Threshold configurations in the presence of Lorentz violating dispersion relations

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A general characterization of lower and upper threshold configurations for two particle reactions is determined under the assumptions that the single particle dispersion relations $E(|\mathbf{p}|)$ are rotationally invariant and monotonic in $|\mathbf{p}|$, and that energy and momentum are conserved and additive for multiple particles. It is found that at a threshold the final particle momenta are always parallel and the initial momenta are always antiparallel. The occurrence of new phenomena not occurring in a Lorentz invariant setting, such as upper thresholds and asymmetric pair production thresholds, is explained, and an illustrative example is given.

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I. INTRODUCTION

The possibility of Lorentz violation (LV) is currently receiving much attention [1]. One of the reasons for this is that high energy observations are now capable of detecting Planck suppressed LV [2–6], which could be a harbinger of quantum gravity effects [7–10]. A straightforward way to probe LV is through the consequences of Lorentz violating dispersion relations for particles. Kinematics with LV can allow new reactions, some with thresholds, or can modify thresholds for reactions. LV threshold effects are of particular interest because they are generically sensitive to Planck suppressed LV at energies far below the Planck energy allowing the use of current astrophysical observations for obtaining good constraints on LV parameters [3,11–14].

The purpose of this paper is to establish some general properties of threshold configurations which are needed for studying reactions in the presence of Lorentz violation. These properties have been used without proof in most of the previous literature on the subject (the only exception being Ref. [3]). We prove a *threshold configuration theorem* which asserts that for arbitrary rotationally invariant, monotonically increasing dispersion relations the final particle momenta of a two-particle reaction are always parallel and the initial momenta are always antiparallel at a threshold. The proof of this theorem involves first establishing that, at any (upper or lower) threshold, the configuration of initial and final particles is such as to minimize the energy of the final particles subject to momentum conservation at fixed values of the incoming particle energies. Our results are a generalization of those found in Ref. [3], which addressed just particle decays with quadratic dispersion relations allowing for a different maximum speed for each particle type. Some related results were also obtained in Ref. [15] for decay of elementary excitations in superfluid helium. After proving the threshold theorem, the occurrence of new phenomena not occurring in a Lorentz invariant setting, such as upper thresholds and asymmetric pair production, is explained, and an illustrative example is given.

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II. THRESHOLD CONFIGURATION THEOREM

We consider reactions with two initial and two final particles. Results for reactions with only one incoming or outgoing particle can be obtained as special cases. We allow each particle to have an independent dispersion relation $E(\mathbf{p})$, where $\mathbf{p}$ is the three momentum and E is the energy, and make the following assumptions:

(1) $E(\mathbf{p})$ is a rotation-invariant function of $\mathbf{p}$.

(2) $E(\mathbf{p})$ is a monotonic increasing function of $|\mathbf{p}|$.

(3) Energy and momentum are both additive for multiple particles and conserved.

Comments on the assumptions: (1) If rotation invariance is not assumed the question of threshold relations is obviously much more complicated. (2) The monotonic assumption is necessary; without it the threshold theorem would not be true. (See Ref. [15] for an example involving the decay of a phonon to a pair of non-parallel rotons.) Note however that the dispersion need only be monotonic in the observationally relevant range of $|\mathbf{p}|$. (3) The third assumption follows from the assumption of space-time translation invariance, with E and $\mathbf{p}$ interpreted as the usual generators of time and space translations. Assumptions (1) and (3) are by no means inevitable, and indeed there is much research on Lorentz violation that does not make some of these assumptions (see e.g. [16–18]).

A number of variables are needed to describe the kinematics of a two particle reaction of the form in Fig. 1. We denote the magnitude of the 3-momenta of the two incoming particles by p_1 and p_2, and the angle between them by α. The final particle momentum magnitudes are p_3 and p_4. The angle between the total incoming momentum vector $\mathbf{p}_{\rm in} = \mathbf{p}_1 + \mathbf{p}_2$ and one of the outgoing momenta $\mathbf{p}_3$ is β. Since the dispersion relations are assumed rotationally invariant there is no loss of generality in assuming all of the momenta lie in a single plane.

A *threshold* is defined relative to a fixed value of p_2 as

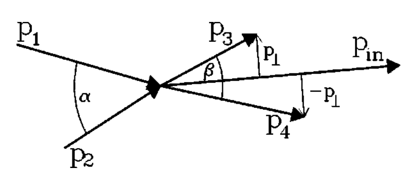

FIG. 1. Geometry of a two particle reaction.

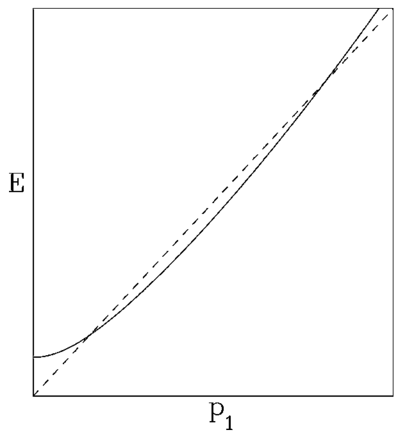

FIG. 2. $E_f^{\alpha,\beta,p_3}(p_1)$. The dashed line is a null line given for reference.

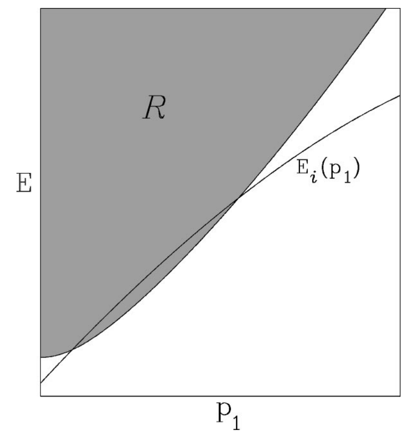

FIG. 3. $\mathcal{R}$ is the region covered by all curves $E_f^{\alpha,\beta,p_3}(p_1)$ for some fixed p_2, assuming momentum conservation holds to determine p_4. The curve $E_i(p_1)$ is the initial energy for the same fixed p_2. Where the latter curve lies inside $\mathcal{R}$ there is a solution to the energy and momentum conservation equations.

follows. With p_2 fixed, we ask for what values of p_1 is there an energy and momentum conserving configuration (α,β,p_3,p_4)? If there exists a p_L such that when $p_1<p_L$ there are no allowed configurations then we call p_L a *lower threshold*. Likewise if there exists a p_U such that when $p_1>p_U$ there are no allowed configurations then we call p_U an *upper threshold*. To prove the threshold configuration theorem it is helpful to understand the solution space of the conservation equations in a graphical manner as follows.

For given values of $(p_1,p_2,\alpha,\beta,p_3)$ momentum conservation determines p_4 and therefore also the energy of the final particles:

$$E_f(p_1,p_2,\alpha,\beta,p_3)=E_3(p_3)+E_4(p_4(p_1,p_2,\alpha,\beta,p_3)). \tag{1}$$

For each value of the configuration variables (p_2,α,β,p_3) we can thus define the final energy function:

$$E_f^{\alpha,\beta,p_3}(p_1)=E_f(p_1,p_2,\alpha,\beta,p_3). \tag{2}$$

Since p_2 is fixed in the definition of a given threshold, that label is suppressed in the notation E_f^{α,β,p_3}. A plot of $E_f^{\alpha,\beta,p_3}(p_1)$ looks for examples like Fig. 2 [assuming that $E_f^{\alpha,\beta,p_3}(0)>0$].

Any choice of α,β,p_3 gives a curve E_f^{α,β,p_3} that is non-negative since each individual particle's dispersion relation is non-negative. Now consider the region $\mathcal{R}$ in the E,p_1 plane covered by plotting E_f^{α,β,p_3} for all possible configurations α,β,p_3. Since each of these curves is bounded below there exists a lower boundary E_B to this region, determined by the configuration of lowest final energy at each p_1.

The initial energy $E_i(p_1,p_2)$ is just a function of p_1 for fixed p_2, so it can be plotted on the same graph. An example is shown in Fig. 3. In the figure it is assumed that the reaction does not happen for $p_1=0$, i.e. that $E_f^{\alpha,\beta,p_3}(0)>E_i(0)$ for all choices of α,β,p_3. If $E_f^{\alpha,\beta,p_3}(0)\leqslant E_i(0)$ then all of the following results still apply, but there is simply no lower threshold.

The plot in Fig. 3 is a graphical representation of the solution space to the conservation equation. There exists a solution when E_i is in $\mathcal{R}$ and no solution when E_i is outside of $\mathcal{R}$. It is evident that a lower threshold occurs when E_i enters $\mathcal{R}$ for the first time and an upper threshold occurs when it leaves $\mathcal{R}$ for the last time. The threshold momenta p_L and p_U therefore correspond to the intersections of E_i with E_B. Hence we have the following important result.

Minimum energy theorem. An upper or lower threshold configuration occurring at incoming momentum p_1 is always the minimum energy configuration conserving momentum at that p_1.

Using this result we can now establish the threshold configuration theorem.

Threshold configuration theorem. At an upper or lower threshold the incoming particle momenta are always antiparallel and the final particle momenta are parallel.

That is, at a threshold, the angles are necessarily given by $\beta=0$ and $\alpha=\pi$. To prove the first statement, note that for any value of β other than 0 or π, momentum conservation can be preserved while simultaneously decreasing p_3 and p_4, since one can subtract out from $\mathbf{p}_3$ and $\mathbf{p}_4$ equal and opposite momenta $\mathbf{p}_\perp$ and $-\mathbf{p}_\perp$ transverse to $\mathbf{p}_{\text{in}}$ (see Fig. 1). For $\beta=\pi$, $\mathbf{p}_3$ and $\mathbf{p}_4$ must be antiparallel, hence momentum conservation can again be preserved while decreasing both their magnitudes. Since all dispersion relations are assumed monotonic, any operation that decreases p_3 and p_4 also decreases the energies E_3 and E_4 of each of the final particles and so decreases $E_f^{\alpha,\beta,p_3}(p_1)$. Only $\beta=0$ can therefore be the lowest energy configuration, hence at a threshold the final particle momenta must always be parallel.

As to the initial particles, if $\alpha<\pi$ then at fixed p_1,p_2, the total incoming momentum can be lowered by increasing α. Since the final particles are necessarily parallel to each other and to the total momentum, monotonicity implies that this allows E_f to be lowered, e.g., by reducing p_4 at fixed $\beta=0$ and p_3. Therefore $\alpha<\pi$ cannot be a threshold configuration, hence the initial particle momenta must always be antiparallel at a threshold.

As an aside, note that E_i can leave and enter $\mathcal{R}$ multiple times. These intermediate intersection points look like thresholds locally in p_1 but are not global thresholds. Nevertheless they could in principle yield interesting phenomenology. Since they do not occur for simple models of Lorentz violating dispersion however, we will not analyze them in detail here. We simply mention that their kinematic configuration will be the same as for global thresholds and their locations in p_1 determined in the same manner.

III. NEW THRESHOLD PHENOMENA FROM LORENTZ VIOLATION

A. Upper thresholds

The possibility of upper thresholds in the presence of Lorentz violating dispersion was noticed in Ref. [3] for particle decays and in Ref. [4] was discussed in the context of photon annihilation to an electron-positron pair $\gamma\gamma \to e^+e^-$. In [14] the analysis was generalized to allow for the possibility of asymmetric pair production (discussed in the next subsection) and a wider class of dispersion relations. Here we make some general remarks about the origin of this peculiar feature of Lorentz violating kinematics.

If Lorentz invariance holds, then there is never an upper threshold. This can be seen by transforming to the center of mass frame, in which the total incoming four-momentum is $(E,0,0,0)$. Suppose the final particles have masses m_1 and m_2, and that $E \geqslant m_1 + m_2$. Then energy and momentum are conserved with the final particle four-momenta $(E_3(|\mathbf{p}|),\mathbf{p})$ and $(E_4(|\mathbf{p}|),-\mathbf{p})$ for some choice of $\mathbf{p}$. As p_1 grows, the condition $E \geqslant m_1 + m_2$ is always met for a head-on collision with Lorentz invariant dispersion, hence there is no upper threshold.

This argument exploited the Lorentz invariant form of the dispersion relations to guarantee that (1) the total four-momentum is timelike, (2) the energy of each particle in the center of mass frame is an unbounded positive function of the momentum p, and (3) the center of mass energy increases without bound as p_1 increases. Any of these properties can fail to hold for non-Lorentz invariant dispersion relations. In particular, if a monotonic unbounded Lorentz violating dispersion relation is transformed from the preferred frame to another frame, it will in general no longer be monotonic or unbounded.

That upper thresholds exist for LV dispersion relations is evident from Fig. 3. Since different particles can have different dispersion, the curve E_i can be varied independently from E_f and hence E_B by choosing various dispersion relations for the initial particles (assuming the initial and final particles are different). If we choose a dispersion relation such that E_i enters and leaves the region $\mathcal{R}$ then this reaction has an upper and lower threshold for that choice of dispersion.

B. Asymmetric pair production

Unlike in Lorentz invariant physics, reactions with identical final particle dispersion can have (both upper and lower) threshold configurations for which the two final momenta are not equal. This was first noticed in Ref. [12] in the context of photon decay and photon annihilation. The example of photon decay will be discussed in the next section. We start by analyzing the Lorentz invariant case and then show how it is modified when Lorentz invariance is violated.

A familiar result in Lorentz invariant physics is that the threshold configuration for pair production of massive particles is symmetric in the final momentum distribution, i.e., $p_3 = p_4$. This follows immediately from the fact that in the center of mass frame the final particles must be created at rest at threshold, so in any other frame they have equal momenta (since they have the same mass). To understand the Lorentz violating case it is useful to re-derive this fact without using the Lorentz transformation to the center of mass frame, since the dispersion relations are not Lorentz invariant and one cannot even always boost to the center of mass frame.

At any threshold the final momenta are parallel and the initial momenta are antiparallel, and momentum conservation holds. Suppose that $p_3 > p_4$. Momentum conservation can be preserved while lowering p_3 and raising p_4 by an equal amount. In a Lorentz invariant theory the dispersion relation $E_o(p_3)$ for each of the outgoing (massive) particles always has positive curvature with respect to p_3. Therefore $\partial E_o/\partial p$ is greater at $p = p_3$ than at $p = p_4$. Since p_3 and p_4 are changed by opposite amounts we have

$$\Delta E_f = -\Delta p \left.\frac{\partial E_o}{\partial p}\right|_{p=p_3} + \Delta p \left.\frac{\partial E_o}{\partial p}\right|_{p=p_4} < 0. \tag{3}$$

Therefore E_f can be lowered, implying that $p_3 > p_4$ is not a threshold. Similarly p_4 cannot be greater than p_3, hence $p_3 = p_4$ at a threshold.

The previous argument does not depend specifically on Lorentz invariance. Hence it establishes a general result.

Condition for symmetry of pair production. Pair production is always symmetric at threshold provided the outgoing particle dispersion relation has positive curvature with respect to momentum.

However, in the Lorentz violating case $E_o(p)$ need not have positive curvature with respect to p, in which case asymmetric pair production is possible. At an asymmetric threshold, the above argument shows that $E_o(p)$ must have negative curvature.

A sufficient condition for the threshold configuration to be *not* symmetric can be found by computing the variation $\Delta E_f = \Delta(E_3 + E_4)$ induced by variations $\Delta p_3 = -\Delta p$ and $\Delta p_4 = +\Delta p$ away from the symmetric solution $p_3 = p_4$. The first order variations of E_3 and E_4 cancel, so we have

$$\Delta E_f = \left[\frac{1}{2}\frac{\partial^2 E_o}{\partial p^2}(-\Delta p)^2 + \frac{1}{2}\frac{\partial^2 E_o}{\partial p^2}(\Delta p)^2\right]_{p=p_3}$$

$$= \left.\frac{\partial^2 E_o}{\partial p^2}\right|_{p=p_3} (\Delta p)^2. \tag{4}$$

Thus E_f can be lowered by moving away from the $p_3=p_4$ configuration if the dispersion relations are such that $\partial^2 E_o/\partial p^2<0$ at $p=(p_1-p_2)/2$, where $E_i(p_1)$ intersects the boundary of $\mathcal{R}$. In such a case the outgoing particle momenta are not symmetric at the threshold. This condition on $\partial^2 E_o/\partial p^2$ is sufficient but not necessary for having an asymmetric threshold, since the energy may be locally minimized by the symmetric configuration but not globally minimized.

IV. EXAMPLE: $\gamma\rightarrow e^+e^-$

To illustrate the general results we now discuss the case of photon decay to an electron-positron pair. The example is chosen for its simplicity, so one can easily see how the threshold phenomena depend on the values of the Lorentz violating parameters. We refer to [14] for analysis of further cases of direct observational relevance.

Consider the following deformed photon and electron dispersion relations respectively:

$$\omega^2=k^2+\epsilon k^2+\xi k^3/M \tag{5}$$

$$E^2=p^2+m^2+\eta p^3/M \tag{6}$$

where ϵ, ξ and η are dimensionless parameters and $M=10^{19}$ GeV is approximately the Planck mass. The physical idea is that these are just the deformations of lowest order in p/M. Observed photons have energies far below M, so we will calculate the threshold in the regime where $p/M\ll 1$. Hereafter we adopt units with $M=1$.

In the Lorentz-invariant case $\epsilon=\xi=\eta=0$ the photon cannot decay. The final energy region $\mathcal{R}$ asymptotes to the line $E_f=p$ in the E-p plane, while the incoming energy curve is just $E_i(p)=\omega(p)=p$ which never enters $\mathcal{R}$. The simplest way to allow photon decay is to let ϵ or ξ be positive and nonzero. In this case, $E_i(p)$ enters $\mathcal{R}$ and never leaves, so there is a lower threshold but no upper threshold. Since the electron/positron energy function $E(p)$ has positive curvature, this lower threshold configuration is symmetric in the final momenta.

To allow for an upper threshold we can take $\epsilon>0$ and $\xi<0$ (with $\eta=0$). At sufficiently large k the k^3 term in the photon dispersion relation will cause the $E_i(k)$ to dip back down and exit $\mathcal{R}$, again at a symmetric configuration. To determine the value of k at these thresholds we invoke the threshold configuration theorem and the positive curvature of $E(p)$ to set the final momenta each equal to $k/2$. Energy conservation then yields

$$\xi k^3+\epsilon k^2=4m^2. \tag{7}$$

The lower and upper thresholds occur at the lower and upper positive roots of Eq. (7). For a numerical example, let us suppose that $\xi=-1$. The maximum of the left-hand side of Eq. (7) occurs at $k=(2/3)\epsilon$, where it is equal to $k^3/2=(4/27)\epsilon^3$. In order to have a threshold this must be larger than $4m^2$, so $\epsilon\geqslant 3m^{2/3}=4.1\times 10^{-15}$. The threshold is at $k=2m^{2/3}=28$ TeV for this critical value of epsilon, and for larger epsilon the lower threshold is below 28 TeV while the upper threshold is above 28 TeV.

To obtain an example with asymmetric thresholds we must introduce negative curvature in the electron dispersion relation, hence we allow for $\eta<0$. The case with $\epsilon=0$ and $\xi\neq 0$ was studied in detail in Ref. [14], where it was found that there is never an upper threshold and there can be a lower threshold which is either symmetric or asymmetric, depending on the values of ξ and η. We now sketch how that conclusion is reached.

Since the final momenta are not necessarily equal at threshold, the algebra is more complicated. To keep it manageable we expand the dispersion relations to lowest order in the small quantities. Using the fact that the final momenta are parallel at threshold we set the positron momentum equal to $k-p$, and the energy conservation equation then becomes

$$\xi k^2=\eta(p^2+(k-p)^2)+\frac{m^2k}{p(k-p)}. \tag{8}$$

This equation is valid as long as both p and $k-p$ are much greater than the mass m, and all momenta are much smaller than M. (For a careful discussion about the validity of the expansion see Ref. [14].)

Were we to incorrectly assume the threshold to be always symmetric, we would set $p=k/2$ in Eq. (8) and conclude that it is given by

$$k_{\mathrm{th}}^{\mathrm{sym}}=\left(\frac{8m^2}{2\xi-\eta}\right)^{1/3}. \tag{9}$$

This would also imply that there is only a threshold when $\xi>\eta/2$. However, this result is incorrect. A threshold occurs when E_f is minimized for a fixed k. While the value $p=k/2$ always corresponds to a stationary point of E_f (since the final particles have equal mass), this point can be a maximum rather than a minimum. (In general it could also be a local minimum that is not the global minimum, but that possibility does not occur in the present example.) For $\eta>0$, the electron dispersion relation has positive curvature, so the symmetric point is the minimum and the symmetric threshold (9) applies. When η is negative, as shown in Ref. [14], the symmetric threshold applies for $\xi>0$, but for $\xi<0$ there is an asymmetric threshold whenever $\xi>\eta$:

$$k_{\mathrm{th}}^{\mathrm{aysm}}=\left[\frac{-8m^2\eta}{(\xi-\eta)^2}\right]^{1/3}. \tag{10}$$

The amount of asymmetry is given by $p-(k/2)=k\sqrt{\xi/\eta}$.

If one incorrectly assumed that the thresholds were always symmetric, one would miss the region $\eta<\xi<\eta/2<0$ where there is an asymmetric threshold and no symmetric solution. Moreover, in the overlapping region $\eta/2<\xi<0$ where the actual threshold is asymmetric but a symmetric solution does exist, the symmetric threshold formula (9) would simply give the wrong result for the threshold. For a numerical example, let $\eta=-1$. Then for $\xi=-0.6$ there is an asymmetric threshold $k_{\mathrm{th}}^{\mathrm{aysm}}=50$ TeV and no symmetric solution. For $\xi=-0.4$ there is again an asymmetric threshold $k_{\mathrm{th}}^{\mathrm{aysm}}=38$ TeV, and a (non-threshold) symmetric solution $k_{\mathrm{th}}^{\mathrm{sym}}=46$ TeV.

V. CONCLUSION

Much of the apparent simplicity of the familiar structure of thresholds in Lorentz invariant theories derives from the fact that one can always transform to the center of mass frame. We have seen that in a Lorentz violating context there can be surprising threshold behavior. Nevertheless, as we have shown, certain basic kinematic relations always pertain to threshold configurations as long as the dispersion relations are rotationally invariant and monotonically increasing with momentum. These relations can be used to systematically determine the threshold behavior.

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