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QUANTUM DYNAMICS OF A MASSLESS RELATIVISTIC STRING

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Abstract: We develop the classical and quantum mechanics of a massless relativistic string, the light string, which is characterized by an action proportional to the area of the world sheet swept out by the string in space time. We show that, classically, there are only D-2 dynamically independent components among the D functions $x^{\mu}(\sigma,\tau)$ which represent the world sheet (D is the dimension of space time). Quantizing only these independent components, we find that the angular momentum operators suggested by the correspondence principle generate $O(D-1,1)$ only when the first excited state is a photon, i.e., a spin-one massless state, and when $D = 26$. By allowing additional degrees of freedom in the quantum mechanics, we are able to quantize the string in a Lorentz covariant manner for any value of D and any mass for the first excited state. In this latter scheme the full Fock space contains negative norm states. However, when $D \leqslant 26$ for a massless first excited state and $D \leqslant 25$ for a real massive first excited state, the physical states span a positive subspace of the Fock space. The excitation spectrum of the light string coincides with the space of *physical* states in the dual resonance model for unit intercept of the leading Regge trajectory. We point out the connection of this work to previous studies of the physical states in dual models.

1. Introduction

Ever since the early days of dual resonance models, people have attempted to ascribe the excitation spectrum of those models to a vibrating relativistic string. In the earliest treatments [1] the space-time interpretation of the string was obscure even at the classical level. The main obstacle to such an interpretation was that the

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time component of the string vibrated along with the space components. Only recently, following an original idea of Nambu [2] has progress been made in giving the string a space-time interpretation [3]. The suggestion was made that the action of the relativistic string should be proportional to the area of the world sheet swept out by the string in space time. The Virasoro gauge conditions [4] which appear in dual models arise naturally in this description because of the arbitrariness of parametrization of the surface swept out by the string: one can always choose an *orthonormal* parametrization such that

$$\frac{\partial x}{\partial \tau} \cdot \frac{\partial x}{\partial \sigma} = 0 \,, \quad \left(\frac{\partial x}{\partial \tau}\right)^2 + \left(\frac{\partial x}{\partial \sigma}\right)^2 = 0 \,.$$

The Fourier components of these equations just say that the Virasoro gauges, L_N, are zero.

This freedom of parametrization can be exploited to choose, as one parameter, the time, x^0, in some Lorentz frame. By choosing the other parameter, σ, to be the fraction of the total energy included between one end of the string and the point in question, the equations of motion simplify to

$$\left(\frac{\partial^2}{\partial t^2} - \frac{1}{(2\pi E)^2}\frac{\partial^2}{\partial \sigma^2}\right) \boldsymbol{x}\,(\sigma, t) = 0 \,, \quad \boldsymbol{x}'(0, t) = \boldsymbol{x}'(1, t) = 0 \,,$$

and the subsidiary conditions

$$\frac{\partial \boldsymbol{x}}{\partial t} \cdot \frac{\partial \boldsymbol{x}}{\partial \sigma} = 0 \,, \quad \left(\frac{\partial \boldsymbol{x}}{\partial t}\right)^2 + \frac{1}{(2\pi E)^2}\left(\frac{\partial \boldsymbol{x}}{\partial \sigma}\right)^2 = 0 \,,$$

where E is the total energy of the string. One consequence of these conditions is that the ends of the string move at the speed of light. Another is the fact that only the motion of the string perpendicular to itself is dynamically significant. Thus the number of independent vibrations of the string is only two: the string is transverse. The string can then be quantized by making $\boldsymbol{x}(\sigma, t)$ quantum mechanical operators and subjecting the physical states to the subsidiary conditions. This procedure clear ly uses a positive definite Hilbert space so there is no problem with negative norm states. However, manifest Lorentz covariance is lost and must be proven before the theory is acceptable.

In the meantime an extensive study of the physical states in the dual resonance model led to proofs of the absence of ghosts (negative-norm states) under certain conditions [5, 6]. These conditions are that the intercept of the leading Regge trajectory must be unity and the dimension of space-time cannot be greater than 26. Further, these proofs established that when the dimension of space time is 26, the spectrum of physical states is purely transverse. Since a set of transverse states had already been explicitly constructed [7] these could be taken as a complete set of physical states. The O(25,1) transformation properties of physical states in 26 dimensions were studied [8] and a representation of O(25), the little group of the total momentum of a particular state on the transverse subspace alone constructed explicitly. This construction failed in less than 26 dimensions because the physical states are then not transverse.

It is our aim in this paper to understand the results described in the previous paragraph in terms of the quantization of the relativistic string. Since the transverse physical states mentioned before are simple in the limit of infinite momentum, we are led to choose one parameter of the surface swept out by the string proportional to $x_+ = \sqrt{\frac{1}{2}}(x^0 + x^3)$ rather than simply x^0. We shall show how previous results on the representations of the little group in 26 dimensions arise naturally from this choice of parametrization. In the process we shall review completely the classical and quantum mechanics of the relativistic string and shall attempt to unify the various treatments.

2. Classical mechanics of the light string

2.1. Equations of motion

We characterize a string mathematically as a finite curve in space which in general is allowed to change its shape and position as a function of time. In a particular Lorentz frame we can describe the string at a time $x^0 = t$ by a set of functions

$$x^i(\sigma, t)\,,$$

where $x^i(\sigma, t)$ is the ith spatial coordinate of the point on the string labeled by the parameter σ. In space-time the propagating string can be characterized by the two-dimensional surface $x^\mu(\sigma, t)$:

$$x^0(\sigma, t) = ct\,, \quad x^i(\sigma, t) = x^i(\sigma, t)\,. \tag{1}$$

Clearly, the knowledge of this surface gives a complete description of the motion of the string: we obtain the configuration of the string at any time t by intersecting the surface with the hyperplane $x^0 = ct$.

The only *a priori* restriction we place on the surface spanned by the string is that in the neighbourhood of each of its points there exists an infinitesimal displacement along the surface which points in the time-like or light-like direction; i.e., we require each point of the string to move at a velocity less than or equal to the velocity of light. We shall call such a surface time-like. Eq. (1) give a possible parametric representation of the surface, but we prefer to use a completely general parametrization.

$$x^\mu = x^\mu(\sigma, \tau)\,. \tag{2}$$

Let us introduce the differential forms

$$\mathrm{d}x^\mu = \frac{\partial x^\mu}{\partial \sigma}\,\mathrm{d}\sigma + \frac{\partial x^\mu}{\partial \tau}\,\mathrm{d}\tau\,, \tag{3}$$

$$\mathrm{d}F^{\mu\nu} = \mathrm{d}\sigma\,\mathrm{d}\tau \left(\frac{\partial x^\mu}{\partial\sigma}\,\frac{\partial x^\nu}{\partial\tau} - \frac{\partial x^\mu}{\partial\tau}\,\frac{\partial x^\nu}{\partial\sigma}\right)$$
$$\equiv \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,. \tag{4}$$

Then the condition that the surface be time-like is just *

$$-\mathrm{d}F^{\mu\nu}\,\mathrm{d}F_{\mu\nu} = (\mathrm{d}\sigma\,\mathrm{d}\tau)^2 \left[\left(\frac{\partial x}{\partial\sigma}\cdot\frac{\partial x}{\partial\tau}\right)^2 - \left(\frac{\partial x}{\partial\sigma}\right)^2\left(\frac{\partial x}{\partial\tau^2}\right)^2\right] \geqslant 0\,. \tag{5}$$

The area element spanned by the two infinitesimal displacements

$$\frac{\partial x^\mu}{\partial\sigma}\,\mathrm{d}\sigma\,,\qquad \frac{\partial x^\mu}{\partial\tau}\,\mathrm{d}\tau\,,$$

is given by

$$\mathrm{d}A = \{-\mathrm{d}F^{\mu\nu}\,\mathrm{d}F_{\mu\nu}\}^{\frac{1}{2}}\,. \tag{6}$$

For our purposes it will be convenient to choose the parameters so that the ends of the string correspond to $\sigma = 0$ and $\sigma = \pi$ and so that the initial and final configurations of the string, after some motion in space time, correspond to fixed values of τ, τ_i and τ_f. In almost all physical problems the initial and final configurations of the string will be those seen by a definite observer at a given instant of time in his Lorentz frame: this means that, initially and finally, the lines of constant τ will be lines of constant time in that frame. But we shall be more general and shall allow as initial and final configurations of the string any two non-intersecting space-like curves, $x^\mu_\mathrm{i}(\sigma)$, $x^\mu_\mathrm{f}(\sigma)$, which can be connected by a time-like surface.

We shall take the dynamics of the string to be given by choosing the action proportional to the area of the surface swept out by the string [9]

$$S = -\frac{1}{2\pi\alpha'\hbar c^2}\int_{\tau_\mathrm{i}}^{\tau_\mathrm{f}}\mathrm{d}\tau\int_0^\pi \mathrm{d}\sigma\left\{\left(\frac{\partial x}{\partial\sigma}\cdot\frac{\partial x}{\partial\tau}\right)^2 - \left(\frac{\partial x}{\partial\sigma}\right)^2\left(\frac{\partial x}{\partial\tau}\right)^2\right\}^{\frac{1}{2}}$$
$$\equiv \int_{\tau_\mathrm{i}}^{\tau_\mathrm{f}}\mathrm{d}\tau\int_0^\pi \mathrm{d}\sigma\; L\,. \tag{7}$$

This is the simplest action one can write down which is intrinsic to the surface, and the choice is closely analogous to choosing the action of a structureless point particle proportional to the length of its world line. The constant α' has dimensions $(\text{energy})^{-2}$ and ensures that S has the dimensions of action. Our theory thus has a fundamental length, which, as we shall see measures the slope of Regge trajectories:

$$\frac{\mathrm{d}J}{\mathrm{d}E^2} = \alpha'\hbar\,.$$

* Our metric is $-g^{00} = +g^{ii} = 1$.

We shall henceforth choose our units so that $\alpha' = \hbar = c = 1$, so that all physical quantities will be dimensionless. At any point, of course, we can restore these factors and convert to conventional units.

The equations of motion for the string follow from the principle of least action: we require that S be stationary under any small variation of the surface that joins the initial and final configurations of the string ($\dot{x} \equiv \partial x/\partial t$, $x' \equiv \partial x/\partial \sigma$):

$$\delta S = \int_{\tau_\mathrm{i}}^{\tau_\mathrm{f}} \mathrm{d}\tau \int_0^\pi \mathrm{d}\sigma \left\{ -\frac{\partial}{\partial \tau}\frac{\partial L}{\partial \dot{x}_\mu} - \frac{\partial}{\partial \sigma}\frac{\partial L}{\partial x'_\mu} \right\} \delta x_\mu$$

$$+ \int_{\tau_\mathrm{i}}^{\tau_\mathrm{f}} \mathrm{d}\tau \left(\delta x_\mu \frac{\partial L}{\partial x'_\mu} \right) \Bigg|_{\sigma=0}^{\sigma=\pi} + \int_0^\pi \mathrm{d}\sigma \left(\delta x_\mu \frac{\partial L}{\partial \dot{x}_\mu} \right) \Bigg|_{\tau=\tau_\mathrm{i}}^{\tau=\tau_\mathrm{f}} = 0 \tag{8}$$

for all $\delta x^\mu(\sigma, \tau)$ such that $\delta x^\mu(\sigma, \tau_\mathrm{i}) = \delta x^\mu(\sigma, \tau_\mathrm{f}) = 0$.

Thus we find the equations of motion *

$$\frac{\partial}{\partial \tau}\frac{\partial L}{\partial \dot{x}^\mu} + \frac{\partial}{\partial \sigma}\frac{\partial L}{\partial x'^\mu} = 0 , \tag{9a}$$

and the boundary condition

$$\frac{\partial L}{\partial x'^\mu}(0, \tau) = \frac{\partial L}{\partial x'^\mu}(\pi, \tau) = 0 . \tag{9b}$$

These equations have a very simple physical interpretation. In some particular Lorentz frame take the parameter τ to be the time t, so that the motion is described by $\boldsymbol{x} = \boldsymbol{x}(t, \sigma)$, $\boldsymbol{x}$ a three-vector. Let $\mathrm{d}s = |\partial \boldsymbol{x}/\partial \sigma| \mathrm{d}\sigma$ be the element of length along the string and $\boldsymbol{v}_\perp = \partial \boldsymbol{x}/\partial t - (\partial \boldsymbol{x}/\partial s)[(\partial \boldsymbol{x}/\partial t) \cdot (\partial \boldsymbol{x}/\partial s)]$ the velocity of the string perpendicular to itself. Then (7) becomes

$$S = \int_{t_\mathrm{i}}^{t_\mathrm{f}} L \,\mathrm{d}t ,$$

$$L = -T_\mathrm{o} \int_{\sigma_1}^{\sigma_2} \mathrm{d}\sigma \frac{\mathrm{d}s}{\mathrm{d}\sigma} \left(1 - \frac{v_\perp^2}{c^2}\right)^{\frac{1}{2}} , \tag{7'}$$

where $T_\mathrm{o} = 1/(2\pi\alpha'\hbar c)$ has the dimensions of force. If we simplify further by choosing σ so that $(\partial \boldsymbol{x}/\partial t) \cdot (\partial \boldsymbol{x}/\partial \sigma) = 0$ i.e., so that the path of a point of constant σ is always perpendicular to the string, then (9) becomes

* If our surface were space-like, eq. (9a) would be the equation for a surface of minimal area, and if $x^\mathrm{o} = t$, the minimal Euclidian area. In the Euclidian case, however, no solution exists unless the boundary is closed: there is no solution statisfying (9b). In our case the eqs. (9b) can be satisfied, and require the ends of the string to move with the velocity of light.

$$\frac{\partial}{\partial t}\left[\frac{T_0}{c^2}\frac{v_\perp}{(1-v_\perp^2/c^2)^{\frac{1}{2}}}\frac{ds}{d\sigma}\right]=\frac{\partial}{\partial\sigma}\left[T_0\left(1-\frac{v_\perp^2}{c^2}\right)^{\frac{1}{2}}\frac{\partial x}{\partial s}\right]. \tag{9a$'$}$$

The obvious interpretation is that we have a string with zero rest mass and rest tension T_0. The mass per unit length is $(T_0/c^2)(1-v_\perp^2/c^2)^{-\frac{1}{2}}$ and the effective tension is $T=T_0(1-v_\perp^2/c^2)^{\frac{1}{2}}$. Only the transverse motion is significant. There is no conservation of material of the string; stretching it makes it more massive. The original interpretation in terms of the area of the world sheet shows that the theory is covariant.

The boundary conditions become

$$(v_\perp)_1=(v_\perp)_2=c\,,\quad \frac{d\sigma_1}{dt}=\frac{d\sigma_2}{dt}=0\,, \tag{9b$'$}$$

so that a free end moves with the speed of light at right angles to the string. This is necessary to make $T=0$ at the ends. It is now simple to verify the obvious expressions for momentum, energy and angular momentum of the string, but we do this using the general covariant expressions in the next paragraph. It is also easy to verify that a simple set of possible motions of the string are rigid rotations of a straight string of length $2a$ with angular velocity ω, where $\omega a=c$. In such a motion the total energy $E\approx a$ and the angular momentum $J\approx a^2$ so that $E^2\approx J$. In fact $E=\pi T_0 a$, $J=(\pi T_0/2c)a^2$ so that $J/\hbar=(\frac{1}{2}\pi\hbar c T_0)E^2=\alpha' E^2$. We shall see that this corresponds to the leading Regge trajectory of the dual resonance model. A tension of 13 tons gives a slope of 1 GeV^{-2}.

Returning to the general formulation, we note that because the Lagrangian density L is invariant under the Poincaré group (since it is a function of Lorentz scalars formed out of the *derivatives* of x^μ), there will be locally conserved quantities associated with infinitesimal transformations in this group. By performing an infinitesimal *translation* we see that the energy momentum flowing across an infinitesimal line element is just

$$dP^\mu=\frac{\partial L}{\partial\dot{x}_\mu}\,d\sigma+\frac{\partial L}{\partial x'_\mu}\,d\tau\,.$$

Thus on the surface we can define an energy momentum current P_i^μ $(i=\tau,\sigma)$ with components

$$P_\tau^\mu=\frac{\partial L}{\partial\dot{x}_\mu}\equiv P^\mu\,,\qquad P_\sigma^\mu=\frac{\partial L}{\partial x'_\mu}\,. \tag{10}$$

Then eq. (9a) expresses the local conservation of energy momentum along the surface, and eq. (9b) expresses the fact that no energy momentum flows from the ends of the string. In particular, the total momentum,

$$P^\mu = \int_C (\mathrm{d}\sigma \, \mathcal{P}^\mu_\tau + \mathrm{d}\tau \, \mathcal{P}^\mu_\sigma) = \int_0^\pi \mathrm{d}\sigma \, \mathcal{P}^\mu_\tau \,, \tag{11}$$

where C is any curve from the boundary line $x^\mu(0, \tau)$ to the boundary line $x^\mu(\pi, \tau)$, is independent of the choice of C and represents the total energy momentum of the string. In a similar way we can perform infinitesimal *Lorentz transformations* and see that the angular momentum flowing across an infinitesimal line element is

$$\mathrm{d}M^{\mu\nu} = (x^\mu \mathcal{P}^\nu_\tau - x^\nu \mathcal{P}^\mu_\tau) \, \mathrm{d}\sigma + (x^\mu \mathcal{P}^\nu_\sigma - x^\nu \mathcal{P}^\mu_\sigma) \, \mathrm{d}\tau \,. \tag{12}$$

So we can define an angular momentum current $\mathcal{M}^{\mu\nu}{}_i \equiv x^\mu \mathcal{P}^\nu_i - x^\nu \mathcal{P}^\mu_i$ which is locally conserved

$$\frac{\partial \mathcal{M}^{\mu\nu}_\tau}{\partial \tau} + \frac{\partial \mathcal{M}^{\mu\nu}{}_\sigma}{\partial \sigma} = 0 \,,$$

$$\mathcal{M}^{\mu\nu}{}_\sigma = 0 \,, \text{ at } \sigma = 0, \pi \,. \tag{13}$$

Eqs. (13) are also a direct consequence of eqs. (9). The (conserved) total angular momentum of the string is, of course,

$$M^{\mu\nu} = \int_0^\pi \mathrm{d}\sigma \, \mathcal{M}^{\mu\nu}{}_\tau \,. \tag{14}$$

Because our action is independent of the way we parametrize the surface, eq. (9a) is form invariant under an arbitrary non-singular reparametrization of the surface and (9b) is form invariant under those reparametrizations for which the lines $x(\tilde{\sigma} = 0, \tilde{\tau}), x(\tilde{\sigma} = \pi, \tilde{\tau})$ coincide with the lines $x(\sigma = 0, \tau)$, $x(\sigma = \pi, \tau)$. This means that if the functions

$$x^\mu(\sigma, \tau)$$

satisfy (9), then so do the functions

$$x^\mu(\tilde{\sigma}(\sigma, \tau), \tilde{\tau}(\sigma, \tau)) \,,$$

provided only $\tilde{\sigma}(0, \tau) = 0$, $\tilde{\sigma}(\pi, \tau) = \pi$. This invariance reflects the fact that these two functions define the same surface and should therefore be regarded as equivalent. There is a continuous infinity of functions which satisfy eq. (9) given the initial and final configurations. We must therefore specify the parametrization in some way before we can solve them.

In fixing our parametrization we would also like to do it in such a way as to simplify eqs. (9). It is a well-known result from the theory of Euclidian surfaces [10] that one can linearize these equations by choosing an orthonormal parametrization

$$\frac{\partial \boldsymbol{x}}{\partial \sigma} \cdot \frac{\partial \boldsymbol{x}}{\partial \tau} = 0 \,, \quad \left(\frac{\partial \boldsymbol{x}}{\partial \sigma}\right)^2 - \left(\frac{\partial \boldsymbol{x}}{\partial \tau}\right)^2 = 0 \,,$$

which can be done for any Euclidean surface. Since our surface is assumed to be time-like, the second of these equations is impossible. However, we can always choose a parametrization so that

$$\frac{\partial x}{\partial\sigma}\cdot\frac{\partial x}{\partial\tau}=0\,,\quad \left(\frac{\partial x}{\partial\sigma}\right)^2+\left(\frac{\partial x}{\partial\tau}\right)^2=0\,. \tag{15}$$

With this parametrization, eq. (9a) becomes simply the one-dimensional wave equation

$$\left(\frac{\partial^2}{\partial\tau^2}-\frac{\partial^2}{\partial\sigma^2}\right)x^\mu(\sigma,\tau)=0\,.$$

We could thus impose eqs. (15) as additional constraint equations merely on geometrical grounds. However, we prefer to proceed from a more physical point of view and fix a parametrization according to the arguments that follow. The orthonormality conditions, eqs. (15) will still result, together with a further constraint.

In our problem we would like to identify τ with some time coordinate, so we start by setting

$$n\cdot x=2(n\cdot P)\,\tau\,, \tag{16}$$

where P is the total momentum of the string and n is a constant vector, such that $n^2\leqslant 0$. We do not specify $n=(1,0,0,0)$ because we want to leave some arbitrariness in the choice of frame and also because, as we shall see, when $n^2=0$, we get special simplifications. The component of momentum along n, $(n\cdot P)$, is inserted because it simplifies some further equations, but this is not fundamental. If we want to identify σ with some physical quantity, and still keep it varying from 0 to π, we must relate it to a conserved quantity. This suggests that we take σ proportional to the relative momentum along n included between the boundary and the point considered, i.e.,

$$(n\cdot P)\,\sigma=\pi\int\limits_{\substack{0\\ \text{fixed }\tau}}^{\sigma}\mathrm{d}\sigma\, n\cdot \mathcal{P}_\tau\,. \tag{17}$$

This equation amounts to saying that $n\cdot\mathcal{P}_\tau$ is constant along σ and since its integral is conserved, it must also be independent of τ, i.e.,

$$n\cdot\mathcal{P}_\tau=\frac{n\cdot P}{\pi}\,. \tag{18}$$

Then the equations of motion (9a), (9b) become

$$\frac{\partial}{\partial\sigma}\,n\cdot\mathcal{P}_\sigma=0\,,\qquad \mathcal{P}_\sigma=0\,,\ \text{at}\ \sigma=0,\pi\,,$$

so that $n\cdot\mathcal{P}_\sigma=0$. On the other hand,

$$n \cdot P_\sigma \propto \frac{\dot{x} \cdot x'}{\{(\dot{x} \cdot x')^2 - \dot{x}^2 x'^2\}^{\frac{1}{2}}}$$

so that $\dot{x} \cdot x' = 0$. Then (18) implies $\dot{x}^2 + x'^2 = 0$, so we find the ortho-normality conditions (15).

Summarizing, our system of equations has become

$$\frac{\partial x}{\partial \sigma} \cdot \frac{\partial x}{\partial \tau} = 0 \,, \quad \left(\frac{\partial x}{\partial \sigma}\right)^2 + \left(\frac{\partial x}{\partial \tau}\right)^2 = 0 \,, \tag{19a}$$

$$\left(\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \sigma^2}\right) x^\mu = 0 \,, \tag{19b}$$

$$\frac{\partial x^\mu}{\partial \sigma} = 0 \,, \quad \text{at } \sigma = 0, \tau \,, \tag{19c}$$

$$n \cdot x = 2(n \cdot P)\,\tau \,. \tag{19d}$$

Taking advantage of (19a) we can evaluate the momentum and angular momentum currents

$$P^\mu_\tau = \frac{1}{2\pi} \frac{\partial x^\mu}{\partial \tau} \,, \quad P^\mu_\sigma = -\frac{1}{2\pi} \frac{\partial x^\mu}{\partial \sigma} \,,$$

$$M^{\mu\nu}_\tau = \frac{1}{2\pi}\left(x^\mu \frac{\partial x^\nu}{\partial \tau} - x^\nu \frac{\partial x^\mu}{\partial \tau}\right), \quad M^{\mu\nu}_\sigma = -\frac{1}{2\pi}\left(x^\mu \frac{\partial x^\nu}{\partial \sigma} - x^\nu \frac{\partial x^\mu}{\partial \sigma}\right).$$

Eq. (18) is therefore a consequence of (19d) and (19a).

By specifying our parameter τ as in (19d) we have given up manifest Lorentz covariance since we have singled out a particular direction n^μ. The quantity $n \cdot x/n \cdot P$ plays the role of a parameter rather than a dynamical variable. If we only require eqs. (19a) – (c), which are manifestly covariant, then the parametrization is not fixed: one can still perform a reparametrization,

$$\sigma = \sigma(\tilde{\sigma}, \tilde{\tau}) \,, \quad \tau = \tau(\tilde{\sigma}, \tilde{\tau}) \,,$$

which preserves (19a), provided that

$$\frac{\partial \sigma}{\partial \tilde{\sigma}} = \frac{\partial \tau}{\partial \tilde{\tau}} \,, \quad \frac{\partial \sigma}{\partial \tilde{\tau}} = \frac{\partial \tau}{\partial \tilde{\sigma}} \,, \quad \frac{\partial \sigma}{\partial \tilde{\tau}} = 0 \,, \text{ at } \tilde{\sigma} = 0, \pi \,.$$

These last two equations determine $\sigma(\tilde{\sigma}, \tilde{\tau})$ (up to an additive constant) from $\tau(\tilde{\sigma}, \tilde{\tau})$ and further require that $\tau(\tilde{\sigma}, \tilde{\tau})$ satisfies

$$\left(\frac{\partial^2}{\partial \tilde{\tau}^2} - \frac{\partial^2}{\partial \tilde{\sigma}^2}\right) \tau(\tilde{\sigma}, \tilde{\tau}) = 0 \,, \quad \frac{\partial \tau}{\partial \tilde{\sigma}} = 0 \,, \text{ at } \tilde{\sigma} = 0, \pi \,. \tag{20}$$

In terms of the new parameters only (19a) – (c) are true; but $\tilde{\tau}$ is no longer proportional to a time variable. Conversely, if we had initiallly required only (19a) then we could always further specialize τ to be proportional to $n \cdot x$ since each component of x satisfies (20b). Because the special reparametrization must satisfy the wave equation and boundary conditions (20b), it is completely fixed once we specify two space-like non-intersecting equal-τ lines on our surface. In particular, once we specify the initial and final configurations to be equal-τ lines, then the parametrization of the surface has been completely specified by requiring orthonormality (19a).

At this point we can draw an analogy with electromagnetism. In this theory, specifying the Lorentz condition $\partial_\mu A^\mu = 0$ does not completely determine the potential, since the gauge transformation

$$A^\mu \to A^{\mathrm{M}} + \partial^\mu \chi ,$$

preserves $\partial_\mu A^\mu = 0$ if $\partial^2 \chi = 0$. However, once we specify the potentials initially and finally (e.g., $A^0 = 0$ at $t = \pm\infty$), then this last special gauge freedom is completely removed. In both cases, the manifestly covariant gauge conditions are sufficient to enable one to solve the equations of motion.

2.2. *Canonical formalism*

For our discussion of the quantum mechanics of the relativistic string we shall need expressions for the Poisson brackets of our dynamical variables. One could think of taking x^μ and $\mathcal{P}^\mu_\tau$ as canonically conjugate variables: however, these quantities are not independent. Using eq. (10) and the expression for L it is simple to check that

$$\frac{\partial x}{\partial \sigma} \cdot \mathcal{P}_\tau = 0 \,, \quad \mathcal{P}_\tau^2 + \frac{1}{(2\pi)^2} \left(\frac{\partial x}{\partial \sigma}\right)^2 = 0 \,, \tag{21}$$

so that our phase space is constrained. The presence of eqs. (21) is related to the arbitrariness in the choice of the parametrization.

In the presence of constraints, one can establish a canonical formalism either by computing all Poisson brackets before the constraints are applied (assuming in this case canonical Poisson brackets for all the components of x^μ and $\mathcal{P}^\mu$), and imposing *afterwards* the constraint equations onto the dynamical system [11], or by solving explicitly the constraints to eliminate some of the variables from the equations of motion.

The quantization procedures that follow these two treatments of the classical system are different, and we shall discuss them both. At the classical level we shall study in detail only the formalism based on the elimination of the redundant variables. (For the other formalism see ref. [11] and also appendix A.) Our procedure is to first specify completely the parametrization and then use eqs. (21) to express all of the dynamical variables in terms of a certain set of independent ones.

Although this method could be followed for any choice of parametrization, we find it particularly convenient to fix the parametrization according to eqs. (16) and (18), with n light-like. We introduce the notation $n_\pm = \sqrt{\tfrac{1}{2}}(n^0 \pm n^3)$ for every

Lorentz vector n^μ, and denote by $\boldsymbol{n}$ its residual space components. For definiteness, we take n of the form $1/\sqrt{2}(1, \mathbf{0}, -1)$; i.e., $n_- = 1, n_+ = \boldsymbol{n} = 0$. Eqs. (18) and (16) tell us that (the parametrization is such that) the density of momentum $\mathcal{P}^+$ does not depend on σ, so that it is proportional to the total momentum P_+:

$$\mathcal{P}_+ = \frac{P_+}{\pi}, \tag{22a}$$

and that x_+ is proportional to τ:

$$x_+ = 2P_+\tau . \tag{22b}$$

Then eqs. (21) give

$$\frac{P^+}{\pi} x'_- = \boldsymbol{x}' \cdot \boldsymbol{\mathcal{P}}, \tag{23a}$$

$$2\frac{P_+}{\pi} \mathcal{P}_- = \boldsymbol{\mathcal{P}}^2 + \frac{\boldsymbol{x}'^2}{(2\pi)^2} . \tag{23b}$$

We introduce the baricentric coordinate

$$q_-(\tau) = \frac{1}{\pi}\int_0^\pi \mathrm{d}\sigma\, x_-(\sigma, \tau) .$$

We can then solve eqs. (23) explicitly to obtain *

$$x_-(\sigma, \tau) = q_-(\tau) + \frac{\pi}{P^+}\int_0^\pi \mathrm{d}\sigma' \left(\frac{\sigma'}{\pi} - \theta(\sigma' - \sigma)\right) \boldsymbol{x}' \cdot \boldsymbol{\mathcal{P}} \tag{24a}$$

$$\mathcal{P}_-(\sigma, \tau) = \frac{\pi}{2P^+} \left(\boldsymbol{\mathcal{P}}^2 + \frac{\boldsymbol{x}'^2}{(2\pi)^2}\right) . \tag{24b}$$

We see that all dynamical variables can be expressed in terms of the transverse variables $\boldsymbol{x}$, $\boldsymbol{\mathcal{P}}$, and the additional two quantities P_+ and q_-.

We should establish Poisson brackets among these quantities so that the equations of motion follow in the Hamiltonian form

$$\dot{f} = \{f, H\} + \frac{\partial f}{\partial \tau} . \tag{25}$$

Notice that H should be given by

$$H = 2P_+P_- = \pi\int_0^\pi \mathrm{d}\sigma \left(\boldsymbol{\mathcal{P}}^2 + \frac{\boldsymbol{x}'^2}{(2\pi)^2}\right) , \tag{26}$$

* $q_-(t)$ must be introduced into eq. (24a) because eq. (23a) contains only $\partial x^-/\partial\sigma$.

since P_- is the generator of infinitesimal translations in the x_+ direction. The equations of motion are the following ones:

$$\dot{P} = \frac{x''}{2\pi}, \tag{27a}$$

$$\dot{x} = 2\pi P, \tag{27b}$$

$$P_+ = \text{constant}, \tag{27c}$$

$$\dot{q}_- = 2P_- = \frac{H}{P_+}. \tag{27d}$$

These equations, together with eqs. (25) and (26), demand that

$$\{x^i, x^j\} = \{P^i, P^j\} = 0, \quad \{x^i(\sigma), P^j(\sigma')\} = \delta^{ij}\delta(\sigma - \sigma'), \tag{28a}$$

and that

$$q_- = q_{0-} + \frac{H}{P_+}\tau,$$

where q_{0-} is a constant of the motion. We can then assume a canonical Poisson bracket between q_{0-} and P_+.

$$\{q_{0-}, P_+\} = -1. \tag{28b}$$

This last ansatz is consistent with the fact that P_+ generates displacements in the minus direction, and, together with $\{P_+, x\} = \{P_+, P\} = \{q_{0-}, x\} = \{q_{0-}, P\} = 0$, completes the specification of the Poisson brackets among the independent dynamical variables.

The Poisson brackets of the x_- variables among themselves and with other variables can be computed from (24) and (28). It is more instructive, however, to expand all of the variables into normal modes and derive the Poisson brackets for these. By virtue of the wave equation and boundary conditions we can expand $x^\mu(\sigma, \tau)$ as follows:

$$x^\mu(\sigma, \tau) = q_0^\mu + \sqrt{2}\,[a_0^\mu \tau + i \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{a_n^\mu}{n} \cos n\sigma\, e^{-in\tau}]\ . \tag{29}$$

Then q_0^μ, a_n^μ will all be constants of the motion. Reality of x^μ implies $q_0 = q_0^*$, $a_0 = a_0^*$, $a_n^* = a_{-n}$. Eqs. (22) and (24) give the following constraints on the normal modes:

$$q_0^+ = a_n^+ = 0, \quad n \neq 0, \quad a_0^+ = \sqrt{2}\,P_+, \tag{30}$$

$$a_n^- = \frac{1}{a_0^+} L_n, \quad \text{where} \quad L_n = \frac{1}{2}\sum_{k=-\infty}^{\infty} a_{-k}\cdot a_{n+k}\ ; \tag{31}$$

a_n^i, q_0^i, a_0^+ and q_0^- are independent variables. The Poisson brackets among these variables follow from (28) and (29):

$$\{a_n^i, a_m^j\} = -in\,\delta_{n,-m}\delta^{ij}, \{q_0^i, a_0^j\} = \sqrt{2}\,\delta^{ij}, \quad \{q_0^i, q_0^j\} = 0, \quad \{q_0^-, a_0^-\} = -\sqrt{2} \tag{32}$$

From these the algebra of the dependent modes follows:

$$\{L_n, L_m\} = -i(n-m)\,L_{n+m}, \quad \{L_n, a_m^j\} = im\,a_{m+n}^j, \quad \{g_0^j, L_m\} = \sqrt{2}\,a_m^i \tag{33}$$

In terms of the normal modes the Hamiltonian is

$$H = 2P_+P_- = L_0 = p^2 + \sum_{n=1}^{\infty} \boldsymbol{a}_n \cdot \boldsymbol{a}_n^* \,, \tag{34}$$

and the invariant (mass)^2 is

$$M^2 = 2P_+P_- - P^2 = \sum_{n=1}^{\infty} |\boldsymbol{a}_n|^2 \,. \tag{35}$$

We can also work out the total momentum and angular momentum:

$$P^\mu = \frac{1}{\sqrt{2}}\,a_0^\mu \,, \tag{36}$$

$$M^{\mu\nu} = \frac{1}{\sqrt{2}}(q_0^\mu a_0^\nu - q_0^\nu a_0^\mu) + i \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{a_n^\mu a_{-n}^\nu}{n} \,. \tag{37}$$

If we consider the case of the string rotating about the 3 axis in the nth mode ($a_n^2 = i\,a_n^1$), then the total angular momentum and invariant mass are:

$$J = M^{12} = i\left(\frac{a_n^1 a_{-n}^2 - a_{-n}^1 a_n^2}{n}\right) = \frac{2}{n}|a_n^2|^2 = \frac{|\boldsymbol{a}_n|^2}{n} \,,$$

$$M = |\boldsymbol{a}_n| \,,$$

so that

$$J = \frac{M^2}{n} \,. \tag{38}$$

We have a leading linear Regge trajectory ($n = 1$) with slope 1 in our units. If we choose conventional units the slope is just α', of course.

2.3. Lorentz covariance properties

Having developed a canonical formalism for the motion of the string in light-cone variables we could proceed immediately to the quantization of the system through the correspondence principle:

$$i\,\{\text{Poisson bracket}\} \to [\text{commutator}] \,.$$

However, it is useful to first analyze the covariance properties of our dynamical variables under Lorentz transformations. This is non-trivial since we have given up manifest Lorentz covariance through eqs. (22). These equations will no longer hold in the frame of a different observer whose x_+ coordinate will be

$$\widetilde{x}_+ = \Lambda_{+\nu} x^\nu \,, \tag{39}$$

where Λ is the Lorentz transformation between his frame and the original one; $\widetilde{x}_+$ will in general depend on both τ and σ, and will not satisfy the transformed eq. (22b)

$$\widetilde{x}_+ \neq 2\widetilde{P}_+ \tau \,.$$

Nevertheless we can always change the parametrization by $\widetilde{\tau} = \widetilde{\tau}(\sigma, \tau)$, $\widetilde{\sigma} = \widetilde{\sigma}(\sigma, \tau)$ so that, in terms of the new variables

$$\widetilde{x}_+(\widetilde{\sigma}, \widetilde{\tau}) = 2\widetilde{P}_+ \widetilde{\tau} \,. \tag{40}$$

The possibility of redefining τ so that (40) is true depends on the fact that, as a function of σ and τ, $\widetilde{x}_+ = \Lambda_+^\nu x_\nu$ satisfies the wave equation as well as the boundary condition $\partial \widetilde{x}_+ / \partial \sigma = 0$. We shall study this procedure in detail for an infinitesimal Lorentz transformation:

$$\widetilde{x}^\mu(\sigma, \tau) = x^\mu(\sigma, \tau) + \epsilon \bar{M}^{\mu\nu} x_\nu(\sigma, \tau) \,, \quad \widetilde{P}^\mu = P^\mu + \epsilon \bar{M}^{\mu\nu} P_\nu \,. \tag{41}$$

Eq. (40) demands

$$\widetilde{\tau}(\sigma, \tau) = \tau + \frac{\epsilon}{2P_+} \bar{M}^+_\nu x^\nu(\sigma, \tau) - \epsilon \tau \bar{M}^+_\nu P^\nu \,. \tag{42}$$

Then $\widetilde{\sigma}$ must be computed from $\partial\widetilde{\sigma}/\partial\tau = \partial\widetilde{\tau}/\partial\sigma$, $\partial\widetilde{\sigma}/\partial\sigma = \partial\widetilde{\tau}/\partial\tau$. We obtain

$$\widetilde{\sigma} = \sigma + \frac{\epsilon}{2P_+} \int_0^\sigma \mathrm{d}\sigma' \, \bar{M}^+_\nu \dot{x}^\nu(\sigma', \tau) - \epsilon \sigma \bar{M}^+_\nu P^\nu \,. \tag{43}$$

The new observer will therefore describe the system through the functions

$$\widetilde{x}_\mu(\sigma(\widetilde{\sigma}, \widetilde{\tau}), \tau(\widetilde{\sigma}, \widetilde{\tau})) \,,$$

so that the difference in descriptions will be given by:

$$\begin{aligned} \delta x_\mu &= \widetilde{x}_\mu(\sigma(\widetilde{\sigma}, \widetilde{\tau}), \tau(\widetilde{\sigma}, \widetilde{\tau})) - x_\mu(\widetilde{\sigma}, \widetilde{\tau}) \\ &= \epsilon \, [\bar{M}^\mu_\nu x^\nu(\sigma, \tau) - \frac{\bar{M}^+_\nu}{2P_+} (x'^\mu \{ \int_0^\sigma \mathrm{d}\sigma' \dot{x}^\nu - 2\sigma P^\nu \} + \dot{x}^\mu \{ x^\nu - 2\tau P^\nu \})] \,, \end{aligned} \tag{44}$$

In terms of the normal modes we have:

$$\delta a^\mu_m = \epsilon \left[\bar{M}^\mu_\nu a^\nu_m - \frac{\bar{M}^+_\nu}{a^+_0} m \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(\frac{a^\mu_{m-n} a^\nu_n}{n} - \frac{i}{\sqrt{2}} q^\nu_0 a^\mu_m \right) \right] \,.$$

$$\delta q_0^\mu = \epsilon \; \bar{M}^\mu_{\ \nu} q_0^\nu - i \frac{\bar{M}^+_{\ \nu}}{a_0^+} \left(\sqrt{2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{a^\mu_{-n} a^\nu_n}{n} - i q_0^\nu a_0^\mu \right) ,$$

$$\delta a_0^\mu = \epsilon \bar{M}^\mu_{\ \nu} a_0^\nu \; . \tag{45}$$

We see that the descriptions of the evolution of the string in two different frames are not related by a mere Lorentz transformation; the Lorentz transformation induced by the change of frame is followed by reparametrization, by which the new observer adjusts the description to his own light-cone parameters. Of course, eqs. (44) provide a representation of the Lorentz group (a *non-linear* and *non-trivial* one) within the space of functions $x(\sigma, \tau), \dot{x}(\sigma, \tau)$ and the numbers q_0^- and a_0^+.

If our prescription for Poisson brackets is consistent, then the transformations (44) or (45) should be generated by taking the Poisson bracket with the total angular momentum (37). We shall show in appendix B that when the expressions (30), (31) for q_0^+, a_n^+, q_0^-, a_n^- are substituted into the expression for the angular momentum (37), then the expression $M = -\frac{1}{2} \bar{M}^{\mu\nu} M_{\mu\nu}$ generates the transformation (44), i.e.,

$$\delta x^\mu(\sigma, \tau) = \epsilon \; \{M, x^\mu(\sigma, \tau)\} \; . \tag{46}$$

It then follows that $M^{\mu\nu}$ obeys the Lorentz algebra:

$$\{M^{\mu\nu}, M^{\rho\sigma}\} = (g^{\mu\rho} M^{\nu\sigma} - g^{\nu\rho} M^{\mu\sigma} - g^{\mu\sigma} M^{\nu\rho} + g^{\nu\sigma} M^{\mu\rho}) \; . \tag{47}$$

3. Quantum mechanics of the light string

3.1. General considerations

In constructing the quantum theory of the string we rely on the correspondence principle. That is, we regard the dynamical variables as operators whose equal-τ commutators are obtained by the rule

$$i \; \{\text{Poisson bracket}\} \rightarrow [\text{commutator}] \; . \tag{48}$$

The presence of constraints among the classical quantities $x^\mu(\sigma, \tau)$ and $\mathcal{P}^\mu(\sigma, \tau)$ requires some care in the construction of the quantum theory. We have seen that, already at the classical level, we can either eliminate some of the variables from the equations of motion, and impose canonical Poisson brackets on the remaining ones, or introduce canonical Poisson brackets for all x^μ and $\mathcal{P}^\mu$, but impose the constraints in the weak sense of Dirac, namely, after all the Poisson brackets have been computed.

Correspondingly, at the quantum level we can assume canonical commutation relations for the transverse operators x_i and $\mathcal{P}_i$, plus the additional operators q_{0-} and P_+, if we choose to eliminate the redundant variables. The other operators x_-, $\mathcal{P}_-$, x_+ and $\mathcal{P}_+$ will be expressed in terms of these. In particular, according to the choice of parametrization, x_+ will be given by the total momentum in the + direction times a c-number.

Alternatively, we can quantize covariantly

$$[x^\mu, P^\nu] = i g^{\mu\nu}\delta(\sigma - \sigma'), \tag{49}$$

and impose eqs. (21) only for matrix elements between *physical* states:

$$\langle \psi_1 | \frac{\partial x}{\partial \sigma} \cdot P | \psi_2 \rangle = \langle \psi_1 | P^2 + \frac{x'^2}{(2\pi)^2} | \psi_2 \rangle = 0 . \tag{50}$$

This last procedure is analogous to the Gupta-Bleuler prescription for imposing the Lorentz gauge in quantum electrodynamics, whereas the other one resembles the method by which electrodynamics was first quantized.

Of course, in the fully covariant method of quantization it is difficult to ascribe a meaning to the time variable along the string $x_0(\sigma, \tau)$, or to the equivalent light-cone variable $x_+(\sigma, \tau)$ which become operators. In our opinion, this difficulty has been the main obstacle in visualizing the quantum string introduced in ref. [1] as imbedded in ordinary space-time. The difficulty is not present if we quantize only the independent variables.

However, as we shall see, it is possible, under certain conditions, to eliminate x_+ as a dynamical variable also in the fully covariant formalism, by requiring that $x_+ = 2P_+\tau$ as a weak equation, i.e.,

$$\langle \psi_1 | (x_+ - 2P_+\tau) | \psi_2 \rangle = 0 , \tag{51}$$

when $|\psi_1\rangle$ and $|\psi_2\rangle$ are physical states.

We shall describe first the non-covariant quantization, and then discuss the other possibility. In both cases, the transition from the classical to the quantum theory involves ambiguities relative to the order of operators in the definition of some quantum variables. We shall show how these ambiguities are resolved, in the non-covariant formalism, by the requirement that the quantum theory be in fact Lorentz covariant; whereas they are resolved by the requirement that eq. (51) be compatible, in the covariant quantization procedure.

3.2. Non-covariant quantization

We have seen in subsect. 2.2. that we can consider as independent dynamical variables $\boldsymbol{x}, \boldsymbol{P}, q_0^-$ and P_+, and express all others in terms of these. We then use the canonical quantization procedure on these independent dynamical variables:

$$[x^i(\sigma), P^j(\sigma')] = i\,\delta^{ij}\delta(\sigma - \sigma'), \qquad [x^i(\sigma), x^j(\sigma')] = [P^i(\sigma), P^j(\sigma')] = 0 ,$$

$$[q_0^-, P^+] = -i, \qquad [q_0^-, x^i] = [q_0^-, P^i] = [P^+, x^i] = [P^+, P^i] = 0 . \tag{52a}$$

It will be much clearer to work with normal modes so that we do not have to worry about continuous indices. In terms of these the quantum conditions are:

$$[a^i_n, a^j_m] = n\,\delta_{n,-m}\delta^{ij}, \qquad [q^i_0, a^j_0] = -i\sqrt{2}\,\delta^{ij}\,, \quad [q^i_0, q^j_0] = 0\,,$$

$$[q^-_0, a^+_0] = -i\sqrt{2}\,, \quad [q^-_0, a^i_n] = [q^-_0, q^i_0] = [a^+_0, a^i_n] = [a^+_0, q^i_0] = 0\,. \tag{52b}$$

(We shall regard a_n $n > 0$ as annihilation operators; $a^+_n = a_{-n}$ are then creation operators for $n > 0$.) The dependent operators $a^\pm_n$, q^+_0 are given in terms of these by eqs. (30) and (31), but now we have an ambiguity in ordering the operators. In fact, the only place this ambiguity occurs is in the operator a^-_0, since this one involves products of operators which do not commute. For the moment we shall leave this ambiguity open and write

$$a^-_0 = \frac{1}{a^+_0}\,[\,L_0 - \alpha_0]\,, \tag{53}$$

where by L_0 we mean the *normal ordered expression.* The physical meaning of α_0 can be seen from the quantum mechanical analogue of (35) which becomes

$$M^2 = L_0 - \alpha_0 - P^2 = \sum_{n=1}^{\infty} a^+_n \cdot a_n - \alpha_0\,, \tag{54}$$

so that $(-\alpha_0)$ is just the $(\text{mass})^2$ of the ground state. We denote by $|0, k\rangle$ the ground state vector with momentum k. The first excited state $a^{+i}_1|0, k\rangle$ has $M^2 = 1 - \alpha_0$. On the other hand it has only transverse degrees of freedom, so if the theory is to be Lorentz covariant this state must be massless, i.e., $\alpha_0 = 1$. We shall see this result again in our detailed study of the Lorentz covariance of our theory, to which we now turn.

In writing down the angular momentum operators, which generate Lorentz transformations, we must again be careful about how we order the operators which occur in products. For a start we symmetrize the classical expression in x and P to ensure that $M^{\mu\nu}$ is a Hermitan operator, so we write:

$$M^{\mu\nu} = \frac{1}{2}\int_0^\pi d\sigma\,(x^\mu P^\nu + P^\nu x^\mu - x^\nu P^\mu - P^\mu x^\nu)\,. \tag{55}$$

Working these out in normal modes separately for the various choices of μ and ν, we have:

$$M^{ij} = \frac{1}{\sqrt{2}}\,(q^i_0 a^j_0 - q^j_0 a^i_0) - i\sum_{n=1}^{\infty}\frac{a^i_{-n}a^j_n - a^j_{-n}a^i_n}{n}\,, \tag{56a}$$

$$M^{i-} = -M^{-i} = \frac{1}{2\sqrt{2}}\left[q^i_0\frac{1}{a^+_0}(L_0-\alpha_0) + \frac{1}{a^+_0}(L_0-\alpha_0)q^i_0\right] - \frac{1}{\sqrt{2}}q^-_0 a^i_0 \tag{56b}$$

$$- \frac{i}{\alpha^+_0}\sum_{n=1}^{\infty}\frac{a^i_{-n}L_n - L_{-n}a^i_n}{n}\,,$$

$$M^{i+} = -M^{+i} = \frac{1}{\sqrt{2}} q_0^i a_0^+ , \tag{56c}$$

$$M^{+-} = -M^{-+} = -\frac{1}{2\sqrt{2}}(q_0^- a_0^+ + a_0^+ q_0^-) . \tag{56d}$$

It is not hard to check, using the known algebra of the operators occurring in (56), that a change of the ordering of any of the operators leads either to no change in the expression or to an expression which is no longer Hermitian, so that hermiticity alone fixes the ordering of the operators.

If we consider for example the state with only nth mode excitations and definite components of angular momentum in the 3 direction:

$$|\psi\rangle = (a_n^{+1} + i\, a_n^{+2})^K |0\rangle ,$$

then

$$J_3 |\psi_n\rangle = M^{12} |\psi_n\rangle = \tfrac{1}{2}\,(a_n^{+1} + i\, a_n^{+2})(a_n^1 - i\, a_n^2)|\psi_n\rangle$$

$$= \frac{M^2 + \alpha_0}{n} |\psi_n\rangle = \left(K + \frac{\alpha_0}{n}\right) |\psi_n\rangle .$$

We see that there is a leading Regge trajectory ($n = 1$) given by

$$\alpha(s) = s + \alpha_0 .$$

We still must prove that the expressions (56) form a representation of the Lorentz group. We have already seen there is no hope unless $\alpha_0 = 1$ but let us compute the algebra of the expressions (56) first with α_0 arbitrary. The algebra of M^{ij} among themselves is all right since they involve no non-canonical expressions. However, the expression for M^{i-} involves the non-canonical operators L_n which have an algebra similar but not quite the same as the classical variables:

$$[L_n, L_m] = (n-m)\, L_{n+m} + \tfrac{1}{12}(D-2)\,\delta_{n,-m}(n^3 - n) , \tag{57a}$$

$$[L_n, a_m^j] = -m\, a_{m+n}^j , \tag{57b}$$

$$[q_0^i, L_n] = i\sqrt{2}\, a_n^i , \tag{57c}$$

where D is the dimension of space time. The appearance of the second term in (57a) is a purely quantum mechanical effect which is due to the fact that the operator L_0 is in normal ordered form and on commuting say L_n with L_{-n} one gets terms which are not normally ordered. The term is proportional to $D-2$, which is the total number of transverse operators of a given mode, because each component clearly contributes the same amount.

In appendix B we shall compute $[M^{i-}, M^{j-}]$ which should be zero if our theory is Lorentz covariant. In general it is non-zero, in fact:

$$[M^{i-}, M^{j-}] = -\frac{1}{4a_0^{+2}} \sum_{m=1}^{\infty} [m(1 - \tfrac{1}{24}(D-2)) + \frac{1}{m}(\tfrac{1}{24}(D-2) - \alpha_0)]\,(a_{-m}^i a_m^j - a_{-m}^j a_m^i) . \tag{58}$$

For arbitrary values of D and α_0, then, the theory is not covariant. For the particular values

$$\alpha_0 = 1 ,$$
$$D = 26 , \tag{59}$$

and only for these values does $[M^{i-}, M^{j-}] = 0$. None of the other commutators among the expressions (56) give trouble as can be easily seen by direct computation. Thus the quantum mechanics given by the correspondence principle is consistent only when the intercept of the leading trajectory and the dimension of space time are quantized according to (59). A particular consequence of (59) is that the ground state is a tachyon, $M^2 = -1$.

All these results have been obtained in ref. [8] where we studied the Lorentz covariance of physical states in the dual resonance model. In that paper we constructed the generators of O(25), (the little group of the total momentum of the state), on the transverse physical states in 26 space-time dimensions constructed by Del Giudice, Di Vecchia and Fubini [7] which, in fact, coincide with what one would get from the expressions for these generators constructed from (56). We found, from essentially the same algebra used here, that the construction only worked for $D = 26$. However, the form the generators took and their action on the transverse states were mysterious and with the present work we see how they come out naturally in terms of a particular parametrization of the relativistic string. We also have constructed the remaining generators of O(25,1) which we had not done in our previous work.

3.3. The covariant quantization

The covariant quantization is performed by assuming

$$[x^\mu(\sigma), P^\nu(\sigma')]_{\text{equal}\,\tau} = i\, g^{\mu\nu}\delta\,(\sigma - \sigma') ,$$

and imposing eq. (50) on the physical states.

Resolving eqs. (49) and (50) into normal modes, we see that they are equivalent to

$$[a_n^\mu, a_m^\nu] = g^{\mu\nu}\delta_{n,-m} , \quad [q_0^\mu, a_0^\nu] = i\sqrt{2}\, g^{\mu\nu} , \tag{60}$$

$$\langle \psi_1 | L_N | \psi_2 \rangle = 0 , \qquad N \neq 0 , \tag{61a}$$

$$\langle \psi_1 | L_0 | \psi_2 \rangle = \alpha_0 \langle \psi_1 | \psi_2 \rangle , \tag{61b}$$

where

$$L_N = \frac{1}{2} \sum_{l=-\infty}^{\infty} : a_{-l} \cdot a_{l+N} : . \tag{62}$$

The operators L_N satisfy the algebra

$$[L_N, L_M] = (N-M)\,L_{N+M} + \tfrac{1}{12} D\,\delta_{N,-M}(N^3 - N)\,, \tag{63a}$$

$$[L_N, a_n^\mu] = -n\,a_{n+N}^\mu\,, \tag{63b}$$

$$[q_0^\mu, L_N] = i\sqrt{2}\,a_N^\mu\,. \tag{63c}$$

We therefore impose the subsidiary conditions *

$$L_N|\psi\rangle = 0\,, \qquad N > 0\,, \tag{64a}$$

$$L_0|\psi\rangle = \alpha_0|\psi\rangle\,. \tag{64b}$$

Because of the indefinite metric in (60) we are not sure that the solutions of eq. (64) contain no negative norm state. The solutions of these equations have been studied by Brower [5] and two of us [6]. The results are that for $\alpha_0 = 1$ and $1 \leqslant D \leqslant 26$ and for $\alpha_0 < 1$ and $1 \leqslant D \leqslant 25$ there are no negative norm solutions to (64a) and (64b). For the dual resonance model in which the physical states satisfy eq. (64) with $\alpha_0 = 1$ these results have the significance that there are no negative width resonances (ghosts) coupling to physical states when $1 \leqslant D \leqslant 26$.

So, if we only require the positiveness of the physical subspace we can relax the conditions $D = 26$, $\alpha_0 = 1$. The price we pay, as can be seen from refs. [5, 6] is that the number of independent degrees of freedom in the quantum system is larger than in the classical system. The quantum system no longer reflects that last degree of freedom allowed after imposing (19a) in the classical system. An analogy from QED is helpful here. In the covariant quantization the Lorentz gauge condition becomes in momentum space (on shell $k^2 = 0$)

$$k \cdot a(k)|\psi\rangle = 0\,.$$

Now, this equation has three solutions:

$$k \cdot a^+|0\rangle\,, \qquad a_\perp^+|0\rangle\,,$$

where $a_\perp^+$ are the transverse components. However, $k \cdot a^+|0\rangle$ has zero norm:

$$\langle 0|a \cdot k\, a^+ \cdot k|0\rangle \propto k^2 = 0\,,$$

and decouples from the theory. The existence of such zero norm states reflects an arbitrariness in the definition of physical states, namely the state

* The operator L_0 which appears in eq. (64b) can be considered the Hamiltonian of the system, so that, in the Heisenberg picture,

$$i\,\dot{x}^\mu = [x_\mu, L_0]\,.$$

This identification specifies a particular parametrization, and corresponds to the constraints (19a) in the classical formulation. The fact that the choice of a parametrization (of a gauge for the string) is equivalent to the specification of a particular form for the Hamiltonian within a class of equivalent ones is illustrated in ref. [11] and, for our case, also in appendix A.

$$|\psi'\rangle = |\psi\rangle + \lambda\, k \cdot a^{+}|0\rangle\,,$$

where $|\psi\rangle$ is physical is also physical and further has identical couplings to $|\psi\rangle$. Effectively the only physical states which couple in the theory are the finite norm ones and there are only two such states so the photon is really transverse.

In our theory a similar phenomenon may happen. It may be that there are zero norm solutions to (64). If there are then the states of physical interest, the finite normed states, are not uniquely determined. It is this ambiguity which reflects the further freedom of parametrization allowed after (19a) are imposed. We may then formulate the possibility of imposing (19d) as follows: we should always be able to choose the finite normed solutions of (64) in such a way that

$$\langle\overline{\psi}_1^{\mathrm{F}}|n\cdot x|\overline{\psi}_2^{\mathrm{F}}\rangle = 2\,\langle\overline{\psi}_1^{\mathrm{F}}|n\cdot P|\overline{\psi}_2^{\mathrm{F}}\rangle\,\tau\,, \tag{65}$$

for any light-like vector $n(n^2 = 0)$. In terms of normal modes, this means

$$\langle\overline{\psi}_1^{\mathrm{F}}|n\cdot a_p|\overline{\psi}_2^{\mathrm{F}}\rangle = 0\,, \qquad l\neq 0\,. \tag{66}$$

In other words $|\overline{\psi}^{\mathrm{F}}\rangle$ must satisfy

$$n\cdot a_l|\overline{\psi}^{\mathrm{F}}\rangle = 0\,, \qquad l>0\,, \tag{67}$$

in addition to

$$L_l|\overline{\psi}^{\mathrm{F}}\rangle = 0\,, \qquad l>0\,.$$

In ref. [6] it has been shown that all the solutions of (67) are precisely the states constructed by DDF for $n = k$. The results of refs. [5, 6] show that these states span the finite norm subspace of the solutions of (64) if and only if $D = 26$ and $\alpha_0 = 1$.

For $D < 26$ and $\alpha_0 = 1$ effectively only the modes a_1 are transverse and all of the others have three independent components. For $D < 25$ and $\alpha_0 < 1$ all of the modes effectively have three independent components. There are some analogies of the more general solutions of (64) and the Higgs mechanism for ordinary gauge theories. By means of the Higgs mechanism one is able to give a mass to the photon in such a way as to preserve enough of the gauge invariance of the theory to ensure the absence of ghosts. Of course, in the process one must also supply a third degree of freedom because a massive vector particle cannot be transverse. Similarly, as one departs from $D = 26$, $\alpha_0 = 1$, maintaining (64a), one must introduce into the theory extra degrees of freedom in a well-defined way. Of course, at present we only have an S-matrix for the case $\alpha_0 = 1$, the conventional dual resonance model. The possibility remains open of constructing an S-matrix for different values of α_0.

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Appendix A

In this appendix we would like to describe a method for treating the variational problem in terms of the phase space variables x^μ, p^μ, rather than in terms of x^μ, $\dot{x}^\mu$ *. The action can be written simply

$$S = \int \mathrm{d}\sigma\, \mathrm{d}\tau\, \dot{x} \cdot P . \tag{A.1}$$

However, we cannot vary x and p independently because of the phase space constraints (21). We, therefore, introduce two Lagrange multipliers λ_1, λ_2, define

$$S' = \int \mathrm{d}\sigma\, \mathrm{d}\tau\, [\dot{x} \cdot P - \tfrac{1}{2}\lambda_1 \left(P^2 + \frac{x'^2}{(2\pi)^2}\right) - \tfrac{1}{2}\lambda_2\, x' \cdot P] . \tag{A.2}$$

and require S' to be stationary under independent variations of x, p, λ_1 and λ_2.

Varying λ_1 and λ_2 we get the constraints

$$P^2 + \frac{x'^2}{(2\pi)^2} = 0 , \tag{A.3}$$

$$x' \cdot P = 0 . \tag{A.4}$$

Varying P we obtain

$$\dot{x}^\mu - \lambda_1 P^\mu - \tfrac{1}{2}\lambda_2 x'^\mu = 0 . \tag{A.5}$$

Finally varying x, we obtain

$$\dot{P} - \frac{\partial}{\partial\sigma}\left(\frac{\lambda_1}{(2\pi)^2}\, x'\right) - \frac{\partial}{\partial\sigma}\left(\tfrac{1}{2}\lambda_2 P\right) = 0 , \tag{A.6}$$

$$\frac{\lambda_1}{(2\pi)^2}\, x' + \tfrac{1}{2}\lambda_2 P = 0 , \quad \text{at the ends;}$$

λ_1 and λ_2 are determined by substituting P as determined from (A.5) into (A.3) and (A.4). On doing this, one finds (A.5) and (A.6) become

$$P = \frac{\partial L(\dot{x}, x')}{\partial \dot{x}} ,$$

$$\frac{\partial}{\partial\tau}\frac{\partial L}{\partial \dot{x}} + \frac{\partial}{\partial\sigma}\frac{\partial L}{\partial x'} = 0 ,$$

$$\frac{\partial L}{\partial x'} = 0 \qquad \text{at the ends,}$$

in agreement with (9a) and (9b).

* We thank K. Johnson for an instructive conversation on this formulation of the variational problem.

One virtue of formulating the variational problem this way is that we can impose our particular parametrization choice (22) *before* finding the stationary points of S'. Substituting, we find

$$S' = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \int_0^{\pi} \mathrm{d}\sigma [2(P^+ + \dot{P}^+\tau)\, P_- + \dot{x}_- \frac{P_+}{\pi} - \tfrac{1}{2}\lambda_1 \left(\frac{2P^+}{\pi} P^- - P^2 - \frac{x'^2}{(2\pi)^2}\right)$$

$$- \dot{x}\cdot P - \tfrac{1}{2}\lambda_2 (x'_- \frac{P^+}{\pi} - x'\cdot P)] \,. \tag{A.7}$$

Requiring that S' be stationary under arbitrary small variation in $P^+(\tau)$, $x^-(\sigma,\tau)$, $P^-(\sigma, \tau)$, $x(\sigma, \tau)$, $P(\sigma, \tau)$, λ_1 and λ_2, it is straightforward algebra to verify the equations

$$\dot{P}^+ = 0\,, \qquad \dot{x} = 2\pi\, P, \qquad \dot{x}_- = 2\pi\, P^-\,,$$

$$\frac{P_+}{\pi} x'_- = x'\cdot P, \qquad \frac{2P_+}{\pi} P^- = P^2 + \frac{x'^2}{(2\pi)^2}\,,$$

$$\ddot{x} - x'' = 0\,, \qquad x' = 0\,, \quad \text{at} \quad \sigma = 0, \pi\,,$$

in agreement with our previous results (23) and (25).

Going over to the Hamiltonian formalism from (A.2) we find a Hamiltonian

$$H' = \int \mathrm{d}\sigma \left[\tfrac{1}{2}\lambda_1 \left(P^2 + \frac{x'^2}{(2\pi)^2}\right) + \tfrac{1}{2}\lambda_2 x'\cdot P\right] \tag{A.8}$$

where λ_1 and λ_2 are arbitrary. Following Dirac [11] one then takes *canocical* Poisson brackets

$$\{x^{\mu}(\sigma),\, P^{\nu}(\sigma')\,\} = g^{\mu\nu}\delta(\sigma - \sigma')\,, \tag{A.9}$$

and imposes the constraints

$$P^2 + \frac{x'^2}{(2\pi)^2} = 0\,, \qquad x'\cdot P = 0\,, \tag{A.10}$$

only *after* finding Hamilton's equations of motion

$$\dot{x}^{\mu} = \{x^{\mu}, H'\}\,, \qquad \dot{P}^{\mu} = \{P^{\mu}, H'\}\,.$$

The equations of motion depend, of course, on the arbitrary functions λ_1 and λ_2 reflecting the fact that there are many functions $x^{\mu}(\sigma, \tau)$ representing the same state of motion of the system. If one eliminates λ_1 and λ_2 using (A.10) one returns to the equations of motion (9a).

Dirac's quantization prescription is somewhat different from the one we described in the text, eqs. (44) and (50). He would impose the canonical commutation relations (40) and would require on physical states the *phase-space constraints* (A.10)

$$\langle\psi_1|\left(P^2+\frac{x'^2}{(2\pi)^2}\right)|\psi_2\rangle=\alpha_0\langle\psi_1|\psi_2\rangle\,,\qquad \langle\psi_1|x'\cdot P|\psi_2\rangle=0\,.$$

His equations of motion would still involve the arbitrary quantities λ_1 and λ_2. By choosing the Hamiltonian to be L_0 we have restricted ourselves to those particular solutions for which $\lambda_2 = 0$ and $\lambda_1 = 1$. These two restrictions enforce the orthonormality conditions (15) in the weak sense of Dirac.

Appendix B

In this appendix we shall verify some equations quoted in the text. We turn first to eq. (46). It is easiest to verify that the normal modes transform correctly (45). The demonstration is straightforward and we shall illustrate the procedure by considering the case where only $\bar{M}^{i+}$ is non-zero. Then

$$M=\bar{M}^{i+}M^{i-}\;; \tag{B.1}$$

M^{i-} is given by eq. (37):

$$M^{i-}=\frac{1}{\sqrt{2}}\left(q_0^i\frac{1}{a_0^+}L_0-q_0-a_0^i\right)+\frac{i}{a_0^+}\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{a_n^iL_{-n}}{n}\;. \tag{B.2}$$

Computing:

$$\{M^{i-},a_m^j\}=\delta^{ij}\delta_{m,0}\frac{1}{a_0^+}L_0+\frac{im}{\sqrt{2}}q_0^i\frac{1}{a_0^+}a_m^j+\frac{i}{a_0^+}\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{ima_n^ia_{m-n}^j-in\delta^{ij}\delta_{n-m}L_{-n}}{n}$$

$$=\delta^{ij}\frac{L_m}{a_0^+}-\frac{1}{a_0^+}\sum_{n=-\infty}^{\infty}\frac{m}{n}a_n^i\,a_{m-n}^j+\frac{im}{\sqrt{2}}q_0^i\frac{1}{a_0^+}a_m^j\,.$$

From (45)

$$\delta a_m^j=\epsilon\left[\bar{M}^{i+}a_m^- -\frac{\bar{M}^{i+}}{a_0^+}m\left(\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{a_{m-n}^ja_n^i}{n}-i\frac{q_0^i}{\sqrt{2}}a_m^j\right)\right]\,,$$

i.e.,

$$\delta a_m^j=\epsilon\,\{M,a_m^j\}\,. \tag{B.3}$$

Again

$$\{M^{i-},q_0^i\}=-q_0^i\frac{a_0^i}{a_0^+}+\delta^{ij}q_0^- -\frac{i\sqrt{2}}{a_0^+}\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{a_n^i\,a_{-n}^j}{n}\,,$$

whereas

$$\delta q_0^j = \epsilon\ \bar{M}^{j+} q_0^- - i \frac{\bar{M}^{i+}}{a_0^+} \left(\sqrt{2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{a_{-n}^j a_n^i}{n} - i q_0^i a_0^j \right) ,$$

so we have

$$\delta q_0^i = \epsilon\ \{M, q_0^j\ \} . \tag{B.4}$$

From (B.3) and (B.4) it follows that

$$\delta\, x^j(\sigma, \tau) = \epsilon\ \{M, x^j(\sigma, \tau)\} . \tag{B.5}$$

$$\{M^{i-},\ L_m\} = a_m^i \frac{1}{a_0^+} L_0 + \frac{i\,m}{\sqrt{2}} q_0^i \frac{1}{a_0^+} L_m$$

$$+ \frac{i}{a_0^+} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{i\,(n+m)\ L_{m-n} a_n^i - i\,n\,a_{n+m}^i L_{-n}}{n}$$

$$= \frac{i\,m}{\sqrt{2}} q_0^i \frac{1}{a_0^+} L_m + \frac{1}{a_0^+} a_0^i\ L_m - \frac{m}{a_0^+} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{L_{m-n} a_n^i}{n} ,$$

$$\{M^{i-}, \frac{1}{a_0^+}\} = -\frac{a_0^i}{a_0^{+2}} , \tag{B.6}$$

so

$$\therefore \{M^{i-}, a_m^-\ \} = \frac{i\,m}{\sqrt{2}} \frac{q_0^i}{a_0^+} a_m^- - \frac{m}{a_0^+} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{a_{m-n}^- a_n^i}{n} ,$$

whereas

$$\delta\, a_m^- = \epsilon \left[-\frac{\bar{M}^{i+}}{a_0^+}\ m \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{a_{m-n}^- a_n^i}{n} - \frac{i}{\sqrt{2}} q_0^i a_m^- \right) \right] ,$$

so again

$$\delta\, a_m^- = \epsilon\ \{M, a_m^-\} . \tag{B.7}$$

Finally, we compute

$$\{M^{i-}, q_0^-\} = -q_0^i \frac{1}{a_0^{+2}} L_0 - \frac{i\sqrt{2}}{a_0^{+2}} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{a_n^i\ L_{-n}}{n} ,$$

as compared to

$$\delta q_0^- = -\frac{i}{a_0^+}\bar{M}^{i+}\Bigg(\sqrt{2}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{a_n^- a_n^i}{n} - i q_0^i a_0^-\Bigg),$$

which verifies

$$\delta q_0^- = \epsilon\ \{M, q_0^-\}. \tag{B.8}$$

(B.8) and (B.7) imply

$$\delta x^-(\sigma,\tau) = \epsilon\ \{M, x^-(\sigma,\tau)\}. \tag{B.9}$$

Finally that

$$\delta x^+(\sigma,\tau) = \epsilon\ \{M, x^+(\sigma,\tau)\}, \tag{B.10}$$

is readily verified. Thus, we have proved (46) when only $M^{i+} \neq 0$. The other cases are in fact much easier, so we leave it for the reader to complete the proof of (46).

Finally, we must prove eq. (58). The quantum mechanical expression for M^{i-} is given in (56b). The proof is rather tedious and most of the algebra has been done in our previous paper [8]. The main difficulty comes in working out the commutator.

$$\left[\sum_{n=1}^{\infty}\frac{a_{-n}^i L_n - L_{-n}a_n^i}{n}, \sum_{m=1}^{\infty}\frac{a_{-m}^j L_m - L_{-m}a_m^j}{m}\right]. \tag{B.11}$$

If we remove the zero modes from L_m writing

$$L_n = \tilde{L}_n + a_0\cdot a_n, \qquad n\neq 0, \qquad L_0 = \tilde{L}_0 + \tfrac{1}{2}a_0^2.$$

Then in ref. [8] we worked out the commutator

$$\left[\sum_{n=1}^{\infty}\frac{a_{-n}^i \tilde{L}_n - \tilde{L}_{-n}a_n^i}{n}, \sum_{m=1}^{\infty}\frac{a_{-m}^j \tilde{L}_m - \tilde{L}_{-m}a_m^j}{m}\right]$$

$$= -2(\tilde{L}_0 - \tfrac{1}{24}(D-2))\sum_{n=1}^{\infty}\frac{a_{-n}^i a_n^j - a_{-n}^j a_n^i}{n} + 2(\tfrac{1}{24}(D-2)-1)\sum_{n=1}^{\infty} n(a_{-n}^j a_n^i - a_{-n}^i a_n^j) \tag{B.12}$$

We shall not prove this formula again, but only remark that the terms with factors $\frac{1}{24}(D-2)$ and $\frac{1}{24}(D-2)-1$ come from the fact that when one performs the commutator the result has terms which would cancel each other except for the ordering of factors. One then uses the commutation relations to arrange the factors in the same order so they cancel leaving only the contributions from the commutators. The calculation is tedious because one must separate the positive modes from the negative modes as in (B.12) to make sure we keep factors normal ordered throughout. Otherwise one can easily add and subtract infinities and get the wrong result. Using (B.12) it is then only straightforward algebra to verify eq. (58).

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