PHYSICS REPORTS (Review Section of Physics Letters) 68, No. 4 (1981) 189–398. North-Holland Publishing Company

SUPERGRAVITY

P. van NIEUWENHUIZEN

Institute for Theoretical Physics, State University of New York at Stony Brook, L.I., New York 11794, U.S.A.

Received July 1980

To Joel Scherk

Contents:

Single orders for this issue

PHYSICS REPORTS (Review Section of Physics Letters) 68, No. 4 (1981) 189–398.

Copies of this issue may be obtained at the price given below. All orders should be sent directly to the Publisher. Orders must be accompanied by check.

Single issue price Dfl. 95.00, postage included.

SUPERGRAVITY

P. van NIEUWENHUIZEN

Institute for Theoretical Physics, State University of New York at Stony Brook, L.I., New York 11794, U.S.A.

NORTH-HOLLAND PUBLISHING COMPANY – AMSTERDAM

In memoriam Joel Scherk

Joel Scherk died on May 16, 1980. Although 33 years old, he had made many very important discoveries in dual string theory and supergravity, as will be manifest in this review. It is striking to note how logical and sustained the development of his thinking was.

His first important contribution was the renormalization of the one-loop open dual string amplitude. Then he developed the idea of the zero-slope limit when all masses except the lightest tend to infinity, and found that this leads to quantum field theories. In particular the closed string sector yielded general relativity, and from then on he was fascinated by gravitation. The one-loop contributions of the open string model in $d = 26$ dimensions were shown to correspond to a new ghost-free set of states, corresponding to closed strings and interpreted as Pomeron states. One of these states was described by an antisymmetric tensor gauge field, which is nowadays of interest. Then he started wondering what happens with the extra dimensions in our four-dimensional world. This led him to the idea of spontaneous compactification of space-time, extending ideas of Kalusza and Klein.

In supergravity he started by solving the matter coupling problem and worked out the super-Higgs effect. Then he constructed the $N = 3$ supergravity, and both the SO(4) and the SU(4) versions of $N = 4$ extended supergravity. The latter model was suggested by taking the zero slope limit of the $d = 10$ dimensional closed string dual model with fermions. The $d = 10$ open string model yielded in the zero slope limit $N = 4$ supersymmetric Yang–Mills theory. In the SU(4) model he found a noncompact SU(1, 1) global symmetry, and in the $N = 2, 3, 4$ models he gave a complete treatment of the duality and chiral symmetries. He then formulated $N = 1$ $d = 11$ supergravity, which yields the $N = 8$ $d = 4$ model upon dimensional reduction. He also developed the idea of spontaneous symmetry breaking by means of dimensional reduction, and worked out the particle spectrum. There he found a massless vector boson and proposed the idea that it yields the (testable) force of antigravity.

Despite this impressive record, Joel was a relaxed and very friendly person. He was among the best and most pedagogical speakers and his articles were crystal clear. He was always open to discussion and had a very developed sense of humour. He also contributed actively to the popularization of theoretical physics by radio talks, lectures for lay-men and articles in popular scientific magazines.

This short review was written by his closest friends. We will miss him deeply.

Preface

Supergravity was discovered four years ago [234, 120, 235] and interest in this new theory has been great from the beginning. However, the rapid development of the field and various technical barriers (such as properties of Majorana spinors) have prevented many interested physicists from starting active research. This report hopes to provide an introduction which will allow the reader to go from no knowledge at all about supergravity up to the point where present research begins. It is written as a course for graduate students and thus the focus is more on explaining than on reviewing. Most importantly: it is self-contained. We have tried to discuss all major areas of supergravity, including the advanced aspects, but we have placed the emphasis on supergravity as a gauge theory. In this way, this report might be useful as the basis for a graduate course in gauge fields and quantum field theory.

Rather than giving some general discussions or refer to the literature for details, we have tried to be explicit. Thus we have not shied away from working out details, and this should be helpful to the reader. Actually, large parts of this report were tried out on students at Stony Brook, and in this way we discovered where more explicit work was required. We must confess that as a result this report is rather long, and we have had nightmares in which the publisher provided it together with a wheelbarrow.

In order to describe the work of our colleagues in detail, we have visited many of them for long discussions. Thus the preparation of this report took more than two years. It is a pleasure to thank them for their cooperation. In addition, we have asked all of our colleagues to send us a list of their publications as a basis for the references in this report. This report is dedicated to our superfriends, who have created light (and heat) by their enthusiasm. Let us hope that nature is equally enthusiastic and has reserved an important role for this most elegant of all gauge theories.

1. Simple supergravity in ordinary space-time

1.1. Introduction and outlook

The gravitational force is the oldest force known to man and the least understood. Some reasons for this curious fact are that gravitational experiments are difficult to perform, due to the smallness of the gravitational constant κ, while at the theoretical level the dimensionful character of the gravitational constant has prevented the construction of a predictive, i.e. renormalizable, quantum theory of gravity, as we will discuss. All experimental data known at present confirm Einstein's theory of gravity and it seems that any future quantum theory of gravitation should be an extension rather than a replacement of general relativity. In the near future, several new gravitational experiments (the gyroscope experiment, direct detection of gravitational waves, the study of gravitational phenomena in the binary pulsar, light deflection at the sun to second order in κ(!) by optical rather than radio means) will further single out or reject Einstein's theory of gravity as the unique low-energy limit of any classical theory of gravity.

For the nongravitational forces, renormalizable quantum field theories exist which seem to describe nature adequately. The electromagnetic and weak interaction have been unified within the framework of the electroweak interaction (quantum flavor dynamics) of which the minimal $SU_2 \times U_1$ theory of Glashow, Salam, Ward, Weinberg and others is the most attractive. For the strong interactions, a theory of gluons (Yang–Mills bosons) interacting with quarks (quantum chromodynamics) exists which explains the observed phenomena of scaling in deep inelastic lepton scattering and predicts the occurrence of jets in lepton and hadron scattering. There exist even "grand unified schemes" in which these two theories are unified and describe a world of color and flavor but without gravity.

It is one of the main objectives of science to bring order in chaos and to explain the many diverse physical phenomena by one underlying theory. Clearly, one of the central problems for the last quarter of our century is the unification of the gravitational force with the nongravitational forces ("super unified schemes"). But not only a unification of all forces is needed, also a unification of the principles on which the construction of theories is based. At the beginning of our century, three pillars of theoretical physics were discovered: special relativity, quantum mechanics and general relativity. The first two have been successfully combined into quantum field theory, which has provided the renormalizable field theories for quantum flavor-dynamics and quantum chromodynamics. However, the incorporation into these theories of the principle of general relativity has proven up to now to be impossible, because general relativity leads to gravity and gravity has a coupling constant that is dimensionful (due to the equivalence principle) which destroys renormalizability. Thus what is needed is a field theory describing both gravity and the other interactions and based upon the principles of special and general relativity and of quantum mechanics, and that, though nonrenormalizable, still has predictive power.

Supergravity is proposed as such a theory. It is based on a new symmetry principle, namely a symmetry between the bosonic and fermionic fields in Lagrangian field theory. The remarkable thing is that a gauge symmetry between bosons and fermions can only be implemented in field theory if space-time is curved and hence if gravity is present. Since there is in supergravity a symmetry between bosons and fermions, there must be one fermionic companion to the usual bosonic gravitational field, and other fermionic companions to the other bosonic fields. The fermionic companion to the gravitational field is a spin 3/2 field, called the gravitino. If there are N gravitinos in the theory, denoted below by $\psi_\mu^{\alpha,i}$ with $\mu = 1, 4$, $\alpha = 1, 4$, $i = 1, N$ with $N \leq 8$, then they appear symmetrically. In the original formulation of supergravity (simple or $N = 1$ supergravity) there was one real massless gravitino. But in the extended ($N = 2, 3, \ldots, 8$) theories, N gravitinos appear, so that, for instance the $N = 2$ theory has a complex (Dirac) gravitino rather than a Majorana (real) gravitino. Also, it was found that gravitinos could acquire a mass by means of spontaneous symmetry breaking (the "super-Higgs effect") [125, 94, 406]. Discovery of an elementary spin 3/2 particle in the laboratory would be a triumph for supergravity because the only consistent field theory for interacting spin 3/2 fields is supergravity. In fact, in a really unified theory of which the $N = 8$ model [97] is a prototype, the gravitinos would be nothing else than a new kind of quarks or leptons, endowed with all the characteristics of the known quarks (color, flavor, lepton number, etc.), but differing from these by having spin 3/2 rather than spin 1/2.

Whether massive or massless, the new fermionic gravitational fields do not modify the classical predictions of general relativity because the exchange of fermions leads to a short-range potential. (This follows, for example, from dispersion theory and is due to the necessity to exchange *two* fermions to produce a potential.) On the other hand, at short distances and thus at high energies, supergravity differs radically from ordinary general relativity and is a better quantum theory. Infinities in the S-matrix in the first and second order quantum corrections cancel due to the symmetry between fermions and bosons. Whether these cancellations persist in all higher order quantum corrections is an open question at present. Thus supergravity is not finite because infinities are renormalized away, as in the nongravitational theories, but because in the S-matrix infinities cancel. Due to this finiteness, supergravity has predictive power, just as ordinary renormalizable models.

In simple ($N = 1$) supergravity, bosons and fermions occur always in pairs, and these pairs are the irreducible representations of the supersymmetry algebra. One such pair is the graviton and gravitino. Other pairs are spin 1/2 and spin 1 doublets ("photon–neutrino system") and spin 0 spin 1/2 doublets. One can add as many of such matter doublets to the spin 2 spin 3/2 gauge doublet as one wishes. However, a special feature arises when one adds one or more spin 1 spin 3/2 doublets to the spin 2 spin 3/2 doublet. The resulting theories are the so-called extended ($N = 2, \ldots, 8$) supergravity theories which, in addition to the space-time symmetries which any gravitational theory possesses, have as many Fermi–Bose symmetries as there are gravitinos, namely N. In addition, they have $\frac{1}{2}N(N-1)$ spin 1 real vector fields, and lower spins. Most importantly, they have a global $U(N)$ group of combined chiral-dual symmetries which has nothing to do with supersymmetry. Under these symmetries, fermions rotate into themselves or γ_5 times themselves, and curls of gauge fields rotate into their dual curvatures. The $O(N)$ part of this $U(N)$ group rotates the gravitinos into each other according to the orthogonal group $O(N)$, and simultaneously the photons into each other, etc.

The first of the extended supergravities is the $N = 2$ model [511]. It realizes Einstein's dream of unifying electromagnetism and gravity, and does so by adding two real (=one complex) gravitino to the photon and the graviton. It is in this model that the breakthrough in finiteness of quantum supergravity occurred: an explicit calculation of photon–photon scattering which was known to be divergent in the coupled Maxwell–Einstein system yielded a dramatic result [512]: the new diagrams involving gravitinos cancelled the divergencies found previously (see fig. 1).

One should view the N-extended supergravities as extensions of pure general relativity (i.e., without matter) in which the single graviton, N gravitinos, $\frac{1}{2}N(N-1)$ vectors etc., all rotate into each other under either supersymmetry or $U(N)$ symmetries. Since these extensions are irreducible, one calls these theories *pure* extended supergravities. Thus the graviton is replaced by a new superparticle whose "polarizations" are the graviton, gravitinos, quarks, photons, electrons, etc. This unification of all particles into one superparticle leads also to a unification of all forces, because forces arise by exchange of particles. One can also couple matter to these theories, but only N-extended matter to N-extended supergravity (N-extended matter having before coupling N global supersymmetries and a global $U(N)$ invariance).

The importance of the extended supergravities lies in the fact that the global $O(N)$ group can be gauged by the $\frac{1}{2}N(N-1)$ vector fields which are present in the pure N-extended supergravities [108, 225]. This leads then to theories with *two coupling constants*: the gravitational coupling constant κ and the internal $O(N)$ coupling constant g. The latter should describe the nongravitational interactions, but in order that this be possible spontaneous symmetry breaking must first occur such that g splits up

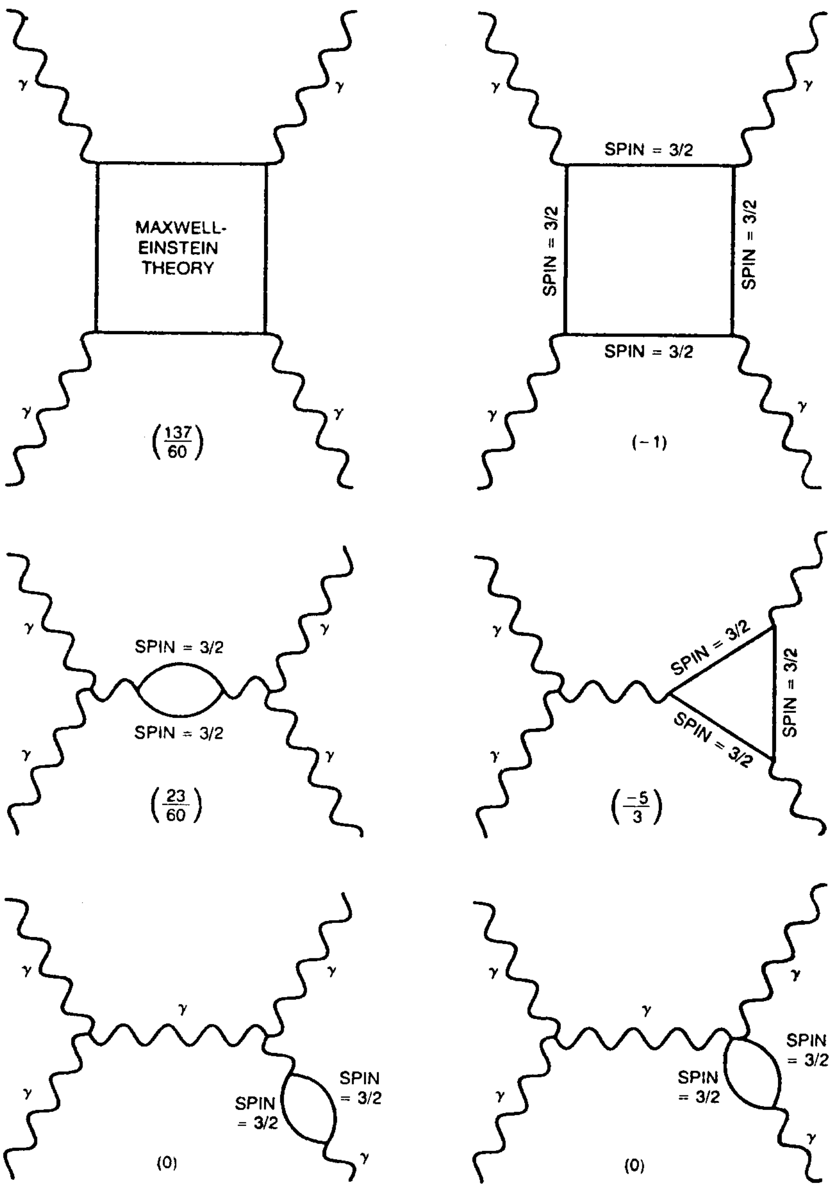


Fig. 1. Finite probability for loop diagrams in a quantum theory of gravity is obtained by including gravitinos in the interaction. The diagrams shown here are those for the interaction between two photons. The first diagram, labeled Maxwell–Einstein theory, consolidates all the one-loop diagrams that involve only gravitons and photons; the contribution from these diagrams is equal to an infinite quantity multiplied by the coefficient 137/60. Five one-loop diagrams involving gravitinos can be constructed; each of them is proportional to the same infinite term multiplied by the coefficients shown in brackets. Only the sum of the diagrams is observable, and adding the coefficients shows that the sum is zero. Hence the infinite contributions of the gravitinos cancel those of the gravitons and the diagrams have a finite probability [249].

into the different couplings of the electroweak and strong interactions. This has not yet been achieved, and is of prime importance for the phenomenological implications of supergravity. It is tempting to speculate that supergravity is the high energy limit of the nongravitational interactions. In that case, one would restore the symmetry by going to high energies and find that bosons and fermions have equal (vanishing?) masses.

In extended supergravity theories with unbroken supersymmetry all the companions of the graviton, including of course the vector gauge fields, but also the "matter fields" (spin 1 and spin 1/2) are (as the graviton) massless. We already mentioned that even in pure extended supergravity models there is a mechanism of symmetry breaking that gives masses to the gravitinos and also to vector and matter fields, leaving the graviton massless [404, 405]. Even in the situation when supersymmetry is broken, the breaking is soft and mass relations exist that insure that quantities such as the one-loop corrections to the cosmological constant, which are finite when supersymmetry is unbroken, still remain finite when spontaneous symmetry breaking is introduced [406, 603]. Whether in these massive models the one- and two-loop corrections to the S-matrix are finite is not known.

The N-extended supergravities are only viable for $0 \leq N \leq 8$ because for $N > 8$ spin 5/2 and higher-spin fields enter onto the stage, while also several spin 2 fields are needed. The reason is that the massless irreducible multiplets of N-extended supersymmetry constrains: one helicity $+2$ (the graviton), N helicities $+3/2$ (the gravitinos) and so on, according to a binomial series. For acceptable field theories, one adds the *CPT* conjugate irreducible representation, starting with one helicity -2, N helicities $-3/2$ and so on. Only for $N = 8$ is one representation enough, since it starts with $\lambda = +2$ and ends with $\lambda = -2$. For $N > 8$ the tower of helicities which starts with $\lambda = +2$ extends below $\lambda = -2$. Recently, interest in higher-spin theory has been revived [258, 259], and acceptable free-field actions for all higher spins have been found. However, the coupling of spin 5/2 to gravity and to other spins is inconsistent [153] and there exists no satisfactory coupling of fields with spins exceeding 2. Thus nature seems to allow only spins ranging from 0 to 2, and we are stuck with the extended supergravities with $0 \leq N \leq 8$.

The $N = 8$ theory with spontaneously broken symmetry comes rather close to being a unified field theory of Nature. In the special case when some of the still arbitrary mass parameters are set equal, its invariance group is $SU_3 \times U_1 \times U_1$ (see fig. 2). The 8 gravitinos are massive and describe a spin 3/2-quark and a spin 3/2 lepton. The spin 1/2 fields are all massive and contain precisely the up, down and charmed quarks as well as an electron and gluinos (spin 1/2 color [i.e., SU_3] octets) and sexy quarks (a spin 1/2 color sextet). (A strange quark is eaten by a massless gravitino with charge $-1/3$ according to the super-Higgs effect.) This $N = 8$ model seems to be the first model that predicts rather than assumes fractional charges (the correct $+2/3$ and $-1/3$) for quarks and integer charges $(1, 0)$ for leptons. Of the 28 vector fields, only 10 are massless, namely the photon and the gluons, while for the last massless vector boson corresponding to the second U(1) factor, several interpretations exist: either it describes the weak neutral current (Z°), or it describes "antigravity". (At short distances the exchange of this vector, being proportional to the gravitational constant κ, cancels the gravitational exchange between two particles and doubles it between two antiparticles. Hence it might describe a gravitational neutral current. It makes testable predictions [391].)

Great excitement has followed the recent discovery in this $N = 8$ model of an extra global E_7 and local SU(8) symmetry. It is speculated that the local SU(8) group could even produce spin 1 and spin 1/2 bound states which are needed for grand unification. (In its simplest form, W^+, W^-, the μ and τ lepton and all neutrinos would appear in this way. Moreover, among the fields which gauge this local SU(8) would be W^+ and W^-, just as the ordinary spin connection viewed as a function of other fields (tetrads, gravitinos) is not itself an independent propagating field, but still gauges Lorentz rotations.) If this can be shown to be the case, the $N = 8$ model comes indeed very near to being "the model of Nature".

It seems then that supergravity is our best hope of unifying the forces of nature and the three principles mentioned before. It is a tight scheme with very high predictive power, which is exciting but might be its downfall because one cannot adjust masses or couplings at will to fit the experimental data

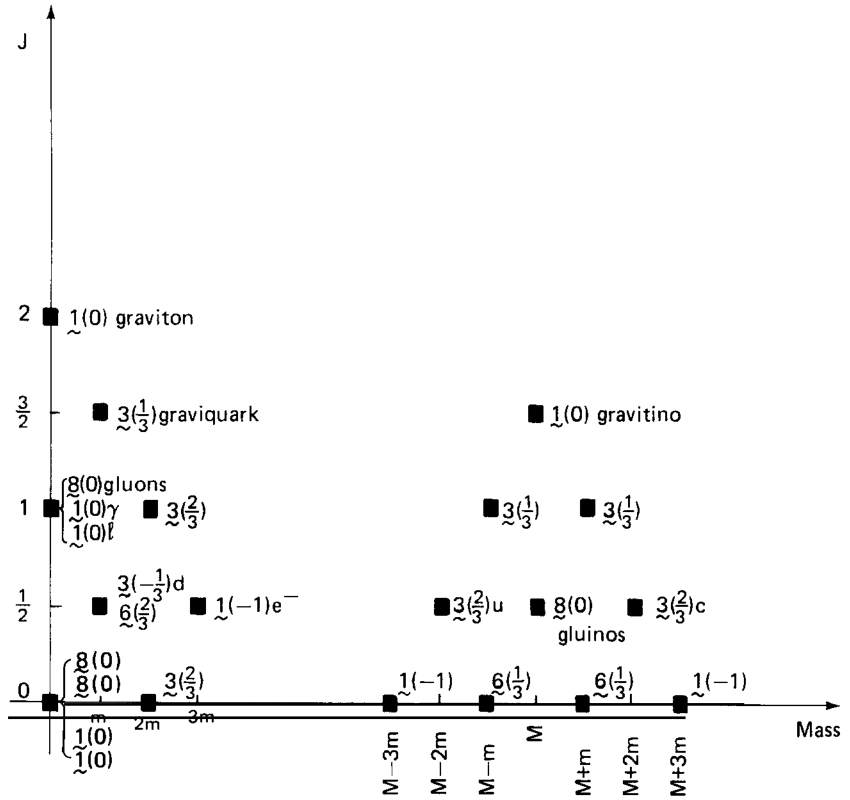


Fig. 2. Spectrum of the spontaneously broken, $N = 8$ model with SU(3)⊗U(1)⊗U(1) invariance. The SU(3) representations are underlined (ex: $\underset{\sim}{3}$); the bracket indicates the electric charge [391].

without destroying local supersymmetry. Once one accepts the principle of a local supersymmetry between bosons and fermions, the theory follows in a unique way, and we shall derive it in this way. It is of course not excluded that supersymmetry is only one attempt at quantization and unification in a long historical row, and that it will eventually fail, so that one must consider another symmetry. All one can do is work out its consequences and hope for the best.

From a theoretical point of view, supergravity is the gauge theory of the space-time symmetries and supersymmetry. The tetrad e_μ^m and spin connection ω_μ^{mn} gauge the space-time symmetries while the gravitino (the massless real spin 3/2 field) gauges supersymmetry. All these symmetries form an irreducible algebra that is an extension of the concept of ordinary Lie algebras because it contains anticommutators as well as commutators. This algebra is called the super Poincaré algebra. The symmetry between bosons and fermions is usually formulated in terms of anticommuting numbers (Grassmann variables, we will give matrix representations), but this is only a matter of convenience since one can formulate the symmetry also without them. A very conservative way of considering supergravity is to interpret it as just another quantum field theory in ordinary space-time, on the same footing as for example quantum electrodynamics, the only new feature being that it contains spin 3/2 fields.

A very interesting and much less conservative description of supergravity it to consider it as a theory in superspace. Superspace has as coordinates not only the ordinary $x-y-z-t$, but in addition $4N$ spinorial anticommuting coordinates $\theta^{\alpha i}$ ($\alpha = 1, 4$ and $i = 1, N$). There are various approaches to superspace, based on different geometrical ideas and using different base and tangent manifolds and groups, but they all have in common that the notion of Grassmann variables (anticommuting c-numbers) as coordinates is essential. The approaches using ordinary $x-y-z-t$ space, are entirely equivalent to the approaches using (x^μ, θ^α) superspace, but superspace methods give a better geometrical insight. They are also less accessible due to algebraic complications and for that reason we devote a fair amount of space to them.

1.2. Physical arguments leading to supergravity

1.2.1. Supersymmetry

Supergravity is the gauge theory of supersymmetry and supersymmetry is the symmetry between bosonic and fermionic fields of certain Lagrangian field theory models. In this section we show how simple physical arguments lead one directly to supergravity.

Let us begin with supersymmetry. In order to be definite, consider the simplest model of global supersymmetry, namely the Wess–Zumino model [581] with a scalar A, a pseudoscalar B and spin 1/2 field λ. All fields are real and massless, and we choose a real representation of the Dirac matrices (i.e., γ^1, γ^2, γ^3, real while $\gamma^4 = \mathrm{i}\gamma^0$ is purely imaginary). We shall later on define Majorana spinors as general representations of the Dirac matrices; however, the Dirac matrices $\gamma^1, \ldots, \gamma^4$ are always hermitian. Thus, for real λ, we define $\lambda^+_\alpha = \lambda^\alpha$ at this point. We work with real (Majorana) spinors and these are more fundamental than complex ones (two real make up a complex spinor). One could, however, also describe supersymmetry in terms of complex spinors; in our example one would need then four real (or two complex) scalar fields.

The action is a sum of the Klein–Gordon actions and the Dirac action

$$\mathscr{L} = -\tfrac{1}{2}(\partial_\mu A)^2 - \tfrac{1}{2}(\partial_\mu B)^2 - \tfrac{1}{2}\bar{\lambda}\not{\partial}\lambda, \qquad \bar{\lambda} = \lambda^\dagger \gamma^4. \tag{1}$$

This action is globally supersymmetric, since it is invariant under the following set of transformation rules

$$\delta A = \tfrac{1}{2}\bar{\epsilon}\lambda, \qquad \delta B = -\frac{\mathrm{i}}{2}\bar{\epsilon}\gamma_5\lambda, \qquad \delta\lambda = \tfrac{1}{2}\not{\partial}(A - \mathrm{i}B\gamma_5)\epsilon$$
$$\bar{\epsilon} = \epsilon^\dagger\gamma^4, \qquad \gamma_5 = \gamma^1\gamma^2\gamma^3\gamma^4 = \text{Hermitian and imaginary}. \tag{2}$$

(In fact this model is even invariant if one drops B everywhere. However, adding mass terms or a self coupling, invariance is lost. Also the algebra without B is different (see below).) Note that the transformation rules preserve the reality:

$$(\bar{\epsilon}\lambda)^* = \bar{\lambda}\epsilon = (\lambda)^\alpha(\gamma^4)^\alpha{}_\beta(\epsilon)^\beta = -\epsilon^\beta(\gamma^4)^\alpha{}_\beta(\lambda)^\alpha = \bar{\epsilon}\lambda \tag{3}$$

since γ^4 is antisymmetric in a real representation. We have already used that ϵ and λ anticommute, about which more below. The action is clearly hermitian, since λ^α and λ^β anticommute. This is required by the spin-statistics theorem, but it is amusing to note that if λ^α and λ^β were commuting, the Lagrangian $\bar{\lambda}\not{\partial}\lambda$ would be a total derivative. The proof that the action $I = \int \mathrm{d}^4x\,\mathscr{L}$ is invariant, is rather simple. Varying the first terms in $\mathscr{L}$, one has $-\tfrac{1}{2}(\partial^\mu A)\,\partial_\mu(\bar{\epsilon}\lambda)$ while varying $\delta\lambda = \tfrac{1}{2}\not{\partial}A\epsilon$ and $\delta\bar{\lambda} = (\delta\lambda)^{\mathrm{T}}\gamma^4 = \tfrac{1}{2}\epsilon^{\mathrm{T}}(\not{\partial}A)^{\mathrm{T}}\gamma^4 = -\tfrac{1}{2}\bar{\epsilon}(\not{\partial}A)$, one find from the last term in (1) $-\tfrac{1}{4}\bar{\lambda}\not{\partial}\not{\partial}A\epsilon + \tfrac{1}{4}\bar{\epsilon}(\not{\partial}A)\,(\not{\partial}\lambda)$. Similarly for B. Hence *the Lagrangian varies into a total derivative*,

$$\delta\mathscr{L} = \partial_\mu K^\mu, \qquad K^\mu = -\tfrac{1}{4}\bar{\epsilon}\gamma^\mu[\not{\partial}(A - \mathrm{i}\gamma_5 B)]\lambda \tag{4}$$

as a little calculation shows, using $\bar{\epsilon}\lambda = \bar{\lambda}\epsilon$, and $\bar{\epsilon}\gamma_5\lambda = \bar{\lambda}\gamma_5\epsilon$ (see appendix C).

The transformation rules in (2) reveal a set of general properties which any globally supersymmetric model has (by global we mean constant ϵ). The general law which rotates a boson into a fermion, is of the form (suppressing indices).

$$\delta B = \bar{\epsilon}F. \tag{5}$$

Requiring covariance of this equation, i.e., requiring that both sides have the same properties, one derives

(1) *Statistics*: Since fermions must be anticommuting because of the spin-statistics theorem, also ϵ^α are anticommuting parameters. An explicit matrix representation can be given in terms of Wigner–Jordan absorption matrices a_j for fermions (not also creation operators) as follows

$$\epsilon^\alpha = \sum_j a_j\beta^\alpha{}_j, \qquad \bar{\epsilon}_\alpha = \sum_j a_j\bar{\beta}_{\alpha j}, \qquad \bar{\beta}_{\alpha j} = (\beta_j^\dagger\gamma^4)_\alpha. \tag{6}$$

The Dirac bar does not act on a_j, the sum over j runs over an infinite set, and the $\beta^\alpha{}_j$ are ordinary numbers. A similar representation can be used for spinor fields λ, and one defines the $\dagger$ operation on the product by $(a_ja_k)^\dagger = a_ka_j$, *not* $a_k^\dagger a_j^\dagger$. Since such spinor fields anticommute, we treat $\lambda(x)$ here as a classical field, and not in second quantization form. Formally this is the $h \to 0$ limit in the quantum mechanical (anti)commutators. From this explicit representation one reads off the following (anti)commutation relations between fermion fields F, boson fields B and parameters ϵ

$$\{\epsilon^\alpha, \epsilon^\beta\} = \{\epsilon^\alpha, \bar{\epsilon}_\beta\} = \{\epsilon^\alpha, F\} = [\epsilon^\alpha, B] = \{F, F\} = 0. \tag{7}$$

Thus the ϵ^α and λ^α span a Grassmann algebra and not a Clifford algebra. (A Clifford algebra has "one" on the right hand side, for example the Dirac matrices span a Clifford algebra. Perhaps one could use Clifford algebras for supersymmetry.)

(2) *Spin*: Since bosons have integer spin and fermions have half-integer spin, the parameters ϵ^α have half-integer spin. The simplest case, spin 1/2, is realized in supergravity. Models where ϵ^α has spin 3/2 have negative energy states [44]. Also the gauge fields for parameters with spin 3/2 are spin 5/2 fields and for these no consistent coupling exists [538].

(3) *Representations*: Since ϵ has spin 1/2, the irreducible representations must involve at least one boson and one fermion. It was shown by Salam and Strathdee that for massless states, this is in fact enough: the representations consist of a boson–fermion doublet with adjacent helicities $(\lambda, \lambda + 1/2)$. For example, in our model there are two irreducible representations, namely $(A + \mathrm{i}B, (1 + \gamma_5)\lambda)$ and $(A - \mathrm{i}B, (1 - \gamma_5)\lambda)$. As the reader may verify using (2), $A + \mathrm{i}B$ and $(1 + \gamma_5)\lambda$ transform into each other; similarly for the second doublet. These doublets are in fact the irreducible representations of a new kind of algebra, namely superalgebra, see below.

(4) *Dimensions*: Boson fields have dimension one and fermion fields have dimension 3/2 in order that the action be dimensionless (in units $\hbar = c = 1$). The reason is, of course, that boson fields have two

derivatives in their action while fermion fields have only one (see eq. (1)). Hence the dimension of ϵ is $-\frac{1}{2}$ according to eq. (5). However, this means that *in the inverse transformation $\delta F \sim B\epsilon$ there is a gap of one unit of dimension.* The only dimensionful object available in flat space with massless fields to fill this gap is a derivative. Thus, *purely on dimensional grounds* (omitting indices)

$$\delta F = (\partial B)\epsilon. \tag{8}$$

The reader may check this relation for the special case of eq. (2).

(5) *Commutator*: Consider now two consecutive infinitesimal global supersymmetry transformations of a boson field B (the same arguments hold for F). The first rotation transforms B into F, but the second rotates F back into $\partial_\mu B$. *Thus two internal supersymmetry transformations have led to a space-time translation.* In any globally supersymmetric model one always finds the same relation

$$[\delta(\epsilon_1), \delta(\epsilon_2)]B = \tfrac{1}{2}(\bar{\epsilon}_2\gamma^\mu\epsilon_1)\partial_\mu B. \tag{9}$$

For F the same result holds, see subsection 8. We leave it as an exercise to verify this relation for eq. (2) using the appendix. Hence, global supersymmetry is the square root of the translation operator, and one expects that *local supersymmetry is the square root of general relativity*, just as the Dirac equation is the square root of the Klein–Gordon equation. This we will show in more detail in subsection 3.5.
We derived this extremely interesting result from the simple fact that bosons and fermions have different dimensions. For a discussion of the complete algebra of symmetries of globally supersymmetric models, see below. For further details of global supersymmetry, see the Physics Report by P. Fayet and S. Ferrara.

1.2.2. Supergravity

We now turn to local supersymmetry and ask what would happen if we make ϵ^α local, hence $\epsilon^\alpha(x)$. For example, in eq. (2) we define

$$\delta\lambda = \tfrac{1}{2}(\not\partial(A - \mathrm{i}\gamma_5 B))\,\epsilon(x) \quad \text{without } \partial_\mu\epsilon \text{ terms in } \delta\lambda$$

since λ is considered as a matter field. (Quite generally, only gauge fields have in their transformation rules derivatives of parameters (in this case $\epsilon(x)$) and only on the parameters to which they belong. For example $\partial_\mu\epsilon$ will only occur in the supersymmetry gauge field (the gravitino) but not in, say, the Yang–Mills field transformation law.) Since for constant ϵ the action in (1) is invariant under (2), it follows that for local ϵ the variation of the action must be proportional to $\partial_\mu\epsilon(x)$

$$\delta I = \int \mathrm{d}^4x\,(\partial_\mu\bar{\epsilon}(x))j^\mu_{\mathrm{N}}. \tag{10}$$

Here j^μ_{N} is the Noether current, and can be obtained from (2) and (4) as follows

$$\bar{\epsilon}j^\mu_{\mathrm{N}} = \frac{\delta\mathcal{L}}{\delta\varphi_{,\mu}}\delta\varphi - K^\mu = -\tfrac{1}{2}\bar{\epsilon}(\not\partial(A + \mathrm{i}\gamma_5 B))\gamma^\mu\lambda. \tag{11}$$

Indeed, for constant ϵ one has $\delta\mathcal{L} = (\partial\mathcal{L}/\partial\varphi_{,\mu})\delta\varphi_{,\mu} + (\partial\mathcal{L}/\partial\varphi)\delta\varphi = \partial_\mu K^\mu$ so that on shell (where

$\partial_\mu(\partial\mathscr{L}/\partial\varphi_{,\mu}) = \partial\mathscr{L}/\partial\varphi)$ the Noether current is conserved. For local ϵ one h᠀ $\delta\mathscr{L} = \partial_\mu K^\mu + (\partial_\mu\bar{\epsilon})s^\mu$ with some s^μ. Hence for local ϵ,

$$\delta\mathscr{L} = \left(\frac{\delta\mathscr{L}}{\delta\varphi} - \partial_\mu \frac{\delta\mathscr{L}}{\delta\varphi_{,\mu}}\right)\delta\varphi + \partial_\mu\left[\frac{\partial\mathscr{L}}{\delta\varphi_{,\mu}}\delta\varphi\right] = \partial_\mu K^\mu + (\partial_\mu\bar{\epsilon})s^\mu \tag{12}$$

and equating the $\partial\epsilon$ terms it follows that $s^\mu = j^\mu_{\mathrm{N}}$. Therefore, as in any gauge theory, one may start an iterative procedure ("Noether method") of adding extra terms to action and transformation laws such that in the end the complete action is invariant. The first term is, as always, the coupling of the gauge field to the Noether current.

Since gauge fields are obtained by adding a world index μ to the parameter, the gauge field of supersymmetry is a vectorial spinor (or spinorial vector, if one prefers) $\psi_\mu{}^a$ called the gravitino. Indeed, its spin content is

$$\text{spin } \psi_\mu{}^a = (1+0)\otimes\tfrac{1}{2} = \tfrac{3}{2}+\tfrac{1}{2}+\tfrac{1}{2} \tag{13}$$

where the two spin 1/2 parts correspond to $\partial\cdot\psi$ and $\gamma\cdot\psi$. (The decomposition into $3/2+1/2+1/2$ is nonlocal. As far as the Lorentz group is concerned, $\psi_\mu{}^a$ transforms as

$$(\tfrac{1}{2},\tfrac{1}{2})\otimes[(\tfrac{1}{2},0)+(0,\tfrac{1}{2})] = (1,\tfrac{1}{2})+(\tfrac{1}{2},1)+(0,\tfrac{1}{2})+(\tfrac{1}{2},0).) \tag{14a}$$

This corresponds to the local decomposition

$$(1\pm\gamma_5)(\psi_\mu - \tfrac{1}{4}\gamma_\mu\gamma\cdot\psi) \quad\text{and}\quad (1\pm\gamma_5)(\gamma_\mu\gamma\cdot\psi). \tag{14b}$$

Thus one adds to the action in eq. (1) the following Noether coupling

$$I^{\mathrm{N}} = \int \mathrm{d}^4x\,(-\kappa\bar{\psi}_\mu j^\mu_{\mathrm{N}}) \tag{15}$$

and requires that $\delta\psi_\mu \sim \partial_\mu\epsilon(x) + \text{more}$. (Note that I^{N} is off-diagonal in the fermionic fields ψ_μ and λ; this is due to the fact that ψ_μ is a gauge field.) However, since fermions have dimension 3/2 and ϵ has dimension $-1/2$, *a dimensional coupling κ appears*. We will write therefore $\delta\psi_\mu = \kappa^{-1}\partial_\mu\epsilon + \text{more}$ where "more" should be fixed at the next iterative stages.

The dimensionful coupling is the first indication that *local supersymmetry needs gravity*. A second argument in this direction follows from the result in eq. (9). For local $\epsilon(x)$, one expects something of the form

$$[\delta(\epsilon_1(x)), \delta(\epsilon_2(x))]B \sim \tfrac{1}{2}(\bar{\epsilon}_2(x)\gamma^\mu\epsilon_1(x))\,\partial_\mu B. \tag{16}$$

(The exact result is given in subsection 9, eq. (8).)

In other words, one expects translations ∂_μ over distances $d^\mu = \tfrac{1}{2}\bar{\epsilon}_2(x)\,\gamma^\mu\,\epsilon_1(x)$ which differ from point to point. This is the notion of a general coordinate transformation and leads one to expect that gravity must be present. Thus *local* supersymmetry should lead to gravity. The reverse is also expected, since if one starts with a constant parameter ϵ and performs a local Lorentz transformation, then this parameter will in general become space-time dependent as a result of this Lorentz transformation. Hence *local supersymmetry and gravity imply each other* and this is the reason for the name supergravity. However, we stress that this is a heuristic argument only, and that, for example, constant translations when written in terms of curvilinear coordinates look also like general coordinate transformations. Nevertheless, the results confirm this heuristic reasoning.

A third argument is more solid but requires a little bit of work. If one considers the variation of I^N in (15) then one finds new terms due to δA, δB, $\delta\lambda$ in δI^N. For the terms quadratic in A and B one finds

$$\delta(I+I^N)=\int d^4x\,\frac{\kappa}{2}(\bar{\psi}_\mu\gamma_\nu\epsilon)(T^{\mu\nu}(A)+T^{\mu\nu}(B)) \tag{17}$$

where $T_{\mu\nu}(A)=\partial_\mu A\,\partial_\nu A-\frac{1}{2}\delta_{\mu\nu}(\partial_\lambda A)^2$ is the energy momentum tensor of the field A. (The AB terms will be discussed in subsection 3, eq. (10).) Such variations can only be canceled by adding a second Noether coupling, of a new field $g_{\mu\nu}$ to the Noether current of translations, $\frac{1}{2}T^{\mu\nu}$, and requiring that under local supersymmetry

$$\delta g_{\mu\nu}=-\frac{\kappa}{2}\,\bar{\psi}_\mu\gamma_\nu\epsilon-\frac{\kappa}{2}\,\bar{\psi}_\nu\gamma_\mu\epsilon. \tag{18}$$

Since 1915 we know how to do this kind of Noether method to all orders in iteration: by covariantizing with respect to gravity ($\partial_\mu\to D_\mu$, $\delta_{\mu\nu}\to g_{\mu\nu}$ and a $\sqrt{g}$ in front of the Lagrangian). Varying in $-\frac{1}{2}\sqrt{g}\,\partial_\mu A\,\partial_\nu A\,g^{\mu\nu}$ the metric, one finds $\frac{1}{2}\delta g_{\mu\nu}\,T^{\mu\nu}(A)$ and the field $g_{\mu\nu}$ in eq. (18) can be identified with the metric tensor.

We have now learned two things: First of all, gravity must be present and since in supersymmetry one necessarily has fermions present, the gravitational field is described by tetrads e_μ^a rather than the metric field $g_{\mu\nu}$. The relation between them is $g_{\mu\nu}=e^m{}_\mu e^n{}_\nu\delta_{mn}$. For details of tetrad formalism, see Weinberg's book. Secondly, the transformation law of the tetrad must satisfy (18). This does not determine δe_μ^m uniquely, but consideration of the $\bar{\lambda}\lambda\epsilon$ terms in δI^N yields as unique result

$$\delta e_\mu^m=\tfrac{1}{2}\kappa\bar{\epsilon}\gamma^m\psi_\mu \tag{19}$$

(see under (32)). For the transformation law of the gravitino one expects

$$\delta\psi_\mu{}^a=\kappa^{-1}D_\mu\epsilon,\qquad D_\mu\epsilon=\partial_\mu\epsilon+\tfrac{1}{2}\omega_\mu{}^{mn}\sigma_{mn}\epsilon \tag{20}$$

since we are in curved space. Here $\omega_\mu{}^{mn}$ is a spin connection (see appendix I).

The gauge fields of supersymmetry should in particular form a representation of global supersymmetry. (More precisely: the physical states with helicities (+2, +3/2) and (−2, −3/2) described by e_μ^m and ψ_μ^a should form two doublets.) This puts another constraint on δe_μ^m and $\delta\psi_\mu^a$, which is in agreement with (19) and (20) as we now discuss.

The helicity (2, 3/2) representation of global supersymmetry is given by

$$\delta g_{\mu\nu}=\frac{\kappa}{2}(\bar{\epsilon}\gamma_\mu\psi_\nu+\bar{\epsilon}\gamma_\nu\psi_\mu),\qquad \delta\psi_\mu=\frac{1}{2\kappa}(\omega_\mu{}^{mn})_L\sigma_{mn}\epsilon \tag{21}$$

where L stands for linearized and ϵ are constant. Since boson fields have dimension 1, one defines the spin 2 field by $h_{\mu\nu} = (g_{\mu\nu} - \delta_{\mu\nu})/\kappa$ in which case the κ in (21) disappear. (Thus $h_{\mu\nu}$ is the quantum gravitational field.) Thus (21) agrees with (19). This result was derived in ref. [281] by using only the algebra of global supersymmetry, the so-called super-Poincaré algebra, which we now derive and discuss.

If one defines for the fields A, B, λ in eq. (1) $\delta_Q(\epsilon)A = [A, \bar{\epsilon}_\alpha Q^\alpha]$ etc., and defines the Poincaré generators by $\delta_P(\xi^m)A = \partial_\mu A = [A, \xi^m P_m]$ and $\delta_M(\lambda^{mn})\lambda = \frac{1}{2}(\lambda^{mn}\sigma_{mn})^\alpha{}_\beta\lambda^\beta = [\lambda, \frac{1}{2}\lambda^{mn}M_{mn}]$ one can find the commutators of Q, P and M using the Jacobi identities. For example

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)]A = \tfrac{1}{2}(\bar{\epsilon}_2\gamma^m\epsilon_1)\,\partial_m A = \tfrac{1}{2}\bar{\epsilon}_2\gamma^m\epsilon_1[A, P_m]$$
$$= [[A, \bar{\epsilon}_2 Q], \bar{\epsilon}_1 Q] - [[A, \bar{\epsilon}_1 Q], \bar{\epsilon}_2 Q] = [A, [\bar{\epsilon}_2 Q, \bar{\epsilon}_1 Q]] \tag{22}$$

where we used the Jacobi-identity in the last equation. Removing $\bar{\epsilon}_2$ and $\bar{\epsilon}_1$ (using that $\bar{\epsilon}_{1,\alpha} = \epsilon_1{}^\beta C_{\beta\alpha}$ and that ϵ and Q anticommute) one finds that the following equation holds when acting on A

$$\{Q^\alpha, Q^\beta\} = \tfrac{1}{2}(\gamma^m)^\alpha{}_\delta(C^{-1})^{\delta\beta}P_m \tag{23}$$

where $C_{\alpha\beta}C^{-1,\beta\gamma} = \delta_\alpha{}^\gamma$ and $C^{\mathrm{T}} = -C$ (see appendix B). In this way one derives the following algebraic commutators and anticommutators when acting on A or B. (For λ one only finds the same results if one adds auxiliary fields, see subsection 8.)

$$[M_{mn}, M_{rs}] = \delta_{nr}M_{ms} + 3 \text{ more terms} \tag{24a}$$
$$[P_m, M_{rs}] = \delta_{mr}P_s - \delta_{ms}P_r, \quad [P_m, P_n] = 0 \tag{24b}$$
$$[Q^\alpha, P_m] = 0, \quad [Q^\alpha, M_{mn}] = (\sigma_{mn})^\alpha{}_\beta Q^\beta \tag{25}$$
$$\{Q^\alpha, Q^\beta\} = \tfrac{1}{2}(\gamma^m C^{-1})^{\alpha\beta}P_m. \tag{26}$$

This is the super-Poincaré algebra. It is a closed algebra since all Jacobi identifies are satisfied (or since an explicit matrix representation will be given in section 3).

The results in (24) are of course the ordinary Poincaré algebra, but (25) states that the charges Q^α are constant in space and time, and carry spin 1/2. An explicit representation of Q^α for the model in (1) is $Q^\alpha = \int d_3x\, j_{\mathrm{N}}^{0,\alpha}$ with the Noether current given in (11). The really interesting relation is (26); it is the algebraic analogue of eq. (9). One may check (24–26) by expressing all charges in terms of their corresponding Noether currents, and using canonical (anti)commutation rules. (For these rules for Majorana spinors, see subsection 2.14.) Since Q^α have spin 1/2, a physicist is not too astonished to see in (26) anticommutators, but the appearance of anticommutators leads one outside the domain of ordinary Lie algebras and into superalgebras.

The matrix C in (26) must satisfy $C\gamma^m C^{-1} = -\gamma^{m,\mathrm{T}}$ in order that the following Jacobi identities are satisfied

$$\{[M_{mn}, Q^\alpha], Q^\beta\} + \{[M_{mn}, Q^\beta], Q^\alpha\} = [M_{mn}, \{Q^\alpha, Q^\beta\}]. \tag{27}$$

Physicists recognize this matrix as the charge conjugation matrix; for further details see appendix B.

One important conclusion is that if eq. (25) holds, one cannot have a term with $M_{\mu\nu}$ in the right hand side of eq. (26) since it would violate eq. (27) with $M_{\mu\nu}$ replaced by P_μ.

Thus, as follows from eq. (21), the law $\delta\psi_\mu = (1/\kappa)D_\mu\epsilon$ combines both the local gauge aspect $\delta\psi_\mu = (1/\kappa)\,\partial_\mu\epsilon + \text{more}$, and transformation law of the global spin (2, 3/2) multiplet. (A similar situation occurs of course in Yang–Mills theory with $\delta W_\mu{}^a = (D_\mu\Lambda)^a = \partial_\mu\Lambda^a + f^{abc}W_\mu{}^b\Lambda^c$.) For the tetrad e_μ^m, one expects with eq. (21) that $\delta e^m{}_\mu \sim \frac{1}{2}\bar\epsilon\gamma^m\psi_\mu$ or $\delta e^m{}_\mu = \frac{1}{2}(\bar\epsilon\gamma^b\psi^m)e_{b\mu}$. The difference is a Lorentz rotation, $\delta e^m{}_\mu = \frac{1}{2}(\bar\epsilon\gamma^m\psi^n - \bar\epsilon\gamma^n\psi^m)e_{b\mu}$, but fixing $\delta\psi_\mu = (1/\kappa)D_\mu\epsilon$, one has no freedom in dropping Lorentz transformation from $\delta e^m{}_\mu$ alone. One argument in favor of $\delta e^m{}_\mu = \frac{1}{2}\bar\epsilon\gamma^m\psi_\mu$ is that it is linear in fields (γ^m is constant, but $\gamma^\mu = \gamma^m e_m{}^\mu$ is not). However, there is a better argument, and that is obtained by gauging the algebra of global supersymmetry in eqs. (24–26) [586]. This we now discuss.

Associating with every generator a gauge field and a parameter as follows

$$V_\mu = e^m{}_\mu P_m - \tfrac{1}{2}\omega_\mu{}^{mn}M_{mn} + \kappa\bar\psi_{\mu,a}Q^a \tag{28}$$

$$\Lambda = \xi^m P_m + \tfrac{1}{2}\lambda^{mn}M_{mn} + \bar\epsilon_a Q^a \tag{29}$$

one defines gauge transformations by

$$\delta_g V_\mu = D_\mu\Lambda = \partial_\mu\Lambda - [V_\mu, \Lambda]. \tag{30}$$

Applying these rules to the super Poincaré algebra, one finds

$$\delta e_\mu^m = \tfrac{1}{2}\bar\epsilon\gamma^m\psi_\mu, \qquad \delta\psi_\mu = \partial_\mu\epsilon + \tfrac{1}{2}\omega_\mu^{mn}\sigma_{mn}\epsilon, \qquad \delta\omega_\mu^{mn} = 0. \tag{31}$$

The minus sign in (28) was chosen such that ω in $\delta\psi_\mu$ has a plus sign. The result $\delta\omega_\mu{}^{mn} = 0$ will be explained when we discuss the 1.5 order formalism. Hence, group theory leads one straight to

$$\delta e^m{}_\mu = \tfrac{1}{2}\kappa\bar\epsilon\gamma^m\psi_\mu. \tag{32}$$

Also the Noether method leads to this result. If one would put the spin 1/2 action in eq. (1) in curved space-time, variation of the tetrad yields

$$\delta I(\lambda) = (\delta e^m{}_\mu)T_m{}^\mu(\lambda). \tag{33}$$

On the other hand, there are also $\bar\lambda\lambda$ terms coming from varying A and B in I^N, and again one is led to (32). It is a good exercise to derive these results.

Finally we come back to the question how to use non-real representations.

Majorana spinors are spinors which are real in a real representation of Dirac matrices. As shown in appendix B, a Majorana spinor is in general complex, but satisfies the following relation

$$\bar\lambda_\alpha = \lambda^\beta C_{\beta\alpha} = \lambda^\dagger_\beta(\gamma_4)^\beta{}_\alpha. \tag{34}$$

For a real representation, $C \sim \gamma_4$, but in general this relation does not hold. For gravitinos, $\bar\psi_\mu = \psi_\mu{}^T C$. From now on we will use general (hermitian) matrices γ_m and Majorana spinors. The diligent reader may check that all results of this section remain true if one replaces real spinors by Majorana spinors.

1.3. The gauge action of simple supergravity

As we have seen from gauging the super Poincaré algebra, there are three fields in the gauge action: the tetrad e_μ^m, the gravitino ψ_μ^a and the spin connection ω_μ^{mn}. Since e_μ^m and ψ_μ^a already form a boson–fermion doublet of states with adjacent helicities ± 2 and $\pm 3/2$, the field ω_μ^{mn} should not be physical, since otherwise there would be too many bosonic states. This is a familiar situation in ordinary relativity where the Einstein–Cartan theory starts out with two independent fields e_μ^m and ω_μ^{mn}, but where subsequently ω_μ^{mn} is eliminated as an independent field by solving its nonpropagating field equation, as a result of which ω_μ^{mn} becomes a function of e_μ^m. This solution $\omega_\mu^{mn}(e)$ happens to satisfy the tetrad postulate ("the tetrad is covariantly constant")

$$\partial_\mu e^m{}_\nu + \omega_\mu{}^{mn}(e) e_{n\nu} - \Gamma^\alpha_{\nu\mu}(g)\, e^m{}_\alpha = 0. \tag{1}$$

(When one defines $\omega(e)$ by (1), one speaks of the tetrad postulate. We, however, do not postulate (1) but solve ω from its field equation.) Also for supergravity we will use the same Einstein–Cartan formalism. The method of finding the dependence of ω_μ^{mn} on other fields by first treating it as an independent field in the action and then solving its (always nonpropagating) field equation is called Palatini formalism.

For the bosonic part of the gauge action of supergravity it thus seems appropriate to take the usual Hilbert action R (and not, for example, R^2, since in this case ω is propagating and cannot be eliminated algebraically). Let us forget for a moment the gravitinos and concentrate on R alone. Usually one defines the Hilbert action in terms of connections $\Gamma^\alpha_{\mu\nu}$ (such that under parallel transport $\delta A_\mu = \Gamma^\alpha_{\mu\nu} A_\alpha\, \mathrm{d}x^\nu$ and under round-transport $\delta A_\nu = \frac{1}{2} R^\mu{}_{\nu\rho\sigma} A_\mu \oint x^\rho\, \mathrm{d}x^\sigma$) as follows

$$\begin{aligned} R &= \delta^\sigma_\mu g^{\nu\rho} R^\mu{}_{\nu\rho\sigma}(\Gamma) \\ R^\mu{}_{\nu\rho\sigma} &= \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\lambda_{\nu\sigma}\Gamma^\mu_{\nu\rho} - \Gamma^\lambda_{\nu\rho}\Gamma^\mu_{\lambda\sigma}. \end{aligned} \tag{2}$$

At this point one can use second order formalism and define $\Gamma^\alpha_{\mu\nu}$ to be the (symmetric) Christoffel symbol $\Gamma^\alpha_{\mu\nu}(g)$, or one can treat $\Gamma^\alpha_{\mu\nu}$ as an independent field (not necessarily symmetric) and solve its field equation $\delta I/\delta\Gamma^\alpha_{\mu\nu} = 0$. The result is then $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu}(g)$ if there is no matter present. If there is matter present, one gets a different answer, and $\Gamma^\alpha_{\mu\nu}$ is no longer symmetric.

From a more fundamental point of view, the connection $\Gamma^\alpha_{\mu\nu}$ has less meaning than the spin connection ω_μ^{mn}. As we shall see, group theory leads to the following curvature

$$R_{\mu\nu}{}^{mn}(\omega) = \partial_\mu \omega_\nu{}^{mn} - \partial_\nu \omega_\mu{}^{mn} + \omega_\mu{}^{mc}\omega_{\nu c}{}^n - \omega_\nu{}^{mc}\omega_{\mu c}{}^n. \tag{3}$$

There is an interesting connection between (2) and (3). If one relates the (nonsymmetric) symbol $\Gamma^\rho_{\mu\nu}$ to the spin connection ω_μ^{mn} (both considered as independent fields) by the same "tetrad postulate" as in eq. (1) (thus formally: the completely covariant derivative of the tetrad field vanishes)

$$\partial_\mu e_\nu^m + \omega_\mu{}^{mn} e_{n\nu} - \Gamma^\alpha_{\nu\mu} e_\alpha^m = 0 \tag{4}$$

then a direct but tedious calculation yields the following interesting result

$$R_{\mu\nu mn}(\omega) = R^\alpha{}_{\tau\mu\nu}(\Gamma)\, e_{m\alpha} e_n{}^\tau. \tag{5}$$

In particular, one can thus write for the Hilbert action with $e = \det e_\mu^m$ *

$$\mathcal{L}^{(2)} = -\frac{1}{2\kappa^2}\sqrt{g}R(g,\Gamma) = -\frac{1}{2\kappa^2}eR(e,\omega)$$
$$R(e,\omega) = e^{m\nu}e^{n\mu}R_{\mu\nu mn}(\omega). \tag{6}$$

The factor $\frac{1}{2}$ is a matter of convention (the kinetic term occurs then with the canonical normalization), but the κ^{-2} is needed to give the action the correct dimension. It is this form, in terms of e and ω, which we will take as bosonic part for the gauge action of supergravity. Let us remark that if one solves from eq. (6) (thus without matter) the field equation for ω_μ^{mn}, then one finds $\omega_\mu^{mn} = \omega_\mu^{mn}(e)$. With matter, the action with $\omega_\mu^{mn}(e)$ yields a different result from the action which one obtains if one starts with ω_μ^{mn} as an independent field and then eliminates it by solving $\delta I/\delta\omega_\mu^{mn} = 0$. There is no strict rule which forbids any one of these choices. However, as we shall see, it is the choice ω_μ^{mn} (thus Palatini formalism) which leads to the simplest description of supergravity.

We now turn to the fermionic part of the gauge action. We expect that its linearized, free-field part should be quadratic in ψ_μ^a and contain only one derivative. As we shall show in subsection 13, there is only one such action which has positive energy and that is the same action as written down in 1941 by Rarita and Schwinger in their Physical Review article

$$\mathcal{L} = -\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu\partial_\rho\psi_\sigma. \tag{7}$$

It is unique up to field redefinitions $\psi_\sigma = \phi_\sigma + \lambda\gamma_\sigma\gamma\cdot\phi$ with $\lambda \neq -\frac{1}{4}$. We also expect that the fermionic part of the gauge action of supergravity should have the local linearized gauge invariance $\delta\psi_\sigma = \partial_\sigma\epsilon(x)$. Indeed, the Rarita–Schwinger action as in eq. (7) (thus with $\lambda = 0$) has this invariance. This leads one to take for the full nonlinear gravitino action in curved space the extension of eq. (7) to curved space

$$\mathcal{L}^{(3/2)} = -\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu D_\rho\psi_\sigma, \qquad D_\rho\psi_\sigma = (\partial_\rho + \tfrac{1}{2}\omega_\rho^{\ mn}\sigma_{mn})\psi_\sigma. \tag{8}$$

Note that $D_\rho\psi_\sigma$ occurs as a curl, due to the ϵ-symbol. Since $\epsilon^{\mu\nu\rho\sigma}$ is a density (hence always equals 0, ±1), we do not need the density $e = \det e_\mu^m$ in front of the action. The matrix $\gamma_5 = \gamma^1\gamma^2\gamma^3\gamma^4$ is constant, but γ_ν depends on the tetrad field. Let us now discuss the curl $D_\rho\psi_\sigma - D_\sigma\psi_\rho$. In principle one needs two connections in $D_\rho\psi_\sigma$, a Lorentz connection for the flat indices and a connection for the curved indices. The role of these connections is to ensure that $D_\rho\psi_\sigma$ transforms as a tensor as indicated by its indices. One has four choices: ω or $\omega(e)$ for Lorentz connection, and Γ or $\Gamma(g)$ for the other connection. The choice appropriate for local supersymmetry is ω and $\Gamma(g)$. Any other choice would do as well, but one would need extra complicated terms in the action. Since $\Gamma_{\rho\sigma}^\alpha(g)$ is symmetric in ρ, σ it cancels from the curl.

It gives faith in this series of arguments, that also the Noether method which we used to couple the Wess–Zumino model to supergravity, leads to these results. We recall that we had arrived at the matter action

$$I + I^{\mathrm{N}} = \int \mathrm{d}^4x - \frac{e}{2}[(\partial_\mu A\partial_\nu A + \partial_\mu B\partial_\nu B)g^{\mu\nu} + \bar{\lambda}\not{D}(\omega(e))\lambda - \kappa\bar{\psi}_\mu(\not{\partial}(A + \mathrm{i}\gamma_5 B))\gamma^\mu\lambda] \tag{9}$$

* A simple way of writing this action is $\mathcal{L}(D^\mu E^m{}_\nu)(D^\nu E^m{}_\mu) - (D_\mu E^{m\mu})(D_\nu E^{m\nu})$ with $E_\mu^m = \sqrt{e}\,e_\mu^m$.

and that we had found that with $\delta\psi_\mu = (1/\kappa)\,\partial_\mu\epsilon + \text{more}$ and $\delta e^m_\mu = (\kappa/2)\bar\epsilon\gamma^m\psi_\mu$ all terms of order κ^0 cancelled, while also at the order κ level the AA and BB terms cancelled. If one now varies λ in the Noether term and considers the AB terms, one finds

$$\delta(I+I^{\mathrm{N}}) = \int \mathrm{d}^4x\, \frac{\mathrm{i}\kappa}{2}(\bar\psi_\mu\gamma_5\gamma_\tau\epsilon^{\rho\sigma\mu\tau})\,(\gamma_5\epsilon)\,(\partial_\rho A\,\partial_\sigma B). \tag{10}$$

According to the Noether method, one must cancel these terms by adding either terms to the action or to the transformation laws. The only way to cancel eq. (10) is to partially integrate $\partial_\rho A$ or $\partial_\sigma B$, and to interpret $D_\rho(\bar\psi_\mu\gamma_5\gamma_\tau)\epsilon^{\rho\sigma\mu\tau}$ as the gravitino field equation. In that case, adding to $\delta\psi_\mu$ a term $(\mathrm{i}\kappa/4)\gamma_5\epsilon(A\partial_\mu B)$, one cancels part of the terms in eq. (10). (The other part, the terms with $\partial\epsilon$, are cancelled by adding a term of the form $\kappa^2\bar\psi\psi A\overleftrightarrow{\partial}B$ to the action and using the $\delta\psi_\mu = (1/\kappa)\,\partial_\mu\epsilon$ leading variation.) In this way one again finds that the gravitino action is given by eq. (8). Strictly speaking, at this point in the Noether procedure, one would have found in eq. (8) the spin connection $\omega(e)$, but not yet $\omega(e,\psi)$. However, at the next levels in κ one would find the extra terms which turn $\omega(e)$ into $\omega(e,\psi)$. This possibility to derive the gauge action from requiring gauge invariance of the matter action is peculiar to supergravity (it is, for example, not the case in Yang–Mills theory). It is due to the fact that matter fields appear in the gauge field transformation laws if one does not use auxiliary fields. In fact, auxiliary fields can be viewed as Lagrange multipliers which eliminate such matter terms, see subsection 9.

Thus we have arrived at the following proposal for the gauge action of simple supergravity. The action is the sum of the Hilbert action for the tetrad field and the Rarita–Schwinger action for the gravitino field. In both, the spin connection is considered a dependent field. We now turn to a proof that this action is in fact invariant under the local supersymmetry transformation $\delta e^m_\mu = (\kappa/2)\bar\epsilon\gamma^m\psi_\mu$ and $\delta\psi_\mu = (1/\kappa)D_\mu\epsilon$ in which one takes again the same spin connection ω. No independent transformation law for $\omega_\mu^{\ mn}$ need be given since ω is a function of tetrad and gravitino obtained by solving its own field equations.

1.4. *Palatini formalism*

In this subsection we solve the field equation of the spin connection. As a result, we will find torsion induced by gravitinos. The analysis is very simple if one uses a trick*: the rewriting of the Hilbert action as

$$\mathcal{L}^{(2)} = \frac{1}{8\kappa^2}\,\epsilon^{\mu\nu\rho\sigma}\epsilon_{mncd}e^m{}_\mu e^n{}_\nu R_{\rho\sigma}{}^{cd}(\omega) \tag{1}$$

where we used that $\epsilon^{\mu\nu\rho\sigma}\epsilon_{mncd}e^m{}_\mu e^n{}_\nu = 2e(e_c{}^\rho e_d{}^\sigma - e_c{}^\sigma e_d{}^\rho)$. Varying the spin connection, one finds

$$\begin{aligned}
&\delta R_{\rho\sigma}{}^{cd}(\omega) = D_\rho\delta\omega_\sigma{}^{cd} - D_\sigma\delta\omega_\rho{}^{cd}\\
&D_\rho\delta\omega_\sigma{}^{cd} = \partial_\rho\delta\omega_\sigma{}^{cd} + \omega_\rho{}^{ce}\delta\omega_{\sigma e}{}^{d} + \omega_\rho{}^{de}\delta\omega_\sigma{}^{c}{}_{e}.
\end{aligned} \tag{2}$$

Partially integrating, one finds (using the remark under (4) on page 224)

* This trick is really writing differential forms with the indices explicitly shown.

$$\delta\mathcal{L}^{(2)} = \frac{1}{2\kappa^2}\epsilon^{\mu\nu\rho\sigma}\epsilon_{mncd}(D_\sigma e^m_\mu)e^n_\nu\delta\omega_\rho{}^{cd}$$

$$D_\sigma e^m_\mu = \partial_\sigma e^m_\mu + \omega_\sigma{}^{mn}e_{n\mu}. \tag{3}$$

Note that the derivative D_σ always contains only the spin connection, but not a connection $\Gamma^\alpha_{\mu\nu}$. As a result, $D_\mu e^m_\nu$ is non-zero.

Next we vary the spin connection in the Rarita–Schwinger action.

$$\delta\mathcal{L}^{3/2} = -\tfrac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar\psi_\mu\gamma_5\gamma_\nu\sigma_{cd}\psi_\sigma)(\delta\omega_\rho{}^{cd}). \tag{4}$$

Since $\bar\psi_\mu\gamma_5\gamma_\nu\sigma_{cd}\psi_\sigma$ can be written as vector terms and axial vector terms by decomposing $\gamma_\nu\sigma_{cd}$ (see appendix A)

$$\bar\psi_\mu\gamma_5\gamma_\nu\sigma_{cd}\psi_\sigma = \tfrac{1}{2}\bar\psi_\mu\gamma_5(e_{c\nu}\gamma_d - e_{d\nu}\gamma_c)\psi_\sigma + \tfrac{1}{2}e^b{}_\nu\epsilon_{bcdm}\bar\psi_\mu\gamma^m\psi_\sigma \tag{5}$$

and since $\bar\psi_\mu\gamma_5\gamma_d\psi_\sigma$ is symmetric in μ and σ while $\bar\psi_\mu\gamma^a\psi_\sigma$ is antisymmetric (see appendix C), we find

$$\delta\mathcal{L}^{3/2} = -\tfrac{1}{8}\epsilon^{\mu\nu\rho\sigma}\epsilon_{ncdm}(\bar\psi_\mu\gamma^m\psi_\sigma)e^n{}_\nu\delta\omega_\rho{}^{cd}. \tag{6}$$

Comparison of eqs. (6) and (3) thus yields

$$D_\mu e^m{}_\nu - D_\nu e^m{}_\mu = \frac{\kappa^2}{2}(\bar\psi_\mu\gamma^m\psi_\nu) \tag{7}$$

for the field equation of the spin connection. Due to our rewriting of $\mathcal{L}^2$ as in eq. (1), one can easily read off this result, since in eq. (6) and in eq. (1) the two ϵ-symbols are in common.

With the first tetrad postulate (which is really a definition of $\omega(e)$) that $\partial_\mu e^m{}_\nu + \omega_\mu{}^m{}_\nu(e) - \Gamma^\alpha_{\mu\nu}(g)e^m_\alpha = 0$, one can proceed to solve from eq. (7) the spin connection ω_μ^{mn} itself. To this purpose it is useful to introduce the contorsion tensor κ_μ^{mn} by

$$\omega_\mu{}^{mn} = \omega_\mu{}^{mm}(e) + \kappa_\mu{}^{mn}$$

$$\omega_{\mu mn}(e) = \tfrac{1}{2}e_m{}^\nu(\partial_\mu e_{n\nu} - \partial_\nu e_{n\mu}) - \tfrac{1}{2}e_n{}^\nu(\partial_\mu e_{m\nu} - \partial_\nu e_{m\mu}) - \tfrac{1}{2}e_m{}^\rho e_n{}^\sigma(\partial_\rho e_{c\sigma} - \partial_\sigma e_{c\rho})e^c{}_\mu. \tag{8}$$

With (7) and the first tetrad postulate, one finds

$$\partial_\mu e^m{}_\nu + \omega_\mu{}^m{}_\nu(e) - (\mu \leftrightarrow \nu) = 0$$

$$\kappa_{\mu m\nu} - \kappa_{\nu m\mu} = \frac{\kappa^2}{2}\bar\psi_\mu\gamma_m\psi_\nu. \tag{9}$$

This equation is solved in the same way as one solves in general relativity for $\Gamma^\alpha_{\mu\nu}(g)$ in terms of $\partial_\mu g_{\nu\rho}$, that is to say, one considers the identity

$$(\kappa_{\mu m\nu} - \kappa_{\nu m\mu}) + (\kappa_{m\mu\nu} - \kappa_{\nu\mu m}) + (\kappa_{m\nu\mu} - \kappa_{\mu\nu m}) = 2\kappa_{\mu m\nu}. \tag{10}$$

Substituting (9), one arrives at

$$\omega_{\mu mn}(e,\psi)=\omega_{\mu mn}(e)+\frac{\kappa^2}{4}(\bar\psi_\mu\gamma_m\psi_n-\bar\psi_\mu\gamma_n\psi_m+\bar\psi_m\gamma_\mu\psi_n). \tag{11}$$

Using the second tetrad postulate, $\partial_\mu e^a_\nu+\omega_\mu{}^a{}_\nu-\Gamma^\alpha_{\nu\mu}e^a_\alpha=0$ one can obtain an expression for $\Gamma^\alpha_{\mu\nu}$ from eq. (11). In general one defines torsion as the antisymmetric part of $\Gamma^\alpha_{\mu\nu}$

$$S^\alpha_{\mu\nu}=\tfrac{1}{2}(\Gamma^\alpha_{\mu\nu}-\Gamma^\alpha_{\nu\mu}). \tag{12}$$

Clearly, $\Gamma^\alpha_{\mu\nu}-\Gamma^\alpha_{\nu\mu}=-\kappa_\mu{}^\alpha{}_\nu+\kappa_\nu{}^\alpha{}_\mu$, so that the torsion is given by

$$S^\alpha_{\mu\nu}=-\frac{\kappa^2}{4}(\bar\psi_\mu\gamma^\alpha\psi_\nu). \tag{13}$$

In a similar manner as in eq. (10), one can solve from $D_\mu(\omega(e))e^m_\nu-\mu\leftrightarrow\nu=0$ the dependence of $\omega^{ab}_\mu(e)$ on the tetrad field. (Another way is to solve directly the tetrad postulate $\partial_\mu e^m{}_\nu+\omega_\mu{}^{mn}(e)e_{n\nu}-\Gamma^\alpha_{\mu\nu}(g)e^m{}_\alpha=0$.) Putting this result together with eq. (11), one finds

$$\omega_{\mu mn}(e,\psi)=\tfrac{1}{2}(R_{\mu n,m}-R_{\mu m,n}+R_{mn,\mu})$$
$$R_{\mu\nu,m}=-\partial_\mu e_{m\nu}+\partial_\nu e_{m\mu}+\frac{\kappa^2}{2}\bar\psi_\mu\gamma_m\psi_\nu. \tag{14}$$

The symbol $R_{\mu b,m}$ is defined to be $e_b{}^\nu R_{\mu\nu,m}$, similarly for the other two R-functions. Clearly, the spin connection $\omega^{mn}_\mu(e,\psi)$ is *supercovariant*, by which is meant that if one transforms it under local supersymmetry transformations, then it contains no $\partial\epsilon$ terms. Indeed, the terms $(-\frac{1}{2}\partial_\mu\bar\epsilon)\gamma^m\psi_\nu$ coming from $\delta(-\partial_\mu e^m{}_\nu)$ cancel against the $\partial\epsilon$ terms from $\delta\psi_\mu$. The symbols $R_{\mu\nu,m}$ are supercovariant by themselves. That the solution of $\delta I/\partial\omega=0$ is supercovariant is an accident. For example, in the extended supergravities this equality does not hold, see subsection 6.2.

1.5. Flat supergravity with torsion

In Einstein–Cartan theory, the Hilbert action has a well-known symmetry $\int d^4x\, eR(e,\omega(e)+\tau)=\int d^4x\, e[R(e,\omega(e))-\tau_{\mu\nu\rho}\tau^{\rho\nu\mu}+(\tau^\lambda{}_{\lambda\mu})^2]$ under $\omega_\mu{}^{ab}\to\omega_\mu{}^{ab}+\tau_\mu{}^{ab}$ with $\tau_\mu{}^{ab}=-\tau_\mu{}^{ba}$ but further arbitrary. The proof is trivial if one writes the terms linear in τ in $R_{\mu\nu ab}(\omega(e)+\tau)$ as $D_\mu(\omega(e))\tau_{\nu ab}-\mu\leftrightarrow\nu$ since one may add a Christoffel connection to $D_\mu\tau_{\nu ab}$ (because it cancels anyhow in the curl) and having thus obtained the full covariant derivative, one can partially integrate to obtain a total derivative.

In supergravity a similar identity holds. It is a matter of a small computation to show that [207]

$$\mathcal{L}^{(2)}(e,\omega(e,\psi)+\tau)+\mathcal{L}^{3/2}(e,\psi,\omega(e,\psi)+\tau)$$
$$=\mathcal{L}^{(2)}(e,\omega(e,\psi))+\mathcal{L}^{3/2}(e,\psi,\omega(e,\psi))+\frac{1}{2\kappa^2}(\tau_{\mu\nu\rho}\tau^{\rho\nu\mu}-(\tau^\lambda{}_{\lambda\mu})^2)+\text{total derivative}. \tag{1}$$

Since $\omega(e,\psi)$ is an extremum of the action, terms linear in τ cancel.

One can now take a particular choice for τ, namely $\tau_\mu{}^{ab} = -\omega_\mu{}^{ab}(e, \psi)$. In this case, the Hilbert action vanishes, and the gravitino action loses its spin connections. From the result for $\omega_\mu{}^{ab}$ in terms of the curls $\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu - (\kappa^2/2)\bar\psi_\mu\gamma^a\psi_\nu$ in eq. (14) of the last section one finds that one can rewrite the action of supergravity as an action without curvature R but with torsion terms

$$I = \int d^4x\, [-\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar\psi_\mu\gamma_5\gamma_\nu\partial_\rho\psi_\sigma - \tfrac{1}{8}R^2_{\mu\nu a} - R_{\mu\nu a}R^{a\nu\mu} + \tfrac{1}{2}R^2_{\nu\lambda\lambda}]. \tag{2}$$

The objects $R_{\mu\nu a} = -\partial_\mu e_{a\nu} + \partial_\nu e_{a\mu} + (\kappa^2/2)\bar\psi_\mu\gamma_a\psi_\nu$ can clearly be interpreted as the supercovariantized torsion tensor $-D_\mu e_{a\nu} + D_\nu e_{a\mu} + (\kappa^2/2)\bar\psi_\mu\gamma_a\psi_\nu$ with vanishing spin connection. In other words, one can reinterpret supergravity (just as general relativity) as a teleparallelism theory: since $\omega_\mu{}^{ab}$ can be considered as being zero, one can define parallel transport over finite distances and not only in an infinitesimal neighborhood. However, this is simply a rewriting of the same theory, and is rather a curiosity than a result of fundamental significance.

1.6. The 1.5 order formalism

The first proof of gauge invariance of the action used second order formalism. By starting with $\omega_\mu^{ab}(e)$, Freedman, van Nieuwenhuizen and Ferrara found that extra $\bar\psi\psi$ terms in $\delta\psi_\mu$ and extra $\kappa^2(\bar\psi\psi)^2$ terms in the action are needed by the Noether method to obtain complete invariance [500]. The action and transformation laws were the same as if one had started with $\mathcal{L} = \mathcal{L}^{(2)} + \mathcal{L}^{(3/2)}$ and $\delta\psi_\mu = \kappa^{-1}D_\mu(\omega)\epsilon$, and replaced $\omega_\mu{}^{ab}$ by the $\omega_\mu{}^{ab}(e, \psi)$ which solves $\delta I/\delta\omega_\mu{}^{ab} = 0$. From the expression derived before for $\omega_{\mu ab}(e, \psi)$, one finds for the variation of $\omega_{\mu ab}(e, \psi)$ according to the chain rule (thus in second order formalism)

$$\delta\omega_{\mu ab}(\text{second order}) = \frac{\kappa}{4}(\bar\epsilon\gamma_b\psi_{\mu a} - \bar\epsilon\gamma_a\psi_{\mu b} - \bar\epsilon\gamma_\mu\psi_{ab}) \tag{1}$$

where $\psi_{ab} = e_a{}^\mu e_b{}^\nu(D_\mu\psi_\nu - D_\nu\psi_\mu)$ and $\delta e^m{}_\mu = (\kappa/2)\bar\epsilon\gamma^m\psi_\mu$.

A reformulation using first order formalism simplifies the action. By taking $\omega_\mu{}^{ab}$ to be an independent field from the start, Deser and Zumino [120] showed that one can find a law for $\delta\omega_{\mu,ab}$ such that $\mathcal{L}^{(2)} + \mathcal{L}^{(3/2)}$ viewed as a function of $e^a{}_\mu$, $\psi^a{}_\mu$ and $\omega_\mu{}^{ab}$ is invariant. In addition to the previous results $\delta e^a{}_\mu = (\kappa/2)\bar\epsilon\gamma^a\psi_\mu$ and $\delta\psi_\mu = \kappa^{-1}D_\mu(\omega)\epsilon$, they found that one needs

$$\delta\omega_{\mu ab}(\text{first order}) = -\tfrac{1}{4}\bar\epsilon\gamma_5\gamma_\mu\tilde\psi_{ab} + \tfrac{1}{8}\bar\epsilon\gamma_5(\gamma^\lambda\tilde\psi_{\lambda b}e_{a\mu} - \gamma^\lambda\tilde\psi_{\lambda a}e_{b\mu}),$$
$$\tilde\psi_{ab} = \epsilon_{ab}{}^{cd}\psi_{cd}. \tag{2}$$

Since, as we shall see, the gravitino field equation implies

$$\psi_{\mu\nu} + \tfrac{1}{2}\gamma_5\tilde\psi_{\mu\nu} = 0, \qquad \gamma^\lambda\tilde\psi_{\lambda\mu} = 0, \tag{3}$$

it follows that on-shell (1) and (2) are equivalent. Off-shell they differ, however.

The simplest formulation of supergravity is undoubtedly a mixed case, combining the virtues of second and first order formalism [517, 586]. The basic observation is so obvious that it often causes problems to understand it. If the spin connection (or any field for that matter!) satisfies $\delta I/\delta\omega_{\mu ab} = 0$, then in the variation of the action one has according to the chain rule

$$\delta I(e, \psi, \omega(e, \psi)) = \delta e \left.\frac{\delta I}{\delta e}\right|_{\psi,\omega(e,\psi)} + \delta\psi \left.\frac{\delta I}{\delta \psi}\right|_{e,\omega(e,\psi)} + \left.\frac{\delta I}{\delta\omega}\right|_{e,\psi} \left(\cancel{\frac{\delta\omega(e, \psi)}{\delta\psi}}\,\delta\psi + \cancel{\frac{\delta\omega(e, \psi)}{\delta e}}\,\delta e\right). \tag{4}$$

Since, however, $\omega_{\mu ab}(e, \psi)$ satisfies its own field equation, one can drop the last term due to $\delta I/\delta\omega \equiv 0$ *identically* after inserting $\omega = \omega(e, \psi)$. Thus, one can replace in the variation of the action the complicated expression which results when one applies the chain rule to $\omega(e, \psi)$ by zero! In other words, *one only needs to vary the explicit tetrad fields and gravitino fields*, but, although $\omega(e, \psi)$ is certainly not invariant, one may put $\delta\omega = 0$ in the action. This is the same result as found in subsection 2 by gauging the super Poincaré algebra, where we found that $\delta\omega = 0$. Hence, it is the 1.5 order formalism that makes contact with group theory.

It should be noted that in the extended supergravities, this observation has been crucial in obtaining the actions. When one deals not with actions, but, for example, with gauge algebras, then one still needs the non-zero law $\delta\omega(e, \psi)$ in eq. (1).

One small technical detail in our notation. When we write ψ_{ab} we mean $e_a{}^\mu e_b{}^\nu(D_\mu\psi_\nu - D_\nu\psi_\mu)$ and not $D_a\psi_b - D_b\psi_a$ with $\psi_b = e_b{}^\mu\psi_\mu$. These two differ since $D_\mu e_\nu$ is non-zero.

The result in (2) will be needed when we discuss constrained Hamiltonian systems, hence we briefly discuss its derivations. Varying the spin connections in the Einstein action, one finds (see subsection 4, eq. (3))

$$\delta\omega_\nu{}^{mn}\,(\epsilon^{\nu\mu\rho\sigma}\epsilon_{mnrs})(D_\mu e^r{}_\rho)(\tfrac{1}{2}e^s{}_\sigma). \tag{5}$$

Varying $\delta\bar\psi_\mu = (1/\kappa)D_\mu\bar\epsilon$ in the Rarita–Schwinger action and partially integrating, one picks up a term

$$\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}(D_\mu e^m{}_\nu)(\bar\epsilon\gamma_5\gamma_m D_\rho\psi_\sigma). \tag{6}$$

The sum of both terms must vanish, and one solves $\delta\omega_{\mu mn}$ from

$$\begin{aligned} &\delta\tilde\omega_{\rho m\sigma} - \delta\tilde\omega_{\sigma m\rho} = -\bar\epsilon\gamma_5\gamma_m\psi_{\rho\sigma}\,, \\ &\delta\tilde\omega_{\rho ab} = \epsilon_{abcd}\delta\omega_\rho{}^{cd} \end{aligned} \tag{7}$$

in the same way as one solves for the Christoffel symbol.

1.7. Explicit proof of gauge invariance

We now present an explicit proof of the gauge invariance of the gauge action. Using the 1.5 order formalism, we must vary the fields indicated by arrows. From now on we will put $\kappa = 1$,

$$\mathcal{L} = -\tfrac{1}{2}(\underset{\uparrow}{e}\,\underset{\uparrow}{e^{a\nu}}\,\underset{\uparrow}{e^{b\mu}})R_{\mu\nu ab}(\omega) - \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\underset{\uparrow}{\bar\psi_\mu}\gamma_5\underset{\uparrow}{\gamma_\nu}D_\rho(\omega)\underset{\uparrow}{\psi_\sigma}. \tag{1}$$

As we shall see, the variation of the Hilbert action yields as always the Einstein tensor times $\delta e^a{}_\mu$. The variation of the gravitino fields yields the commutator $[D_\rho, D_\sigma]\epsilon$, hence a curvature times ϵ. Since there are not enough indices available, one is guaranteed to find again the Einstein tensor times an expression of the form $\psi\epsilon$. Finally, as we shall see, the contribution of $\delta(\gamma_\nu)$ cancels the extra terms due to partially integrating $\delta\bar\psi_\mu = D_\mu\bar\epsilon$. We now give the details.

The variation of the explicit tetrads in the Hilbert action yields with $\delta e^a{}_\mu = \frac{1}{2}\bar\epsilon\gamma^a\psi_\mu$

$$\delta\mathscr{L}^{(2)} = \frac{e}{2}(\bar\epsilon\gamma^\nu\psi^a)G_{\nu a}, \qquad G_{\nu a} = R_{\nu a} - \tfrac{1}{2}e_{a\nu}R \tag{2}$$

where $R_{\nu a} = R_{\mu\nu ab}e^{b\mu}$. We have used that $\delta e^{a\nu} = -\frac{1}{2}\bar\epsilon\gamma^\nu\psi^a$ which follows from $\delta(e_{a\mu}e^{a\nu}) = 0$.

The variation of ψ_σ yields a factor

$$[D_\rho, D_\sigma]\epsilon = R_{\rho\sigma ab}(\omega(e,\psi))\tfrac{1}{2}\sigma^{ab}\epsilon. \tag{3}$$

This one proves easily, using that the Lorentz generators σ^{ab} satisfy $[\sigma^{ab}, \sigma^{cd}] = \delta^{bc}\sigma^{ad}$ + three more terms, obtained by antisymmetrizing in (a, b) and (c, d). Varying $\bar\psi_\mu$ and partially integrating $D_\mu\bar\epsilon$, one finds again a commutator $[D_\mu, D_\rho]\psi_\sigma$ plus an extra term due to the fact that D_μ does not commute with γ_ν. For the two curvature terms one finds

$$\delta\mathscr{L}^{3/2} = -\tfrac{1}{8}\epsilon^{\mu\nu\rho\sigma}[\bar\psi_\mu\gamma_5\gamma_\nu\sigma_{cd}\epsilon - \bar\epsilon\gamma_5\gamma_\nu\sigma_{cd}\psi_\mu]R_{\rho\sigma}{}^{cd} \tag{4}$$

after relabeling of the indices in the second term. Using that for Majorana spinors (see appendix C)

$$\bar\epsilon\gamma_5\gamma_\nu\sigma_{cd}\psi_\mu = -\bar\psi_\mu\gamma_5\sigma_{cd}\gamma_\nu\epsilon \tag{5}$$

one finds with $\gamma_\nu\sigma_{cd} + \sigma_{cd}\gamma_\nu = \epsilon_{cdba}\gamma_5\gamma^a e^b{}_\nu$ and using the identity

$$e^b{}_\nu\epsilon^{\nu\mu\rho\sigma}\epsilon_{bcda} = (\delta^{\mu\rho\sigma}_{cda} + \delta^{\rho\sigma\mu}_{cda} + \delta^{\sigma\mu\rho}_{cda} - c \leftrightarrow d)e \tag{6}$$

where $\delta^{\mu\nu\rho}_{abc} = e_a{}^\mu e_b{}^\nu e_c{}^\rho$, that varying the gravitino fields yields

$$\delta\mathscr{L}^{3/2} = \frac{e}{2}(\bar\psi^\mu\gamma^a\epsilon)G_{a\mu}. \tag{7}$$

Since $\bar\epsilon\gamma^a\psi^\mu = -\bar\psi^\mu\gamma^a\epsilon$ (see appendix C), the terms with the Einstein tensor in eqs. (2) and (7) clearly cancel.

Note that since we use 1.5 order formalism, we did not vary the spin connection in $\mathscr{L}^{(2)}$. All that is left is the terms due to $\delta(\gamma_\nu)$ and the terms with $(D_\mu\gamma_\nu)$ obtained by partially integrating $D_\mu\bar\epsilon$. Since

$$D_\mu\bar\epsilon = \partial_\mu\bar\epsilon - \tfrac{1}{2}\bar\epsilon\sigma^{ab}\omega_{\mu ab} \tag{8}$$

after partial integration D_μ commutes with γ_5 (since $[\gamma_5, \sigma_{ab}] = 0$) but for the commutator of D_μ with γ_ν one finds

$$\begin{aligned} D_\mu(\gamma_\nu D_\rho\psi_\sigma) - \gamma_\nu D_\mu D_\rho\psi_\sigma &= \{\partial_\mu\gamma_\nu + \tfrac{1}{2}\omega_\mu{}^{ab}[\sigma_{ab}, \gamma_\nu]\}D_\rho\psi_\sigma \\ &= (D_\mu e^a{}_\nu)\gamma_a D_\rho\psi_\sigma. \end{aligned} \tag{9}$$

In fact, as illustrated in this example, one has generally $[D_\mu, \gamma_\nu] = (D_\mu e^a{}_\nu)\gamma_a$. Thus the remaining terms are

$$\delta\mathcal{L}^{3/2}(\text{rest}) = -\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\psi}_\mu\gamma_5\gamma_a D_\rho\psi_\sigma)(\tfrac{1}{2}\bar{\epsilon}\gamma^a\psi_\nu) + \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}\gamma_5\gamma_a D_\rho\psi_\sigma)(D_\mu e^a{}_\nu). \tag{10}$$

For the second term we substitute the torsion equation of subsection 4 eq. (7) and find

$$\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}\gamma_5\gamma_a D_\rho\psi_\sigma)(\tfrac{1}{4}\bar{\psi}_\mu\gamma^a\psi_\nu). \tag{11}$$

For the first term in eq. (10) we use a Fierz rearrangement as discussed in appendix D. The general formula yields

$$-\tfrac{1}{4}(\bar{\psi}_\mu O_j\psi_\nu)(\tfrac{1}{2}\bar{\epsilon}\gamma^a)O_j(-\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_a D_\rho\psi_\sigma) \tag{12}$$

but since only for $O_j = \gamma_\tau$ and $2\mathrm{i}\sigma_{\alpha\beta}$ the factor $\bar{\psi}_\mu O_j\psi_\nu$ is antisymmetric in μ and ν while $\gamma^a\sigma_{\alpha\beta}\gamma_a = 0$ (see appendix A), one finds for the first term in eq. (10) using $\gamma^a\gamma_\tau\gamma_5\gamma_a = 2\gamma_\tau\gamma_5$

$$\tfrac{1}{8}(\bar{\psi}_\mu\gamma_\tau\psi_\nu)\epsilon^{\mu\nu\rho\sigma}(\bar{\epsilon}\gamma^\tau\gamma_5 D_\rho\psi_\sigma). \tag{13}$$

Clearly, eq. (13) plus eq. (11) vanish.

This concludes the proof of gauge invariance. We have shown that the action in eq. (1) is invariant under

$$\delta e^a{}_\mu = \tfrac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu, \qquad \delta\psi_\mu = D_\mu(\omega(e,\psi))\epsilon. \tag{14}$$

In the proof we have used 1.5 order formalism twice: Once by not varying $\omega(e,\psi)$ in eq. (1) and once more by using the torsion equation in eq. (10).

One can rewrite supergravity in terms of differential forms, and thus make the structure more transparent. As an example we reproduce the preceding proof.

The action is written as

$$\mathcal{L} = -\tfrac{1}{2}\eta_{kl} \wedge \Omega^{kl} - \tfrac{1}{2}\bar{\psi} \wedge \gamma_5\gamma \wedge D\psi \tag{15}$$

where

$$\eta_{kl} = \tfrac{1}{2}\theta^m \wedge \theta^n(\epsilon_{klmn}\det e), \quad \psi = \psi_n\theta^n, \quad \gamma = \gamma_n\theta^n,$$
$$D\psi = \mathrm{d}\psi + \tfrac{1}{2}\omega_{ab}\sigma^{ab}\psi, \quad \omega_{ab} = \omega_{mab}\theta^m \quad \text{and} \quad \Omega^{kl} = \tfrac{1}{2}R_{mn}{}^{kl}\theta^m \wedge \theta^n.$$

One may think of θ^n as $e^n{}_\mu\,\mathrm{d}x^\mu$.

Since $\delta D\psi = D\delta\psi$ and $\delta\Omega^{kl} = D\delta\omega^{kl}$ one has upon partial integration using $\delta\psi = D\epsilon$,

$$\begin{aligned}\delta\mathcal{L} = &-\tfrac{1}{2}\delta\theta^m \wedge \eta_{klm} \wedge \Omega^{kl} - \tfrac{1}{2}\eta_{klm} \wedge D\theta^m \wedge \delta\omega^{kl} - \bar{\epsilon}\gamma_5\gamma \wedge DD\psi - \tfrac{1}{2}\bar{\epsilon}\gamma_5\gamma_m D\psi \wedge D\theta^m\\ &-\tfrac{1}{2}\bar{\psi} \wedge \gamma_5\gamma_m D\psi \wedge \delta\theta^m - \tfrac{1}{2}(\bar{\psi} \wedge \gamma_5\gamma \wedge \sigma_{kl}\psi)\delta\omega^{kl}\end{aligned} \tag{16}$$

since $\eta_{klm} = \theta^m\eta_{kl}$. Now $DD\psi = \sigma_{ab}\psi \wedge \Omega^{ab}$ and evaluating, as before, $\bar{\psi} \wedge \gamma_5\gamma \wedge \sigma_{ab}\psi$ one finds a term with the Einstein tensor $\eta_{klm} \wedge \Omega^{kl} \wedge \bar{\epsilon}\gamma^m\psi$ and a term with the dual of the Riemann tensor $\bar{\epsilon}\gamma_5\gamma \wedge D\psi \wedge \eta_{ab}\Omega^{ab}$. Using $\delta\theta^m = \bar{\epsilon}\gamma^m\psi$ the terms with the Einstein tensor cancel. The ω field equation expresses the torsion $D\theta^m$ ($\sim D_\mu e^a{}_\nu - D_\nu e^a{}_\mu$) in terms of $\bar{\psi}\gamma\psi$

$$D\theta^m \sim \eta^{klm}\bar{\psi} \wedge \gamma_5\gamma\sigma_{kl}\psi \sim \bar{\psi}\gamma^m\psi. \tag{17}$$

Thus one is left with

$$\delta\mathscr{L} = \bar{\epsilon}\gamma_5\gamma \wedge D\psi \wedge \eta_{ab}\Omega^{ab} - \tfrac{1}{2}\bar{\epsilon}\gamma_5\gamma_m \wedge D\psi D\theta^m - \tfrac{1}{2}\bar{\psi} \wedge \gamma_5\gamma_m D\psi \wedge \bar{\epsilon}\gamma^m\psi. \tag{18}$$

From here on, the proof proceeds in the same way as before.

The Lagrangian density varies into the following total derivative

$$\delta\mathscr{L} = \partial_\mu\kappa^\mu, \qquad \kappa^\mu = -\bar{\epsilon}\gamma^\mu\sigma^{\rho\sigma}D_\rho\psi_\sigma. \tag{19}$$

This shows clearly that supersymmetry is not an internal symmetry, but a spacetime symmetry, just like general coordinate transformations, where $\delta\mathscr{L} = \partial_\mu(\mathscr{L}\xi^\mu)$. In both cases $\delta\mathscr{L} = 0$ on-shell.

Finally, we note that the linearized action is, of course, invariant under global transformations which are linearized in the fields. However, not every action quadratic in fields which is invariant under global transformations which are linear in fields, can always be extended to an action which is invariant under local transformations (which usually then becomes nonlinear in fields). A counter example [478] is $N = 1$ supergravity in $d = 11$ dimensions with as fields a tetrad, a gravitino and a photon $A_{\mu\nu\rho\sigma\nu}$.

1.8. Auxiliary fields in global supersymmetry

In globally supersymmetric models such as the Wess–Zumino model, one needs auxiliary fields in order that the algebra of global symmetries closes. It is only with auxiliary fields that one can obtain a tensor calculus, see section 4, while also the quantum theory becomes much simpler with auxiliary fields, since in this case the transformation laws are linear and one can easily derive Ward identities for one-particle irreducible Green's functions. For supergravity the auxiliary fields are not only needed for the same reasons, but in addition one needs them in order to apply the usual covariant quantization methods of Feyman, De Witt, Faddeev–Popov, 't Hooft and Veltman and others.

Let us begin with global sypersymmetry. That one needs two scalar auxiliary fields for the Wess–Zumino model might be expected by counting field components

4 fermionic components λ^α

2 bosonic components A, B.

Thus one needs at least two extra bosonic fields in order to have equal numbers of bosonic and fermionic field components off mass shell. The action reads

$$\mathscr{L} = -\tfrac{1}{2}[(\partial_\mu A)^2 + (\partial_\mu B)^2 + \bar{\lambda}\not{\partial}\lambda - F^2 - G^2] \tag{1}$$

and it is rather easy to show that it is invariant under

$$\delta A = \tfrac{1}{2}\bar{\epsilon}\lambda, \quad \delta B = -\tfrac{1}{2}\mathrm{i}\bar{\epsilon}\gamma_5\lambda, \quad \delta F = \tfrac{1}{2}\bar{\epsilon}\not{\partial}\lambda, \quad \delta G = \tfrac{1}{2}\mathrm{i}\bar{\epsilon}\gamma_5\not{\partial}\lambda$$
$$\delta\lambda = \tfrac{1}{2}(\not{\partial}(A - \mathrm{i}\gamma_5 B))\epsilon + \tfrac{1}{2}(F + \mathrm{i}\gamma_5 G)\epsilon. \tag{2}$$

Since the dimensions of the nonpropagating fields F and G are 2, one must have a law $\delta F \sim (\partial\lambda)\epsilon$ and since $F = G = 0$ are field equations, and field equations rotate into field equations, one understands the appearance of $\not{\partial}\lambda$ in δF and δG.*

Let us now consider the commutator algebra. There are three global symmetries: translations P, Lorentz rotations L, and global supersymmetry Q. To give an easy example

$$[\delta_Q(\epsilon), \delta_P(\xi^\mu)]A = \tfrac{1}{2}\xi^\mu\partial_\mu(\bar{\epsilon}\lambda) - \tfrac{1}{2}\bar{\epsilon}\xi^\mu\partial_\mu\lambda = 0 \tag{3}$$

reflecting that $[Q^\alpha, P_\mu] = 0$. Next consider

$$[\delta_L(\lambda^{mn}), \delta_Q(\epsilon)]A = \tfrac{1}{2}\bar{\epsilon}(\tfrac{1}{2}\lambda^{mn}\sigma_{mn}\lambda) \tag{4}$$

reflecting that $[Q^\alpha, M_{\mu\nu}] = (\sigma_{\mu\nu})^\alpha{}_\beta Q^\beta$. More explicitly: $\delta_Q(\epsilon)A = [A, \bar{\epsilon}Q]$ and $\delta_L(\lambda^{ab})A = [A, \tfrac{1}{2}\lambda^{ab}M_{ab}]$ and with the Jacobi identity

$$[[A, \bar{\epsilon}Q], \tfrac{1}{2}\lambda^{ab}M_{ab}] - [[A, \tfrac{1}{2}\lambda^{ab}M_{ab}], \bar{\epsilon}Q] = [A, [\bar{\epsilon}Q, \tfrac{1}{2}\lambda^{ab}M_{ab}]]$$

one finds using the (anti) commutators of the super-Poincaré algebra that

$$[\delta_L(\lambda^{mn}), \delta_Q(\epsilon)] = \delta_Q(-\tfrac{1}{2}\lambda^{mn}\sigma_{mn}\epsilon) \tag{5}$$

in agreement with eq. (4).

In general one has for generators X_A and X_B and structure constants $f_{AB}{}^C$ defined by $[X_A, X_B] = f_{AB}{}^C X_C$ a realization of the algebra on fields

$$[\delta_B(\alpha^B), \delta_A(\alpha^A)]\phi = \delta_C(\alpha^C = -f_{AB}{}^C\alpha^B\alpha^A)\phi \tag{6}$$

where ϕ can be any field or combination of fields and α^A are parameters.

For the commutator of two supersymmetry variations one finds without F and G

$$[\delta(\epsilon_1), \delta(\epsilon_2)]A = \tfrac{1}{2}(\bar{\epsilon}_2\gamma^\mu\epsilon_1)\partial_\mu A, \quad \text{idem } B.$$

For λ one finds, however,

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\lambda = \tfrac{1}{4}(\bar{\epsilon}_1\partial_\mu\lambda)(\gamma^\mu\epsilon_2) - \tfrac{1}{4}(\bar{\epsilon}_1\gamma_5\partial_\mu\lambda)(\gamma^\mu\gamma_5\epsilon_2) - 1 \leftrightarrow 2. \tag{7}$$

Upon a Fierz rearrangement, this becomes

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\lambda = -\tfrac{1}{16}(\bar{\epsilon}_1 O_j\epsilon_2)(\gamma^\mu O_j\partial_\mu\lambda - \gamma^\mu\gamma_5 O_j\gamma_5\partial_\mu\lambda) - (1 \leftrightarrow 2). \tag{8}$$

* One can replace A, F and G by one antisymmetric tensor gauge field $t_{\mu\nu}$, which has three components. In that case no auxiliary fields are needed to close the algebra, but mass terms and self-couplings are difficult to write down.

Clearly, only the tensors with $O_j = \gamma_\tau$ survive since $\bar\epsilon_1 O_j \epsilon_2$ is only antisymmetric for $O_j = \gamma_\tau$ or $\sigma_{\mu\nu}$, but $\gamma_5\sigma_{\mu\nu}\gamma_5 = \sigma_{\mu\nu}$. One finds

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\lambda = \tfrac{1}{2}(\bar\epsilon_2\gamma^\mu\epsilon_1)\partial_\mu\lambda - \tfrac{1}{4}(\bar\epsilon_2\gamma^\tau\epsilon_1)(\gamma_\tau\not\partial\lambda). \tag{9}$$

Thus, only on-shell where $\not\partial\lambda = 0$, does one find the same result as for A and B.

In principle one could define a new symmetry $\delta'\lambda = \zeta^\tau\gamma_\tau\not\partial\lambda$, $\delta'A = \delta'B = 0$. This would not be sufficient to close the algebra, since new commutators involving δ' will lead to new symmetries δ'', and so on. The only way to obtain a finite-dimensional closed algebra is to introduce F and G. Indeed, their contribution to (8) is

$$\tfrac{1}{4}(\bar\epsilon_1\not\partial\lambda)\epsilon_2 - \tfrac{1}{4}(\bar\epsilon_1\gamma_5\not\partial\lambda)\gamma_5\epsilon_2 - 1\leftrightarrow 2 \tag{10}$$

and after a Fierz transformation, the $\not\partial\lambda$ terms in eqs. (10) and (9) cancel.

For any set of fields and set of global symmetry operations which closes on-shell, one can try to find a set of extra fields which lead to closure off-shell. In general, these new fields will be nonpropagating in the action, but sometimes (for example in R^2 theories) they propagate. There is some confusion in the literature as to whether auxiliary fields can have derivatives in the action, and can be gauge fields. A simple example due to Ogievetski and Sokatchev suffices to clear this point up. Consider [347]

$$\mathcal{L} = \epsilon^{\mu\nu\rho\sigma}T_{\mu\nu}\partial_\rho A_\sigma. \tag{11}$$

Clearly, the A_σ field equation is $\epsilon^{\mu\nu\rho\sigma}\partial_\rho T_{\mu\nu} = 0$ and can be solved $T_{\mu\nu} = \partial_\mu t_\nu - \partial_\nu t_\mu$. Similarly, the $T_{\mu\nu}$ field equation yields $A_\sigma = \partial_\sigma A$. Also $T_{\mu\nu}$ and A_σ are gauge fields since the action is invariant under $\delta T_{\mu\nu} = \partial_\mu\lambda_\nu - \partial_\nu\lambda_\mu$ or $\delta A_\sigma = \partial_\sigma\Lambda$. In a Hamiltonian formalism, the moments for A_0 and $T_{\mu 0}$ lead to so-called primary constraints $(p_{A_0} - \epsilon^{\mu\nu\rho\sigma}T_{\mu 0} = 0)$ and this shows that there are no dynamical modes associated with A_0 and $T_{\mu 0}$. Also in the propagators one finds no k^{-2} poles. Hence there is no propagation between the sources. Thus, auxiliary fields can be gauge fields and carry derivatives in the action.

In order that the reader may test his understanding, we close this section with a second model of global supersymmetry: the photon–neutrino system. The action is

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}{}^2 - \tfrac{1}{2}\bar\lambda\not\partial\lambda + \tfrac{1}{2}D^2, \quad F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{12}$$

and is invariant under $\delta B_\mu = -\tfrac{1}{2}\bar\epsilon\gamma_\mu\lambda$, $\delta\lambda = \tfrac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}\epsilon + \tfrac{1}{2}\mathrm{i}\gamma_5 D\epsilon$, $\delta D = \tfrac{1}{2}\mathrm{i}\bar\epsilon\gamma_5\not\partial\lambda$. In this case the counting shows that there are four fermionic components λ^α and only three photon components, hence the need for the single auxiliary field D. The reason that there are only three components of B_μ is that there is a gauge invariance in the algebra as follows from the commutator

$$\begin{aligned}&[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)]\phi = \tfrac{1}{2}\bar\epsilon_2\gamma^\mu\epsilon_1\partial_\mu\phi + \delta_{\text{gauge}}(\Lambda = -\tfrac{1}{2}\bar\epsilon_2\not B\epsilon_1)\phi\\ &\delta_{\text{gauge}}(\Lambda)B_\mu = \partial_\mu\Lambda, \quad \delta_{\text{gauge}}(\Lambda)\lambda = \delta_{\text{gauge}}(\lambda)D = 0.\end{aligned} \tag{13}$$

In general there are as many field components absent as there are local gauge parameters. This is of importance for supergravity, to which we now turn.

1.9. Auxiliary fields for the gauge algebra

Auxiliary fields are needed in order that the transformation rules of gauge fields do not depend on matter fields [see subsections 11 and 12]. If they did, one could not add, without further modifications, two matter actions each of which has been coupled to the gauge action in an invariant way, such that the sum is again invariant. The reason is that the gauge field transformation rules of system I would not work for system II and vice-versa. However, if one adds auxiliary fields, then the gauge field transformation rules are always the same, independent of matter fields and valid for any matter coupling system. Let us start by counting how many auxiliary fields we need.

In supergravity there are three local gauge invariances: general coordinate transformations G with parameter ξ^μ, local Lorentz rotations L with parameter λ^{mn} and local supersymmetry transformations Q with parameters ϵ^α. Thus the counting of field components in the gauge action yields the following result

$$\begin{aligned} &16 e^m{}_\mu - 4 \text{ gen. coord.} - 6 \text{ local Lorentz} = 6 \text{ bosonic fields} \\ &16 \psi^a{}_\mu - 4 \text{ local supersymm.} = 12 \text{ bosonic fields.} \end{aligned} \tag{1}$$

Hence there is a mismatch of six bosonic components. This suggests that the gauge algebra will not be closed, and secondly, that one needs $6+n$ bosonic auxiliary field components and n fermionic ones. There exist several sets of auxiliary fields, the most prominent being the set with $n=0$, consisting of an axial vector A_m, a scalar S and a pseudoscalar P.

We begin by giving the transformation rules of this minimal set of auxiliary fields. Then we will show that they close the gauge algebra. Finally, we will discuss how one obtains this result;

$$\begin{aligned} \delta e^m{}_\mu &= \frac{\kappa}{2} \bar\epsilon \gamma^m \psi_\mu \\ \delta \psi_\mu &= \frac{1}{\kappa}\left(D_\mu + \frac{\mathrm{i}\kappa}{2} A_\mu \gamma_5\right)\epsilon - \tfrac{1}{2}\gamma_\mu \eta \epsilon \\ \delta S &= \tfrac{1}{4}\bar\epsilon \gamma \cdot R^{\mathrm{cov}} \\ \delta P &= -\frac{\mathrm{i}}{4} \bar\epsilon \gamma_5 \gamma \cdot R^{\mathrm{cov}} \\ \delta A_m &= \frac{3\mathrm{i}}{4} \bar\epsilon \gamma_5 (R^{\mathrm{cov}}_m - \tfrac{1}{3}\gamma_m \gamma \cdot R^{\mathrm{cov}}) \\ \eta &= -\tfrac{1}{3}(S - \mathrm{i}\gamma_5 P - \mathrm{i}\not{A}\gamma_5). \end{aligned} \tag{2}$$

For once we have explicitly shown the κ dependence; from now on we will again put $\kappa = 1$. The gauge action is invariant under these rules and reads

$$\mathscr{L}(\text{gauge}) = \mathscr{L}^{(2)}(e, \omega) + \mathscr{L}^{3/2}(e, \psi, \omega) - \frac{e}{3}(S^2 + P^2 - A_m^2). \tag{3}$$

In this action S, P, A_m are clearly nonpropagating, and one sees that, as in the Wess–Zumino model, their dimension is 2. Hence they must rotate into the gravitino field equation. The symbol $R^{\mu,\text{cov}}$ is the gravitino field equation $R^\mu = \epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu D_\rho\psi_0$ (which we will discuss in the next subsection) but with the supercovariant derivatives. Hence

$$R^{\mu,\text{cov}} = \epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu\Big(D_\rho\psi_\sigma - \frac{\mathrm{i}}{2}A_\sigma\gamma_5\psi_\rho + \tfrac{1}{2}\gamma_\sigma\eta\psi_\rho\Big). \tag{4}$$

One obtains the supercovariantization $D_\mu{}^{\text{cov}}$ of the derivative $\partial_\mu A$ of some field A transforming as $\delta A = \bar\epsilon B$ by adding the connection $-\bar\psi B$. Indeed, in that case $\delta(D_\mu{}^{\text{cov}}A) = \delta(\partial_\mu A - \bar\psi_\mu B)$ is free from $\partial_\mu\epsilon$ terms. For the gravitino this rule yields all terms in eq. (4), except that we did not try to add connections for the variations $\delta\psi_\alpha = D_\alpha\epsilon$ and $\delta\omega_\rho{}^{mn}$ inside D_ρ for two reasons. First, the curl $[D_\rho, D_\sigma]\epsilon$ is already supercovariant by itself (and covariantizing $\delta\psi_\sigma = D_\sigma\epsilon$ by adding a connection, the whole curl would vanish). Secondly, the spin connection $\omega_\nu{}^{mn}(e,\psi)$ is supercovariant by itself because it was already supercovariant without S, P, A_μ while in $\omega_\mu{}^{mn}(e,\psi)$ no $\partial\psi$ terms appear, so that the extra terms in $\delta\psi_\mu$ cannot produce $\partial\epsilon$ terms in $\delta\omega_\mu{}^{mn}$. One finds upon explicit computation [224]

$$\delta\omega_{\mu ab} = \tfrac{1}{4}\bar\epsilon(\gamma_b\psi_{\mu a}{}^{\text{cov}} - \gamma_a\psi_{\mu b}{}^{\text{cov}} - \gamma_\mu\psi_{ab}{}^{\text{cov}}) + \tfrac{1}{2}\bar\epsilon(\sigma_{ab}\eta + \eta\sigma_{ab})\psi_\mu \tag{5}$$

where $\psi^{\text{cov}}_{\rho\sigma}$ is the curl between parentheses in eq. (4).

Let us now consider the gauge algebra. *Without auxiliary fields*, all commutators on $e^m{}_\mu$, $\psi^a{}_\mu$ close except the commutator of two local sypersymmetry transformations on the gravitino [235]. For this commutator on the tetrad one finds

$$[\delta(\epsilon_1), \delta(\epsilon_2)]e^m{}_\mu = \tfrac{1}{2}\bar\epsilon_2\gamma^m D_\mu\epsilon_1 - 1 \leftrightarrow 2. \tag{6}$$

Guided by the global anticommutator $[\bar\epsilon_1 Q, \bar\epsilon_2 Q] = \tfrac{1}{2}\bar\epsilon_2\gamma^\mu\epsilon_1 P_\mu$, we expect on the right hand side general coordinate transformations with parameters $\xi^\mu = \tfrac{1}{2}\bar\epsilon_2\gamma^\mu\epsilon_1$. Thus we rewrite (6) as

$$\begin{aligned}&\tfrac{1}{2}e^m{}_\lambda\partial_\mu(\bar\epsilon_2\gamma^\lambda\epsilon_1) + \tfrac{1}{2}(\partial_\mu e^n{}_\lambda)(\bar\epsilon_2\gamma^\lambda\epsilon_1) + \tfrac{1}{4}\bar\epsilon_2[\gamma^m, \sigma^{cd}]\epsilon_1\omega_\mu{}^{cd}\\ &\quad = \tfrac{1}{2}\{\partial_\mu(\bar\epsilon_2\gamma^\lambda\epsilon_1)\}e^m{}_\lambda + \tfrac{1}{2}(\bar\epsilon_2\gamma^\lambda\epsilon_1)(\partial_\lambda e^m{}_\mu) + \tfrac{1}{2}(D_\mu e^m{}_\lambda - D_\lambda e^m{}_\mu)\bar\epsilon_2\gamma^\lambda\epsilon_1 + \tfrac{1}{2}\bar\epsilon_2\gamma^\lambda\epsilon_1\omega_\lambda{}^{mn}e_{n\mu}.\end{aligned} \tag{7}$$

The first two terms are clearly a general coordinate transformation on the tetrad, with parameter $\tfrac{1}{2}\bar\epsilon_2\gamma^\lambda\epsilon_1$. The last term we recognize as a local Lorentz rotation with parameter $\tfrac{1}{2}\bar\epsilon_2\gamma^\lambda\epsilon_1\omega_\lambda{}^{mn}$. The remaining terms can be rewritten, using the torsion equation $D_\mu e^m{}_\lambda - D_\lambda e^m{}_\mu = \tfrac{1}{2}\bar\psi_\mu\gamma^m\psi_\lambda$ as $-\tfrac{1}{4}(\bar\epsilon_2\gamma^\lambda\epsilon_1)\bar\psi_\lambda\gamma^m\psi_\mu$ and clearly constitute a local supersymmetry variation of $e^m{}_\mu$ with parameter $-\tfrac{1}{2}(\bar\epsilon_2\gamma^\lambda\epsilon_1)\psi_\lambda$. Hence, we can summarize that, as far as the tetrad is concerned, the following commutator is valid

$$\begin{aligned}&[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_G(\xi^\mu) + \delta_L(\xi^\mu\omega_\mu{}^{mn}) + \delta_Q(-\xi^\mu\psi_\mu),\\ &\xi^\mu = \tfrac{1}{2}\bar\epsilon_2\gamma^\mu\epsilon_1.\end{aligned} \tag{8}$$

This is the local version of the super Poincaré algebra. As one sees, the P in $\{Q, Q\} = P$ is replaced by general coordinate transformations, and this we anticipated in the beginning of this report when we derived supergravity. *But there is more.* One also finds on the right hand side the other two gauge symmetries. We also see that the structure constants defined by this result are field-dependent. One might speak of structure functions. This is a property of supergravity not present in Yang–Mills theory or gravity. The moral of this result is that one cannot simply deduce the local algebra from the global algebra.

We now repeat this calculation for the gravitino, still without auxiliary fields

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\psi_\mu = \tfrac{1}{2}(\sigma_{mn}\epsilon_2)(\delta(\epsilon_1)\omega_\mu{}^{mn}) - 1 \leftrightarrow 2. \tag{9}$$

With the result of subsection 6, eq. (1) for $\delta\omega_\mu{}^{mn}$ without auxiliary fields, one finds after Fierzing so that the parameters ϵ_2 and ϵ_1 are in the same spinor trace

$$-\tfrac{1}{32}(\bar\epsilon_1 O_j \epsilon_2)[2\sigma_{ab}O_j\gamma_b\psi_{\mu a} + \sigma_{ab}O_j\gamma_\mu\psi_{ba}] - 1 \leftrightarrow 2. \tag{10}$$

Using that $\gamma_b\psi_{\mu a} - \gamma_a\psi_{\mu b}$ is equal to $-\gamma_\mu\psi_{ab} + R^\lambda$-terms (see the next subsection) one arrives at

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\psi_\mu = \tfrac{1}{2}(\bar\epsilon_2\gamma^\lambda\epsilon_1)(D_\lambda\psi_\mu - D_\mu\psi_\lambda) + \tfrac{1}{4}(\bar\epsilon_1\gamma^\alpha\epsilon_2)\, V_{\mu\alpha\beta}R^\rho + \tfrac{1}{4}(\bar\epsilon_1\sigma^{\rho\sigma}\epsilon_2)T_{\mu\rho\sigma\lambda}R^\lambda \tag{11}$$

where

$$eV_{\mu\alpha\rho} = \tfrac{1}{4}g_{\mu\rho}\gamma_\alpha + \tfrac{1}{2}e\epsilon_{\mu\alpha\rho\lambda}\gamma_5\gamma^\lambda \tag{12}$$

$$eT_{\mu\rho\sigma\lambda} = g_{\mu\rho}g_{\sigma\lambda} + \tfrac{1}{2}g_{\mu\lambda}\sigma_{\rho\sigma} - \tfrac{1}{2}e\epsilon_{\rho\sigma\mu\lambda}\gamma_5. \tag{13}$$

The functions V and T will be called the nonclosure functions. The first terms in eq. (11) produce again the result in eq. (8). To see this, we rewrite these terms as

$$\tfrac{1}{2}(\bar\epsilon_2\gamma^\lambda\epsilon_1)(\partial_\lambda\psi_\mu) + \tfrac{1}{2}\{\partial_\mu(\bar\epsilon_2\gamma^\lambda\epsilon_1)\}\psi_\lambda - \tfrac{1}{2}\partial_\mu(\bar\epsilon_2\gamma^\lambda\epsilon_1\psi_\lambda) + \tfrac{1}{4}(\bar\epsilon_2\gamma^\lambda\epsilon_1)(\omega_\lambda\cdot\sigma\psi_\mu - \omega_\mu\cdot\sigma\psi_\lambda). \tag{14}$$

One again finds the general result of eq. (8) back, since

$$\begin{aligned} &\delta_G(\xi)\psi_\mu = \xi^\lambda\partial_\lambda\psi_\mu + (\partial_\mu\xi^\lambda)\psi_\lambda, \quad \delta_Q\psi_\mu = D_\mu\epsilon \\ &\delta_L(\lambda)\psi_\mu = \tfrac{1}{2}\sigma\cdot\lambda\psi_\mu, \quad (\sigma\cdot\lambda = \sigma^{mn}\lambda_{mn}). \end{aligned} \tag{15}$$

Thus, without the auxiliary fields S, P, A_μ, the gauge algebra does not close, since there are extra terms proportional to the fermionic field equation in the commutator of two local supersymmetric variations of the fermion. These results are identical to what happened in the Wess–Zumino model.

The other commutators in the gauge algebra close and are given by

$$[\delta_G(\eta^\alpha), \delta_G(\xi^\alpha)] = \delta_G(\xi^\alpha\partial_\alpha\eta^\beta - \eta^\alpha\partial_\alpha\xi^\beta) \tag{16}$$

$$[\delta_L(\omega^{mn}), \delta_G(\eta^\alpha)] = \delta_L(\eta^\alpha\partial_\alpha\omega^{mn}) \tag{17}$$

$$[\delta_L(\omega^{mn}), \delta_L(\Omega^{mn})] = \delta_L(-\omega^m{}_\rho\Omega^{\rho n} + \Omega^m{}_\rho\omega^{\rho n}) \tag{18}$$

$$[\delta_Q(\epsilon^\alpha), \delta_G(\eta^\alpha)] = \delta_Q(\eta^\alpha \partial_\alpha \epsilon) \tag{19}$$

$$[\delta_Q(\epsilon^\alpha), \delta_L(\omega^{mn})] = \delta_Q(\tfrac{1}{2}\omega^{mn}\sigma_{mn}\epsilon). \tag{20}$$

As a little help for the reader, we derive the last relation. $\delta_Q\delta_L e^m{}_\mu = \omega^{mn}\frac{1}{2}\bar{\epsilon}\gamma_n\psi_\mu$ while $-\delta_L\delta_Q e^m{}_\mu = -\frac{1}{2}\bar{\epsilon}\gamma^m\frac{1}{2}\omega\cdot\sigma\psi_\mu$. The commutator $[\delta_Q, \delta_L]e^m{}_\mu$ is equal to $-\frac{1}{4}\bar{\epsilon}\sigma\cdot\omega\gamma^m\psi_\mu$, which is indeed a local supersymmetry variation of the form $\delta e^m{}_\mu = \frac{1}{2}\bar{\epsilon}'\gamma^m\psi_\mu$ with $\bar{\epsilon}' = -\frac{1}{2}\bar{\epsilon}\sigma\cdot\omega$. To find ϵ' we use $\epsilon' = -C^{-1}(\bar{\epsilon}')^{\mathrm{T}}$ and find $\epsilon' = \frac{1}{2}\omega\cdot\sigma\epsilon$. The reader who feels insecure in handling Majorana spinors can always flip $\bar{\epsilon}\sigma\cdot\omega\gamma^m\psi_\mu$ around to $\bar{\psi}_\mu\gamma^m\omega\cdot\sigma\epsilon$ and compare this with $\delta e^m{}_\mu = -\frac{1}{2}\bar{\epsilon}\gamma^m\psi_\mu$. All these structure constants are field independent. With auxiliary fields, *these* commutators remain the same. If one uses first order formalism with $\omega_\mu{}^{mn}$ an independent field (transforming under G and L the same way but under Q with a different law as we discussed before) then eqs. (16)–(20) are unmodified, but for eq. (8) one finds that it closes on the tetrad only modulo the ω field equation, on the gravitino only modulo the ω and ψ equations and on the spin connection only modulo all three field equations.

With auxiliary fields the gauge algebra closes. In this case the $\{Q, Q\}$ anticommutator can be most easily found if one evaluates it on the tetrad; however, the result is valid for all fields

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_G(\xi^\alpha) + \delta_Q(-\xi^\alpha\psi_\alpha) + \delta_L[\xi^\mu\hat{\omega}_\mu{}^{mn} + \tfrac{1}{3}\bar{\epsilon}_2\sigma^{mn}(S - \mathrm{i}\gamma_5 P)\epsilon_1]$$

$$\hat{\omega}_{\mu ab} = \omega_{\mu ab} - \frac{\mathrm{i}}{3}\epsilon_{\mu abc}A^c, \quad \xi^\mu = \tfrac{1}{2}\bar{\epsilon}_2\gamma^\mu\epsilon_1. \tag{21}$$

Hence the structure constants now also depend on the auxiliary fields.

The geometrical meaning of closure of the gauge algebra is the following. Consider all gauge transformations acting in the space spanned by all field components, with arbitrary parameters. Considering gauge transformations as vectors in this function space, they define hypersurfaces. Closure now means that making successive gauge transformations, one never leaves a given hypersurface.

The minimal set of auxiliary fields S, P and A_μ was discovered by Ferrara and van Nieuwenhuizen [525], and by Stelle and West [445]. An earlier nonminimal set by Breitenlohner involved two auxiliary spin 1/2 fields, two axial vector fields, a vector field, and a pseudoscalar and a scalar [68, 69, 160, 70].

Let us discuss how one arrives at this result. The tetrad e^m_μ and gravitino $\psi_\mu{}^a$ fit into a vector superfield (see subsection 4.1) as first observed by Ogievetski and Sokatchev [349]

$$H_\mu(x, \theta) = C_\mu + \bar{\theta}Z_\mu + P_\mu\bar{\theta}\theta + S_\mu\bar{\theta}\mathrm{i}\gamma_5\theta + e^m_\mu\bar{\theta}\gamma_m\mathrm{i}\gamma_5\theta + \bar{\theta}\theta\bar{\theta}\mathrm{i}\gamma_5\psi_\mu + (\bar{\theta}\theta)(\bar{\theta}\theta)A_\mu. \tag{22}$$

The components of H_μ should transform under linear gauge transformations since e^m_μ and $\psi_\mu{}^a$ transform under linear gauge transformations G, L and Q. As we shall explain

$$\delta H_\mu = \bar{D}\gamma_\mu\mathrm{i}\gamma_5\Lambda, \qquad \bar{D}(1+\gamma_5)D\bar{D}(1-\gamma_5)\Lambda = 0 \tag{23}$$

where Λ^α is a spinor superfield and $D^\alpha = \partial/\partial\bar{\theta}_\alpha + \frac{1}{4}(\not{\partial}\theta)^\alpha$. Working out how the components of δH_μ look, one finds after field redefinitions that

$$\delta C_\mu = \hat{C}_\mu, \quad \delta Z_\mu = \hat{Z}_\mu, \quad \delta P_\mu = \hat{P}_\mu \text{ with } \partial_\mu\hat{P}^\mu = 0, \text{ idem } \delta S_\mu,$$

$$\delta e_{m\mu} = \partial_\mu\hat{\xi}_m + \partial_m\hat{\xi}_\mu + \hat{\lambda}_{mn}, \quad \delta\psi = \partial_\mu\hat{\epsilon} \tag{24}$$

where the hatted symbols are functions of the components of Λ^α. One recognizes the expected general coordinate and Lorentz transformation on the tetrad, and the local linear supersymmetry rotation on the gravitino. It follows that one can use the gauge freedom to eliminate in H the components C_μ, Z_μ, and the transversal parts of P_μ, S_μ. Hence, one expects as auxiliary fields, $\partial_\mu S^\mu$, $\partial_\mu P^\mu$ and A_μ.

To find how these fields rotate into each other under linear global supersymmetry transformations, one first performs the transformation $\delta H_\mu = \bar{\epsilon} G H_\mu$ (see subsection 5.3). Since this will produce nonzero C_μ, Z_μ, etc., one subsequently performs a gauge transformation, which restores the gauge $C_\mu = 0$ etc. The sum of both transformations is then the global supersymmetry transformation, which leads to the linearized version of (2).

To explain why local (linear) gauge transformations are of the form $\delta H_\mu = \bar{D} \mathrm{i} \gamma_\mu \gamma_5 \Lambda$ we recall that in linearized general relativity the coupling $\int h_{\mu\nu} T^{\mu\nu}\, \mathrm{d}^4x$ is invariant under $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ since the source satisfies $\partial_\mu T^{\mu\nu} = 0$. In global supersymmetry $T^{\mu\nu}$ is part of a superfield of sources $J^{A\dot{A}}$ (in two-component notation) containing also the axial current and the Noether current of global supersymmetry. These sources satisfy

$$D_A J^{A\dot{A}} = D^{\dot{A}} S \tag{25}$$

where S is a chiral superfield ($D^A S = 0$). (For the definition of D_A, see again subsection 5.3.) If $D_A J^{A\dot{A}} = 0$ would hold, $\int (D_A \Lambda_{\dot{A}} - D_{\dot{A}} \Lambda_A) J^{A\dot{A}} = 0$ and one would expect that $H_{A\dot{A}}$ (the two-component form of H_μ) is invariant under

$$\delta H_{A\dot{A}} = (D_A \Lambda_{\dot{A}} - D_{\dot{A}} \Lambda_A). \tag{26}$$

However, $D_A J^{A\dot{A}} = D^{\dot{A}} S$, hence we must restrict Λ_A such that $\Lambda_{\dot{A}} D^{\dot{A}} S$ vanishes. All these terms appear in the action and using well-known results of global supersymmetry

$$\begin{aligned}\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \Lambda_{\dot{A}}\, D^{\dot{A}} S &= -\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, (D^{\dot{A}} \Lambda_{\dot{A}}) S \\ &= \int \mathrm{d}^4x\, \mathrm{d}^2\bar{\theta}\, D^A D_A [(D^{\dot{A}} \Lambda_{\dot{A}}) S] \\ &= \int \mathrm{d}^4x\, \mathrm{d}^2\bar{\theta}\, (D^A D_A D^{\dot{A}} \Lambda_{\dot{A}}) S\end{aligned} \tag{27}$$

one arrives at the quoted result for δH_μ.

1.10. Field equations and consistency

The field equations for the gauge action are obtained very easily by using 1.5 order formalism. Indeed, in the action

$$\mathcal{L} = -\tfrac{1}{2} e R(e, \omega) - \tfrac{1}{2} \epsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_5 \gamma_\nu D_\rho \psi_\sigma - \frac{e}{3} (S^2 + P^2 - A_m^2) \tag{1}$$

we may again consider $\omega_\mu{}^{mn}(e, \psi)$ as an inert field. Thus one easily derives for the field equations

$$S = P = A_m = 0 \tag{2}$$

$$\frac{\delta I}{\delta e_{a\nu}} = eG^{a\nu} - \tfrac{1}{4}\bar{\psi}_\lambda \gamma_5 \gamma^a \tilde{\psi}^{\lambda\nu} - \frac{e}{3} e^{a\nu}(S^2 + P^2 - A_m^2) \tag{3}$$

$$-\delta I/\delta\bar{\psi}_\mu = \epsilon^{\mu\nu\rho\sigma}(\gamma_5\gamma_\nu D_\rho\psi_\sigma + \tfrac{1}{2}\gamma_5\gamma_a\psi_\sigma(D_\rho e^a{}_\nu)). \tag{4}$$

The latter term is obtained by partially integrating $\bar{\psi}_\mu\gamma_5\gamma_\nu D_\rho\delta\psi_\sigma = (D_\rho\delta\bar{\psi}_\sigma)\gamma_5\gamma_\nu\psi_\mu$ and vanishes (see appendix C). Thus the field equations reduce to

$$S = P = A_m = 0, \qquad R^\mu \equiv \epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu D_\rho\psi_\sigma = 0 \tag{5}$$

$$eG^{a\nu} - \tfrac{1}{4}\bar{\psi}_\lambda\gamma_5\gamma^a\tilde{\psi}^{\lambda\nu} = 0, \qquad \tilde{\psi}^{\lambda\nu} = \epsilon^{\lambda\nu\rho\sigma}(D_\rho\psi_\sigma - D_\sigma\psi_\rho). \tag{6}$$

We now discuss the question of consistency [120]. Since $R^\mu = 0$ has open indices, one can once more differentiate. To show what might go wrong, consider first the case of a complex spin 3/2 field coupled minimally to electromagnetism. The field equation $\epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu(\partial_\rho - ieA_\rho)\psi_\sigma = 0$ yields upon differentiation with $\partial_\mu - ieA_\mu$ that $\tilde{F}^{\mu\nu}\gamma_\mu\psi_\nu = 0$ which puts as extra constraint for consistency that either $\psi_\nu = 0$ or that the photon be a gauge excitation. In supergravity, no such problems arise. From

$$D_\mu R^\mu = \epsilon^{\mu\nu\rho\sigma}\gamma_5[\gamma_\nu D_\mu D_\rho\psi_\sigma + (\gamma_a D_\rho\psi_\sigma)(D_\mu e^a{}_\nu)] \tag{7}$$

and the torsion equation $D_\mu e^a{}_\nu - D_\nu e^a{}_\mu = \tfrac{1}{2}\bar{\psi}_\mu\gamma^a\psi_\nu$ one finds

$$D_\mu R^\mu = \tfrac{1}{4}\epsilon^{\mu\nu\rho\sigma}\gamma_5[\gamma_\nu\sigma_{cd}\psi_\sigma R_{\mu\rho}{}^{cd} + (\gamma_a D_\rho\psi_\sigma)(\bar{\psi}_\mu\gamma^a\psi_\nu)]. \tag{8}$$

If one expands the product $\gamma_\nu\sigma_{cd}$ as explained in appendix A and uses the cyclic identity with torsion

$$\begin{aligned} &(\gamma_5\gamma_\nu\sigma_{cd}\psi_\sigma)\epsilon^{\mu\nu\rho\sigma}R_{\mu\rho}{}^{cd} = -2G_{e\sigma}\gamma^e\psi^\sigma + \gamma^5\gamma_d\psi_\sigma\epsilon^{\mu\nu\rho\sigma}R_{\mu\rho\nu}{}^d \\ &\epsilon^{\mu\nu\rho\sigma}R_{\mu\rho\nu d}(\omega) = -(\bar{\psi}_\lambda\gamma_d D_\alpha\psi_\beta)\epsilon^{\alpha\beta\lambda\sigma} \end{aligned} \tag{9}$$

all terms in $D_\mu R^\mu$ cancel on-shell. (Use the Einstein field equations and Fierz both undifferentiated gravitinos together.) Hence supergravity is consistent.

It is usually believed that consistency is a consequence of gauge invariance. This is not always so; for example [125], adding to the gauge action of simple supergravity a mass term $e\bar{\psi}_\mu\sigma^{\mu\nu}\psi_\nu$ leads again to a consistent theory although it is not locally supersymmetric. (For this one also needs a cosmological term, see subsection 6.1.)

Finally we give a list of different forms of the gravitino field equation

$$\begin{aligned} &R^\mu = \epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu D_\rho\psi_\sigma, \qquad \gamma^\lambda\psi_{\lambda\mu} = R_\mu - \tfrac{1}{2}\gamma_\mu\gamma\cdot R, \qquad \gamma^\lambda\tilde{\psi}_{\lambda\mu} = -2\gamma_5 R_\mu \\ &\gamma\cdot R = 2\sigma^{\mu\nu}\psi_{\mu\nu}, \qquad \psi_{\mu\nu} = D_\mu\psi_\nu - D_\nu\psi_\mu \\ &\gamma_\alpha\psi_{\beta\gamma} + \gamma_\beta\psi_{\gamma\alpha} + \gamma_\gamma\psi_{\alpha\beta} = \epsilon_{\mu\alpha\beta\gamma}\gamma_5 R^\mu \\ &\psi_{\mu\nu} + \tfrac{1}{2}\gamma_5\tilde{\psi}_{\mu\nu} = -\gamma_\alpha\sigma_{\mu\nu}R^\alpha, \qquad \tilde{\psi}_{\mu\nu} = \epsilon_{\mu\nu}{}^{\rho\sigma}\psi_{\rho\sigma} \\ &R_\mu - \tfrac{1}{3}\gamma_\mu\gamma\cdot R = \tfrac{1}{6}\gamma_5\gamma^\nu\tilde{\psi}_{\mu\nu} - \tfrac{2}{3}\gamma^\nu\psi_{\mu\nu}. \end{aligned} \tag{10}$$

To derive these results, use for example that $\gamma_\mu \epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu = -2\sigma_{\mu\nu}\gamma_5\epsilon^{\mu\nu\rho\sigma} = 4\sigma^{\rho\sigma}$. Also use

$$\epsilon_{\mu\alpha\beta\gamma}\epsilon^{\mu\nu\rho\sigma} = \delta^{\nu\rho\sigma}_{\alpha\beta\gamma} + \delta^{\rho\sigma\nu}_{\alpha\beta\gamma} + \delta^{\sigma\nu\rho}_{\alpha\beta\gamma} - (\alpha \leftrightarrow \beta) \tag{11}$$

where $\delta^{\nu\rho\sigma}_{\alpha\beta\gamma} = \delta^\nu_\alpha \delta^\rho_\beta \delta^\sigma_\gamma$.

It follows that on-shell $G_\lambda{}^\lambda = 0$, but one should not conclude from this result that the gravitino energy momentum tensor is traceless. There are extra gravitino terms in $G_{\mu\nu}$ which should be considered as part of the gravitino stress tensor, and their trace does not vanish. Thus, in second-order formalism the action for the gravitino (including the torsion terms from the Hilbert and Rarita–Schwinger actions) is not locally scale invariant on-shell. (We recall that a matter action $\mathscr{L}(e, \phi)$ for a matter field ϕ varies into $T_\lambda{}^\lambda$, under $\delta e^a{}_\mu = \Lambda e^a{}_\mu$ and $\delta\phi = \Lambda^\alpha \phi$ with some power α, provided one is on-shell where the ϕ field equation is satisfied, so that $\delta\mathscr{L}/\delta\phi = 0$. See Weinberg's book.)

Sometimes one considers the gravitino as a fermion in an external gravitational field. The curl $D_\rho\psi_\sigma - D_\sigma\psi_\rho$ contains then only the connection $\omega_\mu{}^{mn}(e)$, not $\omega_\mu{}^{mn}(e, \psi)$. In this case the action is invariant under $\delta\psi_\mu = D_\mu(\omega(e))\epsilon$ provided the external gravitational field satisfies the Einstein equations (we leave the proof as an exercise). Also this action is not locally scale invariant, now because although the gravitino field equation again reads $R^\mu = 0$ (with $\omega(e)$), we can no longer use 1.5 order formalism and there is an extra contribution to the stress tensor coming from varying $\omega(e)$ with respect to the tetrad field. However, in first-order formalism, the action $-\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar\psi_\mu\gamma_5\gamma_\nu(\partial_\rho + \frac{1}{2}\omega_\rho \cdot \sigma)\psi_\sigma$ is locally scale invariant under $\delta e^a{}_\mu = \Lambda e^a{}_\nu$ and $\delta\psi_\mu = \Lambda^{-1/2}\psi_\mu$, $\delta\omega_\mu{}^{mn} = 0$. This is obvious since if the derivative ∂ hits Λ, the rest $(\bar\psi_\mu\gamma_5\gamma_\nu\psi_\sigma)\epsilon^{\mu\nu\rho\sigma}$ vanishes due to the Majorana character of the gravitino. Again, one cannot say that the matter action is invariant, since the terms containing ω coming from the Hilbert action are not invariant.

1.11. *Matter coupling: the supersymmetric Maxwell–Einstein system*

After having discussed the gauge action of simple ($N = 1$) supergravity, we now turn to the second stage: matter coupling. As in any gauge theory, we start with a globally supersymmetric action, and couple it to the gauge action by introducing couplings to the gauge fields. For ordinary gauge theories the prescription is rather simple: replace ordinary derivatives by minimally covariant derivatives ($\partial_\mu - ieA_\mu$ in electromagnetism). In supersymmetry this is not so simple and the analogous solution is best obtained by the tensor calculus which we will discuss later. Here we will start with the Noether method since it is conceptually simpler (though algebraically more tedious).

We will begin with the coupling of the spin (1, 1/2) photon–neutrino system to the gauge action of supergravity. Historically this was the first matter coupling [204]. Another coupling exists between photons and gravitons, namely the coupling of the spin (3/2, 1) systems to the gauge action (so-called $N = 2$ extended supergravity) [203]. This latter coupling comes nearer to a unification of gravity and electromagnetism, since one can connect gravitons and photons by a series of symmetry operations. In the model we are going to discuss, this is not possible. For convenience we consider an abelian vector field, but all results easily generalize to the Yang–Mills case [202, 237].

The photon–neutrino Lagrangian density in curved space reads

$$\mathscr{L}^0 = -\tfrac{1}{4} e F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} - \frac{e}{2}\bar\lambda \not{D}^0 \lambda, \qquad F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{1}$$

where $D^0_\mu\lambda = \partial_\mu\lambda + \frac{1}{2}\omega_\mu{}^{ab}(e)\sigma_{ab}\lambda$ contains not yet ψ-torsion. $\mathscr{L}^0$ varies under the global supersymmetry transformations in flat space

$$\delta B_\mu = -\tfrac{1}{2}\bar{\epsilon}\gamma_\mu\lambda, \qquad \delta\lambda = \tfrac{1}{2}\sigma\cdot F\epsilon, \qquad \sigma\cdot F = \sigma^{\mu\nu}F_{\mu\nu} \tag{2}$$

into a total derivative, $\delta\mathscr{L}^0 = \partial_\mu[\tfrac{1}{2}F^{\mu\nu}\bar{\epsilon}\gamma_\nu\lambda + \tfrac{1}{4}\bar{\lambda}\gamma^\mu\sigma\cdot F\epsilon]$. To show this, use the Bianchi identity $\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$. The Noether current is thus given by $-\tfrac{1}{2}\sigma\cdot F\gamma^\mu\lambda$. Hence, we begin by adding the Noether term

$$\mathscr{L}^{\mathrm{N}} = \frac{e\kappa}{2}\bar{\psi}_\mu\sigma\cdot F\gamma^\mu\lambda. \tag{3}$$

This ensures that the order κ^0 terms in $\delta(I^0 + I^{\mathrm{N}})$ cancel, even in curved space.

To order κ, one has terms coming from $\delta e^a{}_\mu = (\kappa/2)\bar{\epsilon}\gamma^a\psi_\mu$ in I^0 and from δB_μ and $\delta\lambda$ in I^{N}. Consider first the terms of the form $\kappa F^2\epsilon$. The Maxwell action yields $(\kappa/2)(\bar{\epsilon}\gamma^\mu\psi^\nu)T_{\mu\nu}(B)$ while the Noether coupling yields $(\kappa/2)\bar{\psi}_\mu\sigma\cdot F\gamma^\mu\sigma\cdot F\epsilon$. The sum is equal to

$$\frac{\kappa}{4}(\bar{\psi}_\mu\gamma_5\gamma^\tau\epsilon)(F_{\mu\alpha}\tilde{F}_{\tau\alpha} - \tfrac{1}{4}g_{\mu\tau}F_{\alpha\beta}\tilde{F}^{\alpha\beta}). \tag{4}$$

Since, however, the second factor is identically zero, these variations cancel. (This identity can be proved by using that $\delta_\mu{}^\nu\epsilon^{\alpha\beta\rho\sigma}$ antisymmetrized in $(\nu\alpha\beta\rho\sigma)$ vanishes.)

To order κ there are also $\kappa\epsilon\psi\lambda\partial\lambda$ terms in $\delta(I^0 + I^{\mathrm{N}})$. From the Dirac action one finds

$$\delta I^{1/2} = -e\frac{\kappa}{4}\bar{\epsilon}\gamma\cdot\psi(\bar{\lambda}\not{D}^0\lambda) + \frac{e\kappa}{4}(\bar{\epsilon}\gamma^\mu\psi_a)(\bar{\lambda}\gamma^a D_\mu{}^0\lambda) - \frac{e\kappa}{16}(\bar{\lambda}\gamma^\mu\sigma^{ab}\lambda)(2\bar{\epsilon}\gamma_b\psi_{\mu a} - \bar{\epsilon}\gamma_\mu\psi_{ab}). \tag{5}$$

The first term is clearly cancelled if one adds to $\delta\lambda$ a new term $\delta\lambda = -(\kappa/2)\bar{\epsilon}\gamma\cdot\psi\lambda$ since this new variation produces in $\mathscr{L}^{1/2}$ precisely the opposite. Clearly, quite generally *terms in the varied action proportional to a field equation can always be cancelled by adding an extra term to the transformation law of the field whose field equation appears.* Using this observation, we can cancel the last terms in eq. (5), since $\bar{\lambda}\gamma^m\sigma^{ab}\lambda = \tfrac{1}{2}\epsilon^{mabc}\bar{\lambda}\gamma_5\gamma_c\lambda$ and $\epsilon^{mabc}\gamma_a\psi_{bc} = 2\gamma_5 R^m$ is proportional to the gravitino field equation.

A second general mechanism which one uses over and over again in the Noether method is the observation that *terms in the varied action with $\partial\epsilon$ can be cancelled by adding an extra term to the action obtained by replacing $\partial_\mu\epsilon$ by $-\psi_\mu$, provided all gravitinos appear symmetrically in the end result.* For example, the variation of B_μ in the Noether coupling in eq. (3) yields

$$-\tfrac{1}{2}\bar{\psi}\cdot\gamma\sigma^{\alpha\beta}\lambda\partial_\alpha(\bar{\epsilon}\gamma_\beta\lambda) - \tfrac{1}{2}(\bar{\psi}^\mu\gamma^\alpha\lambda)[\partial_\alpha(\bar{\epsilon}\gamma_\mu\lambda) - \partial_\mu(\bar{\epsilon}\gamma_\alpha\lambda)]. \tag{6}$$

Partially integrating the first two terms, they are proportional to R^μ since $\not{\partial}\gamma\cdot\psi - \partial\cdot\psi = \tfrac{1}{2}\gamma\cdot R$ and $\not{\partial}\psi_\mu - \partial_\mu\gamma\cdot\psi = \tfrac{1}{2}\gamma_\alpha\partial_\mu R^\alpha$ so that they are cancelled by an extra $\delta\psi_\mu$. But the last term is equal to

$$\frac{\kappa}{2}(\bar{\psi}^\mu\gamma_\alpha\lambda)(\partial_\mu\bar{\epsilon})\gamma^\alpha\lambda + \frac{\kappa}{2}(\bar{\psi}^\mu\gamma^\alpha\lambda)(\bar{\epsilon}\gamma_\alpha\partial_\mu\lambda) \tag{7}$$

and one clearly sees that the $\partial\epsilon$ terms can be cancelled by adding to the action $-(\kappa^2/4)(\bar{\psi}_\mu\gamma_\alpha\lambda)(\bar{\psi}^\mu\gamma^\alpha\lambda)$ since both gravitinos yield a $\partial_\mu\epsilon$ variation and appear symmetrically.

In this way one adds systematically new terms to action and transformation laws until one arrives at a completely invariant action. There are general arguments that this iterative procedure must always stop except when there are spin 0 fields A present [202]; in that case the action can be infinite series in $(\kappa A)^m$.

The final result obtained in this way is given by

$$\mathscr{L} = -\frac{e}{4} F_{\mu\nu}F^{\mu\nu} - \frac{e}{2}\bar{\lambda}\not{D}\lambda + \frac{e\kappa}{2}\bar{\psi}_\mu\sigma\cdot F\gamma^\mu\lambda + \frac{e\kappa^2}{4}(\bar{\psi}_\mu\sigma^{\rho\sigma}\gamma^\mu\lambda)(\bar{\psi}_\rho\gamma_\sigma\lambda) + \frac{3e}{16}\kappa^2(\bar{\lambda}\lambda)(\bar{\lambda}\lambda)$$

$$\delta\lambda = \tfrac{1}{2}\sigma\cdot F^{\text{cov}}\epsilon, \qquad \delta B_\mu = -\tfrac{1}{2}\bar{\epsilon}\gamma_\mu\lambda \tag{8}$$

$$\delta e^m{}_\mu = \frac{\kappa}{2}\bar{\epsilon}\gamma^m\psi_\mu, \qquad \delta\psi_\mu = \frac{1}{\kappa}D_\mu\epsilon + \frac{\kappa}{8}(\bar{\lambda}\gamma_5\gamma^\rho\lambda)(g_{\rho\mu} + \sigma_{\rho\mu})\gamma_5\epsilon.$$

The spin connection in $\not{D}\lambda$ contains only ψ-torsion, but no λ-torsion. The symbol $F^{\text{cov}}_{\mu\nu}$ is the supercovariant curl

$$F^{\text{cov}}_{\mu\nu} = \partial_\mu B_\nu + \frac{\kappa}{2}\bar{\psi}_\mu\gamma_\nu\lambda - \mu \leftrightarrow \nu. \tag{9}$$

The first suggestion that there should be an axial vector auxiliary field was due to this result [202]. Clearly, the result for $\delta\psi_\mu$ depends on the matter fields λ and suggests replacing them by a new field A^ρ. We know already from the gauge algebra that

$$\delta\psi_\mu = \frac{1}{\kappa}\left(D_\mu + \frac{\mathrm{i}\kappa}{2}A_\mu\gamma_5\right)\epsilon + \tfrac{1}{6}\gamma_\mu(S - \mathrm{i}\gamma_5 P - \mathrm{i}\not{A}\gamma_5)\epsilon \tag{10}$$

and hence one adds to the action the gauge action $\mathscr{L}(\text{gauge}, e, \psi, S, P, A_\mu)$ as well as a coupling

$$+\frac{\mathrm{i}\kappa}{4}(\bar{\lambda}\gamma_5\gamma^\rho\lambda)A_\rho. \tag{11}$$

Solving for A_ρ, and substituting the solution $A_\rho = -(3\mathrm{i}/8)\kappa(\bar{\lambda}\gamma_5\gamma_\rho\lambda)$ back into the action and $\delta\psi_\mu$, one reproduces all λ^4 and $\lambda^2\epsilon$ terms.

Thus, in the supersymmetric Maxwell–Einstein system, only A_μ plays a role, but S and P do not. This is due to the fact that this system is superconformal invariant. As we shall see, A_μ is the gauge field of chiral rotations. (In the example of the spin (0, 1/2) coupling, S and P do play a role.) The action can be written in a suggestive manner as

$$\mathscr{L} = \mathscr{L}^{(2)} + \mathscr{L}^{3/2} - \frac{e}{3}(S^2 + P^2 - A_\mu^2) + \mathscr{L}^1 - \frac{e}{2}\bar{\lambda}\left(\not{D} + \frac{\mathrm{i}\kappa}{2}\not{A}\gamma_5\right)\lambda + \frac{e\kappa}{4}(\bar{\psi}_\mu\sigma^{\alpha\beta}\gamma^\mu\lambda)(F_{\alpha\beta} + F^{\text{cov}}_{\alpha\beta})$$

$$\delta B_\mu = -\frac{\kappa}{2}\bar{\epsilon}\gamma_\mu\lambda, \qquad \delta\lambda = \tfrac{1}{2}\sigma\cdot F^{\text{cov}}\epsilon. \tag{12}$$

Due to the auxiliary fields S, P, A_μ one has only added matter terms to the action without changing the gauge field transformation laws. Since the λ field equation in the absence of A_μ can only rotate into the B_μ field equation and this latter field equation requires two derivatives, the variation of the λ-field equation cannot contain $\partial\epsilon$ terms. Thus the λ-field equation must be supercovariant by itself, and since the terms $\bar{\psi}\lambda F$ in the action contain only one field λ, they require an extra factor of 2 in order that the λ field equation becomes $\not{D}^{\text{cov}}\lambda$.

Finally, we discuss the role of the matter auxiliary fields. The globally supersymmetric spin $(1,\frac{1}{2})$ system has a closed algebra if one adds a pseudoscalar field D (see subsection 8)

$$\delta B_\mu = -\tfrac{1}{2}\bar{\epsilon}\gamma_\mu\lambda, \qquad \delta\lambda = \tfrac{1}{2}\sigma\cdot F\epsilon + \frac{\mathrm{i}}{2}\gamma_5 D\epsilon, \qquad \delta D = \frac{\mathrm{i}}{2}\bar{\epsilon}\gamma_5\not{\partial}\lambda. \tag{13}$$

One finds uniformly for λ and D the usual result $[\delta(\epsilon_1),\delta(\epsilon_2)]D = \frac{1}{2}(\bar{\epsilon}_2\gamma^\mu\epsilon_1)\partial_\mu D$, but for B_μ there is a Maxwell gauge transformation added: $-\partial_\mu(\frac{1}{2}\bar{\epsilon}_2\not{B}\epsilon_1)$. The flat space action becomes

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}^2 - \tfrac{1}{2}\bar{\lambda}\not{\partial}\lambda + \tfrac{1}{2}D^2. \tag{14}$$

In curved spacetime, the Noether current and hence the Noether coupling are unchanged. The matter transformation laws are now given by

$$\delta B_\mu = -\tfrac{1}{2}\bar{\epsilon}\gamma_\mu\lambda, \qquad \delta\lambda = \tfrac{1}{2}(\sigma\cdot F^{\text{cov}} + \mathrm{i}\gamma_5 D)\epsilon$$

$$\delta D = \frac{\mathrm{i}}{2}\bar{\epsilon}\gamma_5\gamma^\mu\left(D_\mu^{\text{cov}}\lambda + \frac{\mathrm{i}\kappa}{2}A_\mu\gamma_5\lambda\right)$$

$$D_\mu^{\text{cov}}\lambda = D_\mu\lambda - \frac{\kappa}{2}(\sigma\cdot F^{\text{cov}} + \mathrm{i}\gamma_5 D)\psi_\mu. \tag{15}$$

Since D is a matter field, one expects δD to vary without $\partial\epsilon$, and indeed one finds a supercovariant derivative in δD.

Summarizing, the total action reads

$$\mathcal{L} = \mathcal{L}^{(2)} + \mathcal{L}^{3/2} + \mathcal{L}^1 + \mathcal{L}^{1/2}\left(D_\mu + \frac{\mathrm{i}\kappa}{2}A_\mu\gamma_5\right) - \frac{e}{3}(S^2 + P^2 - A_\mu^2) + \frac{e}{2}D^2 + \frac{e\kappa}{4}\bar{\psi}_\mu(\sigma\cdot F + \sigma\cdot F^{\text{cov}})\gamma^\mu\lambda \tag{16}$$

and is invariant under eq. (15). The derivatives D_μ contain only ψ-torsion.

1.12. The scalar multiplet coupling

The next example of the coupling of a globally supersymmetric matter system to the gauge action of simple supergravity is the coupling of the spin $(\frac{1}{2}, 0^+, 0^-)$ Wess–Zumino model [202, 509]. The action, already covariantized with respect to curved spacetime, is given by

$$\mathcal{L}^0 = -\tfrac{1}{2}eg^{\mu\nu}(\partial_\mu A\,\partial_\nu A + \partial_\mu B\,\partial_\nu B) - \tfrac{1}{2}e\bar{\lambda}\gamma^\mu D_\mu{}^0\lambda + \tfrac{1}{2}e(F^2 + G^2) \tag{1}$$

where $D_\mu{}^0$ is the gravitationally covariant derivative without torsion. In curved space it varies into

$$\delta A = \tfrac{1}{2}\bar{\epsilon}\lambda, \qquad \delta B = -\frac{\mathrm{i}}{2}\bar{\epsilon}\gamma_5\lambda, \qquad \delta F = \tfrac{1}{2}\bar{\epsilon}\not{D}^0\lambda,$$

$$\delta G = \frac{\mathrm{i}}{2}\bar{\epsilon}\gamma_5\not{D}^0\lambda, \qquad \delta\lambda = \tfrac{1}{2}[\not{\partial}(A - \mathrm{i}\gamma_5 B)]\epsilon + \tfrac{1}{2}(F + \mathrm{i}\gamma_5 G)\epsilon, \tag{2}$$

$$\delta\mathscr{L} = -\frac{e}{2}(D_\mu\bar{\epsilon})(\not{\partial}A + \not{\partial}\mathrm{i}\gamma_5 B)\gamma^\mu\lambda + \partial_\mu\left[-\frac{e}{4}\bar{\epsilon}\gamma^\mu(\not{\partial}A - \not{\partial}\mathrm{i}\gamma_5 B)\lambda + \frac{e}{4}(F + \mathrm{i}\gamma_5 B)\gamma^\mu\lambda\right].$$

From now on we will omit the auxiliary fields F and G, but reinstate them at the end of this subsection.

Since the action I^0 varies for local $\epsilon(x)$ into the Noether current, (see subsection 2) we add the Noether coupling

$$\mathscr{L}^{\mathrm{N}} = \frac{e}{2}\bar{\psi}_\mu[\not{\partial}(A + \mathrm{i}\gamma_5 B)]\gamma^\mu\lambda. \tag{3}$$

The variations of A, B, λ yield

(i) AA and BB terms. These cancel with the vierbein variation in $\mathscr{L}^0$ (both are proportional to the energy-momentum tensor, see subsection 2).

(ii) AB terms. These are of the form $\kappa(\bar{\psi}_\mu\gamma_\nu\epsilon)\epsilon^{\mu\nu\rho\sigma}\partial_\rho A\,\partial_\sigma B$ and upon partial integration the $\partial\epsilon$ terms are cancelled by adding a term $(\kappa^2/4)(\bar{\psi}_\mu\gamma_\nu\psi_\rho)\epsilon^{\mu\nu\rho\sigma}A\overleftrightarrow{\partial}_\sigma B$ while the remainder is proportional to the gravitino field equation and can be absorbed by an extra $\delta\psi_\mu \sim \gamma_5\epsilon A\overleftrightarrow{\partial}_\mu B$ (see subsection 3).

(iii) the $\bar{\lambda}\lambda$ terms coming from $\delta e^m{}_\mu$ in $\mathscr{L}^0$ and from δA and δB in $\mathscr{L}_{\mathrm{N}}$ cancel (after tedious algebra), if one adds $\delta\lambda \sim \kappa\bar{\psi}\lambda\epsilon$ and $\delta\psi \sim \bar{\lambda}\lambda\epsilon$ terms to the action. One needs Fierz rearrangements, the identity $\not{\partial}\gamma\cdot\psi - \partial\cdot\psi = \tfrac{1}{2}\gamma_\mu R^\mu$ and the identity

$$\partial_\rho(\bar{\lambda}\gamma^\rho\gamma_\alpha\gamma_\mu\lambda) = \bar{\lambda}\overleftarrow{\not{\partial}}\gamma_\alpha\gamma_\mu\lambda + \bar{\lambda}\gamma_\alpha\gamma_\mu\not{\partial}\lambda + 2\bar{\lambda}(\partial_\alpha\gamma_\mu - \gamma_\alpha\partial_\mu)\lambda. \tag{4}$$

The new $\delta\lambda$ and $\delta\psi$ variations as well as the old variations must now be used in the new, order κ^2 terms in the action. One finds

(iv) that the new $\delta\psi \sim \kappa A\,\partial B\epsilon$ in $\mathscr{L}_{\mathrm{N}}$ are cancelled by adding a term $\kappa^2\bar{\lambda}\lambda A\,\partial B$ to the action.

(v) that the new $\delta\psi \sim \bar{\lambda}\lambda\epsilon$ in $\mathscr{L}_{\mathrm{N}}$ yields $\kappa^2\lambda^3(\partial A)\epsilon$ variations while also the old $\delta A \sim \bar{\epsilon}\lambda$ in the new term in (iv) yields such terms. The sum is cancelled by adding a $\kappa^2(\bar{\lambda}\lambda)^2$ term to the action, and a $\delta\lambda = \kappa^2\bar{\lambda}\lambda A\epsilon$ term.

(vi) Finally, many sources contribute to $\kappa^2\psi^2\lambda\,\partial A$ variations, but only one source yields a term with $\partial\epsilon$. Partially integrating this term, all variations cancel if one adds a $\delta\psi \sim \kappa^2 A\lambda\psi\epsilon$ term.

The complete answer is given by

$$\mathscr{L} = \mathscr{L}(\text{gauge}, e, \psi) + \mathscr{L}^0 + \mathscr{L}^{\mathrm{N}} + \mathscr{L}(\text{contact})$$

$$\mathscr{L}(\text{contact}) = \kappa^2\epsilon^{\mu\nu\rho\sigma}(\bar{\psi}_\mu\gamma_\nu\psi_\rho)\left[\frac{1}{32}\bar{\lambda}\gamma_5\gamma_\sigma\lambda - \frac{\mathrm{i}}{8}A\overleftrightarrow{\partial}_\sigma B\right]$$
$$+ e\kappa^2(\bar{\lambda}\gamma_5\gamma^\tau\lambda)\left[-\frac{1}{16}\bar{\psi}_\alpha\gamma_5\gamma_\tau\psi^\alpha - \frac{\mathrm{i}}{8}A\overleftrightarrow{\partial}_\tau B - \frac{1}{64}\bar{\lambda}\gamma_5\gamma_\tau\lambda\right] \tag{5}$$

where $\mathscr{L}$(gauge) contains ψ-torsion, but the Dirac action for λ contains no torsion. The transformation rules are

$$\delta A = \tfrac{1}{2}\bar{\epsilon}\lambda, \qquad \delta B = -\frac{i}{2}\bar{\epsilon}\gamma_5\lambda, \qquad \delta e^m{}_\mu = \frac{\kappa}{2}\bar{\epsilon}\gamma^m\psi_\mu$$

$$\delta\psi_\mu = \frac{1}{\kappa}D_\mu\epsilon + \frac{i}{4}\kappa\gamma_5\epsilon A\overleftrightarrow{D}{}^{\mathrm{cov}}_\nu B + \frac{\kappa}{8}\sigma_{\mu\rho}\gamma_5\epsilon(\bar{\lambda}\gamma_5\gamma^\rho\lambda) \tag{6}$$

$$\delta\lambda = \tfrac{1}{2}[\not{D}^{\mathrm{cov}}(A - i\gamma_5 B)]\epsilon + \frac{\kappa^2}{8}\gamma_5\lambda[A\bar{\epsilon}\gamma_5\lambda - iB\bar{\epsilon}\lambda].$$

The derivative D_μ contains ψ-torsion, and D^{cov}_μ is the supercovariant derivative ($D_\mu A = \partial_\mu A - (\kappa/2)\bar{\psi}_\mu\lambda$).

So here we have a second model of matter coupling. By now the reader probably starts asking himself the following questions: how can one eliminate the matter fields from $\delta\psi_\mu$; in particular, could one have found the minimal set of auxiliary fields S, P, A_m in this way. (We recall that the Maxwell–Einstein system was too simple: it is a conformal system and hence S and P did not play a role. *This* system is not conformal.) Indeed, one can rediscover S, P, A_m as well as their transformation rules in this way, but, to be honest, only if one knows already the answer; otherwise one probably would not have thought of all necessary subtleties. The next question is: how unique are such results obtained by the Noether method? Fortunately, the most general coupling of the scalar multiplet to supergravity is known from the tensor calculus, and hence we can answer this question completely.

The most general coupling (eq. (67) of ref. [541]) with canonical gauge action and canonical spin 1/2 action has as scalar kinetic terms $f(A^2+B^2)$ times $(\partial_\mu A)^2 + (\partial_\mu B)^2$ where f is arbitrary, and also a nonpolynomial potential for A and B. Requiring canonical scalar kinetic terms ($f = -\frac{1}{2}$) and a polynomial action (which means $G = -\frac{1}{2}zz^* - \ln\frac{1}{4}|g|^2$ in eq. (67) and letting $g = \text{constant} \to 0$) one finds that eqs. (5, 6) are unique; going back to the point in ref. [541] where one still had S, P, A_μ present, and where $e^m{}_\mu$, $\psi_\mu{}^a$, S, P, A_μ transform as in subsection 9, this action takes a nonpolynomial form. (In going from this version to eqs. (5, 6) complicated field redefinitions eliminate this nonpolynomiality but also destroy the canonical transformation rules of the gauge fields. Also S, P, A_μ have been eliminated through their (complicated) nonpropagating field equations.)

Thus it would have been difficult to deduce the auxiliary fields from (5, 6). The action with S, P, A_μ which is equivalent to (5, 6) and invariant under the transformation rules of the scalar multiplet $\Sigma = (A, B, \lambda, F, G)$ as given by the tensor calculus,

$$\delta A = \frac{\bar{\epsilon}}{2}\lambda, \qquad \delta B = -\frac{i}{2}\bar{\epsilon}\gamma_5\lambda$$

$$\delta\lambda = \tfrac{1}{2}[\not{D}^{\mathrm{cov}}(A - i\gamma_5 B)]\epsilon + \tfrac{1}{2}(F + i\gamma_5 G)\epsilon$$

$$\delta F = \tfrac{1}{2}\bar{\epsilon}\left(\not{D}^{\mathrm{cov}} - \frac{i}{2}\not{A}\gamma_5\right)\lambda + \tfrac{1}{2}\bar{\epsilon}\eta\chi$$

$$\delta G = \tfrac{1}{2}i\bar{\epsilon}\gamma_5\left(\not{D}^{\mathrm{cov}} - \frac{i}{2}\not{A}\gamma_5\right)\lambda - \frac{i}{2}\bar{\epsilon}\gamma_5\eta\chi \tag{7}$$

where $\eta = -\frac{1}{3}(S - i\gamma_5 P - i\not{A}\gamma_5)$, and under the standard transformation rules for S, P, A_μ, $e^m{}_\mu$, and ψ_μ, reads as follows

$$\mathscr{L} = \exp(-x)[\mathscr{L}(\text{gauge}) + \mathscr{L}^0(1-x) + \mathscr{L}^{\mathrm{N}}(1-x) + \mathscr{L}^{\mathrm{N}}_{\mathrm{imp}} + \mathscr{L}(A_\mu) + \mathscr{L}(S,P,F,G)]. \tag{8}$$

Here $x = -\frac{1}{6}(A^2 + B^2)$, $\mathscr{L}$(gauge) is the gauge action of subsection 3 (with ψ-torsion but without S, P, A_μ), and $\mathscr{L}^0$ is the Wess–Zumino action and $\mathscr{L}^{\mathrm{N}}$ the Noether coupling of eq. (3).

Interestingly enough, one finds the improved Noether coupling

$$\mathscr{L}^{\mathrm{N}}_{\mathrm{imp}} = -\tfrac{2}{3}\bar{\lambda}(A - i\gamma_5 B)\sigma^{\mu\nu}(D^0{}_\mu\psi_\nu) \tag{9}$$

(where $D^0{}_\mu$ does not contain ψ-torsion). In flat space it satisfies on-shell

$$\partial_\mu(j^\mu_{\mathrm{N}} + j^\mu_{\mathrm{N,imp}}) = \gamma^\mu(j^\mu_{\mathrm{N}} + j^\mu_{\mathrm{N,imp}}) = 0. \tag{10}$$

Hence we have here a clear signal that the minimal auxiliary fields of supergravity are closely linked to conformal aspects. We will see more about this in the section on conformal supergravity – in fact, we will show there how one can *derive* the tensor calculus from conformal supergravity.

The remaining terms are

$$\mathscr{L}(A_\mu) = \frac{1}{3}A_\mu^2 - \frac{1}{3}A^\mu(A\overleftrightarrow{D}{}^{\mathrm{cov}}_\mu B) - \frac{i}{4}(\bar{\lambda}\not{A}\gamma_5\lambda)(1-x)$$
$$+ \text{4-fermion terms} + (A\overleftrightarrow{\partial}_\sigma B)\left[\frac{-i}{6}(\bar{\lambda}\gamma_5\gamma^\sigma\lambda)(1-\tfrac{1}{2}x) - \frac{i}{8}\epsilon^{\mu\nu\rho\sigma}(\bar{\psi}_\mu\gamma_\nu\psi_\rho)\right]. \tag{11}$$

Finally,

$$\mathscr{L}(S,P,F,G) = -\tfrac{1}{3}(S^2 + P^2) + \tfrac{1}{2}(F^2 + G^2)(1-x) + \tfrac{1}{3}F(AS + BP) + \tfrac{1}{3}G(BS - AP)$$
$$+ \text{terms with } \bar{\lambda}\lambda \text{ and } (\bar{\lambda}\lambda)^2. \tag{12}$$

The addition of masses and a φ^4-coupling in the scalar multiplet coupled to simple supergravity was done in ref. [221]. A super-Higgs effect in this model, as well as the gauging of the $N = 4$ supergravity (the O(4) version) and the scalar sector of this model, and also the coupling of super-QED to supergravity was discussed in ref. [222].

1.13. Free spin 2 and spin 3/2 field theory

The physical fields of supergravity are the tetrad $e^m{}_\mu$ and the gravitino $\psi^a{}_\mu$. It is interesting that the free field limit of the gauge action of supergravity reduces to the sum of the Fierz–Pauli and Rarita–Schwinger actions, which are the unique actions for the spin 2 and 3/2 without ghosts as we shall show. Indeed,

$$\mathscr{L}_{\mathrm{FP}} = -\tfrac{1}{4}h^2_{\mu\nu,\lambda} + \tfrac{1}{2}h^2_\mu - \tfrac{1}{2}h_\mu h_{,\mu} + \tfrac{1}{4}h^2_{,\mu} \tag{1}$$

$$\mathcal{L}_{\mathrm{RS}} = -\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu\partial_\rho\psi_\sigma \tag{2}$$

where $h_\mu = \partial_\lambda h^\lambda{}_\mu$ and $h = h^\lambda{}_\lambda$, and $\mathcal{L}_{\mathrm{FP}}$ is the linearized part of $\int -\sqrt{g}R\ \mathrm{d}^4x$.

We first show that these actions indeed describe two physical modes with helicities ± 2 and $\pm 3/2$. Then we show that the only actions for $h_{\mu\nu}$ and ψ_μ with positive energy are those in eqs. (1) and (2).

For spin 2 the field equations read

$$\Box\chi_{\mu\nu} - \chi_{\mu,\nu} - \chi_{\nu,\mu} + \delta_{\mu\nu}\chi_{\alpha,\alpha} = 0, \qquad \chi_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2}h\delta_{\mu\nu} \tag{3}$$

where $\chi_\mu = \partial_\lambda\chi_{\mu\lambda}$, and $\chi_{\mu\nu}$ and $h_{\mu\nu}$ are symmetric. It is invariant under gauge transformations $\delta\chi_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \delta_{\mu\nu}\xi^\alpha{}_{,\alpha}$. Thus $\delta(\partial_\mu\chi_{\mu\nu}) = \Box\xi_\nu$ and one can choose the de-Donder gauge in which $\partial_\mu\chi_{\mu\nu} = 0$. In this gauge the field equation reduces to $\Box\chi_{\mu\nu} = 0$. The solutions are plane waves and taking these along the z-axis, $\chi_{\mu\nu}(x_3 - x_4)$, one has from the de-Donder gauge that

$$\chi_{3\mu} = \chi_{4\mu}. \tag{4}$$

As always in differential gauges, one can once more use the gauge freedom provided $\Box\xi^\nu = 0$ in order not to undo the de-Donder gauge. Thus $\xi_{\mu\nu} = \chi_{\mu\nu} + \partial_\mu\lambda_\nu + \partial_\nu\lambda_\mu - \delta_{\mu\nu}\lambda_{\alpha,\alpha}$ with $\Box\lambda_\nu = 0$ still satisfies $\partial_\mu\xi^\mu{}_\nu = 0$. Choosing λ_1 such that $\xi_{31} = \chi_{31} + \partial_3\lambda_1 = 0$ one finds from $\partial_\mu\xi^\mu{}_\nu = 0$ that $\xi_{41} = 0$. Similarly, a proper choice of λ_2 yields $\xi_{32} = \xi_{42} = 0$. Finally, λ_3 and λ_4 are fixed by requiring that $\xi_{\lambda\lambda}$ and ξ_{34} vanish. As a consequence, from (4) also $\chi_{33} = \chi_{44} = 0$. Thus, only $\chi_{12} = \chi_{21}$ and $\chi_{11} = -\chi_{22}$ are nonzero. Since the polarization tensors for spin 2 fields with helicities ± 2 are given in terms of photon polarization vectors $\epsilon^\pm_\mu$ by

$$\lambda^{+2}_{\mu\nu} = \epsilon^+_\mu\epsilon^+_\nu, \qquad \lambda^{-2}_{\mu\nu} = \epsilon^-_\mu\epsilon^-_\nu, \qquad \epsilon^+_\mu = \frac{\epsilon^1_\mu + \mathrm{i}\epsilon^2_\mu}{\sqrt{2}} \tag{5}$$

one sees that the Fierz–Pauli action describes two physical modes with helicities ± 2. This follows easily if one writes $\chi_{\mu\nu} = \lambda^\pm_{\mu\nu}a^\pm(k)\mathrm{e}^{\mathrm{i}kx} + \mathrm{h.c.}$ because this shows that only $\chi_{11} - \chi_{22}$ and χ_{12} are nonzero.

We now repeat this analysis for spins 3/2 (ref. [522]). The spin 3/2 field equation in flat space reads

$$R^\mu = \epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu\partial_\rho\psi_\sigma = 0, \qquad \delta R^\mu = 0 \text{ if } \delta\psi_\sigma = \partial_\sigma\epsilon. \tag{6}$$

One chooses the gauge $\gamma\cdot\psi = 0$, which can be solved since $\delta(\gamma\cdot\psi) = \not\partial\epsilon$. From

$$\begin{aligned} &\not\partial\psi_\mu - \partial_\mu\gamma\cdot\psi = R_\mu - \tfrac{1}{2}\gamma_\mu\gamma\cdot R \\ &\not\partial\gamma\cdot\psi - \partial\cdot\psi = 2\sigma^{\mu\nu}\partial_\mu\psi_\nu = \tfrac{1}{2}\gamma\cdot R \end{aligned} \tag{7}$$

it follows that on-shell in this gauge also $\partial\cdot\psi = \not\partial\psi_\nu = 0$, so that $\Box\psi_\mu = 0$. Decomposing ψ_μ in k-space on-shell into the complete set $\epsilon^\pm{}_\mu$, k_μ and $\bar{k}_\mu$

$$\psi_\mu{}^a = \epsilon_\mu{}^+u_a{}^+ + \epsilon_\mu{}^-u_a{}^- + k_\mu u_a{}^0 + \bar{k}_\mu v_a \tag{8}$$

where $\bar{k}_\mu = (\boldsymbol{k}, -k_4)$ is the time reversal of k_μ, it follows from $k\cdot\psi = 0$ that $v_a = 0$. Also, from $\not\partial\psi_\mu = 0$ we see that $\not\partial u^+ = \not\partial u^0 = \not\partial u^- = 0$.

As before, we choose a second gauge, maintaining the previous gauge $\gamma \cdot \psi = 0$

$$\varphi_\mu = \psi_\mu - k_\mu u^0 = \epsilon_\mu^{\ +} u^+ + \epsilon_\mu^- u^- . \tag{9}$$

One still has $k \cdot \varphi = \gamma \cdot \varphi = \not{k}\varphi = 0$ on-shell. Thus, in particular

$$\not{\epsilon}^- \gamma \cdot \varphi = 0 = \not{\epsilon}^- \not{\epsilon}^+ u^+ = \left(1 - \frac{\bar{k} \cdot \boldsymbol{\sigma}}{|\boldsymbol{k}|}\right) u^+ = 0 \tag{10}$$

where we used that $\not{\epsilon}^+ \not{\epsilon}^+ = 0$ and defined $\sigma_k = \epsilon_{ijk} \gamma_i \gamma_j/(2\mathrm{i})$. Thus $\boldsymbol{\sigma}$ is the spin projection operator, and u^+ has helicity $+1/2$. Since ϵ^+ has helicity $+1$, we conclude that φ describes two physical modes with helicities $\pm 3/2$.

We now prove that (1) and (2) are unique. We only treat the spin 3/2 case. For the spin 2 case see for example, Nucl. Phys. B 60 (1973) 478. The most general action with one derivative is given by

$$\mathcal{L} = \bar{\psi}_\mu [\alpha \gamma_\mu \partial_\nu + \beta \gamma_\nu \partial_\mu + \epsilon \gamma_\mu \not{\partial} \gamma_\nu + \zeta \not{\partial} \delta_{\mu\nu}] \psi_\nu . \tag{11}$$

We introduce spin projection operators

$$P^{3/2}_{\mu\nu} = \theta_{\mu\nu} - \tfrac{1}{3} \hat{\gamma}_\mu \hat{\gamma}_\nu, \qquad \hat{\gamma}_\mu = \gamma_\mu - \omega_\mu, \qquad \omega_\mu = \partial_\mu \not{\partial} \Box^{-1},$$

$$(P^{1/2}_{11})_{\mu\nu} = \tfrac{1}{3} \hat{\gamma}_\mu \hat{\gamma}_\nu, \qquad (P^{1/2}_{12})_{\mu\nu} = \frac{1}{\sqrt{3}} \hat{\gamma}_\mu \omega_\nu, \qquad \theta_{\mu\nu} = \delta_{\mu\nu} - \omega_\mu \omega_\nu,$$

$$(P^{1/2}_{21})_{\mu\nu} = \frac{1}{\sqrt{3}} \omega_\mu \hat{\gamma}_\nu, \qquad (P^{1/2}_{22})_{\mu\nu} = \omega_\mu \omega_\nu . \tag{12}$$

They have three properties

(i) orthonormality: $(P^I_{ij})_{\mu\nu} (P^J_{kl})_{\nu\rho} = \delta^{IJ} \delta_{jk} (P^J_{il})_{\mu\rho}$;

(ii) decomposition of unity: $P^{3/2} + P^{1/2}_{11} + P^{1/2}_{22} = I$. Hence the sum of the diagonal projection operators equals the unit operator I;

(iii) completeness: the operators P^I_{ij} span the space of all field equations of the form (11). This is obvious since the five P^I_{ij} span the four-dimensional space in (11) and they must satisfy one linear relation to cancel the $\partial_\mu \partial_\nu / \Box$ terms. We have chosen this set, but one may use any other set satisfying (i), (ii), and (iii) (for example, with $\delta_{\mu\nu} - (k_\mu \bar{k}_\nu + k_\nu \bar{k}_\mu)/(k \cdot \bar{k})$).

We now consider first massive spin 3/2 fields and then the massless case.

Massive fields: On-shell one has $\gamma \cdot \psi = \partial \cdot \psi = 0$, hence on-shell $\psi_\mu = P^{3/2}_{\mu\nu} \psi_\nu$ [use (ii)]. However, one cannot define as field equation $(P^{3/2} \not{\partial} - M)\psi = 0$ since $P^{3/2}$ contains $\Box^{-1}$ singularities. The choice $(P^{3/2} \Box^2 - M^2)\psi = 0$ is local (i.e., free from $\Box^{-1}$ singularities) but a higher derivative equation. However, if one writes for the field equation $(O\not{\partial} - M)\psi = 0$ such that $O\not{\partial}$ is a linear combination of the operators in (12), which is local, then iteration yields $O\not{\partial} O\not{\partial} \psi = M^2 \psi = \Box P^{3/2} \psi$. Hence $O\not{\partial}$ must be a square root of $\Box P^{3/2}$. This method is called the root method and is due to Ogievetski and Sokatchev. The most general square root is

$$0 = P^{3/2} + \alpha (P^{1/2}_{11} + P^{1/2}_{22} + \beta P^{1/2}_{12} + \beta^{-1} P^{1/2}_{21}). \tag{13}$$

(To derive this result, it is convenient to use the identity $\{\hat{\gamma}_\mu, \not\partial\} = 0$.) $0\not\partial$ is local if

$$2(1-\alpha) = -\sqrt{3}\alpha(\beta - \beta^{-1}). \tag{14}$$

(Special cases $\beta = 0$ or $\beta^{-1} = 0$ are included.) Thus one has obtained a one-parameter set of field equations of the form (11) which leads on-shell to $\gamma \cdot \psi = \partial \cdot \psi = 0$. However, one cannot use as action $\mathcal{L} = \bar{\psi}_\mu O_{\mu\nu}\not\partial\psi_\nu$ since the two gravitino fields do not appear symmetrically in this action, so that varying $\bar{\psi}_\mu$ would reproduce the field equations, but variation of ψ_ν would not.

There is a way out of this dilemma of finding an action, and that is to *shift the field in the field equation.* With parameter σ in

$$\psi_\mu = \varphi_\mu + \sigma\gamma_\mu\gamma \cdot \varphi \tag{15}$$

one has again a local field equation, and requiring that the new field operator $O'_{\mu\nu}$ is such that the new field equation

$$F_\mu = O'_{\mu\nu}\not\partial\varphi_\nu - M(\varphi_\mu + \sigma\gamma_\mu\gamma \cdot \varphi) = 0 \tag{16}$$

has the property that $\bar{\varphi}_\mu F_\mu$ is symmetric under a Majorana flip (see appendix C) one finds as action $\mathcal{L} = \bar{\psi}_\mu O'_{\mu\nu}\not\partial\psi_\nu - M(\bar{\psi}_\mu\psi_\mu + \sigma\bar{\psi} \cdot \gamma\gamma \cdot \psi) = 0$. The most general solution is again a one-parameter class, one element of which is given by (use $\sigma = (\beta + \beta^{-1})(2\sqrt{3} - \beta - 3\beta^{-1})^{-1}$ and let $\beta \to \infty$)

$$\mathcal{L} = -\frac{1}{2}\bar{\psi}(P^{3/2} - 2P^{1/2}_{11})\not\partial\psi - \frac{M}{2}\bar{\psi}(P^{3/2} - 2P^{1/2}_{11} - \sqrt{3}(P^{1/2}_{12} + P^{1/2}_{21}))\psi \tag{17}$$

while the general solution is obtained if one *shifts the field in this action* according to (15). The result in eq. (17) yields the massive Rarita–Schwinger action. Adding a coupling to an external source in order to obtain below propagators, one has

$$\mathcal{L} = -\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu\partial_\rho\psi_\sigma - \tfrac{1}{2}M\bar{\psi}_\mu\sigma^{\mu\nu}\psi_\nu - \bar{\psi}_\mu J^\mu. \tag{18}$$

It is clear that the mass terms in the action in eq. (18) cannot be brought in the form $\bar{\psi}_\mu\psi_\mu$ by the redefinition in eq. (15), but that one can redefine the fields in the field equation such that *there* one does find a diagonal mass term M.

From eq. (17) one immediately finds the propagator corresponding to eq. (18), since one only needs to invert separate spin blocks. One inverts the field equation $F_\mu = J_\mu = (P^{3/2} + P^{1/2}_{11} + P^{1/2}_{22})_{\mu\nu}J_\nu$ and finds

$$\psi_\mu = \left[\frac{P^{3/2}(\not\partial - M)}{\Box - M^2} - \frac{1}{\sqrt{3}M}(P^{1/2}_{12} + P^{1/2}_{21}) - \frac{2}{3M^2}P^{1/2}_{22}(\not\partial - M)\right]_{\mu\nu} J^\nu. \tag{19}$$

Hence, the propagator is given by

$$[(\delta_{\mu\nu} - \partial_\mu\partial_\nu M^{-2})(\not\partial - M) + \tfrac{1}{3}(\gamma_\mu - \partial_\mu M^{-1})(\not\partial + M)(\gamma_\nu - \partial_\nu M^{-1})](\Box - M^2)^{-1}. \tag{20}$$

Physically this means that $\Box$ and $\not\partial$ in $P^{3/2}$ have been replaced by M^2 and M, so that one finds exactly spin 3/2 on-shell. Similar results hold for spin 1, 2, 5/2, etc. This is a general result, valid for any massive spin. (Note that $(\not\partial + M)\not\partial = (\not\partial + M)M$ if $\square = M^2$.)

Massless fields: For massless fields there are local gauge invariances and hence some of the spin block matrices are singular. One finds the propagator which is sandwiched between sources by inverting the maximal nonsingular submatrices of the spin blocks. Each gauge invariance implies a source constraint and these source constraints are needed to cancel ghosts in the propagator. To prove that the linearized Rarita–Schwinger action for real massless gravitinos is the only action (up to field redefinitions as in eq. (15)) which is free from ghosts, one must consider all possible cases contained in eq. (11): the rank of the 2×2 spin 1/2 matrix 0, 1, 2 and the rank of the 1×1 spin 3/2 matrix being 0 or 1. If one requires that at the pole $k^2 = 0$ the residue is positive definite, then only the Rarita–Schwinger action is found. We shall not do this straightforward but tedious computation. Instead we show that (2) is without ghosts.

From eq. (17) with $M = 0$ one finds the field equation with external source J_μ

$$(P^{3/2} - 2P^{1/2}_{11})\not\partial\psi_\mu = J_\mu. \tag{21}$$

Clearly, there is the gauge invariance $\delta\psi_\mu = (P^{1/2}_{2j})_{\mu\nu}\chi_\nu$ where χ_ν is arbitrary, since replacing ψ_μ by $\delta\psi_\mu$ the left hand side vanishes. Thus there is the gauge invariance $\delta\psi_\mu = \partial_\mu\epsilon$. Acting with $P^{1/2}_{j2}$ on this equation, the left hand side again vanishes, and one finds the source constraint $P^{1/2}_{j2}J = 0$ (for $j = 1, 2$), which is equivalent to

$$\partial_\mu J^\mu = 0. \tag{22}$$

Inverting the spinblocks, the propagator becomes in k-space

$$\mathrm{i}J^{\mathrm{T}}_\mu(-k)C(P^{3/2} - \tfrac{1}{2}P^{1/2}_{11})_{\mu\nu}\frac{\not k}{k^2}J_\nu(k) = \bar{J}_\mu\,\frac{(-\frac{1}{2}\gamma_\nu\not\partial\gamma_\mu)}{\square}\,J_\nu. \tag{23}$$

For a Majorana source $\bar{J}(x) = J^{\mathrm{T}}C$, one has $J^{\mathrm{T}}(-k)C = (J(k))^\dagger\gamma_4$ which one may define to be $\bar{J}(k)$. To show that the residue $\bar{J}_\mu(k)\gamma_\nu\not k\gamma_\mu J_\nu(k)$ is positive definite, we use the spin 1/2 result that $\not k = u^+\bar{u}^+ + u^-\bar{u}^-$, and the decomposition of the Kronecker delta function

$$\delta_{\mu\nu} = \epsilon^+_\mu(\epsilon^+_\nu)^* + \epsilon^-_\mu(\epsilon^-_\nu)^* + (k_\mu\bar{k}_\nu + \bar{k}_\mu k_\nu)(k\cdot\bar{k})^{-1} \tag{24}$$

valid on-shell where $k^2 = 0$. We rewrite the residue as $\bar{J}_{\mu'}\delta_{\mu'\mu}\gamma_\nu\not k\gamma_\mu\delta_{\nu\nu'}J_{\nu'}$. The terms in eq. (24) with k_ν or k_μ do not contribute since $\not k u^\pm = 0$ and $k\cdot J = 0$. For the rest one finds, using

$$\not\epsilon^+u^+ = 0, \qquad \not\epsilon^-u^- = 0 \tag{25}$$

that the residue is given by

$$|\bar{u}^-\not\epsilon^+\epsilon^+\cdot J|^2 + |\bar{u}^+\not\epsilon^-\epsilon^-\cdot J|^2. \tag{26}$$

Hence, there are no ghosts in linearized Rarita–Schwinger theory, and two physical modes.

1.14. Higher spin theory

Only for $N>8$ do the extended supergravity theories have a $SU_3\times SU_2\times U_1$ internal symmetry group. This is obvious since O(9) contains $O(6)\times O(3)$ and O(6) is isomorphic to SU(4) which contains $SU_3\times U_1$. However, for $N>8$ one has also spin 5/2 fields and more than one graviton. The question arises whether one can repeat history and construct consistent field theories for spin 5/2 and higher. Free field actions for any higher spin do indeed exist (for massive fields, see Hagen and Singh; for massless fields see Fang and Fronsdal [258, 259]).

We consider as spin 5/2 field a symmetric tensor spinor $\psi^a_{\mu\nu}=\psi^a_{\nu\mu}$. Coupling it to an external source $J^a_{\mu\nu}$, one can ask whether an action exists which leads to a propagator without ghosts or tachyons. (This is equivalent to requiring positive energy of the classical theory, and easier to prove.) The result is that there is an action, given for massless Majorana spinors by

$$\mathscr{L}=-\tfrac{1}{2}\bar\psi_{\mu\nu}\not\partial\psi_{\mu\nu}-\bar\psi_{\mu\nu}\gamma_\nu\not\partial\gamma_\lambda\psi_{\lambda\mu}+2\bar\psi_{\mu\nu}\gamma_\nu\partial_\lambda\psi_{\lambda\mu}+\tfrac{1}{4}\bar\psi_{\lambda\lambda}\not\partial\psi_{\mu\mu}-\bar\psi_{\lambda\lambda}\partial_\mu\gamma_\nu\psi_{\mu\nu}+\bar\psi_{\mu\nu}J_{\mu\nu}. \tag{1}$$

It is invariant under the following local gauge transformations

$$\delta\psi_{\mu\nu}=\partial_\mu\epsilon_\nu+\partial_\nu\epsilon_\mu \quad \text{with} \quad \gamma^\mu\epsilon_\mu=0, \tag{2}$$

as first found by J. Schwinger.

In fact, this action is unique as an analysis based on spin projection operators shows. The field equation can be written in a Christoffel-like way as

$$\gamma^\mu(\partial_\mu\psi_{\alpha\beta}-\partial_\alpha\psi_{\mu\beta}-\partial_\beta\psi_{\mu\alpha})=0. \tag{3}$$

Since the free field action is invariant under local spin 3/2 gauge transformations, one might repeat the analysis of supergravity and see whether minimal coupling to gravity by means of $\partial_\mu\to D_\mu$ leads to a variation of the action proportional only to the Einstein tensor. In that case one can add the Einstein action and define the tetrad variation such that the total system is invariant to lowest order (afterwards one should then follow the Noether procedure). However, a simple calculation [153] shows that under $\delta\psi_{\mu\nu}=D_\mu\epsilon_\nu+D_\nu\epsilon_\mu$ with $\gamma\cdot\epsilon=0$ one finds

$$\begin{aligned}\delta I=G_{\mu b}\big[&\tfrac{3}{2}\bar\epsilon_\nu\gamma^b\psi^{\mu\nu}-\bar\epsilon^\mu\gamma^b\psi^\lambda_{\ \lambda}+\tfrac{1}{2}\bar\epsilon_\mu\gamma^\lambda\psi_{\lambda b}+\tfrac{1}{2}\bar\epsilon_\nu\gamma_5\gamma_\sigma\psi_{\mu\rho}\epsilon^{b\nu\rho\sigma}-\tfrac{4}{3}\bar\epsilon_\lambda\gamma_\nu\psi^{\nu\lambda}e^{b\mu}\big]\\ &+\bar\epsilon_\nu\gamma_a\psi_{\lambda b}C^{\nu\lambda ab}-\tfrac{1}{2}\bar\epsilon_\nu\gamma_5\gamma_a\psi_{\lambda b}\epsilon^{\nu\lambda\rho\sigma}C_{\rho\sigma ab}.\end{aligned} \tag{4}$$

The last two terms are proportional to the Weyl tensor and seem to exclude a consistent coupling between spin 2 and spin 5/2.

Approaching the problem from the matter end, no matter systems have been found which are invariant under transformations with constant parameters ϵ_μ satisfying $\gamma\cdot\epsilon=0$ [153] (for such systems one could couple $\psi_{\mu\nu}$ to the Noether current and iterate), while also taking for spin 5/2 a vierbein spinor (nonsymmetric $\psi^a_{m\mu}$) did not improve matters [144].

The massive spin 5/2 system contains a mass term for $\psi_{\mu\nu}$ plus an extra spin 1/2 field χ and is, again, unique. Quite generally, for any higher spins, these extra auxiliary fields χ couple only in the mass term to the ψ field; for spin 5/2 on has

$$\mathscr{L}(\text{mass}) = -M(\bar{\psi}_{\mu\nu}\psi_{\mu\nu} - \tfrac{3}{4}\bar{\psi}\cdot\gamma\gamma\cdot\psi - \tfrac{7}{4}\bar{\psi}\psi - \tfrac{16}{3}\bar{\chi}\psi - \tfrac{32}{9}\bar{\chi}\chi). \tag{5}$$

Hence here there is already a so-called van Dam–Veltman mass-discontinuity in the action.

For spin 3, a massless and massive theory exist, too. In this theory the limit $m \to 0$ cannot even be taken in the sandwiched propagator. (For spin 2 there is a *finite* discontinuity.)

Our conclusion is that it seems, at this moment, that Nature stops at spin 2.

2. Quantum supergravity

2.1. Introduction

Supergravity can be covariantly quantized by the same methods as any other gauge theory, provided that the gauge algebra closes. This closure is obtained by adding auxiliary fields to the classical action and these fields remain auxiliary fields in the effective quantum action, but become propagating in the counter terms. The covariant quantization methods can be justified by a Hamiltonian path-integral approach as we shall see, and, once quantized, unitarity and gauge invariance can be proven. The simplest way to do so is to use the Becchi–Rouet–Stora–Tyutin symmetry (BRST-symmetry) of the quantum action which is the quantum extension of the classical gauge invariance. We will also consider the BRST formalism for gauge theories whose gauge algebra does not close, for example simple supergravity without S, P, A_m.

Gravitational theories have a dimensional coupling constant, Newton's constant, and this fact alone is sufficient to rule out ordinary renormalizability for the following reasons. Due to the dimensional character of the gravitational coupling constant, one-loop (and higher loop) divergences have a different *functional* form than the quantum action so that one cannot absorb these divergences back into the original quantum action by rescaling of the physical parameters. For example, in Einstein gravity the one-loop divergences are on-shell proportional to $S^{\text{div}} = \alpha R_{\mu\nu}^2 + \beta R^2$ and the two-loop divergences are proportional to $(R_{\mu\nu\alpha\beta}R^{\alpha\beta\rho\sigma}R_{\rho\sigma}{}^{\mu\nu})$ etc., whereas the original action is proportional to $\kappa^{-2}(g^{1/2}R)$. Clearly, one cannot reabsorb the two-loop divergences back into $\mathscr{L}$ by rescaling of $g_{\mu\nu}$ and κ alone. (Expanding S^{div} into powers of $h_{\mu\nu}$ where $\kappa h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, and subsequently replacing each field $h_{\mu\nu}$ by an infinite set of tree graphs with physical momenta and physical polarization tensors at the end of the trees, one finds the divergences of the n-point Green's functions on-shell.) One also finds that any n-point Green's function is divergent from the two-loop level on. At the one-loop level, $\Delta\mathscr{L} = 0$ since $R_{\mu\nu} = 0$, where $R_{\mu\nu} = 0$ because the fields $g_{\mu\nu}$ satisfy $R_{\mu\nu}(g_{\mu\nu}) = 0$ since they represent infinite sets of tree graphs with physical end-momenta and polarization tensors. For more details, we refer to ref. [606], where the relation between normal field theory and the background field method is given.

Thus gravitational theories are never renormalizable and only the two extremal cases are possible: finiteness of the S-matrix or the infinities in the S-matrix cannot be removed. The latter case is usually called "non-renormalizability of the S-matrix" but we stress that it is not the counterpart of renormalizability but of finiteness. Finiteness of the S-matrix is of course a much stronger property than renormalizability and one can expect it to occur only when "miraculous" cancellations of divergences occur due to a symmetry of the theory. It is here that supergravity has had impressive successes. First discovered by a direct calculation and later explained by various equivalent theoretical proofs, it was found that pure simple supergravity as well as the pure-extended supergravities (irreducible supergravities with one graviton, N gravitinos and specified numbers of lower spin fields) are one-loop finite. They are even two-loop finite. This is a better result than obtained so far for pure ordinary Einstein gravity which is known to be one-loop finite (but which might be two-loop finite; this constitutes one of the major unsolved problems in quantum gravity). Thus supergravity has taken the lead in the loop race and the reason is that the extra Fermi–Bose symmetry allows one to prove extra cancellations. Even if ultimately pure Einstein gravity will turn out to be two-loop finite, supergravity has the advantage that the pure extended supergravities contain not only gravitons but also gravitinos, vectors, spinors and scalars. Thus one has the possibility of a finite quantum theory describing gravity as well as the other interactions which might be phenomenologically correct.

About the need to quantize gravity, and now also supergravity, various opinions exist. Some people feel that one should not quantize at all. Others believe that Einstein's theory of gravity is a phenomenological theory such as thermodynamics and that one should try to quantize the underlying microscopical more fundamental theory. In this connection there have been endeavours to consider the graviton as a bound state of two photons or of two gravitinos [310, 311]. Another point of view is that gravity is something like van der Waals forces, present where matter is but not existing as free radiation modes. The recent results from a binary pulsar, which seems to lose energy through gravitational radiation, seem to refute this proposal. Yet other ideas are to replace the points in spacetime by twistors and to try to quantize those. A point of ordinary spacetime is described by the intersection of two twistors, and hopefully the quantum fluctuations give rise to "fluttering" lightcones rather than to fuzzy points in spacetime. Even further goes the proposal to use path-integral quantization in the space of all geometries. In this connection by far the most appealing has been Hawking's spacetime foam: very small strongly curved regions of spacetime which average out to flat spacetime over larger distances.

Interesting as these suggestions are in their own right, it would be preferable if the same ideas of quantum field theory which have worked so well in quantum electrodynamics and, later, in unified theories of the weak and electromagnetic interactions, would also turn out to describe gravity with the same success. Of course there are two major differences between gravity and other interactions. First of all, the metric determines which points are spacelike and which are timelike and it is not at all clear that one ends up with the same light cones due to the quantized metric as those due to the classical metric one starts with. A second problem is that only in special classical spacetimes (those with a timelike Killing vector field) can one define the absorption and creation operators of second quantization. These two, and other related problems are serious, but our strategy is to just go ahead and see how far one gets. As it turns out, and we will see, elegant results are obtained, and sitting at one's desk and seeing the cancellations occur, one cannot resist the feeling that something important is going on. Maybe what follows is not the ultimate way of interpreting quantum gravity but it may be the solution in disguise.

2.2. General covariant quantization of gauge theories

We begin by reviewing the covariant quantization of gauge theories. Suppose one has given a classical action $I(\mathrm{cl})$ which depends on a set of gauge fields ϕ^i and which is invariant under local gauge transformations with parameters ξ^α

$$\delta\phi^i = R^i{}_\alpha(\phi)\xi^\alpha. \tag{1}$$

We use DeWitt's summation convention in which i and α denote indices as well as spacetime points and refer to refs. [514, 303, 304]. The symbol R may contain derivatives and depend on gauge fields, for example it could represent the Yang–Mills law $\delta W_\mu{}^a = (\partial_\mu \delta^a_c + gf^a{}_{bc} W_\mu{}^b)\Lambda^c$. The parameters ξ^α depend on x^μ and represent the parameters of general coordinate, local Lorentz and local supersymmetry.

One adds a gauge fixing term for each local gauge invariance. We will consider this I(fix) to be quadratic in the gauge choices $F_\alpha(\phi)$

$$I(\text{fix}) = \tfrac{1}{2} F_\alpha(\phi) \gamma^{\alpha\beta} F_\beta(\phi). \tag{2}$$

This is not necessary, but useful in order to be able to define propagators. The matrix $\gamma^{\alpha\beta}$ is taken to be independent of the quantum gauge fields ϕ^i. It may depend on external fields, namely when one considers a quantized gravitino in a background gravitational field, but the correct theory for the case that $\gamma^{\alpha\beta}$ depends on ϕ^i is not yet known. The gauge fixing term is needed to remove the degeneracy in the path integral. For practical purposes, its use is that the kinetic terms of the sum $I(\text{cl}) + I(\text{fix})$ are nonsingular, so that one can invert them to obtain the propagators of the theory.

By varying the gauge choices one obtains the ghost action I(ghost):

$$I(\text{ghost}) = C^{*\alpha} F_{\alpha,j} R^j{}_\beta C^\beta. \tag{3}$$

Thus one varies the gauge function, replaces the gauge parameter in eq. (1) by a ghost field and multiplies from the left with another ghost field which is denoted by $C^{*\alpha}$. In a path-integral formalism, this result is obtained for bosonic symmetries by exponentiating the normalization determinant, see subsection 4, in front of the gauge-fixing delta function $\det(F_{\alpha,j} R^j{}_\beta)\, \delta(F_\alpha - b_\alpha)$ as follows

$$\det F_{\alpha,j} R^j{}_\beta = \int \prod_{\gamma,\delta} \mathrm{d}C_2^\delta\, \mathrm{d}C_1^\gamma \exp\!\left(\frac{\mathrm{i}}{2} C_1^\alpha F_{\alpha,j} R^j{}_\beta C_2^\beta\right).$$

Integration over anticommuting variables being defined by $\int \mathrm{d}C^\alpha = 0$, $\int \mathrm{d}C^\alpha\, C^\beta = \delta^{\alpha\beta}$, one then finds, upon expanding the exponential, that only the term with four C_1 and four C_2 fields is nonzero and yields indeed the determinant of $(F_{\alpha,j} R^j{}_\beta)$. The normalization factor i/2 cancels in computations. If one has both bosonic and fermionic symmetries, one defines a superdeterminant (see appendix) and still arrives at eq. (3). Since, as we shall see, ghost fields C^β have opposite statistics from the parameters ξ^β, one must specify where to put the C^β. We will always put C^β in (3) to the far right.

The effective quantum action is thus given by

$$I(\text{qu}) = I(\text{cl}) + \tfrac{1}{2} F_\alpha \gamma^{\alpha\beta} F_\beta + C^{*\alpha} F_{\alpha,j} R^j{}_\beta C^\beta. \tag{4}$$

It should be observed that the sum over α and β is over the whole gauge group. Thus, for example, in simple supergravity Lorentz ghosts interact with supersymmetry antighosts, etc.

The quantum equivalent of the classical gauge invariance given by eq. (1) is the global nonlinear Becchi–Rouet–Stora–Tyutin transformations.† This *quantum symmetry* uses a constant anticommuting

† See Proc. 1975 Erice Conf., eds Velo and Wightman, pages 331, 332.

parameter Λ and reads

$$\delta\phi^i = R^i{}_\alpha C^\alpha \Lambda \tag{5a}$$

$$\delta C^\alpha = -\tfrac{1}{2} f^\alpha{}_{\beta\gamma} C^\gamma \Lambda C^\beta \tag{5b}$$

$$\delta C^{*\alpha} = \Lambda F_\beta \gamma^{\beta\alpha}. \tag{5c}$$

The invariance of (4) under (5) is easy to demonstrate. I(cl) is invariant under eq. (5a). The variation of I(fix) and of $C^{*\alpha}$ yield

$$F_\alpha \gamma^{\alpha\beta}(F_{\beta,j} R^j{}_\beta C^\beta \Lambda) + (\Lambda F_\delta \gamma^{\delta\alpha}) F_{\alpha,j} R^j{}_\beta C^\beta = 0 \tag{6}$$

which vanishes since the variation of an action is a bosonic object. The remaining terms come from varying $F_{\alpha,j} R^j{}_\beta C^\beta$. The variation of $F_{\alpha,j}$ in (4) vanishes since

$$F_{\alpha,jk} = (-1)^{jk} F_{\alpha,kj}, \qquad R^k{}_\delta C^\delta \Lambda R^j{}_\beta C^\beta = (-1)^{(k+1)(j+1)+k+1+j+1}(k \leftrightarrow j) \tag{7}$$

so that the symmetries of both terms in (j, k) are opposite. For the variation of $R^j{}_\beta C^\beta$ one finds

$$R^j{}_{\beta,k} R^k{}_\gamma C^\gamma \Lambda C^\beta - \tfrac{1}{2} f^\beta{}_{\rho\sigma} R^j{}_\beta C^\sigma \Lambda C^\rho = 0. \tag{8}$$

Closure of the gauge algebra implies that the commutator of the local gauge transformations is again a local gauge transformation, thus again of the form (1)

$$\begin{aligned}[][\delta(\eta), \delta(\xi)]\phi^i &= R^i{}_{\alpha,k} R^k{}_\beta \eta^\beta \xi^\alpha - \xi \leftrightarrow \eta \\ &= R^i{}_\gamma f^\gamma{}_{\rho\sigma}(\phi) \eta^\sigma \xi^\rho.\end{aligned} \tag{9}$$

The structure constants may depend on ϕ^i and will be called *structure functions*. These equations are the local equivalent of the (anti) commutators which define superalgebras. It is now clear why (8) vanishes. Replacing in eq. (9) η^β by $C^\beta \Lambda$ and ξ^α and $C^\alpha \Lambda_1$ one finds the combination

$$(C^\beta \Lambda)(C^\alpha \Lambda_1) - (C^\beta \Lambda_1)(C^\alpha \Lambda) = 2(C^\beta \Lambda C^\alpha)\Lambda_1. \tag{10}$$

(Note that interchanging ξ and η in eq. (9) does *not* mean interchanging also their indices β and α!) Thus the first term in eq. (8) can be written as a commutator, and applying eq. (9), one completes the proof that eq. (4) is invariant under eq. (5).

It is instructive to verify that indeed relations such as eq. (10) follow from the formal symmetries of the ghost fields for the case of Yang–Mills theory. We leave it as an exercise to verify that $f^{abc}(D_\mu C)^b \Lambda C^c$ is indeed antisymmetric in both ghosts although one of them is differentiated.

BRST transformations are nilpotent. Applying eq. (5a) and eq. (5b) to (5a) one finds zero due to eq. (8). Variation of δC^α yields

$$-\tfrac{1}{2} f^\alpha{}_{\beta\gamma,k} R^k{}_\delta C^\delta \Lambda_1 C^\gamma \Lambda C^\beta + \tfrac{1}{4} f^\alpha{}_{\beta\gamma} f^\gamma{}_{\rho\sigma} C^\sigma \Lambda_1 C^\rho \Lambda C^\beta + \tfrac{1}{4} f^\alpha{}_{\beta\gamma} C^\gamma \Lambda (f^\beta{}_{\rho\sigma} C^\sigma \Lambda_1 C^\rho). \tag{11}$$

Since the structure functions $f^{\beta}{}_{\rho\sigma}$ are antisymmetric in $(\rho\sigma)$ except when both denote fermionic symmetries, one finds that eq. (11) is equal to

$$\tfrac{1}{2}(-f^{\alpha}{}_{\beta\gamma,k}R^{k}{}_{\delta}+f^{\alpha}{}_{\beta\lambda}f^{\lambda}{}_{\gamma\delta})C^{\delta}\Lambda_{1}C^{\gamma}\Lambda C^{\beta}=0. \tag{12}$$

That this expression vanishes follows from the double commutator of gauge transformations and will be derived in subsection 8.

For antighosts, the BRST transformation is not nilpotent as given by eq. (5), but one can obtain also here nilpotency by introducing auxiliary fields once again. For example, in Yang–Mills theory one defines

$$\mathscr{L}(\mathrm{qu})=\mathscr{L}(\mathrm{cl})+\tfrac{1}{2}\boldsymbol{g}^{2}+\boldsymbol{g}(\partial\cdot\boldsymbol{W})+\boldsymbol{C}^{*}\partial_{\mu}D_{\mu}\boldsymbol{C} \tag{13}$$

and one defines $\delta\boldsymbol{g}=0$, $\delta\boldsymbol{C}^{*}=\Lambda\boldsymbol{g}$. The general case is obvious.

Thus, for any theory with a closed gauge algebra one can define nilpotent BRST transformations which are the quantum equivalent of the classical gauge invariances. By means of them one can derive the Ward identities which are needed for the proofs of unitarity and gauge invariance. The nilpotency of BRST transformations of C^{α} and ϕ^{i} is used to derive Ward identities for one-particle irreducible Green functions. The anticommuting nature of the constant Λ has led some authors to speculate on a deep relation between supergravity and BRST transformations, but the two have really nothing to do with each other: *any* gauge theory is BRST invariant at the quantum level.

2.3. *Covariant quantization of simple supergravity*

In this subsection we apply the general formalism of the previous subsection to simple supergravity. Since we assumed that the gauge algebra closes, we consider as gauge fields not only the graviton and gravitino but also the minimal set of auxiliary fields S, P, A_m.

There are three local symmetries and hence three pairs of ghost-antighost fields [236]

(i) general coordinate ghosts C^{ν}, C^{*}_{ν}

(ii) local Lorentz ghosts C^{mn}, C^{*}_{mn}

(iii) local supersymmetry ghost C^{a}, C^{*}_{a}.

The complex vector C^{ν} and the complex tensor C^{mn} are anticommuting with themselves but the complex four-component spin 1/2 ghost C^{a} is *commuting* with itself. This not only follows from general principles but can also be checked by verification of a Ward identity, as we shall show (page 257).

As gauge fixing term a convenient choice is

$$\mathscr{L}(\mathrm{fix})=-\tfrac{1}{4}(\partial_{\mu}\sqrt{g}g^{\mu\nu})^{2}+\alpha(e_{a\mu}-e_{b\nu}\delta^{b}_{\mu}\delta^{\nu}_{a})^{2}+\tfrac{1}{4}\bar{\psi}^{\mu}\gamma_{\mu}\partial\!\!\!/(\gamma^{\nu}\psi_{\nu}). \tag{1}$$

The advantage of this choice is that the propagators for graviton and gravitino become very simple. The kinetic terms of the Hilbert action$+\mathscr{L}(\mathrm{fix})$ read in terms of $h_{\mu\nu}$ with $g_{\mu\nu}=\delta_{\mu\nu}+\kappa h_{\mu\nu}$

$$\mathscr{L}^{(2)}_{\mathrm{kin}}=-\tfrac{1}{8}h^{2}_{\mu\nu,\rho}+\tfrac{1}{16}h^{2}_{\mu\mu,\rho}+\alpha(c_{a\mu}-c_{\mu a})^{2} \tag{2}$$

where $e_{a\mu}=\delta_{a\mu}+\kappa c_{a\mu}$. From $h_{\mu\nu}=c_{\mu\nu}+c_{\nu\mu}+\mathscr{O}(c^{2})$ one therefore finds for the $c_{a\mu}$ propagator for

$\alpha \to \infty$

$$P^{(2)}_{\mu\nu,\rho\sigma} = \langle c_{\mu\nu} c_{\rho\sigma} \rangle = \frac{-\mathrm{i}}{2}(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\nu}\delta_{\rho\sigma})k^{-2}. \tag{3}$$

For comparison we note that the spin 0 propagator reads $P^0 = -\mathrm{i}k^{-2}$. The gravitino propagator is obtained from the terms bilinear in $\psi_\mu{}^a$

$$\mathscr{L}^{3/2}_{\text{kin}} = -\frac{e}{4}\bar{\psi}_\mu[\gamma^\mu\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho\gamma^\mu]\,\partial_\rho\psi_\sigma + \frac{e}{4}\bar{\psi}_\mu\gamma^\mu\gamma^\rho\gamma^\sigma\partial_\rho\psi_\sigma$$
$$= \frac{e}{4}\bar{\psi}_\mu\gamma^\sigma\gamma^\rho\gamma^\mu\partial_\rho\psi_\sigma \tag{4}$$

where e is the determinant of e^a_μ. The field equation is thus $\frac{1}{2}\gamma^\sigma \not\partial \gamma^\mu \psi_\sigma$ and its inverse yields the propagator [510]

$$P^{3/2}_{\mu\nu} = \langle \psi_\mu \bar{\psi}_\nu \rangle = \tfrac{1}{2}\gamma_\nu \not k \gamma_\mu k^{-2}. \tag{5}$$

For comparison we note that the spin 1/2 propagator reads $P^{1/2} = -\not k k^{-2}$.

Instead of (1), one sometimes uses the part linear in $h_{\mu\nu}$ of $\sqrt{g}g^{\mu\nu}$. Also one often chooses instead of $\gamma^\mu\psi_\mu$ the linearization $\gamma^a\delta^\mu{}_a\psi_\mu$ with constant γ^a. They yield the same propagators. For $\alpha \to \infty$, the antisymmetric part of the vierbein is frozen out. However, as we shall see, the *Lorentz ghosts derived from the antisymmetric vierbein should not be neglected*, even in the limit that α tends to infinity.

In n dimensions with the indices of $h_{\mu\nu}$ running from 1 to n, the graviton propagator acquires a factor $2/(n-2)$ in front of $\delta_{\mu\nu}\delta_{\rho\sigma}$. The gravitino propagator in eq. (4) in n-dimensions yields the n-dependent result [556]

$$P^{3/2}_{\mu\nu} = \frac{1}{n-2}[\gamma_\nu \not k \gamma_\mu + (4-n)\{\delta_{\mu\nu}\not k - 2k_\mu k_\nu k^{-2}\not k\}]k^{-2}. \tag{6}$$

However we keep all indices of $\psi_\mu{}^a$ and $h_{\mu\nu}$ always four-dimensional and only regularize by letting x^μ and momenta p^μ become n-dimensional (see subsection 7).

We now turn to the ghost action. From $\mathscr{L}$(fix) we find the gauge choices

$$F_\alpha = \{-\partial_\mu(\sqrt{g}g^{\mu\nu}),\, e_{a\mu} - e_{b\nu}\delta^b_\mu\delta^\nu_a,\, -\gamma\cdot\psi\}. \tag{7}$$

The ghost action is obtained by varying F_α with respect to all gauge symmetries and sandwiching the resulting matrix with an antighost row and a ghost column. Thus one finds for infinitesimal variations

$$\delta(\sqrt{g}g^{\mu\nu}) = -\sqrt{g}[(\partial_\alpha\xi^\mu)g^{\alpha\nu} + (\partial_\alpha\xi^\nu)g^{\mu\alpha}] + \partial_\alpha(\xi^\alpha\sqrt{g}g^{\mu\nu})$$
$$-\tfrac{1}{2}\sqrt{g}(\bar{\epsilon}\gamma^\mu\psi^\nu + \bar{\epsilon}\gamma^\nu\psi^\mu) + \tfrac{1}{2}\bar{\epsilon}\gamma\cdot\psi\sqrt{g}g^{\mu\nu}. \tag{8}$$

$$\delta(\gamma\cdot\psi) = \xi^\alpha\partial_\alpha(\gamma\cdot\psi) + \left(\not D + \frac{\mathrm{i}}{2}\not A\gamma_5 - 2\eta\right)\epsilon + \tfrac{1}{2}\lambda\cdot\sigma\gamma\cdot\psi. \tag{9}$$

$$\delta(e_{a\mu} - e_{b\nu}\delta^b_\mu\delta^\nu_a) = \{(\partial_\mu\xi^\alpha)e_{a\alpha} + \xi^\alpha\partial_\alpha e_{a\mu} + \lambda_a{}^b e_{b\mu} + \tfrac{1}{2}\bar{\epsilon}\gamma_a\psi_\mu\} - \{a \to b, \mu \to \nu\}\delta^b_\mu\delta^\nu_a. \tag{10}$$

For the kinetic terms of the ghosts one clearly obtains after replacing ξ^α by C^α, ϵ^a by C^a and λ^{ab} by C^{ab}

$$\mathscr{L}(\text{ghost})^{\text{kin}} = C^*_\nu \Box C^\nu - \bar{C}\not{D}C + 2C^*_{ab}(C_{ab} + \partial_b C_\nu \delta_a{}^\nu). \tag{11}$$

Thus the general coordinate vector ghost is itself no longer a gauge field. The bar on $\bar{C}$ denotes Dirac's bar: $C = C^\dagger\gamma_4$ in our conventions. Clearly, *one must rediagonalize the Lorentz ghost field* $C_{ab} + \frac{1}{2}\partial_b C_a - \frac{1}{2}\partial_a C_b \equiv C'_{ab}$. One then finds new vertices originating from the last term in eq. (9) [556] of the form $(\bar{C}\not{\psi})\partial_\mu C_\nu$. The ghost propagators are easily found

$$\langle C^\nu C^*_\rho\rangle = -\mathrm{i}\delta^\nu{}_\rho k^{-2} \qquad \langle C^a C^*_b\rangle = \delta^a{}_b(-\not{K})k^{-2}$$
$$\langle C'_{cd}C^*_{ab}\rangle = (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}). \tag{12}$$

The vertices of quantum simple supergravity can be worked out from the foregoing.

Thus we have obtained at this point the propagators and vertices of the theory, and this defines quantum supergravity in a perturbation expansion. We note that in $I(\text{fix}) = \frac{1}{2}F_\alpha\gamma^{\alpha\beta}F_\beta$ we indeed did choose $\gamma^{\alpha\beta}$ to be field-independent

$$\gamma^{\alpha\beta} = \{-\tfrac{1}{4}\delta^{\nu\nu'}, \alpha\delta_{aa'}\delta^{\mu\mu'}, \tfrac{1}{4}\not{\partial}^{aa'}\}. \tag{13}$$

We now consider the role of the auxiliary fields in somewhat more detail. They appear in $I(\text{qu})$ due to (9) as follows

$$-\tfrac{1}{3}(\det e)(S^2 + P^2 - A_m^2) - \bar{C}\left[\frac{\mathrm{i}}{2}\not{A}\gamma_5 C + \frac{2}{3}(S - \mathrm{i}\gamma_5 P - \mathrm{i}\not{A}\gamma_5)\right]C. \tag{14}$$

Clearly, S, P, A_m are still nonpropagating at the quantum level and can be eliminated using their quantum field equations

$$eS = -\bar{C}C, \qquad eP = \bar{C}\mathrm{i}\gamma_5 C, \qquad eA_m = \tfrac{1}{4}\bar{C}\mathrm{i}\gamma_5\gamma_m C. \tag{15}$$

Reinserting the quantum field equations of the auxiliary fields in the quantum action, new four-ghost couplings emerge,

$$\mathscr{L}(\text{four ghost}) = \frac{e}{3}(S^2 + P^2 - A_m^2) = \frac{e}{3}\left[(\bar{C}C)(\bar{C}C) - (\bar{C}\gamma_5 C)(\bar{C}\gamma_5 C) + \frac{1}{16}(\bar{C}\gamma_5\gamma_m C)(\bar{C}\gamma_5\gamma_m C)\right]. \tag{16}$$

Let us now bring ghosts and antighosts together in the same spinor covariants. To this purpose we define the Majorana conjugates of C by $\hat{C} = C^{\mathrm{T}}\mathscr{C}$, where $\mathscr{C}$ is the charge conjugation matrix. Similarly, the Majorana conjugate of the spinor $\bar{C}$ is defined by $\hat{\bar{C}} = \bar{C}\mathscr{C}^{-1}$. In this case, $\bar{C}C = -\hat{C}\hat{\bar{C}}$ since these ghosts commute. After a Fierz rearrangement, not forgetting an extra $-$ sign since the C commute, one finds the very simple result

$$\mathscr{L}(\text{four-ghost}) = -\frac{5}{32e}(\bar{C}\gamma_m\hat{\bar{C}})(\hat{C}\gamma^m C). \tag{17}$$

One can also obtain BRST invariance of $I(\mathrm{qu})$ without S, P, A_m. In addition to eq. (17), one only needs to substitute eq. (15) into the transformation rules. This we will discuss in subsection 2.8.

2.4. Path-integral quantization of gauge theories

The approach to covariant quantization sketched up to this point used the notion of BRST symmetry which can be used to derive Ward identities from which one may prove unitarity and gauge invariance. An alternative approach is path-integral quantization. Since it exhibits some quantization aspects in a clear way, and explains why $I(\mathrm{qu})$ is as it is, we will also consider this approach.

One starts with the path-integral

$$Z = \int [\mathrm{d}\phi^i] \exp(\mathrm{i}I(\mathrm{cl}))$$

$$\phi^i = (e^m{}_\mu, \psi_\mu{}^a, S, P, A_m) \tag{1}$$

and multiplies by unity

$$1 = \int \prod_\alpha \mathrm{d}\xi^\alpha \prod_\beta \delta(F_\beta(\phi^i(\xi)) - a_\beta)\,\mathrm{sdet}(F_{\alpha,j}R^j{}_\beta) \tag{2}$$

where $\phi^i(\xi) = \phi^i + R^i{}_\alpha\xi^\alpha$ and sdet denotes the superdeterminant (see appendix). Using gauge invariance and replacing in eq. (1) $[\mathrm{d}\phi^i]$ by $[\mathrm{d}\phi^i(\xi)]$, one obtains, after interchanging the $[\mathrm{d}\phi^i(\xi)]$ and $\mathrm{d}\xi^\alpha$ integrations and dropping a field-independent factor, writing $[\mathrm{d}\phi^i]$ instead of $[\mathrm{d}\phi^i(\xi)]$

$$Z = \int [\mathrm{d}\phi^i\, \mathrm{d}C^\alpha\, \mathrm{d}C^{*\alpha}] \exp\{\mathrm{i}[I(\mathrm{cl}) + I(\mathrm{ghost})]\}\delta(F_\beta - a_\beta). \tag{3}$$

Again one may multiply by unity

$$1 = \int [\mathrm{d}b_\alpha] \exp\left(\frac{\mathrm{i}}{2} b_\alpha \gamma^{\alpha\beta} b_\beta\right) (\mathrm{sdet}\,\gamma)^{1/2}. \tag{4}$$

If one can show that eq. (3) is independent of a_α and if the matrix $\gamma^{\alpha\beta}$ is independent of quantum fields, one may replace $\delta(F_\alpha - a_\alpha)$ by $\delta(F_\alpha - b_\alpha)$ and integrating over $[\mathrm{d}b_\alpha]$ one arrives at the same result as before

$$Z = \int [\mathrm{d}\phi^i\, \mathrm{d}C^\alpha\, \mathrm{d}C^*_\alpha] (\mathrm{sdet}\,\gamma)^{1/2} \exp(\mathrm{i}I(\mathrm{qu})). \tag{5}$$

Thus the restriction that $\gamma^{\alpha\beta}$ be independent of ϕ^i clearly emerges from this formalism.

We now prove that (3) is indeed independent of a_α. A necessary and sufficient criterion is that under a gauge transformation $\delta\phi^i = R^i{}_\alpha N^{\alpha\beta}\lambda_\beta$ with arbitrary λ_β and $(F_{\alpha,j}R^j{}_\beta)N^{\beta\delta} = \delta^\delta_\alpha$ the product of Jacobian and the variation of the Faddeev–Popov (super) determinant equals unity. As for example shown in ref. [607], this is equivalent to

$$\delta(R^i{}_\alpha\xi^\alpha)/\delta\phi^i + f^\alpha{}_{\alpha\beta} = 0. \tag{6}$$

A direct calculation shows that (6) is indeed true, provided one uses as basic fields in addition to $e^m{}_\mu$, $\psi_\mu{}^a$, S, P the contravariant vector field A^μ and not the world-scalar A_m [446, 556]. Perhaps a more significant basis is

$$\phi^i = (e^m{}_\mu, \psi_m{}^a, S, P, A_m), \qquad \psi_m{}^a = e_m{}^\mu \psi_\mu{}^a. \tag{7}$$

Also on this basis (6) holds, since

$$\delta(\delta\psi_m{}^a)/\delta\psi_m{}^a = \delta(\delta\psi_\mu{}^a)/\delta\psi_\mu{}^a + \delta(\delta e_m{}^\mu)/\delta e_m{}^\mu \tag{8}$$

with a similar relation with $\psi^a_m \to A_m$ and $\psi^a_\mu \to A^\mu$ so that the vierbein-terms cancel. In equations such as (6), traces over fermionic symmetries or fermionic fields acquire an extra minus sign. Right-differentiations $\delta/\delta\phi^i$ in (6) are equivalent to left-differentiations, see appendix H.

On the basis $(e^m{}_\mu, \psi_\mu{}^a, S, P, A_m)$ there is thus a measure in the path-integral given by $(\det e)^{-1}$ since $\delta(\ln e^{-1}) = (-\delta e^m{}_\mu)e_m{}^\mu$ while $\delta e^m{}_\mu = -e^m{}_\nu(\delta e_n{}^\nu)e^n{}_\mu$. If one integrates out S, P, A_m one finds the integration measure

$$[\mathrm{d}e^m{}_\mu\, \mathrm{d}\psi_\mu{}^a (\det g)^{-2}]. \tag{9}$$

The precise form of the measure has no physical significance, but it is important that one can choose a measure at all. For example, if one would not have a closed gauge algebra (by dropping S, P, A_m altogether), (6) does not hold and four-ghost terms must be added to the action.

Later on we will derive Ward identities from the BRST invariance of $I(\mathrm{qu})$. To that purpose one must show that also the Jacobian equals unity for a BRST transformation [514]. It is not always appreciated that the criterion which allows Slavnov–Taylor identities which was given in (6) is also the criterion for unit BRST Jacobian. Indeed, the first term is the contribution to the Jacobian from ϕ^i if one replaces ξ^α by $C^\alpha \Lambda$ while the second term is precisely the contribution of the ghost fields. The anti-ghosts do not contribute, since their BRST-law does not contain antighost fields. In Yang–Mills theory each term in (6) vanishes, but in supergravity only the sum vanishes, but it vanishes for all three local symmetries separately.

Finally we come to an interesting aspect of path-integral quantization. If one considers a quantized gravitino in a background gravitational field, one integrates only over $[\mathrm{d}\psi_\mu{}^a]$. The classical action is then

$$\mathscr{L}(\mathrm{cl}) = -\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar\psi_\mu \gamma_5 \gamma_\nu D_\rho \psi_\sigma$$

and it is invariant under $\delta\psi_\sigma = D_\sigma\epsilon$ alone, provided the gravitational field satisfies $R_{\mu\nu} = 0$ and the spin connection $\omega_\mu{}^{mn}(e)$ is free from torsions. Under these circumstances one may choose as gauge fixing term

$$\mathscr{L}(\mathrm{fix}) = \frac{e}{4}\bar\psi \cdot \gamma \not{D} \gamma \cdot \psi$$

so that at all stages one has kept the space-time symmetries manifestly preserved. In this case the normalization factor $(\det \not{D})^{-1/2}$ leads to a new ghost, first discovered by Nielsen and Kallosh [343–345, 304]. Since a commuting Majorana spinor has vanishing action, one circumvents problems with spin and statistics by replacing $(\det \not{D})^{-1/2}$ by $(\det \not{D})^{-1}$ $(\det \not{D})^{1/2}$. In this case one finds a real anticommuting and a complex commuting ghost. The net effect is that there is one Majorana ghost. This Majorana ghost is crucial, for example, in order to obtain the correct axial anomaly.

2.5. Gauge independence of supergravity

Coupling the gauge fields ϕ^i and ghost and antighost fields to sources as follows

$$I(\text{total}) = I(\text{qu}) + \mathrm{i}(\phi^i J_i + K_\beta C^{*\beta} + L_\beta C^\beta) \tag{1}$$

one finds, changing the integration variables of the path-integral by an infinitesimal BRST variation (whose Jacobian equals unity as we saw),

$$\langle \delta\phi^i J_i + K_\beta \delta C^{*\beta} + L_\beta \delta C^\beta \rangle = 0. \tag{2}$$

Left-differentiating with respect to K_β and right-differentiating with respect to J_k, and then putting K_β and L_β equal to zero, one finds

$$\langle \mathrm{i}C^{*\beta}(R^i{}_\alpha C^\alpha \Lambda)(\delta_i^k + \mathrm{i}J_i\phi^k) + \Lambda F_\alpha \gamma^{\alpha\beta}\mathrm{i}\phi^k \rangle = 0. \tag{3}$$

In the absence of S, P, A_m, $\delta\phi^i$ contains extra three-ghost terms which should be added to $R^i{}_\alpha C^\alpha \Lambda$. In that case, (3) not only relates the longitudinal part of the gauge field propagator to the ghost propagator, but also to extra four-ghost terms.

For linear $F_\alpha = F_{\alpha j}\phi^j$, this Ward identity can be rewritten as (assuming that $F_{\alpha i}$ is a commuting constant)

$$\langle (\delta Z/\delta F_{\beta k})\mathrm{i}\Lambda - C^{*\beta}\delta\phi^i J_i \phi^k \rangle = 0. \tag{4}$$

On-shell the second term does not contribute, since either $\delta\phi^i$ contains no one-particle pole (for example for $\delta e^m{}_\mu = -\bar{\psi}_\mu \gamma^m C\Lambda$) or otherwise contraction with a physical polarization tensor yields zero (for example the term $\delta\psi_\mu = \partial_\mu C\Lambda$). Thus the S-matrix is gauge invariant under changes in $F_{\alpha i}$ [304].

For general F_α which are nonlinear in ϕ^i, the proof may be extended as follows [Townsend and the author, unpublished]. Consider changing F_α into $F_\alpha + \epsilon G_\alpha$ with infinitesimal ϵ. Then we add to (1) a term $\mathrm{i}G_\alpha \hat{J}^\alpha$, and obtain after right-differentiation with respect to $\hat{J}^\beta$

$$\langle \mathrm{i}C^{*\beta}(G_{\beta,i}R^i{}_\alpha C^\alpha \Lambda + \Lambda F_\alpha \gamma^{\alpha\beta}\mathrm{i}G_\beta) + \mathrm{i}C^{*\beta}(R^i{}_\alpha C^\alpha \Lambda)\mathrm{i}J_i G_\beta \rangle = 0. \tag{4a}$$

The first two terms are $(\delta Z/\delta\epsilon)\mathrm{i}\Lambda$ while the last term does not contribute to the S-matrix for the same reasons as before. Thus, on-shell the S-matrix is independent of the choice of gauge fixing term F_α.

We must now decide whether the S-matrix is also independent of $\gamma^{\alpha\beta}$ [304]. Consider the following Ward identity

$$\int [\mathrm{d}\phi^i\, \mathrm{d}C^\alpha\, \mathrm{d}C^{*\alpha}\, (\operatorname{sdet}\gamma)^{1/2}] \frac{\partial}{\partial C^{*\epsilon}} C^{*\beta} \exp \mathrm{i}I(\text{qu}) = 0. \tag{5}$$

For one anticommuting variable, $(\partial/\partial C)f(C)$ is C-independent because $f(C)$ is at most linear in C, and

(5) follows from $\int \mathrm{d}C = 0$ (only $\int \mathrm{d}C\, C = 1$). Thus

$$\langle \delta^\beta_\epsilon + (-1)^{(\beta+1)(\epsilon+1)} C^{*\beta} \mathrm{i} F_{\epsilon j} R^j{}_\alpha C^\alpha \rangle = 0. \tag{6}$$

As usual, $\beta = 1$ for fermionic symmetries and $\beta = 0$ for bosonic symmetries.

Let us, for later use, return to (3) and multiply it by $F_{\epsilon k}$ and $-\frac{1}{2}\mathrm{i}(\gamma^{-1})_{\beta\delta}$. Then on-shell again

$$\langle \tfrac{1}{2} C^{*\beta} (\gamma^{-1})_{\beta\delta} F_{\epsilon k} R^k{}_\alpha C^\alpha \Lambda + \tfrac{1}{2} \Lambda F_\delta F_{\epsilon k} \phi^k \rangle = 0. \tag{7}$$

Rewriting (6) as

$$\langle (-1)^{(\epsilon+1)} \delta^\beta_\epsilon + C^{*\beta} \mathrm{i} F_{\epsilon k} R^k{}_\alpha C^\alpha \rangle = 0 \tag{8}$$

one finds for the first term in (7)

$$\left\langle \frac{1}{2} C^{*\beta} (\gamma^{-1})_{\beta\delta} F_{\epsilon k} R^k{}_\alpha C^\alpha \Lambda \right\rangle = \left\langle \frac{\mathrm{i}}{2} (\gamma^{-1})_{\epsilon\delta} (-)^{\epsilon+1} \Lambda \right\rangle. \tag{9}$$

Hence, on-shell we find the following Ward identity if one would not add a measure sdet $\gamma^{1/2}$ in the path integral

$$\langle \delta Z / \delta \gamma^{\delta\epsilon} + \tfrac{1}{2} (\gamma^{-1})_{\epsilon\delta} (-)^\epsilon \rangle = 0. \tag{10}$$

The last term is, however, precisely the variation of $(\mathrm{sdet}\, \gamma)^{1/2}$ because

$$\delta (\mathrm{sdet}\, \gamma^{\alpha\beta})^{1/2} = \tfrac{1}{2} (\gamma^{-1})_{\epsilon\delta} (\delta \gamma^{\delta\epsilon}) (-)^\epsilon (\mathrm{sdet}\, \gamma^{\alpha\beta})^{1/2}. \tag{11}$$

Thus the S-matrix of supergravity is indeed gauge-invariant under changes in $\gamma^{\alpha\beta}$ if one chooses the measure as sdet $\gamma^{1/2}$, as we did before.

Thus also gauge invariance requires that one take into account the factor $(\mathrm{sdet}\, \gamma^{\alpha\beta})^{1/2}$. This factor is of course the Nielsen–Kallosh ghost which we found before. It plays a crucial role in obtaining the correct axial anomaly when one uses the Adler–Rosenberg method.

2.6. *Unitarity of supergravity*

Closely related to gauge invariance is unitarity of the S-matrix [522]. Unitarity means that unphysical modes in the propagators do not contribute to S-matrix elements. Let us begin with the easier example of Yang–Mills theory. In the gauge $\mathscr{L}(\mathrm{fix}) = -\frac{1}{2}(\partial \cdot W)^2$, the renormalizable propagator is $\delta_{\mu\nu}(-\mathrm{i}k^{-2})$. The unitary propagator, on the other hand, reads

$$(-\mathrm{i}k^{-2})(\epsilon^+_\mu (\epsilon^+)^*_\nu + \epsilon^-_\mu (\epsilon^-)^*_\nu) = (-\mathrm{i}k^{-2}) \left(\delta_{\mu\nu} - \frac{k_\mu \bar{k}_\nu + k_\nu \bar{k}_\mu}{k \cdot \bar{k}} \right). \tag{1}$$

This equation holds on-shell only, where $k^2 = 0$. The problem is now to show that the terms with k_μ and its time reversal $\bar{k}_\mu$ cancel with the ghost contributions. Consider a two-particle cut

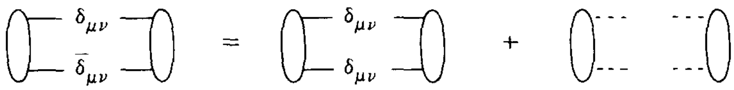


Fig. 1. Unitarity in Yang–Mills theory.

The proof of unitarity is based on the following Ward identities, which one derives by making a BRST transformation of the integration variables in the path-integral

$$\delta_{\mathrm{BRST}}\langle C^{*a}(x)\partial\cdot W^b(y)\rangle = 0 = \langle \partial\cdot W^a(x)\,\partial\cdot W^b(y)\rangle \tag{2}$$

$$\delta_{\mathrm{BRST}}\langle C^{*a}(x)\,W^b_\nu(y)\rangle = 0 = \langle \partial_\mu W^a_\mu(x)\,W^b_\nu(y) - C^{*a}(x)\,\partial_\nu C^b(y)\rangle. \tag{3}$$

Graphically they are represented in fig. 2. In (2) we have omitted a term $\Box_y$ since we are on momentum-shell.

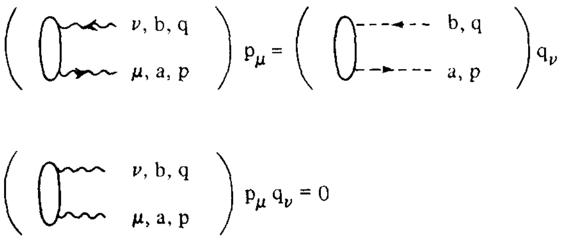


Fig. 2. Ward identities for unitarity.

The second identity describes a conservation law, but note that one must hit all non-physical (but with physical momenta) lines by momenta. The first identity shows what happens if one only hits one gauge field: the gauge field becomes a ghost field which emits a momentum at the other line.

Let us now return to fig. 1. Using

$$\bar{\delta}_{\mu\nu} = \delta_{\mu\nu} - (k_\mu \bar{k}_\nu + \bar{k}_\mu k_\nu)(k\cdot\bar{k})^{-1} \tag{4}$$

we omit the k_μ terms in *one* of the two propagators, since we can use yet another Ward identity

$$\langle \partial\cdot W^a(x)\,W^{b_1}_{\mu_1}(y_1)\dots W^{b_n}_{\mu_n}(y_n)\rangle = 0 \qquad \text{(conservation)} \tag{5}$$

where all $W^{b_i}_{\mu_i}(y_i)$ fields are on-shell with physical polarizations. Thus we consider fig. 1 with $\delta_{\mu\nu}$ upstairs but $\bar{\delta}_{\mu\nu}$ downstairs in the first diagram. Using the first identity in fig. 2 once to cycle clockwise on the left, and once more to cycle clockwise on the right, one makes a full circle and ends up with a factor $-\bar{k}_\nu(k\cdot\bar{k})^{-1}$ times the final ghost momentum k_ν. This is exactly the Faddeev–Popov ghost plus minus sign with counter clockwise orientations. Similarly, the term with $-\bar{k}_\mu k_\nu(k\cdot\bar{k})^{-1}$ can be used to make a full circle counter clockwise and yields the other orientation of the Faddeev–Popov ghost. Thus, as expected, although after fixing the gauge all four modes of W^a_μ propagate for a given a, the complex world-scalar ghost subtracts two modes, and one ends up with two physical modes in the quantum theory as well.

Let us now turn to supergravity. There are 16 real fields $\psi_\mu{}^a$, so one expects after gauge fixing that there are 8 gravitino modes propagating. Indeed, there are 8 modes in the propagator as an explicit decomposition shows [522]. However, the complex spin 1/2 supersymmetry ghost is expected to subtract only 4 modes. (Fermions have always half as many modes as bosons.) Where do the remaining 2 modes go?

The solution of this puzzle can be found in the proof of unitarity of ref. [522]. To see what happens, we begin by stating that the unitary gravitino propagator is given by

$$\tfrac{1}{2}\bar{\delta}_{\nu\alpha}(\gamma_\alpha \not{k} \gamma_\beta)\bar{\delta}_{\beta\mu} k^{-2}. \tag{6}$$

The Ward identities now become (considering a gravitino loop)

$$\delta_{\rm BRST}\langle C^*\gamma\cdot\psi\rangle = 0 = \langle -(\tfrac{1}{2}\bar{\psi}\cdot\gamma\bar{\not{D}})(\gamma\cdot\psi)\rangle + \langle C^*\not{\partial}C\rangle, \qquad \text{cf. (2)} \tag{7}$$

hence $\langle\gamma\cdot\psi(x)\bar{\psi}(y)\cdot\gamma\rangle = 0$ on-shell, since there are extra factors $\not{\partial}$ in $\langle C^*\not{\partial}C\rangle$, and

$$\delta_{\rm BRST}\langle C^{*a}(x)\psi_\mu^b(y)\rangle = 0 = \langle\gamma\cdot\psi(x)\,\psi_\mu^b(y)\rangle + \langle C^{*a}(x)\,\partial_\mu C^b(y)\rangle, \qquad \text{cf. (3).} \tag{8}$$

Notice now that $\gamma^\mu\langle\psi_\mu\bar{\psi}_\nu\rangle = 2k_\nu k^{-2}$ so that we may use conservation to replace in one of the two cut propagators the unitary propagator in (6) by the renormalizable propagator $\frac{1}{2}\gamma_\nu\not{k}\gamma_\mu k^{-2}$. (We also use that a term $\bar{k}_\nu k_\alpha$ in (6) vanishes on-shell.) In the other propagator one has left

$$k^2(P^{3/2}_{\mu\nu}(\text{unit}) - P^{3/2}_{\mu\nu}(\text{ren})) = -\tfrac{1}{2}(k_\nu\bar{\not{k}}\not{k}\gamma_\mu + \gamma_\nu\not{k}\bar{\not{k}}k_\mu - 2k_\nu\bar{\not{k}}k_\mu)(k\cdot\bar{k})^{-1}. \tag{9}$$

The second term can be written as k_μ times a factor $(-\bar{k}_\alpha)(k\cdot k)^{-1}$ times the gravitino propagator $\frac{1}{2}\gamma_\alpha\not{k}\gamma_\mu$. Cycling as before with k_μ once on the left, and once on the right, both times clockwise, one ends up with k_α times $-\bar{k}_\alpha(k\cdot\bar{k})^{-1}$, hence with precisely one ghost loop. Notice that contracting a gravitino with its own momentum on one side of a blob first of all flips this fermion line into a ghost line, but, as is clear from (8), it also obliterates the numerator of the gravitino propagator at the other end, replacing it by the numerator of the ghost propagator.

Thus the first two terms in (9) are just equal to the ghost contributions. That leaves us with the last term. It represents indeed *two modes in the* propagator (since $\bar{\not{k}}$ can be written as $\omega^+\bar{\omega}^+ + \omega^-\bar{\omega}^-$ with spinors ω^+, ω^- orthogonal to the physical spinors u^+ and u^-); but they *decouple from the S-matrix*. This follows easily if one cycles once with the factor k_μ on the left and then uses (7).

The above sketch of the unitarity proof has been extended to general N-particle cuts, taking into account all ghosts and antighosts. The results are unchanged. One can choose different commutation properties between ghosts without violating unitarity [553].

2.7. Supersymmetric regularization

All preceding manipulations only make sense if one would use a regularization scheme to give meaning to the divergent loops. Preferably, but not necessarily, a scheme should be used which maintains manifest supersymmetry at all stages. Such a scheme exists. It is an extension of dimensional regularization [417, 425], but there are also some differences. It is based on the idea of dimensional reduction.

Consider as an example how the scheme works, quantum electrodynamics. The action is given by

$$\mathcal{L} = -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \bar{\lambda}\gamma^\mu(\partial_\mu - \mathrm{i}eA_\mu)\lambda. \tag{1}$$

We now continue *only coordinates and momenta to n dimensions where* $n \le 4$. However, the indices μ of A_μ remain 4-dimensional, while also λ remains a four-component spinor. Thus one can write the action as

$$\mathcal{L} = -\tfrac{1}{4}(\partial_i A_j - \partial_j A_i)^2 - \bar{\lambda}\gamma^j(\partial_j - \mathrm{i}eA_j)\lambda - \tfrac{1}{2}(\partial_i A_\sigma)^2 + \mathrm{i}e\bar{\lambda}\gamma^\sigma\lambda A_\sigma. \tag{2}$$

In other words, we have here normal QED in n-dimensions, plus extra scalars coupled to the spinors by a Yukawa coupling. Since $0 \le i \le n$ but $n < \sigma \le 4$, there are ϵ scalars. They are unphysical in the sense that for $\epsilon = 4 - n$ tending to zero they disappear, but they can yield factors ϵ in the numerators which cancel poles in ϵ such as to yield a finite nonvanishing contribution.

If one were to quantize (1), one adds the gauge fixing term $-\frac{1}{2}(\partial_j A_j)^2$. Since ∂_j is contracted with A_j and derivatives clearly only run from 0 to n, this term does not contain scalars. (In true democratic fashion, the smallest wins.) The model does have a gauge invariance

$$\delta A_j = \partial_j \Lambda, \qquad \delta A_\sigma = 0 \tag{3}$$

which explains that this kind of regularization does not violate those Ward identities which follow from ordinary local symmetries such as Yang–Mills or gravity.

The great advantage of this scheme is that it keeps the indices of fields 4-dimensional. Thus, even when regularizing, one still has the usual 4-dimensional Dirac algebra. In ordinary dimensional regularization things are different. Going to n dimensions, one has an n-dependent number of Dirac covariants, and ordinary Fierz rearrangements are very different in n-dimensions.

To give an example, consider supersymmetric Yang–Mills theory. The action

$$\mathcal{L} = -\tfrac{1}{4}(G^a_{\mu\nu})^2 - \tfrac{1}{2}\bar{\lambda}^a\gamma^\mu D^{ab}_\mu\lambda^b \tag{4}$$

is invariant under $\delta\lambda^a = \sigma \cdot G^a\epsilon$, $\delta W^a{}_\mu = -\bar{\epsilon}\gamma_\mu\lambda^a$ provided the following identity holds

$$(\bar{\lambda}^a\gamma_\mu\lambda^b)(\bar{\lambda}^c\gamma_\mu\epsilon)f^{abc} = 0. \tag{5}$$

The identity is true in four dimensions, and also when x^μ and p^μ become n-dimensional, since it only amounts to a reshuffling property of the four components of λ^a. But in dimensional regularization it is not true in n dimensions if one has not four matrices γ_μ.

One may show that Ward identities of gauge symmetry as well as supersymmetry are manifestly preserved by this scheme [550]. Note that in supergravity the field $\psi_\mu{}^a$ remains a 4×4 matrix so that its propagator is given by $\frac{1}{2}\gamma_\mu \not{k}\gamma_\nu k^{-2}$, even in the regularized theory, and not by an n-dependent result.

As an example, consider the gauge $F(\alpha) = \gamma \cdot \psi + (1-\alpha)\gamma^a\delta^\mu_a\psi_\mu$ in simple supergravity. In this gauge one has the following Ward identity

$$\langle F^a(\alpha)\,\bar{F}^b(\alpha)\rangle = 0. \tag{6}$$

If one manipulates the Dirac algebras as follows

$$\begin{aligned} &\gamma_\sigma \not{K} \gamma^\sigma = -\epsilon \not{K}, \\ &\gamma_i \not{K} \gamma_i = (2-n) \not{K} \\ &\gamma_\mu \not{K} \gamma_\mu = -2\not{K} \end{aligned} \tag{7}$$

one finds that (6) is satisfied at the one-loop level to order ϵ^{-1} if one uses either dimensional regularization or dimensional reduction, but to order ϵ^0 only the latter scheme preserves (6) [556].

In ordinary renormalizable models, this scheme respects renormalizability only if these theories are supersymmetric. For example, the two coupling constants e in (2) are rescaled by different Z-factors. Thus for supersymmetric theories this scheme makes sense, but for non-supersymmetric ones it does not. One can also use it for supergravity, where the issue of renormalizability does not exist. One can use this scheme to calculate anomalies.

There exist other regularization schemes which are claimed to preserve global (and sometimes local) supersymmetry. One such scheme is the higher derivative scheme of Slavnov. For example, in Yang–Mills theory one adds to the usual Yang–Mills action a term $\Lambda^{-2}(D_\lambda G_{\mu\nu}{}^a)^2$ where Λ ultimately tends to infinity. Propagators now fall off like k^{-4} and this regularizes all loops except single loops. For those one substracts infinities by adding to the action gauge-invariant Pauli–Villars terms

$$\sum_j (\delta^2 \mathcal{L}(A_\mu, \Lambda)/\delta A_\mu \delta A_\nu) B^j_\mu B^j_\nu - m_j^2 (B^j_\mu)^2 \tag{8}$$

(satisfying, as usual, $\Sigma\, a_i = \Sigma\, a_i m_i^2 = 0$) as well as anticommuting ghosts c^j for these new gauge fields B_μ corresponding to $D_\mu(A) B^{\mu,j} = 0$

$$\mathcal{L} = D_\mu(A)\, \bar{c}^j\, D_\mu(A)\, c^j - m_j^2 \bar{c}_j c_j. \tag{9}$$

For global supersymmetry one can use superfields and find globally supersymmetric Yang–Mills-gauge fixing terms. For supergravity it is not clear whether this scheme can be applied.

There is also the BPHZ-scheme, as applied to supersymmetry by Clark, Piguet, Sybold, where one expands Green's functions in terms of the external momenta. Each coefficient turns out to be supersymmetric, but to prove this requires detailed analysis.

2.8. *BRST invariance and open algebras*

Supergravity without auxiliary fields has an open gauge algebra, by which we mean that it closes only on the classical mass shell. On the other hand, for quantum calculations it is much easier to work without auxiliary fields. Thus the problem arises what new features arise at the quantum level for gauge theories with open algebras. In particular, unitarity and gauge invariance of the S-matrix can be derived from Ward identities, and to obtain these Ward identities one needs BRST invariance of the quantum action. In this subsection we see how to modify the transformation rules and action to obtain BRST invariance if the gauge algebra is open [304, 162]. For closed gauge algebras we refer back to subsection 2.

The gauge commutator for open algebras has the form

$$R^i{}_{\alpha,k}R^k{}_\beta - (-)^{\alpha\beta}R^i{}_{\beta,k}R^k{}_\alpha = R^i{}_\gamma f^\gamma{}_{\alpha\beta} + I^{\rm cl}_{,j}\eta^{ij}_{\alpha\beta} \tag{1}$$

where $I^{\rm cl}_j = \delta I^{\rm cl}/\delta\phi^j$ with $I^{\rm cl}$ the classical action, and we always consider in this section *right*-derivatives. In the variation of the usual quantum action of subsection 2, it is only at the last step that not all terms cancel, but one is left with

$$\delta I(\text{quantum}) = C^{*\alpha}F_{\alpha,j}(\tfrac{1}{2}I^{\rm cl}_{,l}\eta^{jl}_{\beta\gamma}C^\gamma\Lambda)C^\beta. \tag{2}$$

As we discussed in the section on matter coupling, such terms proportional to $I^{\rm cl}_{,l}$ can always be canceled by adding an extra term to $\delta\phi^l$. After bringing $I^{\rm cl}_{,l}$ to the left, one finds

$$\delta\phi^i(\text{extra}) = -\tfrac{1}{2}C^{*\alpha}F_{\alpha,j}\eta^{ji}_{\beta\gamma}C^\gamma\Lambda C^\beta(-)^{(j+1)i}. \tag{3}$$

This new variation of ϕ^i leads to extra terms in δI (quantum). Keeping the same gauge fixing term, we find for the ghost action by variation of the gauge fixing terms and sandwiching the result with ghosts and antighosts the following result

$$I = I^{\rm cl} + \tfrac{1}{2}F_\alpha\gamma^{\alpha\beta}F_\beta + C^{*\alpha}F_{\alpha,j}R^j{}_\beta C^\beta - \tfrac{1}{4}C^{*\alpha}F_{\alpha,j}C^{*\beta}F_{\beta,k}\eta^{kj}_{\gamma\delta}C^\delta C^\gamma(-)^{(k+1)j+\gamma+1}. \tag{4}$$

The new four-ghost coupling is due to $\delta\phi^i$ (extra) and one has a factor $-1/4$ rather than $-1/2$ since both antighosts appear symmetrically due to symmetry properties of η^{ij} which we now discuss.

From the gauge commutator it follows that $\hat{\delta}\phi^i \equiv I^{\rm cl}_{,j}\eta^{ij}_{\alpha\beta}\eta^\beta\xi^\alpha$ is again a local gauge invariance of the action. Hence

$$I^{\rm cl}_{,i}I^{\rm cl}_{,j}\eta^{ij}_{\alpha\beta}\eta^\beta\xi^\alpha = 0. \tag{5}$$

This means that the nonclosure function $\eta^{ij}_{\alpha\beta}$ is a sum of terms either with the symmetry

$$\eta^{ij}_{\alpha\beta} = (-)^{ij+1}\eta^{ji}_{\alpha\beta} \tag{6}$$

(where $i = 1$ for fermionic fields and $i = 0$ for bosonic fields), or with $\hat{\delta}\phi^i$ a gauge transformation itself, which then of course vanishes on-shell,

$$\hat{\delta}\phi^i = R^i{}_\lambda(X^{\lambda j}I^{\rm cl}_{,j}) \tag{7}$$

where $X^{\lambda j}$ is some matrix which follows from (5). In supergravity we calculated the nonclosure function $\eta^{ij}_{\alpha\beta}$ in subsection 1.9 and found that it is nonzero for i, j, α, β all supersymmetry indices, since only the commutator of two local supersymmetry transformations on a gravitino gave a field equation, namely the field equation of the gravitino. Thus for the tetrad $\hat{\delta}\phi^i = 0$, from which we conclude that $X^{\lambda j}$ in (7) vanishes. Hence the nonclosure function has the symmetries in (6), and this justifies the factor $-1/4$ in (4). This symmetry follows also directly from eqs. (12, 13) of subsection 1.9 if one replaces R^ρ by $-C^{-1}\bar{R}^{\rho,\mathrm{T}}$ and uses that $\gamma_\alpha C^{-1}$ is symmetric but $\gamma_5\gamma_\alpha C^{-1}$ antisymmetric. In general one has to add the local symmetries $\hat{\delta}\phi^i$ to the total set of gauge invariances in theories with open gauge algebras when $\delta\hat{\phi}^i$ is nonzero.

It should be stressed that these new four-ghost couplings in (4) can no longer be written as a Faddeev–Popov determinant and hence that the usual Slavnov–Taylor method for deducing Ward identities is not applicable here. Instead one should use BRST invariance, as we shall do.

We must now use the new ϕ variation in the old two-ghost action, and the new and old ϕ variation as well as the ghost variation in the new four-ghost action. From now one we specialize to supergravity where $\eta^{ij}_{\alpha\beta}$ depends only on tetrads and the indices i, j, α, β refer to gravitinos and supersymmetry parameters. In this case the new ϕ variation does not contribute in the new ghost action since

$$\eta^{kl}_{\gamma\delta,i}\eta^{ji}_{\rho\sigma} = 0 \tag{8}$$

and since $F_{\alpha,j}$ and $F_{\beta,k}$ in (4) do not depend on the gravitino field. (We exclude gauge choices F_α quadratic in ψ from consideration. This is not necessary, but covers all useful cases.) For the remaining three variations one finds

$$\begin{aligned}&C^{*\alpha}F_{\alpha,j}R^j{}_{\beta,l}(-\tfrac{1}{2}C^{*\gamma}F_{\gamma,k}\eta^{kl}_{\rho\sigma}C^\sigma\Lambda C^\rho)C^\beta - \tfrac{1}{4}C^{*\alpha}F_{\alpha,j}C^{*\beta}F_{\beta,k}\eta^{kj}_{\gamma\delta,l}(R^l{}_\lambda C^\lambda\Lambda)C^\delta C^\gamma\\ &-\tfrac{1}{4}C^{*\alpha}F_{\alpha,j}C^{*\beta}F_{\beta,k}\eta^{kj}_{\rho\sigma}(-\tfrac{1}{2}f^\sigma{}_{\gamma\delta}C^\delta\Lambda C^\gamma)C^\rho.\end{aligned} \tag{9}$$

Since $C^{*\gamma}$ in the first term is commuting, we symmetrize the first term in both antighosts and find

$$\delta I(\text{quantum}) = -\tfrac{1}{4}C^{*\lambda}F_{\lambda,l}C^{*\tau}F_{\tau,i}P^{il}_{\gamma\beta\alpha}C^\alpha\Lambda C^\beta C^\gamma \tag{10}$$

where the expression for $P^{il}_{\gamma\beta\alpha}$ looks very much like a commutator of an ordinary gauge transformation $\delta\phi^i \sim R^i{}_\alpha$ with a transformation $\delta\phi^i \sim \eta^{il}_{\gamma\beta}$ since

$$P^{il}_{\gamma\beta\alpha} = \eta^{il}_{\gamma\beta,k}R^k{}_\alpha + R^l{}_{\gamma,k}\eta^{ki}_{\beta\alpha} + R^i{}_{\gamma,k}\eta^{kl}_{\beta\alpha} - \eta^{il}_{\gamma\delta}f^\delta{}_{\beta\alpha}. \tag{11}$$

Since the transformation $\delta\phi^i \sim \eta^{il}_{\gamma\beta}$ looks like one of the terms one finds in the commutator of two gauge transformation, one is led to consider the Jacobi identities for three consecutive gauge transformations.

Consider the Jacobi identity

$$\delta(\zeta)[\delta(\eta), \delta(\xi)] - [\delta(\eta), \delta(\xi)]\delta(\zeta) + \text{cyclic in } \zeta\eta\xi = 0. \tag{12}$$

Substituting (1), and substituting in the result once more the identity (1), one finds

$$R^i{}_\lambda A^\lambda{}_{\alpha\beta\gamma}\zeta^\gamma\eta^\beta\xi^\alpha + I^{\text{cl}}_{,l}B^{li} + \text{cyclic in } \zeta\eta\xi = 0 \tag{13}$$

where A and B are defined by

$$A^\lambda{}_{\alpha\beta\gamma} = -f^\lambda{}_{\alpha\delta}f^\delta{}_{\beta\gamma} + f^\lambda{}_{\alpha\beta,k}R^k{}_\gamma \tag{14}$$

$$B^{li} = -\eta^{il}_{\gamma\delta}f^\delta{}_{\alpha\beta}\eta^\beta\xi^\alpha\zeta^\gamma + \eta^{il}_{\alpha\beta,k}R^k{}_\gamma\zeta^\gamma\eta^\beta\xi^\alpha - R^l{}_{\gamma,k}\zeta^\gamma\eta^{ik}_{\alpha\beta}\eta^\beta\xi^\alpha(-)^{\gamma k} + R^i{}_{\gamma,k}\eta^{kl}_{\alpha\beta}\eta^\beta\xi^\alpha\zeta^\gamma(-)^{l(i+\gamma)}. \tag{15}$$

If the algebra closes, $B=0$. In the second and third terms of B^{li} one can bring ζ^γ to the right without change in sign. Moreover, since $\eta^{ik}(-)^{\gamma k}=\eta^{ik}(-)^{\gamma i}$, it follows that

$$B^{li}=B^{il}(-)^{il+1}. \tag{16}$$

We now write $\xi^\alpha=C^\alpha\Lambda_1$, $\eta^\beta=C^\beta\Lambda_2$ and $\zeta^\gamma=C^\gamma\Lambda_3$ in order to make contact with (10). One may verify a very useful identity

$$\eta^\beta\xi^\alpha\zeta^\gamma+\text{two cyclic terms}=C^\beta\Lambda_3C^\alpha C^\gamma(-)^\gamma(3\Lambda_1\Lambda_2). \tag{17}$$

In other words, using ghost fields the cyclicity is automatic. With $\eta^{\,il}_{\gamma\delta}(-)^\gamma=-\eta^{\,il}_{\gamma\delta}$ and

$$\eta^{\,il}_{\beta\alpha,k}R^k{}_\gamma C^\beta\Lambda C^\alpha C^\gamma(-)^\gamma=-\eta^{\,il}_{\gamma\beta,k}R^k{}_\alpha C^\alpha\Lambda C^\beta C^\gamma \tag{18}$$

one finds upon contracting B^{li} with two antighosts and dropping the common factor $3\Lambda_1\Lambda_2$, the following result (after substantial relabeling of indices)

$$C^{*\lambda}F_{\lambda,i}C^{*\tau}F_{\tau,l}B^{li}=C^{*\lambda}F_{\lambda,i}C^{*\tau}F_{\tau,l}[\eta^{\,il}_{\gamma\delta}f^\delta{}_{\beta\alpha}-\eta^{\,il}_{\gamma\beta,k}R^k{}_\alpha-R^l{}_{\gamma,k}\eta^{\,ik}_{\beta\alpha}+R^i{}_{\gamma,k}\eta^{\,kl}_{\beta\alpha}(-)^{il}]C^\alpha\Lambda C^\beta C^\gamma. \tag{19}$$

This result looks indeed very much like (10) if one compares (11) with the expression between square brackets in (19). In fact, these expressions are equal. To see this note that $i+l$ is even since B has the symmetry in (16) but the two antighosts in (19) have symmetry $(-)^{(i+1)(l+1)}$. Since always either i or j in B^{ij} is fermionic, see (15), $il=1$. Thus the terms with B^{li} in (19), coming from the triple commutator of gauge transformations, are indeed the same terms as what is left of δI (quantum).

If one could argue that $B^{li}=0$, then one would have completed the proof of BRST invariance. However, B is nonzero and we proceed to solve it from (13). On-shell $R^i{}_\lambda A^\lambda{}_{\dots}=0$ and defining $R^i{}_\lambda$ such that $R^i{}_\lambda\neq 0$ on-shell (parts vanishing on-shell are proportional to $I_{,m}$ and can be incorporated into the non-closure function η), it follows that $A^\lambda{}_{\dots}=I(\mathrm{cl})_{,l}A^{\lambda l}{}_{\dots}$. Hence (13) becomes $I(\mathrm{cl})_{,l}(R^i{}_\lambda A^{\lambda l}{}_{\dots}+B^{li}+\text{cycl.})=0$ which tells us that the second factor is a gauge invariance of the classical action. Thus

$$R^i{}_\lambda A^{\lambda l}{}_{\dots}+B^{li}+\text{cycl.}=R^l{}_\lambda X^{li}+I(\mathrm{cl})_{,j}M^{jli}$$

with $I(\mathrm{cl})_{,l}I(\mathrm{cl})_{,i}M^{jli}=0$ and this yields finally

$$B^{li}=(R^l{}_\lambda X^{i\lambda}+I^{\mathrm{cl}}_{,j}M^{jli})+(-)^{il+1}(i\leftrightarrow l) \tag{20}$$

where M^{jli} has the same symmetry in (jl) and (il), namely that of η^{li}. Thus M^{jli} is completely symmetric.

We will now show that in supergravity the function M vanishes while X is only nonzero when λ refers to local Lorentz rotations. From (15) it follows that B^{li} does not contain $\partial\psi$ or $\partial\partial e$ terms. Indeed, $\eta^{\,ij}_{\alpha\beta}$ and $\eta^{\,ij}_{\alpha\beta,k}$ and $R^j{}_{\gamma,k}$ do not contain any derivatives, while $R^k{}_\gamma$ does not contain a derivative of fields and $f^\tau{}_{\alpha\beta}$ contains only a derivative of a field (of the tetrad) if τ refers to local Lorentz rotations and α, β to local supersymmetry (in this case the commutator is proportional to $\omega_\mu{}^{mn}(e,\psi)$ as we saw). Thus M^{jli} vanishes. The Jacobi identity thus becomes

$$[R^i{}_\lambda A^\lambda{}_{\alpha\beta\gamma}+I^{\mathrm{cl}}_{,l}R^i{}_\lambda X^{l\lambda}_{\alpha\beta\gamma}(-)^{il+1}]C^\gamma\Lambda C^\beta C^\alpha(-)^\alpha=0 \tag{21}$$

where $X^{l\lambda}_{\alpha\beta\gamma}$ follows from (13), (15), (17). Since the $R^i{}_\lambda$ are independent, this implies

$$[-f^\lambda{}_{\alpha\delta}f^\delta{}_{\beta\gamma}+f^\lambda{}_{\alpha\beta,k}R^k{}_\gamma-I^{\rm cl}_{,l}X^{l\lambda}_{\alpha\beta\gamma}](\zeta^\gamma\eta^\beta\xi^\alpha+\text{cyclic terms})=0. \tag{22}$$

This is the open gauge commutator for fields which do not transform as ϕ^i but which transform in the adjoint representation. For closed gauge algebras $X=0$ and we find the identity which is needed for nilpotency of BRST transformations of ghost fields (eq. (12) of subsection 2.2).

Since in supergravity $R^i{}_\lambda$ does not vanish on-shell, $A^\lambda{}_{\alpha\beta\gamma}$ must vanish when $I^{\rm cl}_{,j}=0$. Inspection of (14) shows that this is only possible when λ refers to local Lorentz invariance and α, β, γ to local supersymmetry. In that case $f^\lambda{}_{\alpha\beta}$ is proportional to the spin connection (as we already discussed), which rotates under local supersymmetry into the gravitino field equation plus terms which cancel against the product of the two structure functions in (14). Thus also the index l in (13) refers to the gravitino.

The final conclusion is that the remaining terms in the variations of (4) under the usual BRST laws plus (3) is given by (see (10), (19) and (20))

$$\delta I(\text{quantum})=\tfrac{1}{4}C^{*\lambda}F_{\lambda,l}C^{*\tau}F_{\tau,i}(2R^l{}_\lambda X^{i\lambda}_{\alpha\beta\gamma})C^\alpha\Lambda C^\beta C^\gamma. \tag{23}$$

Since $i+l=$ even and λ = local Lorentz, one may place $R^l{}_\lambda$ next to $F_{\lambda,l}$ without sign change. Hence we can cancel (23) by *adding an extra term to the ghost transformation law* of the Lorentz ghost field,

$$\delta(\text{extra})C^\alpha=-\tfrac{1}{2}f^\alpha{}_{\beta\gamma}(\text{extra})C^\gamma\Lambda C^\beta$$

$$f^\alpha{}_{\beta\gamma}(\text{extra})=\tfrac{1}{2}C^{*\tau}F_{j,\tau}X^{j\alpha}_{\gamma\beta\delta}C^\delta.$$

This new term in the variation of the Lorentz ghost cancels δI (quantum) when substituted in the two-ghost action, but it does not yield new variations in the four-ghost action since that depends on supersymmetry ghosts only.

Thus supergravity without auxiliary fields, i.e. with open gauge algebra, is indeed BRST invariant. The total quantum action is given in (4), and the transformation rules are the usual BRST rules for closed gauge algebras plus the extra terms in (3) for the gauge fields and an extra term in the transformation law of the Lorentz ghost. Those results are the same as if one had started with auxiliary fields and eliminated them from the *quantum* action. In particular, *the extra terms in the BRST law for the gravitino come from elimination of the auxiliary fields.* Also the extra term in the Lorentz ghost transformation law are understood from this point of view, namely only in the Lorentz rotation which is produced by two local supersymmetry transformations are there auxiliary fields.

For the extended supergravities one might reverse these arguments, and obtain information about the auxiliary fields by first establishing BRST invariance. Since this is one of the central problems in supergravity, we have been rather detailed in this section.

We close with referring the reader to the literature for the most general case of theories with open gauge algebras [162]. In these cases, the process of adding extra terms to the ghost action and to the gauge and ghost fields does not stop after one cycle, but one can get 6, 8, etc. ghost-couplings. At each level one considers a Jacobi identity of two gauge transformations and one new transformation of ϕ^i (the first new case being $\delta\phi^i\sim\eta^{ij}_{\alpha\beta}$). These new transformations are no new invariances of the action, but lead to results similar to the A, B analysis given above. One gets each time new M terms in $\delta\phi^i$ and new X terms in δC^α, but $\delta C^{*\alpha}$ remains unchanged. These new BRST laws are again nilpotent upon use of the quantum field equations (now also of the ghosts) as derived from the *full* quantum action. For example, two BRST transformations on the antighost $C^{*\alpha}$ yield a result proportional to $\delta(F_\beta\gamma^{\beta\alpha}) = F_{\beta,i}R^i{}_\gamma C^\gamma$ which is indeed proportional to $\delta\mathscr{L}(\mathrm{qu})/\delta C^{*\alpha}$. (Nilpotency is important in renormalizable theories to analyze the higher order quantum correction.) It is not known whether such theories with 6 and higher ghost-couplings exist, and, if they exist, whether one can eliminate these ghost couplings by introducing auxiliary fields.

2.9. *Finiteness of quantum supergravity?*

Quantum supergravity is more finite as far as ultraviolet divergences are concerned than ordinary general relativity. Since this subject has been discussed in detail at many conferences, we refer for details to these sources as well as the original literature. For a review, see ref. [286].

The first breakthrough [512] occurred in the $N=2$ model, where an explicit computation of the one-loop correction to photon–photon scattering revealed that all divergences in the process cancel. A theoretical explanation was given [512], based on the observation that

(i) graviton–graviton scattering is finite because possible counter terms are of the form $\Delta\mathscr{L} = \alpha R^2_{\mu\nu} + \beta R^2$ and on-shell $R_{\mu\nu} = 0$.

(ii) Using (global) supersymmetry of the S-matrix, one can "rotate" gravitons away and replace by gravitinos, etc. This then shows that all four-point processes are finite. (Higher point processes have also been shown to be finite in this way.)

Of course one must show that starting from graviton–graviton scattering one can indeed reach *all* other processes. This has been done.

An alternative proof, valid for all n-point functions, was given [123] by showing that no $\Delta\mathscr{L}$ exists on-shell which is invariant under the on-shell transformations of supergravity. In particular it was shown that no $\Delta\mathscr{L}$ can start with gravitino terms but must start with $R_{\mu\nu}{}^2$ and R^2. Not all details of this analysis have been published.

At the two-loop level, again both proofs of finiteness have been used. Since the only bosonic $\Delta\mathscr{L}$ which does not vanish on-shell is given by [503]

$$\Delta\mathscr{L} = \alpha\kappa^2 R_{\alpha\beta}{}^{\mu\nu} R_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma}{}^{\alpha\beta} \tag{1}$$

and describes helicity-flipping of graviton–graviton scattering, whereas supersymmetry (global as well as local) conserves helicity, $\alpha = 0$. Hence, as before, graviton–graviton scattering is two-loop finite in supergravity, and rotating gravitons away and replacing them by other particles, one shows that also at the two-loop level supergravity is finite [283]. The counter term analysis has matched also this result by claiming that one cannot extend $\Delta\mathscr{L}$ in (1) to a full on-shell supersymmetric invariant [123].

For $N=1$ supergravity, the tensor calculus [189], or, equivalently, superspace methods [191] have superseded these proofs and prove one- and two-loop finiteness. In particular, the counter term analysis was performed only to order ψ^2, whereas the tensor calculus and superspace give complete results with considerable less algebra. For $N=2$ supergravity similar results have recently been obtained.

Possible problems (but no more than that!) arise at the three-loop level. Namely, it seems that in all N-extended pure supergravities on-shell a supersymmetric invariant three-loop counter term exists [129]. Whether it actually will have nonvanishing coefficient is the big question. This $\Delta\mathscr{L}$ has in its bosonic sector the Bel–Robinson tensor.

The Bel–Robinson tensor is defined by

$$T^{\alpha\beta\lambda\mu} = R^{\alpha\rho\lambda\sigma}R^{\beta}{}_{\rho}{}^{\mu}{}_{\sigma} + {}^*R^{\alpha\rho\lambda\sigma}\,{}^*R^{\beta}{}_{\rho}{}^{\mu}{}_{\sigma} + R^{*\alpha\rho\lambda\sigma}R^{*\beta}{}_{\rho}{}^{\mu}{}_{\sigma} + {}^*R^{*\alpha\rho\lambda\sigma}\,{}^*R^{*\beta}{}_{\rho}{}^{\mu}{}_{\sigma} \tag{2}$$

where the stars denote left and right duals. Since ${}^*R^*{}_{\alpha\beta\rho\sigma} + R_{\alpha\beta\rho\sigma}$ is proportional to $R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R$ and $** = -1$, it follows that on-shell (i.e., $R_{\mu\nu} + \Lambda g_{\mu\nu} = 0$), left and right dual of the Riemann tensor coincide. Also ${}^*R^{*\alpha\beta}{}_{\alpha\nu}$ is on-shell proportional to the Einstein tensor. The B.R. tensor is symmetric in $\alpha\beta$, in $\lambda\mu$, under pair exchange $(\alpha\beta) \leftrightarrow \lambda\mu$ and on-shell it is conserved, $D_\alpha T^{\alpha\beta\lambda\mu} = 0$. (See the thesis of L. Bel, for example.) If there is no cosmological constant Λ, one has on-shell that $T^{\alpha\beta\lambda\mu}$ is completely symmetric, traceless and conserved.

It is now possible, either by the Noether procedure or (to all orders in κ and all orders in ψ_μ) by the tensor calculus, to show that one can extend the invariant $T_{\alpha\beta\gamma\delta}T^{\alpha\beta\gamma\delta}$ to a full locally supersymmetric invariant. Thus, at the three loop level there is a $\Delta\mathcal{L}$ for $N = 1$ supergravity on-shell which is on-shell invariant under local supersymmetry and which poses a danger for the finiteness of supergravity.

The crucial question whether $N = 8$ supergravity is finite beyond two loops has been analyzed by means of superspace methods by Howe and Lindström. Just as for $N = 1$, 2, 3, 4 supergravity, all torsions and curvatures are on-shell functions of only one superfield. For $N = 1$ this is a 3-spinor W_{ABC} (see section 5), for $N = 2$ a 2-spinor W_{AB}, for $N = 3$ a one-spinor W_{A}, for $N = 4$ a scalar W, while for $N = 8$ one has a scalar W_{ijkl} where $i, \ldots, l$ are SU(8) indices. Moreover, W_{ijkl} is totally antisymmetric and self-dual. The $\theta = 0$ components of W_{ijkl} represent the scalars in the $N = 8$ model after fixing the local SU(8) gauge (see section 6). The constraint $D^i_\alpha W_{jklm} = \delta^i_{[j}\Lambda_{klm]\alpha}$ puts W on-shell. These results are linearizations of results obtained by Brink and Howe.

The Weyl tensor C appears in W as

$$D^i_A D^j_B D^k_C D^l_D W_{mnpq} = 4!\,\delta^i_{[m}\delta^j_n\delta^k_p\delta^l_{q]}C_{ABCD}. \tag{3}$$

(In two-component notations, C is totally symmetric, see the appendix.) It is now easy to write down a supersymmetric invariant

$$I = \kappa^{12}\int \mathrm{d}^4x\,\mathrm{d}^{32}\theta (W_{mnpq}\bar{W}^{mnpq})^4. \tag{4}$$

Note that from (3) it follows that W is dimensionless, so that one finds in I a term of the form $\kappa^{20}\partial^8 h^8$ where $g_{\mu\nu} = \gamma_{\mu\nu} + \kappa h_{\mu\nu}$. Performing the θ-integration one ends up with a nonvanishing product of Weyl tensors (together with its supersymmetric extensions). Thus it would seem that $N = 8$ has nonvanishing invariants which could serve as infinities in the S-matrix from 7 loops onwards. Increased understanding of how to build invariant actions with measures $\mathrm{d}^n\theta$ where $n < 32$ might lower the onset of nonfiniteness down to 3-loops.

For $N = 1$ the complete set of 3- and higher loop counter terms, as presented to order ψ^2 and κ^2 in ref. [123], the complete result being obtained from the tensor calculus in ref. [531], shows that already at the 3-loop level such $\Delta\mathcal{L}$ exist. For $N = 2$ the recently established tensor calculus seems also to give this result. For $N = 8$ the above results seem to leave little hope short of a miracle, but let us discuss possible escape routes.

First of all, the invariant I is a globally supersymmetric invariant (though the use of superfields yields exact results to all orders in ψ and κ), and it might not be possible to extend it to a local invariant. This seems unlikely. Secondly, other $N = 8$ formulations might exist, notably formulations involving antisymmetric tensor fields. However, it is known that different field representations for a given spin do lead to the same S-matrices (although they lead to different anomalies [560]). It should, however, not be overlooked, that using antisymmetric tensor fields, the trace anomalies cancel "miraculously" in the $N = 8$ model [609]. Something is going on here.

Extra symmetries usually help to eliminate possible invariants. For $N = 8$, the maximal on-shell symmetry group is U(8) (see subsection 3.3), and W_{ijkl} is manifestly SU(8) covariant, while a U(1) invariance would violate the self duality property. Thus there seem to be no on-shell extra symmetries left. (The chiral-dual symmetries first discovered in "miraculous" cancellations were later shown to be part of the U(N) invariance. In addition, the $N = 8$ model has a self-duality, see section 6.)

In the presence of central charges, the integration measure is no longer $d^{32}\theta$ since in that case the highest component of a multiplet varies into a total derivative *plus* something more. However, central charges usually vanish on-shell (section 6).

Yet another approach is to pin one's hope on the spontaneously broken version of the $N = 8$ model (section 6). However, mimicking in superspace the dimensional reduction from the massless $d = 5$, $N = 8$ theory to the massive $d = 4$, $N = 8$, one would expect to end up with a dangerous invariant $\Delta\mathscr{L}$ in $d = 4$ dimensions. In fact, the reverse might happen: it might be that massive $d = 4$, $N = 8$ is not even one-loop finite. This is presently unknown and under study.

Of course, miracles do sometimes happen, but they would have to happen at every loop order. It seems better to accept a fundamental property of gravitons: that no nearby massless theory exist (the van Dam–Veltman theorem) and to use nonperturbative methods. Perhaps this is the way quantum supergravity should go in the future, but it is easier to say that one should do nonperturbative calculations than to do them.

2.10. Explicit calculations

2.10.1. Green's functions are not finite

As a first application of the Feynman rules derived in the last section we calculate the graviton energy in simple supergravity at the one-loop level. This yields at once an important result of quantum supergravity: Green's functions are not finite in general [510]. Later we shall see that S-matrix elements are sometimes finite.

The general amputated two point function receives contributions from a spin 2 loop, a general coordinate (spin 1) ghost, a spin 3/2 loop and a supersymmetry (spin $\frac{1}{2}$) ghost, see fig. 1. The divergent parts are local and the general form is (using dimensional regularization)

$$D_{\mu\nu,\rho\sigma} = \frac{\kappa^2}{n-4}[Ap_\mu p_\nu p_\rho p_\sigma + (2B\delta_{\mu\rho}\delta_{\nu\sigma} + C\delta_{\mu\nu}\delta_{\rho\sigma})p^4 + (2Dp_\mu p_\nu\delta_{\rho\sigma} + 4Ep_\mu p_\rho\delta_{\nu\sigma})p^2]$$

where one should still symmetrize in $(\mu\nu)$, in $(\rho\sigma)$ and under $(\mu\nu)\leftrightarrow(\rho\sigma)$. As regularization scheme we

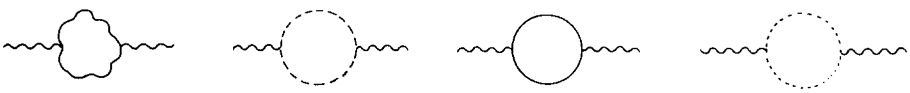


Fig. 1. Graviton self energy graphs.

may use either dimensional regularization or dimensional reduction for the one-loop divergences. The latter scheme should be used in general (see subsection 7).

For the results one finds

	spin 2	spin 1	spin 3/2	spin 1/2
A	13/15	−1/6	−3/40	1/12
B	71/240	−1/24	23/480	0
C	27/80	−7/48	−7/80	0
D	−19/60	1/8	7/80	0
E	−11/40	1/48	−13/480	−1/48

We have added an extra minus sign to the spin 1 (ghost) and spin 3/2 (physical) loops, but not to the spin $\frac{1}{2}$ (ghost).

Two conclusions can be drawn from these results:

(i) *The supersymmetry ghost field commutes with itself*, since *only* then is the Ward identity

$$p^{\mu}D_{\mu\nu\rho\sigma}p^{\sigma}=0$$

satisfied. It is of course satisfied separately in the purely gravitational and in the extra half-integer-spin sector.

(ii) *Green's functions are not finite.* For example, the B coefficients do not sum up to zero. In S-matrix elements the two point functions vanish since $p^2=0$ and polarization tensors are transverse. In supergravity, the S-matrix elements of higher point vertices are finite if one adds self energies, triangles, boxes. However, the sum of self energy graphs is separately not zero. In fact one finds

$$B(\text{photon})=\frac{1}{40}\ ,\quad B(\text{photon ghost})=0$$

$$B(\text{real electron})=\frac{1}{160}\ ,\quad B(\text{real scalar})=\frac{1}{480}\ .$$

Clearly, all B are positive as one might expect from a unitarity argument and no magical combination of fields of different spins can yield zero for the graviton self energy. We have restricted our attention here to B because if one couples the gravitons to a traceless conserved source (photons) only B contributes.

As a natural counterpart one may consider the gravitino self energy. Again the four-gravitino couplings do not contribute, nor do the four-ghost couplings due to elimination of S, P, A_m from the quantum action. One finds the results in fig. 2. In the gauge $\gamma\cdot\psi$ the ghost loop is zero separately and the self energy is again transverse. In more general gauges, the self energy graphs of the gravitino are transverse independently of whether the supersymmetry and coordinate ghost commute or anticommute with each other. This freedom has been explained theoretically [553].

2.10.2. *Matter couplings are not finite*

Since Green's functions are divergent, the next question is whether S-matrix elements are finite. We

Fig. 2. Gravitino self energy corrections.

first consider matter coupling theories (i.e., not the extended supergravities) and we will later come back to the pure (simple and extended) supergravities.

Only an explicit calculation can demonstrate that matter couplings diverge, as a moment of thought will convince most readers. As an example of such a calculation, consider photon–photon scattering at the one-loop in supersymmetric Maxwell–Einstein theory. The spin content of this system is $(2, 3/2)+(1, 1/2)$, and the action reads

$$\mathscr{L} = \mathscr{L}(\text{gauge}) + \mathscr{L}(\text{kinetic}) + \frac{\kappa}{2} e\bar{\psi}_\mu \sigma \cdot F\gamma^\mu \lambda + (\psi^4, \psi^2\lambda^2, \lambda^4)\text{terms}.$$

The process photon–photon scattering is considered since it yields the simplest diagrams; these are depicted in fig. 3. The diagram "ME" stands for all diagrams which are already present in the ordinary Einstein–Maxwell system and which contain only gravitons and photons. These one-loop divergences were evaluated with the background field method and with normal field theory. As expected, the same result was obtained. This result was rather amusing: (137/60) times $(n-4)^{-1}$ times the energy-momentum tensor of the photon squared. The other diagrams in fig. 3 are what supergravity adds, and three conclusions are drawn [510]:

(i) *The supersymmetric Maxwell–Einstein system is invariant under combined duality-chirality transformations* up to order κ^2 at least

$$\delta F_{\mu\nu} = \alpha \frac{\mathrm{i}}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}, \qquad \delta\lambda = -\alpha \mathrm{i}\gamma_5 \lambda.$$

The duality invariance of the ordinary Maxwell–Einstein system and the chirality invariance of the massless Dirac system is well-known. However, the Noether coupling is invariant under combined duality-chirality transformations. This explains why all divergences are of the form $T_{\mu\nu}{}^2$, since $F_{\mu\nu}{}^4$ and

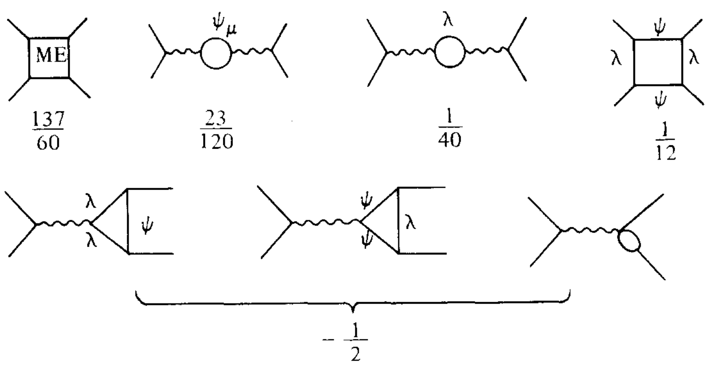


Fig. 3.

$(F_{\mu\nu}{}^2)^2$ are not duality invariant. The combined chirality-duality invariance first found in this system (as a result of "miraculous regularities" in an explicit calculation!) has been found also in the extended supergravities as a complete invariance to all orders in κ (see section 6).

(ii) *Matter couplings are not finite.* The sum of the coefficients in fig. 3 is 25/12 and hence the S-matrix diverges. Also all other matter couplings have been found to be divergent. These include

(a) in the $(2, \frac{3}{2}) + (\frac{1}{2}, 0^+, 0^-)$ system photon–photon and scalar–scalar scattering is not finite [516].

(b) in $N = 4$ extended supergravity with spin content $(2, 3/2^4, 1^6, 1/2^4, 0^2)$ coupled to m matter O(4) multiplets with spin content $(1, 1/2^4, 0^6)$ one finds that gauge–photon gauge–photon scattering is one-loop finite (although *virtual* matter contributes!) [223]. This can be explained by helicity arguments [512, 515]. ***Any process whose external fields are related to the graviton by symmetries is one and two-loop finite, even though virtual matter fields contribute.*** However, matter-photon matter-photon scattering is not finite [223].

(iii) It is incorrect to state that unitarity prevents cancellation of all divergences (as the finiteness of extended supergravities shows). In a background field calculation, the coefficient of $R_{\mu\nu}{}^2$, R^2, $F_{\mu\nu}{}^4$ etc. are indeed positive, but on-shell $G_{\mu\nu} = T_{\mu\nu}$ and for example $R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha}$ can become negative. (For a discussion about the meaning of $G_{\mu\nu} = T_{\mu\nu}$ for diagrams, see ref. [607].)

Thus we conclude that it seems likely that no matter coupled system exists with a finite S-matrix at the one-loop level.

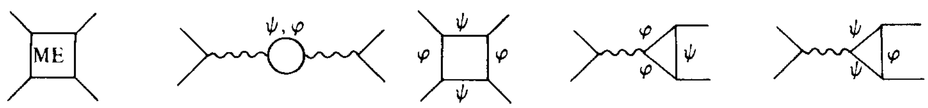
2.10.3. *Calculations of finite processes*

For pure $N = 1$ supergravity, the simplest and at the same time only rigorous proof on one- and two-loop finiteness is by means of the tensor calculus. For $N > 1$, no rigorous proofs exist, but it is rather probable that also here one- and two-loop finiteness exists. Historically, though, it was only through explicit calculations of certain processes that one-loop finiteness was discovered. Since these calculations have been done up to $N = 8$, it is very probable that all extended supergravities are one-loop finite. About two-loop finiteness no rigorous results beyond order κ^2 are known because here no tensor calculus is yet available. Also two-loop computations in supergravity are hard. Hence here a tensor calculus might bring relief.

N = 2. The first successful result in quantum supergravity was a calculation of photon–photon scattering in $N = 2$ extended supergravity [512]. The only terms in the action which contribute in addition to the $N = 2$ gauge action is the Noether coupling

$$\mathcal{L}_{\mathrm{N}} = \frac{\kappa}{\sqrt{2}}\bar{\psi}_\mu(F^{\mu\nu} + \tfrac{1}{2}\gamma_5\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma})\varphi_\nu, \qquad (\psi_\mu \text{ and } \varphi_\nu \text{ are the two gravitons}). \tag{1}$$

The set of diagrams is given in fig. 4.



$\frac{137}{60}$ $\quad 2\left(\frac{23}{120}\right)$ $\quad (-1)$ $\quad -\frac{5}{6}$ $\quad -\frac{5}{6}$

Fig. 4.

All other diagrams are zero, for example the *two* diagrams with gravitino ghost loops.

Again all divergences are proportional to $T_{\mu\nu}{}^2$ and again an exact chirality-duality invariance is the explanation. We have indicated in fig. 4 the coefficients of these graphs.

Since there are two local supersymmetries associated with the two gravitinos ψ_μ and φ_μ, we choose two gauge conditions $\gamma \cdot \psi = 0$ and $\gamma \cdot \varphi = 0$. Since

$$\delta\psi_\mu = D_\mu \epsilon_\psi - \frac{1}{2\sqrt{2}} \sigma \cdot \hat{F} \gamma_\mu \epsilon_\varphi \tag{2}$$

one finds that $\delta(\epsilon_\varphi)\gamma \cdot \psi = 0$, so that there is no mixing between ghosts related to ψ and ghosts related to φ. In fact, these ghosts do not contribute. For example, the vertices due to the spin connection in $\bar{C}\not{D}C$ yield $(\bar{C}\gamma^\mu \sigma^{ab} C)\partial_b e_{a\mu}$. Only the vector terms in $\bar{C}\gamma^\mu\sigma^{ab}C$ survive, but those yield $e_{\lambda\lambda}$ or $\partial_\lambda e_{\lambda\mu}$ and this in turn yields zero since $T_{\mu\nu}$ (photon) is conserved and traceless. There is no mixing between bosonic and fermionic ghosts since in that case a gravitino should emerge from the ghost loop and lead to a second loop. All gravitational ghost contributions are already contained in "ME". Also the photon self energy is contained in "ME".

N = 3. The $N = 3$ pure extended supergravity theory contains both the $N = 2$ case and the supersymmetric Maxwell–Einstein case as consistent truncations (meaning that one can set some fields equal to zero, such that they not appear in the transformation rules of the rest of the fields). Again photon–photon scattering is considered [516] and again only the Noether coupling contributes in addition to the kinetic terms

$$\mathcal{L} = \tfrac{1}{2} e\kappa \bar{\psi}_\mu{}^i \sigma \cdot F^i \gamma^\mu \lambda - \epsilon^{ijk} \frac{\kappa}{2\sqrt{2}} \bar{\psi}_\mu{}^i (eF^{\mu\nu,j} + \tfrac{1}{2}\gamma_5 \tilde{F}^{\mu\nu,j})\psi_\nu{}^k . \tag{3}$$

Again one finds finiteness, both for photons with the same index i and for $i + j \to i + j$ scattering. This checks also the nonzero result for the supersymmetric Maxwell–Einstein system.

N = 4, 5, 6, 7, 8. Photon–photon scattering with all possible internal indices ("i" and "j") as well as scalar–scalar scattering is one-loop finite [223]. These calculations serve as a check on all other calculations in gravity and in supergravity. Some extra "miraculous cancellation" in the calculations for $N = 6$ and $N = 8$ seem to indicate new symmetries, but these have not yet been found.

We conclude by stating that, as in other areas of supergravity, there are very many details which the reader cannot find in the literature, nor could these details be explained here. The only way to discover them is to do calculations oneself. The reader be warned – it is addictive.

2.11. Higher-order invariants, multiplets and auxiliary fields

The action of simple and extended supergravity should describe states which constitute multiplets of global supersymmetry. In general, one does not know the auxiliary fields of the extended supergravities, but by assuming that they exist, one can get some knowledge about what they are by considering the physical states of the following action [194]

$$\mathcal{L}(\text{kin}) = -\tfrac{1}{2}\gamma(R + \text{more}) + \alpha(R^2 + \text{more}) + \beta(R_{\mu\nu}{}^2 - \tfrac{1}{3}R^2 + \text{more}). \tag{1}$$

Here, "more" denotes terms with gravitinos, auxiliary fields, vector fields, etc., which are needed to make each term separately locally supersymmetric.

For $N = 1$ supergravity, one finds that the R^2-type invariants as far as kinetic terms are concerned, are given by (either by the Noether method or by the tensor calculus)

$$\begin{aligned}\mathscr{L}_2 &= R^2 + \bar{R}\cdot\gamma\partial\!\!\!/\gamma\cdot R - 4(\partial_\mu S)^2 - 4(\partial_\mu P)^2 + 4(\partial\cdot A)^2\\ \mathscr{L}_3 &= (R_{\mu\nu}{}^2 - \tfrac{1}{3}R^2) - \bar{\psi}_\mu(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu)(R^\nu - \tfrac{1}{3}\gamma^\nu\gamma\cdot R) - \tfrac{1}{3}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2.\end{aligned} \tag{2}$$

For the gauge action we have, of course,

$$\mathscr{L}_1 = R + \bar{\psi}\cdot R + \tfrac{2}{3}(S^2 + P^2 - A_m^2).$$

To find the physical states in this system, it is advantageous to use spin projection operators in order to describe the terms bilinear in fields. In this way one finds with (see subsection 1.13)

$$\begin{aligned} R_{\mu\nu} &= \tfrac{1}{2}(\Box h_{\mu\nu} - h_{\mu,\nu} - h_{\nu,\mu} + h_{,\mu\nu}) = \tfrac{1}{2}(P^2 + \sqrt{3}P^{0,\mathrm{ts}} + P^{0,\mathrm{s}})_{\mu\nu\rho\sigma}h_{\rho\sigma}\\ eR &= \Box\theta_{\mu\nu}h_{\mu\nu} - \tfrac{1}{4}h_{\mu\nu}(\Box P^2 - 2\Box P^{0,\mathrm{s}})_{\mu\nu\rho\sigma}h_{\rho\sigma}\\ R^2 &= 3h(\Box^2P^{0,\mathrm{s}})h, \qquad R_{\mu\nu}{}^2 - \tfrac{1}{3}R^2 = \tfrac{1}{4}h(\Box^2P^2)h\end{aligned} \tag{3}$$

that the graviton propagator becomes $P^2(\frac{1}{8}\gamma\Box + \frac{1}{4}\beta\Box^2)^{-1} + P^{0,\mathrm{s}}(-\frac{1}{4}\gamma\Box + 3\alpha\Box^2)^{-1}$, so that the graviton states are:

(i) a massive spin 2 with $M^2 = -\frac{1}{2}\gamma/\beta$ and ghost if $\beta < 0$
(ii) a massive spin 2 and physical if $\gamma > 0$
(iii) a massive spin 0 with $M^2 = \frac{1}{2}\gamma/\alpha$ and physical if $\alpha > 0$.

For the gravitino part one finds (see subsection 1.13)

$$\begin{aligned}\bar{\psi}\cdot R &= \bar{\psi}(P^{3/2} - 2P^{1/2,\mathrm{s}})\partial\!\!\!/\psi\\ \bar{R}\cdot\gamma\partial\!\!\!/\gamma\cdot R &= -12\bar{\psi}(\Box P^{1/2,\mathrm{s}})\partial\!\!\!/\psi\\ \bar{\psi}_\mu\Box\theta^{\mu\nu}(R_\nu - \tfrac{1}{2}\gamma_\nu\gamma\cdot R) &= \bar{\psi}(\Box P^{3/2})\partial\!\!\!/\psi\end{aligned} \tag{4}$$

that the propagator becomes $-P^{3/2}[\partial\!\!\!/(\beta\Box + \frac{1}{2}\gamma)]^{-1} + P^{1/2}[\partial\!\!\!/(\alpha\Box + \frac{1}{2}\gamma)]^{-1}$. Thus the gravitino states are:

(i) two massive spin 3/2 with $M^2 = -\frac{1}{2}\gamma/\beta$ and both ghosts if $\beta < 0$
(ii) a massless spin 3/2 and physical if $\gamma > 0$
(iii) two massive spin 1/2 with $M^2 = \frac{1}{12}\gamma/\alpha$ and both physical if $\alpha > 0$.

Note that at $\Box = 0$ both P^2 and $P^{0,\mathrm{s}}$ (and $P^{3/2}$ and $P^{1/2,\mathrm{s}}$) define the usual tetrad and gravitino states.

To check these results, note that $\mathscr{L}_3$ contains only highest spin (as expected from a conformal invariant) while $\mathscr{L}_1$ yields the usual spin 2 and 3/2 propagators if sandwiched between conserved currents. The relative factors between the graviton and gravitino projection operators in $\mathscr{L}_2$ and $\mathscr{L}_3$ are the same.

If one considers the spin 1 and spin 0 states of the auxiliary fields (which propagate in $\mathscr{L}_2$ and $\mathscr{L}_3$!) one finds for the A_μ propagator $P^1(\frac{1}{3}\gamma + \frac{2}{3}\beta\Box)^{-1} + P^0(\frac{1}{3}\gamma - 4\alpha\Box)$. For the fields S and P one finds two more massive spin 0 states. All these fields clearly constitute

(i) the usual $(2, \frac{3}{2})$ massless physical multiplet;

(ii) a massive $(2, \frac{3}{2}, \frac{3}{2}, 1)$ ghost multiplet with $M^2 = -\frac{1}{2}\gamma/\beta$

(iii) two massive $(\frac{1}{2}, 0^+, 0^-)$ multiplets, the first containing S, P and one mode of $\gamma \cdot R$, the other containing $\partial \cdot A$, the scalar in $R + R^2$ and the second mode in $\gamma \cdot R$. Both have $M^2 = \frac{1}{12}\gamma/\alpha$.

If one would not have known these auxiliary fields, one would have predicted the existence of auxiliary fields which lead to the encircled states

$$(2, \tfrac{3}{2}), (2, \tfrac{3}{2}, \tfrac{3}{2}, 1), (\tfrac{1}{2}, 0^+, 0^-), (\tfrac{1}{2}, 0^+, 0^-). \tag{5}$$

Of course, this does not uniquely determine which field representation should yield these states, but the results are so simple that one can hardly miss A_μ, S and P. It also explains that a chiral gauge field is needed in $N = 1$ conformal supergravity.

One can turn these arguments around and deduce some knowledge about auxiliary fields from the requirement that all states should fill up multiplets [168]. In the $N = 2$ model, one of the linearized invariants contains $R^2 + \bar{R} \cdot \gamma \not\partial \gamma \cdot R + 2[\partial_\mu(t_{\mu\nu} - 2^{-1/2}(\partial^\mu A^\nu - \partial^\nu A^\mu))]^2$ where the auxiliary field $t_{\mu\nu}$ rotates into the gravitino field equation! [169] (A similar thing happens in the $N = 8$ model in 11 dimensions [559].) Adding the Poincaré action, the states constitute one massless $(2, \frac{3}{2}, \frac{3}{2}, 1)$ multiplet and the massive $(1, (\frac{1}{2})^4, 0^5)$ multiplet. Adding the invariant which starts with $R^2_{\mu\nu}$, a massive $(2, (\frac{3}{2})^4, (1)^6, (\frac{1}{2})^4, 0^+)$ multiplet is produced. A curious feature is that an entire $(1, (\frac{1}{2})^4, (0)^5)$ multiplet develops, generated by the auxiliary fields. All $N = 2$ auxiliary fields occur in these multiplets (see section 6).

2.12. On-shell counterterms and non-linear invariances

One approach to renormalizability has been to investigate which supersymmetric counterterms are non-zero on-shell, based on the assumption that counterterms must be supersymmetric. This assumption may not be correct as the following counter examples show. In Yang–Mills theory, the Lorentz invariant $\boldsymbol{W}_\mu \cdot (\boldsymbol{W}_\nu \wedge \boldsymbol{F}_{\mu\nu})$ is not a Yang–Mills invariant, nor does it vanish on-shell, but its variation vanishes on-shell. If it would be produced in one-loop corrections as a counterterm, it would clearly be incorrect to first write down all possible off-shell invariants and then to study which of them vanish on-shell.

As a second example [165], consider the globally supersymmetric Wess–Zumino model

$$\begin{aligned} &\mathcal{L} = \mathcal{L}^{\text{kin}} + g\mathcal{L}^{g}, \qquad \mathcal{L}^{\text{kin}} = -\tfrac{1}{2}(\partial_\mu A)^2 - \tfrac{1}{2}(\partial_\mu B)^2 - \tfrac{1}{2}\bar{\lambda}\not\partial\lambda + \tfrac{1}{2}F^2 + \tfrac{1}{2}G^2 \\ &\mathcal{L}^{g} = -F(A^2 + B^2) + 2GAB + \bar{\lambda}(A + \mathrm{i}\gamma_5 B)\lambda. \end{aligned} \tag{1}$$

Both $\mathcal{L}^{\text{kin}}$ and $\mathcal{L}^{g}$ are separately invariant under $\delta(A + \mathrm{i}B) = \bar{\epsilon}(1 + \gamma_5)\lambda$, $\delta(F + \mathrm{i}G) = \bar{\epsilon}\not\partial(1 + \gamma_5)\lambda$, and $\delta\lambda = \not\partial(A - \mathrm{i}\gamma_5 B)\epsilon + (F + \mathrm{i}\gamma_5 G)\epsilon$. The one-loop divergences $\Delta\mathcal{L}$ are proportional to only $\mathcal{L}^{\text{kin}}$; this is a peculiarity of this model which has been explained in several ways, but which fact we will take for granted. It is now clear that if one eliminates F and G from $\mathcal{L}^{\text{kin}} + \mathcal{L}^{g}$, from $\Delta\mathcal{L}$ and from the transformation rules by substituting the equations of motion obtained from $\mathcal{L}^{\text{kin}} + \mathcal{L}^{g}$, then $\mathcal{L}^{\text{kin}} + \mathcal{L}^{g}$ is still invariant but $\Delta\mathcal{L}$ is no longer invariant. In fact one finds

$$\Delta\mathcal{L} = -\tfrac{1}{2}(\partial_\mu A)^2 - \tfrac{1}{2}(\partial_\mu B)^2 - \tfrac{1}{2}\bar{\lambda}\not\partial\lambda + \tfrac{1}{2}g^2(A^2 + B^2)^2 \tag{2}$$

$$\delta\,\Delta\mathcal{L} = \bar{\epsilon}g(A - \mathrm{i}\gamma_5 B)^2 I_{,\lambda} \quad \text{with} \quad I_{,\lambda} = \not\partial\lambda - 2g(A + \mathrm{i}\gamma_5 B)\lambda. \tag{3}$$

One can also add masses and consider non-renormalizable models such as the Lang–Wess model. In all these cases one finds the same phenomenon. Thus there are three possibilities for counterterms:

(i) $\Delta\mathscr{L}$ is off-shell invariant;

(ii) $\Delta\mathscr{L}=0$ on-shell but not off-shell invariant;

(iii) $\delta(\Delta\mathscr{L})=0$ but $\Delta\mathscr{L}\neq 0$ on-shell, nor off-shell invariant.

It is easy to construct many examples of case (iii). Take any invariant action, thus $\delta I=\delta\phi^j I_{,j}=0$. Differentiating this *identity* with respect to some parameter g in the action, one finds $\delta I_{,g}+(\delta\phi^j{}_{,g})I_{,j}=0$ so that $\delta I_{,g}$ vanishes on-shell but $I_{,g}$ itself is not invariant in general.

Thus to prove the finiteness of the S-matrix it is incorrect to demonstrate that all invariant counterterms vanish on-shell. It is correct to show that on-shell no $\Delta\mathscr{L}$ exists which is on-shell invariant. In that case, one must use the on-shell transformation rules and, for example, to show that no $\Delta\mathscr{L}$ exists without purely gravitational sector, requires very detailed analysis. Details of such analysis have not yet been published.

From a more general point of view [165], the generating functional in the background field method

$$Z(\phi)=\int \mathrm{d}\chi\, \exp \mathrm{i}[I(\phi+\chi)-I(\phi)-I(\phi)_{,\phi}\chi] \tag{4}$$

explains these examples. If the original action is invariant under $\phi\to g(\phi)$ then in general the background functional is invariant under background symmetry transformations $\phi\to g(\phi)$ and $\chi\to g(\phi+\chi)-g(\phi)$, provided $g(\phi)$ is linear in ϕ. (In that case $I_{,\phi}\chi$ is again an invariant.) In our examples, however, $g(\phi)$ was after elimination of auxiliary fields no longer linear in fields.

In order to connect these considerations to supergravity, one must find arguments which show that case (iii) counterterms do not occur there. In most of the analysis of on-shell invariants, auxiliary fields were not considered since they were unknown at the time. Now, however, one can discuss the influence of nonclosure of the algebra on invariance of counterterms. Since the commutator of two symmetry transformations is again a symmetry of the original action, in supergravity with open algebra (without auxiliary fields) there are infinitely many field-dependent transformations under which any true invariant of the theory should be invariant. In most cases this leaves the original action as the only true invariant, and dangerous counterterms should then be of type (iii) with respect to these symmetries.

It should be noted that in supergravity the transformation laws with auxiliary fields are still non-linear. However, the one- and two-loop finiteness remains true, since it was proved on-shell using helicity arguments [512, 283] or Noether methods [123].

2.13. Canonical decomposition of the action

The Hamiltonian approach to supergravity, using path-integrals, first demonstrated four-ghost couplings in supergravity (Fradkin and Vasiliev [226]). As an introduction we discuss the "p, q" decomposition of the action. This action analysis is the counterpart of the analysis of the field equations, and, of course, we will again find that there are two physical modes. However, unlike in the field equations, one does not need to choose a gauge for the actions.

We consider the free linear spin 3/2 action first, and make a spacetime split [126]

$$\mathscr{L}=\tfrac{1}{2}\epsilon^{ijk}\bar{\psi}_i\gamma_5(\gamma_j\partial_4-\gamma_4\partial_j)\psi_k+\epsilon^{ijk}\bar{\psi}_4\gamma_5\gamma_i\partial_j\psi_k. \tag{1}$$

The Lagrange multiplier leads to the constraint

$$\epsilon^{ijk}\gamma_5\gamma_i\partial_j\psi_k = -2\gamma_4\sigma^{jk}\partial_j\psi_k = 0. \tag{2}$$

Thus only ψ_j^{T} appears in the action, since the longitudinal part $\psi_k \sim \partial_k\Box^{-1}\partial\cdot\psi$ vanishes in the first term in (1) after partial integration and use of (2). If we had worked with field equations, we would at this point have needed to introduce a gauge choice. Next we note that since we may replace ψ by ψ^{T}

$$\not{\partial}\boldsymbol{\gamma}\cdot\boldsymbol{\psi}^{\mathrm{T}} = 0, \qquad \psi_i^{\mathrm{T}} = (\delta_i^j - \partial_i\partial^j\nabla^{-2})\psi_j \tag{3}$$

again due to the constraint in (2). Thus $\boldsymbol{\gamma}\cdot\boldsymbol{\psi}^{\mathrm{T}} = 0$ since $\not{\partial} \equiv \partial_i\gamma^i$ is elliptic (Latin indices run always from (1, 3)). Hence, only $\boldsymbol{\psi}$ appears in the action and is transverse both in ordinary space ($\partial\cdot\boldsymbol{\psi} = 0$) and in spinor space ($\boldsymbol{\gamma}\cdot\boldsymbol{\psi} = 0$). Thus

$$\psi_\mu{}^a = (P^{3/2}_{\mu\nu})^a{}_b\psi_\nu{}^b \equiv (\psi_\mu{}^a)^{\mathrm{TT}} \tag{4}$$

where $P^{3/2}$ is the spin 3/2 spin projection operator of subsection 1.13. The fields ψ^{TT} and ψ^{T} are gauge invariant (i.e. invariant under $\delta\psi_\mu = \partial_\mu\epsilon$) so that the action has been decomposed into gauge invariant dynamical modes.

The action becomes equal to

$$I = \int \mathrm{d}^4x - \tfrac{1}{2}\bar{\psi}_i^{\mathrm{TT}}(\gamma^4\partial_4 + \gamma^j\partial_j)\psi_i^{\mathrm{TT}} \tag{5}$$

if one uses the expansion of $\gamma_i\gamma_j\gamma_k$ and the double transversability of ψ_i^{TT}. Thus there are two physical modes, given already explicitly in subsection 1.13.

Using electromagnetic notation, one can also rewrite the action as

$$\mathscr{L} = -\tfrac{1}{2}\bar{\boldsymbol{\psi}}\mathrm{i}\gamma_5(\boldsymbol{\gamma}\wedge\boldsymbol{E} + \gamma_0\boldsymbol{B}) \tag{6}$$

where

$$E_i = \dot{\psi}_i^{\mathrm{TT}}, \qquad B^i = \epsilon^{ijk}\partial_j\psi_k^{\mathrm{TT}} \tag{7}$$

so that the field equations read $\boldsymbol{E} + \gamma_5\boldsymbol{B} = 0$.

2.14. *Dirac quantization of the free gravitino*

Although a Majorana field is its own conjugate momentum, there is no problem in determining uniquely the canonical commutation relations, if one uses Dirac brackets $\{\ \}_{\mathrm{D}}$ rather than Poisson brackets in order to go from the classical to the quantum theory. Poisson brackets are defined as usual, but, as we will see, the Dirac bracket $\{\psi_\mu{}^a, \psi_\nu{}^b\}_{\mathrm{D}}$ is nonzero and in agreement with $\{\psi_\mu{}^a, \pi_\nu{}^b\}_{\mathrm{D}}$. In second quantized form there is no problem at all, since one finds in $\psi_\mu{}^a$ absorption operators $a_\lambda(\boldsymbol{k})$ and creation operators $a_\lambda^\dagger(\boldsymbol{k})$ (and not $b_\lambda^\dagger(\boldsymbol{k})$ since $\psi_\mu{}^a$ is a Majorana field) so that $\{\psi_\mu{}^a, \psi_\nu{}^b\}$ evaluated at the quantum level is nonzero. For example, in the gauge $\gamma \cdot \psi = 0$ one has

$$\psi_\mu{}^a(x) = \sum_{k,\lambda} \frac{1}{\sqrt{2V}} [a_\lambda(k)\, u^\lambda_\mu(k)\, e^{ikx} + a^\dagger_\lambda(k)\, u^\lambda_\mu(k)^*\, e^{-ikx}] \tag{1}$$

where $u^\lambda_\mu = \epsilon^\lambda_\mu u_\lambda$ and $\lambda = +$ or $-$.

Consider a theory with fermionic variables. We define canonical momenta as left derivatives of the Lagrangian

$$p = \partial^\ell L / \partial \dot{q}. \tag{2}$$

Then the Hamiltonian is independent of $\dot{q}$ if one defines it as

$$H_c = \dot{q}p - L, \qquad \delta H_c = \dot{q}\, \delta p - \delta q\, (\partial^\ell L / \partial q). \tag{3}$$

The equations of motion read (r = right)

$$\partial^r H_c / \partial p = \dot{q}, \qquad \partial^\ell H_c / \partial q = -\partial^\ell L / \partial q = -\dot{p} \tag{4}$$

where in the second equation we have used the Euler–Lagrange equations

$$\partial^\ell L / \partial q = \frac{d}{dt} (\partial^\ell L / \partial \dot{q}) = \dot{p}. \tag{5}$$

A functional $F(q, p)$ evolves then in time as

$$dF/dt = (\partial^r H_c / \partial p)(\partial^\ell F / \partial q) - (\partial^r F / \partial p)(\partial^\ell H_c / \partial q) = \{F, H_c\}. \tag{6}$$

This defines the Poisson bracket for two variables f and g if at least one of them is bosonic.

In the general case, the Poisson bracket is defined as

$$\{f, g\} = -(\partial^r f / \partial p)(\partial^\ell g / \partial q) + (-)^{fg} (\partial^r g / \partial p)(\partial^\ell f / \partial q) \tag{7}$$

where $f = 0(1)$ if f is bosonic (fermionic). The extra sign $(-)^{fg}$ is needed in order that for two fermionic f and g one has $\{f, g\} = \{g, f\}$. The overall sign is fixed by requiring that $\{f, gh\} = \{f, g\}h + (-)^{fg}\{g, fh\}$. We now consider the free spin $\frac{3}{2}$ field, following P. Senjanovič [608].

Thus, for example, for the gravitino field $\{\psi_\mu{}^a, \psi_\nu{}^b\} = \{\bar{\pi}^\mu{}_a, \bar{\pi}^\nu{}_b\} = 0$ where we denote the conjugate momentum by $\bar{\pi}^\mu{}_a$ rather than $\pi^\mu{}_a$ in order to obtain manifestly Lorentz covariant results. In particular

$$\{\psi_\mu{}^a(x, t), \bar{\pi}^\nu{}_b(y, t)\} = -\delta^a_b \delta^\nu_\mu \delta^3(x - y). \tag{8}$$

The Hamiltonian is (as usual for fermionic systems) independent of time derivatives

$$H_c = \int d_3x\, \tfrac{1}{2} \epsilon^{\mu\nu r\sigma} \bar{\psi}_\mu \gamma_5 \gamma_\nu \partial_r \psi_\sigma \qquad (r = 1, 2, 3). \tag{9}$$

All sixteen momentum definitions in (2) are primary constraints, and can be written as

$$\bar{\phi}^k \equiv \bar{\pi}^k - \frac{\mathrm{i}}{2}\epsilon^{klm}\bar{\psi}_l\gamma_5\gamma_m \approx 0, \qquad \bar{\phi}^0 \equiv \bar{\pi}^0 \approx 0. \tag{10}$$

Thus, as stated above, the field variables are equal to their conjugate momenta.

Adding the primary constraints to the canonical Hamiltonian leads to the total Hamiltonian

$$H_{\mathrm{T}} = H_{\mathrm{c}} + \bar{\phi}^\mu c_\mu. \tag{11}$$

In terms of H_{T} one has the usual Hamilton equations valid for unconstrained systems

$$\partial^e H_{\mathrm{T}}/\partial\psi = -\partial^e L/\partial\psi = -\dot{\bar{\pi}}, \qquad \partial^e H_{\mathrm{T}}/\partial\bar{\pi} = \dot{\psi}. \tag{12}$$

The Lagrange multipliers c_μ are anticommuting functions. Consistency, i.e., $\bar{\phi}^\mu \approx 0$ for all time, thus $\{\bar{\phi}^\mu, H_{\mathrm{T}}\} \approx 0$, then leads to new, so-called secondary, constraints. From $(\mathrm{d}/\mathrm{d}t)\bar{\phi}^k \approx 0$ one finds

$$\epsilon^{krs}(\gamma_4\partial_r\psi_s - \gamma_s\partial_r\psi_4) + \mathrm{i}\epsilon^{krs}\gamma_r c_s = 0. \tag{13}$$

This equation fixes c_k but does not lead to secondary constraints. The solution is unique and can be found by multiplication by $\epsilon^{kmn}\gamma_m$

$$c_k = -\mathrm{i}\partial_k\psi_4 - \epsilon_{kmn}\left(\frac{\mathrm{i}}{2}\gamma_5\partial^m\psi^n\right) - \frac{\mathrm{i}}{2}\gamma^n\gamma_4(\partial_n\psi_k - \partial_k\psi_n). \tag{14}$$

From $(\mathrm{d}/\mathrm{d}t)\bar{\phi}^0 \approx 0$ one finds with $\{\bar{\phi}^0_\alpha, \bar{\phi}^0_\beta\} = 0$ four secondary constraints

$$\bar{\chi}_\alpha = \{\bar{\phi}^0_\alpha, H_{\mathrm{c}}\} = \epsilon^{lmn}\mathrm{i}\partial_l(\bar{\psi}_m\gamma_5\gamma_n)_\alpha \approx 0. \tag{15}$$

These constraints hold whether or not one has substituted the solution for c_k. Since $\bar{\chi}_\alpha$ and H_{c} do not depend on momenta, consistency of $\bar{\chi} = 0$ leads to $\{\bar{\chi}_\alpha, \bar{\phi}^k\}c_k \approx 0$. This seems to lead to yet new constraints, but one finds

$$\{\chi^\alpha, \bar{\phi}^k c_k\} = -\mathrm{i}\epsilon^{klm}\gamma_4\gamma_5\gamma_k\partial_l c_m \tag{16}$$

which is just ∂_k times (13). Hence the total set of constraints consists of the sixteen $\bar{\phi}^\mu_\alpha$ in (10) and the four $\bar{\chi}_\alpha$ in (15).

One now divides these constraints into constraints which commute with all other constraints (first class constraints) and the rest (second class constraints). One finds eight first class constraints

$$\bar{\phi}^0_\alpha = \pi^0_\alpha, \qquad \bar{\chi}_\alpha + \partial_k\bar{\phi}^k_\alpha = \partial_k\bar{\pi}^k_\alpha + \frac{\mathrm{i}}{2}\epsilon^{klm}\,\partial_k\bar{\psi}_l\gamma_5\gamma_m \tag{17}$$

and twelve second class constraints (namely the $\bar{\phi}^k$ plus any linear combination of (17)). To check this result note that

$$\{\bar{\phi}^k_\alpha, \bar{\phi}^l_\beta\} = -\mathrm{i}\epsilon^{klm}(C\gamma_5\gamma_m)_{\alpha\beta}. \tag{18}$$

First class constraints generate gauge transformations. For each first class constraint one chooses a gauge condition. For our case we choose

$$(\psi^{0,\alpha}, \partial_k\psi^{k,\alpha}) = \text{gauge conditions.} \tag{19}$$

Defining $\Lambda_i = (\varphi_a, \zeta_b, \theta_c)$ as the first class constraints, gauge choices and second class constraints, the matrix $C_{ij} = \{\Lambda_i, \Lambda_j\}$ must be nonsingular. In that case one defines the Dirac bracket by

$$\{A, B\}_\mathrm{D} = \{A, B\} - \{A, \Lambda_i\}C^{-1,ij}\{\Lambda_j, B\}. \tag{20}$$

Clearly $\{\Lambda_i, B\}_\mathrm{D} = 0$ for arbitrary B. Thus, *as far as the Dirac bracket is concerned, it does not matter that for Majorana fields the conjugate momenta are proportional to the fields* (see (10)).

Since the pair $(\psi^{0,\alpha}, \bar{\pi}^0_\beta)$ commutes with all other constraints, we may drop them. To evaluate the Dirac bracket, one first evaluates it with respect to the $\bar{\phi}^k$, and then with respect to the other variables λ_i in Λ_i. Denoting the first bracket by $\{\quad\}_\mathrm{I}$, one has

$$\{A, B\}_\mathrm{D} = \{A, B\}_\mathrm{I} - \{A, \lambda_i\}_\mathrm{I}(\{\lambda, \lambda\}_\mathrm{I})^{-1,ij}\{\lambda_j, B\}_\mathrm{I}. \tag{21}$$

In this way one finds for the Dirac brackets

$$\{\psi^\alpha_i(x, t), \bar{\pi}^j_\beta(y, t)\}_\mathrm{D} = -\tfrac{1}{2}(P^{3/2})^\alpha{}_\beta\delta^j_i\delta^3(x-y) \tag{22}$$

where $P^{3/2}$ is the spin 3/2 projection operator. The anticommutator for two ψ fields or momenta π follows from (21) and $\{\Lambda_i, B\}_\mathrm{D} = 0$; in particular, the result for $\{\psi, \psi\}_\mathrm{D}$ is the same as when evaluated using (1). (In the gauge (19), the constraints imply $\gamma \cdot \psi = 0$ [107]. For comparison one must use in (1) the polarization tensors satisfying $\gamma \cdot u = \not{\partial} u_\mu = 0$.) Thus the total result is that replacing the Poisson bracket by the Dirac bracket replaces $\delta^a{}_b$ by $\tfrac{1}{2}(P^{3/2})^a{}_b$. (The factor $\tfrac{1}{2}$ is needed in order that $\dot{\psi}^\alpha_i = \{\psi^\alpha_i, H_\mathrm{T}\}_\mathrm{D}$.)

For path-integral quantization, one starts with

$$Z = \int \mathrm{d}\psi_k\, \mathrm{d}\bar{\pi}^k\, \mathrm{d}\psi_0\, \mathrm{d}\bar{\pi}^0\, \delta(\varphi_a)\,\delta(\zeta_a)\,\delta(\theta_a)\,\mathrm{sdet}\{\zeta_a, \varphi_b\}\,(\mathrm{sdet}\{\theta_a, \theta_b\})^{1/2} \exp \mathrm{i}\int \mathrm{d}^3x\, (\dot{\psi}_\mu\bar{\pi}^\mu - \mathcal{H}_\mathrm{T}) \tag{23}$$

if the gauge conditions are in convolution, $\{\zeta_a, \zeta_b\} = 0$. The product of both superdeterminants is just $(\mathrm{sdet}\{\Lambda_i, \Lambda_j\})^{1/2}$ as one easily verifies, using that the Poisson bracket of φ with θ and itself vanishes.

2.15. Anomalies

In global supersymmetry, the axial and trace and supersymmetry anomalies ($\partial \cdot j^A$, $T_\lambda{}^\lambda$ and $\gamma \cdot j^S$) are part of a multiplet [299]. We discuss below the first two anomalies for a gravitino in an external gravitational field.

Axial anomaly – The action of a gravitino in a background gravitational field

$$\mathcal{L} = -\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu \gamma_5 \gamma_\nu D_\rho(\omega(e))\psi_\sigma \tag{1}$$

is gauge invariant under $\delta\psi_\sigma = D_\sigma(\omega(e))\epsilon$, if the gravitational field satisfies the Einstein equations $G_{\mu\nu}(e) = 0$. Thus one must fix the gauge. Two obvious choices are

$$\mathcal{L}^{\mathrm{I}}(\mathrm{fix}) = \tfrac{1}{4}\bar{\psi}\cdot\gamma\partial\!\!\!/\gamma\cdot\psi, \qquad \mathcal{L}^{\mathrm{II}}(\mathrm{fix}) = \frac{e}{4}\bar{\psi}\cdot\gamma D\!\!\!\!/(\omega(e))\gamma\cdot\psi. \tag{2}$$

In the second choice, the quantum action is manifestly general coordinate invariant, but one must add Nielsen–Kallosh ghosts (see subsection 2.4). In this gauge one can apply the well-known Adler–Rosenberg method for obtaining the axial anomaly. Denoting the matrix element for the axial current going to two gravitons (α, β, p_1) and (γ, δ, p_2) by $M^{\mu}{}_{\alpha\beta\gamma\delta}(p_1, p_2)$, general coordinate invariance $p_1^{\mu}M_{\alpha\beta\gamma\delta} = 0$ relates divergent to convergent form factors, and the axial anomaly $A = (p_{1,\mu} + p_{2,\mu})M^{\mu}{}_{\alpha\beta\gamma\delta} = 0$ can be expressed in terms of finite integrals. Clearly, the contributions of the complex and real spin 1/2 ghosts (both commuting) are known from the spin 1/2 axial anomaly, and one only has to apply the Adler–Rosenberg method to the gravitino loop. The answer is -21 times the anomaly for a real anticommuting spin 1/2 field, and was first obtained by Duff and Christensen by a topological method [172]. Here we will follow the Adler–Rosenberg method. For comparison between the various existing methods, see ref. [530].

We must begin by writing down the axial currents. The classical axial current is the Noether current for the global symmetry $\delta\psi_\sigma = -\mathrm{i}\gamma_5\psi_\sigma$

$$j^{\mu}_{\mathrm{cl}} = -\frac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\nu\gamma_\rho\psi_\sigma \tag{3}$$

and transforms under gauge transformations $\delta\psi_\sigma = D_\sigma(\omega(e))\epsilon$ into

$$\delta j^{\mu}_{\mathrm{cl}} = -\mathrm{i}\partial_\sigma\epsilon^{\mu\nu\rho\sigma}(\bar{\psi}_\nu\gamma_\rho\epsilon) - \mathrm{i}\bar{\epsilon}\gamma_5 R^{\mu} \tag{4}$$

so that the classical axial charge is gauge invariant on-shell. At the quantum level the gravitino equation acquires an extra term from the gauge fixing term [344]

$$-\delta I^{\mathrm{qu}}/\delta\bar{\psi}_\mu = \epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu D_\rho\psi_\sigma - \frac{e}{2}\gamma^{\mu}D\!\!\!\!/\gamma\cdot\psi. \tag{5}$$

At the quantum level we also use BRST transformations rather than gauge transformations, and with $\delta\bar{C} = \Lambda(-\tfrac{1}{2}\bar{\psi}\cdot\gamma\overleftarrow{D\!\!\!\!/})$ one finds that the axial current due to the Faddeev–Popov ghost action (which corresponds to $F_\alpha = -\gamma\cdot\psi$) transforms under BRST transformations as follows

$$j^{\mu}_{\mathrm{FP}} = \mathrm{i}e\bar{C}\gamma^{\mu}\gamma_5 C, \qquad \delta j^{\mu}_{\mathrm{FP}} = -\frac{\mathrm{i}}{2}e\bar{\psi}\cdot\gamma D\!\!\!\!/\gamma^{\mu}\gamma_5 C\Lambda \tag{6}$$

and the extra term in (5) is produced by (6) so that again $j^{\mu}_{\mathrm{cl}} + j^{\mu}_{\mathrm{FP}}$ gives on-shell a BRST-invariant charge. Although the chiral charge of the ghost C^α is fixed by requiring this on-shell invariance, it agrees with the chiral charge which keeps the effective quantum action invariant (note that there are terms $(\bar{C}\partial_\mu\gamma\cdot\psi)C^\mu$ in the Faddeev–Popov ghost action).

One is left with the contribution from the gauge fixing term and the new Nielsen–Kallosh ghost. The first yields $j^\mu_{\text{fix}} = (i/4)\bar{\psi}\cdot\gamma\gamma^\mu\gamma_5\gamma\cdot\psi$. At this point we use the Ward identity derived in subsection 6 which states that constructing two gravitinos with gamma matrices is equivalent to minus twice the same process with a spin 1/2 field. (Insert in (8) of subsection 6 a factor $\partial_\mu\gamma^\mu\gamma^5\Box^{-1}$ and integrate over $x = y$. To check sign and factors consider free propagators.) *Thus, at the one-loop level the chiral charge is BRST invariant if the chiral charge of the Nielsen–Kallosh ghost is opposite to that of the Faddeev-Popov ghosts.* (Note that both closed ghost loops do not require a minus sign.)

The result is now easy. For the gravitino loop one straightforwardly finds $-20A$, for the Faddeev–Popov ghost $-2A$ and for the Nielsen–Kallosh ghost $+1A$, where A is the axial anomaly for a real electron. It would be interesting to analyze what happens at the two-loop level, but this requires first to solve the problem how to define BRST invariance with $\gamma^{\alpha\beta}$ in $\frac{1}{2}F_\alpha\gamma^{\alpha\beta}F_\beta$ being dependent on dynamical fields. It would also be interesting to redo the calculation for different gauge choices, to check that the axial anomaly is gauge-invariant.

Trace anomaly [172] – This anomaly can, in principle, be calculated by using again the Adler–Rosenberg method. It has also been calculated by using 't Hooft's lemma for one-loop divergences, using

$$T^\lambda{}_\lambda = -2g^{-1/2}g^{\mu\nu}\,\delta W/\delta g^{\mu\nu} = \epsilon\,\Delta\mathcal{L} \tag{7}$$

where W is the total one-loop effective action (W but not $T^\lambda{}_\lambda$ contains nonlocal terms). The $\epsilon\,\Delta\mathcal{L}$ result in (7) is valid for massless theories. If the classical action is not invariant under local scale transformations, as for example supergravity, which is only globally scale invariant as far as the terms quadratic in quantum fields are concerned, then the $T^\lambda{}_\lambda$ in (7) includes the non-anomalous part as well and is infinite (in fact its infinity is proportional to $\epsilon^{-1}\Box R$). Using dimensional regularization, one obtains the anomaly as the difference between first letting n tend to 4 and then taking the trace, and the reverse order of these two operations.

Here, however, we will present a method which gives the trace anomaly for all spins at the same time. The method yields $T^\lambda{}_\lambda$ off-shell, but only on-shell (with respect to the background field which may include the extended supergravity fields) is $T^\lambda{}_\lambda$ proportional to a total derivative (since $\Delta\mathcal{L}$ in (7) is quadratic in field equations up to total derivatives). To obtain the anomaly due to a physical massless or massive spin J particle, one considers fields $\phi(A, B)$ which transform irreducibly under Lorentz transformations as the (A, B) representation and takes linear combinations such as to end up with the correct number of helicities. For example, for spin 3/2 one considers $\phi(1, \frac{1}{2})$ and subtracts twice the contribution from $\phi(\frac{1}{2}, 0)$. Working with $\phi(1, \frac{1}{2})$ means choosing the unweighted gauge $\gamma\cdot\psi = 0$, while $2\phi(\frac{1}{2}, 0)$ is the usual Faddeev–Popov ghost. One obtains the desired anomaly by computing the contributions from the fields $\phi(A, B)$ as obtained from the asymptotic expansion of the heat kernel; for details see ref. [172, 174].

The results for N massless spin J physical fields are

$$\int d^4x\,\sqrt{g}\,T^\lambda{}_\lambda = \frac{1}{24}\left[\frac{1}{32\pi^2}\int d^4x\,\sqrt{g}\,{}^*R_{\mu\nu\rho\sigma}\,{}^*R^{\mu\nu\rho\sigma}\right]$$
$$\times\left[-\frac{22}{15}(N_2 - N_{3/2} + N_1 - N_{1/2} + \tfrac{1}{2}N_0) + (58N_2 - 17N_{3/2} - 2N_1 - N_{1/2} + N_0)\right] \tag{8}$$

$$\int d^4x\,\sqrt{g}\,D_\mu j_5^{\mu,A} = \frac{1}{24}\left[\frac{1}{16\pi^2}\int d^4x\,\sqrt{g}\,{}^*R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right][21N_{3/2} - N_{1/2}]. \tag{9}$$

The result in (8) holds for $R_{\mu\nu} = 0$ but (9) is valid even off-shell. Readers who are not sympathetic to heat kernels could have derived these results by direct though tedious computations. For supersymmetric theories, the first term in (8) cancels, and one finds that O(4) supermatter ($N_1 = n$, $N_{1/2} = 4n$, $N_0 = 6n$) and O(3) supergravity have vanishing trace anomaly (note that although for theories which are not locally scale invariant $T^\lambda{}_\lambda(x)$ is infinite, its infinity, proportional to $\Box R$ disappears upon integration). We remark that for self-dual spaces the left-hand side of (8) should be an integer, so that the Euler number χ between square brackets in (8) is an integer times 24; also the numbers $\frac{58}{24}\chi$ are the number of zero spin 2 modes in a self-dual compact gravitational background. Vanishing of the axial anomaly is harder to achieve (although one might define appropriate chiral weights) and one notes in particular that $\int T^\lambda{}_\lambda = 0$ does not imply $\int D_\mu J^{\mu,A} = 0$ even in supersymmetric theories.

2.16. *Chirality, self-duality and counter terms*

The gauge action of $N = 1$ supergravity has a global chiral invariance under $\delta\psi_\mu = i\gamma_5\psi_\mu$. This can be used to find restrictions on possible counter terms [175]. Since the curls $D_\mu\psi_\nu - D_\nu\psi_\mu = \psi_{\mu\nu}$ satisfy on-shell $\psi_{\mu\nu} + \frac{1}{2}\gamma_5\tilde{\psi}_{\mu\nu} = 0$, the combinations $\psi^\pm_{\mu\nu} = \psi_{\mu\nu} \pm \frac{1}{2}\gamma_5\tilde{\psi}_{\mu\nu}$ transform as follows

$$\psi^\pm_{\mu\nu} \to (\exp \mp i\eta)\psi^\pm_{\mu\nu} \qquad \text{if } \psi_\mu \to \exp(i\eta\gamma_5)\psi_\mu. \tag{1}$$

If the leading fermionic terms are only products of $\psi_{\mu\nu}$ (which is the case in all known models), then this chirality symmetry only allows as counter terms $(\psi^+)^k(\psi^-)^k$. In superspace, the possible counter terms are constructed from the Weyl superfield W_{ABC} (symmetric in the three dotted or undotted indices), and introducing again W^+ and W^- as above, one finds for the generic counterterms

$$\Delta\mathscr{L} = W_+^k W_-^k + (DW_+)^p(DW_-)^q + W_+^k W_-^k[(DW_+)^n + \cdots]. \tag{2}$$

This result is due to the fact that DW_+ is invariant under (1). This follows from the fact that W_+ scales by a factor, as in (1), while the derivative D scales in the opposite direction (see [498]). Since the superfields W_+ contain only self-dual Weyl tensors (and W_- only anti-self-dual Weyl tensors), it is clear that if $W_+ = 0$ then the only possible counter terms are of the form $(DW_+)^k$ with any number of derivatives D. These, it is claimed, are total derivatives, and hence the only possible counter terms have as leading bosonic parts the following product of (anti) self-dual Weyl tensors

$$C_+^k C_-^n + C_-^k C_+^n, \qquad k \geq 2 \tag{3}$$

possibly with arbitrarily many derivatives (parity requires the sums in (3)).

If the theory would also have a chirality invariance of the Weyl tensor

$$C_+ \to C_+, \qquad C_- \to -C_- \tag{4}$$

then in (3) one only has k and n even. However, there is no reason to suppose that such an invariance is present, the only duality invariance being the one for $\psi_{\mu\nu}$.

We now discuss the reason that in (3) one has $k>1$. This is because in two-components notation the Weyl tensor has four dotted or undotted indices and is completely symmetric, so that one can only contract self-dual and anti-self-dual Weyl tensor among themselves. Thus, since the Weyl tensor is traceless, one needs at least two of them to obtain a nonvanishing invariant.

From these superspace arguments it follows that there are in $N=1$ supergravity possible dangerous counter terms at 3, 4, 5, etc. loops. All these counter terms vanish when the superfield W_{abc} is self-dual or anti-self-dual (except the one-loop topological counter term which starts with $R^2_{\mu\nu\rho\sigma}-4R^2_{\mu\nu}+R^2$ which is nonvanishing in topologically nontrivial spaces). The result seems to be connected with the phenomenon of helicity conservation in supergravity [178]. These dangerous counterterms also follow from the tensor calculus [189].

2.17. *Super index theorems*

Index theorems relate the number of zero eigenvalue modes of certain elliptic differential operators, such as the Euclidean Dirac operator and the Laplace operator to the topological invariants

$$\begin{aligned} \chi &= \frac{1}{32\pi^2}\int \mathrm{d}^4x\,\sqrt{g}\;{}^*R^*{}_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\ P &= \frac{1}{16\pi^2}\int \mathrm{d}^4x\,\sqrt{g}\;{}^*R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \end{aligned} \tag{1}$$

where χ is the Euler number and P the Pontrjagin number (while the Hirzebruch signature is equal to $\tau=\frac{1}{3}P$ in four-dimensional compact space without boundary).

Let λ_i be the nonzero eigenvalue of the operator Δ, then $\zeta(s)=\Sigma\,\lambda_i^{-s}$ is finite at $s\to 0^+$ and counts the number m of nonzero eigenvalues. On the other hand

$$\mathrm{Tr}\,\mathrm{e}^{-\Delta t}=\sum \mathrm{e}^{-\lambda_i t}\simeq\sum_{k=0}^{\infty}B_k t^{(k-4)/2} \tag{2}$$

where $\simeq$ denotes an asymptotic expansion for small t. Thus, the t-independent terms yield $B_4=n+m$ where n is the number of zero eigenvalues. The B_k are integral invariants formed from the Riemann tensor and its covariant derivatives, involving k derivatives. Thus

$$B_4=\int \mathrm{d}^4x\,\sqrt{g}[\alpha\,{}^*R_{\mu\nu\rho\sigma}{}^*R^{\mu\nu\rho\sigma}]+\beta R^2_{\mu\nu}+\gamma R^2+\epsilon\,{}^*R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \tag{3}$$

In order to obtain relations between the number of zero eigenvalues of two differential operators with the same nonzero eigenvalues, one subtracts the two corresponding B_4 coefficients so that $B_4-B_4'=n-n'$ [174].

One considers fields $\phi(A,B)$ which are irreducible (A,B) representations of the Lorentz group, and fields $\phi[A,B]$ which have $2A$ dotted and $2B$ undotted indices but are reducible. Next one introduces a set of operators $\Delta(A,B)$ acting on $\phi(A,B)$. For example, $\Delta(0,0)$ is the Klein–Gordon operator $-D_\mu D^\mu$, $\Delta(\frac{1}{2},0)=-D_\mu D^\mu+\frac{1}{4}R$ while $\Delta(0,\frac{1}{2})$ differs because the connection in D_μ is different. Furthermore, $\Delta(\frac{1}{2},\frac{1}{2})=-D_\lambda D^\lambda g_{\mu\nu}+R_{\mu\nu}$. It is more convenient to consider $\Delta[A,B]$ acting on $\phi[A,B]$, because one can show that the number of nonzero eigenvalues in $[A,B]$ depends only on $S=A+B$ but not $T = A - B$. In this way one finds index theorems by considering for example $B_4[A, B] - B_4[A - \frac{1}{2}, B + \frac{1}{2}] = n[A, B] - n[A - \frac{1}{2}, B + \frac{1}{2}]$ since one can also compute the α, β, γ, ϵ in (3) as functions of A and B. This is the method for ordinary index theorems.

For super index theorems, one can show that when the Riemann tensor is (anti) self-dual, then $m[A, B]$ is proportional to 4^S. Linear combination of B_4 functions for *different* spins (hence different S) can still give under these conditions index theorems, now relating zero bosonic and fermionic modes. For example

$$B_4[S, T] - 4^{S-S'}B_4[S', T'] = n[S, T] - 4^{S-S'}n[S', T']. \tag{4}$$

To give a few examples of ordinary index theorems

$n[\frac{1}{2}, 0] - n[0, \frac{1}{2}] = -\frac{1}{24}P$ (Atiyah–Singer)

$n[1, 0] - n[\frac{1}{2}, \frac{1}{2}] = \frac{1}{2}\chi + \frac{1}{6}P$ Hirzebruch signature

and

$n[\frac{1}{2}, \frac{1}{2}] - n[0, 1] = -\frac{1}{2}\chi + \frac{1}{6}P$ Gausz–Bonnet theorem

$n[1, \frac{1}{2}] - n[\frac{1}{2}, 1] = \frac{5}{6}P$ spin 3/2 axial anomaly.

A few super index theorems are

$$\left.\begin{aligned} &n[\tfrac{1}{2}, 0] - 2n[0, 0] = -\tfrac{1}{24}(\chi + \tfrac{1}{2}P) \\ &n[\tfrac{1}{2}, \tfrac{1}{2}] - 2n[0, \tfrac{1}{2}] = -\tfrac{1}{12}(\chi + \tfrac{1}{2}P) \end{aligned}\right\} \text{Christensen–Duff.}$$

Notice that on the left hand side one finds the same field content as in supersymmetric theories.

3. Group theory

3.1. *The superalgebras*

The algebraic basis of supergravity is formed by the super algebras. The simplest kind (the only ones we will discuss) contain even and odd generators, and one has always commutators in the algebra, except between two odd elements in which case one has an anticommutator. Since this is entirely as a physicist would expect, we will forego an axiomatic definition. We will only consider algebras whose structure constants are ordinary c-numbers. More general cases (anticommuting structure constants or commuting structure constants containing nilpotent terms) will not be considered (see a forthcoming book by B. DeWitt, P.C.West and the author.)

The simple super algebras consist of two main families, the orthosymplectic OSp(n/m) and the super-unitary SU(n/m). In addition there are several exceptional (and hyper exceptional) algebras corresponding to some (but not all!) of the exceptional Lie algebras G_2, F_4, E_6, E_7, E_8. In addition to simple (super) algebras, there are semisimple super algebras and non-semisimple super algebras. An example of the latter we have already encountered, it is the super Poincaré algebra which has an ideal (an invariant subalgebra) spanned by the generators P_m. Simple (semisimple) super-algebras have no ideal (Abelian ideal).

In general we consider algebras with even (E) and odd (O) generators. The commutations relations are

$$[E_i, E_j] = f_{ij}{}^k E_k, \qquad [E_i, O_\alpha] = g_{i\alpha}{}^\beta O_\beta, \qquad \{O_\alpha, O_\beta\} = h_{\alpha\beta}{}^i E_i. \tag{1}$$

In order that this is a consistent system, one must satisfy the Jacobi identities. Those for three even elements are the usual Jacobi identities for ordinary Lie algebras. In addition there are these Jacobi *identities*

$$[E_i, [E_j, O_\alpha]] - [E_j, [E_i, O_\alpha]] = [[E_i, E_j], O_\alpha] \tag{2}$$

$$[E_i, \{O_\alpha, O_\beta\}] = \{[E_i, O_\alpha], O_\beta\} + \{[E_i, O_\beta], O_\alpha\} \tag{3}$$

$$[O_\alpha, \{O_\beta, O_\gamma\}] = [\{O_\alpha, O_\beta\}, O_\gamma] + [\{O_\alpha, O_\gamma\}, O_\beta]. \tag{4}$$

The first of these three identities says that the O_α form a representation of the ordinary Lie algebra spanned by E_i. (Consider the O_α as vectors on which the E_i act.) The second is equivalent to the first if the Killing form is nonsingular in which case $g_{i\alpha\beta} = h_{\alpha\beta i}$.* The last identity restricts the possible representations O_α of the ordinary Lie algebra. This relation is the reason that not every ordinary Lie algebra can be extended to a superalgebra.

At this point we wish to make a distinction between graded algebras and superalgebras (the two are often confused). A graded algebra is an algebra in which a grading exists, such as for example in the Lorentz group, where the rotation generators, denoted by $\{L_0\}$, and the boosts $\{L_1\}$ define a Z_2 grading:

$$[L_i, L_j] \subset L_{i+j \bmod 2}.$$

A superalgebra is an algebra with a Z_2 grading ("even" and "odd" elements) such that (i) the bracket of two generators is always antisymmetric except for two odd elements where it is symmetric and (ii) the Jacobi identities (2, 3, 4) are satisfied. The bracket relation between two generators need not always be represented by a commutator or anticommutator (see (31) for an example), but one can always find a representation with this property (namely the regular (adjoint) representation).

In order to familiarize the reader a little with super algebras, we first give some explicit representations of the super Poincaré algebra. The defining representation is a 5×5 matrix representation

$$P_m = \begin{pmatrix} & & & & 0 \\ & -\tfrac{1}{2}\gamma_m(1-\gamma_5) & & & 0 \\ & & & & 0 \\ & & & & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad M_{mn} = \begin{pmatrix} & & & & 0 \\ & \sigma_{mn} & & & 0 \\ & & & & 0 \\ & & & & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(Q^\alpha)^m{}_n = 0, \qquad (Q^\alpha)^m{}_5 = [-\tfrac{1}{2}(1+\gamma_5)C^{-1}]^{m\alpha}$$

$$(Q^\alpha)^5{}_m = \tfrac{1}{2}(1-\gamma_5)^\alpha{}_m. \tag{5}$$

The matrix C is the charge conjugation matrix, satisfying $C\gamma_m C^{-1} = -\gamma_m^{\mathrm{T}}$.

* The Killing form has only even–even and odd–odd parts. If it is nonsingular, the superalgebra is nonsingular (the converse is not true).

One easily verifies that the matrices in (5) satisfy the usual Poincaré algebra, plus $[Q^\alpha, P_m] = 0$, $[Q^\alpha, M_{mn}] = (\sigma_{mn})^\alpha{}_\beta Q^\beta$ and $\{Q^\alpha, Q^\beta\} = \frac{1}{2}(\gamma^m C^{-1})^{\alpha\beta} P_m$. Note that this matrix representation uses only ordinary c-numbers, but anticommutators as well as commutators enter in the algebra. One can define the algebra only in terms of commutators by introducing anticommuting elements in the matrix Q, but this we shall not do.

A representation in terms of differential operators is given by (see subsection 5.3)

$$P_m = \partial/\partial x^m, \qquad Q^\alpha = \partial/\partial\bar{\theta}_\alpha - \tfrac{1}{4}(\not{\partial}\theta)^\alpha$$

$$M_{mn} = x_m\partial_n - x_n\partial_m + \bar{\theta}\sigma_{mn}\partial/\partial\bar{\theta}. \tag{6}$$

The variables θ^α are anticommuting variables, and derivatives are left derivatives. Thus

$$\frac{\partial}{\partial\bar{\theta}_\alpha} = (C^{-1})^{\alpha\beta}\frac{\partial}{\partial\theta^\beta}, \qquad \frac{\partial}{\partial\theta^\alpha} = C_{\alpha\beta}\frac{\partial}{\partial\bar{\theta}_\beta}. \tag{7}$$

The multiplets of the tensor calculus are simply irreducible nonlinear representations of the superalgebra considered. Acting with (6) on (x^μ, θ^α) yields an 8-dimensional nonlinear representation.

Casimir operators for the super Poincaré algebra can be defined: the supermass operator P_m^2 and the superspin operator $(K_\mu P_\nu - K_\nu P_\mu)^2$ where

$$K^\mu = \epsilon^{\mu\nu\rho\sigma} M_{\nu\rho} P_\sigma + \bar{Q}\gamma^\mu\gamma^5 Q.$$

By adding a chiral generator A whose only nontrivial bracket is $[Q^\alpha, A] = \mathrm{i}\gamma_5{}^\alpha{}_\beta Q^\beta$ one can define another Casimir operator: $A - \mathrm{i}\bar{Q}\gamma^m\gamma^5 Q P_m (P^k P_k)^{-1}$. These three Casimir operators completely specify any representation [442] of the $N = 1$ super Poincaré algebra without central charges.

The orthosymplectic super algebra OSp(N/M) contains as bosonic part the ordinary Lie algebras Sp(M) and SO(N). It can be defined as those linear transformations which leave invariant the bilinear real form

$$F = x^i y^j \delta_{ij} + \theta^\alpha \chi^\beta C_{\alpha\beta} \qquad (i, j = 1, N;\ \alpha, \beta = 1, M) \tag{8}$$

where θ and χ are anticommuting objects. (If one takes also θ and χ as real ordinary numbers, then x and θ must transform differently from y and χ.) The symbol C denotes an antisymmetric real metric. The first term is of course invariant under SO(N). The second term is invariant under Sp(M). For $M = 4$ (the case of most interest), it is invariant under $\theta' = (\alpha^j O_j)\theta$ where O_j are the 16 Dirac matrices, if

$$(\alpha^j O_j)^{\mathrm{T}} C + C(\alpha^j O_j) = 0. \tag{9}$$

Clearly $O_j = \{\gamma_m, \sigma_{mn}\}$. Since the form F is real, we need real O_j. In a Majorana representation we thus find that Sp(4) is generated by

$$\left(\sigma_{kl}, \mathrm{i}\sigma_{k4}, \frac{1}{2}\gamma_k, \frac{\mathrm{i}}{2}\gamma_4\right), \qquad (k, l = 1, 3). \tag{10}$$

This set of generators generates also the algebra O(3, 2). Indeed, denoting the above generators by

$M_{AB}(A<B)$, they satisfy

$$[M_{AB}, M_{CD}] = g_{BC}M_{AD} + 3\text{ terms}, \qquad g_{AB} = (+,+,+,-,-). \tag{11}$$

This local isomorphism of Sp(4) and O(3, 2) is well-known. (Sometimes there is some confusion that all Lie algebras are compact, while O(3, 2) is not. *This* form of Sp(4) is noncompact; only if one defines Sp(M) by quaternions it is compact.)

For the algebras SO(N) one can also use the diagonal form $F = x^2$ to define SO(N). For Sp(M), one can do the same thing, but then one must require that θ^α are anticommuting objects. In that case one considers

$$F = x^i x^j \delta_{ij} + \theta^\alpha \theta^\beta C_{\alpha\beta} \equiv x^2 + \bar{\theta}\theta. \tag{12}$$

The orthosymplectic algebra OSp(N/M) consists of all real transformations, leaving F invariant. As we saw, the diagonal parts are SO(N) and Sp(M); let us now consider the off-diagonal terms. The most general transformation leaving F in (8) invariant and mixing x and θ (and y and χ) is of the form

$$\delta x^i = -\epsilon^{i\alpha} C_{\alpha\beta}\theta^\beta, \qquad \delta\theta^\alpha = \epsilon^{i\alpha}\delta_{ij}x^j \tag{13}$$

or, (defining $\bar{\theta}_\alpha = \theta^\beta C_{\beta\alpha}$) $\delta x^i = -\bar{\epsilon}^i\theta$ and $\delta\theta = x\cdot\epsilon$. The parameters $\epsilon^{i\alpha}$ are anticommuting. If one considers the θ^α and x^i as a vector in an $(M+N)$ dimensional space, then a matrix representation for the odd generators of OSp(N/M) is given by $(M+N)\times(M+N)$ matrices with anticommuting entries. Clearly, the set of transformations leaving F invariant forms a group, and thus, as one easily checks, the set of generators of SO(N), Sp(M) and the $M\times N$ odd generators form a closed algebraic system under *commutators* only.

One can, however, define a set of $(M+N)\times(M+N)$ matrices with ordinary numbers only, which still forms a closed algebraic system, but now under anticommutators as well as commutators. To this purpose one extracts the parameters $\bar{\epsilon}^i{}_\alpha$ out of the generators. In this way one finds the following representation for OSp(1/4),

$$M_{AB} = \begin{pmatrix} & & & & 0 \\ & m_{AB} & & & 0 \\ & & & & 0 \\ & & & & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad S^\alpha = \frac{1}{\sqrt{2}}\begin{pmatrix} & & & & -C^{-1,1\alpha} \\ & & 0 & & -C^{-1,2\alpha} \\ & & & & -C^{-1,3\alpha} \\ & & & & -C^{-1,4\alpha} \\ \delta_1^\alpha & \delta_2^\alpha & \delta_3^\alpha & \delta_4^\alpha & 0 \end{pmatrix} \tag{14}$$

$$[S^\alpha, M_{AB}] = -(m_{AB})^\alpha{}_\beta S^\beta$$

$$\{S^\alpha, S^\beta\} = -(m^{AB}C^{-1})^{\alpha\beta} M_{AB}, \qquad (A<B)$$

where $m^{AB} = g^{AA'}g^{BB'}m_{A'B'}$ and the matrices m_{AB} were given in (10). Since, as we saw, Sp(4) ~ O(3, 2), one calls this particular super algebra sometimes the super de-Sitter algebra because O(3, 2) is the de-Sitter algebra. Note that although Sp(4) represents the spacetime algebra, it acts on fermionic coordinates θ^α in this explicit representation. (These four θ^α have of course no relations to the four coordinates of superspace.)

One can make a so-called Wigner–Inönü contraction which leads one from OSp(1/4) down to the super Poincaré algebra. To this purpose one writes the de-Sitter algebra as

$$
\begin{aligned}
&[M, M] \sim M, \qquad [M, \Pi] \sim \Pi, \qquad [\Pi, \Pi] \sim M \\
&[M, S] \sim S, \qquad [\Pi, S] \sim S, \qquad \{S, S\} \sim M + \Pi
\end{aligned}
\tag{15}
$$

(with $\Pi \sim M_{\mu 5}$ and $M \sim M_{\mu\nu}$) and scales $\alpha\Pi = P$ and $\beta S = Q$. (One does not scale M since one wants to keep the $M, M \sim M$ Lorentz algebra.) Two possibilities arise:

(i) $\alpha = \beta^2$. This leads to $\{Q, Q\} \sim P$ and $[P, P] = [P, Q] = 0$ for $\alpha \to 0$. Thus one recovers the super Poincaré algebra.

(ii) $\alpha = \beta$. Now $\{Q, Q\} = [Q, P] = [P, P] = 0$. In this case the Q are "outside charges" which rotate as spinors under M, but for the rest have nothing to do with spacetime groups.

Above we have discussed some representations of OSp(1/4). The defining representation was 5×5 dimensional. The adjoint representation (the one in terms of structure constants) is of course 14×14 dimensional since there are 14 generators. However, not every representation of $\mathrm{SO}(N) \times \mathrm{Sp}(M)$ can be extended to a representation of $\mathrm{OSp}(N/M)$. For example, the set of matrices (γ_m, σ_{mn}) and the set $(\gamma_m\gamma_5, \gamma_5)$ both are a representation of Sp(4). Probably the second set cannot be extended to a representation of OSp(1/4).

Supercanonical transformations.† Canonical transformations between creation and absorption operators for bosons define the group $\mathrm{Sp}(M)$. (We recall that M is always even for $\mathrm{Sp}(M)$.) For fermions one finds the group $\mathrm{SO}(N, N)$. As one might expect, the most general canonical transformations mixing bosons and fermions lead to $\mathrm{OSp}(N, N/M)$. This we now prove.

Consider the infinitesimal transformation

$$
\begin{aligned}
\hat{a}_{B,i} &= a_{B,i} + \varphi_{ik} a_{B,k} + \psi_{ik} a^{\dagger}_{B,k} + \epsilon_{ik} a_{F,k} + \eta_{ik} a^{\dagger}_{F,k} \\
\hat{a}_{F,i} &= a_{F,i} + F_{ik} a_{F,k} + P_{ik} a^{\dagger}_{F,k} + E_{ik} a_{B,k} + H_{ik} a^{\dagger}_{B,k}
\end{aligned}
\tag{16}
$$

where ϵ, η, E, H are Grassmann variables. The bosonic sector is symplectic (ψ symmetric, φ antiHermitean)

$$
A \equiv \begin{pmatrix} \varphi & \psi \\ \psi^* & \varphi^* \end{pmatrix}; \qquad AJ + JA^{\mathrm{T}} = 0, \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.
\tag{17}
$$

(A star denotes complex conjugation and T denotes transposition.) Also well-known is that the purely fermionic sector is orthogonal (P antisymmetric, F antiHermitean):

$$
D \equiv \begin{pmatrix} F & P \\ P^* & F^* \end{pmatrix}, \qquad DS + SD^{\mathrm{T}} = 0, \qquad S = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.
\tag{18}
$$

S can be diagonalized by the matrix $I \otimes (\sigma_1 + \sigma_3)$ and has N eigenvalues $+1$ and N eigenvalues -1. Thus one finds the group $\mathrm{SO}(N, N)$ for the canonical transformations between fermionic variables.

The transformations which mix bosonic and fermionic operators are canonical if $H = -\eta^{\mathrm{T}}$ and $E = -\epsilon^{\dagger}$. Introducing matrices as before, one must not forget that in $\hat{a}^{\dagger}_{B,i}$ one finds $(\epsilon_{ij} a_{F,j})^{\dagger} = -\epsilon^{*}_{ij} a^{\dagger}_{F,j}$.

† I thank Professor Berezin for showing me these results and B. Voronov for discussions.

One finds

$$B \equiv \begin{pmatrix} \epsilon & \eta \\ -\eta^* & -\epsilon^* \end{pmatrix}, \qquad C \equiv \begin{pmatrix} E & H \\ H^* & E^* \end{pmatrix}, \qquad BS + JC^{\mathrm{T}} = 0. \tag{19}$$

Thus the canonical transformations are generated by OSp($N, N/M$) and the generators can be represented by $(M+2N)\times(M+2N)$ matrices satisfying

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} J & 0 \\ 0 & S \end{pmatrix} + \begin{pmatrix} J & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\mathrm{T}} = 0 \tag{20}$$

where the supertransposition replaces A, B, C, D, by A^{T}, C^{T}, $-B^{\mathrm{T}}$, D^{T} (see appendix).

The superunitarity algebras, to which we now turn, are the other main family of super algebras. They are denoted by SU(N/M) and contain as bosonic part the algebras $\mathrm{SU}(N)\times\mathrm{SU}(M)\times\mathrm{U}(1)$. In an $(N+M)\times(N+M)$ representation, the SU(N) and SU(M) algebras lie in the first $N\times N$ and last $M\times M$ submatrices, while the U(1) lies along the $(N+M)\times(N+M)$ diagonal. As for the orthosymplectic case, one can define the superunitary algebras as leaving invariant the form

$$F = (x^i)^* x^j \delta_{ij} + (\theta^\alpha)^* \theta^\beta g_{\alpha\beta}, \qquad (g_{\alpha\beta} = \pm\delta_{\alpha\beta}) \tag{21}$$

where $i = 1, N$ and $\alpha = 1, M$. Since the θ^β are complex and F is real, one can always diagonalize the θ-metric and thus M and N can be even or odd. Hence, for SU(N/M) we do not need an antisymmetric metric $C_{\alpha\beta}$ but we can use two diagonal metrics δ_{ij} and $g_{\alpha\beta}$. (If one takes θ to be commuting, one simply finds the algebra SU($N+M$); thus one must take θ as anticommuting.) It is clear that again these transformations form a closed algebra under ordinary commutation relations provided the $x\to\theta$ transformations contain anticommuting entries. Again we go over to representations with ordinary numbers only by extracting the anticommuting parameters.

The most interesting case is the superconformal algebra SU($2, 2/N$). A representation of SU(2, 2) in terms of 4×4 matrices is well-known. It is the set of matrices which leaves $\bar\theta\theta = \theta_1^*\theta^1 + \theta_2^*\theta^2 - \theta_3^*\theta^3 - \theta_4^*\theta^4$ invariant. (For this representation of γ_4 see appendix A.) This is the set of 15 Dirac matrices satisfying $O_j^\dagger\gamma_4 + \gamma_4 O_j = 0$. Thus the generators of SU(2, 2) are

$$P_m = -\tfrac{1}{2}\gamma_m(1-\gamma_5), \qquad K_m = \tfrac{1}{2}\gamma_m(1+\gamma_5), \qquad D = -\tfrac{1}{2}\gamma_5, \qquad M_{mn} = \sigma_{mn}. \tag{22}$$

A representation of the generators G^i of the other unitary group and of the U(1) is given by the antiHermitean generators τ^i for SU(N) and all matrices have zero supertrace. Further

$$A = -\frac{\mathrm{i}}{4}\operatorname{diag}\left(1, 1, 1, 1, \frac{4}{N}, \frac{4}{N}, \dots, \frac{4}{N}\right) \tag{23}$$

for U(1).* Finally there are $2\; N\times4$ generators $Q^{\alpha i}$ and $S^{\alpha i}$ which are the usual square roots of P_m and K_m. They have entries only in the $(4+i)$th columns and rows, and are given by

$$\begin{aligned} &(Q^{\alpha i})^k{}_{4+i} = [-\tfrac{1}{2}(1+\gamma_5)C^{-1}]^{k\alpha}, \qquad (S^{\alpha i})^k{}_{4+i} = [\tfrac{1}{2}(1-\gamma_5)C^{-1}]^{k\alpha} \\ &(Q^{\alpha i})^{4+i}{}_k = \tfrac{1}{2}(1-\gamma_5)^\alpha{}_k, \qquad (S^{\alpha i})^{4+i}{}_k = -\tfrac{1}{2}(1+\gamma_5)^\alpha{}_k. \end{aligned} \tag{24}$$

* For $N = 4$ A is a central charge which may be omitted (and is absent in the S-matrix, see subsection 3.3) without violating the Jacobi identities.

The superconformal algebra SU(2, 2/1) will be discussed in subsection 4.1. In order to obtain the extension to SU(2, 2/N), one endows Q^α and S^α with an extra index i, as in (24). If one evaluates the commutators of $Q^{\alpha i}$ (and $S^{\alpha i}$) with G^j, one finds structure constants which are a product of γ_5 times the symmetric matrices τ^i (the antisymmetric matrices are multiplied by the unit matrix). Hence it is advantageous to introduce chiral charges. One finds when with $Q_{\rm L}^{\alpha i} = \frac{1}{2}(1+\gamma_5)^\alpha{}_\beta Q^{\beta i}$, etc.

$$\left[\begin{matrix} Q^i_{\rm L} \\ S^i_{\rm R}\end{matrix},\ G^j\right] = (\tau^j)^i{}_k \begin{pmatrix} Q^k_{\rm L} \\ S^k_{\rm R}\end{pmatrix}, \left[\begin{matrix} Q_{{\rm R}i} \\ S_{{\rm L}i}\end{matrix},\ G^j\right] = -(\tau^j)^k{}_i \begin{pmatrix} Q_{{\rm R}k} \\ S_{{\rm L}k}\end{pmatrix}. \tag{25}$$

Thus, $Q_{\rm L}$ and $S_{\rm R}$ transform in the vector representation of SU(N), while $Q_{\rm R}$ and $S_{\rm L}$ transform in the complex conjugate of this representation. As explained in appendix E, we therefore have written the indices i of the (N) representation upstairs but those of the N^* representation as subscripts. Moreover, $Q^i_{\rm L}$ anticommutes with $S^i_{\rm R}$ (and $Q_{{\rm R},i}$ with $S_{{\rm L},i}$). Since there are no generators which transform as tensors under G^i, this is to be expected. However, the product of (N) and (N^*) contains scalars and vectors, as shown by

$$\begin{aligned} \{Q^i_{\rm L}, S_{{\rm L}j}\} &= \tfrac{1}{2}(1+\gamma_5)[\delta^i_j(\tfrac{1}{2}C^{-1}D - \sigma^{mn}C^{-1}M_{mn} - {\rm i}C^{-1}A) + 2(\tau_k)^i{}_j C^{-1}G^k] \\ \{Q_{{\rm R}j}, S^i_{\rm R}\} &= \tfrac{1}{2}(1-\gamma_5)[\delta^i_j(\tfrac{1}{2}C^{-1}D - \sigma^{mn}C^{-1}M_{mn} + {\rm i}C^{-1}A) - 2(\tau_k)^i{}_j C^{-1}G^k]. \end{aligned} \tag{26}$$

Also this is clear, since the invariant tensors δ^i_j and $(\tau_k)^i{}_j$ are the Clebsch–Gordon coefficients for the two irreducible representations contained in $(N)\times(N^*)$. The anticommutator with right-handed charges follows from complex conjugation. (Using $Q^* = -S$, $\gamma_5^* = -\gamma_5$ and that the τ^k are antiHermitean.)

Of course the super Poincaré algebra is contained in the superconformal algebra; therefore we have chosen the representations for P_m, M_{mn} and $Q^{\alpha i}$ the same. However, also the de-Sitter algebra is contained in SU(2, 2/N) [197]. To see this, take a general linear combination of all charges

$$\hat{Q}^i = \alpha Q^i_{\rm L} + \beta Q_{{\rm R}i} + \gamma S_{{\rm L}i} + \delta S^i_{\rm R} \tag{27}$$

and require $\{\hat{Q}^i, \hat{Q}^j\} = \frac{1}{2}(\gamma^m C^{-1})\hat{P}_m\delta^{ij} + \text{more}$. One finds $\hat{P}_m = \alpha\beta P_m - \gamma\delta K_m$. Next require that $[\hat{P}, \hat{Q}]$ is again proportional to $\hat{Q}$. One finds only one condition, namely $\alpha\gamma = \beta\delta$. However, if one now evaluates completely $\{\hat{Q}, \hat{Q}\}$ one finds that only the antisymmetric generators $\tau^{\rm R}_A$ appear on the right-hand side

$$\{\hat{Q}^{\alpha i}, \hat{Q}^{\beta j}\} = \{\tfrac{1}{2}\gamma^m C^{-1}\hat{P}_m\delta^{ij} - (\alpha\gamma+\beta\delta)\sigma^{mn}C^{-1}M_{mn}\delta^{ij} + 4\alpha\gamma(\tau_{k,A})^{ij}C^{-1}G^k_A\}^{\alpha\beta}. \tag{28}$$

This suggests that the maximal internal symmetry group of the super de-Sitter algebra is an SO(N) and not an SU(N) group. It also explains why the de-Sitter models are of the O(3, 2) type and not O(4, 1): because the commutator $[\hat{P}_m, \hat{P}_n] = kM_{mn}$ has $k = 4\alpha\beta\gamma\delta > 0$. The spin 1 fields of N-extended de-Sitter supergravities gauge these O(N) subgroups, and one has in these models always a cosmological constant and a masslike term for the gravitinos.

If one now goes from de-Sitter space to Minkovski space (i.e., $[\tilde{Q}^{\alpha i}, \tilde{P}_m] = 0$) by means of a group contraction, the O(N) charges can become central charges which commute with all other generators. Thus, as explained in the section on how to gauge algebras, the gauge coupling of the vector fields disappears and one finds $\frac{1}{2}N(N-1)$ Abelian vector fields. The process of contraction leads one to the super Poincaré algebra. To this purpose one multiplies the $\{\hat{Q}, \hat{Q}\}$ relation by ϵ^2, defines new charges $\tilde{Q} = \epsilon\hat{Q}$ and has then two possibilities for $\epsilon \to 0$ and $\tilde{P} = \epsilon^2 \hat{P}$

$$\begin{aligned} &\{(\epsilon\hat{Q}),(\epsilon\hat{Q})\} \sim (\epsilon^2\hat{P}) + \epsilon^2(M) + \epsilon^2(G), \qquad [(\epsilon\hat{Q}), G^k] \sim (\epsilon\hat{Q}) \\ &\{(\epsilon\hat{Q}),(\hat{\epsilon}\hat{Q})\} \sim (\epsilon^2\hat{P}) + \epsilon^2(M) + (\epsilon^2 G), \qquad [(\epsilon\hat{Q}),(\epsilon^2 G^k)] \sim \epsilon^2(\epsilon\hat{Q}). \end{aligned} \tag{29}$$

In the first case the internal charges G become *outside charges*: they are not reproduced on the right-hand sides, but they rotate the $\tilde{Q}$ under internal symmetry transformations. In the second case one has *central charges*: now the G commute with all other generators, but they are produced on the right-hand sides. For example, OSp(2/4) has an O(2) charge which rotates the two $Q^{\alpha i}$ into each other. Contracting to the $N = 2$ super Poincaré algebra with central charge, this central charge still produces the $\delta A_\mu \sim \bar{\epsilon}^i \psi^j_\mu \epsilon_{ij}$ transformations of the photon under supersymmetry, but all physical states become inert under the central charge (only $\delta A_\mu = \partial_\mu \Lambda$ remains after contraction).

Other super algebras. There exist other super algebras. Since they are little used in supergravity we will be brief. The matrices with vanishing supertrace form of course a closed system since str $M_1M_2 = \mathrm{str}\, M_2M_1$ (considering entries with anticommuting variables one has only commutators in the algebra). This is the super algebra SL(M/N) which can be viewed as acting on a carrier space with M fermionic and N bosonic coordinates. If $M = N$, the unit element has zero super trace, and one can omit it to obtain SSL(N/N) = SL(N/N)/λI. These algebras are simple.

An example of a non semisimple algebra is for example given by the fermion operators E, a, $a^\dagger$, $a^\dagger a$. This is GL(1/1) and the unit element E is the ideal (an invariant subalgebra). Again one can construct SL(1/1) by omitting $a^\dagger a$, and SSL(1/1) by omitting the unit E.

Some exceptional simple super algebras are F_4 with $3+21+2\times 8$ generators (with bosonic part $SU_2 \times$ spin (7) where spin (7) are the twentyone 8×8 matrices σ_{KL} constructed from the gamma matrices in 7 dimensions), G_3 with $3+14+2\times 8$ generators (its bosonic part is $SU_2 \times G_2$; the ordinary exceptional Lie algebra G_2 is obtained by fixing one component of spin (7) and has thus 14 generators). Also there is an algebra D(2/1; α) (=OSp(2/4) if $\alpha = -1$), as well as W, S, $\tilde{\mathrm{S}}$, H.

In addition there are hyperexceptional algebras P(n), Q(n). For example Q(3) contains as even generators the SU_3 generators along the diagonal, and as odd generators again the SU_3 matrices

$$\mathrm{Even} = \begin{pmatrix} \lambda^i & 0 \\ 0 & \lambda^i \end{pmatrix}, \qquad \mathrm{Odd} = \begin{pmatrix} 0 & \lambda^i \\ \lambda^i & 0 \end{pmatrix}. \tag{30}$$

The bracket relations are usual commutators, except that the bracket for two odd generators is given by

$$\langle O^1, O^2 \rangle = \{O^1, O^2\} - \tfrac{2}{3}\,\mathrm{tr}(O^1 O^2) \tag{31}$$

(which yields the d_{ijk} coefficients of SU_3). We have given this example to show that the bracket relations need not be only commutators or anticommutators, when one has a representation in terms of matrices.

For further literature, see

L. Corwin, Y. Ne'eman and S. Sternberg, Rev. Mod. Phys. 47 (1975) 573.
P. Freund and I. Kaplansky, J. Math. Phys. 17 (1976) 228.
V.G. Kac, Funkc. analiz 9 (1975) 91 and Comm. math. Phys. 53 (1977) 31.
Lecture notes in Math. 676, Advances in Math. 26 (1977) 8.

W. Nahm, V. Rittenberg and M. Scheunert, Phys. Lett. 61 B (1976) 383 and papers in J. Math. Phys.
A. Pais and V. Rittenberg, J. Math. Phys. 16 (1975) 2062.

3.2. *The gauging of (super)algebras*

(Conformal) supergravity is the gauge theory of the super(conformal) Poincaré algebra. In order to be complete we will therefore first discuss how one gauges in general a superalgebra.

Let the algebra be given by generators X_A which satisfy the following commutators or anticommutators

$$[X_A, X_B] = f_{AB}{}^C X_C. \tag{1}$$

The structure constants $f_{AB}{}^C$ are ordinary numbers.

Corresponding to any generator, one introduces a gauge field, as shown by the following superalgebra-valued vector field for the super Poincaré algebra

$$h_\mu = h_\mu{}^A X_A = e^m{}_\mu P_m + \omega_\mu{}^{mn} M_{mn} + \bar{\psi}_{\mu,\alpha} Q^\alpha \quad (m < n). \tag{2}$$

In order that Q^α transforms as a spinor, its associated gauge field is $\bar{\psi}_{\mu,\alpha} = \psi_\mu{}^\beta C_{\beta\alpha}$ rather than $\psi_\mu{}^\alpha$, and in order that one counts every generator once, we restrict the sum over $m < n$. Each gauge field corresponds to a local gauge parameter ϵ^A, as shown by the following superalgebra valued scalar field

$$\epsilon = \epsilon^A X_A = \xi^m P_m + \lambda^{mn} M_{mn} + \bar{\epsilon}_\alpha Q^\alpha \quad (m < n). \tag{3}$$

One requires that covariant derivatives $D_\mu = \partial_\mu + h_\mu^A X_A$ (base manifold indices) commute with generators X_A of the tangent space, hence $[D_\mu, X_A] = 0$. This implies that the gauge fields transform in the adjoint representation. Thus $[D_\mu, \epsilon^A X_A] = (D_\mu \epsilon^A) X_A$ and one obtains the result that gauge transformations are defined as covariant derivatives *with respect to the full algebra*

$$\delta(\text{gauge}) h_\mu{}^A = (D_\mu \epsilon)^A \equiv \partial_\mu \epsilon^A + h_\mu{}^B \epsilon^C f_{CB}{}^A. \tag{4}$$

Curvatures are defined by the commutator of two covariant derivatives

$$R_{\mu\nu}{}^A = \partial_\nu h_\mu{}^A - \partial_\mu h_\nu{}^A + h_\nu{}^B h_\mu{}^C f_{CB}{}^A. \tag{5}$$

They rotate homogeneously under gauge transformations

$$\delta(\text{gauge}) R_{\mu\nu}{}^A = R_{\mu\nu}{}^B \epsilon^C f_{CB}{}^A \tag{6}$$

which follows easily from the Jacobi identity

$$[X_Q, [X_S, X_P\}\}(-)^{Q+S} + [[X_S, X_Q\}, X_P\} = [X_S, [X_Q, X_P\}\}. \tag{7}$$

(The extra minus sign occurs when one interchanges X_Q and X_S and both are fermionic.)

Under *arbitrary* variations, curvatures transform as

$$\delta R_{\mu\nu}{}^{A} = D_\nu \delta h_\mu{}^{A} - D_\mu \delta h_\nu{}^{A}. \tag{8}$$

Consider now a Yang–Mills type of action,

$$I = \int \mathrm{d}^4x\, R_{\mu\nu}{}^{A} R_{\rho\sigma}{}^{B} Q^{\mu\nu\rho\sigma}_{AB} \tag{9}$$

where Q may depend on fields. Under local gauge transformations

$$\delta I = \int \mathrm{d}^4x\, \{R_{\mu\nu}{}^{A} \epsilon^{B} R_{\rho\sigma}{}^{C}(-f_{CB}{}^{D} Q^{\mu\nu\rho\sigma}_{AD} + f_{BA}{}^{D} Q^{\mu\nu\rho\sigma}_{DC}) + R_{\mu\nu}{}^{A} R_{\rho\sigma}{}^{B} \delta_{\text{gauge}} Q^{\mu\nu\rho\sigma}_{AB}\}. \tag{10}$$

Under arbitrary variations

$$\delta I = \int \mathrm{d}^4x\, [4(D_\nu \delta h_\mu)^{A} R_{\rho\sigma}{}^{B} Q^{\mu\nu\rho\sigma}_{AB} + R_{\mu\nu}{}^{A} R_{\rho\sigma}{}^{B} \delta Q^{\mu\nu\rho\sigma}_{AB}]. \tag{11}$$

Partially integrating $D_\nu \delta h_\mu{}^{A}$ and using that always $f_{CD}{}^{A} h_\nu{}^{D} \delta h_\mu{}^{C} = -\delta h_\mu{}^{C} h_\nu{}^{D} f_{DC}{}^{A}$ one finds [308] for arbitrary variations

$$\begin{aligned}\delta I = \int \mathrm{d}^4x\, [&4\delta h_\mu{}^{A} h_\nu{}^{B} R_{\rho\sigma}{}^{C}(f_{CB}{}^{D} Q^{\mu\nu\rho\sigma}_{AD} - f_{BA}{}^{D} Q^{\mu\nu\rho\sigma}_{DC}) \\ &+ R_{\mu\nu}{}^{A} R_{\rho\sigma}{}^{B}\, \delta Q^{\mu\nu\rho\sigma}_{AB} - 4\delta h_\mu{}^{A} (D_\nu R_{\rho\sigma})^{B} Q^{\mu\nu\rho\sigma}_{AB} - 4\delta h_\mu{}^{A} R_{\rho\sigma}{}^{B}\, \partial_\nu Q^{\mu\nu\rho\sigma}_{AB}].\end{aligned} \tag{12}$$

In the frequent case that $Q^{\mu\nu\rho\sigma}_{AB}$ is a constant times $\epsilon^{\mu\nu\rho\sigma}$, the last three terms vanish, due to the Bianchi identity $D_\nu R_{\rho\sigma} \epsilon^{\nu\rho\sigma\tau} = [D_\nu, [D_\rho, D_\sigma]]\epsilon^{\nu\rho\sigma\tau} = 0$.

Finally we derive a theorem, which is useful when constraints on curvatures are not gauge invariant under all local symmetries [523]. Suppose one can solve a constraint $R^{A}_{\mu\nu} \Gamma^{\mu\nu}_{A} = 0$ for a given field $h_\mu{}^{A_0}$. Then $h_\mu{}^{A_0}$ is a functional of other fields, and transforms no longer as $(D_\mu \epsilon)^{A_0}$, but according to the chain rule. However, after solving the constraint, its variation vanishes identically. Denoting by δ' the difference between the actual transformation law (using the chain rule) of $h_\mu{}^{A_0}$ and the group law δ(gauge), one has the following

Theorem:

$$(\delta' h_\mu{}^{A_0}) \frac{\delta}{\delta h_\mu{}^{A_0}} (R_{\mu\nu}{}^{A} \Gamma^{\mu\nu}_{A}) = [-f_{CB}{}^{A} R_{\mu\nu}{}^{B} \epsilon^{C} - R_{\mu\nu}{}^{A}\, \delta(\text{gauge})] \Gamma^{\mu\nu}_{A}. \tag{13}$$

In particular, if the constraint is invariant under δ(gauge), then $h_\mu{}^{A_0}$, although no longer an independent gauge field, still transforms according to δ(gauge) just as when it was an independent field.

Using the formalism of this subsection, McDowell and Mansouri [308] showed that $N = 1$ supergravity with cosmological constant follows from gauging OSp(1/4) and Townsend and the author [517] showed the same for $N = 2$ and OSp(2/4). In the latter case, however, a term $\delta\psi^{i}_{\lambda} \sim F_{\mu\nu}\epsilon^{i}$ did not follow from group theory alone. It might be that elimination of auxiliary fields would lead to such a term; this is at present an open question.

An interesting aspect of the $N = 2$ model is that it contains a central charge in the limit of vanishing cosmological constant (see section 6). This is due to the generator T in

$$\{Q_i^\alpha, Q_j^\beta\} = \tfrac{1}{2}\delta_{ij}(\gamma^m C^{-1})^{\alpha\beta} P_m + \epsilon_{ij}(C^{-1})^{\alpha\beta} T. \tag{14}$$

Gauging of this OSp(2/4) superalgebra leads to a theory where the photon gauges T, and where physical states (namely the gravitinos) rotate under T. Making a group contraction, T no longer acts on physical states (it only acts on the photon, and only as a Maxwell gauge transformation) and T becomes in the global algebra a central charge. In the gauge algebra the commutator of two local supersymmetry transformations now also contains a Maxwell transformation with parameter $(\bar\epsilon_2^i \epsilon_1^j \epsilon_{ij})$.

The actions for $N = 1$ supergravity takes in this approach the form

$$I = \int \mathrm{d}^4x\, [R_{\mu\nu}(M)^{mn} R_{\rho\sigma}(M)^{pq} \epsilon^{\mu\nu\rho\sigma} \epsilon_{mnpq} + R_{\mu\nu}(Q)^\alpha R_{\rho\sigma}(Q)^\beta (\gamma_5 C)_{\alpha\beta}] \tag{15}$$

and invariance is obtained if one imposes the constraint $R_{\mu\nu}(P)^m = 0$, which is solved by $\omega = \omega(e, \psi)$. Hence, here we encounter the same result as obtained from 1.5 order formalism, but now by a group theoretical approach. The action for $N = 2$ is obtained by summing the Q-terms over $i = 1, 2$ and adding a Maxwell action.

It is interesting to note that the Yang–Mills type of action in (14) actually only contains terms linear in the curvatures. This comes about because

$$R_{\mu\nu}(M)^{mn} = \text{Riemann tensor} + (e_\mu^m e_\nu^n - e_\nu^m e_\mu^n) + \bar\psi_\mu \sigma^{mn} \psi_\nu. \tag{16}$$

The term quadratic in the Riemann tensor yields in (15) a total derivative (the Gausz–Bonnet invariant), while the cross terms yield the Einstein action. The remaining terms contain the cosmological constant. Similarly, after partial integration, one finds that the higher derivative terms in the gravitino action cancel, and one finds the mass-like terms. (Without gravitinos, one thus also can write ordinary general relativity in Yang–Mills form.) To go to theories without cosmological terms, one performs a Wigner–Inönü group contraction.

Finally we note that also the action and transformation rules for $N = 1$ conformal supergravity can be obtained [523] by gauging SU(2, 2/1), see subsection 4.2.

A slightly different version is due to Regge and Ne'eman. They need a symmetric decomposition of the de-Sitter (not the Poincaré) superalgebra: $(Q_\mathrm{L} + M) + (Q_\mathrm{R} + P)$ and start out with a group manifold (see next subsection). They contract the horizontal part $(Q_\mathrm{R} + P)$ and find thus as field equations $R(P) = R_\mathrm{R}(Q) = 0$ (omitting indices). Subsequently they add a new set of generators and fields $(Q'_\mathrm{R} + M') + (Q'_\mathrm{L} + P')$ and do the same. At this point all torsions are zero, while only curvatures with two horizontal legs (see next subsection) are nonzero. The constraints $R(Q)$ eliminate two of the four gravitino fields ψ_L, ψ_R, ψ'_L, ψ'_R and one finally ends up with the same action as MacDowell and Mansouri. Thus this approach does their construction twice: once in a left and once in a right basis. The doubling of generators is very reminiscent of the chiral superspace approach and we refer to subsection 5.6.

3.3. *The Haag–Lopuszanski–Sohnius theorem*

One can, to a large extent, determine the possible extra symmetries in supersymmetric theories by analyzing the commutator algebra. We stress that in what follows we do not analyze the gauge algebra, but only field-independent super algebras. Thus, for example, central charges which vanish on shell are not discovered in this way. Nevertheless, it is amazing how much can be deduced from (seemingly) rather weak assumptions.

Consider a super algebra which is an extension of the super Poincaré algebra. It contains the Poincaré algebra itself, and N conserved spin $\frac{1}{2}$ charges ($i = 1, N$)

$$[Q^{\alpha}{}_i, P_m] = 0, \qquad [Q^{\alpha}{}_i, M_{mn}] = (\sigma_{mn})^{\alpha}{}_{\beta} Q^{\beta}{}_i. \tag{1}$$

In addition, the following (anti) commutators are present, using two-component notation (see appendix)

$$\{Q^A{}_i, Q^B{}_j\} = \epsilon^{AB} \Omega^l_{ij} Z_l \tag{2}$$

$$\{Q^A{}_i, \bar{Q}_{\dot{B}}{}^j\} = \delta_i^j \left(\frac{\mathrm{i}}{2}\sigma^m\right)^A{}_{\dot{B}} P_m, \quad \bar{Q}_{\dot{B}}{}^j \equiv (Q^A{}_j \epsilon_{AB})^* \tag{3}$$

$$[Q^A{}_i, B_l] = (b_l)_i{}^j Q^A{}_j. \tag{4}$$

The Z_l are a subset of the internal charges B_l and, according to (2) these Z_l commute with P_m. We choose these Z_l to be a basis for the $Z_{ij} = \Omega_{ij}{}^l Z_l$, hence there are in general fewer Z_l than Z_{ij}. In fact, it follows quite generally from the Coleman–Mandula theorem that all B_l are Lorentz scalars which commute with P_m. This theorem states that the most general *bosonic* extension of the Poincaré algebra is a *direct sum* of the Poincaré algebra, a semisimple Lie algebra and an Abelian algebra. Since the Coleman–Mandula theorem holds only for symmetries of the S-matrix, the only symmetries discussed in this subsection are those of the S-matrix. Since the Z_l carry no Lorentz indices, the invariant tensor ϵ^{AB} must appear in (2) (in four-component notation this means that central charges appear as $U_l + \mathrm{i}\gamma_5 V_l$) and it follows that $\Omega^l{}_{ij}$ is antisymmetric in (ij). We choose all B_l anti-Hermitean in order to preserve unitarity.

From now on we will use the Jacobi identities to analyze the algebra further. We will denote these identities for three generators A, B and C by (ABC). Let us consider first $(Q, \bar{Q}, P)$ and note that in (3) one cannot add a term with M_{mn} on the right-hand side. (In a de-Sitter space, $[Q^{\alpha}, P_m]$ is non-vanishing and one does indeed find terms with M_{mn}. Here we consider non-Sitter spaces where $[P_m, P_n] = 0$ and *hence* $[Q^{\alpha}, P_m] = 0$.)

From (QBB) it follows that the b_l are a representation of B_l. From $(BQ\bar{Q})$ it follows that

$$(b_l)_i{}^k = -((b_l)_k{}^i)^* \Rightarrow b_l \text{ anti-Hermitean} \tag{5}$$

assuming that P_m and B_l (and hence Z_l) are anti-Hermitean.

The notation $(Q^B{}_j)^* = \bar{Q}^j{}_{\dot{B}}$ will now be justified. If a tensor T_i transforms under a given group as $M_i{}^j T_j$ let us define a tensor T^i to transform as $(M^{-1,\mathrm{T}})^i{}_j T^j$. (Note that $M^{-1,\mathrm{T}}$ is again a representation of the same algebra of which M is a representation.) One could define two more kinds of tensors, transforming under M^* and under $M^{-1,\dagger}$. (These four sets of tensors correspond in two-component formalism to χ_A, χ^A, $\chi_{\dot{A}}$ and $\chi^{\dot{A}}$.) Since according to (5) the group elements generated by b_l are unitary,

$M^{-1,\mathrm{T}} = M^*$ so that $(Q_i^A)^*$ transforms as a tensor with index i up. This is the reason one writes for $(Q_j^B)^*$ the symbol $\bar{Q}^j_{\dot{B}}$. For the spinor index $\dot{B}$, see appendix E.

The Kronecker delta function in (3) is already the most general possibility. If one would have written a matrix N_i^j instead, complex conjugation of (3) would imply that N is Hermitean. Diagonalizing N by means of a unitarity transformation of the Q_i, and noting that on *states* $\{Q_i^A, (Q_i^A)^*\}$ is positive definite, one can rescale Q such that N_i^j is replaced by δ_i^j.

The $(QQ\bar{Q})$ Jacobi identity tells us that $\Omega^l b_l$ vanishes

$$[\Omega^l{}_{ij}Z_l, Q_k^A] = \Omega^l{}_{ij}(b_l)_k{}^{k'}Q_{k'}^A = 0 \tag{6}$$

for any values of i, j, k, k'. Thus $Z_{ij} \equiv \Omega^l{}_{ij}Z_l$ commutes with Q_k^A. It also commutes with $\bar{Q}^k_{\dot{A}}$ as follows from (5). From (QQB) we find the interesting matrix equation

$$b_l\Omega^m + \Omega^m b_l^{\mathrm{T}} = c_{kl}{}^m\Omega^k \tag{7}$$

where the structure constants c of the Lie algebra generated by B_l are defined by $[B_k, B_l] = c_{kl}{}^m B_m$.

We now turn to the commutator

$$[\Omega^k{}_{ij}Z_k, B_l] = \Omega^k{}_{ij}c_{kl}{}^m B_m. \tag{8}$$

Using (7), the right-hand side is rewritten as

$$[Z_{ij}, B_l] = (b_l\Omega^m + \Omega^m b_l^{\mathrm{T}})_{ij}Z_m \tag{9}$$

since Ω^m vanishes for B_m not equal to any of the Z_m according to (2). This proves that the Z_{ij} form an invariant subalgebra of the internal symmetry algebra generated by the B.

Consider now the (Q, Q, Z_{ij}) Jacobi identity where we recall the definition $Z_{ij} = \Omega^l{}_{ij}Z_l$. It leads to $[Z_{ij}, Z_{kl}] = 0$ due to (6). Clearly, the Z_{ij} form an Abelian subalgebra of B_l. Going back to the Coleman–Mandula theorem, it follows that the Z_{ij} cannot be part of the semisimple Lie algebra so that they must be part of the Abelian Lie algebra in this theorem. Consequently, the Z_{ij} commute with all B_l. Also Z_{ij} commute with Q_k^A and $\bar{Q}^k_{\dot{A}}$ according to (6), and with B_l as shown above, and with the Poincaré algebra according to the Coleman–Mandula theorem. Consequently, Z_{ij}, and hence Z_l *are central charges.*

Let us now return to (8). Since the index k refers to T_k (since Ω^k vanishes for B_k not equal to Z_k, see (2)), it follows that the structure constants $c_{kl}{}^m$ in (8) must vanish. Thus one obtains the important result

$$b_l\Omega^m + \Omega^m b_l^{\mathrm{T}} = 0. \tag{10}$$

In other words, $\Omega^m{}_{ij}$ is an invariant tensor under the internal symmetry group.

For the particular (but frequently arising) case that Ω^m is real we make an orthogonal transformation such that

$$\hat{\Omega}_{ij}{}^m = O_i{}^k O_j{}^l \Omega_{kl}{}^m \tag{11}$$

is of the form of a degenerate symplectic metric

$$\hat{\Omega}=\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}. \tag{12}$$

It now follows from (10) and (5) that the internal symmetry representation b is

$$\mathrm{USp}(2n, 2n) \otimes \mathrm{U}(m). \tag{13}$$

Thus, for example, the $N=2$ supergravity with U(2) invariance cannot (and indeed does not) have a central charge *on-shell.*

For super de-Sitter algebras we are not aware of similar complete results.

As far as the super conformal algebras are concerned, if one considers only internal charges which are Poincaré scalars again, then in the $\{Q, Q\}$ and $\{S, S\}$ relations one finds no B_l for dimensional reasons alone, but in the $\{Q, S\}$ anticommutators a group U(N) (but SU(4) for $N=4$) *must* be produced and *central charges are absent.* For $N=4$, an outside charge U(1) (i.e., a charge which appears on the left-hand side but not on the right-hand side) can be present, so that Q and S still rotate under a full U(4). This special role of the $N=4$ case may explain certain cancellations of infinities in the $N=4$ globally supersymmetric Yang–Mills theory.

For literature, see:

S. Coleman and J. Mandula, Phys. Rev. 159 (1967) 1251.
R. Haag, J. Lopuszanski and M. Sohnius, Nucl. Phys. B 88 (1975) 257.

3.4. *Supergravity as a gauge theory on the group manifold*

An approach to the gauging of superalgebras which differs from the approach in subsection 2 is based on the notion of the group manifold [334, 1]. In this approach one considers as "base manifold" a space with as many (bosonic and fermionic) coordinates as there are generators in the full superalgebra G. The superalgebra G is arbitrary, and can be non-semi-simple. Fields depend initially on all coordinates, but as a consequence of the field equations they actually turn out to depend on fewer coordinates than are present in the group manifold. One chooses a particular sub(super) algebra H of the full algebra G and requires that the action be invariant under general coordinate transformations in the full base manifold and under local gauge transformations generated by H. It then turns out that these symmetries become symmetries of the final theory in a smaller base manifold [608].

It is at present not known when the program works and when not. Also, one knows at present only how to obtain the final theory on-shell, but no guiding principle is known how to obtain the auxiliary fields. Nevertheless, the approach seems promising.

General coordinate transformations can be written as gauge transformations plus curvature-terms*

$$\begin{aligned} \delta_{\mathrm{gc}}(\xi) h_\Lambda{}^A &= \xi^\Pi \partial_\Pi h_\Lambda{}^A + \partial_\Lambda \xi^\Pi h_\Pi{}^A \\ &= D_\Lambda(\xi^\Pi h_\Pi{}^A) + \xi^\Pi R_{\Pi\Lambda}{}^A \\ (D_\Pi \epsilon)^A &= \partial_\Pi \epsilon^A + h_\Pi{}^B \epsilon^C f_{CB}{}^A . \end{aligned} \tag{1}$$

* This result seems due to F.W. Hehl, P. von der Heyde and G.D. Kerlick, Rev. Mod. Phys. 48 (1976) 393.

Introducing one-forms $h^A = dy^\Lambda h_\Lambda{}^A$ and flat superparameters $\xi^A = \xi^\Lambda h_\Lambda{}^A$ one has

$$\delta_{gc}(\xi)h^A = (D\xi)^A + h^C\xi^B R_{BC}{}^A. \tag{2}$$

These transformations will become in the end the local supersymmetry and general coordinate transformations in ordinary four-dimensional spacetime.

As we shall see, it sometimes happens as a result of the field equations that curvatures with one or both lower indices in the subalgebra H vanish ($R_{\mathrm{H}}{}^{\cdot} = 0$, *horizontality*). Moreover, sometimes $R_{0\cdot}$ can be expressed in terms of $R_{II}{}^{\cdot}$ (*rheonomic symmetry*) where I are the generators in the final base manifold and 0 the generators which are neither in H nor in I. In such cases the transformations δ_{gc} acting on $h_I{}^A$ become transformations which involve only these fields themselves and their I-derivatives and are thus ordinary symmetries defined in the final base manifold I. For example, in $N = 1$ supergravity, G is the super Poincaré algebra and the coordinates of the group manifold are $y^\Lambda = (x^\mu, \theta^\alpha, \eta^{ab})$. The subgroup H is the Lorentz group and $I = P_m$, $0 = Q^\alpha$. In this case one has horizontality and rheonomic symmetry and δ_{gc} becomes indeed the sum of general coordinate and local supersymmetry transformations.

Those final symmetries can be systematically traced, once G, H and the action are given. One chooses as action an expression *linear in curvatures* (this is thus an essential difference with the method of subsection 2),

$$I = \int_{\mathrm{M}} R^A \wedge h^{K_1} \wedge \cdots h^{K_n} C_{A,K_1,\ldots,K_n} \tag{3}$$

where A ranges over the whole (super) algebra G and K_i lie in G/H. The C are field-independent constants such as ϵ_{ijkl} and $\gamma_5\gamma_m$. The fields $h_\Lambda{}^A$ are, for example, equal to $e^m{}_\mu(x, \theta, \eta)$. In order that the action be H-invariant, the C must be invariant tensors under H.

Varying h^{K_1} the $h^{K_1} \wedge \cdots C_{A\ldots K_n}$ do not transform in the coadjoint representation of the full group G. (Thus, for a tensor T_A one has $\delta T_A = f_{AB}{}^C T_C \epsilon^B$ so that $T^A T_A$ is invariant.) Note that in general $D_\mu C_{A,\ldots,K_n}$ is not zero since the C's are not invariant tensors under G.

The manifold M over which one integrates is $(n+2)$-dimensional since the integrand is clearly an $(n+2)$ form. One has $n+2 < \dim K = \dim \mathrm{G/H}$. One uses a *generalized variational principle* to obtain the field equations: one varies both fields and surface. (Thus one considers all possible surfaces M in the group manifold, and not one particular surface.)

The heart of the matter are the constants C. These are determined by requiring that the vacuum be a solution of all field equations, since this fixes the C to be the coefficients of a cohomology class of the Lie algebra G. The equation for C to be solved is thus

$$D[(g^{-1}\,dg)^{K_1} \wedge \cdots (g^{-1}\,dg)^{K_n} C_{A,K_1,\ldots,K_n}] = 0. \tag{4a}$$

Indeed, for the vacuum $h = g^{-1}\,dg$ and R^A vanishes, so that one need only to vary R^A yielding $\delta R^A = D\delta h^A$, and if (4a) holds, then $\delta I = 0$. (One may partially integrate to obtain (4a) if the object in (4a) is in the coadjoint representation.) The solution of (4a) is obtained if the C in the integrand in (3) form a cohomology class of the G superalgebra. In fact, cohomology class means that

$$D(h^{K_1} \wedge \cdots \wedge h^{K_n} C_{A,K_1,\ldots,K_n}) = 0 \tag{4b}$$

is proportional to curvatures and is not itself the exterior covariant derivative of some other forms. The first condition guarantees that while varying the action, one finds field equations linear and homogeneous in curvatures, while the second condition guarantees that the action does not vanish as a consequence of the Bianchi identities. The useful mathematical result is that cohomology theory gives all solutions to (4b) and that there are only few such solutions.

The $h = h^A X_A$ are super Lie algebra valued one-forms, but h are not yet the usual connections ("principal connections on a fiber bundle") because in that case the H-part of h should have the form

$$h^{\mathrm{H}} = (g^{-1}\,\mathrm{d}g)^{\mathrm{H}} + \mathrm{d}y^{\mathrm{K}} h_{\mathrm{K}}{}^{\mathrm{H}} \tag{5}$$

with the curved K in G/H, meaning simply that the H-part of h^{H} is a gauge transformation under H itself (gauge transformations with flat parameters ϵ^A are defined as usual, $\delta h^A = (D\epsilon)^A$). One calls the h^A therefore pseudoconnections. If (5) holds, h^{H} defines a principal connection on the fiber bundle G(G/H, H) and the rest of h, namely h^{K} (with flat K) is what is called the soldering form ("the super-vielbein").

One now proceeds as follows. Varying the action in (3) with respect to all $h_\Lambda{}^A$, one finds the complete set of field equations. From here on we discuss the particular case of supergravity; this should enable the reader to deduce also the general case. Assume that one has found the coefficients C such that the action reads

$$I = \int_{\mathrm{M}_4} (R^{ij} \wedge E^k \wedge E^l \epsilon_{ijkl} + \bar{\psi} \wedge \gamma_5 \gamma_i \mathscr{D}\psi \wedge E_i). \tag{6}$$

The R^{ij} are the Lorentz curvatures $\mathrm{d}\omega^{ij} + \frac{1}{2}\omega^{ij} \wedge \omega^{jk}$, E^k is the vielbein one-form, ψ^a the gravitino one-form and $\mathscr{D}$ is covariant with respect to H, i.e., it contains only the Lorentz connection. The integration is over a four-dimensional manifold since for the super Poincaré algebra there are only three cohomology classes of the algebra (not to be confused with the cohomology classes of manifolds), one of which does not contain gravity, a second gives a theory with only the vacuum solution while the third is the one used in (6). This proves the uniqueness of supergravity from the point of view of differential geometry.

The field equations become

$$\delta I/\delta\omega^{ij} = \epsilon_{ijkl} R^k \wedge E^l = 0 \tag{7}$$

$$\delta I/\delta E^i = \epsilon_{ijkl} R^{jk} \wedge E^l + \tfrac{1}{2}\bar{\psi} \wedge \gamma_5\gamma_i R = 0 \tag{8}$$

$$\delta I/\delta\bar{\psi}_a = (\gamma_5\gamma_i R)^a \wedge E^i - \tfrac{1}{2}(\gamma_5\gamma_i\psi)^a \wedge R^i = 0. \tag{9}$$

The symbols R^a and R^k denote the curvatures of ψ^a and E^k (the latter contains a $\bar{\psi}\gamma\psi$ term). In deriving (9) we used the identity which we also used in the proof of the gauge invariance of the action in section 1

$$\gamma_i\psi \wedge \bar{\psi}\gamma^i\psi = 0. \tag{10}$$

The interesting point now is that one can project (7, 8, 9) onto the various components of $\mathrm{d}y^\Lambda$. After some lengthy algebra one finds the following results:

(i) *Horizontality*: all curvatures $R_{AB}{}^{C}$ with at least one vertical leg vanish. In other words, if A and/or B equals H, $R_{AB}{}^{C}=0$. The indices A and B are flat, due to contracting $R_{\Pi\Lambda}{}^{C}$ with inverse group vielbeins $h_A{}^{\Lambda}$. From this result one derives factorization.

(ii) *Factorization*: the fields h^A with A in G/H have no components along $\mathrm{d}\eta^{ij}$ while for $A=\mathrm{H}$ the component of h^A along $\mathrm{d}\eta^{ij}$ is a gauge transformation

$$\omega^{ij}=(g^{-1}\,\mathrm{d}g)^{ij}+\mathrm{d}\theta^{\alpha}\,\omega_{\alpha}{}^{ij}+\mathrm{d}x^{\mu}\,\omega_{\mu}{}^{ij} \tag{11}$$

$$E^{i}=0\qquad\qquad +\mathrm{d}\theta^{\alpha}\,E_{\alpha}{}^{i}+\mathrm{d}x^{\mu}\,E_{\mu}{}^{i} \tag{12}$$

$$\psi^{b}=0\qquad\qquad +\mathrm{d}\theta^{\alpha}\,\psi_{\alpha}{}^{b}+\mathrm{d}x^{\mu}\,\psi_{\mu}{}^{b}. \tag{13}$$

This result states that after one has removed any dependence on η^{ij} (or its integral g^{ij}) of all components $\omega_{\alpha}{}^{ij}\ldots\psi_{a}{}^{b}$ by a suitable local Lorentz rotation (as one can do always), the $\mathrm{d}\eta^{ij}$ coefficients of E^i and ψ^b vanish, while those of ω^{ij} are those of a left-invariant one-form on H (the Lorentz group). This basically means that at this point one has a tangent and base space, i.e. we are down to superspace, and that H is a true symmetry because it acts only on the indices of the fields.

Factorization also implies that g^{kj} only depends on η^{ij} but not on x^{μ}, θ^{α}. The reverse is also true: if $g^{kj}=g^{kj}(\eta^{ij})$ and the $\mathrm{d}\eta^{ij}$ components of E^i and ψ^a vanish, then horizontality holds (as is proved in, for example, Kobayashi and Nomitsu's book on differential geometry).

(iii) *Field equations in the flat θ directions*: These will enable us to come down from superspace to ordinary space. As a result of horizontality we only need consider curvatures with legs in the (flat) x and θ directions. The results are (in the normalization of the Torino group)

$$R^{i}{}_{\alpha\beta}=0,\qquad R^{i}{}_{\alpha m}=0 \tag{14}$$

$$R^{ij}{}_{\alpha\beta}=0,\qquad R^{ij}{}_{\alpha m}=\text{constant}\times(B^{ij}{}_{\alpha m}+\tfrac{1}{2}\delta^{i}_{m}B^{jn}{}_{\alpha n}-\tfrac{1}{2}\delta^{j}_{m}B^{in}{}_{\alpha n}) \tag{15}$$

$$R^{\alpha}{}_{\beta\gamma}=0,\qquad R^{\alpha}{}_{\beta m}=0 \tag{16}$$

where $B^{ij}{}_{\alpha m}=\epsilon^{ijkl}(\gamma_5\gamma_m)_{\alpha\gamma}R^{\gamma}{}_{kl}$. (As usual, spinor indices are raised by C^{-1} and lowered by C where C is the charge conjugation matrix.) Looking back at (1) and inverting (14)–(16) we see that with flat parameters $\xi^A=(\xi^i,\xi^a,\xi^{ij})$ one has

$$\mathrm{d}\omega^{ij}=-\mathscr{D}\xi^{ij}+2E^{m}\xi^{n}R^{ij}{}_{mn}+\text{constant}\times E^{m}\xi^{\alpha}(B^{ij}{}_{\alpha m}+\cdots) \tag{17}$$

$$\mathrm{d}E^{i}=(D\xi)^{i}-E_{k}\xi^{ki}-\frac{\mathrm{i}}{2}\,\bar{\xi}\gamma^{i}\psi \tag{18}$$

$$\mathrm{d}\psi^{\alpha}=(D\xi)^{\alpha}-2E^{m}\xi^{n}R^{\alpha}{}_{mn}+\tfrac{1}{2}(\sigma_{ij}\psi)^{\alpha}\xi^{ij}. \tag{19}$$

(iv) *Final base-manifold*: *ordinary spacetime*. The transformation rules in (17)–(19) look very much like the transformation rules of supergravity in ordinary spacetime. In principle, however, a problem might arise, namely it could be that the flat-index curvatures (in particular those in the tensor B) contain curvatures with curved indices also in the θ directions (namely, $R^{A}{}_{mn}=h_m{}^{\Lambda}h_n{}^{\Pi}R_{\Pi\Lambda}{}^{A}$) and one might find in (17)–(19) curvatures $R^{A}_{\mu\alpha}$ with curved fermionic α. In that case one would not only find derivatives $\partial/\partial x^{\mu}$ in the symmetries of supergravity in ordinary spacetime, but also $\partial/\partial\theta^{\alpha}$ derivatives and one would not have arrived at ordinary spacetime. Explicit evaluation shows that this does not happen, fortunately. In fact, this is not accidental, but follows if one chooses a coordinate system in superspace such that in (12) $E^i{}_\alpha = 0$ at $\theta = 0$. (A similar "gauge" we will discuss in superspace later.) This concludes the discussion of the action in (6).

The results are likely to hold in more general cases. Whenever G/H is larger than the manifold M over which one integrates, it may happen (as, for example, in the above case) that the curvatures with at least one flat index out of M but in G/H ("outer directions") are algebraic functions (as a consequence of the equation of motion) of those curvatures whose flat indices are in M ("inner directions"). When this happens, one calls the theory rheonomically symmetric, since it has an extra symmetry on spacetime

$$\delta h^A = D(\epsilon^{k_{\text{out}}} h^A_{k_{\text{out}}}) + h^B \epsilon^{k_{\text{out}}} R^A_{k_{\text{out}}B} \tag{20}$$

so that δh^A only depends on fields with indices in M. This was pointed out for the first time by Regge and Ne'eman.

At this moment it is not yet understood when this program works and when not; samples are known when a given $C_{AK_1\ldots K_n}$ does not lead to a consistent theory and the program crashes.

Since one starts out with a closed algebra (general coordinate transformations and gauge transformations in the gauge manifolds), one might say that one has the maximal rather than the minimal set of auxiliary fields (most of the $h_\Lambda{}^A$ are auxiliary). In descending to fewer fields by eliminating them by means of the field equations there comes a point where closure is lost. Where precisely closure is lost is not known (although one might discover this by explicit calculation). Perhaps the greatest virtue of the program is that one need not introduce constraints by hand from the outside. Some aspects are under further study: cohomology theory for non-compact groups and the notion of a manifold with anticommuting variables.

Another approach using fiber bundles has been developed by Mansouri and Schaer [326–328] and by McDowell [309]. To start with, they take the base manifold to be superspace and the fiber the full group G. Then they make a 1-to-1 identification of the G/H part of the fiber with the base manifold. In other words, P and Q in the fiber and in the base manifold appear as separate entities to begin with, but they eliminate this "double representing" not by imposing constraints on curvatures but by "soldering". In this way they obtain non-linear realizations of local gauge symmetries, in which all the transformation laws of the fields (including general coordinate transformations) follow from gauge transformations (defined by covariant derivatives, as usual). The method is equally applicable to Poincaré, de Sitter, and conformal supergravity.

The strength of this approach rests in its manifest covariance with respect to all local (super-) symmetries. To take full advantage of this, one writes down invariant actions directly in superspace. The variation of each action leads to a set of field equations. Then to make contact with models in 4-dimensional spacetime, one looks for θ-independent solutions of these equations. In other words, one regards the 4-dimensional models as particular classes of solutions of superspace field equations. Thus, the correspondence between superspace and spacetime is established not directly between actions but through these solutions which extremize both actions.

3.5. Supergravity and spinning space

It is possible to find the local analogue of the commutator algebra of global supersymmetry by using Hamiltonian methods [457]. The action of supergravity reads in Hamiltonian form (using Diracs

Hamiltonian)

$$I = \int d^4x\,[\dot{q}p - \lambda^m C_m(p, q)]$$

$$q = \{e^m{}_i, \psi_i^a, \omega_i{}^{mn}\}, \qquad \lambda^m = \{e^m{}_0, \bar{\psi}_{0,a}, \omega_0{}^{mn}\}. \tag{1}$$

The p are the conjugate momenta, while $C(p, q)$ denote the first class constraints (after eliminating the second class constraints by solving the latter) which generate gauge transformations

$$C_m(p, q) = \{\mathcal{H}_m, Q^a, J_{mn}\}. \tag{2}$$

All generators have flat indices. The proof that the action can be written in this way is well-known for pure gravity (due to work by Dirac) but the extension to supergravity which we discuss below is due to Teitelboim and Pilati.

If the action is to be invariant under $\delta p = \{p, C_m\}\epsilon^m$ and $\delta q = \{q, C_m\}\epsilon^m$ (where ϵ^m are field-independent local parameters), one finds after partially integrating (discarding surface terms)

$$\delta(\dot{q}p) = \dot{q}\delta p - (\delta q)\dot{p} = -\epsilon^m\, dC_m/dt = \dot{\epsilon}^m C_m. \tag{3}$$

Variation of C_m yields the Poisson bracket of the constraints

$$\delta C_m = \{C_m, C_n\}\epsilon^n. \tag{4}$$

Defining the field-dependent structure constant $f_{mn}{}^p$ by

$$\{C_m, C_n\} = f_{mn}{}^p C_p \qquad \text{(strong equality)} \tag{5}$$

it follows that in order for I to be invariant one must transform the Lagrange multipliers as

$$\delta\lambda^p = \dot{\epsilon}^p + f_{mn}{}^p \epsilon^n \lambda^m. \tag{6}$$

From the known transformation rules of $e^m{}_0$, $\psi_0{}^a$, $\omega_0{}^{mn}$ as given in section 1 one can thus read off what $f_{mn}{}^p$ are. This is a very simple and elegant way to find the algebra of constraints, but one can, of course, also obtain the result by directly determining the charges C_m and then compute the anticommutators.

In order that $\delta\lambda^m$ has indeed the form as given above, one must combine a general coordinate transformation with parameter ξ^m with a local Lorentz transformation with parameter $\xi^m\omega_m{}^{kl}$ and a local supersymmetry transformation with parameter $-\xi^m\psi_m$ (the same combination as found by Freedman and the author in the commutator of two local supersymmetry variations). We will call these transformations covariant translations.

The transformation laws of the fields of supergravity in first order formalism as discussed in section 1 read

$$\delta e^m{}_\mu = D_\mu \xi^m + \tfrac{1}{2}\bar{\epsilon}\gamma^m \psi_\mu - \lambda^m{}_n e^n{}_\mu \tag{7}$$

$$\delta\psi_\mu{}^a = \xi^m e_m{}^\nu (D_\nu \psi_\mu{}^a - D_\mu \psi_\nu{}^a) + D_\mu \epsilon^a - \tfrac{1}{2}\lambda \cdot \sigma\psi_\mu{}^a \tag{8}$$

$$\delta\omega_\mu{}^{mn} = \xi^m e_m{}^\nu R^{\text{cov},\,mn}_{\nu\mu} + \delta_S\omega_\mu{}^{mn} + D_\mu\lambda^{mn} - \xi^m e_m{}^\nu \bar\psi_\mu \frac{\delta}{\delta\bar\epsilon}\delta_S\omega_\nu{}^{mn} \tag{9}$$

$$\delta_S\omega_\mu{}^{mn} = \text{given in subsection 1.6, eq. (2).} \tag{10}$$

Since the *variations* of the objects on the right hand sides are independent of $\dot\epsilon$ if $\mu \neq 4$, it follows that the expressions themselves in the right hand sides must be independent of the Lagrange multipliers for $\mu = 1, 2, 3$. This follows from the general rule of varying an arbitrary function

$$\delta G = \frac{\delta G}{\delta\lambda^k}(\dot\epsilon^k + f_{mn}{}^k\epsilon^n\lambda^m) + \frac{\delta G}{\delta q^k}\epsilon^j\frac{\partial C_j}{\partial p^k} - \frac{\partial G}{\partial p^k}\epsilon^j\frac{\partial C_j}{\partial q^k} \tag{11}$$

and observing that only the explicit $\dot\epsilon^k$ contains a time-derivative of ϵ^k.

The reader who is not familiar with Hamiltonian methods, may at first wonder how, for example, H_{ab} in $\delta\psi_i = \xi^m e^n{}_i H_{mn}$ manages to depend only on $\psi_i{}^a$ and $e^m{}_i$, but not on the Lagrange multipliers $\psi_4{}^\alpha$ and $e^m{}_4$. We now show this in detail,

$$H_{ab} = e_a{}^i e_b{}^j H_{ij} + (e_a{}^4 e_b{}^j - e_a{}^j e_b{}^4)H_{4j}. \tag{12}$$

One should now express $\partial_4\psi_j$ in terms of p and q by evaluating $\{\psi_j, H_\mathrm{T}\}$. An equivalent and quicker way is to use the field equation $\gamma^\mu(D_\mu\psi_\nu - D_\nu\psi_\mu) = 0$. In this way one finds with $H_{mn} = A_{mn}{}^{ij}H_{ij}$

$$A_{ab}{}^{ij} = e_a{}^i e_b{}^j - (e_a{}^4 e_b{}^j - e_b{}^4 e_a{}^j)\,\gamma^4\gamma^i/g^{44} \tag{13}$$

with curved γ^4 and γ^i.

Now, $e_a{}^4(g^{44})^{-1/2}$ is the normal n_a, satisfying $n_a e^a{}_i = 0$ and $n_a n^a = 1$. Hence, n_a depends only on $e^a{}_i$. Clearly

$$A_{ab}{}^{ij} = (e_a{}^i e_b{}^j - e_b{}^i e_a{}^j) - (n_a e_b{}^j - n_b e_a{}^j)(e_c{}^i n_d)\gamma^d\gamma^c \tag{14}$$

with flat γ^c, γ^d. Decomposing $e_m{}^i$ into the complete orthonormal set n_a, e_{ai} as follows,

$$e_a{}^i = \alpha^i n_a + \beta^{ij}e_{aj} \tag{15}$$

it follows upon multiplication by e_{bi} that $\beta^{ij} = {}^3g^{ij}$ (the inverse of g_{ij}) and therefore "good". The α^i are "bad" but cancel from $A_{ab}{}^{ij}$ if one uses $\gamma^d\gamma^c = \delta^{dc} + 2\sigma^{dc}$ and replaces $e^i{}_c$ by $\alpha^i n_c$ for the δ^{dc} term. For the term with σ^{dc} one can replace $e_c{}^i$ by $\beta^{ij}e_{cj}$. Hence, $A_{ab}{}^{ij}$ only depends on e_{ai} but not on the Lagrange multipliers e_{a4}. Hence,

$$\delta\psi_\mu = \xi^m e^b{}_\mu H_{mb} \tag{16}$$

indeed only depends on (p, q) for $\mu = j$.

We can now read off the algebra of constraints from the laws (7–10). One finds for the nontrivial (anti)commutators

$$\{Q^a(x), \bar Q_b(y)\} = (\gamma^m)^a{}_b\mathcal{H}_m(x)\,\delta^3(x-y) \tag{17}$$

$$\{\mathcal{H}_m(x), \mathcal{H}_n(y)\} = [\tfrac{1}{2}\Omega_{mn}{}^{pq}J_{pq} + \bar{Q}(D_m\psi_n - D_n\psi_m)]\, \delta^3(x-y) \tag{18}$$

$$\{Q^a(x), \mathcal{H}_m(y)\} = (\bar{\epsilon}\gamma_5\gamma_m D_a\psi_b)\epsilon^{abcd}J_{cd}\, \delta^3(x-y) \tag{19}$$

where Ω contains the supercovariantized Riemann curvature. Indeed, in the flat space limit one finds back the global algebra. We stress that these results could have been found by a straightforward Hamiltonian analysis, in which case one would have found that in all these (anti)commutators only p's and q's appear.

The truly interesting thing is the off-shell closure of the gauge algebra. To establish its relation to the gauge algebra when evaluated on fields F, we note that the Jacobi identity yields

$$\{F, \{C_m, C_n\}\} = \{F, f_{mn}{}^p\}C_p + f_{mn}{}^p\{F, C_p\}. \tag{20}$$

Whenever $\{F, f_{mn}{}^p\}$ is nonzero, one finds extra terms in the gauge algebra proportional to C_p. These are the equation of motion terms known from the Lagrangian approach (which can only be gotten rid of by introducing the auxiliary fields into the theory). Apparently, in supergravity the structure constants depend on the canonical momenta. (In gravity they depend only on the canonical coordinates in the costumary parallel-transverse basis, but they depend also on momenta if one uses the same $\mathcal{H}_m$ as one used above.)

In the anticommutator $\{Q, Q\} \sim \mathcal{H}$, no fields appear in the structure functions as a result of defining instead of general coordinate transformations the covariant translations with parameters ξ^m. There are quite a number of fine points. For example, the brackets are really Dirac brackets, but we refer to the literature for further study.

Let us now discuss why the $\{Q, Q\} \sim \mathcal{H}$ relation implies that supergravity is the theory of spinning space. We are led to consider the Dirac equation as a square root of the Klein–Gordon equation

$$(\gamma^m\partial_m + m)\psi = 0, \qquad (\Box - m^2)\psi = 0. \tag{21}$$

Multiplying by γ_5 and defining $\gamma_5\gamma^m = \theta^m$, $\gamma_5 = \theta^5$, we can introduce, in addition to the usual canonical variables (x^m, p_n) five anticommuting variables (θ^m, θ^5) satisfying

$$\{x^m, p_n\} = \delta^m_n, \qquad \{\theta^m, \theta^n\} = -2\delta^{mn}, \qquad \{\theta^5, \theta^m\} = 0, \qquad \{\theta^5, \theta^5\} = 2. \tag{22}$$

One now finds a set of first class (i.e. commuting) constraints

$$S = \theta^m p_m + m\theta^5, \qquad \mathcal{H} = p^2 + m^2 \tag{23}$$

$$\{S, S\} = 2\mathcal{H}, \qquad \{S, H\} = 0, \qquad \{\mathcal{H}, \mathcal{H}\} = 0. \tag{24}$$

An action for this system leading to these (anti)commutators can also be given. Hence one can take a square root of the constraint $\mathcal{H}$.

Just as taking the square root of the Klein–Gordon field leads to a spinning particle, taking the square root of (the generators H_m of) ordinary space leads to (the generators Q^a of) spinning space.

4. Conformal simple supergravity

4.1. Conformal supersymmetry

Conformal supergravity is the supersymmetric extension of Weyl's action $R^2{}_{\mu\nu}-\frac{1}{3}R^2$ which is invariant under local scale transformations $\delta g_{\mu\nu}(x)=\Lambda(x)\,g_{\mu\nu}(x)$. The theory is thus a higher derivative theory, and for the time being it does not seem to be of physical interest. However, that may be due to our present underdeveloped knowledge how to treat higher derivative field theories. At the more formal level, conformal supergravity is of the utmost interest, since it explains the structure of ordinary supergravity: where the auxiliary fields come from and how to obtain a tensor calculus. Since (super)conformal methods are little known, we will give here a rather complete treatment. We begin by discussing conformal rigid symmetry.

At the basis lies the superconformal algebra [581, 519]. It contains as bosonic part the conformal algebra with (P_m, M_{mn}, K_m, D). Since P_m and the conformal boosts K_m play a rather symmetrical role, it comes perhaps as no surprise that there are two fermionic spinorial charges, namely the usual square root of P_m called Q^α, and a square root of K_m, called henceforth S^α

$$\{Q^\alpha, Q^\beta\}=+\tfrac{1}{2}(\gamma^m C^{-1})^{\alpha\beta}P_m \tag{1}$$

$$\{S^\alpha, S^\beta\}=-\tfrac{1}{2}(\gamma^m C^{-1})^{\alpha\beta}K_m. \tag{2}$$

(In the literature on conformal supergravity one often finds the special representation where $(C^{-1})^{\alpha\beta}=-C_{\alpha\beta}$, see appendix.) The rather unexpected feature is that if one adds Q^α and S^α, one needs at the same time one more bosonic generator A for chiral rotations

$$\{Q^\alpha, S^\beta\}=+\tfrac{1}{2}C^{-1,\alpha\beta}D-(\sigma^{mm}C^{-1})^{\alpha\beta}M_{mm}-(\mathrm{i}\gamma_5 C^{-1})^{\alpha\beta}A \qquad (m>n). \tag{3}$$

The conformal algebra is given by the ordinary Poincaré algebra, discussed in subsection 2.1, eqs. (24) together with

$$[P_m, D]=P_m,\qquad [K_m, D]=-K_m,\qquad [K_m, P_n]=-2(\delta_{mn}D+M_{mn}). \tag{4}$$

The nonvanishing commutators linear in fermionic charges are

$$\left[\begin{matrix}Q^\alpha\\ S^\alpha\end{matrix}, M_{mn}\right]=(\sigma_{mn})^\alpha{}_\beta\begin{pmatrix}Q^\beta\\ S^\beta\end{pmatrix},\qquad \left[\begin{matrix}Q^\alpha\\ S^\alpha\end{matrix}, A\right]=\frac{3\mathrm{i}}{4}(\gamma_5)^\alpha{}_\beta\begin{pmatrix}-Q^\beta\\ S^\beta\end{pmatrix}$$

$$\left[\begin{matrix}Q^\alpha\\ S^\alpha\end{matrix}, D\right]=\tfrac{1}{2}\begin{pmatrix}Q^\alpha\\ -S^\alpha\end{pmatrix},\qquad \begin{matrix}[S^\alpha, P_m]=(\gamma_m)^\alpha{}_\beta Q^\beta\\ [Q^\alpha, K_m]=-(\gamma_m)^\alpha{}_\beta S^\beta.\end{matrix} \tag{5}$$

Rather than check explicitly the Jacobi identities, we prove that this is a closed algebraic system by giving an explicit matrix representation. We construct 5×5 matrices, of which the 4×4 part contains the conformal group, in the fifth row and fifth column one finds Q and S, and along the diagonal there is A,

$$M_{mn} = \sigma_{mn}, \qquad P_m = -\tfrac{1}{2}\gamma_m(1-\gamma_5), \qquad K_m = \tfrac{1}{2}\gamma_m(1+\gamma_5)$$

$$A = \operatorname{diag}\frac{-\mathrm{i}}{4}(1,1,1,1,4), \qquad D = -\tfrac{1}{2}\gamma_5$$

$$(Q^\alpha)^j{}_5 = -\tfrac{1}{2}[(1+\gamma_5)C^{-1}]^{j\alpha}, \qquad (Q^\alpha)^5{}_j = +\tfrac{1}{2}(1-\gamma_5)^\alpha{}_j$$

$$(S^\alpha)^j{}_5 = +\tfrac{1}{2}[(1-\gamma_5)C^{-1}]^{j\alpha}, \qquad (S^\alpha)^5{}_j = -\tfrac{1}{2}(1+\gamma_5)^\alpha{}_j. \tag{6}$$

We will now write down a representation of this superalgebra on fields. Consider the scalar multiplet $\Sigma = [A, B, \chi, F, G]$. (We hope the reader will not confuse the scalar field $A(x)$ with the chiral generator A.) We already have a representation for the global super Poincaré algebra. Defining $\delta_Q A(x) = [A, \bar\epsilon Q] = \frac{1}{2}\bar\epsilon\chi$, one finds with $\bar\epsilon_\alpha = \epsilon^\beta C_{\beta\alpha}$

$$\begin{aligned}&\delta_Q A = \tfrac{1}{2}\bar\epsilon\chi, \qquad \delta_Q B = -\tfrac{1}{2}\mathrm{i}\bar\epsilon\gamma_5\chi, \qquad \delta_Q F = \tfrac{1}{2}\bar\epsilon\not\partial\chi, \qquad \delta_Q G = \tfrac{1}{2}\mathrm{i}\bar\epsilon\gamma_5\not\partial\chi\\ &\delta_Q\chi = \tfrac{1}{2}\not\partial(A-\mathrm{i}\gamma_5 B)\epsilon + \tfrac{1}{2}(F+\mathrm{i}\gamma_5 G)\epsilon.\end{aligned} \tag{7}$$

Thus $[\delta(\epsilon_1), \delta(\epsilon_2)]A = [A, [\bar\epsilon_2 Q, \bar\epsilon_1 Q]]$ so that if one defines

$$\delta_P A(x) = [A, \xi^m P_m] = \xi^m \partial_m A \tag{8}$$

then the $\{Q, Q\} \sim P$ relation is satisfied.

In order to find the representation of M_{mn} on fields we first define how M_{mn} acts on fields at the origin. The only nontrivial result is

$$\delta_M \chi^\alpha(0) = [\chi^\alpha(0), \lambda^{mn}M_{mn}] = \lambda^{mn}(\sigma_{mn})^\alpha{}_\beta\chi^\beta \qquad (m<n). \tag{9}$$

From this one finds, using $\chi(x) = \mathrm{e}^{-P\cdot x}\chi(0)\,\mathrm{e}^{P\cdot x}$ and $[\mathrm{e}^{-P\cdot x}, M_{mn}] = [-P\cdot x, M_{mn}]\,\mathrm{e}^{-P\cdot x}$ the extra orbital parts

$$[\chi^\alpha(x), \lambda^{mn}M_{mn}] = \lambda^{mn}(\sigma_{mn})^\alpha{}_\beta\chi^\beta(x) + \lambda^{mn}(x_m\partial_n - x_n\partial_m)\chi^\alpha(x). \tag{10}$$

Similarly the variations of (A, B, F, G) are found to have only orbital parts. Thus one has at this point a representation of the super Poincaré algebra on the scalar multiplet Σ.

Next we define the conformal weight λ of Σ by $\delta_D A = [A, \lambda_D D] = \lambda\lambda_D A$. Using the $[Q^\alpha, D]$ and $[D, P_m]$ commutators one derives for the spin parts of δ_D

$$\delta_D\begin{pmatrix}A\\B\end{pmatrix} = \lambda\begin{pmatrix}A\\B\end{pmatrix}\lambda_D, \qquad \delta_D\chi = (\lambda+\tfrac{1}{2})\chi\lambda_D, \qquad \delta_D\begin{pmatrix}F\\G\end{pmatrix} = (\lambda+1)\begin{pmatrix}F\\G\end{pmatrix}\lambda_D. \tag{11}$$

We now consider conformal boosts. For dimensional reasons, $[K_m, F]$ can be proportional to A, while K_m on A, B, χ must vanish. However, one cannot match the indices in $[K_m, F] \sim A$ so that $[K_m, \Sigma] = 0$. It should be noted that for example on the global vector multiplet $V = [C, Z, H, K, B_a, \Lambda,$

$\hat{D}]$ the boost generator can be nonzero. Specifically, with $\delta_D C = \lambda C \lambda_D$, we shall see that

$$[\hat{D}, K_m] = -2\lambda \partial_m C, \qquad [\Lambda, K_m] = -\lambda \gamma_m Z. \tag{12}$$

In order to avoid confusion with the dilation generator D, the field $\hat{D}$ carries a hat.

Consider next S-supersymmetry. From $S = \frac{1}{4}[K_m, (\gamma^m Q)^\alpha]$ one finds that A, B are S-inert. But for χ one finds a nonzero result using the Jacobi identity

$$[K_m, \partial_b A] = [K_m, [A, P_b]] = [[K_m, A], P_b] - [[K_m, P_b], A]. \tag{13}$$

One finds

$$\delta_S \begin{pmatrix} A \\ B \end{pmatrix} = 0, \qquad \delta_S \chi = \lambda (A + \mathrm{i}\gamma_5 B)\epsilon_S, \qquad \delta_S \begin{pmatrix} F \\ G \end{pmatrix} = (1-\lambda)\begin{pmatrix} \bar{\epsilon}_S \chi \\ -\mathrm{i}\bar{\epsilon}_S \gamma_5 \chi \end{pmatrix}. \tag{14}$$

Since we know now δ_Q and δ_S, we determine δ_A from $\{Q, S\}$ anticommutator. One finds

$$\delta_A \begin{pmatrix} A \\ B \end{pmatrix} = \frac{\lambda}{2}\begin{pmatrix} -B \\ A \end{pmatrix}\lambda_A, \qquad \delta_A \chi = \left(\frac{-3}{4} + \frac{\lambda}{2}\right)\mathrm{i}\gamma_5 \chi \lambda_A, \qquad \delta_A \begin{pmatrix} F \\ G \end{pmatrix} = \left(\frac{-3}{2} + \frac{\lambda}{2}\right)\begin{pmatrix} -G \\ F \end{pmatrix}\lambda_A. \tag{15}$$

These results for D, K_a, A are again only for the fields at $x = 0$. Just as for M_{mn}, one can find the transformation rules of the fields away from zero by using the commutation rules of P_m with these generators. However, these results we will not need, because in order to construct covariant derivatives, one only needs the transformations of fields at the origin ("the spin parts"; the orbital parts are needed to prove, for example, the invariance of the Dirac action under global Lorentz transformations). Thus we have derived a representation of the superconformal algebra on the scalar multiplet.

A representation of the superconformal algebra on the coordinates of superspace (x^μ, θ^α) is given by

$$\begin{aligned}
&M_{mn} = x_m \partial_n - x_n \partial_m + \bar{\theta}\sigma_{mn}\partial/\partial\bar{\theta}, \qquad P_m = \partial_m \\
&Q^\alpha = \tfrac{1}{2}(\partial/\partial\bar{\theta} - \not{\partial}\theta)^\alpha, \qquad D = x \cdot \partial + \tfrac{1}{2}\bar{\theta}\partial/\partial\bar{\theta} \\
&A = -\tfrac{3}{4}\mathrm{i}\bar{\theta}\gamma_5 \partial/\partial\bar{\theta} \\
&K_m = 2x_m x \cdot \partial - x^2 \partial_m - \bar{\theta}\hat{x}\gamma_m \partial/\partial\bar{\theta} - \tfrac{1}{2}(\bar{\theta}\theta)^2 \partial_m - \bar{\theta}\theta(\bar{\theta}\gamma_m \partial/\partial\bar{\theta}) \\
&S^\alpha = \tfrac{1}{2}(\bar{\theta}\theta\partial/\partial\bar{\theta} + \bar{\theta}\gamma_5\theta\gamma_5\partial/\partial\bar{\theta})^\alpha + \tfrac{1}{4}(\bar{\theta}\gamma_5\gamma_m\theta)(\gamma_5\gamma^m\partial/\partial\bar{\theta})^\alpha + (-\tfrac{1}{2}\hat{x}\partial/\partial\bar{\theta} + \tfrac{1}{2}\hat{x}\not{\partial}\theta + \tfrac{1}{2}\bar{\theta}\theta\not{\partial}\theta)^\alpha.
\end{aligned} \tag{16}$$

The derivatives are always left derivatives and $\hat{x} = \gamma^m x_m$. One can always choose the scale of θ and the sign of the charge conjugation matrix such that Q^α is as above. (Taylor expanding as $\varphi(x, \theta) = \varphi(x, 0) + \theta^\alpha(\partial/\partial\theta^\alpha)\varphi(x, 0) + \cdots$ and using $\partial/\partial\theta^\alpha = C_{\alpha\beta}\partial/\partial\bar{\theta}_\beta$.)

For completeness we also determine the representation of the superconformal algebra on the vector multiplet $V = [C, Z, H, K, B_m, \Lambda, \hat{D}]$. The transformation rules follow from $\delta(\text{supsym})V(x, \theta) = (\bar{\epsilon}\mathrm{i}\gamma_5 G)V(x, \theta)$ with $G^\alpha = \partial/\partial\bar{\theta}_\alpha - (\not{\partial}\theta)^\alpha$ (see subsection 5.3; we have rescaled θ here by a factor 2).

Using

$$V(x,\theta) = C + \bar\theta Z + \tfrac{1}{2}\bar\theta\theta H + \tfrac{1}{2}\bar\theta \mathrm{i}\gamma_5\theta K + \tfrac{1}{2}\bar\theta \mathrm{i}\gamma^m\gamma_5\theta B_m + \bar\theta\theta\bar\theta(\Lambda + \tfrac{1}{2}\not\partial Z) + \tfrac{1}{4}(\bar\theta\theta)^2(\hat D + \tfrac{1}{2}\Box C) \tag{17}$$

one finds easily

$$\begin{aligned}
&\delta_Q C = \tfrac{1}{2}\bar\epsilon \mathrm{i}\gamma_5 Z\\
&\delta_Q Z = \tfrac{1}{2}(\mathrm{i}\gamma_5 H - K - \not B + \not\partial C \mathrm{i}\gamma_5)\epsilon\\
&\delta_Q H = \tfrac{1}{2}\bar\epsilon \mathrm{i}\gamma_5\not\partial Z + \tfrac{1}{2}\bar\epsilon \mathrm{i}\gamma_5\Lambda\\
&\delta_Q K = -\tfrac{1}{2}\bar\epsilon\not\partial Z - \tfrac{1}{2}\bar\epsilon\Lambda\\
&\delta_Q B_m = -\tfrac{1}{2}\bar\epsilon\partial_m Z - \tfrac{1}{2}\bar\epsilon\gamma_m\Lambda\\
&\delta_Q\Lambda = \tfrac{1}{2}(\sigma\cdot F + \mathrm{i}\gamma_5\hat D)\epsilon, \qquad F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu\\
&\delta_Q\hat D = +\tfrac{1}{2}\bar\epsilon \mathrm{i}\gamma_5\not\partial\Lambda.
\end{aligned} \tag{18}$$

Indeed, the $\{Q, Q\} \sim P$ relation holds with $\delta_P C = \xi^m\partial_m C$. The Lorentz transformation rules of V are obvious. The conformal weight is defined by $\delta_D C = [C, \lambda_D D] = \lambda\lambda_D C$ (where C is the first component of V and D denotes the dilation generation). Thus the scale of Z is $\lambda + \frac{1}{2}$, that of H, K, B_a is $\lambda + 1$, that of Λ is $\lambda + \frac{3}{2}$ and that of $D = \lambda + 2$.

Under conformal boosts the only possible nonzero relations are

$$[\hat D, K_m] = \alpha\partial_m C, \qquad [\Lambda, K_m] = \beta\gamma_m Z. \tag{19}$$

For example, $[\hat D, K_m] \sim B_m$ or $[B_m, K_n] \sim \delta_{mn}C$ have wrong parity. We will fix α, β below but continue first.

For S-supersymmetry one finds with the $[K, Q] \sim S$ commutator and using $[K, P] \sim D + M$ as before

$$\begin{aligned}
&\delta_S C = 0, \qquad \delta_S Z = (-\mathrm{i}\gamma_5\epsilon_S)\lambda C\\
&\delta_S H = (\lambda - 1 + \beta/2)(\bar\epsilon_S \mathrm{i}\gamma_5 Z)\\
&\delta_S K = (\lambda - 1 + \beta/2)(\bar\epsilon_S Z)\\
&\delta_S B = \tfrac{1}{4}(\lambda + 2 - \beta)(\bar\epsilon_S\gamma_m Z)\\
&\delta_S\Lambda = \left[\frac{\mathrm{i}}{2}\beta\gamma_5 H + \frac{\beta}{2}K - \left(\frac{3\lambda}{4} + \frac{\beta}{4}\right)\not B + \left(\frac{\alpha}{2} - \frac{\beta}{4}\right)\not\partial C \mathrm{i}\gamma_5\right]\epsilon_S\\
&\delta_S\hat D = -\lambda(\bar\epsilon_S \mathrm{i}\gamma_5\Lambda) + \tfrac{1}{4}(\beta - \alpha/2)(\bar\epsilon_S \mathrm{i}\gamma_5\not\partial Z).
\end{aligned} \tag{20}$$

Under chiral rotations C, B_m and D are inert, but from the $\{Q, S\}$ anticommutator one finds

$$\delta_A\begin{pmatrix}H\\K\end{pmatrix} = \left(\frac{3}{2}-\frac{\lambda}{2}-\frac{\beta}{4}\right)\begin{pmatrix}K\\-H\end{pmatrix}, \qquad \delta_A Z = (\tfrac{1}{4}(\lambda+\beta)-\tfrac{3}{4})\mathrm{i}\gamma_5 Z\lambda_A. \tag{21}$$

Also from $\{Q, S\}$ one finds finally

$$\beta = -\lambda, \qquad \alpha = 2\lambda. \tag{22}$$

4.2. *The action of conformal simple supergravity* [523]

The superconformal algebra consisting of the conformal generators P_m, M_{mn}, K_m, D, the spinorial charges Q^α, S^α and the axial generator A was discussed in the last subsection. We now apply the results of subsection 3.2 to it.

We define gauge fields and parameters by

$$h_\mu = e^m{}_\mu P_m + \omega_\mu{}^{mn} M_{mn} + \bar\psi_{\mu,\alpha} Q^\alpha + f^m{}_\mu K_m + b_\mu D + \bar\varphi_{\mu,\alpha} S^\alpha + A_\mu A \tag{1}$$

$$\epsilon = \xi^m P_m + \lambda^{mn} M_{mn} + \bar\epsilon_{Q,\alpha} Q^\alpha + \xi^m_K K_m + \lambda_D D + \bar\epsilon_{S,\alpha} S^\alpha + \lambda_A A \tag{2}$$

with $m > n$ and construct the curvatures,

$$\begin{aligned}
R_{\mu\nu}{}^{mn}(M) &= 2\partial_\nu\omega_\mu{}^{mn} - 2\omega_\nu{}^{mc}\omega_{\mu c}{}^n - 4(e^m{}_\mu f^n{}_\nu - e^n{}_\mu f^m{}_\nu) - 2\bar\psi_\mu\sigma^{mn}\varphi_\nu \\
R_{\mu\nu}{}^m(P) &= 2\partial_\nu e^m{}_\mu - 2\omega_\nu{}^{mn} e_{n\mu} + \tfrac{1}{2}\bar\psi_\mu\gamma^m\psi_\nu + 2e^m{}_\mu b_\nu \\
R_{\mu\nu}{}^m(K) &= 2\partial_\nu f^m{}_\mu - 2\omega_\nu{}^{mn} f_{n\mu} - \tfrac{1}{2}\bar\phi_\mu\gamma^m\phi_\nu - 2f^m{}_\mu b_\nu \\
R_{\mu\nu}(D) &= 2\partial_\nu b_\mu + 4e_{m\mu} f^m{}_\nu + \bar\psi_\mu\phi_\nu \\
R_{\mu\nu,\alpha}(Q) &= (2D_\nu\bar\psi_\mu + 2\bar\phi_\mu\gamma_\nu + b_\nu\bar\psi_\mu - \tfrac{3}{2}\mathrm{i}A_\nu\bar\psi_\mu\gamma_5)_\alpha \\
R_{\mu\nu,\alpha}(S) &= \left(2D_\nu\bar\phi_\mu - 2\bar\psi_\mu\gamma_m f^m{}_\nu - b_\nu\bar\phi_\mu + \frac{3\mathrm{i}}{2}A_\nu\bar\phi_\mu\gamma_5\right)_\alpha \\
R_{\mu\nu}(A) &= 2\partial_\nu A_\mu - 2\mathrm{i}\bar\psi_\mu\gamma_5\phi_\nu.
\end{aligned} \tag{3}$$

All these curvatures are still to be antisymmetrized in $(\mu\nu)$. As we will see, there are constraints on these curvatures. These have been found by considering the (unique) action for conformal simple supergravity, but the results are so simple and suggestive, that it seems clear that a deeper geometrical origin exists, independent from any dynamical model.

The most general action, bilinear in curvatures, parity conserving and without dimensional constants and affine (i.e., without explicit gauge fields) is given by

$$I = \int d^4x\, [\alpha R_{\mu\nu}{}^{mn}(M)\, R_{\rho\sigma}{}^{pq}(M)\epsilon_{mnpq} + \beta R_{\mu\nu}(Q)\gamma_5(-C^{-1})R_{\rho\sigma}(S)$$

$$+ \gamma R_{\mu\nu}(A)R_{\rho\sigma}(D) + \zeta R_{\mu\nu}(P)R_{\rho\sigma}(K)]\epsilon^{\mu\nu\rho\sigma} \tag{4}$$

(note that $R_{\rho\sigma}(S)$ starts with $-\partial_\rho\bar{\varphi}_\sigma + \partial_\sigma\bar{\varphi}_\rho$). For internal symmetries such as electromagnetism, the action is unique and non-affine, namely the Maxwell action. Hence, since the internal symmetries consist here of only the U(1) chiral invariance, we add its action

$$I = \int d^4x\, [\delta R_{\mu\nu}(A)\, R_{\rho\sigma}(A)\, g^{\mu\rho}g^{\nu\sigma}\sqrt{g}]. \tag{5}$$

Under local M, D, A gauge transformations (given by $\delta h^A{}_\mu = (D_\mu\epsilon)^A$ with ϵ equal to λ^{mn}, λ_D, λ_A) the action is invariant if one uses the rule of transformation of curvatures. Under local K-gauges, the action is invariant if the P curvature vanishes and if the Q-curvature is chiral-dual

$$R_{\mu\nu}{}^m(P) = 0, \qquad R_{\mu\nu}(Q) + \tfrac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}(Q)\gamma_5 = 0. \tag{6}$$

Local S-invariance follows then if

$$\beta = 2i\gamma = \delta = -8\alpha, \qquad \zeta = 0. \tag{7}$$

As we will discuss shortly, local Q-invariance follows if one imposes one more constraint

$$R_{\mu\nu}(Q)\sigma^{\mu\nu} = 0. \tag{8}$$

Thus all constraints are *derived* from a dynamical model by requiring various gauge invariances. In principle, the transformation rules of fields are modified when one solves these constraints. However, as one easily verifies, all constraints are M, K, D, A, S invariant, and thus for these symmetries all fields keep their transformation rules $\delta h_\mu{}^A = (D_\mu\epsilon)^A$, and the action remains M, K, D, A, S invariant.

One can solve the constraints above. The P-curvature yields the torsion equation,

$$\omega_\mu{}^{mn} = -\omega_\mu{}^{mn}(e) - \tfrac{1}{4}(\bar{\psi}_\mu\gamma^m\psi^n - \bar{\psi}_\mu\gamma^n\psi^m + \bar{\psi}^m\gamma_\mu\psi^n) + (e^n{}_\mu b^m - e^m{}_\mu b^n). \tag{9}$$

Thus there is the same torsion as in ordinary supergravity, induced by gravitinos, plus torsion induced by the dilation field.

The solution of the Q-constraints eliminates the conformal supersymmetry gauge field

$$\phi_\mu = \tfrac{1}{3}\gamma^\nu(S_{\mu\nu} + \tfrac{1}{4}\gamma_5\tilde{S}_{\mu\nu}), \qquad S_{\mu\nu} = \left(D_\nu + \tfrac{1}{2}b_\nu - \frac{3i}{4}A_\nu\gamma_5\right)\psi_\mu - \mu \leftrightarrow \nu$$

$$\tilde{S}_{\mu\nu} = e\epsilon_{\mu\nu\rho\sigma}S^{\rho\sigma}. \tag{10}$$

Note that $D_\nu\psi_\mu = (\partial_\nu - \tfrac{1}{2}\omega_\nu{}^{mn}\sigma_{mn})\psi_\mu$ but that substituting $\omega_\nu{}^{mn} = -\omega_\nu{}^{mn}(e)\ldots$, one recovers the usual sign.

The constraints are not Q invariant (nor P-invariant, as we will discuss) so that extra terms in the transformation rules for the dependent fields arise after solving the constraints (see subsection 3.2)

$$\delta' \omega_\mu{}^{mn} = \tfrac{1}{2} R^{mn}(Q) \gamma_\mu \epsilon_Q \tag{11}$$

$$\delta' \phi_\nu = \frac{i}{4} \gamma^\mu (\gamma_5 R_{\mu\nu}(A) + \tfrac{1}{2} \tilde{R}_{\mu\nu}(A)) \epsilon_Q. \tag{12}$$

These results and a similar result for $\delta' f^m{}_\mu$ will be derived in subsection 4.

We now discuss the Q-invariance of the action. If all fields would transform as given by the structure constants alone, one would have

$$\begin{aligned} \delta I = 8\alpha \int d^4x\, [&\tilde{R}^{\mu\nu}(Q)\, \gamma_5 \gamma_m \epsilon_Q\, R_{\mu\nu}{}^m(K) + 2i R^{\mu\nu}(S)\, (\gamma_5 R_{\mu\nu}(A) + \tfrac{1}{2}\tilde{R}_{\mu\nu}(A))\epsilon_Q \\ &- \tfrac{1}{2}(\bar{\epsilon}_Q \gamma \cdot \psi) R_{\mu\nu}(A) R^{\mu\nu}(A) + 2\bar{\epsilon}_Q \gamma^\nu \psi_\sigma R_{\mu\nu}(A)\, R^{\mu\sigma}(A)]. \end{aligned} \tag{13}$$

The extra terms $\delta' \omega_\mu{}^{mn}$ and $\delta' \phi_\mu$ give an extra $\delta' I$, whose general form was derived on page 281. However, since $Q_{AB}{}^{\mu\nu\rho\sigma}$ is constant and proportional to $\epsilon^{\mu\nu\rho\sigma}$ for $A = M$, S, the Bianchi identity holds so that the $D_\nu R_{\rho\sigma}$ terms vanishes, and also the $\partial_\nu Q$ terms vanish. One is left with

$$\delta' I = 8\alpha \int d^4x [2\delta' \omega_\mu{}^{mn}\, \tilde{R}^{\mu\nu,p}(K)\, \epsilon_{\nu mnp} + 4\tilde{R}^{\mu\nu}(S)\, \gamma_\mu \gamma_5 \delta' \phi_\nu + 4i \bar{\psi}_\mu (\gamma_5 R^{\mu\gamma}(A) + \tfrac{1}{2}\tilde{R}^{\mu\nu}(A))\, \delta' \phi_\nu]. \tag{14}$$

The $R(Q)\, R(K)$ terms in δI cancel against the $R(K)$ term in $\delta' I$ if and only if $\delta' \omega_\mu{}^{mn}$ is as given above. For this to be the case, one needs both Q-constraints; with the dual-chiral constraint alone no cancellation would occur. Furthermore, substituting $\delta' \phi_\mu$ all other terms in $\delta I + \delta' I$ cancel at once.

We now show that the dilation field cancels from the action. The remaining independent fields are $e^m{}_\mu$, $\psi^a{}_\mu$, A_μ, b_μ and $f^m{}_\mu$ while $f^m{}_\mu$ is nonpropagating and is expressed in terms of other fields by solving its own field equation (just as $\omega_\mu{}^{mn}$ in ordinary supergravity). Let us consider how these fields transform under K-gauges. Only b_μ transforms, and since $\delta_K b_\mu = 2\xi_{K,\mu}$ one can gauge away b_μ altogether. This one might have expected since there are as many K-gauge degrees of freedom as b-components. However, since the action was K-invariant to begin with, it must therefore be independent of b_μ.

The action is thus unique and determines in turn the constraints [523]. In fact, as we shall see, the solution of the $f^m{}_\mu$-field equation is at the same time the solution of an extra constraint [535]. At the linearized level, these results were obtained using superspace methods [191].

Let us close this subsection by showing that there are equal numbers of boson and fermion states. Since the graviton and gravitino have higher derivative actions, one has two spin 2 states ($\Box^2$ counts as two $\Box$) and three spin $\frac{3}{2}$ states ($\not{\partial}\Box$ counts as three $\not{\partial}$). Indeed, one finds one massless $(2, \frac{3}{2})$ multiplet and one massive $(2, \frac{3}{2}, \frac{3}{2}, 1)$ multiplet [191, 526].

4.3. Constraints and gauge algebra

For ordinary supergravity, one can formulate the theory in second order formalism either by imposing a constraint on curvatures ($R_{\mu\nu}{}^m(P) = 0$) or by solving a field equation belonging to the gauge action (namely $\delta I/\delta\omega_\mu{}^{mn}=0$). Both methods lead to the same result, but the former has the advantage that it is independent of any particular action. In conformal supergravity, the spin connection is propagating, and the torsion is here fixed by the constraint $R_{\mu\nu}{}^m(P)=0$, and not by a field equation. However, for the conformal boost field $f^m{}_\mu$ this duality is again present: solving the field equation $\delta I/\delta f^n{}_\mu=0$ for $f^m{}_\mu$ one finds a result which can also be obtained as a constraint on the curvatures. This constraint is

$$R_{\nu\lambda mn}(M)e^{m\lambda}e^n{}_\mu-\tfrac{1}{2}R_{\lambda\mu}(Q)\,\gamma_\nu\psi^\lambda+\frac{i}{4}\epsilon_{\mu\nu\rho\sigma}R^{\rho\sigma}(A)=0 \tag{1}$$

and it is again K, M, D, S, A invariant. Thus under these symmetries, $\delta' f=0$ but as we shall derive shortly, $\delta'_Q f\neq 0$. (It should be noted that as far as invariance of the action is concerned, all $\delta' f$ are allowed to be non-vanishing since f satisfies its own field equation. See subsection 1.6.) Eq. (1) is the covariantization of the "Einstein equations" $R_{\mu\nu}(M)=0$. The term with $R(Q)$ is the connection needed to covariantize the derivative $\partial\omega$ inside the M-curvature. If one introduces a connection

$$\hat\omega_\mu{}^{mn}=\omega_\mu{}^{mn}+\frac{i}{2}e_\mu{}^k\epsilon_{kmnc}A^c\ (A^c=\text{chiral gauge field}) \tag{2}$$

then the constraint even simplifies to

$$R^{\mathrm{cov}}_{\mu\nu}(M,\hat\omega)=0. \tag{3}$$

Thus there are three constraints in conformal simple supergravity:

(i) *Torsion*: $R^m{}_{\mu\nu}(P)=0$, not a field equation.

(ii) *Duality-chirality and tracelessness*: $R_{\mu\nu}(Q)\gamma^\nu=0$ which is equivalent to $R_{\mu\nu}(Q)+\frac{1}{2}\tilde R_{\mu\nu}\gamma_5=0$ plus $R_{\mu\nu}(Q)\sigma^{\mu\nu}=0$.

(iii) *Einstein equation*: $R^{\mathrm{cov}}_{\mu\nu}(M,\hat\omega)=0$.

It is probably not a coincidence that the generators for which the curvatures are constrained (P, Q, M) form the super Poincaré algebra.

All constraints are gauge invariant under all 24 local symmetries, except under P-gauge and Q-gauge transformations. The extra terms $\delta'_Q\omega_\mu{}^{mn}$ and $\delta'_Q\phi_\mu$ were mentioned before. In a similar way one finds

$$\delta'_Q f^m{}_\mu=-\tfrac{1}{2}R^{\mathrm{cov}}_{\rho\mu}(S)\,\sigma^{m\rho}\epsilon_Q-\tfrac{1}{8}R^{m,\mathrm{cov}}_{\mu}(S)\,\gamma_5\epsilon_Q \tag{4}$$

where the Q-covariantization is due to the $\partial\phi$ terms in $R(S)$.

One can now understand geometrically why these constraints are needed. The global commutator $\{Q,Q\}\sim P$ must turn into $\{Q,Q\}\sim$ general coordinate transformation plus more if one evaluates the commutator of the two ordinary supersymmetry transformations on *fields*. This is to be expected since our action was by construction invariant under general coordinate transformation, and invariance under also P-gauges would be double-counting. How does the theory manage to do this? Note that a general coordinate transformation on a gauge field $h^A{}_\mu$ is related to local gauge transformation by the following *identity*

$$\delta_{\mathrm{gen.\,coord}}(\xi^\lambda)h^A{}_\mu=(D_\mu(\xi\cdot h))^A+\xi^\lambda R^A{}_{\mu\lambda}. \tag{5}$$

Consider now the $\{Q, Q\}$ anticommutator acting on $e^m{}_\mu$. Since only physical fields appear at all stages, and physical fields have never a nonzero δ' transformation law, the only way the theory manages to make P-gauges an invariance, is by requiring that $R(P)$ in (5) vanishes. Thus

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)]e^m{}_\mu = \delta_P(f^A{}_{BC}\epsilon_1^B\epsilon_2^C = \tfrac{1}{2}\bar{\epsilon}_2\gamma^m\epsilon_1) \tag{6}$$

and since $R^m(P) = 0$, this is according to (5) a sum of a general coordinate transformation plus other local symmetries. This explains also why in the superconformal algebra the parameters in the $\{Q, Q\}$ anticommutator are all of the form $\bar{\epsilon}_2\gamma^\mu\epsilon_1$ times the gauge field h_μ^A.

The other constraints play a similar role. For example, in the $\{Q, Q\}$ anticommutator on ψ_μ one finds now an unphysical field, namely $\omega_\mu{}^{mn}$ due to $\delta\psi_\mu = D_\mu\epsilon$. Thus now the theory has a choice: either choosing an appropriate $\delta'\omega$ (which again leads to the constraint $R(P) = 0$), or by putting $R(Q) = 0$ as a constraint. The latter choice is not possible since no field can be solved from $R(Q) = 0$, while the former choice is the correct one:

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)]\bar{\psi}_\mu^\alpha = \tfrac{1}{2}\bar{\epsilon}_2\gamma^\lambda\epsilon_1 R_{\mu\lambda}^\alpha(Q) + \delta_P(\tfrac{1}{2}\bar{\epsilon}_2\gamma^m\epsilon_1)\bar{\psi}^\alpha{}_\mu \tag{7}$$

due to $\delta'_Q\omega_\mu{}^{mn} = \frac{1}{2}R^{mn}(Q)\gamma_\mu\epsilon_Q$. Thus, for ordinary supergravity, the constraint $R_{\mu\nu}{}^m(P) = 0$ is enough. But for conformal supergravity one has one extra independent field to consider, namely A_μ. Now with $\delta_Q A_\mu = -\mathrm{i}\bar{\epsilon}_Q\gamma_5\phi_\mu$

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)]A_\mu = \delta_P(\tfrac{1}{2}\bar{\epsilon}_2\gamma^m\epsilon_1)A_\mu - (\mathrm{i}\bar{\epsilon}_2\gamma_5\delta'_Q(\epsilon_1)\phi_\mu - 1 \leftrightarrow 2) \tag{8}$$

and indeed $\delta'_Q\phi_\mu$ is such that the last term turns into a term with $R_{\mu\nu}(A)$ such that the sum is again a general coordinate transformation.

Thus the constraints on curvatures have the geometrical significance of turning P-gauge transformations, defined by the group in the tangent space, into general coordinate transformations in the base manifold.

The gauge algebra is thus given by

$$[\delta(\epsilon_1^A), \delta(\epsilon_2^A)]h_\mu^A = \delta(f^A{}_{BC}\epsilon_1^B\epsilon_2^C) + f^A{}_{BC}\delta'(\epsilon_1^A)h_\mu^B\epsilon_2^C - 1 \leftrightarrow 2. \tag{9}$$

Since only δ'_Q is nonzero, only the (Q, Q), (Q, K), (Q, S), (Q, D), (Q, M) or (Q, A) relations might be modified. However, K, S, D, M, A rotate physical fields into physical fields so that only the (Q, Q) anticommutator is modified. One finds uniformly on e_μ^m, ψ_μ^α, A_μ and b_μ, and hence (because of the Jacobi identities) also on any function of these fields (in particular on ϕ_μ^α, f_μ^m, ω_μ^{mn})

$$\begin{aligned}[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = {}& \delta_{\text{gen. coord}}(\xi^\lambda) + \delta_{\text{Lorentz}}(-\xi^\lambda\omega_\lambda{}^{mn}) + \delta_{\text{chiral}}(-\xi^\lambda A_\lambda) \\ & + \delta_Q(-\xi^\lambda\bar{\psi}_\lambda) + \delta_S(-\xi^\lambda\bar{\phi}_\lambda)\end{aligned} \tag{10}$$

with $\xi^\lambda = \frac{1}{2}\bar{\epsilon}_2\gamma^\lambda\epsilon_1$. If one puts b_μ equal to zero, as well as δb_μ, the gauge algebra still closes but now an extra S-supersymmetry transformation is present,

$$\delta_S^{\text{extra}}(\tfrac{1}{4}(\bar{\phi}_\mu\epsilon_2)\bar{\epsilon}_1\gamma^\mu - 1 \leftrightarrow 2). \tag{11}$$

Thus the gauge algebra of $N = 1$ conformal supergravity closes without the need to introduce auxiliary fields [523]. This is no longer true for the $N > 1$ case.

We now show how to derive $\delta'\omega_\mu{}^{mn}$ and $\delta'\phi_\mu$ without much work, using group theory. Since $R_{\mu\nu}{}^\mu(P) = 0$ identically, its variation vanishes, too. This variation consists of two contributions: namely, if all fields would transform according to the group, one could use the homogeneous rotations of curvatures

$$\delta(\text{group})R_{\mu\nu}{}^m(P) = -\tfrac{1}{2}R_{\mu\nu}(Q)\gamma^m\epsilon. \tag{12}$$

However, $\delta\omega_\mu{}^{mn}$ is modified because $\omega_\mu{}^{mn}$ is expressed in terms of other fields by solving $R_{\mu\nu}{}^m(P) = 0$. Thus there is an extra term $\delta'\omega_\mu{}^{mn}e_{n\nu} - \delta'\omega_\nu{}^{mn}e_{n\mu}$. Solving in the same way as one solves for the Christoffel symbol, one finds $\delta'\omega_\mu{}^{mn}$.

For $\delta'\phi_\mu$ we proceed as follows [169]. From the Bianchi identity

$$\epsilon^{\mu\nu\rho\sigma}D_\nu R_{\rho\sigma}(P) = 0 = -\epsilon^{\mu\nu\rho\sigma}(R_{\nu\rho\sigma}{}^m(M) + e_\nu{}^m R_{\rho\sigma}(D)) \tag{13}$$

one finds two relations, using that $R_{\mu\nu}(Q)\sigma^{\mu\nu}\gamma^\tau = \tfrac{1}{2}R_{\mu\nu}(Q)\gamma_5\gamma_\lambda\epsilon^{\mu\nu\tau\lambda} = 0$,

$$R_{\mu\nu}(M) - R_{\nu\mu}(M) = -2R_{\mu\nu}(D) \tag{14}$$

$$R_{\alpha\beta\gamma\tau}(M) + 2 \text{ terms cyclic in } \alpha\beta\gamma = -2\delta_{\alpha\tau}R_{\beta\gamma}(D) + 2 \text{ terms}. \tag{15}$$

From the latter result one deduces

$$R_{\alpha\beta\gamma\delta}(M) - R_{\gamma\delta\alpha\beta}(M) = \delta_{\alpha\gamma}R_{\beta\delta}(D) + (\alpha \leftrightarrow \beta, \gamma \leftrightarrow \delta). \tag{16}$$

As before, we now use that the variation of a solved constraint vanishes ($\hat{R} = -C^{-1}R^{\mathrm{T}}$)

$$\delta\gamma^\mu\hat{R}_{\mu\nu}(Q) = 0 = \gamma^\mu\left[\frac{3\mathrm{i}}{4}\gamma_5\epsilon R_{\mu\nu}(A) - \frac{1}{2}\epsilon R_{\mu\nu}(D) + \frac{1}{2}\sigma^{mn}\epsilon R_{\mu\nu mn}(M)\right] + [2\delta'\phi_\nu + \gamma_\nu\gamma\cdot\delta'\phi]. \tag{17}$$

It is now easy to deduce the result for $\delta'\phi_\nu$ quoted in subsection 2.

The result for $\delta' f^m{}_\mu$ in eq. (4) is obtained in a similar way, by varying the third constraint (the Einstein equations) and using the results for $\delta'\omega_\mu{}^{mn}$ and $\delta'\phi_\mu$, as well as the Bianchi identity $\epsilon^{\mu\nu\rho\sigma}D_\nu R_{\rho\sigma}(Q) = 0$. (This identity is needed since varying the Lorentz curvature one produces terms of the form $\partial_\mu\partial\omega'_\nu{}^{mn}$.)

4.4. *Conformal tensor calculus*

Since all fields in conformal supergravity are gauge fields which couple minimally, it is straightforward to obtain a tensor calculus for conformal supergravity: all one has to do is to replace ordinary derivatives by covariant derivatives. In order to obtain a tensor calculus for ordinary supergravity, one has to do two things. First of all, the transformation rules of ordinary supergravity are obtained from

"$Q+S$ rule" of subsection 4.5. Since the results then still depend on the conformal weight λ, whereas the notion of a weight is an alien concept in *ordinary* (nonconformal) theories, one then redefines fields such that the final results are independent of λ.

To illustrate the first step, consider for example the spin $(1,\frac{1}{2})$ scalar multiplet with canonical weight $\lambda=\frac{3}{2}$ [527]

$$W_\alpha = [\bar\lambda_\alpha, -\mathrm{i}(\bar\lambda\gamma_5)_\alpha, (-\sigma^{mn}F_{mn}+\mathrm{i}\gamma_5 D)_{\beta\alpha}, -(\bar\lambda\overleftarrow{\partial\!\!\!/})_\alpha, \mathrm{i}(\bar\lambda\overleftarrow{\partial\!\!\!/}\gamma_5)_\alpha]. \tag{1}$$

Since $\delta V_\mu = -\bar\epsilon\gamma_\mu\lambda$, $\delta\lambda = (\sigma^{\mu\nu}F_{\mu\nu}+\mathrm{i}\gamma_5 D)\epsilon$, $\delta D = +\mathrm{i}\bar\epsilon\gamma_5\partial\!\!\!/\lambda$, the local version W_α is obtained by replacing ∂_μ by the superconformally covariant derivative $D_\mu{}^{c}$

$$\begin{aligned} D_\mu{}^{c}\lambda &= (D_\mu - \tfrac{1}{2}\mathrm{i}A_\mu\gamma_5 - \tfrac{3}{2}b_\mu)\lambda - \tfrac{1}{2}(\sigma^{\mu\nu}F_{\mu\nu}{}^{c} + \mathrm{i}\gamma_5 D)\psi_\mu \\ F_{\mu\nu}{}^{c} &= (\partial_\mu V_\nu + \tfrac{1}{2}\bar\psi_\mu\gamma_\nu\lambda) - \mu \leftrightarrow \nu. \end{aligned} \tag{2}$$

This $D_\mu{}^{c}$ follows from subsection 1. As a rule, it is better to use flat indices; for example, it is F_{mn} which transforms simply, not $F_{\mu\nu} = e^m{}_\mu e^n{}_\nu F_{mn}$.

As a second example we consider the local analogue of the "kinetic multiplet" $T(\Sigma)$ [531]. *This* multiplet will be used in the next subsection. The details which follow have not been published before.

It is easy to show that the following is a scalar multiplet of rigid conformal supersymmetry

$$T(\Sigma) = [F, -G, \partial\!\!\!/\chi, \Box A, -\Box B]. \tag{3}$$

To make it a local multiplet, we replace ordinary derivatives by covariant derivatives and $D_\mu{}^{c}\chi$ is already obtained from subsection 1. Since under S-supersymmetry $\delta_S F = (1-\lambda)\bar\epsilon\chi$ while the first component should be S-inert, it follows that one should take $\lambda=1$, i.e., $T(\Sigma)$ has canonical weight 2. We will now derive the conformal dalembertian $\Box^{c}A$. For completeness we will derive the result for arbitrary λ, although in $T(\Sigma)$ one only can admit $\lambda=1$ as we showed.

From subsection 1 we know the global superformal transformation of Σ; in particular

$$D_\mu{}^{c}A = \partial_\mu A - \tfrac{1}{2}\bar\psi_\mu\chi - \lambda b_\mu A + \frac{\lambda}{2}BA_\mu{}^{c} \tag{4}$$

where $A_\mu{}^{c}$ is the chiral gauge field ($A_\mu{}^{c} = -\frac{2}{3}A_\mu{}^{\mathrm{aux}}$). In order to take the covariant derivative of this covariant derivative, we begin with the following well-known simple expression for the dalembertian in general relativity

$$\tfrac{1}{e}\partial_\nu(g^{\mu\nu}eD_\mu{}^{c}A). \tag{5}$$

This takes care of the Lorentz and general coordinate connections (the latter replace as always P-gauge). We now write the other connection terms for $\partial_\mu(g^{\mu\nu}eD_\mu{}^{c}A)$. First we state an important property: *Ordinary derivatives in local objects must never be written as commutators with* P_a if one wants to construct connections. For example, if one wants to determine the K-connection for the derivative of $\partial_\mu A$ then one finds zero, and it would be wrong to argue that $[K_\nu, \partial_\mu A] = [K_\nu, [A, P_\mu]] = -[[K_\nu, P_\mu], A]$.

For the connections one now finds

$$S:\quad \frac{1}{2}\bar{\phi}\cdot\gamma\chi+\frac{\lambda}{2}\bar{\psi}^{\mu}(A+\mathrm{i}\gamma_5 B)\phi_\mu-\frac{\lambda}{2}\bar{\psi}\phi A-\frac{\mathrm{i}\lambda}{2}\bar{\phi}_\mu\gamma_5\psi^\mu B. \tag{6}$$

Here we have used that $\delta_S\chi=\lambda(A+\mathrm{i}\gamma_5B)\epsilon_S$ while $\delta_S b_\mu=-\frac{1}{2}\bar{\psi}_\mu\epsilon_S$ as follows from $\delta h_\mu{}^A=f_{CB}{}^A h_\mu{}^B\epsilon^C$ and $\{Q^\alpha,S^\beta\}=\frac{1}{2}C^{-1,\alpha\beta}D$ +more,

$$A:\quad \frac{\lambda}{2}A^{\mathrm{c}}\cdot\partial B+\frac{3\mathrm{i}}{8}A^{\mathrm{c}}\cdot\bar{\psi}\gamma_5\chi+\frac{1}{2}\bar{\psi}\cdot A^{\mathrm{c}}\mathrm{i}\gamma_5\chi\left(-\frac{3}{4}+\frac{\lambda}{2}\right)-\frac{\lambda^2}{2}b\cdot A^{\mathrm{c}}B-\frac{\lambda^2}{4}(A_\mu{}^{\mathrm{c}})^2A. \tag{7}$$

$$D:\quad (2-\lambda)b^\mu D_\mu{}^{\mathrm{c}}A. \tag{8}$$

Here we have used that $\delta_D e^m{}_\mu=-\lambda_D e^m{}_\mu$ and that $\delta_D D_\mu{}^{\mathrm{c}}A=\lambda_D D_\mu{}^{\mathrm{c}}A$. Thus under *local* scale transformation, the scale of the derivative is zero. (This makes sense, ∂_μ is not a field and one only transforms fields under local scale transformations.)

$$K:\quad -2\lambda A f^m{}_\mu e_m{}^\mu=-2\lambda A(-\tfrac{1}{12}R-\tfrac{1}{6}\bar{\psi}_\mu\sigma^{\mu\nu}\phi_\nu). \tag{9}$$

The connection comes from $\delta_K b_\mu=-2\xi_K^m e_{m\mu}$. All other fields are K-inert. In particular $\partial_\mu A$ is inert under local K-gauges (under global K-transformations, $\partial_\mu A$ is not inert since $[K_\mu,P_\nu]\neq 0$!). The symbol R denotes the scalar curvature with $\omega(e,\psi,b)$.

The Q-connection is obtained by varying $eg^{\mu\nu}$ and $D_\mu{}^{\mathrm{c}}A$. For the latter one finds

$$\begin{aligned}\delta_Q^{\mathrm{c}}(\epsilon)D_\mu{}^{\mathrm{c}}A&=\delta_Q^{\mathrm{c}}\left(\partial_\mu A-\frac{1}{2}\bar{\psi}_\mu\chi+\frac{\lambda}{2}BA_\mu-\lambda b_\mu A\right)\\&=\frac{1}{2}\bar{\epsilon}\left(\partial_\mu+\frac{1}{2}\omega_\mu{}^{mn}\sigma_{mn}+\frac{3\mathrm{i}}{4}A_\mu{}^{\mathrm{c}}\gamma_5-\frac{1}{2}b_\mu\right)\chi-\frac{1}{2}\bar{\psi}_\mu\delta_Q{}^{\mathrm{c}}\chi\\&\quad-\frac{\mathrm{i}\lambda}{4}\bar{\epsilon}\gamma_5\chi A_\mu{}^{\mathrm{c}}+\frac{\lambda}{2}B(-\bar{\epsilon}\mathrm{i}\gamma_5\varphi_\mu)-\frac{\lambda}{2}(\bar{\epsilon}\varphi_\mu)A-\frac{\lambda}{2}(\bar{\epsilon}\chi)b_\mu.\end{aligned} \tag{10}$$

Now one can write

$$-\tfrac{1}{2}\bar{\psi}_\mu\delta_Q{}^{\mathrm{c}}\chi=-\tfrac{1}{4}\bar{\psi}_\mu(\not{D}^{\mathrm{c}}(A-\mathrm{i}\gamma_5B)+F+\mathrm{i}\gamma_5G)\epsilon \tag{11}$$

as

$$\bar{\epsilon}(\not{D}^{\mathrm{c}}(A-\mathrm{i}\gamma_5B)+F+\mathrm{i}\gamma_5G)(-\tfrac{1}{4}\psi_\mu)+\tfrac{1}{2}\bar{\epsilon}(\not{D}^{\mathrm{c}}A)\psi_\mu. \tag{12}$$

Thus

$$\delta_Q^{\mathrm{c}}(D_\mu{}^{\mathrm{c}}A)=\tfrac{1}{2}\bar{\epsilon}D_\mu{}^{\mathrm{c}}\chi+\tfrac{1}{2}\bar{\epsilon}(\not{D}^{\mathrm{c}}A)\psi_\mu. \tag{13}$$

To this one must add the variation of $eg^{\mu\nu}$, which yields an extra term

$$(\delta_Q{}^c(eg^{\mu\nu}))(D_\mu{}^cA) = (\tfrac{1}{2}\bar\epsilon\gamma\cdot\psi g^{\mu\nu} - \tfrac{1}{2}\bar\epsilon\gamma^\mu\psi^\nu - \tfrac{1}{2}\bar\epsilon\gamma^\nu\psi^\mu)D_\mu{}^cA. \tag{14}$$

Interestingly, the non-covariantizable terms $\bar\epsilon \not{D}^c A\psi_\mu$ cancel, and the complete Q-connection is easily found to be

$$Q:\quad -\tfrac{1}{2}\bar\psi^\mu D_\mu{}^c\chi - \tfrac{1}{2}\bar\psi^\mu\gamma\cdot\psi D_\mu{}^cA. \tag{15}$$

Thus the full superconformal dalembertian is given by

$$\begin{aligned}\Box^c A = {}& \frac{1}{e}\partial_\nu(g^{\mu\nu}eD_\mu{}^cA) + \frac{1}{2}\bar\phi\cdot\gamma\chi - \frac{\lambda}{3}A^c\cdot\partial B - \frac{\mathrm{i}\lambda}{6}\bar\psi\cdot A^c\gamma_5\chi + \frac{\lambda^2}{3}b\cdot A^cB \\ & - \frac{\lambda^2}{9}(A_\mu{}^c)^2A + (2-\lambda)b^\mu D_\mu{}^c\chi + \frac{\lambda}{6}AR + \frac{\lambda}{3}A\bar\psi_\mu\sigma^{\mu\nu}\phi_\nu - \frac{1}{2}\bar\psi^\mu D_\mu{}^c\chi - \frac{1}{2}\bar\psi^\mu\gamma\cdot\psi D_\mu{}^cA.\end{aligned} \tag{16}$$

Two interesting remarks are in order. First of all, although the separate connections are not inert under those symmetries to which they do not refer, *the* sum (thus $\Box^cA$) is really covariant under all symmetries. Secondly, as we have stressed so often, flat indices simplify transformation properties, and indeed,

$$\delta_Q{}^c(D_a{}^cA) = \tfrac{1}{2}(\bar\epsilon D_a{}^c\chi). \tag{17}$$

One can now at once write down the conformal coupling of the scalar multiplet to the gauge fields of conformal supergravity. It is given by the action for the multiplet $\Sigma\otimes T(\Sigma)$. The multiplication of two local multiplets is the same as the multiplication of two global multiplets of conformal sypersymmetry

$$\begin{aligned}\Sigma_1\otimes\Sigma_2 = {}&(A_1A_2 - B_1B_2,\ A_1B_2 + A_2B_1,\ (A_1 + \mathrm{i}\gamma_5B_1)X_2 + (A_2 + \mathrm{i}\gamma_5B_2)X_1, \\ & -\bar X_1X_2 + A_1F_2 + A_2F_1 - B_1G_2 - B_2G_1,\ A_1G_2 + A_2G_1 + B_1F_2 + B_2F_1 + \mathrm{i}\bar X_1\gamma_5X_2).\end{aligned} \tag{18}$$

One may verify explicitly that the product of two superconformal scalar multiplets with weights λ_1 and λ_2 respectively, as given by (18), is again a superconformal scalar multiplet with weight $\lambda_1+\lambda_2$. In superfields this is immediately clear. In the next subsection we show how to construct an action for a superconformal scalar multiplet.

4.5. *The origins of the auxiliary fields S, P, A_m* [471, 472]

The independent (physical) fields of conformal supergravity are the tetrad $e^m{}_\mu$, the gravitino (ordinary supersymmetry gauge field) $\psi^a{}_\mu$, the chiral gauge field A_μ and the dilation b_μ. Just as in ordinary supergravity, one can again count *field* components in the action, taking into account that each degree of gauge invariance removes one field component. The count is as follows:

$16e^{m}{}_{\mu}$ − 10 spacetime − 1 dilational gauge = 5 Bose fields

$16\psi^{a}{}_{\mu}$ − (4 + 4) ordinary and conformal supersymmetries = 8 Fermi fields

$4A_{\mu}$ − 1 chiral gauge = 3 Bose fields

$4b_{\mu}$ − 4 conformal boosts = 0.

Thus there are equal numbers of Bose and Fermi fields off-shell, and no auxiliary fields are needed from this point of view. Indeed, as discussed in subsection 4.3, the gauge algebra of conformal simple supergravity closes by itself. This has an important consequence. *Since all fields of conformal simple supergravity are gauge field, they couple minimally to matter.*

Thus, one can at once write down the coupling of the globally superconformal multiplet discussed in subsection 1 to conformal supergravity. All one has to do is replace ordinary derivatives by superconformal derivatives, where the gauge connections are given in the usual minimal way.

To see the relevance of this fact for ordinary supergravity, let us recall that the Hilbert action of general relativity (R) can be written in a locally scale invariant form by adding a scalar kinetic term with unphysical signs

$$\mathscr{L}(\text{Brans–Dicke}) = -\frac{e}{2}R\varphi^2 + \frac{e}{12}\partial^{\mu}\phi\partial_{\mu}\phi. \tag{1}$$

Fixing the new degree of gauge freedom by fixing $\varphi = \kappa^{-1}$, one regains the Hilbert action. Or equivalently, one can choose a gauge in which the regauged metric $g_{\mu\nu}\varphi^{-2}$ is locally scale invariant.

In supergravity the analogue of a scalar field φ is a scalar multiplet $\Sigma = [A, B, X, F, G]$ which we take to be a superconformal scalar multiplet with weight $\lambda = 1$ in order to interpret A as the analogue of φ. The action for Σ is in flat space

$$\mathscr{L}(\Sigma) = -\tfrac{1}{2}[(\partial_{\mu}A)^2 + (\partial_{\mu}B)^2 + \bar{\lambda}\not{\partial}\lambda - F^2 - G^2] \tag{2}$$

and it is invariant under $\delta(A + \mathrm{i}B) = \bar{\epsilon}(1 + \gamma_5)\chi$,

$$\delta\chi = \tfrac{1}{2}\not{\partial}(A - \mathrm{i}\gamma_5 B)\epsilon + \tfrac{1}{2}(F + \mathrm{i}\gamma_5 G)\epsilon \qquad \text{and} \qquad \delta(F + \mathrm{i}G) = \tfrac{1}{2}\bar{\epsilon}\not{\partial}(1 + \gamma_5)\chi.$$

To couple we first obtain the supercovariant derivatives $D_{\mu}{}^{\mathrm{c}}$. From subsection 1 we find easily

$$\begin{aligned}
D_{\mu}{}^{\mathrm{c}}A &= \partial_{\mu}A - \frac{1}{2}\bar{\psi}_{\mu}\chi - \lambda b_{\mu}A + \frac{\lambda}{2}A_{\mu}B \\
D_{\mu}{}^{\mathrm{c}}B &= \partial_{\mu}B + \frac{\mathrm{i}}{2}\bar{\psi}_{\mu}\gamma_5\chi - \lambda b_{\mu}B - \frac{\lambda}{2}A_{\mu}A \\
D_{\mu}{}^{\mathrm{c}}\chi &= D_{\mu}\chi - \frac{1}{2}\not{D}^{\mathrm{c}}(A - \mathrm{i}\gamma_5 B)\psi_{\mu} - \frac{1}{2}(F + \mathrm{i}\gamma_5 G)\psi_{\mu} \\
&\quad - \left(\lambda + \frac{1}{2}\right)b_{\mu}\chi - \left(\frac{\lambda}{2} - \frac{3}{4}\right)A_{\mu}(\mathrm{i}\gamma_5\chi) - \lambda(A + \mathrm{i}\gamma_5 B)\varphi_{\mu}.
\end{aligned} \tag{3}$$

To obtain the supersymmetric extension of (2), one replaces ∂_μ by $D_\mu{}^c$ and uses the same product rules for multiplets as in global supersymmetry (since covariant derivatives satisfy the Leibniz rule, just as ordinary derivatives). Thus one constructs the multiplet $\Sigma \otimes T\Sigma$ and uses the following action formula, valid for any superconformal scalar multiplet Σ (in particular $\Sigma \otimes T\Sigma$) with weight $\lambda = 3$ (so that the F component has weight 4) [527]

$$I = \int \mathrm{d}^4x\, [eF + \tfrac{1}{2}e\bar{\psi}\cdot\gamma\chi + \tfrac{1}{2}e\bar{\psi}_\mu\sigma^{\mu\nu}(A - \mathrm{i}\gamma_5 B)\psi_\nu], \qquad (\lambda = 3). \tag{4}$$

In this way one obtains supersymmetric Brans–Dicke theory.

An alternative way to derive this action is as follows. The action $2\mathscr{L}(\Sigma)$ of global (conformal) supersymmetry can be obtained by first writing Σ as a vector multiplet [448]

$$V(\Sigma) = [-A, \mathrm{i}\gamma_5\chi, F, G, \partial_m B, 0, 0] \tag{5}$$

then multiplying $V(\Sigma)$ times itself, and finally taking the last component (the $\hat{D}$-component) which is given by

$$2C\hat{D} - 2\bar{Z}\Lambda + H^2 + K^2 - B_m^2 - (\partial_m C)^2 - \bar{Z}\not{\partial}Z. \tag{6}$$

Replacing ∂_μ by $D_\mu{}^c$, and using the following action formula for a vector multiplet with weight $\lambda = 2$ (so that $\hat{D}$ has weight 4) [447]

$$I = \int \mathrm{d}^4x\, e[\hat{D} - \tfrac{2}{3}\kappa^2 C(-\tfrac{1}{2}R - \tfrac{1}{2}\bar{\psi}_\mu R^\mu) + \text{more}] \tag{7}$$

one finds the action from $V(\Sigma)\times V(\Sigma) + V(\text{par}\,\Sigma)\times V(\text{par}\,\Sigma)$, where par Σ is the parity reflected multiplet $(B, -A, -\mathrm{i}\gamma_5\chi, G', -F')$. The result is

$$\mathscr{L}(\text{supersymmetric Brans–Dicke}) = (-4e)\times$$
$$[-\tfrac{1}{12}(A^2 + B^2)(R + \bar{\psi}_\mu R^\mu) + \tfrac{1}{2}(D_\mu{}^c A)^2 + \tfrac{1}{2}(D_\mu{}^c B)^2 - \tfrac{1}{2}F^2 - \tfrac{1}{2}G^2 + \cdots]. \tag{8}$$

To reduce to the case of ordinary supergravity, one fixes the local scale invariance by $A = $ constant, the local chiral invariance by $B = 0$, the local boosts by $b_\mu = 0$, and a linear combination of Q and S supersymmetry such that $\chi = 0$ as well as $\delta\chi = 0$. The only linear combination of Q and S supersymmetry which satisfies $\delta\chi = 0$ is $\delta_Q{}^c(2\epsilon) + \delta_S{}^c((-F - \mathrm{i}\gamma_5 G - (\mathrm{i}/2)\not{A}\gamma_5 A)\epsilon)$ and the action reduces to

$$\mathscr{L} = -\frac{A^2}{12}(R + \bar{\psi}_\mu R^\nu) - \frac{1}{2}(F^2 + G^2) + \frac{1}{8}A_\mu^2 A^2. \tag{9}$$

If one identifies the chiral gauge field A_μ with $-2/3$ times the auxiliary field A_m^{aux}, and further $F = \frac{1}{3}S$, $G = -\frac{1}{3}P$, $A = 1$ then one recovers exactly ordinary supergravity. For example, $\delta_Q(2\epsilon)F = \bar{\epsilon}\not{D}^c\chi$ and since $\delta_S{}^c F = 0$ for $\lambda = 1$, one finds, using that $\gamma\cdot\varphi = -\frac{1}{6}\gamma_\mu R^\mu + (\mathrm{i}/3)\sigma^{\mu\nu}A_\nu\gamma_5\psi_\mu$ the result for δS in ordinary supergravity. If we had initially included the locally scale (Weyl) invariant coupling (eA^4 + more), then one would have found Poincaré supergravity plus cosmological term.

In particular, the supersymmetry transformation of ordinary supergravity is a linear combination of the two supersymmetries of conformal supergravity [525]

$$\delta^P_Q(\epsilon) = \delta_Q{}^c(\epsilon) + \delta_S{}^c(\tfrac{1}{2}\eta\epsilon + \tfrac{1}{2}\not{b}\epsilon), \qquad \eta = -\tfrac{1}{3}(S - \mathrm{i}\gamma_5 P - \mathrm{i}\not{A}^{\mathrm{aux}}\gamma_5). \tag{10}$$

Moreover, the chiral gauge field turns into the axial auxiliary field A_m and the auxiliary fields F, G of the scalar multiplet become the auxiliary fields S, P [471].

These results are not only valid for matter fields but hold generally. For example, for the field $e^m{}_\mu$, $\psi^a{}_\mu$, A_μ one has

$$\delta_Q{}^c e^m{}_\mu = \tfrac{1}{2}\bar{\epsilon}_Q\gamma^m\psi_\mu, \qquad \delta_Q{}^c A_\mu{}^c = -\mathrm{i}\bar{\epsilon}_Q\gamma_5\varphi_\mu$$

$$\delta_Q{}^c\psi_\mu = \left(D_\mu + \tfrac{1}{2}b_\mu - \frac{3\mathrm{i}}{4}A_\mu{}^c\gamma_5\right)\epsilon_Q \tag{11}$$

$$\delta_S{}^c e^m{}_\mu = 0, \qquad \delta_S{}^c\psi_\mu = -\gamma_\mu\epsilon_S, \qquad \delta_S{}^c A_\mu{}^c = \mathrm{i}\bar{\epsilon}_S\gamma_5\psi_\mu$$

and using the "$Q+S$ rule" one finds the usual transformation rules for the tetrad and gravitino in ordinary supergravity. (In particular the dilation terms in $\omega_\mu{}^{mn}$ and the explicit b_μ in $\delta_Q{}^c\psi_\mu$ cancel against the $\not{b}$ in $\delta_S{}^c$.)

It is thus clear also, why S^2 and P^2 have negative signs and A^2_m has a positive sign in the gauge action of ordinary simple supergravity: because the Brans–Dicke scalars have unphysical kinetic sign.

4.6. Tensor calculus for ordinary supergravity

In the preceding subsection it was shown how to obtain a tensor calculus for conformal simple supergravity. In this section we show how one derives from these results a tensor calculus for ordinary, i.e. Poincaré, supergravity.

Consider, as an example, a scalar multiplet with conformal weight λ. Under conformal supergravity one has

$$\begin{aligned}&\delta_Q{}^c(\epsilon)A = \tfrac{1}{2}\bar{\epsilon}\chi, \qquad \delta_Q{}^c(\epsilon)B = \frac{-\mathrm{i}}{2}\bar{\epsilon}\gamma_5\chi\\ &\delta_Q{}^c(\epsilon)\chi = \tfrac{1}{2}(\not{D}^c(A - \mathrm{i}\gamma_5 B) + F + \mathrm{i}\gamma_5 G)\epsilon\end{aligned} \tag{1}$$

and

$$\begin{aligned}&\delta_S{}^c(\epsilon)A = \delta_S{}^c(\epsilon)B = 0, \qquad \delta_S{}^c(\epsilon)\chi = \lambda(A + \mathrm{i}B\gamma_5)\epsilon\\ &\delta_S{}^c(\epsilon)F = (1-\lambda)\bar{\epsilon}\chi, \qquad \delta_S{}^c(\epsilon)G = -\mathrm{i}(1-\lambda)\bar{\epsilon}\gamma_5\chi.\end{aligned} \tag{2}$$

These results follow from covarianting the results of global supersymmetry, which is unique as we have stressed already several times.

Using the "$Q+S$ rule", the transformation rules of ordinary supergravity are given by

$$\begin{aligned}&\delta^P_Q(\epsilon)A=\tfrac{1}{2}\bar{\epsilon}\chi,\qquad \delta^P_Q(\epsilon)B=-\tfrac{1}{2}\mathrm{i}\bar{\epsilon}\gamma_5\chi\\ &\delta^P_Q(\epsilon)\chi=\tfrac{1}{2}(\not{D}^P(A-\mathrm{i}\gamma_5B)+F+\mathrm{i}\gamma_5G)\epsilon+\tfrac{1}{2}\lambda(A+\mathrm{i}B\gamma_5)\eta\epsilon+\tfrac{1}{4}\lambda\not{A}(B+\mathrm{i}\gamma_5A)\epsilon\end{aligned}\tag{3}$$

plus results for F and G which we will discuss in a moment. The derivatives $D^P_\mu A$ and $D^P_\mu B$ are covariant only with respect to the super-Poincaré algebra, as required in ordinary supergravity, and all axial auxiliary field terms cancel since $\eta=-\frac{1}{3}(S-\mathrm{i}\gamma_5P-\mathrm{i}\not{A}^{\mathrm{aux}}\gamma_5)$ and $A(\text{gauge})=-\frac{2}{3}A(\text{aux})$. However, the result for $\delta^P_Q\chi$ is λ-dependent. From the structure of $\delta^P_Q\chi$ it is clear that a redefinition of fields is possible such that the λ-dependence disappears.

$$F'=F-\tfrac{1}{3}\lambda(AS+BP),\qquad G'=G-\tfrac{1}{3}\lambda(BS-AP).\tag{4}$$

The final task is now to show that the variation of these fields F' and G' is again λ-independent. One has

$$\delta^P_Q(\epsilon)F'=\bar{\epsilon}\not{D}^c\chi+(1-\lambda)(\bar{\epsilon}\eta+\bar{\epsilon}\not{b})\chi-\frac{\lambda}{3}\bar{\epsilon}(S-\mathrm{i}\gamma_5P)\chi-\frac{1}{3}\lambda\frac{1}{2}\bar{\epsilon}(A-\mathrm{i}\gamma_5B)\gamma\cdot R^{\mathrm{cov}}.\tag{5}$$

Since

$$D_\mu{}^c\chi=D^P_\mu\chi-\frac{\lambda}{4}(\not{A}B+\mathrm{i}\gamma_5\not{A}A)\psi_\mu+\left(\frac{\lambda}{2}-\frac{3}{4}\right)A_\mu\mathrm{i}\gamma_5\chi-\lambda(A+\mathrm{i}\gamma_5B)\phi_\mu$$

and (subsection 4.2, eq. (10))

$$\gamma\cdot\phi=-\frac{1}{6}\gamma\cdot R+\frac{\mathrm{i}}{3}\sigma^{\mu\nu}A_\nu\gamma_5\psi_\mu\tag{6}$$

one finds that F' is indeed λ-independent, provided one uses also F', G' on the right hand side. One finds for F' and G'

$$\delta^P_Q(\epsilon)F'=\bar{\epsilon}\left(\not{D}^P-\frac{\mathrm{i}}{2}\not{A}^{\mathrm{aux}}\gamma_5\right)\chi+\bar{\epsilon}\eta\chi,\qquad \delta^P_Q(\epsilon)G'=\mathrm{i}\bar{\epsilon}\gamma_5\left(\not{D}^P-\frac{\mathrm{i}}{2}\not{A}^{\mathrm{aux}}\gamma_5\right)\chi-\mathrm{i}\bar{\epsilon}\gamma_5\eta\chi.\tag{7}$$

These are the results for a local scalar multiplet in ordinary supergravity [527]. (In the literature one often omits the primes on F and G, but we use them to distinguish F from F', see (4).) If one would have known a priori that elimination of λ by redefinition of fields was possible at all, then one could have obtained these results straight away by merely putting $\lambda=0$. The action is (cf. p. 307)

$$I=\int\mathrm{d}^4x\,[eF'+\tfrac{1}{2}e\bar{\psi}\cdot\gamma\chi+\tfrac{1}{2}e\bar{\psi}_\mu\sigma^{\mu\nu}(A-\mathrm{i}\gamma_5B)\psi_\nu+e(SA+PB)].\tag{8}$$

One can apply these considerations for example to the kinetic multiplet $T(\Sigma)$. Since its first component is the same in Poincaré as in conformal supergravity, one expects that $\Box^c A$ does not contain b_μ, since b_μ plays no role in Poincaré supergravity. Indeed, b_μ cancels in $\Box^c A$.

The tensor calculus for supergravity was developed by Ferrara and the author [526, 527, 531] and by Stelle and West [583, 584, 585, 445] as well as others [297, 167, 534]. There are much more details of the tensor calculus, but since an exhaustive review is available [539] we refer the reader to that source.

5. Superspace

5.1. Introduction to superspace

One way to describe supergravity is to consider it as just another gauge theory in ordinary spacetime. The local symmetry between bosons and fermions can even be formulated without anticommuting parameters (for example, $\delta A = \lambda^\beta$ and $\delta\lambda^\alpha = \not{\partial}(A - i\gamma_5 B)^{\alpha\beta}$) and invariance of the action can then still be investigated; it has, in this case, an open index β. Such a theory can be quantized in the usual way and one can compute S-matrix elements. This is the approach we have been following up to now, and it leads to all results in which one is interested.

There is, however, another approach to supergravity theory, and that is superspace. Actually, there are many approaches, and we will discuss them in this order: (i) the approach which builds a bridge between the ordinary space and superspace, (ii) the Wess–Zumino approach, which is geometrical but has too many field components in superspace and needs therefore constraints on the torsion from the outside imposed. These constraints also serve to eliminate fields with spins exceeding two, (iii) the chiral superspace approach of Ogievetski, Sokatchev, Siegel, Gates and others, which is economical in that it deals with two small chirally related superspaces but whose geometry is less usual, and might need constraints for $N > 1$, (iv) gauge supersymmetry of Arnowitt and Nath which was the first superspace approach, but which contains ghosts and higher spin fields.

A major advantage of superspace approaches is their application to quantum gravity, in particular to explicit calculations. In the ordinary space approach one has gauged away many of the fields present in the superspace approach, and consequently one cannot use globally supersymmetric gauges. Also, in the ordinary space approach one cannot use a supersymmetric background field formalism, in which the effective quantum action is invariant under locally supersymmetric background field transformations. In the superspace approaches, these goals can be met. Finally, superspace methods allow the use of supergraphs, which contain many ordinary Feynman graphs, and are easier to use than ordinary Feynman diagrams. However, as with any new formalism, many new problems had to be solved before it could be used which explains that up to now the ordinary space approach has yielded all results first.

Another advantage of superspace methods is that it explains many results of ordinary space in a simple way. To mention a typical example, if one wants to know how a scalar multiplet with external Lorentz indices transforms, one can obtain the result directly in ordinary space [224], but a method based on the use of covariant derivatives in superspace gives more insight. However, if one needs explicit formulae in terms of component fields, for example to check whether there exist supersymmetric extensions of topological invariants [534], then, in the author's opinion, ordinary space methods are more appropriate.

We now discuss general properties of superspace in the remainder of this subsection.

Consider a base manifold consisting of four bosonic coordinates x^μ and $4N$ fermionic coordinates $\theta^{\alpha i}$ where $\alpha = 1, 4$ and $i = 1, N$. The index N refers to N-extended supergravity. For simplicity (and since for $N > 1$ only partial results are known) we restrict ourselves to $N = 1$. As in general relativity, one erects at each point $(x^\mu, \theta^{\alpha i})$ a local tangent frame. In the base manifold one considers general coordinate transformations with parameters Ξ^Λ where $\Lambda = (\mu, \alpha)$. As we shall see, Ξ^μ generate the ordinary general coordinate transformations but Ξ^α *generate local supersymmetry transformations*! The difference between the various superspace approaches is often that a different symmetry group is chosen in tangent space. For example, in the theory of Arnowitt and Nath the tangent group is Osp(3, 1/4N), while in most other approaches the tangent group is the purely bosonic Lorentz group times unitary (or orthogonal) transformations acting on the index i, hence $O(3,1) \times SU(N)$.

There are two geometrical superfields in superspace, namely the supervielbein (viel = many in German) and the superconnection. (In the chiral approach, both are functions of an axial vector superfield. In the Wess–Zumino approach, the superconnections can be expressed in terms of the supervielbein after imposing constraints, see subsection 5. All these approaches use thus second order formalism.) In addition, there may or may not be matter superfields. The supervielbein $V_\Lambda{}^A(x, \theta)$ depends on all 8 coordinates x^μ, θ^α, and the curved (world) index Λ runs over $\Lambda = (\mu, \alpha)$, while the flat index A runs over $A = (m, a)$. As the indices indicate, $V_\mu{}^m$, and $V_\alpha{}^a$ are bosonic objects, and $V_\mu{}^a$ and $V_\alpha{}^m$ are fermionic. That is to say, they are even and odd elements of a Grassmann algebra, respectively. The superconnection $h_\Lambda{}^{mn}(x, \theta)$ is bosonic for $\Lambda = \mu$ and fermionic for $\Lambda = \alpha$. The two flat indices mn refer to the Lorentz group. For $N > 1$ one has also an $O(N)$ connection. It follows that $h_\Lambda{}^{mn}$ is antisymmetric in (mn).

The base manifold and tangent manifold symmetries act on these two geometrical fields in a straightforward generalization of general relativity. Under general coordinate transformations G with parameters $\Xi^\Lambda(x, \theta)$ one has

$$\begin{aligned} \delta_G(\Xi) V_\Lambda{}^A &= (\partial_\Lambda \Xi^\pi) V_\pi{}^A + \Xi^\pi \partial_\pi V_\Lambda{}^A \\ \delta_G(\Xi) h_\Lambda{}^{mn} &= (\partial_\Lambda \Xi^\pi) h_\pi{}^{mn} + \Xi^\pi \partial_\pi h_\Lambda{}^{mn} \end{aligned} \tag{1}$$

where $\partial_\Lambda = \partial/\partial x^\Lambda$ is a *left derivative*. Contractions are defined without extra signs if the upper index is on the left. Each contraction is thus a sum over four bosonic and four (or $4N$) fermionic indices, and ∂_Λ stands for $(\partial/\partial x^\mu, \partial/\partial\theta^\alpha)$. A scalar density D transforms as $\delta D = \partial_\Lambda(\Xi^\Lambda D)(-)^\Lambda$ as one easily checks for sdet V.

The local Lorentz rotations L in the tangent frames with parameters $L^{rs}(x, \theta)$ act on the two basic superfields as

$$\begin{aligned} \delta_L(L) V_\Lambda{}^m &= L^m{}_n V_\Lambda{}^n \\ \delta_L(L) V_\Lambda{}^a &= \tfrac{1}{2}(L^{mn}\sigma_{mn})^a{}_b V_\Lambda{}^b \\ \delta_L(L) h_\Lambda{}^{mn} &= -(\partial_\Lambda L^{mn} + h_\Lambda{}^{mt} L_t{}^n + h_\Lambda{}^{nt} L^m{}_t). \end{aligned} \tag{2}$$

Tensors with indices Λ upstairs and indices A downstairs transform per definition such that $T^A T_A$ and $T^\Lambda T_\Lambda$ are scalars. Thus if $\delta T^A = L^A{}_B T^B$ and $\delta T_A = L_A{}^B T_B$, then $L^A{}_B = -L_B{}^A$. Using our contraction rule we rewrite this as $\delta T^A = -T^B L_B{}^A$. The matrices $L_A{}^B$ are diagonal (have no Bose–Fermi parts) and

$L^A{}_B = \frac{1}{2}L^{mn}(X_{mn})^A{}_B$ act on tensors as in (2). One can now easily show that $\delta^A{}_B$ is an invariant tensor. (Since the parameters $L^{rs}(x, \theta)$ are bosonic it does not matter whether one writes them in front or in the back. In the theory of gauge supersymmetry with tangent group Osp(3, 1/4N) there are fermionic parameters, which leads to complicated extra signs.)

There are $8\times 8\times 16$ ordinary fields in $V_\Lambda{}^A$ (since expansion in θ yields sixteen components) and $8\times 6\times 16$ ordinary fields in $h_\Lambda{}^{mn}$. The number of local symmetries is 8×16 (for ξ^Λ) plus 6×16 (for Λ^{mn}). These numbers are much larger than for the ordinary space approach to supergravity, where one has $22+16$ fields (or $22+16+4\times 6$ fields if $\omega_\mu{}^{mn}$ is an independent field) and $4+4+6$ local symmetries. The question now arises what the relation between the superspace approach and the ordinary space approach is. As we shall see, if one chooses a certain gauge in superspace in which to order θ^0

$$V_\mu{}^m = e_\mu{}^m, \qquad V_\mu{}^a = \psi_\mu{}^a, \qquad h_\mu{}^{mn} = \omega_\mu{}^{mn}(e, \psi) \tag{3}$$

and if one imposes in addition constraints on the supertorsions by hand (i.e., from the outside; they are *not* field equations), then one recovers the ordinary space approach.

Let us conclude this introduction by defining torsion and curvature. The inverse of the supervielbein is defined by

$$V_A{}^\Lambda V_\Lambda{}^B = \delta_A{}^B. \tag{4}$$

As a consequence, also $V_\Lambda{}^B V_B{}^\Pi = \delta_\Lambda{}^\Pi$. However,

$$V_B{}^\Pi V_\Lambda{}^B = (-)^{(B+\Pi)(\Lambda+B)} V_\Lambda{}^B V_B{}^\Pi \tag{5}$$

where in the exponent $\Pi = 0$ for bosonic values and $\Pi = 1$ for fermionic values.

Covariant derivatives are defined by

$$D_\Lambda = \partial_\Lambda + \tfrac{1}{2}h_\Lambda{}^{mn}X_{mn}, \qquad D_A = V_A{}^\Lambda D_\Lambda \tag{6}$$

where X_{mn} are the Lorentz generators. These derivatives are covariant with respect to local Lorentz rotations, but only with respect to general coordinate transformations if one considers curls. They are the kind of derivatives first introduced by Cartan, and one can introduce p-forms in superspace which avoid any extra minus signs. We prefer to exhibit indices explicitly for pedagogical reasons.

It should be stressed that these Cartan covariant derivatives are not the same as the covariant derivatives which one sometimes introduces in gauging groups. For example, in gauging the super Poincaré algebra, we defined in section 3

$$\delta h_\mu{}^A = (D_\mu \epsilon)^A \equiv \partial_\mu \epsilon^A + h_\mu{}^B \epsilon^C f_{CB}{}^A \tag{7}$$

and one sums over all connections $h_\mu{}^B$, not only over the Lorentz connections. Thus, in this way of gauging groups there is double counting: translations appear both in the tangent group and in the base manifold (as general coordinate transformations).

Torsion and curvature are defined by

$$[D_A, D_B\} = R_{AB}{}^{mn}(\tfrac{1}{2}X_{mn}) - 2T_{AB}{}^C D_C. \tag{8}$$

(The factor 1/2 is due to our summing over $m > n$ and $m < n$.) The symbol [} is always a commutator, except when A and B are both fermionic, in which case one has an anticommutator. These definitions agree in the purely bosonic sector with the usual definitions if one notes that

$$D_A V_B{}^\Lambda = -(-)^{A(B+\Pi)} V_B{}^\Pi (D_A V_\Pi{}^C) V_C{}^\Lambda . \tag{9}$$

(This result follows from $D_A(V_B{}^\Lambda V_\Lambda{}^C) = 0$, but one can check signs faster by noting that bringing indices on the right hand side in the same order as on the left, one must pull A to the left of B and Π, yielding the sign $A(B+\Pi)$. In general, one can in this way write equations correctly, and this is the reason differential forms will not be needed below.)

Explicit evaluation yields for the supertorsion

$$2T_{AB}{}^C = (-)^{\Lambda(B+\Pi)} V_A{}^\Lambda V_B{}^\Pi (D_\Lambda V_\Pi{}^C - (-)^{\Lambda\Pi} D_\Pi V_\Lambda{}^C). \tag{10}$$

Note that $T_{ab}{}^C = T_{ba}{}^C$ but $T_{mn}{}^C = -T_{mn}{}^C$ while $T_{ma}{}^C = -T_{am}{}^C$. For the supercurvature one finds

$$R_{AB}{}^{mn} = (-)^{\Lambda(B+\Pi)} V_A{}^\Lambda V_B{}^\Pi \, [\partial_\Lambda h_\Pi{}^{mn} - (-)^{\Lambda\Pi} \partial_\Pi h_\Lambda{}^{mn} + h_\Lambda{}^{mt} h_{\Pi t}{}^n - (-)^{\Lambda\Pi} h_\Pi{}^{mt} h_{\Lambda t}{}^n]. \tag{11}$$

Finally we recapitulate our index convention

	vectors	spinors	vectors & spinors
tangent space	l, m, n	a, b, c	$A = (a, m), B, C$
base space	λ, μ, ν	α, β, γ	$\Lambda = (\alpha, \mu), \Pi, \Sigma$

5.2. Superspace from ordinary space

The Caltech group has shown how to build a bridge between ordinary space and superspace on-shell [398–402]. They use an idea due to Arnowitt and Nath (who called it gauge completion [24] in their application to gauge supersymmetry). The method was extended to the off-shell case by Ferrara [545], van Nieuwenhuizen [551], West [549] and Gates and Castellani [555], who translated ordinary space results into superspace language. Wess and Zumino went the opposite way and showed how the tensor calculus in ordinary space follows from superspace [577].

One begins by choosing a *gauge* in which

$$V_\mu{}^m(x, \theta = 0) = e_\mu{}^m, \qquad V_\mu{}^a(x, \theta = 0) = \psi_\mu{}^a, \qquad h_\mu{}^{mn}(x, \theta = 0) = \omega_\mu{}^{mn}. \tag{1}$$

(We will take $\omega_\mu{}^{mn} = \omega_\mu{}^{mn}(e, \psi)$ as in second order formalism.) This is really a gauge because although one can of course always call $V_\mu{}^m(\theta = 0)$ by the name of $e_\mu{}^m$, one also requires *compatibility*, that is to say that the transformations of left- and right-hand side agree, order by order in θ. Thus one equates the order $\theta = 0$ terms in (capital letters refer to superspace quantities)

$$\Delta V_\mu{}^m = \partial_\mu \Xi^\Lambda V_\Lambda{}^m + \Xi^\Lambda \partial_\Lambda V_\mu{}^m + L^m{}_n V^n{}_\mu \tag{2}$$

with the ordinary space transformation

$$\delta e_\mu{}^m = \partial_\mu \xi^\nu e_\nu{}^m + \xi^\nu \partial_\nu e_\mu{}^m + \tfrac{1}{2}\bar{\epsilon}\gamma^m \psi_\mu + \lambda^m{}_n e_\mu{}^n. \tag{3}$$

In order to do so, one *identifies* the parameters as follows

$$\Xi^\mu(\theta=0) = \xi^\mu, \qquad \Xi^\alpha(\theta=0) = \epsilon^\alpha, \qquad L^{mn}(\theta=0) = \lambda^{mn}. \tag{4}$$

Compatibility for the terms with ξ^μ and λ^{mn} is manifest, but for the terms in ϵ^α one finds

$$(\partial_\mu \epsilon^\alpha) V_\alpha{}^m(\theta=0) + \epsilon^\alpha (\partial_\alpha V_\mu{}^m)_{\theta=0} = \tfrac{1}{2}\bar{\epsilon}\gamma^m \psi_\mu. \tag{5}$$

Thus, $V_\alpha{}^m(\theta=0)=0$ and $V_\mu{}^m = e_\mu{}^m + \frac{1}{2}\bar{\theta}\gamma^m\psi_\mu + \cdots$. One must thus choose a gauge such that these results hold. Whether this can be done in all order θ cases considered below, has not been proven but seems plausible.

Repeating the same procedure for the gravitino, we have (see page 217)

$$\begin{aligned}&(\Delta V_\mu{}^a)_{\theta=0} = \delta\psi_\mu{}^a, \qquad \delta\psi_\mu{}^a = \left(D_\mu + \frac{\mathrm{i}}{2} A_\mu \gamma_5\right)\epsilon - \tfrac{1}{2}\gamma_\mu \eta \epsilon \\ &(\Delta V_\mu{}^a) = (\partial_\mu \Xi^\nu) V_\nu{}^a + (\partial_\mu \Xi^\alpha) V_\alpha{}^a + \Xi^\nu \partial_\nu V_\mu{}^a + \Xi^\alpha \partial_\alpha V_\mu{}^a + \tfrac{1}{2}(L\cdot\sigma)^a{}_b \psi_\mu{}^b.\end{aligned} \tag{6}$$

In this case, $\delta(\sup)\psi_\mu$ must be equal to the $\theta=0$ terms in $(\partial_\mu \Xi^\alpha)V_\alpha{}^a + \Xi^\alpha \partial_\alpha V_\mu{}^a$, and one finds

$$V_\alpha{}^a(\theta=0) = \delta_\alpha{}^a, \qquad V_\mu{}^a = \psi_\mu{}^a + \tfrac{1}{2}(\hat{\omega}_\mu{}^{mn}\sigma_{mn} - \tilde{\eta}\gamma_\mu)\theta^a + \cdots \tag{7}$$

where $\tilde{\eta} = -\frac{1}{3}(S + \mathrm{i}\gamma_5 P - 2\mathrm{i}\not{A}\gamma_5)$ and $\hat{\omega}_\mu{}^{mn}$ is the improved spin connection (the algebra simplifies if one uses $\tilde{\eta}$ and $\tilde{\omega}$ instead of η and ω)

$$\hat{\omega}_\mu{}^{mn} = \omega_\mu{}^{mn}(e,\psi) - \frac{\mathrm{i}}{3}\epsilon_\mu{}^{mnp} A_p. \tag{8}$$

Similarly one can find $h_\mu{}^{mn}$ to order θ, and finds $h_\alpha{}^{mn}(\theta=0)=0$. The gauges $V_\alpha{}^A(\theta=0) = h_\alpha{}^{mn}(\theta=0)=0$ can always be implemented by choosing the order θ parts of Ξ^A and Λ^{mn} approximately.

Since $V_\alpha{}^a(\theta=0)=\delta_\alpha{}^a$ and $V_\alpha{}^m(\theta=0) = h_\alpha{}^{mn}(\theta=0)=0$, one can require also compatibility for these components. One finds then to order $\theta=0$

$$\Delta V_\alpha{}^a = 0 = (\partial_\alpha \Xi^\mu)\psi_\mu{}^a + (\partial_\alpha \Xi^\beta)\delta_\beta{}^a + \xi^\mu \partial_\mu \delta_\alpha{}^a + \epsilon^\beta \partial_\beta V_\alpha{}^a + \tfrac{1}{2}\lambda^{mn}(\sigma_{mn})^a{}_b \delta_\alpha{}^b. \tag{9}$$

Clearly one needs to know first the parameters to order θ, and then one can determine $V_\alpha{}^a$ to order θ from (9).

The superparameters to order θ^{k+1} follow from the knowledge of the superparameters to order θ^k by requiring *compatibility of the parameter composition rules* in ordinary space and in superspace. In superspace a symmetry operation with $(\Xi_2{}^\Lambda, L_2{}^{mn})$ followed by an operation with $(\Xi_1{}^\Lambda, L_1{}^{mn})$ minus $1\leftrightarrow 2$ yields again a symmetry operation with composite parameters

$$\begin{aligned}\Xi_{12}{}^{\Pi} &= (\Xi_2{}^{\Lambda}\partial_{\Lambda}\Xi_1{}^{\Pi} + \bar{\delta}_1\Xi_2{}^{\Pi}) - (1 \leftrightarrow 2)\\ L_{12}{}^{mn} &= (\Xi_2{}^{\Lambda}\partial_{\Lambda}L_1{}^{mn} + L_2{}^{mt}L_{1,t}{}^{n} + \bar{\delta}_1 L_2{}^{mn}) - (1 \leftrightarrow 2).\end{aligned} \tag{10}$$

Since we work in a special gauge, the superparameters will in general depend on $e^m{}_\mu$, $\psi_\mu{}^\alpha$, etc., and one must take the variation of these fields as dictated by the ordinary space approach. This is indicated by the symbol $\bar{\delta}_1$. In ordinary space the parameter composition law is as discussed in section 1 (pages 219, 220)

$$\begin{aligned}\xi_{12}{}^{\mu} &= (\xi_2{}^{\nu}\partial_\nu\xi_1{}^{\mu} + \tfrac{1}{4}\bar{\epsilon}_2\gamma^\mu\epsilon_1) - (1 \leftrightarrow 2)\\ \epsilon_{12}{}^{\alpha} &= (\xi_2{}^{\nu}\partial_\nu\epsilon_1{}^{\alpha} + \tfrac{1}{2}\lambda_2{}^{mn}(\sigma_{mn}\epsilon_1)^\alpha - \tfrac{1}{4}\bar{\epsilon}_2\gamma^\mu\epsilon_1\psi_\mu{}^\alpha) - (1 \leftrightarrow 2)\\ \lambda_{12}{}^{mn} &= (\xi_2{}^{\nu}\partial_\nu\lambda_1{}^{mn} + \lambda_2{}^{mt}\lambda_{1,t}{}^{n} + \tfrac{1}{4}\bar{\epsilon}_2\gamma^\mu\epsilon_1\hat{\omega}_\mu{}^{mn} + \tfrac{1}{6}\bar{\epsilon}_2\sigma^{mn}(S - \mathrm{i}\gamma_5 P)\epsilon_1) - (1 \leftrightarrow 2).\end{aligned} \tag{11}$$

Thus we need in ordinary space a closed algebra, and since this has only been achieved in second order formalism, we use $\omega = \omega(e, \psi)$.

As a result one finds the superparameters to order θ by equating $\Xi_{12}{}^\mu(\theta = 0) = \xi_{12}{}^\mu(\theta = 0)$ and similarly for $\Xi_{12}{}^\alpha$ and $L_{12}{}^{mn}$, and solving these equations. Using these order θ parameters, one can obtain the remaining order θ components of the supervielbein. The results are given in the following table:

Table

$$\Xi^\mu = \xi^\mu + \tfrac{1}{4}\bar{\theta}\gamma^\mu\epsilon, \qquad \Xi^\alpha = \epsilon^\alpha + \tfrac{1}{4}\bar{\epsilon}\gamma^\mu\theta\psi_\mu{}^\alpha - \tfrac{1}{2}(\lambda\cdot\sigma\theta)^\alpha$$

$$L^{mn} = \lambda^{mn} + \tfrac{1}{4}\bar{\theta}[\gamma^\mu\hat{\omega}_\mu{}^{mn} + \tfrac{2}{3}\sigma^{mn}(S - \mathrm{i}\gamma_5 P)]\epsilon$$

$$V_\Lambda{}^A = \begin{pmatrix} e^m{}_\mu + \tfrac{1}{2}\bar{\theta}\gamma^m\psi_\mu & \psi_\mu{}^a + \tfrac{1}{2}(\hat{\omega}_\mu\cdot\sigma - \bar{\eta}\gamma_\mu)\theta \\ -\tfrac{1}{4}(\bar{\theta}\gamma^m)_\alpha & \delta_\alpha{}^a \end{pmatrix}.$$

In order to obtain parameters to order θ^2, one evaluates $[\Delta_2, \Delta_1]$ on, say, the field $V_\mu{}^m$, and in $\Delta_2\Delta_1 V_\mu{}^m$ the variation Δ_2 not only acts on $V_\mu{}^m$ as dictated by superspace but also on the fields of ordinary space according to δ_2. It is essential to *consider always the full groups in superspace and ordinary space.* For example, $[\Delta(\Xi^\alpha), \Delta(L^{mn})]$ is again a Lorentz rotation in superspace, but in ordinary space $[\delta(\epsilon^\alpha), \delta(\lambda^{mn})]$ is a supersymmetry transformation.

The general procedure is to obtain first the parameters to order θ^{k+1} from those to order θ^k. Then $V_\mu{}^A$ and $h_\mu{}^{mn}$ to order θ^{k+1} follow from the superparameters to order θ^k and $V_\alpha{}^A$ and $h_\alpha{}^{mn}$ follow from the superparameters to order θ^{k+1}. It is remarkable that in solving for the superparameters one finds differential equations whose equations must satisfy certain fermionic integrability conditions. They always do. Also, the superparameter solutions are not unique but contain arbitrary integration constants. For example, solving

$$\epsilon_2{}^\alpha\partial_\alpha\Xi_1{}^\mu - \epsilon_1{}^\alpha\partial_\alpha\Xi_2^\mu = \tfrac{1}{2}\bar{\epsilon}_2\gamma^\mu\epsilon_1 \tag{12}$$

the general solution is

$$\Xi^\mu(\theta) = -\tfrac{1}{4}\bar{\epsilon}\gamma^\mu\theta + \bar{\epsilon}\theta H_\mu + \bar{\epsilon}\gamma_5\theta K_\mu + \bar{\epsilon}\gamma_5\gamma^\nu\theta L_{\mu\nu} \tag{13}$$

with arbitrary H, K and L. We will discuss below an example of conformal superspace supergravity where these integration constants are nonzero and become fixed at the next θ-level. In most cases, however, they are zero.

Suppose one has obtained the supervielbein to order θ, then one can construct the supertorsion to order $\theta = 0$. Similarly, knowing the superconnection to order θ, one can find the supercurvatures to order $\theta = 0$. One can now find tensor relations between supertorsion components to all order in θ by simply first determining their relations to order $\theta = 0$ and then using that such relations must hold to all order in θ due to the following theorem.

Theorem: a *tensor* which vanishes at $\theta = 0$ vanishes for all θ.

The proof is simple: in the transformation rule of the tensor all terms vanish, except the fermionic transport term ($\epsilon^\alpha \partial_\alpha$ times the tensor). Thus, by induction, the tensor vanishes at all order in θ. Physically this important theorem is clear: the origin in one coordinate system is not the origin in another, and by requiring that a tensor vanishes at the origin of any coordinate system, it vanishes everywhere.

It turns out that (to order $\theta = 0$ in any case) the supertorsion and supercurvature are independent of the integration constants. This is to be expected, but a proof is lacking. If one chooses $\omega_\mu{}^{mn} = \omega_\mu{}^{mn}(e, \psi)$ as in second order formalism, one finds, first to order $\theta = 0$ and thus to all order in θ, the following *tensor relations*

$$T_{ab}{}^{c} = T_{ar}{}^{s} = T_{rs}{}^{t} = T_{ab}{}^{r} + \tfrac{1}{4}(C\gamma^r)_{ab} = 0. \tag{14}$$

These are the constraints Wess and Zumino postulated for $N = 1$ superspace supergravity. If one uses instead the improved spin connection $\hat{\omega}_\mu{}^{mn}$, one finds the constraints of Ogievetski and Sokatchev. If one were to be diligent, one could compute $V_\Lambda{}^A$ to all orders in θ (i.e., to order θ^4), then obtain sdet V, and then obtain from the action $\int d^4x\, d^4\theta$ sdet V the ordinary space action (by taking the θ^4 component of sdet V). This action should then agree with the action of the ordinary space approach. (Incidentally, one can directly obtain sdet V to all orders in θ, since sdet V is also a tensor. Similarly one can determine the supertorsions and supercurvatures to higher order in θ once they are known to order $\theta = 0$, without having to determine $V_\Lambda{}^A$, since also these geometrical objects are tensors in superspace.)

In this approach we have translated results of the ordinary space approach into results for the superspace approach (one could also go the other way). Crucial was the identification of $\Xi^\Lambda(\theta = 0)$ with the *field-independent* parameters $(\xi^\mu, \epsilon^\alpha)$. In other approaches [578] one sometimes chooses $\Xi^A = V^A{}_\Lambda \Xi^\Lambda$ as field-independent. We have obtained the constraints of $N = 1$ superspace supergravity, assuming that the gauge algebra in ordinary space closes, i.e., with the presence of the auxiliary fields. For $N = 2$ the constraints were in a similar way found from the auxiliary fields for $N = 2$ supergravity. Some of them read

$$T_{rs}{}^{t} = T_{ai,s}{}^{t} = T^{\dot{C}k}_{\dot{A}i,Bj} = T^{Ck}_{\dot{A}i,Bk} = 0 \tag{15}$$

but at this moment it seems that constraints quadratic in T's appear [555].

For conformal supergravity in superspace, one can play the same game [549]. In superspace one has now, in addition to the spacetime symmetries $\Delta_G(\Xi)$ and $\Delta_L(L)$, two more symmetries, namely local chiral rotations $\Delta_A(\Lambda_A)$ and local scale transformations $\Delta_D(\Lambda_D)$. By requiring compatibility of the gauge algebras in ordinary space and superspace and identifying $\Lambda_A(\theta = 0) = \lambda_A$ and $\Lambda_D(\theta = 0) = \lambda_D$, one finds also Λ_A and Λ_D to order θ. One must keep here integration constants proportional to ϵ_Q. One finds that the S-supersymmetry parameter ϵ_S of the ordinary space approach appears at the order θ level in Λ_A

and Λ_D, but $\xi_K{}^m$ does not seem to appear in the superparameters. The ϵ_S parameters appear in Λ_A and Λ_D such that for these terms $\Lambda_D \pm (\mathrm{i}/2)\Lambda_A$ are chiral, i.e.,

$$(1 \mp \gamma_5)^\alpha{}_\beta \frac{\partial}{\partial \bar{\theta}_\beta}\left(\Lambda_D \pm \frac{\mathrm{i}}{2}\Lambda_A\right) = 0. \tag{16}$$

Requiring that this relation is generally true at the order θ-level (and not only for the ϵ_S parameters) one finds compatibility for $\Delta V_\mu{}^m$ and $\delta e^m{}_\mu$ if one defines $\Delta(\Lambda_A, \Lambda_D) V_\Lambda{}^m = -\Lambda_D V_\Lambda{}^m$. Compatibility for $\Delta V_\mu{}^a$ and $\delta\psi_\mu{}^a$ as far as the ϵ_S terms are concerned, can only be achieved if one defines chiral rotations by

$$\begin{aligned}\Delta(\Lambda_A, \Lambda_D) V_\Lambda{}^a = {} & V_\Lambda{}^m \gamma_m \tfrac{1}{2}(1+\gamma_5)\frac{\partial}{\partial\bar{\theta}}\left(-\Lambda_D - \frac{\mathrm{i}}{2}\Lambda_A\right) + \left(\frac{3\mathrm{i}}{4}\Lambda_A\gamma_5 - \frac{1}{2}\Lambda_D\right) V_\Lambda{}^a \\ & + V_\Lambda{}^m \gamma_m \frac{1}{2}(1-\gamma_5)\frac{\partial}{\partial\bar{\theta}}\left(-\Lambda_D + \frac{\mathrm{i}}{2}\Lambda_A\right).\end{aligned} \tag{17}$$

These transformations are just the transformations which Howe and Tucker [498] found in superspace by requiring that they leave the Wess–Zumino constraints in (14) invariant (provided one replaces $\partial/\partial\bar{\theta}_\alpha$ by D^α). Thus the kinematics of $N = 1$ superspace supergravity is even conformal invariant, but the dynamics may or may not be, depending on whether one takes an action which is invariant under these Howe–Tucker transformations or not. For the superconnection one finds analogous results

$$\Delta(\Lambda_D, \Lambda_A) h_\Lambda{}^{mn} = -\tfrac{1}{2}(V_\Lambda{}^m V^{n\Pi}\partial_\Pi \Lambda_D - m \leftrightarrow n) + V_\Lambda{}^{\cdot} C\sigma^{mn}\tfrac{1}{2}(1+\gamma_5)\frac{\partial}{\partial\bar{\theta}}\left(-\Lambda_D - \frac{\mathrm{i}}{2}\Lambda_A\right) + \text{c.c.} \tag{18}$$

For the superdeterminant one finds with $\delta\,\mathrm{sdet}\, V = \mathrm{sdet}\, V(V_A{}^\Lambda \delta V_\Lambda{}^A)(-)^A$ that $\delta(\mathrm{sdet}\, V) = -2\Lambda_D(\mathrm{sdet}\, V)$.

Since torsions and curvatures can be expressed in terms of $V_\Lambda{}^A$ and $h_\Lambda{}^{mn}$, one can also determine how they transform under $\Delta(\Lambda_A, \Lambda_D)$. In particular, since, as we shall see, all torsions and curvatures are functions of three arbitrary superfields R, G_m and W_{ABC} only, one can find how these three superfields transform. One finds*

$$\begin{aligned}&\delta R = (2\Sigma^* - 4\Sigma)R - \tfrac{1}{8}\bar{D}(1-\gamma_5)D\Sigma^* \\ &\delta G_m = -(\Sigma + \Sigma^*)G_m + \mathrm{i}D_m(\Sigma^* - \Sigma) \\ &\delta W_{ABC} = -3\Sigma W_{ABC}, \qquad \Sigma = -\frac{1}{2}\Lambda_D - \frac{\mathrm{i}}{4}\Lambda_A.\end{aligned} \tag{19}$$

The superfields R, G_m, W_{ABC} are the superspace equivalents of three similar multiplets in the ordinary space approach, and are discussed further in subsection 5.

A slightly different approach is due to Lindström and Roček [365, 367], who constructed $V_a{}^\Lambda$, Ξ^Λ and $(D^{\dot{A}} D_{\dot{A}} + \frac{1}{3}R)$ to all orders in θ in terms of chiral coordinates by translating the kinetic multiplet of ordinary space discussed in subsection 4.4 into superspace.

* Note that $D_{\dot{A}}R = D_{\dot{A}}\Sigma = D_{\dot{A}}(D^{\dot{B}}D_{\dot{B}} - 8R)U = 0$ for any U.

*5.3. Flat superspace**

Supertorsion and supercurvatures are defined by the (anti) commutator of two covariant derivatives with flat indices. In global supersymmetry one can also define covariant derivatives, and from their (anti) commutators one finds that *global supersymmetry has torsion but no curvature.* This flat space limit of curved superspace was obtained in the last subsection by using gauge completion; in particular we found that $V_\alpha{}^m = -\frac{1}{4}(\bar\theta\gamma^m)_\alpha$. Here we will obtain the same result by deriving the covariant derivatives in global supersymmetry and by using the method of induced representations, following Salam and Strathdee.

Consider a group element of the form

$$g_{\mathrm{L}}(x, \theta, \eta) = \exp(\bar\theta Q + x^m P_m + \tfrac{1}{2}\eta^{mn} M_{mn}). \tag{1}$$

Consider next a new group element, obtained by left multiplication

$$g_{\mathrm{L}}(x', \theta', \eta') = \exp(\bar\epsilon Q + \xi^m P_m + \tfrac{1}{2}\lambda^{mn} M_{mn})\, g_{\mathrm{L}}(x, \theta, \eta). \tag{2}$$

Using the Baker–Hausdorff formula, one finds an induced representation of the generators P, Q, M on the coordinates x, θ, η of the group manifold, which is 14-dimensional and infinitely nonlinear.

To find an induced representation on superspace with coordinates x^m, θ^α we consider the coset spaces

$$\exp(\bar\theta Q + x^m P_m) \quad \mathrm{mod}\, M. \tag{3}$$

By mod M we mean $\exp \frac{1}{2}\lambda^{mn} M_{mn}$ with any λ^{mn}. Any group element of the form (1) belongs to a coset. To see this explicitly, note that, using

$$\mathrm{e}^{A+B} = \mathrm{e}^{-[A,B]/2+\cdots}\mathrm{e}^A \mathrm{e}^B \tag{4}$$

where $A = \bar\theta Q + x\cdot P$ and $B = \frac{1}{2}\eta\cdot M$, we can factor the M dependence out. This is possible, since in the multiple commutators $[[A, B], B]\ldots$ one always encounters only P and Q while $[M, M] \sim M$. (In other words, $M \oplus (P+Q)$ is a semidirect sum. The decomposition $(M + Q_{\mathrm{L}}) \oplus (P + Q_{\mathrm{R}})$ yields a symmetric algebra.) We now consider

$$g_{\mathrm{L}}(x', \theta') = \exp(\bar\epsilon Q + \xi\cdot P + \tfrac{1}{2}\lambda\cdot M)\exp(\bar\theta\cdot Q + x\cdot P) \quad \mathrm{mod}\, M \tag{5}$$

with antihermitian Q, P, M. Using†

$$\mathrm{e}^A \mathrm{e}^B = \mathrm{e}^{A+B+[A,B]/2+\cdots} \qquad \text{if } \lambda^{mn} = 0 \tag{6}$$

$$\mathrm{e}^A \mathrm{e}^B = \mathrm{e}^{B+[A,B]+\cdots}\mathrm{e}^A \qquad \text{if } \epsilon = \xi = 0 \tag{7}$$

* I thank Dr. Roček for useful discussions concerning this subsection.

† Eq. (6) is the usual form of the Baker–Hausdorff theorem. Eq. (7) follows if one expands $\mathrm{e}^A\mathrm{e}^B\mathrm{e}^{-A} = \exp(\mathrm{e}^A B\, \mathrm{e}^{-A})$ in terms of B. Eq. (4) follows from (6) if one uses (6) twice.

one finds the following induced representation of the generators P, Q, M on the coordinates x, θ

$$
\begin{aligned}
&\delta^{\mathrm{L}}_{Q}(\epsilon)\theta^{\alpha} = \epsilon^{\alpha}, && \delta^{\mathrm{L}}_{Q}(\epsilon)\,x^{m} = \tfrac{1}{4}\bar{\epsilon}\gamma^{m}\theta \\
&\delta^{\mathrm{L}}_{P}(\xi)\theta^{\alpha} = 0, && \delta^{\mathrm{L}}_{P}(\xi)\,x^{m} = \xi^{m} \\
&\delta^{\mathrm{L}}_{M}(\lambda)\theta^{\alpha} = \tfrac{1}{2}\lambda^{mn}(\sigma_{mn}\theta)^{\alpha}, && \delta^{\mathrm{L}}_{M}(\lambda)\,x^{m} = \lambda^{m}{}_{n}x^{n}.
\end{aligned}
\tag{8}
$$

We have used the anticommutator $\{Q^{\alpha}, Q^{\beta}\} = \frac{1}{2}(\gamma^{m}C^{-1})^{\alpha\beta}P_{m}$. Note that (8) is an 8-dimensional nonlinear transitive representation of the supersymmetry algebra on (x^{m}, θ^{α}). The stability subgroup of any fixed point $(x_{0}^{\mu}, \theta_{0}^{\alpha})$ of this representation space is isomorphic to the Lorentz generators M_{mn} (as one easily sees by considering the origin (0, 0)). Consequently, the representation space x, θ is isomorphic to the coset space G/M $\sim P + Q$. *This coset approach forces one to the Lorentz group as tangent group in supergravity.*

Define now a scalar superfield $\phi(x, \theta)$ by

$$\phi'(x', \theta') = \phi(x, \theta). \tag{9}$$

Let $\phi'(x, \theta)$ be related to $\phi(x, \theta)$ by a unitary transformation with operators Q^{L}, P^{L}, M^{L} which at this stage have nothing to do with Q, P, M. Then, infinitesimally, the commutator

$$\delta^{\mathrm{L}}\phi(x, \ = \ [\bar{\epsilon}Q^{\mathrm{L}} + \xi^{m}P^{\mathrm{L}}_{m} + \tfrac{1}{2}\lambda^{mn}M^{\mathrm{L}}_{mn}, \phi(x, \theta)] \tag{10}$$

is, according to (9), equal to

$$\delta\phi = -(\delta\theta^{\alpha}\,\partial/\partial\theta^{\alpha} + \delta x^{m}\,\partial/\partial x^{m})\phi. \tag{11}$$

Substituting (8), one finds

$$
\begin{aligned}
&\delta^{\mathrm{L}}_{Q}\phi = [\bar{\epsilon}Q^{\mathrm{L}}, \phi] = -\bar{\epsilon}\left(\frac{\partial}{\partial\bar{\theta}} + \tfrac{1}{4}\gamma^{m}\theta\partial_{m}\right)\phi = -(\bar{\epsilon}D)\phi \\
&\delta^{\mathrm{L}}_{P}\phi = [\xi^{m}P^{\mathrm{L}}_{m}, \phi] = -\xi^{m}\,\partial_{m}\phi \\
&\delta^{\mathrm{L}}_{M}\phi = [\tfrac{1}{2}\lambda^{mn}M^{\mathrm{L}}_{mn}, \phi] = -\left(\tfrac{1}{2}\lambda^{mn}(\sigma_{mn}\theta)^{\alpha}\,\frac{\partial}{\partial\theta^{\alpha}} + \lambda^{m}{}_{n}x^{n}\,\frac{\partial}{\partial x^{m}}\right)\phi.
\end{aligned}
\tag{12}
$$

Before interpreting these results, we repeat the preceding analysis, but now using right multiplication. Thus

$$g_{\mathrm{R}}(x', \theta') = (\mathrm{mod}\,M)\exp(\bar{\theta}Q + x\cdot P)\exp(-\bar{\epsilon}Q - \xi\cdot P - \tfrac{1}{2}\lambda\cdot M). \tag{13}$$

Now one finds

$$
\begin{aligned}
&\delta^{\mathrm{R}}_{Q}(\epsilon)\theta^{\alpha} = -\epsilon^{\alpha}, && \delta^{\mathrm{R}}_{Q}(\epsilon)x^{m} = +\tfrac{1}{4}\bar{\epsilon}\gamma^{m}\theta \\
&\delta^{\mathrm{R}}_{P}(\xi)\theta^{\alpha} = 0, && \delta^{\mathrm{R}}_{P}(\xi)x^{m} = -\xi^{m} \\
&\delta^{\mathrm{R}}_{M}(\lambda)\theta^{\alpha} = +\tfrac{1}{2}\lambda\cdot\sigma\theta, && \delta^{\mathrm{R}}_{M}(\lambda)x^{m} = +\lambda^{m}{}_{n}x^{n}
\end{aligned}
\tag{14}
$$

which again constitutes a representation of the supersymmetry algebra on (x^m, θ^α). Finally, with (9) and introducing Q^R, P^R, M^R as in (10) one finds

$$\begin{aligned}
\delta^R\phi(x,\theta) &= [\bar\epsilon Q^R + \xi^m P^R_m + \tfrac{1}{2}\lambda^{mn}M^R_{mn}, \phi(x,\theta)] \\
\delta^R_Q\phi &= [\bar\epsilon Q^R, \phi] = +\bar\epsilon\Big(\frac{\partial}{\partial\bar\theta} - \frac{1}{4}\gamma^m\theta\partial_m\Big)\phi = -(\bar\epsilon G)\phi \\
\delta^R_P\phi &= [\xi^m\cdot P^R_m, \phi] = +\xi^m\partial_m\phi \\
\delta^R_M\phi &= \Big[\frac{1}{2}\lambda\cdot M^R, \phi\Big] = -\Big(\frac{1}{2}\lambda\cdot\sigma\theta\frac{\partial}{\partial\theta} + \lambda^m{}_n x^n\frac{\partial}{\partial x^m}\Big)\phi.
\end{aligned} \tag{15}$$

As one may verify, the operators Q^R, P^R, M^R satisfy the super-Poincaré algebra. In particular

$$\{G^\alpha, G^\beta\} = \tfrac{1}{2}(\not\partial C^{-1})^{\alpha\beta} = \tfrac{1}{2}(\gamma^m C^{-1})^{\alpha\beta}P^R_m. \tag{16}$$

Also the operators in (12) form a representation of the supersymmetry algebra on fields. In particular

$$\{D^\alpha, D^\beta\} = -\tfrac{1}{2}(\gamma^m C^{-1})^{\alpha\beta}\partial_m = +\tfrac{1}{2}(\gamma^m C^{-1})^{\alpha\beta}P^L_m. \tag{17}$$

In fact, D^α and G^β anticommute, as one may easily verify directly. This is no accident but due to the fact that the induced representations on the coset space are associative

$$[(\exp\bar\epsilon^L Q)g]\exp\bar\epsilon^R Q = \exp\bar\epsilon^L Q[g(\exp\bar\epsilon^R Q)]. \tag{18}$$

Hence on fields

$$(\exp\bar\epsilon^R G)(\exp-\bar\epsilon^L D)\phi(\exp\bar\epsilon^L D)(\exp-\bar\epsilon^R G) = (\text{same with } D\leftrightarrow G) \tag{19}$$

from which it indeed follows that $\{D^\alpha, G^\beta\} = 0$.

The interpretation of these results is the following. G^α generates supersymmetry transformations. (Note that on ordinary fields $\delta_P A = [A, P_\mu] = \delta_\mu A$.) For example, on a superfield $V(x,\theta) = C + \bar\theta Z + \frac{1}{2}\bar\theta\theta H$ etc. one has from $\delta(\epsilon)V = \bar\epsilon G V$, upon equating the coefficients of definite powers of θ

$$\delta C = \bar\epsilon Z, \qquad \delta Z = \tfrac{1}{4}\not D C\epsilon \tag{20}$$

$$[\delta(\epsilon_1), \delta(\epsilon_2)]C = \tfrac{1}{2}\bar\epsilon_2\gamma^m\epsilon_1\partial_m C \tag{21}$$

and one has indeed a representation of the algebra of global supersymmetry on the components of $V(x,\theta)$. Since D^α commutes with G^β, *we identify D^α with the spinor part of the covariant derivative.* Indeed, covariant derivatives are per definition derivatives such that, for any tensor T, $D^\alpha T$ transforms as T, and this means that D^α must commute with G^β. For the vector part of the covariant derivative one easily finds that $D_\mu = \partial/\partial x^\mu$. (There are better but more complicated arguments.)

The connection with the covariant derivatives D_A of curved superspace is now easily established

$$D_A = \left(\frac{\partial}{\partial\theta^a} + \frac{1}{4}(\bar{\theta}\not{\partial})_a, \frac{\partial}{\partial x^m}\right), \qquad D_A = \left(\frac{\partial}{\partial\theta^\alpha}, \frac{\partial}{\partial x^\mu}\right). \tag{22}$$

Note that we consider (x, θ) as "cartesian" coordinates for which no connection is needed in the base manifold. (Even for rigid translations in curvilinear coordinates one needs connections.) The flat supervielbein reads thus

$$V_A{}^A(x, \theta) = \begin{pmatrix} \delta_m{}^\mu & 0 \\ \frac{1}{4}(\bar{\theta}\gamma^\mu)_a & \delta_a{}^\alpha \end{pmatrix}. \tag{23}$$

Since supervielbeins can be considered as determining the orientations of local superframes with respect to the base manifold, this result states that a particular orientation (which is supercoordinate dependent) leads to derivatives D_A which are very convenient (they commute with supersymmetry). This is indeed the flat limit of the supervielbein as obtained by gauge completion. This flat supervielbein and the vanishing spin connection are a solution of the fields equations of supergravity (but not of those of gauge supersymmetry at the classical level). If one computes the supertorsion and supercurvature according to the formulae in subsection 1, then one finds that there is a nonvanishing component of supertorsion (see page 316)

$$T_{ab}{}^m = -\tfrac{1}{4}(C\gamma^m)_{ab}. \tag{24}$$

Thus, flat supersymmetry has torsion but no curvature.

5.4. Construction of multiplets using the closed gauge algebra

We show here how one can use superspace methods to extend a tensor in ordinary space with external Lorentz or internal indices to a full multiplet of which this tensor is the first component. For example, in the supersymmetric Maxwell–Einstein system there is a scalar multiplet Σ_α which starts with $\bar{\lambda}_\alpha$, but the other components do not rotate into each other in exactly the same way as the components of the canonical multiplet $\Sigma = (A, B, \chi, F', G')$ of subsection 4.6. This one might already expect from the fact that the commutator of two local supersymmetry variations contains a local Lorentz rotation which does not act on A but does act on $\bar{\lambda}_\alpha$. Hence local supersymmetry transformations for Σ_α must be modified.

The method is based on the known anticommutation relations of the covariant derivatives D_A. Since to order $\theta = 0$ in global supersymmetry D^α is equal to the supersymmetry generator G^α (see the previous subsection) while in local supersymmetry this is still true for superfield tensors with flat indices, one defines, to begin with

$$\delta_{\text{sup}}(\epsilon)\, A(x, \theta = 0) = (\bar{\epsilon}C^{-1}D)\, A(x, \theta)|_{\theta=0}. \tag{1}$$

We use complex notations where A stands for $A - \mathrm{i}B$, and repeat that A may have external indices. Using a Fierz rearrangement (see appendix) one finds for the commutator of two spinorial covariant derivatives, not forgetting while Fierzing that D_a and D_b do not anticommute

$$[D_a\epsilon^a, D_b\eta^b] = -\tfrac{1}{2}(\bar{\epsilon}\gamma^m\eta)(D\gamma_m C^{-1}D) + (\bar{\epsilon}\sigma^{mn}\eta)(D\sigma_{mn}C^{-1}D)$$

$$= \epsilon^a(2T_{ab}{}^cD_c - R_{ab}{}^{mn}\tfrac{1}{2}X_{mn})\eta^b + \text{internal symmetries}. \tag{2}$$

We recall that D_a starts with $\partial/\partial\theta^a$ and $\sigma^{mn} = \frac{1}{4}[\gamma^m, \gamma^n]$. From the explicit expressions for the supertorsions and super constraints of $N=1$ superspace supergravity one finds upon equating the coefficients of $\bar{\epsilon}O_j\eta$ (using that $h_\alpha{}^{mn} = [(-\mathrm{i}/12)\bar{\theta}\gamma_s A_t\epsilon^{mnst} + \frac{1}{6}\bar{\theta}\sigma^{mn}(S - \mathrm{i}\gamma_5 P)]_\alpha$ and

$$R_{ab}{}^{mn} = \frac{1}{3}(C\sigma^{mn})(S - \mathrm{i}\gamma_5 P)_{ab} - \frac{\mathrm{i}}{6}(C\gamma_r)_{ab}A_s\epsilon^{mnrs}$$

as follows easily from gauge completion) that

$$D\gamma_m C^{-1}D = +D_m - \frac{\mathrm{i}}{6}\epsilon_{mnpq}G^nX^{pq} \tag{3}$$

$$D\sigma_{mn}\tfrac{1}{2}(1+\gamma_5)C^{-1}D = -\tfrac{1}{6}RX^{(\mathrm{R})}_{mn} \qquad \text{with } X^{(\mathrm{R})}_{mn} = \tfrac{1}{2}X_{mn} - \tfrac{1}{4}\epsilon_{mnpq}X^{pq} \tag{4}$$

$$R = S - \mathrm{i}P + \mathcal{O}(\theta), \qquad G_m = A_m + \mathcal{O}(\theta) \tag{5}$$

and a similar result with R replaced by R^* and the + and − signs interchanged.

From now on we consider chiral multiplets and chiral superfields. The superfield A is then restricted by

$$[(1+\gamma_5)C^{-1}D]^a A(x, \theta) = 0. \tag{6}$$

From (4) one immediately deduces with (5) that

$$X^{(\mathrm{R})}_{mn}A(x, \theta) = 0. \tag{7}$$

Since $(1-\gamma_5)C^{-1}D$ is not constrained, we define a new superfield χ^a which has the same extra indices as A

$$\chi^a = [\tfrac{1}{2}(1-\gamma_5)C^{-1}D]^a A. \tag{8}$$

Clearly $\chi = \frac{1}{2}(1-\gamma_5)\chi$. From two superfields $A(1)$ and $A(2)$ one can also construct the product $A(12) = A(1)A(2)$. Then (8) leads to $\chi^a(12) = A(1)\chi^a(2) + A(2)\chi^a(1)$. This is the beginning of the product formula of two multiplets.

We now apply (3) to A. Since A is chiral, (3) yields with (7)

$$D\gamma_m \frac{1}{2}(1-\gamma_5)\chi = +D_m A - \frac{\mathrm{i}}{6}(X^{(\mathrm{L})}_{mn}A)G^n. \tag{9}$$

Acting with the complex conjugate of (4) on A yields

$$D\sigma_{mn}\tfrac{1}{2}(1-\gamma_5)\chi = -\tfrac{1}{6}R^* X^{(\mathrm{L})}_{mn}A. \tag{10}$$

From (9) and (10) one knows how D_a acts on $\gamma_m\chi$, $\gamma_m\gamma_5\chi$ and $\sigma_{mn}\chi$. Thus, since χ is chiral, we can define a new superfield F as follows

$$F(x,\theta) = +D\tfrac{1}{2}(1-\gamma_5)\chi \tag{11}$$

where the complex $F(x, \theta = 0)$ corresponds to $F - \mathrm{i}G$ of the multiplet Σ of Poincaré supergravity. For the product rule we now find

$$F(12) = A(1)\,F(2) + A(2)\,F(1) - \bar{\chi}(1)\,(1-\gamma_5)\,\chi(2). \tag{12}$$

We can now determine how χ^a transforms by using (1). Combining (9), (10) and the definition in (11), we get

$$\bar{\epsilon}C^{-1}D\tfrac{1}{2}(1-\gamma_5)\chi = \tfrac{1}{2}\Big(D_mA - \frac{\mathrm{i}}{6}G^n X^{(\mathrm{L})}_{mn}A\Big)\gamma^m\tfrac{1}{2}(1+\gamma_5)\epsilon + \tfrac{1}{2}\Big(F + \frac{1}{12}R^* X^{(\mathrm{L})}_{mn}\sigma^{mn}\Big)\tfrac{1}{2}(1-\gamma_5)\epsilon. \tag{13}$$

One sees here clearly that if A has extra Lorentz indices, then $\delta_{\mathrm{sup}}(\epsilon)\chi$, as given by the $\theta = 0$ part of (13), is modified by terms proportional to Lorentz rotations on A.

Next we determine how F transforms. As before we apply (3) and the complex conjugate of (4) to $\tfrac{1}{2}(1-\gamma_5)\chi$. We do this by acting on (13) with $D\gamma^m$ and $D\sigma_{mn}\tfrac{1}{2}(1-\gamma_5)$ respectively and find

$$(D\gamma^m C^{-1}D)\tfrac{1}{2}(1-\gamma_5)\chi = [D\gamma^m(-\gamma^n D_nA + \cdots F)\tfrac{1}{2}(1-\gamma_5)\,(\tfrac{1}{2}C^{-1})]^{\mathrm{T}} \tag{14}$$

$$(D\sigma^{mn}\tfrac{1}{2}(1\pm\gamma_5)C^{-1}D)\tfrac{1}{2}(1-\gamma_5)\chi = [D\sigma^{mn}\tfrac{1}{2}(1\pm\gamma_5)\,(\gamma^n D_nA + \cdots)\,(-\tfrac{1}{2}C^{-1})]^{\mathrm{T}}. \tag{15}$$

Using on the left-hand side (3) and (4), one finds that D_aF is fully determined (already by (14) alone, (15) yields a consistency test). The only unknown at this point is the double derivative on A, which, however, can also be obtained using that

$$D_aD_mA = D_mD_aA + [D_a, D_m]A. \tag{16}$$

From $[D_a, D_m] = -2T_{am}{}^bD_b + R_{am}{}^{pq}\tfrac{1}{2}X_{pq}$ one sees that in the transformation law of χ a term enters, proportional to the gravitino curl when A has external Lorentz indices.

In this way we have a multiplet of superfields which transform into each other under the action of D_a. The correspondence with the multiplets in ordinary spacetime is simply that the order $\theta = 0$ components of the superfields are the components while the action of D_a is a supersymmetry transformation. Only if A has no external indices, one can see that $F^* + RA$ again satisfies the same chiral constraint as A does, namely eq. (6). Hence, it can be viewed as the lowest component of a new multiplet, namely the kinetic multiplet.

Similar ideas can be used for vector multiplets (non-chiral superfields A) or for linear multiplets corresponding to superfields and satisfying

$$DC^{-1}DA = (D\gamma_5 C^{-1}D)A = (D\sigma_{mn}C^{-1}D)A = 0. \tag{17}$$

For applications of these ideas to supergravity see P. Breitenlohner [68, 71].

5.5. The Wess–Zumino approach to superspace

In this approach, the tangent group is the ordinary Lorentz group. The geometry is thus: general supercoordinate transformation in the base manifold, local Lorentz rotations in the tangent manifold, and one has supervielbeins $V_\Lambda{}^A$ and superconnection $h_\Lambda{}^{mn}$. These superfields are not arbitrary, but satisfy the following constraints (as we deduced earlier from the ordinary spacetime approach but now postulate)

$$T_{rs}{}^t = T_{as}{}^t = T_{ab}{}^c = T_{ab}{}^m + \tfrac{1}{4}(C\gamma^m)_{ab} = 0. \tag{1}$$

We now discuss how one might arrive at postulating them. First of all, notice [605] that for a tensor w^A

$$\partial_\Lambda(\text{sdet } Vw^A V_A{}^\Lambda)(-)^\Lambda = \text{sdet } V(D_A w^A + 2w^B T_{BA}{}^A)(-)^A. \tag{2}$$

To prove this relation, note that $\delta(\text{sdet } V) = (\text{sdet } V)(\delta V_\Lambda{}^A)V_A{}^\Lambda(-)^\Lambda$ so that for a general coordinate transformation

$$\delta(\text{sdet } V) = \partial_\Lambda(\Xi^\Lambda \text{ sdet } V)(-)^\Lambda. \tag{3}$$

This expression can be written as

$$\begin{aligned}(\delta V_\Lambda{}^A)V_A{}^\Lambda(-)^\Lambda &= (\Xi^\Pi\partial_\Pi V_\Lambda{}^A + \partial_\Lambda \Xi^\Pi V_\Pi{}^A)V_A{}^\Lambda(-)^\Lambda \\ &= \partial_\Lambda(\Xi^\Pi V_\Pi{}^A)V_A{}^\Lambda(-)^\Lambda + \Xi^\Pi(\partial_\Pi V_\Lambda{}^A - (-)^{\Lambda\Pi}\partial_\Lambda V_\Pi{}^A)V_A{}^\Lambda(-)^\Lambda.\end{aligned} \tag{4}$$

One can replace ∂_Λ by D_Λ in both terms, while the extra terms needed in replacing ∂_Π by D_Π cancel since $(X_{mn})^A{}_B$ is traceless in (AB). Identifying w^A with $\Xi^\Pi V_\Pi{}^A$ the result in (2) follows.

Hence in order that one has just as in general relativity

$$\int d^4x\, d^4\theta \text{ sdet } V D_A w^A (-)^A = 0 \tag{5}$$

one has the requirement that $T_{BA}{}^A(-)^A = 0$. For $B = b$ this follows from the constraints, for $B = m$ it is true if $T_{ma}{}^a = 0$. This is indeed the case and follows from the Bianchi identities [578] if one assumes the constraints on the torsion. We will discuss this below.

Another interpretation is that chiral superfields can live in superspace [224, 421, 452, 453]. Indeed, if

$$(1-\gamma_5)^a{}_b D^b \phi(x,\theta) \equiv D_{\dot{A}}\phi(x,\theta) = 0 \tag{6}$$

then also $\{D_{\dot{A}}, D_{\dot{B}}\}\phi = 0$. From the definition of torsion and curvature this implies

$$T_{\dot{A}\dot{B}}{}^C = T_{\dot{A}\dot{B}}{}^r = 0. \tag{7}$$

The first constraint is part of $T_{ab}{}^c = 0$, the second follows from $T_{ab}{}^r + \frac{1}{4}(C\gamma^r)_{ab} = 0$. Actually, if $D_{\dot{A}}\phi_{CD} = 0$ one also finds $R_{\dot{A}\dot{B}}{}^{mn} = 0$ and also this constraint follows from the torsion constraints as a consequence of the Bianchi identities. Thus the set of constraints imposed on the supertorsion seems reasonable, but they are postulated, not derived from some general principle.

The Bianchi identities follow from

$$[[D_A, D_B\}, D_C\}(-)^{AC} + 2 \text{ cyclic terms} = 0 \tag{8}$$

by taking separately the terms proportional to D_E or $D_{\dot{E}}$. Thus one finds

$$\begin{aligned} &D_A R_{BC}{}^{mn} + 2T_{AB}{}^D R_{DC}{}^{mn} = 0 \\ &D_A T_{BC}{}^D + 2T_{AB}{}^E T_{EC}{}^D + \tfrac{1}{2}R_{ABC}{}^D = 0 \end{aligned} \tag{9}$$

where one must add terms cyclic in ABC with the proper signs. The symbol $R_{ABC}{}^D D_D$ is $R_{ABm}{}^n D_n$ if D_D is bosonic and $R_{AB}{}^{mn} D_D(-\frac{1}{2}\sigma_{mn})^D{}_C$ if D_D is fermionic. The minus signs are due to the fact that the index D of D_D is a lower index.

As an example, consider the following identity

$$[\{D_A, D_B\}, D_{\dot{C}}] + [\{D_B, D_{\dot{C}}\}, D_A] + [\{D_{\dot{C}}, D_A\}, D_B] = 0 \tag{10}$$

where A, B, $\dot{C}$ are 2-component spinor indices (see appendix). The terms proportional to $D_{\dot{D}}$ yield with $(C\gamma^r)_{AB} = 0$

$$R_{AB}{}^{mn}\tfrac{1}{2}[X_{mn}, D_{\dot{C}}] + [(C\gamma^m)_{B\dot{C}} T_{mA}{}^{\dot{D}} + A \leftrightarrow B] D_{\dot{D}}. \tag{11}$$

Since $[X_{mn}, D_a] = -(\sigma_{mn})^b{}_a D_b$ one has in 2-component notation

$$[X_{mn}, D_{\dot{C}}] = \tfrac{1}{4}(\sigma_{m,\,\dot{C}E}\sigma_n{}^{E\dot{F}} - m \leftrightarrow n) D_{\dot{F}}. \tag{12}$$

Moreover

$$\begin{aligned} &R_{AB}{}^{mn} = \tfrac{1}{4}\sigma^{m,E\dot{E}}\sigma^{n,F\dot{F}} R_{AB,E\dot{E},F\dot{F}} \\ &R_{AB,E\dot{E},F\dot{F}} = \epsilon_{\dot{E}\dot{F}} R_{AB,EF} + \epsilon_{EF} R_{AB,\dot{E}\dot{F}}. \end{aligned} \tag{13}$$

We arrive at (symmetric parts are underlined)

$$R_{\underline{A}\dot{B}\underline{C}\dot{D}} \sim T_{B\dot{C},A,\dot{D}} + T_{A\dot{C},B,\dot{D}}. \tag{14}$$

Decomposing the torsion as

$$T_{\dot{D}\dot{G}GA} = T_{\underline{\dot{D}\dot{G}GA}} + T_{\underline{\dot{D}\dot{G}}}\epsilon_{GA} + T_{\underline{GA}}\epsilon_{\dot{D}\dot{G}} + \epsilon_{\dot{D}\dot{G}}\epsilon_{GA}(-2\mathrm{i}R) \tag{15}$$

and substituting this back and multiplying by $\epsilon^{\dot{A}\dot{G}}\epsilon^{AG}$ one finds $T_{\dot{D}\dot{G}} = 0$. Similar manipulations lead to $T_{GA} = T_{AB\dot{C}\dot{D}} = 0$. Finally, with $D_{\dot{C}}$ in (10) replaced by D_m, a similar analysis yields

$$\begin{aligned} &R_{\underline{\dot{E}\dot{D}\dot{G}\dot{A}}} \sim (\epsilon_{\dot{E}\dot{A}}\epsilon_{\dot{D}\dot{G}} + \epsilon_{\dot{D}\dot{A}}\epsilon_{\dot{E}\dot{G}})R \\ &R_{\dot{E}\dot{D}GA} = 0. \end{aligned} \tag{16}$$

This example shows how to use Bianchi identities; much work in superspace is based on deducing consequences of Bianchi identities.

After working out all Bianchi identities, one arrives at the following remarkable result [578]: *all supertorsions and supercurvatures can be expressed in terms of three superfields*

$$R, \quad G_{A\dot{A}}, \quad W_{ABC}.$$

Moreover $G_{A\dot{A}}$ is Hermitean, W_{ABC} totally symmetric while

$$\begin{aligned} &D_{\dot{A}}R = D_{\dot{A}}W_{BCD} = 0, \qquad D^{A}G_{A\dot{A}} = D_{\dot{A}}R^{*} \\ &D^{A}W_{ABC} = D_{B}{}^{\dot{E}}G_{C\dot{E}} + B \leftrightarrow C. \end{aligned} \tag{17}$$

The conditions with $D_{\dot{A}}$ state that R and W_{ABC} are chiral superfields; the last condition shows that differentiating W_{ABC} (so that for example the curl of A_μ enters without θ factor) yields the same result as differentiating $G_{A\dot{B}}$. (Indeed, $G_\mu\ (x, \theta = 0) = A_\mu$ and again its curl is produced.) The second relation has a similar meaning; it corresponds in ordinary space-time to the fact that the trace of the Einstein tensor is proportional to the scalar curvature.

The formulas expressing $T_{AB}{}^{C}$ and $R_{AB}{}^{mn}$ in terms of R, G_m and W_{ABC} are complicated and will not be reproduced here.

In Einstein theory the geometry is specified by stating that it is Riemannian. In superspace supergravity we defined the geometry by the restriction that the tangent space parameters are the Lorentz parameters (to appreciate how drastic this choice is, see subsection 7)

$$\epsilon_m{}^n = L^m{}_n, \qquad \epsilon_a{}^b = \tfrac{1}{2}L^{mn}(\sigma_{mn})^a{}_b, \qquad \epsilon_m{}^a = \epsilon_a{}^m = 0 \tag{18}$$

and by the torsion constraints.

Since the tangent group is degenerate the constraints in (1) on the supertorsion lead to two results

(i) all spin connections can be expressed in terms of supervielbeins. Thus enough components of $T_{AB}{}^{C}$ are given to solve the torsion equation. Specifically, from $T_{rs}{}^{t} = 0$ one obtains $h_{rs}{}^{t}$, just as in general relativity, while from $T_{AB}{}^{C} = 0$ one finds $h_{AB}{}^{C}$ (and from $T_{\dot{A}\dot{B}}{}^{\dot{C}} = 0$ one finds $h_{\dot{A}\dot{B}}{}^{\dot{C}}$). From the six constraints $T_{A\dot{B}\dot{C}} + T_{A\dot{C}\dot{B}} = 0$ one finds the six $h_{A\dot{B}\dot{C}}$, using that $h_{\dot{B}A\dot{C}} = h_{\dot{C}A\dot{B}} = 0$.

(ii) The other constraints are therefore constraints on the supervielbein. (In general relativity such a result is not present since there the tangent group is non-degenerate.) In particular one can express $V_m{}^{\Lambda}$ in terms of $V_a{}^{\Lambda}$. The $V_a{}^{\Lambda}$ can in turn be expressed into prepotentials [273].

We now turn to dynamics. The action of $N = 1$ superspace supergravity is simply [574]

$$I = \int \mathrm{d}^4x\ \mathrm{d}^4\theta\ \mathrm{sdet}\ V. \tag{19}$$

Let us see what its field equations are. From sdet $V = \exp \operatorname{str} \ln V$ one has

$$\delta I = \int d^4x\, d^4\theta\, (V_A{}^\Lambda \delta V_\Lambda{}^A)(-)^a \operatorname{sdet} V, \tag{20}$$

but the variations $\delta V_\Lambda{}^A$ must respect the torsion constraints. From the definition of $T_{BC}{}^A$ given earlier it is straightforward to derive (using the index convention of page 313)

$$\delta T_{BC}{}^A = \tfrac{1}{2}D_B H_C{}^A - \tfrac{1}{2}(-)^{BC} D_C H_B{}^A + T_{BC}{}^D H_D{}^A - H_B{}^D T_{DC}{}^A + (-)^{BC} H_C{}^D T_{DB}{}^A + \Omega_{BC}{}^A - (-)^{BC}\Omega_{CB}{}^A \tag{21}$$

where $H_A{}^B = V_A{}^\Lambda \delta V_\Lambda{}^B$ while $\Omega_{BC}{}^A = V_B{}^\Lambda \delta\phi_{\Lambda C}{}^A$

$$\begin{aligned} D_\Lambda V_\Pi{}^A &= \partial_\Lambda V_\Pi{}^A + (-)^{\Lambda(B+\Pi)} V_\Pi{}^B \phi_{\Lambda B}{}^A \\ D_\Lambda V_A{}^\Pi &= \partial_\Lambda V_A{}^\Pi - \phi_{\Lambda A}{}^B V_{B\Pi}. \end{aligned} \tag{22}$$

The signs follow by first making the purely bosonic sector agree, and then moving indices such that they appear on the left and right hand side in the same order. As a consequence, the Leibniz rule

$$\partial_\Lambda (v^A u_A) = (\partial_\Lambda v^A) u_A + (-)^{\Lambda A} v^A \partial_\Lambda u_A \tag{23}$$

holds also for covariant derivatives. Of course the $\phi_{\Lambda B}{}^A$ can be expressed in terms of the earlier defined spin connections $h_\Lambda{}^{mn}$ (compare with (18)).

One can solve this equation for $H_A{}^B$ and finds then

$$\delta \int d^4x\, d^4\theta \operatorname{sdet} V = \int d^4x\, d^4\theta \operatorname{sdet} V[v^m T_{ma}{}^a - RU - R^* U^*] \tag{24}$$

where v^m and U are arbitrary superfields. Since $T_{ma}{}^a$ is proportional to G_m, one has as field equations

$$G_m = R = 0. \tag{25}$$

These are the $N = 1$ superspace supergravity field equations. Since, as we shall show presently, $R = S + iP + \cdots$ and $G_m = A_m^{\text{aux}} + \cdots$ they agree with the ordinary space approach.

To show how the Wess–Zumino approach can reproduce the ordinary space approach we establish the bridge by requiring compatibility as before [577].

For any superfield V (with any number of tangent space indices) a transformation δ is defined by

$$\delta V = \xi^A D_A V = \xi^A V_A{}^\Lambda \partial_\Lambda V + \xi^A h_A{}^{mn}(\tfrac{1}{2}X_{mn} V). \tag{26}$$

Hence, δ is the sum of a general coordinate transformation with field-dependent parameter $\xi^\Lambda = \xi^A V_A{}^\Lambda$ and a local Lorentz transformation with field-dependent parameter $\xi^A h_A{}^{mn}$. The advantage of this definition is that the commutator is proportional to only supertorsion and super curvature

$$[\delta_1, \delta_2] V = \xi_2^B \xi_1^A [D_A, D_B\} = \xi_2^B \xi_1^A (R_{AB}{}^{mn} \tfrac{1}{2} X_{mn} - 2T_{AB}{}^C D_C) V. \tag{27}$$

For the vielbein itself one has an extra term due to the index Λ,

$$\delta V_\Lambda{}^A = D_\Lambda \xi^A + 2 V_\Lambda{}^C \xi^B T_{BC}{}^A. \tag{28}$$

As in subsection 2, we choose at $\theta = 0$ a gauge with

$$V_\mu{}^m = e^m{}_\mu, \qquad V_\mu{}^a = \psi_\mu{}^a, \qquad V_\alpha{}^a = \delta_\alpha{}^a, \qquad V_\alpha{}^m = 0. \tag{29}$$

Using the constraints on the supertorsion, one finds that $h_\alpha{}^{mn}(\theta = 0) = 0$ while $h_\mu{}^{mn}(\theta = 0)$ agrees with the connection in ordinary supergravity.

Supersymmetry transformations are now defined by $\xi^m(\theta = 0) = 0$ and $\xi^a(\theta = 0) = \epsilon^a$ with $\epsilon^a(x)$ the usual four-component spinorial parameter of ordinary supergravity. These transformations maintain the gauge $V_\alpha{}^m = 0$ if

$$\partial_\alpha \xi^m + V_\alpha{}^C \xi^b T_{bC}{}^m = 0 \tag{30}$$

hence with $T_{ba}{}^m = -\frac{1}{4}(C\gamma^m)_{ba}$ and $T_{bn}{}^m = 0$ one finds

$$\xi^m(x, \theta) = \xi^m(x) + \tfrac{1}{4}\bar{\theta}\gamma^m \epsilon + \mathcal{O}(\theta^2) \tag{31}$$

precisely as before. On the other hand, the condition $V_\alpha{}^a = \delta_\alpha{}^a$ is maintained provided the order θ term in $\xi^a(x, \theta)$ vanishes. This also coincides with the Caltech approach, since $\xi^a = \xi^\Lambda V_\Lambda{}^a = \xi^\mu \psi_\mu{}^a + \delta_\alpha{}^a \xi^\alpha$ and $\xi^\mu = -\frac{1}{4}\bar{\epsilon}\gamma^\mu\theta$ while $\xi^\alpha(x, \theta) = \epsilon^\alpha + \frac{1}{4}\psi_\mu{}^\alpha(\bar{\epsilon}\gamma^\mu\theta)$.

Let us now see how the tetrad and gravitino at order θ transform. One finds for the tetrad:

$$\delta V_\mu{}^m = 2\psi_\mu{}^c \epsilon^b T_{bc}{}^m + 2 e_\mu{}^n \epsilon^b T_{bn}{}^m. \tag{32}$$

With the constraints $T_{bc}{}^m = -\frac{1}{4}(C\gamma^m)_{bc}$ and $T_{bn}{}^m = 0$ this precisely reproduces the law $\delta e^m{}_\mu = \frac{1}{2}\bar{\epsilon}\gamma^m\psi_\mu$.

For the gravitino one finds an interesting result

$$\delta\psi_\mu{}^a = (D_\mu \epsilon)^a + 2 e^m{}_\mu \epsilon^b T_{bm}{}^a. \tag{33}$$

The torsion components $T_{bm}{}^a$ can be expressed by means of the Bianchi identities in terms of the super fields R and G_m

$$\begin{aligned} &T_{\dot{B},M\dot{M},A} \sim \epsilon_{\dot{B}\dot{M}}\epsilon_{MA} R \\ &T_{B,M\dot{M},A} \sim \epsilon_{MA} G_{B\dot{M}} - 3\epsilon_{BA} G_{M\dot{M}} - 3\epsilon_{BM} G_{A\dot{M}}. \end{aligned} \tag{34}$$

Identifying at $\theta = 0$ the auxiliary fields by $R \sim S + \mathrm{i}P$ and $G_m \sim A_\mu^{\mathrm{aux}}$ one finds the result of ordinary supergravity back

$$\delta\psi_\mu{}^a = \left(D_\mu + \frac{\mathrm{i}}{2} A_\mu \gamma_5\right)\epsilon + \tfrac{1}{6}\gamma_\mu (S - \mathrm{i}\gamma_5 P - \mathrm{i}\not{A}\gamma_5)\epsilon. \tag{35}$$

Thus the minimal set of auxiliary fields S, P, A_m follows from the superspace approach if one imposes the constraints on supertorsion. Moreover, the constraints themselves follow if one requires that supersymmetry transformations maintain the gauge at $\theta = 0$, and if one requires that superspace reproduces the transformation rules of ordinary supergravity.

5.6. Chiral superspace

The auxiliary fields of ordinary $N = 1$ supergravity in ordinary space fit into a real vector superfield $H^\mu(x, \theta)$

$$H^\mu(x, \theta) = C^\mu + \bar{\theta} Z^\mu + \tfrac{1}{2}\bar{\theta}\theta \hat{H}^\mu + \tfrac{1}{2}\bar{\theta}\mathrm{i}\gamma_5\theta K^\mu + \tfrac{1}{2}\bar{\theta}\mathrm{i}\gamma^m\gamma^5\theta e_m{}^\mu + \bar{\theta}\theta\bar{\theta}\psi^\mu + \bar{\theta}\theta\bar{\theta}\theta A^\mu. \tag{1}$$

Namely $P = \partial_\mu \hat{H}^\mu$, $S = \partial_\mu K^\mu$ and $C^\mu = Z^\mu = 0$ in a special gauge. (To avoid confusion between the superfield $H(x, \theta)$ and the ordinary field $\hat{H}^\mu(x)$, we have put a hat on the latter.) Thus an axial vector superfield seems to have some significance for supergravity. The superspace approaches discussed before have the drawback of involving a large number of fields and a large symmetry group (general coordinate transformations in superspace in addition to ordinary Lorentz rotations) so that one must choose constraints and a particular gauge to establish compatibility with ordinary supergravity. (For some purposes, for example for the construction of invariant objects, the larger symmetry may be useful.) An approach which uses from the start fewer fields and a smaller symmetry group is due to Ogievetski and Sokatchev [346–360], and Siegel and Gates [407–428]. (See also Roček and Lindström [368].) They consider two chiral complex superspaces which are related by complex conjugation. Such spaces were also considered by Brink, Gell-Mann, Ramond and Schwarz [401]. The coordinates in each space are complex and the symmetry group in each separate space is complex general coordinate transformations

$$(x^\mu{}_\mathrm{L}, \theta^\mu{}_\mathrm{L}), \qquad \delta x^\mu{}_\mathrm{L} = \lambda^\mu(x_\mathrm{L}, \theta_\mathrm{L}), \qquad \delta\theta^\mu{}_\mathrm{L} = \lambda^\alpha(x_\mathrm{L}, \theta_\mathrm{L}). \tag{2}$$

This approach is typically tailored for four dimensions, since it uses dotted and undotted indices. It is not clear how to extend it to higher dimensions. The right-handed coordinates do not transform independently but as the complex conjugate of these rules. One now identifies

$$\begin{aligned} &\theta_\mathrm{L} = \tfrac{1}{2}(1+\gamma_5)\theta, \qquad \theta_\mathrm{R} \equiv (\tfrac{1}{2}(1-\gamma_5)\theta)^*, \qquad \tfrac{1}{2}(x^\mu{}_\mathrm{L} + x^\mu{}_\mathrm{R}) = x^\mu \\ &x^\mu{}_\mathrm{L} - x^\mu{}_\mathrm{R} = 2\mathrm{i}H^\mu(x, \theta) \end{aligned} \tag{3}$$

where $H^\mu(x, \theta)$ is a function of $x^\mu{}_\mathrm{L} + x^\mu{}_\mathrm{R}$, θ_L and θ_R. Thus, the imaginary part of the coordinates x_L and x_R is interpreted as an axial super field while the real part is identified with true spacetime. Of course these transformations form a group (two general coordinate transformations lead again to a general coordinate transformation). However, for Einstein supergravity one restricts the group to the super volume preserving transformations in each of the chiral spaces, i.e., one requires that both super Jacobians equal unity (see appendix),

$$\frac{\partial}{\partial x^\mu{}_\mathrm{L}} \lambda^\mu(x_\mathrm{L}, \theta_\mathrm{L}) - \frac{\partial}{\partial \theta^\alpha{}_\mathrm{L}} \lambda^\alpha(x_\mathrm{L}, \theta_\mathrm{L}) = 0. \tag{4}$$

Since super Jacobians satisfy the product rule, one can require that only the product or quotient of the super Jacobians of each chiral space be constraint. This adds global chiral or scale transformations to the symmetry group. If one has no constraint on λ^μ and λ^α at all, one finds that this leads to conformal supergravity.

Thus the transformations in the chiral superspaces can be written as (see appendix E)

$$x'^\mu = x^\mu + \tfrac{1}{2}\lambda^\mu(x^\nu + \mathrm{i}H^\nu(x,\theta), \tfrac{1}{2}(1+\gamma_5)\theta) + \tfrac{1}{2}\lambda^{\mu *}(x^\nu - \mathrm{i}H^\nu(x,\theta), \theta_\mathrm{R}) \tag{5}$$

and similarly for θ'^α. For the axial super field one has

$$H'^\mu(x',\theta') = \frac{1}{2\mathrm{i}}(x'^\mu{}_\mathrm{L} - x'^\mu{}_\mathrm{R}) = \frac{1}{2\mathrm{i}}(x^\mu{}_\mathrm{L} - x^\mu{}_\mathrm{R}) + \cdots$$

$$= H^\mu(x,\theta) + \left\{\frac{1}{2\mathrm{i}}\lambda^\mu(x+\mathrm{i}H(x,\theta), \tfrac{1}{2}(1+\gamma_5)\theta) + \text{c.c.}\right\}. \tag{6}$$

The crucial point is now that by expressing x' and θ' in terms of x, θ one gets an equation for $\delta H^\mu(x,\theta)$ which determines how the components of $H^\mu(x,\theta)$ transform. After choosing a gauge which puts the first two components of $H^\mu(x,\theta)$ equal to zero, there remains still a class of transformations which maintains this gauge. As we shall see, these transformations are just general coordinate, local Lorentz and local supersymmetry transformations, while the remaining components of H^μ are the fields $e^m{}_\mu$, $\psi^a{}_\mu$, S, P, A_m of simple pure supergravity with auxiliary fields. Thus the minimal set of auxiliary fields arises naturally from this chiral superspace approach. However, although this approach is perhaps the closest possible to the ordinary approach in normal space-time, one still has to fix the gauge such that H^μ takes the form mentioned above. On the other hand, no constraints are needed here, whereas in other superspace approaches they are needed (for one reason, to exclude spin 5/2, 3 etc.).

The geometric picture is thus that one has in the combined $8+4$ dimensional space $(x_\mathrm{L}, x_\mathrm{R}, \theta_\mathrm{L}, \theta_\mathrm{R})$ a $4+4$ dimensional hypersurface defined by

$$x^\mu{}_\mathrm{L} - x^\mu{}_\mathrm{R} = 2\mathrm{i}H^\mu(x^\mu{}_\mathrm{L} + x^\mu{}_\mathrm{R}, \theta_\mathrm{L}, \theta_\mathrm{R}). \tag{7}$$

When one shifts points by a general coordinate transformation, the hypersurface itself is deformed according to (6) such that the new points lie on the new hypersurface.

The idea to consider the imaginary part of the bosonic coordinates of the chiral superspaces as a dynamical field, i.e. eq. (7), comes from global supersymmetry. There, a chiral superfield is defined by

$$D_a(1-\gamma_5)^a{}_b\phi(x,\theta) = 0, \qquad D_a = (\partial/\partial\theta^a + \tfrac{1}{4}(\bar\theta\not\partial)_a) \tag{8}$$

whose general solution is easily seen to be

$$\phi = \phi(x_\mathrm{L}, \theta_\mathrm{L}), \qquad x^\mu{}_\mathrm{L} = x^\mu - \tfrac{1}{8}\bar\theta\gamma^\mu\gamma^5\theta, \qquad \theta_\mathrm{L} = \tfrac{1}{2}(1+\gamma_5)\theta. \tag{9}$$

Usually one considers the $x^\mu{}_\mathrm{L}$ and $x^\mu{}_\mathrm{R} = x^\mu + \tfrac{1}{8}\bar\theta\gamma^\mu\gamma^5\theta$ as some convenient basis inside the $4+4$ dimensional space (x^μ, θ^α). However, in order to generalize

$$x^\mu{}_\mathrm{L} - x^\mu{}_\mathrm{R} = -\tfrac{1}{4}\bar\theta\gamma^\mu\gamma^5\theta \tag{7a}$$

to curved space, we consider (7a) as a flat hypersurface in the $4+8$ dimensional space spanned by $x^\mu{}_L$, $x^\mu{}_R$, θ_L, θ_R. The global supersymmetry transformations on the coordinates (x^μ, θ^α) in eq. (8) of subsection 3 induce the transformations

$$\begin{aligned}
\delta\theta_L &= \epsilon_L + \tfrac{1}{2}\lambda\cdot\sigma\theta_L, &\quad \delta x^m{}_L &= \tfrac{1}{4}\bar\epsilon\gamma^m\theta_L + \xi^m + \lambda^m{}_n x^n{}_L\\
\delta\theta_R &= \epsilon_R + \tfrac{1}{2}(\lambda\cdot\sigma)^*\theta_R, &\quad \delta x^m{}_R &= \tfrac{1}{4}\bar\epsilon\gamma^m\theta_R + \xi^m + \lambda^m{}_n x^n{}_R.
\end{aligned} \tag{10}$$

By rewriting these results in two-component notation one finds that $\delta x^m{}_R = (\delta x^m{}_L)^*$ and $\delta\theta_R = (\delta\theta_L)^*$. To go to the local case, one replaces these combined global transformations by one local function (just as in the case of ordinary gravity) as in (2). One requires also in the local case that right-handed and left-handed transformations are related by complex conjugation. Finally, $\bar\theta\gamma^m\gamma^5\theta$ being the flat space limit of $H^\mu(x,\theta)$ in (1), it is natural to generalize (7a) to (7).

Let us now describe how this chiral superspace approach indeed describes ordinary supergravity. First one performs a general super coordinate transformation such that the components of $H^\mu(x,\theta)$ with no and one θ factor are zero. *The set of transformations which maintain this gauge have parameters which are field-dependent.* The same phenomenon one encounters in the previous approaches to superspace. The advantage of this gauge choice for $H^\mu(x,\theta)$ is that then the transformation rule,

$$\begin{aligned}
\delta H^\mu(x,\theta) = {}& \frac{1}{2\mathrm{i}}(\lambda^\mu(x+\mathrm{i}H,\theta_L) - \lambda^{*\mu}(x-\mathrm{i}H,\theta_R)) - \frac{1}{2}(\lambda^\nu(x+\mathrm{i}H,\theta_L) + \lambda^{*\nu}(x-\mathrm{i}H,\theta_R))\partial_\nu H^\mu(x,\theta)\\
&- \left[\lambda^\alpha(x+\mathrm{i}H,\theta_L)\frac{\partial}{\partial\theta^\alpha{}_L} + \lambda^{*\alpha}(x-\mathrm{i}H,\theta_R)\frac{\partial}{\partial\theta^\alpha{}_R}\right]H^\mu(x,\theta)
\end{aligned} \tag{11}$$

which is a nonlinear realization of the left- and right-handed supergroup on $H^\mu(x,\theta)$, becomes polynomial (though still nonlinear) in $H^\mu(x,\theta)$. The parameters are expanded in terms of θ_L (or θ_R) as

$$\begin{aligned}
\lambda^\mu &= -a^\mu + \mathrm{i}b^\mu + \bar\theta_L\varphi^\mu{}_L + \bar\theta_L\theta_L(s^\mu + \mathrm{i}p^\mu), \qquad \text{with } \bar\theta_L\theta_L = \theta_L{}^{\mathrm{T}}C\theta_L\\
\lambda^\alpha &= \epsilon^\alpha{}_L + \omega_L{}^\alpha{}_\beta\theta^\beta{}_L + \bar\theta_L\theta_L\eta^\alpha{}_L
\end{aligned} \tag{12}$$

where $a^\mu,\dots,\eta^\alpha$ still depend on $x^\mu{}_L = x^\mu + \mathrm{i}H^\mu$. The bispinor $\omega_L{}^\alpha{}_\beta$ is equal to $\frac{1}{2}(1+\gamma_5)^\alpha{}_\rho\omega_L{}^\rho{}_\sigma\frac{1}{2}(1+\gamma_5)^\sigma{}_\beta$; in other words, viewed as a 4×4 matrix it only has nonzero elements in the first two rows and columns in a representation in which γ_5 is diagonal.

Let us now consider the field-independent terms in δH^μ

$$\delta H^\mu(x,\theta) = \frac{1}{2\mathrm{i}}[2\mathrm{i}b^\mu + \bar\theta_L\varphi^\mu - \bar\theta_R\varphi^{*\mu} + \bar\theta_L\theta_L(s^\mu+\mathrm{i}p^\mu) - \bar\theta_R\theta_R(s^\mu - \mathrm{i}p^\mu)] + \cdots, \qquad \bar\theta_{L,\alpha} = \theta_L{}^\beta C_{\beta\alpha} \tag{13}$$

where all $b^\mu,\dots,p^\mu$ now only depend on x. Clearly there is enough freedom in parameters to choose a gauge such that $C^\mu(x)$ and $Z^\mu(x)$ in $H^\mu(x,\theta)$ are zero. Let us assume this has been done. Then the full δH^μ is polynomial, since third powers of H^μ vanish.

The set of transformations which maintain this gauge have $b^\mu = 0$ while for the vanishing of the order θ terms in $\delta H^\mu(x,\theta)$ one finds the conditions

$$\bar{\theta}_{\mathrm{L}}\varphi_{\mathrm{L}}-\bar{\theta}_{\mathrm{R}}\varphi_{\mathrm{R}}=2\mathrm{i}\left[\epsilon^{\alpha}{}_{\mathrm{L}}\frac{\partial}{\partial\theta^{\alpha}{}_{\mathrm{L}}}+\epsilon^{\alpha}{}_{\mathrm{R}}\frac{\partial}{\partial\theta^{\alpha}{}_{\mathrm{R}}}\right]\left[\frac{1}{2}\bar{\theta}\theta\hat{H}^{\mu}\text{ plus }K^{\mu}\text{ and }e_{m}{}^{\mu}\text{ terms}\right].\tag{14}$$

Hence, the parameters φ^{μ} are expressed in terms of dynamical fields. These conditions are part of the total set of gauge conditions in this approach.

The condition that volumes are preserved under coordinate transformations expresses some parameters in terms of others. One finds that

$$\begin{aligned}&\frac{\partial}{\partial x^{\mu}{}_{\mathrm{L}}}s^{\mu}(x_{\mathrm{L}})=\frac{\partial}{\partial x^{\mu}{}_{\mathrm{L}}}p^{\mu}(x_{\mathrm{L}})=0\\&\frac{\partial}{\partial x^{\mu}{}_{\mathrm{L}}}\varphi^{\alpha}{}_{\mathrm{L}}{}^{\mu}(x_{\mathrm{L}})+2\eta^{\alpha}{}_{\mathrm{L}}(x_{\mathrm{L}})=0\\&\frac{\partial}{\partial x^{\mu}{}_{\mathrm{L}}}(-a^{\mu}(x_{\mathrm{L}})+\mathrm{i}b^{\mu}(x_{\mathrm{L}}))-\omega_{\mathrm{L}}{}^{\alpha}{}_{\alpha}=0.\end{aligned}\tag{15}$$

(If one requires that only the product (quotient) of both superdeterminants is constant, then one finds in the trace of $\omega_{\mathrm{L}}{}^{\alpha}{}_{\beta}$ in 2-component notation the extra *constant* term $\mathrm{i}b(b)$ which can be shown to generate global chiral (scale) transformations.) Thus s^{μ} and p^{μ} are transverse, and these transverse fields appear in δH^{μ} in the order $\bar{\theta}\theta$ terms as

$$\delta H^{\mu}(x,\theta)=\tfrac{1}{2}\bar{\theta}\theta p^{\mu}-\tfrac{1}{2}\bar{\theta}\mathrm{i}\gamma_{5}\theta s^{\mu}+\cdots.\tag{16}$$

Hence, we can gauge away also the transversal parts of $\hat{H}^{\mu}$ and K^{μ}. This completes the set of gauge conditions. We introduce new symbols for the longitudinal parts of $\hat{H}^{\mu}$ and K^{μ}

$$P=\partial_{\mu}\hat{H}^{\mu},\qquad S=\partial_{\mu}K^{\mu}.\tag{17}$$

At this point all fields that remain are S, P, A^{μ}, $e_{m}{}^{\mu}$ and $\psi^{\alpha\mu}$. The remaining independent gauge parameters are thus ϵ^{α}, a^{μ} and traceless part of $\omega_{\mathrm{L}}{}^{\alpha}{}_{\beta}$. The latter is an element of SL(2, C), hence we can expand it on a complete basis

$$\omega_{\mathrm{L}}{}^{\alpha}{}_{\beta}-\tfrac{1}{2}\delta^{\alpha}{}_{\beta}\omega_{\mathrm{L}}{}^{\gamma}{}_{\gamma}=(\sigma_{mn})^{\alpha}{}_{\beta}\Omega^{mn}\tag{18}$$

with real coefficients Ω^{mn} (see appendix). These Ω^{mn} generate local Lorentz rotations. Thus local Lorentz rotations are not independent from but induced by the general coordinate transformations (just as in the Wess–Zumino approach local supersymmetry but not local Lorentz rotations are induced by general coordinate transformations).

After complicated field redefinitions one finds then the following results. The parameters ϵ^{α}, a^{μ} and Ω^{mn} are just the parameters of local supersymmetry, general coordinate and local Lorentz rotations of ordinary supergravity. The fields S, P, A_{μ}, $e_{\mu}{}^{m}$, $\psi_{\mu}{}^{\alpha}$ transform precisely as the minimal set of auxiliary fields.

The advantage of super space approaches is that the closure of the gauge algebra is evident. In this approach, the product of two volume preserving general coordinate transformations, each of which keeps the gauge in which the axial superfield $H^{\mu}(x,\theta)$ has only terms of order θ^{2} and higher and has longitudinal components $\hat{H}^{\mu}$ and K^{μ}, is again of this kind.

Spinor covariant derivatives are defined, and vector covariant derivatives follow then from the anticommutator of two spinor covariant derivatives. Invariant actions can be defined and general counter terms have been obtained [411, 191].

Summarizing one might say that the chiral approach uses fewer fields (64 in $H^\mu(x,\theta)$ as compared to 1024 in $V_A{}^A(x,\theta)$) and fewer parameters (40 volume preserving λ^μ and λ^α as against 224 parameters $\Xi^A(x,\theta)$ and $L^m{}_n(x,\theta)$). Moreover, one only needs to fix 26 more gauges to find back the 38 fields $e^m{}_\mu$, $\psi_\mu{}^a$, S, P, A_μ while in the non-chiral approaches one simply does not have enough gauge parameters to do this (1024 > 224). Therefore, one must impose in the non-chiral approach additional algebraic constraints. These are satisfied automatically in the chiral approach.

All supervielbeins and connections are expressed in terms of the axial $H^\mu(x,\theta)$ above, and H^μ is an unconstrained field. As action one takes again the superdeterminant sdet $V_\Lambda{}^A$. Varying with respect to H^μ, the field equations read $G_{A\dot{A}}(x,\theta)=0$, but not $R=0$, as in the Wess–Zumino approach. R being chiral, of course satisfies $D_{\dot{A}}R=0$. Defining $G_{A\dot{A}}$ to be determined by the supertorsion just as in the Wess–Zumino approach

$$T_{A,B\dot{B}}{}^C=\epsilon_{AB}G_{\dot{B}}{}^C \tag{19}$$

one finds that (off-shell)

$$D^A G_{A\dot{A}}=D_{\dot{A}}R^*. \tag{20}$$

Hence, on-shell, $D_A R = D_{\dot{A}}R=0$, so that $R=$ constant. In order to understand this constant, one can consider component fields. In the action the term $eS^2=e(e^{-1}\partial_m S^m+(\mathrm{i}/8)\bar{\psi}_m\sigma^{mn}\psi_n)^2$ appears. Varying with respect to S^m (not S!) one finds $\partial_m(e^{-1}\partial_n S^n+(\mathrm{i}/8)\bar{\psi}_m\sigma^{mn}\psi_n)=0$. Hence $S=$ constant which is actually the same constant as found before since R starts with S. Inserting this result into the field equations for gravitino and graviton, one finds mass and cosmological terms. Note that the cosmological terms of the component approach $e(S-(\mathrm{i}/8)\bar{\psi}_m\sigma^{mn}\psi_n)$ turns here into a total derivative if one expresses S in terms of S_m.

At this point we anticipate our discussion of the work of Siegel and Gates, and discuss how in their approach a cosmological constant appears in the action. A chiral scalar density superfield V is introduced such that gauge transformations now have an unconstrained superfield parameter, as opposed to the case we discussed in subsection 1.9, eq. (23). (The parameter now generates superconformal transformations.) The cosmological term in the action is now given by $\Lambda\int \mathrm{d}^4x\,\mathrm{d}^2\theta\, V+\text{h.c.}=\Lambda\int\mathrm{d}^4x\,\mathrm{d}^4\theta\,(E/R)+\text{h.c.}$ Varying with respect to V directly yields the equation of motion $R=\Lambda$ (because in the action supdet E the supervielbein is V-dependent).

Both the chiral and the non-chiral approach can reproduce the results of the non-superspace approach to simple supergravity. Although the initial starting points look very different, each approach can lead to the other.

In fact, one can choose a special gauge (just as in general relativity) in which the components of $H^\mu(x,\theta)$ take on a particularly simple form at a given point. One finds that the first few components vanish, and the rest is a function of R, $G_{A\dot{A}}$ and W_{ABC} alone. Using this special form of H^μ, it is easy to show that the supertorsions and supercurvatures (which depend on $V_\Lambda{}^A$, which depend in turn only on derivatives of H^μ) satisfy identically the Wess–Zumino constraints.

We discuss now the closely related superspace approach of Siegel and Gates. They consider *complex* coordinate transformations. A scalar superfield $\phi(x,\theta)$ transforms as

$$\phi'(x,\theta) = \exp(\lambda^{\Pi}\partial_{\Pi})\,\phi(x,\theta) = \phi + \lambda^{\Pi}\partial_{\Pi}\phi + \cdots \tag{21}$$

and

$$\lambda^{\Pi}(x,\theta) \neq \lambda^{*}(x,\theta)^{\Pi}.$$

Chiral superfields are defined by

$$\partial_{\dot{A}}\phi \equiv \frac{1}{2}(1-\gamma_5)\frac{\partial}{\partial\bar{\theta}}\phi(x,\theta) = 0 \tag{22}$$

and depend thus on x and θ^{A} only (not yet on x_{L} and θ_{L}). Consider those λ^{Π} which maintain chirality. Thus, with (21),

$$\partial_{\dot{B}}(\delta\phi) = \partial_{\dot{B}}(\lambda^{A}\partial_{A} + \lambda^{\dot{A}}\partial_{\dot{A}} + \lambda^{\mu}\partial_{\mu})\phi = 0. \tag{23}$$

It follows that λ^{A} and λ^{μ} are chiral but $\lambda^{\dot{A}}$ is unrestricted since $\partial_{\dot{A}}\phi = 0$ from (22).

Similarly, ϕ^{*} is antichiral, $\frac{1}{2}(1+\gamma_5)(\partial/\partial\bar{\theta})\phi^{*} = \partial^{A}\phi^{*} = 0$, and requiring that also antichirality is maintained, one finds $\partial^{B}\lambda^{*\dot{A}} = \partial^{B}\lambda^{*\mu} = 0$. Again λ^{*A} is unrestricted. (The star operation is in a general representation not equal to complex conjugation C, but rather $\chi^{*} = C^{-1}\gamma_4^{\mathrm{T}}\chi^{C}$. In a Majorana representation $* = C$.)

The chiral scalar superfields of Ogievetski and Sokatchev are obtained as follows

$$\begin{aligned} &\phi(x_{\mathrm{L}},\theta_{\mathrm{L}}) = \exp(\mathrm{i}U)\,\phi(x,\theta^{A}) \\ &\phi^{*}(x_{\mathrm{R}},\theta_{\mathrm{R}}) = \exp(-\mathrm{i}U)\,\phi(x,\theta_{\dot{A}}) \end{aligned} \tag{24}$$

where $U = U^{\Lambda}(x,\theta)\partial_{\Lambda}$ with $(U^{\Lambda})^{*} = U^{\Lambda}$. This field U transforms per definition as

$$(\exp\mathrm{i}U)' = (\exp\lambda^{*\Pi}\partial_{\Pi})(\exp\mathrm{i}U)(\exp - \lambda^{\Pi}\partial_{\Pi}). \tag{25}$$

To lowest order in λ and U one has

$$\delta U^{\Lambda} = \mathrm{i}(\lambda^{\Lambda} - \lambda^{*\Lambda}) + \cdots \tag{26}$$

so that *one can gauge away the spinor superfields U^{A} and $U^{\dot{A}}$* by fixing $\lambda^{\dot{A}}$ and λ^{*A} appropriately. Clearly, (25) preserves the reality (Majorana character) of U^{Λ}. In this gauge (24) yields

$$x^{\mu}{}_{\mathrm{L}} = (\exp\mathrm{i}U)x^{\mu}, \qquad x^{\mu}{}_{\mathrm{R}} = (\exp - \mathrm{i}U)x^{\mu} \tag{27}$$

as one may easily check. It follows that $x^{\mu}{}_{\mathrm{L}} - x^{\mu}{}_{\mathrm{R}} = 2\mathrm{i}U^{\mu} + \cdots$. Hence, to lowest order, $U^{\mu} = H^{\mu}$ is the axial vector superfield discussed before. This is superconformal superspace supergravity.

To obtain Poincaré superspace supergravity, consider a chiral scalar density V

$$\delta V = \lambda^{\Pi}\partial_{\Pi}V + (\partial_{\mu}\lambda^{\mu} - \partial_{A}\lambda^{A})V. \tag{28}$$

The gauge choice $V =$ constant leads then to further constraints on λ^{Π}

$$\partial_\mu \lambda^\mu - \partial_A \lambda^A = 0. \tag{29}$$

Clearly, $V\phi$ is again chiral, and $\int d^4x\, d^2\theta\, V\phi$ is an invariant action.

Siegel and Gates construct explicit constraint-free supervielbeins, and, after a conventional choice of the superconnection, they find that the Wess–Zumino constraints are satisfied identically.

The motivation for writing the fields such that they appear in an exponential is to have a formalism which is analogous to the super Yang–Mills case. For the extended supergravities it seems unavoidable that one needs constraints on H^μ itself. Thus here the distinction between the chiral and non-chiral approach seems less clear even at the starting point.

5.7. Gauge supersymmetry

The first approach to local supersymmetry is the theory of Arnowitt and Nath [17–43], called gauge supersymmetry. Its tangent group is not, as in supergravity, $O(3,1) \times O(N)$, but the larger supergroup $Osp(3,1/4N)$. The dynamical equations of motion are the Einstein equations with a cosmological constant, generalized to a space with four bosonic and $4N$ fermionic coordinates. Due to the cosmological constant, the metric $g_{\mu\nu} = \delta_{\mu\nu}$ is not a solution of the field equations, while global supersymmetry (see subsection 3) is only a solution at the tree level when $N = 2$. When quantum corrections are taken into account, global supersymmetry could also be a solution for other values of N. The main problem of gauge supersymmetry is that it contains higher spin fields and ghosts. As such it is not a good particle theory. Nevertheless, we feel it important to discuss also this theory here because it is different from supergravity in its geometrical structure.*

We will begin by defining tensors. This is more complicated than in supergravity, since the tangent group has Bose–Fermi parts in gauge supersymmetry, but not in supergravity.

The base manifold is $z^A = x^\mu, \theta^{\alpha i}$ where $i = 1, N$ and N is arbitrary. The basic field is the metric tensor $g_{A\hat{\Pi}}(z)$ where the caret indicates that the two indices transform differently as we shall see. Coordinates transform as

$$\bar{z}^A = z^A - \xi^A, \qquad d\bar{z}^A = dz^A - \xi^A{}_{,\hat{\Sigma}}\, dz^\Sigma \tag{1}$$

where in this subsection we follow the literature and always use *right derivatives*. Requiring that the line element

$$(ds)^2 = dz^A g_{A\hat{\Pi}}\, dz^\Pi \qquad \text{with} \quad g_{A\hat{\Pi}} = (-)^{A+\Pi+A\Pi} g_{\Pi\hat{A}} \tag{2}$$

is invariant, one finds that the metric transforms under infinitesimal general supercoordinate transformations as

$$\delta g_{A\hat{\Pi}} = g_{A\hat{\Sigma}}\xi^\Sigma{}_{,\hat{\Pi}} + (-)^{(\Sigma+1)A}\xi^\Sigma{}_{,\hat{A}} g_{\Sigma\hat{\Pi}} + g_{A\hat{\Pi},\hat{\Sigma}}\xi^\Sigma. \tag{3}$$

Note that left-up to right-below contractions are the usual contractions and are free from carets, but that the (unusual) other contractions carry carets on the lower index. In fact, changing an index A to $\hat{A}$ introduces a factor $(-)^A$ as we shall discuss.

* The following is an excellent exercise in tensor analysis. Dynamics starts at eq. (33).

Contravariant vectors with flat and curved indices transform per definition as follows

$$\delta\phi^A = \phi^B\epsilon_B{}^A + \phi^A{}_{,\hat\Lambda}\xi^{\hat\Lambda} \tag{4}$$

$$\delta\phi^\Lambda = -\xi^\Lambda{}_{,\hat\Sigma}\phi^\Sigma + \phi^\Lambda{}_{,\hat\Sigma}\xi^\Sigma \tag{5}$$

where $\epsilon_B{}^A$ are completely arbitrary functions of z. The law for $\delta\phi^\Lambda$ follows from requiring that $\phi^\Lambda g_{\Lambda\Pi}\psi^\Pi$ be a scalar. The internal symmetry group $\epsilon_B{}^A$ is at this point arbitrary.

Covariant vectors are defined by requiring that $\psi_{\hat A}\phi^A$, $\phi^A\psi_A$, $\psi_{\hat\Lambda}\phi^\Lambda$ and $\phi^\Lambda\psi_\Lambda$ be scalars. A scalar S transforms as $\delta S = \xi^\Lambda\partial_\Lambda S = (\partial_{\hat\Lambda}S)\xi^\Lambda$. One finds, for example,

$$\begin{aligned}\delta\phi_A &= -\epsilon_A{}^B\phi_B + \phi_{A,\hat\Lambda}\xi^\Lambda\\ \delta\phi_{\hat A} &= -\phi_{\hat B}\epsilon_A{}^B(-)^{(A+B)A} + \phi_{\hat A,\hat\Lambda}\xi^\Lambda.\end{aligned} \tag{6}$$

Note that $(-)^A\phi_A$ transforms as $\phi_{\hat A}$. This allows one to switch to and from hatted indices, by multiplication by $(-)^A$.

Next a tangent space metric $\eta_{A\hat B}$ is introduced by requiring that

$$\phi^A\eta_{A\hat B}\psi^B \qquad \text{with} \quad \eta_{A\hat B} = (-)^{A+B+AB}\eta_{B\hat A} \tag{7}$$

be a tangent and world scalar. This determines $\delta\eta_{A\hat B}$ and shows that $\eta_{A\hat B}$ is a tensor, but requiring moreover that the tangent metric be an invariant tensor, one finds

$$\delta\eta_{A\hat B} = 0 = -\epsilon_A{}^C\eta_{C\hat B} - (-)^{B(B+C)}\eta_{A\hat C}\epsilon_B{}^C + \eta_{A\hat B,\hat\Sigma}\xi^\Sigma. \tag{8}$$

Clearly, $\eta_{A\hat B}$ is a constant tensor, and clearly, the tangent space parameters ϵ_{AB} which are defined by $\epsilon_{AB} = \epsilon_A{}^C\eta_{C\hat B}(-)^B$, must satisfy the following symmetry

$$\epsilon_{AB} + \epsilon_{BA} = 0. \tag{9}$$

This thus restricts the parameters ϵ for given flat metric.

One can lower flat and curved indices of contravariant tensors in two ways, by putting $g_{\Pi\hat\Lambda}$ and $\eta_{A\hat B}$ on the left or on the right of a given tensor. We define

$$\begin{aligned}\phi_{\hat\Lambda} &= \phi^\Sigma g_{\Sigma\hat\Lambda}, \qquad \phi_{\hat A} = \phi^B\eta_{B\hat A}\\ \phi_\Lambda &= g_{\Lambda\hat\Pi}\phi^\Pi, \qquad \phi_A = \eta_{A\hat B}\phi^B.\end{aligned} \tag{10}$$

Clearly, $\phi_{\hat\Lambda} = (-)^\Lambda\phi_\Lambda$. These objects are good covariant tensors as defined above (6). For completeness we give the transformation rules of curved covariant tensors

$$\begin{aligned}\delta\phi_{\hat\Lambda} &= \phi_{\hat\Sigma}\xi^\Sigma{}_{,\hat\Lambda} + \phi_{\hat\Lambda,\hat\Sigma}\xi^\Sigma\\ \delta\phi_\Lambda &= \phi_{\Lambda,\hat\Sigma}\xi^\Sigma + \phi_\Sigma\xi^\Sigma{}_{,\hat\Lambda}(-)^{\Lambda+\Sigma}.\end{aligned} \tag{11}$$

The last term can be rewritten as $\phi_{\hat\Sigma}\xi^\Sigma{}_{,\Lambda}$ in which case all indices match on both sides of the equation.

In general one contracts indices as in the four contractions above (6). One first writes down an equation so that it is correct in the bosonic sector, and then adds signs so that it is correct for all sectors. As an exercise the reader may derive

$$\phi_{\hat{\Lambda}\hat{\Pi}} = (-)^{\Delta(\Sigma+\Lambda)}\phi^{\Sigma\Delta}g_{\Sigma\hat{\Lambda}}g_{\Delta\hat{\Pi}}. \tag{12}$$

We now come to the important point of specifying *the group in tangent space*, i.e., the determination of ϵ_{AB}. At this point a (weak) assumption is made: the tangent group contains ordinary constant Lorentz rotations. Under ordinary Lorentz rotations $\delta v^m = \lambda^m{}_n v^n$ and $\delta v^a = \frac{1}{2}(\lambda\cdot\sigma)^a{}_b v^b$. In superspace $\delta s^A = s^B\epsilon_B{}^A$ per definition. Hence for the bosonic part $\lambda^m{}_n = \epsilon_n{}^m$ while for the fermionic part $\epsilon_b{}^a = \frac{1}{2}(\lambda\cdot\sigma)^a{}_b$. Notice that we always use gamma matrices with indices as in $(\gamma^m)^c{}_d$.

Taking in $\delta\eta_{A\hat{B}}$ in (8), A and B equal to m and n, one easily derives that η_{mn} is a multiple of the Minkowski metric. On the other hand, taking $A = m$ and $B = b$ one finds that the Bose–Fermi part of $\eta_{A\hat{B}}$ vanishes. For the Fermi–Fermi part of the tangent metric one finds with $A = a$, $B = b$ from (8) with these special $\epsilon_b{}^a$

$$\eta_{bc}(\sigma^{mn})^c{}_a - \eta_{ac}(\sigma^{mn})^c b = 0. \tag{13}$$

Hence, η must commute with the Lorentz generator. The general solution is therefore

$$\eta_{cd} = a[C(1+\mathrm{i}b\gamma_5)]_{cd} \tag{14}$$

where C_{ab} is the charge conjugation matrix.

Hence, there is a 3-parameter class of solutions of $\eta_{A\hat{B}}$ in (8) for the special case that ϵ is equal to constant Lorentz rotations, namely [$\mathrm{d}\eta_{mn}$ (Mink.), $aC_{\alpha\beta} + \mathrm{i}b(C\gamma_5)_{\alpha\beta}$]. We now pick one element of this class as the metric in tangent space. It is at this point that the difference with supergravity occurs, because in supergravity one keeps all 3 invariant metrics, as a result of which the tangent group of supergravity is the Lorentz group. Also the coset approach of page 319 leads to this result.

Substituting this form for the tangent metric, one finds that the tangent group is defined by leaving invariant $\mathrm{d}\phi^m\,\eta_{mn}(\text{Mink.})\psi^n + a\bar{\phi}(1+\mathrm{i}\gamma_5 b)\psi$. This shows that the local tangent group is Osp(3, 1/4N). (For $b = 0$ this is the standard definition for $b \neq 0$, see below.)

In gauge supersymmetry, the only field is the metric $g_{\Lambda\hat{\Pi}}$. There are no independent matter fields; the component matter fields appear in the θ expansion of the metric. Therefore one can describe the theory equally well in terms of the metric or of supervielbeins. However, in order to make contact with other formalisms, we introduce a supervielbein $V_\Lambda{}^A$ by the definition

$$g_{\Lambda\hat{\Pi}} = V_\Lambda{}^A\eta_{A\hat{B}}(-)^{\Pi(1+B)}V_\Pi{}^B. \tag{15}$$

The transformation properties of $V_\Lambda{}^A$ are as indicated by the indices

$$\delta V_\Lambda{}^A = (-)^{(\Lambda+\Sigma)(A+1)}V_\Sigma{}^A\xi^\Sigma{}_{,\Lambda} + V_\Lambda{}^A{}_{,\Sigma}\xi^\Sigma + V_\Lambda{}^B\epsilon_B{}^A. \tag{16}$$

Knowing how $V_\Lambda{}^A$, $g_{\Lambda\hat{\Pi}}$ and $\eta_{A\hat{B}}$ transform, one finds that the relation between g and V is a correct tensor relation. One can now scale the γ_5 in the tangent metric away by rescaling the super-vielbeins. To see this, note that $(a+\mathrm{i}b\gamma_5) = (a^2+b^2)^{1/2}\exp(\mathrm{i}\alpha\gamma_5)$ with $\cos\alpha = a(a^2+b^2)^{-1/2}$, and redefine $\phi =$

$(\exp -(\mathrm{i}/2)\alpha\gamma_5)\phi'$. The rescaled super-vielbein is still arbitrary, except when $a^2+b^2=0$ which case we exclude. We also rescale $V_\Lambda{}^m$ such that the Bose–Bose metric becomes strictly equal to the Minkowski metric. Thus the tangent metric takes on the simple form

$$\eta_{A\hat{B}} = [\eta_{mn}(\text{Mink.}), kC_{ab}], \qquad k>0. \tag{17}$$

(If $k<0$, a γ_5 rotation reverses the sign of k.)

Thus the tangent group is the set of transformations which leaves invariant

$$\phi^m\phi_m + k\bar{\phi}\phi, \qquad k>0. \tag{18}$$

In other words, *the tangent group is* Osp(3, 1/4N) (not Osp(N/4)). By restricting the tangent group to the local Lorentz group, one can go over to supergravity.

Up to this point we have exclusively considered the case $N=1$. For $N>1$ the flat indices are $A=(m, ai)$ and a similar analysis shows that

$$\eta_{A\hat{B}} = [\eta_{mn}(\text{Mink.}), kC_{ab}\delta_{ij}], \qquad (i,j=1,N). \tag{19}$$

We now will first define what gauge supersymmetry is, then discuss its field equations and finally consider in what way global supersymmetry is part of gauge supersymmetry.

A Riemannian space is defined by

(i) the covariant derivative of the super-vielbein vanishes. The latter is defined by

$$V_\Lambda{}^A{}_{;\Sigma} = V_\Lambda{}^A{}_{,\Sigma} - (-)^{\Sigma(\Lambda+\Delta)}\Gamma_\Lambda{}^\Delta{}_\Sigma V_\Delta{}^A + V_\Lambda{}^B\Omega_B{}^A{}_\Sigma. \tag{20}$$

Contrary to general usage, the index $\hat{\Sigma}$ in the spin connection is written on the right. This ensures that no extra signs appear in the term in (20) with Ω. (In supergravity, one can equally well write Σ on the left of AB since A and B are there both bosonic or fermionic. Not so here.) Similarly, one defines for covariant tensors T_B and $T_{\hat{B}}$ the covariant derivative by requiring that T^BT_B and $T_BT^{\hat{B}}$ are scalars for which the covariant derivative equals the ordinary derivative, and by requiring that the Leibniz rule holds.

(ii) the affinity $\Gamma_\Lambda{}^\Sigma{}_{\hat{\Pi}}$ is symmetric (i.e., no torsion)

$$\Gamma_\Lambda{}^\Sigma{}_{\hat{\Pi}} = (-)^{\Lambda+\Pi+\Sigma(\Lambda+\Pi)+\Lambda\Pi}\Gamma_\Pi{}^\Delta{}_\Sigma. \tag{21}$$

(We might note that defining $\Gamma_{\Lambda\hat{\Pi}}{}^\Sigma$ to be the Christoffel connection would simplify the formulae.)

As a consequence of (20), also the metric is covariantly constant

$$g_{\Lambda\hat{\Pi};\Sigma} = g_{\Lambda\Pi,\Sigma} - (-)^{\Lambda+\Pi+\Lambda\Pi+\Sigma(\Lambda+\Delta)}\Gamma_\Pi{}^\Delta{}_\Sigma g_{\Delta\hat{\Lambda}} - (-)^{\Sigma(\Pi+\Delta)}\Gamma_\Lambda{}^\Delta{}_\Sigma g_{\Delta\hat{\Pi}} = 0. \tag{22}$$

To prove this, use (15) and (20) and deduce that (22) implies that $\eta_{A\hat{B};\Sigma}=0$. This means that also the tangent space metric must be covariantly constant, and this is the case provided

$$\Omega_{AB\Sigma} = -(-)^{AB}\Omega_{BA\Sigma} \quad \text{with } \Omega_{AB\Sigma} = \Omega_A{}^C{}_\Sigma\eta_{B\hat{C}}. \tag{23}$$

From $V_\Lambda{}^A{}_{;\hat\Sigma} = 0$ and the transformation properties of tensors the transformation laws of the two affinities in (20) follow (see Nucl. Phys. B122 (1977) 301, eq. A(15)). Just as in general relativity without torsion one can solve the Christoffel symbol from (22) and the spin connection from (20). We leave it as an exercise to write down immediately the correct result for $\Gamma_\Pi{}^\Sigma{}_{\hat\Lambda}$ in terms of $g_{\Pi\hat\Delta}$ by adding signs to the Christoffel symbol of ordinary general relativity in the way discussed before.

The inverse metric is defined by

$$g_{\Lambda\hat\Sigma} g^{\Sigma\Pi} = \delta_\Lambda{}^\Pi \quad \text{where } g^{\Sigma\Pi} = (-)^{\Sigma\Pi} g^{\Pi\Sigma}. \tag{24}$$

(Incidentally, for the tensor $g_{\Lambda\Pi}$ one has $g_{\Lambda\Pi} = (-)^{\Lambda\Pi} g_{\Pi\Lambda}$, too.) As a consequence a little calculation shows that also

$$g^{\Pi\Sigma} g_{\Sigma\hat\Lambda} = \delta_\Lambda{}^\Pi = \delta^\Pi{}_{\hat\Lambda}. \tag{25}$$

Next one can solve $\Omega_B{}^A{}_C - (-)^{AB+BC+CA} \Omega_C{}^A{}_B$ from (20) but in order to find Ω itself one must lower the upper index by means of $\eta_{A\hat B}$. Defining

$$\Omega_{ABC} = \Omega_A{}^D{}_{\hat\Lambda} V_C{}^\Lambda (-)^{CA} \eta_{B\hat D} \tag{26}$$

and using $\Omega_{ABC} = -(-)^{AB} \Omega_{BAC}$ one finds for the connection

$$\Omega_{ABC} = \tfrac{1}{2}(R_{ABC} + (-)^{BC} R_{ACB} - (-)^{A(B+C)} R_{BCA}) \tag{27}$$

$$R_{ABC} = V_A{}^\Lambda (-)^{C+BC} [V_\Lambda{}^D{}_{,\hat\Sigma} - (-)^{D(\Lambda+\Sigma)+\Lambda+\Sigma+\Lambda\Sigma} V_\Sigma{}^D{}_{,\hat\Lambda}] (-)^{B\Sigma} V_B{}^\Sigma \eta_{DC}. \tag{28}$$

The inverse super vielbein is defined by

$$V_A{}^\Lambda V_\Lambda{}^B = \delta_A{}^B. \tag{29}$$

A remark about the signs: they follow easily by noting that one contracts as ${}_{\hat\Lambda} \quad {}^\Lambda$ and ${}^\Lambda \quad {}_\Lambda$. There are more signs in the Riemannian geometry formulae than in the corresponding supergravity formulae since the tangent space group is Osp(3, 1/4N) and not O(3, 1)$\otimes$O(N) (only the latter is truly bose).

The curvature tensor is defined by the commutator of two covariant derivatives on, say, U_Λ

$$U_{\Lambda;\hat\Pi\hat\Sigma} - (-)^{\Pi\Sigma} U_{\Lambda;\hat\Sigma\hat\Pi} = -(-)^{\Pi\Delta+\Sigma\Delta} R_\Lambda{}^\Delta{}_{\hat\Pi\hat\Sigma} U_\Delta \tag{30}$$

and reads

$$R_\Lambda{}^\Delta{}_{\hat\Pi\hat\Sigma} = -(-)^{\Pi\Sigma} \Gamma_\Lambda{}^\Delta{}_{\hat\Sigma,\hat\Pi} + \Gamma_\Lambda{}^\Delta{}_{\hat\Pi,\hat\Sigma} + (-)^{\Pi(\Delta+\Omega)} \Gamma_\Lambda{}^\Omega{}_{\hat\Pi} \Gamma_\Omega{}^\Delta{}_{\hat\Sigma} - (-)^{\Sigma(\Delta+\Omega)+\Pi\Sigma} \Gamma_\Lambda{}^\Omega{}_{\hat\Sigma} \Gamma_\Omega{}^\Delta{}_{\hat\Pi}. \tag{31}$$

(This definition differs by (−) from earlier definitions.) There are only two independent contractions, as in general relativity

$$R_{\Lambda\hat\Pi} = (-)^\Delta R_\Lambda{}^\Delta{}_{\Delta\hat\Pi} \quad \text{and} \quad R = g^{\Lambda\Pi} R_{\Pi\hat\Lambda} (-)^\Lambda. \tag{32}$$

We can now define the dynamics of gauge supersymmetry. It consists of the following equations of motion

$$R_{\Lambda\hat{\Pi}} = \lambda g_{\Lambda\hat{\Pi}}. \tag{33}$$

They follow from the following action

$$I = \int \mathrm{d}^m x\, \mathrm{d}^n\theta \sqrt{-g}\,[R + (n - m + 2)\lambda]. \tag{34}$$

The symbol $\sqrt{-g}$ is the superdeterminant (see appendix) and we used that $g_{\Pi\hat{\Lambda}} g^{\Lambda\Pi} = m - n$. There is also the familiar Bianchi identity

$$(-)^{\Lambda}(R^{\Pi\Lambda} - \tfrac{1}{2}g^{\Pi\Lambda}R)_{;\hat{\Lambda}} = 0. \tag{35}$$

Clearly (for λ non-zero) flat space ($g_{\Lambda\Pi} \sim \delta_{mn}$) is no solution.

We turn now to the question whether gauge supersymmetry contains global supersymmetry. This can be analyzed by looking at the vacuum expectation value of the field equations. Consider that $V_{\Lambda}{}^{A(0)} = \langle 0|V_{\Lambda}{}^{A}|0\rangle$ is given by

$$V_{\mu}{}^{m(0)} = \delta_{\mu}^{m}, \quad V_{\mu}{}^{ai} = 0, \quad V_{\alpha\iota}{}^{m(0)} = -\tfrac{1}{4}(\bar{\theta}\Gamma^{m})_{\alpha\iota}, \quad V_{\alpha\iota}{}^{aj(0)} = \delta_{\alpha}^{a}\delta_{\iota}^{j}. \tag{36}$$

This is the most general form for $V_{\Lambda}{}^{A}$ which is invariant under constant transformations with

$$\xi^{A(0)} = (\tfrac{1}{4}\bar{\epsilon}^{i}\Gamma_{ij}^{\mu}\theta^{j}, \epsilon^{\alpha\iota}). \tag{37}$$

The functions $(C\Gamma_{ij}^{\mu})_{\alpha\beta}$ must be symmetric in $(\alpha i, \beta j)$ in order that the metric in $g^{(0)}{}_{\Lambda\hat{\Pi}}$ satisfy the Einstein equations (33). (A tree level statement.) Hence

$$\Gamma_{ij}^{\mu} = \gamma^{\mu}\Gamma_{ij}(s) + \mathrm{i}\gamma^{\mu}\gamma^{5}\Gamma_{ij}(a). \tag{38}$$

Any global tensor $Q_{\Lambda\Pi}$ (i.e., a constant tensor invariant under $\xi^{A(0)}$) is of the form

$$\begin{aligned} &Q_{\mu\nu} = k_1\eta_{\mu\nu}, \qquad Q_{\mu,\alpha i} = k_1(\bar{\theta}\Gamma_{\mu})_{\alpha i}, \\ &Q_{\alpha i\beta j} = k_{2,ij}C_{\alpha\beta} + k_1(\bar{\theta}\Gamma^{\nu})_{\alpha i}(\bar{\theta}\Gamma_{\nu})_{\beta j}. \end{aligned} \tag{39}$$

Squaring $V_{\mu}{}^{m(0)}$ according to $g = V\eta V$ one finds for $g^{(0)}_{\Lambda\hat{\Pi}}$ the same result as in $Q_{\mu\nu}$ but with $k_1 = 1$ and $k_{2,ij} = k\delta_{ij}$ where k is the k in $\eta_{A\hat{B}}$. In the tree approximation $g^{(0)}_{\Lambda\hat{\Pi}}$ obeys

$$R_{\Lambda\hat{\Pi}}(g^{(0)}_{\Sigma\hat{\Delta}}) = \lambda g^{(0)}_{\Lambda\hat{\Pi}} \tag{40}$$

provided

$$-2k^{-2}\Gamma_{\mu}\Gamma^{\mu} = \lambda, \qquad -k^{-2}\,\mathrm{Tr}(\Gamma_{\mu}\Gamma_{\nu}) = \lambda\eta_{\mu\nu}. \tag{41}$$

These equations follow from the Bose–Bose and Fermi–Fermi part of (40); the Bose–Fermi content is redundant.

If one remains at the tree level, then this implies that $N = 2$ (trace the second equation). However, in the presence of quantum loop correction, calculated with the globally supersymmetric gauge fixing term, the corrections to $\langle 0|g_{\Lambda\Pi}|0\rangle$ are obtained by differentiating the effective potential with respect to $g^{(0)}_{\Lambda\Pi}$. The result must again be of the form of $Q_{\mu\nu}$ but also $R_{\Lambda\Pi}$ must be of this form (with calculable coefficients). The only difference is that in (37) one must replace λ by $\lambda - K'$ and $\lambda - \lambda'$ respectively. In this case it may turn out that tracing of (37) selects particular N values or perhaps none. In any case, (37) represents $N(N+1)+1$ equations for N^2+1 unknowns (namely, the elements of $\Gamma(s)$ and $\Gamma(a)$, and λ). Hence, if there are solutions, there must be extra symmetries.

To investigate the residual symmetries of the vacuum, one solves the Killing equations

$$\delta g^{(0)}_{\Lambda\Pi} = 0 = \xi_{\Lambda;\Pi} + (-)^{\Lambda+\Pi+\Lambda\Pi}\xi_{\Pi;\Lambda}. \tag{42}$$

The general solution is (see Phys. Rev. 10 (1978) 2759):

$$\begin{aligned} \xi^{\mu} &= a^{\mu} + \lambda^{\mu}{}_{\nu}x^{\nu} + \bar{\epsilon}\Gamma^{\mu}\theta \\ \xi^{\alpha i} &= \tfrac{1}{2}(\lambda\cdot\sigma)^{\alpha}{}_{\beta}\theta^{\beta i} + \epsilon^{\alpha i} - M^{\alpha i}{}_{\beta j}\theta^{\beta j} \end{aligned} \tag{43}$$

where the matrix M must satisfy $(CM)+(M^{\mathrm{T}}C)=0$, and $[M, \Gamma^{\mu}]=0$. Lorentz covariance implies $M^{\alpha i}{}_{\beta j} = (M_0)^i{}_j\delta^{\alpha}_{\beta} + (M_1)^i{}_j(\mathrm{i}\gamma_5)^{\alpha}{}_{\beta}$ so that the two $N\times N$ matrices M_0 and M_1 are antisymmetric in i and j. The important point is now that corresponding to the global symmetries of the vacuum metric given by M, the full dynamical theory has the corresponding local symmetries. In particular, the component fields are found inside the metric tensors as follows

$$\begin{aligned} g_{\mu\nu}(z) &= g_{\mu\nu}(x) + \bar{\theta}\psi_{\mu\nu}(x) + \cdots \\ g_{\mu,\alpha i}(z) &= \psi_{\mu,\alpha i}(x) + (\bar{\theta}M^{I})\alpha_i\phi^{I}_{\mu} + (\bar{\theta}\gamma^{m})_{\alpha i}e_m{}^{\mu}(x) + \cdots . \end{aligned} \tag{44}$$

The fields ϕ^I_μ are Yang–Mills fields and I denotes the generators of the internal group M. Also, upon explicit evaluation, it is found that the action in (34) has as unbroken symmetries only the spacetime and local supersymmetries, plus the Yang–Mills symmetries associated with M. All other symmetries contained in $\xi^{\Lambda}(z)$ are broken spontaneously, hence there are no fields with vanishing masses with spins exceeding 2. One might call the symmetry group which is given by the Killing vectors in (39) the group $\mathrm{Gsp}(N/4)_{R\to\infty}$ since it contains the contracted symplectic group Sp(4) (hence the spacetime symmetries) plus the group G of Yang–Mills invariances plus its grading.

Let us repeat the analysis performed above on the metric now on the super-vielbein. Requiring that $\delta V_{\Lambda}{}^{A(0)} = 0$ with $V_{\Lambda}{}^{A(0)}$ and $\xi^{\Lambda(0)}$ given before one finds that

$$\begin{aligned} &\epsilon_m{}^{n}(z)^{(0)} = \lambda_m{}^{n}(x), \qquad \epsilon_m{}^{ai}(z)^{(0)} = 0 \\ &\epsilon_{ai}{}^{m}(z)^{(0)} = 0, \qquad \epsilon_{ai}{}^{bj}(z)^{(0)} = \tfrac{1}{2}(\lambda\cdot\sigma)^{b}{}_{a}\delta^{j}{}_{i} + M_{ai}{}^{bj}. \end{aligned} \tag{45}$$

In other words, of the whole original tangent symmetry group Osp(3, $1/N$) only the diagonal part $\mathrm{O}(3,1)\times\mathrm{G}$ remains unbroken.

We come now to the well known $k \to 0$ limit. In this limit the group G(N) is expanded into the group O(N), and one hopes to find the N-extended supergravity models. To see how this limit is obtained, we state without proof that one may place Γ^μ by γ^μ in (36) and (39) provided one replaces in $g^{(0)}_{\alpha i,\beta j}$ the matrix δ_{ij} by K_{ij}. In that case the condition becomes $[M, \gamma^\mu] = 0$ and $[M, kK] = 0$. Thus $M^{\alpha i}{}_{\beta j} = \delta^\alpha{}_\beta (M_0)^i{}_j$ where $(M_0)^i{}_j$ is antisymmetric, so that the internal symmetry group expands into O(N).

We conclude our discussion of gauge supersymmetry with a short discussion of four topics.

Vanishing cosmological term. While the supercosmological constant λ in (30) is nonzero, the ordinary cosmological constant (in the Einstein sector) vanishes as a consequence of the global supersymmetry of the vacuum metric below (35). (It may be stressed that other vacuum solutions might not be globally supersymmetric, but this might induce an enormous cosmological constant if such breakdown occurred at the tree or the first loop level.) The reason is that if one substitutes in (30) $g = g^{(0)} + h$, due to (36) only terms at least linear in the quantum fields h survive. In more detail

$$R_{\mu\nu}(z) = R^{\text{Einstein}}_{\mu\nu}(x) + \Gamma^{\alpha(0)}_{\mu\beta} \Gamma^{\beta(0)}_{\nu\alpha} \quad \text{at } \theta = 0$$

and the second term cancels $\lambda g^{(0)}_{\mu\nu}$.

Ultraviolet finiteness. One quantizes with the globally supersymmetric gauge fixing term

$$g^{AB(0)} C_B(z)\, C_A(z) \tag{46}$$

where $C_A(z)$ is the linearized harmonic gauge. The Faddeev–Popov ghost action is then obtained as usual. The propagator of the metric is obtained by solving the Ward identity

$$\langle \delta h_{\Pi\Lambda}(z_1)\, h_{\Sigma\Omega}(z_2) + h_{\Pi\Lambda}(z_1)\, \delta h_{\Sigma\Omega}(z_2) \rangle = 0. \tag{47}$$

The result is

$$\Delta_{\Pi\Lambda,\Sigma\Omega}(p, z_1, z_2) = \exp(\bar{\omega} p_\mu \Gamma^\mu \xi) \sum_{m=0}^{4} F^{(m)}_{\Lambda\Pi\Sigma\Omega}(p, \omega^\alpha)\, P_m(\xi^\alpha) \tag{48}$$

where P_m is a polynomial or order m in $\xi^\alpha = \frac{1}{2}(\theta_1^\alpha + \theta_2^\alpha)$ and $\omega^\alpha = \theta_1^\alpha - \theta_2^\alpha$. Furthermore

$$F^m(p, \omega^\alpha) \sim \sum_{n=0}^{4N} A^m_n p^{-(2+4N-n)} \omega^{\alpha_1} \dots \omega^{\alpha_n}. \tag{49}$$

Let us now consider the n-polygon in fig. 1. There are n vertices, and each vertex goes like k^2.

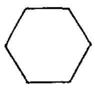


Fig. 1.

Its degree of divergence is given by

$$d_n = 4 + 2n - \sum_{i=1}^{n} (2 + 4N + n_i). \tag{50}$$

Since $\Sigma n_i \leq 4N(n-1)$ because there are only $n-1$ independent $\omega_{12}, \omega_{23}, \ldots, \omega_{n1}$, it follows that

$$d \leq 4 \mp 4N. \tag{51}$$

In other words, for $N \geq 2$, polygons are finite. For the general case the result is as follows: the overall degree of divergence of an n-point m-loop function is

$$d_{n,m} \leq m(4-4N) - 2(m-1). \tag{52}$$

Thus all n-point functions are finite even off-shell in this (and several but not all) gauges. One remark about spinor derivatives in the vertices acting on propagators. If

$$V(p) = V_2(p) + V_1^{\alpha}(p)\,\partial_\alpha + V_0^{\alpha\beta}(p)\,\partial_\alpha\,\partial_\beta \tag{53}$$

then the ∂_α act as one power of p while $V_i(p)$ contain $(p)^i$. Hence all vertices really go like p^2.

The question of ghosts. The entire $g_{\Lambda\Pi}$ to all order in θ and for arbitrary N contains the following component fields: (i) massless gauge fields, (ii) gauge fields whose mass is due to the (super) Higgs effect, (iii) other fields.

As a preliminary study, to examine the particle content prototype models have been studied with the following features: (i) a higher derivative theory, or equivalently, a nondiagonal kinetic matrix, (ii) global supersymmetry, (iii) they possess mass terms or mass parameters. In general, higher derivative massless theories lead to dipole ghosts. However, in the massive models examined, one finds that the dipole ghosts are replaced by complex conjugate Lee–Wick ghosts plus non-ghost real particles. In such theories one can define an S-matrix which is unitary and Lorentz-invariant. However, there is a drawback, namely causality is violated. Although the amount of violation is proportional to M^{-1} (10^{-40} seconds) one might say that one has exchanged causality for unitarity. Lee–Wick ghosts also introduce nonrenormalizability (in spite of finiteness!) because at each order in perturbation theory one needs to introduce parameters to tell which way to integrate around poles/cuts (see original literature by Lee, Wick and others).

Gauge completion. A general method for constructing a tensor which is covariant with respect to a given transformation group, called gauge completion, has been developed by Arnowitt and Nath. The method enabled them to construct the metric tensor in terms of the fields of $N=1$ supergravity in ordinary space-time, such that the superspace transformation laws of the metric correctly reproduced the transformation laws of tetrad and gravitino. In the limit $k \to 0$ the equations $R_{\mu\nu} = \lambda g_{\mu\nu}$ reproduced the gravitino field equation. Also the tetrad field equation was obtained. To go to higher orders in θ they went on-shell since at that time there were not auxiliary fields known.

6. Extended supergravities

6.1. *The N = 2, 3, 4 models*

The extended supergravities contain more than one gravitino. They are usually called N-extended supergravities where N is the number of real gravitinos. ($N=0$ is then general relativity and $N=1$ is simple supergravity.) There are only eight viable supergravities, namely $1 \leq N \leq 8$ since for $N > 8$ one has particles with spins larger than two and more than one graviton (for $N=9$ one has 10 gravitons, an "embarras de richesse"). Since it has been shown that the free field action for spin 5/2 is unique and cannot be coupled consistently to either gravity or to simple matter systems, it seems as if nature stops at spin 2, and supergravity at $N = 8$ (see subsection 1.14).

The simplest extended supergravity is the $N = 2$ model (Ferrara and van Nieuwenhuizen [203]). It was obtained by coupling the (2, 3/2) gauge action to the (3/2, 1) matter multiplet by means of the Noether method. It was only afterwards found that the model had a larger symmetry, namely (to begin with) a manifest O(2) invariance which rotates the two gravitinos into each other. This model unifies electromagnetism with gravity. The action reads

$$\mathscr{L} = -\frac{e}{2}R(e,\omega) - \frac{e}{2}\bar{\psi}^i_\mu \Gamma^{\mu\rho\sigma} D_\rho(\omega)\psi^i_\sigma - \frac{e}{4}F^2_{\mu\gamma} + \frac{\kappa}{4\sqrt{2}}\bar{\psi}^i_\mu[e(F^{\mu\nu} + \hat{F}^{\mu\nu}) + \tfrac{1}{2}\gamma_5(\tilde{F}^{\mu\nu} + \hat{\tilde{F}}^{\mu\nu})]\psi^j_\nu \epsilon^{ij} \tag{1}$$

and is invariant under general coordinate, local Lorentz and Maxwell transformations ($\delta A_\mu = \partial_\mu \Lambda$) as well as under the following *two* local supersymmetries

$$\begin{aligned} &\delta e^m{}_\mu = \frac{\kappa}{2}\bar{\epsilon}^i \gamma^m \psi^i_\mu, \qquad \delta A_\mu = \frac{\kappa}{\sqrt{2}}\bar{\epsilon}^i \psi^j_\mu \epsilon^{ij} \\ &\delta\psi^i_\mu = D_\mu(\omega)\epsilon^i + \frac{\kappa}{2\sqrt{2}}\epsilon^{ij}(\hat{F}_{\mu\lambda}\gamma^\lambda + \tfrac{1}{2}e\hat{\tilde{F}}_{\mu\lambda}\gamma^\lambda\gamma_5)\epsilon^j. \end{aligned} \tag{2}$$

Indeed, each gravitino gauges one local supersymmetry as one sees from $\delta\psi_\mu{}^i = \partial_\mu \epsilon^i + \text{more}$. The symbol $\tilde{F}_{\mu\nu}$ equals $\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ and $\hat{F}_{\mu\nu}$ is the supercovariant photon curl

$$\hat{F}_{\mu\nu} = \left(\partial_\mu A_\nu - \frac{\kappa}{2\sqrt{2}}\bar{\psi}^i_\mu \psi^j_\nu \epsilon^{ij}\right) - (\mu \leftrightarrow \nu). \tag{3}$$

One may always take the spin connection ω in the Hilbert action as that function of tetrads and gravitinos which solves $\delta I/\delta\omega = 0$ (1.5 order formalism). If one would have started in the Hilbert action with $\omega + \tau$ where τ is arbitrary, then one finds only $\tau^2 = \psi^4$ terms upon expanding about ω. These ψ^4 terms are absorbed into the $\hat{F}_{\mu\nu}$ tensors, and it is easy to argue why this must be possible (as of course has been checked explicitly). Since fermionic field equations rotate into bosonic field equations and the former have only one derivative while the latter have two derivatives, no derivative can act on ϵ. (Since otherwise there would no longer be two derivatives available for the bosonic field equations. This argument breaks down when there are nonpropagating (auxiliary) fields.) The gravitino field equation is obtained by varying all ψ fields, and since the terms with $\hat{F}$ have four ψ's which appear symmetrically, one must take one half of the former in order that the field equation only contains the supercovariant $\hat{F}_{\mu\nu}$, but not, for example, bare $F_{\mu\nu}$'s. Note that this argument also tells one that the spin connection in the gravitino field equation must be supercovariant. As it happens, ω is itself supercovariant, hence both in Hilbert and gravitino action one finds ω. (In other cases, for example in $d = 11$ or $d = 5$ dimensions, one finds in the gravitino action $\frac{1}{2}(\omega + \hat{\omega})$ instead of ω, but, based on our argument given before, one always chooses ω in the Hilbert action.)

The same kind of argument shows that if auxiliary fields are absent the gauge algebra must close on bosonic fields, and this is also a helpful criterion in constructing theories. (It is difficult to credit one particular person, but certainly J. Scherk gave an important contribution.)

This model was the model where finite quantum corrections were found for the first time (see ref.

[282]). Some chiral–dual symmetries were first found in another model which was not finite but displayed miraculous cancellations [510]. For a complete treatment of dual–chiral symmetries, see ref. [382]. An explanation of this finiteness was given by using these symmetries in this model which we now discuss.

According to the analysis of Haag, Lopuszanski and Sohnius (subsection 3.3), the maximal symmetry group of the S-matrix of supersymmetry algebras containing the Poincaré algebra is U(2). In fact, the field theory has this global symmetry only on-shell (i.e., only the field equations have an U(2) global symmetry). Off-shell, only an SU(2) remains. These symmetries read [382]

$$\begin{aligned} &\delta\psi_\mu{}^{\mathrm{L}} = \mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\tau}\psi_\mu{}^{\mathrm{L}} \qquad \psi_\mu = \begin{pmatrix}\psi_\mu^1\\ \psi_\mu^2\end{pmatrix}, \qquad \psi_\mu{}^{\mathrm{L}} = \tfrac{1}{2}(1+\gamma_5)\psi_\mu \\ &\delta\psi_\mu{}^{\mathrm{R}} = -\mathrm{i}\boldsymbol{\omega}\cdot\boldsymbol{\tau}\psi_\mu{}^{\mathrm{R}}, \qquad\qquad\qquad \psi_\mu{}^{\mathrm{R}} = \tfrac{1}{2}(1-\gamma_5)\psi_\mu \\ &\mathrm{U}(1)\colon\ \delta\psi_\mu = -\mathrm{i}\gamma_5\psi_\mu \quad \text{and} \quad \delta\hat{F}_{\mu\nu} = \mathrm{i}e\epsilon_{\mu\nu\rho\sigma}\hat{F}^{\rho\sigma}. \end{aligned} \tag{4}$$

Thus the SU(2) part rotates ψ^{L} as (2) and ψ^{R} as $(\bar{2})$, while the U(1) part is a combined chirality-duality transformation. To prove the U(2) invariance, note that in the gravitino action ψ_{L} couples to $\psi_{\mathrm{R}}^{\mathrm{T}}C$ so that these terms are SU(2) and of course chirally (U(1)) invariant. The torsion is separately U(2) invariant, and, finally, the remaining terms can be analyzed by eliminating the $F_{\mu\nu}$ kinetic terms through the equation of motion for $F_{\mu\nu}$. This cancels half of the Noether coupling with bare $F_{\mu\nu}$ and the result is that the action is the sum of the tetrad and gravitino actions plus the following term

$$\frac{\kappa}{4\sqrt{2}}\bar{\psi}^i_\mu[e\hat{F}^{\mu\nu} + \tfrac{1}{2}\gamma_5\hat{\tilde{F}}^{\mu\nu}]\psi_\nu{}^j\epsilon^{ij}. \tag{5}$$

Clearly, also this last term is U(2) invariant. The SU(2) invariance holds off-shell, but for $N>2$ only the O(N) invariance holds off-shell.

An interesting connection between the coupling of photons and the cosmological constant was found by Das and Freedman [108] and Fradkin and Vasiliev [225]. When one couples the photons minimally to the fermions one *needs* at the same time a cosmological constant and a mass like term in the action. Thus it would seem that electromagnetism is due to the curvature of a de-Sitter universe. Actually, this cosmological constant is much too large, and can be eliminated altogether by spontaneous symmetry breaking, as we shall discuss in subsection 5.

For the $N=2$ model one can find this extension of the action by adding a cosmological constant to the action and finding the further modifications by the Noether method. The complete set of extra terms is given by

$$\begin{aligned} &\mathscr{L}(\text{cosm.}) = 6eg^2 + 2e\bar{\psi}_\mu{}^i\sigma^{\mu\nu}\psi_\nu{}^i - \tfrac{1}{2}\bar{\psi}_\mu{}^i\Gamma^{\mu\rho\sigma}(D_\rho\psi_\sigma{}^i + g\epsilon^{ik}A_\rho\psi_\sigma{}^k) \\ &\delta\psi_\mu{}^i = D_\mu(\omega(e,\psi))\epsilon^i + g\gamma_\mu\epsilon^i + g\epsilon^{ik}A_\mu\epsilon^k \end{aligned} \tag{6}$$

where we added the terms with D_μ for comparison. Clearly, g is a dimensionless gauge coupling, but it is remarkable that also a mass-like term is needed (as well as a cosmological term) when one couples the photon to the gravitinos in an electromagnetic manner. Note that the theory still has the same number of local invariances. The mass term is actually needed in order that $\psi_\mu{}^i$ still be massless in de-Sitter space [125], and for $g \neq 0$ there is only an SO_2 symmetry on and off-shell, since the mass term breaks the SU_2 invariance.

In fact, even in simple ($N = 1$) supergravity one needs a mass term if one adds a cosmological term even though here there is no photon [474]. As the reader may easily verify, the action

$$\mathcal{L} = -\frac{e}{2} R(e, \omega) - \tfrac{1}{2}\bar{\psi}_\mu \Gamma^{\mu\rho\sigma} D_\rho \psi_\sigma + 6eg^2 + 2eg\bar{\psi}_\mu \sigma^{\mu\nu} \psi_\nu \tag{7}$$

is invariant under $\delta e^m{}_\mu = \frac{1}{2}\bar{\epsilon}\gamma^m \psi_\mu$ and $\delta\psi_\mu = (D_\mu + g\gamma_\mu)\epsilon$. This cosmological term plus mass term follows easily from the tensor calculus;* it is simply the action belonging to the unit scalar multiplet $\Sigma = (1, 0, 0, 0, 0)$. This cosmological constant is positive because in the $N = 1$ gauge action one finds $-\frac{1}{3}S^2 + \alpha S$ with α arbitrary. On the other hand, in the Wess–Zumino model one finds a term $+\frac{1}{2}F^2$ (with F the usual auxiliary field) and by coupling this model to supergravity, one can cancel the cosmological term [532, 541].

The next extended supergravity we discuss is the $N = 3$ model. It was obtained by Freedman [240] and Ferrara, Scherk and Zumino [201]. Here a new feature is that the Abelian photon gauge invariance turns into a non-Abelian Yang–Mills invariance in de-Sitter space. In other words, as in the $N = 2$ case, coupling the triplet of photons to the gravitinos, one needs at the same time a cosmological constant and a mass term. The action reads

$$\begin{aligned}\mathcal{L} = {} & \mathcal{L}(\text{kinetic terms for } e^m{}_\mu,\ \psi_\mu{}^i,\ A_\mu{}^i,\ \lambda \qquad \text{with } i = 1, 3) \\ & + \tfrac{1}{2}(\mathcal{L}^{\text{Noether}}_{\text{bare}} + \mathcal{L}^{\text{Noether}}_{\text{supercov.}}) + (\text{eq. (6) with } \epsilon^{ik} A_\mu \to \epsilon^{ijk} A_\mu{}^j)\end{aligned} \tag{8}$$

where the spin connection depends on $\psi_\mu{}^i$ and λ and solves $\delta I/\delta\omega = 0$. (We thus again use 1.5 order formalism.) The bare (i.e. non-supercovariantized) Noether coupling is a sum of the Noether couplings of the $(\frac{3}{2}, 1)$ and $(1, \frac{1}{2})$ systems to the $(2, \frac{3}{2})$ system.

$$\mathcal{L}^{\text{Noether}}_{\text{bare}} = -\frac{1}{2\sqrt{2}} \bar{\psi}_\mu{}^i (eF^{\mu\nu,j} + \tfrac{1}{2}\gamma_5 \tilde{F}^{\mu\nu,j}) \psi_\nu{}^k \epsilon^{ijk} + \tfrac{1}{2}(\bar{\psi}_\mu{}^i \sigma^{\alpha\beta}\gamma^\mu \lambda)(F^i{}_{\alpha\beta}). \tag{9}$$

The supercovariantized Noether coupling follows from the local supersymmetry transformation rules, which read

$$\begin{aligned}
&\delta e^m{}_\mu = \tfrac{1}{2}\bar{\epsilon}^i \gamma^m \psi_\mu{}^i, \qquad \delta\lambda = \tfrac{1}{2}(\sigma^{\mu\nu}\epsilon^i)(\hat{F}_{\mu\nu}{}^i) \\
&\delta A_\mu{}^i = \frac{1}{\sqrt{2}} \epsilon^{ijk} \bar{\epsilon}^j \psi_\mu{}^k - \tfrac{1}{2}\bar{\epsilon}^i \gamma_\mu \lambda \\
&\delta\psi_\mu{}^i = D_\mu(\omega(e, \psi, \lambda))\epsilon^i + \frac{1}{2\sqrt{2}} \epsilon^{ijk} (\sigma^{\rho\sigma}\gamma_\mu \epsilon^k)(\hat{F}_{\rho\sigma}{}^j) + \frac{1}{4\sqrt{2}} \epsilon^{ijk} [(\bar{\psi}_\mu{}^j \gamma_\rho \lambda)(\gamma^\rho \epsilon^k) + (\bar{\psi}_\mu{}^j \gamma_\rho \gamma_5 \lambda)(\gamma_5 \gamma^\rho \epsilon^k)] \\
&\qquad + \tfrac{1}{8}(\bar{\lambda}\gamma_5 \gamma^\rho \lambda)(\gamma_\rho \gamma_\mu \gamma_5 \epsilon^i) + g\epsilon^{ijk} A_\mu{}^j \epsilon^k + g\gamma_\mu \epsilon^i.
\end{aligned} \tag{10}$$

Note that the expression for $\delta\psi_\mu$ is itself no longer supercovariant. The noncovariant terms involve the spin $\frac{1}{2}$ fields, and this is a general (but not understood) feature: without spin $\frac{1}{2}$ fields, all

* Use eq. (8) on page 307.

transformation rules are always supercovariant. This means that there must be extra terms in the commutator of two local supersymmetry transformations. These terms are: an extra local Lorentz transformation with parameter

$$(2\sqrt{2})^{-1}\epsilon^{ijk}\left(\bar{\epsilon}_2^i\epsilon_1^k\hat{F}^j_{mn}+\frac{e}{2}\bar{\epsilon}_2^i\gamma_5\epsilon_1^k\hat{\tilde{F}}^j_{mn}\right) \tag{11}$$

and an extra local supersymmetry transformation with parameter

$$(2\sqrt{2})^{-1}\epsilon^{ijk}(\bar{\epsilon}_2^j\epsilon_1^k\lambda-\bar{\epsilon}_2^j\gamma_5\epsilon_1^k\gamma_5\lambda) \tag{12}$$

and an O(3) Yang–Mills gauge transformation with parameter

$$\begin{aligned} &\Lambda^i=(2\sqrt{2})^{-1}\epsilon^{ijk}(\bar{\epsilon}_2^j\epsilon_1^k)-\tfrac{1}{2}\bar{\epsilon}_2^j\gamma^\mu\epsilon_1^jA^i_\mu \\ &\delta A_\mu=\partial_\mu\Lambda^i+g\epsilon^{ijk}A^j_\mu\Lambda^k, \qquad \delta e^m{}_\mu=0 \\ &\delta\psi^i_\mu=g\epsilon^{ijk}\psi^j_\mu\Lambda^k, \qquad \delta\lambda=0. \end{aligned} \tag{13}$$

Note that the parameter composition rules do not depend on g (or only through F^j_{mn} as in (11)) although the transformation rules depend on g explicitly.

There is a global U(3) invariance for $g=0$ on-shell. Off-shell only the O(3) invariance remains, since all other symmetries involve not only chiral transformations but also duality transformations. The precise rules follow by truncation from the U(4) group of the SO(4) version of the $N=4$ model (see below). For $g\neq 0$, the O(3) invariance remains valid off-shell, but since the Yang–Mills coupling involves bare A^i_μ fields and the mass term breaks chiral invariance, the U(3) invariance is again lost for $g\neq 0$ on and off-shell.

The gauge parameter in (13) contains a nonvanishing term in the global limit (when fields are set equal to zero). Hence, there is a central charge in the super Poincaré algebra (i.e. for $g=0$), but this charge does not act on physical states. Also in the super de-Sitter algebra (with $g\neq 0$) there is an "electric" charge in the $\{Q, Q\}$ anticommutator, but now this charge is no longer a central charge, since it does not commute with supersymmetry. As a result δ(gauge) $\psi_\mu\neq 0$, so that for $g\neq 0$ the electric charge does act on physical states.

From the $N=3$ theory (or any other theory) one can obtain other theories by consistent truncation. A truncation means putting certain fields equal to zero, and consistency requires that also their variations then vanish. Two consistent truncations are: $A^1_\mu=A^2_\mu=\psi^3_\mu=\lambda=0$ which yields the $N=2$ model [511], while $A^2_\mu=A^3_\mu=\psi^2_\mu=\psi^3_\mu=0$ yields the $(2,\frac{3}{2})+(1,\frac{1}{2})$ Maxwell–Einstein systems [507].

We now discuss the $N=4$ extended supergravity. New here is the occurrence of scalars, for example A, which appear in a nonpolynomial way. This is possible since κA is dimensionless. For photons, κA_μ is also dimensionless, but Maxwell gauge invariance forbids an infinite series in photons [202] while one can also show that fermion fields cannot appear nonpolynomially [202].

There are actually two versions of the $N=4$ model known, the SO(4) model [109, 90, 91] with fields (e, ψ'^i, $V_\mu{}^{ij}$, λ'^i, A, B') and the SU_4 model [190] with fields (e, ψ^i, $A_\mu{}^k$, $B_\mu{}^k$, λ^i, ϕ, B). The six fields $V_\mu{}^{ij}$ are all vector fields, but the three $A_\mu{}^k$ are vector fields while the $B_\mu{}^k$ are axial vector fields. One can obtain the SU(4) model by reduction of the $N=1$ model in $d=10$ dimensions. Spinors in $d=10$ can satisfy both the Majorana condition $\bar{\lambda}^D=\bar{\lambda}^M$ *and* the Weyl condition $\lambda=\frac{1}{2}(1\pm\gamma_{11})\lambda$. Thus, in $d=10$, ψ_μ has 16 components and becomes in $d=4$ equal to four gravitinos. However, one finds in $d=4$ matter multiplets, and splitting in $d=4$ these matter couplings from the pure $N=4$ gauge action is rather hard and has not been attempted. Rather, the SU(4) model was constructed directly. The appearance of an off-shell SU(4) symmetry is obvious since the Lorentz group O(9, 1) splits into the usual Lorentz group O(3, 1) of $d=4$ and an O(6) which is equivalent to SU(4). Since the pseudoscalar field B is obtained by a duality transformation on an antisymmetric tensor field (which can only appear polynomially in $d=10$ and $d=4$), B necessarily has opposite parity and appears polynomially; however, ϕ does appear nonpolynomially.

At the classical level the two theories are equivalent because one can find a transformation law of the field which turns the one action into the other. For the axial and vector fields the correspondence is the following. Denoting by $G_{\mu\nu}{}^{ij}$ the tensor which gives the equation of motion of V^{ij}_μ, hence $D^\mu G_{\mu\nu}{}^{ij}=0$ (since only the curls $F^{ij}_{\mu\nu}$ of V^{ij}_μ appear in the action, $G_{\mu\nu}{}^{ij}$ can be defined) one has for the transformation laws of the curls of $A_\mu{}^{ij}$ and $B_\mu{}^{ij}$

$$\begin{aligned} \alpha^{ij}_n A^n_{\mu\nu} &= \frac{1}{\sqrt{2}}(F^{ij}_{\mu\nu} + \tfrac{1}{2}F^{kl}_{\mu\nu}\epsilon^{ij}{}_{kl}) \\ \beta^{ij}_n B^n_{\mu\nu} &= \frac{1}{\sqrt{2}}(G^{ij}_{\rho\sigma} - \tfrac{1}{2}G^{kl}_{\rho\sigma}\epsilon^{ij}{}_{kl})\epsilon_{\mu\nu}{}^{\rho\sigma} \end{aligned} \tag{14}$$

where α and β are certain numerical matrices. The first equations can be carried over directly onto the gauge fields themselves, but the second equation together with the Bianchi identity $D^\mu \tilde{B}_{\mu\nu}{}^n = 0$ yields

$$(D^\mu G^{ij}_{\mu\nu}) - \tfrac{1}{2}\epsilon^{ij}{}_{kl}(D^\mu G^{kl}_{\mu\nu}) = 0. \tag{15}$$

Hence, one-half of the V^{ij}_μ field equations must be satisfied and one is partly on-shell. It is now also clear why in the SU(4) model there are three axial vector fields, namely due to the $\epsilon_{\mu\nu}{}^{\rho\sigma}$ symbol in (14).

The SU(4) model is the simpler model. In the action one only finds nonpolynomiality in the scalar field ϕ, and only by means of exponentials of ϕ. In the SO(4) model, on the other hand, the scalar kinetic term reads $[1-\kappa^2(A^2+B^2)]^{-1}[(\partial_\mu A)^2 + (\partial_\mu B)^2]$, which, incidentally, limits the range of κA and κB between -1 and $+1$ [221, 222].

In the SU(4) model there is a group of global SU(4) invariances *off-shell*, and a further global SU(1, 1) on-shell. The off-shell SU(4) transformations transform the *fields* A^k_μ and B^k_μ into each other and contains also chiral rotations of $\psi_\mu{}^i$, and of χ^i, while ϕ, and B and $e^m{}_\mu$ are inert.

The SU(1, 1) global symmetry group leaves the scalar kinetic terms invariant. These are given by $(\partial_\mu Y \partial^\mu \bar{Y})\,(Y+\bar{Y})^{-2}$ where $Y=\exp(-2\phi)-2\mathrm{i}B$ and the transformations $Y \to (\alpha Y + \mathrm{i}\beta)(\mathrm{i}\gamma Y + \delta)^{-1}$ with $\alpha\delta + \beta\gamma = 1$ leave it invariant. To see that these transformations are really SU(1, 1), note that the transformations

$$\delta(z) = \delta\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \alpha & \mathrm{i}\beta \\ \mathrm{i}\gamma & \delta \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

can be written as $\exp(\mathrm{i}\omega_1\tau_1 + \omega_2\tau_2 + \omega_3\tau_3)$ and leave $z^*\tau_1 z$ invariant. *Since this noncompact group is realized nonlinearly it does not lead to ghosts.* Again, this SU(1, 1) contains duality transformations of the form $\delta A_{\mu\nu}{}^k = \tilde{G}_{\mu\nu}{}^k(A)$ and idem for $B_\mu{}^k$.

We now turn to the SO(4) version of the $N = 4$ model. Here the global SU(4) consists of a manifest SO(4) symmetry plus transformations generated by symmetric antihermitian matrices $\mathrm{i}\Lambda^{jk}$. More in detail

$$\begin{aligned}
&\delta\psi_\mu^{\mathrm{L},j} = \mathrm{i}\Lambda^j{}_k\psi_\mu^{\mathrm{L},k} \\
&\delta\chi^{\mathrm{L},j} = -\mathrm{i}(\Lambda^j{}_k - \delta^j{}_k \operatorname{tr}\Lambda)\chi^k \\
&\delta A = B\operatorname{tr}\Lambda, \qquad \delta B = -A\operatorname{tr}\Lambda \\
&\delta F_{\mu\nu}^{ij} = \tfrac{1}{2}\Lambda^i{}_k\tilde{G}_{\mu\nu}{}^{jk} - i\leftrightarrow j.
\end{aligned} \tag{16}$$

The tensor $D^\mu G_{\mu\nu}{}^{jk}$ is again the field equation of V_ν^{jk}.

All SU(4) symmetries except the O(4) hold only on-shell. The same is true for the $N = 3$ model, as one easily sees by reduction (i.e., $A = B = 0$, etc.). There does not seem to be a consistent truncation to $N = 2$ or $N = 3$ supergravity theories of the SU(4) version of the $N = 4$ theory.

One can gauge both the SO(4) and the SU(4) models. The SO(4) model was gauged by Das, Fischler and Roček [111] and has a potential with an indifferent equilibrium. Of the SU(4) model, having only 6 spin 1 fields, one can at best gauge an SU(2) × SU(2) subgroup and this was done by Freedman and Schwarz [244]. However, in this case the potential has no stationary point.

Let us now jump to the $N = 8$ theory. One might expect that this theory is prohibitively complicated – but one can formulate it in a very simple way in $d = 11$ dimensions as $N = 1$ (simple) supergravity. If one then uses dimensional reduction to $d = 4$ dimensions, the full theory emerges automatically.

6.2. *The* $N = 8$ *model in* 11 *dimensions* [93]

Dimensional reduction means that all fields are assumed to depend only on $x^1, \ldots, x^4$ instead of $x^1, \ldots, x^{11}$. A tetrad in $d = 11$ splits up then in $d = 4$ into one tetrad e_μ^m ($m = 1, 4$), 7 vectors e_μ^a ($a = 5, 11$) and 7×7 scalar fields e_α^a ($\alpha = 5, 11$) which describe $\frac{1}{2}8 \times 7$ scalar particles (because the local O(7) symmetry, the residue of the O(10, 1) Lorentz symmetry, eliminates the antisymmetric parts of e_α^a). Dimensional reduction has been extremely fruitful in supergravity (and in global supersymmetry).

We now explain why the simplest form of $N = 8$ model is in $d = 11$ dimensions, and then construct this theory in $d = 11$ in the remainder of this subsection. Dimensional reduction and spontaneous symmetry breaking will be discussed in following subsections.

The example of the tetrad just given shows that by going to higher dimensions, supergravity theories can be reformulated in terms of fewer fields. The $N = 8$ theory in 4 dimensions has 8 gravitinos, hence one expects that its simplest form appears in $d = 10$ or $d = 11$ dimensions. Indeed, in $d = 10$ or $d = 11$, spinors have 32 components. The simplest version is the $d = 11$ theory, with only an 11×11 "elfbein" e_μ^m, a 32×11 gravitino $\psi_\mu{}^a$ and a "photon" $A_{\mu\nu\rho}$ (antisymmetric in $\mu\nu\rho$). That one needs $A_{\mu\nu\rho}$ follows from counting of states

e_μ^m transversal and traceless: $\frac{1}{2}9 \times 10 - 1 = 44$
$\psi_\mu{}^a$ transversal in gauge $\gamma \cdot \psi = 0$: $(9 \times 32 - 32) \times \frac{1}{2} = 128$
$A_{\mu\nu\rho}$ transversal: $\binom{9}{3} = 84$.

(A choice $A_{\mu_1\ldots\mu_6}$ has also $\binom{9}{6}=\binom{9}{3}=84$ states, but does not lead to a consistent theory [559].) Thus there are indeed equal numbers of bosons and fermions.

Having decided upon 11 dimensions we note that the signature of this space must be (10, 1) in order that one does not find ghosts in the dimensionally reduced action. With more than one time coordinate, some of the scalars e^a_α would have kinetic terms with the wrong sign.

Since $A_{\mu\nu\rho}$ is transversal it must be a gauge field with gauge invariance $\delta A_{\mu\nu\rho} = \partial_\mu \Lambda_{\nu\rho}$ + cyclic terms, with $\Lambda_{\nu\rho} = -\Lambda_{\rho\nu}$. Hence, the action must start with

$$\mathscr{L} = -\frac{e}{2}R(e,\omega) - \frac{e}{2}\bar{\psi}_\mu \Gamma^{\mu\rho\sigma} D_\rho(\omega)\psi_\sigma - \frac{e}{48}F^2_{\mu\nu\rho\sigma} \tag{1}$$

where $F_{\mu\nu\rho\sigma} = \partial_\mu A_{\nu\rho\sigma}$ plus 23 terms. (The symbol $\Gamma^{\mu\rho\sigma}$ has strength one, $\Gamma^{\mu\rho\sigma} = \Gamma^\mu\Gamma^\rho\Gamma^\sigma$ antisymmetrized.)

The gravitino must be real, in order that the counting of states works, or rather it must satisfy a Majorana condition. (Since $d = 11$ is odd, we cannot invoke a Weyl condition to halve the number of complex fields.)

One may prove in general or by giving an explicit representation that in $10+1$ dimensions there exists a real representation and that $C\gamma_\mu C^{-1} = -\gamma_\mu^{\mathrm{T}}$. It follows that in $d = 11$ just as in $d = 4$

$$(\bar{\lambda}\Gamma^{A_1}\cdots\Gamma^{A_n}\chi) = (-)^n(\bar{\chi}\Gamma^{A_n}\cdots\Gamma^{A_1}\lambda). \tag{2}$$

To construct the action, we begin by postulating

$$\delta e^m{}_\mu = \tfrac{1}{2}\bar{\epsilon}\gamma^m\psi_\mu, \qquad \delta\psi_\mu = D_\mu(\omega)\epsilon. \tag{3}$$

Using 1.5 order formalism, one finds for the variation of the Hilbert action

$$\tfrac{1}{2}e(\bar{\epsilon}\gamma^\nu\psi^a)(R_{a\nu} - \tfrac{1}{2}e_{a\nu}R). \tag{4}$$

For variation of ψ_σ in the gravitino action, we get

$$-\frac{e}{8}(\bar{\psi}_\mu\Gamma^{\mu\rho\sigma}\tfrac{1}{2}\Gamma^{mn}\epsilon)R_{\rho\sigma mn}(\omega), \qquad (\Gamma^{mn} = 2\sigma^{mn}). \tag{5}$$

On the other hand, varying $\bar{\psi}_\mu$ and partially integrating, neglecting torsion terms

$$D_\mu e^m{}_\rho - D_\rho e^m{}_\mu$$

since these are of higher order in ψ, we find

$$+\frac{e}{8}(\bar{\epsilon}\Gamma^{\mu\rho\sigma}\tfrac{1}{2}\Gamma^{mn}\psi_\sigma)R_{\mu\rho mn}(\omega). \tag{6}$$

With the symmetry in (2), the sum of these two variations equals

$$\frac{e}{16}\bar{\epsilon}\{\Gamma^{\mu\rho\sigma}, \Gamma^{mn}\}\psi_\sigma R_{\mu\rho mn}. \tag{7}$$

In the anticommutator only the terms with five and one gamma survive but the three gamma terms cancel. The five gamma term leads to a complete antisymmetrization of $R_{\mu\rho mn}$ and hence we again neglect these terms, since they are of higher order in ψ (without torsion the cyclic identity says that these terms vanish). The one-gamma terms are

$$(\delta^{\sigma m}\delta^{\rho n}\Gamma^\mu - \delta^{\sigma m}\delta^{\mu n}\Gamma^\rho + \delta^{\mu m}\delta^{\rho n}\Gamma^\sigma) - (m \leftrightarrow n) \tag{8}$$

and one finally finds that these terms again lead to an Einstein tensor. In fact, all terms cancel! Thus, this justifies our choice for δe_μ^m. This is a nontrivial result because it relies on the fact that no three gamma terms are present. If, for example, in $d = 11$ one would have $C\gamma_\mu C^{-1} = +\gamma_\mu{}^{\mathrm{T}}$, then one would have found a commutator $[\Gamma^{\mu\rho\sigma}, \Gamma^{mn}]$ and no invariance to order ψ could have been achieved. (In $3+2$ dimensions, $C\Gamma^A C^{-1} = +\Gamma^{A,\mathrm{T}}$ and one finds a commutator. Hence, although one can define in $3+2$ dimensions a Majorana spinor as usual, no supergravity exists. In $4+1$ dimensions one still has $C\Gamma^A C^{-1} = +\Gamma^{A,\mathrm{T}}$ but here one defines a Majorana spinor with internal indices and one finds now an anticommutator.)

To find the higher order terms, we recall that if ω is a solution of $\delta I(\text{total})/\delta\omega = 0$, then

$$I^{(2)}(\omega + \tau) + I^{(3/2)}(\omega + \tau) = I^{(2)}(\omega) + I^{(3/2)}(\omega) + \tau^2 \text{ terms}. \tag{9}$$

Hence, without loss of generality, we can take the Hilbert action as $eR(e, \omega)$ but we must then use 1.5 order formalism (i.e., $\omega = \omega(e, \psi)$ is a solution of $\delta I/\delta\omega = 0$). The τ^2 terms are of order ψ^4 and must be found later.

Taking the most general laws for $\delta A_{\mu\nu\rho} \sim \kappa\bar{\epsilon}\psi$ and $\delta\psi \sim \kappa F\epsilon$ and the most general terms for $\kappa\psi^3 F$ in the action, one finds a unique solution. A peculiar term bilinear in the Maxwell curls and linear in a bare photon field is needed as well. It is gauge invariant! Thus one has the theory up to terms trilinear in ψ in the variation laws and up to terms quartic in ψ in the action. The remaining trilinear terms are fixed by requiring that $\delta\psi_\mu$ be supercovariant. Indeed, since in $[\delta(\epsilon_1), \delta(\epsilon_2)]\psi_\mu$ the $\partial_\mu\epsilon$ terms cancel one must make $\delta\psi_\mu$ itself supercovariant. (In the commutator one finds a general coordinate contribution $(\partial_\mu\xi^\nu)\psi_\nu$ and a supersymmetry contribution $\partial_\mu(-\xi^\nu\psi_\nu)$ with $\xi^\nu = \frac{1}{2}\bar{\epsilon}_2\gamma^\nu\epsilon_1$. From the $\{Q, Q\}$ anticommutator on vierbeins we know this. The $\partial_\mu\xi^\nu$ terms cancel.) This yields

$$\delta\psi_\mu = D_\mu(\hat{\omega})\epsilon + (\text{gamma matrices})\epsilon\hat{F}_{\alpha\beta\gamma\delta} \tag{10}$$

where $\hat{F}$ is supercovariant, while $\hat{\omega}$ is the supercovariant extension of $\omega(e)$, which is thus the same expression as in $d = 4$

$$\hat{\omega}_{\mu mn} = \omega_{\mu mn}(e) + \tfrac{1}{4}(\bar{\psi}_\mu\gamma_m\psi_n - \bar{\psi}_\mu\gamma_n\psi_m + \bar{\psi}_m\gamma_\mu\psi_n). \tag{11}$$

Finally the quartic terms in the action are fixed by requiring that the gravitino field equation be itself supercovariant.

The final result for the $N = 8$ action in $d = 11$ is very simple. It is polynomial since a power series in

$\kappa A_{\mu\nu\rho}$ is not allowed (it would violate Maxwell gauge invariance)

$$\begin{aligned}
\mathscr{L} &= -\frac{e}{2} R(e,\omega) - \frac{e}{2}\bar{\psi}_\mu \Gamma^{\mu\rho\sigma} D_\rho\left(\frac{\omega+\hat{\omega}}{2}\right)\psi_\sigma - \frac{e}{48} F^2_{\mu\nu\rho\sigma} \\
&\quad - \frac{3D}{4} e\kappa [\bar{\psi}_\mu \Gamma^{\mu\alpha\beta\gamma\delta\nu}\psi_\nu + 12\bar{\psi}^\alpha \Gamma^{\beta\gamma}\psi^\delta][F_{\alpha\beta\gamma\delta} + \hat{F}_{\alpha\beta\gamma\delta}] + C\kappa\epsilon^{\mu_1\ldots\mu_{11}} F_{\mu_1\ldots\mu_4} F_{\mu_5\ldots\mu_8} A_{\mu_9\mu_{10}\mu_{11}} \\
\delta e^m{}_\mu &= \frac{\kappa}{2}\bar{\epsilon}\gamma^m\psi_\mu \\
\delta\psi_\mu &= \kappa^{-1} D_\mu(\hat{\omega})\epsilon + D(\Gamma^{\alpha\beta\gamma\delta}{}_\mu - 8\delta^\alpha_\mu \Gamma^{\beta\gamma\delta})\epsilon \hat{F}_{\alpha\beta\gamma\delta} \\
\delta A_{\mu\nu\rho} &= E\bar{\epsilon}\Gamma_{[\mu\nu}\psi_{\rho]}. \qquad (C, D, E \text{ are constants.})
\end{aligned} \tag{12}$$

Remarkably enough, the ψ^4 terms are obtained by putting in the gravitino action $\frac{1}{2}(\omega+\hat{\omega})$ as connection and also taking $\frac{1}{2}(F+\hat{F})$ in the Noether-type coupling. I do not know why this "minimal" solution yields all ψ terms. (In $N=2$, $d=4$ supergravity, and also in the $d=4$ Maxwell–Einstein system one finds the same features.) Roughly speaking, the covariantization terms in $\hat{F}$ have twice as many ψ's as the terms with F, so one needs half of them to obtain a supercovariant field equation, but unlike the $N=2$ model, the four ψ's in the F terms do not appear symmetrically, so that this argument is not complete.

The spin connection ω solves $\delta I/\delta\omega = 0$ and is given by

$$\omega_{\mu mn} = \hat{\omega}_{\mu mn} - \tfrac{1}{8}(\bar{\psi}^\alpha \Gamma_{\alpha\mu mn\beta}\psi^\beta). \tag{13}$$

In $d=4$, $\omega = \hat{\omega}$, but here they differ. This illustrates once again that by going to more complicated theories one finds out which equalities are a coincidence and which are not.

The gauge algebra is the usual one except that the Lorentz parameter is

$$\lambda^{mn} = \xi^\nu \hat{\omega}_\nu{}^{mn} + \bar{\epsilon}_2(\Gamma^{mn\alpha\beta\gamma\delta} - 24 e^{m\alpha} e^{n\beta}\Gamma^{\gamma\delta})\hat{F}_{\alpha\beta\gamma\delta} \tag{14}$$

while two supersymmetry transformations on $A_{\mu\nu\rho}$ lead to a gauge transformation $\delta A_{\mu\nu\rho} = \partial_\mu \Lambda_{\nu\rho}$ + cyclic terms, where

$$\Lambda_{\mu\nu} = -\tfrac{1}{2}\bar{\epsilon}_2 \Gamma_{\mu\nu}\epsilon_1 - \xi^\sigma A_{\sigma\mu\nu}. \tag{15}$$

Terms such as $-\xi^\sigma A_{\sigma\mu\nu}$ always come from converging P in $\{Q, Q\} = P +$ more, into

$$\delta_P(\xi) = \delta_{\text{gen.coord.}}(\xi^\nu) - \delta_{\text{gauge}}(\xi^\lambda h_\lambda{}^A). \tag{16}$$

The gauge algebra reads

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_{\text{gen. coord.}}(\xi^\nu) + \delta_Q(-\xi^\nu\psi_\nu) + \delta_{\mathrm{L}}(\lambda^{mn} \text{ in (14)}) + \delta_{\text{Maxwell}}(\Lambda_{\mu\nu} \text{ in (15)}) \tag{17}$$

and this suggests strongly that the $N=1$, $d=11$ model can be obtained by gauging Osp(1/32) since Sp(32) is spanned by Γ^m, Γ^{mn}, Γ^{mnrst}. (These are the matrices satisfying the definition of symplectic matrices $CM + M^{\mathrm{T}}C = 0$ with antisymmetric C. We recall that $C\Gamma^A = -\Gamma^{A,\mathrm{T}}C$.) One would like to consider e_μ^m and $\omega_\mu{}^{mn}$ as the gauge fields of Γ^m, and Γ^{mn}. However, $A_{\mu\nu\rho}$ cannot be considered as the gauge field of Γ^{mnrst} (it does not have enough indices and it is totally antisymmetric), and this is a problem for future research [559].

6.3. The $N=8$ model in 4 dimensions by dimensional reduction

In this subsection we discuss the important and pioneering work of Cremmer and Julia [96, 99]. As we saw, in ordinary dimensional reduction all fields depend only on $x^1, \ldots, x^4$ instead of on all eleven coordinates. One then proceeds to compute the various quantities appearing in the action: inverse vielbein and connections. One writes the four-dimensional part of the elfbeins as $\delta^\gamma e^m{}_\mu$ where δ is the determinant of the 7×7 part of the elfbein, and by choosing γ appropriately, the Einstein action of the reduced theory has the usual form.

In this section we discuss how the massless $N=8$ model in $d=4$ dimensions is obtained by ordinary (i.e. not accompanied by spontaneous symmetry breaking) dimensional reduction. This analysis will also lead us to deduce that the $d=4$ theory has as internal symmetry an E(7) global $\times$ SU(8) local. However, the SU(8) local has not 63 gauge fields, so that this SU(8) cannot serve as a grand unification of $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$. Rather, the connections are nonlinear functions of other fields, in the same way as the spin connection $\omega(e)$ gauges SO(3, 1) but is not an independent gauge field.

In $d=11$ dimensions, one has general coordinate transformations

$$\delta V_\Lambda = \xi^\Pi \partial_\Pi V_\Lambda + (\partial_\Lambda \xi^\Pi) V_\Pi, \qquad (\Lambda, \Pi = 1, 11). \tag{1}$$

Splitting $\Lambda = (\lambda, \alpha)$ with $\lambda, \mu, \nu = 1, 4$ and $\alpha, \beta, \gamma = 5, 11$ one has under ordinary dimensional reduction with fields $\phi(x^1, \ldots, x^{11})$ restricted to $\phi(x^1, \ldots, x^4)$ a global symmetry group generated by

$$\xi^\Pi = (\xi^\mu = 0, \xi^\alpha = M^\alpha{}_\beta x^\beta). \tag{2}$$

Indeed, δV_Λ again depends only on $x^1, \ldots, x^4$ as follows from

$$\delta V_\mu = 0, \qquad \delta V_\alpha = (\partial_\alpha \xi^\beta) V_\beta = M^\beta{}_\alpha V_\beta. \tag{3}$$

One must in fact require that $\partial_\alpha \xi^\alpha = M^\alpha{}_\alpha = 0$ in order that the reduced Hilbert action is still invariant under this special class of general coordinate transformations. To see this, note that $\delta(eR) = \partial_\Lambda(\xi^\Lambda eR)$ since the action is a scalar density. Usually one discards this total derivative since coordinates tend to infinity, but now 7 coordinates have become little circles and one must require that $\partial_\alpha \xi^\alpha = 0 = M^\alpha{}_\alpha$. Thus, in $d=4$ one expects a global group SL(7, R).

The Lorentz group O(10, 1) in $d=11$ acts on tetrads as

$$\delta e_\Lambda{}^M = \omega^M{}_N e_\Lambda{}^N. \tag{4}$$

Splitting the flat index M into (m, a) with $m, n, \cdots = 1, 4$ and $a, b, c, \cdots = 5, 11$, we have

$$\begin{aligned} \delta e^m{}_\Lambda &= \omega^m{}_n e^n{}_\Lambda + \omega^m{}_a e^a{}_\Lambda \\ \delta e^a{}_\Lambda &= \omega^a{}_n e^n{}_\Lambda + \omega^a{}_b e^b{}_\Lambda. \end{aligned} \tag{5}$$

Thus we can fix the parameters $\omega^a{}_n = -\omega_n{}^a$ by putting $e_\alpha^n = 0$. In this case the vielbein has the form

$$e_\Lambda{}^M = \begin{pmatrix} e_\mu{}^m & e_\mu{}^a \\ 0 & e_\alpha{}^a \end{pmatrix}. \tag{6}$$

The remaining symmetry group is now $O(3,1)\times O(7)$, and one expects in $d=4$ a local group $O(7)$.

Let us now do the reduction of the $d=11$, $N=1$ fields down to $d=4$. If one reduces from $d=11$ to $d=4$, one finds for the bosonic fields ($a,\alpha,\beta,\dots$ internal indices)

$$
\begin{aligned}
&e^{M}{}_{\Lambda}\to 1 \text{ tetrad}, 7 \text{ photons } e^{a}_{\mu}, \tfrac{1}{2}7\times 8 \text{ scalars } e^{a}_{\alpha}\\
&A_{\Lambda\Pi\Sigma}\to A_{\mu\nu\rho}, 21 \text{ photons } A_{\mu\alpha\beta}, 7 \text{ scalars } A_{\mu\nu\alpha}, \qquad \binom{7}{3}=35 \text{ scalars } A_{\alpha\beta\gamma}.
\end{aligned} \tag{7}
$$

The field $A_{\mu\nu\rho}$ in $d=4$ is pure gauge, while $A_{\mu\nu}$ is a scalar since $F_{\mu\nu\rho}=\epsilon_{\mu\nu\rho\sigma}\psi^{\sigma}$ so that $\partial^{\mu}F_{\mu\nu\rho}=0$ implies $\psi_{\sigma}=\partial_{\sigma}\psi$. This is indeed the bosonic sector of the $N=8$, $d=4$ model, but all bosons have only a manifest $O(7)$ symmetry (from $O(10,1)\to O(3,1)\times O(7)$) while the $O(8)$ invariance is hidden.

The fermionic sector is reduced as follows. One splits the $d=11$ gravitino ψ^{A}_{Λ} into eight gravitinos with $\Lambda=\mu=1,4$ and $A=(1,32)=ai$ with $a=1,4$ and $i=1,8$ and 56 spin $\frac{1}{2}$ fields $\lambda_{\alpha}{}^{a}$ with $\alpha=5,11$. One finds thus also the fermionic fields on $N=8$, $d=4$ theory in this way. The numbers (1, 8, 28, 56, 70) are the dimensions of antisymmetric tensor representations of SU(8). (Under SO(8) the scalars A^{ijkl} and B^{ijkl}, with A self dual and B anti-self dual, transform into themselves, but the complex $\phi^{ijkl}\equiv(A+\mathrm{i}B)^{ijkl}$ form an irreducible representation of SU(8) and satisfy $(\phi^{*})_{ijkl}=\phi_{ijkl}$ with $\phi_{i_1\dots i_4}=\epsilon_{i_1\dots i_8}\phi^{i_5\dots i_8}$ and $\phi^{*}_{ijkl}=A^{ijkl}-\mathrm{i}B^{ijkl}$.) Since one expects that there are no central charges present on-shell, one expects that there is on-shell a global SU(8) symmetry.*

The first nontrivial extension of the expected $SL(7,\boldsymbol{R})$ global $\times$ O(7) local occurred when it was found that there was a set of global scale invariances for the action in $d=4$ which extended $SL(7,\boldsymbol{R})$ to $GL(7,\boldsymbol{R})$. Secondly, it was found that the action had a local SU(8) invariance, of which the local O(7) was a part. (Historically, Cremmer and Julia conjectured, when they tried to extend O(7) local to an expected O(8) local, that in fact an SU(8) local would be possible. Upon reduction to the $N=4$ model (the SU(4) version of the $N=4$ model we have discussed) this gauge gets fixed and no SU(4) local is left.)

General theory of nonlinear realization tells us that the scalars can be written as an exponential with in the exponential the generators of a global group. Since one can eliminate 63 scalars by fixing the local SU(8), and there are 70 scalar particles in the $N=8$ model, this global group must have $63+70=133$ generators. To make these arguments more familiar, we compare with the usual tetrad $e^{m}{}_{\mu}$. It describes 10 particles (if we forget the general coordinate invariance for a moment). The local group is SO(3, 1) with 6 generators, and hence the global group has $10+6=16$ generators. This group is of course $GL(4,\boldsymbol{R})$. The local group must be a subgroup of the global noncompact group in order to eliminate ghosts.

Cremmer and Julia found the Lie algebra with 133 generators in Cartan's work: E(7)! This algebra is defined by the real matrices Λ and Σ

$$
\begin{aligned}
\delta x^{ij}&=\Lambda^{i}{}_{k}x^{kj}-\Lambda^{j}{}_{k}x^{ki}+\tfrac{1}{24}\epsilon^{ij\alpha_1\dots\alpha_6}\Sigma_{\alpha_1\dots\alpha_4}y_{\alpha_5\alpha_6}\\
\delta y_{ij}&=\Lambda_{i}^{\ k}y_{kj}-\Lambda_{j}^{\ k}y_{ki}+\Sigma_{ijkl}x^{kl}
\end{aligned} \tag{8}
$$

with $\Lambda^{i}{}_{k}=-\Lambda_{k}{}^{i}$, $\Lambda^{i}{}_{i}=0$. (The $\Lambda_{k}{}^{i}$ are not the transposed of $\Lambda^{i}{}_{k}$.) The Σ_{ijkl} are totally antisymmetric. The vectors $x^{ij}=-x^{ji}$ and $y^{ij}=-y^{ji}$ have each 28 components, and this 56 dimensional representation is in fact the fundamental representation of E(7). The 63 Λ's generate $SL(8,\boldsymbol{R})$ (and the previous $GL(7,\boldsymbol{R})$ is thus

* Not a U(8) symmetry since the ϵ-symbol is only an SU(8) invariant tensor.

contained in E_7) while the remaining $\binom{8}{4} = 70\Sigma$'s generate the rest of E_7. Also the expected SU(8) global can be found by considering $(x^{jk} + iy_{jk})$ (not to be confused with the SU(8) local). This SU(8) is the maximal compact subgroup of this form of E_7.

The global E(7) symmetry acts on the curls of the photons, not on the photon fields themselves. It is an on-shell symmetry only, of the same kind as the dual-chiral symmetries in the $N = 2, 3, 4$ models. In fact, truncating the $N = 8$, $d = 4$ model to the $N = 4$, $d = 4$ model, one finds back the $SU(4) \times SU(1, 1)$ global symmetry of the so-called SU(4) model.

The reader may have wondered what happens with the general coordinate transformations whose parameters are in $d = 11$ given by $\xi^\alpha(x^1, \ldots, x^4)$ where $\alpha = 5, 11$. The answer is that they become Maxwell gauge parameters for the photons $A_\mu{}^\alpha$ which are defined by $e_\mu{}^a = A_\mu{}^\alpha e_\alpha{}^a$. As one easily checks, $\delta A_\mu{}^\alpha = \partial_\mu \xi^\alpha$, while $\delta e^m{}_\mu = \delta e_\alpha^a = 0$.

There are much more fascinating details. We regretfully refer to the literature for this massless $N = 8$, $d = 4$ model, and now turn to a massive version of it.

6.4. Spontaneous symmetry breaking in the N = 8 model by dimensional reduction

The work which we now discuss is due to Cremmer, Scherk and Schwarz [404, 405, 406]. It solves a long-standing puzzle of how to obtain spontaneous symmetry breaking in the extended supergravities. For simple supergravity, spontaneous symmetry breaking can occur as soon as S or P are nonzero (which may or may not imply that the scalar fields A and B have nonzero vacuum expectation values [541]). However, for $N > 1$, many supergravity practitioners were afraid that no analogous results would be obtained. As usual, matters turned out to be better than expected.

A most interesting way to give masses to the particles of the $N = 8$ model in $d = 4$ dimensions is to first reduce the $d = 11$, $N = 1$ theory to the $d = 5$, $N = 8$ massless theory. Then, in $d = 5$ dimensions, one writes fields as

$$\phi(x^1, \ldots, x^5) = \exp(Mx^5)\psi(x^1, \ldots, x^4) \tag{1}$$

where M is in general a matrix; see below. In this way, for example, a massless scalar field in $d = 5$ can become massive in $d = 4$

$$\varphi^* \bigcirc \varphi = \psi^*(\Box - M^2)\psi. \tag{2}$$

Note that this kind of symmetry breaking is not such that the theory itself determines the minimum of a Higgs potential, but, rather, it is we who (arbitrarily) prescribe that fields must have the factored form as in (1). Since coordinates are dimensional, the substitution in (1) introduces masses in a natural way.

In general, the procedure for constructing a D-dimensional theory with spontaneously broken symmetry is to first consider a theory with unbroken symmetry in $D + E$ dimensions and to wrap the extra E dimensions into circles with such small radii that only the lowest Fourier coefficients have finite mass. If we write fields as in (1), then the Lagrangian density in $D + E$ dimensions is independent of M (and hence remains gauge invariant in $D + E$ and in D dimensions) as long as the transformation $\phi = \exp(Mx^5)\psi$ corresponds to a symmetry. In practice, the useful symmetries are the global symmetries.

How do we find global symmetries in $D + E$ dimensions? A very interesting idea is to consider yet a higher dimension, say $D+E+F$, and to use local symmetries there to obtain global symmetries in $D+E$ dimensions. For our applications we start with the $N=1$ model in $D+E+F=11$ dimensions, and want to obtain a global symmetry of as high a rank as possible in $D+E=5$ dimensions. The local symmetry we use is general coordinate invariance in $d=11$. (In order that the method works also for systems without fermions, we do not study whether local Lorentz invariance can be used.) As we discussed in the last section, the expected global symmetry group is SL(6, R). Now no extra scale transformations are present which extend this SL(6, R) to GL(6, R). (Note that E_6 does not contain GL(6, R) but that E(7) contains GL(7, R.) Basically the reason is that there are two types of terms in the action which cannot be made scale-invariant at the same time, namely $\delta^{1/2}(F^{\alpha\beta}_{\mu\nu})^2$ and $\epsilon^{\mu\nu\rho\sigma\tau}\epsilon_{abcdef}F^{ab}_{\mu\nu}F^{cd}_{\rho\sigma}A^{ef}_{\tau}$. Thus the expected global symmetry is SL(6, R). We refer to the previous section.

The Lorentz group SO($D+E+F-1$, 1) splits up into SO($D+E-1$, 1) $\times$ SO(F) if we fix part of this local symmetry by casting the vielbein in triangular form. This we discussed in the previous section. Then, naively we expect for the $N=8$ model in $d=5$ dimensions as internal symmetry group SO(6) local $\otimes$ SL(6, R) global.

However, things are more interesting. The particle spectrum in $d=5$ consists of irreducible representations of Sp(8) rather than of SO(6). This particle spectrum is: one tetrad e^m_μ ($m, \mu = 1, 5$), eight gravitinos $\psi_\mu{}^a$ ($a=1, 8$), 27 vector fields, 48 spin $\frac{1}{2}$ fields and 42 scalar fields. Indeed, all fields can be represented by completely antisymmetric tensors which are traceless with respect to the symplectic metric. For example, the photons are described by the 27-dimensional Sp(8) representation $A_\mu{}^{\alpha\beta} = -A_\mu{}^{\beta\alpha}$ with $\Omega_{\alpha\beta}A_\mu{}^{\alpha\beta}=0$ ($\alpha, \beta = 1, 8$) which is not an irreducible SO(6) representation.

This particle spectrum is obtained from the fields $e_\Lambda{}^M$, $\psi_\Lambda{}^B$, $A_{\Lambda\Pi\Sigma}$ ($M, \Lambda, \Pi, \Sigma = 1, 11$ and $B=1, 32$) by ordinary dimensional reduction as follows:

$$
\begin{aligned}
&e^M_\Lambda \to e^m_\mu,\ 6 \text{ vectors } e^a_\mu,\ \tfrac{1}{2}(6\times 7)=21 \text{ scalars } e^a_\alpha\\
&\psi^B_\Lambda \to 8\psi^a_\mu \text{ and } 48 \text{ spin } \tfrac{1}{2} \text{ fields } \psi^i_\alpha\ (i=1,8;\ \alpha=5,11)\\
&A_{\Lambda\Pi\Sigma} \to 1 \text{ scalar } A_{\mu\nu\rho},\ 6 \text{ vectors } A_{\mu\nu\alpha},\ 15 \text{ vectors } A_{\mu\alpha\beta},\\
&20 \text{ scalars } A_{\alpha\beta\gamma}.
\end{aligned}
\tag{3}
$$

The field $A_{\mu\nu\rho}$ has as field equation $D^\mu F_{\mu\nu\rho\sigma}=0$ and since $F_{\mu\nu\rho\sigma} = \partial_{[\mu}A_{\nu\rho\sigma]} = \epsilon_{\mu\nu\rho\sigma\tau}\varphi^\tau$, it follows that $\varphi_\tau = \partial_\tau\varphi$ and $A_{\mu\nu\rho}$ describe vectors (since $\partial_{[\mu}A_{\nu\rho]a} = F_{\mu\nu\rho a} = \epsilon_{\mu\nu\rho\sigma\tau}\varphi^{\sigma\tau}_a$ one has $\varphi^{\sigma\tau}_a = \partial^{[\sigma}\varphi_a{}^{\tau]}$. Inserting this into F yields a Maxwell action). As a check one may verify that, as in $d=11$, there are 128 bosonic and 128 fermionic states (in $d=5$ the little group of a massless particle is SO(3) and hence a particle of spin J has $2J+1$ polarizations).

Thus one expects that one can write the fields as completely antisymmetric and symplectic traceless tensors e^m_μ, ψ^a_μ, A^{ab}_μ, χ^{abc}, ϕ^{abcd} where $\Omega_{ab}A^{ab}_{\mu\nu}=0$ and $\Omega = I_4\times \mathrm{i}\tau_2$ is the antisymmetric 8×8 metric of Sp(8).

The reality conditions of the bosonic fields are $(A_\mu{}^{ab})^* = A_{\mu ab}$ and idem for ϕ^{abcd} where indices are raised and lowered by Ω. Note that under Sp(8) a vector V^a transforms as $\delta V^a = \Lambda^a{}_b V^b$ and that $\delta(V^a\Omega_{ab}V^b)=0$. Hence, $\delta V_a = -V_b\Lambda^b{}_a$ where $V_b = V^a\Omega_{ab}$. Only if $(V^a)^*$ transforms as V_a can we impose the reality conditions mentioned above. Now $\delta(V^a)^* = (\Lambda^a{}_b)^*(V^b)^*$. Hence one must require that $(\Lambda^a{}_b)^* = -\Lambda^b{}_a$. In other words, in addition to being symplectic, the matrices must be antihermitian. *The local symmetry group is Usp(8), not Sp(8).* However, they have the same generators.

Symplectic matrices satisfy $M^{\mathrm{T}}\Omega + \Omega M = 0$. Multiplying by Ω one has also $\Omega M^{\mathrm{T}} + M\Omega = 0$ since $\Omega^2 = -1$ and taking the real and imaginary parts of the sum and differences of these equations, one sees that Usp(N, C) has as many generators as Sp(N, R). In fact, for $N=8$, Usp(N, C) = Sp(N, R).

In order to discuss the reality conditions for the spinors, we want to define Majorana spinors. In $d=5$ one has $C\Gamma^A C^{-1} = +\Gamma^{A,\mathrm{T}}$ with $A=1, 5$. (In a Majorana representation $C=\gamma^4\gamma^5$. In $d=5$, C is antisymmetric.) Thus we define

$$(\bar{\lambda}^{\mathrm{D}})^a{}_\beta = (\lambda^\alpha_a)^\dagger(\gamma^4)^\alpha{}_\beta, \qquad (\bar{\lambda}^{\mathrm{M}})^a_\beta = \lambda^\alpha_b C_{\alpha\beta}\Omega^{ab} \tag{4}$$

since, as we discussed, $(\lambda_a)^*$ transforms as λ^a under Usp(8), and since we can only equate tensors with indices in the same position. We now define a Majorana spinor by $\bar{\lambda}^{\mathrm{D}} = \alpha\bar{\lambda}^{\mathrm{M}}$ with α arbitrary. We leave as an exercise to show that taking the complex conjugate of this relation and reinserting in this result again that $\bar{\lambda}^{\mathrm{D}} = \alpha\bar{\lambda}^{\mathrm{M}}$, one finds indeed a consistent result for $|\alpha|=1$. Crucial for this result is that $\Omega^*\Omega = -1$. (In fact, the most general solution of such Ω is probably equivalent to the symplectic metric we have assumed so far.) Thus the reality conditions for the fermions are

$$(\chi^{abc})^* = \chi^{\mathrm{T}}_{abc}C, \quad \text{idem } \psi^a_\mu. \tag{5}$$

We can now again use the theory of nonlinear representations to predict what the global symmetry group will be. Since Sp(8) has $\frac{1}{2}(8\times 9) = 36$ generators and there are 42 scalars, one needs a Lie algebra with 78 generators: E_6!

The Lie algebra E_6 is most easily defined by giving its fundamental 27-dimensional representation

$$\delta z^{ij} = \Lambda^i{}_k z^{kj} - \Lambda^j{}_k z^{ki} + \Sigma^{ij}{}_{kl} z^{kl} \tag{6}$$

where $\Lambda^i{}_k$ is antiHermitian and symplectic ($\Omega\Lambda + \Lambda^{\mathrm{T}}\Omega = 0$ meaning that $\Lambda_{ij} = \Lambda_{ji}$) and Σ_{ijkl} is totally antisymmetric, traceless with respect to Ω_{ij} and $(\Sigma_{ijkl})^* = \Sigma^{ijkl}$. The tetrad is of course a scalar under both groups (Sp(8) local and E_6 global), but gravitinos are in the 8 representation of Sp(8) (and denoted by ψ^a_μ) while photons are in the 27 of E_6 and denoted by $A^{\alpha\beta}_\mu$ (and thus Sp(8) scalars). Finally, the scalars are written as $\mathcal{V}_{\alpha\beta}{}^{ab}$ where the indices α, β are E_6 indices and a, b are Sp(8) indices.

One writes $\mathcal{V}_{\alpha\beta}{}^{ab}$ as an exponential of E_6 generators

$$\mathcal{V} = [\exp(c^{de}\Lambda_{de} + c_{ijkl}\Sigma^{ijkl})]_{\alpha\beta}{}^{ab} \tag{7}$$

where c are coefficients and Λ, Σ indicate the various generators which are matrices. For example

$$(\Lambda_{cd})_{\alpha\beta}{}^{ab} = (\Lambda_{cd})_{[\alpha}{}^{[a}\delta^{b]}_{\beta]}, \qquad (\Lambda_{cd})_\alpha{}^a = \Omega_{[c\alpha}\delta^a_{d]}. \tag{8}$$

Since, in general, $\mathcal{V}^{-1}\partial_\mu\mathcal{V}$ lies in the algebra (it is *not* a group element), one has $\mathcal{V}^{-1}\partial_\mu\mathcal{V} = Q_\mu + P_\mu$ where $(Q_\mu)_{cd}{}^{ab} = Q_\mu{}^{[a}{}_{[c}\delta^{b]}{}_{d]}$ is a linear combination of Sp(8) generators. Since

$$\delta Q_\mu{}^a{}_b = \partial_\mu\Lambda^a{}_b + \Lambda^a{}_c Q_\mu{}^c{}_b - Q_\mu{}^a{}_c\Lambda^c{}_b \tag{9}$$

Q_μ is the Sp(8) gauge field, and one can define Sp(8) covariant derivatives. For example, $\mathcal{V}^{-1}D_\mu\mathcal{V} = P_\mu$. (Just as for the spin connection in $d=4$, $N=1$ theory, one can find the exact form of Q by solving its field equation $\delta I/\delta Q_\mu = 0$. This is again 1.5 order formalism.)

We will not write down the complete action; for that we refer to the literature. However, let us now turn to the mass generation. One writes only the fields with curved indices in the form of (1), namely

$$\begin{aligned} A_\mu{}^{\alpha\beta}(x, x^5) &= U^\alpha{}_{\alpha'}(x^4) U^\beta{}_{\beta'}(x^5) A_\mu{}^{\alpha'\beta'}(x) \\ U^\alpha{}_\beta(x^5) &= \exp(Mx^5) \end{aligned} \tag{10}$$

where M is a Sp(8) matrix, which can be taken as

$$M = \begin{pmatrix} 0 & m_1 & & & \\ -m_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & m_4 \\ & & & -m_4 & 0 \end{pmatrix}. \tag{11}$$

In principle any E_6 matrix could have been chosen as mass matrix, but in order to exclude a cosmological constant, one selects the maximal compact subgroup of E_6. As it happens, this is also a Sp(8) group. (This is thus a global Sp(8) and has nothing to do with the Sp(8) local which extended the O(6) local.) Since Sp(8) has rank 4, there are precisely four arbitrary mass parameters generated.

Inserting (10) into the Maxwell action, one finds as mass term for the photons

$$\mathcal{L}^{\text{mass}}(\text{photons}) = A^\mu_{\alpha\beta} A^{\alpha\beta}_\mu ((-)^\alpha m_{[\alpha/2]} + (-)^\beta m_{[\beta/2]}). \tag{12}$$

This leads in $d = 4$ dimensions to 3 massless photons (A_{12}, A_{34}, A_{56} and A_{78}, satisfying $\Omega^{\alpha\beta} A_{\alpha\beta} = 0$) and 24 massive vector fields with masses $(m_i \pm m_j)^2$ where $i, j = 1, 4$ (each mass is doubly degenerate). A fourth massless photon is obtained by the reduction of the vielbein from 5 to 4 dimensions. One can take $m_1 = m_2 = m_3$ in which case one finds six more massless photons, yielding a total of 10. These represent the 8 gluons, the ordinary photon and the Z^0 or the antigraviton, see Scherk in ref. [391].

For the fermions, masses are due to the Sp(8) connections Q_5 in the covariant derivatives, while the scalars obtain masses through P_5. We recall that

$$\mathcal{V}_{\alpha\beta}{}^{ab}(x, x^5) = \mathcal{V}_{\gamma\delta}{}^{ab}(x) (U^{-1}(x^5))^\gamma{}_\alpha (U^{-1}(x^5))^\delta{}_\beta \tag{13}$$

so that one finds for $\mathcal{V}^{-1}\partial_5\mathcal{V}$ the result

$$-\mathcal{V}^{-1}{}_{cd}{}^{\alpha'\beta'}(2M^\alpha{}_{\alpha'}\delta^\beta{}_{\beta'})\mathcal{V}_{\alpha\beta}{}^{ab} = e^{-c\Sigma} M \cdot \Lambda \, e^{c\Sigma} = M \cdot \Lambda + [M \cdot \Lambda, c\Sigma] + \cdots \tag{14}$$

in the special Sp(8) gauge where there are no Λ generators in the exponent of $\mathcal{V}$. Clearly, Q_5 starts with $M \cdot \Lambda$ and hence Q_5 yields masses to the fermions.

The masses of the scalars come from P_5 because the scalar kinetic matrix is $\operatorname{tr} D_\mu \mathcal{V}^{-1} D^\mu \mathcal{V} = -\operatorname{tr} P_\mu P^\mu$ and because P_5 is linear in masses *and* scalar fields: $P_5 = [M \cdot \Lambda, c\Sigma] + \cdots$. In fact, the E_6 algebra has a Z_2 grading: $[\Lambda, \Lambda] \sim \Lambda$, $[\Lambda, \Sigma] \sim \Sigma$, $[\Sigma, \Sigma] \sim \Lambda$. (This is thus a Cartan grading of an ordinary Lie algebra. Superalgebras have also a Z_2 grading, but the bracket relation of two odd elements is for superalgebras symmetric whereas it is here still, as usual, antisymmetric.) Thus the Q_μ are even in the number of scalar fields c, while the P_μ are odd.

This concludes our discussion of the way in which one can introduce masses into the $N = 8$ model in four dimensions. One obtains a massive action without cosmological constant (which previously was always disastrously large) and with a potential which is bounded from below.* It is a beautiful method, but it may not be the last word. Hence (in the author's opinion at least) it is not impossible that one can also construct different mass spectra. Perhaps not everything we need for physical applications descends upon us from higher dimensions. Therefore we refer the interested reader to a result in the literature which gives the most general spontaneous symmetry breaking which can take place in the $N = 1$ model in 4 dimensions.

As shown by Cremmer, Julia, Scherk, Ferrara, Girardello and the author [95], one can indeed break supersymmetry spontaneously (at least in $N = 1$ supergravity) such that the super-Higgs effect occurs, the potential is bounded from below, there is no cosmological constant (this fixes only part of the freedom in the action) and a mass formula $\Sigma(2J+1)\,(-)^J\,(m_J)^2 = 0$ is found. All masses are further totally free, and the model still contains an arbitrary *function* of *two* variables. Clearly, local supersymmetry is much less restrictive than was thought in the beginning of supergravity.

Finally, we mention one of the outstanding problems for the $N = 8$ model: in how far are the quantum corrections still finite if one adds masses? It has been shown that all one-loop divergences in the corrections to the cosmological constant vanish, leaving only a finite constant correction. However, whether also physical processes are finite is an open question. If they are, one might call the symmetry breaking discussed in this section spontaneous, since for spontaneously broken theories the theory behaves in the ultraviolet region as if there are no masses. Let us conclude with an optimistic note. Perhaps the $N = 8$ model with masses is the ultimate finite quantum theory of gravity, and it will be *shown* that all particles of nature appear as bound states of this model.

6.5. Auxiliary fields for $N = 2$ supergravity

Recently, sets of auxiliary fields for $N = 2$ supergravity have been found. A first set for Poincaré supergravity was found by Fradkin and Vasiliev [228, 229], and de Wit and van Holten [164, 169], by direct means. Breitenlohner and Sohnius found a set by starting with the causal group (Poincaré plus local scale transformation) [72]. All this work has been clarified by superconformal methods. Interestingly, these superconformal methods are quite similar to the corresponding $N = 1$ case which we discussed in section 4, and it is hoped that this approach will also be useful for the $N = 8$ case.

The original $N = 2$ Poincaré auxiliary fields constituted the following set

$$\begin{aligned}&\text{Bose fields:}\quad A_m,\ A_m{}^{\overline{ij}},\ V_m{}^{\hat{ij}},\ S,\ P^{\overline{ij}},\ t_{mn}{}^{\hat{ij}},\ V_m,\ M^{\hat{ij}},\ N^{\hat{ij}}\\ &\text{Fermi fields:}\quad \chi^i,\ \lambda^i.\end{aligned} \tag{1}$$

The bars denote symmetric and traceless tensors, the hooks antisymmetric tensors. In the action they appear as

$$\begin{aligned}\mathcal{L} = {}& -S^2 - \tfrac{1}{2}(P^{ij})^2 + \tfrac{1}{8}(t^{ij}_{mn})^2 + A_m^2 + \tfrac{1}{4}(A^{ij}_m)^2 + \tfrac{1}{4}(V^{ij}_m)^2 - \tfrac{1}{2}(V_m)^2\\ & - \tfrac{1}{4}(M^{ij})^2 - \tfrac{1}{4}(N^{ij})^2 + 2\bar{\lambda}^i(\chi^i + \not{D}\lambda^i) + \text{trilinear terms.}\end{aligned} \tag{2}$$

In order to make the reduction to $N = 1$ supergravity, one starts with $\delta e^m{}_\mu = \bar{\epsilon}_1\gamma^m\psi^1_\mu + \bar{\epsilon}_2\gamma^m\psi^2_\mu$, and sees how two multiplets develop:

* The one-loop corrections to the cosmological term are finite [406] due to the mass relations $\Sigma(-)^J(2J+1)m_J^{2n}$ for $n = 0, 1, 2, 3$ [208].

$$(e^m{}_\mu,\ \psi^1_\mu,\ \chi^1,\ \lambda^1,\ S,\ P^{11},\ V_m,\ A^{11}_m,\ A_m)$$

$$(\psi^2_\mu,\ \chi^2,\ \lambda^2,\ B^{12}_\mu,\ t^{12}_{mn},\ P^{12},\ M^{12},\ N^{12},\ V^{12}_m,\ A^{12}_m)\,. \tag{3}$$

The first multiplet is, in fact, the nonminimal set of auxiliary fields of refs. [68, 69, 160] for $N = 1$ supergravity. The second multiplet is a spin $(\frac{3}{2}, 1)$ multiplet, which, however, does not coincide with the spin $(\frac{3}{2}, 1)$ multiplet of ref. [347], one characteristic difference being that in the latter there are auxiliary gauge fields (see subsection 1.8, eq. (11)). A check on the set in (1) is that by covariantly quantizing the theory with auxiliary fields according to the procedure of subsection 2.2, and then eliminating the auxiliary fields by means of their field equation, one obtains the same result as when one quantizes the theory without auxiliary fields [473] (see subsection 2.8). This equivalence was the basis of the construction of refs. [228, 229].

As expected from the $N = 2$ model without auxiliary fields, the transformation rules as well as the Poincaré action (but not the de-Sitter action) are manifestly SU(2) invariant. From now on we use SU(2) notation.

Based on the auxiliary fields one can define the following multiplets

(i) scalar multiplets: $(A^\alpha{}_i,\ \chi^\alpha,\ F^\alpha{}_i)$. These multiplets have nonvanishing central charges ($\alpha = 1, 2$);

(ii) chiral multiplets (in superspace: $(1+\gamma_5)D^\alpha{}_i\phi = 0$) with complex and chiral $(A_i,\ B_{ij},\ C,\ \tilde{F}_{mn} = -F_{mn},\ \psi_i,\ \lambda_i)$;

(iii) vector multiplets (the sum of $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ multiplets) containing $(A^{\hat{ij}},\ B^{\hat{ij}},\ F^{\hat{ij}},\ G^{\hat{ij}},\ \psi^i,\ W_\mu)$;

(iv) linear multiplets: which are a sum of an ordinary $(0, \frac{1}{2})$ multiplet and a $(0, \frac{1}{2})$ multiplet in which an antisymmetric tensor represents the spin 0 field (the simplest case is $(T^{\hat{ij}}_{\mu\nu},\ A,\ B^{\overline{ij}},\ \psi^i,\ F,\ G)$);

(v) nonlinear multiplets: which is the first known case where matter fields transform into squares of matter fields.

Due to space limitations, we cannot discuss these multiplets in further detail, see refs. [170, 171].

We now discuss how the notion of $N = 2$ superconformal gravity sheds light on the structure of these results. In $N = 2$ conformal supergravity one expects as fields which gauge the algebra SU(2, 2/2)

$$(e^m{}_\mu,\ \omega_\mu{}^{mn},\ b_\mu,\ f^m_\mu,\ \psi^i_\mu,\ \varphi^i_\mu,\ V_\mu{}^i{}_j,\ A_\mu) \quad \text{with } V_\mu{}^i{}_j = -(V_\mu)^{*j}{}_i. \tag{4}$$

One imposes the same constraints on the (modified) $R(P)$, $R(M)$, $R(Q)$ curvatures as discussed in section 4. Elementary counting then reveals that one lacks 1 auxiliary spinor field (at least). The auxiliary fields which one must add, turn out to be: $T_{ab}{}^{ij}$ (6 fields) + D_c (1 field) + χ^i_c (-8 fields). Thus one has at this point a closed algebra for $N = 2$ conformal supergravity with (24 + 24) fields. (The Weyl multiplet.) All previously mentioned multiplets can be coupled to this gauge action in a locally $N = 2$ supercovariant way. If one couples the vector multiplet, one finds (32 + 32) fields and a field-dependent central charge in the $\{Q, Q\}$ anticommutator which only acts on the photon (as a Maxwell transformation). One can then also couple a scalar multiplet (which needs a central charge). Fixing the non-Poincaré symmetries by fixing most of the components of the scalar multiplet and some of the Weyl multiplet, one finds a different set of auxiliary fields from the one in (1), although it also contains (40 + 40) fields. As action one finds the sum of the vector multiplet action and the kinetic multiplet action for the scalar multiplet (see section 4). The de-Sitter $N = 2$ action results if one adds the mass term of the scalar multiplet.

The original set in (1) results if one couples a vector and a nonlinear multiplet to superconformal $N = 2$ gravity.

In both, inequivalent, (40 + 40) sets, Poincaré supersymmetry follows a Q + S rule (except that for (1) it becomes a Q + S + K rule and for the set which yields the de-Sitter action, it becomes even a Q + S + K + U(1) rule).

A conformal tensor calculus has been developed for $N = 2$, from which one deduces the $N = 2$ Poincaré tensor calculus. Similar results for the Poincaré tensor calculus based on the set (1) have been obtained by Breitenhohner and Sohnius from superspace methods [72].

It is gratifying to see how starting from a larger symmetry ($N = 2$ superconformal theory) with fewer fields, one can descend to smaller symmetries with more fields in a systematic fashion. Although rather technical, all these results reflect the $N = 1$ results, and the reader should have no conceptual problems after studying section 4.

Subsection 6.6 is added as an addendum

7. Appendices

A. *Gamma matrices*

The Dirac matrices we use satisfy $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ with $\mu, \nu = 1, 4$ and $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ with $\gamma_5^2 = 1$. All five matrices are Hermitean. A basis for 4×4 matrices is given by the 16 elements

$$O_j = (1, \gamma_\mu, 2\mathrm{i}\sigma_{\mu\nu}, \mathrm{i}\gamma_5\gamma_\mu, \gamma_5) \quad \text{with} \quad \mu < \nu \tag{1}$$

which satisfies $\mathrm{tr}(O_iO_j) = 4\delta_{ij}$. We define $\sigma_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]$ so that they satisfy the Lorentz algebra

$$[\sigma_{\mu\nu}, \sigma_{\rho\sigma}] = g_{\nu\rho} \cdot \sigma_{\mu\sigma} + 3\text{ other terms.} \tag{2}$$

Furthermore, we define $\not{D} = D_\mu\gamma^\mu$, and $\lambda \cdot \sigma = \lambda^{mn}\sigma_{mn}$. Gamma matrices with Latin indices are constant, but with Greek indices one has $\gamma_\mu \equiv \gamma^a e_{a\mu}$. The matrix γ_5 is always constant.

Some useful formulae are

$$\epsilon^{abcd}\sigma_{cd} = -2\gamma_5\sigma_{cd} \tag{3}$$

$$\gamma_a\gamma_b\gamma_c + \gamma_c\gamma_b\gamma_a = 2(\delta_{ab}\gamma_c + \delta_{bc}\gamma_a - \delta_{ac}\gamma_b) \tag{4}$$

$$\gamma_a\gamma_b\gamma_c - \gamma_c\gamma_b\gamma_a = 2\epsilon_{abcd}\gamma_5\gamma^d, \qquad (\epsilon_{1234} = +1). \tag{5}$$

The result for $\gamma_a\gamma_b\gamma_c$ follows by adding the last two equations. Thus

$$\gamma_\mu\sigma_{ab} = \tfrac{1}{2}(e_{a\mu}\gamma_b - e_{b\mu}\gamma_a + e^c{}_\mu\epsilon_{abcd}\gamma_5\gamma^d). \tag{6}$$

Other identities often used are

$$\gamma^a\sigma_{cd}\gamma_a = \sigma^{cd}\gamma_a\sigma_{cd} = 0 \tag{7}$$

$$\sigma^{ab}\sigma_{ab} = -3, \qquad \sigma^{ab}\sigma_{cd}\sigma_{ab} = \sigma_{cd}. \tag{8}$$

An often used matrix representation of the gamma matrices is

$$\gamma_k = \begin{pmatrix} 0 & -\mathrm{i}\sigma_k \\ \mathrm{i}\sigma_k & 0 \end{pmatrix}, \qquad \gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

For two-component spinor formalism one uses another representation, see appendix E. A Majorana representation is, for example,

$$\gamma_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & -\mathrm{i}\sigma_2 \\ \mathrm{i}\sigma_2 & 0 \end{pmatrix}, \qquad \gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma_4 = \begin{pmatrix} 0 & -\mathrm{i}\sigma_3 \\ \mathrm{i}\sigma_3 & 0 \end{pmatrix}.$$

B. *Majorana spinors*

A four-component spinor $\lambda^\alpha (\alpha = 1, 4)$ transforms under Lorentz transforms actions as

$$(\lambda')^\alpha = (\exp \tfrac{1}{2}\omega^{\mu\nu}\sigma_{\mu\nu})^\alpha{}_\beta \lambda^\beta, \qquad \sigma_{\mu\nu} = \tfrac{1}{4}[\gamma_\mu, \gamma_\nu] \tag{1}$$

where ω^{ij} are real and ω^{i4} are purely imaginary (in our conventions μ, $\nu = 1, 4$ and $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$). The Dirac conjugate spinor $\bar\lambda_\alpha \equiv (\lambda^\dagger)_\beta (\gamma^4)^\beta{}_\alpha$ transforms thus as

$$(\bar\lambda)_\alpha = \bar\lambda_\beta (\exp - \tfrac{1}{2}\omega^{\mu\nu}\sigma_{\mu\nu})^\beta{}_\alpha \tag{2}$$

because $\sigma_{\mu\nu}$ are antihermitian (γ_μ is always taken to be hermitian). The *Majorana conjugate spinor* $\hat\lambda$ is defined by

$$\hat\lambda_\alpha = \lambda^\beta C_{\beta\alpha} \tag{3}$$

and is required to transform in the same way as the Dirac conjugate spinor. Note that λ and $\hat\lambda$ are in general complex. We will always write the index of λ as a superscript and that of $\bar\lambda$ as a subscript. From

$$\delta\hat\lambda_\alpha = \hat\lambda_\beta (C^{-1})^{\beta\gamma} (\exp \tfrac{1}{2}\omega^{\mu\nu}(\sigma_{\mu\nu})^{\mathrm{T}})_\gamma{}^\delta C_{\delta\alpha} \tag{4}$$

it follows that the matrix $C_{\alpha\beta}$ must satisfy (note the position of the indices)

$$C_{\alpha\beta}(\sigma_{\mu\nu})^\beta{}_\gamma (C^{-1})^{\gamma\delta} = -(\sigma^{\mathrm{T}}_{\mu\nu})_\alpha{}^\delta. \tag{5}$$

The most general solution of this equation is

$$C = \alpha(1+\gamma_5)C_0 + \beta(1-\gamma_5)C_0 \tag{6}$$

where C_0 is a special solution, usually called "the charge conjugation matrix", and discussed below, and α and β are arbitrary constants. In two-component formalism, C_0 is diagonal and equal to $(\epsilon_{AB}, \epsilon^{\dot A\dot B})$, and since dotted and undotted spinors transform independently, while $\sigma_{\mu\nu}$ is also diagonal in dotted–undotted space, the appearance of *two* arbitrary constants is clear.

A Majorana spinor is per definition a spinor whose Dirac conjugate is proportional to its Majorana conjugate (note again the position of indices)

$$\bar{\lambda}_\alpha = \lambda^\beta C_{\beta\alpha} b, \qquad b \text{ arbitrary.} \tag{7}$$

Hence $\lambda^{\mathrm{T}} = \lambda^\dagger C^* \gamma_4^* b^* = \lambda^{\mathrm{T}} C\gamma_4 C^* \gamma_4^* (b^* b)$ from which it follows that $C\gamma_4 C^* \gamma_4^*$ must be a positive number times the unit matrix. This is indeed the case in four dimensions (see below), but, for example, in five dimensions this is not the case and there one needs an internal index in order to define Majorana spinors. We will always scale C such that

$$\bar{\lambda} = \lambda^{\mathrm{T}} C. \tag{8}$$

In order that the relation $\bar{\lambda} = \lambda^{\mathrm{T}} C$ is maintained in time, one has for a spinor whose field equation is the Dirac equation $(\gamma^\mu \partial_\mu + M)\lambda = 0$

$$\left.\begin{array}{r} \bar{\lambda}(\gamma^\mu \overleftarrow{\partial}_\mu - M) = 0 \\ \hat{\lambda}(C^{-1}\gamma^{\mu\mathrm{T}} C \overleftarrow{\partial}_\mu + M) = 0 \end{array}\right\} \quad C\gamma_\mu C^{-1} = -\gamma_\mu^{\mathrm{T}}. \tag{9}$$

For massless spinors, one finds the weaker condition

$$C\gamma_\mu C^{-1} = a\gamma_\mu^{\mathrm{T}}, \qquad a^2 = +1. \tag{10}$$

(One easily proves that $a^2 = +1$ from (5).) In four dimensions one finds solutions with $a = +1$ and -1, but, for example, in five dimensions $a = +1$. However, although in five dimensions a matrix C exists, one needs more than one spinor in order to define a Majorana spinor, as we have seen above.

Since the dotted and undotted components of λ are mixed in the Dirac equation, one now gets the stronger condition $C\gamma_\mu C^{-1} = \pm\gamma_\mu^{\mathrm{T}}$ for C, and it is clear that if a C exists, it is unique up to a constant. *This matrix C is the charge conjugation matrix.* That a C exists follows from the fact that the matrices $(-\gamma_\mu^{\mathrm{T}})$ (as well as $+\gamma_\mu^{\mathrm{T}}$) satisfy the same group multiplication table as γ_μ. Hence, since according to Schur's lemma there is only one inequivalent four-dimensional representation of the finite group with 32 elements spanned by γ-matrices, γ_μ and $-\gamma_\mu^{\mathrm{T}}$ are indeed equivalent, as are γ_μ and $+\gamma_\mu^{\mathrm{T}}$.

For spin 3/2 fields one can essentially repeat their steps, and the same results are obtained.

In a general representation of $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ with, however, hermitian γ_μ, *the charge conjugation matrix C is antisymmetric*

$$C^{\mathrm{T}} = -C. \tag{11}$$

However, only in special representations is it true that $(C^{-1})^{\alpha\beta} = -C_{\alpha\beta}$ (as should already be anticipated from the position of the indices). To see this, note that using $C\gamma_\mu C^{-1} = a\gamma_\mu^{\mathrm{T}}$ twice, one finds

$$(C^{-1})^{\mathrm{T}} C\gamma_\mu C^{-1} C^{\mathrm{T}} = \gamma_\mu. \tag{12}$$

Hence $C^{-1}C^{\mathrm{T}}$ commutes with γ_μ. Thus $C^{-1}C^{\mathrm{T}} = kI$, or $C^{\mathrm{T}} = kC$ so that $(C^{\mathrm{T}})^{\mathrm{T}} = C = k^2 C$ and $k = \pm 1$. Thus $(C\gamma_\mu)^{\mathrm{T}} = ak(C\gamma_\mu)$, $(C\sigma_{\mu\nu})^{\mathrm{T}} = -k(C\sigma_{\mu\nu})$, $(C\gamma_5\gamma_\mu)^{\mathrm{T}} = -akC\gamma_5\gamma_\mu$ and $(C\gamma_5)^{\mathrm{T}} = kC\gamma_5$. Clearly $a = +1$ and $a = -1$ are both allowed ($C = \gamma_4\gamma_5$ and $C = \gamma_4$ in a Majorana representation). Since the 16 Dirac matrices are linearly independent and there can be at most 6 antisymmetric matrices, $k = -1$ and C is antisymmetric.

Taking the hermitian conjugate of $C\gamma_\mu C^{-1} = a\gamma_\mu^{\mathrm{T}}$ one finds that $C^\dagger C = lI$ and clearly $l > 0$. Thus.

one can scale C such that C is unitarity. It follows that $C^* = -C^{-1}$. Only if $a = -1$ is $C\gamma_4 C^* \gamma_4^*$ a positive multiple of the unit matrix. For representations in which γ_μ is symmetric or antisymmetric, one has $C(C\gamma_\mu C^{-1})C^{-1} = \gamma_\mu$ so that one can choose the scale of C such that $C^{-1} = -C$. However, if γ_a does not have this symmetry then this is not true in general. Summarizing

$$C^{\mathrm{T}} = -C, \qquad C^\dagger C = CC^\dagger = I, \qquad C^* C = CC^* = -I, \qquad C\gamma_\mu C^{-1} = -\gamma_\mu{}^{\mathrm{T}} \tag{13}$$

but $C^{-1} = -C$ only in special cases.

In two-component formalism (see below), the Majorana condition $\bar\lambda_\alpha = \lambda^\beta C_{\beta\alpha}$ becomes

$$(\lambda_{\dot A})^* = \lambda^B \epsilon_{BA} \tag{14}$$

which is sometimes written as $\bar\lambda_A = \lambda_A$.

C. *Symmetries of covariants built from Majorana spinors*

From $C\gamma_\mu C^{-1} = -\gamma_\mu{}^{\mathrm{T}}$ and $\bar\lambda_\alpha = \lambda^\beta C_{\beta\alpha}$ and idem for χ, one may derive the following symmetries

$$\begin{aligned} &\bar\lambda\chi = \bar\chi\lambda, \qquad \bar\lambda\gamma_m\chi = -\bar\chi\gamma_m\lambda, \qquad \bar\lambda\sigma_{mn}\chi = -\bar\chi\sigma_{mn}\lambda \\ &\bar\lambda\gamma_5\gamma_m\chi = \bar\chi\gamma_5\gamma_m\lambda, \qquad \bar\lambda\gamma_5\chi = \bar\chi\gamma_5\lambda. \end{aligned} \tag{1}$$

In general

$$\bar\lambda\gamma_{m_1}\cdots\gamma_{m_n}\chi = \bar\chi\gamma_{m_n}\cdots\gamma_{m_1}\lambda(-)^n. \tag{2}$$

These relations are easily proven. For example,

$$\begin{aligned} \bar\lambda\chi &= \bar\lambda_\alpha\chi^\alpha \\ &= \lambda^\beta C_{\beta\alpha}\chi^\alpha = -\chi^\alpha C_{\beta\alpha}\lambda^\beta \quad \text{(since } \lambda \text{ and } \chi \text{ anticommute)} \\ &= \chi^\alpha C_{\alpha\beta}\lambda^\beta \qquad \text{(since } C \text{ is antisymmetric)} = \bar\chi\lambda. \end{aligned}$$

Note that if $C\gamma_m C^{-1} = +\gamma_m{}^{\mathrm{T}}$, then no extra minus signs appear.

D. *Fierz rearrangements*

Consider four Majorana spinors λ, χ, ψ, φ (they may have external indices, for example ψ_μ). Let them be coupled as follows (again suppressing possible extra indices)

$$I = (\bar\lambda_\alpha M^\alpha{}_\beta\chi^\beta)(\bar\psi_\gamma N^\gamma{}_\delta\varphi^\delta). \tag{1}$$

We will show that

$$I = -\tfrac{1}{4}\sum_{j=1}^{16} (\bar{\lambda} M O_j N \varphi)(\bar{\psi} O_j \chi) \tag{2}$$

where the $16 O_j$ were given before. The proof is based on the observation that the matrix $M^\alpha{}_\beta N^\gamma{}_\delta$ can be viewed for fixed α and δ as an ordinary 4×4 matrix which can be expanded into the complete set of O_j

$$M^\alpha{}_\beta N^\gamma{}_\delta = c_j{}^\alpha{}_\delta (O_j)^\gamma{}_\beta. \tag{3}$$

The coefficients c follow from the orthonormality property

$$\mathrm{tr}(O_i O_j) = 4\delta_{ij}. \tag{4}$$

Thus

$$I = (\bar{\lambda}_\alpha \chi^\beta \bar{\psi}_\gamma \varphi^\delta)(\tfrac{1}{4} M O^j N)^\alpha{}_\delta (O_j)^\gamma{}_\beta \tag{5}$$

and one finds the desired results if one takes into account that all spinors anticommute.

As an application one may show by "Fierzing" twice that

$$\begin{aligned}
&(\bar{\lambda}\lambda)\lambda^\alpha = -(\bar{\lambda}\gamma_5\lambda)(\gamma_5\lambda)^\alpha = \tfrac{1}{4}(\bar{\lambda}\gamma_5\gamma_m\lambda)(\gamma_5\gamma^m\lambda)^\alpha \\
&(\bar{\lambda}\gamma_5\lambda)\lambda = -(\bar{\lambda}\lambda)(\gamma_5\lambda)
\end{aligned} \tag{6}$$

and

$$(\bar{\lambda}\gamma_5\gamma_m\lambda)\lambda = -(\bar{\lambda}\lambda)\gamma_5\gamma_m\lambda.$$

An important identity which follows by Fierzing once is

$$(\bar{\psi}_\mu \gamma^\alpha \psi_\nu)(\gamma_\alpha \psi_\rho)\epsilon^{\mu\nu\rho\sigma} = 0. \tag{7}$$

It is used to show that torsion terms in the gravitino field equation cancel.

E. *Two-component formalism*

Since $SO(3,1) \sim SL(2,C)$ and four-component spinors transform as the $(\tfrac{1}{2},0)+(0,\tfrac{1}{2})$ representation of $SL(2,C)$, it is advantageous to work with the irreducible 2-dimensional spinor representations. One can always go from four-component to two component $\chi^A = \tfrac{1}{2}(1+\gamma_5)^A{}_\alpha \lambda^\alpha$, $\zeta_{\dot{A}} = \tfrac{1}{2}(1-\gamma_5)_{\dot{A}\beta}\lambda^\beta$, but in order to solve algebraical constraints like those in superspace, two-component formalism is indispensable.

Consider the particular hermitian representation of the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ $(\mu, \nu = 1, 4)$ in which γ_5 is diagonal and in which the rotation generators $\sigma_{ij} = \tfrac{1}{4}(\gamma_i\gamma_j - \gamma_j\gamma_i)$ are proportional to the usual SO_3 generators $(\sigma_1, \sigma_2, \sigma_3)$. This representation is unique up to the signs of γ_4 and γ_5. We choose

$$\gamma_k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix}, \qquad \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Lorentz generators $\sigma_{\mu\nu}$ are diagonal, and under Lorentz transformations $(\lambda')^a = (\exp \frac{1}{2}\omega \cdot \sigma)^a{}_b \lambda^b$

$$(\chi')^A = [\exp(\boldsymbol{\omega} + i\boldsymbol{v})(i\boldsymbol{\sigma}/2)]^A{}_B \chi^B, \qquad (\zeta')_{\dot{A}} = [\exp(\boldsymbol{\omega} - i\boldsymbol{v})(i\boldsymbol{\sigma}/2)]_{\dot{A}}{}^{\dot{B}} \zeta_{\dot{B}}. \tag{1}$$

We define $(\eta')_A$ and $(\lambda')^{\dot{A}}$ by $(\chi')^A(\eta')_A = \chi^A \eta_A$ and $(\lambda')^{\dot{A}}(\zeta')_{\dot{A}} = \lambda^{\dot{A}}\zeta_{\dot{A}}$. Note that $\omega^i = \frac{1}{2}\epsilon^{ijk}\omega_{jk}$ and $v^i = i\omega_4$ are real. Thus $(\chi')^A = L^A{}_B \chi^B$ with $L \in SL(2, C)$ and $\zeta_{\dot{A}}$ transforms as $\zeta'_{\dot{A}} = \bar{l}_{\dot{A}}{}^{\dot{B}} \zeta_{\dot{B}}$ with $\bar{l}_{\dot{A}}{}^{\dot{B}}$ the Hermitian inverse of L.

It is straightforward to prove that ϵ^{AB}, ϵ_{AB}, $\epsilon^{\dot{A}\dot{B}}$, $\epsilon_{\dot{A}\dot{B}}$ are invariant tensors under $SL(2, C)$. (A fortiori, they are invariant tensors of the subgroup SU(2), and $\zeta_A = \zeta^B \epsilon_{BA}$ transforms as $(\zeta^A)^*$. For SU(2) one therefore denotes tensors which transform as the complex conjugate of t^A by t_A, and for SU(N) a similar convention is adopted although for $N > 2$ no ϵ_{BA} exists. Instead one has the same result that $(\zeta^A)^*$ transforms as $\zeta_A = \zeta^B \Omega_{BA}$ if one considers the groups USp(N) with symplectic metric Ω_{BA}. For $SL(2, C)$ the transformation properties of $(\zeta^A)^* \equiv \bar{\zeta}^{\dot{A}}$ differ from those of ζ_A, but here another relation exists, namely $(\chi^A)^*$ transforms as $\epsilon^{\dot{A}\dot{B}}\zeta_{\dot{B}}$.) We make the at first sight unusual normalization

$$\epsilon_{AB} = \epsilon^{AB} = -\epsilon_{\dot{A}\dot{B}} = -\epsilon^{\dot{A}\dot{B}} = 1 \quad \text{for } A = 1,\ B = 2. \tag{2}$$

Thus $(\epsilon_{AB})^\dagger = \epsilon_{\dot{B}\dot{A}} = \epsilon_{AB}$ and ϵ is a Hermitian matrix. The ϵ tensors are what the metric $g_{\mu\nu}$ is in four dimensions. They raise and lower indices as follows $\chi_A = \chi^B \epsilon_{BA}$, $\chi^A = \epsilon^{AB}\chi_B$ (idem with $\dot{A}$, $\dot{B}$). In order that $\epsilon_{AB} = \epsilon^{CD}\epsilon_{CA}\epsilon_{DB}$, one finds that $\epsilon^{12} = \epsilon_{12}$, and we always contract as indicated: from left-above to right-below (defining $\chi^A = \chi_B \epsilon^{BA}$ would give the opposite sign).

It is straightforward to prove that χ_A transforms as $(\zeta_{\dot{A}})^*$, and $\zeta^{\dot{A}}$ as $(\chi^A)^*$. This is really the justification of denoting complex conjugated spinors by dotting their indices. Thus we define general tensors $\chi^{\dot{A}}$ and ζ_A by

$$\begin{aligned} (\chi')^{\dot{A}} &= (L^A{}_B)^* \chi^{\dot{B}} \equiv \bar{L}^{\dot{A}}{}_{\dot{B}} \chi^{\dot{B}} \\ (\zeta')_A &= (\bar{l}_{\dot{A}}{}^{\dot{B}})^* \zeta_B \equiv l_A{}^B \zeta_B. \end{aligned} \tag{3}$$

Another invariant tensor is $\delta_A{}^B$ and $\delta_{\dot{A}}{}^{\dot{B}}$. In fact $\epsilon^{AB}\epsilon_{BC} = -\delta^A{}_C$, so that the δ and ϵ tensors are really the same abstract tensors. The charge conjugation matrix in this special representation is diagonal

$$C = \gamma_4 \gamma_2 = \begin{pmatrix} \epsilon_{AB} & 0 \\ 0 & \epsilon^{\dot{A}\dot{B}} \end{pmatrix}. \tag{4}$$

Thus a Majorana spinor satisfies $\zeta_{\dot{A}} = (\zeta_A)^*$.

The gamma matrices define 2×2 matrices which map an upper dotted index into lower undotted index, and vice-versa,

$$\gamma_\mu = -i \begin{pmatrix} 0 & \sigma_\mu{}^{A\dot{B}} \\ -\sigma_{\mu,\dot{A}B} & 0 \end{pmatrix}. \tag{5}$$

One thus finds

$$\sigma_\mu{}^{A\dot{B}} = (\boldsymbol{\sigma}, \mathrm{i}I)^{A\dot{B}} \quad \text{and} \quad (\sigma_\mu)_{\dot{A}B} = (\boldsymbol{\sigma}, -\mathrm{i}I)_{AB}. \tag{6}$$

Since $(\sigma_\mu)^{A\dot{B}}$ are Hermitian (which means antiHermitian for $\mu = 4$ in our conventions), lowering its indices to $(\sigma_\mu)_{C\dot{D}}$ one finds the transpose (or complex conjugate) of $(\sigma_\mu)_{\dot{A}B}$, namely

$$(\sigma^\mu)_{C\dot{D}} \equiv (\sigma^\mu)^{A\dot{B}}\epsilon_{AC}\epsilon_{\dot{B}\dot{D}} = \sigma^\mu{}_{\dot{D}C}. \tag{7}$$

Note that

$$(\sigma_\mu)^{A\dot{B}}(\sigma^\mu)_{\dot{B}C} = \delta^A{}_C \qquad \text{(no sum over } \mu). \tag{8}$$

Raising of the index μ is done by $\delta^{\mu\nu} = (+,+,+,+)$, with $\mu, \nu = 1, 4$ in our conventions.

The matrices $(\sigma_\mu)^{A\dot{B}}$ and $(\sigma_\mu)_{A\dot{B}}$ are invariant tensors of SL(2, C). That is to say that, for example,

$$(\sigma^{\mu,A\dot{B}})' = L^A{}_C\bar{L}^{\dot{B}}{}_{\dot{D}}L^\mu{}_\nu\sigma^{\nu,C\dot{D}} = \sigma^{\mu,A\dot{B}}. \tag{9}$$

One can *thus* express tensors with Lorentz indices into tensors with spinor indices as follows

$$T^{\mu\cdots}_{\rho\cdots} \to T^{M\dot{M}\cdots}_{R\dot{R}\cdots} \equiv (\sigma_\mu)^{M\dot{M}}(\sigma^\rho)_{R\dot{R}}T^{\mu\cdots}_{\rho\cdots}. \tag{10}$$

Often one writes $T_{R\dot{R}S\dot{S}}$ as $T_{RS\dot{R}\dot{S}}$, but one should not confuse this with $T_{SR\dot{R}\dot{S}}$.

The connections $(\sigma_\mu)^{A\dot{B}}$ satisfy the following completeness relations

$$\begin{aligned}
&(\sigma_\mu)_{\dot{A}B}(\sigma^\mu)^{C\dot{D}} = 2\delta^{\dot{D}}{}_{\dot{A}}\delta^C{}_B \\
&(\sigma_\mu)_{\dot{A}B}(\sigma^\mu)_{\dot{C}D} = 2\epsilon_{\dot{A}\dot{C}}\epsilon_{BD} = (\sigma_\mu)_{B\dot{A}}(\sigma^\mu)_{D\dot{C}} \qquad \text{(see (7))} \\
&(\sigma_\mu)_{\dot{A}B}(\sigma^\nu)^{B\dot{A}} = 2\delta^\nu_\mu = (\sigma_\mu)_{A\dot{B}}(\sigma^\nu)^{\dot{B}A} = (\sigma_\mu)_{A\dot{B}}(\sigma^\nu)^{A\dot{B}} \\
&(\sigma_\mu)^{A\dot{B}}(\sigma^\mu)^{C\dot{D}} = 2\epsilon^{AC}\epsilon^{\dot{B}\dot{D}} = (\sigma_\mu)^{\dot{B}A}(\sigma^\mu)^{\dot{D}C}.
\end{aligned} \tag{11}$$

The reason we defined $\epsilon^{\dot{A}\dot{B}} = -1$ for $\dot{A} = 1$, $\dot{B} = 2$, is that with this definition no minus signs occur in these relations. One can use them to go back and forth:

$$\begin{aligned}
&F_{\mu\nu} = \tfrac{1}{4}\sigma_\mu{}^{A\dot{B}}\sigma_\nu{}^{C\dot{D}}F_{A\dot{B}C\dot{D}} \\
&F_{A\dot{B}C\dot{D}} = F_{\mu\nu}\sigma^\mu{}_{A\dot{B}}\sigma^\nu{}_{C\dot{D}} = F_{\dot{B}A\dot{D}C}
\end{aligned} \tag{12}$$

where $F_{\dot{B}A\dot{D}C} = F_{\mu\nu}(\sigma^\mu)_{\dot{B}A}(\sigma^\nu)_{\dot{D}C}$. In fact, quite generally $V_{A\dot{B}} = V_{\dot{B}A}$, as one easily proves using

$$\sigma^\mu{}_{\dot{A}B} = \sigma^\mu{}_{B\dot{A}} \quad \text{and} \quad \sigma_\mu{}^{A\dot{B}} = \sigma_\mu{}^{\dot{B}A}. \tag{13}$$

An antisymmetric tensor $G_{AB} = -G_{BA}$ is clearly proportional to ϵ_{AB}. Hence, for an antisymmetric tensor $F_{\mu\nu} = -F_{\nu\mu}$ one has

$$F_{\mu\nu} \to F_{A\dot{B}C\dot{D}} = \epsilon_{AC}\rho_{\dot{B}\dot{D}} + \epsilon_{\dot{B}\dot{D}}\tau_{AC} \tag{14}$$

where $\rho_{\dot{B}\dot{D}}$ and τ_{AC} are symmetric. Repeated application yields for antisymmetric tensor ϵ^{abcd} with $\epsilon^{1234} = +1$ the result

$$\epsilon^{M\dot{M}N\dot{N}R\dot{R}S\dot{S}} \equiv \epsilon^{\mu\nu\rho\sigma}(\sigma_\mu)^{M\dot{M}} \cdots (\sigma_\sigma)^{S\dot{S}} = 4(\epsilon^{MN}\epsilon^{RS}\epsilon^{\dot{M}\dot{R}}\epsilon^{\dot{N}\dot{S}} - \epsilon^{MR}\epsilon^{NS}\epsilon^{\dot{M}\dot{N}}\epsilon^{\dot{R}\dot{S}}). \tag{15}$$

To check this expression, note that it vanishes when two pairs of 2-component indices are equal, and that it agrees for a particular choice of $M, \dot{M}, \dots, \dot{S}$. Also one may verify that δ^ν_μ goes over into $2\delta^N_M\delta^{\dot{N}}_{\dot{M}}$.

We now give a few applications. If $F_{\mu\nu}$ is self dual

$$F_{\mu\nu} + \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} = 0 \qquad \text{then} \quad \rho_{\dot{B}\dot{D}} = 0. \tag{16}$$

The proof is simple: multiply by $(\sigma^\mu)_{\dot{A}B}(\sigma^\nu)_{\dot{C}D}$ and use the completeness relations to prove that

$$F^{R\dot{R}S\dot{S}} \equiv F^{\rho\sigma}(\sigma_\rho)^{R\dot{R}}(\sigma_\sigma)^{S\dot{S}} = \epsilon^{RR'}\epsilon^{SS'}\epsilon^{\dot{R}\dot{R}'}\epsilon^{\dot{S}\dot{S}'}F_{\dot{R}'R'\dot{S}'S'} \tag{17}$$

where $F_{\dot{R}R\dot{S}S} = F_{\mu\nu}(\sigma^\mu)_{\dot{R}R}(\sigma^\nu)_{\dot{S}S}$. Inserting the expression for $\epsilon^{\mu\nu\rho\sigma}$ one finds indeed that

$$F_{\mu\nu} + {}^*F_{\mu\nu} = 0, \qquad \text{then} \quad F_{M\dot{M}N\dot{N}} = \epsilon_{\dot{M}\dot{N}}\tau_{MN}. \tag{18}$$

For anti selfdual tensors it is of course $\rho_{\dot{M}\dot{N}}$ which survives.

If $F_{\mu\nu}$ is real, then it follows with the Hermiticity property $[(\sigma^\mu)_{\dot{A}B}]^* = (-)^{\delta_{\mu 4}}(\sigma^\mu)_{\dot{B}A}$ that

$$\tau_{AC} = -(\rho_{\dot{A}\dot{C}})^*. \tag{19}$$

The minus sign is due to $(\epsilon_{AB})^* = -\epsilon_{\dot{A}\dot{B}}$.

Consider now a tensor spinor $R^\alpha_{\mu\nu}$ with the spinor index α running from 1 to 4. Substituting the expressions for γ_5 and γ_μ one finds with $R^\alpha_{\mu\nu} = R^A_{\mu\nu} + R_{\mu\nu,\dot{A}}$ that

$$\begin{aligned} &\tfrac{1}{2}(1+\gamma_5)(\gamma^\mu)^\alpha{}_\beta R^\beta_{\mu\nu} = (-\mathrm{i}\sigma^\mu)^{A\dot{A}}\tfrac{1}{4}\sigma_\mu{}^{M\dot{M}}\sigma_\nu{}^{N\dot{N}}R_{\dot{A},M\dot{M}N\dot{N}} \\ &R_{\dot{A},M\dot{M}N\dot{N}} = \epsilon_{MN}\rho_{\dot{A},\dot{M}\dot{N}} + \epsilon_{\dot{M}\dot{N}}\tau_{\dot{A},MN} \qquad \text{(idem with } A\text{).} \end{aligned} \tag{20}$$

Thus the constraint $\frac{1}{2}(1+\gamma_5)\gamma^\mu R_{\mu\nu} = 0$ is equivalent to

$$\epsilon_{AN}\epsilon^{\dot{A}\dot{M}}\rho_{\dot{A},\dot{M}\dot{N}} - \delta^{\dot{A}}_{\dot{N}}\tau_{\dot{A},AN} = 0. \tag{21}$$

Hence, the constraint $\gamma^\mu R_{\mu\nu} = 0$ says that $R^\alpha_{\mu\nu}$ is in two-component formalism a sum of a function with three undotted indices, and a function with three dotted indices, both completely symmetric.

Two weaker constraints one often encounters are now discussed. If $\sigma^{\mu\nu}R_{\mu\nu} = 0$ this implies that ρ and τ are symmetric

$$\sigma^{\mu\nu}R_{\mu\nu} = 0 \qquad \text{then} \quad \rho_{\dot{A}}{}^{\dot{A}\dot{B}} = \tau^A{}_{,AB} = 0. \tag{22}$$

The chirality-duality condition $R_{\mu\nu} + \frac{1}{2}\gamma_5 \tilde{R}_{\mu\nu} = 0$ yields two relations if one projects with $\frac{1}{2}(1 \pm \gamma_5)$

$$\begin{aligned} &R^A_{\mu\nu} - \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma} R^{\rho\sigma,A} = 0 \Rightarrow \rho^A{}_{,M\dot{N}} = 0 \\ &R_{\dot{A},\mu\nu} + \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma} R^{\rho\sigma}_{\dot{A}} = 0 \Rightarrow \tau_{\dot{A},MN} = 0. \end{aligned} \tag{23}$$

Hence these two conditions together yield the same result as $\gamma^\mu R_{\mu\nu} = 0$ alone. In four-component formalism this is easier to see: if $\gamma^\mu R_{\mu\nu} = 0$ then from $\sigma^{\alpha\beta}\sigma^{\mu\nu} R_{\mu\nu} = 0$ it follows upon expanding $\sigma^{\alpha\beta}\sigma^{\mu\nu}$ as

$$2\sigma^{\alpha\beta}\sigma^{\mu\nu} = \{\sigma^{\alpha\beta}, \sigma^{\mu\nu}\} + [\sigma^{\alpha\beta}, \sigma^{\mu\nu}] \tag{24}$$

that indeed $R_{\mu\nu} + \frac{1}{2}\gamma_5 \epsilon_{\mu\nu\rho\sigma} R^{\rho\sigma} = 0$.

F. *Supermatrix algebra*

The following subsections deal with algebra and analysis using anticommuting variables. For more details, see a forthcoming book by B.S. DeWitt, P.C. West and P. van Nieuwenhuizen.

Supermatrices are matrices $M = \binom{A\,B}{C\,D}$ whose Bose–Bose parts A and Fermi–Fermi parts D are even elements of a Grassmann algebra and whose Bose–Fermi parts B and Fermi–Bose parts C are odd elements. They are multiplied as ordinary matrices; this multiplication rule is associative and the product is again supermatrix. There is the obvious unit element. In order to see whether they form a group, we construct an inverse by first decomposing M as

$$M = ST, \qquad S = \begin{pmatrix} A & 0 \\ C & I \end{pmatrix}, \qquad T = \begin{pmatrix} I & A^{-1}B \\ 0 & D - CA^{-1}B \end{pmatrix}$$

$$M = UV, \qquad U = \begin{pmatrix} I & B \\ 0 & D \end{pmatrix}, \qquad V = \begin{pmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & I \end{pmatrix}$$

$$M^{-1} = T^{-1}S^{-1} = \begin{pmatrix} I & -A^{-1}B(D - CA^{-1}B)^{-1} \\ 0 & (D - CA^{-1}B)^{-1} \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ -CA^{-1} & I \end{pmatrix}$$

$$M^{-1} = V^{-1}U^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & 0 \\ -D^{-1}C(A - BD^{-1}C)^{-1} & I \end{pmatrix} \begin{pmatrix} I & -BD^{-1} \\ 0 & D^{-1} \end{pmatrix}.$$

Note that $M^{-1}{}_{\mathrm{ff}} = (D - CA^{-1}B)^{-1}$. Thus an inverse exists if M_{bb} and $M^{-1}{}_{\mathrm{ff}}$ are non-singular, or if $M^{-1}{}_{\mathrm{bb}}$ and M_{ff} are nonsingular. Let us define $A(\mathrm{ord})$ as A minus its nilpotent part $A(\mathrm{nil})$. Thus $A = A(\mathrm{ord}) + A(\mathrm{nil})$. Then the inverse of A exists if $A(\mathrm{ord})$ has an inverse because one can expand A^{-1} as $[A(\mathrm{ord})]^{-1}$ times a power series in the nilpotent elements (this power series is in fact a finite series):

$$A^{-1} = A(\mathrm{ord})^{-1}[1 - A(\mathrm{ord})\,A(\mathrm{nil}) + [A(\mathrm{ord})\,A(\mathrm{nil})]^2 \cdots]. \tag{2}$$

Since in $D - CA^{-1}B$ the terms $CA^{-1}B$ are nilpotent, it follows that an inverse exists if and only if $A(\mathrm{ord})$ and $D(\mathrm{ord})$ have an inverse. In other words, *M has an inverse if and only if $M(\mathrm{ord})$ has an inverse.*

The superdeterminant is defined by

$$\begin{aligned}\operatorname{sdet} M &= \det M_{\mathrm{bb}} \det M^{-1}{}_{\mathrm{ff}} = \det A \det^{-1}(D - CA^{-1}B) \\ &= \det^{-1} M^{-1}{}_{\mathrm{bb}} \det^{-1} M_{\mathrm{ff}} = \det(A - BD^{-1}C) \det^{-1} D.\end{aligned} \tag{3}$$

It is sometimes called the Berezinian. In other words, the superdeterminant is the determinant of the Bose–Bose part of M times the determinant of the Fermi–Fermi part of the inverse of M. This definition should of course be shown to be the one needed in the applications. We shall show below that in path integrals the Faddeev–Popov determinant is indeed this superdeterminant. One usually replaces this definition by generalizing the well-known formula $\det A = \exp(\operatorname{tr} \ln A)$

$$\begin{aligned}&\operatorname{sdet} M = \exp(\operatorname{str} \ln M) \\ &\operatorname{str} N = \Sigma(-)^a N^A{}_A \\ &\ln M = \ln(1 + M - 1) = \Sigma(M-1)^k\,(-)^{k-1}\,k^{-1}\end{aligned} \tag{4}$$

where $a = 0$ for $A =$ bosonic and $a = 1$ for $A =$ fermionic. (These minus signs for the trace over fermionic elements are reminiscent of the minus signs for closed loops of fermions in Feynman diagrams.) With this definition of the supertrace one has $\operatorname{str} M_1M_2 = \operatorname{str} M_2M_1$. If $M_{1,2} = \exp N_{1,2}$, one can use the Baker–Hausdorff rule to obtain $M_1M_2 = \exp(N_1 + N_2 + \text{multiple commutators})$ and it follows from the definitions of the supertrace that

$$\operatorname{sdet} M_1 \operatorname{sdet} M_2 = \operatorname{sdet} M_1M_2. \tag{5}$$

Using the decomposition $M = ST$ and computing $\operatorname{str}\ln S = \operatorname{tr}\ln A$, $\operatorname{str}\ln T = -\operatorname{tr}\ln(D - CA^{-1}B)$, one finds the result for $\operatorname{sdet} M$. From the definition of $\operatorname{sdet} M$ in terms of $\operatorname{str}\ln M$, it follows that

$$\delta(\operatorname{sdet} M) = (\operatorname{sdet} M)\operatorname{str}(M^{-1}\delta M) \tag{6}$$

as long as M is a supermatrix. ($\ln M = \ln M(\text{ord}) + \ln(1 + M^{-1}(\text{ord})\,M(\text{nil}))$ exists if $M^{-1}(\text{ord})$ exists.)

Another nice way of deriving these results is to start from

$$M = \begin{pmatrix} A & 0 \\ 0 & D\end{pmatrix}\left[\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix} + K\right], \qquad K = \begin{pmatrix} 0 & A^{-1}B \\ D^{-1}C & 0\end{pmatrix} \tag{7}$$

so that $\operatorname{sdet} M = \det A(\det D)^{-1}\operatorname{sdet}(1 + K)$. Since K is nilpotent, $\ln(1 + K)$ is well-defined, and again with $\operatorname{sdet}(1 + K) = \exp\operatorname{str}\ln(1 + K)$ (easy to check directly in this case) one finds

$$\operatorname{sdet}(1 + K) = \det(1 - A^{-1}BD^{-1}C). \tag{8}$$

Thus one finds again the result for $\operatorname{sdet} M$.

For supermatrices one may define a transition rule satisfying $(M_1M_2)^{\mathrm{T}} = M_2{}^{\mathrm{T}}M_1{}^{\mathrm{T}}$. This relation holds if one defines

$$M^{\mathrm{T}}=\begin{pmatrix} A^{\mathrm{T}} & \sigma C^{\mathrm{T}} \\ -\sigma B^{\mathrm{T}} & D^{\mathrm{T}} \end{pmatrix}, \qquad \sigma=\pm 1. \tag{9}$$

It follows that $\operatorname{sdet} M^{\mathrm{T}}=\operatorname{sdet} M$ and $(M^{-1})^{\mathrm{T}}=(M^{\mathrm{T}})^{-1}$. This transposition rule follows naturally from $\bar{x}^i=M^i{}_j x^j$ and $\bar{x}^i=x^j(M^{\mathrm{T}})_j{}^i$. In our conventions, A is the Bose–Bose part of M and $\sigma=+1$. (If one would have defined that D is the Bose–Bose part then one would have found that $\sigma=-1$.)

G. *Superintegration*

In path-integrals, the integration over anticommuting physical fields must be translation invariant in order that the expectation of the field equation vanishes. For example, for a Dirac electron ψ

$$\begin{aligned} &Z=\int \mathrm{d}\psi\, \mathrm{d}\bar{\psi}\, \exp I(\eta,\bar{\eta}) \qquad \text{with } I(\eta,\bar{\eta})=-\bar{\psi}\not{\partial}\psi+\bar{\eta}\psi+\bar{\psi}\eta \\ &-\left(\not{\partial}\frac{\delta}{\delta\eta}-\eta\right)Z=\int \mathrm{d}\psi\, \mathrm{d}\bar{\psi}\, \frac{\delta}{\delta\bar{\psi}}(\exp I)=0. \end{aligned} \tag{1}$$

Thus, for anticommuting variables θ one requires $\int \mathrm{d}\theta(\mathrm{d}/\mathrm{d}\theta)f(\theta)=0$, and since quite generally $f(\theta)=a+b\theta$, one has

$$\int \mathrm{d}\theta=0, \qquad \int \mathrm{d}\theta\, \theta=1. \tag{2}$$

The number 1 in the second integral is an arbitrary normalization. Indeed, the integral is now translationally invariant

$$\int \mathrm{d}\theta\, f(\theta)=b=\int \mathrm{d}\theta\, f(\theta+\epsilon). \tag{3}$$

Thus, *on physical grounds*, one requires translation of invariance of the θ-integral.

A closely related argument is that if translational invariance holds, then one can complete squares in the exponent in the path integral and perform the integration. One is then left with $\bar{\eta}(\not{\partial})^{-1}\eta$ in the exponent, and functionally differentiating with respect to η, one recovers the usual Feynman rules.

Anticommuting ghosts appear in path integrals when one exponentiates a Faddeev–Popov determinant as follows

$$\det M=\int \prod_{i=1}^{N} \mathrm{d}C_2^i\, \mathrm{d}C_1^i\, \exp(\mathrm{i}C_1^i M_{ij} C_2^j). \tag{4}$$

If one defines C_j^i to be anticommuting and defines the integrals as for θ, only the term $(C_1 M C_2)^N$ in the expansion of the exponential contributes and yields $\det M$ (up to an irrelevant constant). (The minus signs in the expression for $\det M$ come about correctly because the C's anticommute.) Thus, also for Faddeev–Popov anticommuting ghosts the same integration rules hold but now derived from a different physical requirement.

In what follows we will show that

(i) fixing the gauge in the path integral one finds as a generalization of the Faddeev–Popov determinant a superdeterminant;

(ii) exponentiating this superdeterminant by means of a super Gaussian integral one finds the generalized ghost actions.

Consider a path-integral with bosonic and fermionic symmetries. We want to multiply by unity writing as usual

$$1=\int \mathrm{d}\xi^{\alpha}\,\mathrm{d}\eta^{\beta}\,\delta(F_{\alpha}-f_{\alpha})\,\delta(B_{\beta}-b_{\beta})J \tag{5}$$

where $\xi(\eta)$ are the bosonic (fermionic) gauge parameters, $B(F)$ the gauge choices and $b(f)$ ordinary (Grassmann) variables. We must first define the Dirac delta function for fermionic arguments. Since $\int \mathrm{d}\theta\,\theta=1$, it follows that $\delta(\theta)=\theta$. Thus $\delta(F_{\alpha}-f_{\alpha})$ is equal to $(F_{\alpha}-f_{\alpha})$ for fermionic $F_{\alpha}-f_{\alpha}$. Making a change of integration variables $(\xi,\eta)\to(B,F)$, it follows that J is the Jacobian for the change of integration variables $(B,F)\to(\xi,\eta)$.

Theorem: The Jacobian for $x'^{i}\to x^{i}(x',\theta')$, $\theta'^{\alpha}\to\theta^{\alpha}(x',\theta')$ is given by the superdeterminant

$$J=\frac{\det[\partial x'^{i}/\partial x^{j}-(\partial x'^{i}/\partial\theta^{\alpha})\,(\partial\theta'/\partial\theta)^{-1,\alpha}{}_{\beta}(\partial\theta'^{\beta}/\partial x^{j})]}{\det(\partial\theta'^{\alpha}/\partial\theta^{\beta})} \tag{6}$$

where it does not matter whether one uses left or right derivatives except that $\partial x'^{i}/\partial\theta^{\alpha}$ is a right derivative. (In all other cases we use left derivatives.)

We will begin by considering linear transformation

$$\begin{pmatrix}x'\\ \theta'\end{pmatrix}=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}x\\ \theta\end{pmatrix} \tag{7}$$

and derive that $J=\operatorname{sdet}M$. In this case it is clear that $\partial x'^{i}/\partial\theta^{\alpha}$ is a right derivative. Consider first the case of linear changes of anticommuting variables θ^{i} ($i=1,N$). From

$$\begin{aligned}\int \mathrm{d}\theta^{1}\cdots\mathrm{d}\theta^{N}\,\theta^{N}\cdots\theta^{1}=1&=\int\mathrm{d}\theta'^{1}\cdots\mathrm{d}\theta'^{N}\,\theta'^{N}\cdots\theta'^{1}\\ &=\int\mathrm{d}\theta^{1}\cdots\mathrm{d}\theta^{N}\,J\theta'^{N}\cdots\theta'^{1}\end{aligned} \tag{8}$$

with $\theta'^{i}=D^{i}{}_{j}\theta^{j}$ it follows that $J=\det D^{-1}$, just the inverse of what one has for bosonic variables. For general linear transformations involving commuting x^{i} and anticommuting θ^{α} one has

$$\begin{pmatrix}x'\\ \theta'\end{pmatrix}=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}x\\ \theta\end{pmatrix}=\begin{pmatrix}I & BD^{-1}\\ 0 & I\end{pmatrix}\begin{pmatrix}A-BD^{-1}C & 0\\ 0 & D\end{pmatrix}\begin{pmatrix}I & 0\\ D^{-1}C & I\end{pmatrix}\begin{pmatrix}x\\ \theta\end{pmatrix}. \tag{9}$$

The first and last transformations are shifts of integration variables θ^{α} and x^{i} respectively, and since the integrals are translationally invariant these Jacobians equal unity. The second change of integration variables is diagonal in x and θ, and one finds indeed that for linear changes of integration variables the super Jacobian is equal to the superdeterminant.

For nonlinear changes of integration variables, the super Jacobian is given by the result previously given, or by the equivalent result

$$J = \det(\partial x''/\partial x^j)\det^{-1}(\partial\theta'^\alpha/\partial\theta^\beta - \partial\theta'^\alpha/\partial x^i\,(\partial x'/\partial x)^{-1i}{}_j\,\partial x''/\partial\theta^\beta) \tag{10}$$

and this result is derived in the next subsection.

Finally we discuss how to exponentiate the superdeterminant. We claim

$$\operatorname{sdet} M = \int \mathrm{d}C \exp \mathrm{i}C_1^i M_{ij} C_2^j, \qquad \mathrm{d}C \equiv \prod_k \mathrm{d}C_2^k\,\mathrm{d}C_1^k \tag{11}$$

where C_i^k denote commuting as well as anticommuting ghosts, namely

$$C_1^i M_{ij} C_2^j = C_1^\mu A_{\mu\nu} C_2^\nu + C_1^\mu B_{\mu\alpha} C_2^\alpha + C_1^\alpha C_{\alpha\nu} C_2^\nu + C_1^\alpha D_{\alpha\beta} C_2^\beta \tag{12}$$

and C_i^μ are anticommuting while C_i^α are commuting. The proof is based on translational invariance

$$\begin{aligned}
0 &= (-)^j(\delta M^{-1})^{ji} \int \mathrm{d}C \frac{\vec{\partial}}{\partial C_1^i}(\exp(\mathrm{i}C_1 M C_2))\frac{\overleftarrow{\partial}}{\partial C_2^j} \\
&= (-)^j(\delta M^{-1})^{ji} \int \mathrm{d}C\,[\mathrm{i}M_{ij} - M_{ik} C_2^k C_1^l M_{lj}] \exp(\mathrm{i}C_1 M C_2) \\
&= \int \mathrm{d}C\,[\mathrm{i}\operatorname{str}(\delta M^{-1} M) - C_1^l\,\delta M_{lk}\,C_2^k] \exp(\mathrm{i}C_1 M C_2)
\end{aligned} \tag{13}$$

since interchanging $[(\delta M^{-1})MC_2]^j$ and $(C_1 M)^j$ cancels the factor $(-)^j$ and yields an extra factor (-1) since ghosts have different statistics.

With

$$I = \int \mathrm{d}C \exp(\mathrm{i}C_1 M C_2) \quad \text{and} \quad \operatorname{str}(\delta M^{-1} M) = -\operatorname{str}(\delta M M^{-1}) \tag{14}$$

one finds thus that

$$\delta \ln I = \delta \ln \operatorname{sdet} M \tag{15}$$

which is indeed what we intended to prove.

One often rewrites the ghost action as

$$\mathscr{L}(\text{ghost}) = C^{*i} M_{ij} C^j. \tag{16}$$

Indeed, the Feynman rules are the same, but since (for example in Yang–Mills theory) M is not

Hermitian, it is not true that the extra terms $C_1MC_1 + C_2MC_2$ vanish (their kinetic parts vanish). Note that we did not have to assume any special properties for M.

Summarizing, we have the following results

$$1 = \int \prod_A \mathrm{d}z^A\, \delta(F_A(z) - a_A)\, \mathrm{sdet}(\partial F_A/\partial z^B), \qquad z^A = (x^m, \theta^\alpha)$$
$$\mathrm{sdet}(\partial F_A/\partial z^B) = \int \prod_A \mathrm{d}C_2^A\, \mathrm{d}C_1^A\, \exp(\mathrm{i}C_1^A(\partial F_A/\partial z^B)C_2^B). \tag{17}$$

H. *The super Jacobian is the superdeterminant*

In the covariant quantization of supergravity using path integrals, an essential ingredient is the super Jacobian. We show here that for a general change of integration variables (x, θ) to $\bar{x}(x, \theta)$, $\bar{\theta}(x, \theta)$ the super Jacobian is given by the superdeterminant:

$$J = \det(\partial x^i(\bar{x}, \bar{\theta})/\partial \bar{x}^j)\, \det(\partial \bar{\theta}^m(x, \theta)/\partial \theta^n). \tag{1}$$

Defining the supermatrices M and M^{-1} by

$$M = \begin{pmatrix} \partial x/\partial \bar{x} & \partial x/\partial \bar{\theta} \\ \partial \theta/\partial \bar{x} & \partial \theta/\partial \bar{\theta} \end{pmatrix}, \qquad M^{-1} = \begin{pmatrix} \partial \bar{x}/\partial x & \partial \bar{x}/\partial \theta \\ \partial \bar{\theta}/\partial x & \partial \bar{\theta}/\partial \theta \end{pmatrix} \tag{2}$$

one need specify only for $\partial x/\partial \bar{\theta}$ and $\partial \bar{x}/\partial \theta$ whether one has left or right directives. One needs right derivatives, in order that $(\partial x/\partial \bar{x})(\partial \bar{x}/\partial x) + (\partial x/\partial \bar{\theta})(\partial \bar{\theta}/\partial x)$ be the unit matrix. Hence the super Jacobian is the determinant of the Bose–Bose part of M (the usual result) times the determinant of the Fermi–Fermi part of the *inverse* of M. For infinitesimal changes one has $J = \Sigma\, \partial x^i/\partial \bar{x}^i - \partial \theta^m/\partial \bar{\theta}^m$, and a proof that J equals the superdeterminant has been given by DeWitt, based on the principle that exponentiation of this linearized result gives the superdeterminant. Here we give a direct proof valid for finite transformations and due to Fung.

We consider the change of variables $(x, \theta) \to (\bar{x}, \bar{\theta})$ in two steps

(i) we change $\theta^m, \dots, \theta^1$ to $\bar{\theta}^m, \dots, \bar{\theta}^1$, keeping all x fixed;

(ii) we change $x^1, \dots, x^k$ to $\bar{x}^1, \dots, \bar{x}^k$, keeping all $\bar{\theta}$ fixed.

The second change of variables has an ordinary Jacobian

$$\det(\partial x^i(\bar{x}, \bar{\theta})/\partial \bar{x}^j).$$

Notice that it is essential that this change of variables is performed when the Fermi variables are already barred – otherwise one would find the mixed determinant $\det(\partial x^i(x, \bar{\theta})/\partial \bar{x}^j)$. We now consider (i).

Consider an integral of the form

$$I = \int \mathrm{d}x^1 \cdots \mathrm{d}x^k\, \mathrm{d}\theta^m \cdots \mathrm{d}\theta^1\, P(x, \theta)$$
$$= \int \mathrm{d}\bar{x}^1 \cdots \mathrm{d}\bar{x}^k\, \mathrm{d}\bar{\theta}^m \cdots \mathrm{d}\bar{\theta}^1\, J P(x(\bar{x}, \bar{\theta}), \theta(\bar{x}, \bar{\theta})) \tag{3}$$

where P is a polynomial in θ and an arbitrary function of x, and J the super Jacobian to be determined. One can for any given i decompose $P(x, \theta) = A_i + \theta^i B_i$ where A_i and B_i are independent of θ^i. Thus, with $i = 1$

$$\begin{aligned}
&\int \mathrm{d}x\, \mathrm{d}\theta (A_1 + \theta^1 B_1) = \int \mathrm{d}x\, \mathrm{d}\theta^m \cdots \mathrm{d}\theta^2 B_1 \\
&\int \mathrm{d}x\, \mathrm{d}\theta^m \cdots \mathrm{d}\theta^2\, \mathrm{d}\bar{\theta}^1 (A_1 + \tilde{\theta}^1 B_1) = \int \mathrm{d}x\, \mathrm{d}\theta^m \cdots \mathrm{d}\theta^2 (\partial\tilde{\theta}^1/\partial\bar{\theta}^1) B_1
\end{aligned} \tag{4}$$

where

$$\theta^1 = \tilde{\theta}^1(x^1, \ldots, x^k, \theta^m, \ldots, \theta^2, \bar{\theta}^1) \tag{5}$$

is obtained by inverting $\bar{\theta}^1 = \bar{\theta}^1(x^1, \ldots, x^k, \theta^m, \ldots, \theta^1)$ for fixed x and $\theta^m, \ldots, \theta^2$. Thus, in going from $\mathrm{d}\theta^1$ to $\mathrm{d}\bar{\theta}^1$, the Jacobian equals $(\partial\bar{\theta}^1/\partial\tilde{\theta}^1)^{-1}$.

Next we change θ^2 to $\bar{\theta}^2$ keeping x, $\theta^m, \ldots, \theta^3$ and $\bar{\theta}^1$ fixed. Again one can write $(\partial\tilde{\theta}^1/\partial\bar{\theta}^1)P$ as $A_2 + \theta^2 B_2$ where A_2 and B_2 do not depend on θ^2, and defining

$$\theta^2 = \tilde{\theta}^2(x^1, \ldots, x^k, \theta^m, \ldots, \theta^3, \bar{\theta}^2, \bar{\theta}^1) \tag{6}$$

one finds a Jacobian $(\partial\bar{\theta}^2/\partial\tilde{\theta}^2)^{-1}$. The total Jacobian for (i) is thus

$$[(\partial\tilde{\theta}^m/\partial\bar{\theta}^m) \cdots (\partial\tilde{\theta}^1/\partial\bar{\theta}^1)]^{-1}. \tag{7}$$

It is now a general theorem that for any change of variables $y^i = y^i(\bar{y})$ with $i = 1, m$ one can write the Jacobian as follows

$$\det(\partial y^i/\partial\bar{y}^j) = (\partial\tilde{y}^m/\partial\bar{y}^m) \cdots (\partial\tilde{y}^1/\partial\bar{y}^1) \tag{8}$$

where $\tilde{y}^m(\bar{y}) = y^m(\bar{y})$ and $\tilde{y}^{m-1}, \ldots, \tilde{y}^1$ are obtained by successively inverting $y = y(\bar{y})$ as follows

$$\begin{aligned}
&y^1 = \tilde{y}^1(\bar{y}^1, y^2, \ldots, y^m) \\
&y^2 = \tilde{y}^2(\bar{y}^1, \bar{y}^2, y^3, \ldots, y^m) \\
&\ldots\ldots\ldots\ldots \\
&y^m = \tilde{y}^m(\bar{y}^1, \ldots, \bar{y}^m).
\end{aligned} \tag{9}$$

The proof is simple. Define $F^i(y^1, \ldots, y^m, \bar{y}^1, \ldots, \bar{y}^m) = \tilde{y}^i - y^i$. Then

$$\begin{aligned}
\det(\partial y^i/\partial\bar{y}^j) &= (-)^m \det(\partial F^i/\partial\bar{y}^j) \det{}^{-1}(\partial F^i/\partial y^j) \\
&= (-)^m \begin{vmatrix} \partial\tilde{y}^1/\partial\bar{y}^1 & 0 & 0\cdots \\ \partial\tilde{y}^2/\partial\bar{y}^1 & \partial\tilde{y}^2/\partial\bar{y}^2 & 0\cdots \\ \cdots & \cdots & \cdots\cdot \end{vmatrix} \div \begin{vmatrix} -1 & \partial\tilde{y}^1/\partial y^2 & \cdots \\ 0 & -1 & \partial\tilde{y}^2/\partial y^3 \\ \cdot & \cdots & \cdots \end{vmatrix} \\
&= (\partial\tilde{y}^1/\partial\bar{y}^1) \cdots (\partial\tilde{y}^m/\partial\bar{y}^m).
\end{aligned} \tag{10}$$

Applying this theorem to the transformations under (i), the proof of the super Jacobian is completed. (Incidentally – the general theorem is in fact a proof that the ordinary Jacobian equals the ordinary determinant.)

I. *Elementary general relativity*

The ϵ-tensors $\epsilon^{\mu\nu\rho\sigma}$ and $\epsilon_{\mu\nu\rho\sigma}$ are always $+1$ or -1 ($\epsilon^{1234} = \epsilon_{1234} = +1$). They are a density and an antidensity, respectively. Thus $\epsilon_{\mu\nu\rho\sigma}$ is obtained from by lowering the indices by means of the metric and dividing by ($\det g_{\mu\nu}$). The tensors ϵ^{abcd} and ϵ_{abcd} are also ± 1 and have density-weight zero. Thus ϵ_{abcd} is obtained from ϵ^{abcd} by lowering the indices with the Minkovski metric ($\delta_{\mu\nu} = ++++$) in our conventions).

Consider a four-dimensional manifold with

(i) connections which define parallel transport

$$\delta v^\alpha = -\Gamma_{\mu\nu}{}^\alpha v^\mu \, \mathrm{d}x^\nu,$$

(ii) a metric $g_{\mu\nu}$ defining distances $(\mathrm{d}s)^2 = g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu$.

Hence the covariant derivative is $D_\rho v^\alpha = \partial_\rho v^\alpha + \Gamma_{\beta\rho}{}^\alpha v^\beta$. Length is preserved under parallel transport if and only if

$$D_\rho g_{\mu\nu} = 0 = \partial_\rho g_{\mu\nu} - \Gamma_{\mu\rho}{}^\alpha g_{\alpha\nu} - \Gamma_{\nu\rho}{}^\alpha g_{\mu\alpha}. \tag{1}$$

Parallel transporting an infinitesimal vector u^α along an infinitesimal v^β, and v^β along u^α, the parallelogram closes if and only if $\Gamma_{\mu\nu}{}^\alpha - \Gamma_{\nu\mu}{}^\alpha = 0$. Torsion is defined by the torsion tensor $S^\alpha_{\mu\nu} = \frac{1}{2}(\Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu})$. One usually requires in general relativity that length is preserved and torsion is absent. In this case $\Gamma^\alpha_{\mu\nu}$ equals the Christoffel symbol $\{{}^{\alpha}_{\mu\nu}\}$ but let us relax these restrictions.

If length is preserved, but torsion is present, one can solve (1) to obtain

$$\begin{aligned} \Gamma^\lambda_{\mu\nu} &= \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} + K^\lambda_{\mu\nu} \\ K^\lambda_{\mu\nu} &= g^{\lambda\rho}(S^\alpha_{\mu\rho} g_{\alpha\nu} + S^\alpha_{\mu\nu} g_{\alpha\rho} + S^\alpha_{\nu\rho} g_{\mu\alpha}). \end{aligned} \tag{2}$$

The tensor $K^\lambda_{\mu\nu}$ is often called the contorsion tensor. Conversely, if $\Gamma^\lambda_{\mu\nu}$ is given by (2) with arbitrary $S^\lambda_{\mu\nu}$, then length is preserved.

If length is not preserved but changes proportional to $\mathrm{d}x^\mu$ and to the length itself, one has

$$\delta(\mathrm{d}s)^2 = (\phi_\mu \, \mathrm{d}x^\mu)(\mathrm{d}s)^2. \tag{3}$$

The covariant vector ϕ_μ is the Weyl vector and is by us interpreted as the dilation gauge field. Now

$$D_\rho g_{\mu\nu} = \phi_\rho g_{\mu\nu} \tag{4}$$

and

$$\Gamma^{\lambda}_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\} + K^{\lambda}_{\mu\nu} + \tfrac{1}{2} g^{\lambda\rho}(\phi_\mu g_{\nu\rho} + \phi_\nu g_{\mu\rho} - \phi_\rho g_{\mu\nu}). \tag{5}$$

Conversely, for such $\Gamma^{\lambda}_{\mu\nu}$ length is changed according to (3).

So far, nullness is preserved. The most general relation reads

$$D_\rho g_{\mu\nu} = \phi_\rho g_{\mu\nu} + T_{\rho,\mu\nu} \tag{6}$$

and does not preserve $(\mathrm{d}s^2) = 0$. One can clearly require that $T_{\mu\nu\rho} = T_{\mu\rho\nu}$ and $g^{\nu\rho} T_{\mu\nu\rho} = 0$ by redefining the Weyl field. $T_{\mu\nu\rho}$ is called the shear tensor; it stretches and shrinks length. The connection is given by (5) with $\phi_\mu g_{\nu\rho}$ replaced by $\phi_\mu g_{\nu\rho} + T_{\mu\nu\rho}$. (Instead of $T_{\rho\mu}{}^{\mu} = 0$ one can make $T_{\mu\nu\rho}$ completely symmetric.)

Under parallel round transport a vector v^α is changed into

$$\delta v^\alpha = -\tfrac{1}{2} R^{\alpha}{}_{\nu\rho\sigma} v^\nu \oint x^\rho \, \mathrm{d}x^\sigma$$

$$R^{\alpha}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\alpha}_{\nu\sigma} + \Gamma^{\alpha}_{\beta\rho} \Gamma^{\beta}_{\nu\sigma} - (\rho \leftrightarrow \sigma). \tag{7}$$

When spinors are present, covariant derivatives are defined by adding connections for local Lorentz transformations. For a spin 1/2 field $D_\mu \chi = \partial_\mu \chi + \frac{1}{2}\omega_\mu{}^{mn} \sigma_{mn} \chi$ see Weinberg's book. All that is required is that $\omega_\mu{}^{mn}$ transforms as a world tensor under general coordinate transformations

$$\delta(\xi^\nu)\omega_\mu{}^{mn} = \xi^\nu \partial_\nu \omega_\mu{}^{mn} + (\partial_\mu \xi^\nu)\omega_\nu{}^{mn} \tag{8a}$$

and as a connection under local Lorentz rotations

$$\delta(\lambda^{mn})\omega_\mu{}^{mn} = -(\partial_\mu \lambda^{mn} + \omega_\mu{}^{mp}\lambda_p{}^{n} + \omega_\mu{}^{np}\lambda^{m}{}_{p}). \tag{8b}$$

In this case, $(D_\mu \chi)^a$ transforms as a tensor $T_\mu{}^a$. There are many ω's satisfying (8a) and (8b). The particular connection $\omega_\mu{}^{ab}(e, \psi)$ used in supergravity has the advantage that it is a tensor under local supersymmetry transformations, i.e., its variation contains no $\partial\epsilon$ terms.

In supergravity one need never specify $\Gamma^{\alpha}_{\mu\nu}$; one only needs $\omega_\mu{}^{mn}$. This is possible since one always deals with world curls such as $D_\mu \psi_\rho - D_\rho \psi_\mu$ for which no general coordinate connection is needed (one could, however, add one. This would not lead to supergravity). However, one can define $\Gamma^{\alpha}_{\mu\nu}$ in terms $\omega_\mu{}^{mn}$, just as one defines in ordinary Einstein–Cartan theory $\{{\alpha \atop \mu\nu}\}$ in terms of $\omega_\mu{}^{mn}(e)$. Namely, one postulates that the tetrad is covariantly constant

$$D_\rho e^{m}{}_{\mu} = \partial_\rho e^{m}{}_{\mu} - \Gamma^{\alpha}_{\mu\rho} e^{m}{}_{\alpha} + \omega_\rho{}^{mn} e_{n\mu} = 0. \tag{9}$$

If this postulate is assumed, then one can rewrite the Riemann tensor as in subsection 1.3.

The derivative D_μ satisfies the Leibniz rule and $D_\mu \gamma_5 = 0$. In the presence of torsion one still has $D_\mu \epsilon_{abcd} = D_\mu \epsilon_{\nu\rho\sigma\tau} = 0$.

J. *Duffin–Kemmer–Petiau formalism for supergravity*

In this formalism the action for bosons has one derivative only, so that bosons and fermions have the same dimension. One might expect that this is the formalism best suited for supersymmetry. Let us take a closer look.

The action for a real spin 0 field is given by

$$\mathcal{L} = -\tfrac{1}{2}\bar{A}(\beta^\mu \partial_\mu + M)A, \qquad \bar{A} = A^{\mathrm{T}}\eta. \tag{1}$$

The matrices β^μ are supposed to be Hermitian and to satisfy

$$\beta^\mu\beta^\nu\beta^\lambda + \beta^\lambda\beta^\nu\beta^\mu = \delta^{\mu\nu}\beta^\lambda + \delta^{\nu\lambda}\beta^\mu. \tag{2}$$

Just as the equation $(a_\rho u^\rho - 1)(b_\sigma u^\sigma + 1) = u_\tau u^\tau - 1$ is solved by the Dirac matrices $a_\rho = b_\rho = \gamma_\rho$, in the same way the β^ρ are a particular solution of*

$$(a_{\rho\sigma}u^\rho u^\sigma + c_\rho u^\rho - 1)(b_\lambda u^\lambda + 1) = u_\tau u^\tau - 1. \tag{3}$$

The Lorentz generators are $\sigma^{mn} = \frac{1}{4}[\beta^m, \beta^n]$ and satisfy the Lorentz algebra as a consequence of (2). One can define $\bar{A} = A^\dagger\zeta$ by requiring that $\bar{A}A$ is Lorentz invariant. This leads to

$$[\zeta, \sigma_{mn}] = 0 \quad \text{and} \quad \{\zeta, \sigma_{m4}\} = 0 \quad \text{with } m, n = 1, 3.$$

A Majorana conjugate boson $\hat{A} = A^{\mathrm{T}}\eta$ is defined by requiring that $\hat{A}$ transform as $\bar{A}$. This leads to $\eta\sigma_{\mu\nu}\eta^{-1} = -\sigma_{\mu\nu}{}^{\mathrm{T}}$. A Majorana boson is defined by $\hat{A} = \bar{A}$. In order that this relation is maintained in time under (1), one needs $\eta^{-1}\beta^{\mu,\mathrm{T}}\eta = \zeta^{-1}\beta^\mu\zeta(-)^{\delta_{\mu 4}}$.

There exists a 5×5 representation of (2) for spin 0 fields and a 10×10 representation for spin 1 fields. A Majorana representation of the former is given by

$$(\beta^k)_{5j} = (\beta^k)_{j5} = \delta^k_j, \qquad (\beta^4)_{54} = -(\beta^4)_{45} = \mathrm{i}. \tag{4}$$

In this representation $\eta = \zeta = 2\beta^4\beta^4 + 1 = \mathrm{diag}(1, 1, 1, -1, -1)$ commutes with β^4 and anticommutes with β^k. Also $\eta\beta^\mu\eta^{-1} = -\beta^{\mu,\mathrm{T}}$. However, $\eta^{\mathrm{T}} = +\eta$ whereas for fermions $C^{\mathrm{T}} = -C$.

The field equations of (1) read in this representation $MA_k + \partial_k A_5 = 0$ and $MA_4 - \mathrm{i}\partial_4 A_5 = 0$ so that A equals the five-vector $(\partial_k A_5, -\mathrm{i}\partial_4 A_5, -MA_5)$. Inserting these equations into the action, one finds the Klein–Gordon action for $\varphi = \mathrm{i}A_5$

$$\mathcal{L} = \frac{1}{2M}\varphi(\Box - M^2)\varphi, \qquad \varphi = \mathrm{i}A_5. \tag{5}$$

Consider now as a model of supersymmetry a Majorana boson A and a Majorana fermion ψ. The action reads

$$\mathcal{L} = -\tfrac{1}{2}\bar{A}(\beta^\mu\partial_\mu + M)A - \tfrac{1}{2}\bar{\psi}(\gamma^\mu\partial_\mu + M)\psi. \tag{6}$$

It is invariant under

* I thank Dr. Dresden for pointing this out to me.

$$\delta A = (\bar{\epsilon}\partial_\mu\psi)(\beta^\mu u) + (\bar{\epsilon}\gamma^\mu\partial_\mu\psi)u$$
$$\delta\psi = -(\bar{u}\partial_\mu A)(\gamma^\mu\epsilon) - (\bar{u}\beta^\mu\partial_\mu A)\epsilon. \tag{7}$$

The five-vector $u = \mathrm{diag}(0,0,0,0,1)$ is Lorentz invariant. To prove the invariance, one may use $\bar{A}B = \bar{B}A$, $\bar{A}\beta^\mu B = -\bar{B}\beta^\mu A$ for Majorana bosons, and $\beta^\mu\beta^\nu u = \delta^{\mu\nu}u$ while $\bar{u}\beta^\mu\beta^\nu = \bar{u}\delta^{\mu\nu}$.

The algebra closes only on-shell, even if one adds a pseudoscalar B just as in the usual formulation of the Wess–Zumino model. Off-shell closure requires six auxiliary spin 0 fields and two auxiliary Majorana spinors. Since this number is considerably higher than the two auxiliary fields F and G in the usual formulation, it is to be expected that Kemmer–Duffin–Petiau formalism does not lead to a simpler formulation of supergravity.

For further literature see

R. Gastmans and W. Troost, Leuven preprint.
P. Roman, Theory of elementary particles (North-Holland, Amsterdam, 1961).
H. Umezawa, Quantum field theory (North-Holland, Amsterdam, 1956).
A. I. Akhiezer and V. B. Berestetskii, Quantum Electrodynamics (Interscience, New York, 1965).
N. G. Deshpande and P. C. MacNamee, Phys. Rev. D5 (1972) 1389.

K. *Glossary*

Supersymmetric transformation laws (*SSTL*) – A set of continuous transformations which transform classical, commuting, Bose, real or complex, integer spin fields, into classical, anticommuting, Fermi, real (Majorana) or complex (Dirac), half integer spin fields and vice versa. In the usual technical sense, a supersymmetric generator transforms a real Bose field into a real (Majorana) Fermi field.

Supersymmetric model (*SM*) – A classical Lagrangian field theory in flat space-time, whose action is invariant under one-global (x-independent) SSTL.

Extended supersymmetric model (*ESM*) – The same as before, with "one" replaced by N ($N \geq 2$).

Supersymmetry – The domain of Mathematical Physics which studies SSTL's, SM's and ESM's.

Superspace – A set of coordinates z^A which generalizes ordinary space-time through the introduction of extra Fermionic anticommuting coordinates $\theta^{i\alpha}$ ($\alpha = 1, \ldots, 4$; $i = 1, \ldots, N$) carrying spin $\frac{1}{2}$. This generalizes the usual concept of a "point" to the concept of a "spinning point", referring "spin" to spacetime rather than to the fields.

Superfield – A function $\theta(z^A)$ with or without indices.

Supermultiplet – An irreducible representation of simple ($N = 1$) or extended ($N \geq 2$) supersymmetry in terms of Poincaré covariant fields.

Matter supermultiplet – A supermultiplet with maximal spin $J_{\mathrm{MAX}} = 1$ or $\frac{1}{2}$. For $J_{\mathrm{MAX}} = 1$, they are called vector multiplets. For $J_{\mathrm{MAX}} = \frac{1}{2}$, they are called scalar multiplets.

Supersymmetric Yang–Mills theories (*SSYM*) – The self-interacting field theory based on a vector supermultiplet, with dimensionless gauge coupling constant g, containing as a special case the Yang–Mills theory.

Goldstino – The massless, spin $\frac{1}{2}$ fermion associated with the spontaneous breaking of supersymmetry invariance.

Gluinos – Supersymmetric partners of the gluons, having spin $\frac{1}{2}$ (colour octets of zero electric charge).

Majorana representation – A representation of the γ matrices where they are all real or imaginary (depending upon the metric convention) such that the Dirac operator is real.

Majorana spinor – A spinor whose Majorana and Dirac conjugate are equal. In a Majorana representation for the γ matrices, it is a real spinor.
Weyl spinors – Left- or right-handed spinors, i.e. eigenfunctions of $(1 \pm \gamma_5)$ with $\gamma_5^2 = +1$.
Majorana–Weyl spinors – Have both of the previous properties. This is possible only for space-times of dimension $D = 2$ modulo 8.
Rigid supersymmetry – Supersymmetry with constant parameters. The more familiar term global supersymmetry has the drawback that it is not meant to refer to global topological properties.
Global supersymmetry – See above.
Local supersymmetry – Supersymmetry with spacetime dependent parameters.
Simple supergravity models – Supersymmetric models in curved space-time, invariant under one SSTL.
Extended supergravity models – Same as before, with "one" replaced by N ($N \geq 2$).
Supergravity – The domain of Mathematical Physics which studies simple ($N = 1$) or extended ($N \geqslant 2$) supergravity models.
Gravitinos – The gauge particles of spin $\frac{3}{2}$, associated with local, simple or extended supersymmetry.
Gauge supermultiplet – The supermultiplet containing the graviton, the gravitinos and eventually (for $N \geq 2$), lower spin fields.
Pure supergravity – Simple or extended supergravity models based on a gauge supermultiplet with a self coupling K, of dimension the inverse of a mass, uncoupled to any matter multiplet.
Super Higgs effect – The supersymmetric analog of the Higgs effect whereby, when supersymmetry is spontaneously broken, a gravitino (or several of them) "eats up" a goldstino (or several) and becomes massive.
Antigravity – A vectorial force, due to the exchange of a vector particle which is a member of an extended ($N \geq 2$) gauge supermultiplet and which between two particles exactly cancels the gravitational force in the static limit, and doubles it between a particle and an antiparticle.
Tetrad — The gauge fields of spin 2 which must be used instead of the metric when fermions are present.
Vierbein – Same as tetrad.
Supervielbein – Vierbein in superspace (viel = many in German).
Graded Lie algebras – A Lie algebra with a grading. For example, the Lorentz group with $\boldsymbol{k}$ and $\boldsymbol{j}$ has a Z_2 grading.
Superalgebras – A Z_2 graded Lie algebra whose elements are "even" or "odd" such that the bracket relation for two odd elements is symmetric, while also the super-Jacobi identity is satisfied. (Hence a superalgebra is a graded algebra, but not the reverse.)
Fierzing – Performing a Fierz rearrangement on four spinors (see appendix D).
Vector multiplet – A multiplet containing a vector field. In $N = 1$ supergravity, it contains a real vector, a Majorana spinor and an (auxiliary) scalar D.
Scalar multiplet – A multiplet containing scalar fields. In $N = 1$ supergravity it contains a Majorana spinor and two propagating and two auxiliary spin 0 fields.
Multiplets – Irreducible representations of superalgebras. In global supersymmetry, these representations are linear, in supergravity they are nonlinear in fields.
Supercovariant – An object whose supersymmetry variation does not contain terms with $\partial_\mu \epsilon$.

L. *Conventions*

In order to facilitate comparison between the conventions used in this report and those used by other authors, we give here a table of translations

Ours	WZ	SW	LR
$\bar{\chi}\eta$	$\bar{\chi}_\alpha\bar{\eta}^\alpha + \chi^\alpha\eta_\alpha$	$\mathrm{i}\bar{\chi}\eta$	$\bar{\chi}_{A'}\bar{\eta}^{A'} + \chi_A\eta^A$
$\bar{\chi}\mathrm{i}\gamma_5\eta$	$\mathrm{i}(\bar{\chi}_\alpha\bar{\eta}^\alpha - \chi^\alpha\eta_\alpha)$	$-\mathrm{i}\bar{\chi}\gamma_5\eta$	$-\mathrm{i}(\bar{\chi}_{A'}\bar{\eta}^{A'} - \chi_A\eta^A)$
$\bar{\chi}\left(\frac{1+\gamma_5}{2}\right)\gamma_\mu\eta$	$-\mathrm{i}\eta\sigma_\mu\bar{\chi}$	$\mathrm{i}\bar{\chi}\left(\frac{1+\mathrm{i}\gamma_5}{2}\right)\gamma_\mu\eta$	$\sqrt{2}\chi_M\bar{\eta}_{M'}$
$\bar{\chi}\left(\frac{1-\gamma_5}{2}\right)\gamma_\mu\eta$	$-\mathrm{i}\chi\sigma_\mu\bar{\eta}$	$\mathrm{i}\bar{\chi}\left(\frac{1+\gamma_5}{2}\right)\gamma_\mu\eta$	$\sqrt{2}\eta_M\bar{\chi}_{M'}$
$\bar{\chi}\sigma_{\mu\nu}\eta$	$2(\chi\sigma_{\mu\nu}\eta + \bar{\chi}\bar{\sigma}_{\mu\nu}\bar{\eta})$	$\mathrm{i}\bar{\chi}\sigma_{\mu\nu}\eta$	
$\eta^{\mu\nu} = (++++)$	$(+---)$	$(-+++)$	$-\epsilon^{MN}\epsilon^{M'N'}$
$\epsilon_{\mu\nu\rho\sigma}$		$-\mathrm{i}\epsilon_{\mu\nu\rho\sigma}$	
$R_{\mu\nu}$		$-R_{\mu\nu}$	$R_{\mu\nu}$
${\omega_\mu}^{mn}$	$-{\omega_\mu}^{mn}$	${\omega_\mu}^{mn}$	${\omega_\mu}^{mn}$

The abbreviations stand for WZ = Wess–Zumino, SW = Stelle West, LR = Lindstrom–Roček. The conventions of deWit et al. agree with ours.

References

[1] A. D'Adda, R. D'Auria, P. Fre and T. Regge, Geometrical Formulation of Supergravity Theories on Orthosymplectic Supergroup Manifolds, IFTT, Torino preprint (to be published in Rivista del Nuovo Cimento).

[2] P.C. Aichelburg and T. Dereli, Exact Plane-wave solutions of supergravity field equations, Phys. Rev. D18 (1978) 1754.

[3] P.C. Aichelburg and T. Dereli, First non-trivial exact solution of supergravity, Czech. J. Phys. B29 (1979).

[4] V.P. Akulov, D.V. Volkov and V.A. Soroka, Gauge Fields on Superspace, with different Holonomy Groups, JETP Lett. 22 (1975) 187.

[4a] D.V. Volkov and V.P. Akulov, On possible universal interaction of neutrino, Pis'ma Zh. Eksp. Teor. Fiz. 16 (1972) 621;
D.V. Volkov and V.P. Akulov, Is the neutrino a Goldstone particle? Phys. Lett. 46B (1973) 109;
V.P. Akulov and D.V. Volkov, Goldstone fields with a spin one half, Teor. Mat. Fiz. 18 (1974) 39;
D.V. Volkov and V.A. Soroka, Higgs effect for Goldstone particles with spin $\frac{1}{2}$, Pis'ma Zh. Eksp. Teor. Fiz. 18 (1973) 529;
D.V. Volkov and V.A. Soroka, Gauge fields for symmetry group with spinor parameters, Teor. Mat. Fiz. 20 (1974) 291;
V.P. Akulov, D.V. Volkov and V.A. Soroka, Gauge fields on superspaces with different holonomy groups, Pis'ma Zh. Eksp. Teor. Fiz. 22 (1975) 396.
V.P. Akulov, D.V. Volkov and V.A. Soroka, On general covariant theories of gauge fields on superspace, Teor. Mat. Fiz. 31 (1977) 12;
V.P. Akulov and D.V. Volkov, On Riemannian superspaces of minimal dimensionality, Teor. Mat. Fiz. 41 (1979) 147.

[5] C. Aragone, J. Chela and A. Restuccia, Local Geometry of Superconformal Gravity, Phys. Lett. 82B (1979) 377.

[6] C. Aragone and S. Deser, Consistency Problems of Hypergravity, Phys. Lett. 86B (1979) 161.

[7] C. Aragone and S. Deser, Hamiltonian Form For Massless Higher Spin Fermions, Phys. Rev. D 21 (1980) 352.

[8] C. Aragone, Son Universales Las Gravedades?, Einstein Sinposium, ed. J. Chela-Flores (Editorial Equinioccio, Caracas).

[9] C. Aragone and A. Restuccia, The Baker–Campbell–Hausdorff Formula For the Chiral SU(2) Supergroup, Lett. Math. Phys. 3 (1979) 29.

[10] C. Aragone and S. Deser, Consistency Problems of Hypergravity, Phys. Lett. 86B (1979) 161.

[11] C. Aragone and S. Deser, Consistency Problems of Spin-2 Gravity Couplings, Nuovo Cimento B (1980).

[12] C. Aragone and S. Deser, Higher Spin Vierbein Gauge Fermions and Hypergravities, Nucl. Phys. B (1980).

[13] C. Aragone and S. Deser, Consistency Requirements in Hypergravity and Spin-2 Matter Gravity Coupling Problems, Proc. Supergravity Workshop (North-Holland, 1980).

[14] C. Aragone and S. Deser, Massless Vierbein spin 5/2 fields and Hypergravity, Proc. Marcel Grossmann Conf. (North-Holland, 1980).

[15] C. Aragone, J. Chela and A. Restuccia, The BCH formula for SU(2) supergroups, J. Math. Phys. (1980) (to be published).

[16] C. Aragone, Euclidean hypergravities and vierbein gauge fermions, Proc. of the Gravity, Gravities and Supersymmetry Symposium, ed. J. Chela (Equinoccio, 1980) (to be published).

[17] R. Arnowitt, P. Nath and B. Zumino, Superfield Densities and Action Principle in Curved Superspace, Phys. Lett. 56 (1975) 81.

[18] P. Nath, Supersymmetry and Gauge Supersymmetry, Invited talk at Conf. on Gauge Theories and Modern Field Theory at Northeastern Univ., Sept. 26–27, 1975 (M.I.T. Press, Cambridge, Mass., 1976).

[19] P. Nath, Local Supersymmetries, Invited paper presented at the American Physical Society Meeting at Washington, D.C., April 1977, Northeastern Univ. Report NUB #2329.

[20] P. Nath and R. Arnowitt, Generalized Super-Gauge Symmetry as a New Framework for Unified Gauge Theories, Phys. Lett. 56B (1975) 177.

[21] R. Arnowitt and P. Nath, Riemannian Geometry in Spaces with Grassman Coordinates, Proc. Conf. The Riddle of Gravitation, on the occasion of the 60th birthday of Peter G. Bergmann, Syracuse, N.Y., 1975 (Invited talk); General Relativity and Gravitation 7 (1976) 89.
[22] P. Nath and R. Arnowitt, Gauge Supersymmetry, Proc. French Physical Society Meeting at Dijon, France, July 1975 (Invited talk); Journal de Physique 2 (1976) C2-85.
[23] R. Arnowitt and P. Nath, Spontaneous Symmetry Breaking of Gauge Supersymmetry, Phys. Rev. Lett. 36 (1976) 1526.
[24] P. Nath and R. Arnowitt, Supergravity and Gauge Supersymmetry, Phys. Lett. 65B (1976) 73.
[25] R. Arnowitt and P. Nath, Origin of Internal Symmetry, Phys. Rev. D15 (1977) 1033.
[26] P. Nath and R. Arnowitt, Structure of Spontaneously Broken Local Gauge Supersymmetry, Nucl. Phys. B122 (1977) 301.
[27] R. Arnowitt and P. Nath, Local Supersymmetry and Interactions, Proc. Orbis Scientiae, University of Miami, Coral Gables, Fla., 1977 (Invited talk).
[28] R. Arnowitt, New Directions in Supersymmetries, Proc. 5th Intern. Conf. on Experimental Meson Spectroscopy, 1977 (Invited talk) p. 158.
[29] P. Nath and R. Arnowitt, Quantum Effects on Vacuum Symmetries of Gauge Supersymmetry, Phys. Rev. D 18 (1978) 2759.
[30] R. Arnowitt and P. Nath, Preserved Gauges, Broken Gauges and Fictitious Goldstone Fields in Gauge Supersymmetry, Nucl. Phys. B144 (1978) 363.
[31] R. Arnowitt and P. Nath, Superspace Formulation of Supergravity, Phys. Lett. 78B (1978) 581.
[32] R. Arnowitt and P. Nath, Local Supersymmetries in Superspace, Proc. Seoul Symposium on Elementary Particle Physics, Seoul (1978) p. 159.
[33] R. Arnowitt and P. Nath, Globally Supersymmetric Green's Functions in Gauge Supersymmetry (Invited paper) for FESTSCHRIFT for the 60th birthday of Julian Schwinger, Physica 96A (1979) 111.
[34] P. Nath and R. Arnowitt, Ultraviolet Finiteness of all Quantum Loops in Gauge Supersymmetry, Phys. Rev. Lett. 42 (1979) 138.
[35] P. Nath and R. Arnowitt, Geometrization in Superspace and Local Supersymmetry, Proc. VII Integrative Conf. on Group Theory and Mathematical Physics, Austin, Texas (1978).
[36] R. Arnowitt and P. Nath, Supersymmetry Formulated in Superspace, Proc. Orbis Scientiae, University of Miami, Coral Gables, Florida (1979).
[37] R. Arnowitt and P. Nath, Supergraph Analysis of the Ultraviolet Finiteness of Gauge Supersymmetry, Nucl. Phys. B161 (1979) 321.
[38] P. Nath and R. Arnowitt, Supergravity Geometry in Superspace (sub. to) Nucl. Phys. (NUB #2405).
[39] P. Nath, Local Supersymmetry, Invited talk in Proc. 19th Intern. Conf. on High Energy Physics, Tokyo 1978.
[40] R. Arnowitt and P. Nath, Superspace Geometry of Supergravity, in: Proc. Stony Brook Conf. on Supergravity, eds. P. van Nieuwenhuizen and D.Z. Freedman (North-Holland, 1979).
[41] P. Nath and R. Arnowitt, Supergraphs and Ultraviolet Finiteness, in: Gauge Supersymmetry, Proc. Stony Brook Conf. on Supergravity, eds. P. van Nieuwenhuizen and D.Z. Freedman (North-Holland, 1979).
[42] R. Arnowitt and P. Nath, Superconnections of Extended Supergravity, Phys. Rev. Lett. 44 (1980) 223.
[43] P. Nath and R. Arnowitt, Supergravity and Gauge Supersymmetry, in: Proc. Europhysics Study Conference in Erice, Italy, eds. J. Ellis, S. Ferrara, P. van Nieuwenhuizen (Plenum, to be published).
[44] S. Baaklini, New Superalgebra And Goldstone Spin 3/2 Particle, Phys. Lett. 67B (1977) 335.
[45] S. Baaklini, Spin 3/2 Field And Cartan's Geometry, Lett. Math. Phys. 2 (1977) 43.
[46] S. Baaklini, Cartan's Geometrical Structure of Supergravity, Lett. Math. Phys. 2 (1977) 115.
[47] S. Baaklini, Non-Linear Realizations And Higgs Mechanism For Extended Supersymmetry, J. Phys. C: Nucl. Phys. 4 (1978) 1.
[48] S. Baaklini, Supersymmetry And Classical Solutions, Nucl. Phys. B129 (1977) 354.
[49] S. Baaklini, S. Ferrara and P. van Nieuwenhuizen, Classical Solutions In Supergravity, Lett. Nuovo Cimento 20 (1977) 113.
[50] S. Baaklini, Solitons And Supertranslations, Nucl. Phys. B134 (1978) 169.
[51] S. Baaklini, Nonlinear de Sitter Symmetry And Restricted Space-time Regions In Gravity And Supergravity, Gen. Rel. Grav. 10 (1979) 365.
[52] S. Baaklini, Nonlinear de Sitter Symmetry And Restricted Propagation Universes, Lett. Nuovo Cimento 22 (1978) 219.
[53] S. Baaklini, SL(2, C) Exterior Forms And Quantum Gravity, Phys. Rev. D19 (1979) 467.
[54] S. Baaklini, Quantum Spinors And The Singularity Theorems Of General Relativity, Phys. Lett. 66A (1978) 357.
[55] S. Baaklini, Is Supergravity The Fundamental Supergeometrodynamic Theory Of The Hadron? Hadronic J. 2 (1979) 135.
[56] S. Baaklini and M. Tuite, Dirac Quantization Of Massive Spin 3/2 Field, J. Phys. A 6 (1978) L139.
[57] S. Baaklini, Quantization Around Superparametrized Finite-Energy Classical Solutions, J. Phys. A11 (1978) 2083.
[58] S. Baaklini, Quantization Of Superparametrized Monopoles, Nucl. Phys. B142 (1978) 510.
[59] S. Baaklini and M. Tuite, Dirac Quantization of Spin 2 Field, J. Phys. A (1978).
[60] S. Baaklini and H.E. Bouzouita, Quantization of The Nonlinear Super Poincaré Fermion In A Bag-like Formulation, J. Phys. A (1979).
[61] S. Baaklini and Abdus Salam, Non-metrical Supergravity For A Spin Two Nonet, Nucl. Phys. B153 (1979) 237.
[62] N.S. Baaklini and M. Tuite, Dirac Quantization of Spin-2 Field, J. Phys. A12 (1979) L13.
[63] N.S. Baaklini and H.E. Bousouita, Quantization of the Nonlinear Super Poincaré Fermion in a Bag-like Formulation, J. Phys. A12 (1979) 2169.
[64] I. Bars and S. MacDowell, Spinor Theory of General Relativity Without Elementary Gravitons, Phys. Lett 71B (1977) 111.
[65] I. Bars and S. MacDowell, A Spin-3/2 Theory of Gravitation, Gen. Rel. and Grav. 10 (1979) 205.
[66] I. Bars and M. Günaydin, Construction of Lie Algebras and Lie Superalgebras From Ternary Algebras, J.M.P. (September 1979).
[67] I. Bars, Ternary Algebras as Building Blocks of Lie Algebras and Lie Superalgebras, in: Proc. 8th Int. Colloq. on Group Theoretical Methods in Physics, Israel, March 1979, to be published.
[68] P. Breitenlohner, A Geometric Interpretation of Local Supersymmetry, Phys. Lett. B67 (1977) 49.

[69] P. Breitenlohner, Some invariant Lagrangians for Local Supersymmetry, Nucl. Phys. B124 (1977) 500–510.
[70] P. Breitenlohner, On the Auxiliary Fields of Supergravity, Phys. Lett. 80B (1979) 217.
[71] P. Breitenlohner and A. Kabelschacht, The Auxiliary Fields of $N = 2$ Extended Supergravity in 5 and 6 Space-Time Dimensions, Nucl. Phys. B148 (1979) 96–106.
[72] P. Breitenlohner and M. Sohnius, Superfields, Auxiliary Fields, and Tensor Calculus for $N = 2$ Extended Supergravity, Submitted to Nucl. Phys. B (Sept. 1979).
[73] L. Brink, S. Deser, B. Zumino, P. di Vecchia and P. Howe, Local Supersymmetry for Spinning Particles, Phys. Lett. 64B (1976) 435.
[74] L. Brink, P. di Vecchia and P. Howe, A Lagrangian Formulation of the Classical and Quantum Dynamics of Spinning Particles, Nucl. Phys. B118 (1977) 76.
[75] L. Brink and J. Schwarz, Local Complex Supersymmetry in 2 Dimensions, Nucl. Phys. B121 (1977) 285.
[76] L. Brink, P. di Vecchia and P. Howe, A Locally Supersymmetric and Reparametrization Invariant Action for the Spinning String, Phys. Lett. 65B (1976) 471.
[77] L. Brink, M. Gell-Mann, P. Ramond and J. Schwarz, Supergravity as Geometry of Superspace, Phys. Lett. 74B (1978) 336.
[78] L. Brink, M. Gell-Mann, P. Ramond and J. Schwarz, Extended Supergravity as Geometry of Superspace, Phys. Lett. 76B (1978) 417.
[79] L. Brink, M. Gell-Mann, P. Ramond and J. Schwarz, Prepotentials in a Superspace Formulation of Supergravity, Nucl. Phys. B145 (1978) 93.
[80] L. Brink and P. Howe, Minimal Set of Superspace Equations for Supergravity and Supersymmetric Yang-Mills Theories, submitted to Phys. Lett.
[81] L. Brink and P. Howe, A superspace Action for Supergravity, Phys. Lett. 88B (1979) 81.
[82] L. Brink and P. Howe, The $N = 8$ Supergravity in Superspace, Phys. Lett 88B (1979) 268.
[83] L. Brink and P. Howe, Eleven-dimensional Supergravity on the Mass-shell in Superspace, Phys. Lett. (to be published).
[84] S. Christensen, S. Deser, M. Duff and M. Grisaru, Chirality, Self-Duality and Supergravity Counterterms, Phys. Lett. 84B (1979) 411.
[85] S.M. Christensen and M.J. Duff, Quantizing Gravity with a Cosmological Constant (to be published).
[86] S.M. Christensen, Second and Fourth Order Invariants on Curved Manifolds with Torsion, J. Phys. A (in press).
[87] S.M. Christensen, M.J. Duff, G.W. Gibbons and M. Roček, Vanishing One-loop β Function in Gauged $N > 4$ Supergravity (to be published).
[88] S.M. Christensen, M.J. Duff, G.W. Gibbons and M. Roček, One-Loop Effects in Supergravity with a Cosmological Constant (to be published).
[89] E. Cremmer, J. Scherk and S. Ferrara, $U(N)$ Invariance in Extended Supergravity, Phys. Lett. 68B (1977) 234.
[90] E. Cremmer and J. Scherk, Modified Interaction of the Scalar Multiplet Coupled to Supergravity, Phys. Lett. 69B (1977) 97.
[91] E. Cremmer and J. Scherk, Algebraic simplification in supergravity theories, Nucl. Phys. B127 (1977) 259.
[92] E. Cremmer and J. Scherk, The Supersymmetric non linear σ model in 4 dimensions and its coupling to supergravity, Phys. Lett. 74B (1978) 341.
[93] E. Cremmer, B. Julia and J. Scherk, Supergravity in 11 dimensions, Phys. Lett. 76B (1978) 409.
[94] E. Cremmer, B. Julia, J. Scherk, P. van Nieuwenhuizen, S. Ferrara and L. Girardello, The super Higgs effect in supergravity with general scalar interactions, Phys. Lett. 79B (1978) 231.
[95] E. Cremmer, B. Julia, J. Scherk, P. van Nieuwenhuizen, S. Ferrara and L. Girardello, Spontaneous symmetry breaking and Higgs effect in supergravity without cosmological constant, Nucl. Phys. B147 (1979) 105.
[96] E. Cremmer and B. Julia, The $N = 8$ supergravity theory. The Lagrangian, Phys. Lett. 80B (1978) 48.
[97] E. Cremmer, $N = 8$ supergravity, in: Unification of the fundamental interactions, Europhysics Conf., March 1980.
[98] E. Cremmer, Supergravities in 5 dimensions, Nuffield gravity workshop, Cambridge, England (June 1980).
[99] E. Cremmer and B. Julia, SO(8) Supergravity, Nucl. Phys. B159 (1979) 141.
[100] E. Cremmer, J. Scherk and J. Schwarz, Spontaneously broken $N = 8$ Supergravity, Phys. Lett. 84B (1979) 83.
[101] E. Cremmer and S. Ferrara, Formulation of 11-dimensional supergravity in Superspace, Phys. Lett. 91B (1980) 61.
[102] T.L. Curtright and G.I. Ghandour, Stability and Supersymmetry: General Formalism And Explicit Two-Loop Applications, Ann. Phys. (NY) 106 (1977) 209.
[103] T.L. Curtright and G.I. Ghandour, Stability and Supersymmetry: Models With Local Gauge Symmetry, Ann. Phys. (NY) 112 (1978) 237.
[104] T.L. Curtright, Conformal Spinor Current Anomalies, Phys. Lett. 71B (1977) 185.
[105] T.L. Curtright, Massless Field Supermultiplets With Arbitrary Spin, Phys. Lett. 85B (1979) 219.
[106] T.L. Curtright and P.G.O. Freund, SU(8) Unification And Supergravity, Proc. Stony Brook Supergravity Workshop.
[107] A. Das and D.Z. Freedman, Gauge Quantization for spin 3/2 fields, Nucl. Phys. B114 (1979) 271.
[108] A. Das and D.Z. Freedman, Gauge Internal Symmetry in Extended Supergravity, Nucl. Phys. B120 (1977) 221.
[109] A. Das, SO(4) Invariant Extended Supergravity, Phys. Rev. D15 (1977) 2805.
[110] A. Das, M. Fischler and M. Roček, Massive, Self-interacting Scalar Multiplet Coupled to Supergravity, Phys. Lett. 69B (1977) 186.
[111] A. Das, M. Fishler and M. Roček, Super Higgs Effect in a Class of Scalar Models and a Model of Super Q.E.D., Phys. Rev. D16 (1977) 3427.
[112] A. Das, M. Kaku and P. Townsend, A Unified Approach to Matter Coupling in Weyl and Einstein Supergravity, Phys. Rev. Lett. 40 (1978) 1215.
[113] D. Derbes, The failure of minimal coupling in supergravity, Phys. Lett. 79B (1978) 219.
[114] D. Derbes, The Higgs mechanism in $1 + 1$ supergravity, Phys. Lett. 77B (1978) 59.
[115] T. Dereli and S. Deser, Supergravity coupling to Non-Linear Realizations in two-dimensions, J. Phys. A10 (1977) L149; Addendum: A10 (1977) 2189.
[116] T. Dereli and S. Deser, Fermionic Goldstone-Higgs effect in $(2 + 1)$-dimensional supergravity, J. Phys. A11 (1978) L27.

[117] P.C. Aichelburg and T. Dereli, Exact plane-wave Solutions of Supergravity Field Equations, Phys. Rev. D18 (1978) 1754.
[118] T. Dereli and P.C. Aichelburg, Exact plane-wave solutions of O(2) extended Supergravity, Phys. Lett. 80B (1979) 357.
[119] T. Dereli and R. Guven, Exact Solutions of Supergauge Invariant Yang–Mills Equations, Phys. Lett. 84B (1979) 201.
[120] S. Deser and B. Zumino, Consistent Supergravity, Phys. Lett. 62B (1976) 335.
[121] S. Deser, L. Brink, P. DiVecchia, P. Howe and B. Zumino, Local Supersymmetry for Spinning Particles, Phys. Lett. 64B (1976) 435.
[122] S. Deser and B. Zumino, A Complete Action for the Spinning String, Phys. Lett. 65B (1976) 369.
[123] S. Deser, J. Kay and K. Stelle, Renormalizability Properties of Supergravity, Phys. Rev. Lett. 38 (1977) 527.
[124] S. Deser and C. Teitelboim, Supergravity Has Positive Energy, Phys Rev. Lett. 39 (1977) 249.
[125] S. Deser and B. Zumino, Broken Supersymmetry and Supergravity, Phys. Rev. Lett. 38 (1977) 1433.
[126] S. Deser, J. Kay and K. Stelle, Hamiltonian Form and Duality Properties of Supergravity, Phys. Rev. D16 (1977) 2448.
[127] S. Deser and T. Dereli, Supergravity Coupling to Non-Linear Realization in Two Dimensions, J. Phys. A10 (1977) L149.
[128] S. Deser and T. Dereli, Fermionic Goldstone-Higgs Effect in (2 + 1) dimensional Supergravity, J. Phys. A11 (1978) L27.
[129] S. Deser and J.H. Kay, Three-Loop Counterterms for Extended Supergravity, Phys. Lett. 76B (1978) 400.
[130] S. Deser, Supergravity, in: Proc. Intern. Conf. on Differential Geometric Methods in Mathematical Physics, 1978 (Springer Verlag, Berlin, NY, 1978).
[131] S. Deser, From Einstein's Gravity to Supergravity, in: Sixty Years of General Relativity (Crown Publ., NY, 1979).
[132] S. Deser, The Dynamics of Supergravity in Gravitation: Recent Developments (Plenum Press, NY, 1979).
[133] S. Deser, From Gravity to Supergravity, in: Einstein Centennial Volume (Plenum Press, NY, 1979).
[134] S. Deser, J.H. Kay and D.G. Boulware, Supergravity from Self-Interaction, Physica (special issue in honor of J. Schwinger) 96A (1979) 141.
[135] S. Deser, Exclusion of Static Solutions in Gravity-Matter Coupling, B. Jouvet Memorial Volume (D. Reidel Publ. Co., Dordrecht, Holland, 1979).
[136] S. Deser, Absence of Tachyons in Supergravity, Phys. Rev. D19 (1979) 3165.
[137] S. Deser, Current Status of Supergravity, Springer Lecture Notes in Math (1979).
[138] S. Deser, Space-time Properties of Supergravity, Springer Lecture Notes in Physics (1979), 94, Group Theoretical Methods in Physics.
[139] S. Deser, On Consistency of Spin 3/2 Couplings, J. Phys. A12 (1979) L97.
[140] S. Deser, Les Bases Quantiques de la Relativite-Generale et de la Supergravite, in: Centenaire D'Einstein (CNRS Paris, 1979).
[141] S. Deser, What can Supergravity Teach us About Gravity (to be published) in Springer tracts in mathematics (1979).
[142] S. Deser, Supergravity: A post-Einstein unification, in: On the Path of Albert Einstein (Plenum Press, NY, 1979).
[143] S. Deser and U. Lindstrom, Extended Supersymmetry Invariants By Dimensional Reduction, Brandeis Preprint (1979).
[144] S. Deser and C. Aragone, Vierbein Description Of Higher Spin Fermions, Brandeis preprint (1979).
[145] S. Deser and C. Aragone, Inconsistency Of Gravitationally Coupled Tensor Fields, Brandeis preprint (1979).
[146] S. Deser, Supergravity without supergravity Brandeis Preprint (1979).
[147] S. Deser, Massless Limit of Vector Multiplets in Supergravity, Phys. Rev. D (1980).
[148] S. Deser, Hypergravities and Supergravity – a Unique Self-Interacting Theory, Springer Lectures in Physics (1980).
[149] S. Deser and R. Puzolowski, Supersymmetric Nonpolynomial Vector Multiplets and Causal Propagation, Journal of Physics A (1980).
[150] S. Deser, Supergravities: Successes and Problems, in: Unification of the Fundamental Interactions, eds. J. Ellis, S. Ferrara and P. van Nieuwenhuizen (Plenum, 1980).
[151] B. de Wit, F.A. Berends, J.W. van Holten and P. van Nieuwenhuizen, On Spin 5/2 gauge fields, Phys. Lett. 83B (1979) 188.
[152] B. de Wit, F.A. Berends, J.W. van Holten and P. van Nieuwenhuizen, On Field Theory For Massive And Massless Spin-5/2 Particles, Nucl. Phys. B154 (1979) 261.
[153] B. de Wit, F.A. Berends, J.W. van Holten and P. van Nieuwenhuizen, On Spin-5/2 Gauge Fields, Journal of Physics A., to be published.
[154] B. de Wit, Supersymmetry Currents In The Presence Of Symmetry Breaking, Phys. Rev. D11 (1975) 898.
[155] B. de Wit, Quantization Of Supersymmetric Gauge Theories, Phys. Rev. D12 (1975) 1628.
[156] B. de Wit and D.Z. Freedman, Phenomenology Of Goldstone Neutrino's, Phys. Rev. Lett. 35 (1975) 827.
[157] B. de Wit and D.Z. Freedman, Combined Supersymmetric And Gauge-Invariant Field Theories, Phys. Rev. D12 (1975) 2286.
[158] B. de Wit, On A Locally Supersymmetric Extension Of Quantum Electrodynamics, Phys. Lett. 66B (1977) 77.
[159] B. de Wit and D.Z. Freedman, On SO(8) Extended Supergravity, Nucl. Phys. B130 (1977) 105.
[160] B. de Wit and M.T. Grisaru, Covariant Quantization Of Supergravity, Phys. Lett. 74 (1978) 57.
[161] B. de Wit and M.T. Grisaru, Auxiliary Fields and Ultraviolet Divergences in Supergravity, Nucl. Phys. B139 (1978) 531.
[162] B. de Wit and J.W. van Holten, Covariant Quantization Of Gauge Theories With Open Gauge Algebra, Phys. Lett. 79B (1978) 389.
[163] B. de Wit, Properties Of SO(8)-Extended Supergravity, Nucl. Phys. B158 (1979) 189.
[164] B. de Wit and J.W. van Holten, Multiplets Of Linearized SO(2) Supergravity, Nucl. B155 (1979) 530.
[165] B. de Wit and M.T. Grisaru, On-shell Counterterms And Nonlinear Invariances, Phys. Rev. D20 (1979) 2082.
[166] B. de Wit and D.Z. Freedman, Systematics Of Higher-spin Gauge Fields, Phys. Rev. D21 (1980).
[167] B. de Wit and P. van Nieuwenhuizen, The auxiliary field structure in chirally extended supergravity, Nucl. Phys. B139 (1979) 531.
[168] B. de Wit and S. Ferrara, On higher-order invariants in extended supergravity, Phys. Lett. 81B (1979) 317.
[169] B. de Wit, J.W. van Holten and A. Van Proeyen, Transformation Rules of $N = 2$ Supergravity Multiplets, Nucl. Phys. B (to be published).
[170] M. de Roo, J.W. van Holten, B. de Wit and A. Van Proeyen, Chiral Superfields in $N = 2$ Supergravity, Leiden preprint, March 1980.
[171] B. de Wit, Formulations of $N = 2$ Supergravity Theories, Erice Europhysics Workshop (Plenum, to be published).

[172] M. Duff and S.M. Christensen, Axial and Conformal Anomalies for Arbitrary Spin in Gravity and Supergravity, Phys. Lett. 76B (1978) 571.
[173] M. Duff and S.M. Christensen, Quantum Gravity in $2+\epsilon$ Dimensions, Phys. Lett. 79B (1978) 213.
[174] M. Duff and S.M. Christensen, New Gravitational Index Theorems and Supertheorems, Nucl. Phys. B154 (1979) 301.
[175] M. Duff, S.M. Christensen, S. Deser and M. Grisaru, Chirality, Self-Duality and Supergravity Counterterms, Phys. Lett. 84B (1979) 411.
[176] M. Duff and C.J. Isham, Self-Duality, Helicity, and Supersymmetry: The Scattering of Light by Light, Phys. Lett. 86B (1979) 157.
[177] M. Duff, Index Theorems in Gravity and Supergravity, Proc. 2nd Marcel Grossman Meeting to celebrate 100 years of Albert Einstein, July 1979.
[178] M. Duff, Self-Duality and Helicity in Supergravity, Proc. of the Supergravity Workshop at SUNY, Stony Brook, September 1979 (North-Holland).
[179] M. Duff, Quantizing Supergravity with a Cosmological Constant, in: Proc. European Study Conf. on Unification of the Fundamental Interactions, Eric, March 1980.
[180] M. Duff, S.M. Christensen, G.W. Gibbons and M. Roček, Vanishing One-loop β Function in Gauged $N>4$ Supergravity (to be published).
[181] M. Duff, S.M. Christensen, G.W. Gibbons and M. Roček, One-loop Effects in Supergravity with a Cosmological Constant (to be published).
[182] P. Fayet, Mixing between Gravitational And Weak Interactions Through The Massive Gravitino, Phys. Lett. 70B (1977) 461.
[183] P. Fayet, Supersymmetric Theories Of Particles, Proc. Orbis Scientiae, Coral Gables, Florida, USA, Jan. 1978 (Plenum Publishing Corporation, NY) p. 413.
[184] P. Fayet, Weak Interactions Of A Light Gravitino: A Lower Limit On The Gravitino Mass From The Decay $\psi\to$ gravitino + antiphotino, Phys. Lett. 84B (1979) 421.
[185] P. Fayet, Scattering Cross-sections of The Photino And The Goldstino(gravitino) on Matter, Phys. Lett. 84B (1979) 273.
[186] S. Ferrara and B. de Wit, On Higher Invariants In Extended Supergravity, Phys. Lett. 81B (1979) 317.
[187] S. Ferrara, E. Cremmer, B. Julia, J. Scherk, P. van Nieuwenhuizen and L. Girardello, Spontaneous Symmetry Breaking And Higgs Effect In Supergravity Without Cosmological Term, Nucl. Phys. B147 (1979) 105.
[188] S. Ferrara, E. Cremmer, B. Julia, J. Scherk, P. van Nieuwenhuizen and L. Girardello, Super Higgs Effect In Supergravity With General Scalar Interactions, Phys. Lett. 79B (1978) 231.
[189] S. Ferrara and P. van Nieuwenhuizen, Structure Of Supergravity, Phys. Lett. 78B (1978) 573.
[190] S. Ferrara, E. Cremmer and J. Scherk, SU(4)-Invariant Supergravity Theory, Phys. Lett. 74B (1978) 61.
[191] S. Ferrara and B. Zumino, Structure Of Linearized Supergravity And Conformal Supergravity, Nucl. Phys. B134 (1978) 301.
[192] S. Ferrara and P. van Nieuwenhuizen, Tensor Calculus For Supergravity, Phys. Lett. 76B (1978) 404.
[193] S. Ferrara and P. van Nieuwenhuizen, The Auxiliary Fields Of Supergravity, Phys. Lett. 74B (1978) 333.
[194] S. Ferrara, M.T. Grisaru and P. van Nieuwenhuizen, Poincaré and Conformal Supergravity Models With Closed Algebras, Nucl. Phys. 138B (1978) 430.
[195] S. Ferrara, M. Kaku, P.K. Townsend and P. van Nieuwenhuizen, Unified Field Theories With U(N) Internal Symmetries: Gauging the Superconformal Group, Nucl. Phys. B129 (1977) 125.
[196] S. Ferrara, N.S. Baaklini and P. van Nieuwenhuizen, Classical Solutions In Supergravity Lett. Nuouo Cim. 20 (1977) 113.
[197] S. Ferrara, Algebraic Properties of Extended Supergravity Theories in de Sitter Space, Phys. Lett. 69B (1977) 481.
[198] S. Ferrara, E. Cremmer and J. Scherk, U(N) Invariance In Extended Supergravity, Phys. Lett. 68B (1977) 234.
[199] S. Ferrara, J. Scherk and B. Zumino, Algebraic Properties of Extended Supergravity Theories, Nucl. Phys. B121 (1977) 393.
[200] S. Ferrara, D.Z. Freedman, P. van Nieuwenhuizen, P. Breitenlohner, F. Gliozzi and J. Scherk, Scalar Multiplet Coupled To Supergravity, Phys. Rev. D15 (1977) 1013.
[201] S. Ferrara, J. Scherk and B. Zumino, Supergravity and Local Extended Supersymmetry, Phys. Lett. 66B (1977) 35.
[202] S. Ferrara, F. Gliozzi, J. Scherk and P. van Nieuwenhuizen, Matter Coupling in Supergravity Theory, Nucl. Phys. B117 (1976) 333.
[203] S. Ferrara and P. van Nieuwenhuizen, Consistent Supergravity With Complex Spin 3/2 Gauge Fields, Phys. Rev. Lett. 37 (1976) 1669.
[204] S. Ferrara, J. Scherk and P. van Nieuwenhuizen, Locally Supersymmetric Maxwell–Einstein Theory, Phys. Rev. Lett. 37 (1976) 1035.
[205] S. Ferrara, D.Z. Freedman and P. van Nieuwenhuizen, Progress Towards A Theory of Supergravity, Phys. Rev. D13 (1976) 3214.
[206] S. Ferrara, L. Girardello and F. Palumbo, General Mass Formula in Broken Supersymmetry, Phys. Rev. D20 (1979) 403.
[207] S. Ferrara and P. van Nieuwenhuizen, Simplifications of Einstein Supergravity, Phys. Rev. D20 (1979) 2079.
[208] S. Ferrara and B. Zumino, The Mass Matrix of $N=8$ Supergravity, Phys. Lett. 91B (1980) 61.
[209] S. Ferrara, Recent Devel. in Gravitation, Cargese 1978, Massive Gravitinos, p. 563.
[210] S. Ferrara, Formulation of Supergravity Without Superspace, in: Proc. Intern. Conf. High Energy Physics, EPS, Geneva, Switzerland (CERN editor) Vol. II, p. 963.
[211] S. Ferrara and P. van Nieuwenhuizen, Supergravity With And Without Superspaces, ITP-SB 79-78.
[212] S. Ferrara, Supersymmetric Theories of Fundamental Interactions, CERN-TH 2514 (1978) invited lectures at the Intern. School of Subnuclear Physics, Erice, Sicily (August 1978), to be published by Plenum Press.
[213] S. Ferrara, Supergravity In The Physics of Particles And Fields, invited Lecture at the VI Int. Colloquium on Group-Theoretical Methods in Physcis, Tubingen, July 1978 (Springer Verlag, 1978) p. 22.
[214] S. Ferrara and P. Fayet, Supersymmetry, Physics Reports 32C (1977) 250.
[215] S. Ferrara, D.Z. Freedman and P. van Nieuwenhuizen, Local Supersymmetry And Gravitation, Proc. XVII Intern. Conf. on High Energy Physics, Tbilisi (July 1976) Vol. T. 34.

[216] S. Ferrara, D.Z. Freedman and P. van Nieuwenhuizen, Local Supersymmetry And Supergravity, Proc. APS-DPF meeting, published by Brookhaven National Lab. (October 1976).
[217] S. Ferrara, Spontaneous Supersymmetry Breaking In Supergravity, invited talk at the Second Marcel Grossmann Meeting ICTP, Trieste, 5–11 July 1979, 1C/79/97 (Trieste Int. Rep.).
[218] S. Ferrara, Superspace Aspects of Supersymmetry and Supergravity. Lecture Notes in Physics Math. Problems in Theoretical Physics, vol. 116, p. 358.
[219] R. Finkelstein, A Simple Approach To Supergravity, Physica 96A (1979) 223.
[220] R. Finkelstein and S. Einstein, Solutions of the Rarita–Schwinger Equation in the Kerr–Newman Space, UCLA preprint UCLA/78/TEP/1, to be Published in J. Math. Phys.
[221] M. Fischler, A. Das and M. Roček, Massive, Self-Interacting Scalar Multiplet Coupled to Supergravity, Phys. Lett. 69B (1977) 186.
[222] M. Fischler, A. Das and M. Roček, Super Higgs Effect in a New Class of Scalar Models and a Model of Super QED, Phys. Rev. D16 (1977) 3427.
[223] M. Fischler, Finiteness Calculations for O(4) Matter, Phys. Rev. D20 (1979) 396.
[224] M. Fischler, Globally Supersymmetric Multiplets Without Local Extensions, Phys. Rev. D20 (1979) 1842.
[225] E.S. Fradkin and M.A. Vasiliev, SO(2) Supergravity with Minimal Electromagnetic interaction, 1.5-order formalism and the explicit Form of the Structure Coefficient, Lebedev Institute preprint N 197 (1976).
[226] E.S. Fradkin and M.A. Vasiliev, Hamiltonian Formalism Quantization, S-matrix (four-ghost term) for Supergravity and "open algebras" of O(1) Supergravity, Phys. Lett. 72B (1977) 70.
[227] E.S. Fradkin and M.A. Vasiliev, The Minimal Set of Auxiliary Fields in SO(1) Supergravity, General expression of the S-matrix in arbitrary gauge (including gauge depending on ghost fields) for the theories with closed algebra, Four-ghost interaction in the S-matrix for theories that admit closure of the algebra with the aid of auxiliary fields, Lett. Nuovo Cimento 22 (1978) 651.
[228] E.S. Fradkin and M.A. Vasiliev, The Minimal Set Of Auxiliary Fields in Linearized SO(2) Supergravity, S-matrix (four-ghost term) for SO(2) Supergravity, one parameter class of ways to close algebra in SO(1) Supergravity, Lett. Nuovo Cimento 25 (1979) 79.
[229] E.S. Fradkin and M.A. Vasiliev, The Closing Of The Algebra in SO(2) Supergravity (non-linear) with Explicitly SU(2) Invariant Technique, Phys. Lett. 85B (1979) 47.
[230] E.S. Fradkin and T.E. Fradkina, Quantization of relativistic systems with boson and fermion first and second-class constraints, Phys. Lett. 72B (1978) 343.
[231] E.S. Fradkin, Quantization and S-matrix in gravity and supergravity – asymptotical freedom, invited talk at the Second Marcel Grossmann Meeting ICTP 5–11 July, 1979 1C/79/67 Trieste.
[232] D.Z. Freedman and B. de Wit, Phenomenology of Goldstone Neutrinos, Phys. Rev. Lett. 35 (1975) 827.
[233] D.Z. Freedman and B. de Wit, Combined Supersymmetric and Gauge Invariant Field Theories, Phys. Rev. D12 (1975) 2286.
[234] D.Z. Freedman, P. van Nieuwenhuizen and S. Ferrara, Progress Toward a Theory of Supergravity, Phys. Rev. D13 (1976) 3214.
[235] D.Z. Freedman and P. van Nieuwenhuizen, Properties of Supergravity Theory, Phys. Rev. D14 (1976) 912.
[236] D.Z. Freedman and A. Das, Gauge Quantization for Spin 3/2 Fields, Nucl. Phys. B114 (1976) 271.
[237] D.Z. Freedman and J. Schwarz, Unification of Supergravity and Yang–Mills Theory, Phys. Rev. D15 (1977) 1007.
[238] D.Z. Freedman, S. Ferrara, P. van Nieuwenhuizen, P. Breitenlohner, F. Gliozzi and J. Scherk, Scalar Multiplet Coupled to Supergravity, Phys. Rev. D15 (1977) 1013.
[239] D.Z. Freedman, Supergravity with Axial Gauge Invariance, Phys. Rev. D15 (1977) 1173.
[240] D.Z. Freedman, SO(3) Invariant Extended Supergravity, Phys. Rev. Lett. 38 (1977) 105.
[241] D.Z. Freedman and A. Das, Gauge Internal Symmetry in Extended Supergravity, Nucl. Phys. B120 (1977) 221.
[242] D.Z. Freedman and B. de Wit, On SO(8) Extended Supergravity, Nucl. Phys. B130 (1977) 105.
[243] D.Z. Freedman, M.T. Grisaru and P. van Nieuwenhuizen, The Axial Anomaly for Spin 3/2 Fields, Phys. Lett. 71B (1977) 377.
[244] D.Z. Freedman and J.H. Schwarz, $N = 4$ Supergravity Theory with Local SU(2) Invariance, Nucl. Phys. B137 (1978) 333.
[245] D.Z. Freedman, S. Ferrara and P. van Nieuwenhuizen, Local Supersymmetry and Gravitation, Proc. XVII Intern. Conf. on High Energy Physics, Tbilisi (July 1976), Vol. II, T. 34.
[246] D.Z. Freedman, S. Ferrara and P. van Nieuwenhuizen, Local Supersymmetry and Supergravity, BNL Report No. BNL-50598, eds. H. Gordon and R.F. Peierls, Proc. American Physical Society Meeting, Division of Particles and Fields, Upton, NY, 1976.
[247] D.Z. Freedman, Supergravity Field Theories and the Art of Constructing Them, in: Deeper Pathways in High Energy Physics, eds. B. Kursunoglu, A. Perlmutter and L. Scott (Plenum Press, New York, 1977).
[248] D.Z. Freedman, Supergravity, in: Particles and Fields – 1977 (American Institute of Physics, 1978) p. 65.
[249] D.Z. Freedman and P. van Nieuwenhuizen, Supergravity and the Unification of the Laws of Physics, in: Scientific American 238, No. 2 (1978) p. 126.
[250] D.Z. Freedman, Irreducible Representations of Supersymmetry, in: Recent Developments in Gravitation, Cargese 1978, eds. M. Levy and S. Deser (Plenum Press, 1979).
[251] D.Z. Freedman, Review of Supersymmetry and Supergravity, Proc. of the XIX Intern. Conf. in High Energy Physics, Tokyo, Japan – 1978, eds. S. Homma, M. Kawaguchi and H. Miyazawa (Physical Society of Japan, 1978).
[252] D.Z. Freedman, Unification in Supergravity, Proc. Jerusalem Einstein Centennial, March 1979 (to be published).
[253] D.Z. Freedman, Conformal Invariance and Supergravity, preprint SUSB-ITP-76-39.
[254] D.Z. Freedman and B. de Wit, Systematics of Higher Spin Gauge Fields, Phys. Rev. D21 (1980) 358.

[255] D.Z. Freedman and L. Alvarez-Gaumé, Kahler Geometry and the Renormalization of Supersymmetric σ-models, SUSB-ITP-80-13.
[256] D.Z. Freedman and L. Alvarez-Gaumé, Ricci-flat Kahler Manifolds and Supersymmetry, SUSB-ITP-80-14 (to be published in Physics Letters B).
[257] P.G.O. Freund, Quark Parastatistics And Color Gauging, Phys. Rev. D13 (1976) 2322.
[258] S. Fronsdal, Massless Fields With Integer Spin, Phys. Rev. D18 (1978) 3624.
[259] S. Fronsdal and J. Fang, Massless Fields With Half-Integral Spin, Phys. Rev. D18 (1978) 3630.
[260] Y. Fujii and H. Nishino, Some Phenomenological Consequences of the Super Higgs Effect, Zeitschrift für Physik C (1979).
[261] S.J. Gates Jr. and J.A. Shapiro, Local Supersymmetry in Superspace, Phys. Rev. D18, No. 8 (15 Oct. 1978).
[262] S.J. Gates Jr. and W. Siegel, Understanding Constraints in Superspace Formulations of Supergravity, Nucl. Phys. B163 (1980) 519.
[263] S.J. Gates Jr. and W. Siegel, (3/2, 1) Superfield of O(2) Supergravity, Nucl. Phys. B164 (1980) 484.
[264] S.J. Gates Jr., On the Geometry of Superspace, M.I.T. Preprint CTP-621 (taken from thesis M.I.T. Physics Department) April 1977, unpublished.
[265] S.J. Gates Jr., Geometry of Superspace and Local Supersymmetry, Phys. Rev. D 17 (1978) 3188.
[266] S.J. Gates Jr., A Note on the Geometry of Local Supersymmetry, Harvard preprint HUTP-78/A001 (January 1978), unpublished.
[267] S.J. Gates Jr. and J.A. Shapiro, Local Supersymmetry in Superspace, Phys. Rev. D18 (1978) 2768.
[268] W. Siegel and S.J. Gates Jr., Superfield Supergravity, Nucl. Phys. B147 (1979) 77.
[269] S.J. Gates Jr., Geometry of Superspace and Local Supersymmetry (II), Harvard preprint HUTP-78/A028 (July 1978), unpublished.
[270] M. Brown and S.J. Gates Jr., Superspace Bianchi Identities and the Supercovariant Derivative, Ann. of Phys. 122 (1979) 443.
[271] S.J. Gates Jr., A Comment on Superspace Bianchi Identities and Six-Dimensional Spacetime, Phys. Lett. 84B (1979) 205.
[272] S.J. Gates Jr., Superconformal Transformation and Six-Dimensional Spacetime, Nucl. Phys. B162 (1980) 79.
[273] S.J. Gates Jr. and W. Siegel, Understanding Constraints in Superspace Formulations of Supergravity, Nucl. Phys. B163 (1980) 519.
[274] S.J. Gates Jr., "Super conformal" symmetry breakdown as a guide to supergravity constraints, Proc. Supergravity Workshop at Stony Brook.
[275] M. Brown and S.J. Gates Jr., Superspace Geometry and $N = 1$ Nonminimal Supergravity, M.I.T. preprint CTP-811, to appear in Nucl. Phys. B.
[276] S.J. Gates Jr., "Superconformal" Symmetry Breakdown as a Guide to Supergravity Constraints, Harvard preprint HUTP-79/A069 (Invited talk at the Stony Brook Supergravity Workshop).
[277] S.J. Gates Jr., K.S. Stelle and P.C. West, Algebraic Origins of Superspace Constraints in Supergravity, Imperial College preprint ICTP/79–80/8 (December 1979).
[278] L. Castellani, P. van Nieuwenhuizen and S.J. Gates Jr., The Constraints for $N = 2$ Superspace from Extended Supergravity in Ordinary Space, Stony Brook preprint ITP-SB-80-9 (January 1980).
[279] S.J. Gates Jr., Supercovariant Derivatives, Super-Weyl Groups, and $N = 2$ Supergravity, Harvard preprint HUTP-80/A011 (March 1980).
[280] L. Girardello and John Iliopoulos, Quantum Corrections to a mass Formula in Broken Supersymmetry, Phys. Lett. 88B (1979) 85.
[281] M. Grisaru, H.N. Pendleton and P. van Nieuwenhuizen, Supergravity and the S-matrix, Phys. Rev. D15 (1977) 996.
[282] M. Grisaru, P. van Nieuwenhuizen and J.A.M. Vermaseren, One-Loop Renormalizability of Supergravity and of Einstein–Maxwell Theory in Extended Supergravity, Phys. Rev. Lett. 37 (1976) 1662.
[283] M. Grisaru, Two-loop Renormalizability of Supergravity, Phys. Lett. B66 (1977) 75.
[284] M. Grisaru, Anomalies, Field Transformations and the Relation Between SU(4) and SO(4) Supergravity, Phys. Lett 79B (1978) 225.
[285] M. Grisaru, Anomalies in Supersymmetric Theories, in: Proc. NATO Advanced Study Institute on Gravitation, Cargese 1978 (to be published.)
[286] M. Grisaru and P. van Nieuwenhuizen, Renormalizability of Supergravity, in: New Pathways in Theoretical Physics (Coral Gables, 1977), eds. B.B. Kursunoglu and A. Perlmutter.
[287] M. Grisaru and H.N. Pendleton, Soft Spin 3/2 Fermions Require Gravity and Supersymmetry Phys. Lett. 67B (1977) 323.
[288] M. Grisaru and H.N. Pendleton, Some Properties of Scattering Amplitudes in Supersymmetric Theories, Nucl. Phys. B124 (1977) 81.
[289] M. Grisaru, L.F. Abbott and H.J. Schnitzer, Possible Supersymmetry Breaking by Instantons, Phys. Rev. D16 (1977) 3002.
[290] M. Grisaru, L.F. Abbott and H.J. Schnitzer, Supercurrent Anomaly in a Supersymmetric Gauge Theory, Phys. Rev. D16 (1977) 2995.
[291] M. Grisaru, L.F. Abbott and H.J. Schnitzer, Cancellation of the Supercurrent Anomaly in Supersymmetric Gauge Theory, Phys. Lett. 71B (1977) 161.
[292] M. Grisaru, L.F. Abbott and H.J. Schnitzer, A Supercurrent Anomaly in Supergravity, Phys. Lett. 73B (1978) 71.
[293] M. Grisaru, D.Z. Freedman and P. van Nieuwenhuizen, The Axial Anomaly for Spin 3/2 Fields, Phys. Lett. 71B (1977) 377.
[294] M. Grisaru, Positivity of the Energy in Einstein Theory, Phys. Lett. 73B (1978) 207.
[295] M. Grisaru and B. de Wit, Covariant Quantization of Supergravity, Phys. Lett. 74B (1978) 57.
[296] M. Grisaru and L.F. Abbott, Local Supersymmetry Transformations and Fermion Solutions in the Presence of Instantons, Phys. Rev. D17 (1978) 2809.
[297] M. Grisaru, S. Ferrara and P. van Nieuwenhuizen, Poincaré and Conformal Supergravity Models with Closed Algebras, Nucl. Phys. B138 (1978) 430.
[298] M. Grisaru and B. de Wit, Auxiliary Fields and Ultraviolet Divergences in Supergravity, Nucl. Phys. B139 (1978) 531.
[299] M. Grisaru, Anomalies in Supersymmetric Theories, in: Proc. NATO Advanced Study Ins. on Gravitation, Cargese 1978 (to be published).
[300] M. Grisaru, S. Christensen, S. Deser and M.J. Duff, Chirality, Self-Duality, and Supergravity Counterterms, Phys. Lett. 84B (1979) 411.
[301] P.S. Howe, Superspace And The Spinning String, Phys. Lett. 70B (1977) 453.

[302] P.S. Howe, Superweyl Transformations In Two Dimensions, Journal of Physics A12 (1979) 393.
[303] R.E. Kallosh, Gauge Invariance in Supergravity, Zh. Eksp. Teor. Fiz. Pis. Red. 26 (1977) 575 (JETP Lett. 26 (1977) 427).
[304] R.E. Kallosh, Modified Feynman Rules in Supergravity, Nucl. Phys. B141 (1978) 141.
[305] R.E. Kallosh, Super Self-duality, Zh. Eksp. Teor. Fiz. Pis. Red. 29 (1979) 192 (JETP Lett. 29 (1979) 172).
[306] R.E. Kallosh, Structure of Counterterms in Supergravity, Zh. Eksp. Teor. Fiz. Pis. Red. 29 (1979) 493 (JETP Lett. 29 (1979) 449).
[307] R.E. Kallosh, Self-duality in Superspace, Nucl. Phys. B165 (1980) 119.
[308] S.W. MacDowell and F. Mansouri, Unified Geometric Theory of Gravity and Supergravity, Phys. Rev. Lett. 38 (1977) 739.
[309] S.W. MacDowell, Algebraic Approach to Supergravity in Superspace, Phys. Lett. 80B (1979) 212.
[310] S.W. MacDowell and I. Bars, A Spinor Theory of General Relativity Without Elementary Gravitons, Phys. Lett. B71 (1977) 111.
[311] S.W. MacDowell and I. Bars, A Spin-3/2 Theory of Gravitation, General Relativity and Gravitation 10 (1979) 205.
[312] S.W. MacDowell, The Transformation Algebra of Fields in Supergravity, 8th Intern. Conf. on General Relativity and Gravitation, University of Waterloo, Ontario, Canada (1977).
[313] F. Mansouri, Geometry of Superspaces with Bose and Fermi Coordinates and Applications to Graded Lie Bundles and Supergravity, J. of Math. Phys. 18 (1977) 52–57.
[314] F. Mansouri, A New Class of Superalgebras and Local Gauge Groups in Superspace, J. of Math. Phys. 18 (1977) 2395.
[315] L. Chang, K. Macrae and F. Mansouri, A New Supersymmetric String Model and the Supergauge Constraints in the Dual Resonance Model, Phys. Lett. 57B (1975) 59. [This is how Kaluza–Klein found its way into gauge theories and supergravity!]
[316] L.N. Chang, K. Macrae and F. Mansouri, Geometrical Approach to Local Gauge and Supergauge Invariance, Phys. Rev. D13 (1976) 235.
[317] F. Mansouri, Super Unified Theories Based on the Geometry of Local (Super-) Gauge Invariance, Phys. Rev. D16 (1977) 2456.
[318] F. Mansouri and L.N. Chang, Non-Linear Gauge Fields and the Structure of Gravity and Super-gravity Theories, Phys. Rev. D17 (1978).
[319] F. Mansouri, Non-linear Gauge Fields and Unified Theories, Proc. 2nd Johns Hopkins Workshop on Current Problems in Particle Phys., eds. G. Domokos and S. Kovesi-Domokos (April 1978).
[320] F. Mansouri, Locally Supersymmetric Theories with Model Independent Transformation Laws, Phys. Lett. 78B (1978) 274.
[321] F. Mansouri, Conformal Gravity as a Gauge Theory, Phys. Rev. Lett. 42 (1979) 1021.
[322] F. Mansouri, Superunified Theories in Superspace, Proc. Integrative Conf. on Group Theory and Math. Phys, Austin, Texas, eds. W. Beiglbock, A. Bohm, E. Takasugi (Springer-Verlag, 1978).
[323] F. Mansouri, Geometrical Transformation Laws of Fields in Superunified Theories, with Schaer, Phys. Lett. 83B (1979) 327.
[324] F. Mansouri, Dynamical Criteria for a Unified Gauge Theory (including gravity), Invited talk given at the Third Workshop on Current Problems in High Energy Particle Theory, Florence, Italy, May, 1979, to be published in Proceedings.
[325] F. Mansouri, Dynamical Criteria in Superunified Theories, Talk presented at the Second Marcel Grossmann Meeting on the Recent Developments of General Relativity, Trieste, Italy, 1979, to be published in the Proceedings.
[326] F. Mansouri and C. Schaer, in: Supergravity, eds. D.Z. Freedman and P. van Nieuwenhuizen (North-Holland, 1979) p. 169.
[327] F. Mansouri, Superspace Actions for Conformal and Extended Conformal Supergravity Theories, Yale preprint, YTP 80-13, 1980, to be published.
[328] F. Mansouri and C. Schaer, SU(N)-Extended Conformally Invariant Actions in Superspace, Yale preprint, YTP 80-12, 1980, to be published.
[329] W. Nahm, Supersymmetries and their Representations, Nucl. Phys. B135 (1978) 149.
[330] M.A. Namazie and D. Storey, Supersymmetric Quantization of Linearized Supergravity, Nucl. Phys. B157 (1979) 170.
[331] Y. Ne'eman, L. Corwin and S. Sternberg, Graded Lie Algebras in Mathematics and Physics (Bose–Fermi Symmetry), Rev. Mod. Phys. 47 (1975) 573–604.
[332] Y. Ne'eman, Algebraic Incompatibilities between Arnowitt–Nath Gauges and Supersymmetrized Gravity, Transactions of the New York Academy of Sciences (I.I. Rabi Festschrift), Series II, Vol. 38, ed. L. Motz (1977) pp. 106–115. [First issued as Tel Aviv Univ. TAUP 549–76, 2nd version as Caltech preprint 68/570.]
[333] Y. Ne'eman and T. Regge, Gauge Theory of Gravity and Supergravity on a Group Manifold, Rivista Del Nuovo Cimento 1, N.5. (1978) (series 3 first issued as IAS Princeton and U. Texas ORO 3993 328 preprints) pp. 1–42.
[334] Y. Ne'eman and T. Regge, Gravity and Supergravity as Gauge Theories on a Group Manifold, Phys. Lett. 74B (1978) pp. 54–58.
[335] Y. Ne'eman, T. Regge and J. Thierry-Mieg, Ghost Fields, BRS and Extended Supergravity as Applications of Gauge Geometry (to be published) in: Proc. XIX Intern. Conf. on High Energy Physics, Tokyo 1978, pp. 552–554.
[336] Y. Ne'eman, Quantum Gravity and Supergravity (to be published) in: Proc. Princeton Einstein Centennial Symposium (Addison-Wesley Pub.).
[337] Y. Ne'eman and J. Thierry-Mieg, Extended Supergravity on Group Manifolds with Spontaneous Fibration, Ann. of Physics. 112 (1979).
[338] Y. Ne'eman, Gauge Theories of Gravity and its Extensions, including Supergravity (Erice 1977 Abstracts) Prog. Sc. Culture (1978).
[339] Y. Ne'eman, Soft-Group Manifold Structure of Supergravity and the Proof of Unitarity (in coll. with J. Thierry-Mieg), Proc. 2nd Marcel Grossmann Symp. (Trieste, 1979) (to be published).
[340] Y. Ne'eman, Soft-Group Manifold Becchi-Rouet-Stora transformations and Unitarity for Gravity, Supergravity and Extensions (in coll. with E. Takasugi) to be publ. in Phys. Rev. D.
[341] Y. Ne'eman, Minimal and Centered Graded Spin-Extensions of the SL(3, R) Algebra (in coll. with Dj. Sijacki), J. Math. Phys. (to be published).
[342] N.K. Nielsen, M. Grisaru, H. Römer and P. van Nieuwenhuizen, Approaches to the Gravitational Spin 3/2 Axial Anomaly, Nucl. Phys. B140 (1978) 477–498.

[343] N.K. Nielsen, Ghost Counting in Supergravity, Nucl. Phys. B140 (1978) 499–509.
[344] N.K. Nielsen, On the Quantization of the Axial Current for the Massless Rarita–Schwinger Field, Nucl. Phys. B142 (1978) 306–314.
[345] N.K. Nielsen, Implications of Non Linear Invariances for Spinor Theories in Curved Space-time, Nucl. Phys. B151 (1979) 536–546.
[346] V. Ogievetsky and L. Mezincescu, Boson Fermion Symmetries and Superfields, Usp. Fiz. Nauk. 117 (1975) 637 (English translation: Sov. Phys. Usp. 18 (1976) 960).
[347] V. Ogievetsky and E. Sokatchev, Equations of Motion for Superfields, J. Phys. A10 (1977) 2021.
[348] V. Ogievetsky and E. Sokatchev, Proc. Fourth Intern. Conf. on Nonlocal and Nonlinear Field Theory, April 1976, Alushta, USSR, JINR-Dubna, D2, 9788 (1976) p. 183.
[349] V. Ogievetsky and E. Sokatchev, On a Vector Superfield Generated by the Supercurrent, Nucl. Phys. B124 (1977) 309.
[350] V. Ogievetsky and E. Sokatchev, Supercurrent, Yad. Fiz. 28 (1978) 825 (English Translation available) Preprint JINR Dubna, E2-11528 (1978).
[351] V. Ogievetsky and E. Sokatchev, Structure of Supergravity Group, Phys. Lett. 79B (1978) 222.
[352] V. Ogievetsky and E. Sokatchev, The Axial Superfield and the Supergravity Group, Yad. Fiz. 28 (1978) 1631.
[353] V. Ogievetsky and E. Sokatchev, Talk presented at the Seminar on Quantum Gravity, Moscow, December 1978.
[354] V. Ogievetsky and E. Sokatchev, The Simplest Group of Einstein Supergravity, JINR preprint. E2-12469, Yad. Fiz. 31 (1980) 264.
[355] V. Ogievetsky and E. Sokatchev, The Axial Gravitational Superfield and the Formalism of Differential Geometry, JINR preprint E2-12513, Yad. Fiz. 31 (1980) 821.
[356] V. Ogievetsky and E. Sokatchev, Geometry, Axial Superfield and Supergravity in Problems of Field Theory, JINR-Dubna P2-12462 (1979) 148.
[357] V. Ogievetsky and E. Sokatchev, The Simplest Geometric Approach to Supergravity, preprint JINR-Dubna E2-80-26.
[358] V. Ogievetsky and E. Sokatchev, Normal Gauge in Supergravity, preprint JINR-Dubna E2-80-127 (to appear in Yad. Fiz.)
[359] V. Ogievetsky and E. Sokatchev, Torsion and Curvature in Terms of the Axial Gravitational Superfield, preprint JINR-Dubna E2-80-138 (to appear in Yad. Fiz.)
[360] V. Ogievetsky and E. Sokatchev, Equation of Motion for the Axial Gravitational Superfield, preprint JINR-Dubna E2-80-139 (to appear in Yad. Fiz.).
[361] D. Olive, F. Gliozzi and J. Scherk, Supersymmetry Supergravity Theories and the Dual Spinor Model, Nucl. Phys. B122 (1977) 253.
[362] D. Olive, F. Gliozzi and J. Scherk, Supergravity and the Spinor Dual Model, Phys. Lett. 65B (1976) 282.
[363] D. Olive and E. Witten, Supersymmetry Algebras that Include Topological Charges, Phys. Lett. 78B (1978) 97.
[364] D. Olive, The Electric and Magnetic Charges as Extra Components of four Momentum, Nucl. Phys. B153 (1979) 1.
[365] M. Roček and U. Lindstrom, Components Of Superspace, Phys. Lett. 79B (1978) 217.
[366] M. Roček, Linearizing The Voluov–Akulov Model, Phys. Rev. Lett. 41 (1978) 451.
[367] M. Roček and U. Lindstrom, More Components Of Superspace, Phys. Lett. 83B (1979) 179.
[368] M. Roček and U. Lindstrom, Constrained Local Superfields, Phys. Rev. D19:8 (1979) 2300.
[369] H. Römer, N.K. Nielsen and B. Schroer, Classical Anomalies and Local Version of the Atiyah–Singer Theorem, CERN TH 2352, Phys. Lett. 70B (1977) 445.
[370] H. Römer and A.J. Hanson, Gravitational Instanton Contribution to Spin 3/2 Axial Anomaly, CERN TH 2564 (Sept. 1978), Phys. Lett. 80B (1978) 58.
[371] H. Römer, Axial Anomaly and Boundary Terms for General Spinor Fields, CERN TH 2624, Phys. Lett. 83B (1979) 172.
[372] H. Römer, G.W. Gibbons and C.N. Pope, Index Theorem Boundary Terms for Gravitational Instantons, DAMTP preprint (1979).
[373] D.K. Ross, Geodesics In Gauge Supersymmetry, Phys. Rev. D16 (1977) 1717.
[374] D.K. Ross, Spherically Symmetric Solution of Gauge Supersymmetry, J. Phys. A10 (1977) 2031.
[375] D.K. Ross, Fiber Bundles, Superspace and the Gravitational Field, submitted to Phys. Rev. (Iowa State University preprint).
[376] D.C. Ross, Gauge Group in a Fiber Bundle Approach to Supergravity, submitted to Phys. Rev. (Iowa State University preprint).
[377] J. Scherk, S. Ferrara, F. Gliozzi and P. van Nieuwenhuizen, Matter Couplings in Supergravity Theory, Nucl. Phys. B117 (1978) 333.
[378] J. Scherk, S. Ferrara and P. van Nieuwenhuizen, Locally Supersymmetric Maxwell Einstein Theory, Phys. Rev. Lett. 37 (1976) 1035.
[379] J. Scherk, S. Ferrara, D.Z. Freedman et al., Scalar Multiplet Coupled to Supergravity, Phys. Rev. D 15, Nr 4 (1977) 1013.
[380] J. Scherk, F. Gliozzi and D. Olive, Supergravity and the Dual Spinor Model, Phys. Lett. 65B (1976) 282.
[381] J. Scherk, S. Ferrara and B. Zumino, Supergravity and Extended Supersymmetry ($N = 3$ Ext. SG), Phys. Lett. 66B (1977) 35.
[382] J. Scherk, S. Ferrara and B. Zumino, Algebraic Properties of Extended Supergravity Theories (dual invariance $O(N) \to SU(N)$ central charges), Nucl. Phys. B121 (1977) 393.
[383] J. Scherk, F. Gliozzi and D. Olive, Supersymmetry, Supergravity and the Dual Spinor Model, Nucl. Phys. B122 (1977) 253.
[384] J. Scherk, L. Brink and J.H. Schwarz, Supersymmetry Yang–Mills Theories, Nucl. Phys. B121 (1977) 77.
[385] J. Scherk, E. Cremmer and S. Ferrara, $U(N)$ Invariance in Extended Supergravity, Phys. Lett. 68B (1977) 235.
[386] J. Scherk and E. Cremmer, Algebraic Simplifications in Supergravity Theories, Nucl. Phys. B127 (1977).
[387] J. Scherk and E. Cremmer, Modified Interaction of the Scalar Multiplet Coupled to Supergravity, Phys. Lett. 69B (1977) 97.
[388] J. Scherk, E. Cremmer and S. Ferrara, SU(4) Invariant Supergravity Theory, Phys. Lett. 74B (1978) 61.
[389] J. Scherk and E. Cremmer, The Supersymmetric Non Linear σ Model in 4 Dimensions and its Coupling to Supergravity, Phys. Lett. 74B (1978) 341.
[390] J. Scherk, E. Cremmer and B. Julia, Supergravity Theory in 11 Dimensions, Phys. Lett. 76B (1978) 409.

[391] J. Scherk, in: Proc. Stony Brook Conf. (North-Holland, 1979).
[392] J. Scherk, Extended Supersymmetry and Extended Supergravity Theory, Cargese Lecture Notes, LPTENS 78/21, Sept. 1978.
[393] J.H. Schwarz and D.Z. Freedman, Unification Of Supergravity and Yang–Mills Theory, Phys. Rev. D15 (1977) 1007.
[394] J.H. Schwarz and L. Brink, Local Complex Supersymmetry in Two Dimensions, Nucl. Phys. B121 (1977) 285.
[395] J.H. Schwarz, L. Brink and J. Scherk, Supersymmetric Yang–Mills Theories, Nucl. Phys. B121 (1977) 77.
[396] J.H. Schwarz, Spinning String From A Modern Perspective, in: Proc. Orbis Scientiae 1978 on New Frontiers in High-Energy Physics (Plenum Press, New York, 1978) p. 431.
[397] J.H. Schwarz and D.Z. Freedman, $N = 4$ Supergravity With Local $SU(2) \times SU(2)$ Invariance, Nucl. Phys. B137 (1978) 333.
[398] J.H. Schwarz, L. Brink, M. Gell-Mann and P. Ramond, Supergravity as Geometry Of Superspace, Phys. Lett. 74B (1978) 336, erratum 76B (1978) 664.
[399] J.H. Schwarz, L. Brink, M. Gell-Mann and P. Ramond, Extended Supergravity As Geometry Of Superspace, Phys. Lett. 76B (1978) 417.
[400] J.H. Schwarz, M. Gell-Mann and P. Ramond, Lagrangian Formulation Of Supergravity in Superspace, Proc. 19th Intern. Conf. on High-Energy Physics, Tokyo 1978) (The Physical Society of Japan 1979) p. 560.
[401] J.H. Schwarz, L. Brink, M. Gell-Mann and P. Ramond, Prepotentials in a Superspace Formulation of Supergravity, Nucl. Phys. B145 (1978) 93.
[402] J.H. Schwarz, M. Gell-Mann and P. Ramond, Elimination Of Redundancy in Superspace Equations Of Supergravity, CALT-68-679 (Dirac Symposium, 1978).
[403] J.H. Schwarz, Supersymmetry Approach to the Unification of Interactions, Lectures presented at Erice workshop on "Hadronic Matter at Extreme Energy Density". Oct. 1978 (to be published).
[404] J.H. Schwarz and J. Scherk, Spontaneous Breaking Of Supersymmetry Through Dimensional Reduction, Phys. Lett. 82B (1979) 60.
[405] J.H. Schwarz and J. Scherk, How To Get Masses From Extra Dimensions, Nucl. Phys. B153 (1979) 61.
[406] J.H. Schwarz, E. Cremmer and J. Scherk, Spontaneously Broken $N = 8$ Supergravity, Phys. Lett. 84B (1979) 83.
[407] W. Siegel, Supergravity Superfields without a Supermetric, Harvard preprint HUTP-77/A068 (November 1977).
[408] W. Siegel, The Superfield Supergravity Action, Harvard preprint HUTP-77/A080 (December 1977).
[409] W. Siegel, A Polynomial Action for a Massive, Self-Interacting Chiral Superfield Coupled to Supergravity, Harvard preprint HUTP-77/A077 (December 1977).
[410] W. Siegel, A Derivation of the Supercurrent Superfield, Harvard preprint HUTP-77/A089 (December 1977).
[411] W. Siegel, Solution to Constraints in Wess–Zumino Supergravity Formalism, Nucl. Phys. B142 (1978) 301.
[412] W. Siegel, Superfield Supergravity (with S.J. Gates Jr.), Nucl. Phys. B147 (1979) 77.
[413] W. Siegel, Superfields in Higher-Dimensional Spacetime, Phys. Lett. 80B (1979) 220.
[414] W. Siegel, Superconformal Invariance of Superspace with Nonminimal Auxiliary Fields, Phys. Lett. 80B (1979) 224.
[415] W. Siegel, Superfield Supergravity, in: Group Theoretical Methods in Physics, Proc., Austin, Texas, September 1978 (Springer-Verlag, Berlin, 1979) p. 487.
[416] W. Siegel, Supergravity Supergraphs, Phys. Lett. 84B (1979) 197.
[417] W. Siegel, Supersymmetric Dimensional Regularization via Dimensional Reduction, Phys. Lett. 84B (1979) 193.
[418] W. Siegel, Gauge Spinor Superfield as Scalar Multiplet, Phys. Lett. 85B (1979) 333.
[419] W. Siegel, Unextended Superfields in Extended Supersymmetry, Nucl. Phys. B156 (1979) 135.
[420] W. Siegel, Improved Methods for Supergraphs (with M.T. Grisaru and M. Roček), Nucl. Phys. B159 (1979) 429.
[421] W. Siegel, Understanding Constraints in Superspace Formulations of Supergravity (with S.J. Gates Jr.), Nucl. Phys. B 63 (1980) 519.
[422] W. Siegel, ($\frac{3}{2}$, 1) Superfield of O(2) Supergravity (with S.J. Gates Jr.), Nucl. Phys. B 64 (1980) 484.
[423] W. Siegel, Constraints on Superspace, in: Supergravity, Proc., Stony Brook, September 1979 (North-Holland, Amsterdam, 1979).
[424] W. Siegel, Supergravity (with M.T. Grisaru) in preparation.
[425] W. Siegel, Inconsistency of Supersymmetric Dimensional Regularization, Phys. Lett. 94B (1980) 37.
[426] W. Siegel, Hidden Ghosts, Phys. Lett. 93B (1980) 170.
[427] W. Siegel, Off-Shell Central Charges, Nucl. Phys. B 173 (1980) 51.
[428] W. Siegel, On-Shell $O(N)$ Supergravity in Superspace, to appear in Nucl. Phys. B.
[429] M.F. Sohnius and P. Dondi, Supersymmetry Transformations with Internal SU(2)-Symmetry, Nucl. Phys. B81 (1974) 317.
[430] M.F. Sohnius, R. Haag and J. Lopuszanski, All Possible Generators of Supersymmetries of the S-Matrix, Nucl. Phys. B88 (1975) 61.
[431] M.F. Sohnius, Supersymmetry, Internal Symmetry, and the Conformal Group, Ph.D. Thesis, Karlsruhe Univ., 1976.
[432] M.F. Sohnius, The Conformal Group in Superspace, in: Quantum Theory and the Structures of Time and Space, eds. L. Castell, M. Drieschner and C.F. von Weizäcker (Hanser Verlag, Munich, 1977).
[433] M.F. Sohnius, A Supersymmetric Extension of the Weinberg–Salam Model, Nucl. Phys. B122 (1977) 291.
[434] M.F. Sohnius, R. Grimm and J. Wess, Extended Supersymmetry and Gauge Theories, Nucl. Phys. B133 (1978) 275.
[435] M.F. Sohnius, Bianchi-Identities for Supersymmetric Gauge Theories, Nucl. Phys. B136 (1978) 461.
[436] M.F. Sohnius, Supersymmetry and Central Charges, Nucl. Phys. B138 (1978) 109.
[437] M.F. Sohnius, The Multiplet of Currents for $N = 2$ Extended Supersymmetry, Phys. Lett. 81B (1979) 8.
[438] M.F. Sohnius, Supergravity, (invited lecture) at the Einstein Centenary Memorial Symp. in Ahmedabad, India, Jan. 1979, to be published in the Proceedings.

[439] M.F. Sohnius, K.S. Stelle and P.C. West, Off-Mass-Shell Formulation of Extended Supersymmetric Gauge Theories, Preprint ICTP/79-80/17, submitted to Phys. Lett.
[440] M.F. Sohnius, K.S. Stelle and P.C. West, Dimensional Reduction by Legendre Transformation Generates Off-Shell Supersymmetric Yang–Mills Theories, Preprint ICTP/79-80/22, submitted to Nucl. Phys.
[441] P. Breitenlohner and M. F. Sohnius, Superfields, Auxiliary Fields, and Tensor Calculus for $N = 2$ Extended Supergravity, to be published in Nucl. Phys. B.
[442] E. Sokatchev, Projection Operators and Supplementary Conditions for Superfields with Arbitrary Spin, Nucl. Phys. B99 (1975) 96.
[443] K.S. Stelle, S. Deser and J. Kay, Renormalizability Properties of Supergravity, Phys. Rev. Lett. 38 (1977) 527.
[444] K.S. Stelle, S. Deser and J. Kay, Hamiltonian Formulation of Supergravity, Phys. Rev. D16 (1977) 2448.
[445] K.S. Stelle and P.C. West, Minimal Auxiliary Fields for Supergravity, Phys. Lett. 74B (1978) 330.
[446] K.S. Stelle and P.C. West, Matter Coupling and BRS Transformations with Auxiliary Fields in Supergravity, Nucl. Phys. B140 (1978) 285.
[447] K.S. Stelle and P.C. West, Tensor Calculus for the Vector Multiplet Coupled to Supergravity, Phys. Lett. B77 (1978) 376.
[448] K.S. Stelle and P.C. West, Relation between Vector and Scalar Multiplets and Gauge Invariance in Supergravity, Nucl. Phys. B145 (1978) 175.
[449] K.S. Stelle and P.C. West, de Sitter Gauge Invariance and the Geometry of the Einstein–Cartan Theory, J. Phys. A: Math. Gen. 8 (1979) 1819.
[450] K.S. Stelle and P.C. West, Spontaneously Broken de Sitter Symmetry and the Gravitational Holonomy Group, Phys. Rev. D, in press.
[451] K.S. Stelle and P.C. West, de Sitter Gauge Invariance and Gravitational Geometry, to be published in Proc. Second Marcel Grossman Meeting, July 1979, Trieste, Italy.
[452] K.S. Stelle and P.C. West, Algebraic Understanding of Constraints, in: Supergravity, Proc. Stony Brook Supergravity Workshop, eds. P. van Nieuwenhuizen and D.Z. Freedman (North-Holland, Amsterdam, 1979).
[453] K.S. Stelle, S.J. Gates Jr. and P.C. West, Algebraic Origins of Superspace Constraints in Supergravity, Nucl. Phys. B. (in press).
[454] K.S. Stelle and P.C. West, Algebraic Derivation of $N = 2$ Supergravity Constraints, Phys. Lett. 90B (1980) 393.
[455] K.S. Stelle, M.F. Sohnius and P.C. West, Off-Mass-Shell Formulation of Extended Supersymmetric Gauge Theories, Phys. Lett. B (in press).
[456] K.S. Stelle, M.F. Sohnius and P.C. West, Dimensional Reduction by Legendre Transformation Generates Off-Shell Supersymmetric Yang–Mills Theories, Nucl. Phys. B. (in press).
[457] C. Teitelboim, Supergravity and Square Roots of Constraints, Phys. Rev. Lett. 38 (1977) 1106.
[458] C. Teitelboim, Surface Integrals as Symmetry Generators in Supergravity Theory, Phys. Rev. Lett. 69B (1977) 240.
[459] C. Teitelboim and R. Tabensky, The Square Root of General Relativity, Phys. Lett. 69B (1977) 453.
[460] C. Teitelboim, The Hamiltonian Structure of Spacetime, 1978; to appear in a volume published in commemoration of the hundredth anniversary of the birth of Albert Einstein under the auspices of the Intern. Committee on General Relativity and Gravitation (Plenum Publ, New York, 1979).
[461] C. Teitelboim and P. Cordero, Remarks on Supersymmetric Black Holes, Phys. Lett. 78B (1978) 80.
[462] C. Teitelboim, Spin, Supersymmetry, and Square Roots of Constraints, Proc. of Current Trends in the Theory of Fields, Conf. on the 50th anniversary of the Dirac Equation, April 1978, AIP Conf. Proc. #48, Particles and Fields #15, eds. J.E. Lannutti and P.K. Williams (New York, 1978) pp. 134–139.
[463] C. Teitelboim and P. Cordero, No-Hair Conjecture in Supergravity, Report presented at the Third Latin American Symp. on Relativity and Gravitation, SILARG III, México City, August 1978.
[464] C. Teitelboim, Surface Deformations, Their Square Root and the Signature of Spacetime (to appear) in: Proc. 1978 Integrative Conf. on Group Theory and Mathematical Physics, Lecture Notes in Physics (Springer, Berlin).
[465] C. Teitelboim and C.A.P. Galvao, Classical Supersymmetric Particles, to appear in J. Math. Phys.
[466] C. Teitelboim, Proper Time Approach to the Quantization of the Gravitational Field (to appear).
[467] C. Teitelboim, Proper Time Approach to Quantized Supergravity (to appear).
[468] J. Thierry-Mieg, Pseudo Invariant gauge Lagrangians, Letts. Nuovo Cim. 23, 13 (1978) 489–493.
[469] J. Thierry-Mieg, Extended Supergravity on group Manifold, Annals of Physics 123 (1979) 247 (with Ne'eman).
[470] J. Thierry-Mieg and Y. Ne'eman, Proc. 2nd Marcel Grossman Meeting.
[471] P.K. Townsend and M. Kaku, Poincaré Supergravity as Broken Superconformal Gravity, Phys. Lett. B76 (1978) 54.
[472] P.K. Townsend, M. Kaku and A. Das, Unified Approach to Matter Coupling in Weyl and Einstein Supergravity, Phys. Rev. Lett. 40 (1978) 1215.
[473] P.K. Townsend, Quartic Ghost Interactions of O(N) Supergravity, Phys. Lett. B85 (1979) 328.
[474] P.K. Townsend, Cosmological Constant in Supergravity, Phys. Rev. D15 (1977) 2808.
[475] P.K. Townsend, Local Supersymmetry without Gravity?, in: Supergravity, Proc. Stony Brook Conf. (North-Holland, 1979).
[476] H. Nicolai and P.K. Townsend, Anomalies and Supersymmetric Regularization by Dimensional Reduction, Phys. Lett. B.
[477] H. Nicolai and P.K. Townsend, Supersymmetric regularization and superconformal anomalies in Unification of the Fundamental Interactions, Erice Proceedings (Plenum Press, 1980).
[478] H. Nicolai, P.K. Townsend and P. van Nieuwenhuizen, Comments on 11-dimensional supergravity, Nuovo Cim. Lett.
[479] L.F. Urrutia and R. Finkelstein, Constructive Approach to Supergravity, UCLA preprint UCLA/78/TEP/26 (to be published in General Relativity and Gravitation).
[480] L.F. Urrutia, Supergravity as an Example of a Constructive Approach to Supersymmetries, UCLA preprint UCLA/79/TEP/8 (to be published in Proc. First Chilean Symp. of Theor. Phys.).
[481] L.F. Urrutia, An Example of the No-Hair Conjecture in Supergravity, UCLA preprint UCLA/79/TEP/10 (to be published in Phys. Lett. B).

[482] J.G. Taylor, The Local Symmetry of the Space-time of Supergravity, Phys. Lett. 79B (1978) 399–402.
[483] J.G. Taylor, S. Bedding and S. Downes-Martin, A Superfield Analysis of Supergravity, Annals of Physics (to appear).
[484] J.G. Taylor, S. Bedding, S. Downes-Martin and C. Pickup, A Critique of Superfield Theories of Supergravity, Phys. Lett. 83B (1979) 59–62.
[485] J.G. Taylor and S. Bedding, The Components of a Superfield Version of Supergravity, King's College preprint (Feb. 1979).
[486] J.G. Taylor, Quantised Superfield Supergravity, King's College preprint (May–July 1979).
[487] J.G. Taylor, Light-Plane Integrability Conditions for Extended Supergravity, King's College preprint (August 1979).
[488] J.G. Taylor, The Ultra-Violet Divergences of Superfield Supergravity, Proc. E.P.S. meeting, June 1979 (to appear).
[489] J.G. Taylor, A Superfield Formulation of Supergravity, Phys. Lett. 80B (1978) 52–54.
[490] J.G. Taylor, Quantising Super-Riemannian Space-Time, Proc. Roy. Soc. A 362 (1978) 493–507.
[491] J.G. Taylor, Flat Superspace with Supertorsion, Phys. Lett. 78B (1978) 577–580.
[491a] J.P. Thierry-Mieg and B. Morel, Harvard preprint, 1980, see also V. Kac, Lecture notes in Math. 67b.
[492] R.W. Tucker and P.A. Collins, An Action Principle and Canonical Formalism for the Neveu–Schwarz–Ramond String, Phys. Lett. 64B (1976) 207–210.
[493] R.W. Tucker and P.A. Collins, An Action Principle for the Neveu–Schwarz–Ramond String and Other Systems Using Supernumerary Variables, Nucl. Phys. B121 (1977) 307–325.
[494] R.W. Tucker and P.S. Howe, A Locally Supersymmetric and Reparametrization Invariant Action for a Spinning Membrane, J. Phys. A10 (1978) 155–158.
[495] R.W. Tucker and P.S. Howe, Local Supersymmetry in (2 + 1)-Dimensions, I. Supergravity and Differential Forms, J. Math. Phys. 19 (1978) 869–873.
[496] R.W. Tucker and P.S. Howe, Local Supersymmetry in (2 + 1)-Dimensions, II. An Action for a Spinning Membrane, J. Math. Phys. 19 (1978) 981–984.
[497] R.W. Tucker and P.S. Howe, A Note on the Linearisation of Supergravity, Journ. Phys. A. (Math) 12 (1979) 21–24.
[498] R.W. Tucker and P.S. Howe, Scale Invariance in Superspace, Phys. Lett. 80B (1978) 138–141.
[499] R.W. Tucker, Affine Transformations and the Geometry of Superspace, J. Math. Phys. (to be published).
[500] P. van Nieuwenhuizen, D.Z. Freedman and S. Ferrara, Progress toward a Theory of Supergravity, Phys. Rev. D13 (1976) 3214.
[501] P. van Nieuwenhuizen and D.Z. Freedman, Properties of Supergravity Theory, Phys. Rev. D14 (1976) 912.
[502] P. van Nieuwenhuizen, Local Supersymmetry and Gravitation, in: Asymptotic Structure of Spacetime, Proc. Cincinnati Conf. 1976 (Plenum Press, 1977).
[503] P. van Nieuwenhuizen and C.C. Wu, On the Integral Relations for Invariants Constructed from Three Riemann Tensors and their Applications in Quantum Gravity, J. Math. Phys. 18 (1977) 182.
[504] P. van Nieuwenhuizen, On the Renormalization of Quantum Gravity without Matter, Ann. of Physics 104 (1977) 197.
[505] P. van Nieuwenhuizen, M.T. Grisaru and H. Pendleton, Supergravity and the S-Matrix, Phys. Rev. D15 (1977) 996.
[506] P. van Nieuwenhuizen, Local Supersymmetry and Gravitation, Proc. Tbilisi Conf., U.S.S.R., July 15–21, 1976.
[507] P. van Nieuwenhuizen, S. Ferrara and J. Scherk, Locally Supersymmetric Maxwell–Einstein Theory, Phys. Rev. Lett. 37 (1976) 1035.
[508] P. van Nieuwenhuizen, S. Ferrara, F. Gliozzi and J. Scherk, Matter Couplings in Supergravity Theory, Nucl. Phys. B117 (1977) 333.
[509] P. van Nieuwenhuizen, S. Ferrara, D.Z. Freedman, F. Gliozzi and J. Scherk, Scalar Multiplet coupled to supergravity, Phys. Rev. D15 (1977) 1013.
[510] P. van Nieuwenhuizen and J.A. Vermaseren, One-loop divergences in the quantum theory of supergravity, Phys. Lett. 65B (1976) 263.
[511] P. van Nieuwenhuizen and S. Ferrara, Consistent supergravity with complex spin 3/2 gauge fields, Phys. Rev. Lett. 37 (1976) 1669.
[512] P. van Nieuwenhuizen, M.T. Grisaru and J.A.M. Vermaseren, One-loop renormalizability of pure supergravity and the Maxwell–Einstein theory in extended supergravity, Phys. Rev. Lett. 37 (1976) 1662.
[513] P. van Nieuwenhuizen, Local supersymmetry and supergravity, Proc. 1976 Brookhaven Conf.
[514] P. van Nieuwenhuizen and P. Townsend, BRS gauge and ghost field supersymmetry in gravity and supergravity, Nucl. Phys. B120 (1977) 301.
[515] P. van Nieuwenhuizen and M.T. Grisaru, Renormalizability of supergravity, Proc. Coral Gables Conf., 1977.
[516] P. van Nieuwenhuizen and J.A.M. Vermaseren, Finiteness of SO_3 and SO_4 supergravity and nonrenormalizability of the scalar multiplet coupling in explicit one-loop examples, Phys. Rev. D16 (1977) 298.
[517] P. van Nieuwenhuizen and P.K. Townsend, Geometrical interpretation of extended supergravity, Phys. Lett. B67 (1977) 439.
[518] P. van Nieuwenhuizen, M. Kaku and P.K. Townsend, Gauge theory of the conformal and graded conformal group, Phys. Lett. 69B (1977) 304.
[519] P. van Nieuwenhuizen, S. Ferrara, M. Kaku and P.K. Townsend, Gauging the graded conformal group with unitary internal symmetry, Nucl. Phys. B129 (1977) 125.
[520] P. van Nieuwenhuizen, N. Baaklini and S. Ferrara, Classical solutions in supergravity, Lett. Nuovo Cimento 20 (1977) 113.
[521] P. van Nieuwenhuizen, M. Kaku and P.K. Townsend, Superconformal unified field theory, Phys. Rev. Lett. 39 (1977) 1109.
[522] P. van Nieuwenhuizen, G. Sterman and P.K. Townsend, Unitary Ward identities and BRS invariance, Phys. Rev. D17 (1978) 1501.
[523] P. van Nieuwenhuizen, M. Kaku and P.K. Townsend, Properties of Conformal Supergravity Phys. Rev. D17 (1978) 3179.
[524] P. van Nieuwenhuizen and D.Z. Freedman, Supergravity, Scientific American, Feb. 1978.
[525] P. van Nieuwenhuizen and S. Ferrara, The Auxiliary fields of Supergravity, Phys. Lett. B74 (1978) 333.
[526] P. van Nieuwenhuizen, S. Ferrara and M.T. Grisaru, Poincaré and conformal supergravity models with closed algebras, Nucl. Phys. B138 (1978) 430.
[527] P. van Nieuwenhuizen and S. Ferrara, Tensor calculus for supergravity, Phys. Lett. 76 (1978) 404.
[528] P. van Nieuwenhuizen, Covariant quantization of supergravity, Proc. 1978 Coral Gable Conf. (Plenum Press, 1978).

[529] P. van Nieuwenhuizen and B. de Wit, The auxiliary field structure in chirally extended supergravity, Nucl. Phys. B139 (1978) 216.
[530] P. van Nieuwenhuizen, M.T. Grisaru, H. Römer and N. Nielsen, Approaches to the gravitational spin 3/2 axial anomaly, Nucl. Phys. B140 (1978) 477.
[531] P. van Nieuwenhuizen and S. Ferrara, Structure of supergravity, Phys. Lett. 78B (1978) 573.
[532] P. van Nieuwenhuizen, (with E. Cremmer, S. Ferrara, B. Julia, J. Scherk and L. Girardello), Super-Higgs effect in supergravity with general scalar interactions, Phys. Lett. 79B (1978) 231.
[533] P. van Nieuwenhuizen, Introductory review of supergravity, Waterloo Conf. 1977, General Relativity and Gravitation 10 (1979) 211.
[534] P. van Nieuwenhuizen and P.K. Townsend, Anomalies, topological invariants and the Gauss–Bonnet theorem in supergravity, Phys. Rev. D19 (1979) 3592.
[535] P. van Nieuwenhuizen and P.K. Townsend, Simplifications of conformal supergravity, Phys. Rev. D19 (1979) 3166.
[536] P. van Nieuwenhuizen, F.A. Berends, J.W. van Holten and B. de Wit, On spin 5/2 gauge fields, Phys. Lett. 83B (1979) 188.
[537] P. van Nieuwenhuizen, F.A. Berends, J.W. van Holten and B. de Wit, On field theory for massive and massless spin 5/2 fields, Nucl. Phys. B154 (1979) 261.
[538] P. van Nieuwenhuizen, On spin 5/2 gauge fields, Sanibel meeting 1979, J. Phys. A13 (1980) 1643.
[539] P. van Nieuwenhuizen, Lectures in supergravity, Cargese Summerschool 1978 (Plenum Press).
[540] P. van Nieuwenhuizen and S. Ferrara, Simplifications of Einstein supergravity, Phys. Rev. D20 (1979) 2079.
[541] P. van Nieuwenhuizen, W. Cremmer, B. Julia, J. Scherk, S. Ferrara and L. Girardello, Spontaneous symmetry breaking and Higgs effect in Supergravity without cosmological term, Nucl. Phys. B147 (1979) 105.
[542] P. van Nieuwenhuizen and P.K. Townsend, Dimensional regularization and supersymmetry at the two-loop level, Phys. Rev. D. 20 (1979) 1832.
[543] P. van Nieuwenhuizen and S. Ferrara, Supergravity: an odyssey through spacetime and superspace, ed. A. Held, in the Einstein Memorial volume.
[544] P. van Nieuwenhuizen, Four lectures at the 1979 Erice school on spin, torsion, rotation and supergravity, ed. P. Bergmann (Plenum Press).
[545] P. van Nieuwenhuizen and S. Ferrara, Supergravity with and without superspace, Ann. of Phys. 127 (1980) 274.
[546] P. van Nieuwenhuizen, Supergravity, Proc. Princeton Einstein Centennial Symp.
[547] P. van Nieuwenhuizen, Supergravity as a gauge theory derived from matter coupling, Proc. Math. Phys. Conf., Lausanne, 1979.
[548] P. van Nieuwenhuizen. Simple supergravity in ordinary spacetime, Proc. 2nd Marcel Grossmann meeting, Trieste, 1979.
[549] P. van Nieuwenhuizen and P.C. West, From conformal supergravity in ordinary space to its superspace constraints, Nucl. Phys. B172 (1980).
[550] P. van Nieuwenhuizen, D.M. Capper and D.R.T. Jones, Regularization by dimensional reduction of supersymmetric and non-supersymmetric gauge theories, Nucl. Phys. B (1980).
[551] S. Ferrara and P. van Nieuwenhuizen, Supergravity with and without superspace, Proc. Stony Brook Supergravity Conf. 1979 (North-Holland, 1979).
[552] P. van Nieuwenhuizen and D.Z. Freedman, Proc. 1979 Stony Brook Supergravity Workshop (North-Holland, 1979).
[553] P. van Nieuwenhuizen, Unitarity of supergravity and Z_2 or $Z_2 \times Z_2$ or $Z_2 \times Z_2 \times Z_2$ gradings of gauge and ghost fields, J. Math. Phys. 21 (1980) 2562.
[554] P. van Nieuwenhuizen, Supergrawitacja a unifitacja prow fizyki, translation of 46. intr Postepy Fizyki, Tom 30, Zeszyt 5, 1979.
[555] L. Castellani, S.J. Gates and P. van Nieuwenhuizen, The constraints for $N = 2$ supergravity.
[556] M.K. Fung, D.R.T. Jones and P. van Nieuwenhuizen, Transversality of the gravitino self-energy, regularization, Phys. Rev. D22 (1980) 2364.
[557] P. van Nieuwenhuizen, Supergravity, lectures at the Cambridge Workshop (1980).
[558] P. van Nieuwenhuizen, Supergravitia y unifikatia Sakonov physiki, translation into Uspekhi (1979).
[559] H. Nicolai, P.K. Townsend and P. van Nieuwenhuizen, Comments on 11-dimensional Supergravity, Nuovo Cimento Lett.
[560] M. Duff and P. van Nieuwenhuizen, Quantum inequivalence of different field Representations, Phys. Lett. 94B (1980) 179.
[561] B. de Wit, J.W. van Holten and A. Van Proeyen, Transformation Rules of $N = 2$ Supergravity Multiplets, Leuven preprint (1979), to be published in Nucl. Phys. B.
[562] E. Bergshoeff, M. de Roo and B. de Wit, Extended conformal supergravity.
[563] A. Van Proeyen, Massive Vectormultiplets in Supergravity, Nucl. Phys. B162 (1980) 376.
[564] A. Van Proeyen, Massive Vectormultiplets in Supergravity, in: Supergravity, Proc. Supergravity Workshop at Stony Brook (North-Holland, 1979).
[565] B. de Wit, J.W. van Holten and A. Van Proeyen, Transformation rules of $N = 2$ Supergravity Multiplets, to be published in Nucl. Phys. B.
[566] A. Van Proeyen, Introduction to the gauge theory of Supergravity, Preprint KUL-TR-80/8, Leuven (1980).
[567] A. Van Proeyen, $N = 2$ Supergravity multiplets, preprint KUL-TF-80/7, Leuven (1980), Proc. Karpacz winter school.
[568] M. de Roo, J.W. van Holten, B. de Wit and A. Van Proeyen, Chiral Superfields in $N = 2$ supergravity, Nucl. Phys. B (to be published).
[570] J. Wess, Supersymmetry, Acta Physica Austriaca, Supplementum XV (1976) 475.
[571] J. Wess and B. Zumino, Superspace Formulation Of Supergravity, Phys. Lett. 66B (1977).
[572] J. Wess, Supergravity, A Geometrical Approach, Hadron Constituents and Symmetry, Vol. 3 (Bratislava, 1978) p. 11.
[573] J. Wess, Supersymmetry-Supergravity, Lectures given at VIII GIFT Intern. Seminar on Theor. Phys., Salamanca June 13–19, 1977; Lectures notes in Phys. 77 (Springer-Verlag).
[574] J. Wess and B. Zumino, Superfield Lagrangian For Supergravity, Phys. Lett. 74B (1978) 51.
[575] J. Wess, R. Grimm and M. Sohnius, Extended Supersymmetry And Gauge Theories, Nucl. Phys. B133 (1978) 275.
[576] J. Wess, R. Grimm and B. Zumino, Consistency Checks On The Superspace Formulation Of Supergravity, Phys. Lett. 73B (1978) 415.
[577] J. Wess and B. Zumino, The Component Formalism Follows From The Superspace Formulation Of Supergravity, Phys. Lett. 79B (1978) 394.

[578] J. Wess, R. Grimm and B. Zumino, A Complete Solution Of The Bianchi Identities In Superspace With Supergravity Constraints, Nucl. Phys. B152 (1979) 255.
[579] J. Wess, Supergravity, talk given at the IX Texas Symp. on Relativistic Astrophysics, Munich 1978; Karlsruhe preprint 1979, to be published.
[580] J. Wess, Methods of Differential Geometry In Gauge Theories and Gravitational Theories, talk given at the Einstein Symp., Berlin 1979, Karlsruhe preprint, to be published.
[581] J. Wess, Lectures given at the Boulder Summer School 1979; Karlsruhe preprint, to be published.
[581a] J. Wess and B. Zumino, Nucl. Phys. B70 (1974) 39.
[582] P.C. West and K.S. Stelle, Auxiliary Fields For Supergravity, Phys. Lett. B74 (1978) 330.
[583] P.C. West and K.S. Stelle, Vector Multiplet Coupled to Supergravity, Phys. Lett. B77 (1978) 376.
[584] P.C. West and K.S. Stelle, Relation Between Vector And Scalar Multiplet And Gauge Invariance In Supergravity, Nucl. Phys. B145 (1978) 175.
[585] P.C. West and K.S. Stelle, Maxwell Multiplet Coupled To Supergravity and B.R.S. Invariance, Nucl. Phys. B140 (1978) 285.
[586] P.C. West and A.H. Chamseddine, Supergravity As A Gauge Theory Of Supergravity, Nucl. Phys. B129 (1977) 39.
[587] P.C. West, A Geometric Gravity Lagrangian, Phys. Lett. 76B (1978) 569.
[588] P.C. West and K.S. Stelle, Spontaneously Broken de Sitter Symmetry And The Gravitational Holonomy Group, preprint ICTP/78-79/19.
[589] P.C. West and K.S. Stelle, de Sitter Gauge Invariance And The Geometry Of The Einstein–Cartan Theory, J. Phys. A, No. 8, L205 (1979).
[590] T. Yoneya, Background Metric In Supergravity Theories, Phys. Rev. D17 (1978) 2567.
[591] C.K. Zachos, Extended S.G. With A Gauged Central Change, Cal. Tech. Ph.D. thesis, April 1979.
[592] C.K. Zachos, $N = 2$ Supergravity With A Gauged Central Change, Phys. Lett. 76B (1978) 329.
[593] B. Zumino, Symmetry, Proc. Conf. at Northern University, Boston, Sept. 26 and 27, 1975, eds. R. Arnowitt and P. Nath (the MIT Press Cambridge, Mass. and London, England).
[594] B. Zumino, Non-Linear Realization Of Supersymmetry In Anti De Sitter Space, Nucl. Phys. B127 (1977) 189.
[595] B. Zumino, Supersymmetry And The Vacuum, Nucl. Phys. B89 (1975) 535.
[596] B. Zumino, Supergravity, Annals of the NY Academy of Sciences 302 (1977) 545.
[597] B. Zumino, Supersymmetry and Supergravity, Physica 96A (1979) 99.
[598] B. Zumino, Some Recent Developments In Supergravity, Czech. J. Phys. B29 (1979) 245.
[599] B. Zumino, Supergravity, Colloques Nationaux du Centre de la Recherche Scientifique, No. 937.
[600] B. Zumino, Relativistic Strings And Supergauges, reprint from Renormalization and Invariance In Quantum Field Theory, ed. E.R. Caianiello (Plenum Publishing).
[601] B. Zumino, Euclidean Supersymmetry And The Many-Instanton Problem, Phys. Lett. 69B (1977) 369.
[602] B. Zumino, Supersymmetry And Kahler Manifolds, to be published in Phys. Lett. B.
[603] B. Zumino and S. Ferrara, The Mass Matrix of $N = 8$ Supergravity, to be published in Phys. Lett. B.
[604] B. Zumino, Topics In Supergravity And Supersymmetry, Coral Gables Conf., Jan. 1977.
[605] B. Zumino, in: Proc. 1979 Cargèse school (Plenum Press).
[606] M.T. Grisaru, P. van Nieuwenhuizen and C.C. Wu, Phys. Rev. D12 (1975) 3203.
[607] B. Lee and J. Zinn-Justin, Phys. Rev. D7 (1973) 1049.
[608] G. Senjanovic, Phys. Rev. D16 (1977) 307.

Addendum

6.6. Supergravity and phenomenology

The nongravitational forces around 10^2 GeV seem to be adequately described by SU_3 (color) $\times SU_2 \times U_1$. If one uses the renormalization group techniques to determine the dependence of the corresponding three running coupling constants on Q^2, they seem to come together at 10^{15} GeV (the "grand unified mass" M). Although this is not far from the Planck mass $(\hbar/G)^{1/2}\ c^{5/2} = 10^{19}$ GeV, it is commonly assumed that the nongravitational forces first combine at M into the simple group SU_5. Since SU_5 (or other candidates like E_6 and SO_{10}) have many free parameters, while also gravity is left out, these results cannot be the conclusive end point of unification.

Supergravity opens the possibility to include also gravity; moreover, it has very few free parameters. The massless $N = 8$ model has only a global SU_8 symmetry of the physical particles on-shell, while it is not even clear whether the subgroup SO_8 can be gauged off-shell. Even if it could, this local SO_8 is too small to contain $SU_3 \times SU_2 \times U_1$. However, there are now several recent proposals to circumvent this. One fascinating proposal by Ellis, Gaillard, Maiani and Zumino, though containing many loose ends and based on rather uncertain assumptions, nevertheless comes up with the result that we have seen all quarks (except, at this moment of writing the top quark). The argument goes as follows.

The basic ansatz is that the local off-shell SU_8 symmetry of the classical (preon) action in $d = 4$ dimensions (comparable to the local Usp(8) symmetry in $d = 5$ dimensions, see section 6.3) can give rise at the nonperturbative quantum level to bound states which form an $N = 8$ supersymmetry multiplet at the Planck mass, in which particles of a given spin form SU_8 multiplets, such that the theory with bound states plus the graviton (which is a preon) has a *local* SU_8 symmetry. At the classical level there is only a local SU_8 symmetry if one adds to the 70 physical spin 0 fields 63 extra nonphysical fields (just as one can add to the metric $g_{\mu\nu}$ extra antisymmetric fields such that one ends up with tetrads). This is thus a fake symmetry. The gauge fields of this local SU_8 are composites of scalar fields (compare with the spin connection $\omega(e)$). However, it is known that $\mathcal{L} = |(\partial_\mu + iA_\mu)\phi_i|^2$, which has the same features (one can solve A_μ from its algebraic field equation, but with A_μ the theory has a local U(1) symmetry), yields a spin 1 massless bound state (poles in the 2-point Green's function in a $1/N$ expansion in the path-integral) and that the fermions also form bound states. Actually, in this model the fermionic bound states have the global SU(N) symmetry of the classical action, not only the local symmetry, which for the $N = 8$ model would mean that the bound states form E_7 multiplets. Since E_7 is noncompact, these multiplets would be infinite dimensional; we will not discuss this possibility any further.

Thus one assumes that the bound states form an $N = 8$ massless supersymmetry multiplet. Which multiplet? To answer this question, note that the $N = 1$ massive multiplet [2, 3/2, 3/2, 1] splits into the massless [2, 3/2] + [3/2, 1] multiplets, while for $N = 2$ the massive [2, $(3/2)^4$, $(1)^6$, $(1/2)^4$, 0] contains a massless [$(3/2)^2$, $(1)^4$, $(1/2)^2$]. These cases suggest the following $N = 8$ supersymmetry multiplet in which the vectors are in the adjoint representation of SU(8) rather than U(8) (see subsection 6.3)

helicity	3/2	1	1/2	0	−1/2	−1	−2	−5/2
field	G^A	$V^A{}_B$	$F^A{}_{BC}$	$S^A{}_{BCD}$	$F^A{}_{BCDE}$			
components	8	63	216	420	504			
		(+1)	(+8)	(+28)	(+56)			

Since one expects SU_8 rather than U_8 multiplets, one drops the traces; these are indicated in brackets.

To this multiplet one adds the *CTP* conjugate multiplet which starts at helicity +5/2, as well as the graviton which is not considered as a bound state but rather as a preon. (Since it has no SU_8 quantum numbers, it will presumably not be confined inside bound states.) One already sees that there are more than the expected 63 Yang–Mills fields, but the second spin 1 multiplet will be argued away below.

The idea is now to consider for which subgroup of SU_8 there are spin 0 fields in $S^A{}_{BCD}$ (and $\bar{S}_A{}^{BCD}$) which transform as scalars under this subgroup. This is a necessary condition for SU_8 to be able to be broken at (or below) the Planck mass down to this subgroup by means of the acquisition of a nonzero vacuum expectation value of one or more spin 0 fields. We will show in detail below that the first possible grand unified group is SU_5!

Having found SU_5, one then also decomposes the SU_8 multiplets for spin 1/2 and spin 1 into SU_5 multiplets. One drops the spin 3/2, 2, 5/2 fields, as well as the vector fields outside the 24 of SU_5 on the basis of Veltman's theorem. Actually, Veltman's theorem only says that if one has a renormalizable theory to begin with, and one considers a subset of light particles and looks at low (10^{15} GeV) energies, then only those particles can remain light which form a renormalizable theory. The rest disappears to high energies (the Planck mass). In our case, the theory with higher spins is not strictly renormalizable (instead, supergravity is finite) but Veltman's theorem is nevertheless used. On the basis of this theorem one also drops those spin 1 fields which do not gauge SU_5, as we already said.

The theorem is used a third time for the spin 1/2 sector. One considers only those sets of SU_5 spin 1/2 multiplets in which the sum of all axial anomalies cancels. Since one can consider all multiplets as left-handed by replacing right-handed multiplets in a representation R by their left-handed complex-conjugated $\bar{R}$, one can easily calculate these axial anomalies. Axial anomolies spoil renormalizability, and those spin 1/2 which cause axial anomalies will disappear again to the Planck mass. There remain several possible anomaly-free sets of spin 1/2 SU_5 multiplets, but one must require more. The QCD and EM interactions are purely vector-like (which means that fermions couple to the gauge fields of SU_3 (color) and U_1 (em) only with γ^μ but not with $\gamma^\mu\gamma_5$). Thus one must require that in the anomaly-free sets, when decomposed with respect to SU_3 (color) and U_1 (em), there exists for every complex representation R an $\bar{R}$ representation. Under these conditions one finds a unique largest set of spin 1/2 SU_5 multiplets. This largest set contains many fermions which can (and therefore do so according to the party line) acquire masses (left-handed fermions can acquire a Majorana mass $\psi_L(\bar{R})^T C\psi_L(R)$ if for every R there is an $\bar{R}$, or if R is real). However, there are precisely 3 families $(10+\bar{5})$ of leptons which cannot acquire masses ("chiral leptons"), and these are massless at 10^{15} GeV, and are the leptons we see at 10^2 GeV! (In the first family $\bar{5}$ contains $\bar{d}_L$, $e^-{}_L$ and $\nu^e{}_L$, while 10 contains u_L, $\bar{u}_L$, d_L and $e^+{}_L$, where u and d are the up and down quark.)

Let us now show in more detail how SU_5 arrives from an analysis of $S^A{}_{BCD}$, and how one calculates axial anomalies for the fields in $F^A{}_{BC}$.

The spin 0 SU_8 multiplet $S^A{}_{BCD}$ can be decomposed under SU_5 by splitting each index $A = 1, 8$ into $a = 1, 5$ and $j = 6, 8$. Thus one easily finds for the $SU_5 \times SU_3$ multiplets (SU_3 is a bookkeeping device, but might be the generation group)

$$420 = S^A{}_{BCD} = (24, \bar{3}) + (5, 3) + (5, \bar{6}) + (45, 3) + (\overline{40}, 1) + (\overline{10}, \bar{3}) + (\bar{5}, 1) + (10, 1) + (10, 8) + (1, \bar{3}).$$

Thus there are indeed SU_5 scalars, namely the three $S^j{}_{678}$ (note that $S^i{}_{i78} = -S^6{}_{678}$ since we dropped the trace parts in $S^A{}_{BCD}$). If one decomposes SU_8 under SU_7 or SU_6, one finds no scalars. One cannot consider E_6 or SO_{10}, since they are not subgroups of SU_8 (because E_6 has more (78) generators than

SU_8, while if SO_{10} were a subgroup of SU_8 then the 8 of SU_8 would have to split up in ten 1's of SO_{10} which is impossible). So supergravity rules out SU_6, SU_7, SU_8, E_6 and SO_{10}, but allows SU_5 as the grand unified group.

Let us finally discuss how to determine which subsets of SU_5 spin 1/2 multiplets have vanishing axial anomaly. The axial anomaly of a triangle graph (this is enough to consider) due to fermions in an SU_5 representation R which are minimally coupled to the SU_5 gauge fields, is given by

$$A_{abc}(R) = \mathrm{tr}[\{T_a(R), T_b(R)\}T_c(R)] = d_{abc}A(R)$$

where $d_{abc} = \mathrm{tr}[\{\lambda_a, \lambda_b\}\lambda_c]$ and $T_a(R)$ and λ_a are the SU_5 generators in the R and adjoint representation, respectively. To compute $A(R)$ for the various spin 1/2 multiplets in

$$216 = (10,\bar{3}) + (\bar{5},\bar{3}) + (\overline{45},1) + (1,3) + (5,1) + (1,\bar{6}) + (24,3) + (5,8)$$

$$\overline{504} = (45,3) + (40,\bar{3}) + (24,1) + (15,1) + (10,1) + (10,8) + (\overline{10},6) + (\overline{10},\bar{3}) + (5,3) + (\bar{5},\bar{3})$$

one uses that $A(R_1 + R_2) = A(R_1) + A(R_2)$, while $A(R_1 \otimes R_2) = A(R_1) \times \dim R_2 + \dim R_1 \times A(R_2)$. Since all representations of compact Lie algebras can be taken unitary, $A(\bar{R}) = -A(R)$. With these rules one can express all $A(R)$ in terms of $A(5)$ and $A(10)$. For example, $A(10 \otimes 5) = A(\overline{10}) + A(40)$, where $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array} \otimes \begin{array}{|c|}\hline \ \\ \hline\end{array} = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array} + \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ and

$$\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} = 40 = \bar{F}_a^{\ bcd8} - \text{trace } (5 \times 10 - 10 = 40 \text{ components}).$$

The reader easily computes the other $A(R)$, and relates them to elements of $F^A{}_{BC}$ or $F_A{}^{BCDE}$. As is well known, for SU_5 one has $A(5) = A(10)$. To prove this by elementary means, note that 5 splits up under SU_3 into $3+1+1$, while 10 ($=5\times 5$ antisymmetrized) splits up under SU_3 into $3+3+\bar{3}+1$. Consider now in $A_{abc}(R)$ the indices a, b, c to be of the subgroup SU_3, and make a similarity transformation such that in $T_a(R)$ the SU_3 matrices appear in the left upper corner. The d_{abc} are, of course, invariant under this change of basis, but $d_{abc}\ (SU_5) = d_{abc}\ (SU_3)$ for these a, b, c. Consequently, $A(5, SU_5) = A(3, SU_3)$ and $A(10, SU_5) = A(3, SU_3)$.

With these results one finds several subsets which are free from axial anomalies (one of which is given below). However, decomposing each set under SU_3 (color) $\times\ U_1$ (em) and requiring vector-like couplings to these groups, there is a unique subset which has more chiral fermions than all others

$$(45+\overline{45}) + 4(24) + 9(10+\overline{10}) + 3(5+\bar{5}) + 3(5+\overline{10}) + 9(1).$$

All fermions in this set can acquire masses (between 10^{15} and 10^{19} GeV), except the three chiral $(5+\overline{10})$ multiplets. This is the result mentioned above.

There are other approaches to phenomenology. One of them (due to E. Witten) starts from the $d = 11$ dimensional model of section 6.2, and assumes that the $11-4=7$ dimensional internal space is a coset space G/H. If one takes $G = SU_3 \times SU_2 \times U_1$, the minimal H which still allows a nontrivial action of each of the factors SU_3, SU_2 and U_1 of G on G/H is $H = SU_2 \times U_1 \times U_1$. Remarkably, this space is exactly 7-dimensional. If one decomposes the elfmetric g_{AB} in the way we have discussed in section 6.3, and replaces the 7 vector fields $A_{\mu\alpha}(x, y)$ by $A_\mu(x)^a K^a{}_\alpha(y)$, where the rectangular matrix $K^a{}_\alpha(y)$ are the Killing vectors of the 7-dimensional y-space ($a = 1, 12$ and $\alpha = 1, 7$), and if one substitutes this ansatz back into the Einstein action, one finds the required number of Yang–Mills actions needed to gauge $SU_3 \times SU_2 \times U_1$ in $d = 4$. The y-dependence drops from the action after integration over y, but the nontrivial dependence of K on y provides the extra spin 1 modes. In ordinary dimensional reduction, K is a y-independent square matrix, where as in section 6.4 K was equal to the square matrix $U(y) \sim \exp My$. This is a different approach to phenomenology, which also circumvents the problem that SO_8 is too small to contain $SU_3 \times SU_2 \times U_1$ (*Proof*: if it did, the 8 of SO_8 would split up under SU_3 into $3 + \bar{3} + 1 + 1$ since it must be real, but then SU_2 could not act on these SU_3 multiplets), but in this approach one does not need to invoke bound states. Whether these ideas, which have been worked out for the metric only, can be extended to supergravity is not known.

This concludes our report on supergravity. It is certainly (at least in the author's mind) three-loop finite and possibly all-loop finite, which means something, although what exactly is not clear since nonperturbative effects should be important. It may unify grand unified theories or the 10^2 GeV interactions with gravity, leading to a model with almost no free parameters. It is the unique theory with a local gauge symmetry between fermions and bosons. It is the most beautiful gauge theory known, so beautiful, in fact, that Nature should be aware of it!