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Citation: Journal of Mathematical Physics **14**, 1651 (1973);
View online: https://doi.org/10.1063/1.1666240
View Table of Contents: http://aip.scitation.org/toc/jmp/14/11
Published by the American Institute of Physics

Quark structure and octonions*

Murat Günaydin and Feza Gürsey†

Physics Department, Yale University, New Haven, Connecticut 06520
(Received 14 May 1973)

The octonion (Cayley) algebra is studied in a split basis by means of a formalism that brings out its quark structure. The groups $SO(8)$, $SO(7)$, and G_2 are represented by octonions as well as by 8×8 matrices and the principle of triality is studied in this formalism. Reduction is made through the physically important subgroups $SU(3)$ and $SU(2) \otimes SU(2)$ of G_2, the automorphism group of octonions.

1. INTRODUCTION

Octonions made their appearance in physics as a by-product of an early attempt to generalize quantum mechanics through the associativity condition for physical observables.[1,2] In their algebraic approach to quantum mechanics, Jordan, von Neumann, and Wigner focused on the properties of Hermitian density matrices. Such matrices close under the commutative "Jordan" product which can be defined as their anticommutator. Thus, in switching from the matrix algebra of density matrices, we trade associativity for commutativity. The two formulations are equivalent except in the case of octonion hermitian 3×3 density matrices which form an exceptional Jordan Algebra.[2] In the latter case the nonassociativity is intrinsic and cannot be removed by going over to a corresponding operator algebra in a finite Hilbert space. In fact, it originates in the structure of the underlying octonion algebra which is a not commutative, not associative, but, alternative division algebra.

The Jordan approach has proved to be more fruitful in mathematics than physics. It has since been quietly dropped in favor of the associative Dirac algebra of operators in Hilbert space,[3] which generalizes the algebra of finite matrices.

The story took a new turn when the charge space made its appearance two decades ago in the Gell-Mann-Nishijima[4] scheme based on isospin and strangeness. A decade later, this led to the quark structure of elementary particles, revealing the underlying $SU(3)$ symmetry.[5] Meantime another group of rank two, namely G_2 was tried[6] and abandoned. Now, G_2 is the automorphism group of the octonion algebra and it admits $SU(3)$ as a subgroup. In fact, $SU(3)$ is the automorphism group of the multiplication rules among six of the octonion units. In terms of this subgroup the generators of G_2 split into an $SU(3)$ octet, a triplet and an antitriplet. Furthermore G_2 has a $SU(2) \times SU(2)$ subgroup under which the generators decompose as $(1, 0)$, $(0, 1)$, and $(1/2, 3/2)$. One of the $SU(2)$ is the isospin, while the other is a generalization of hypercharge to a rotation group. Hence the quark structure is manifest in G_2 and also in the other exceptional groups which are all related to octonions[7] and admit G_2 as a subgroup. An example is the exceptional Jordan algebra which has the exceptional group F_4 as its automorphism group.[8] The quark structure of this algebra was pointed out by Gamba.[9] Other possible connections of the Cayley algebra with internal symmetries were discussed by Pais[10] and others[11] while the admissibility of the elements of the exceptional Jordan algebra as observables was considered by Sherman[12] following the general algebraic framework of Segal.[13] Finally, we have shown recently[14] that the Poincaré group possesses an octonionic representation that leads to a quark structure arising from the breaking of G_2 with $SU(3)$ as the surviving subgroup.

An independent, and perhaps related line of research concerns the construction of Weinberg type renormalizable models[15] based on groups that do not give rise to triangular anomalies, including G_2, $SO(7)$, and $SO(8)$. The attempts enumerated above seem to provide sufficient motivation for a reformulation of the octonion algebra, and the groups $SO(8)$, $SO(7)$, and G_2 connected with it, in a manner which manifestly exhibits its quark structure and its $SU(3)$ content in charge space. Although a vast mathematical literature exists on these subjects,[7,16] it is not presented in a form directly usable by the particle physicist. The object of the present paper is to recast the mathematical theory in a quark language, in direct correspondence with Gell-Mann's treatment of $SU(3)$,[5] to develop an 8×8 matrix formalism, initiated by Seligman,[17] which allows us to treat $SO(8)$, $SO(7)$, G_2 in a unified way and prepare the ground for the investigation of the properties of an octonionic Hilbert space.[18]

The features which seem to be new consist of a matrix form for the octonion multiplication, the representation of G_2 through purely octonionic multiplication in a split basis and the reduction of $SO(8)$, $SO(7)$, and G_2 with respect to their physically important subgroups $SU(3)$ and $SU(2) \times SU(2)$. It is this reduction which exhibits the quark structure of the algebra.

The contents of the paper are as follows. The octonion algebra in the split basis is introduced in Sec 2 and its automorphism group G_2 derived in Sec. 3. The Lie algebra of G_2 and its imbedding in $SO(7)$ are given in Sec. 4. The following section 5 covers the $SU(3)$ and $SU(2) \times SU(2)$ subgroups of G_2. Section 6 is devoted to split octonions and split G_2 while the quark structure in split basis emerges in Sec. 7. A purely octonion representation of split G_2 appears in Sec. 8. The 8×8 matrix formulation of the Cayley algebra forms the object of Sec. 9. The imbedding of G_2 in $SO(7)$ and $SO(8)$ and its reduction with respect to its $SU(2) \times SU(2)$ subgroups are discussed respectively in Secs. 10 and 11. Finally the principle of triality is discussed within the formalism developed previously in Sec. 12. Additional details such as the structure constants of G_2, Zorn's vector-matrix method, theorems pertaining to triality and the realization of the Cayley algebra by means of Gell-Mann's 3×3 λ-matrices, and Dirac's 4×4 γ-matrices appear in the appendixes.

2. THE OCTONION ALGEBRA AND ITS SPLIT BASIS

A composition algebra is defined as an algebra A with identity and with a nondegenerate quadratic form Q defined over it such that Q permits composition, i.e. for $x, y \in A$.

$$Q(xy) = Q(x)Q(y). \tag{2.1}$$

According to the celebrated Hurwitz theorem, there

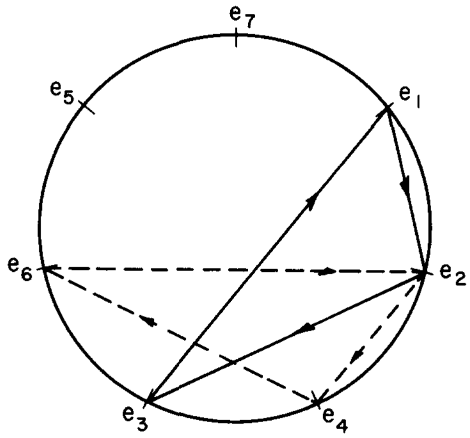

FIG. 1.

exist only four different composition algebras over the real or complex number fields. These are the real numbers **R** of dimension 1, complex numbers **C** of dimension 2, quaternions **H** of dimension 4, and octonions **O** of dimension 8. Of these algebras, the quaternions **H** are not commutative and the octonions **O** are neither commutative nor associative.[19] A composition algebra is said to be a division algebra if the quadratic form Q is anisotropic i.e.,

if $Q(x) = 0$ implies that $x = 0$.

Otherwise the algebra is called *split*.

Assuming that the reader is familiar with the algebras **R, C,** and **H,** we shall review briefly the properties of octonion algebra **O**[20] (sometimes called the Cayley algebra).

A basis for the real octonion **O** will contain eight elements including the identity

$$1, e_A, \quad A = 1, \ldots, 7, \quad \text{where } e_A^2 = -1.$$

For later application to the $SU(3)$ symmetry in particle physics, we label the elements e_A such that they satisfy the following multiplication table:

$$e_1e_2 = e_3, \quad e_5e_1 = e_6, \quad e_6e_2 = e_4, \quad e_4e_3 = e_5,$$
$$e_4e_7 = e_1, \quad e_6e_7 = e_3, \quad e_5e_7 = e_2$$

and

$$e_Ae_B + e_Be_A = -2\delta_{AB}$$

more concisely,

$$e_Ae_B = a_{ABC}e_C - \delta_{AB} \tag{2.2}$$

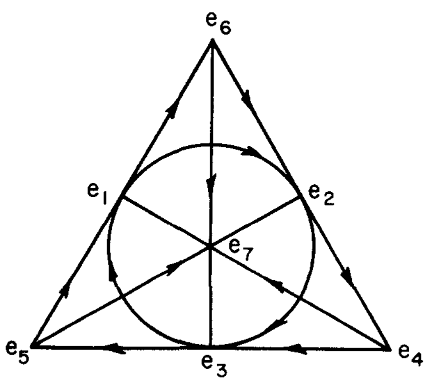

FIG. 2.

where a_{ABC} is totally antisymmetric and

$$a_{ABC} = +1 \text{ for } ABC = 123, 516, 624, 435, 471, 673, 572.$$

Note here the cyclic symmetry obtained by ordering seven points clockwise on a circle with the numbering (1243657) as given in Fig. 1. Then a triangle ABC is obtained from (123) by 6 successive rotations of angle $2\pi/7$. In Fig. 1 the elements corresponding to the corners of the triangle form a basis of a quaternion subalgebra (together with the identity element). Another convenient way of representing the multiplication table by singling out one of the elements is provided by the triangular diagram given in Fig. 2, where arrows show the directions along which the multiplication has a positive sign, e.g.

$$e_5e_1 = e_6, \quad e_1e_5 = -e_6, \quad e_6e_1 = -e_5, \quad e_6e_5 = e_1.$$

From the above multiplication table it is clear that the algebra **O** is not associative. Yet it satisfies a weaker condition than associativity, namely alternativity, i.e., the associator $[x, y, z]$ of the elements x, y, z defined as

$$[x, y, z] = (xy)z - x(yz) \tag{2.3}$$

is an alternating function of x, y, z:

$$[x, y, z] = [z, x, y] = [y, z, x] = -[y, x, z) = -[x, z, y].$$

This property if trivially satisfied by associative algebras **R, C,** and **H.**

The octonion algebra **O** with the above basis considered over the real numbers **R** is a division algebra with the quadratic form Q defined by

$$Q(x) = \bar{x}x = x\bar{x},$$

where $\bar{x}$ is the octonion conjugate of x obtained by replacing e_A in x by $-e_A$.

$$x = x_0 + x_Ae_A, \quad \bar{x} = x_0 - x_Ae_A.$$

This quadratic form is also called the norm form and denoted by $N(x)$. Then

$$N(x) = x\bar{x} = \bar{x}x = x_0^2 + \sum_{A=1}^{7} x_A^2. \tag{2.4}$$

For the split octonion algebra we choose the following basis:

$$\begin{aligned}
u_1 &= \tfrac{1}{2}(e_1 + ie_4), & u_2 &= \tfrac{1}{2}(e_2 + ie_5),\\
u_3 &= \tfrac{1}{2}(e_3 + ie_6), & u_0 &= \tfrac{1}{2}(1 + ie_7)\\
u_1^* &= \tfrac{1}{2}(e_1 - ie_4), & u_2^* &= \tfrac{1}{2}(e_2 - ie_5),\\
u_3^* &= \tfrac{1}{2}(e_3 - ie_6), & u_0^* &= \tfrac{1}{2}(1 - ie_7),
\end{aligned}$$

where $i = \sqrt{-1}$ and is assumed to commute with all e_A. These basis elements satisfy the multiplication table:

$$\begin{aligned}
&u_iu_j = \epsilon_{ijk}u_k^*, \quad i, j, k = 1, 2, 3,\\
&u_i^*u_j^* = \epsilon_{ijk}u_k,\\
&u_iu_j^* = -\delta_{ij}u_0, \quad u_i^*u_j = -\delta_{ij}u_0^*,\\
&u_iu_0 = 0, \quad u_iu_0^* = u_i, \quad u_i^*u_0 = u_i^*, \quad u_i^*u_0^* = 0,\\
&u_0u_i = u_i, \quad u_0^*u_i = 0, \quad u_0u_i^* = 0, \quad u_0^*u_i^* = u_i^*,\\
&u_0^2 = u_0, \quad u_0^{*2} = u_0^*, \quad u_0u_0^* = u_0^*u_0 = 0.
\end{aligned}$$

Clearly, the split octonion algebra contains divisors of zero and hence is not a division algebra. In Appendix B, we give a realization of split octonion algebra in terms of Zorn's vector matrices.

3. G_2 AS THE AUTOMORPHISM GROUP OF OCTONIONS

An automorphism of an algebra A is defined as an isomorphism of A onto itself. Under the automorphism, multiplication table of A is left invariant, i.e.,

$$x, y \in A, \qquad T \in \mathrm{Aut}A,$$

then

$$T(xy) = T(x)T(y)$$

and the automorphisms map the identity 1 into itself.

The set of all automorphisms of composition algebras form a group. For the real numbers **R** and complex numbers **C**, the groups of automorphisms are the trivial identity mapping and the cyclic group C_2, respectively. The automorphism group of quaternions is the $SU(2)$ group.[21] Below we shall investigate the automorphism group of the octonions, which is the exceptional Lie group G_2.

We use the following results of M. Zorn[22] as our starting point: Each automorphism of the Cayley algebra **O** is completely defined by the images of 3 "independent" basis elements.[23] Consider one such set, say $\{e_1, e_2, e_4\}$. Then there exists an automorphism σ such that

$$\sigma(e_1) = e_1,$$
$$\sigma(e_2) = \cos\phi_1 e_2 + \sin\phi_1 e_3,$$
$$\sigma(e_4) = \cos\phi_2 e_4 + \sin\phi_2 e_7, \tag{3.1}$$

The images of the other basis elements are determined by the conditions:

$$\sigma(e_2)\sigma(e_4) = \sigma(e_6), \qquad \sigma(e_1)\sigma(e_2) = \sigma(e_3),$$
$$\sigma(e_1)\sigma(e_4) = \sigma(e_7), \qquad \sigma(e_4)\sigma(e_3) = \sigma(e_5).$$

It can easily be checked that $\sigma(e_A)$ satisfy the same multiplication table as e_A and hence σ is an automorphism. Conversely, one has the very important result that *each automorphism of* **O** *belongs in this manner to at least one Cayley basis*.

Now let us write down all the images of all basis elements under σ explicitly and observe some general patterns:

$$\sigma(e_1) = e_1,$$
$$\begin{pmatrix}\sigma(e_2)\\ \sigma(e_3)\end{pmatrix} = \begin{pmatrix}\cos\phi_1 & \sin\phi_1\\ -\sin\phi_1 & \cos\phi_1\end{pmatrix}\begin{pmatrix}e_2\\ e_3\end{pmatrix},$$
$$\begin{pmatrix}\sigma(e_4)\\ \sigma(e_7)\end{pmatrix} = \begin{pmatrix}\cos\phi_2 & \sin\phi_2\\ -\sin\phi_2 & \cos\phi_2\end{pmatrix}\begin{pmatrix}e_4\\ e_7\end{pmatrix}, \tag{3.2}$$
$$\begin{pmatrix}\sigma(e_6)\\ \sigma(e_5)\end{pmatrix} = \begin{pmatrix}\cos(\phi_1+\phi_2) & -\sin(\phi_1+\phi_2)\\ \sin(\phi_1+\phi_2) & \cos(\phi_1+\phi_2)\end{pmatrix}\begin{pmatrix}e_6\\ e_5\end{pmatrix}.$$

We see that under the automorphism σ we have three invariant planes (e_2, e_3), (e_4, e_7), (e_6, e_5) that undergo rotations through angles ϕ_1, ϕ_2, ϕ_3, respectively, such that

$$\phi_1 + \phi_2 + \phi_3 = 0 \bmod 2\pi.$$

We shall call the automorphisms of the form above *canonical automorphisms*. Each canonical automorphism has a fixed point and 3 invariant planes. If we denote the fixed point by e_k then the invariant planes (e_i, e_j) are determined by the conditions $e_i e_j = e_k$.

For each Cayley basis, there are seven independent canonical automorphisms. The canonical automorphism given above can be written more concisely as:

$$\begin{pmatrix}\sigma(e_2) + i\sigma(e_3)\\ \sigma(e_4) + i\sigma(e_7)\\ \sigma(e_6) + i\sigma(e_5)\end{pmatrix} = e^{(\alpha^1\lambda_3 + \beta^1\lambda_8)e_1}\begin{pmatrix}e_2 + ie_3\\ e_4 + ie_7\\ e_6 + ie_5\end{pmatrix}, \tag{3.3}$$

where λ_3 and λ_8 are the Gell-Mann matrices

$$\lambda_3 = \begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix}$$

and α^1 and β^1 are related to ϕ_1 and ϕ_2 as:

$$\phi_1 = \alpha^1 + 3^{-1/2}\beta^1, \qquad \phi_2 = -\alpha^1 + 3^{-1/2}\beta^1.$$

This can be generalized to all the canonical automorphisms. First define seven octonionic 3-spinors $\psi(e_A)$:

$$\psi(e_1) = \frac{1}{2}\begin{pmatrix}e_6 + ie_5\\ e_2 + ie_3\\ e_4 + ie_7\end{pmatrix} = \frac{(1 + ie_1)}{2}\begin{pmatrix}e_6\\ e_2\\ e_4\end{pmatrix},$$
$$\psi(e_2) = \frac{1}{2}\begin{pmatrix}e_4 + ie_6\\ e_3 + ie_1\\ e_5 + ie_7\end{pmatrix} = \frac{(1 + ie_2)}{2}\begin{pmatrix}e_4\\ e_3\\ e_5\end{pmatrix},$$
$$\psi(e_3) = \frac{1}{2}\begin{pmatrix}e_5 + ie_4\\ e_1 + ie_2\\ e_6 + ie_7\end{pmatrix} = \frac{(1 + ie_3)}{2}\begin{pmatrix}e_5\\ e_1\\ e_6\end{pmatrix},$$
$$\psi(e_4) = \frac{1}{2}\begin{pmatrix}e_3 + ie_5\\ e_6 + ie_2\\ e_7 + ie_1\end{pmatrix} = \frac{(1 + ie_4)}{2}\begin{pmatrix}e_3\\ e_6\\ e_7\end{pmatrix}, \tag{3.4}$$
$$\psi(e_5) = \frac{1}{2}\begin{pmatrix}e_1 + ie_6\\ e_4 + ie_3\\ e_7 + ie_2\end{pmatrix} = \frac{(1 + ie_5)}{2}\begin{pmatrix}e_1\\ e_4\\ e_7\end{pmatrix},$$
$$\psi(e_6) = \frac{1}{2}\begin{pmatrix}e_2 + ie_4\\ e_5 + ie_1\\ e_7 + ie_3\end{pmatrix} = \frac{(1 + ie_6)}{2}\begin{pmatrix}e_2\\ e_5\\ e_7\end{pmatrix},$$
$$\psi(e_7) = \frac{1}{2}\begin{pmatrix}e_1 + ie_4\\ e_2 + ie_5\\ e_3 + ie_6\end{pmatrix} = \frac{(1 + ie_7)}{2}\begin{pmatrix}e_1\\ e_2\\ e_3\end{pmatrix}.$$

A canonical automorphism leaving the element e_A fixed is defined by its action on $\psi(e_A)$:

$$\sigma^A: \psi(e_A) \to \sigma^A\psi(e_A) = \psi'(e_A) = e^{(\alpha^A\lambda_3 + \beta^A\lambda_8)e_A}\psi(e_A)$$
$$= e^{-i(\alpha^A\lambda_3 + \beta^A\lambda_8)}\psi(e_A), \tag{3.5}$$

no sum over A.

The rows $(e_B + ie_C)$ of $\psi(e_A)$ are determined by the invariant planes (e_B, e_C) of the automorphism σ^A and the ordering of the rows is such that the first elements along the column define imaginary units of a quaternion subalgebra in a positive sense. Hence the cyclic permutation of the rows of $\psi(e_A)$ is immaterial for the subsequent discussion. Canonical automorphisms involve two independent parameters each and hence generate a 14-parameter Lie group. That this is the complete automorphism group of octonions follows from the well-known result that the automorphism group of octonions is a 14-parameter Lie group of type G_2.

Consequently, every automorphism of Cayley numbers can be written as a product of canonical automorphisms and; as stated above, every automorphism can be reduced to the canonical form in a suitably chosen basis.

4. THE LIE ALGEBRA OF G_2 AND ITS IMBEDDING IN $SO(7)$

Using the result that canonical automorphisms generate the Lie group G_2, let us now find its Lie algebra. As parameters corresponding to the generators of G_2, we shall take α^A and β^A rather than the angles ϕ_1^A and ϕ_2^A. Now consider a canonical automorphism σ_A with $\beta^A = 0$, then

$$\sigma_A\psi(e_A) = \psi'(e_A) = e^{\alpha^A\lambda_3 e_A}\begin{pmatrix} e_B + ie_C \\ e_D + ie_E \\ e_F + ie_G \end{pmatrix} = \begin{pmatrix} e'_B + ie'_C \\ e'_D + ie'_E \\ e'_F + ie'_G \end{pmatrix}$$

which gives

$$\begin{pmatrix} e'_B \\ e'_C \end{pmatrix} = \begin{pmatrix} \cos\alpha^A & \sin\alpha^A \\ -\sin\alpha^A & \cos\alpha^A \end{pmatrix}\begin{pmatrix} e_B \\ e_C \end{pmatrix},$$

$$\begin{pmatrix} e'_D \\ e'_E \end{pmatrix} \begin{pmatrix} \cos\alpha^A & -\sin\alpha^A \\ \sin\alpha^A & \cos\alpha^A \end{pmatrix}\begin{pmatrix} e_D \\ e_E \end{pmatrix}.$$

Therefore the group action with parameter α^A induces rotations in the invariant planes (e_B, e_C) and (e_D, e_E) through angles α^A and $-\alpha^A$, respectively. Hence the generator of this group action is

$$(J_{BC} - J_{DE}),$$

where J_{BC} and J_{DE} are the anti-Hermitian rotation generators.

$$J_{BC} = -J_{CB} = -J_{BC}^\dagger.$$

Similarly, the generator corresponding to the group action with parameter β^A is

$$(1/\sqrt{3})(J_{BC} + J_{DE} - 2J_{FG}).$$

Since the indices go from 1 to 7, the 14 generators thus constructed will form a subalgebra of $SO(7)$ if they close under commutation. As will be shown below, they indeed close under commutation and hence establish the known result that G_2 is a subgroup of $SO(7)$. The remaining generators of $SO(7)$ can be taken as

$$(J_{BC} + J_{DE} + J_{FG})$$

which are generated by the mappings:

$$\psi(e_A) \to e^{\gamma^A I_3 e_A}\psi(e_A) = e^{-i\gamma^A I_3}\psi(e_A),$$

where I_3 is the 3×3 identity matrix.

For reasons that will be clear later, we shall modify the above basis for G_2 and $SO(7)$ and consider the following Hermitian basis:

$$\begin{aligned}
F_1 &= -i(J_{24} - J_{51}), & M_1 &= (i/\sqrt{3})(J_{24} + J_{51} - 2J_{73}), & N_1 &= i(J_{24} + J_{51} + J_{73}),\\
F_2 &= i(J_{54} - J_{12}), & M_2 &= (-i/\sqrt{3})(J_{54} + J_{12} - 2J_{67}), & N_2 &= i(J_{54} + J_{12} + J_{67}),\\
F_3 &= -i(J_{14} - J_{25}), & M_3 &= (i/\sqrt{3})(J_{14} + J_{25} - 2J_{36}), & N_3 &= i(J_{14} + J_{25} + J_{36}),\\
F_4 &= -i(J_{16} - J_{43}), & M_4 &= (i/\sqrt{3})(J_{16} + J_{43} - 2J_{72}), & N_4 &= i(J_{16} + J_{43} + J_{72}),\\
F_5 &= -i(J_{46} - J_{31}), & M_5 &= (i/\sqrt{3})(J_{46} + J_{31} - 2J_{57}), & N_5 &= i(J_{46} + J_{31} + J_{57}),\\
F_6 &= -i(J_{35} - J_{62}), & M_6 &= (i/\sqrt{3})(J_{35} + J_{62} - 2J_{71}), & N_6 &= i(J_{35} + J_{62} + J_{71}),\\
F_7 &= i(J_{65} - J_{23}), & M_7 &= (-i/\sqrt{3})(J_{65} + J_{23} - 2J_{47}), & N_7 &= i(J_{65} + J_{23} + J_{47}).
\end{aligned}$$

Note that the subscript A in M_A and N_A does not refer to the basis element left invariant by the corresponding canonical automorphism. We have used the above numbering to be consistent with Gell-Mann's notation for $SU(3)$, which is a subgroup of G_2 as is shown in the next section. The generators F_A and M_A, $A = 1, \ldots, 7$ close under commutation and form the Lie algebra of G_2. Denoting the Lie algebra of a group G by $\mathcal{L}G$ we have

$$\mathcal{L}G_2 = F_A \oplus M_A,$$
$$\mathcal{L}SO(7) = F_A \oplus M_A \oplus N_A.$$

The structure constants of $\mathcal{L}G_2$ are given in Appendix A. In the following sections, we shall denote the generators of $SO(7)$ by capital Latin letters F_A, M_A, and N_A and the corresponding $n \times n$ matrix representation of these generators by $\Lambda_A^{(n)}$, $\mu_A^{(n)}$, and $\nu_A^{(n)}$, respectively. The parameters corresponding to the generators M_A, N_A, and F_A will be denoted by m_A, n_A, and f_A, respectively.

5. THE $SU(3)$ AND $SU(2) \times SU(2)$ SUBGROUPS OF G_2

From the above table of the generators of G_2 one can easily observe that there are eight generators annihilating[24] a given basis element e_A. The generators annihilating, say, e_7 are F_A, $A = 1, \ldots, 7$, and $F_8 = -M_3$. They close under commutation and form the Lie algebra of $SU(3)$:

$$[F_a, F_b] = 2if_{abc}F_c, \qquad a, b, c = 1, \ldots, 8,$$

where f_{abc} are the usual structure constants of Gell-Mann. Hence the automorphisms of the Cayley algebra leaving a basis element e_A invariant form a subgroup $SU(3)$ of G_2. Since G_2 has only real representations,[25] only the real representations of $SU(3)$ can occur in the representations of G_2. For example, in the seven-dimensional representations of G_2, the only nontrivial real representation of $SU(3)$ that can occur is the 6-dimensional representation $3 \oplus \bar{3}$. This six-dimensional representation can be constructed from Gell-Mann's matrices as follows:

$$
\begin{aligned}
\Lambda_1^{(6)} &= \sigma_2 \otimes \lambda_1 = -i(\Sigma_{24} - \Sigma_{51}),\\
\Lambda_2^{(6)} &= 1_2 \otimes \lambda_2 = i(\Sigma_{54} - \Sigma_{12}),\\
\Lambda_3^{(6)} &= \sigma_2 \otimes \lambda_3 = -i(\Sigma_{14} - \Sigma_{25}),\\
\Lambda_4^{(6)} &= \sigma_2 \otimes \lambda_4 = -i(\Sigma_{16} - \Sigma_{43}),\\
\Lambda_5^{(6)} &= 1_2 \otimes \lambda_5 = -i(\Sigma_{46} - \Sigma_{31}),\\
\Lambda_6^{(6)} &= \sigma_2 \otimes \lambda_6 = -i(\Sigma_{35} - \Sigma_{62}),\\
\Lambda_7^{(6)} &= 1_2 \otimes \lambda_7 = i(\Sigma_{65} - \Sigma_{23}),\\
\Lambda_8^{(6)} &= \sigma_2 \otimes \lambda_8 = \frac{-i}{\sqrt{3}}(\Sigma_{14} + \Sigma_{25} - 2\Sigma_{36}),
\end{aligned}
\tag{5.1}
$$

where $\otimes$ denotes the direct product of matrices and λ_a are the Gell-Mann's λ matrices and σ_2 is the Pauli matrix

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and Σ_{mn} are the 6×6 matrix representation of the generators J_{mn} of $SO(6)$. This construction shows clearly that $\Lambda_a^{(6)}$ can be imbedded into the seven-dimensional representations of G_2 as the representations of the generators F_a.

G_2 also has an $SU(2) \times SU(2)$ subgroup. The $SU(2) \times SU(2)$ subgroup involving the isospin subgroup of $SU(3)$ is generated by F_1, F_2, F_3 and M_1, M_2, M_3

$$
\begin{aligned}
&[F_i, F_j] = 2i\epsilon_{ijk}F_k, \qquad i, j, k = 1, 2, 3,\\
&[F_i, M_j] = 0, \qquad [M_i, M_j] = (2i/\sqrt{3})\epsilon_{ijk}M_k.
\end{aligned}
\tag{5.2}
$$

$SU(2) \times SU(2)$ subgroup of G_2 arises from the fact that octonions can be constructed from two quaternions.[26]

The $SU(3)$ subgroup can be imbedded in G_2 in seven different ways and for each imbedding of $SU(3)$ there are three different imbeddings of $SU(2) \times SU(2)$ involving I, U, V spin subgroup of $SU(3)$.

Now consider the seven-dimensional action of G_2' on the octonion units e_A as the automorphism action. Then under the $SU(3)$ subgroup, six of the basis elements e_A transform like the six-dimensional real representation of $SU(3)$ $(3 \oplus \bar{3})$ and the seventh element is an $SU(3)$ scalar. Under $SU(2) \times SU(2)$, four of the elements e_A transform like the $(1/2, 1/2)$ representations and the remaining three transform as $(0, 1)$ representations.

6. SPLIT OCTONIONS AND SPLIT G_2

Above we have considered the automorphism group of real octonions with basis $1, e_A$. We saw that if we denote the parameters corresponding to the generators F_A and M_A by f_A and $\sqrt{3}m_A$ and the seven-dimensional representation of these generators by $\Lambda_A^{(7)}$ and $\mu_A^{(7)}$ then the most general automorphism of real octonions are given by the transformation

$$[e] \to [e'] = \exp[-if_A\Lambda_A^{(7)} - i\sqrt{3}m_A\mu_A^{(7)}][e] = e^X[e], \tag{6.1}$$

where

$$[e] = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{pmatrix} \quad \text{and} \quad X = -i(f_A\Lambda_A^{(7)} + \sqrt{3}m_A\mu_A^{(7)}).$$

X is given explicitly by

$$X = \begin{pmatrix}
0 & -(f_2+m_2) & -(f_5+m_5) & (m_3-f_3) & -(f_1+m_1) & (m_4-f_4) & 2m_6\\
(f_2+m_2) & 0 & -(f_7+m_7) & (m_1-f_1) & (f_3+m_3) & -(f_6+m_6) & 2m_4\\
(f_5+m_5) & (f_7+m_7) & 0 & -(f_4+m_4) & (m_6-f_6) & -2m_3 & 2m_1\\
(f_3-m_3) & (f_1-m_1) & (f_4+m_4) & 0 & (m_2-f_2) & (m_5-f_5) & 2m_7\\
(f_1+m_1) & -(f_3+m_3) & (f_6-m_6) & (f_2-m_2) & 0 & (m_7-f_7) & -2m_5\\
(f_4-m_4) & (f_6+m_6) & 2m_3 & (f_5-m_5) & (f_7-m_7) & 0 & 2m_2\\
-2m_6 & -2m_4 & -2m_1 & -2m_7 & 2m_5 & -2m_2 & 0
\end{pmatrix}$$

$$X^\dagger = -X$$

If we transform the real basis $[e]$ into what we shall call the split basis $[d]$ where

$$[d] = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_1^* \\ u_2^* \\ u_3^* \\ (i/\sqrt{2})e_7 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(e_1 + ie_4) \\ \frac{1}{2}(e_2 + ie_5) \\ \frac{1}{2}(e_3 + ie_6) \\ \frac{1}{2}(e_1 - ie_4) \\ \frac{1}{2}(e_2 - ie_5) \\ \frac{1}{2}(e_3 - ie_6) \\ (i/\sqrt{2})e_7 \end{bmatrix} = \begin{pmatrix} \psi(e_7) \\ \psi^*(e_7) \\ ie_7/\sqrt{2} \end{pmatrix}$$

then the automorphisms are generated by the mapping

$$[d] \to [d'] = e^{iZ}[d] \tag{6.2}$$

where Z is

$$Z = \begin{bmatrix} (f_3-m_3) & (f_1+if_2) & (f_4+if_5) & 0 & -(m_1-im_2) & (m_4+im_5) & -\sqrt{2}(m_6+im_7) \\ (f_1-if_2) & -(f_3+m_3) & (f_6+if_7) & (m_1-im_2) & 0 & -(m_6-im_7) & -\sqrt{2}(m_4-im_5) \\ (f_4-if_5) & (f_6-if_7) & 2m_3 & -(m_4+im_5) & (m_6-im_7) & 0 & -\sqrt{2}(m_1+im_2) \\ 0 & (m_1+im_2) & -(m_4-im_5) & -(f_3-m_3) & -(f_1-if_2) & -(f_4-if_5) & -\sqrt{2}(m_6-im_7) \\ -(m_1+im_2) & 0 & (m_6+im_7) & -(f_1+if_2) & (f_3+m_3) & -(f_6-if_7) & -\sqrt{2}(m_4+im_5) \\ (m_4-im_5) & -(m_6+im_7) & 0 & -(f_4+if_5) & -(f_6+if_7) & -2m_3 & -\sqrt{2}(m_1-im_2) \\ -\sqrt{2}(m_6-im_7) & -\sqrt{2}(m_4+im_5) & -\sqrt{2}(m_1-im_2) & -\sqrt{2}(m_6+im_7) & -\sqrt{2}(m_4-im_5) & -\sqrt{2}(m_1+im_2) & 0 \end{bmatrix}$$

$$Z^\dagger = Z,$$

and Z can be written in the form

$$Z = \begin{pmatrix} U_3 & O^\dagger & \mathbf{x}^T \\ O & -U_3^* & \mathbf{x}^\dagger \\ \mathbf{x}^* & \mathbf{x} & 0 \end{pmatrix},$$

where

$$U_3 = \begin{pmatrix} (f_3 - m_3) & (f_1 + if_2) & (f_4 + if_5) \\ (f_1 - if_2) & -(f_3 + m_3) & (f_6 + if_7) \\ (f_4 - if_5) & (f_6 - if_7) & 2m_3 \end{pmatrix},$$

$$O = \begin{pmatrix} 0 & (m_1 + im_2) & -(m_4 - im_5) \\ -(m_1 + im_2) & 0 & (m_6 + im_7) \\ (m_4 - im_5) & -(m_6 + im_7) & 0 \end{pmatrix},$$

$$\mathbf{x} = -\sqrt{2}[(m_6 + im_7)(m_4 - im_5)(m_1 + im_2)]$$

and $O_{ij} = -(1/\sqrt{2})\epsilon_{ijk}x_k$

$$i, j, k = 1, 2, 3.$$

Note that U_3 and O are the three-dimensional representations of the Lie algebras of $SU(3)$ and complex $SO(3)$.

If we further split the identity and consider the split octonions with basis $u_1u_2u_3u_0u_1^*u_2^*u_3^*u_0^*$, defined above, then the automorphism group G_2 will act on this basis by an 8-dimensional reducible representation.

$$G_2: [s] \to [s'] = e^{iY}[s], [s] = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_0 \\ u_1^* \\ u_2^* \\ u_3^* \\ u_0^* \end{bmatrix}, \tag{6.3}$$

where Y is given explicitly as

$$Y = \begin{bmatrix} (f_3-m_3) & (f_1+if_2) & (f_4+if_5) & -(m_6+im_7) & 0 & -(m_1-im_2) & (m_4+im_5) & (m_6+im_7) \\ (f_1-if_2) & -(f_3+m_3) & (f_6+if_7) & -(m_4-im_5) & (m_1-im_2) & 0 & -(m_6-im_7) & (m_4-im_5) \\ (f_4-if_5) & (f_6-if_7) & 2m_3 & -(m_1+im_2) & -(m_4+im_5) & (m_6-im_7) & 0 & (m_1+im_2) \\ -(m_6-im_7) & -(m_4+im_5) & -(m_1-im_2) & 0 & -(m_6+im_7) & -(m_4-im_5) & -(m_1+im_2) & 0 \\ 0 & -(m_1+im_2) & -(m_4-im_5) & -(m_6-im_7) & -(f_3-m_3) & -(f_1-if_2) & -(f_4-if_5) & (m_6-im_7) \\ -(m_1+im_2) & 0 & (m_6+im_7) & -(m_4+im_5) & -(f_1+if_2) & (f_3+m_3) & -(f_6-if_7) & (m_4+im_5) \\ (m_4-im_5) & -(m_6+im_7) & 0 & -(m_1-im_2) & -(f_4+if_5) & -(f_6+if_7) & -2m_3 & (m_1-im_2) \\ (m_6-im_7) & (m_4+im_5) & (m_1-im_2) & 0 & (m_6+im_7) & (m_4-im_5) & (m_1+im_2) & 0 \end{bmatrix}$$

$$Y^\dagger = Y,$$

and Y can be written in the form

$$Y = \begin{bmatrix} U_3 & \mathbf{x}^T/\sqrt{2} & O^\dagger & -\mathbf{x}^T/\sqrt{2} \\ \mathbf{x}^*/\sqrt{2} & 0 & \mathbf{x}/\sqrt{2} & 0 \\ O & \mathbf{x}^\dagger/\sqrt{2} & -U_3^* & -\mathbf{x}^\dagger/\sqrt{2} \\ -\mathbf{x}^*/\sqrt{2} & 0 & -\mathbf{x}/\sqrt{2} & 0 \end{bmatrix}$$

or alternatively as

$$Y = \begin{pmatrix} D & -E^* \\ E & -D^* \end{pmatrix},$$

where

$$D = \begin{pmatrix} U_3 & \mathbf{x}^T/\sqrt{2} \\ \mathbf{x}^*/\sqrt{2} & 0 \end{pmatrix}, \quad E = \begin{pmatrix} O & \mathbf{x}^\dagger/\sqrt{2} \\ -\mathbf{x}^*/\sqrt{2} & 0 \end{pmatrix}.$$

Note that the matrices D and E are not independent. D involves all the parameters of E. Keeping this point in mind, we see that $E \in$ complex $\mathcal{L}SO(4)$, $D \in \mathcal{L}SU(4)$. Matrices E close under the Lie product. Matrices D need one more generator to close under Lie product to form the four-dimensional representation of the Lie algebra of $SU(4)$.

The above form of G_2 as the automorphism group of split octonions is called the split G_2. Under the $SU(3)$ subgroup of split G_2 leaving u_0 and u_0^* invariant, the three split octonions (u_1, u_2, u_3) transform like a unitary triplet (quarks) and the complex conjugate octonions (u_1^*, u_2^*, u_3^*) transform like a unitary antitriplet (antiquarks). This property of split octonions is physically very important and plays a crucial role in obtaining a quark structure from the octonionic representations of Poincaré group.[14]

7. QUARK STRUCTURE IN THE SPLIT BASIS

To see another physically interesting property of split G_2 let us define the following basis for its Lie algebra[27]:

$$\begin{aligned}
&E_{12} = \tfrac{1}{2}(F_1 + iF_2), \quad E_{21} = \tfrac{1}{2}(F_1 - iF_2),\\
&E_{13} = \tfrac{1}{2}(F_4 + iF_5), \quad E_{31} = \tfrac{1}{2}(F_4 - iF_5),\\
&E_{23} = \tfrac{1}{2}(F_6 + iF_7), \quad E_{32} = \tfrac{1}{2}(F_6 - iF_7),\\
&F_3 = (E_{11} - E_{22}), \quad F_8 = (1/\sqrt{3})(E_{11} + E_{22} - 2E_{33}), \qquad (7.1)\\
&Q_1 = \tfrac{1}{2}(M_6 + iM_7), \quad Q_1^\dagger = \tfrac{1}{2}(M_6 - iM_7),\\
&Q_2 = \tfrac{1}{2}(M_4 - iM_5), \quad Q_2^\dagger = \tfrac{1}{2}(M_4 + iM_5),\\
&Q_3 = \tfrac{1}{2}(M_1 + iM_2), \quad Q_3^\dagger = \tfrac{1}{2}(M_1 - iM_2),
\end{aligned}$$

where the expressions for Λ_3 and Λ_8 are purely formal at this point and will be explained shortly. In this basis commutation relations of split G_2 have the form:

$$\begin{aligned}
&[Q_i, Q_j] = -(2/\sqrt{3})\epsilon_{ijk}Q_k^\dagger,\\
&[Q_i, Q_j^\dagger] = T_{ij}, \qquad i,j = 1,2,3,\\
&[E_{ij}, Q_k] = \delta_{jk}Q_i,\\
&[T_{ii}, T_{jj}] = 0, \qquad [E_{ij}, E_{ji}] = (T_{ii} - T_{jj}),\\
&[T_{ii}, E_{ij}] = E_{ij}, \qquad [T_{jj}, E_{ij}] = -E_{ij},\\
&[E_{ij}, E_{jk}] = E_{ik}, \qquad [E_{ji}, E_{kj}] = -E_{ki}
\end{aligned} \qquad (7.2)$$

where T_{ij} is defined as

$$T_{ij} = E_{ij}, \qquad i \neq j,$$

and

$$\begin{aligned}
&T_{11} = \tfrac{1}{2}F_3 + (1/2\sqrt{3})F_8 = \tfrac{1}{3}(2E_{11} - E_{22} - E_{33}),\\
&T_{22} = -\tfrac{1}{2}F_3 + (1/2\sqrt{3})F_8 = \tfrac{1}{3}(-E_{11} + 2E_{22} - E_{33}),\\
&T_{33} = -(1/\sqrt{3})F_8 = \tfrac{1}{3}(-E_{11} - E_{22} + 2E_{33}). \qquad (7.3)
\end{aligned}$$

The generators T_{ij} form the subalgebra $SU(3)$. The generators F_3 and F_8 form a Cartan subalgebra of both $SU(3)$ and G_2. If we assign quantum numbers to the generators of split G_2, i.e., to its adjoint representation, using as the generators of third component of isospin and hypercharge the generators $I_3 = \frac{1}{2}F$ and $Y = (1/\sqrt{3})F_8$, we find that three quarks, three antiquarks, and eight mesons can be imbedded in the adjoint representation of split G_2, i.e., we can have the correspondence

$$\begin{aligned}
&Q_1 \leftrightarrow p \text{ quark}, \qquad Q_1^\dagger \leftrightarrow \bar{p},\\
&Q_2 \leftrightarrow n \text{ quark}, \qquad Q_2^\dagger \leftrightarrow \bar{n}\\
&Q_3 \leftrightarrow \lambda \text{ quark}, \qquad Q_3^\dagger \leftrightarrow \bar{\lambda}\\
&E_{12} \leftrightarrow \pi^+ \text{ (or } \rho^+), \qquad E_{21} \leftrightarrow \pi^- \text{ (or } \rho^-),\\
&E_{13} \leftrightarrow K^+ (K^{*+}), \qquad E_{31} \leftrightarrow K^- (K^{*-}),\\
&E_{23} \leftrightarrow K^0 (K^{*0}), \qquad E_{32} \leftrightarrow \overline{K^0}\, (\overline{K^{*0}}),\\
&\Lambda_3 \leftrightarrow \pi^0 (\rho^0), \qquad \Lambda_8 \leftrightarrow \eta\, (\omega_8),
\end{aligned}$$

This identification agrees with Gell-Mann's quark model in the assignment of the quantum numbers I_3 and Y and differs from it in the assignment of baryon number. If one uses the generator N_3 of $SO(7)$ as the baryon number generator, one gets the result that mesons are assigned zero baryon number as they must be but that the generators Q_i ($\leftrightarrow$ quarks) and the generators $Q_i^\dagger$ ($\leftrightarrow$ antiquarks) do not have well-defined baryon numbers.[28] These (pseudo-quark) generators Q_i have the interesting property that they generate the (anti-pseudo-quarks) $Q_i^\dagger$ under commutation, i.e.,

$$[Q_i, Q_j] = -(2/\sqrt{3})\epsilon_{ijk}Q_k^\dagger$$

and the $SU(3)$ subalgebra (mesons) under Lie triple product, i.e.

$$[Q_i, [Q_j, Q_k]] = -(2/\sqrt{3})\epsilon_{jkl}T_{il}. \qquad (7.4)$$

8. AN OCTONIONIC REPRESENTATION OF SPLIT G_2

The split octonions

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

transform as the three-dimensional irreducible representation of the $SU(3)$ subgroup of split G_2. But the elements

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_0 \end{pmatrix}$$

do not form a four-dimensional irreducible representation of split G_2. In fact the lowest nontrivial representation of G_2 is seven dimensional. Yet the action of G_2 on the basis

$$[s] = \begin{pmatrix} u \\ u^* \end{pmatrix}$$

is completely defined by its action on u, i.e., if

$$G_2 : \begin{matrix} u \to u' \\ u^* \to (u^*)' \end{matrix},$$

then

$$(u^*)' = (u')^*.$$

The action of G_2 generators on u can be represented by multiplication with octonion units in the following compact form:

$$E_{ij}u = u_i(u_j^* u), \qquad u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_0 \end{pmatrix}, \quad i,j = 1,2,3,$$

$$\begin{aligned}
&Q_i u = -(1/\sqrt{3})uu_i + (1/\sqrt{3})(u_i u_0)u = -(1/\sqrt{3})uu_i,\\
&Q_i^\dagger u = -(1/\sqrt{3})uu_i^* + (1/\sqrt{3})(u_i^* u_0)u = -(1/\sqrt{3})[u, u_i^*].
\end{aligned}$$

with

$$\begin{aligned}
F_3 u &= (E_{11} - E_{22})u = u_1(u_1^* u) - u_2(u_2^* u),\\
F_8 u &= (1/\sqrt{3})(E_{11} + E_{22} - 2E_{33})u\\
&= (1/\sqrt{3})\{u_1(u_1^* u) + u_2(u_2^* u) - 2u_3(u_3^* u)\},
\end{aligned}$$

which justifies the formal expressions for F_3 and F_8 given above. Thus the above form of the Lie algebra action of G_2 generates an octonionic representation of split $\mathcal{L}G_2$. The automorphism group of real octonions can also be shown in this form because a real octonion

$$\Phi = \Phi_0 + \Phi_A e_A$$

can be written as

$$\begin{aligned}\Phi = 2\,\mathrm{Re}\{&(\phi_0 - i\phi_7)\tfrac{1}{2}(1 + ie_7)\\ &+ (\phi_1 - i\phi_4)\tfrac{1}{2}(e_1 + ie_4)\\ &+ (\phi_2 - i\phi_5)\tfrac{1}{2}(e_2 + ie_5)\\ &+ (\phi_3 - i\phi_6)\tfrac{1}{2}(e_3 + ie_6)\},\end{aligned}$$

where Re refers to the real part with respect to the complex unit i. Then

$$\Phi = \phi^\dagger u + \text{c.c.} = \phi^\dagger u + \phi^T u^*,$$

where

$$\phi = \begin{pmatrix}\phi_1 + i\phi_4\\ \phi_2 + i\phi_5\\ \phi_3 + i\phi_6\\ \phi_0 + i\phi_7\end{pmatrix} \qquad u = \begin{pmatrix}u_1\\ u_2\\ u_3\\ u_0\end{pmatrix}$$

Therefore, the action of Aut**O** on e_A is completely defined by its action on u.

9. 8 x 8 MATRIX FORMULATION OF THE CAYLEY ALGEBRA

In Appendix D, we give two constructions of Cayley algebra in terms of 3×3 λ-matrices and 4×4 γ-matrices. In this section, we shall study the 8×8 matrix construction of octonions. Consider the column matrix

$$[s] = \begin{bmatrix}u\\ u^*\end{bmatrix}$$

of split octonions. Define the conjugate matrix $[\bar{s}]^\dagger$ as

$$\begin{aligned}[\bar{s}]^\dagger &= [\bar{s}]^{*T}\\ &= (-u_1^*, -u_2^*, -u_3^*, u_0, -u_1, -u_2, -u_3, u_0^*),\end{aligned}$$

where the overbar denotes octonion conjugation, * denotes complex conjugation, and T is the usual transposition. Then the product $[s][\bar{s}]^\dagger$ can be written in the form

$$[s][\bar{s}]^\dagger = \tfrac{1}{2}(1 - i\Gamma_A e_A), \quad A = 1, \ldots, 7, \tag{9.1}$$

where Γ_A are 8×8 matrices given by:

$$\begin{aligned}\Gamma_1 &= -\sigma_1 \otimes \sigma_1 \otimes \sigma_2 = -\tau_1\rho_1\sigma_2,\\ \Gamma_2 &= -\sigma_1 \otimes \sigma_2 \otimes I = -\tau_1\rho_2,\\ \Gamma_3 &= \sigma_1 \otimes \sigma_3 \otimes \sigma_2 = \tau_1\rho_3\sigma_2,\\ \Gamma_4 &= \sigma_2 \otimes \sigma_2 \otimes \sigma_1 = \tau_2\rho_2\sigma_1,\\ \Gamma_5 &= -\sigma_2 \otimes \sigma_2 \otimes \sigma_3 = -\tau_2\rho_2\sigma_3,\\ \Gamma_6 &= \sigma_2 \otimes I \otimes \sigma_2 = \tau_2\sigma_2,\\ \Gamma_7 &= -\sigma_3 \otimes I \otimes I = -\tau_3\end{aligned} \tag{9.2}$$

($\otimes$ denotes direct product of the Pauli matrices $\sigma_1, \sigma_2, \sigma_3, I$),

where we have defined

$$\begin{aligned}I \otimes I \otimes \sigma_i &= \sigma_i,\\ I \otimes \sigma_i \otimes I &= \rho_i,\\ \sigma_i \otimes I \otimes I &= \tau_i,\end{aligned}$$

and chosen a representation in which $\Gamma_1, \Gamma_2, \Gamma_3$ are imaginary and $\Gamma_4, \Gamma_5, \Gamma_6, \Gamma_7$ are real. These seven matrices Γ_A are Hermitian and satisfy the anticommutation relations

$$\{\Gamma_A, \Gamma_B\} = 2\delta_{AB}, \quad \Gamma_A^\dagger = \Gamma_A. \tag{9.3}$$

Now number the rows of the column vector $[s]$ from 1 to 8 and define a mapping L_{u_i} on $[s]$ as the mapping induced by multiplication from the left by the element u_i.[29] Then a simple calculation gives the result that

$$\begin{aligned}L_{u_1}^{(8)} + L_{u_1^*}^{(8)} \quad L_{u_1+u_1^*}^{(8)} &= L_{e_1}^{(8)} = \bar{\Sigma}_{18} - \bar{\Sigma}_{27} + \bar{\Sigma}_{36} - \bar{\Sigma}_{45},\\ L_{u_1-u_1^*}^{(8)} &= L_{ie_4}^{(8)} = \bar{E}_{81} + \bar{E}_{18}\\ &\qquad - \bar{E}_{54} - \bar{E}_{45} - \bar{E}_{36} - \bar{E}_{63} + \bar{E}_{27} + \bar{E}_{72},\\ L_{u_2+u_2^*}^{(8)} &= L_{e_2}^{(8)} = \bar{\Sigma}_{28} - \bar{\Sigma}_{46} + \bar{\Sigma}_{53} + \bar{\Sigma}_{17},\\ L_{u_2-u_2^*}^{(8)} &= L_{ie_5}^{(8)} = -\bar{E}_{71} - \bar{E}_{17}\\ &\qquad + \bar{E}_{53} + \bar{E}_{35} - \bar{E}_{46} - \bar{E}_{64} + \bar{E}_{28} + \bar{E}_{82},\\ L_{u_3+u_3^*}^{(8)} &= L_{e_3}^{(8)} = \bar{\Sigma}_{61} - \bar{\Sigma}_{52} - \bar{\Sigma}_{47} + \bar{\Sigma}_{38},\\ L_{u_3-u_3^*}^{(8)} &= L_{ie_6}^{(8)} = \bar{E}_{61} + \bar{E}_{16}\\ &\qquad - \bar{E}_{52} - \bar{E}_{25} - \bar{E}_{47} - \bar{E}_{74} + \bar{E}_{38} + \bar{E}_{83},\\ L_{u_4-u_4^*}^{(8)} &= L_{ie_7}^{(8)} = \bar{E}_{11} + \bar{E}_{22}\\ &\qquad + \bar{E}_{33} + \bar{E}_{44} - \bar{E}_{55} - \bar{E}_{66} - \bar{E}_{77} - \bar{E}_{88},\end{aligned} \tag{9.4}$$

where $\bar{E}_{ij}$ are the 8×8 matrix units and $\bar{\Sigma}_{ab} = \bar{E}_{ab} - E_{ba}$

Comparing the matrices Γ_A with the mappings L_{e_A} considered as matrices acting on the basis $[s]$ we have

$$\begin{aligned}L_{e_1}^{(8)} &= -i\Gamma_1, & L_{e_4}^{(8)} &= i\Gamma_4,\\ L_{e_2}^{(8)} &= -i\Gamma_2, & L_{e_5}^{(8)} &= i\Gamma_5, & L_{e_7}^{(8)} &= i\Gamma_7 \quad \text{on} \begin{bmatrix}u\\ u^*\end{bmatrix}.\\ L_{e_3}^{(8)} &= -i\Gamma_3, & L_{e_6}^{(8)} &= i\Gamma_6,\end{aligned} \tag{9.5}$$

From these equalities, the anticommutation relations of Γ_A follow automatically, since

$$L_{e_A}L_{e_B} + L_{e_B}L_{e_A} = L_{e_Ae_B + e_Be_A} = L_{-2\delta_{AB}} \tag{9.6}$$

which in turn follows from the identity

$$O_1(O_2O_3) + O_2(O_1O_3) = (O_1O_2 + O_2O_1)O_3, \qquad O_1, O_2, O_3 \in O \tag{9.7}$$

for octonions.

Now define the matrices Γ_{AB} as

$$\begin{aligned}\Gamma_{AB} &= (1/2i)[\Gamma_A, \Gamma_B],\\ \Gamma_{AB} &= -\Gamma_{BA}, \quad \Gamma_{AB}^\dagger = \Gamma_{AB}.\end{aligned} \tag{9.8}$$

Twenty-one matrices Γ_{AB} form the Lie algebra of Spin (7) and $\Gamma_{AB} \oplus \Gamma_A$ form the Lie algebra of $SO(8)$.

Having constructed the matrices $\mathbf{\Gamma}_A$ and $\mathbf{\Gamma}_{AB}$ from the spinor $[s]$, we can forget about the octonionic character of $[s]$ and consider an eight-component spinor Ψ. Then

$$\{V_8 = \Psi^\dagger \Psi,\ V_A = \Psi^\dagger \mathbf{\Gamma}_A \Psi\}$$

transform like a vector under $SO(8)$. Under the subgroup Spin (7), $\Psi^\dagger\Psi$ is a scalar and $\Psi^\dagger \mathbf{\Gamma}_A \Psi = V_A$ is a vector. To characterize G_2, we need one more condition in addition to the requirement that $\Psi^\dagger\Psi$ be a scalar. Now G_2 is the automorphism group of octonions and it leaves the identity invariant. Therefore we would expect the G_2 subgroup of $SO(7)$ to leave $(\Psi_4^* + \Psi_8^*)(\Psi_4 + \Psi_8)$ invariant, since $(\Psi_4 + \Psi_8)$ corresponds to the identity of the octonions in the nonoctonionic formulation considered here. Thus, under G_2 both $\Psi^\dagger\Psi$ and $\Psi^\dagger K \Psi$ are scalars, where K is the matrix

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In fact, the assertion that $\Psi^\dagger K \Psi$ is a scalar under G_2 can be rigorously proved by showing that only those linear combinations of the generators $\mathbf{\Gamma}_{AB}$ of Spin (7) that belong to G_2 commute with the matrix K. To do this it is convenient to use the following expression for K

$$K = \tfrac{1}{4}[\mathbf{1} - i(1/3!) a_{ABC} \mathbf{\Gamma}_A \mathbf{\Gamma}_B \mathbf{\Gamma}_C], \qquad A, B, C, = 1, \ldots, 7, \tag{9.9}$$

where a_{ABC} is a totally antisymmetric tensor and satisfies

$$a_{ABC} = 1 \text{ for } ABC = 123, 246, 435, 516, 572, 471, 673.$$

The matrix K is related to the octonion conjugation matrix O^c defined by $O^c[s] = [\bar{s}]$ as

$$O^c = K - \mathbf{1} \quad \text{or} \quad K = \mathbf{1} + O^c.$$

Therefore the conditions that $\Psi^\dagger K \Psi$ be a scalar is equivalent to the condition that $\Psi^\dagger O^c \Psi$ be a scalar.

The conditions that characterize G_2, i.e., that $\Psi^\dagger\Psi$ and $\Psi^\dagger K \Psi$ be invariant, are equivalent to saying that G_2 is the common subgroup of $SO(7)$ and Spin (7),[30] i.e.,

$$G_2 = \text{Spin }(7) \cap SO(7).$$

Above we showed that the matrices $\mathbf{\Gamma}_A$ correspond to the left multiplication by octonion units e_A acting on $[s]$. This does not mean that e_A can be represented by matrices $\mathbf{\Gamma}_A$ to form a Cayley algebra under the usual matrix multiplication. To get a Cayley algebra from $\mathbf{\Gamma}_A$, we define the product of two $\mathbf{\Gamma}$ matrices as

$$\mathbf{\Gamma}_A \circ \mathbf{\Gamma}_B = \tfrac{1}{2}\{\mathbf{\Gamma}_A, \mathbf{\Gamma}_B\} + \tfrac{1}{2}\{[\mathbf{\Gamma}_A, \mathbf{\Gamma}_B], M\} + \mathbf{\Gamma}_A M \mathbf{\Gamma}_B - \mathbf{\Gamma}_B M \mathbf{\Gamma}_A, \tag{9.10}$$

where

$$M = K - \tfrac{1}{4}\mathbf{1} = -\tfrac{1}{4} i (1/3!) a_{ABC} \mathbf{\Gamma}_A \mathbf{\Gamma}_B \mathbf{\Gamma}_C,$$

which gives

$$\mathbf{\Gamma}_A \circ \mathbf{\Gamma}_B = \delta_{AB} + i a_{ABC} \mathbf{\Gamma}_C, \qquad A, B, C = 1, \ldots, 7.$$

Hence defining the multiplication by a multiple c of the identity $\mathbf{1}$ as multiplication by the scalar c, we get a Cayley algebra with basis

$$\begin{aligned} &e_A \equiv -i\mathbf{\Gamma}_A, \qquad 1 \equiv \mathbf{1}, \\ &(-i\mathbf{\Gamma}_A) \circ (-i\mathbf{\Gamma}_B) = -\delta_{AB} + a_{ABC}(-i\mathbf{\Gamma}_C). \end{aligned} \tag{9.11}$$

10. IMBEDDING IN $SO(7)$ AND $SO(8)$

In the above, we have decomposed the Lie algebra of $SO(7)$ as

$$SO(7) = F_A \oplus M_A \oplus N_A,$$

where $F_A \oplus M_A$ generate the subgroup G_2.

The 8-dimensional representation of G_2 as the automorphism group of octonions acting on the basis

$$[s] = \begin{bmatrix} u \\ u^* \end{bmatrix}$$

will induce an 8-dimensional spinor representation of $SO(7)$: In fact, after some algebra, one finds that the action of N_A on $[s]$ can be represented as

$$\begin{aligned} \nu_1^{(8)} &= \tfrac{1}{2} i (L_{e_6}^{(8)} - R_{e_6}^{(8)}), & \nu_2^{(8)} &= -\tfrac{1}{2} i (L_{e_3}^{(8)} - R_{e_3}^{(8)}), \\ \nu_3^{(8)} &= \tfrac{1}{2} i (L_{e_7}^{(8)} - R_{e_7}^{(8)}), & \nu_5^{(8)} &= -\tfrac{1}{2} i (L_{e_2}^{(8)} - R_{e_2}^{(8)}), \\ \nu_4^{(8)} &= \tfrac{1}{2} i (L_{e_5}^{(8)} - R_{e_5}^{(8)}), & \nu_7^{(8)} &= -\tfrac{1}{2} i (L_{e_1}^{(8)} - R_{e_1}^{(8)}), \\ \nu_6^{(8)} &= \tfrac{1}{2} i (L_{e_4}^{(8)} - R_{e_4}^{(8)}), \end{aligned} \tag{10.1}$$

where L_{e_A} and R_{e_A} are left and right multiplications by the element e_A, respectively. Explicit matrix form of L_{e_A} was given above. For the R_{e_A} we have

$$\begin{aligned} R_{e_1}^{(8)} &= \bar{\Sigma}_{27} + \bar{\Sigma}_{58} + \bar{\Sigma}_{14} + \bar{\Sigma}_{63}, \\ R_{e_2}^{(8)} &= \bar{\Sigma}_{71} + \bar{\Sigma}_{35} + \bar{\Sigma}_{24} + \bar{\Sigma}_{68}, \\ R_{e_3}^{(8)} &= \bar{\Sigma}_{78} + \bar{\Sigma}_{52} + \bar{\Sigma}_{34} + \bar{\Sigma}_{16}, \\ R_{e_4}^{(8)} &= -i\{(\bar{E}_{14} + \bar{E}_{41}) \\ &\quad + (\bar{E}_{36} + \bar{E}_{63}) - (\bar{E}_{58} + \bar{E}_{85}) - (\bar{E}_{27} + \bar{E}_{72})\}, \\ R_{e_5}^{(8)} &= -i\{(\bar{E}_{17} + \bar{E}_{71}) \\ &\quad + (\bar{E}_{24} + \bar{E}_{42}) - (\bar{E}_{35} + \bar{E}_{53}) - (\bar{E}_{68} + \bar{E}_{86})\}, \\ R_{e_6}^{(8)} &= -i\{(\bar{E}_{25} + \bar{E}_{52}) \\ &\quad + (\bar{E}_{34} + \bar{E}_{43}) - (\bar{E}_{16} + \bar{E}_{61}) - (\bar{E}_{78} + \bar{E}_{87})\}, \\ R_{e_7}^{(8)} &= -i\{(\bar{E}_{44} + \bar{E}_{55} \\ &\quad + \bar{E}_{66} + \bar{E}_{77}) - (\bar{E}_{11} + \bar{E}_{22} + \bar{E}_{33} + \bar{E}_{88})\}. \end{aligned} \tag{10.2}$$

We have shown that the 8×8 matrices $\mathbf{\Gamma}_A$ and $\mathbf{\Gamma}_{AB}$ form an eight-dimensional representation of the Lie algebra of $SO(8)$. Since $\mathbf{\Gamma}_A$ correspond to the left multiplication by e_A acting on $[s]$, we have the result that the eight-dimensional representation of $\mathcal{L}SO(8)$ can also be decomposed as:

$$\mathcal{L}SO(8) = \Lambda_A^{(8)} \oplus \mu_A^{(8)} \oplus \nu_A^{(8)} \oplus \xi_A^{(8)},$$

where $\xi_A^{(8)} = \frac{1}{2}i(L_{e_A}^{(8)} + R_{e_A}^{(8)})$ corresponding to the generator

$$Z_A = \tfrac{1}{2}i(L_{e_A} + R_{e_A}).$$

Since the group $SO(8)$ has rank four, its Cartan subalgebra will be four dimensional. One can redefine the generators of $SO(8)$ such that F_3, M_3, N_3, and Z_3 form a Cartan subalgebra.

If we take $I_3 = \frac{1}{2}F_3$, $Y_3 = -(1/\sqrt{3})M_3$, $B = -\frac{1}{3}N_3$, $\frac{1}{2}Z_3$ as the Cartan subalgebra generators, we can assign the following quantum numbers to the basis elements:

	I_3	Y_3	B	$\frac{1}{2}Z_3$
u_1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	0
u_1^*	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{3}$	0
u_2	$-\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	0
u_2^*	$\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{3}$	0
u_3	0	$-\frac{2}{3}$	$\frac{1}{3}$	0
u_3^*	0	$+\frac{2}{3}$	$-\frac{1}{3}$	0
u_0	0	0	0	-1
u_0^*	0	0	0	1

Therefore, under the correspondence

$(u_1, u_2, u_3) \leftrightarrow (p, n, \lambda)$ quarks

$(u_1^*, u_2^*, u_3^*) \leftrightarrow (\bar{p}, \bar{n}, \bar{\lambda})$ antiquarks

$(u_0, u_0^*) \leftrightarrow$ (core, anticore)

we have the result that I_3, Y_3, and B act like the generators of third component of isospin, hypercharge, and baryon number. Subscript 3 in Y_3 refers to the fact that within G_2, Y is the generator of the third component of an $SU(2)$ subgroup just as I_3 is.

11. REDUCTION WITH RESPECT TO THE $SU(2) \times SU(2)$ [I-SPIN-G-SPIN] SUBGROUP OF G_2

The generators $I_i = F_i$, $G_i = \sqrt{3}M_i$, $i = 1, 2, 3$ form an $SU(2) \times SU(2)$ subalgebra of $\mathcal{L}G_2$:

$$[I_i, I_j] = 2i\epsilon_{ijk}I_k, \quad [G_i, G_j] = 2i\epsilon_{ijk}G_k,$$
$$[I_i, G_j] = 0, \quad i, j, k = 1, 2, 3. \tag{11.1}$$

I_i is the isospin subalgebra of the $SU(3)$ subalgebra of $\mathcal{L}G_2$ annihilating the basis element e_7. Now the spinors

$$\psi = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \leftrightarrow \begin{pmatrix} p \\ n \end{pmatrix}$$

and

$$\psi^* = \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} \leftrightarrow \begin{pmatrix} \bar{p} \\ \bar{n} \end{pmatrix}$$

correspond to isospin doublets and the elements u_3, u_3^*, $(1/\sqrt{2})ie_7$ are isospin scalars.

Consider the infinitesimal group action generated by G_i

$$G: \psi \to \psi' = (1 - im^3)\psi - (m^2 + im^1)\psi^G,$$
$$\psi^{G'} = (m^2 - im^1)\psi + (1 + im^3)\psi^G,$$

where ψ^G is the G parity conjugate spinor defined by

$$\psi^G = i\tau_2\psi^* \quad \text{and} \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Therefore under the G-spin subgroup (generated by G_1) of G_2 the spinor ψ and ψ^G form a G-spin doublet and transform as

$$G: \begin{pmatrix} \psi \\ \psi^G \end{pmatrix} \to \begin{pmatrix} \psi' \\ \psi^{G'} \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}\begin{pmatrix} \psi \\ \psi^G \end{pmatrix}, \tag{11.2}$$

where

$$|a|^2 + |b|^2 = 1.$$

Similarly we find that

$$\phi = \begin{pmatrix} u_3 \\ ie_7/\sqrt{2} \\ u_3^* \end{pmatrix}$$

forms a G-spin triplet which transforms infinitesimally as

$$G: \phi \to \phi' = \begin{pmatrix} 1 + 2im^3 & -i\sqrt{2}(m^1 + im^2) & 0 \\ -i\sqrt{2}(m^1 - im^2) & 1 & -i\sqrt{2}(m^1 + im^2) \\ 0 & -i\sqrt{2}(m^1 - im^2) & 1 - 2im^3 \end{pmatrix}\begin{pmatrix} u_3 \\ ie_7/\sqrt{2} \\ u_3^* \end{pmatrix},$$

the global form of which is

$$G: \phi \to \phi' = \begin{pmatrix} a^2 & \sqrt{2}ab & b^2 \\ -\sqrt{2}ab^* & |a|^2 - |b|^2 & \sqrt{2}a^*b \\ b^{*2} & -\sqrt{2}a^*b^* & a^{*2} \end{pmatrix}\phi. \tag{11.3}$$

An important property of G-spin is that its third component is proportional to hypercharge Y, i.e.,

$$Y = -\tfrac{1}{3}G_3$$

and hence it should properly be called hypercharge spin. This hypercharge-spin subgroup of G_2 commutes with the isospin subgroup generated by F_1, F_2, F_3. The isospin and hypercharge spin groups together generate a four-dimensional rotation group $SU(2) \times SU(2)$ which has been considered before[31] as applied to an isotopic doublet such as the nucleon or the p and n quarks. The multiplets $(u_1, u_2, u_2^*, -u_1^*)$ and $(u_3, (ie_7/\sqrt{2}), u_3^*)$ form the (1/2, 1/2) and (0, 1) representations of the subgroup $SU(2)_I \otimes SU(2)_Y$, respectively. The $SU(3)$ singlet $(ie_7/\sqrt{2})$ is not an hypercharge spin singlet. It transforms like the third component of an hypercharge triplet. We shall call it the vacuum v. Therefore the lowest-dimensional representation of G_2 has the root system shown is Fig. 3. Above we defined the G-parity conjugate spinor ψ^G of an isospin doublet ψ as

$$\psi^G = Ce^{i\pi I_2}\psi,$$

where I_2 is the second component of isospin and C is charge conjugation which in our case is taken as complex conjugation. We will generalize this G-parity concept to other charge space $SU(2)$ groups as follows: Write the above equation as

$$\Psi_I^{G^I} = Ce^{i\pi I_2}\psi_I; \tag{11.3a}$$

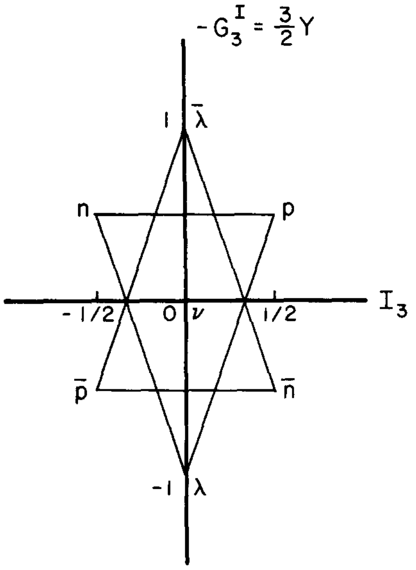

FIG. 3.

then for U and V spins we can define

$$\psi_U^{G^U} = Ce^{i\pi U_2}\psi_U \tag{11.3b}$$

$$\psi_V^{G^V} = Ce^{i\pi V_2}\psi_V \tag{11.3c}$$

Then under the $SU(2)_U \otimes SU(2)_{G^U}$ subgroup of G_2 generated by

$$\begin{aligned} &U_1 = \tfrac{1}{2}F_6, \quad U_2 = \tfrac{1}{2}F_7, \\ &U_3 = \tfrac{1}{4}(-\sqrt{3}M_3 - F_3), \\ &G_1^U = (\sqrt{3}/2)M_6, \quad G_2^U = (\sqrt{3}/2)M_7, \\ &G_3^U = (\sqrt{3}/4)\left(\frac{1}{\sqrt{3}}F_3 - M_3\right). \end{aligned} \tag{11.4a}$$

The spinor

$$\begin{pmatrix} \psi_U \\ \psi_U^{G^U} \end{pmatrix}$$

transforms like the (1/2, 1/2) representation, where

$$\psi_U = \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} \leftrightarrow \begin{pmatrix} n \\ \lambda \end{pmatrix},$$

and the spinor

$$\phi_U = \begin{pmatrix} u_1 \\ (i/\sqrt{2})e_7 \\ u_1^* \end{pmatrix}$$

will transform as the (0, 1) representation. Same thing applies for the $SU(2)_V \otimes SU(2)_{G^V}$ subgroup generated by

$$\begin{aligned} &V_1 = \tfrac{1}{2}F_4, \quad V_2 = \tfrac{1}{2}F_5, \\ &V_3 = \tfrac{1}{4}(F_3 - \sqrt{3}M_3), \\ &G_1^V = (\sqrt{3}/2)M_4, \quad G_2^V = (\sqrt{3}/2)M_5, \end{aligned} \tag{11.4b}$$

$$G_3^V = (\sqrt{3}/4)(M_3 + \sqrt{3}F_3), \tag{11.4c}$$

except we have to replace ψ_U by

$$\psi_V = \begin{pmatrix} u_3 \\ u_1 \end{pmatrix} \leftrightarrow \begin{pmatrix} \lambda \\ p \end{pmatrix}$$

and ϕ_U by

$$\phi_V = \begin{pmatrix} u_2 \\ (i/\sqrt{2})e_7 \\ u_2^* \end{pmatrix}.$$

Hence, we have the result that just as the group $SU(3)$ contains three overlapping $SU(2)$ groups corresponding to I, U, V spins, the group G_2 contains three overlapping $SU(2) \times SU(2)_G$ groups corresponding to I, U, V spins together with their generalized G-parity extensions. We shall call the $SU(2)_G$ groups corresponding to I, U, V spins, the hypercharge spin, charge spin, and hypocharge spin groups, respectively. Under the $SU(2) \times SU(2)_G$ subgroup, the adjoint representation of G_2 decomposes as

$$14 = (1, 0) \oplus (0, 1) \oplus (1/2, 3/2).$$

So far we have considered the decomposition of $\mathcal{L}SO(8)$ in terms of seven anticommutating matrices Γ_A, which correspond to left multiplication by the basis elements e_A of octonions acting on the basis

$$[s] = \begin{vmatrix} u \\ u^* \end{vmatrix}.$$

A more consistent approach is to consider matrices corresponding to multiplication by the split octonions: Since

$$L_{u_i} = L_{1/2(e_i + ie_{i+3})} = \tfrac{1}{2}(L_{e_i} + iL_{e_{i+3}}), \qquad i = 1, 2, 3,$$

we have

$$\begin{aligned} &L_{u_i}^{(8)} = U_i = -\tfrac{1}{2}(\Gamma_{i+3} + \Gamma_i), \\ &L_{u_i^*}^{(8)} = \tilde{U}_i = \tfrac{1}{2}(\Gamma_{i+3} - i\Gamma_i), \\ &L_{u_0}^{(8)} = U_0 = \tfrac{1}{2}(1 - \Gamma_7), \\ &L_{u_0^*}^{(8)} = \tilde{U}_0 = \tfrac{1}{2}(1 + \Gamma_7). \end{aligned} \tag{11.5}$$

One can also change the basis on which the octonion units act and consider the real octonion basis on which real or split octonions may act. In any case, octonion multiplication operators L_a or R_a, $a \in \mathbf{O}$ is uniquely defined and choosing different bases on which they can act changes their matrix representations.

12. LIE MULTIPLICATION ALGEBRA OF OCTONIONS AND THE PRINCIPLE OF TRIALITY

A derivation D of an algebra A is defined as a linear transformation satisfying the property:

$$D(xy) = (Dx)y + x(Dy) \quad \text{for all } x, y \in A. \tag{12.1}$$

Derivations of an algebra A form a Lie algebra under Lie product, i.e.,

$$[D_i, D_j]_- = -[D_j, D_i]$$

and for all $D_i, D_j, D_k \in \mathrm{Der}A$,

$$[D_i,[D_j,D_k]] + [D_k,[D_i,D_j]] + [D_j,[D_k,D_i]] = 0,$$

Jacobi identity.

Derivation algebra of an algebra A is isomorphic to the Lie algebra of the automorphism group of A,[32] i.e., if $D \in \mathrm{Der}\ A$

$$D(xy) = (Dx)y + x(Dy) \Rightarrow e^D(xy) = (e^Dx)(e^Dy).$$

Therefore the derivation (Lie) algebra of octonions is isomorphic to the Lie algebra of G_2. Lie multiplication algebra of the octonions is defined as the Lie algebra with elements:

$$\mathfrak{L}M\mathbf{O} = \operatorname{Der}\mathbf{O} \oplus L_{\mathbf{O}_0} \oplus R_{\mathbf{O}_0},$$

where $L_{\mathbf{O}_0}$ and $R_{\mathbf{O}_0}$ correspond to multiplication from the left and the right by traceless (or imaginary) octonion units. Since the octonions are not associative left and right multiplications do not commute. As was shown above, the Lie multiplication algebra of octonions is isomorphic to the Lie algebra of the group $SO(8)$.

$$\mathfrak{L}SO(8) = F_A \oplus M_A \oplus N_A \oplus Z_A, \tag{12.2}$$

where

$$\operatorname{Der}\mathbf{O} \cong F_A \oplus M_A.$$

The usual real octonionic norm is invariant under the group $SO(8)$. Denoting the parameters corresponding to the generators $\Lambda_A^{(8)}, \mu_A^{(8)}, \nu_A^{(8)}, \xi_A^{(8)}$ by f_A, m_A, n_A, z_A, we can represent the action of $SO(8)$ on the split octonion basis $[s]$ by

$$SO(8):[s] \to e^{iL}[s], \qquad [s] = \begin{bmatrix} u \\ u^* \end{bmatrix}, \qquad u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_0 \end{pmatrix},$$

$$L = \begin{pmatrix} U & V \\ V^\dagger & -U_4^* \end{pmatrix},$$

where

$$U_4 = \begin{pmatrix} (f_3 - m_3 - n_3) & (f_1 + if_2) & (f_4 + if_5) & (\zeta_1 + i\zeta_2) \\ (f_1 - if_2) & -(f_3 + m_3 + n_3) & (f_6 + if_7) & (\zeta_3 + i\zeta_4) \\ (f_4 - if_5) & (f_6 - if_7) & (2m_3 - n_3) & (\zeta_5 + i\zeta_6) \\ (\zeta_1 - i\zeta_2) & (\zeta_3 - i\zeta_4) & (\zeta_5 - i\zeta_6) & -2z_3 \end{pmatrix}, \tag{12.3}$$

where

$$\zeta_1 = -m_6 + \tfrac{1}{2}(n_6 - z_6), \qquad \zeta_4 = m_5 - \tfrac{1}{2}(n_5 + z_5),$$
$$\zeta_2 = -m_7 - \tfrac{1}{2}(n_7 + z_7), \qquad \zeta_5 = -m_1 + \tfrac{1}{2}(n_1 - z_1),$$
$$\zeta_3 = -m_4 + \tfrac{1}{2}(n_4 - z_4), \qquad \zeta_6 = -m_2 - \tfrac{1}{2}(n_2 + z_2).$$

Matrices U_4 close under commutation and form the four-dimensional representation of $\mathfrak{L}U(4)$.

The matrices V are antisymmetric.

$V_{\mu\nu} = -V_{\nu\mu}$ and form the Lie algebra of complex $SO(4)$:

$$V_{12} = -(m_1 + n_1) + i(m_2 - n_2),$$
$$V_{13} = (m_4 + n_4) + i(m_5 + n_5),$$
$$V_{14} = [m_6 - \tfrac{1}{2}(n_6 + z_6)] + i[m_7 + \tfrac{1}{2}(n_7 - z_7)],$$
$$V_{23} = -(m_6 + n_6) + i(m_7 - n_7),$$
$$V_{24} = [m_4 - \tfrac{1}{2}(n_4 + z_4)] - i[m_5 - \tfrac{1}{2}(n_5 - z_5)],$$
$$V_{34} = [m_1 - \tfrac{1}{2}(n_1 + z_1)] + i[m_2 + \tfrac{1}{2}(n_2 - z_2)].$$

Denoting U_4 as U, we have that U and V can be decomposed as

$$\begin{aligned} U &= U_{G_2} \oplus U_{SO(8)/G_2} \\ L &= L_{G_2} \oplus L_{SO(8)/G_2}, \\ V &= V_{G_2} \oplus V_{SO(8)/G_2} \end{aligned} \tag{12.4}$$

where $\oplus$ refers to vector space direct sum and V_{G_2}, U_{G_2} involve only the parameters f_A and m_A and $U_{SO(8)/G_2}$ and $V_{SO(8)/G_2}$ involve only n_A and z_A. Below we will construct the $\mathfrak{L}SO(8)$ matrices that are in local triality with each other (see Appendix C for the principle of triality.) The principle of local triality states that for a given matrix $T^L \in \mathfrak{L}SO(8)$ acting on the 8-dimensional space of octonions and which is skew with respect to the natural bilinear form (x, y) defined over the octonions, there exist uniquely determined matrices T^R and T^P belonging to the Lie algebra of $SO(8)$ (i.e., which are skew with respect to the norm form) such that

$$T^P(xy) = (T^L x)y + x(T^R y), \qquad x, y \in \mathbf{O},$$
$$T^P, T^L, T^R \in \mathfrak{L}SO(8). \tag{12.5}$$

Decomposing T^P and T^R and T^L as above,

$$T^L = T^L_{G_2} \oplus T^L_{SO(8)/G_2},$$
$$T^R = T^R_{G_2} \oplus T^R_{SO(8)/G_2},$$
$$T^P = T^P_{G_2} \oplus T^P_{SO(8)/G_2},$$

we have

$$T^P_{G_2}(xy) + T^P_{SO(8)/G_2}(xy) = (T^L_{G_2}x)y + (T^L_{SO(8)/G_2}x)y + x(T^R_{G_2}y) + x(T^R_{SO(8)/G_2}y).$$

Now since G_2 is the automorphism group of octonions, its Lie algebra will be the derivation algebra of octonions satisfying

$$D(xy) = (Dx)y + x(Dy), \qquad D \in \text{Lie algebra of } G_2 = \mathfrak{L}G_2.$$

Hence it follows that

$$D = T^P_{G_2} = T^L_{G_2} = T^R_{G_2} \in \mathfrak{L}G_2.$$

In other words under the triality mappings

$$\begin{tikzcd} L \arrow[rr] & & R \arrow[dl] \\ & P \arrow[ul] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} L \arrow[dr] & & R \arrow[ll] \\ & P \arrow[ur] & \end{tikzcd}$$

$\mathfrak{L}G_2$ subalgebra of $\mathfrak{L}SO(8)$ remains fixed. If we let

$$T^L_{SO(8)/G_2} = \begin{pmatrix} U_{SO(8)/G_2} & V_{SO(8)/G_2} \\ V^\dagger_{SO(8)/G_2} & -U^*_{SO(8)/G_2} \end{pmatrix}, \tag{12.6a}$$

$$T^R_{SO(8)/G_2} = \begin{pmatrix} A_{SO(8)/G_2} & B_{SO(8)/G_2} \\ B^\dagger_{SO(8)/G_2} & -A^*_{SO(8)/G_2} \end{pmatrix}, \tag{12.6b}$$

$$T^P_{SO(8)/G_2} = \begin{pmatrix} C_{SO(8)/G_2} & D_{SO(8)/G_2} \\ D^\dagger_{SO(8)/G_2} & -C^*_{SO(8)/G_2} \end{pmatrix}, \tag{12.6c}$$

where

$$U_{SO(8)/G_2}=\begin{pmatrix} -n_3 & 0 & 0 & [\frac{1}{2}(n_6-z_6)-\frac{1}{2}i(n_7+z_7)] \\ 0 & -n_3 & 0 & [\frac{1}{2}(n_4-z_4)-\frac{1}{2}i(n_5+z_5)] \\ 0 & 0 & -n_3 & [\frac{1}{2}(n_1-z_1)-\frac{1}{2}i(n_2+z_2)] \\ [\frac{1}{2}(n_6-z_6)+\frac{1}{2}i(n_7-z_7)] & [\frac{1}{2}(n_4-z_4)+\frac{1}{2}i(n_5-z_5)] & [\frac{1}{2}(n_1-z_1)+\frac{1}{2},i(n_2-z_2)] & -z_3 \end{pmatrix}, \tag{12.7a}$$

$$V_{SO(8)/G_2}=$$
$$\begin{pmatrix} 0 & -(n_1+in_2) & (n_4+in_5) & -[\frac{1}{2}(n_6+z_6)-\frac{1}{2}i(n_7-z_7)] \\ (n_1+in_2) & 0 & -(n_6+in_7) & -[\frac{1}{2}(n_4+z_4)-\frac{1}{2}i(n_5-z_5)] \\ -(n_4+in_5) & (n_6+in_7) & 0 & -[\frac{1}{2}(n_1+z_1)-\frac{1}{2}i(n_2-z_2)] \\ [\frac{1}{2}(n_6+z_6)-\frac{1}{2}i(n_7-z_7)] & [\frac{1}{2}(n_4+z_4)-\frac{1}{2}i(n_5-z_5)] & [\frac{1}{2}(n_1+z_1)-\frac{1}{2}i(n_2-z_2)] & 0 \end{pmatrix}. \tag{12.7b}$$

Then we find, after some calculation,

$$A_{SO(8)/G_2}=\begin{pmatrix} \frac{1}{2}(n_3+z_3) & 0 & 0 & -(n_6-in_7) \\ 0 & \frac{1}{2}(n_3+z_3) & 0 & -(n_4-in_5) \\ 0 & 0 & \frac{1}{2}(n_3+z_3) & -(n_1-in_2) \\ -(n_6+in_7) & -(n_4+in_5) & -(n_1+in_2) & -\frac{1}{2}(3n_3-z_3) \end{pmatrix}, \tag{12.7c}$$

$$B_{SO(8)/G_2}=$$
$$\frac{1}{2}\begin{pmatrix} 0 & [(n_1+z_1)+i(n_2-z_2)] & -[(n_4+z_4)+i(n_5-z_5)] & -[(n_6-z_6)-i(n_7+z_7)] \\ -[(n_1+z_1)+i(n_2-z_2)] & 0 & [(n_6+z_6)+i(n_7-z_7)] & -[(n_4-z_4)-i(n_5+z_5)] \\ [(n_4+z_4)+i(n_5-z_5)] & -[(n_6+z_6)+i(n_7-z_7)] & 0 & -[(n_1-z_1)-i(n_2+z_2)] \\ [(n_6-z_6)-i(n_7+z_7)] & [(n_4-z_4)-i(n_5+z_5)] & [(n_1-z_1)-i(n_2+z_2)] & 0 \end{pmatrix}, \tag{12.7d}$$

and

$$C_{SO(8)/G_2}=\begin{pmatrix} +\frac{1}{2}(n_3-z_3) & 0 & 0 & -(n_6-in_7) \\ 0 & \frac{1}{2}(n_3-z_3) & 0 & -(n_4-in_5) \\ 0 & 0 & \frac{1}{2}(n_3-z_3) & -(n_1-in_2) \\ -(n_6+in_7) & -(n_4+in_5) & -(n_1+in_2) & -\frac{1}{2}(3n_3+z_3) \end{pmatrix}, \tag{12.7e}$$

$$D_{SO(8)/G_2}=$$
$$\frac{1}{2}\begin{pmatrix} 0 & [(n_1-z_1)+i(n_2+z_2)] & -[(n_4-z_4)+i(n_5+z_5)] & -[(n_6+z_6)-i(n_7-z_7)] \\ -[(n_1-z_1)+i(n_2+z_2)] & 0 & [(n_6-z_6)+i(n_7+z_7)] & -[(n_4+z_4)-i(n_5-z_5)] \\ [(n_4-z_4)+i(n_5+z_5)] & -[(n_6-z_6)+i(n_7+z_7)] & 0 & -[(n_1+z_1)-i(n_2-z_2)] \\ [(n_6+z_6)-i(n_7-z_7)] & [(n_4+z_4)-i(n_5-z_5)] & [(n_1+z_1)-i(n_2-z_2)] & 0 \end{pmatrix}. \tag{12.7f}$$

We had shown earlier that the action of $\mathcal{L}G_2 \cong$ **DerO** on the octonion units can be represented by octonion multiplication and the action on the split octonion basis

$$[s]=\begin{pmatrix} u \\ u^* \end{pmatrix}$$

is uniquely determined by the action on u. Similarly, the action of $SO(8)$ on split octonions can be represented by octonion multiplication and the action on u uniquely determines the action on $[s]$. Below we give the expressions for the action of $\mathcal{L}SO(8)/G_2$ matrices that are in triality with each other in terms of octonion multiplication acting on u:

$$u=\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_0 \end{pmatrix},$$

$$\begin{aligned} T^L_{SO(8)/G_2}u = {} & \tfrac{1}{2}n_1([u_3^*,u]+(u+uu_0^*)u_3) \\ & -\tfrac{1}{2}in_2([u_3^*,u]-(u+uu_0^*)u_3) \\ & -n_3(uu_0^*) \\ & +\tfrac{1}{2}n_4([u_2^*,u]+(u+uu_0^*)u_2) \\ & -\tfrac{1}{2}in_5([u_2^*,u]-(u+uu_0^*)u_2) \\ & +\tfrac{1}{2}n_6([u_1^*,u]+(u+uu_0^*)u_1) \\ & -\tfrac{1}{2}in_7([u_1^*,u]-(u+uu_0^*)u_1) \\ & +z_1\tfrac{1}{2}(\{u_3^*,u\}-(uu_0)u_3) \\ & -iz_2\tfrac{1}{2}(-\{u_3^*,u\}-(uu_0)u_3) \\ & -z_3(uu_0) \\ & +z_4\tfrac{1}{2}(\{u_2^*,u\}-(uu_0)u_2) \\ & -iz_5\tfrac{1}{2}(-\{u_2^*,u\}-(uu_0)u_2) \\ & +z_6\tfrac{1}{2}(\{u_1^*,u\}-(uu_0)u_1) \\ & -iz_7\tfrac{1}{2}(-\{u_1^*,u\}-(uu_0)u_1), \end{aligned} \tag{12.8}$$

$$
\begin{aligned}
T^R_{SO(8)/G_2} u = {} & n_1(uu_3^* + \tfrac{1}{2}u_3^* u - (u - \tfrac{1}{2}uu_0^*)u_3) \\
& - in_2(uu_3^* + \tfrac{1}{2}u_3^* u + (u - \tfrac{1}{2}uu_0^*)u_3) \\
& + \tfrac{1}{2}n_3(uu_0^*) - \tfrac{3}{2}n_3(uu_0) \\
& + n_4(uu_2^* + \tfrac{1}{2}u_2^* u - (u - \tfrac{1}{2}uu_0^*)u_2) \\
& - in_5(uu_2^* + \tfrac{1}{2}u_2^* u + (u - \tfrac{1}{2}uu_0^*)u_2) \\
& + n_6(uu_1^* + \tfrac{1}{2}u_1^* u - (u - \tfrac{1}{2}uu_0^*)u_1) \\
& - in_7(uu_1^* + \tfrac{1}{2}u_1^* u + (u - \tfrac{1}{2}uu_0^*)u_1) \\
& + z_1\tfrac{1}{2}(-(uu_0^*)u_3 - u_3^* u) \qquad (12.9) \\
& - iz_2\tfrac{1}{2}(-(uu_0^*)u_3 + u_3^* u) \\
& + \tfrac{1}{2}z_3 u \\
& + z_4\tfrac{1}{2}(-(uu_0^*)u_2 - u_2^* u) \\
& - iz_5\tfrac{1}{2}(-(uu_0^*)u_2 + u_2^* u) \\
& + z_6\tfrac{1}{2}(-(uu_0^*)u_1 - u_1^* u) \\
& - iz_7\tfrac{1}{2}(-(uu_0^*)u_1 + u_1^* u),
\end{aligned}
$$

$$
\begin{aligned}
T^P_{SO(8)/G_2} u = {} & n_1(uu_3^* + \tfrac{1}{2}u_3^* u - (u - \tfrac{1}{2}uu_0^*)u_3) \\
& - in_2(uu_3^* + \tfrac{1}{2}u_3^* u + (u - \tfrac{1}{2}uu_0^*)u_3) \\
& + n_3(\tfrac{1}{2}u - 2uu_0) \\
& + n_4(uu_2^* + \tfrac{1}{2}u_2^* u - (u - \tfrac{1}{2}uu_0^*)u_2) \\
& - in_5(uu_2^* + \tfrac{1}{2}u_2^* u + (u - \tfrac{1}{2}uu_0^*)u_2) \\
& + n_6(uu_1^* + \tfrac{1}{2}u_1^* u - (u - \tfrac{1}{2}uu_0^*)u_1) \\
& - in_7(uu_1^* + \tfrac{1}{2}u_1^* u + (u - \tfrac{1}{2}uu_0^*)u_1) \\
& + z_1\tfrac{1}{2}((uu_0^*)u_3 + u_3^* u) \qquad (12.10) \\
& - iz_2\tfrac{1}{2}((uu_0^*)u_3 - u_3^* u) \\
& - \tfrac{1}{2}z_3 u \\
& + z_4\tfrac{1}{2}((uu_0^*)u_2 + u_2^* u) \\
& - iz_5\tfrac{1}{2}((uu_0^*)u_2 - u_2^* u) \\
& + z_6\tfrac{1}{2}((uu_0^*)u_1 + u_1^* u) \\
& - iz_7\tfrac{1}{2}((uu_0^*)u_1 - u_1^* u)
\end{aligned}
$$

ACKNOWLEDGMENTS

The authors gratefully acknowledge useful and stimulating conversations with Professor N. Jacobson, Professor G. Seligman, and Dr. S. Orfanidis.

APPENDIX A: STRUCTURE CONSTANTS OF G_2

Consider the basis of $\mathcal{L}G_2$ given in Sec. 1

$$\mathcal{L}G_2 = F_A \oplus M_A, \quad A = 1, \ldots, 7.$$

As was pointed out in Sec. 2, the generators F_A and $F_8 = -M_3$ form the $SU(3)$ subalgebra of $\mathcal{L}G_2$, i.e.,

$$[F_a, F_b] = 2if_{abc}F_c, \quad a, b, c = 1, 2, \ldots, 8, \tag{A1}$$

where f_{abc} are the totally antisymmetric structure constants of Gell-Mann, the nonzero elements of which are given in Table A1.

Now

$$\mathcal{L}G_2 = F_a \oplus M_s, \quad a = 1, 2, \ldots, 8,$$
$$m_s = s = 1, 2, 4, 5, 6, 7,$$

$$F_a \cong \mathcal{L}SU(3).$$

TABLE A1.

abc	f_{abc}
123	1
147	$\frac{1}{2}$
156	$-\frac{1}{2}$
246	$\frac{1}{2}$
257	$\frac{1}{2}$
345	$\frac{1}{2}$
367	$-\frac{1}{2}$
458	$\sqrt{3}/2$
678	$\sqrt{3}/2$

TABLE A2.

a	$m_s m_t$	$C_{am_sm_t}$
1	m_4m_7	1/2
1	m_5m_6	1/2
2	m_4m_6	$-1/2$
2	m_5m_7	1/2
3	m_4m_5	1/2
3	m_6m_7	1/2
4	m_1m_7	1/2
4	m_2m_6	$-1/2$
5	m_1m_6	$-1/2$
5	m_2m_7	$-1/2$
6	m_1m_5	$-1/2$
6	m_2m_4	$-1/2$
7	m_1m_4	$-1/2$
7	m_2m_5	1/2
8	m_4m_5	$-1/2\sqrt{3}$
8	m_6m_7	$1/2\sqrt{3}$
8	m_1m_2	-1

TABLE A3.

$m_sm_tm_u$	$C_{m_sm_tm_u}$
$m_1m_4m_7$	$-1/\sqrt{3}$
$m_1m_5m_6$	$1/\sqrt{3}$
$m_2m_4m_6$	$-1/\sqrt{3}$
$m_2m_5m_7$	$-1/\sqrt{3}$

The structure constants of the form C_{abm_s} vanish because F_a form a subalgebra. Hence the remaining nonvanishing structure constants of G_2 are of the form

$$C_{am_sm_t}, \quad a = 1, \ldots, 8, \quad s, t, u = 1, 2, 4, 5, 6, 7$$

or of the form

$$C_{m_sm_tm_u},$$

$$
\begin{aligned}
&[F_a, F_b] = 2if_{abc}F_c, \\
&[F_a, M_s] = 2iC_{am_sm_t}M_t, \\
&[M_s, M_t] = 2i(C_{m_sm_ta}F_a + C_{m_sm_tm_u}M_u,
\end{aligned} \tag{A2}
$$

where all the structure constants are totally antisymmetric. Below we list all the nonvanishing elements of $C_{am_sm_t}$ and $C_{m_sm_tm_u}$ (Tables A2 and A3).

APPENDIX B: ZORN'S VECTOR MATRICES

A realization of the split octonion algebra is via the Zorn's vector matrices

$$\begin{pmatrix} a & \mathbf{x} \\ \mathbf{y} & b \end{pmatrix},$$

where a and b are scalars and $\mathbf{x}$ and $\mathbf{y}$ are 3-vectors, with the product defined as

$$\begin{pmatrix} a & \mathbf{x} \\ \mathbf{y} & b \end{pmatrix}\begin{pmatrix} c & \mathbf{u} \\ \mathbf{v} & d \end{pmatrix} = \begin{pmatrix} ac + \mathbf{x}\cdot\mathbf{v} & a\mathbf{u} + d\mathbf{x} - \mathbf{y}\times\mathbf{v} \\ c\mathbf{y} + b\mathbf{v} + \mathbf{x}\times\mathbf{u} & \mathbf{y}\cdot\mathbf{u} + bd \end{pmatrix}. \tag{B1}$$

$\times$ denotes the usual vector product.

If the basis vectors of the three-dimensional space are $\mathbf{e}_i$, $i = 1, 2, 3$ with $\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk}\mathbf{e}_k$ and $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, then we can relate the split octonions to the vector matrices; namely

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}=u_0^*,\quad \begin{pmatrix}0&-\mathbf{e}_i\\0&0\end{pmatrix}=u_i^*,$$

$$\begin{pmatrix}0&0\\0&1\end{pmatrix}=u_0,\quad \begin{pmatrix}0&0\\\mathbf{e}_i&0\end{pmatrix}=u_i.$$

Octonion conjugation defined above induces a natural involution for the vector matrices, i.e., if

$$A=\begin{pmatrix}a&-\mathbf{x}\\\mathbf{y}&b\end{pmatrix},\quad \overline{A}=\begin{pmatrix}b&+\mathbf{x}\\-\mathbf{y}&a\end{pmatrix},$$

$$A=au_0^*+x_iu_i^*+bu_0+y_iu_i,$$

$$\overline{A}=au_0-x_iu_i^*+bu_0^*-y_iu_i,$$

$$N(A)=A\overline{A}=\overline{A}A=(ab+\mathbf{x}\cdot\mathbf{y}). \tag{B2}$$

APPENDIX C: PRINCIPLE OF TRIALITY

The usual octonionic norm is invariant under the group $SO(8)$ or equivalently the bilinear form induced by the octonionic norm is skew with respect to the Lie algebra of $SO(8)$, i.e.,

$$(x,y)\equiv\tfrac{1}{2}(\bar{x}y+y\bar{x}) \tag{C1}$$

then for $T\in\mathcal{L}SO(8)$

$$(Tx,y)+(x,Ty)=0 \quad \text{for all } x,y\in\mathbf{O} \tag{C2}$$

For the elements D of the derivation algebra of octonions we have

$$D\in \text{Der}\,\mathbf{O}\cong\mathcal{L}G_2,\qquad D(xy)=(Dx)y+x(Dy). \tag{C3}$$

Integrated form of this (local) identity gives us the automorphisms of $\mathbf{O}$, i.e.,

$$e^D(xy)=(e^Dx)(e^Dy) \tag{C4}$$

or

$$d=e^D,$$

$$d(xy)=(dx)(dy)\Rightarrow d\in G_2.$$

The principle of triality is nothing but a generalization of the identities (C3)–(4) and is unique to octonions.[8] According to the principle of local triality (PLT) it is possible to generalize identity (C3) to all the elements of the Lie multiplication algebra $\mathcal{L}SO(8)$. Namely, given an element $T^L\in\mathcal{L}SO(8)$ acting on the octonions there exist *unique* T^R and $T^P\in\mathcal{L}SO(8)$ such that

$$(\text{PLT}):(T^Lx)y+x(T^Ry)=T^P(xy)\quad\text{for all } x,y\in\mathbf{O}. \tag{C5}$$

Just as it is possible to integrate the derivations of octonion algebra to get its automorphisms, one can also integrate the PLT to get the principle of global triality (PGT), which is a generalization of the concept of automorphism. According to the PGT, given $t^l\in SO(8)$ acting on the octonions there exist t^r and $t^p\in SO(8)$, unique up to a sign, such that[33]

$$\text{PGT}:(t^lx)(t^ry)=t^p(xy)\quad\text{for all } x,y\in\mathbf{O}. \tag{C6}$$

Since the group $SO(8)$ is the "Lie multiplication group" of octonions (i.e., that every action of $SO(8)$ on O can be represented by octonion multiplication), one can reformulate the PGT as follows[34]:

Given $d^1\in SO(8)$ $\quad d^2,d^3\in SO(8)$

$$(d^1x)(d^2y)=\overline{d^3\overline{(xy)}}\qquad\text{for all } x,y\in\mathbf{O}, \tag{C7a}$$

where the overbar denotes octonion conjugation.

In this form of the PGT we have cyclic symmetry between d^1, d^2 and d^3, i.e.,

$$(d^1x)(d^2y)=\overline{d^3\overline{(xy)}}$$

implies

$$(d^2x)(d^3y)=\overline{d^1\overline{(xy)}}, \tag{C7b}$$

$$(d^3x)(d^1y)=\overline{d^2\overline{(xy)}}. \tag{C7c}$$

Since givên d^1, d^2, and d^3 are determined uniquely up to a sign, the subgroup of $SO(8)\times SO(8)\times SO(8)$ consisting of elements which are in triality will form a twofold covering group of $SO(8)$, i.e., it will be isomorphic to Spin (8). The group $SO(8)$ has the subgroup $SO(7)$ and given $t\in SO(7)$ there exist $\bar{t}\in SO(8)$

$$(tx)(\bar{t}y)=\bar{t}(xy)\quad\text{for all } x,y\in\mathbf{O} \tag{C8}$$

then the elements t form the covering group Spin (7) of $SO(7)$.

APPENDIX D: REALIZATIONS OF THE CAYLEY ALGEBRA IN TERMS OF GELL-MANN λ MATRICES AND DIRAC'S γ-MATRICES

1. The λ-matrices

We want to define a product between the λ matrices of Gell-Mann such that they will form the nonassociative Cayley algebra. Since there are eight λ matrices and seven imaginary octonion units e_A, the product will be defined between seven of the λ matrices and will involve the eighth λ matrix. In view of the broken $SU(3)$, this eighth λ matrix will be taken to be λ_8. The general form of the product consistent with octonion multiplication can be parameterized as follows:

$$\begin{aligned}\lambda_A\circ\lambda_B=&\tfrac{1}{2}\beta\,\text{Tr}(\lambda_A\lambda_B)1+\tfrac{1}{2}\delta\,\text{Tr}(\lambda_8\{\lambda_A,\lambda_B\})1\\&-(2/\sqrt{3})(\alpha+\tfrac{1}{6}\gamma)\,\text{Tr}(\lambda_8[\lambda_A,\lambda_B])1\\&+\{\alpha 1+\sqrt{3}(\alpha+\tfrac{1}{6}\gamma)\lambda_8,[\lambda_A,\lambda_B]\}\\&+\gamma[\{\lambda_8,\lambda_A\},\{\lambda_8,\lambda_B\}],\end{aligned} \tag{D1}$$

where $\{\ ,\ \}$ and $[\ ,\ ]$ denote anticommutation and commutation, respectively. Then, for $A=1,2,3$ we have

$$\lambda_A\circ\lambda_A=\beta+(2/\sqrt{3})\delta\equiv 1/s^2,\qquad\text{no sum over } A, \tag{D2}$$

and for $A=4,5,6,7$

$$\lambda_A\circ\lambda_A=\beta-(1/\sqrt{3})\delta\equiv 1/t^2. \tag{D3}$$

In addition, the octonion multiplication imposes the conditions:

$$\alpha=-\tfrac{5}{4}(2\gamma/9),\qquad\beta=15(2\gamma/9)^2,$$

$$\delta=5\sqrt{3}(2\gamma/9)^2. \tag{D4}$$

Hence, we get the result that the 3×3 matrices

$$e_i = is\lambda_i, \quad i = 1, 2, 3,$$
$$e_4 = it\lambda_4, \quad e_6 = -it\lambda_6,$$
$$e_5 = it\lambda_5, \quad e_7 = -it\lambda_7 \tag{D5}$$

satisfy the octonion multiplication table of the imaginary units e_A under the product defined above and generate a Cayley algebra with identity being the scalar identity:

$$e_A \circ e_B = -\delta_{AB} + a_{ABC}e_C \tag{D6}$$

An interesting property of this product is that the coefficient multiplying the λ matrices is different for different isospin multiplets.

2. The γ-matrices

Let us define a product between 4×4 Hermitian matrices of the form:

$$A = \begin{pmatrix} \alpha 1_2 & -i\sigma\cdot\mathbf{a} \\ i\sigma\cdot\mathbf{b} & \beta 1_2 \end{pmatrix}, \quad C = \begin{pmatrix} \gamma 1_2 & -i\sigma\cdot\mathbf{c} \\ i\sigma\cdot\mathbf{d} & \delta 1_2 \end{pmatrix}.$$

Such that they form a Cayley algebra. First, note that the matrix A can be written in terms of γ matrices as:

$$A = \tfrac{1}{2}(\alpha+\beta) + \tfrac{1}{2}(\alpha-\beta)\gamma_5 + \tfrac{1}{2}\gamma_5\boldsymbol{\gamma}\cdot(\mathbf{a}-\mathbf{b}) + \tfrac{1}{2}\boldsymbol{\gamma}\cdot(\mathbf{a}+\mathbf{b}), \tag{D7}$$

where γ matrices are taken in the Weyl basis and the parameters $\alpha, \beta, \mathbf{a}, \mathbf{b}$ are all real.

$$\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3) = \rho_2 \otimes \sigma,$$
$$\gamma_4 = \rho_1 \otimes I, \quad \gamma_5 = \rho_3 \otimes I. \tag{D8}$$

To get a product which is not associative, we are led to defining a new operation $\tilde{}$ over the 4×4 matrices:

$$\tilde{M} = \widetilde{\begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}} \equiv \begin{pmatrix} \mathbf{a}^\dagger & \mathbf{b} \\ \mathbf{c} & \mathbf{d}^\dagger \end{pmatrix}, \tag{D9}$$
$$= \tfrac{1}{2}(1+\gamma_5)A^\dagger\tfrac{1}{2}(1+\gamma_5) + \tfrac{1}{2}(1-\gamma_5)A^\dagger\tfrac{1}{2}(1-\gamma_5)$$
$$+ \tfrac{1}{2}(1+\gamma_5)A\tfrac{1}{2}(1-\gamma_5) + \tfrac{1}{2}(1-\gamma_5)A\tfrac{1}{2}(1+\gamma_5),$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are 2×2 matrices.

Then under the product

$$A * C = \tfrac{1}{2}(AC + \widetilde{AC}) + \tfrac{1}{2}\gamma_4(AC^\dagger - \widetilde{AC^\dagger}) \tag{D10}$$

the matrices of the form shown above form a split Cayley algebra equivalent to the Zorn's vector matrices

$$A*C = \begin{pmatrix} (\alpha\gamma + \mathbf{a}\cdot\mathbf{d}) & (-i\alpha\sigma\cdot\mathbf{c} - i\delta\sigma\cdot\mathbf{a} + i\sigma\cdot\mathbf{b}\times\mathbf{d}) \\ (i\gamma\sigma\cdot\mathbf{b} + i\beta\sigma\cdot\mathbf{d} + i\sigma\cdot(\mathbf{a}\times\mathbf{c}) & (\beta\delta + \mathbf{b}\cdot\mathbf{c}) \end{pmatrix}. \tag{D11}$$

Writing A in the form

$$A = \tfrac{1}{2}(1+\gamma_5)(\alpha + \boldsymbol{\gamma}\cdot\mathbf{a}) + \tfrac{1}{2}(1-\gamma_5)(\beta + \boldsymbol{\gamma}\cdot\mathbf{b}),$$
$$= \tfrac{1}{2}(1-ie_7)(\alpha + e_i a_i) + \tfrac{1}{2}(1+ie_7)(\beta + e_i b_i),$$
$$= \alpha u_0^* + u_i^* a_i + \beta u_0 + u_i b_i, \tag{D12}$$

it is easily seen that the split octonion basis u_i, u_0, u_i^*, u_0^* is realized in this case by

$$u_0^* = \tfrac{1}{2}(1+\gamma_5), \quad u_0 = \tfrac{1}{2}(1-\gamma_5),$$
$$u_i^* = \tfrac{1}{2}(1+\gamma_5)\gamma_i, \quad u_i = \tfrac{1}{2}(1-\gamma_5)\gamma_i, \; i = 1, 2, 3. \tag{D13}$$

Therefore, the role played by ie in extending the quaternion algebra $(1, e_1, e_2, e_3)$ into the split octonion algebra is played in the above realization by γ_5, i.e.,

$$ie_7(1, e_1, e_2, e_3) = (ie_7, ie_4, ie_5, ie_6),$$
$$\gamma_5 * (1, \gamma_1, \gamma_2, \gamma_3) = \gamma_5(1, \gamma_1, \gamma_2, \gamma_3)$$
$$= (\gamma_5, \gamma_5\gamma_1, \gamma_5\gamma_2, \gamma_5\gamma_3).$$

$*$ multiplication by γ_5 reduces to the ordinary matrix multiplication. Conversely, the crucial role played by γ_5 in constructing projection operators into lh and rh states is reflected in the octonion algebra by the important role played by u_0 and u_0^* as projection operators into quark and antiquark states in the octonionic representations of the Poincaré group.[14]

*Research (Yale Report 3075-27) supported by the U. S. Atomic Energy Commission under contract AT(11-1)3075.
†Now at the Middle East Technical University, Ankara, on leave from Yale University.

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[19]Note the distinction between the terms nonassociative and not associative. The former is generally used to denote all the composition algebras mentioned above which satisfy the property of alternativity defined below.
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[23]By three "independent" elements we mean any three elements e_i, e_j, e_k such that none of them is proportional to a product of the other two, i.e., $e_k \neq a e_i e_j$.

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[26]For example, take $\{1, e_1, e_2, e_3\}$ as a basis generating a quaternion algebra H. Then each octonion can be written as $z = q_1 + q_2 e_7$, $w = r_1 + r_2 e_7 \in O$, where $q_1, q_2, r_1, r_2 \in H$, with the product defined by $zw = (q_1 r_1 - \bar{r}_2 q_2) + (r_2 q_1 + q_2 \bar{r}_1) e_7$. (The bar denotes quaternion conjugation).

[27]An equivalent form of this basis was first studied by G. Seligman as the derivation algebra of Zorn's vector matrices given in Appendix B. See Ref. 17.

[28]In fact, the generator N_3 extends $SU(3)$ subgroup into $U(3)$ and the group G_2 into $SO(7)$.

[29]Note that we put a bar over the indexed matrices when they act on the split octonions, i.e., under the numbering: $(u_1 u_2 u_3 u_0 u_1^* u_2^* u_3^* u_0^*)$ $(s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8)$ we have $\bar{E}_{ab} s_c = \delta_{bc} s_a$, $a, b, c = 1, \ldots, 8$. For the real octonions we have the numbering $(e_A, 1) \leftrightarrow (e_A, e_8)$, $A = 1, \ldots, 7$. Then $E_{ab} e_c = \delta_{bc} e_a$, $a, b, c, = 1, \ldots, 8$. We also defined Σ_{ab} as $\Sigma_{ab} = E_{ab} - E_{ba}$, $\bar{\Sigma}_{ab} = \bar{E}_{ab} - \bar{E}_{ba}$.

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[32]For a proof and more details see N. Jacobson *Lie Algebras* (Interscience, New York, 1962.)

[33]Here we use capital letters for the elements of the Lie algebra and small letters for the group elements.

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