SOLITON ENERGIES IN SUPERSYMMETRIC THEORIES

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We show that in two-dimensional supersymmetric theories, the energies of solitons do receive non-zero but finite quantum corrections. This is in contrast to the earlier work of D'Adda and Di Vecchia, who have claimed that, to order $\hbar$, these corrections vanish exactly. Our results is also relevant to the quantum energy-bound derived by Witten and Olive.

One of the fascinating features of supersymmetry is that radiative corrections to some quantities produced by fermions and bosons fluctuations cancel [‡1]. In some cases, though not all, this cancellation is complete and the correction is strictly zero. This raises the question as to whether such exact cancellation also takes place in the quantum corrections to the energy of solitons in supersymmetric field theories. The general procedure for quantising solitons and obtaining their energies in a systematic semi-classical expansion has been well known for nearly a decade [2] [‡2]. In typical examples like the "kink" of the $(1+1)$-dimensional ϕ^4 theory, or the soliton of the sine-Gordon theory, the quantum soliton states do receive, in the leading one-loop order $[\mathrm{O}(\hbar)]$, finite and non-zero corrections to their masses. When fermions are coupled to these boson fields, their one-loop contribution to the quantum energy will come with opposite sign to the contribution of boson fluctuations, but, generally, these contributions will not exactly cancel one another. The question before us is, whether this cancellation becomes exact when the lagrangian is chosen to be supersymmetric.

Our paper is addressed to this problem, which, interesting in its own right, assumes added importance in the light of Witten and Olive's work [4]. They derive a quantum theoretic generalization of the classical Bogomolny type bound, for solitons in supersymmetric theories. Their bounds are exactly saturated classically. Whether they continue to be saturated at the quantum level, is clearly related to how much quantum correction the soliton energy obtains.

The problem of quantum corrections to the energy of one-dimensional supersymmetric solitons was already studied by D'Adda and Di Vecchia [5] some years ago. They claimed that to one-loop order, the quantum corrections are indeed zero. However, on closer inspection we find that they have neglected to take into account two important ingredients. One is the difference in density of fluctuation modes between bosons and fermions. The other is the presence of renormalisation counterterms. We present the correct calculation below, incorporating both these ingredients. We find that each of these ingredients gives rise to a substantial (indeed, ultraviolet divergent) correction to the D'Adda–di Vecchia result. Fortunately, the ultraviolet divergence coming from these two sources do cancel. But what is left behind is a non zero $\mathrm{O}(\hbar)$ contribution to the soliton energy, even in these supersymmetric theories.

In any calculation employing renormalisation counter terms, the finite portions of these coun-

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[‡1] For a review, see ref. [1].

[‡2] For a detailed discussion, see ref. [3].

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ter terms are not unique. This is the usual freedom associated with the renormalization prescription. Nevertheless, our statement that the soliton energy receives a non-zero finite quantum correction can be made in an unambiguous way, in terms of renormalized quantities. This we do at the end.

We will illustrate our calculation with a familiar example, viz., the supersymmetric extension of the two-dimensional ϕ^4 theory. But it will be evident that our procedure is general, and can be equally well applied to other two-dimensional theories, such as the supersymmetric sine-Gordon model.

Consider a supersymmetric model in $1+1$ dimensions described by the action [6],

$$\mathscr{A} = \frac{1}{2}\int d^2x[(\partial_\mu\varphi)^2 - S^2 + i\bar{\psi}\not\partial\psi - S'\bar{\psi}\psi] \tag{1}$$

where

$$S = (\lambda/2)^{1/2}(\varphi^2 - \mu^2/\lambda), \tag{2}$$

φ is a real scalar field and ψ is a Majorana Fermi field.

There are two degenerate ground states for this model, $\varphi = \pm\mu\sqrt{\lambda}$. On expanding φ about any one of them, the action (1) takes the following form in terms of $\eta \equiv \varphi - \mu/\sqrt{\lambda}$:

$$\mathscr{A} = \int d^2x[\tfrac{1}{2}(\partial_\mu\eta)^2 - \mu^2\eta^2 + \tfrac{1}{2}\bar{\psi}(i\not\partial - \sqrt{2}\mu)\psi - \tfrac{1}{4}\lambda\eta^4 - \mu\sqrt{\lambda}\,\eta^3 - (\lambda/2)^{1/2}\eta\bar{\psi}\psi]\,. \tag{3}$$

This describes a boson and a Majorana fermion of equal mass, $m_0 = \sqrt{2}\mu$, at the tree level.

This model also admits of soliton solutions, the well known "kink",

$$\varphi_s = (\mu/\sqrt{\lambda})\tanh(\mu x/\sqrt{2})\,, \tag{4}$$

with its classical energy as

$$M_0 = 2\sqrt{2}\mu^3/3\lambda \equiv m_0^3/3\lambda\,. \tag{5}$$

We are interested in calculating the lowest order quantum correction to this soliton energy. This we do using the standard semi-classical techniques [2,3]. The corrections arise from both the boson and fermion fluctuations around the classical soliton as well as the renormalization counter-terms:

$$M_1 = M_0 + \frac{\hbar}{2}\Big(\sum\omega_B - \sum\omega_F\Big) + \Delta M_{ct}\,, \tag{6}$$

where $\frac{1}{2}\hbar\omega_B$ and $\frac{1}{2}\hbar\omega_F$ are the energies of boson and fermion fluctuations and ΔM_{ct} are the counterterm contributions. (The factor 1/2 has to be used for the fermion contributions as well, because the fermion here is a Majorana field).

Boson fluctuations. These obey the equation

$$\begin{aligned}[&-d^2/dx^2 + U''(\varphi_s)]\xi(x)\\ &=\{-d^2/dx^2 + \mu^2[3\tanh^2(\mu x/\sqrt{2}) - 1]\}\xi(x)\\ &=\omega_B^2\xi(x)\,,\end{aligned} \tag{7}$$

where $U(\varphi) = \frac{1}{2}S^2(\varphi)$ is the boson field potential. The solutions of eq. (7) are well known [2,3]. There are two discrete eigenvalues, $\omega_B = 0$, and $\omega_B = (3/2)^{1/2}\mu$ and also a continuum of states whose eigen-functions behave as $\exp\{i[kx \pm \delta_B(k)/2]\}$ as $x \to \pm\infty$, where $\omega_B^2 = k^2 + 2\mu^2$. The phase shifts $\delta_B(k)$ are also known, but we do not need an explicit form for our purpose here. In a box of length L, the $\xi(x)$ must obey boundary conditions appropriate to the second order differential equation

$$\begin{aligned}&\xi(-L/2) = \xi(L/2) \quad \text{and}\\ &d\xi/dx|_{-L/2} = d\xi/dx|_{L/2}\,.\end{aligned} \tag{8}$$

This leads to the usual density of the continuum states

$$dn/dk = (1/2\pi)[L + d\delta_B(k)/dk]\,, \tag{9}$$

and hence, the boson fluctuation energy is

$$\frac{\hbar}{2}\sum\omega_B = \tfrac{1}{2}\hbar(3/2)^{1/2}\mu + \frac{\hbar}{2}\int_{-\infty}^{\infty}\frac{dk}{2\pi}(k^2+2\mu^2)^{1/2}[L + d\delta_B(k)/dk]\,. \tag{10}$$

Fermion fluctuations. We choose the representation for the two-dimensional γ-matrices as $\gamma_0 = \sigma_2$, $\gamma_1 = i\sigma_3$. In this representation, Majorana fermions are real, $\psi^* = \psi$.

The fermion fluctuations are governed by the equation

$$[i\not\partial - S'(\varphi_s)]\psi = 0\,. \tag{11}$$

Writing

$$\psi(x,t) = u(x)\exp(-i\omega_F t) + u^*(x)\exp(i\omega_F t)\,, \tag{12}$$

$$u(x)=\begin{pmatrix}u_+(x)\\ u_-(x)\end{pmatrix}, \tag{13}$$

we can re-express eq. (11) in terms of two coupled equations:

$$\mathrm{i}[\mathrm{d}_x + S'(\varphi_s)]u_-(x) = -\omega_F u_+(x), \tag{14a}$$

$$\mathrm{i}[\mathrm{d}_x - S'(\varphi_s)]u_+(x) = -\omega_F u_-(x). \tag{14b}$$

Multiplying (14a) by $\mathrm{i}[\mathrm{d}_x - S'(\varphi_s)]$, and (14b) by $\mathrm{i}[\mathrm{d}_x + S'(\varphi_s])$, yields

$$\{-\mathrm{d}^2/\mathrm{d}x^2 + \mu^2[3\tanh^2(\mu x/\sqrt{2}) - 1]\}u_-(x) = \omega_F^2 u_-(x), \tag{15a}$$

$$\{-\mathrm{d}^2/\mathrm{d}x^2 + \mu^2[\tanh^2(\mu x/\sqrt{2}) + 1]\}u_+(x) = \omega_F^2 u_+(x), \tag{15b}$$

where we have used eqs. (2) and (4).

We notice the lower component, $u_-(x)$, obeys the same equation as the boson fluctuations, eq. (7). Were it not for subtleties of boundary conditions, it would appear that for every solution ξ of eq. (7) with eigenvalue ω_B^2, we can use the same solution for u_- of eq. (15a) with $\omega_F = \omega_B$. The upper component can then be obtained using eq. (14a), except for the case $\omega_F = 0$, which any way does not give any contribution to the fluctuation energy. On this basis D'Adda and Di Vecchia [5] have asserted that $\Sigma\,\omega_B - \Sigma\,\omega_F = 0$. Also, having neglected the contribution due to counter terms, they have concluded that there are no quantum corrections to the soliton mass. However, cancellation of boson and fermion fluctuation energies, though valid for discrete modes ($\omega_B = \omega_F = (3/2)^{1/2}\mu$, in the present case), is not true for the continuum. This is because the density of these states are different, as explained below.

As befits a first order differential equation the boundary conditions on (11) are

$$u_\pm(-L/2) = u_\pm(+L/2), \tag{16}$$

at the same time, u_+ and u_- are related by eq. (14). To obtain solutions consistent with these boundary conditions, it is helpful to start with solutions for $u_-(x)$, which are odd and even under $x \leftrightarrow -x$, and have the asymptotic forms:

$$u_-^{(1)} \sim \cos(kx \pm \tfrac{1}{2}\delta_B) \quad \text{as } x \to \pm\infty,$$
$$u_-^{(2)} \sim \sin(kx \pm \tfrac{1}{2}\delta_B) \quad \text{as } x \to \pm\infty, \tag{17}$$

where the phase shift δ_B is the same as in the boson case. The associated solutions for $u_+(x)$, obtained using (14a), will have the asymptotic form, as $x \to \pm\infty$,

$$u_+^{(1)} \sim \mathrm{i}\sin[kx \pm \tfrac{1}{2}(\delta_B + \theta)],$$
$$u_+^{(2)} \sim -\mathrm{i}\cos[kx \pm \tfrac{1}{2}(\delta_B + \theta)], \tag{18}$$

where $\tan(\theta/2) = -\sqrt{2}\mu/k$.

Now, $u_-^{(1)}$ and $u_+^{(2)}$ are even and obey the boundary conditions (16). The allowed values of k are, therefore, obtained by enforcing (16) on $u_+^{(1)}$ and $u_-^{(2)}$. This yields the density of states for the two sets as

$$\mathrm{d}n^{(1)}/\mathrm{d}k = (1/2\pi)[L + \mathrm{d}(\delta_B + \theta)/\mathrm{d}k],$$
$$\mathrm{d}n^{(2)}/\mathrm{d}k = (1/2\pi)[L + \mathrm{d}\delta_B/\mathrm{d}k], \tag{19}$$

with $0 \leqslant k < \infty$.

Thus, finally, the contribution due to fermion fluctuations to the soliton mass is

$$-\frac{\hbar}{2}\sum \omega_F = -\tfrac{1}{2}\hbar(3/2)^{1/2}\mu - \frac{\hbar}{2}\int_0^\infty \frac{\mathrm{d}k}{2\pi}(k^2+2\mu^2)^{1/2}(\mathrm{d}n^{(1)}/\mathrm{d}k + \mathrm{d}n^{(2)}/\mathrm{d}k), \tag{20}$$

and

$$\frac{\hbar}{2}\left(\sum \omega_B - \sum \omega_F\right) = -\frac{\hbar}{2}\int_0^\infty \frac{\mathrm{d}k}{2\pi}(k^2+2\mu^2)^{1/2}\frac{\mathrm{d}\theta(k)}{\mathrm{d}k} = \frac{-\mu\hbar}{\sqrt{2}}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\frac{1}{(k^2+2\mu^2)^{1/2}}. \tag{21}$$

We notice that although the δ_B-dependent pieces cancel between $\Sigma\,\omega_B$ and $\Sigma\,\omega_F$, there is a residual non-zero contribution due to the extra phase shift θ suffered by u_+. In fact, this residual contribution in (21) is logarithmically divergent! Fortunately, this logarithmic divergence is removed by the counterterms, as expected in a renormalisable theory.

Counter-terms. In our two-dimensional model, the only divergences present come from the boson self-energy graphs in figs. 1b, 1c, 1e and 1f. One must cancel these divergences by adding the following counter term to the lagrangian:

$$L_{ct} = \tfrac{1}{2}\delta\mu^2(\varphi^2 - \mu^2/\lambda)\,. \tag{22}$$

This counter term is designed to leave the lagrangian supersymmetric. Its contribution to the one-loop vacuum energy is zero. We must remember that $\delta\mu^2$ is required to cancel the divergences, but its finite part is arbitrary. Thus

$$\delta\mu^2 = \hbar\lambda(B + C) \tag{23}$$

where

$$B = \int_{-\infty}^{\infty} \frac{dk}{(4\pi)}\frac{1}{(k^2 + 2\mu^2)^{1/2}}\,, \tag{24}$$

and C is a *finite* arbitrary constant. B is so designed that fig. 1a cancels the divergent part of fig. 1b plus 1c while fig. 1g cancels the divergent part of figs. 1e plus 1f.

The contribution of the counter term (22) to the soliton energy is

$$\Delta M_{ct} = \frac{\delta\mu^2}{2}\int_{-\infty}^{\infty} [\varphi_s^2(x) - \mu^2/\lambda]\,dx$$
$$= \hbar\sqrt{2}\mu(B + C)\,. \tag{25}$$

Adding all the terms in (6), we have, for the soliton energy upto one-loop order,

$$M_1 = m_0^3/3\lambda - \sqrt{2}\hbar\mu\int_{-\infty}^{\infty}\frac{dk}{4\pi}\frac{1}{(k^2 + 2\mu^2)^{1/2}}$$
$$+ \sqrt{2}\hbar\mu(B + C)$$
$$= m_0^3/3\lambda + \sqrt{2}\hbar\mu C = m_0^3/3\lambda + \hbar m_0 C\,. \tag{26}$$

The first term in (26) is the classical soliton mass. The second term yields a *finite* one-loop correction. The logarithmic divergence in $\Sigma\,\omega_B - \Sigma\,\omega_F$ has been cancelled by the piece B in the counter-term.

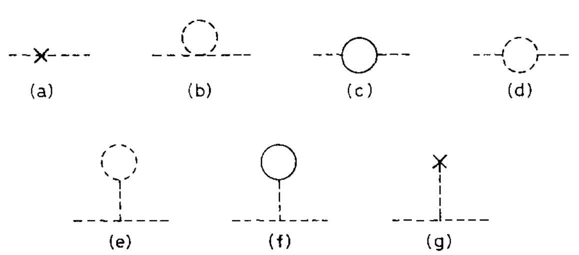

Fig. 1. Boson self energy diagrams to O(λ). The dashed line correspons to the shifted Bose field $\eta = \varphi - \mu\sqrt{\lambda}$, and the heavy line to fermions.

It might appear that we have proved our result, but the quantum correction in (26) is proportional to the arbitrary constant C. To get an unambiguous result independent of this renormalisation freedom, we must eliminate C in favour of some physical quantity. Such a quantity is the renormalised boson mass, which also depends on C. The one-loop boson mass m_1 is obtained by including the complete set of one-loop self-energy graphs in fig. 1. The choice (23)–(24) of counter-terms will ensure that the divergences in these graphs cancel. It is a matter of straightforward algebra to compute these graphs and show that they change the boson propagator from $(p^2 - m_0^2)^{-1}$, by a finite amount, to

$$\{p^2 - m_0^2 - \hbar\lambda[2C + (5m_0^2 + p^2)A(p^2)]\}^{-1} \tag{27}$$

where

$$A(p^2) = \int\frac{d^2k}{(2\pi)^2}\frac{i}{(k^2 - m_0^2)[(k-p)^2 - m_0^2]}\,. \tag{28}$$

This yields a renormalised one-loop boson mass

$$m_1^2 = m_0^2 + \lambda\hbar[2C + 6m_0^2 A(m_0^2)]\,. \tag{29}$$

Use this to elimate m_0 in (26) in favour of m_1 and the arbitrary constant C disappears as expected, yielding for the one-loop soliton mass

$$M_1 = m_1^3/3\lambda + m_1\hbar/\sqrt{12}\pi + O(\hbar^2\lambda)\,. \tag{30}$$

This result is now free of arbitrary constants and bare parameters. The first term is the classical term, but, with the bare boson mass replaced by the physical (one-loop) boson mass. There is, in addition, a finite non-zero correction in the form of the second term. In short, the soliton mass *does* receive a non-zero but finite quantum corection.

Notice that the energy of the *vacuum*, E_{vac}, computed by this method, *will* be zero to one-loop order. For this we would insert the x-independent function $\varphi_s = \mu/\sqrt{\lambda}$ in eqs. (7), (12) and (14). In that case, *both* components $u_+(x)$ and $u_-(x)$ would obey the same equation as the boson fluctuations. The density of states will match and $\Sigma\,\omega_B^{vac} - \Sigma\,\omega_F^{vac} = 0$. The counter-terms

(22) also do not contribute in one-loop order to E_{vac}, which therefore remains zero. In turn, supersymmetry will remain unbroken to one-loop order, consistent with earlier results [7,8] [‡3].

As a final comment, let us recall the very elegant paper by Witten and Olive [4]. One of their results was that for the supersymmetric system in eq. (1), there is a bound on the exact quantum mass of the soliton:

$$M \geqslant \tfrac{1}{2}|\langle \mathrm{Sol}|T|\mathrm{Sol}\rangle| \tag{31}$$

where the operator T is given by,

$$T = 2\int_{-\infty}^{\infty} \mathrm{d}x[S(\varphi)\partial\varphi/\partial x] = [H(\varphi(x))]_{x=-\infty}^{x=+\infty}, \tag{32}$$

with

$$\mathrm{d}H(\varphi)/\mathrm{d}\varphi = 2S(\varphi). \tag{33}$$

Witten and Olive also pointed out that the bound is exactly saturated at the classical level. For instance, in our φ^4-theory, $H(\varphi) = 2(\lambda/2)^{1/2}(\frac{1}{3}\phi^3 - \mu^2\varphi/\lambda)$. At the classical level

$$\begin{aligned}\tfrac{1}{2}T_0 &\equiv \tfrac{1}{2}\langle \mathrm{Sol}|T|\mathrm{Sol}\rangle_{\text{classical}} \\ &= (\lambda/2)^{1/2}[\tfrac{1}{3}\varphi_S^3(x) - \mu^2\varphi_S(x)/\lambda]_{x=-\infty}^{x=+\infty} \\ &= -2\sqrt{2}\mu^3/3\lambda \equiv -M_0,\end{aligned} \tag{34}$$

Now, our result is that M receives non-zero radiative corrections. Thus, it will no longer be equal to T_0. Of course, the bound (i.e. the inequality) in (31) no doubt still holds. Even the saturation of the bound may hold, because $\langle \mathrm{Sol}|T|\mathrm{Sol}\rangle$ may also receive radiative corrections. This in turn is possible if $\langle \mathrm{vac}|\varphi^3|\mathrm{vac}\rangle$ and $\langle \mathrm{vac}|\varphi|\mathrm{vac}\rangle$ develop higher order corrections. Whether the radiative correction to M match the possible radiative corrections to $\frac{1}{2}|\langle \mathrm{Sol}|T|\mathrm{Sol}\rangle|$ requires a careful renormalised evaluation of the latter.

[‡3] Ref. [8] claimed that supersymmetry can be spontaneously broken by radiative corrections. However, their one-loop calculation is contaminated by some $O(\hbar^2)$ terms. When these are removed, we find that their corrected one-loop effective potential preserves supersymmetry. We understand that this has been independently realised by these authors [9]. See also ref. [7].

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