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Quantum gravity and Regge calculus

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For Tullio Regge on his 65th birthday

This is an informal review of the formulation of canonical general relativity and of its implications for quantum gravity; the various versions are compared, both in the continuum and in a discretized approximation suggested by Regge calculus. I also show that the weakness of the link with the geometric content of the theory gives rise to what I think is a serious flaw in the claimed derivation of a discrete structure for space at the quantum level.

1. INTRODUCTION

From chromodynamics, I have inherited the prejudice that a decent quantum field theory must have a credible discrete approximation. Being deep rooted, this prejudice resists counterexamples. In particular, Regge calculus [1] makes the geometric content of general relativity so clear that I find it difficult to believe that a successful quantization scheme would not originate, or translate cleanly into this discretized approximation. And therefore, I am reluctant to add the quantization of Regge calculus to the list of counterexamples, in spite of the modest success which the various attempts have achieved [2–5], and of the unquestionable fact that what progress there has been has been achieved ignoring this point of view.

I shall give a quick sketch of recent work on the canonical quantization of gravity, and a tentative critique of this work in the light of what I claim Regge calculus would suggest, with side comments on the meaning of the results so far obtained. So, in spite of the rather generic title, I shall mention only a small fraction of the work going on.

The idea that gravity should be treated as a connection theory underlies all the work I shall be considering, and this is where I shall begin; the other main ingredient is the idea of the "loop representation" of Rovelli and Smolin, which has evolved into a theory of "Spin nets", and this will be discussed next, with a brief summary of the striking and very recent progress that has taken place. Here I shall try and explain why I think that these results are spoilt by the imperfect link of the theory with the geometric content of general relativity. The crucial importance of this link will be illustrated using Regge calculus as an example.

2. CONNECTIONS AND CONSTRAINTS

If the aim is to formulate canonical general relativity as some sort of a gauge theory on 3–space, we have various options[6].

Option (1) is A.D.M. geometrodynamics rewritten as a connection theory.
One follows A.D.M. taking space slices Σ_t : $t(x)$ =const., with a congruence t^a : $t^a\partial_a t = 1$ related to the unit normal to the slice n^a by $t^a = Nn^a + N^a$, or $n_a = -N\partial_a t$; however A.D.M. start with the Einstein–Hilbert action $S = \frac{1}{2\kappa}\int\sqrt{-g}\ R\ d^4x$, while here one starts with an action $S = \frac{1}{4\kappa}\int \epsilon_{IJKL}e^I\wedge e^J\wedge R^{KL}$ (where R^{KL} is the curvature 2–form of the Levi–Civita connection Ω^{IJ}, and $\kappa := (8\pi G_{Newton})/c^3$), and a further choice is made, to partially fix the $O(3,1)$ gauge freedom setting:

$$e^0_a = -n_a = N\partial_a t \tag{1}$$

This choice ("time gauge") leaves invariance under local $O(3)$ transformations, with e^{ia}, $i =$

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$1,2,3$ space–like to provide a local frame on the slice. The pull–back of Ω^{IJ} to the space slice gives the 3–d L.C. connection $\Gamma_a^i := \frac{1}{2}q_a{}^b\epsilon_{jik}\Omega_b^{jk}$, and the extrinsic curvature $K_a^i := q_a{}^b\Omega_b^{0i} = e^{ib}K_{ab}$. As canonical variables we take the pair:

$$(E^{ia} := \tfrac{1}{2}\epsilon^{abc}\epsilon_{ijk}e_b^j e_c^k = \det(e_a^i)\, e^{ia}\ ,\ K_a^i) \qquad (2)$$

They form a canonical pair, just like the $(q_{ab}, \pi^{ab} := \sqrt{q}(K^{ab} - q^{ab}K)$ variables of A.D.M., because $E^{ia}E^{ib} = q\, q^{ab}$ implies:

$$K_a^i\, \delta E^{ia} = \frac{1}{2\sqrt{q}}K_{ab}\delta(E^{ia}E^{ib}) = -\frac{1}{2}\pi^{ab}\, \delta q_{ab} \qquad (3)$$

The Γ_a^i have curvature R_{ab}^i, and can be expressed in terms of E^{ia} (and its inverse) solving the 9 linear equations they satisfy:

$$D_a E^{ia} = 0 \quad ; \quad \epsilon_{ijk}E_i^a E_j^{(b} D_a E_k^{c)} = 0 \qquad (4)$$

We need to impose the A.D.M. constraint, and to constrain K_a^i to make sure that $K_{ab} = \frac{1}{\sqrt{q}}\, q_{bc}\, E_i^c K_a^i$ is symmetrical; altogether:

$$\begin{aligned}
\mathcal{G}_i &:= \epsilon_{ijk}K_a^j E^{ka} \approx 0 \\
\mathcal{V}_c &:= q_{bc}\nabla_a \pi^{ab} = 2E_i^a D_{[a}K_{c]}^i \approx 0 \\
\mathcal{H} &:= \frac{1}{\sqrt{q}}(\pi^{ab}\pi_{ab} - \tfrac{1}{2}\pi^2 - q\, R) = \\
&\frac{1}{\sqrt{\det E}}(2E_i^{[a}E_j^{b]}K_a^j K_b^i + \epsilon_{ijk}E^{ia}E^{jb}R_{ab}^i) \approx 0
\end{aligned}$$

Everything can be rewitten in terms of 2×2 matrices saturating $i, j, \ldots$ indices with $\tau_i := \sigma_i/(2i)$; so for example the effect of local rotations becomes:

$$\begin{aligned}
E^a &:= \tau_i E^{ia} &\to\quad & U\, E^a\, U^{-1} \\
\Gamma_a &:= \tau_i \Gamma_a^i &\to\quad & U\, (\Gamma_a + \partial_a)U^{-1}
\end{aligned} \qquad (5)$$

There is nothing particularly new about any of this; in particular K_a^i fits into the scheme poorly, like an external field, and the connection is a derived quantity.

Option (2) is a shrewd variation on the theme devised by J. F. Barbero[7,6], in a re–examination of the Ashtekar[8] formulation.

For some β to be fixed, we can change our basic variables as follows:

$$(E^{ia}, K_a^i) \;\to\; (E^{ia}\ ,\ A_a^{(\beta)i} := \Gamma_a^i + \beta\, K_a^i) \qquad (6)$$

which *is* a canonical transformation, because

$$E^{ia}\, \delta A_a^{(\beta)i} = \beta E^{ia}\, \delta K_a^i + \frac{1}{2}\partial_a(\tilde{\epsilon}^{abc}e_b^k\, \delta e_c^k) \qquad (7)$$

so that:

$$\{A_a^{(\beta)i}(x), E^{jb}(y)\} = \beta\kappa\, \delta_j^i \delta_a^b \delta(x,y) \qquad (8)$$

and all other Poisson brackets vanish (in particular, $\{A_a^{(\beta)i}, A_b^{(\beta)j}\} = 0$). However unlike K_a^i, $A_a^{(\beta)i}$ transforms like an $SU(2)$ connection for any β, and one can introduce $D_a^{(\beta)}$ derivatives and a curvature $F_{ab}^{(\beta)i}$. Using the properties of the Levi–Civita connection and a bit of algebra the constraints become:

$$\begin{aligned}
D_a^{(\beta)}E^{ia} &= \beta\mathcal{G}_i \;\approx\; 0 \\
E^{ia}F_{ab}^{(\beta)i} &= \beta\, \mathcal{V}_b + \beta^2\, K_b^j \mathcal{G}_j \;\approx\; 0 \\
\mathcal{H} = &\frac{1}{\sqrt{\det(E)}}\epsilon_{ijk}E^{ia}E^{jb}F_{ab}^{(\beta)k} \\
&-2\frac{(1+\beta^2)}{\sqrt{\det(E)}}E_i^{[a}E_j^{b]}K_a^i K_b^j + \ldots \;\approx\; 0
\end{aligned}$$

where the dots are terms proportional to $\beta\mathcal{G}_i$ and its derivatives.

Clearly the first two constraints are in an acceptable form, and the last one is not: the bit proportional to $(1+\beta^2)$ is a mess, the overall factor $1/\sqrt{\det(E)}$ is unpleasant. As for this last point: we either learn to live with the factor, or we ignore it, absorbing it in the Lagrange multiplier. The messy bit is more troublesome. The quickest way to get rid of it would be to take $\beta = i$. This is Ashtekar's original choice (together with the idea of absorbing the factor $1/\sqrt{\det(E)}$ in the Lagrange multiplier)[8] . The point is that there is much more to be said for this choice: it is (or, one can convince oneself that it is) the one choice that is geometrically and physically well motivated. In fact the connection $A_a^{(i)i}$ is the pull–back to the space slice of the self–dual connection, and E^{ia} the pull–back of the self–dual product of two vierbeins:

$$\begin{aligned}
A_a^{(i)i} &:= q_a{}^b C_{IJ}^i \Omega_b^{IJ} := q_a{}^b(-\tfrac{1}{2}\epsilon_{ijk}\Omega_b^{jk} + i\Omega_b^{0i}) \\
E^{ia} &:= -\epsilon^{abc}C_{IJ}^i e_b^I e_c^J
\end{aligned} \qquad (9)$$

This definition of E^{ia} coincides with (2) if we assume "time gauge", (1); for the definition of

$A_a^{(i)i}$ to coincide with (6) we have to assume "time gauge" and to impose explicitly that the connection Ω_a^{IJ} is Levi-Civita, which follows if we add to the constraints the *real* part of the second (4) as a brand new "reality" condition. The result is, I think, physically splendid, because we retain the full Lorentz group as gauge group of the canonical theory, but technically appalling, because of the obvious complications implicit in the use of complex variables. In particular, the i which now occurs in the Poisson brackets (8) puts all simple minded quantizations on collision course with the desire to have the operators corrsponding to E^{ia} hermitean. More particularly, I have not found a way round this difficulty, simple minded or not.

The alternative that has become most popular is to take $\beta = 1$, the "Barbero connection". I would like to claim that the trouble with this choice is that the connection has no obvious geometric meaning, and that there is nothing special about the value 1. In fact, β can be anything, and I shall leave it arbitrary (but real > 0) in the following. In Euclidean (++++) gravity the second term of $\mathcal{H}$ is proportional to $(1-\beta^2)$, and the choice $\beta = 1$ *is* natural, just like $\beta = i$ was "natural" for the (-+++) signature, but that's not saying much, unless somehow we learn to "Wick-rotate" the theory. For this possibility [6] see later.

There are at least two smart ways out of the inconvenience that K_a^i is a complicated function of $A_a^{(\beta)i}$ and E^{ia} (so that the expression of $\mathcal{H}$ is messy), which come by roundabout arguments devised by T. Thiemann[9]. I begin by listing some identities; let $f(x)$, $h(x)$ be nice test functions on Σ, and

$$\begin{aligned} \mathcal{H}^E[f] &:= \int_\Sigma \frac{f}{\sqrt{\det E}} \epsilon_{ijk} E^{ia} E^{jb} F_{ab}^{(\beta)k} d^3x \\ V[h] &:= \int_\Sigma h\sqrt{\det E} d^3x \end{aligned} \tag{10}$$

then one can work out the following remarkable Poisson brackets:

$$\begin{aligned} \{\mathcal{H}^E[f]\,,\, V[h]\} &= 2\beta^2\kappa \int_\Sigma f\, h\, E^{ia} K_a^i d^3x = \\ &:= 2\beta^2\kappa^2 T[fh] \end{aligned}$$

$$\begin{aligned} \{T[f]\,,\, E^{ia}(x)\} &= +f(x) E^{ia}(x) \\ \{T[f]\,,\, K_a^i(x)\} &= -f(x) K_a^i(x) \\ \{T[f]\,,\, \Gamma_a^i(x)\} &= \tfrac{1}{2} E^{ib} \epsilon_{abc} q^{cd} \partial_d f \\ \left\{A_a^{(\beta)i}\,,\, V[h]\right\} &= \frac{\beta\kappa h}{4\sqrt{\det E}} \epsilon_{abc} \epsilon_{ijk} E^{ia} E^{jb} \end{aligned} \tag{11}$$

I shall ignore possible problems with boundary terms, write simply $V := V[1]$, $T := T[1]$, $H^E := \mathcal{H}^E[1]$, and summarize the above as:

$$\begin{aligned} \{H^E, V\} = 2\beta^2\kappa^2 T \quad ; \quad \{T, E^{ia}\} = E^{ia} \\ \{T, \Gamma_a^i\} = 0 \quad ; \quad \{T, K_a^i\} = -K_a^i \end{aligned} \tag{12}$$

One use of these identities could be to rewrite $\mathcal{H}$ in the (possibly more manageable) form:

$$\begin{aligned} \frac{\beta\kappa}{2}\mathcal{H} = \{A_a^{(\beta)i}, V\}\epsilon^{abc} F_{bc}^{(\beta)i} \\ -\frac{1+\beta^2}{\beta^2}\{A_a^{(\beta)i}, V\}\epsilon^{abc}\epsilon_{ijk}\{A_b^{(\beta)j}, T\}\{A_c^{(\beta)k}, T\} \end{aligned} \tag{13}$$

This is the line followed by T. Thiemann in [9], and claimed to be a runaway success.

A more fancy approach starts from the observation that the Hamiltonian evolution induced by some H on a function f on phase space is given by the map:

$$W_H(t) \circ f := f + t\{f, H\} + \frac{t^2}{2!}\{\{f, H\}, H\} + .. \tag{14}$$

which preserves Poisson brackets etc.; furthermore, from (12):

$$\begin{aligned} W_T(t) \circ E^{ia} &= e^{-t} E^{ia} \\ W_T(t) \circ A_a^{(\beta)i} &= \Gamma_a^i + \beta e^t K_a^i \end{aligned} \tag{15}$$

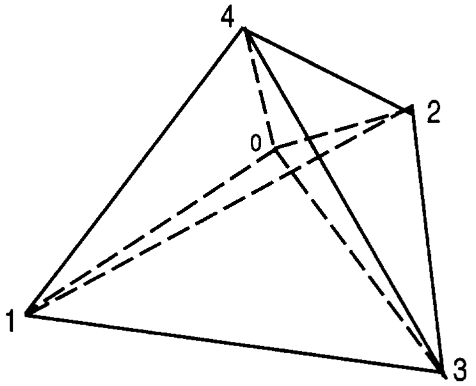
One may say that this relation shows explicitly that the value of β does not matter, since it can be changed at will (which is the point I am trying to make). More boldly, that it gives us a way to go from the Barbero to the Ashtekar connection, setting $t = i\pi/2$ if we had $\beta = 1$ [10,6] . This idea of a "Wick rotation" is striking, but I find it very difficult to articulate. In particular, I find difficult to understand how the transformation brings about a change of (gauge) symmetry group. I nevertheless regard this as the most promising clue to the construction of a satisfactory quantum theory within this set of ideas.

3. SPIN NETS

To quantize the theory we may use the connection representation, in which states are functionals of $A_a^{(\beta)i}(x)$, and

$$E^{ia} \quad \rightarrow \quad \hat{E}^{ia} := \frac{\beta\kappa\hbar}{i}\frac{\delta}{\delta A_a^{(\beta)i}} \tag{16}$$

The important states turn out to be "spin net states", which are an obvious generalization of Wilson loops.



Take for example the net in the picture, which has 5 vertices and 10 lines, and lives in a space with a connection $A_a^{(\beta)i}$; assign to each line γ_l an orientation, hence a "transporter" $g_l := P\exp\int_l \tau_i A_a^{(\beta)i} d\gamma_l^a$, and a spin $s_l = 0, \frac{1}{2}, 1, \ldots$, so that $\gamma_l \rightarrow \mathcal{D}^{s_l}_{mm'}(g_l)$; assign to each vertex v an $SU(2)$ invariant tensor $C^v_{m_a,..,m_d}$, and we shall have that:

$$\psi_{\underline{n}}(g_1, ..., g_{10}) = \sum_{\{m\}}\prod_v C^v_{...}\prod_l \mathcal{D}^{s_l}_{..}(g_l) \tag{17}$$

is gauge invariant. If $g_l \in SU(2)$, and we indicate by dg_l the Haar measure, these states form a (smallish) Hilbert space with the scalar product:

$$< \psi_{\underline{n}'}|\psi_{\underline{n}} >= \int \overline{\psi}_{\underline{n}'}\ \psi_{\underline{n}} dg_1 \ldots dg_{10} \tag{18}$$

Ashtekar and collaborators have discovered and emphasized again and again (see e.g.[11]) that if one considers all possible nets, then the (very large) set of these states is *dense* in the Hilbert space of gauge invariant functionals of $A^{(\beta)}$. In this sense (18) induces a measure $DA^{(\beta)}$ in this space.

The gauge invariant functionals of $A^{(\beta)}$ which might represent physical states are invariant under diffeomorphism. Given one such state $\Psi[A^{(\beta)}]$, for every spin net state $\psi_n[A^{(\beta)}]$ we can in principle calculate its "loop transform":

$$\Psi(\underline{n}) := \int DA^{(\beta)}\ \overline{\psi}_{\underline{n}}[A^{(\beta)}]\ \Psi[A^{(\beta)}] \tag{19}$$

that will represent the same state, and is diffeomorphic invariant <u>if</u> it depends only on the structure of the net and on our assigments of spins and invariant tensors. This is the latest incarnation of the "loop representation" idea of Rovelli and Smolin[12]. This form goes particularly well with the idea of the "weave"[13]: that the world that (we think) we know is likely to be described by states $\Psi(\underline{n})$ with support on huge, immensely complicated and fine meshed nets, in fact with mesh sizes of the order of the Planck length.

If we keep in mind the weave idea, it is quite sensible to look at net states to find the spectrum of the operators that correspond to the area of a surface S and to the volume of a region R, and to study the operator form of the constraints.

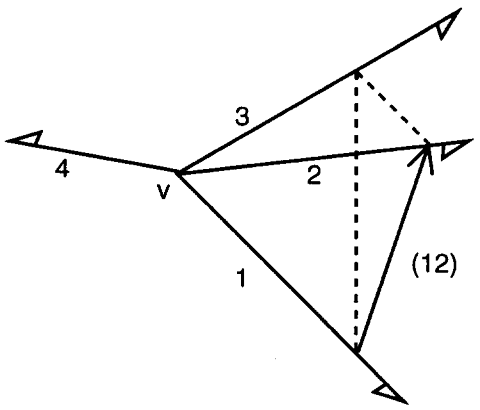
Very briefly: if the surface S intersects a subset $\mathcal{L}$ of lines of the net, does not touch the vertices, has no line lying on it, carefully regularizing the operator (*first* taking the square root, *then* removing the regulator), one finds[11,14] :

$$\begin{aligned}\hat{A}(S)\psi_{\underline{n}} &= :\int_s d^2\sigma\sqrt{n_a n_b \hat{E}^{ia}\hat{E}^{ib}}\ :\ \psi_{\underline{n}} \\ &= (\beta\hbar\kappa)\sum_{l\in\mathcal{L}}\sqrt{s_l(s_l+1)}\ \psi_{\underline{n}}\end{aligned} \tag{20}$$

If the region R contains a subset $\mathcal{V}$ of vertices of the net, one finds[14–16] by a similar procedure that a suitably regularized volume operator $\hat{V}(R)$ mixes the states $\psi_{\underline{n}}$ with coefficients proportional to $(\beta\hbar\kappa)^{\frac{3}{2}}$. This operator can be diagonalized[14] , and has a complicated, but discrete spectrum. Before going further, it is important to notice that nothing works for the theory based on the Ashtekar connection $\beta = i$: all operators have the wrong hermiticity; on this, see later.

However, I claim that the discrete spectra one gets for areas and volume <u>cannot</u> at this stage be interpreted as evidence for a discrete structure of space, because of the arbitrariness of β; we are faced with a "β crisis". Unless we find some good reason to fix β, the commutation relations (8) will be unable to fix the scale of the theory. My feeling is that this requires a group larger than $SU(2)$; a bit like in good old current algebra. A larger group may come from the need to implement the "Wick rotation", whether interpreted as a passage from the Euclidean to the Minkowskian or from the Barbero to the Ashtekar connection I do not know.

Alternatively, it may be that looking at the Wheeler De Witt equation $\hat{\mathcal{H}} \cdot \Psi = 0$, following Thiemann's work [9] , we shall find that solutions exist only for particular values of β. Given the way in which (13) depends on β, this is actually rather likely; but then, it may be just another way of saying the same thing. At the same time, if it were true it would be splendid: because we would have derived the discreteness of space from dynamics, and not from a kinematic fiddle.



Thiemann's work [9] on the Hamiltonian constraint is the latest thing. It is a massive and difficult piece of work, so that my comments, based on a very limited understanding, are obviously very tentative. The main effort is aimed at defining a satisfactory quantum operator corresponding to $\mathcal{H}^E[f]$, (10), from which, in view of (11,12) the rest follows (more or less). A regularization is obtained noticing that for a vertex v of a net and a choice of three (outgoing) lines, say 1, 2, 3, adding lines (IJ) (see the figure), in the naïve limit $g_I \approx 1 + A_a\Delta^a_I + \dots$ one has:

$$\frac{4}{3\beta\kappa}\sum_{I,J,K}\epsilon^{IJK}\mathrm{Tr}(\,g_I g_{IJ} g_J^{-1}\,g_K\,\{g_K^{-1}, V\}) \approx \frac{1}{\sqrt{\det E}}\epsilon_{ijk}E^{ia}E^{jb}F^{(\beta)k}_{ab}\;\mathcal{V}_{(123)} \tag{21}$$

where $\mathcal{V}_{(123)} := \frac{1}{6}\epsilon_{abc}\Delta^a_1\Delta^b_2\Delta^c_3$ is the coordinate volume of the tetrahedron shown. The action of the quantum $\hat{\mathcal{H}}^E[f]$ on net states is defined summing over all vertices and all choices of three lines the corresponding operator:

$$\hat{\mathcal{H}}^E[f]\cdot\psi_{\underline{n}} = \sum_{v,\{IJK\}}\frac{4}{3i\beta\hbar\kappa}f(v)\cdot \cdot\sum_{I,J,K}\epsilon^{IJK}\mathrm{Tr}(\,g_I g_{IJ} g_J^{-1}\,g_K\,[g_K^{-1},\hat{V}])\cdot\psi_{\underline{n}} \tag{22}$$

This is only the beginning, the definition is sharpened and modified several times en route. I understand that this summer it has been the object of intensive discussions by lots of specialists at the ESI workshop, and no doubt you will hear more and in more detail about it.

4. DISCRETIZING

The pattern I have described so far occurs again if you try and discretize the theory, except that there are more variants: from the variable mesh discretizations of the groups that solve problems in the classical theory numerically (an industry in rapid expansion), to emulatations of lattice gauge theory or to the theory of random triangulations. I shall restrict myself drastically, and try to keep close in spirit to the original Regge idea.

Divide space in tetrahedra with a flat inside: the natural variables corresponding to the E^{ia} would be the areas of the triangles, oriented outwards with respect to a frame local to the tetrahedron:

$$S := \tau_i S^i := \tau_i E^{ia}\epsilon_{abc}\tfrac{1}{2}(x_1x_2+x_2x_3+x_3x_1)^{bc} \tag{23}$$

The area square of a triangle will be S^iS^i, and of course tetrahedra must close; for the tetrahedron $A = (1234)$, labeling each triangle with the number of the vertex opposite, this gives the constraint:

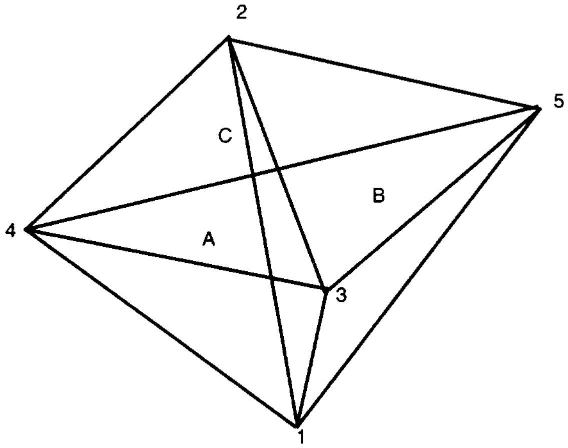
$$S_1^i + S_2^i + S_3^i + S_4^i = 0 \tag{24}$$

The volume square of the tetrahedron will be :

$$V_A^2 = \frac{\epsilon_{ijk}}{18}(S_2^i S_3^j S_4^k + S_1^i S_4^j S_3^k + S_2^i S_4^j S_1^k + S_3^i S_2^j S_1^k)$$

The variables S must be paired with variables g that link the frame attached to tetrahedron A to the one of its neighbour B across the triangle, with the basic condition that the triangle looks the same from both sides:

$$S_B = -g_{AB}^{-1} S_A g_{AB} \tag{25}$$



If g is a rotation, we will discover the local geometry of our space going round an edge, from tetrahedron to tetrahedron, until we get back: then if we are not in the same frame we started from, we conclude that space is curved. Changes of the frames, i.e. local rotations, are "gauge" transformations, which translate (5) to:

$$S_A \to g_A S_A g_A^{-1} \quad ; \quad S_B \to g_B S_B^{-1} g_B$$
$$g_{AB} \to g_A g_{AB} g_B^{-1} \tag{26}$$

An explicit parametrization for the triangle (123) would be :

$$S_A := u_A \cdot \tau_3 s \cdot u_A^{-1} \quad ; \quad S_B := u_B \cdot \tau_3 s \cdot u_B^{-1}$$
$$\text{with} : u_A = e^{\alpha\tau_3} e^{\beta\tau_2} \quad ; \quad u_B := e^{\gamma\tau_3} e^{\delta\tau_2}$$
$$g_{AB} := u_A e^{\Phi\tau_3} 2\tau_2 u_B^{-1} \tag{27}$$

with $s > 0$; the angle Φ is arbitrary at this stage.

Suppose that we are given the lengths of all edges, like in Regge's original scheme for "general relativity without coordinates": then we are in a Riemannian space, and we have conditions for the triangles that share an edge; for the edge (12) they read:

$$[[S_{3B}, S_{5B}], g_{AB}^{-1}[S_{3A}, S_{4A}]\, g_{AB}] = 0 \tag{28}$$

I omit the details: the point is that from these relations we can calculate all the angles Φ, and for the curvature around the edge (12) we find:

$$R_A := e^{F_A} = \tag{29}$$
$$= u_{4A} e^{\phi_A \tau_3}\, e^{(\theta_A+\theta_B+\theta_C-2\pi)\tau_2}\, 2\tau_2\, e^{-\phi_A\tau_3} u_{4A}^{-1}$$

Curvature is essentially the "defect angle" in the rotation; the connection is Levi–Civita.

But what can we make of the K_a^i? To make the connection "dynamic", we must give up (28), and therefore (29), and leave the angles Φ arbitrary, because that is the only place where the extrinsic curvature can go. Omitting details again, by various plausibility arguments, and insisting that in the limit (8) is recovered, one finds for the Liouville form Θ (the $\int pdq$ bit of the action) and for the Poisson brackets:

$$\begin{aligned} \Theta &= -\frac{1}{\beta\kappa}\sum_T \sum_{t\in T} \mathrm{Tr}(S_t \delta g_t g_t^{-1})_T \\ \{S^i, S^j\} &= \beta\kappa\, \epsilon_{ijk} S^k \;;\; \{g, S^i\} = \beta\kappa\, \tau_i\, g \\ \{g^{(1)}, g^{(2)}\} &= 0 \end{aligned} \tag{30}$$

In the explicit parametrization (27) each triangle contributes to the Liouville form:

$$-\frac{2}{\beta\kappa}\mathrm{Tr}(S\,\delta g\, g^{-1}) =$$
$$= \frac{1}{\beta\kappa}(s\,\delta\Phi + s\cos\beta\,\delta\alpha + s\cos\delta\,\delta\gamma) \tag{31}$$

I have not tried to write the constraints for real β. The idea of squeezing in dynamics by fiddling Φ may appear ugly and artificial, but it is the sense of the "Barbero connection". The Ashtekar choice makes much more sense: one replaces the Φ we calculated above with a complex variable:

$$\Phi \quad \to \quad \Phi + i\zeta \tag{32}$$

so that g_{AB} is promoted to a Lorentz transformation, with rapidity ζ, that links the different inertial frames attached to tetrahedra A and B. One may say that this is physically well motivated, and follows straight from the principle of equivalence. However, extending the gauge transformations (26) from $SU(2)$ to $SL(2,\mathbb{C})$ inevitably creates complications: we have to impose explicitly that within a tetrahedron $S^i_I S^i_J$ is real and positive definite, and (28) has to be modified. On the other hand we know that the Hamiltonian constraint simplifies to the $\mathcal{H}^E$ form, for which it is not difficult to guess discrete versions, for example:

$$\sum_{I<J\in T} \frac{1}{V_T}\mathrm{Tr}([S_I,S_J]F_{IJ})_T \approx 0 \tag{33}$$

in words: for each tetrahedron, the sum of the lengths of the edges times the corresponding defect angle must vanish. This comes about because for the edge (12) of A we can see that

$$l^i_{(12)} := \frac{2}{3V_A}\epsilon_{ijk}S^j_4 S^k_3 \approx e^i_a(x^a_2 - x^1_a) \tag{34}$$

In practice to make any use of (33), which is certainly not the only expression possible, we must approximate F with some function of R, e.g.$F \approx (R - R^{-1})/2$. The same or a similar expression can be used for the $\mathcal{H}^E$ part of the Hamiltonian constraint for real β. It is unfortunate that, as far as I can see, the regularized form used by Thiemann is just about the least natural in this scheme. This is because, at the first step one finds, for example:

$$\begin{aligned} g_4\{g_4^{-1}, V_A\} &= \frac{\beta\kappa\tau_i\epsilon_{ijk}}{36V}(S^j_2 S^k_3 + S^j_3 S^k_1 + S^j_1 S^k_2) \\ &= \frac{\beta\kappa}{24}\tau_i(l^i_{(41)} + l^i_{(42)} + l^i_{(43)}) \end{aligned}$$

a rather uncooperative expression; or, said more generally, because Thiemann's tetrahedra live in a lattice dual to the Regge lattice[1], in which to each of our triangles corresponds a line, and to each tetrahedron a (4–valent) vertex.

[1] notice that in (21) the volume operator $\hat{V}$ acts on the vertices of the net, that correspond to Regge tetrahedra, while $\mathcal{V}_{(123)}$ is the volume of a tetrahedron in the dual lattice.

Now for quantization. At first sight, from what I said about nets, it would appear that discretization, and Regge calculus in particular, offers the perfect tool; in fact, I became interested in Regge calculus because I was trying with spin nets. Spin nets naturally live in the dual of the Regge lattice. One can immediately envisage putting on a (small) computer a finite, simple lattice like a five tetrahedra division of S^3. In a real $SU(2)$ formulation the quantization of areas follows directly from (31): the area variable s is conjugate to an angle Φ. This is just the way 't Hooft argued in his discretized 2+1 gravity[17] , but with a very refined argument to Wick-rotate the theory. However, the direct use of the Ashtekar connection for quantization is made impossible by a muddle over the measure. I shall explain this in detail because I still hope that the muddle has a simple solution that I cannot see because of some selective blindness.

One can see what the problem is quite simply by writing the quantum version of (30) for $\beta = i$:

$$[\hat{S}^i,\hat{S}^j] = -\kappa\,\epsilon_{ijk}\hat{S}^k \; ; \quad [\hat{g},\hat{S}^i] = -\kappa\,\tau_i\,\hat{g} \tag{35}$$

$$[\hat{g}^{(1)},\hat{g}^{(2)}] = 0 \tag{36}$$

so that if $\psi = \psi(\{g\})$, $\hat{S}^i = -\kappa\hat{T}^i_L$, the (holomorphic) generator of the left–regular representation of $SL(2,\mathbb{C})$. The problem is that now there is no way to juggle the measure to make $\hat{S}^i$ hermitean; worse, $\hat{S}^2$ is negative, with eigenvalues $-j(j+1)$. Of course one did expect troubles: there are always troubles with non-compact groups, e.g. in 2+1 gravity[18], and even linearized gravity turns out to be quite tricky[19]; in QED the scalar product is well defined only for gauge invariant states. So the cure should come from considering the Hamiltonian constraint; in an interesting analysis of the linearized case[20] the suggestion is that one should gauge fix it. In our case, the idea of "Wick rotation" seems much more attractive, but notice that in any case it depends on solving the Hamiltonian constraint.

The idea[10] is that one can apply to this transformation the formalism of "coherent state transformations" developed by B.C. Hall[21] , suitably generalized. These are isometries between Hilbert spaces on e.g. $SU(2)$ and on its complexification

$SL(2,\mathbb{C})$; in the simplest case one defines heat kernels on the two groups by:

$$\frac{\partial \rho_t}{\partial t} + \tfrac{1}{2}\hat{\mathbf{J}}^2 \rho_t = 0 \quad , \quad \rho_0 = \delta(x)$$
$$\frac{\partial \mu_t}{\partial t} + \tfrac{1}{4}(\hat{\mathbf{J}}^2 + \hat{\mathbf{K}}^2)\mu_t = 0 \quad , \quad \mu_0 = \delta(g) \qquad (37)$$

(notice that $(\hat{\mathbf{J}}^2 + \hat{\mathbf{K}}^2)$ is an elliptic operator, but not the Casimir). Then the transformation B_t:

$$f \to (B_t f)(g) := \int_{SU(2)} f(x)\rho_t(x^{-1}g)dx \qquad (38)$$

maps functions on $SU(2)$ which are L^2 with the measure $\rho_t dx$ to functions on $SL(2,\mathbb{C})$ which are holomorphic and L^2 with the measure $\mu_t dg$. The map can be proved to be invertible and isometric, i.e.

$$\int_{SL(2,C)} \overline{(B_t f)(g)}\ (B_t h)(g)\ \mu_t(g)dg = \\ = \int_{SU(2)} \overline{f(x)}\ h(x)\rho_t(x)dx$$

This seems to be just what we need, but the connection with the "Wick rotation" is still to be understood.

REFERENCES

1. Tullio Regge, Il Nuovo Cimento **19** (1962) 558.
2. Paul Renteln and Lee Smolin, Class. and Quantum Grav.**6** (1989) 275.
3. Renate Loll, Nucl. Phys. **B244** (1995) 619; Phys. Rev. Lett. **75** (1995) 3048; gr-qc/9602041.
4. Giorgio Immirzi, Class. and Quantum Grav. **11** (1994) 1971; **13** (1996) 2385.
5. Josè A. Zapata, Class. and Quantum Grav. **13** (1996) 2617.
6. Abhay Ashtekar, Phys. Rev. **D53** (1996) 2865.
7. Jesus Fernando Barbero, Phys. Rev. **D51** (1995) 5507.
8. Abhay Ashtekar, Phys. Rev. Lett. **57** (1986) 2244.
9. Thomas Thiemann, Phys. Lett. **B380** (1996) 257, gr-qc/9606089, 90.
10. Thomas Thiemann, Class. and Quantum Grav. **13** (1996) 1383.
11. Abhay Ashtekar and Jerzy Lewandoski, gr-qc/9602046.
12. Carlo Rovelli and Lee Smolin, Phys. Rev. Lett. **61** (1988) 1155.
13. A. Ashtekar, C. Rovelli and L. Smolin, Phys. Rev. Lett. **69** (1992) 237.
14. Roberto De Pietri and Carlo Rovelli, Phys. Rev. **D54** (1996) 2664.
15. Jerzy Lewandoski, gr-qc/9602035
16. Thomas Thiemann, gr-qc/9606091
17. Gerardt 't Hooft, Class. and Quantum Grav. **10** (1993) 1653.
18. A. Ashtekar and R. Loll, Class. and Quantum Grav. **11** (1996) 2417.
19. A. Ashtekar, C. Rovelli and L. Smolin, Phys. Rev. **D44** (1991) 1740.
20. Richard P. Woodard, Class. and Quantum Grav. **10** (1993) 483.
21. Brian C. Hall, Journal of Functional Analysis **122** (1994) 103.