when presumably they would be able to start growing again. Unfortunately, oscillations in a real fluid are damped out by photons diffusing out of the oscillating wave.

This is a very important process which damps out all perturbations on scales smaller than

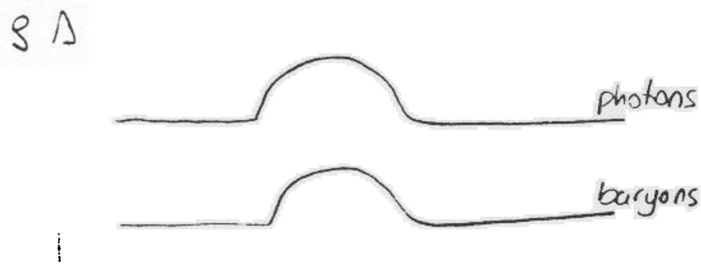
$$M_D \approx 2\, 10^{23} (\Omega_b h^2)^{-5/4} (1+z)^{-15/4} M_\odot \underset{\text{at } z \sim 1000}{\approx} 10^{12} (\Omega_b h^2)^{-5/4} M_\odot$$

$\boxed{\text{So, only \underline{adiabatic} baryonic perturbations on scales larger than galaxies survive until recombination — Galaxies in this picture must form via fragmentation of large scale structures that survive until } z < 1000}$

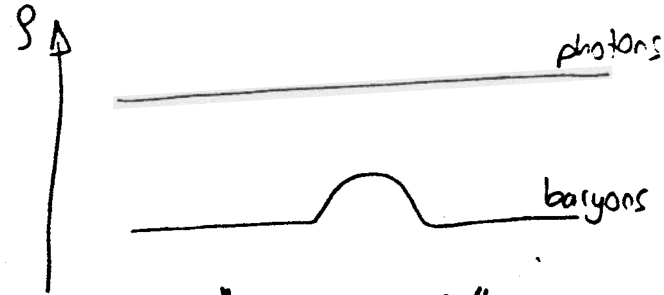
The formation of galaxies therefore occurs in a "top-down" fashion, after the formation of large "pancakes".

# ISOTHERMAL PERTURBATIONS






"adiabatic"




"isothermal"

A second "kind" of perturbations are those in which there are fluctuations in the baryon density which take place against a uniform cosmic background radiation field. Since the internal temperature of the gas is the same as that of the radiation background, which is not "coupled" to the perturbation, there is no difference in temperature inside or outside of the perturbation, hence the name "isothermal".

Let us go back to our original wave equation for the development of fluctuations

$$\frac{d^2\Delta}{dt^2} + 2\left(\frac{\dot{R}}{R}\right)\frac{d\Delta}{dt} = \Delta\left(4\pi G\rho - k^2 c_s^2\right)$$

Now the perturbation is driven by the baryon overdensity and there is no pressure support from the radiation, so in the eq. above

$$\rho = \rho_b$$
$$c_s^2 \approx 0$$

which implies

$$\frac{d^2\Delta}{dt^2} + 2\left(\frac{\dot{R}}{R}\right)\frac{d\Delta}{dt} = 4\pi G \rho_b \Delta$$

Introducing a parameter $y = \frac{\rho_b}{\rho_{rad}} = \frac{R}{R_{eq}}$ we find, for

the matter - dominated phase, when $R = \left(\frac{3}{2}H_0 t\right)^{2/3}$ that

and $\dot{R} \simeq \frac{H_0}{R_{eq}^{1/2}} \frac{(1+y)^{1/2}}{y}$

and substituting y for t $\frac{d^2\Delta}{dy^2} + \frac{2+3y}{2y(1+y)}\frac{d\Delta}{dy} - \frac{3\Delta}{2y(1+y)} = 0$

whose growing solution is

$$\Delta \propto 1 + \frac{3y}{2}$$

From $y=0$ to $y=1$ (the entire radiation dominated phase) the perturbation grows only by a factor of 2.5 in amplitude.

This suppression of growth during radiation domination is known as the "Meszaros effect".

Growth rates of matter perturbations are significantly retarded by the radiation drag force acting on the plasma.

- Isothermal perturbations scarcely grow at all until recombination
- There no Silk damping, since the background radiation is uniform. After recombination, now there are some perturbations left on all scales. Those more massive than $\sim 10^6 M_\odot$ (Jeans' mass after recombination) will grow and collapse to form bound structures.

Galaxies are formed in this scenario by the aggregation and merger small subunits $\Rightarrow$ a "bottom-up" scenario of galaxy formation. This is the progenitor of "hierarchical" theories of structure formation. In a different guise, they constitute the standard paradigm of structure formation today.

## The problems

- Most of the matter in the universe is non-baryonic, dark matter.

From Big Bang nucleosynthesis : $\Omega_b \simeq 0.0125\, h^{-2}$

From CNOC1 survey $\quad \Omega_0 \simeq 0.2 - 0.3$
+ baryon fraction in clusters.

- Inflation predicts $K = 0 \Rightarrow \Omega_0 = 1$ if $\Lambda = 0$

$\Rightarrow \Omega_0 + \Omega_\Lambda = 1$ if $\Lambda \neq 0$

- CMB fluctuations - Since fluctuations can only grow after recombination, and they stop growing at $z \approx \frac{1}{\Omega_0}$; this implies that fluctuations were quite large at recombination -

$$\frac{\delta T}{T} \approx 10^{-3}$$

Actual observed fluctuations are almost two orders of magnitude smaller than this -

We are forced then to consider a scenario where structure forms in a universe dominated by Dark Matter -

$\boxed{\text{Dark Matter and Galaxy Formation}}$

- Observational Evidence for Dark Matter -

  - Dark Matter in the galactic disk

    - the "Oort limit" => important, remember DM must be dissipative if it collects into a disk like structure -

  - Massive Halos around Galaxies

    - Self-graviting disks are bar unstable - Ostriker + Peebles suggested that disks are stabilized by a massive sphercial halo -

- Disk rotation curves

- Timing Argument

- Application of virial theorem to galaxy clusters

- Hydrostatic equilibrium of X-ray gas in clusters -

- Virgo "infall" -

- Peculiar Motions -

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All of these methods give a rough estimate of the total mass associated with a certain amount of light. This mass-to-light ratio, which increases as a function of scale, is combined with the luminosity density of the universe,

$$\mathcal{L} = \frac{L}{V} = \int L\,\phi(L)\,dL \approx 2\times10^{8}\,h\,\frac{L_\odot}{Mpc^3} \quad (\text{in } B)$$

The critical mass-to-light ratio is that needed to have critical mass density

$$\left(\frac{M}{L}\right)_{crit} \approx 1500\left(\frac{M}{L}\right)_\odot \quad , \text{i.e. } \rho_{crit} = \left(\frac{M}{L}\right)_{crit}\mathcal{L}$$

So $\Omega = \dfrac{M/L}{(M/L)_{crit}}$ The following table roughly gives the results of the methods outlined above.

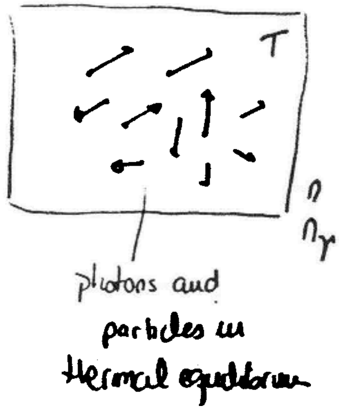
| Object/Method | Scale | $\Omega_0$ estimate |
|---|---|---|
| Visible regions of galaxies | $10 h^{-1}$ Kpc | 0.005 |
| Extended disk rotation curves | $30 h^{-1}$ Kpc | 0.02 |
| Binary galaxies | $50 h^{-1}$ Kpc | 0.02 - 0.07 |
| Galaxy groups | $500 h^{-1}$ Kpc | 0.15 |
| Rich clusters | $1 h^{-1}$ Mpc | 0.2 - 0.3 |
| Virgocentric infall | 15 Mpc | 0.1 - 0.4 |
| Large scale motions | $20-40 h^{-1}$ Mpc | 0.2 - 1.0 |

There is a very clear trend for $M/L$ to increase with scale, and there is some (weak) indication that on the largest scales $\Omega \to 1$, although the most solid evidence indicates $\Omega_0 \simeq 0.2 - 0.4 \lesssim 1$. This is much larger than the maximum baryon density allowed by Big Bang nucleosynthesis calculations, $\Omega_b \simeq 0.0125 h^{-2}$, so Universe must be filled with non-baryonic dark matter.

The most common candidates are "thermal relics from the Big Bang.

As it turns out, it is straightforward to compute the abundance of these relics as a function of the mass of the particle candidate

Take, for example, a box full of particles and photons




Interactions are, in general, short scale, "collisions", and therefore, the interaction rate is given by

$$\Gamma = \underbrace{n}_{\text{number density}} \langle \underbrace{\sigma}_{\text{interaction cross section}} \underbrace{v}_{\text{typical relative velocity.}} \rangle$$

On time scales much longer than $\Gamma$ the box will be in thermal equilibrium, and the distribution function will be

$$f(\vec{p})\, d^3\vec{p} = \frac{g}{(2\pi)^3} \frac{1}{e^{E/kT} \pm 1} d^3\vec{p}$$

$+ \Rightarrow$ fermions

$- \Rightarrow$ bosons

This is a "Bose" or "Fermi" ideal gas, and, of course, is a more general version of the Planck function.

Since for photons $|\vec{P}| = \frac{1}{3}\rho c^2 \Rightarrow d^3\vec{p} \propto p^2 dp \propto E^2 dE \propto \nu^2 d\nu$

$(E = h\nu)$

Now $dE \propto E f(\vec{p})\, d^3\vec{p} \propto \dfrac{\nu\, \nu^2\, d\nu}{e^{h\nu/kT} - 1} \propto \dfrac{\nu^3 d\nu}{e^{h\nu/kT} - 1}$ $\boxed{\text{PLANCK.}}$

photons are bosons

Consider now the temperature in the box to be much higher than the rest mass of the particle $\boxed{kT \gg mc^2}$, i.e., particles are "relativistic".

In this case, photons in the box can create particles as they "collide", and the number density of particles and photons will be roughly comparable. $(n \approx n_\gamma)$. This is because annihilations must balance pair creation.

In this case, using $E^2 = |\vec{p}|^2 + m^2$, $(c=1)$

$$n = \int f(\vec{p})\, d^3\vec{p} = \frac{g}{2\pi^2} \int \frac{(E^2 - m^2)^{1/2}\, E\, dE}{e^{E/kT} \pm 1} \simeq$$

$$\simeq \frac{g}{2\pi^2} \int \frac{E^2\, dE}{e^{E/kT} \pm 1} \propto T^3$$

This is the same dependence on T as the number density of photons,

energy density in photons $= h\nu\, n_\gamma \propto T^4 \propto n_\gamma T \Rightarrow \boxed{n_\gamma \propto T^3}$

Stefan-Boltzmann

$\nu \propto T$ (Wien's displacement law)

So $n \simeq n_\gamma$ as long as particle remains relativistic.

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At lower temperatures, $kT \ll mc^2$, photons cannot create particle pairs and annihilations lead to a sharp reduction in the number density of particles.

Since in this case $e^{E/kT} \pm 1 \simeq e^{E/kT}$

and $E = \sqrt{p^2 + m^2} = m\sqrt{1+(p/m)^2} \simeq m + \frac{p^2}{2m}$

and then

$$n = \int f(\vec{p})\, d^3p \simeq \frac{g}{2\pi^2} \int p^2 dp\, e^{-m/kT}\, e^{-\frac{p^2}{2mkT}} =$$

$$= g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/kT} \qquad \left(\text{Boltzmann's distribution}\right)$$

So as T drops the number density of particles gets exponentially suppressed.

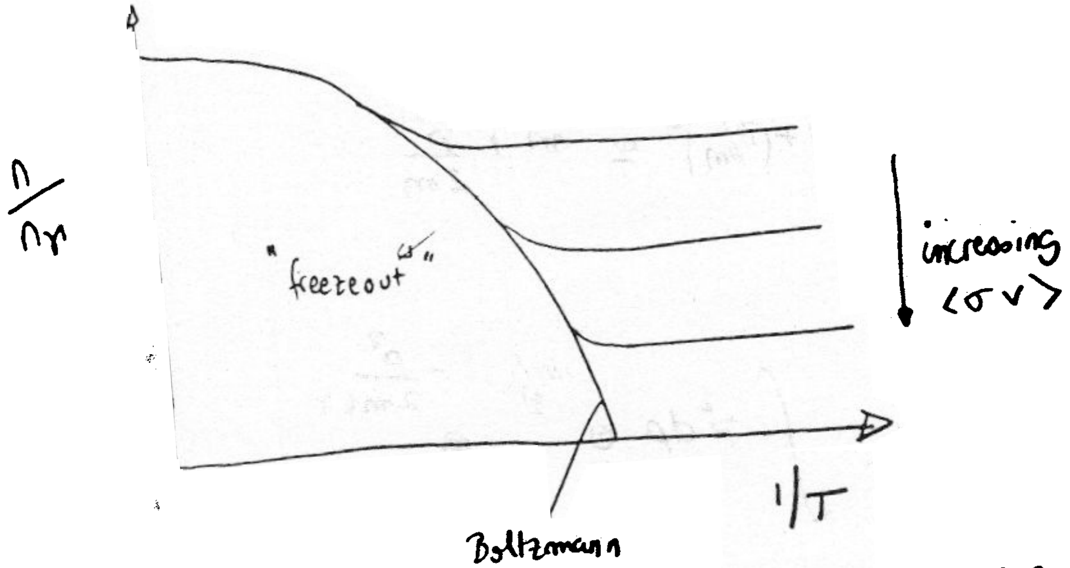
This is all true and well while thermal equilibrium exists between particles and photons, i.e., while the interaction timescale is shorter than the expansion timescale, $\Gamma \ll H(t)$.

But $\Gamma \propto n \propto R^{-3}$

$H(t) \propto R^{-2}$ so eventually they should cross at "decoupling", i.e. at $t_d$, where $\Gamma = H(t_d)$)

radiation dominated era

Number densities of particles are "frozen" at the time of decoupling and they stop following the exponential decrease mandated by Boltzmann's law. (This is the analog of "opening the box" in page II.23)




Note from here that interactions must be weak, otherwise $n$ would be quite large.

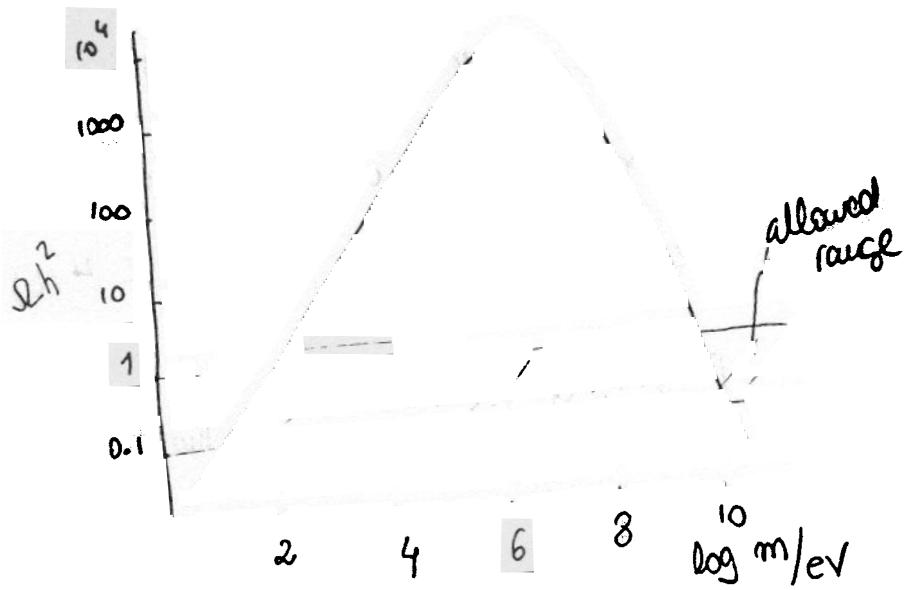
( Using these two concepts; number densities derived from thermal equilibrium and freeze-out, one can compute the contribution from a given particle (with some assumed interaction law) )

For "reasonable" (weak) interaction laws

Thus if one adopt as reasonable limits that $\Omega h^2 < 1$ then thermal relics must have

$m \lesssim 100\ eV$

or $m \gtrsim 1\ GeV$



So either particles are low mass and "hot" (relativistic at decoupling and with large thermal velocities still today) or massive and "cold" (non-relativistic at decoupling and low thermal velocities).

These are the basis of the "hot" and "cold" dark matter scenarios of structure formation.

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## Instabilities in a Universe with collisionless non-baryonic matter

- FREE STREAMING:

If particles "decouple" and "freeze out" when they are relativistic, they will continue to travel on "straight lines" at the speed of light. This will erase all perturbations on scales smaller than the particle horizon until the particles' momentum decays enough to become non relativistic. Or, in other words, until their kinetic energy becomes low enough for them to be "captured" by the binding energy of the perturbation.

In summary, perturbations on mass scales smaller than the horizon at $z_{nr}$, the time at which particles become non relativistic, are erased. For a 10 eV neutrino this is

$$z_{nr} \simeq z_{eq} \approx 2\times10^4 \Rightarrow M_{FS} \sim 2\,10^{15} M_\odot$$

The Jeans' criterion derived for collisional gas remains unchanged provided that the sound speed of the gas is replaced by a suitable characteristic speed of the collisionless component

$$v_*^{-2} = \frac{\int v^{-2} f(v) d^3v}{\int f(v) d^3v} = \text{rms velocity dispersion if } f(v) \text{ Maxwellian}$$

- Dark Matter - baryon - Radiation coupled evolution -

— Ordinary matter + radiation are coupled to DM only through gravity

Let us recall the eq. describing the evolution of perturbations under the simplifying assumption that the internal pressure of the fluctuation can be neglected (i.e. $\lambda \gg \lambda_J$) -

$$\frac{d^2\Delta}{dt^2} + 2\left(\frac{\dot{R}}{R}\right)\frac{d\Delta}{dt} = A \rho \Delta$$

where $A = 4\pi G$ — non-relativistic limit, valid during matter domination after baryons and photons decouple -

$A = \frac{32\pi G}{3}$ — relativistic limit, valid on scales larger than the horizon during the radiation dominated era -

During matter-domination $\rho \approx \rho_{DM} + \rho_b$, and then

$$\frac{d^2 \Delta_b}{dt^2} + 2\left(\frac{\dot{R}}{R}\right)\frac{d}{dt}\Delta_b = A\left(\rho_b \Delta_b + \rho_{DM}\Delta_{DM}\right)$$

$$\frac{d^2 \Delta_{DM}}{dt^2} + 2\left(\frac{\dot{R}}{R}\right)\frac{d}{dt}\Delta_{DM} = A\left(\rho_b \Delta_b + \rho_{DM}\Delta_{DM}\right)$$

If $\Omega = 1$ and $\rho_{DM} \gg \rho_b$

$$\Rightarrow \quad \Delta_{DM} = \underset{\text{const.}}{B}\, R$$

and $\ddot{\Delta}_b + 2\left(\frac{\dot{R}}{R}\right)\dot{\Delta}_b = 4\pi G \rho_{DM} B R$

Using $R = \left(\frac{3H_0 t}{2}\right)^{2/3}$ and $3H_0^2 = 8\pi G \rho_{DM}$ we have

$$R^{3/2}\frac{d}{dR}\left(R^{-1/2}\frac{d\Delta_b}{dR}\right) + 2\frac{d\Delta_b}{dR} = \frac{3}{2}B$$

with solution $\Delta_b = B(R - R_0)$, with the interesting property that, at $R = R_0$ the baryon perturbations are negligible. This is important.

Assume that at some $z = z_0$ the amplitude of baryon fluctuations is very small. Since $\Delta_b \approx \Delta_{DM} \approx BR$ for $R \gg R_0$, we have that

$$\Delta_b = \Delta_{DM}\left(1 - \frac{z}{z_0}\right)$$

so baryon perturbations quickly "catch up" with those in the dark matter component.

Is this really a relevant calculation? Can one think of cases where the fluctuations in dark matter are very different from those in the baryons?

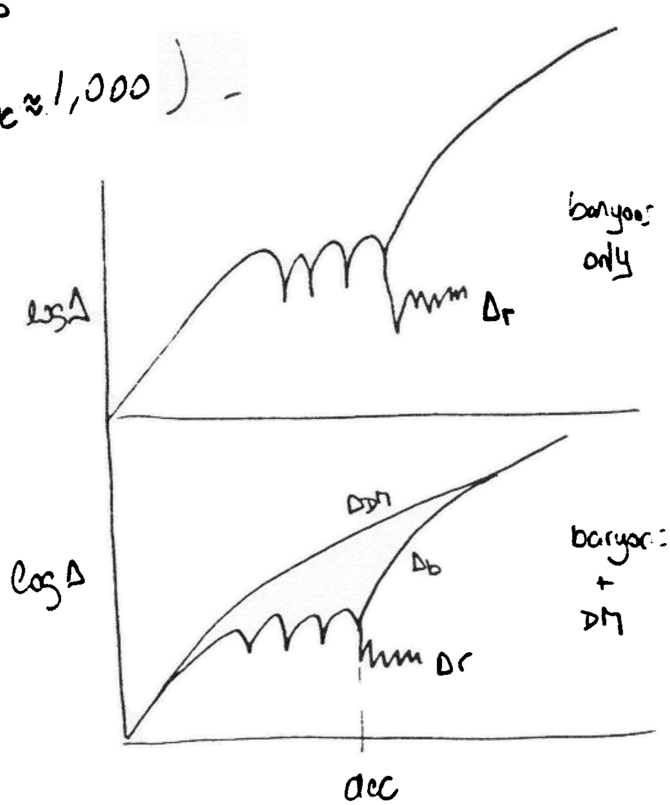
Yes. Imagine an adiabatic perturbation that enters the horizon during radiation domination. At that time,

$$\frac{\Delta \rho_b}{\rho_b} \approx \frac{\Delta \rho_{DM}}{\rho_{DM}} \qquad z > z_{eq}$$

Jeans' length is of the same order as the horizon, so the radiation-dominated plasma can pressure support the perturbation. $\Delta\rho_b/\rho_b$ remains more or less constant until the Jeans mass drops precipitously at recombination. But $\frac{\Delta \rho_{DM}}{\rho_{DM}}$ can grow between $z_{eq}$ and $z_{dec}$ and their amplitudes can exceed $\frac{\Delta\rho_b}{\rho_b}$ at $z_{dec}$ by a factor of 10 or 20

($\delta_{DM} \propto R$ and $z_{eq} \approx 20{,}000$; $z_{dec} \approx 1{,}000$).

This has an important implication. CMB fluctuations measure the perturbations in the baryon-photon fluid. Perturbations in the DM component may be much greater.



Of course, large differences in $\delta_b$ and $\delta_{DM}$ at recombination are seen on scales smaller than the horizon at $z_{eq}$. Fluctuations entering the horizon at recombination have $\delta_b \approx \delta_{DM}$.

The important news here is that the introduction of non-baryonic matter has allowed us to reconcile the small fluctuations in the CMB with the highly structured universe we see today.

We have also discussed the evolution of perturbations on different mass scales depending on whether the dark matter is "hot" or "cold".

We need now consider what are the original fluctuations as a function of mass in each of these models. This is the question we turn our attention now to.

## Spectrum of Initial Density Fluctuations

One simple description of the distribution of galaxies on large scales is the two-point correlation function, defined as $\xi(r)$,

$$dN(r) = N_0 \left[ 1 + \xi(r) \right] dV$$


which describes the number of galaxies in the volume $dV$ at a distance $r$ from another galaxy. $\xi(r)$ describes the excess # of galaxies at distance $r$ from another one.

One can also write $\xi(r)$ in terms of the probability of finding pairs of galaxies separated by distance $r$

$$dN_{pair} = N_0^2 \left[1+\xi(r)\right] dV_1\, dV_2$$

Using the "density contrast" of galaxies $\Delta(\vec{x}) = \delta\rho/\rho$ ; $\rho = \rho_0\left[1+\Delta(x)\right]$

we have

$$dN_{pair} = \rho(\vec{x})\, dV_1\, \rho(\vec{x}+\vec{r})\, dV_2 = \rho_0^2\left[1+\Delta(\vec{x})\right]\left[1+\Delta(\vec{x}+\vec{r})\right] dV_1\, dV_2$$

Averaging over a large volume and noting that $\langle\Delta\rangle = 0$ by definition,

$$dN_{pair} = \rho_0^2\left[1+\langle\Delta(\vec{x})\Delta(\vec{x}+\vec{r})\rangle\right] dV_1\, dV_2$$

which implies that

$$\xi(r) = \langle\Delta(\vec{x})\,\Delta(\vec{x}+\vec{r})\rangle$$

This function can actually be measured from observations, and to very good approximation it shows that

$$\xi(r) = \left(\frac{r}{r_0}\right)^{-\gamma}$$

is a power law. This is valid for

$100\, h^{-1}\, Kpc < r < 10\, h^{-1}\, Mpc$

$\gamma \simeq 1.8 \qquad r_0 \approx 5 h^{-1}\, Mpc$

A few points to note here:

① deviations from a power law are small, which means that the process of clustering galaxies has no obvious scale. This suggests gravity as the main culprit –

(2) Fluctuations must be present on a very wide range of scales.
(have been)

(3) The characteristic scale $r_0 \simeq 5h^{-1}Mpc$ indicates roughly on what scales perturbations are non linear; i.e. $\xi > 1$ for $r < r_0$.

Let us now introduce the mathematical machinery (and jargon!) needed to compare these statistical measures of the observed distribution with theoretical models of inhomogeneous patterns in the universe driven by gravitational instability.