SOME ASPECTS OF ANOMALIES OF GLOBAL CURRENTS

M.B. PARANJAPE [1]

Institut für Theoretische Physik, ETH – Hönggerberg, Zurich, Switzerland

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We show that the covariant form for the anomaly can be obtained for global fermionic currents through the addition of renormalization counter terms to the generating functional. This allows for a fully gauge covariant theory.

The study of anomalies of fermionic currents [1,2] has recently enjoyed a revival. They have been evaluated in gauge and gravitational settings, in any number of spacetime dimensions [3,4]. A remarkable connection between the abelian anomaly and the non-abelian anomaly in two lower dimensions has been discovered [5,6]. Novel contributions to the low energy effective action that summarize their influence on the low energy dynamics have also been discovered [7–9].

In this letter we study the anomalies of global fermionic currents in the presence of gauge fields in four dimensions. By global currents we mean the following. In most gauge theory models with fermions, in elementary particle physics, it is the prevailing situation not to gauge every symmetry of the fermionic kinetic term. The global currents correspond to those symmetries which are not gauged. We find that the global currents, and hence their anomalies, can be made to transform covariantly under the group of gauge symmetries. We achieve this through addition of local counter terms to the generating functional, in the spirit of renormalization [10].

These counter terms depend on gauge field sources introduced for the global currents, in the usual sense, to define the generating functional. The sources are to be put to zero at the end of any calculation. Since we only add local counter terms to the generating functional the Wess–Zumino consistency conditions [7] are automatically satisfied. Covariant and consistent forms for the anomaly have recently been discussed by Bardeen and Zumino [11]. They show how to modify the current to obtain the covariant anomaly. However within their framework it is not possible to implement this through renormalization counter terms. This is because they do not consider the possibility of maintaining covariance only under a subgroup which is to be gauged.

The usual abelian anomaly follows just as a special case of the covariant global anomaly. Its form and normalization have nothing to do with the fact that the generator is abelian, but follow only from the fact that the current is global. Therefore we clarify the use in the literature of the various adjectives abelian, non-abelian, covariant and consistent, for describing anomalies. For us, there exist only covariant anomalies for global currents. These are also consistent in the sense that they satisfy the Wess–Zumino consistency conditions. For gauged currents there exist only consistent anomalies, which cannot be brought to a form covariant under the gauge transformations [‡1], to which they correspond, through adding counter terms to the generating functional.

Without loss of generality we specialize to left-handed fermions. All right-handed fermions can be replaced with their charge conjugate left-handed fields without altering the content of the theory. Consider the general case where we have two sets of anti-hermitean generators t_a and t'_α. The t_a generate a closed Lie algebra and correspond to

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[‡1] The anomaly of a gauged chiral U(1) current is in fact gauge invariant. But this anomaly does not correspond to the usual abelian anomaly. Its normalization is different and cannot be brought to the usual form through adding counter terms.

the gauged symmetries of the lagrangian

$$[t_a, t_b] = f_{abc} t_c .$$

The t'_α correspond to global symmetries and transform amongst themselves under the action of the group generated by the t_a

$$[t_a, t'_\alpha] = f'_{a\alpha\beta} t'_\beta .$$

We dispense with terms explicitly violating the global symmetries as these have no bearing on anomalies. The commutators $[t'_\alpha, t'_\beta]$ are not specified, and in general will contain generators of both kinds, t_a and t'_α. We assume for simplicity all generators to be trace orthogonal and trace normalized to -1.

Introduce gauge fields A^a_μ and A'^{α}_μ coupled to the currents

$$j^a_\mu \equiv \bar{\psi}_L \gamma_\mu t_a \psi_L , \quad j'^{\alpha}_\mu \equiv \bar{\psi}_L \gamma_\mu t'_\alpha \psi_L ,$$

respectively. The gauge fields A'^{α}_μ however, are to be set to zero at the end of any calculation. If we let λ stand for a generator of either kind, we start with the anomaly

$$-\mathrm{Tr}\{\lambda \mathrm{D}{*j}\} = -\tfrac{1}{3}(1/8\pi^2)\,\mathrm{Tr}\{\lambda \mathrm{d}((A+A')\,\mathrm{d}(A+A') + \tfrac{1}{2}(A+A')^3)\}$$

obtained by Bardeen [10], while treating the left-handed sector independently. Here we have used the form notation [6], where the gauge field becomes the Lie algebra valued connection one form

$$A + A' = (A^a_\mu t_a + A'^{\alpha}_\mu t'_\alpha)\,\mathrm{d}x^\mu ,$$

the current is a matrix valued one form

$$j = (j^a_\mu t_a + j'^{\alpha}_\mu t'_\alpha)\,\mathrm{d}x^\mu ,$$

d refers to exterior differentiation while D refers to covariant exterior differentiation

$$\mathrm{D}\omega = \mathrm{d}\omega + (A+A')\omega - (-1)^p\,\omega(A+A') ,$$

where p is the degree of the form ω, the trace is with respect to internal indices and $*$ is the Hodge star operation.

This form of the anomaly is not covariant under a gauge transformation

$$A + A' \to A + A' - \mathrm{D}\Lambda , \quad \Lambda = V^a t_a + V'^{\alpha} t'_\alpha ,$$

however, as has been recently demonstrated [6], the Wess–Zumino consistency conditions are satisfied. This anomaly can be modified by adding local counter terms in the fields A^a_μ and A'^{α}_μ to the generating functional [10]. The Wess–Zumino consistency conditions are preserved since the anomalies are still given by the variation of a now renormalized generating functional under a gauge transformation. The freedom to add counter terms corresponds exactly to the freedom to add any exact differential to the form ω^0_{2n+1} in the work of Wu, Zee and Zumino [6]. We will explicitly demonstrate that the anomalies of the currents corresponding to the generators t'_α, through adding appropriate counter terms, can be made to transform covariantly under a gauge transformation within the group generated by the t_a, even in the presence of the gauge fields A'. When the A' fields are set to zero we obtain the anomalies

$$-\mathrm{Tr}\{t_a \mathrm{D}*j\} = -\tfrac{1}{3}(1/8\pi^2)\,\mathrm{Tr}\{t_a\,\mathrm{d}(A\,\mathrm{d}A + \tfrac{1}{2}A^3)\} , \quad -\mathrm{Tr}\{t'_\alpha \mathrm{D}*j\} = -(1/8\pi^2)\,\mathrm{Tr}\{t'_\alpha F^2\} ,$$

where

$$F = \mathrm{d}A + A^2 = \tfrac{1}{2}F_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$$

is the curvature two form for the gauge field A.

To obtain these results [‡2] we examine the variation of the primed current under an unprimed gauge transfor-

[‡2] These results were originally obtained in collaboration with Professor J. Goldstone. See also ref. [11].

mation δ_V such that

$$A \to A - \mathrm{d}V^a t_a - [A, V^a t_a] \;, \quad A' \to A' - [A', V^a t_a] \;.$$

We assume a generating functional $W(A,A')$ and all expectation values are obtained from $W(A,A')$ with A' set to zero at the end. The primed current is

$$\mathrm{i}j'^{\alpha}_{\mu}(x) = [\delta/\delta A'^{\alpha}_{\mu}(x)]\, W(A,A')$$

where $\delta/\delta A'^{\alpha}_{\mu}(x)$ is the functional derivative. Then

$$\mathrm{i}\delta_V j'^{\alpha}_{\mu}(x) = \delta_V [\delta/\delta A'^{\alpha}_{\mu}(x)]\, W(A,A') = [\delta/\delta A'^{\alpha}_{\mu}(x)]\, \delta_V W(A,A') + [\delta_V, [\delta/\delta A'^{\alpha}_{\mu}(x)]\,]\, W(A,A') \,.$$

But

$$\delta_V = \int \mathrm{d}^4 z\, V^a(z) \{\partial_\mu [\delta/\delta A^a_\mu(z)] + f_{abc} A^b_\mu(z) [\delta/\delta A^c_\mu(z)] + f'_{a\alpha\beta} A'^{\alpha}_{\mu}(z) [\delta/\delta A'^{\beta}_{\mu}(z)]\} \,,$$

so

$$[\delta_V, [\delta/\delta A'^{\alpha}_{\mu}(x)]\,]\, W(A,A') = V^a(x)\, f'_{\alpha a\beta} [\delta/\delta A'^{\beta}_{\mu}(x)]\, W(A,A') \,.$$

Therefore

$$\delta_V j'^{\alpha}_{\mu}(x) = f'_{\alpha a\beta} V^a(x) j'^{\beta}_{\mu}(x) - \mathrm{i}[\delta/\delta A'^{\alpha}_{\mu}(x)]\, \delta_V W(A,A') \,.$$

Evidently the covariant transformation property is maintained if the last term vanishes,

$$[\delta/\delta A'^{\alpha}_{\mu}(x)]\, [\delta_V W(A,A')] = 0 \,.$$

Simply stated this condition means that the anomalies of the unprimed currents must be independent of the primed gauge fields. We must achieve this by adding counter terms to $W(A,A')$.

We remark that our condition also guarantees that all primed higher moments of the generating functional, that is for example the primed current–primed current correlation function and so on, are also gauge covariant under δ_V.

We start with the Bardeen expression for the anomaly

$$-\mathrm{i}\delta_V W(A,A') = -\frac{1}{3}\frac{1}{8\pi^2} \int \mathrm{Tr}\{V\mathrm{d}((A+A')\,\mathrm{d}(A+A') + \tfrac{1}{2}(A+A')^3)\} \,.$$

With the counter term

$$R(A,A') = -\frac{\mathrm{i}}{3}\frac{1}{8\pi^2} \int \mathrm{Tr}\{\mathrm{d}AA'A - AA'\mathrm{d}A + \tfrac{1}{2}(A'\mathrm{d}A'A + A'A\mathrm{d}A') - \tfrac{3}{2}A^3A' + \tfrac{3}{4}A'AA'A - \tfrac{1}{2}AA'^3\}$$

we find

$$-\mathrm{i}\delta_V R(A,A') = \frac{1}{3}\frac{1}{8\pi^2} \int \mathrm{Tr}\{V\mathrm{d}((A+A')\mathrm{d}(A+A') + \tfrac{1}{2}(A+A')^3 - (A\mathrm{d}A + \tfrac{1}{2}A^3))\} \,.$$

Therefore the renormalized generating functional

$$W_R(A,A') = W(A,A') + R(A,A')$$

has the variation

$$-\mathrm{i}\delta_V W_R(A,A') = -\frac{1}{3}\frac{1}{8\pi^2} \int \mathrm{Tr}\{V\mathrm{d}(A\mathrm{d}A + \tfrac{1}{2}A^3)\} \,,$$

which is independent of A' as desired. We note that $R(A,A')$ is not unique. Any term which is gauge invariant under δ_{V} can be added. However, no such term which is linear in A' exists. Therefore the expression for the primed current, which requires functionally differentiating with respect to A' once and then setting A' to zero, is unique.

Now to obtain the expressions for the anomalies of the primed current we simply perform a primed gauge transformation δ_{V}, and calculate the variation of the renormalized generating functional. The resulting anomalies will transform covariantly under an unprimed gauge transformation even in the presence of A'. To calculate these anomalies one must specify the commutators $[t'_\alpha, t'_\beta]$. However, if A' is to be set to zero we do not require these commutators, as we only want the term in the variation $\delta_{V'} W_R(A, A')$ which is independent of A'. We can use a truncated form of the primed gauge transformation

$$A \to A + o(A')\,, \quad A' \to A' - dV' - [A, V'] + o(A')\,,$$

where $V' = V'^{\alpha} t'_\alpha$ and $o(A')$ denotes terms which vanish when A' is set to zero. This easily yields

$$-i\delta_{V'} W_R(A, A')|_{A'=0} = -\frac{1}{8\pi^2}\int \mathrm{Tr}\{V'F^2\}\,.$$

Converting to the usual notation the anomaly is

$$D^\mu j'_{L\mu} = -(1/32\pi^2)\epsilon_{\mu\nu\sigma\tau}\,\mathrm{Tr}\,\{t'_\alpha\, F^{\mu\nu}F^{\sigma\tau}\}\,,$$

which obviously transforms covariantly.

As a specific case we consider the anomaly of global U(1) currents when we gauge first the vector SU(N) subgroup and secondly the left- and right-handed SU(N) sectors independently. The SU(N) vector subgroup appears naturally in the Dirac formalism. But replacing the right-handed fermions with their left-handed charge conjugate fields we find that it is in fact an SU(N) subgroup of U($2N$), which is the actual symmetry group of the fermionic kinetic term. The vector SU(N) subgroup is generated by

$$T_a = \begin{pmatrix} t_a & 0 \\ 0 & t_a^* \end{pmatrix}\,,$$

where the t_a are the generators of SU(N) under which the original Dirac fermions transform. The usual baryon number corresponds to the generator

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the axial vector current corresponds to

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If we now insist on vector gauge covariance it is easy to see that the vector currents are anomaly free, the anomaly of the baryon number current vanishes and the axial vector current assumes the usual abelian anomaly. In obvious notation

$$\partial_\mu j^\mu = 0\,, \quad \partial_\mu j_5^\mu = (1/32\pi^2)\epsilon_{\mu\nu\sigma\tau}\,\mathrm{Tr}\,\{F^{\mu\nu}F^{\sigma\tau}\}\,,$$

where $F^{\mu\nu}$ is the field strength of the vector gauge field. Now if we choose to gauge left- and right-handed SU(N) subgroups independently we find the baryon number current and the axial vector current anomalies to be [12]

$$\partial_\mu j^\mu = (1/32\pi^2)\epsilon_{\mu\nu\sigma\tau}\,\mathrm{Tr}\,\{F_R^{\mu\nu}F_R^{\sigma\tau} - F_L^{\mu\nu}F_L^{\sigma\tau}\}\,, \quad \partial_\mu j_5^\mu = (1/32\pi^2)\epsilon_{\mu\nu\sigma\tau}\,\mathrm{Tr}\,\{F_R^{\mu\nu}F_R^{\sigma\tau} + F_L^{\mu\nu}F_L^{\sigma\tau}\}\,,$$

where $F_R^{\mu\nu}$ and $F_L^{\mu\nu}$ are the field strengths of the right and left gauge fields respectively.

To study the physical consequences of anomalies of global chiral currents we insist that the covariant form for these anomalies must be used. We have shown here that it is possible to add counter terms to the generating functional so that the global currents, and therefore their anomalies, transform covariantly. This means that it is possible to regulate the theory in such a way as to preserve the gauge covariance of the theory. It is absolutely essential to be able to maintain the gauge covariance since it is required for renormalizability. Of course the anomalies of the gauged currents must vanish of their own accord for a renormalizable theory; and this also means the gauged currents will transform covariantly, through straightforward generalization of our argument [11].

When using the anomaly to construct Wess–Zumino–Witten type effective lagrangians [7–9], the gauge fields at first are all external flavour gauge fields. Eventually some will become dynamical gauge fields say of the electroweak interaction, and the others will possibly correspond to vector mesons [9,13]. However, the low energy effective lagrangian obtained from integrating the Wess–Zumino consistency conditions depends on which form for the anomaly one starts with. The correct low energy lagrangian is physically chosen, and will correspond to the theory covariant under the dynamical gauge group. Therefore one must start with the anomaly which is covariant under the correct dynamical gauge group.

In conclusion, we have shown that it is possible to maintain full covariance of a gauge theory with fermions under an arbitrary anomaly free gauge group. The anomalies of the global fermionic currents can be made to transform covariantly under the gauge group. This clarifies the role of covariant and consistent anomalies. The usual abelian anomaly is just a special case of a covariant global anomaly while the usual non-abelian anomaly is just the consistent anomaly. However covariant forms of the global anomalies are still consistent, since they are obtained by adding renormalization counter terms to the generating functional. We anticipate our result is easily generalizable to higher dimensions.

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