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Bäcklund Transformations for Nonlinear Sigma Models with Values in Riemannian Symmetric Spaces*

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Abstract. This work deals with Bäcklund transformations for the principal $SL(n,\mathbb{C})$ sigma model together with all reduced models with values in Riemannian symmetric spaces. First, the dressing method of Zakharov, Mikhailov, and Shabat is shown, for the case of a meromorphic dressing matrix, to be equivalent to a Bäcklund transformation for an associated, linearly extended system. Comparison of this multi-Bäcklund transformation with the composition of ordinary ones leads to a new proof of the permutability theorem. A new method of solution for such multi-Bäcklund transformations (MBT) is developed, by the introduction of a "soliton correlation matrix" which satisfies a Riccati system equivalent to the MBT. Using the geometric structure of this system, a linearization is achieved, leading to a nonlinear superposition formula expressing the solution explicitly in terms of solutions of a single Bäcklund transformation through purely linear algebraic relations. A systematic study of all reductions of the system by involutive automorphisms is made, thereby defining the multi-Bäcklund transformations and their solution for all Riemannian symmetric spaces.

1. Introduction

In two previous papers [1, 2], the solution of iterated sequences of Bäcklund transformations (BT's) for principal $U(n)$ and $SL(n,\mathbb{C})$ sigma models were found. The equations defining such models are:

$$A_\eta + B_\xi = 0, \tag{1.1}$$

and

$$A = g_\xi g^{-1}, \quad B = g_\eta g^{-1}, \tag{1.2}$$

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where $g(\xi, \eta)$ is a group [$SL(n, \mathbb{C})$ or $U(n)$] valued function and (ξ, η) are light-cone coordinates on two-dimensional Minkowski space. The Bäcklund transformation [1–3] determines a new solution, $\tilde{g}$, from a given one through the first order system:

$$\begin{aligned} \tilde{g}_\xi \tilde{g}^{-1} - g_\xi g^{-1} &= -\lambda_0 (\tilde{g} g^{-1})_\xi, \\ \tilde{g}_\eta \tilde{g}^{-1} - g_\eta g^{-1} &= \lambda_0 (\tilde{g} g^{-1})_\eta, \end{aligned} \tag{1.3}$$

subject to the constraints

$$\lambda_0 \tilde{g} g^{-1} + \mu_0 g \tilde{g}^{-1} = (\mu_0 + \lambda_0) \mathbb{1}, \tag{1.4}$$

where μ_0, λ_0 are arbitrary complex parameters for $SL(n, \mathbb{C})$ and $\mu_0 = \overline{\lambda_0}$ for $U(n)$.

The method of solution developed in [1, 2] consisted of converting the BT into pseudopotential equations of matrix Riccati type, interpreting these geometrically in terms of the action of $SL(2n, \mathbb{C})$ on the Grassman manifold $G_n(\mathbb{C}^{2n})$, and linearizing through an algebraic procedure based upon the subgroup reduction $SL(2n, \mathbb{C}) \supset SL(n, \mathbb{C}) \times SL(n, \mathbb{C})$. The iterated sequence was solved through a recursive procedure leading to a nonlinear superposition formula expressing the resulting solution directly in terms of solutions to (1.3), (1.4), with various values of the input parameters (μ_0, λ_0).

The linearized equations are exactly those of Zakharov and Mikhailov [4, 6] and Zakharov and Shabat [5], (henceforth ZMS)

$$\psi_\xi = \frac{A}{1+\lambda} \psi, \qquad \psi_\eta = \frac{B}{1-\lambda} \psi, \tag{1.5}$$

which, in their approach, form the starting point of the "dressing method." The equivalence of the two methods for the generation of solitons was also established in [1, 2].

In another previous work, one of the authors [7] applied a reduction procedure to pass from the principal $O(n)$ sigma model to real Grassman manifolds $SO(p+q)/SO(p) \times SO(q)$, obtaining the Bäcklund transformations and multisoliton solutions for that case. This required the introduction of a new type of BT in order to satisfy nontrivially the appropriate reality conditions. Within the ZMS approach, this corresponded to the introduction of a pair complex conjugate poles in the dressing matrix.

In the present work, we generalize and extend the results of [1, 2, 7] in several ways. First, we show how the correspondence between the ZMS procedure and the Bäcklund transformation approach may be extended beyond the level of single poles or complex conjugate pairs through the introduction of a "multi-Bäcklund transformation" which has the effect of generating, in a single step, a new solution identical to that obtained by solving an iterated sequence of simple BT's. This multi-Bäcklund transformation may be obtained either by composition of simple BT's or by replacing the original system of Eqs. (1.1) and (1.2) by an equivalent, extended system obtained by evaluating all derivatives of Eq. (1.5) at $\lambda = 0$ up to the number of iterations involved. The transformation equations which extend Eqs. (1.3) and (1.4) are obtained by repeated differentiation of the ZMS dressing matrix at $\lambda = 0$ and may be interpreted as a simple Bäcklund transformation for the extended system. This procedure is developed in Sect. 2 and used to give a new direct proof of the permutability theorem for BT's which does not require the explicit integrated form of the solution to an iterated sequence.

In Sect. 3, a new method for analysis of multi-soliton solutions (in any background field) is developed by introducing what we call the "soliton-correlation matrix." This matrix, which is constructed by using the ZMS dressing matrix and its inverse as creation and annihilation operators for solitons, is shown to satisfy a matrix Riccati equation corresponding to the action of $SL(2nK,\mathbb{C})$ on $G_{nK}(\mathbb{C}^{2nK})$, where K is the number of solitons. Generalizing the procedure applied to single solitons in [1, 2] we obtain in Sect. 4 a linearization based upon a subgroup reduction from $SL(2nK,\mathbb{C})$ to products of $SL(n,\mathbb{C})$'s and $SL(2n,\mathbb{C})$'s (depending upon parameter degeneracies), thereby determining the solution again in terms of the ZMS equations (1.5) through a linear fractional transformation of the initial data. This approach has the virtue of giving an explicitly constructed nonlinear superposition formula without the need for a recursive procedure. Moreover, the various degenerate cases involving coincident poles in the dressing matrix and its inverse are treated uniformly, the distinction arising only in the Jordan normal forms defining the subgroup reductions. The method of solution thus reduces the problem to elementary linear algebra and is somewhat simpler than the one used by Zakharov and Mikhailov [6]. Moreover, it lends itself conveniently to the formulation of the reduction procedure.

In Sect. 5 we recall how reductions of the sigma model to all possible Riemannian symmetric spaces may be obtained from the Cartan immersions of such spaces into their group of isometries through the use of involutive automorphisms [8]. Applying these reductions to the soliton correlation matrix, we derive corresponding constraints within the Grassman manifold $G_{nK}(\mathbb{C}^{2nK})$, and prove through the geometrical interpretation that these are consistent with the matrix Riccati equations defining the evolution of the system. Each such reduction involves a constraint determining a submanifold of $G_{nK}(\mathbb{C}^{2nK})$, consisting either of totally isotropic subspaces of $\mathbb{C}^{2nK}$ under some hermitian, quadratic or symplectic form, or of subspaces invariant under certain linear maps, and a corresponding reduction of the group $SL(2nK,\mathbb{C})$ preserving these submanifolds. The constraints are also expressed in the notation of [6], and the results for all reductions defining the irreducible classical Riemannian symmetric spaces are given in Tables 1 and 2.

2. Multi-Bäcklund Transformations

Consider the system (1.5), whose integrability conditions are Eqs. (1.1) and (1.2). Now define the sequence of matrix functions

$$g_l=\frac{1}{l!}\frac{d^l}{d\lambda^l}\psi(\lambda)|_{\lambda=0}, \qquad l=0,1,\dots,K-1. \tag{2.1}$$

In particular, $\psi(\lambda)$ may be normalized so that

$$g\equiv g_0=\psi(0) \tag{2.2}$$

is the solution of Eqs. (1.1) and (1.2). The remaining functions satisfy the following linear equations, obtained by differentiating (1.5) repeatedly at $\lambda=0$

$$g_{l,\xi}+g_{l-1,\xi}=Ag_l, \quad g_{l,\eta}-g_{l-1,\eta}=Bg_l, \quad l=1,\ldots,K-1. \tag{2.3}$$

These equations determine the extended system $\{g_0,\ldots,g_{K-1}\}$ from the original one (1.1) and (1.2). Since the integrability conditions for (2.3) are just (1.1) and (1.2), given a solution g to the latter, the remaining $\{g_l\}_{l=1,\ldots,K-1}$ are determined up to arbitrary initial conditions by the linear system (2.3). Now, following ZMS, we introduce a dressing matrix $\chi(\lambda)$ which together with its inverse $\chi^{-1}(\lambda)$ is meromorphic in λ with simple poles at $\{\lambda_i\}_{i=1,\ldots,K}$ and $\{\mu_i\}_{i=1,\ldots,K}$, respectively, and normalized such that

$$\chi(\infty)=\chi^{-1}(\infty)=\mathbb{1}.$$

In terms of the residues

$$Q_i=\frac{1}{2\pi i}\oint_{\lambda_i}\chi(\lambda)\,d\lambda, \quad R_i=\frac{1}{2\pi i}\oint_{\mu_i}\chi^{-1}(\mu)\,d\mu \tag{2.4}$$

(the contours taken so as to include only the pole indicated). We have:

$$\chi(\lambda)=\mathbb{1}+\sum_{i=1}^{K}\frac{Q_i}{\lambda-\lambda_i}, \quad \chi^{-1}(\lambda)=\mathbb{1}+\sum_{i=1}^{K}\frac{R_i}{\lambda-\mu_i}. \tag{2.5}$$

A new solution to (1.5) is obtained from a given one (A,B,ψ) by the transformation:

$$\psi\to\tilde{\psi}=\chi\psi, \tag{2.6a}$$

$$A\to\tilde{A}=\chi(-1)A\chi^{-1}(-1), \quad B\to\tilde{B}=\chi(+1)B\chi^{-1}(+1), \tag{2.6b}$$

provided the dressing matrix satisfies the differential equations:

$$\chi_\xi(\lambda)=\frac{\tilde{A}\chi(\lambda)}{1+\lambda}-\frac{\chi(\lambda)A}{1+\lambda}, \quad \chi_\eta(\lambda)=\frac{\tilde{B}\chi(\lambda)}{1-\lambda}-\frac{\chi(\lambda)B}{1-\lambda}. \tag{2.7}$$

This is equivalent to the system of equations:

$$Q_{i\xi}=\frac{\tilde{A}Q_i}{1+\lambda_i}-\frac{Q_iA}{1+\lambda_i}, \quad Q_{i\eta}=\frac{\tilde{B}Q_i}{1-\lambda_i}-\frac{Q_iB}{1-\lambda_i} \tag{2.8}$$

or, equivalently

$$R_{i\xi}=\frac{AR_i}{1+\mu_i}-\frac{R_i\tilde{A}}{1+\mu_i}, \quad R_{i\xi}=\frac{BR_i}{1-\mu_i}-\frac{R_i\tilde{B}}{1-\mu_i}, \tag{2.8'}$$

together with the set of constraints:

$$Q_i+R_i+\sum_{\substack{j=1\\ j\neq i}}^{K}\frac{R_jQ_i}{\lambda_i-\mu_j}+\sum_{\substack{j=1\\ j\neq i}}^{K}\frac{R_iQ_j}{\mu_i-\lambda_j}=0, \tag{2.9a}$$

$$R_iQ_i=(\lambda_i-\mu_i)\left[R_i+\sum_{\substack{j=1\\ j\neq i}}^{K}\frac{R_iQ_j}{\mu_i-\lambda_j}\right]=-(\lambda_i-\mu_i)\left[Q_i+\sum_{\substack{j=1\\ j\neq i}}^{K}\frac{R_jQ_i}{\lambda_i-\mu_j}\right]. \tag{2.9b}$$

The latter are written in a form valid whether λ_i and μ_i are distinct or not. The other $\{\mu_i\}$, $\{\lambda_j\}$ for $i \neq j$ may without loss of generality be taken as distinct.

The system (2.8) with constraints (2.9) may be expressed in a different form which defines a Bäcklund transformation from the extended system $\{g_0, g_1, \ldots, g_{K-1}\}$ to a new extended system $\{\tilde{g}_0, \tilde{g}_1, \ldots, \tilde{g}_{K-1}\}$ satisfying the same Eqs. (1.1), (1.2), and (2.3) as follows. A meromorphic function on the Riemann sphere which is analytic at 0 and ∞ and has K simple poles at given points is uniquely determined by its value at 0 and ∞, together with its first $K-1$ derivatives at 0. We therefore introduce the K matrix functions:

$$V_l = \frac{1}{l!}\frac{d^l}{d\lambda^l}\chi(\lambda)|_{\lambda=0}, \qquad l=0, \ldots, K-1 \tag{2.10}$$

which, because of the normalization $\chi(\infty)=1$, may be expressed in terms of the residues $\{Q_i\}$ as:

$$V_0 = 1 - \sum_{i=1}^{K} \frac{Q_i}{\lambda_i} = \chi(0), \qquad V_l = -\sum_{i=1}^{K} \frac{Q_i}{\lambda_i^{l+1}}, \qquad l=1, \ldots, K-1. \tag{2.11}$$

By evaluating derivatives of the system (2.7) at the origin, we find:

$$V_{l\xi} = \sum_{i=0}^{l} (-1)^{l-i}[\tilde{A}V_i - V_i A], \qquad V_{l\eta} = -\sum_{i=0}^{l} [\tilde{B}V_i - V_i B], \qquad l=0, \ldots, K-1. \tag{2.12}$$

Furthermore, from the contour integral of (2.7) around ∞, we have:

$$\sum_{i=1}^{K} Q_{i\xi} = \tilde{A} - A, \qquad \sum_{i=1}^{K} Q_{i\eta} = -(\tilde{B} - B). \tag{2.13}$$

The pair of relations (2.12) and (2.13) are thus equivalent to (2.6b) and (2.7). Equation (2.13) may be re-expressed in terms of the V_l's, using the following identity, which is a consequence of Eq. (2.11):

$$\sum_{i=1}^{K} Q_i = \sum_{\substack{j=1 \\ i_1 < \ldots < i_j}}^{K} (-1)^j \lambda_{i_1}, \ldots, \lambda_{i_j} V_{j-1}. \tag{2.14}$$

The resulting system is:

$$\sum_{\substack{j=1 \\ i_1 < \ldots < i_j}}^{K} (-1)^j \lambda_{i_1}, \ldots, \lambda_{i_j} V_{j-1,\xi} = \tilde{A} - A, \tag{2.15a}$$

$$-\sum_{\substack{j=1 \\ i_1 < \ldots < i_j}}^{K} (-1)^j \lambda_{i_1}, \ldots, \lambda_{i_j} V_{j-1,\eta} = \tilde{B} - B, \tag{2.15b}$$

$$V_{l,\xi} + V_{l-1,\xi} = \tilde{A}V_l - V_l A, \tag{2.15c}$$

$$V_{l,\eta} - V_{l-1,\eta} = \tilde{B}V_l - V_l B. \tag{2.15d}$$

Conversely, by re-summing Eq. (2.15) and making use of Eq. (2.11), we can derive Eq. (2.8) with $(\tilde{A}, \tilde{B})$ given by (2.6b).

The significance of this linear change of coordinates is that the extended system $\{\tilde{g}_0, \ldots, \tilde{g}_{K-1}\}$ obtained from the new solution $\tilde{\psi}$ given by Eq. (2.6a) by successive derivations:

$$\tilde{g}_l = \frac{1}{l!} \frac{d^l \tilde{\psi}}{d\lambda^l}\bigg|_{\lambda=0}, \qquad l=0, \ldots, K-1, \tag{2.16}$$

is simply expressible in terms of the $\{V_l\}$:

$$\tilde{g}_l = \sum_{i=0}^{l} V_{l-i} g_i . \tag{2.17}$$

Therefore, Eq. (2.15) may be interpreted as a Bäcklund transformation from $\{g_0, \ldots, g_{K-1}\}$ to $\{\tilde{g}_0, \ldots, \tilde{g}_{K-1}\}$, where

$$\tilde{A} \equiv \tilde{g}_\xi \tilde{g}^{-1}, \qquad \tilde{B} \equiv \tilde{g}_\eta \tilde{g}^{-1}, \qquad (\tilde{g} \equiv \tilde{g}_0), \tag{2.18}$$

and $\{\tilde{g}, \ldots, \tilde{g}_{K-1}\}$ is determined by Eq. (2.17).

Equation (2.17) may be explicitly inverted to express the $\{V_l\}$ in terms of $\{g_l\}$, $\{\tilde{g}_l\}$ by solving:

$$V_l = \tilde{g}_l g^{-1} - \sum_{i=1}^{l} V_{l-i} g_i g^{-1} \tag{2.17'}$$

recursively. Therefore, Eqs. (2.15a)–(2.15d) may be regarded as a differential system relating $\{g_0, \ldots, g_{K-1}\}$ and $\{\tilde{g}_0, \ldots, \tilde{g}_{K-1}\}$. It is immediate from Eqs. (2.15a) and (2.15b) that if $\{g, A, B\}$ satisfies Eqs. (1.1) and (1.2), then $\{\tilde{g}, \tilde{A}, \tilde{B}\}$ does as well, and from Eq. (2.15c) and (2.15d) it follows that if $\{g_l\}_{l=0,\ldots,K-1}$ satisfy Eq. (2.3), then so does $\{\tilde{g}_l\}_{l=0,\ldots,K-1}$ as defined by Eq. (2.17). Summarizing these results, we have:

Theorem 1.1. *The system* (2.15), *subject to the constraints* (2.9*a*) *and* (2.9*b*) *defines a Bäcklund transformation for the extended system* $\{g_0, \ldots, g_{K-1}\}$ *and is equivalent to Eqs.* (2.8) *and* (2.6b) *for the residues of the dressing matrix.*

Note that, whereas the constraints (2.9a) and (2.9b) involve both the residues $\{Q_i\}$ and $\{R_i\}$, the latter may also be expressed explicitly in terms of $\{g_l\}$ and $\{\tilde{g}_l\}$ through relations of the same form as Eq. (2.11) derived from $\chi^{-1}(\lambda)$. In terms of the original system $\tilde{A}$, $\tilde{B}$, Eqs. (2.15) determine a new solution equivalent to that obtained by solving an iterated sequence of K simple Bäcklund transformations and therefore should be interpreted as a multi-Bäcklund transformation. The case $K=1$ reduces to Eq. (1.3) with constraints (1.4). The relationship between the case $K=2$ and the composition of two successive Bäcklund transformations may be used to prove the following:

Theorem 2.2 (Permutability Theorem). *If two successive Bäcklund transformations are applied to a given solution* $\{g, A, B\}$ *of Eqs.* (1.1) *and* (1.2),

$$(g, A, B) \xrightarrow{\lambda_1 \mu_1} (\hat{g}_1, \hat{A}_1, \hat{B}_1) \xrightarrow{\lambda_2 \mu_2} (\tilde{g}, \tilde{A}, \tilde{B}),$$

there exists another sequence

$$(g, A, B) \xrightarrow{\lambda_2 \mu_2} (\hat{g}_2, \hat{A}_2, \hat{B}_2) \xrightarrow{\lambda_1 \mu_1} (\tilde{g}, \tilde{A}, \tilde{B})$$

with the same resulting solution $(\tilde{g}, \tilde{A}, \tilde{B})$.

Proof. The equations defining the first BT are:

$$-\lambda_1 U_{1\xi} = \hat{A}_1 - A_1, \quad \lambda_1 U_{1\eta} = \hat{B}_1 - B_1, \quad \lambda_1 U_1 + \mu_1 U_1^{-1} = (\lambda_1 + \mu_1)\mathbb{1}, \tag{2.19}$$

where

$$U_1 \equiv \hat{g}_1 g^{-1}. \tag{2.20}$$

And similarly the second one is defined by

$$-\lambda_2 U_{2\xi} = \tilde{A} - \hat{A}_1, \quad \lambda_2 U_{2\eta} = \tilde{B} - \hat{B}_1, \quad \lambda_2 U_2 + \mu_2 U_2^{-1} = (\lambda_2 + \mu_2)\mathbb{1}, \tag{2.21}$$

where

$$U_2 \equiv \tilde{g}\tilde{g}_1^{-1}. \tag{2.22}$$

Define the new quantities:

$$V_0 \equiv U_2 U_1 = \tilde{g} g^{-1}, \quad V_1 \equiv \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) V_0 - \frac{1}{\lambda_2} V_1 - \frac{1}{\lambda_1} V_2. \tag{2.23}$$

Then Eqs. (2.14) and (2.21) are equivalent to the following:

$$\begin{aligned} -(\lambda_1 + \lambda_2) V_{0,\xi} + (\lambda_1 \lambda_2) V_{1,\xi} &= \tilde{A} - A, \\ (\lambda_1 + \lambda_2) V_{0,\eta} - (\lambda_1 \lambda_2) V_{1,\eta} &= \tilde{B} - B, \\ V_{0,\xi} + V_{1,\xi} &= \tilde{A} V_1 - V_1 A, \\ -V_{0,\eta} + V_{1,\eta} &= \tilde{B} V_1 - V_1 B, \end{aligned} \tag{2.24}$$

together with the constraints (2.9) for $k=2$, where

$$\begin{aligned} V_0 &\equiv 1 - \frac{Q_1}{\lambda_1} - \frac{Q_2}{\lambda_2}, \\ V_1 &\equiv \frac{Q_1}{\lambda_1^2} - \frac{Q_2}{\lambda_2^2}, \\ V_0^{-1} &\equiv 1 - \frac{R_1}{\mu_1} - \frac{R_2}{\mu_2}, \\ V_0^{-1} V_1 V_0^{-1} &\equiv \frac{R_1}{\mu_1^2} + \frac{R_2}{\mu_2^2} \end{aligned} \tag{2.25}$$

define Q_1, Q_2, R_1, and R_2. Thus, the composition of two simple Bäcklund transformations is equivalent to the $K=2$ double Bäcklund transformation which is manifestly symmetric under the interchange of parameters $(\mu_1 \lambda_1) \leftrightarrow (\mu_2 \lambda_2)$. Therefore, by suitable choice of initial conditions, the two successive transformations with parameters exchanged gives rise to the same solution $(\tilde{g}, \tilde{A}, \tilde{B})$.

3. The Soliton Correlation Matrix

In order to solve the multi-Bäcklund transformation (2.15) or, equivalently, the system (2.8), it is convenient to introduce a new quantity, the soliton correlation matrix M, which is an $nK \times nK$ dimensional matrix whose entries of $n \times n$ blocks M_{ij} are defined in terms of the residues of the product $\chi^{-1}(\mu)\chi(\lambda)$:

$$M_{ij} \equiv \frac{1}{(2\pi i)^2} \oint_{\mu_i} d\mu \oint_{\lambda_j} d\lambda \frac{\chi^{-1}(\mu)\chi(\lambda)}{\mu-\lambda}. \tag{3.1}$$

In the case that $\mu_i \neq \lambda_j$, the evaluation of these contour integrals gives:

$$M_{ij} \equiv \frac{R_i Q_j}{\mu_i - \lambda_j}. \tag{3.2}$$

If $\mu_i = \lambda_j$, the expression (3.2) is undefined, but (3.1) is still valid provided the integration contours are chosen as disjoint, i.e., one contained in the other. The choice of inner and outer contours in this case does not affect the result of the integration in (3.1). The residues Q_i, R_j may be recovered from the M matrix by summing over the row blocks or column blocks:

Lemma 3.1.

$$Q_i = \sum_{j=1}^{K} M_{ji}, \tag{3.3a}$$

$$R_i = -\sum_{j=1}^{K} M_{ij}. \tag{3.3b}$$

Proof. If $\mu_i \neq \lambda_j$, these relations are equivalent to the constraints (2.9), but in general they are shown to be valid by deforming the integration contours, choosing the λ_i contour to be the outer one in the case of a degeneracy $\mu_j = \lambda_i$:

$$\begin{aligned}\sum_{j=1}^{K} M_{ji} &= \sum_{j=1}^{K} \frac{1}{(2\pi i)^2} \oint_{\mu_j} d\mu \oint_{\lambda_i} d\lambda \frac{\chi^{-1}(\mu)\chi(\lambda)}{\mu-\lambda} \\ &= \frac{1}{(2\pi i)^2} \oint_{\lambda_i} d\lambda \left[\oint_{\infty} d\mu \cdot \frac{\chi^{-1}(\mu)}{\mu-\lambda} - \oint_{\lambda} d\mu \frac{\chi^{-1}(\mu)}{\mu-\lambda} \right] \chi(\lambda) \\ &= \frac{1}{2\pi i} \oint_{\lambda_i} d\lambda [\mathbb{1} - \chi^{-1}(\lambda)]\, \chi(\lambda) \\ &= \frac{1}{2\pi i} \oint_{\lambda_i} d\lambda\, \chi(\lambda) = Q_i,\end{aligned}$$

and similarly for the other relation.

The constraints may be expressed equivalently in terms of the M-matrix alone.

Lemma 3.2. *The M-matrix satisfies the following constraints, which together with* (3.3) *are equivalent to* (2.9)

$$\mu_i M_{ij} - \lambda_j M_{ij} = -\sum_{kl} M_{ik} M_{lj}. \tag{3.4}$$

Proof.

$$-\sum_{kl} M_{ik} M_{lj} = \frac{1}{(2\pi i)^2} \oint_{\mu_i} d\mu \oint_{\lambda_j} d\lambda \frac{\chi^{-1}(\mu)\chi(\lambda)}{\mu-\lambda}(\mu_i-\lambda_j) = \mu_i M_{ij} - \lambda_j M_{ij}.$$

Conversely Eqs. (3.3) and (3.4) imply Eq. (3.2) for $\mu_i \neq \lambda_j$. Substituting this relation in (3.3a) and (3.3b) and adding gives Eq. (2.9a). If $\mu_i \neq \lambda_i$, Eqs. (3.3a) and (3.3b) is also equivalent to (2.9b), while if $\mu_i = \lambda_i$, Eq. (3.4) implies $R_i Q_i = 0$, and hence (2.9b) is still implied.

One further form of the algebraic constraints on M will also be useful in the following section.

Lemma 3.3. *There exists a block diagonal $nK \times nK$ dimensional matrix S, consisting of $n \times n$ blocks S_i, $i=1, \ldots, K$ with*

$$S_i = 0, \quad \text{if} \quad \lambda_i \neq \mu_i,$$

such that the following relation holds:

$$M(D+S)M = M, \tag{3.5}$$

where the $nK \times nK$ dimensional matrix D has as ij^{th} block:

$$\begin{aligned} D_{ij} &= \frac{1}{\lambda_i - \mu_j} \mathbb{1} \quad \text{if} \quad \lambda_i \neq \mu_j \ \ (\text{in particular, if } i \neq j), \\ D_{ii} &= 0 \qquad\qquad \text{if} \quad \lambda_i = \mu_i . \end{aligned} \tag{3.6}$$

Proof. If $\lambda_i \neq \mu_j$, $\forall i,j$, then S vanishes identically and (3.5) is another form of the constraints (3.4). In general, the proof is as follows. The following relation is an identity which is derived by performing the inner two integrations and deforming the summed contours to infinity:

$$\frac{1}{(2\pi i)^4} \sum_{kl} \oint_{\mu_i} d\mu \oint_{\lambda_k} d\lambda \oint_{\mu_l} d\mu' \oint_{\lambda_j} d\lambda' \frac{\chi^{-1}(\mu)\chi(\lambda)\chi^{-1}(\mu')\chi(\lambda')}{(\mu-\lambda)(\lambda-\mu')(\mu'-\lambda')} = M_{ij}.$$

We can evaluate all the terms with $\lambda_k \neq \mu_l$ explicitly to get:

$$\sum_{kl}{}' \frac{M_{ik} M_{lj}}{\lambda_k - \mu_l} + \frac{1}{(2\pi i)^4} \sum_{\mu_l = \lambda_l} \oint_{\mu_i} d\mu \oint_{\lambda_l} d\lambda \oint_{\lambda_l} d\mu' \oint_{\lambda_j} d\lambda' \frac{\chi^{-1}(\mu)\chi(\lambda)\chi^{-1}(\mu')\chi(\lambda')}{(\mu-\lambda_l)(\lambda-\mu')(\lambda_l-\lambda')} = M_{ij},$$

where the first sum is over all terms with $\lambda_k \neq \mu_l$ and equals $(MDM)_{ij}$, and the second is over those diagonal terms with $\mu_l = \lambda_l$. Evaluating the interior integrals in the second sum, we have

$$\begin{aligned} \frac{1}{(2\pi i)^2} \oint_{\lambda_l} d\lambda \oint_{\lambda_l} d\mu' \frac{\chi(\lambda)\chi^{-1}(\mu')}{\lambda-\mu'} &= \frac{Q_l}{2\pi i} \oint_{\lambda_l} d\mu' \frac{\chi^{-1}(\mu')}{\lambda_l - \mu'} \\ &= \frac{1}{2\pi i} \oint_{\lambda_l} d\lambda \frac{\chi(\lambda)}{\lambda - \lambda_l} R_l , \end{aligned}$$

and hence there exists a matrix S_l such that:

$$\begin{aligned} \frac{1}{(2\pi i)^2} \oint_{\lambda_l} d\lambda \oint_{\lambda_l} d\mu' \frac{\chi(\lambda)\chi^{-1}(\mu')}{\lambda-\mu'} &= Q_l S_l R_l \\ &= \frac{1}{(2\pi i)^2} \oint_{\lambda_l} d\lambda \oint_{\lambda_l} d\mu' \chi(\lambda) S_l \chi^{-1}(\mu') . \end{aligned}$$

Substituting gives

$$\frac{1}{(2\pi i)^4}\sum_l \oint_{\mu_i} d\mu \oint_{\lambda_l} d\lambda \oint_{\lambda_l} d\mu' \oint_{\lambda_j} d\lambda' \frac{\chi^{-1}(\mu)\chi(\lambda)}{(\mu-\lambda_l)} S_l \frac{\chi^{-1}(\mu')\chi(\lambda')}{(\lambda_l-\lambda')} = \sum_l M_{il} S_l M_{lj},$$

which proves the relation (3.5).

We now derive the equations satisfied by M which are equivalent to the system (2.8) and hence also the multi-Bäcklund transformation.

Theorem 3.4. *The M-matrix evolves according to the following system of matrix Riccati equations:*

$$M_\xi = p^+ M - M s^+ - M r^+ M, \tag{3.7a}$$

$$M_\eta = p^- M - M s^- - M r^- M, \tag{3.7b}$$

where

$$\left.\begin{aligned} p^+ &= \operatorname{diag}\left\{\frac{A}{1+\mu_i}\right\}, & p^- &= \operatorname{diag}\left\{\frac{B}{1-\mu_i}\right\} \\ s^+ &= \operatorname{diag}\left\{\frac{A}{1+\lambda_i}\right\}, & s^- &= \operatorname{diag}\left\{\frac{B}{1-\lambda_i}\right\} \\ (r_+)_{ij} &= \frac{A}{(1+\lambda_i)(1+\mu_j)}, & (r_-)_{ij} &= \frac{-B}{(1-\lambda_i)(1-\mu_j)}. \end{aligned}\right\} \tag{3.8}$$

The integrability conditions for (3.7) *are Eqs.* (1.1), (1.2), *and the constraints Eq.* (3.4) *are preserved.*

Proof. Differentiating $\chi^{-1}(\mu)\chi(\lambda)$, and using (2.7) gives:

$$[\chi^{-1}(\mu)\chi(\lambda)]_\xi = \frac{(\mu-\lambda)\chi^{-1}(\mu)\tilde{A}\chi(\lambda)}{(1+\mu)(1+\lambda)} + \frac{A\chi^{-1}(\mu)\chi(\lambda)}{1+\mu} - \frac{\chi^{-1}(\mu)\chi(\lambda)A}{1+\lambda}, \tag{3.9a}$$

$$[\chi^{-1}(\mu)\chi(\lambda)]_\eta = -\frac{(\mu-\lambda)\chi^{-1}(\mu)\tilde{B}\chi(\lambda)}{(1-\mu)(1-\lambda)} + \frac{B\chi^{-1}(\mu)\chi(\lambda)}{1-\mu} - \frac{\chi^{-1}(\mu)\chi(\lambda)B}{1-\lambda}. \tag{3.9b}$$

Integrating $\dfrac{[\chi^{-1}(\mu)\chi(\lambda)]_\xi}{\mu-\lambda}$ around the poles μ_i and λ_j, the first term gives:

$$\frac{1}{(2\pi i)^2}\oint_{\mu_i} d\mu \oint_{\lambda_j} d\lambda \frac{\chi^{-1}(\mu)\chi(-1)A\chi^{-1}(-1)\chi(\lambda)}{(1+\mu)(1+\lambda)},$$

where the expression (2.6b) has been substituted for $\tilde{A}$. The integrals are evaluated by deforming the σ, τ contours in:

$$\begin{aligned} \frac{1}{2\pi i}\oint_{\mu_i} d\mu \oint_{-1} d\sigma \frac{\chi^{-1}(\mu)\chi(\sigma)}{(\sigma-\mu)(1+\sigma)} &= -\frac{1}{2\pi i}\sum_k \oint_{\mu_i} d\mu \oint_{\lambda_k} d\sigma \frac{\chi^{-1}(\mu)\chi(\sigma)}{(\sigma-\mu)(1+\sigma)} \\ &\quad - \frac{1}{2\pi i}\oint_{\mu_i} d\mu \oint_{\mu} d\sigma \frac{\chi^{-1}(\mu)\chi(\sigma)}{(\sigma-\mu)(1+\sigma)} \\ &= -\sum_k \frac{M_{ik}}{1+\lambda_k}, \end{aligned}$$

and, similarly

$$\frac{1}{2\pi i}\oint_{-1} d\tau \oint_{\lambda_j} d\lambda \frac{\chi^{-1}(\tau)\chi(\lambda)}{(\tau-\lambda)(1+\sigma)} = -\sum_l \frac{M_{lj}}{(1+\mu_l)}.$$

Evaluating the remaining terms in (3.9a) directly gives:

$$M_{ij,\xi} = -\sum_{kl} \frac{M_{ik}AM_{lj}}{(1+\lambda_k)(1+\mu_l)} + \frac{AM_{ij}}{1+\mu_i} - \frac{M_{ij}A}{1+\lambda_j},$$

and similarly

$$M_{ij,\eta} = +\sum_{kl} \frac{M_{ik}BM_{lj}}{(1-\lambda_k)(1-\mu_l)} + \frac{BM_{ij}}{1-\mu_i} - \frac{M_{ij}B}{1-\lambda_j},$$

which is exactly Eqs. (3.7a) and (3.7b) split into blocks. The verification that Eqs. (1.1) and (1.2) are indeed the integrability conditions for (3.7) and that the constraints (3.4) are preserved relies upon the geometrical interpretation of Eqs. (3.7a) and (3.7b) and will be left to the next section.

4. Geometrical Structure and Linearization

Matrix Riccati systems of the type (3.7) have a natural geometrical interpretation in terms of group actions on Grassman manifolds. This has been formulated in detail in [1, 2, 9] and may be summarized briefly as follows. The M-matrix may be regarded as defining a function on $\mathbb{R}^2$ with values in the Grassman manifold $G_{nK}(\mathbb{C}^{2nK})$ of $n\times n$ planes in $\mathbb{C}^{2nK}$, expressed in affine coordinates. The group $\mathrm{SL}(2nK,\mathbb{C})$ acts in a natural way on $G_{nK}(\mathbb{C}^{2nK})$, this action being expressed in affine coordinates by the linear fractional transformation:

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} M \mapsto (PM+Q)(RM+S)^{-1},$$
$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in \mathrm{SL}(2nK,\mathbb{C}),\quad P,Q,R,S,M \in \mathbb{C}^{nK\times nK}. \tag{4.1}$$

The $\mathrm{SL}(2nK,\mathbb{C})$-valued 1-form

$$\omega = -\begin{pmatrix} p^+ & 0 \\ r^+ & s^+ \end{pmatrix} d\xi - \begin{pmatrix} p^- & 0 \\ r^- & s^- \end{pmatrix} d\eta \tag{4.2}$$

defines a connection on the trivial principal $\mathrm{SL}(2nK,\mathbb{C})$ bundle

$$\mathbb{R}^2 \times \mathrm{SL}(2nK,\mathbb{C}) \xrightarrow{\pi} \mathbb{R}^2,$$
$$\pi : (\xi,\eta,\mathscr{G}) \to (\xi,\eta),$$
$$\mathscr{G} \in \mathrm{SL}(2nK,\mathbb{C}),$$

with connection form:

$$\omega_{(\xi,\eta,\mathscr{G})} = \mathrm{Ad}\,\mathscr{G}^{-1}\omega + \mathscr{G}^{-1}\,d\mathscr{G}. \tag{4.3}$$

The system (3.6) may be regarded as determining a covariant constant section of the associated trivial Grassmannian bundle $\sigma:\mathbb{R}^2\to\mathbb{R}^2\times G_{nK}(\mathbb{C}^{2nK})$,

$$\sigma(\xi,\eta)=((\xi,\eta),M(\xi,\eta)), \tag{4.4}$$

expressed in affine coordinates. The integrability condition for arbitrary initial conditions is the vanishing of the curvature:

$$d\omega+\tfrac{1}{2}[\omega,\omega]=0, \tag{4.5}$$

which reduces to the relation:

$$\mathscr{A}\otimes A_\eta-\mathscr{B}\otimes B_\xi+\mathscr{A}\mathscr{B}\otimes AB-\mathscr{B}\mathscr{A}\otimes BA=0, \tag{4.6}$$

where $\mathscr{A},\mathscr{B}$ are the constant $2K\times 2K$ dimensional matrices

$$\begin{gathered}\mathscr{A}=\left(\begin{array}{c|c}\operatorname{diag}\dfrac{1}{1+\mu_i} & 0\\ \hline \dfrac{1}{1+\lambda_i}\cdot\dfrac{1}{1+\mu_j} & \operatorname{diag}\dfrac{1}{1+\lambda_i}\end{array}\right),\\ \mathscr{B}=\left(\begin{array}{c|c}\operatorname{diag}\dfrac{1}{1-\mu_i} & 0\\ \hline -\dfrac{1}{1-\lambda_i}\cdot\dfrac{1}{1-\mu_j} & \operatorname{diag}\dfrac{1}{1-\lambda_i}\end{array}\right),\end{gathered} \tag{4.7}$$

in terms of which

$$\begin{pmatrix}p^+ & 0\\ r^+ & s^+\end{pmatrix}=\mathscr{A}\otimes A,\qquad \begin{pmatrix}p^- & 0\\ r^- & s^-\end{pmatrix}=\mathscr{B}\otimes B. \tag{4.8}$$

Equation (4.6) reduces to the field Eqs. (1.1) and (1.2), because of the identities

$$\mathscr{A}\mathscr{B}=\tfrac{1}{2}(\mathscr{A}+\mathscr{B})=\mathscr{B}\mathscr{A}. \tag{4.9}$$

The general solution to Eq. (3.7) may be expressed in terms of the solution to the corresponding problem of determining a covariant constant section in the principal bundle

$$\begin{gathered}\sigma_{\mathscr{G}}:\mathbb{R}^2\to\mathbb{R}^2\times \mathrm{SL}(2nK,\mathbb{C}),\\ \sigma_{\mathscr{G}}(\xi,\eta)\to(\xi,\eta,\mathscr{G}(\xi,\eta)),\end{gathered} \tag{4.10}$$

$$\mathscr{G}_\xi=\mathscr{A}\otimes A\mathscr{G},\qquad \mathscr{G}_\eta=\mathscr{B}\otimes B\mathscr{G}. \tag{4.11}$$

If we choose initial conditions at (ξ_0,η_0)

$$M(\xi_0,\eta_0)=m,\qquad \mathscr{G}(\xi_0,\eta_0)=\mathbb{1}, \tag{4.12}$$

then $M(\xi,\eta)$ is determined by the linear fractional transformation:

$$M(\xi,\eta)=[P(\xi,\eta)m][R(\xi,\eta)m+S(\xi,\eta)]^{-1}, \tag{4.13}$$

where

$$\mathscr{G}(\xi,\eta)\equiv\begin{pmatrix}P(\xi,\eta) & 0\\ R(\xi,\eta) & S(\xi,\eta)\end{pmatrix} \tag{4.14}$$

has lower block triangular form because of Eqs. (4.8), (4.11), and (4.12). The form of the solution (4.13) remains unchanged if Eq. (4.12) is replaced by the more general conditions:

$$M(\xi_0,\eta_0)=S_1^{-1}mS_2, \quad \mathscr{G}(\xi_0,\eta_0)=\begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} \tag{4.15}$$

for arbitrary $S_1, S_2 \in \mathrm{SL}(2nK, \mathbb{C})$.

To verify that Eq. (3.6) is consistent with the constraints (3.4), we introduce the linear map $T:\mathbb{C}^{2nK}\to\mathbb{C}^{2nK}$ with matrix representation:

$$T\equiv\tau\otimes\mathbb{1}, \tag{4.16}$$

where

$$\tau=\begin{pmatrix} \operatorname{diag}\{\mu_i\} & 0 \\ -E & \operatorname{diag}\{\lambda_i\} \end{pmatrix}\in\mathbb{C}^{2K\times 2K}, \tag{4.17}$$

and

$$E\equiv\begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix}. \tag{4.18}$$

This determines a map, also denoted T, on the Grassman manifold $T:G_{nK}\to G_{nK}$, which, expressed in terms of affine coordinates is the solution to the linear system

$$\lambda_j T(M)_{ij}-\sum_{kl} T(M)_{ik}M_{lj}=\mu_i M_{ij}. \tag{4.19}$$

Thus, the constraints (3.4) express the geometrical condition that M defines an invariant point in $G_{nK}(\mathbb{C}^{2nK})$ under the map T:

$$T(M)=M. \tag{4.20}$$

The submanifold of $G_{nK}(\mathbb{C}^{2nK})$ consisting of T-invariant nK-planes is an orbit of the subgroup of $\mathrm{SL}(2nK,\mathbb{C})$ consisting of elements which commute with T:

$$T\mathscr{G}T^{-1}=\mathscr{G}. \tag{4.21}$$

The fact that $\mathscr{G}(\xi,\eta)$ takes values in this subgroup and hence the constraints (3.4) are preserved follows from the fact that Eq. (4.11) determines $\mathscr{G}$ from the $\mathrm{SL}(2nK,\mathbb{C})$ algebra valued functions $\mathscr{A}\otimes A$ and $\mathscr{B}\otimes B$ which themselves commute with T:

$$[T,\mathscr{A}\otimes A]=0, \qquad [T,\mathscr{B}\otimes B]=0, \tag{4.22}$$

or equivalently,

$$[\tau,\mathscr{A}]=[\tau,\mathscr{B}]=0. \tag{4.23}$$

The commutativity of $\tau, \mathscr{A}, \mathscr{B}$ is a consequence of the identities:

$$\tau=\mathscr{A}^{-1}-\mathbb{1}=-\mathscr{B}^{-1}+\mathbb{1}. \tag{4.24}$$

This completes the proof of Theorem (3.1) and also suggests the procedure for reducing the Riccati system (3.6) to a system of linear matrix equations in n or $2n$ dimensions. Namely, choose a basis of eigenvectors or generalized eigenvectors of

τ, thereby simultaneously transforming τ, $\mathscr{A}$, and $\mathscr{B}$ into Jordan normal form. The parameters $\{\mu_i, \lambda_i\}$ are the eigenvalues of τ and we may assume that the $\{\mu_i\}$ and $\{\lambda_i\}$ are separately distinct and $\mu_i \neq \lambda_j$ for $i \neq j$ by suitable ordering. Therefore the various cases are determined by whether or not the pairs (λ_i, μ_i) are equal and the Jordan blocks are all either 1 or 2 dimensional. It is sufficient to examine the two cases where either:

(i) $\mu_i \neq \lambda_j \ \forall i, j$,

or

(ii) $\mu_i \neq \lambda_j$ if $i \neq j$, $\mu_i = \lambda_i \ \forall i$,

since the general case is a direct sum of these two.

Case (i). $\mu_i \neq \lambda_j, \ \forall i, j$.

The matrix of eigenvectors is of the form:

$$S = \begin{pmatrix} \mathbb{1} & 0 \\ d & \mathbb{1} \end{pmatrix}, \tag{4.25}$$

where

$$d_{ij} = \frac{1}{\lambda_i - \mu_j}. \tag{4.26}$$

The diagonalized forms of τ, $\mathscr{A}$, $\mathscr{B}$ are

$$S^{-1}\tau S = \begin{pmatrix} \operatorname{diag}\{\mu_i\} & 0 \\ 0 & \operatorname{diag}\{\lambda_i\} \end{pmatrix}, \tag{4.27a}$$

$$S^{-1}\mathscr{A}S = \begin{pmatrix} \operatorname{diag}\left\{\dfrac{1}{1+\mu_i}\right\} & 0 \\ 0 & \operatorname{diag}\left\{\dfrac{1}{1+\lambda_i}\right\} \end{pmatrix}, \tag{4.27b}$$

$$S^{-1}\mathscr{B}S = \begin{pmatrix} \operatorname{diag}\left\{\dfrac{1}{1-\mu_i}\right\} & 0 \\ 0 & \operatorname{diag}\left\{\dfrac{1}{1-\lambda_i}\right\} \end{pmatrix}. \tag{4.27c}$$

It follows that $\mathscr{A} \otimes A$, $\mathscr{B} \otimes B$ and hence $\mathscr{G}$ may be block diagonalized by the same transformation:

$$\mathscr{G} = S\mathscr{G}_0 S^{-1} \tag{4.28}$$

(using S here as abbreviated notation for $S \otimes \mathbb{1}$), where $\mathscr{G}_0$ satisfies the equations:

$$\begin{aligned} \mathscr{G}_{0,\xi} &= \begin{pmatrix} \operatorname{diag}\dfrac{A}{1+\mu_i} & 0 \\ 0 & \operatorname{diag}\dfrac{A}{1+\lambda_i} \end{pmatrix} \mathscr{G}_0, \\ \mathscr{G}_{0,\eta} &= \begin{pmatrix} \operatorname{diag}\dfrac{A}{1-\mu_i} & 0 \\ 0 & \operatorname{diag}\dfrac{A}{1-\lambda_i} \end{pmatrix} \mathscr{G}_0. \end{aligned} \tag{4.29}$$

Choosing initial conditions such that $\mathcal{G}_0(\xi_0,\eta_0)$ is block diagonal, this is just $2K$ copies of the ZMS Eqs. (1.5) evaluated at $\{\lambda_i,\mu_i\}$ and hence:

$$\mathcal{G}_0=\begin{pmatrix}\tilde{\Psi} & 0\\ 0 & \Psi\end{pmatrix}, \tag{4.30}$$

where

$$\tilde{\Psi}\equiv\operatorname{diag}\{\psi(\mu_i)\}, \qquad \Psi\equiv\operatorname{diag}\{\psi(\lambda_i)\} \tag{4.31}$$

are determined by $\psi(\lambda)$ at the various eigenvalues. Substitution in (4.28) gives

$$\mathcal{G}=\begin{pmatrix}\tilde{\Psi} & 0\\ D\tilde{\Psi}-\Psi D & \Psi\end{pmatrix}, \tag{4.32}$$

where

$$D\equiv d\otimes\mathbb{1}_{n\times n}, \tag{4.33}$$

and hence the solution $M(\xi,\eta)$ as expressed by the linear fractional transformation (4.13) is:

$$M(\xi,\eta)=\tilde{\Psi}m[(D\tilde{\Psi}-\Psi D)m+\Psi]^{-1}. \tag{4.34}$$

Case (ii). $\mu_i\neq\lambda_j$, if $i\neq j$; $\lambda_i=\mu_i$.

The matrix of generalized eigenvectors is:

$$S_0=\begin{pmatrix}\mathbb{1} & 0\\ d^0 & \mathbb{1}\end{pmatrix}, \tag{4.35}$$

where

$$\begin{aligned} d_{ij}^0&=\frac{1}{\lambda_i-\lambda_j} \quad \text{if} \quad i\neq j\\ &=0 \qquad\quad \text{if} \quad i=j. \end{aligned} \tag{4.36}$$

The Jordan normal form (up to normalization) is:

$$S_0^{-1}\tau S_0=\begin{pmatrix}\operatorname{diag}\{\lambda_i\} & 0\\ -\mathbb{1} & \{\operatorname{diag}\lambda_i\}\end{pmatrix}, \tag{4.37}$$

$$S_0^{-1}\mathcal{A}S_0=\begin{pmatrix}\operatorname{diag}\dfrac{1}{1+\lambda_i} & 0\\ \operatorname{diag}\left(\dfrac{1}{1+\lambda_i}\right)^2 & \operatorname{diag}\dfrac{1}{1+\lambda_i}\end{pmatrix}, \tag{4.38}$$

$$S_0^{-1}\mathcal{B}S_0=\begin{pmatrix}\operatorname{diag}\dfrac{1}{1-\lambda_i} & 0\\ \operatorname{diag}-\left(\dfrac{1}{1-\lambda_i}\right) & \operatorname{diag}\dfrac{1}{1-\lambda_i}\end{pmatrix}. \tag{4.39}$$

Once again $\mathcal{G}$ is determined by

$$\mathcal{G}=S_0\mathcal{G}_0S_0^{-1}, \tag{4.40}$$

where $\mathscr{G}_0$ is now of the form:

$$\mathscr{G}_0 = \begin{pmatrix} \Psi & 0 \\ \Phi & \Psi \end{pmatrix}, \tag{4.41}$$

with

$$\Phi = \operatorname{diag}\{\phi(\lambda_i)\}, \tag{4.42}$$

and $\phi(\lambda)$ is determined by the linear $2n \times 2n$ matrix equations:

$$\begin{pmatrix} \psi & 0 \\ \phi & \psi \end{pmatrix}_\xi = \begin{pmatrix} \frac{A}{1+\lambda} & 0 \\ \frac{A}{(1+\lambda)^2} & \frac{B}{1+\lambda} \end{pmatrix} \begin{pmatrix} \psi & 0 \\ \phi & \psi \end{pmatrix}, \tag{4.43a}$$

$$\begin{pmatrix} \psi & 0 \\ \phi & \psi \end{pmatrix}_\eta = \begin{pmatrix} \frac{B}{1-\lambda} & 0 \\ \frac{-B}{(1-\lambda)^2} & \frac{B}{1-\lambda} \end{pmatrix} \begin{pmatrix} \psi & 0 \\ \phi & \psi \end{pmatrix}. \tag{4.43b}$$

These equations are obtained from the ZMS Eqs. (1.5) by differentiating with respect to the λ-parameter, and the general solution for ϕ is given by:

$$\phi(\lambda) = -\frac{d\psi(\lambda)}{d\lambda} + \psi(\lambda)\, C(\lambda), \tag{4.44}$$

where $\psi(\lambda)$ is the general solution to (1.5) and $C(\lambda)$ is an arbitrary constant $n \times n$ matrix.

Substitution in (4.40) gives

$$\mathscr{G} = \begin{pmatrix} \Psi & 0 \\ D_0\Psi - \Psi D_0 + \Phi & \Psi \end{pmatrix}, \tag{4.45}$$

where

$$D_0 \equiv d^0 \otimes \mathbb{1}_{n\times n}, \tag{4.33'}$$

and hence, the solution $M(\xi,\eta)$ as expressed by the linear fractional transformation (4.13) is:

$$M(\xi,\eta) = \Psi m[(D_0\Psi - \Psi D_0 + \Phi)m + \Psi]^{-1}. \tag{4.46}$$

Combining these two cases, we may summarize the general result as follows:

Theorem 4.1. *The general solution of Eq. (3.7) is given by the linear fractional transformation:*

$$M = \tilde{\Psi} m[(D\tilde{\Psi} - \Psi D) + \Phi)m + \Psi]^{-1}, \tag{4.47}$$

where

$$\Psi \equiv \operatorname{diag}\{\psi(\lambda_i)\}, \tag{4.48a}$$

$$\tilde{\Psi} \equiv \operatorname{diag}\{\psi(\mu_i)\}. \tag{4.48b}$$

D is as defined in Lemma 3.3, Φ is the block diagonal matrix with $n \times n$ blocks

$$\begin{aligned} \phi_i &\equiv -\psi'(\lambda_i)+\psi(\lambda_i)c_i \quad \text{if} \quad \mu_i=\lambda_i \\ &\equiv 0 \qquad\qquad\qquad\qquad \text{if} \quad \mu_i \neq \lambda_i , \end{aligned} \tag{4.49}$$

$c_i \in \mathbb{C}^{n\times n}$ are arbitrary constant matrices and $m \in \mathbb{C}^{nK\times nK}$ satisfies the constraints (3.4) *of Lemma 3.2.*

The solution may be expressed in another form which corresponds to that determined by Zakharov and Mikhailov for U(n), O(n), and SP(n) in [6].

Theorem 4.2. *The residues Q_i, R_i defining the dressing matrix $\chi(\lambda)$ and its inverse $\chi^{-1}(\lambda)$, respectively, are of the form:*

$$\begin{gathered} Q_i = X_i F_i^+ , \\ R_i = H_i K_i^+ , \\ X_i, F_i \in \mathbb{C}^{n\times q_i}, \qquad H_i, K_i \in \mathbb{C}^{n \times r_i}, \\ q_i = \operatorname{rank} Q_i , \qquad r_i = \operatorname{rank} R_i , \end{gathered} \tag{4.50}$$

where the rectangular matrices F_i, H_i are determined from their initial values f_i, h_i by

$$F_i = \psi^{+-1}(\lambda_i) f_i , \qquad H_i = \psi(\mu_i) h_i , \tag{4.51}$$

and X_i, K_i are solutions to the linear system

$$\sum_{i=1}^{K} X_i \Gamma_{ij} = H_j , \tag{4.52a}$$

$$\sum_{i=1}^{K} K_i \Gamma_{ij}^+ = -F_j , \tag{4.52b}$$

with

$$\Gamma_{ij} \equiv \frac{F_i^+ H_j}{\lambda_i - \mu_j}, \quad \text{if} \quad \lambda_i \neq \mu_j \quad (\text{in particular}, i \neq j) , \tag{4.53a}$$

and

$$\Gamma_{ii} \equiv -F_i^+ \psi'(\lambda_i)\psi^{-1}(\lambda_i) H_i + f_i^+ c_i h_i \quad \text{if} \quad \lambda_i = \mu_i , \tag{4.53b}$$

where $c_i \in \mathbb{C}^{n\times n}$ is arbitrary and $f_i^+ h_i = 0$ in the latter case.

Proof. Let V_i stand for $\mathbb{C}^n$ for $i=1, \ldots, K$, and write $\mathbb{C}^{nK} = \bigoplus_{i=1}^{K} V_i$. We prove below that if we assume that for the initial value $m = M(\xi_0, \eta_0)$ the kernel $\hat{K}$ of M splits as a sum $\hat{K} = \bigoplus \hat{K}_i$ with $\hat{K}_i \subset V_i$ and similarly for the range $\hat{R} = \bigoplus \hat{R}_i$ with $\hat{R}_i \subset V_i$, then this is true for all (ξ, η). Under this assumption let F_i be an $n \times q_i$ matrix representing a basis for the orthogonal complement of $\hat{K}_i$ with respect to the standard hermitian structure on $\mathbb{C}^n$; let H_i be an $n \times r_i$ matrix representing a basis for $\hat{R}_i$, and let $p = \operatorname{rank} M = \sum r_i = \sum q_i$. Then M can be represented

$$M = HWF^+ = \begin{pmatrix} H_1 & 0 & \ldots & 0 \\ 0 & H_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & H_k \end{pmatrix} W \begin{pmatrix} F_1^+ & 0 & \ldots & 0 \\ 0 & F_2^+ & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & F_k^+ \end{pmatrix} \tag{4.54}$$

for an invertible $p \times p$ matrix W.

By Theorem 4.1, M is determined as the solution to the linear system

$$M[(D\tilde{\Psi}-\Psi D+\Phi)m+\Psi]=\tilde{\Psi}m, \tag{4.55}$$

as long as the quantity in brackets multiplying M is invertible. Under this assumption we will show that the kernel and range of M evolve according to the block diagonal matrices Ψ and $\tilde{\Psi}$, respectively, and that if M is represented as in (4.54) the matrix W is the inverse of (Γ_{ij}) as defined in (4.53).

By Lemma (3.3), there exists a block diagonal matrix $s=\operatorname{diag}\{s_i\}$, $s_i\in\mathbb{C}^{n\times n}$ with $s_i=0$ if $\mu_i\neq\lambda_i$, such that

$$m(D+s)m=m. \tag{4.56}$$

[Note that this is valid whether m is actually the initial value of M or is just related to it by a transformation of type (4.15) with $S_1=S_2$ an arbitrary block matrix determining the initial values of $\{\psi(\lambda_i),\psi(\mu_i)\}$.] Multiplying (4.55) on the right by $(D+s)m$ and using (4.56) shows that it splits into the two equations:

$$M[D+\hat{\Phi}\tilde{\Psi}^{-1}]\tilde{\Psi}m=\tilde{\Psi}m, \tag{4.57a}$$

$$M\psi(D+s)m=M\Psi, \tag{4.57b}$$

where $\hat{\Phi}\equiv\Phi+\Psi s$ is the block diagonal matrix with $n\times n$ blocks also given by Eq. (4.49), where the constant matrices s_i are absorbed into the arbitrary matrices c_i. It follows from (4.57a) and (4.57b) that $\Psi^{-1}\operatorname{Ker}M=\operatorname{Ker}M\Psi\supset\operatorname{Ker}m$ and $\operatorname{Rng}M\supset\tilde{\Psi}\operatorname{Rng}m$. But

$$\begin{aligned}\dim\Psi^{-1}\operatorname{Ker}M+\dim\operatorname{Rng}M&=\dim\operatorname{Ker}M+\dim\operatorname{Rng}M=nk\\&=\dim\tilde{\Psi}\operatorname{Rng}m+\dim\operatorname{Ker}m\end{aligned}$$

implying equality $\operatorname{Ker}M=\Psi\operatorname{Ker}m$, $\operatorname{Rng}M=\tilde{\Psi}\operatorname{Rng}m$. Therefore if the kernel of m and the range of m split as direct sums the same is true for M.

If H_i and F_i are defined as in the beginning of the proof, and h_i and f_i are the initial values at (ξ_0,η_0), then

$$F_i=\psi^{+-1}(\lambda_i)f_i, \tag{4.58a}$$

$$H_i=\psi(\mu_i)h_i. \tag{4.58b}$$

Let H and F be the block diagonal matrices with entries H_i and F_i, respectively, and let h and f be their initial values. Set $M=HWF^+$, $m=hwf^+$ and substitute in (4.57a)

$$HWF^+[D+\hat{\Phi}\tilde{\Psi}^{-1}]\tilde{\Psi}hwf^+=\tilde{\Psi}hwf^+=Hwf^+,$$

which implies $WF^+[D+\hat{\Phi}\tilde{\Psi}^{-1}]H=\mathbb{1}$, $W=\Gamma^{-1}$. The invertibility of Γ is equivalent to the solution considered as a function with values in $G_{nK}(\mathbb{C}^{2nK})$ remaining in the domain of a fixed affine coordinate system. Let (γ_{ij}) represent the matrix W:

$$M_{ij}=H_i\gamma_{ij}F_j^+. \tag{4.59}$$

Summing along the columns,

$$Q_j=\sum_i M_{ij}=\Big(\sum_i H_i\gamma_{ij}\Big)F_j^+. \tag{4.60}$$

Summing along the rows,

$$R_i = -\sum_j M_{ij} = -H_i\Big(\sum_j \gamma_{ij} F_j^+\Big). \tag{4.61}$$

These equations are equivalent to (4.52), completing the proof.

The constraints (3.4) imply

$$F_i^+ H_i = 0,\ f_i^+ h_i = 0 \quad \text{if} \quad \mu_i = \lambda_i. \tag{4.62}$$

This form of the solution reduces to that given by Zakharov and Mikhailov in [6] if the reality conditions $\lambda_i = \mu_i$ and $R_i = Q_i^+$ are added and the poles are chosen in complex conjugate pairs. This amounts to a reduction of the $\mathrm{SL}(n, \mathbb{C})$ problem to $\mathrm{SU}(n)$ and is an example of the general reduction problem dealt with in the next section.

5. Reduction to Riemannian Symmetric Spaces (RSS)

The reduction problem involves finding particular solutions to the system (1.1), (1.2) taking values in a submanifold (real or complex) of $\mathrm{SL}(n, \mathbb{C})$. If the submanifold is itself a Lie subgroup, this defines the principal sigma model for the given group. More generally, it was shown by Eichenherr and Forger [8] that the Cartan immersion of Riemannian symmetric spaces in their isometry groups determines consistent reductions of Eqs. (1.1) and (1.2) which satisfy the field equations defining the corresponding sigma model. By a Riemannian symmetric space, we shall understand a Riemannian manifold with transitive isometry group G, and an involutive automorphism of $\sigma: G \to G$ such that the isotropy group H at some arbitrarily chosen origin satisfies

$$(G_\sigma)_0 \subseteqq H \subseteqq G_\sigma, \tag{5.1}$$

where G_σ is the subgroup of σ-invariant elements and $(G_\sigma)_0$ is its identity component. This group theoretical characterization is shown in standard texts [10] to coincide locally with various geometrical characterizations, such as covariant constancy of the curvature tensor. There also exist decomposition theorems (de Rham) which reduce all RSS to products of irreducible symmetric spaces and complete classifications of the latter in terms of the involutive automorphisms σ.

The Cartan immersion of G/H in G is defined by

$$i: G/H \to G, \qquad i(gH) \mapsto \sigma(g) g^{-1}. \tag{5.2}$$

The image is a totally geodesic submanifold $\Sigma \subset G$ defined by the relations

$$\Sigma = \operatorname{Im} i = \{g \in G \mid \sigma(g) = g^{-1}\}, \tag{5.3}$$

and is covered a finite number of times. The constraint (5.3) defining $i(G/H) \sim \Sigma$ as a submanifold of G is compatible with the Eqs. (1.1) and (1.2), and, as shown by Eichenherr and Forger [8], solutions lying in Σ coincide under the identification i with the solutions of the sigma model with values in G/H.

In view of the decomposition theorems, it is sufficient to study sigma fields with values in irreducible RSS's, and we shall limit ourselves here to those where the isometry group G is one of the classical groups. These are all listed in the first column of Table 1, following the notations of [10]. As quotient spaces G/H these are identified by first giving the constraints determining G as a subgroup of $\mathrm{SL}(n,\mathbb{C})$ and then the involution defining $i(G/H)$ as a submanifold of G. The subgroup conditions determining $G \subset \mathrm{SL}(n,\mathbb{C})$ may also be defined in terms of involutive automorphisms, since they involve one or two constraints of the form:

$$G = \{g \in \mathrm{SL}(n,\mathbb{C}) \mid \sigma(g) = g\}. \tag{5.4}$$

In Column 2 of Table 1 all the involutions which are needed to determine $i(G/H)$ as a submanifold of $\mathrm{SL}(n,\mathbb{C})$ are listed, the subgroup conditions of type (5.4) being denoted σ_+ and the quotient conditions of type (5.3), σ_-. Note that all the involutions are of one of the following types;

$$\begin{aligned} \sigma_1(g) &= tgt^{-1}, \qquad & \sigma_2(g) &= t\bar{g}t^{-1}, \\ \sigma_3(g) &= tg^{T-1}t^{-1}, \qquad & \sigma_4(g) &= tg^{+-1}t^{-1}, \end{aligned} \tag{5.5}$$

where t is either a symmetric, antisymmetric, hermitian or antihermitian matrix, which may up to conjugation be chosen as (following the notation of [10]):

$$\left.\begin{aligned}
\mathbb{1}_{p,q} &= \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\ \ddots\\ 1\end{matrix}}\right\} q \\ \left.\vphantom{\begin{matrix}1\\ \ddots\\ 1\end{matrix}}\right\} q \end{matrix} && \text{symmetric/hermitian} \\
J_n &= \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} && \text{antisymmetric/antihermitian} \\
K_{p,q} &= \begin{pmatrix} \mathbb{1}_p & & & \\ & -\mathbb{1}_q & & \\ & & \mathbb{1}_p & \\ & & & -\mathbb{1}_q \end{pmatrix} && \text{symmetric/hermitian}
\end{aligned}\right\} \tag{5.6}$$

In order to apply the results of the preceding sections to these cases, we must determine corresponding reductions of the ZMS system (1.5), and hence also of the Bäcklund transformations. We follow a reduction procedure suggested by Mikhailov [11], slightly generalized to include the involutive automorphisms.

Let s be a linear fractional transformation of the complex plane which satisfies the two conditions that it leaves invariant the two poles ± 1 of the ZMS system (1.5) and $s^2 = 1$. Then either

$$\begin{aligned} &\text{(i)} && s(\lambda) = \lambda \\ &\text{or} && \\ &\text{(ii)} && s(\lambda) = 1/\lambda. \end{aligned} \tag{5.7}$$

Table 1. Irreducible Riemannian symmetric spaces

Type		RSS	Involution	Minimal set of poles	Minimal set of poles when $\lvert\lambda_i\rvert=1$
I	AI	$SU(n)/O(n)$	$\sigma_+(g)=g^{+-1}$ $\sigma_-(g)=\bar{g}$	$\lambda_i, 1/\bar{\lambda}_i$ $\bar{\lambda}_i, 1/\lambda_i$	λ_i $\bar{\lambda}_i$
	AII	$SU(2n)/Sp(n)$	$\sigma_+(g)=g^{+-1}$ $\sigma_-(g)=J_n\bar{g}J_n^{-1}$	$\lambda_i, 1/\bar{\lambda}_i$ $\bar{\lambda}_i, 1/\lambda_i$	λ_i $\bar{\lambda}_i$
	AIII	$SU(p+q)/S(U(p)\times U(q))$	$\sigma_+(g)=g^{+-1}$ $\sigma_-(g)=I_{pq}gI_{pq}$	$\lambda_i, 1/\lambda_i$ $\bar{\lambda}_i, 1/\bar{\lambda}_i$	$\lambda_i, \bar{\lambda}_i$ $\lambda_i, \bar{\lambda}_i$
	BDI	$SO(p+q)/SO(p)\times SO(q)$	$\sigma_+(g)=g^{+-1}$ $\sigma_+(g)=\bar{g}$ $\sigma_-(g)=I_{pq}gI_{pq}$	$\lambda_i, \bar{\lambda}_i, 1/\lambda_i, 1/\bar{\lambda}_i$ $\lambda_i, \bar{\lambda}_i, 1/\lambda_i, 1/\bar{\lambda}_i$	$\lambda_i, \bar{\lambda}_i$ $\lambda_i, \bar{\lambda}_i$
	DIII	$SO(2n)/U(n)$	$\sigma_+(g)=g^{+-1}$ $\sigma_+(g)=\bar{g}$ $\sigma_-(g)=J_ngJ_n^{-1}$	$\lambda_i, \bar{\lambda}_i, 1/\lambda_i, 1/\bar{\lambda}_i$ $\lambda_i, \bar{\lambda}_i, 1/\lambda_i, 1/\bar{\lambda}_i$	$\lambda_i, \bar{\lambda}_i$ $\lambda_i, \bar{\lambda}_i$
	CI	$Sp(n)/U(n)$	$\sigma_+(g)=g^{+-1}$ $\sigma_+(g)=J_n\bar{g}J_n^{-1}$ $\sigma_-(g)=J_ngJ_n^{-1}$	$\lambda_i, \bar{\lambda}_i, 1/\lambda_i, 1/\bar{\lambda}_i$ $\lambda_i, \bar{\lambda}_i, 1/\lambda_i, 1/\bar{\lambda}_i$	$\lambda_i, \bar{\lambda}_i$ $\lambda_i, \bar{\lambda}_i$
	CII	$Sp(p+q)/Sp(p)\times Sp(q)$	$\sigma_+(g)=g^{+-1}$ $\sigma_+(g)=J_n\bar{g}J_n^{-1}$ $\sigma_-(g)=K_{pq}gK_{pq}$	$\lambda_i, \bar{\lambda}_i, 1/\lambda_i, 1/\bar{\lambda}_i$ $\lambda_i, \bar{\lambda}_i, 1/\lambda_i, 1/\bar{\lambda}_i$	$\lambda_i, \bar{\lambda}_i$ $\lambda_i, \bar{\lambda}_i$
II	a_n	$SU(n)\simeq SU(n)\times SU(n)/SU(n)_D$	$\sigma_+(g)=g^{+-1}$	λ_i $\bar{\lambda}_i$	λ_i $\bar{\lambda}_i$
	b_n, d_n	$SO(n)\simeq SO(n)\times SO(n)/SO(n)_D$	$\sigma_+(g)=g^{+-1}$ $\sigma_+(g)=\bar{g}$	$\lambda_i, \bar{\lambda}_i$ $\lambda_i, \bar{\lambda}_i$	$\lambda_i, \bar{\lambda}_i$ $\lambda_i, \bar{\lambda}_i$
	c_n	$Sp(n)\simeq Sp(n)\times Sp(n)/Sp(n)_D$	$\sigma_+(g)=g^{+-1}$ $\sigma_+(g)=J_n\bar{g}J_n^{-1}$	$\lambda_i, \bar{\lambda}_i$ $\lambda_i, \bar{\lambda}_i$	$\lambda_i, \bar{\lambda}_i$ $\lambda_i, \bar{\lambda}_i$
III	AI	$Sl(n,\mathbb{R})/SO(n)$	$\sigma_+(g)=\bar{g}$ $\sigma_-(g)=g^{T-1}$	$\lambda_i, \bar{\lambda}_i$ $1/\lambda_i, 1/\bar{\lambda}_1$	$\lambda_i, \bar{\lambda}_i$ $\lambda_i, \bar{\lambda}_i$
	AII	$SU^*(2n)/Sp(n)$	$\sigma_+(g)=J_n\bar{g}J_n^{-1}$ $\sigma_-(g)=g^{+-1}$	$\lambda_i, \bar{\lambda}_i$ $1/\lambda_i, 1/\bar{\lambda}_i$	$\lambda_i, \bar{\lambda}_i$ $\lambda_i, \bar{\lambda}_i$
	AIII	$SU(p,q)/S(U(p)\times U(q))$	$\sigma_+(g)=I_{pq}g^{+-1}I_{pq}$ $\sigma_-(g)=I_{pq}gI_{pq}$	$\lambda_i, 1/\lambda_i$ $\bar{\lambda}_i, 1/\bar{\lambda}_i$	$\lambda_i, \bar{\lambda}_i$ $\lambda_i, \bar{\lambda}_i$
	BDI	$SO_0(p,q)/SO(p)\times SO(q)$	$\sigma_+(g)=\bar{g}$ $\sigma_+(g)=I_{pq}g^{T-1}I_{pq}$ $\sigma_-(g)=I_{pq}gI_{pq}$	$\lambda_i, \bar{\lambda}_i, 1/\lambda_i, 1/\bar{\lambda}_i$ $\lambda_i, \bar{\lambda}_i, 1/\lambda_i, 1/\bar{\lambda}_i$	$\lambda_i, \bar{\lambda}_i$ $\lambda_i, \bar{\lambda}_i$
	DIII	$SO^*(2n)/U(n)$	$\sigma_+(g)=g^{T-1}$ $\sigma_+(g)=J_n\bar{g}J_n^{-1}$ $\sigma_-(g)=g^{+-1}$	$\lambda_i, \bar{\lambda}_i, 1/\lambda_i, 1/\bar{\lambda}_i$ $\lambda_i, \bar{\lambda}_i, 1/\lambda_i, 1/\bar{\lambda}_i$	$\lambda_i, \bar{\lambda}_i$ $\lambda_i, \bar{\lambda}_i$
	CI	$Sp(n,\mathbb{R})/U(n)$	$\sigma_+(g)=\bar{g}$ $\sigma_+(g)=J_ng^{+-1}J_n^{-1}$ $\sigma_-(g)=g^{+-1}$	$\lambda_i, \bar{\lambda}_i, 1/\lambda_i, 1/\bar{\lambda}_i$ $\lambda_i, \bar{\lambda}_i, 1/\lambda_i, 1/\bar{\lambda}_i$	$\lambda_i, \bar{\lambda}_i$ $\lambda_i, \bar{\lambda}_i$
	CII	$Sp(p,q)/Sp(p)\times Sp(q)$	$\sigma_+(g)=J_ng^{T-1}J_n^{-1}$ $\sigma_+(g)=K_{pq}g^{+-1}K_{pq}$ $\sigma_-(g)=K_{pq}gK_{pq}$	$\lambda_i, \bar{\lambda}_i, 1/\lambda_i, 1/\bar{\lambda}_i$ $\lambda_i, \bar{\lambda}_i, 1/\lambda_i, 1/\bar{\lambda}_i$	$\lambda_i, \bar{\lambda}_i$ $\lambda_i, \bar{\lambda}_i$
IV	"a_n"	$Sl(n,\mathbb{C})/SU(n)$	$\sigma_-(g)=g^{+-1}$	λ_i $1/\bar{\lambda}_i$	λ_i λ_i
	"b_n, d_n"	$SO(n,\mathbb{C})/SO(n)$	$\sigma_+(g)=g^{T-1}$ $\sigma_-(g)=\bar{g}$	$\lambda_i, 1/\bar{\lambda}_i$ $\lambda_i, 1/\bar{\lambda}_i$	λ_i λ_i
	"c_n"	$Sp(n,\mathbb{C})/Sp(n)$	$\sigma_+(g)=J_ng^{T-1}J_n^{-1}$ $\sigma_-(g)=J_n\bar{g}J_n^{-1}$	$\lambda_i, 1/\bar{\lambda}_i$ $\lambda_i, 1/\bar{\lambda}_i$	λ_i λ_i

The automorphism σ may either be holomorphic (σ_1, σ_3) or antiholomorphic (σ_2, σ_4). We impose the invariance condition

$$\psi(\lambda) = f\sigma\psi(s(\tilde{\lambda})), \tag{5.8}$$

where either

$$\begin{aligned} &\text{(i)} && s(\lambda)=\lambda, && f=1 \quad \text{for} \quad \sigma_+, \\ &\text{(ii)} && s(\lambda)=1/\lambda, && f=g \quad \text{for} \quad \sigma_-, \end{aligned} \tag{5.9}$$

and $\tilde{\lambda}=\lambda$ if σ is holomorphic, $\tilde{\lambda}=\bar{\lambda}$ if σ is anti-holomorphic.

Since $\psi(0)=g$, Eq. (5.8) implies the correct constraint on g:

$$\sigma_+(g)=g, \qquad \sigma_-(g)=g^{-1}, \tag{5.10}$$

and the ZMS equation is compatible with this reduction, since Eq. (5.10) implies, in terms of the differential σ_* at the identity,

$$\sigma_*\left(\frac{A}{1+\lambda}\right)=\frac{A}{1+\tilde{\lambda}}, \qquad \sigma_*\left(\frac{B}{1-\lambda}\right)=\frac{B}{1-\tilde{\lambda}} \quad \text{for} \quad \sigma_+, \tag{5.11a}$$

$$\sigma_*\left(\frac{A}{1+\lambda}\right)=\frac{-g^{-1}Ag}{1+\tilde{\lambda}}, \qquad \sigma_*\left(\frac{B}{1-\lambda}\right)=\frac{-g^{-1}Ag}{1-\tilde{\lambda}} \quad \text{for} \quad \sigma_-. \tag{5.11b}$$

The dressing matrix $\chi(\lambda)$ preserves the reduction under the transformation

$$\psi(\lambda) \Rightarrow \tilde{\psi}(\lambda)=\chi(\lambda)\psi(\lambda), \tag{5.12}$$

if

$$\chi(\lambda)=\tilde{f}[\sigma\chi(s(\tilde{\lambda}))]f^{-1}, \tag{5.13}$$

where $\tilde{f}$ is defined analogously to f in terms of the new solution $\tilde{g}=\tilde{\psi}(0)$.

We shall now express these conditions in terms of equivalent ones on the M-matrix defined by Eq. (3.1). To satisfy the invariance condition (5.13), it is necessary that the set of poles $\{\lambda_i\}$ and $\{\mu_i\}$ of χ and χ^{-1}, respectively, be invariant under $\lambda \to s(\tilde{\lambda})$ for σ_1 and σ_2, and that they be mapped into each other for σ_2 and σ_4. We define:

$$M_{\tilde{s}(i)\tilde{s}(j)}=\frac{1}{(2\pi i)^2}\oint_{s(\tilde{\mu}_i)} d\mu \oint_{s(\tilde{\lambda}_j)} d\lambda \frac{\chi^{-1}(\mu)\chi(\lambda)}{\mu-\lambda} \quad \text{if} \quad \sigma=\sigma_1 \quad \text{or} \quad \sigma_2, \tag{5.14a}$$

$$M_{\tilde{s}(j)\tilde{s}(i)}=\frac{1}{(2\pi i)^2}\oint_{s(\tilde{\lambda}_j)} d\mu \oint_{s(\tilde{\mu}_i)} d\lambda \frac{\chi^{-1}(\mu)\chi(\lambda)}{\mu-\lambda} \quad \text{if} \quad \sigma=\sigma_3 \quad \text{or} \quad \sigma_4. \tag{5.14b}$$

With these as preliminaries, we have the following theorem.

Theorem 5.1. *The invariance condition* (5.13) *is equivalent to one of the following conditions on the M-matrix:*

(i) *If σ is of type σ_1 or σ_2 (linear over $\mathbb{R}$), then either:*

$$M_{ij}=\sigma(M_{\tilde{s}(i)\tilde{s}(j)}) \qquad \text{if} \quad s(\lambda)=\lambda, \tag{5.15a}$$

or

$$M_{ij}=-\mu_i\lambda_j g\sigma(M_{\tilde{s}(i)\tilde{s}(j)})g^{-1} \quad \text{if} \quad s(\lambda)=1/\lambda. \tag{5.15b}$$

(ii) *If σ is of type σ_3 or σ_4, then, defining the $\mathbb{R}$-linear anti-automorphism*

$$\hat{\sigma}=\sigma\circ\mathscr{I}, \tag{5.16}$$

where $\mathscr{I}$ is inversion

$$\mathscr{I}(g)=g^{-1}, \tag{5.17}$$

either

$$M_{ij}=-\hat{\sigma}M_{\bar{s}(j)\bar{s}(i)} \qquad \text{if} \quad s(\lambda)=\lambda \tag{5.18a}$$

or

$$M_{ij}=\mu_i\lambda_j g\hat{\sigma}M_{\bar{s}(j)\bar{s}(i)}g^{-1} \quad \text{if} \quad s(\lambda)=1/\lambda, \tag{5.18b}$$

where the right-hand sides of Eqs. (5.15) *and* (5.18) *are defined by Eqs.* (5.14*a*) *and* (5.14*b*), *respectively.*

[Note that whereas σ and $\hat{\sigma}$ are defined as automorphisms of $\mathrm{SL}(n,\mathbb{C})$, they extend naturally to any $n\times n$ matrices and in fact are, up to a sign, equal to the differential σ_*, at the identity element.]

Proof. The condition (5.13) on χ is equivalent to

$$\chi^{-1}(\mu)\chi(\lambda)=f\sigma(\chi(s(\tilde{\mu}))^{-1}\chi(s(\tilde{\lambda})))f^{-1}. \tag{5.19}$$

Clearly (5.13) implies (5.19) and moving the μ independent terms to one side we get

$$\sigma(\chi(s(\tilde{\mu})))f^{-1}\chi^{-1}(\mu)=\sigma(\chi(s(\lambda)))f^{-1}\chi^{-1}(\lambda), \tag{5.20}$$

which implies that both sides are independent of λ or μ. Letting $\lambda\to\infty$, we find that the common value is $\tilde{f}^{-1}$, giving Eq. (5.13). Now, integrating (5.14) around the poles $\{\mu_i,\lambda_j\}$ gives:

$$\begin{aligned} M_{ij}&=\frac{1}{(2\pi i)^2}\oint_{\mu_i}d\mu\oint_{\lambda_j}d\lambda\frac{\chi^{-1}(\mu)\chi(\lambda)}{\mu-\lambda}\\ &=\frac{f}{(2\pi i)^2}\oint_{\mu_i}d\mu\oint_{\lambda_j}d\lambda\frac{\sigma[\chi(s(\tilde{\mu}))^{-1}\chi(s(\tilde{\lambda}))]f^{-1}}{\mu-\lambda}. \end{aligned} \tag{5.21}$$

We shall first prove that the equality (5.21) implies (5.19) and hence is equivalent to the reduction condition (5.13), and second that it reduces to Eqs. (5.15a) and (5.15b) or (5.18a) and (5.18b) for the four types of involutions σ_1, σ_2, σ_3, and σ_4. To prove the first implication, define

$$h(\mu,\lambda)=\frac{\chi^{-1}(\mu)\chi(\lambda)-f\sigma(\chi(s(\mu))^{-1}\chi(s(\lambda))f^{-1}}{\mu-\lambda}. \tag{5.22}$$

The function $h(\mu,\lambda)$ is holomorphic in (μ,λ) for $\mu\neq\mu_i$, $\lambda\neq\lambda_j$, and $\mu\neq\lambda$. In fact the singularity at $\mu=\lambda$ for $\mu\neq\mu_i$, $\lambda\neq\lambda_j$ is removable, since h may be re-written as:

$$-\chi^{-1}(\mu)\left[\frac{\chi(\lambda)-\chi(\mu)}{\lambda-\mu}\right]+f\sigma(\chi(s(\mu)))^{-1}\left[\frac{\sigma(\chi(s(\lambda)))-\sigma(\chi(s(\mu)))}{\lambda-\mu}\right]f^{-1}, \tag{5.23}$$

and we can take the limit as $\lambda-\mu\to 0$ as long as the poles of χ and χ^{-1} are avoided. To show that (5.21) implies that $h(\mu,\lambda)$ vanishes identically, consider the functions of one variable:

$$h_i(\lambda) = \oint_{\mu_i} d\mu\, h(\mu, \lambda),$$
$$k_j(\mu) = \oint_{\lambda_j} d\lambda\, h(\mu, \lambda), \tag{5.24}$$

which are meromorphic in λ and μ, respectively. Equation (5.21) implies that all the finite residues are zero. From (5.22) we have an approximation

$$|h(\mu, \lambda)| \leqq \frac{M}{\lambda}, \tag{5.25a}$$

uniformly for large λ if μ is in a compact set containing none of the poles of χ or χ^{-1} and similarly

$$|h(\mu, \lambda)| \leqq \frac{M}{\mu} \tag{5.25b}$$

for large μ. These estimates show

$$\lim_{\lambda \to \infty} h_i(\lambda) = 0 \quad \text{and} \quad \lim_{\mu \to \infty} k_j(\mu) = 0.$$

For λ fixed and not equal to any λ_i, $h(\mu, \lambda)$ is meromorphic in μ, and if $\lambda \neq \mu_i$, it has simple poles. Since all the residues are zero and the limit $\mu \to \infty$ is zero, we conclude $h(\mu, \lambda)$ vanishes for $\lambda \neq \lambda_i$, μ_j and all μ, and similarly for $\mu \neq \lambda_i$, μ_j and all λ. By continuity as a map into the Riemann sphere, it therefore vanishes identically, implying Eq. (5.19).

To show that Eq. (5.21) is equivalent to Eqs. (5.15a) and (5.15b) or (5.18a) and (5.18b) of the theorem, there are eight subcases to consider. Since the proofs are all similar, we shall only give two representative cases: (5.15b) for σ_1 and (5.18a) for σ_4. Making the change of variables $\lambda \to s(\lambda)$ and $\mu \to s(\mu)$ in the right hand side of Eq. (5.21), we get:

$$\begin{aligned} &\frac{f}{(2\pi i)^2} \oint_{s(\mu_i)} d\mu \oint_{s(\lambda_j)} d\lambda \frac{\sigma(\chi(\mu))^{-1}\chi(\lambda))}{s(\mu) - s(\lambda)} s'(\mu)\, s'(\lambda)\, f^{-1} \\ &= \frac{f}{(2\pi i)^2} \oint_{s(\mu_i)} d\mu \oint_{s(\lambda_j)} d\lambda \frac{\sigma(\chi^{-1}(\mu)\chi(\lambda))}{(\lambda - \mu)\mu\lambda} f^{-1}. \end{aligned}$$

For the case (5.15b), $f = g^{-1}$ and $\sigma = \sigma_1$ is linear, hence this equals:

$$\begin{aligned} &-g\sigma\left[\frac{1}{(2\pi i)^2} \oint_{s(\mu_i)} d\mu \oint_{s(\lambda_j)} d\lambda \frac{\chi^{-1}(\mu)\chi(\lambda)}{(\mu - \lambda)\mu\lambda}\right] g^{-1} \\ &= -g\sigma\left[\frac{M_{s(i)s(j)}}{s(\mu_i)\, s(\lambda_j)}\right] g^{-1} = -\mu_i\lambda_j g\sigma(M_{s(i)s(j)}) g^{-1}. \end{aligned}$$

Now consider the case (5.18a) for σ of type σ_4. In this case $\hat{\sigma}$ is anti-linear, $s(\tilde{\lambda}) = \bar{\lambda}$, $f = 1$, and $\{s(\tilde{\mu}_i)\} = \{\lambda_j\}$. The right hand side of Eq. (5.21) is

$$\frac{1}{(2\pi i)^2} \oint_{\mu_i} d\mu \oint_{\lambda_j} d\lambda \frac{\hat{\sigma}((\chi(\bar{\lambda}))^{-1}\chi(\bar{\mu}))}{\mu - \lambda} = \hat{\sigma} \frac{1}{(2\pi i)^2} \oint_{\mu_i} d\bar{\mu} \oint_{\lambda_j} d\bar{\lambda} \frac{\chi^{-1}(\bar{\lambda})\chi(\bar{\mu})}{\bar{\mu} - \bar{\lambda}}$$

by the anti-linearity of $\hat{\sigma}$. Make the change of variables $\mu \to \bar{\mu}$, $\lambda \to \bar{\lambda}$ which changes the sign of each contour integral, but not the product, and reverse the order of integration to get:

$$-\hat{\sigma}\left(\frac{1}{(2\pi i)^2}\oint_{\bar{\lambda}_j} d\lambda \oint_{\bar{\mu}_i} d\mu \frac{\chi(\lambda)^{-1}\chi(\mu)}{\lambda-\mu}\right) = -\hat{\sigma} M_{\bar{s}(j)\bar{s}(i)},$$

which proves the result for this case.

The interpretation of Theorem 5.1 in terms of Bäcklund transformations is as follows. Suppose that ϕ is a given solution to the sigma model with values either in a Lie subgroup $H \subset G$ of a given group or a Riemannian symmetric space G/H. Denote by $X \subset G$ the submanifold of G corresponding either to the embedding of H or the image Σ of G/H under (5.2). The Bäcklund transformation for the principal sigma model with values in G, which is solved by the M-matrix given in Theorem 4.1, restricts, by the imposition of constraints given in Theorem 5.1 to one preserving X. Denoting both the inclusion map $i: H \to G$ and the immersion $i: G/H \to G$ by i, the situation is summarized in the diagram below:

σ Model with Values in H or G/H		σ Model with Values in G
Initial solution ϕ	$\xrightarrow{\quad i \quad}$	$g \equiv i \circ \phi \subset X$ for all (ξ, η)
$\Downarrow$		$\downarrow$
New solution $\tilde{\phi}$	$\xleftarrow[i^{-1}]{\text{(locally defined inverse)}}$	$\tilde{g} \subset X$ for all (ξ, η)

The conditions of Theorem 5.1 may be given a simple geometrical interpretation if, as in the previous section, the values of the M-matrix are interpreted as affine coordinates of a point in $G_{nK}(\mathbb{C}^{2nK})$. When σ is linear or anti-linear (i.e. σ_1 or σ_2), the conditions (5.15a) or (5.15b) are equivalent to the fact that the point is fixed under a linear or anti-linear transformation, which is defined in Table 2 by the $2nK \times 2nK$ dimensional matrices L and $\tilde{L}$, respectively. The explicit form of L depends on whether the involution defines a subgroup reduction (σ_+) or a quotient reduction (σ_-), but in all cases, the constraints on M may be expressed as

$$L\begin{bmatrix} M \\ \mathbb{1} \end{bmatrix} = \begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}, \tag{5.26a}$$

or

$$\tilde{L}\begin{bmatrix} \bar{M} \\ \mathbb{1} \end{bmatrix} = \begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}, \tag{5.26b}$$

where

$$\begin{bmatrix} M \\ \mathbb{1} \end{bmatrix} = \begin{bmatrix} MG \\ G \end{bmatrix}, \qquad G \in \mathrm{GL}(nK, \mathbb{C}) \tag{5.27}$$

denotes the equivalence class of points in $\mathbb{C}^{2nK \times nK}$ defining the homogeneous coordinates of a point in $G_{nK}(\mathbb{C}^{2nK})$. When $\sigma = \sigma \circ \mathscr{I}$, as defined in Theorem 5.1 is linear or anti-linear (i.e. σ_3 or σ_4), the conditions (5.18a) or (5.18b) are equivalent to

Table 2. Constraints associated with involutions $\sigma_{\pm}$

Involution	Poles related	Constraints on $\begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}$	Constraints on F_i, H_i, and Γ_{ij}
(i) *σ is linear anti-linear*			
$\sigma_+(g)=tgt^{-1}$ σ_{+*} is linear	λ_i μ_i	$L=\left(\begin{array}{cc} \mathbb{1} & \\ \hline & \mathbb{1} \end{array}\right)\otimes t$ $L\begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}=\begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}$	$[tF_i]=[F_i],\ [tH_i]=[H_i]$ no constraints on Γ_{ij}
$\sigma_+(g)=t\bar{g}t^{-1}$ σ_{+*} is anti-linear	$\lambda_i, \bar{\lambda}_i$ $\mu_i, \bar{\mu}_i$	$\tilde{L}=\left(\begin{array}{cccc} & \mathbb{1} & & \\ \mathbb{1} & & & \\ \hline & & & \mathbb{1} \\ & & \mathbb{1} & \end{array}\right)\otimes t$ $\tilde{L}\begin{bmatrix} \bar{M} \\ \mathbb{1} \end{bmatrix}=\begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}$	$[t\bar{F}_i]=[F_{\bar{i}}],\ [t\bar{H}_i]=[H_{\bar{i}}]$ $\Gamma_{ij}=\bar{\Gamma}_{\bar{i}\bar{j}}$
$\sigma_-(g)=tgt^{-1}$ σ_{-*} is linear	$\lambda_i, 1/\lambda_i$ $\mu_i, 1/\mu_i$	$L=\left(\begin{array}{cccc} & \{\mu\} & & \\ \{\mu^{-1}\} & & & \\ \hline & & & -\{\lambda^{-1}\} \\ & & -\{\lambda\} & \end{array}\right)\otimes gt$ $L\begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}=\begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}$	$[g^{-1+}tF_i]=[F_{\hat{i}}],\ [gtH_i]=[H_{\hat{i}}]$ $\Gamma_{ij}=\dfrac{-\Gamma_{\hat{i}\hat{j}}}{\lambda_i\mu_j}$
$\sigma_-(g)=t\bar{g}t^{-1}$ σ_{-*} is anti-linear	$\lambda_i, 1/\bar{\lambda}_i$ $\mu_i, 1/\bar{\mu}_i$	$\tilde{L}=\left(\begin{array}{cccc} & \{\mu\} & & \\ \{\bar{\mu}^{-1}\} & & & \\ \hline & & & -\{\lambda^{-1}\} \\ & & -\{\bar{\lambda}\} & \end{array}\right)\otimes gt$ $\tilde{L}\begin{bmatrix} \bar{M} \\ \mathbb{1} \end{bmatrix}=\begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}$	$[g^{-1+}t\bar{F}_i]=[F_{\hat{\bar{i}}}],\ [gt\bar{H}_i]=[H_{\hat{\bar{i}}}]$ $\Gamma_{ij}=\dfrac{-\bar{\Gamma}_{\hat{\bar{i}}\hat{\bar{j}}}}{\lambda_{\bar{i}}\mu_{\bar{j}}}$
(ii) $(\sigma\circ\mathscr{I})$ is linear or anti-linear			
$\sigma_+(g)=tg^{T-1}t^{-1}$ σ_{+*} is linear	λ_i λ_i	$\left(\begin{array}{cc} & \mathbb{1} \\ \hline \mathbb{1} & \end{array}\right)\otimes t^{-1}=\begin{cases} s & \text{if } t=\mathbb{1}, I_{pq}, K_{pq} \\ a & \text{if } t=J_n \end{cases}$ $[M^T\quad \mathbb{1}]\, s\begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}=0$ or $[M^T\quad \mathbb{1}]\, a\begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}=0$	$[t\bar{F}_i]=[H_i],\ F_i^+H_i=0$ $\Gamma_{ij}=-\varepsilon_t\Gamma^T_{ji}$
$\sigma_+(g)=tg^{+-1}t^{-1}$ σ_{+*} is anti-linear	λ_i $\bar{\lambda}_i$	$\left(\begin{array}{cc} & \mathbb{1} \\ \hline \mathbb{1} & \end{array}\right)\otimes t^{-1}=\begin{cases} h & \text{if } t=\mathbb{1}, I_{pq}, K_{pq} \\ a_h & \text{if } t=J_n \end{cases}$ $[M^+\quad \mathbb{1}]\, h\begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}=0$ or $[M^+\quad \mathbb{1}]\, a_h\begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}=0$	$[tF_i]=[H_{\bar{i}}]$ $\Gamma_{i\bar{j}}=-\varepsilon_t\Gamma^+_{j\bar{i}}$
$\sigma_-(g)=tg^{T-1}t^{-1}$ σ_{-*} is linear	λ_i $1/\lambda_i$	$\left(\begin{array}{cc} & \{\hat{\lambda}\} \\ \hline -\{\lambda\} & \end{array}\right)\otimes(gt)^{-1}=\begin{cases} a_s & \text{if } t=\mathbb{1}, I_{pq}, K_{pq} \\ s & \text{if } t=J_n \end{cases}$ $[M^T\quad \mathbb{1}]\, a\begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}=0$	$[gt\bar{F}_i]=[H_{\hat{i}}]$

Table 2 (continued)

Involution	Poles related	Constraints on $\begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}$	Constraints on F_i, H_i, and Γ_{ij}
		or	$\Gamma_{i\hat{j}}=\varepsilon_t\Gamma^T_{j\hat{i}}\dfrac{\lambda_j}{\lambda_i}$
		$[M^T \quad \mathbb{1}]\, s \begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}=0$	
$\sigma_-(g)=tg^{+\,-1}t^{-1}$ σ_{-*} is anti-linear	λ_i $1/\bar{\lambda}_i$	$\left(\begin{array}{c\|c} & +\{\lambda\} \\ \hline -\{\bar{\lambda}\} & \end{array}\right) \otimes (gt)^{-1} = \begin{cases} a_h & \text{if } t=\mathbb{1}, I_{pq}, K_{pq} \\ h & \text{if } t=J_n \end{cases}$	$[gtF_i]=[H_{\hat{i}}]$
		$[M^+ \quad \mathbb{1}]\, a_h \begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}=0$	
		or	$\Gamma_{i\hat{j}}=\varepsilon_t\Gamma^+_{j\hat{i}}\dfrac{\bar{\lambda}_j}{\lambda_i}$
		$[M^+ \quad \mathbb{1}]\, h \begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}=0$	

the fact that M determines a point in the submanifold of $G_{nK}(\mathbb{C}^{2nK})$ consisting of completely isotropic nK-planes under a bilinear or sesquilinear form. These forms depend again upon the specific involution and type of reduction, but they are all defined by a nonsingular matrix which is either symmetric, anti-symmetric, hermitian or anti-hermitian. Denoting these, respectively, as s, a, h, a_h, the condition of total isotropy may be expressed as:

$$[M^+\mathbb{1}]\, s \begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}=0, \qquad s=s^T, \tag{5.28a}$$

$$[M^+\mathbb{1}]\, a \begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}=0, \qquad a=-a^T, \tag{5.28b}$$

$$[M^+\mathbb{1}]\, h \begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}=0, \qquad h=h^+, \tag{5.28c}$$

$$[M^+\mathbb{1}]\, a_h \begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}=0, \qquad a_h=-a_h^+ . \tag{5.28d}$$

The second column of Table 2 gives the minimal set of eigenvalues which are invariant under the associated involution s and the third column indicates the matrices L, $\tilde{L}$, a, s, h, and a_h which define the constraints (5.26) or (5.28) upon M. The matrix entries are ordered according to the following conventions. Denote the various residues in $\chi(\lambda)$ and $\chi^{-1}(\mu)$ as:

$$\begin{aligned}
&\text{residue in } \chi(\lambda) \text{ at } \lambda_i : Q_i; && \text{residue in } \chi^{-1}(\mu) \text{ at } \mu_i : R_i;\\
&\text{residue in } \chi(\lambda) \text{ at } \bar{\lambda}_i : Q_{\bar{i}}; && \text{residue in } \chi^{-1}(\mu) \text{ at } \bar{\mu}_i : R_{\bar{i}};\\
&\text{residue in } \chi(\lambda) \text{ at } \frac{1}{\lambda_i} : Q_{\hat{i}}; && \text{residue in } \chi^{-1}(\mu) \text{ at } \frac{1}{\mu_i} : R_{\hat{i}};\\
&\text{residue in } \chi(\lambda) \text{ at } \frac{1}{\bar{\lambda}_i} : Q_{\hat{\bar{i}}}; && \text{residue in } \chi^{-1}(\mu) \text{ at } \frac{1}{\bar{\mu}_i} : R_{\hat{\bar{i}}}.
\end{aligned} \tag{5.29}$$

The matrix M is the submatrix of the following $4nl \times 4nl$ matrix, with $k = l$, $2l$ or $4l$, obtained by deleting those $ln \times ln$ dimensional blocks corresponding to eigenvalues which do not appear:

$$\begin{pmatrix}
M_{ij} & M_{i\bar{j}} & M_{i\hat{j}} & M_{i\hat{\bar{j}}}\\
M_{\bar{i}j} & M_{\bar{i}\bar{j}} & M_{\bar{i}\hat{j}} & M_{\bar{i}\hat{\bar{j}}}\\
M_{\hat{i}j} & M_{\hat{i}\bar{j}} & M_{\hat{i}\hat{j}} & M_{\hat{i}\hat{\bar{j}}}\\
M_{\hat{\bar{i}}j} & M_{\hat{\bar{i}}\bar{j}} & M_{\hat{\bar{i}}\hat{j}} & M_{\hat{\bar{i}}\hat{\bar{j}}}
\end{pmatrix}. \tag{5.30}$$

The fact that the conditions (5.15) and (5.18) of Theorem 5.1 coincide with the constraints (5.26) and (5.28) for the eight cases listed in Table 2 follows by directly substituting the various types of involutions σ_1, σ_2, σ_3, and σ_4 as defined in Eq. (5.5). Using the explicit involutions which define the RSS's as listed in Column 2 of Table 1, we arrive at the minimal set of eigenvalues listed in Column 3 for the general case and Column 4 if the eigenvalues are chosen on the unit circle. Examples of the use of these two tables to reconstruct all constraints defining a particular RSS will be given later.

In order to verify that the reduction procedure is consistent with the multi-Bäcklund transformation and hence that the solutions determined by Theorem 4.1 actually give rise to solutions of Eqs. (1.1) and (1.2) within the correct submanifold, provided the input solution g is within it, we must demonstrate the consistency of the constraints on M with the matrix Riccati system (3.7a) and (3.7b) determining it. This result is proved easily through the geometrical interpretation of the system and constraints.

Theorem 5.2. *The reduction by involutions is propagated by the differential equations for the dressing matrix. That is, let m be the initial value of the M-matrix at (ξ_0, η_0). If m satisfies the conditions of Theorem* 5.1, *then $M(\xi, \eta)$ does for all (ξ, η).*

Proof. The dependence on (ξ, η) of M is, according to Eq. (4.13) completely determined by the $\mathrm{SL}(2nK, \mathbb{C})$-valued function $\mathscr{G}(\xi, \eta)$ satisfying Eq. (4.11), through:

$$\begin{bmatrix} M \\ \mathbb{1} \end{bmatrix} = \mathscr{G} \begin{bmatrix} m \\ \mathbb{1} \end{bmatrix}. \tag{5.31}$$

According to the geometric formulation given above, the conditions of Theorem 5.1 may be expressed, for case (i) (σ_1 or σ_2) as:

$$L_0\begin{bmatrix} m \\ \mathbb{1} \end{bmatrix} = \begin{bmatrix} m \\ \mathbb{1} \end{bmatrix} \Rightarrow L\begin{bmatrix} M \\ \mathbb{1} \end{bmatrix} = \begin{bmatrix} M \\ \mathbb{1} \end{bmatrix},$$

or

$$\tilde{L}_0\begin{bmatrix} \bar{m} \\ \mathbb{1} \end{bmatrix} = \begin{bmatrix} m \\ \mathbb{1} \end{bmatrix} \Rightarrow \tilde{L}\begin{bmatrix} \bar{M} \\ \mathbb{1} \end{bmatrix} = \begin{bmatrix} M \\ \mathbb{1} \end{bmatrix},$$

where L_0, $\tilde{L}_0$ denote the initial values of L and $\tilde{L}$, respectively, at (ξ_0, η_0).

In view of Eq. (5.31), it is sufficient to prove:

$$L\mathscr{G} = \mathscr{G}L_0 \tag{5.32a}$$

or

$$\tilde{L}\bar{\mathscr{G}} = \mathscr{G}\tilde{L}_0 . \tag{5.32b}$$

Similarly, for Cases (ii) (σ_3 or σ_4), the constraints on M are satisfied provided:

$$\left.\begin{aligned} \mathscr{G}^T s\mathscr{G} &= s_0 \quad &(5.33a) \\ \mathscr{G}^T a\mathscr{G} &= a_0 \quad &(5.33b) \end{aligned}\right\}\sigma_3 ,$$

$$\left.\begin{aligned} \mathscr{G}^+ h\mathscr{G} &= h_0 \quad &(5.33c) \\ \mathscr{G}^+ a_k\mathscr{G} &= a_{h_0} \quad &(5.33d) \end{aligned}\right\}\sigma_4 ,$$

where s_0, k_0, h_0, and a_{h_0} are the initial values of the symmetric, anti-symmetric, hermitian, and anti-hermitian forms s, a, h, a_h, respectively, as defined in Table 2. Since the function $\mathscr{G}$ defines a covariant constant section of the bundle $\mathbb{R}^2 \times \mathrm{SL}(2nK, \mathbb{C})$ and the functions L, $\tilde{L}$, s, a, h, a_h may similarly be interpreted as defining maps and forms on the associated vector bundle $\mathbb{R}^2 \times \mathbb{C}^{2nK}$, the conditions (5.32) and (5.33) have the simple interpretation that these quantities are invariant under parallel transport. The relations (5.32) and (5.33), for given L, $\tilde{L}$, s, a, h or a_h determine a reduction of the principal bundle to the subgroup determined by the relations. Thus the verification of consistency with Eqs. (4.11) determining $\mathscr{G}$ amounts to proving that the holonomy group of the connection (4.2) is contained in the appropriate subgroup.

The proof for the eight cases is quite similar, so we again only give two representative examples:

(A) $$\sigma_+(g) = t\bar{g}t^{-1} = g ,$$

or (5.34)

(B) $$\sigma_-(g) = tg^{T-1}t^{-1} = g^{-1} .$$

Note that for σ_+ reductions like (A), the appropriate map or form is itself constant and hence the subgroup defined by (5.32) or (5.33) is a fixed subgroup of $\mathrm{SL}(2nK, \mathbb{C})$, while for σ_- reductions like (B), it varies by a conjugation in $\mathrm{SL}(2nK, \mathbb{C})$ depending on the point (ξ, η), of the form $g(\xi, \eta) \otimes \mathbb{1}_{2K}$, where $g(\xi, \eta)$ is the input solution.

Case (A). Since $\mathscr{G}(\xi_0, \eta_0) = \mathbb{1}$ satisfies the appropriate condition (5.32b) and $\tilde{L}$ is constant, it is sufficient to prove the differentiated relations:

$$\begin{aligned} -(\mathscr{A} \otimes A)\tilde{L} + \tilde{L}\overline{(\mathscr{A} \otimes A)} &= 0, \\ -(\mathscr{B} \otimes B)\tilde{L} + \tilde{L}\overline{(\mathscr{B} \otimes B)} &= 0, \end{aligned} \tag{5.35}$$

where $\mathscr{A}$ and $\mathscr{B}$ are, according to Eq. (4.7)

$$\mathscr{A} = \left(\begin{array}{c|c|c|c} \operatorname{diag}\left\{\frac{1}{1+\mu_i}\right\} & & & \\ \hline & \operatorname{diag}\left\{\frac{1}{1+\bar{\mu}_i}\right\} & & \\ \hline \frac{1}{1+\lambda_i} \cdot \frac{1}{1+\mu_j} & \frac{1}{1+\lambda_i} \cdot \frac{1}{1+\bar{\mu}_j} & \operatorname{diag}\left\{\frac{1}{1+\lambda_i}\right\} & \\ \hline \frac{1}{1+\bar{\lambda}_i} \cdot \frac{1}{1+\mu_j} & \frac{1}{1+\bar{\lambda}_i} \cdot \frac{1}{1+\bar{\mu}_j} & & \operatorname{diag}\left\{\frac{1}{1+\bar{\lambda}_i}\right\} \end{array}\right), \tag{5.36a}$$

$$\mathscr{B} = \left(\begin{array}{c|c|c|c} \operatorname{diag}\left\{\frac{1}{1-\mu_i}\right\} & & & \\ \hline & \operatorname{diag}\left\{\frac{1}{1-\bar{\mu}_i}\right\} & & \\ \hline -\frac{1}{1-\lambda_i} \cdot \frac{1}{1-\mu_j} & -\frac{1}{1-\lambda_i} \cdot \frac{1}{1-\bar{\mu}_j} & \operatorname{diag}\left\{\frac{1}{1-\lambda_i}\right\} & \\ \hline -\frac{1}{1-\bar{\lambda}_i} \cdot \frac{1}{1-\mu_j} & -\frac{1}{1-\bar{\lambda}_i} \cdot \frac{1}{1-\bar{\mu}_j} & & \operatorname{diag}\left\{\frac{1}{1-\bar{\lambda}_i}\right\} \end{array}\right). \tag{5.36b}$$

According to Eq. (5.11a) for $\lambda = 0$, the reduction condition (5.34a) has the following infinitesimal form:

$$\begin{aligned} -At + t\bar{A} &= 0, \\ -Bt + t\bar{B} &= 0, \end{aligned} \tag{5.37}$$

and, from Table 2, the matrix $\tilde{L}$ has the form

$$\tilde{L} = \Lambda \otimes t, \tag{5.38}$$

where

$$\Lambda = \left(\begin{array}{cc|cc} 0 & \mathbb{1} & & \\ \mathbb{1} & 0 & & \\ \hline & & 0 & \mathbb{1} \\ & & \mathbb{1} & 0 \end{array}\right). \tag{5.39}$$

Equation (5.35) therefore reduces to the commutation relations:

$$[\mathscr{A}, \Lambda] = [\mathscr{B}, \Lambda] = 0, \tag{5.40}$$

which are easily verified to hold.

Case (B). The appropriate condition is (5.33a) or (5.33b) which, differentiating with respect to ξ and η, is equivalent to

$$(\mathscr{A}^T \otimes A^T)s + s_\xi + s(\mathscr{A} \otimes A) = 0, \tag{5.41a}$$

$$(\mathscr{B}^T \otimes B^T)s + s_\eta + s(\mathscr{B} \otimes A) = 0 \tag{5.41b}$$

(or the same relation with s replaced by a) where, according to Eq. (4.7)

$$\mathscr{A} = \left(\begin{array}{c|c} \operatorname{diag}\left\{\dfrac{1}{1+\dfrac{1}{\lambda_i}}\right\} & \\ \hline \dfrac{1}{1+\lambda_i} \cdot \dfrac{1}{1+\dfrac{1}{\lambda_j}} & \operatorname{diag}\left\{\dfrac{1}{1+\lambda_i}\right\} \end{array}\right),$$
$$\mathscr{B} = \left(\begin{array}{c|c} \operatorname{diag}\left\{\dfrac{1}{1-\dfrac{1}{\lambda_i}}\right\} & \\ \hline -\dfrac{1}{1-\lambda_i} \cdot \dfrac{1}{1-\dfrac{1}{\lambda_j}} & \operatorname{diag}\left\{\dfrac{1}{1-\lambda_i}\right\} \end{array}\right), \tag{5.42}$$

and, according to Table 2:

$$s = \Sigma \otimes (gt)^{-1}, \tag{5.43}$$

where

$$\Sigma = \left(\begin{array}{c|c} & \operatorname{diag}\{\lambda_i\} \\ \hline \operatorname{diag}\{-\lambda_i\} & \end{array}\right). \tag{5.44}$$

The reduction (5.34b) has the infinitesimal form given by Eq. (5.11b) at $\lambda = 0$:

$$A^T(gt)^{-1} + (gt)^{-1}A = 0, \qquad B^T(gt)^{-1} + (gt)^{-1}B = 0. \tag{5.45}$$

The matrices $\mathscr{A}$ and $\mathscr{B}$ do not commute with Σ but satisfy the following relation:

$$\mathscr{A}^T\Sigma - \Sigma + \Sigma\mathscr{A} = 0, \qquad \mathscr{B}^T\Sigma - \Sigma + \Sigma\mathscr{B} = 0, \tag{5.46}$$

as a direct computation shows. The left-hand sides of Eq. (5.41a) and (5.41b) are thus:

$$\begin{aligned}(\mathscr{A}^T \otimes A^T)s + s_\xi + s(\mathscr{A} \otimes A) &= \mathscr{A}^T\Sigma \otimes A^T(gt)^{-1} - \Sigma \otimes (gt)^{-1}A + \Sigma\mathscr{A} \otimes (gt)^{-1}A \\ &= (\mathscr{A}^T\Sigma - \Sigma + \Sigma\mathscr{A}) \otimes (gt)^{-1}A \\ &= 0, \\ (\mathscr{B}^T \otimes B^T)s + s_\xi + s(\mathscr{B} \otimes B) &= \mathscr{B}^T\Sigma \otimes B^T(gt)^{-1} - \Sigma \otimes (gt)^{-1}B + \Sigma\mathscr{B} \otimes (gt)^{-1}B \\ &= (\mathscr{B}^T\Sigma - \Sigma + \Sigma\mathscr{B}) \otimes (gt)^{-1}B \\ &= 0,\end{aligned}$$

which concludes the proof for Case (B).

The content of Theorems 5.1 and 5.2 may be re-expressed in terms of the data determining the residues Q_i and R_i of the ZMS dressing matrix $\chi(\lambda)$ and its inverse according to Theorem 4.2. These are: the rectangular matrices $\{F_i, H_i\}$ determined from their initial values $\{f_i, h_i\}$ by Eq. (4.51) and, in the case of a degeneracy $\lambda_i = \mu_i$, the diagonal block Γ_{ii} determined from its initial value $\{f_i^+ c_i h_i\}$ by Eq. (4.53b). The last column of Table 2 gives the constraints on F_i, H_i, and Γ_{ij}, which are equivalent to the constraints of Theorem 5.1. The equalities between the different rectangular matrices $\{F_i, H_i\}$ are to be understood as equalities between the subspaces that the columns of each matrix span. For example, the relation

$$[gt\bar{F}_i] = [H_i] \tag{5.47}$$

should be interpreted as:

$$gt\bar{F}_i = H_i A_i \tag{5.47'}$$

for some constant $A_i \in GL(r_i, \mathbb{C})$, $r_i = rkH_i$. The constraints on Γ_{ij} should be understood accordingly, in view of Eq. (4.53a) and (4.53b). These are redundant for $\lambda_i \neq \mu_j$, since they are reduced from Eq. (4.53a), but for $\lambda_i = \mu_i$, since the initial value of Γ_{ii} is undetermined up to the additive factor $\{f_i^+ c_i h_i\}$, the constraint must be added to those on $\{F_i, H_i\}$ together with the orthogonality relation:

$$F_i^+ H_i = 0, \tag{5.48}$$

which is not explicitly listed in Table 2 unless the minimal set of poles implies such a degeneracy. The number ε_t is

$$\begin{aligned} &\varepsilon_t = +1, \quad \text{if} \quad t = \mathbb{1}, I_{p,q} \quad \text{or} \quad K_{p,q}, \\ &\varepsilon_t = -1, \quad \text{if} \quad t = J_n. \end{aligned} \tag{5.49}$$

Theorem 5.3. *The constraints on the matrix M defined in Theorem 5.1 are equivalent to the constraints on the matrices $\{F_i, H_i\}$ and Γ_{ij} given in Table 2. The latter are valid at all values (ξ, η) if they hold at the initial data point.*

Proof. Although the explicit constraints on M and on F_i, H_i, and Γ_{ij} are different in each of the eight cases, the proofs are identical, therefore we shall only treat one case; namely, the involution

$$\sigma_-(g) = tg^{+-1}t^{-1} = g^{-1}. \tag{5.50}$$

Denoting by $M_{\hat{i}j}$ the block corresponding to the eigenvalues $\left\{\mu_i = \frac{1}{\bar{\lambda}_i}, \lambda_j\right\}$, the constraint (5.18b) becomes:

$$M_{\hat{i}j} = \frac{\lambda_j}{\bar{\lambda}_i}(gt) M_{\hat{j}i}^+ (gt)^{-1}. \tag{5.51}$$

According to the Eq. (4.59), $M_{\hat{i}j}$ has the form:

$$M_{\hat{i}j} = H_{\hat{i}} \gamma_{\hat{i}j} F_j^+, \tag{5.52}$$

where

$$\gamma = \Gamma^{-1}, \tag{5.53}$$

and therefore

$$H_{\hat{i}}(\gamma_{\hat{i}j}F_j^+) = \frac{\lambda_j}{\bar{\lambda}_i}(gtF_i)(\gamma_{\hat{j}i}^+ H_{\hat{j}}^+(gt)^{-1}). \tag{5.54}$$

Since F_j is of maximal rank, we have

$$H_{\hat{i}}\gamma_{\hat{i}j} = \frac{\lambda_j}{\bar{\lambda}_i} gtF_i\gamma_{\hat{j}i}^+ H_{\hat{j}}^+(gt)^{-1}(F_j^+F_j)^{-1}. \tag{5.55}$$

Multiplying on the right by Γ_{ji} and summing over j gives

$$H_{\hat{i}} = gtF_i\Lambda_i \tag{5.56}$$

for some matrix Λ_i. A similar process involving left multiplication gives

$$H_{\hat{i}}\tilde{\Lambda}_i = gtF_i, \tag{5.56'}$$

and therefore F_i, $H_{\hat{i}}$ are of the same rank and span the same space:

$$[H_{\hat{i}}] = [gtF_i]. \tag{5.57}$$

To prove that Λ_i in Eq. (5.56) may be assumed constant, note that $H_{\hat{i}}$ and F_i are determined from their initial values by

$$\begin{aligned} H_{\hat{i}} &= \psi\left(\frac{1}{\bar{\lambda}_i}\right)h_{\hat{i}}, \\ F_i &= \psi^{+-1}(\lambda_i)f_i, \end{aligned} \tag{5.58}$$

where, by Eq. (5.9)

$$gt\psi^{+-1}(\lambda_i) = \psi\left(\frac{1}{\bar{\lambda}_i}\right)t, \tag{5.59}$$

and hence

$$h_{\hat{i}} = tf_i\Lambda_i. \tag{5.60}$$

Since $h_{\hat{i}}$, f_i are constants, Λ_i may be taken as constant. From (5.56) we can deduce the constraints satisfied by Γ_{ij}, provided $\mu_i \neq \lambda_j$, since by the definition (4.53a) of Γ_{ij}, we have:

$$\Gamma_{i\hat{j}} = \frac{F_i^+H_{\hat{j}}}{\lambda_i - \dfrac{1}{\bar{\lambda}_j}} = \frac{F_i^+gtF_j\Lambda_j}{\lambda_i - \dfrac{1}{\bar{\lambda}_j}}, \tag{5.61}$$

and therefore, in view of Eq. (5.56)

$$\Gamma_{i\hat{j}} = \pm\Lambda_i^{+-1}\Gamma_{j\hat{i}}^+\Lambda_j\frac{\bar{\lambda}_j}{\lambda_i} \tag{5.62}$$

(with + if t is hermitian, − if t is anti-hermitian). This derivation is not valid, however, if $\lambda_i = 1/\bar{\lambda}_j$. To prove Eq. (5.62) in general, we again use Eq. (5.52), from which follows

$$\gamma_{\hat{i}j} = (H_{\hat{i}}^+H_{\hat{i}})^{-1}H_{\hat{i}}^+M_{\hat{i}j}F_j(F_j^+F_j)^{-1}, \tag{5.63}$$

since $H_{\hat{i}}$, F_j are of maximal rank. From (5.51), (5.52), and (5.56) this implies

$$\gamma_{\hat{i}j} = \pm \frac{\lambda_j}{\bar{\lambda}_i} \Lambda_i^{-1} \gamma_{\hat{j}i}^{+} \Lambda_j^{+}, \tag{5.64}$$

which is equivalent to Eq. (5.62) for all eigenvalues, whether $\lambda_i = \frac{1}{\bar{\lambda}_j}$ or not.

Conversely, suppose Eqs. (5.56) and (5.62) and hence also (5.64) are satisfied. From Eq. (5.52) we have:

$$\begin{aligned}\frac{\lambda_j}{\bar{\lambda}_i}(gt)\, M_{\hat{j}i}^{+}(gt)^{-1} &= \frac{\lambda_j}{\bar{\lambda}_i} gtF_i \gamma_{\hat{j}i}^{+} H_{\hat{j}}^{+}(gt)^{-1} = \pm gtF_i \Lambda_i \gamma_{\hat{i}j} \Lambda_j^{+\,-1} H_{\hat{j}}^{+}(gt)^{-1} \\ &= H_{\hat{i}} \gamma_{\hat{i}j} F_j^{+} = M_{\hat{i}j},\end{aligned}$$

proving the constraint (5.51).

Finally, to illustrate how the data in Tables 1 and 2 combine to determine the reductions corresponding to all Riemannian symmetric spaces, we shall give the full details for two illustrative examples:

(i) real Grassman manifolds $G_p(\mathbb{R}^{p+q}) = \mathrm{SO}(p+q)/\mathrm{SO}(p) \times \mathrm{SO}(q)$ and

(ii) the space $\mathrm{SO}(n,\mathbb{C})/\mathrm{SO}(n)$, which is the non-compact analogue of the principal $\mathrm{SO}(n)$ sigma model.

Example (i). $\mathrm{SO}(p+q)/\mathrm{SO}(p) \times \mathrm{SO}(q)$.

From Table 1, the minimal set of eigenvalues when $|\lambda_i| \neq 1$ involves all four terms $\left\{\lambda_i, \bar{\lambda}_i, \frac{1}{\lambda_i}, \frac{1}{\bar{\lambda}_i}\right\}$, and hence the dressing matrix and its inverse are of the form:

$$\begin{aligned}\chi(\lambda) &= \mathbb{1} + \sum_{i=1}^{l} \left\{ \frac{Q_i}{\lambda - \lambda_i} + \frac{Q_{\bar{i}}}{\lambda - \bar{\lambda}_i} + \frac{Q_{\hat{i}}}{\lambda - \frac{1}{\lambda_i}} + \frac{Q_{\hat{\bar{i}}}}{\lambda - \frac{1}{\bar{\lambda}_i}} \right\}, \\ \chi^{-1}(\lambda) &= \mathbb{1} + \sum_{i=1}^{l} \left\{ \frac{R_i}{\lambda - \lambda_i} + \frac{R_{\bar{i}}}{\lambda - \bar{\lambda}_i} + \frac{R_{\hat{i}}}{\lambda - \frac{1}{\lambda_i}} + \frac{R_{\hat{\bar{i}}}}{\lambda - \frac{1}{\bar{\lambda}_i}} \right\}.\end{aligned} \tag{5.65}$$

Ordering the $ln \times ln$ dimensional blocks in the M-matrix as in Eq. (5.30), the three involutions involved in this reduction give rise to the following constraints according to Table 2:

$\sigma_+(g) = g^{+\,-1} = g$

$$[M^+\ \mathbb{1}] \left(\begin{array}{cc|cc|cc|cc} & & & & & \mathbb{1} & & \\ & & & & \mathbb{1} & & & \\ \hline & & & & & & & \mathbb{1} \\ & & & & & & \mathbb{1} & \\ \hline & \mathbb{1} & & & & & & \\ \mathbb{1} & & & & & & & \\ \hline & & & \mathbb{1} & & & & \\ & & \mathbb{1} & & & & & \end{array}\right) \begin{bmatrix} M \\ \mathbb{1} \end{bmatrix} = 0, \tag{5.66}$$

or, equivalently,

$$
\begin{aligned}
F_{i} &= H_{\bar{\imath}}, & F_{\bar{\imath}} &= H_{i}, \\
F_{\hat{\imath}} &= H_{\hat{\bar{\imath}}}, & F_{\hat{\bar{\imath}}} &= H_{\hat{\imath}}, \\
\Gamma_{ii} &= -\Gamma^{+}_{\bar{\imath}\bar{\imath}}, & \Gamma_{\hat{\imath}\hat{\imath}} &= -\Gamma^{+}_{\hat{\bar{\imath}}\hat{\bar{\imath}}},
\end{aligned}
\tag{5.67}
$$

(up to a change of basis in the spaces spanned by F_i, $F_{\bar{\imath}}$, $F_{\hat{\imath}}$, $F_{\hat{\bar{\imath}}}$, H_i, $H_{\bar{\imath}}$, $H_{\hat{\imath}}$, $H_{\hat{\bar{\imath}}}$) together with the orthogonality conditions:

$$
F_{i}^{+} H_{i} = F_{\bar{\imath}}^{+} H_{\bar{\imath}} = F_{\hat{\imath}}^{+} H_{\hat{\imath}} = F_{\hat{\bar{\imath}}}^{+} H_{\hat{\bar{\imath}}} = 0 . \tag{5.68}
$$

$\sigma_{+}(g) = \bar{g} = g$

$$
\left(\begin{array}{cc:cc:cc:cc}
 & \mathbb{1} & & & & & & \\
\mathbb{1} & & & & & & & \\
\hdashline
 & & & \mathbb{1} & & & & \\
 & & \mathbb{1} & & & & & \\
\hdashline
 & & & & & \mathbb{1} & & \\
 & & & & \mathbb{1} & & & \\
\hdashline
 & & & & & & & \mathbb{1} \\
 & & & & & & \mathbb{1} &
\end{array}\right)
\begin{bmatrix} \bar{M} \\ \mathbb{1} \end{bmatrix} = \begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}, \tag{5.69}
$$

or, equivalently

$$
\begin{aligned}
F_{i} &= \bar{F}_{\bar{\imath}}, & H_{i} &= \bar{H}_{\bar{\imath}}, \\
F_{\hat{\imath}} &= \bar{F}_{\hat{\bar{\imath}}}, & H_{\hat{\imath}} &= \bar{H}_{\hat{\bar{\imath}}}, \\
\Gamma_{ii} &= \bar{\Gamma}_{\bar{\imath}\bar{\imath}}, & \Gamma_{\hat{\imath}\hat{\imath}} &= \bar{\Gamma}_{\hat{\bar{\imath}}\hat{\bar{\imath}}}.
\end{aligned}
\tag{5.70}
$$

$\sigma_{-}(g) = I_{pq} g I_{pq} = g^{-1}$

$$
\left(\begin{array}{cc:cc:cc:cc}
 & & \{\lambda_i\} & & & & & \\
 & & & \{\bar{\lambda}_i\} & & & & \\
\hdashline
\{\lambda_i^{-1}\} & & & & & & & \\
 & \{\bar{\lambda}_i^{-1}\} & & & & & & \\
\hdashline
 & & & & & & \{-\lambda_i^{-1}\} & \\
 & & & & & & & \{-\bar{\lambda}_i^{-1}\} \\
\hdashline
 & & & & \{-\lambda_i\} & & & \\
 & & & & & \{-\bar{\lambda}_i\} & &
\end{array}\right)
$$

$$
\otimes g I_{pq} \begin{bmatrix} M \\ \mathbb{1} \end{bmatrix} = \begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}, \tag{5.71}
$$

or, equivalently,

$$
\begin{aligned}
g I_{pq} F_{i} &= F_{\hat{\imath}}, & g I_{pq} H_{i} &= H_{\hat{\imath}}, \\
g I_{pq} F_{\bar{\imath}} &= F_{\hat{\bar{\imath}}}, & g I_{pq} H_{\bar{\imath}} &= H_{\hat{\bar{\imath}}}, \\
\Gamma_{ii} &= -\frac{\Gamma_{\hat{\imath}\hat{\imath}}}{\lambda_i^{2}}, & \Gamma_{\bar{\imath}\bar{\imath}} &= -\frac{\Gamma_{\hat{\bar{\imath}}\hat{\bar{\imath}}}}{\bar{\lambda}_i^{2}}.
\end{aligned}
\tag{5.72}
$$

The conditions (5.67), (5.68), (5.70), and (5.72) reduce to the two constraints:

$$F_i^T F_i = 0 \quad \text{and} \quad \Gamma_{ii} = -\Gamma_{ii}^T, \tag{5.73}$$

with all other terms determined by:

$$\begin{aligned} F_i = \bar{F}_{\bar{i}} = gI_{pq} F_{\bar{i}} = gI_{pq} \bar{F}_{\hat{i}} = \bar{H}_i = H_{\bar{i}} = gI_{pq} \bar{H}_{\bar{i}} = gI_{pq} H_{\hat{i}}, \\ \Gamma_{ii} = \bar{\Gamma}_{\bar{i}\bar{i}} = -\frac{\Gamma_{\bar{i}i}}{\lambda_i^2} = -\frac{\bar{\Gamma}_{\hat{i}\bar{i}}}{\lambda_i^2}. \end{aligned} \tag{5.74}$$

Assume now that all eigenvalues λ_i are on the unit circle: $|\lambda_i| = 1$. In this case, the minimal set of poles consists of $\{\lambda_i, \bar{\lambda}_i\}$ and the dressing matrix reduces to:

$$\begin{aligned} \chi(\lambda) &= \mathbb{1} + \sum_{i=1}^{l} \left\{ \frac{Q_i}{\lambda - \lambda_i} + \frac{Q_{\bar{i}}}{\lambda - \bar{\lambda}_i} \right\}, \\ \chi^{-1}(\lambda) &= \mathbb{1} + \sum_{i=1}^{l} \left\{ \frac{R_i}{\lambda - \lambda_i} + \frac{R_{\bar{i}}}{\lambda - \bar{\lambda}_i} \right\}, \\ K &= 2l. \end{aligned} \tag{5.75}$$

The M matrix now consists of only four $ln \times ln$ dimensional blocks:

$$M = \begin{pmatrix} M_{ij} & M_{i\bar{j}} \\ M_{\bar{i}j} & M_{\bar{i}\bar{j}} \end{pmatrix}, \tag{5.76}$$

and the constraints (5.66), (5.69), and (5.71) reduce to:

$$[M^+ \mathbb{1}] \left(\begin{array}{cc|cc} & & & \mathbb{1} \\ & & \mathbb{1} & \\ \hline & \mathbb{1} & & \\ \mathbb{1} & & & \end{array} \right) \begin{bmatrix} M \\ \mathbb{1} \end{bmatrix} = 0, \tag{5.66'}$$

$$\left(\begin{array}{cc|cc} & \mathbb{1} & & \\ \mathbb{1} & & & \\ \hline & & & \mathbb{1} \\ & & \mathbb{1} & \end{array} \right) \begin{bmatrix} \bar{M} \\ \mathbb{1} \end{bmatrix} = \begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}, \tag{5.69'}$$

and

$$\left(\begin{array}{c|c|c|c} & \{\lambda_i\} & & \\ \hline \{\bar{\lambda}_i\} & & & \\ \hline & & & \{-\bar{\lambda}_i\} \\ \hline & & \{-\lambda_i\} & \end{array} \right) \otimes gI_{pq} \begin{bmatrix} M \\ \mathbb{1} \end{bmatrix} = \begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}. \tag{5.71'}$$

Equivalently the constraints (5.73) and (5.74) reduce to the relations:

$$\begin{aligned} F_i^T F_i = 0, \quad \bar{F}_i = gI_{pq} F_i, \\ \Gamma_{ii} = -\Gamma_{ii}^T, \quad \Gamma_{ii} = -\frac{\bar{\Gamma}_{ii}}{\lambda_i^2}, \end{aligned} \tag{5.73'}$$

with the other terms determined by:

$$F_i = \bar{F}_{\bar{i}} = \bar{H}_i = H_{\bar{i}}, \qquad \Gamma_{ii} = \Gamma_{\bar{i}\bar{i}}. \tag{5.74'}$$

The particular case of the above with $\Gamma_{ii} = \Gamma_{\bar{i}\bar{i}} = 0$ and F_i, H_i of rank one was reported previously in [7] and reduces for the case $S^p = SO(p+1)/SO(p)$ to the Bäcklund transformation of Pohlmeyer [12]. A different type of reduction for this case was also given in [3]. The general case treated here has not previously been analyzed.

Example (ii). $SO(n, \mathbb{C})/SO(n)$.

According to Table 1, the minimal set of eigenvalues for $|\lambda_i| \neq 1$ is $\{\lambda_i, 1/\bar{\lambda}_i\}$ and the dressing matrix is thus of the form:

$$\begin{aligned}\chi(\lambda) &= \mathbb{1} + \sum_{i=1}^{l} \left\{ \frac{Q_i}{\lambda - \lambda_i} + \frac{Q_{\hat{i}}}{\lambda - \frac{1}{\bar{\lambda}_i}} \right\}, \\ \chi^{-1}(\lambda) &= \mathbb{1} + \sum_{i=1}^{l} \left\{ \frac{R_i}{\lambda - \lambda_i} + \frac{R_{\hat{i}}}{\lambda - \frac{1}{\bar{\lambda}_i}} \right\},\end{aligned} \tag{5.77}$$

with $K = 2l$.

The M-matrix consists of four $ln \times ln$ dimensional blocks:

$$M = \begin{pmatrix} M_{ij} & M_{i\hat{j}} \\ M_{\hat{i}j} & M_{\hat{i}\hat{j}} \end{pmatrix}. \tag{5.78}$$

There are, from Table 1, two involutions giving rise, according to Table 2, to the following constraints:

$\sigma_+(g) = g^{T-1} = g$

$$[M^T \mathbb{1}] \left(\begin{array}{cc|cc} & & \mathbb{1} & 0 \\ & & 0 & \mathbb{1} \\ \hline \mathbb{1} & 0 & & \\ 0 & \mathbb{1} & & \end{array} \right) \begin{bmatrix} M \\ \mathbb{1} \end{bmatrix} = 0, \tag{5.79}$$

or equivalently

$$\begin{aligned} \bar{F}_i &= H_i, \qquad \bar{F}_{\hat{i}} = H_{\hat{i}}, \\ \Gamma_{ii} &= -\Gamma_{ii}^T, \qquad \Gamma_{\hat{i}\hat{i}} = -\Gamma_{\hat{i}\hat{i}}^T, \end{aligned} \tag{5.80}$$

together with the orthogonality conditions:

$$F_i^+ H_i = 0 \qquad F_{\hat{i}}^+ H_{\hat{i}} = 0. \tag{5.81}$$

$\sigma_-(g) = \bar{g} = g^{-1}$

$$\left(\begin{array}{cc|cc} & \{\lambda_i\} & & \\ \{\bar{\lambda}_i^{-1}\} & & & \\ \hline & & & \{-\lambda_i^{-1}\} \\ & & \{-\bar{\lambda}_i\} & \end{array} \right) \otimes g \begin{bmatrix} \bar{M} \\ \mathbb{1} \end{bmatrix} = \begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}, \tag{5.82}$$

or equivalent:

$$\begin{aligned} &\bar{g}\bar{F}_i = F_{\hat{\imath}}, \qquad g\bar{H}_i = H_{\hat{\imath}}, \\ &\Gamma_{ii} = -\frac{\bar{\Gamma}_{\hat{\imath}\hat{\imath}}}{\lambda_i^2}. \end{aligned} \tag{5.83}$$

The conditions (5.80), (5.81), and (5.83) reduce to the two constraints

$$F_i^+ F_i = 0 \quad \text{and} \quad \Gamma_{ii} = -\Gamma_{ii}^T, \tag{5.84}$$

with all other terms determined by:

$$F_i = \bar{H}_i = \bar{g}\bar{F}_{\hat{\imath}} = \bar{g}H_{\hat{\imath}}, \qquad \Gamma_{\hat{\imath}\hat{\imath}} = -\lambda_i^2 \Gamma_{ii}. \tag{5.85}$$

Again, if $|\lambda_i| = 1$, the minimal set of poles is reduced to just $\{\lambda_i\}$ and the dressing matrix is of the form:

$$\begin{aligned} \chi(\lambda) &= \mathbb{1} + \sum_{i=1}^{l} \frac{Q_i}{\lambda - \lambda_i}, \\ \chi^{-1}(\lambda) &= \mathbb{1} + \sum_{i=1}^{l} \frac{R_i}{\lambda - \lambda_i}, \end{aligned} \tag{5.86}$$

with $K = l$.

The M-matrix consists of just one $ln \times ln$ dimensional block (M_{ij}) and the constraints (5.79) and (5.82) become:

$$[M^T \mathbb{1}] \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{bmatrix} M \\ \mathbb{1} \end{bmatrix} = 0, \tag{5.79'}$$

and

$$\left(\begin{array}{c|c} \{\lambda_i\} & \\ \hline & \{-\bar{\lambda}_i\} \end{array} \right) \otimes g \begin{bmatrix} \bar{M} \\ \mathbb{1} \end{bmatrix} = \begin{bmatrix} M \\ \mathbb{1} \end{bmatrix}. \tag{5.82'}$$

Equivalently, the relations (5.84) and (5.85) reduce to:

$$F_i^+ F_i = 0, \qquad F_i = \bar{g}\bar{F}_i, \qquad \Gamma_{ii} = 0, \tag{5.84'}$$

with

$$H_i = \bar{F}_i. \tag{5.85'}$$

All other cases listed in Table 1 may be similarly treated using the data defining the reductions in Table 2.

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