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A saddle-point solution in the Weinberg-Salam theory

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We give a close approximation to a static, but unstable, solution of the classical field equations of the Weinberg-Salam theory, where the weak mixing angle Θ_w is considered to be small. Its energy increases from ~ 8 TeV to ~ 14 TeV as the Higgs coupling λ runs from 0 to ∞. Furthermore, it has a large magnetic dipole moment and its baryonic (and leptonic) charge is $\frac{1}{2}$. The possible physical relevance of this solution is discussed.

I. INTRODUCTION

Recently it was shown that the field configuration space of the classical Weinberg-Salam theory[1] without fermions has noncontractible loops passing through the vacuum configuration.[2] This makes it likely that there is a static classical solution of the field equations, which is a saddle point of the energy functional, and therefore unstable. The solution would be the maximal-energy configuration on some noncontractible loop, and all other loops homotopic to this one would pass through configurations of equal or greater energy. It would therefore be at the top of the energy barrier for going from the vacuum to the vacuum along a topologically nontrivial path.

One believes that there is such a saddle-point solution by analogy with the work of Taubes,[3] who has shown rigorously that in a slightly different context, namely the zero-monopole sector of the SO(3) gauge theory with an adjoint Higgs field and vanishing Higgs potential, one can apply Morse-theory arguments to relate topological information about the space of field configurations to the existence of stationary points of the energy functional. Forgács and Horváth[4] have reviewed a number of other field theories, in one, two, and three spatial dimensions, where explicitly known saddle-point solutions are related to the topology of the field configuration space.

We have coined the word "sphaleron"[5] to describe any classical solution of this type in a relativistic field theory. A sphaleron, being static and localized in space, is particlelike, but since it is unstable, we do not want to call it a soliton. Unlike a soliton, a sphaleron almost certainly does not correspond to a stable particle state in the quantum theory.

We are still unable to make rigorous the topological argument for the existence of a sphaleron in the Weinberg-Salam theory, where the gauge group is $SU(2)\times U(1)$ and the Higgs field is a complex doublet. However, we have found a more direct approach which appears to lead to such a solution, and can now estimate quite accurately some of its properties. The crucial point is that in the limit that the weak mixing angle Θ_w vanishes, the U(1) field decouples and may consistently be set to zero in the field equations. The Weinberg-Salam theory then reduces to an SU(2) theory with a doublet Higgs field. Here, as pointed out in Ref. 4, there really is a sphaleron. It is the solution found numerically by Dashen, Hasslacher, and Neveu (DHN),[6] and rediscovered in the context of a nuclear physics model by Boguta.[7] The solution has finite energy, and the energy density is localized and spherically symmetric. The fields, strictly speaking, are only axially symmetric. Burzlaff[8] proved recently that this solution rigorously exists, and also proved that it is unstable, by presenting a one-parameter family of field configurations, among which it is the configuration of maximal energy. To make contact with the topological approach discussed above, we shall demonstrate that there is a noncontractible loop in the configuration space passing through the vacuum and the DHN sphaleron, on which the sphaleron is the configuration of maximal energy. This loop is essentially the one discussed in Ref. 2.

Unfortunately, neither Refs. 6 nor 7 presented their solutions in much detail; for example, neither gave the value of the energy. We have not attempted to recalculate the DHN solution, but have rather used variational *Ansätze* to find approximations for the two radial functions which occur. Unlike in Refs. 6 and 7, we have done this over the whole range of values of the quartic Higgs coupling λ, which determines the mass of the physical scalar particle. These *Ansätze* give upper bounds on the energy of the sphaleron. The experimental values of the parameters[9] that occur in the Weinberg-Salam theory make our best upper bound approximately 8 TeV if λ is zero, increasing to approximately 14 TeV if λ is infinite. Subsequently, we have reconsidered the numerical solution, which was not quite straightforward, and have obtained an estimate of the sphaleron energy of 7.6 TeV (for $\lambda=0$), which is more reliable than the value obtained from our best *Ansatz*. However, it is only 3% lower, so for nonzero λ it is likely that the energies obtained from the *Ansatz* are only a few percent too high.

What interested us most was to use the DHN sphaleron (or rather, our approximation to it) to find the approximate form of the sphaleron in the Weinberg-Salam theory when $\Theta_w\neq 0$.[10] It was shown in Ref. 2 that the presence of the U(1) field makes it impossible for the solution to remain spherically symmetric in any sense. It would, therefore, be quite hard to find the true solution, even with numerical methods. However, the field configuration space is independent of the value of Θ_w, and the energy functional changes smoothly with Θ_w. Since the solution is an isolated saddle point of the energy functional when $\Theta_w=0$ (provided one quotients out the translational and rotational degrees of freedom and fixes the gauge), presumably it cannot just disappear as Θ_w increases from zero. Instead, it too should change smoothly. This argument suggests that there will continue to be a solution for small but nonvanishing values of Θ_w, and that a perturbation expansion in Θ_w can be used. It will remain a saddle point because, by continuity, the eigenvalue of the unstable mode will remain imaginary at least for small-enough Θ_w.

It turns out to be quite easy to find the changes in the sphaleron's properties to leading order in Θ_w. To first order, the SU(2) gauge field and Higgs field are unchanged, but they produce a U(1) current density which is axially symmetric and which acts as a source for the U(1) gauge field. The axis of symmetry can be changed by a rotation, of course. The $O(\Theta_w)$ axially symmetric U(1) field has a back reaction on the other fields and produces $O(\Theta_w{}^2)$ changes in them, still preserving the axial symmetry. Such changes are hard to compute and we will neglect them. The asymptotic form of the sphaleron's U(1) field is that of a dipole, whose strength is proportional to Θ_w. To leading order in Θ_w the U(1) field can be identified asymptotically with the electromagnetic field. Therefore, the sphaleron has a magnetic dipole moment. The energy of the sphaleron decreases quadratically with Θ_w, but only by about 1% even for Θ_w as large as the experimental value of 0.50 rad.

The outline of this article is as follows. In Sec. II we review the classical Weinberg-Salam theory and discuss the general form of the sphaleron, both when $\Theta_w=0$ and for small, but nonzero, values of Θ_w. In Sec. III we present *Ansätze* for the radial functions that occur in the $\Theta_w=0$ sphaleron and our estimates for its energy. We also verify that this sphaleron is the maximal-energy configuration on a noncontractible loop. In Sec. IV we calculate to leading order in Θ_w the effects of the U(1) field, from which we deduce the sphaleron's magnetic moment and the shift in its energy relative to the $\Theta_w=0$ value. In Sec. V, we argue that the sphaleron has a baryon number and a lepton number of $\frac{1}{2}$. This is a consequence of the anomalies in the fermionic currents of the Weinberg-Salam theory, whose significance was first discussed by 't Hooft.[11] Section VI contains a discussion of our results on the sphaleron, together with some remarks on its possible relevance for particle physics.

II. GENERAL PROPERTIES OF THE SPHALERON

We consider static classical fields in the Weinberg-Salam theory. Both the fermion fields and the time component of the gauge fields are set to zero. The energy functional is[12]

$$E=\int\left[\tfrac{1}{4}F_{ij}^aF_{ij}^a+\tfrac{1}{4}f_{ij}f_{ij}+(D_i\varphi)^\dagger(D_i\varphi)+V(\varphi)\right]d^3x\ , \tag{1}$$

where

$$F_{ij}^a=\partial_i W_j^a-\partial_j W_i^a+g\epsilon^{abc}W_i^bW_j^c\ , \tag{2a}$$

$$f_{ij}=\partial_i a_j-\partial_j a_i\ , \tag{2b}$$

$$D_i\varphi=\partial_i\varphi-\tfrac{1}{2}ig\sigma^aW_i^a\varphi-\tfrac{1}{2}ig'a_i\varphi\ , \tag{2c}$$

and

$$V(\varphi)=\lambda(\varphi^\dagger\varphi-\tfrac{1}{2}v^2)^2\ . \tag{2d}$$

The weak mixing angle is given by $\tan\Theta_w=g'/g$. The semiclassical masses of the W boson and Higgs boson are, respectively,

$$M_W=\tfrac{1}{2}gv,\quad M_H=\sqrt{2\lambda}v\ . \tag{3}$$

The field equations are

$$(D_jF_{ij})^a=-\tfrac{1}{2}ig[\varphi^\dagger\sigma^aD_i\varphi-(D_i\varphi)^\dagger\sigma^a\varphi]\ , \tag{4a}$$

$$\partial_jf_{ij}=-\tfrac{1}{2}ig'[\varphi^\dagger D_i\varphi-(D_i\varphi)^\dagger\varphi]\ , \tag{4b}$$

and

$$D_iD_i\varphi=2\lambda(\varphi^\dagger\varphi-\tfrac{1}{2}v^2)\varphi\ , \tag{4c}$$

where

$$(D_iF_{ij})^a=\partial_jF_{ij}^a+g\epsilon^{abc}W_j^bF_{ij}^c\ . \tag{5}$$

When $g'=0$, the U(1) gauge potential a_i decouples and may be consistently set to zero. Then the field equations reduce to (4a) and (4c), where now

$$D_i\varphi=\partial_i\varphi-\tfrac{1}{2}ig\sigma^aW_i^a\varphi\ . \tag{6}$$

A solution to these SU(2) equations is possible with fields of the form

$$W_i^a\sigma^adx^i=-\frac{2i}{g}f(gvr)dU^\infty(U^\infty)^{-1}\ , \tag{7a}$$

$$\varphi=\frac{v}{\sqrt{2}}h(gvr)U^\infty\begin{bmatrix}0\\1\end{bmatrix}\ , \tag{7b}$$

where

$$U^\infty=\frac{1}{r}\begin{bmatrix}z & x+iy\\ -x+iy & z\end{bmatrix}\ . \tag{7c}$$

A physically equivalent *Ansatz* is obtained by replacing U^∞ by

$$(U^\infty)'=U_LU^\infty U_R\ , \tag{8}$$

U_L and U_R being constant SU(2) matrices. The effect of U_L on the field is simply that of a rigid gauge transformation. On the other hand, U_R has the effect of a physical rotation. W_i^a is unchanged by U_R and is spherically symmetric; however, φ changes and is not. Perhaps surprisingly, some gauge-invariant quantities, including the energy density, are spherically symmetric for the *Ansatz* (7). This can be understood by considering these fields as fields of an SO(4) gauge theory.[2] In this larger context, the spherical symmetry is manifest.

The DHN *Ansatz*[6] is not identical to Eq. (7), but is related to it by a transformation of the type (8), with

$$U_L=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad U_R=\begin{bmatrix}0 & -i\\ -i & 0\end{bmatrix}. \tag{9}$$

This transformation will be used in Sec. V.

It is convenient now to introduce the dimensionless radial distance $\xi=gvr$. The radial functions f and h which appear in (7) are functions of ξ, and the use of ξ simplifies many of the expressions for physical quantities that we shall consider.

For the fields (7), the energy functional becomes

$$E=\frac{4\pi v}{g}\int_0^\infty \left[4\left[\frac{df}{d\xi}\right]^2+\frac{8}{\xi^2}[f(1-f)]^2+\tfrac{1}{2}\xi^2\left[\frac{dh}{d\xi}\right]^2 +[h(1-f)]^2+\frac{1}{4}\left[\frac{\lambda}{g^2}\right]\xi^2(h^2-1)^2\right]d\xi\,. \tag{10}$$

The field equations reduce to

$$\begin{gathered}\xi^2\frac{d^2f}{d\xi^2}=2f(1-f)(1-2f)-\frac{\xi^2}{4}h^2(1-f)\,,\\ \frac{d}{d\xi}\left[\xi^2\frac{dh}{d\xi}\right]=2h(1-f)^2+\frac{\lambda}{g^2}\xi^2(h^2-1)h\,.\end{gathered} \tag{11}$$

The boundary conditions on the functions f and h are the following. Near $\xi=0$, $f=\alpha\xi^2$ and $h=\beta\xi$. As $\xi\to\infty$, $f=1-\gamma\exp(-\frac{1}{2}\xi)$ and $h=1-(\delta/\xi)\exp(-\sqrt{2\lambda/g^2}\xi)$. α, β, γ, and δ are constants of order unity which can only be determined by finding the complete solution.

The solution of Eqs. (11) minimizes the energy functional (10); the instability of the solution involves field fluctuations outside the class allowed by the *Ansatz* (7). Further, by regarding the solution for one value of λ as an *Ansatz* for a lower value of λ, it is clear that the energy of the solution increases with λ.

Let us consider next the case of nonzero Θ_w. Now it is inconsistent to suppose a_i is zero because the U(1) current

$$j_i=-\tfrac{1}{2}ig'[\varphi^\dagger D_i\varphi-(D_i\varphi)^\dagger\varphi] \tag{12}$$

is generally nonvanishing and acts as a source for a_i. To find the approximate sphaleron solution when g'/g, and hence Θ_w, is small, we suppose W_i^a and φ are unchanged from their $\Theta_w=0$ values, and we compute this current and then the U(1) field it induces through Eq. (4b). Compared with W_i^a, the magnitude of a_i is smaller by a factor Θ_w. One could compute the changes in W_i^a and φ that are induced by the nonzero a_i by linearizing Eqs. (4a) and (4c) about the background $\Theta_w=0$ solution. Such changes are of order ${\Theta_w}^2$ compared with the background values, and we have not attempted to find them.

The change in the sphaleron energy is of order ${\Theta_w}^2$ compared with the energy value when $\Theta_w=0$. This energy shift comes entirely from the terms involving a_i and is

$$\Delta E=\int(\tfrac{1}{4}f_{ij}f_{ij}-a_ij_i)d^3x\,. \tag{13}$$

When Eq. (4b) is satisfied

$$\Delta E=-\int \tfrac{1}{4}f_{ij}f_{ij}d^3x \tag{14a}$$

$$=-\int \tfrac{1}{2}a_ij_id^3x\,, \tag{14b}$$

so we see that ΔE is negative.

With W_i^a and φ given by (7), and a_i vanishing, the current j_i has the form

$$\vec{j}=\tfrac{1}{2}g'v^2\frac{h^2(gvr)[1-f(gvr)]}{r^2}(-y,x,0)\,. \tag{15}$$

Clearly this is axially symmetric about the z axis, but not spherically symmetric. Since the current only involves $L=1$ spherical harmonics, it is simple to find the induced U(1) field, which is a pure dipole field outside the region where the current is concentrated. We parametrize a_i as

$$\vec{a}=\tfrac{1}{2}g'v^2p(gvr)(-y,x,0)\,. \tag{16}$$

This U(1) gauge potential is divergenceless, so Eq. (4b) is

$$-\nabla^2\vec{a}=\vec{j}\,. \tag{17}$$

When everything is expressed in terms of the dimensionless distance ξ, this reduces to

$$\xi^2\frac{d^2p}{d\xi^2}+4\xi\frac{dp}{d\xi}=-h^2(1-f)\,. \tag{18}$$

p must satisfy the boundary conditions

$$\lim_{\xi\to0}\xi^3p(\xi)=0,\quad \lim_{\xi\to\infty}p(\xi)=0\,. \tag{19}$$

The solution of (18) which satisfies these boundary conditions is easily obtained. It is

$$p(\xi)=\frac{1}{3\xi^3}\int_0^\xi d\eta\,\eta^2h^2(\eta)[1-f(\eta)] +\int_\xi^\infty d\eta\frac{1}{3\eta}h^2(\eta)[1-f(\eta)]\,. \tag{20}$$

The asymptotic behavior of p determines the magnetic dipole moment of the sphaleron. Since a pure dipole of moment $\vec{\mu}$ has a field

$$\vec{a}=\frac{\vec{\mu}\times\vec{x}}{4\pi r^3}\,, \tag{21}$$

the dipole here has a moment $\vec{\mu}=(0,0,\mu)$, with strength

$$\mu=\frac{2\pi}{3}\frac{g'}{g^3v}\int_0^\infty \xi^2h^2(\xi)[1-f(\xi)]d\xi\,. \tag{22}$$

$\vec{\mu}$ is truly a magnetic moment for the following reason. In the unitary gauge, and to first order in Θ_w, the electromagnetic vector potential A_i equals $a_i+\Theta_wW_i{}^3$, using the usual mixing formula. But the electromagnetic field is only well defined far from the sphaleron, where the Higgs field has its vacuum value, and there $A_i\simeq a_i$. This is because $W_i{}^3$ is proportional to $1-f$ in the unitary gauge, as can be deduced from (7), so it vanishes rapidly asymptotically.

The energy shift ΔE may be similarly expressed, using our parametrization of the fields, as

$$\Delta E=-\frac{\pi}{3}\frac{g'^2v}{g^3}\int_0^\infty \xi^2h^2(\xi)[1-f(\xi)]p(\xi)d\xi\,. \tag{23}$$

In the following two sections we shall estimate the functions f and h and with them determine the magnetic moment and energy shift.

III. THE SU(2) SPHALERON

In this section we shall find the approximate form of the $\Theta_w=0$ sphaleron, and its energy, by a variational method. We have shown already that the true solution may be expressed in terms of two radial functions f and h, which minimize the energy functional (10). Here we use trial functions for f and h, which each depend on a free scale parameter and which have the correct behavior at $r=0$ and as $r\to\infty$. By minimizing the energy with respect to these parameters we obtain an approximation to the true solution and an upper bound on the energy of the sphaleron.

A particularly simple *Ansatz,* labeled a, for the radial functions is

$$f^a(\xi)=\begin{cases}\left[\dfrac{\xi}{\Xi}\right]^2, & \xi\le\Xi\\ 1, & \xi\ge\Xi\,,\end{cases}$$
$$h^a(\xi)=\begin{cases}\dfrac{\xi}{\Omega}, & \xi\le\Omega\\ 1, & \xi\ge\Omega\,.\end{cases} \tag{24}$$

Recall that ξ is the dimensionless radial distance gvr. The *Ansatz* (24) is similar to the one used in Ref. 2, but there Ξ and Ω were set equal. Its energy is given by

$$E^a=\frac{4\pi v}{g}\frac{1}{210}\left\{\frac{1248}{\Xi}+\left[112-105\left[\frac{\Omega}{\Xi}\right]+56\left[\frac{\Omega}{\Xi}\right]^3-12\left[\frac{\Omega}{\Xi}\right]^5\right]\Xi+4\left[\frac{\lambda}{g^2}\right]\Omega^3\right\}. \tag{25}$$

The scale parameters Ξ and Ω that minimize (25), together with the energy value, are presented in Table I for a few values of λ/g^2. Since the optimal Ξ and Ω are approximately equal when $\lambda=0$, we obtain essentially the same upper bound as in Ref. 2 on the sphaleron energy. As $\lambda\to\infty$, the optimal Ξ has a nonzero limiting value, whereas $\Omega\to 0$. The energy remains finite because the Higgs field takes it vacuum expectation value almost everywhere and, being an upper bound on the sphaleron energy, that remains finite too as $\lambda\to\infty$. Note that the 't Hooft-Polyakov monopole has similar behavior when the Higgs coupling is large.[13]

Ansatz a is useful for order-of-magnitude estimates of the sphaleron properties, but it treats the asymptotic behavior of the radial functions too roughly. A much more accurate approximation to the sphaleron is provided by the following *Ansatz,* labeled b:

$$f^b(\xi)=\begin{cases}\dfrac{\xi^2}{\Xi(\Xi+4)}, & \xi\le\Xi\\ 1-\dfrac{4}{\Xi+4}\exp[\tfrac{1}{2}(\Xi-\xi)], & \xi\ge\Xi\,,\end{cases}$$
$$h^b(\xi)=\begin{cases}\dfrac{\sigma\Omega+1}{\sigma\Omega+2}\dfrac{\xi}{\Omega}, & \xi\le\Omega\\ 1-\dfrac{\Omega}{\sigma\Omega+2}\dfrac{1}{\xi}\exp[\sigma(\Omega-\xi)], & \xi\ge\Omega\,.\end{cases} \tag{26}$$

The parameter σ equals $\sqrt{2\lambda/g^2}$. These functions have the correct behavior near $\xi=0$ and for $\xi\to\infty$. Also, at the crossover points $\xi=\Xi$ and $\xi=\Omega$, the functions and their first derivatives are continuous.

We have determined numerically, for several values of λ/g^2, the values of the scale parameters Ξ and Ω which minimize the energy. These are shown in Table II, together with the energy values. The corresponding radial functions for $\lambda=0$ and ∞ are shown in Figs. 1(a) and 1(b). For all values of λ/g^2, the energy is lower than that given by *Ansatz a,* so *Ansatz b* is certainly better. The energy increases with λ/g^2, as does the true energy of the sphaleron. The minimal energies obtained with our two *Ansätze a* and b are presented in Fig. 2.

The energy values presented in Tables I and II are in dimensionless units. The physical energy is obtained by multiplying by $4\pi v/g$. The experimental data[9] give a value for this quantity of 5.0 TeV. Our estimate of the sphaleron energy, for $\Theta_w=0$, using *Ansatz b,* therefore varies between 7.9 TeV for $\lambda=0$ and 13.7 TeV for $\lambda=\infty$.

The conversion factor between the dimensionless distance ξ and physical distances is $(gv)^{-1}$, whose experimental value is 1.23×10^{-3} fm. From Figs. 1(a) and 1(b) we see that the sphaleron core has a radius of approximately 10^{-2} fm.

It is interesting to compare the results of these variational calculations with the numerical solution[14] of Eqs. (11), for $\lambda=0$. The radial functions obtained numerically are shown together with those from *Ansatz b* in Fig. 1(a). The resulting sphaleron energy is

TABLE I. The minimal energy E and the optimal values of the scale parameters Ξ and Ω for the *Ansatz a.* The energy is given in units of $4\pi v/g$, which has a value of 5.0 TeV in the Weinberg-Salam theory.

λ/g^2	E	Ω	Ξ
0	2.40	4.83	4.98
1	2.88	1.90	3.79
∞	3.56	0	3.34

TABLE II. The minimal energy E and the optimal values of the scale parameters Ξ and Ω for the *Ansatz b*. The energy is in units of $4\pi v/g = 5.0$ TeV. The values of Ξ and Ω are correct to within ± 0.03 or better; the values of E are correct to the accuracy given.

λ/g^2	E	Ω	Ξ
0	1.566	2.60	2.66
10^{-3}	1.61	2.52	2.45
10^{-2}	1.67	2.29	2.12
10^{-1}	1.83	1.90	1.65
1	2.10	1.25	1.15
10	2.41	0.62	0.82
10^2	2.61	0.22	0.74
10^3	2.68	0.07	0.73
∞	2.722	0	0.728

$$E^{\mathrm{num}}(\lambda=0)=1.52\frac{4\pi v}{g}\ , \tag{27a}$$

whose value is 7.6 TeV. This is 3% lower than the *Ansatz-b* value, so it is likely that *Ansatz b* gives an estimate of the sphaleron energy which is only a few percent too high for all λ. Using the same numerical solution we can also verify that the sphaleron, for $\lambda=0$, is the maximal-energy configuration on a noncontractible loop in the field configuration space, and therefore unstable. To do this we make use of the loop constructed in Ref. 2, which is parametrized by an angle $\mu\in[0,\pi]$. The loop starts and ends at the vacuum and passes through the sphaleron (7) when $\mu=\frac{1}{2}\pi$. The energy is given in terms of μ, by the integral (3.3) of Ref. 2. Integrating numerically, we find the energy to be

$$E^{\mathrm{num}}(\lambda=0,\mu)=\frac{4\pi v}{g}(1.28\sin^2\mu+0.24\sin^4\mu)\ . \tag{27b}$$

This energy increases monotonically from zero to a maximum at $\mu=\frac{1}{2}\pi$, and then decreases to zero.

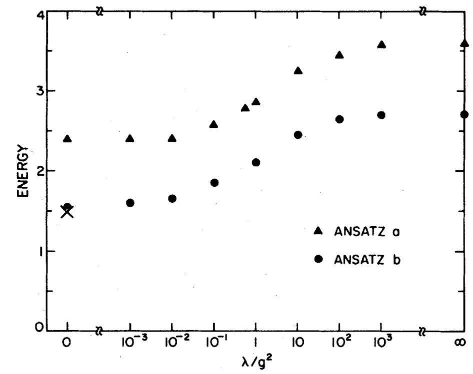

FIG. 2. Upper bounds on the energy of the $\Theta_w=0$ sphaleron, obtained from the optimized *Ansätze a* and *b*. The true sphaleron energy is probably a few percent lower than the *Ansatz-b* bounds. The cross represents a numerical estimate of the energy for $\lambda=0$. The energies are in units of $4\pi v/g$, which is 5.0 TeV in the Weinberg-Salam theory.

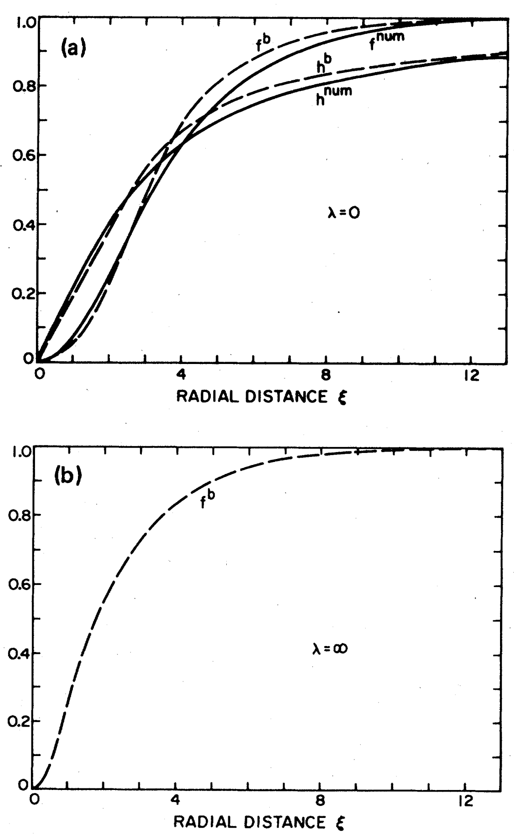

FIG. 1. (a) Radial functions f and h for $\lambda=0$ from the optimized *Ansatz b* (dashed curves) and from numerical integration (Ref. 14) of Eqs. (11) (solid curves). ξ is the dimensionless distance gvr. (b) Radial function f for $\lambda=\infty$ from the optimized *Ansatz b*. h now is a step function: $h(\xi)=1$ for $\xi>0$, $h(0)=0$.

IV. EFFECTS OF THE U(1) FIELD

In Sec. II we discussed the way in which the sphaleron changes as the weak mixing angle increases from 0. To lowest order in Θ_w, the effects are due entirely to the U(1) field. Recall that to find the magnetic moment we need to compute the integral

$$K\equiv\int_0^\infty \xi^2h^2(\xi)[1-f(\xi)]d\xi\ . \tag{28}$$

The shift of the sphaleron's energy relative to its $\Theta_w=0$ value is proportional to

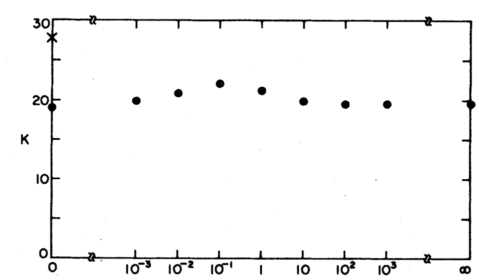

FIG. 3. Estimates of the integral K obtained using the optimized *Ansatz b*. For small Θ_w, the sphaleron has a magnetic dipole moment proportional to $\Theta_w K$. The cross is a numerical estimate of K for $\lambda=0$.

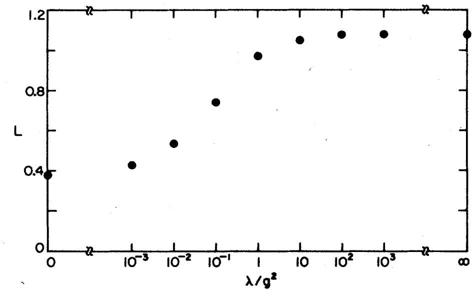

FIG. 4. Estimates of the integral L obtained using the optimized *Ansatz b*. For small Θ_w, the energy shift ΔE is proportional to $-\Theta_w{}^2 L$.

$$L \equiv \int_0^\infty \xi^2 h^2(\xi)[1-f(\xi)]p(\xi)d\xi \,, \tag{29}$$

where the function $p(\xi)$ itself is the integral (20). In this section we shall present estimates of these integrals.

For the radial functions of *Ansatz a* we have calculated the integrals analytically, but the results are only correct to within a factor of 4, so we do not give them here. Better results are obtained numerically using *Ansatz b* with the optimal values of the scale parameters as given in Table II. Our estimates for the integrals K and L, for several values of λ/g^2, are presented in Figs. 3 and 4. Figure 5 shows the function p for $\lambda=0$, 0.1, and ∞. The estimate of K is not very good because it depends critically on f and h in the intermediate region ($\xi\sim 6$), where *Ansatz b* is rather inaccurate. From the numerical solution for $\lambda=0$, we obtain $K^{\rm num}=27.8\pm0.3$ (cross in Fig. 3), which is 40% larger than the *Ansatz b* value.

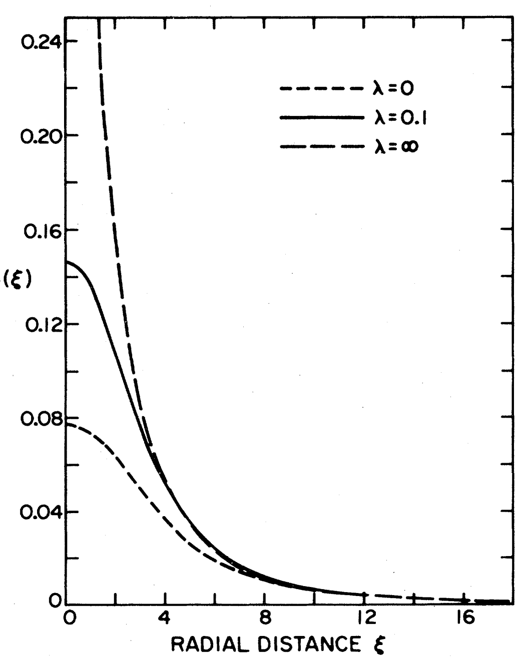

FIG. 5. The function p, for $\lambda=0$, 0.1, and ∞, estimated using the optimized *Ansatz b*. The U(1) field of the sphaleron depends on this function. When $\lambda=\infty$, $p(\xi)\sim(\text{constant}-\frac{1}{3}\ln\xi)$ for small ξ.

The physical magnetic moment of the sphaleron μ is obtained by multiplying K by $2\pi g'/3g^3 v$. This factor has the value[9] 0.0113 GeV^{-1}, so, assuming the value $K^{\rm num}$ above, μ is 0.314 GeV^{-1} for $\lambda=0$ and approximately the same for other λ.

It is interesting to compare this magnetic moment with the magnetic moment of the W boson,[12] $\mu_W=e/M_W$, whose value is 0.0038 GeV^{-1}. The ratio μ/μ_W is 83 when $\lambda=0$. On the other hand, the ratio of the sphaleron energy to the W-boson mass is ~ 100. Note that the magnetic moment of the sphaleron is very large compared with e/E, where E is the sphaleron energy.

Finally, let us consider the energy shift of the sphaleron $\Delta E=-(\pi v g'^2/3g^3)L$. The factor multiplying the integral L has the experimental value -125 GeV. This implies that, within our approximations, the sphaleron energy is lower than its $\Theta_w=0$ value by 48 GeV when $\lambda=0$ and by 135 GeV when $\lambda=\infty$. These energy shifts are not immediately significant, as we have not determined the energy at $\Theta_w=0$ to such accuracy, but it is surprising that they are so small, given that $\Theta_w=0.50$ is not really small. It raises the question of the nature of the sphaleron when Θ_w is larger, and in particular in the limit $\Theta_w\to\pi/2$. Does the energy tend to zero or not?

V. FRACTIONAL CHARGE OF THE SPHALERON

It was pointed out by 't Hooft[11] that there are anomalous fermionic currents in the Weinberg-Salam theory. Both the baryon and lepton currents have Abelian anomalies, but their difference is anomaly free. For each family of quarks and leptons, which we assume to be massless, there is a contribution to the divergence of the baryon current of the form

$$\partial_\mu j_B^\mu = \frac{1}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\left(\tfrac{1}{2}g^2 F^a_{\mu\nu}F^a_{\rho\sigma}+\tfrac{1}{2}g'^2 f_{\mu\nu}f_{\rho\sigma}\right) , \tag{30}$$

and equally for the divergence of the lepton current. For N_f families the divergence of the baryon current is N_f times the right-hand side of (30). For simplicity we will only consider the lightest family, which consists of the u and d quarks, the electron and the electron-type neutrino. Then the baryon current is

$$j_B^\mu = \tfrac{1}{3}(\bar{u}_\alpha\gamma^\mu u_\alpha + \bar{d}_\alpha\gamma^\mu d_\alpha) , \tag{31}$$

where α is the SU(3) color index and the $\frac{1}{3}$ occurs because each quark has a baryon number of $\frac{1}{3}$.

A consequence of (30) is that if the gauge and Higgs fields are time dependent, then the baryonic charge is time dependent, too. If the fields traverse a noncontractible loop in configuration space, starting and finishing at the vacuum, with time as the parameter, then the baryonic charge changes by an integer. This integer is the Pontryagin index of the time-dependent gauge field, which equals the winding number of the loop in configuration space. It is clear from (30) that the Higgs field has no direct effect on the baryonic charge.

Since the sphaleron represents a field configuration at the top of the energy barrier between the vacuum in one gauge and the vacuum in a topologically inequivalent gauge, one expects, if things are symmetrical, that the sphaleron has baryonic charge $\frac{1}{2}$. This appears to be the case, as the following calculation shows.

The baryonic charge is

$$Q_B = \int d^3x\, j_B^0 \,, \tag{32}$$

so

$$\frac{d}{dt}Q_B = \int d^3x\, \partial_t j_B^0 = \int d^3x \left[\vec{\nabla}\cdot\vec{\mathrm{j}}_B + \frac{g^2}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} \right] . \tag{33}$$

Note that only the SU(2) field occurs in (33). Obviously this is correct for the $\Theta_w = 0$ sphaleron, since the U(1) field vanishes. But even when $\Theta_w \neq 0$, the U(1) field does not contribute to Q_B, because as the sphaleron turns on, the electric and magnetic fields are perpendicular.

Consider now a time-dependent gauge field (and Higgs field) with finite energy at all times, starting at the trivial vacuum when $t = -\infty$ and arriving at the sphaleron configuration when $t = t_0$. We suppose that $Q_B = 0$ at $t = -\infty$. Then

$$Q_B(\text{sphaleron}) = \int_{-\infty}^{t_0} dt \int d^3x \left[\vec{\nabla}\cdot\vec{\mathrm{j}}_B + \frac{g^2}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} \right] . \tag{34}$$

Next, we assume that the fields rapidly approach vacuum fields at spatial infinity, i.e., pure gauge, and that $\vec{\mathrm{j}}_B$ vanishes there. Physically there could be a nonvanishing baryon current at infinity, but we expect that it has no topologically unusual properties and just consists of regular particles (baryons). The integrated current flow at infinity would therefore be integral. Since we are only interested here in fractional parts of the baryonic charge, we neglect the current at infinity and drop the $\vec{\nabla}\cdot\vec{\mathrm{j}}_B$ term in (34). This leaves

$$Q_B(\text{sphaleron}) = \frac{g^2}{32\pi^2}\int_{-\infty}^{t_0} dt \int d^3x\, \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} \,. \tag{35}$$

It is well known that $\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}$ can be written as a total divergence $\partial_\mu K^\mu$, where

$$K^\mu = \epsilon^{\mu\nu\rho\sigma}(F^a_{\nu\rho}W^a_\sigma - \tfrac{2}{3}g\epsilon_{abc}W^a_\nu W^b_\rho W^c_\sigma) \,. \tag{36}$$

Therefore,

$$Q_B(\text{sphaleron}) = \frac{g^2}{32\pi^2}\int_{-\infty}^{t_0} dt \int_S \vec{\mathrm{K}}\cdot d\vec{\mathrm{S}} + \frac{g^2}{32\pi^2}\int_{t=t_0} d^3x\, K^0 \,, \tag{37}$$

where S denotes the sphere at spatial infinity. There is no contribution to Q_B at $t = -\infty$ because $W^a_\mu = 0$ there, so $K^0 = 0$.

We know that at spatial infinity, W^a_μ is a pure gauge,

$$W^a_\mu \sigma^a dx^\mu = -\frac{2i}{g}(dU)U^{-1} \,, \tag{38}$$

but it is always possible to choose the gauge so that $U \equiv 1$ there. This is the gauge in which one speaks of an instanton interpolating between topologically distinct vacua. In this gauge, $\vec{\mathrm{K}} = \vec{0}$ at spatial infinity, so finally

$$Q_B(\text{sphaleron}) = \frac{g^2}{32\pi^2}\int_{t=t_0} d^3x\, K^0 \,. \tag{39}$$

This formula shows that Q_B depends just on the sphaleron configuration (although one must work in the correct gauge) and not on the path used to reach it. On the other hand, Eq. (35) makes it clear that Q_B is gauge invariant.

The gauge potential of the sphaleron, as given by (7a), is not yet in the correct gauge, since it only falls off inversely with distance from the origin. We transform to a better gauge in two steps. First, for convenience, we perform the transformation (8) with U_L and U_R given by (9) so that

$$W^a_i = -\frac{2f(gvr)}{gr^2}\epsilon_{iab}x_b \,. \tag{40}$$

Such a transformation has no effect on any contribution to Q_B. Next we perform a gauge transformation of the form

$$\mathrm{U}(\vec{\mathrm{x}}) = \exp[\tfrac{1}{2}i\Theta(r)\vec{\sigma}\cdot\hat{\vec{\mathrm{x}}}] \,, \tag{41}$$

where $\Theta(r)$ increases from 0 to π as r runs from 0 to ∞. As noted by Witten,[15] this changes W^a_i to

$$W^a_i = \frac{[1-2f(gvr)]\cos\Theta(r) - 1}{gr^2}\epsilon_{iab}x_b + \frac{[1-2f(gvr)]\sin\Theta(r)}{gr^3}(\delta_{ia}r^2 - x_i x_a) + \frac{1}{g}\frac{d\Theta}{dr}\frac{x_i x_a}{r^2} \,. \tag{42}$$

Provided $\Theta(r)$ approaches π sufficiently rapidly as $r \to \infty$, this gauge potential approaches zero faster than $1/r$. This ensures that the integral of $\vec{\mathrm{K}}$ vanishes at spatial infinity, so now we are in the correct gauge. It is straightforward to show, by calculating K^0 for the gauge potential (42), and evaluating the integral (39), that

$$Q_B(\text{sphaleron}) = \tfrac{1}{2} \tag{43}$$

as we anticipated. The leptonic charge of the sphaleron is the same as this baryonic charge. Recall that these charges were calculated only for the lightest family of quarks and leptons.

VI. DISCUSSION

In this article we have found a good approximation to a classical static solution of the field equations of the Weinberg-Salam theory. It has finite energy and is localized in space. The solution is axially symmetric and unique up to rotations of the axis of symmetry and translations. Because it is classically unstable, we have called it a sphaleron. A solution of this type is expected for topological reasons, but even our present results do not rigorously establish its existence, although it seems much more plausible now. We hope that an accurate numerical solution for a range of values of λ and Θ_w will be found in the near future.

Let us now summarize the specific properties of the sphaleron established in this article. The energy of the sphaleron E is determined by the scale $4\pi v/g \sim 5.0$ TeV, and its actual value increases from ~ 8 TeV to ~ 14 TeV as λ runs from 0 to ∞. Also, it has a classical magnetic dipole moment along the axis of symmetry. For $\lambda=0$ the dipole strength is ~ 80 times that of the W boson, which is e/M_W. The energy density of the sphaleron is approximately spherically symmetric for small values of the $SU(2)\times U(1)$ mixing angle Θ_w and extends $\sim 10^{-2}$ fm, but we do not know what the sphaleron would look like if Θ_w approached $\pi/2$. Another property is that its baryon and lepton charge is $\frac{1}{2}$, as a consequence of the anomaly equations.

This last property brings us to the place of the sphaleron in the physical picture. There (probably) is a noncontractible loop in configuration space such that its highest energy configuration is that of the sphaleron; other noncontractible loops pass through configurations of higher energies. This loop runs between "topologically distinct vacua" and the energy barrier separating these vacua thus has a definite height, namely the sphaleron energy. This happens because the Weinberg-Salam theory has a mass scale (the Higgs vacuum expectation value), in contrast to the case of a pure-gauge theory such as QCD. These two vacua have baryon charges 0 and 1, so that our result of $\frac{1}{2}$ for the charge of the sphaleron nicely fits the picture outlined here.

It has been claimed by 't Hooft that tunneling between topologically distinct vacua is negligible in the weak interactions because the (Euclidean) action is so large, namely $> 8\pi^2/g^2$. But this argument only applies to a virtual quantum process. If there were sufficient real energy available, more than the sphaleron energy, the tunneling process might be enhanced. There seem to be two situations, at least, where this could happen. The first is in high-energy collisions of particles from a very powerful accelerator, and the signature of the process would be the violation of baryon and lepton-number conservation. The other situation is for a system at very high temperature ($kT \sim E$). Thermal fluctuations might then produce the baryon-number violating process via the sphaleron at a substantial rate. This could be important in the Universe at early times ($t \sim 10^{-15}$ sec, $kT \sim 10$ TeV), where these processes could wash out any baryon number created earlier or even play a crucial role in the generation of the baryon number as observed today, cf. Ref. 16. But, before one can address these problems, a better understanding of the role of the sphaleron and other solutions is required.

We do not know in detail how the sphaleron can be produced and subsequently decay, but energy considerations give some idea. For definiteness, let us consider the case where $\lambda=0$ and Θ_w is small. Then the W^+, W^-, and Z^0 bosons all have a mass of 80 GeV, the Higgs particle is massless, and the sphaleron has an energy of 7.6 TeV. It is easy to verify, within *Ansatz b*, that the energy coming from the term $\frac{1}{2}\xi^2(dh/d\xi)^2$ in (10) is 28% of the total. This should be interpreted as the energy in the physical Higgs field. The remaining 72% of the energy is the gradient and mass energy in the SU(2) gauge field, and the energy in the U(1) field is negligible. It is plausible that when the sphaleron decays, the energy in these fields turns into the energy of the corresponding particles. (Here we have shifted to a quantum-mechanical picture. Classically the sphaleron will decay into outgoing waves, but on quantization these become particles.) Since the diameter of the sphaleron is of the order of $2\pi\hbar c/M_W$, one expects that the outgoing gauge bosons have kinetic energy roughly equal to their rest energy. Dividing the field energy equally between the three types of gauge bosons, which is sensible because of the symmetry of the fields (7a), we conclude that the sphaleron is likely to decay into approximately 11 each of W^+, W^-, and Z particles. The number of Higgs particles is harder to estimate because they are massless.

It is probably difficult to create a sphaleron in a single high-energy collision of particles, because a large number of weak gauge bosons must first be produced and then assembled into a highly coherent state. The strength of the sphaleron's magnetic moment is a measure of this coherence.[17] Only if the energy available were far in excess of the sphaleron energy might this phase-space constraint become less severe. On the other hand, in an equilibrium situation at sufficiently high temperature, large fluctuations of the fields occur and the baryon-number-violating processes connected to the sphaleron configuration may have a substantial rate.

Note added in proof. After completing this article, we integrated Eqs. (11) numerically for $\lambda/g^2=0$, 1, and ∞. The SU(2) sphaleron energy, in units of $4\pi v/g$, is 1.52, 2.07, and 2.70, respectively. The corresponding values of the integral K are 28, 18, and 16.

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