S-MATRIX IN SEMICLASSICAL APPROXIMATION *

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It is shown that the tree approximation currently utilized in connection with phenomenological chiral Lagrangians corresponds to a semiclassical approximation applied to the S-matrix in quantum field theory. Some invariance properties of the S-matrix are derived in this framework.

Recently the method of effective Lagrangians has been widely used in formulating the dynamical contents of chiral symmetry [1]. One writes down in this approach a non-linear Lagrangian invariant under the chiral $SU(2) \times SU(2)$ (or $SU(3) \times SU(3)$) group, in which the pion field (or its SU(3) extention) transforming non-linearly under the group plays a crucial role [2]. It is characteristic of such a Lagrangian that the invariance does not apply to the free and the interaction part separately; these two will mix under the group transformation. Thus the asymptotic one-particle states defined by (appropriately renormalized) free fiels do not form a representation of the group, in sharp contrast to the ordinary case in which a symmetry implies a multiplet structure and associated quantum numbers for one-particle states. This situation may be interpreted in terms of a "spontaneous breakdown" of chiral symmetry; each time a chiral transformation is performed, one has to redefine the free and interaction parts, corresponding to a change of the vacuum state [3].

For the actual construction of the S-matrix, in chiral dynamics, one carries out a perturbation expansion with respect to a coupling parameter characterizing the degree of nonlinearity, collecting consistently all terms that contribute to a specific n-particle process in the lowest possible order. The corresponding Feynman diagrams turn out to be "tree diagrams", or simply connected diagrams having no closed loops [1]. Obviously it is an approximation which does not take into account unitarity (rescattering), self-energy and other higher order corrections to a given process. This may be reasonable in the low energy (soft pion) limit. On the other hand it is not clear whether the highly singular Lagrangian one uses here is really meaningful beyond such an approximation.

It is the purpose of this note to point out that the above procedure may be characterized as a semiclassical approximation to the S-matrix, which is an analog in quantum field theory of the WKB approximation to the Schrödinger equation. This statement is entirely general, being independent of the Lagrangian chosen. In the following we present the proof, and discuss its implications.

We start from the S-matrix in interaction representation, expressed as a time-ordered (T-) product ‡

$$S = T \exp[(i/\hbar c) \int L_{\mathrm{int}}(\phi)\, d^4x] . \qquad (1)$$

Its contribution to individual processes can be obtained by converting it to a normal product, for which there is a compact formula [4] †

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‡ When L_{int} contains derivatives, one should perform the differentiation after time-ordering.

† Eq. (10') could have been derived directly from Feynman's path integral formulation, or from the matrix representation

$$\langle \phi' | U | \phi'' \rangle \sim \exp[(i/2\hbar c) \int (\phi' - \phi'') K (\phi' - \phi'')\, d^4x] .$$

Applying this to eq. (2), we get

$$S(\phi) = U \exp[(i/\hbar c) \bar{L}_{\mathrm{int}}(\phi)]$$

$$= \int \langle \phi | U | \tilde{\phi} \rangle\, D[\phi] \exp[(i/\hbar c) \bar{L}_{\mathrm{int}}(\tilde{\phi})]$$

which is an exact result. Approximating the functional integral by a stationary "path" determined from the classical Euler equation for $\tilde{\phi}$, we arrive at the formulas (9') and (10'). However, their relation to the tree approximation would have been obscure.

$$T\ \exp[(\mathrm{i}/\hbar c)\int L_{\mathrm{int}}(\phi)\,\mathrm{d}^4x] =$$
$$= :U\exp[(\mathrm{i}/\hbar c)\int L_{\mathrm{int}}(\phi)\,\mathrm{d}^4x]: \equiv :S:,$$
$$U \equiv \exp[\tfrac{1}{2}\hbar c\iint \frac{\delta}{\delta\phi(x)}\Delta(x-x')\frac{\delta}{\delta\phi(x')}\,\mathrm{d}^4x\,\mathrm{d}^4x'] . \quad (2)$$

Here we have chosen for simplicity a neutral scalar field $\phi(x)$; $\Delta(x-x')$ is the corresponding free Feynman propagator $(1/\hbar c)\langle T(\phi(x),\phi(x'))\rangle_0$ ‡. The functional differentiation is carried out regarding all $\phi(x)$ with different x as if they were independent c-numbers; i.e., not subject to a free field equation. This formula can be readily generalized to all types of field. For a complex field ϕ_α, we replace U with

$$\exp[\hbar c\iint \frac{\delta}{\delta\phi_\alpha}\Delta_{\alpha\beta}(x-x')\frac{\delta}{\delta\phi^*_\beta(x')}\,\mathrm{d}^4x\,\mathrm{d}^4x'] .$$

When ϕ_α is a Fermi field, we make the further stipulation that $\phi_\alpha(x)$, $\delta/\delta\phi_\alpha(x)$, etc., at different points all anticommute. In order to built up the operation with U step by step, we may supply a fictitious parameter η in front of $\Delta(x-x')$, which will vary from 0 to 1. Differentiating with respect to η, we obtain

$$\frac{\partial}{\partial\eta}:S_\eta: = \quad (3)$$
$$= :\left(\tfrac{1}{2}\hbar c\iint\frac{\delta}{\delta\phi(x)}\Delta(x-x')\frac{\delta}{\delta\phi(x')}\,\mathrm{d}^4x\,\mathrm{d}^4x'\right)S_\eta: .$$

Now let us write

$$:S: = :\exp[-(\mathrm{i}/\hbar c)\Theta(\phi)]: . \quad (4)$$

It is not difficult to see that $\Theta(\phi)$ corresponds to a collection of all connected Feynman diagrams; this will in fact become clear below. Substituting in eq. (3), we may drop the normal product symbol and manipulate S as a mere c-number functional. We get thus

$$\frac{\partial\Theta}{\partial\eta} + \tfrac{1}{2}\mathrm{i}\iint\frac{\delta\Theta}{\delta\phi(x)}\Delta(x-x')\frac{\delta\Theta}{\delta\phi(x')}\,\mathrm{d}^4x\,\mathrm{d}^4x + \quad (5)$$
$$-\tfrac{1}{2}\hbar c\iint\frac{\delta^2\Theta}{\delta\phi(x)\,\delta\phi(x'x)}\Delta(x-x')\,\mathrm{d}^4x\,\mathrm{d}^4x' = 0 ,$$

‡ The dimensional convention adopted here is such that the field equations are "classical" (i.e., use wave numbers rather than momenta), with the free Lagrangian given by

$$-\tfrac{1}{2}(\partial_\mu\phi\partial_\mu\phi+\kappa^2\phi^2) \quad \text{for a scalar, and}$$
$$-(\bar\psi\gamma_\mu\partial_\mu\psi+\kappa\bar\psi\psi) \quad \text{for a spinor, where}$$
$$x_\mu = (x_1,x_2,x_3,x_4=\mathrm{i}ct) , \qquad \kappa = mc/\hbar .$$

with the initial condition

$$\Theta = \int L_{\mathrm{int}}(\phi)\,\mathrm{d}^4x \equiv -\bar{L}_{\mathrm{int}} \quad \text{for} \quad \eta = 0 . \quad (6)$$

If Θ is expanded in a power series in η, the second term in eq. (5) means the contraction of two disconnected diagrams by means of a propagator to form a single connected diagram, whereas the third term corresponds to a contraction within a connected diagram, creating an internal loop. If we drop this third term, which is multiplied by $\hbar c$, eq. (5) takes the form of a classical "Hamilton-Jacobi equation" (writing Θ_c for Θ)

$$\frac{\partial\Theta_c}{\partial\eta} + \bar{H}(\pi(x)) = 0 ,$$
$$\bar{H} = \tfrac{1}{2}\mathrm{i}\iint\pi(x)\Delta(x-x')\pi(x')\,\mathrm{d}^4x\,\mathrm{d}^4x' ,$$
$$\pi(x) \equiv \delta\Theta_c/\delta\phi(x) . \quad (7)$$

where $\phi(x)$ and $\pi(x)$ are canonical conjugates, and η plays the role of time. It is obvious that we generate only tree diagrams.

The "Hamiltonian" contains only the "kinetic energy" dependent on $\pi(x)$, so the solution can be immediately obtained by integrating the equations of motion

$$\frac{\mathrm{d}\pi(x)}{\mathrm{d}\eta} = 0 , \quad \frac{\mathrm{d}\phi(x)}{\mathrm{d}\eta} = \mathrm{i}\int\Delta(x-x')\pi(x')\,\mathrm{d}^4x ,$$
$$\phi(x) = \eta\,\mathrm{i}\int\Delta(x-x')\pi(x')\,\mathrm{d}^4x' + F(\pi) . \quad (8)$$

$F(\pi)$ is determined from the initial condition

$$\pi(x) = \delta\Theta/\delta\phi(x) = -\delta\bar{L}_{\mathrm{int}}/\delta\phi(x) \quad \text{for} \quad \eta = 0 .$$

Thus

$$\pi(x) = -\delta\bar{L}_{\mathrm{int}}(\tilde\phi)/\delta\tilde\phi ,$$
$$\tilde\phi(x) \equiv \phi(x) - \eta\,\mathrm{i}\int\Delta(x-x')\pi(x')\,\mathrm{d}^4x' , \quad (9)$$

which is an intrinsic equation relating ϕ and π. The Hamilton-Jacobi functional Θ_c is now determining from integration of the relation

$$\delta\Theta_c(\phi,\eta) = \int\pi(x)\delta\phi(x)\,\mathrm{d}^4x - \bar{H}\delta\eta$$

We find readily

$$\Theta_c = -\bar{L}_{\mathrm{int}}(\tilde\phi) + \eta\bar{H}(\pi) . \quad (10)$$

This, together with eq. (9), gives the solution Θ_c when it is expressed in terms of $\phi(x)$, and η put equal to 1. Eqs. (9) and (10) can be brought to a more transparent form. From eq. (9) follows

$$\pi(x) = K(\tilde\phi(x) - \phi(x)) = -\delta\bar{L}_{\mathrm{int}}/\delta\tilde\phi(x) ,$$

or

$$\delta \bar{L}_{tot}(\tilde{\phi})/\delta\tilde{\phi}(x) = K\phi(x) ;$$

$$K \equiv i\Delta^{-1}, \quad \bar{L}_{tot}(\tilde{\phi}) \equiv \bar{L}_{int}(\tilde{\phi}) + \bar{L}_0(\tilde{\phi}) \qquad (9')$$
$$\equiv \bar{L}_{int}(\tilde{\phi}) + \tfrac{1}{2}\int \tilde{\phi} K \tilde{\phi}\, d^4x,$$

and eq. (10) becomes (with $\eta = 1$)

$$\Theta_c(\phi, \tilde{\phi}) = -\bar{L}_{int}(\tilde{\phi}) - \bar{L}_0(\tilde{\phi} - \phi) \qquad (10')$$
$$= -\bar{L}_{tot}(\tilde{\phi}) + \int \phi K \tilde{\phi}\, d^4x - \tfrac{1}{2}\int \phi K \phi\, d^4x .$$

Here K is the Klein-Gordon operator $\Box - \kappa^2$, so $\bar{L}_{tot}$ is the total Lagrange functional. Eq. (10') is nothing but a Lagrange functional for the field $\tilde{\phi}$, with ϕ regarded as an external source field, and eq. (9') is the resulting Euler equation. When the latter is solved for $\tilde{\phi}$ in terms of ϕ with the boundary condition characterized by the Feynman kernel Δ, and in the final result we let $K\phi = 0$, then we get a solution of the classical field equation which approaches asymptotically the free field ϕ. Θ_c becomes a functional of ϕ as the dummy (internal) field $\tilde{\phi}$ has been eliminated †.

This proves our contention about the nature of the approximation involved. Eqs. (9') and (10'), or (9) and (10), provide a concise formula for the construction of Θ_c. When there is more than one field, we can simply add a respective source term to each field. The whole thing is treated as a quasi-c number theory, in the sense that for Fermi fields anticommutatively is to be respected. We will now make use of the formalism to examine invariance properties of the S-matrix in our approximation.

1) Suppose

$$\Theta_c(\phi_\alpha, \tilde{\phi}) = -\bar{L}_{int}(\tilde{\phi}_\alpha) - \sum_\alpha \bar{L}_{0\alpha}(\tilde{\phi}_\alpha - \phi_\alpha) \qquad (11)$$

is invariant under an infinitesimal change $d\phi_\alpha(x)$, $d\tilde{\phi}_\alpha(x)$, $\alpha = 1, 2, \ldots,$ of the fields. This will be true of the free and interaction parts of $\bar{L}$ are separately invariant. After the $\tilde{\phi}$'s have been eliminated, $\Theta_c(\phi_\alpha)$ and the S-matrix remain invariant under the change $d\phi_\alpha$.

2) It is possible that $\Theta_c(\phi_\alpha, \tilde{\phi}_\alpha)$ is invariant under a change of the ϕ_α's alone due to a special feature of the free Langrangian. The S-matrix will again be invariant. A notable example is the familiar gauge invariance of the S-matrix under $dA_\mu(x) = \partial_\mu \lambda(x)$ (without making a phase transformation on charged fields). This is because $L_0(A_\mu)$ depends only on the antisymmetric tensor $F_{\mu\nu}$ *.

The case of a massless spin-0 field does not belong to this class, although $L_0(\phi)$ is invariant under a constant displacement $d\phi(x) = c$. The reason is that the Klein-Gordon (or rather the d'Alembertian) operator appearing in the interference term in eq. (10') is spurious, and cannot be counted on to kill a constant, as actually $\tilde{\phi}$ has a singularity $\sim \Delta = iK^{-1}$ according to eq. (9).

3) Next let us assume that $\bar{L}_{tot}(\tilde{\phi}_\alpha)$ as a whole is invariant, but not necessarily for its parts L_{int} and L_0 separately. We have then

$$\delta \bar{L}_{tot}(\tilde{\phi}_\alpha) = \int d\tilde{\phi}_\alpha(x)\, \delta \bar{L}_{tot}/\delta\tilde{\phi}_\alpha(x)\, d^4x = 0 .$$

From the Euler equation for $\tilde{\phi}_\alpha$: $\delta \bar{L}_{tot}/\delta\tilde{\phi}_\alpha(x) = K_\alpha \phi_\alpha(x)$, it follows that

$$\sum_\alpha \int d\tilde{\phi}_\alpha K_\alpha \phi_\alpha\, d^4x = 0 .$$

As a result, the entire $\Theta_c(\phi_\alpha)$ is invariant. This, however, is not a statement about the symmetry of the S-matrix, as the asymptotic fields ϕ_α are not varied. It is rather a statement about the independence of the S-matrix (as well as its off-shell extrapolation $K_\alpha\phi_\alpha \neq 0$) on a redefinition of the dummy fields, which is a symmetry operation on L_{tot}.

4) We can go even further, and subject the dummy field variables to an arbitrary substitution which is not a symmetry operation:

$$\tilde{\phi}_\alpha(x) = F_\alpha(\tilde{\phi}'_\beta(x)) = \tilde{\phi}'_\alpha(x) + f_\alpha(\tilde{\phi}'_\beta(x)) , \qquad (12)$$

where f contains only nonlinear terms in the fields. We claim that the S-matrix (on the mass shell) is again invariant under such a redefinition of the $\tilde{\phi}$'s. To be more precise, we get the same result if we start from

$$\Theta_c(\phi_\alpha, \tilde{\phi}'_\alpha) = -\bar{L}_{tot}(F_\alpha(\tilde{\phi}'_\beta)) - \sum_\alpha \bar{L}_{0\alpha}(\tilde{\phi}'_\alpha - \phi_\alpha) \; (13)$$

instead of eq. (11), provided that we construct the solution as a power series in the fields. It will suffice to show this for infinitesimal f.

* We know from the Ward-Takahashi identity that gauge invariance in the present sense does not hold off the mass shell of charged particles in general, whereas the present argument would seem to apply also off the mass shell. The latter, however, is not true. In order for the Maxwell equations to be consistent, their currents have to be conserved. This would not be true if the external charged fields were arbitrary.

† See footnote † on page 626.

Comparing eqs. (11) and (13), we find

$$\Theta_c'(\phi, \tilde{\Phi}') - \Theta_c(\phi, \tilde{\Phi}) =$$

$$= \int \sum_\alpha f_\alpha(\tilde{\Phi}')K_\alpha \phi_\alpha = \Theta_c'(\phi) - \Theta_c(\phi)$$

when the respective Euler equations are satisfied (stationary values of $\tilde{\Phi}'$ and $\tilde{\Phi}$). But the f_α's have no mass shell singularities in a perturbative solution because they are non-linear. Thus the difference vanishes as $K_\alpha \phi_\alpha \to 0$. (The condition on the f's may be further relaxed.)

Summarizing, we found that the S-matrix in the "tree" approximation corresponds to a semiclassical (WKB) approximation applied to field theory, and that invariance properties expected of the exact S-matrix are also valid here. In particular, the item 4) above can be interpreted as the irrelevance of the choice of the "interpolating fields" $\tilde{\Phi}_\alpha$, a property implied by the Lehmann-Symanzik-Zimmermann formalism of quantum field theory [5]. This property has been explicitly noted in chiral dynamics [6] where the definition of the pion field is not unique, as there is no obvious principle by which to choose a particular nonlinear transformation.

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Actual differences that appear in π-π scattering, for example, are due to different choices of symmetry breaking terms in the Lagrangian, and not due to different choices of field variables in the present context.

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