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The CP^{N-1} Model with Quarks: Effective Action, $1/N$ Expansion and Chiral Symmetry.

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Summary. — The effective action for the 2-dimensional CP^{N-1} model with quarks is explicitly calculated in the low-energy limit, its $1/N$ expansion and properties under chiral symmetry are discussed.

1. – Introduction.

Among the two-dimensional models one of the most interesting is the CP^{N-1} model (1). Its interest resides mainly in the fact that it has many important properties (2) in common with quantum chromodynamics (QCD), as for instance

1) conformal invariance,
2) nontrivial topology and instanton solutions for any N,
3) chiral invariance.

The quantum structure of this model has been studied (3) by using the powerful technique of the $1/N$ expansion and the following important properties have been found:

(1) H. EICHENHERR: *Nucl. Phys. B*, **146**, 215 (1978); E. CREMMER and J. SCHERK: *Phys. Lett. B*, **74**, 341 (1978); A. GOLO and A. PERELOMOV: ITEP preprint (1978).
(2) For a discussion of the analogy of the CP^{N-1} model with QCD see P. DI VECCHIA: in *Field Theory and Strong Interactions*, edited by P. URBAN (Wien, 1980).
(3) A. D'ADDA, P. DI VECCHIA and M. LÜSCHER: *Nucl. Phys. B*, **146**, 63 (1978); **152**, 125 (1979); E. WITTEN: *Nucl. Phys. B*, **149**, 285 (1979).

1) asymptotic freedom and dimensional transmutation,
2) states with nonzero triality are confined if « quarks » are massive,
3) no U_1 problem.

The study [3] of the chiral properties of the CP^{N-1} model has been very useful for solving [4] the U_1 problem in large-N QCD. In particular the low-energy dynamics of the pseudoscalar mesons can be summarized by an effective Lagrangian [5] that satisfies explicitly the anomalous and nonanomalous chiral Ward identities. Unlike in QCD, in the CP^{N-1} model one can explicitly perform the $1/N$ expansion and construct the « hadrons » in terms of the constituent quarks and gluons. In this paper, using the large-N expansion and a method due to Schwinger for computing determinants at low energy, we compute explicitly the low-energy dynamics of the lowest « hadrons » from the Lagrangian of the CP^{N-1} model involving « quarks » and « gluons ». The effective Lagrangian for the lowest « hadrons » that we construct is in complete agreement with the one that has been derived in large-N QCD, provided that one makes suitable identifications.

The paper is organized as follows.

In sect. **2** we review how one performs the functional integral over the « quark » and « gluon » fields getting a Lagrangian only in terms of the « hadrons ».

In sect. **3** we compute explicitly the determinant at low energy that comes out from the integration over the bosonic (« gluon ») fields. Section **4** is devoted to the calculation of the determinant obtained from the integration over the fermion fields (« quarks »). Finally in sect. **5** we discuss the $1/N$ expansion of the previous determinants and we write down explicitly an effective Lagrangian at low energy and for large N for the lowest « hadron » states.

2. – Notations. The effective action.

We consider the two-dimensional Euclidean CP^{N-1} model with quarks, following the notations of ref. [2,3]. The model has a set of complex scalar fields $z_\alpha(x)$, $\alpha = 1, \ldots, n$, and quark fields $\psi^a_\alpha(x)$, where $a = 1, \ldots, L$ is the flavour index and $\alpha = 1, \ldots, n_F$ is the fermionic colour index. As in ref. [2], we do not assume at the start any relation between n, n_F and L, and we do not put any constraint on the field ψ.

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(4) E. WITTEN: *Nucl. Phys. B*, **156**, 269 (1979); G. VENEZIANO: *Nucl. Phys. B*, **159**, 213 (1979); P. DI VECCHIA: *Phys. Lett. B*, **35**, 357 (1979). See also R. ARNOWITT and P. NATH: NUB 2417 (1979).

(5) C. ROSENZWEIG, J. SCHECHTER and C. G. TRAHERN: *Phys. Rev. D*, **21**, 3388 (1980); P. DI VECCHIA and G. VENEZIANO: *Nucl. Phys. B*, **171**, 253 (1980); E. WITTIN: Harvard University preprint HUTP-80/A005 (1980).

The total action then reads

$$S = \int \mathrm{d}^2x \left\{ \overline{D_\mu z} \cdot D_\mu z + \bar{\psi}(\tilde{D} - M_{\mathrm{B}})\psi + \frac{ef}{2n}(\bar{\psi}\gamma_\mu\psi)^2 - \right. \qquad (2.1)$$

$$\left. - \frac{g}{2n_{\mathrm{F}}}[(\bar{\psi}\tau^i\psi)^2 + (\bar{\psi}\tau^i\gamma^5\psi)^2] \right\}$$

with the constraint $|z|^2 = n/2f$.

In (2.1) covariant derivatives D_μ act in a different way on z and ψ fields:

$$\begin{cases} D_\mu z_\alpha = \partial_\mu z_\alpha - \dfrac{f}{n}(\bar{z}\cdot \overleftrightarrow{\partial}_\mu z) z_\alpha \,, \\[2ex] D_\mu \psi^a_\alpha = \partial_\mu \psi^a_\alpha - \dfrac{ef}{n}(\bar{z}\cdot \overleftrightarrow{\partial}_\mu z)\psi^a_\alpha \,. \end{cases} \qquad (2.2)$$

M_{B} is the (bare) quark mass wich is supposed to be independent of the colour index and diagonal over flavour indices.

τ^i, $i = 0, 1, \ldots, L^2 - 1$, form a complete set of Hermitian flavour matrices normalized such that

$$\tau_0 = \frac{1}{\sqrt{L}}\mathbf{1} \,, \qquad \mathrm{tr}(\tau^i\tau^j) = \delta_{ij} \,.$$

The action is invariant under U_1 gauge transformations:

$$z_\alpha(x) \to \exp[i\Lambda(x)]\, z_\alpha(x) \,, \qquad \psi^a_\alpha(x) \to \exp[ie\Lambda(x)]\,\psi^a_\alpha(x) \,. \qquad (2.3)$$

We are interested in the generating functional for the Euclidean Green's functions, which is formally

$$Z(j, \bar{j}, \eta, \bar{\eta}) = \int \mathscr{D}z\,\mathscr{D}\bar{z}\,\mathscr{D}\psi\,\mathscr{D}\bar{\psi} \prod_x \left[\delta\left(|z(x)|^2 - \frac{n}{2f}\right)\right] \cdot \qquad (2.4)$$

$$\cdot \exp\left[-S + \int \mathrm{d}^2x\,[\bar{j}\cdot z + \bar{z}\cdot j + \bar{\eta}\cdot\psi + \bar{\psi}\cdot\eta]\right] .$$

It is easy to make the action quadratic in the fields z and ψ, by the introduction of suitable auxiliary fields α, λ_μ, φ^i and φ^i_5, and then to perform the (Gaussian) integration over the fields ψ and z (see, for instance, ref. [3]). The result is

$$Z(j, \bar{j}, \eta, \bar{\eta}) = \int \mathscr{D}\alpha\,\mathscr{D}\lambda_\mu\,\mathscr{D}\varphi\,\mathscr{D}\varphi_5 \exp\left[-S_{\mathrm{eff}} + \int \mathrm{d}^2x\,[\bar{\eta}\cdot\Delta_{\mathrm{F}}^{-1}\eta + \bar{j}\cdot\Delta_{\mathrm{B}}^{-1}j]\right] , \qquad (2.5)$$

where

$$\begin{cases} \Delta_{\mathrm{B}} = -\Delta_\mu\Delta_\mu + m^2 - \dfrac{i}{\sqrt{n}}\alpha \,, & D_\mu = \partial_\mu + \dfrac{i}{\sqrt{n}}\lambda_\mu \,, \\[2ex] \Delta_{\mathrm{F}} = \tilde{D} - M_{\mathrm{B}} - \dfrac{1}{\sqrt{n_{\mathrm{F}}}}(\varphi^i + \varphi^i_5\gamma_5)\tau^i \,, & \tilde{D} = \gamma_\mu\left(\partial_\mu + \dfrac{ie}{\sqrt{n}}\lambda_\mu\right) \end{cases} \qquad (2.6)$$

and the effective action S_{eff} is given by

$$S_{\text{eff}} = n \operatorname{tr}\log \Delta_{\text{B}} - n_{\text{F}} \operatorname{tr}\log \Delta_{\text{F}} + \int d^2x \left[\frac{i\sqrt{n}\alpha}{2f} + \frac{1}{2g}(\varphi^i\varphi^i + \varphi_5^i\varphi_5^i)\right]. \tag{2.7}$$

Notice that, by performing the functional integration over the z and ψ fields, we have eliminated the « quark » and « gluon » fields obtaining an effective Lagrangian that is a function of the « hadron » fields, as, for instance,

$$\varphi^i = \frac{g}{\sqrt{n_{\text{F}}}}\bar{\psi}\tau^i\psi\,, \qquad \varphi_5^i = \frac{g}{\sqrt{n_{\text{F}}}}\bar{\psi}\gamma_5\tau^i\psi\,. \tag{2.8}$$

3. – The bosonic part of S_{eff}.

We have to evaluate the following expression:

$$S_{\text{eff}}^{\text{B}} = n \operatorname{tr}\log \Delta_{\text{B}} + \int d^2x \frac{i\sqrt{n}}{2f}\alpha \tag{3.1}$$

with Δ_{B} defined in eq. (2.6).

tr log Δ_{B} is, of course, ill defined and divergent, and it must be regularized, as we will show later on.

We are interested in the low-energy dynamics, where we can treat the fields α and $F = \varepsilon_{\mu\nu}\partial_\mu\lambda_\nu$ as constant. In this case we can use the trick developed long ago by SCHWINGER in electrodynamics [6].

We can rewrite tr log Δ_{B} as follows:

$$\operatorname{tr}\log \Delta_{\text{B}} = -\int_0^\infty [s^{-1} \operatorname{tr}\exp[-s\Delta_{\text{B}}]]\, ds + \text{const}\,. \tag{3.2}$$

Of course, the right-hand side of eq. (3.2) is still formal, because the trace diverges for constant fields and, furthermore, the integrand has a pole in $s = 0$. We shall see that in our approximation the divergence of the trace can be factorized out as a multiplicative constant, proportional to space-time volume. On the other hand, the integral in (3.2) can be regularized by introducing a cut-off ε at the lower limit of integration.

We need also the following formula, valid for slowly varying fields [6]:

$$\operatorname{tr}\exp[sD^2] = \frac{1}{4\pi s}\int d^2x \frac{Bs}{\sinh(Bs)}\,, \tag{3.3}$$

where

$$D^2 = D_\mu D_\mu\,, \qquad D_\mu = \partial_\mu + iA_\mu\,, \qquad B = \varepsilon_{\mu\nu}\partial_\mu A_\nu\,.$$

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[6] J. SCHWINGER: *Particles Sources and Fields*, Vol. II (Reading, Mass., 1973), p. 123.

Using eqs. (3.2) and (3.3), we obtain

$$(3.4)\qquad n\,\mathrm{tr}\log\Delta_{\mathrm B}=-\int \mathrm{d}^2x\,\frac{\sqrt{n}F}{4\pi}\int_0^\infty \mathrm{d}s\left\{\exp\left[-s\left(m^2-\frac{i\alpha}{\sqrt{n}}\right)\right]\left(s\sinh\frac{Fs}{\sqrt{n}}\right)^{-1}\right\},$$

that can be rewritten as

$$(3.5)\qquad n\,\mathrm{tr}\log\Delta_{\mathrm B}=-\int \mathrm{d}^2x\,\frac{\sqrt{n}\,|F|}{4\pi}\int_0^\infty \mathrm{d}t\left\{t^{-1}\exp\left[-\frac{\sqrt{n}}{|F|}t\left(m^2-\frac{i\alpha}{\sqrt{n}}\right)\right]\cdot\right.$$
$$\left.\cdot\left(\frac{1}{\sinh t}-\frac{1}{t}\right)\right\}-\int \mathrm{d}^2x\,\frac{n}{4\pi}\int_0^\infty\frac{\mathrm{d}s}{s^2}\left\{\exp\left[-s\left(m^2-\frac{i\alpha}{\sqrt{n}}\right)\right]-\right.$$
$$\left.-\exp[-s]\left[1-s\left(m^2-\frac{i\alpha}{\sqrt{n}}-1\right)\right]\right\}-$$
$$-\int \mathrm{d}^2x\,\frac{n}{4\pi}\int_0^\infty\frac{\mathrm{d}s}{s^2}\exp[-s]\left[1-s\left(m^2-\frac{i\alpha}{\sqrt{n}}-1\right)\right].$$

If we disregard the infinite multiplicative factor $\int \mathrm{d}^2x$, the first and second integral in (3.5) are convergent, whereas the third one diverges. However, the divergence of $n\,\mathrm{tr}\log\Delta_{\mathrm B}$ will be exactly cancelled in $S^{\mathrm B}_{\mathrm{eff}}$ by the term $\int \mathrm{d}^2x\,(i\sqrt{n}\alpha/2f)$ if we assume that the coupling constant f has the following dependence on ε:

$$(3.6)\qquad \frac{1}{2f}-\frac{1}{4\pi}\int_\varepsilon^\infty\frac{\mathrm{d}s}{s}\exp[-s]=c\,.$$

Furthermore, the constant c in (3.6) has to be fixed equal to $(\log m^2)/4\pi$, because we want a vanishing expectation value for the field α.

If we apply in the second integral of (3.5) the following formula:

$$(3.7)\qquad \int_0^\infty\frac{\mathrm{d}t}{t^2}\{\exp[-at]-\exp[-t][1-(a-1)t]\}=a\log\frac{a}{e}+1\,,$$

we obtain

$$(3.8)\qquad S^{\mathrm B}_{\mathrm{eff}}=-\int \mathrm{d}^2x\left\{\frac{n}{4\pi}\left(m^2-\frac{i\alpha}{\sqrt{n}}\right)\log\frac{m^2-i\alpha/\sqrt{n}}{em^2}+\right.$$
$$\left.+\frac{\sqrt{n}\,|F|}{4\pi}\int_0^\infty\frac{\mathrm{d}t}{t}\exp\left[-\frac{\sqrt{n}\,t}{|F|}\left(m^2-\frac{i\alpha}{\sqrt{n}}\right)\right]\left(\frac{1}{\sinh t}-\frac{1}{t}\right)\right\}.$$

The integral in (3.8) can be explicitly computed by applying the identity

$$\int_0^\infty \frac{\mathrm{d}t}{t}\exp[-at]\left(\frac{1}{\sinh t}-\frac{1}{t}\right)=2\left[\int_0^\infty \frac{\mathrm{d}t}{t}\exp[-at]\left(\frac{1}{\exp[t]-1}-\frac{1}{t}+\frac{1}{2}\right)-\right.$$
$$\left.-\int_0^\infty \frac{\mathrm{d}t}{t}\exp[-at/2]\left(\frac{1}{\exp[t]-1}-\frac{1}{t}+\frac{1}{2}\right)\right] \tag{3.9}$$

and the well-known formula

$$\log\Gamma(a)=\left(a-\frac{1}{2}\right)\log a-a+\frac{1}{2}\log 2\pi+$$
$$+\int_0^\infty \frac{\mathrm{d}t}{t}\exp[-at]\left(\frac{1}{\exp[t]-1}-\frac{1}{t}+\frac{1}{2}\right). \tag{3.10}$$

We obtain

$$S^{\mathrm{B}}_{\mathrm{eff}}=-\int \mathrm{d}^2x\left\{\frac{n}{4\pi}\left(m^2-\frac{i\alpha}{\sqrt{n}}\right)\log\frac{|F|}{2m^2\sqrt{n}}+\frac{\sqrt{n}\,|F|}{4\pi}\log 2+\right.$$
$$\left.+\frac{\sqrt{n}\,|F|}{2\pi}\log\frac{\Gamma(\sqrt{n}m^2/|F|-i\alpha/|F|)}{\Gamma(\sqrt{n}m^2/2|F|-i\alpha/2|F|)}\right\}. \tag{3.11}$$

4. – The fermionic part of S_{eff}.

We have to evaluate the following expression:

$$S^{\mathrm{F}}_{\mathrm{eff}}=-n_{\mathrm{F}}\,\mathrm{tr}\log\Delta_{\mathrm{F}}+\frac{1}{2g}\int\mathrm{d}^2x\,(\varphi^i\varphi^i+\varphi^i_5\varphi^i_5) \tag{4.1}$$

with Δ_{F} given by eq. (2.6). We can conveniently use matrix notation and define

$$A=\frac{1}{\sqrt{n_{\mathrm{F}}}}(\varphi^i+i\varphi^i_5)\tau^i\,,\qquad A^+=\frac{1}{\sqrt{n_{\mathrm{F}}}}(\varphi^i-i\varphi^i_5)\tau^i\,,\qquad B=M_{\mathrm{B}}+A\,. \tag{4.2}$$

Then

$$\begin{cases}\Delta_{\mathrm{F}}=\tilde{D}-M_{\mathrm{B}}-(\gamma_-A+\gamma_+A^+)=\tilde{D}-(\gamma_-B+\gamma_+B^+)\,,\\ \varphi^i\varphi^i+\varphi^i_5\varphi^i_5=n_{\mathrm{F}}\,\mathrm{tr}\,(AA^+)\end{cases} \tag{4.3}$$

with $\gamma_\pm=\frac{1}{2}(1\pm i\gamma_5)$. Let us also define $\Delta^*_{\mathrm{F}}=-\tilde{D}-(\gamma_-B^++\gamma_+B^-)$ and $K=(e/\sqrt{n})F$. Then we observe that

$$\Delta_{\mathrm{F}}\Delta^*_{\mathrm{F}}=\gamma_-(-D^2+K+BB^+)+\gamma_+(-D^2-K+B^+B)\,, \tag{4.4}$$

since the fields F and B are constant.

To evaluate $\operatorname{tr}\log \Delta_F$ we consider its first variation

$$\delta \operatorname{tr}\log \Delta_F = \operatorname{tr}(\Delta_F^{-1}\delta\Delta_F) = \tfrac{1}{2}\operatorname{tr}[\delta\Delta_F\Delta_F^*(\Delta_F\Delta_F^*)^{-1} + \Delta_F^*\delta\Delta_F(\Delta_F^*\Delta_F)^{-1}]\,. \tag{4.5}$$

We use now the known properties of γ-matrices to obtain

$$\begin{aligned}\delta \operatorname{tr}\log \Delta_F = \tfrac{1}{2}\operatorname{tr}\{&\delta(-\tilde{D}\tilde{D})[\gamma_-(-D^2+K+BB^+)^{-1}+\\&+\gamma_+(-D^2-K+BB^+)^{-1}]+\gamma_-[\delta BB^+(-D^2+K+BB^+)^{-1}+\\&+B^+\delta B(-D^2+K+B^+B)^{-1}]+\gamma_+[\delta B^+B(-D^2-K+B^+B)^{-1}+\\&+B\,\delta B^+(-D^2-K+BB^+)^{-1}]\}\end{aligned} \tag{4.6}$$

and, therefore, recalling that $\tilde{D}\tilde{D} = D^2 + i\gamma_5 K$ and computing the trace of γ-matrices, we find

$$\begin{aligned}\delta \operatorname{tr}\log \Delta_F = \tfrac{1}{2}\operatorname{tr}\{&\delta(-D^2+K+BB^+)(-D^2+K+BB^+)^{-1}+\\&+\delta(-D^2-K+B^+B)(-D^2-K+B^+B)^{-1}+\\&+B^+\delta B[(-D^2+K+B^+B)^{-1}-(-D^2-K+B^+B)^{-1}]+\\&+B\,\delta B^+[(-D^2-K+BB^+)^{-1}-(-D^2+K+BB^+)^{-1}]\}\,.\end{aligned} \tag{4.7}$$

Now the first two terms in eq. (4.7) can be immediately integrated; the other terms can be rewritten as follows:

$$\begin{aligned}&\tfrac{1}{2}\operatorname{tr}\{B^+\delta B[(-D^2+K+B^+B)^{-1}-(-D^2-K+B^+B)^{-1}]+\\&\quad+B\,\delta B^+[(-D^2-K+BB^+)^{-1}-(-D^2+K+BB^+)^{-1}]\}=\\&\quad=\tfrac{1}{2}\operatorname{tr}\int_0^\infty \mathrm{d}s\{B^+\delta B[\exp[-s(-D^2+K+B^+B)]-\\&\quad-\exp[-s(-D^2-K+B^+B)]]+B\,\delta B^+\cdot\\&\quad\cdot[\exp[-s(-D^2-K+BB^+)]-\exp[-s(-D^2+K+BB^+)]]\}=\\&=\int_0^\infty \mathrm{d}s\,\{\sinh(sK)\operatorname{tr}(\exp[sD^2])\operatorname{tr}[B\,\delta B^+\exp[-sBB^+]-B^+\delta B\exp[-sB^+B]]\}.\end{aligned} \tag{4.8}$$

The last expression can be computed by formula (3.3) and we obtain

$$\begin{aligned}&\int_0^\infty \mathrm{d}s\left\{\sinh(sK)\operatorname{tr}(\exp[sD^2])\cdot\right.\\&\left.\cdot\operatorname{tr}[B\delta B^+\exp[-sBB^+]-B^+\delta B\exp[-sB^+B]]\right\}=\int \mathrm{d}^2x\frac{K}{4\pi}\,\delta\operatorname{tr}\log(B^+B^{-1})\,.\end{aligned} \tag{4.9}$$

The integration of eq. (4.7) is now straightforward and so we can write

$$\mathrm{tr}\log \Delta_{\mathrm{F}} = \frac{1}{2}\,\mathrm{tr}\left[\log(-D^2 + K + BB^+) + \log(-D^2 - K + B^+B)\right] + \int \mathrm{d}^2x\,\frac{K}{4\pi}\,\mathrm{tr}\log(B^+B^{-1}) + \mathrm{const}\,. \tag{4.10}$$

We still have a trace to compute. This can be done in the same way as for the bosonic part of S_{eff}:

$$\frac{1}{2}\,\mathrm{tr}\left[\log(-D^2 + K + BB^+) + \log(-D^2 - K + B^+B)\right] = \tag{4.11}$$

$$= -\int \mathrm{d}^2x\,\frac{|K|}{4\pi}\int_0^\infty \frac{\mathrm{d}t}{t}\left(\mathrm{ctgh}\,t - \frac{1}{t}\right)\mathrm{tr}\exp\left[-t\,\frac{BB^+}{|K|}\right] -$$

$$-\int \mathrm{d}^2x\,\frac{1}{4\pi}\int_0^\infty \frac{\mathrm{d}s}{s^2}\,\mathrm{tr}\left[\exp[-sBB^+] - \exp[-s](1 - sBB^+ + s)\right] +$$

$$+\frac{1}{4\pi}\int \mathrm{d}^2x\int_0^\infty \frac{\mathrm{d}s}{s}\exp[-s]\,\mathrm{tr}(BB^+) + \mathrm{const}\,.$$

The first and the second term of the last relation are convergent (if we disregard the infinite constant factor $\int \mathrm{d}^2x$); the third one is divergent, but it is cancelled in $S^{\mathrm{F}}_{\mathrm{eff}}$ (see eq. (4.1)) by the term $(n_{\mathrm{F}}/2g)\int \mathrm{d}^2x\,\mathrm{tr}\,(AA^+)$ if we assume the following ε-dependence of g and M_{B}:

$$\frac{2\pi}{g} - \int_\varepsilon^\infty \frac{\mathrm{d}s}{s}\exp[-s] = c\,, \qquad \frac{2\pi}{g}\,M_{\mathrm{B}} = \Sigma\,. \tag{4.12}$$

Then the final result is

$$S^{\mathrm{F}}_{\mathrm{eff}} = \frac{n_{\mathrm{F}}}{4\pi}\int \mathrm{d}^2x\left\{\frac{|eF|}{\sqrt{n}}\int_0^\infty \frac{\mathrm{d}t}{t}\left(\mathrm{ctgh}\,t - \frac{1}{t}\right)\mathrm{tr}\exp\left[-t\,\frac{\sqrt{n}}{|eF|}\,BB^+\right] + \right. \tag{4.13}$$

$$\left. + \mathrm{tr}\left[BB^+\log\frac{BB^+}{eM^2} - BB^+\right] - \mathrm{tr}\left[\Sigma(B + B^+)\right] - \frac{eF}{\sqrt{n}}\,\mathrm{tr}\left[\log B^+ - \log B\right]\right\},$$

where $M^2 = \exp[-c]$.

The integral in (4.13) can be easily done via Stirling's formula (3.10) by using the identity

$$\int_0^\infty \frac{\mathrm{d}t}{t}\exp[-at]\left(\mathrm{ctgh}\,t - \frac{1}{t}\right) = 2\int_0^\infty \frac{\mathrm{d}t}{t}\exp\left[-\frac{a}{2}\,t\right]\left[\frac{1}{\exp[t] - 1} - \frac{1}{t} + \frac{1}{2}\right]. \tag{4.14}$$

5. – Effective Lagrangian at large N.

Summing the contributions of the fermionic and the bosonic determinant computed in the previous sections, one gets the following effective Lagrangian:

$$\begin{aligned}(5.1)\qquad \mathscr{L}_{\text{eff}} = &-\frac{n}{4\pi}\beta\left(\log\frac{\beta^2}{m^2}-1\right)-\frac{n}{2}q(x)\int\limits_0^\infty\frac{\mathrm{d}t}{t}\exp\left[-t\frac{\beta}{2\pi q}\right]\left(\frac{1}{\sinh t}-\frac{1}{t}\right)+\\ &+\frac{n_F e}{2}q(x)\int\limits_0^\infty\frac{\mathrm{d}t}{t}\left(\operatorname{ctgh} t-\frac{1}{t}\right)\operatorname{tr}\exp\left[-\frac{2tM^2V^+V}{n_F e q(x)}\right]+\\ &+\operatorname{tr} M^2\left[V^+V\left(\log\frac{2V^+V}{F_\pi^2}-1\right)\right]+\frac{F_\pi}{2\sqrt{2}}\operatorname{tr}(\mathscr{M}V+\mathscr{M}^+V^+)+\\ &+\frac{n_F e}{2}q(x)\operatorname{tr}(\log V-\log V^+)+\frac{1}{2}\operatorname{tr}(\partial_\mu V\,\partial_\mu V^+)\,,\end{aligned}$$

where

$$(5.2)\qquad \left\{\begin{aligned}&\beta(x)=m-\frac{i\alpha(x)}{\sqrt{n}}\,,\qquad q(x)=\frac{1}{2\pi\sqrt{n}}F(x)\,,\\ &V=\frac{1}{2M\sqrt{\pi}}\sqrt{n_F}B\,,\qquad \mathscr{M}=m_\pi^2\mathbb{1}\end{aligned}\right.$$

and, for the sake of simplicity, we have taken the mass matrix proportional to the unit matrix. Remember that $q(x)$ is the topological charge density.

F_π in two dimensions is dimensionless and is given by

$$(5.3)\qquad F_\pi=\frac{\sqrt{n_F}}{\sqrt{2\pi}}\,.$$

The kinetic term has been computed in ref. (3) and has been here added by hand. Notice that in the chiral limit (5.1) is invariant under $SU_L\times SU_L$ chiral transformations. It is, however, not invariant under U_1 chiral transformations, but because of the term with the logarithmic interaction one gets that

$$(5.4)\qquad \mathscr{L}_{\text{eff}}\to\mathscr{L}_{\text{eff}}+ie\varphi L n_F q(x)$$

when one performs the following transformation:

$$(5.5)\qquad q(x)\to q(x)\,,\qquad \beta(x)\to\beta(x)\,,\qquad V\to\exp[i\varphi]\,V\,.$$

The transformation property (5.4) ensures that the effective Lagrangian for the lowest « hadrons » satisfies all the anomalous Ward identities.

In order to compute the large-n and -n_F expansion, one needs to expand L_{eff} around a saddle point. It is easy to check that one must expand around the following values for the fields:

$$\langle\beta\rangle = m^2\,,\qquad \langle q(x)\rangle = 0\,,\qquad \langle V\rangle = \frac{F_\pi}{\sqrt{2}}\,. \tag{5.6}$$

The last vacuum expectation value corresponds to the spontaneous breaking of chiral symmetry.

By making the $1/n$ expansion of (5.1) and taking for simplicity $n_F = e = 1$, one gets the following expression:

$$\mathscr{L}_{eff} = \frac{1}{2}\,\mathrm{tr}\,(\partial_\mu V\,\partial_\mu V) + \frac{F_\pi}{2\sqrt{2}}\,\mathrm{tr}\,(\mathscr{M}V + \mathscr{M}^+ V^+) + \tag{5.7}$$

$$+ M^2\,\mathrm{tr}\left\{V^+ V\left(\log\frac{2V^+V}{F_\pi^2} - 1\right)\right\} + \frac{e}{2}\,q(x)\,\mathrm{tr}\,[\log V - \log V^+] + \frac{1}{aF_\pi^2}\,q^2(x) + \ldots\,,$$

where

$$a = \frac{12m^2}{n}\,. \tag{5.8}$$

One gets, therefore, an effective Lagrangian that has the same form of the one recently proposed in the large-N QCD. For a discussion of the physical implications of a Lagrangian of type (5.7), see ref. [4,5,7].

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(7) P. DI VECCHIA, F. NICODEMI, R. PETTORINO and G. VENEZIANO: CERN TH-2898 (1980).

● RIASSUNTO

Si calcola l'azione efficace, nel limite di bassa energia, per il modello CP^{N-1} bidimensionale con quark. Se ne discute inoltre lo sviluppo $1/N$ e le proprietà nella simmetria chirale.

CP^{N-1} модель с кварками: эффективное действие, $1/N$ разложение и киральная симметрия.

Резюме (*). — В пределе низких энергий в явном виде вычисляется эффективное действие для двумерной CP^{N-1} модели с кварками. Обсуждаются $1/N$ разложение и свойство модели относительно киральной симметрии.

(*) *Переведено редакцией.*