Conservation-law violation at high energy by anomalies

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The time evolution of a quantum Fermi field is investigated in the background of a Minkowski-space, Yang-Mills field configuration with nonvanishing topological charge. The Fermi system is assumed to possess a current $j_\mu(x)$ conserved up to an axial-vector anomaly: $\partial^\mu j_\mu = (g^2/32\pi^2)N_{ij}F^i_{\mu\nu}\tilde{F}^{j\mu\nu}$. It is shown explicitly that the time-dependent Yang-Mills field $A_\mu(\vec{x},t)$ creates and destroys fermions in such a way that the total fermionic charge $\int j_0(\vec{x},t)d^3x$ present in the final state differs from that in the initial state by precisely the amount predicted by the anomaly equation. If $A_\mu(\vec{x},t)$ approaches a gauge transformation sufficiently rapidly for large t, this change in charge can be identified with the number of zero crossings present in the energy spectrum of the time-dependent Dirac Hamiltonian. Finally, it is demonstrated that the change in the charge carried by the fermions will differ from that predicted by the axial-vector anomaly if the large-time limit of A_μ contains physical radiation.

I. INTRODUCTION

It has been known for some time that in gauge theories containing axial-vector currents the usual Ward identities may contain new anomalous terms.[1] More recently, 't Hooft[2] observed that gauge field configurations with nonvanishing topological charge can actually cause explicit violation of the conservation law corresponding to an anomalous Ward identity. As a striking example, 't Hooft considered the baryon-number current $j_\mu(x)$ which, in the conventional Weinberg-Salam model, contains such an anomaly[3]

$$\partial^\mu j_\mu = \frac{1}{32\pi^2}\frac{N_B}{3}(-g^2F^i_{\mu\nu}\tilde{F}^{i\mu\nu} + g'^2F'_{\mu\nu}\tilde{F}'^{\mu\nu}). \qquad (1.1)$$

Here N_B is the number of left-handed SU(2) baryon doublets, $F^i_{\mu\nu}$, $1 \leq i \leq 3$, and $F'_{\mu\nu}$ are the four weak SU(2)×U(1) field strengths and $\tilde{F}^i_{\mu\nu}$, $\tilde{F}'_{\mu\nu}$ are their duals, e.g., $\tilde{F}^i_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{i\rho\sigma}$. Thus one might expect a violation of $(N_B/3)\,k$ units of baryon number associated with a weak Yang-Mills field configuration with topological charge:

$$k = \frac{g^2}{32\pi^2}\int d^4x\, F^i_{\mu\nu}\tilde{F}^i_{\mu\nu}. \qquad (1.2)$$

In the papers mentioned above 't Hooft outlines a calculation of deuteron decay through semiclassical "barrier" penetration using Euclidean, instanton field configurations. The necessary neglect of the Higgs couplings requires that these configurations exist for very brief periods of time.

In a Euclidean-space calculation of that sort, the connection between the change in total baryon number and the topological charge k required by Eq. (1.1) is realized as a consequence of the Atiyah-Singer index theorem: The change in baryon number is directly related to the number of solutions to the Dirac equation obeyed by the weakly interacting Fermi fields in a background Yang-Mills field. The number of these solutions is determined by the topological charge of the Yang-Mills field using the Atiyah-Singer theorem.[4]

Let us now consider the complementary physical situation—a collision process of sufficiently high energy that the masses of the Yang-Mills quanta can be neglected. Under these conditions a semiclassical calculation of baryon-number violation should be performed in Minkowski space and describes passage over the barrier referred to above. This Minkowski-space calculation may be separated into two parts: first, the creation of a classical Yang-Mills field by the collision of very energetic particles with weak interactions (this radiation field dissipates with the passage of time); second, the creation of weakly interacting fermions by this background Yang-Mills field.

In this paper we study the second part of this Minkowski-space calculation, the creation of fermions by a time-dependent background Yang-Mills field. We begin in Sec. II by determining the time evolution of the second-quantized fermion vacuum in terms of the S matrix obtained from scattering solutions of the Dirac equation in the presence of the time-dependent Yang-Mills field. In particular, if we divide the scattering matrix S into four blocks corresponding to transitions between asymptotic eigenstates of initially and finally positive or negative energy,

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad (1.3)$$

then at $t = +\infty$ the initial vacuum state becomes a superposition of particle and antiparticle states with a net charge that can be determined explicitly:

$$\Delta Q = \mathrm{tr}(P_{\ker A}Q) - \mathrm{tr}(P_{\ker A^\dagger}Q)$$
$$= \mathrm{tr}(P_{\ker D^\dagger}Q) - \mathrm{tr}(_{\ker D}Q). \quad (1.4)$$

Here

$$Q = \int j_0(\vec{x}, t)d^3x \quad (1.5)$$

can be any charge conserved by the original Dirac equation while $P_{\ker O}$ is the projection operator onto the kernel of the operator O.

In Sec. III we show that if the quantum-mechanical current operator $j_\mu(x)$ corresponding to the charge Q obeys an anomalous conservation law

$$\partial^\mu j_\mu(x) = N_{ij}\frac{g^2}{32\pi^2}F^i_{\mu\nu}\tilde{F}^{j\mu\nu} \quad (1.6)$$

and if $A_\mu(\vec{x}, t)$ becomes a pure gauge transformation for sufficiently large t then the elements of S obey the further constraint

$$\mathcal{Q} \equiv \frac{g^2}{32\pi^2}N_{ij}\int d^4x F^i_{\mu\nu}\tilde{F}^{j\mu\nu}$$
$$= \mathrm{tr}(QC^\dagger C) - \mathrm{tr}(QB^\dagger B). \quad (1.7)$$

It is then demonstrated, using the unitarity of S, that the right-hand sides of Eqs. (1.4) and (1.7) are equal so that the total charge of the fermions produced is precisely that expected from the anomalous conservation law.

Furthermore, the quantity $\mathrm{tr}(P_{\ker A}Q) - \mathrm{tr}(P_{\ker A^\dagger}Q)$ on the right-hand side of Eq. (1.4) is shown to have a simple connection with the time-dependent enerby spectrum of the fermion Hamiltonian $H(t)$. Consider only those eigenstates of $H(t)$ with charge q_l and define n_l as the number of times the corresponding energy eigenvalues cross zero from below minus the number of zero crossings from above. Then, as is demonstrated in Sec. III,

$$\mathcal{Q} = \Delta Q = \mathrm{tr}(P_{\ker A}Q) - \mathrm{tr}(P_{\ker A^\dagger}Q)$$
$$= \sum_l q_l n_l \,. \quad (1.8)$$

This relationship between the topological charge and the "spectral flow" of $H(t)$ was recognized previously in the adiabatic approximation by Callan, Dashen, and Gross,[5] and in fact is a direct consequence of a generalization of the Atiyah-Singer index theorem by Atiyah, Patodi, and Singer.[6] The relationship between this mathematical result and the Minkowski-space conservation-law violations considered here is in close analogy with the role of the Atiyah-Singer theorem in the original Euclidean-space tunneling analysis mentioned above.

Finally, in Sec. IV we briefly discuss the creation of the background Yang-Mills field and then turn to the application of the previous analysis in the situation where physical radiation is present. In that case, the slowly dissipating radiation field invalidates Eq. (1.7) and shifts the charge of the final vacuum state by an amount

$$q^{\text{out}} = \frac{g^2}{32\pi^2}N_{ij}\int d^3x\,\epsilon_{0\mu\nu\rho}A^{i\mu}(\partial^\nu A^{j\rho} + \tfrac{1}{3}gf^{jkl}A^{k\nu}A^{l\rho}), \quad (1.9)$$

where f^{jkl} are the structure constants of the gauge group. This equation for q^{out} is valid in any gauge for which the Yang-Mills field $A_\mu(\vec{x}, t)$ approaches zero for large time. In the familiar case of ordinary Abelian electromagnetism where topological charge can be easily produced by the interference of electric and magnetic radiation, this final vacuum charge accounts for the total anomalous charge creation. There is no anomalous production of particles by an Abelian radiation field. In general, this will not be true in the non-Abelian case.

II. FERMION CREATION IN A BACKGROUND FIELD

Let us consider a fermion field operator $\psi(x)$ in the Heisenberg representation which obeys the equation of motion

$$i\gamma^\mu(\partial_\mu - igA^i_\mu T^i)\psi = 0, \quad (2.1)$$

where $A^i_\mu(x)$ is a specified classical Yang-Mills field and T^i are the Hermitian group generators for the representation to which $\psi(x)$ belongs. In this section we will compute explicitly the particle creation implied by Eq. (2.1) in terms of the solutions to the unquantized version of that equation.

A. Description of the classical solutions

For simplicity we will treat the three spatial variables, $\vec{x}$, as lying within a large box on whose boundaries the fields A^i_μ and ψ are required to be periodic (up to a gauge transformation). The time variable t of course runs from minus to plus infinity. In addition, the gauge field $A_\mu(x)$ is assumed to reduce to a time-independent gauge transformation for $t \to \pm\infty$:

$$\lim_{t\to+\infty\;(-\infty)} A^i_\mu(\vec{x}, t)T^i = \frac{i}{g}R^{\text{out(in)}}(\vec{x})\partial_\mu R^{\text{out(in)}}(\vec{x})^{-1}. \quad (2.2)$$

Thus, we can define two complete orthonormal sets of functions $\psi^{\text{in}\pm}_n(\vec{x}, t)$ and $\psi^{\text{out}\pm}_n(\vec{x}, t)$ which obey

$$i\gamma^\mu(\partial_\mu - igA^i_\mu T^i)\psi^{\text{out(in)}\pm}_n(x) = 0 \quad (2.3)$$

and

$$\lim_{t\to-\infty} \psi^{\text{in}\pm}_n(\vec{x}, t) = R^{\text{in}}(\vec{x})\psi^\pm_n(\vec{x})e^{\mp iE_n t},$$
$$\lim_{t\to+\infty} \psi^{\text{out}\pm}(\vec{x}, t) = R^{\text{out}}(\vec{x})\psi^\pm_n(\vec{x})e^{\mp iE_n t}, \quad (2.4)$$

where the functions $\psi^\pm_n(\vec{x})$ are eigenstates of the free Dirac Hamiltonian $\vec{\alpha}\cdot(-i\vec{\nabla})$ with positive- or negative-energy eigenvalue $\pm E_n$. We will include with the eigenfunctions ψ_n^+ those with zero energy. Next, define a scattering matrix S for the Dirac Eq. (2.1) from the overlap of the solutions $\psi_n^{\text{in}\,\pm}$, $\psi_n^{\text{out}\pm}$:

$$S=\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2.5}$$

where the matrices A, B, C, and D are defined by

$$\begin{pmatrix} A_{mn} & B_{mn} \\ C_{mn} & D_{mn} \end{pmatrix} = \begin{pmatrix} \langle \psi_m^{\text{in}+}|\psi_n^{\text{out}+}\rangle & \langle \psi_m^{\text{in}+}|\psi_n^{\text{out}-}\rangle \\ \langle \psi_m^{\text{in}-}|\psi_n^{\text{out}+}\rangle & \langle \psi_m^{\text{in}-}|\psi_n^{\text{out}-}\rangle \end{pmatrix} \tag{2.6}$$

with, for example,

$$\langle \psi_m^{\text{in}-}|\psi_n^{\text{out}+}\rangle = \int d^3x\, \psi_m^{\text{in}-\dagger}(\vec{x},t)\psi_n^{\text{out}+}(\vec{x},t). \tag{2.7}$$

The unitarity of the matrix S implies the following relations between A, B, C, D:

$$AA^\dagger+BB^\dagger=1, \tag{2.8a}$$

$$CA^\dagger+DB^\dagger=0, \tag{2.8b}$$

$$AC^\dagger+BD^\dagger=0, \tag{2.8c}$$

$$CC^\dagger+DD^\dagger=1, \tag{2.8d}$$

$$A^\dagger A+C^\dagger C=1, \tag{2.9a}$$

$$B^\dagger A+D^\dagger C=0, \tag{2.9b}$$

$$A^\dagger B+C^\dagger D=0, \tag{2.9c}$$

$$B^\dagger B+D^\dagger D=1. \tag{2.9d}$$

B. Evolution of quantum-mechanical states

It is now a straightforward matter to express the particle production in the second-quantized theory in terms of the matrix S defined above. First, define the annihilation and creation operators $a_n^{\text{out(in)}}$, $b_n^{\text{out(in)}\dagger}$ by expanding the fermion field operator $\psi(\vec{x},t)$ in terms of the two complete sets of solutions $\psi_n^{\text{out(in)}\pm}$

$$\begin{aligned} \psi(x)&=\sum_n \left[a_n^{\text{in}}\psi_n^{\text{in}+}(\vec{x},t)+b_n^{\text{in}\dagger}\psi_n^{\text{in}-}(\vec{x},t)\right], \\ \psi(x)&=\sum_n \left[a_n^{\text{out}}\psi_n^{\text{out}+}(\vec{x},t)+b_n^{\text{out}\dagger}\psi_n^{\text{out}-}(\vec{x},t)\right]. \end{aligned} \tag{2.10}$$

We can then determine the initial vacuum state $|0^{\text{in}}\rangle$, defined by the requirement

$$a_n^{\text{in}}|0_{\text{in}}\rangle=b_n^{\text{in}}|0^{\text{in}}\rangle=0 \text{ for all } n, \tag{2.11}$$

in terms of out states by using the relation between a_n^{in}, $b_n^{\text{in}\dagger}$, and a_n^{out}, $b_n^{\text{out}\dagger}$ provided by S and the orthonormality for fixed t of each of the sets $\{\psi_n^{\text{out}\pm}\}$ and $\{\psi_n^{\text{in}\ \pm}\}$:

$$\begin{pmatrix} a_m^{\text{in}} \\ b_m^{\text{in}\dagger} \end{pmatrix} = \begin{pmatrix} A_{mn} & B_{mn} \\ C_{mn} & D_{mn} \end{pmatrix} \begin{pmatrix} a_n^{\text{out}} \\ b_n^{\text{out}\dagger} \end{pmatrix}. \tag{2.12}$$

Thus in terms of out operators, the definition (2.11) becomes

$$(A_{mn}a_n^{\text{out}}+B_{mn}b_n^{\text{out}\dagger})|0^{\text{in}}\rangle=0, \tag{2.13a}$$

$$(a_n^{\text{out}\dagger}C_{nm}^\dagger+b_n^{\text{out}}D_{nm}^\dagger)|0^{\text{in}}\rangle=0. \tag{2.13b}$$

These equations can be used to uniquely determine $|0^{\text{in}}\rangle$ in terms of out states. First observe that conjugation with the operator $\exp(a_k^\dagger M_{kl}b_l^\dagger)$ translates a_n:

$$\exp(-a_k^\dagger M_{kl}b_l^\dagger)a_n\exp(+a_k^\dagger M_{kl}b_l^\dagger)=a_n+M_{nl}b_l^\dagger. \tag{2.14}$$

Thus, if we use this operator to define a Bogoliubov-transformed state

$$|\mathcal{B}\rangle=\exp(-a_k^{\text{out}\dagger}M_{kl}b_l^{\text{out}\dagger})|0^{\text{in}}\rangle, \tag{2.15}$$

then Eqs. (2.13) become

$$[A_{mn}a_n^{\text{out}}+(A_{mk}M_{kn}+B_{mn})b_n^{\text{out}\dagger}]|\mathcal{B}\rangle=0, \tag{2.16a}$$

$$[a_n^{\text{out}\dagger}(C_{nm}^\dagger-M_{nl}D_{lm}^\dagger)+b_n^{\text{out}}D_{nm}^\dagger]|\mathcal{B}\rangle=0. \tag{2.16b}$$

We will now try to choose M and $|\mathcal{B}\rangle$ so that each of the two terms on the left-hand sides of Eqs. (2.16a) and (2.16b) vanish separately.

Let us first examine the vanishing of the second term in Eq. (2.16a)

$$(A_{mk}M_{kn}+B_{mn})b_n^{\text{out}\dagger}|\mathcal{B}\rangle=0. \tag{2.17}$$

If we consider linear combinations of this equation for various m which lie in the image of A and require the vanishing of the coefficient of each $b_n^\dagger$, the resulting equation for M,

$$AM+P_{\text{im}\,A}B=0, \tag{2.18}$$

has the solution

$$M=-A^{-1}P_{\text{im}\,A}B. \tag{2.19}$$

Here $P_{\text{im}\,A}$ is the projection operator onto the image of A. Because the operator A may possess some null eigenvectors, the expression (2.19) for M is not well defined. It is actually only the combination $P_{\text{im}\,A^\dagger}M$ which is determined by Eq. (2.18)

$$P_{\text{im}\,A^\dagger}M=-P_{\text{im}\,A^\dagger}A^{-1}P_{\text{im}\,A}B. \tag{2.20}$$

(Recall that the image of $A^\dagger$ is the orthogonal complement of the kernel of A.) Similarly, the vanishing of the first term in Eq. (2.16b)

$$a_n^{\text{out}\dagger}(C_{nm}^\dagger-M_{nl}D_{lm}^\dagger)|\mathcal{B}\rangle=0 \tag{2.21}$$

becomes

$$MP_{\text{im}\,D^\dagger}=C^\dagger P_{\text{im}\,D}(D^\dagger)^{-1}P_{\text{im}\,D^\dagger}, \tag{2.22}$$

where this time we have taken linear combinations of Eq. (2.21) for various values of m orthogonal to the kernel of $D^\dagger$ (i.e., lying in the image of D). Thus Eqs. (2.20) and (2.22) are two constraints which M must obey. They are consistent if

$$P_{\mathrm{im}A^\dagger}[C^\dagger P_{\mathrm{im}D}(D^\dagger)^{-1}P_{\mathrm{im}D^\dagger}] = (-P_{\mathrm{im}A^\dagger}A^{-1}P_{\mathrm{im}A}B)P_{\mathrm{im}D^\dagger} \quad (2.23)$$

and this equation is a direct consequence of the unitarity condition (2.8c). Equations (2.20) and (2.22) do not determine M_{mn} uniquely; it contains undertermined terms of the form

$$s_m t_n^*, \quad (2.24)$$

where s lies in the kernel of A and t in the kernel of D,

$$A_{lm}s_m = D_{kn}t_n = 0. \quad (2.25)$$

However, we have considered only special linear combinations of Eqs. (2.17) and (2.21). To ensure their general validity we must also require

$$\begin{aligned} &u_m^{k*}B_{mn}b_n^{\mathrm{out}\dagger}|\mathcal{B}\rangle = 0, \quad 1\le k\le n_A \\ &a_n^{\mathrm{out}\dagger}C_{nm}^\dagger v_m^l|\mathcal{B}\rangle = 0, \quad 1\le l\le n_D, \end{aligned} \quad (2.26)$$

where the vectors $\{u^k, 1\le k\le n_A\}$ and $\{v^l, 1\le l\le n_D\}$ form a complete orthonormal basis for the kernel of $A^\dagger$ and the kernel of $D^\dagger$. This requirement and the vanishing of the first term in Eq. (2.16a) and the second term in Eq. (2.16b),

$$\begin{aligned} &A_{mn}a_n^{\mathrm{out}}|\mathcal{B}\rangle = 0, \\ &b_n^{\mathrm{out}}D_{nm}^\dagger|\mathcal{B}\rangle = 0, \end{aligned} \quad (2.27)$$

then determine the state $|\mathcal{B}\rangle$ uniquely, such that

$$|\mathcal{B}\rangle = \prod_{k=1}^{n_A}[u_m^{k*}B_{mn}b_n^{\mathrm{out}\dagger}]\prod_{l=1}^{n_A}[a_n^{\mathrm{out}\dagger}C_{nm}^\dagger v_m^l]|0^{\mathrm{out}}\rangle. \quad (2.28)$$

Equations (2.27) require that $|\mathcal{B}\rangle$ can only contain particles which lie in the kernel of A and antiparticles the complex conjugate of whose wave functions lie in the kernel of D. However, Eqs. (2.26) require that all the particle states corresponding to the vectors $C^\dagger v^l$ and antiparticle states corresponding to the complex conjugate of the vectors $B^\dagger u^k$ be filled. Fortunately, the unitarity Eqs. (2.8b) and (2.8c) imply

$$\begin{aligned} &DB^\dagger u^k = -CA^\dagger u^k = 0, \\ &AC^\dagger v^l = -BD^\dagger v^l = 0, \end{aligned} \quad (2.29)$$

where the right-hand sides of Eq. (2.29) vanish by the definition of the vectors u^k and v^l. Thus $B^\dagger u^k$ and $C^\dagger v^l$ lie in the kernels of D and A, respectively, so that the choice (2.28) for $|\mathcal{B}\rangle$ is consistent with the requirement (2.27). Furthermore, the unitarity Eqs. (2.8a), (2.8d), (2.9a), and (2.9d) imply that the vectors $B^\dagger u^k$ and $C^\dagger v^l$ span the kernels of D and A, respectively. Consequently, the ambiguous particle content permitted by Eq. (2.27) (because of the nonvanishing kernels of A and D) is completely determined by the conditions (2.26). Similarly, the undetermined terms in the operator $a_m^\dagger M_{mn}b_n^\dagger$, following from the ambiguity (2.24) in our solution for M, vanish when applied to the state $|\mathcal{B}\rangle$ in which *all* particle states with wave functions in the kernel of A and antiparticle states with complex-conjugate wave functions in the kernel of D are filled.

Thus we can express the state which corresponds to the ground state at $t=-\infty$ in terms of states with a definite number of particles at $t=+\infty$;

$$|0^{\mathrm{in}}\rangle = \mathfrak{N}\exp(a_m^\dagger M_{mn}b_n^\dagger)\prod_{k=1}^{n_a}[u_n^{k*}B_{nm}b_m^{\mathrm{out}\dagger}] \times\prod_{l=1}^{n_d}[a_n^{\mathrm{out}\dagger}C_{nm}^\dagger v_m^l]|0^{\mathrm{out}}\rangle, \quad (2.30)$$

where the normalization factor $\mathfrak{N}$ is given by $\det(P_{\ker A}+A^\dagger A)$. This result can be combined with Eq. (2.12) to express any state containing a specific configuration of incoming fermions as a linear combination of states each with a definite number of particles at $t=+\infty$.

As will be shown in Sec. III the exponential factor in Eq. (2.30) is completely consistent with the classical conservation laws. However, the operators enclosed in square brackets allow for the anomalous production of particles as required by the axial-vector anomalies present in the quantum-mechanical version of the conservation laws.

III. IMPLICATIONS OF THE AXIAL-VECTOR ANOMALY

We will now assume that the differential equation (2.1) implies a classically conserved fermion current

$$j_\mu(x) = \bar\psi\gamma_\mu Q\psi, \quad (3.1)$$

where Q is a Hermitian matrix which acts on the multicomponent Dirac spinor ψ but commutes with proper Lorentz transformations and the gauge generators T^i,

$$[Q, T^i] = 0. \quad (3.2)$$

In general, when this classical Dirac field theory is quantized, the operator current $j_\mu(x)$ acquires an anomalous divergence

$$\partial_\mu j_\mu(x) = N_{ij}\frac{g^2}{32\pi^2}F^i_{\mu\nu}(x)F^{j\mu\nu}(x), \quad (3.3)$$

provided $j_\mu(x)$ is defined to be invariant under the non-Abelian gauge transformations of the background Yang-Mills field. The matrix N_{ij} is given by

$$N_{ij}=\tfrac{1}{2}\,\mathrm{tr}[\gamma_5 Q T_i T_j]\,, \tag{3.4}$$

where tr indicates a trace over Dirac and group indices.

A. Definition of the charge operator

Because the current $j_\mu(x)$ is conserved classically, the classical charge $\mathcal{Q}(t)$ obtained by integrating the zeroth component of Eq. (3.1) over space

$$\mathcal{Q}(t)=\int d^3x\,\psi^\dagger(x,t)Q\psi(x,t) \tag{3.5}$$

is independent of time if $\psi(x,t)$ is a classical solution to the Dirac equation (2.1). More generally, the expression on the right-hand side of Eq. (3.5) remains time independent if $\psi^+(x,t)$ is replaced by the Hermitian conjugate of a second classical Dirac solution $\psi'(x,t)$. Using $\psi_k^{\mathrm{out}\,\pm}(x,t)$ for ψ and $\psi_l^{\mathrm{in}\,\pm}(x,t)$ for ψ' and equating the limits $t=\pm\infty$, we deduce that Q commutes with the scattering matrix S or, in the notation of Sec. II,

$$[A,Q^+]=[D,Q^-]=0, \tag{3.6a}$$

$$BQ^- - Q^+B = CQ^+ - Q^-C = 0, \tag{3.6b}$$

where

$$Q_{lk}^{\pm}=\int d^3x\,\psi_l^{\pm}(x)^\dagger Q\psi_k^{\pm}(x). \tag{3.7}$$

Next, let us relate the axial-vector anomaly present in the second-quantized current $j_\mu(x)$ to the elements A, B, C, and D of the S matrix. If we define the operator $j_\mu(x,t)$ by the usual normal-ordering procedure at $t=-\infty$, then the corresponding charge operator at $t=-\infty$ is a simple combination of the a_k^{in} and b_k^{in} operators

$$\lim_{t\to-\infty}\mathcal{Q}(t)=\sum_{l,k}(a_l^{\mathrm{in}\,\dagger}Q_{lk}^{+}a_k - b_l^{\mathrm{in}\,\dagger}Q_{lk}^{-}b_k^{\mathrm{in}}). \tag{3.8}$$

However, the situation at $t=+\infty$ is completely symmetrical with that at $t=-\infty$ if the operators a_k^{in} and b_k^{in} are replaced by a_k^{out} and b_k^{out}. Thus, the same subtraction which gives $\mathcal{Q}(t)$ the simple limit (3.8) at $t=-\infty$ also yields

$$\lim_{t\to+\infty}\mathcal{Q}(t)=\sum_{l,k}(a_l^{\mathrm{out}\,\dagger}Q_{lk}^{\dagger}a_k^{\mathrm{out}} - b_l^{\mathrm{out}\,\dagger}Q_{lk}^{-}b_k^{\mathrm{out}}). \tag{3.9}$$

Now we can use S to relate these two quantities. If Eq. (2.12) is used to express the in operators of Eq. (3.8) in terms of out operators and the two limits subtracted, one obtains

$$\begin{aligned}\Delta Q&=\lim_{t\to+\infty}\mathcal{Q}(t)-\lim_{t\to-\infty}\mathcal{Q}(t)\\&=\mathrm{tr}(Q^-CC^\dagger)-\mathrm{tr}(Q^+BB^\dagger).\end{aligned} \tag{3.10}$$

It is precisely this quantity which is given by the space-time integral of the anomalous operator Eq. (3.3). Defining the axial-vector anomaly

$$\mathcal{A}=N_{ij}\frac{g^2}{32\pi^2}\int d^4x\,F_{\mu\nu}^i(x)F^{i\mu\nu}(x) \tag{3.11}$$

and combining Eqs. (3.3) and (3.10) we find

$$\mathcal{A}=\mathrm{tr}(Q^-CC^\dagger)-\mathrm{tr}(Q^+BB^\dagger). \tag{3.12}$$

It is shown in Appendix A that the two traces in Eq. (3.12) are each well defined if the classical background field obeys appropriate conditions.

B. Consistency of the operator and state-vector analysis

Next we should relate the quantity $\mathrm{tr}(Q^-CC^\dagger)-\mathrm{tr}(Q^+BB^\dagger)$, which appears naturally from the charge operator, with the actual difference in the charge of the state $|0^{\mathrm{in}}\rangle$ at $t=\pm\infty$ implied by the analysis of Sec. II. Clearly, the definition (3.8) of the charge operator $\mathcal{Q}(t)$ yields

$$\mathcal{Q}(-\infty)|0^{\mathrm{in}}\rangle=0. \tag{3.13}$$

Similarly, we can find the charge of $|0^{\mathrm{in}}\rangle$ at $t=+\infty$ if we use Eq. (3.9) for $\mathcal{Q}(+\infty)$ our expression

$$|0^{\mathrm{in}}\rangle=\mathfrak{N}\exp(a_m^{\mathrm{out}\,\dagger}M_{mn}b_n^{\mathrm{out}\dagger})\prod_{k=1}^{n_a}[u_m^{k*}B_{mn}b_n^{\mathrm{out}\,\dagger}]\times\prod_{l=1}^{n_d}[a_n^{\mathrm{out}\,\dagger}C_{nm}^\dagger v_m^l]|0^{\mathrm{out}}\rangle \tag{2.30}$$

for $|0^{\mathrm{in}}\rangle$ in terms of outgoing states. Recalling that u^k and v^l span the kernel of $A^\dagger$ and $D^\dagger$, respectively, and that Q^+ commutes with A and Q^- with D we can choose the states u^k and v^l to be eigenstates of Q. Then Eq. (3.6b) can be used to show that the states

$$a_n^{\mathrm{out}\,\dagger}C_{nm}^{+}v_m^l|0^{\mathrm{out}}\rangle$$

and (3.14)

$$u_m^{k*}B_{mn}b_n^{\mathrm{out}\,\dagger}|0^{\mathrm{out}}\rangle$$

have the same charge at $t=+\infty$ as v^l and minus the charge of u^k, respectively. Finally, the definition of M, Eq. (2.19), and the commutation relation (3.6b) imply that the exponential operator in Eq. (2.30) does not affect the charge of the state. Thus we can deduce directly from the explicit formula (2.30) that between $t=-\infty$ and $t=+\infty$ there has been a net production of charge

$$\Delta Q=\mathrm{tr}[Q^-P_{\ker D^\dagger}]-\mathrm{tr}[Q^+P_{\ker A^\dagger}]\,. \tag{3.15}$$

It is not difficult to see that this expression for ΔQ equals the quantity (3.10) obtained directly from the charge operator and given by the axial-vector anomaly (3.12). One begins by observing that the unitarity equation (2.8a) implies that, when restricted to the kernel of $A^\dagger$, the operator $BB^\dagger$ becomes unity. Consequently,

$$\mathrm{tr}[Q^+P_{\ker A^\dagger}]=\mathrm{tr}[Q^+BB^\dagger P_{\ker A^\dagger}]$$
$$=\mathrm{tr}[Q^+BB^\dagger]-\mathrm{tr}[Q^+BB^\dagger P_{\mathrm{im}\,A}], \quad (3.16)$$

where we have used the fact that

$$P_{\ker A}=I-P_{\mathrm{im}\,A^\dagger}. \quad (3.17)$$

Next rewrite the second term on the right-hand side of (3.16) using sequentially Eqs. (2.9b) and (2.8c):

$$\mathrm{tr}[Q^+BB^\dagger AP_{\mathrm{im}\,A^\dagger}A^{-1}P_{\mathrm{im}\,A}]$$
$$=-\mathrm{tr}[Q^+BD^\dagger CP_{\mathrm{im}A^\dagger}A^{-1}P_{\mathrm{im}\,A}]$$
$$=\mathrm{tr}[Q^+AC^\dagger CP_{\mathrm{im}\,A^\dagger}A^{-1}P_{\mathrm{im}\,A}]$$
$$=\mathrm{tr}[Q^+C^\dagger CP_{\mathrm{im}\,A^\dagger}], \quad (3.18)$$

where the final step requires commuting A past Q^+ and using the cyclicity of the trace to move A from the left to the right. Finally, we can use Eq. (3.17) and the unitarity condition (2.9a) to rewrite the last expression in Eq. (3.18) as

$$\mathrm{tr}[Q^+C^\dagger C]-\mathrm{tr}[Q^+P_{\ker A}], \quad (3.19)$$

so we obtain

$$\mathrm{tr}[Q^+P_{\ker A^\dagger}]-\mathrm{tr}[Q^+P_{\ker A}]=\mathrm{tr}[Q^+BB^\dagger]-\mathrm{tr}[Q^-CC^\dagger]. \quad (3.20)$$

The unitarity Eqs. (2.9a) and (2.8d) imply that the operators C and $C^\dagger$ put $\ker A$ and $\ker D^\dagger$ in one-to-one correspondence so that, using the commutation relations (3.6b), we have

$$\mathrm{tr}[Q^+P_{\ker A}]=\mathrm{tr}[Q^-P_{\ker D^\dagger}]. \quad (3.21)$$

Equations (3.20) and (3.21) then demonstrate the equivalence of the two expressions (3.10) and (3.15) for ΔQ. Again in Appendix A it is shown under what conditions the above manipulations are well defined. Thus, we have shown that the nonconservation of the charge $\mathcal{Q}(t)$ inherent in the relation between in and out states determined in Sec. II is precisely that predicted by the anomalous conservation Eq. (3.30).

C. Time-dependent Hamiltonian spectrum

It is interesting to note that the quantity $\mathrm{tr}[Q^+P_{\ker A}]-\mathrm{tr}[Q^+P_{\ker A^\dagger}]$ appearing on the left-hand side of Eq. (3.20) and equal to the anomaly $\mathcal{A}$ is essentially the charge-weighted dimension of the kernel of A minus that of its adjoint. If we had restricted ourselves to solutions of the Dirac equation (2.1) with a single charge q_l, that quantity would have been simply the index of the operator A restricted to eigenstates of Q^+ with eigenvalue q_l, $[A]_{q_l}$. Thus,

$$\mathrm{tr}[Q^+P_{\ker A}]-\mathrm{tr}[Q^+P_{\ker A^\dagger}]=\sum_l q_l\,\mathrm{index}[A]_{q_l}, \quad (3.22)$$

where

$$\mathrm{index}[A]_{q_l}=\dim\ker[A]_{q_l}-\dim\ker[A^\dagger]_{q_l}. \quad (3.23)$$

As is well known the index of an operator has the important property of being unchanged by continuous variation of that operator. We can employ this property in the present case to relate the anomaly $\mathcal{A}$ with the spectrum of the time-dependent Hamiltonian.

The Dirac equation (2.1) can be rewritten

$$i\frac{\partial}{\partial t}\psi=H_t\psi, \quad (3.24)$$

where the Dirac Hamiltonian is given by

$$H_t=\alpha_i(-i\partial_i-g\vec{A}_i\cdot\vec{\tau}/2)+\vec{A}_0\cdot\vec{\tau}/2, \quad (3.25)$$

α_i, $1\le i\le 3$, being the usual 4×4 Dirac matrices. Let us define eigenstates $\psi^\pm_{t,n}(x)$ of H_t,

$$H_t\psi^\pm_{t,n}=\pm E^\pm_{t,n}\psi^\pm_{t,n}, \quad (3.26)$$

labeling separately those with positive ($E^+_{t,n}\ge 0$) and negative ($E^-_{t,n}<0$) eigenvalues. We can define a time-development matrix in analogy with Eqs. (2.5) and (2.6),

$$U(t)=\begin{pmatrix}A_t & B_t\\ C_t & D_t\end{pmatrix}, \quad (3.27)$$

where A_t, B_t, C_t, and D_t are defined by

$$\begin{pmatrix}A_{t,mn} & B_{t,mn}\\ C_{t,mn} & D_{t,mn}\end{pmatrix}=\begin{pmatrix}\langle\psi^{\mathrm{in}\,+}_m(t)|\psi^+_{t,n}\rangle & \langle\psi^{\mathrm{in}+}_m(t)|\psi^-_{t,n}\rangle\\ \langle\psi^{\mathrm{in}\,-}_m(t)|\psi^+_{t,n}\rangle & \langle\psi^{\mathrm{in}\,-}_m(t)|\psi^-_{t,n}\rangle\end{pmatrix}. \quad (3.28)$$

Clearly, the time-dependent eigenstates can be defined so that

$$\lim_{t\to+\infty}U(t)=S,$$
$$\lim_{t\to-\infty}U(t)=I. \quad (3.29)$$

The invariance of the index of $[A_t]_q$ under continuous changes in A_t can now be used to relate the index of $[A_\infty]_q=[A]_q$ with the index of $[A_{-\infty}]_q$ which is zero (the identity operator has no kernel). The only changes in the index of $[A_t]_q$ occur when an eigenvalue of H_t crosses zero, necessitating a discontinuous relabeling of the eigenvectors $\psi^\pm_{t,n}$ and an addition or deletion of the corresponding state from the domain of A_t. It is easy to show (see Appendix B) that the index of an operator is increased by one if a vector is added to its domain and decreased by one if a vector is removed from its domain. Therefore, define n_l as the number of times a negative-energy state $\psi_{l,n}^-$ of charge q_l becomes a positive-energy state minus the number of times the reverse occurs as t varies between minus and plus infinity. We can then conclude that

$$\operatorname{index}[A]_{q_l} = n_l \tag{3.30}$$

and

$$\Delta Q = \mathcal{Q} = \sum_l q_l n_l . \tag{3.31}$$

Thus, there is a very simple relationship between the axial-vector anomaly and the signed, charge-weighted number of zero crossings found in the spectrum of the time-dependent Dirac Hamiltonian. This conclusion is closely related to two previous results.

First, Callan, Dashen, and Gross[5] considered the violation of chirality in a background SU(2) Yang-Mills field with topological charge k. For a single Yang-Mills doublet of massless fermions the axial-vector anomaly predicts a violation of chirality equal to $2k$. Callan, Dashen, and Gross observed that in the adiabatic approximation the time evolution of the vacuum can be seen by simply tracing the time development of the originally filled Dirac negative-energy sea and the empty positive-energy states. As time involves each negative-energy state whose energy becomes positive becomes a filled particle state while a positive-energy state crossing zero would be interpreted as creation of an antiparticle. Thus, if in total the energies of k right-handed states cross zero from below while the energies of k left-handed states cross zero from above, there will be a net violation of chirality by $2k$ units. We have seen that this is generally the case even when the adiabatic approximation does not apply.

Second, the relationship between the topological charge and the signed number of zero crossings found in the energy spectrum of the Dirac Hamiltonian which we have established can also be deduced from a generalization of the Atiyah-Singer index theorem by Atiyah, Patodi, and Singer.[6] For simplicity consider a SU(2) gauge theory containing only right-handed fermions. Furthermore, we must specialize to the gauge $A_0 = 0$ and examine a variant of the Dirac equation in which the Minkowski "i" multiplying the time derivative has been removed:

$$\left(\frac{\partial}{\partial t} - H_t\right)\psi = \left(\frac{\partial}{\partial t} - \alpha_i(-i\partial_i - g\vec{A}_i\cdot\vec{\tau}/2)\right)\psi = 0. \tag{3.32}$$

Here $\vec{A}_i(x)$ is the original, Minkowski-space background gauge field (in the $A_0 = 0$ gauge).

Atiyah, Patodi, and Singer consider the index of the elliptic operator $\partial/\partial t - H_t$ with the following boundary conditions imposed on $\psi(\vec{x}, t)$:

$$\lim_{t\to-\infty} \int \psi_n^{+\dagger}(\vec{x})\psi(\vec{x},t)d^3x = \lim_{t\to+\infty} \int \psi_n^{-\dagger}(\vec{x})\psi(\vec{x},t)d^3x = 0 , \tag{3.33}$$

where $\psi_n^{\pm}(\vec{x})$ are the positive- (or, more accurately, non-negative) and negative-energy eigenfunctions of the free Dirac equation. In our case, where $\vec{A}_i(\vec{x}, t)$ becomes a gauge transformation at $t = \pm\infty$, their result for the index is simply

$$k = \frac{g^2}{32\pi^2}\int F\tilde{F}\,d^4x, \tag{3.34}$$

the topological charge of the gauge field A. However, it is not difficult to show that for the boundary conditions (3.33) the index of the operator $\partial/\partial t - H_t$ is precisely the number of times an eigenvalue of H_t crosses zero from below minus the number of times an eigenvalue crosses zero from above.[7] Atiyah, Patodi, and Singer call this quantity the spectral flow of H_t. Thus in analogy with the Euclidean-space tunneling calculations we could deduce the consequences of the axial-vector anomaly by referring instead to this result of Atiyah, Patodi, and Singer. For example, in the case of a single doublet of right-handed fermions, we have shown directly that the net number of fermions created is given by the index of A which in turn is equal to the spectral flow of H_t. Then the theorem of Atiyah, Patodi, and Singer can be used to conclude that this quantity is in fact the topological charge (3.34).

Finally, we should note that our division of the asymptotic states into two groups, those with $E_n \geq 0$ and those with $E_n < 0$, is really quite arbitrary. (Our particular choice makes a discussion of the vacuum simple and agrees, for those states with zero energy, with the conventions of Ref. 7.) All of the arguments presented above would work equally well if the asymptotic states were divided for example, into the two classes $\{\psi_n^+\}_{E_n > \mathcal{E}}$ and $\{\psi_n^-\}_{E_n \leq \mathcal{E}}$, where $\mathcal{E} \geq 0$. The state in which the energy levels $0 \leq E_n \leq \mathcal{E}$ are filled would simply replace the vacuum state in our quantum-mechanical arguments. Similarly, the net flow of eigenvalues between the two groups $E_n \leq \mathcal{E}$ and $\mathcal{E} < E_n$ is the same as the spectral flow defined before: the difference between these two definitions of spectral flow is the net number of states flowing out of the region $0 \leq E_n \leq \mathcal{E}$ which is automatically zero because the initial and final spectra are the same.

IV. CONCLUSION

The main objective of this paper has been to further study the anomalous divergence Eq. (1.6) by examining its consequences in a new physical domain. An actual prediction of the baryon-number nonconservation in very-high-energy collisions implied by Eq. (1.1) in the Weinberg-Salam model has not been made. Such a prediction requires an estimate of the production probability for weak Yang-Mills field configurations with topological charge. Furthermore, the problem of fermion creation by a physical radiation field is somewhat more complex than the situation analyzed in Secs. II and III where the gauge field $A_\mu(\vec{x}, t)$ was assumed to reduce to a pure gauge transformation for sufficiently large t. We will now briefly discuss both of these questions.

Let us consider the production of topological charge semiclassically in a very-high-energy collision in the Weinberg-Salam model. The scattering particles can be represented by a time-dependent SU(2) current density $J_\mu^i(x)$ and the radiated Yang-Mills field obtained by solving

$$\frac{\partial}{\partial x^\mu}\vec{F}^{\mu\nu}+g\vec{A}_\mu\times\vec{F}^{\mu\nu}=\vec{J}^{\,\nu}(x). \tag{4.1}$$

This approximation requires sufficiently high energy that the recoil of the scattering particles from the radiation of weakly interacting $W^\pm$, Z^0, and γ bosons can be ignored. Also, the radiating particles should, in principle, belong to large representations of the weak SU(2) group so that the change in weak quantum numbers caused by the radiation of $W^\pm$, Z^0, and γ quanta can be neglected.

A. Nonperturbative radiation

The usual perturbative solution to Eq. (4.1) is a gauge field of order g and the perturbation-theory calculation of the corresponding fermion scattering matrix S will yield matrices A and D with vanishing kernels so that there will be no net change in the number of baryons. Clearly, a perturbation-theory calculation always gives baryon-number conservation. Since this perturbative calculation of S should be quite reliable ($g^2/4\pi=\alpha/\sin^2\theta_w \simeq 0.03$), we may expect baryon-number violation only under unusual circumstances: (i) It is conceivable that the nonlinear equation (4.1) may have other solutions in addition to the perturbative one. Making an analogy with one-dimensional scattering by a potential barrier, we might speculate that such nonperturbative, topolocially charged solutions should exist representing passage over the barrier that was tunneled through in 't Hooft's instanton calculation of deuteron decay. Presumably, if they exist, such nonperturbative solutions would have much more dramatic effects on high-energy scattering than the creation of a few baryons. For example, one might expect to see a significant fraction of the collision energy radiated by weakly interacting particles—considerably greater than the fractional $\alpha/\sin^2\theta_w$ predicted by perturbation theory. (ii) Another direction which might be chosen to avoid the uninteresting perturbative solution to Eq. (4.1) is to consider the scattering of strong sources of the weak gauge field, e.g., high-Z nuclei or magnetic monopoles.

B. Abelian production of topological charge

Next we examine the case in which, for a suitable gauge, the source $\vec{J}_\mu(x)$ and the solution $\vec{A}_\mu(x)$ in Eq. (4.1) point in a fixed SU(2) direction

$$\begin{aligned} J_\mu^i(x)&=\delta_{i3}J_\mu(x),\\ A_\mu^i(x)&=\delta_{i3}A_\mu(x). \end{aligned} \tag{4.2}$$

Furthermore, in this simplified situation, let us relax the requirement that $A_\mu^i(x, t)$ becomes a pure gauge transformation for sufficiently large t, insisting only that the current $J_\mu(x)$ be localized in space and time.[8] Thus the integral of the topological charge density E_iB_i over the space-time region $-T\leq t\leq T$ can be written as a surface integral of the radiation fields in a source-free region, i.e.,

$$\begin{aligned} \nu&=\frac{g^2}{8\pi^2}\int d^3x\int_{-T}^{T}dtE_iB_i\\ &=-\frac{g^2}{8\pi^2}\int d^3xA_i(x, T)B_i(x, T). \end{aligned} \tag{4.3}$$

Here E_i and B_i are the usual components of the Abelian field strength tensor $\partial_\mu A_\nu-\partial_\nu A_\mu$. For simplicity we assume $A_\mu(\vec{x}, t)$ vanishes for sufficiently negative t so that only the $t=+T$ surface contributes to Eq. (4.3).

Somewhat surprisingly, the Abelian topological charge given by Eq. (4.3) is not, in general, zero. If we follow Jackson's[9] conventions to define the electric and magnetic multipole moments of $J(x)$,

$$\begin{aligned} a_E(l,m,\omega)&=\frac{2\omega^2}{i[l(l+1)]^{1/2}}\int_{-\infty}^{\infty}dt\,e^{i\omega t}\int d^3x\,Y_{lm}^*(\hat{x})\left[J_0(\vec{x},t)\frac{\partial}{\partial r}[rj_l(\omega r)]+i\omega[\vec{x}\cdot\vec{J}(\vec{x},t)]\,j_l(\omega r)\right],\\ a_M(l,m',\omega)&=\frac{2\omega^2}{i[l(l+1)]^{1/2}}\int_{-\infty}^{\infty}dt\,e^{i\omega t}\int d^3x\,Y_{lm}^*(\hat{x})\{\vec{\nabla}\cdot[\vec{x}\times\vec{J}(\vec{x},t)]\,j_l(kr)\}. \end{aligned} \tag{4.4}$$

ν is given by

$$\nu=\frac{g^2}{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_{-\infty}^{\infty}\frac{d\omega}{i\omega^3}a_E(l,m,\omega)a_M(l,-m,-\omega). \tag{4.5}$$

Thus coherently oscillating electric and magnetic dipoles will radiate topological charge.

C. Conservation-law violation in the Abelian case

However, we cannot refer to our previous analysis to argue that a net creation of fermionic charge must accompany such creation of topological charge. In fact, since this radiation of topological charge can occur in a circumstance where perturbation theory is accurate, such a conclusion is obviously wrong. What has happened is that the requirement (3.8) that $\mathcal{Q}(-\infty)$ counts the charges of the incoming particles no longer implies Eq. (3.9), that $\mathcal{Q}(+\infty)$ simply counts the charges of the outgoing particles,

$$\lim_{t\to+\infty}\mathcal{Q}(t)\neq\sum_{l,k}(a_l^{\text{out}\,\dagger}Q_{lk}^{+}a_k^{\text{out}}-b_l^{\text{out}\,\dagger}Q_{lk}^{-}b_k^{\text{out}}), \tag{4.6}$$

because the asymptotic gauge fields at $t=\pm\infty$ no longer differ by only a gauge transformation. Of course, this failure of $A_\mu(\vec{x},t)$ to approach a gauge transformation as t tends to infinity is somewhat subtle. Although we can choose a gauge where

$$\lim_{t\to\infty}A_\mu(\vec{x},t)=0, \tag{4.7}$$

there nevertheless exist gauge-invariant, spatial integrals of local polynomials in $A_\mu(\vec{x},t)$ and its derivatives which do not vanish as t becomes infinite. Thus, more precisely, the inequality (4.6) results from our inability to exchange the $t\to+\infty$ limit with the spatial integral appearing in the definition (3.5) of $\mathcal{Q}(t)$.

However, with some effort, we can correctly evaluate the $t\to+\infty$ limit of the charge operator $\mathcal{Q}(t)$ in terms of the out operators. One begins by determining $\psi_n^{\text{out}\,-}$ by solving iteratively the Yang-Feldman equations

$$\psi_n^{\text{out}\,-}(x)=\psi_n(\vec{x})e^{+iE_nt}-\int\Delta(x,y)^{\text{adv}}[g\gamma^\mu A_\mu(y)]\times\psi_n^{\text{out}\,-}(y)d^4y \tag{4.8}$$

obeyed by the outgoing Dirac solutions of Eq. (2.4). Here $\Delta(x-y)^{\text{adv}}$ is the usual advanced fermion Green's function and the gauge field $A_\mu(x,t)$ must obey the condition (4.7). The difference between the right- and left-hand sides of Eq. (4.6), q^{out}, can be obtained by substituting these iterative solutions for $\psi_n^{\text{out}\,-}$ into the expression

$$q^{\text{out}}=\lim_{t\to+\infty}\int d^3x\sum_{l,k}\Big[\psi_l^{\text{out}\,-}(\vec{x}+\vec{\epsilon},t+\epsilon_0)^\dagger P\Big(\exp\Big(ig\int_x^{x+\epsilon}A_\nu dx^\nu\Big)\Big)Q_{lk}^{-}\psi_k^{\text{out}\,-}(\vec{x},t)-\psi_l(\vec{x}+\vec{\epsilon})^\dagger Q_{lk}^{-}\psi_k^{-}(\vec{x})e^{i(E_k-E_l)t}e^{-iE_l\epsilon_0}\Big] \tag{4.9}$$

where the spacelike separation ϵ_μ has been introduced to regulate the ultraviolet divergence of the sum over l and k and the symbol $P(\;)$ indicates the path ordering of the enclosed exponential. If we let $\epsilon_\mu\to 0$ and replace $\epsilon^\mu\epsilon^\nu/\epsilon^2$ by $\frac{1}{4}g^{\mu\nu}$, the quantity (4.9) can be calculated for the Abelian case (4.2) using $\psi^{\text{out}-}$ obtained from the first two iterations of Eq. (4.8) and expanding the path-ordered exponential in Eq. (4.9) to first order with the result

$$q^{\text{out}}=\lim_{t\to+\infty}N\frac{g^2}{32\pi^2}\int d^3x\,B_iA_i, \tag{4.10}$$

where for SU(2) we have written $N_{ij}=N\delta_{ij}$. Thus we find

$$\lim_{t\to+\infty}\mathcal{Q}(t)=q^{\text{out}}+\sum_{l,k}(a_l^{\text{out}\,\dagger}Q_{lk}^{+}a_k^{\text{out}}-b_l^{\text{out}\,\dagger}Q_{lk}^{-}b_k^{\text{out}}), \tag{4.11}$$

where q^{out}, given by Eq. (4.10), is simply the fermionic charge of the out vacuum. Since in the Abelian case Eq. (4.10) implies $q^{\text{out}}=-N\nu$, the entire conservation-law violation is accounted for in the change of charge of the vacuum state. Hence in the Abélian case there is no anomalous particle production, as should be expected.

Following the procedure outlined above we can compute the final vacuum charge in the non-Abelian case for a general gauge group and anomalously conserved current [in this case Eq. (4.8) must be iterated three times]:

$$q^{\text{out}}=\lim_{t\to+\infty}\frac{g^2}{32\pi^2}N_{ij}\int d^3x(A^{i\mu}\partial^\nu A^{j\rho}+\tfrac{1}{3}gA^{i\mu}f^{jkl}A^{k\nu}A^{l\rho})\epsilon_{0\mu\nu\rho}. \tag{4.12}$$

With the necessary restriction (4.7) on possible gauge transformations the right-hand side of Eq. (4.12) is gauge invariant. Consequently, when there is a radiation field present in the final state, the net violation of the classically conserved charge Q as determined by comparing the number of initial and final particles is not given by the integral of the appropriate axial-vector anomaly but instead by $\mathcal{A}-q^{\text{out}}$. Note, the anomaly $\mathcal{A}$ can be computed from Eq. (1.6) in any gauge and is actually given by the right-hand side of Eq. (4.12) in the $A_0=0$ gauge. However, the charge q^{out} of the final vacuum can be computed from Eq. (4.12) only in a gauge obeying Eq. (4.7) and the vanishing initial conditions appropriate to this radiation problem. Since the condition (4.7) may not hold in the $A_0=0$ gauge and Eq. (4.12) is not invariant under general gauge transformations, q^{out} will in general differ from $\mathcal{A}$.

Note added in proof. Considerations similar to some of those in this paper but in a cosmological context appear in G. W. Gibbons, Phys. Lett. 84B, 431 (1979).

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APPENDIX A

We will now show that with suitable restrictions on the time dependence of the background gauge field $A_\mu(x)$, the traces and indices considered in Sec. III are well defined. Physically, the quantities $BB^\dagger$ and $CC^\dagger$ should have finite trace and A and D finite kernels, if for sufficiently large energies the scattering matrix S becomes the identity, as it should if the time dependence of $A_\mu(\vec{x}, t)$ is sufficiently smooth.

This relationship can be made explicit if we consider the generating function

$$G(z)=\lim_{\substack{t_2\to+\infty\\ t_1\to-\infty}} e^{iH(t_2)(t_2+z)}T\left\{\exp\left[-i\int_{t_1}^{t_2}H(t)dt\right]\right\}e^{-iH(t_1)(t_1+z)}. \tag{A1}$$

Here T denotes the usual time-ordering operation. When $G(z)$ is Taylor expanded in z it generates all multiple commutators of the Hamiltonian with the scattering matrix S defined in Eqs. (2.5) and (2.6):

$$G(z)=S^\dagger+iz[H(+\infty)S^\dagger-S^\dagger H(-\infty)]+\frac{(iz)^2}{2!}\{H(+\infty)[H(+\infty)S^\dagger-S^\dagger H(-\infty)]-[H(+\infty)S^\dagger-S^\dagger H(-\infty)]H(-\infty)\}+\cdots. \tag{A2}$$

By redefining t_1 and t_2, Eq. (A1) can be rewritten

$$G(z)=\lim_{\substack{t_2\to+\infty\\ t_1\to-\infty}} e^{iH(t_2)t_2}T\left\{\exp\left[-i\int_{t_1}^{t_2}H(t-z)dt\right]\right\}e^{-iH(t_1)t_1}=T\left\{S^\dagger\exp\left[-i\int_{-\infty}^{\infty}\sum_{r=1}^{\infty}\frac{(-z)^r}{r!}\frac{d^rH(t)}{dt^r}dt\right]\right\}. \tag{A3}$$

Equation (A3) implies that the nth multiple commutator of $H(\pm\infty)$ with S is a bounded operator if the first n time derivatives of A_μ are bounded and, when integrated over all time, yield a convergent integral.

The boundedness of the multiple commutators appearing in Eq. (A2) can be used to establish the following results:

(i) The kernel of A is finite dimensional. Let $\{\phi_i\}$ be a set of independent null vectors of A. Then Eq. (2.9a) implies that

$$C^\dagger C\phi_i=\phi_i. \tag{A4}$$

However, recall that with respect to the basis used to define S, $H(+\infty)$ and $H(-\infty)$ both have the form

$$\begin{pmatrix}E & 0\\ 0 & -E\end{pmatrix}, \tag{A5}$$

where the matrix E is diagonal

$$(E)_{mn}=\delta_{mn}E_n, \tag{A6}$$

the E_n being the free Dirac positive eigenvalues of Eq. (2.4). Thus if the time integrals of $|\partial A_\mu/\partial t|$ are finite, then the operator $H(+\infty)S-SH(-\infty)$ is bounded and its lower left-hand corner $EC+CE$ is also a bounded operator. When combined with Eq. (A4) this implies that the quantities $\phi_i^\dagger E\phi_i$ are bounded,

$$\begin{aligned}\phi_i^\dagger E\phi_i=\phi_i^\dagger C^\dagger CE\phi_i &\le \phi_i^\dagger C^\dagger(CE+EC)\phi_i\\ &\le \|EC+CE\|,\end{aligned} \tag{A7}$$

where $\|EC+CE\|$ is the norm of the operator $EC+CE$. This bound and the rapidly rising spectrum of E then require that the set of independent null vectors of A, $\{\phi_i\}$, must contain only a finite number of elements. Similar reasoning implies that $A^\dagger$, D, and $D^\dagger$ also have finite-dimensional kernels.

(ii) The kernel of A_{t_0} is finite dimensional. This is established by the same reasoning as above.

One simply replaces the original time-dependent Hamiltonian $H(t)$ by

$$H_{t_0}(t)=\begin{cases}H(t), & t\leq t_0\\ H(t_0), & t>t_0.\end{cases} \tag{A8}$$

The scattering matrix defined by this new Hamiltonian is precisely the time-development operator U_{t_0} in terms of which A_{t_0} was defined

$$U_{t_0}^{\dagger}=\lim_{\substack{t_2\to+\infty\\ t_1\to-\infty}} e^{+it_2H_{t_0}(t_2)}\,T\left\{\exp\left[-i\int_{t_1}^{t_2}H_{t_0}(t)\right]\right\}e^{-it_1H_{t_0}(t_1)}. \tag{A9}$$

Thus, if the time integrals of $|\partial A_\mu/\partial t|$ are finite then this remains true after the replacement implied by Eq. (A8) and the argument in (i) implies that A_{t_0}, $A_{t_0}^{\dagger}$, D_{t_0}, and $D_{t_0}^{\dagger}$ have finite-dimensional kernels.

(iii) The operators B and C are completely continuous and have well-defined traces. This conclusion follows if we assume that the first four time derivatives of $A_\mu(x)$ have finite time integrals. Then the fourth-order multiple commutator of H and S is bounded, which implies that

$$|B_{mn}|\leq M(E_m+E_n)^{-4}, \tag{A10}$$

where M is a positive constant. Consequently the sums

$$\sum_n B_{nn} \text{ and } \sum_{n,m}|B_{mn}|^2 \tag{A11}$$

are convergent so that B, and similarly C, are completely continuous and have a well-defined trace.

Finally, we can combine this result with the unitarity of S to show that the operator $P_{(\ker A)_\perp}A^{-1}P_{\mathrm{im}\,A}$, used in Eq. (3.18), is bounded so that the trace manipulations appearing in Eq. (3.18) make sense. Assume to the contrary that there exists a sequence of unit vectors ω_n in the orthogonal compliment of the kernel of A with the property that

$$\lim_{n\to\infty} A\omega_n=0. \tag{A12}$$

Consider the unitarity Eq. (2.9a) applied to ω_n:

$$A^{\dagger}A\omega_n+C^{\dagger}C\omega_n=\omega_n. \tag{A13}$$

Since C is completely continuous we can find a subsequence ω_{n_i} so that $C^{\dagger}C\omega_{n_i}$ converges to a limiting vector ω. Then Eqs. (A12) and (A13) require

$$\lim_{i\to\infty}\omega_{n_i}=\omega. \tag{A14}$$

Thus ω_{n_i} is a sequence of unit vectors orthogonal to the kernel of A with a limit annihilated by A: $A\omega=0$, a contradiction. We conclude that $P_{(\ker A)}A^{-1}P_{\mathrm{im}\,A}$ is a bounded operator.

APPENDIX B

In this appendix we show that the index of an operator A increases (decreases) if a vector is added to (deleted from) the domain of A.

Assume that we are given an operator A mapping a Hilbert space U into a Hilbert space V. Next we add a vector ψ to U and extend the definition of A by defining $A\psi=\phi$, a vector in V. Let us decompose ϕ into a piece in the original image of A and a piece orthogonal to the original image

$$\phi=A\psi'+\phi_\perp, \tag{B1}$$

where ψ' is an element of U. Consider two cases: (i) If $\phi_\perp=0$, $\psi-\psi'$ is a new vector in the kernel of A while $\ker A^{\dagger}=(\mathrm{im}A)_\perp$ is unchanged so the index of A has been increased by one. (ii) If $\phi_\perp\neq 0$ then the kernel of A is unchanged by the addition of ψ while $\phi_\perp$ is a nonvanishing vector previously in the kernel of $A^{\dagger}$ which is not annihilated by the adjoint of the newly defined A. The dimension of $\ker A^{\dagger}$ has been decreased by one and the index of A again increased by one.

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[7]M. F. Atiyah, V. K. Patodi, and I. M. Singer, Math. Proc. Cambridge Philos. Soc. 79, 71 (1976). This connection can be seen directly by multiplying the Hamiltonian H_t in Eq. (3.32) by a factor λ and examining the limit $\lambda \to \infty$. For large λ the adiabatic approximation can be applied and the index of the operator $\partial/\partial t - \lambda H_t$ with the boundary conditions (3.33) can be readily recognized as the spectral flow of λH_t. Since the index and the spectral flow are both independent of λ, this is also true for $\lambda=1$. Similarly, we can construct an alternative argument that the index of A is the spectral flow of H_t—simply repeat the above reasoning using a scattering matrix S_λ computed in the $A_0=0$ gauge with the Hamiltonian λH_t. Again, in the limit of large λ, the adiabatic approximation implies the index $[A]$ equals the spectral flow of H_t.

[8]Even this condition is not strictly obeyed when weakly charged, scattering particles are present in the initial or final state. However, the contribution of the asymptotic trajectories of these particles to the gauge field falls sufficiently rapidly with distance that the arguments to follow are not affected.

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