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Physics for Algebraists: Non-commutative and Non-cocommutative Hopf Algebras by a Bicrossproduct Construction

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The initial part of this paper presents "Physics for Algebraists" in the context of quantum mechanics combined with gravity. Such physical notions as the Yang–Baxter Equations, position observables, momentum space, momentum and position quantization, etc., are described. Many readers may wish to just read this initial part of the paper. The physics leads to the search for self-dual algebraic structures and finally to non-commutative and non-cocommutative Hopf algebras by a bicrossproduct construction. The entire paper contains numerous examples. The non-commutative and non-cocommutative Hopf algebras are obtained as a simultaneous smash product and smash coproduct and denoted $H_1 \bowtie H_2$. Among the examples is one obtained by modifying the Weyl algebra. We also give the context in which the compatibility requirements on the structure maps reduce to the Classical Yang–Baxter Equations, and an example related to Drinfel'd's double Hopf algebra $D(H)$.

1. INTRODUCTION AND PRELIMINARIES

Readers who are primarily interested in the presentation of physics for algebraists should turn immediately to Section 1.1, "Physics for Algebraists." Such readers should skip the present section, which concentrates on motivating the purely algebraic sections of the paper.

From a purely algebraic point of view, this paper is motivated by the search for examples of self-dual algebraic structures. Heuristically, this search means in the first place to find a category with a dualising endofunctor such that some kind of Pontryagin duality theorem holds. The dual object should correspond in some way to *representations* of the original object. Then, in such a situation one would like to find examples of self-

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dual objects. In general, one does not expect such nice results as in the categories of locally compact Abelian groups and finite Abelian groups.

Hopf algebras provide a well-known example of a category with some of these features. The fundamental examples are obtained from the category of finite groups via functors $k[\ ]$ and $k(\ )$. Here k is a field, $k[\ ]$ the group convolution Hopf algebra, and $k(\ )$ the Hopf algebra of functions. One may check that in these cases, Hopf algebra duality correctly reduces to Pontryagin duality. Details are recalled among the preliminaries below.

What are the self-dual objects in the category of k-Hopf algebras? The case of commutative or cocommutative Hopf algebras is well studied. These correspond closely to groups via the above functors. We are therefore primarily interested in non-commutative and non-cocommutative examples. A fundamental problem in such a search is a general shortage of such examples, let alone self-dual ones.

Section 1.1 motivates the search for self-dual algebraic structures in general, and Hopf algebras in particular. The context is the search for simple toy models of *quantum mechanics* combined with *gravity*. Related to this is a broader mathematical search for examples of *non-commutative geometries*. This is explained in the second section, Section 1.2, "Non-commutative geometry and the Yang–Baxter Equations." These sections are supplied entirely for motivation and not used explicitly in the paper.

The basis of the physical picture amounts to a novel interpretation of semidirect product algebra as quantum mechanics on homogeneous spacetimes. In this way semidirect product algebras naturally arise. We will denote them by $k[G_1] \ltimes_\alpha k(G_2)$. Here G_1 is a finite group and α an action on the set G_2. Since we are looking for self-dual Hopf algebra structures it is natural to try to define the algebra structure on the dual linear space also as a semidirect product structure. For this we assume that G_2 is also a group and acts back by an action β on G_1 as a set. Equivalently one can view that β induces a coaction of $k(G_2)$ on $k[G_1]$ and define the corresponding semidirect coproduct coalgebra which we denote $k[G_1]^\beta \rtimes k(G_2)$.

We then ask that the semidirect product by α and the semidirect coproduct by β fit together to form a Hopf algebra. This puts a constraint on the pair (α, β). When the constraint is satisfied, the set $k[G_1] \otimes k(G_2)$ with the semidirect product algebra structure and semidirect coproduct coalgebra structure will be denoted $k[G_1]^\beta \bowtie_\alpha k(G_2)$. We call it the *bisemidirect product* or *bicrossproduct* Hopf algebra. Hopf algebraists also refer to the semidirect product as a *smash product*. The bicrossproduct Hopf algebra is of self-dual type (Theorem 2.2) and is non-commutative when α is non-trivial, and non-cocommutative when β is non-trivial. Therefore when α and β are non-trivial these Hopf algebras are not in the image of the functors $k[\ ]$ or $k(\ )$. The construction is in Section 2 along with some examples of suitable groups G_1, G_2 and actions α, β.

It should be mentioned that the bicrossproduct structure has a little previous and independent history. Namely, following work of W. Singer it was used by [1] in the context of algebraic groups to show that a certain Hopf algebra of Taft and Wilson was of this form. The reference was communicated to the author some time after the original manuscript was written. Reference [1] should be consulted for further references. The reason for the revived interest is that the quantum mechanical interpretation leads to many new examples of Hopf algebras by exploiting the bicrossproduct construction. They are very different from the Hopf algebras already well known in connection with solving the *Quantum Yang–Baxter Equations* [2]. These equations arise in a very different physical context but some speculation about a possible connection is included in Section 1.1.

In Section 3 we ask the same question in the abstract category of k-Hopf algebras. When do the smash product algebra and smash coproduct coalgebra from a Hopf algebra (the bicrossproduct)? We obtain simple necessary and sufficient conditions for two Hopf algebras H_1, H_2 to have a bicrossproduct $H_1 \bowtie H_2$. These are understood via an abstract *factorization* property in terms of the universal properties of smash products and coproducts. We recover the case of H_1 cocommutative and H_2 commutative as for example in [1]. We also recover the case of trivial action and its dual case previously studied in [3].

We are then able in the final section to describe related examples based on Lie algebras. We also describe a general example of the form $H \bowtie H^{\mathrm{op}}$. Here H is any Hopf algebra with skew antipode, and H^{op} is the associated opposite Hopf algebra (this is H with the opposite product). The construction is related to Drinfel'd's "quantum double" Hopf algebra $D(H)$. The latter is reviewed in Section 1.2.

e denotes the neutral element of a group. 1 denotes both the identity element of an algebra A and the linear identity map. τ denotes the twist map $A \otimes A \to A \otimes A$. Elements of a tensor product space are denoted by the formal $\sum$ notation; cf. [4].

Preliminaries

By a k-coalgebra C we mean a vector space over k with a *coproduct* map $\Delta: C \to C \otimes C$ and a *counit* $\varepsilon: C \to k$. These obey axioms dual to the axioms of an algebra. $\otimes$ always means $\otimes_k$. A vector space H which is both an algebra and a coalgebra such that ε and Δ are algebra maps is called a *bialgebra*. Here $H \otimes H$ has the tensor product algebra structure. By Hopf algebra we shall mean a bialgebra H with a certain *antipode* map $S: H \to H$. This is required to obey $\sum (S(h_{(1)}))h_{(2)} = \sum h_{(1)} S(h_{(2)}) = 1\varepsilon(h)$, $\forall h \in H$. Here $\Delta h \equiv \sum h_{(1)} \otimes h_{(2)}$. We also need to assume on occasion a *skew-antipode* S'. This is defined by $\sum (S'(h_{(2)}))h_{(1)} = \sum h_{(2)} S'(h_{(1)}) = 1\varepsilon(h)$ so that it is equivalent to the assumption that the bialgebra H^{op}, defined with the opposite product, is a Hopf algebra, and S' is its antipode. H has a skew-antipode iff S is bijective. This holds if H is finite dimensional. During proofs the summation signs and excess brackets are omitted for clarity. The rest of the preface is a precise review of the functors $k[\ ]$ and $k(\ )$.

Let G be a finite group. For the purposes of exposition we assume k is algebraically closed of characteristic zero. $k[G]$ is a cocommutative Hopf algebra defined on the vector space over k whose basis is G. We shall write this vector space as the set of formal linear combinations of elements of G, $\{f=\sum_{u\in G} f_u u \mid f_u \in k\}$. The algebra structure is defined by the group product, extended by linearity. The remaining structure is defined by

$$\Delta u = u\otimes u, \qquad \varepsilon u = 1_k, \qquad Su = u^{-1}, \qquad 1 = e, \qquad \forall u \in G.$$

Then $k[\ ]$ is the covariant functor defined by $G \mapsto k[G]$ and morphisms mapped by linear extension of their values on $\{u\in G\}$. From $k[G]$ the group G can be recovered as the group-like elements. For H a Hopf algebra, the group-like elements, $\mathscr{G}(H)$, are defined by

$$\mathscr{G}(H) = \{f\in H \mid \Delta f = f\otimes f, \varepsilon(f) = 1_k\}.$$

For $H=k[G]$: $\mathscr{G}(k[G])\cong G$. This provides a covariant equivalence between the category of finite groups and that of cosemisimple cocommutative finite dimensional k-Hopf algebras [5, Sect. 3.4.1].

$k(G)$ is a commutative Hopf algebra defined on $\mathrm{Map}(G, k)$ by

$$f, g\in k(G), \qquad (fg)(s) = f(s)\, g(s), \qquad (\Delta f)(s, t) = f(st)$$
$$\varepsilon(f) = f(e), \qquad (Sf)(s) = f(s^{-1}), \qquad 1(s) = 1_k.$$

$k(\ )$ is the contravariant functor defined by $G\mapsto k(G)$. From $k(G)$ the group G can be recovered as the k-algebra morphisms into k; for H a Hopf algebra consider the group

$$(\mathrm{Mor}_{k\text{-algebra}}(H, k), (\phi\psi)(f) = (\phi\otimes\psi)(\Delta f)).$$

For $H=k(G)$: $\mathrm{Mor}_{k\text{-algebra}}(k(G), k)\cong G$. Here and elsewhere "Mor" stands for *morphism* in the appropriate category. This provides a contravariant equivalence between the category of affine algebraic k-groups and that of reduced commutative finitely generated k-Hopf algebras [5, Sect. 4.2.1].

These two Hopf algebras are dual as follows. In the present finite dimensional case, if H is a Hopf algebra, the dual Hopf algebra is the dual linear space H^* with the structure

$$\forall f, g\in H, \quad \phi, \psi\in H^*, \quad (\phi\psi)(f) = (\phi\otimes\psi)(\Delta_H f), \quad (\Delta\phi)(f\otimes g) = \phi(f\cdot_H g)$$
$$(S\phi)(f) = \phi(S_H f), \qquad \varepsilon(\phi) = \phi(1_H), \qquad 1(f) = \varepsilon_H(f),$$

where we have adopted the identification

$$\rho\colon H^*\otimes H^* \hookrightarrow (H\otimes H)^*, \qquad (\rho(\phi\otimes\psi))(f\otimes g)=\phi(f)\,\psi(g).$$

This is an isomorphism in the case of H finite dimensional.

For example,

$$k[G]^*\cong k(G), \qquad k(G)^*\cong k[G]$$

and if G is Abelian then

$$\mathscr{G}(k(G))\cong \hat{G}\cong \mathrm{Mor}_{k\text{-algebra}}(k[G], k),$$

where $\hat{G}$ is the Pontryagin dual group. Hence $k(G)\cong k[\hat{G}]$ and $k[G]\cong k(\hat{G})$ (Fourier transforms). We see that Hopf algebra duality properly restricts to Pontryagin duality for the Hopf algebras corresponding to Abelian groups.

After this it is natural to seek other examples of non-cocommutative and non-commutative Hopf algebras and to know the self-dual objects. In response to this question an obvious answer is

$$k[G]\otimes k(G)\cong k[G\times\hat{G}]\cong k(\hat{G}\times G) \qquad (\text{if } G \text{ Abelian}),$$

where the left hand side now makes sense for non-Abelian G.

In the remainder of the Introduction we shall need to borrow some ideas from the theory of C^* algebras. A C^* algebra is a type of topological algebra over $\mathbb{C}$. The topology should be determined by a norm $\|\ \|$ with respect to which it is a complete. In addition, there should be an antilinear antialgebra map $*$ (called the *adjoint* operation or involution) such that $\|x^*x\|=\|x\|^2$. Every C^* algebra has a faithful representation as bounded operators on a Hilbert space [6].

1.1. *Physics for Algebraists*

This section provides the physical motivation for looking for self-dual algebraic structures. We define the term *quantize* used in the preface and give an interesting physical interpretation of Hopf algebra duality in this context. These ideas are not used directly in the Hopf algebra constructions in the paper. They are, however, the underlying motivation for the constructions. Regarding quantization, the terms that we define formalise those used by physicists in the case of flat spacetime, but are defined in a coordinate invariant way that extends to the simplest curved spacetimes. This generalization is based on original work of the author [9]. Readers may wish to proceed directly to Section 1.2, where the Introduction continues with a description of some mathematical results of Drinfel'd and a discussion of non-commutative geometry.

1.1.1. *Classical Mechanical System* (G_1, M, α)

The physics to be formulated here is that of a particle moving on a homogeneous spacetime. The first step of the formulation is called *classical mechanics*. Here we employ the framework of Einstein in which gravitational forces are encoded in terms of Riemannian geometry. This means a manifold M (spacetime) equipped with an indefinite Riemannian metric g [7]. A classical particle is defined as a point $(s, v) \in TM$. Here $s \in M$ is called the particle *position* and $v \in T_sM$ is called the particle *velocity*. By definition, the particle evolves in time along geodesics of the metric. If the geodesic through s in direction v is parametrized according to $s(\tau) = \exp_s(\tau v)$ then τ is called *proper time*.

A simple class of metrics, the ones on homogeneous Riemannian manifolds, arises as follows. Let M be a smooth manifold and α a smooth action of a Lie group G_1 on M. If G_1 is semisimple and α is transitive, then there is a natural metric g on M such that the geodesics through s are of the form

$$s(\tau) = \alpha_{\exp \tau\xi}(s), \qquad \xi \in g_1, \quad \tau \in \mathbb{R}.$$

Here g_1 denotes the Lie algebra of G_1 and the element ξ corresponds to a velocity $v = \alpha_*(s)\xi$. Thus elements of g_1 label the geodesics of the motion and are called *momentum observables*. The labelling is degenerate since $\alpha_*(s)$ need not be injective. This can be fixed by orthogonal projection to the complement of $\ker \alpha_*(s)$ with respect to a suitable inner product on g_1. Some relevant references are [7, Vol. II, Chap. X, Theorem 3.5, Corollary 3.6, Theorem 2.10, Proposition 2.4, and Corollary 2.5].

For example, let $M = G_1 = G$ be a semisimple Lie group and let α be the left action defined by $\alpha_u(s) = us$ for all $u, s \in G$. Then the corresponding induced metric is the intrinsic metric on G [7, Vol. I, Chap. IV]. The above reduces to the well-known fact that the geodesics through the identity are just the one-parameter subgroups generated by elements of the Lie algebra. A similar construction works in the non-semisimple case of $\mathbb{R}^n$. For example, if $G = \mathbb{R}^n$ considered as an additive group, the Euclidean metric is induced by

$$s(\tau) = \alpha_{\tau\xi/m\hbar}(s), \qquad \alpha_u(s) = s + \hbar u.$$

Here $\hbar$ (Planck's constant divided by 2π) and m (particle mass) are scale parameters introduced to allow greater flexibility in choice of units. If we wish to describe more than one particle, (s_1, ξ_1), (s_2, ξ_2), ..., then these can be assigned different m_i so that the particle geodesics flow at different rates for the same position and momentum data. This example $(\mathbb{R}^n, \mathbb{R}^n, \alpha)$ is called *rectilinear motion*.

If α is not transitive, $\alpha_*(s)$ is not surjective. Physically this just means that the particle is constrained to move only on some subset of M. For example, let $M = \mathbb{R}^3$, $G_1 = SU(2)$, and α be the action by the standard projection of $SU(2)$ to $SO(3)$ acting in the defining representation on $\mathbb{R}^3$ (i.e., the action by rotations). Then the orbits are spheres and the induced metric is the standard spherical one. g_1 in this case is called the space of "angular momentum observables."

We shall only work with this simple class of metrics. For in this case the geometrical data are all encoded in the algebraic data (G_1, M, α). In this purely algebraic setting we will be able to drop the restriction that G_1 be a Lie group and that M be a manifold. To do this, consider only $\tau_n = n \in \mathbb{Z}$. The geodesics at such τ_n take the form

$$s(n) = \alpha_{u^n}(s), \qquad u \in G_1, \quad n \in \mathbb{Z},$$

where $u = \exp \xi$ and $\xi \in g_1$ the Lie algebra of G_1. Thus we think of the points generated by u an element of finite group G_1 as a "geodesic" through s in the finite set M.

One final point of notation remains to be explained. Instead of working directly with points in the manifold (or set) M, it is convenient to work dually with $C^\infty(M)$ (or $k(M)$). This is equivalent for if the values of all $f \in C^\infty(M)$ are known then there is a unique $s \in M$ such that the known value of f is $f(s)$ for all $f \in C^\infty(M)$. The elements of $C^\infty(M)$ are called *position observables*. We continue to denote by α the induced action on this linear space. One can also regard the momentum observables $g_1 \subseteq (g_1^*)^*$ as (linear) functions in this way. g_1^* is then called *momentum space*. The pair $g_1^* \times M$ is called the *phase space*.

In the case where $G_1 = M = G$ is a Lie group, one has that the phase space is $g^* \times G = T^*G$. This is the starting point of a more conventional formulation based on the notion of *Poisson structures* on phase space. These are defined in Section 1.2. A reference is [8]. In this alternative formulation the phase space would be T^*M. $C^\infty(T^*M)$ is called the "classical algebra of observables." In our case $C^\infty(M) \subset C^\infty(T^*M)$ (by pullback via the bundle projection) and $g_1 \to C^\infty(T^*M)$ by mapping $\xi \in g_1$ to each fiber T_sM by $\alpha_*(s)$. Our designation of momentum space as g_1^* corresponds to a preferred choice of co-ordinates. The concept of phase space does not play a direct role in our formulation, which is more closely tied to the geodesics on spacetime M.

1.1.2. *Quantization of* (G_1, M, α)

The formulation of classical mechanics described above does not, however, fit observations on a sub-microscopic scale. One finds that it is impossible to measure both the position s and velocity v (or s, and momentum ξ) of a particle to an arbitrary accuracy: measuring one invalidates any values previously obtained for the other. A more sophisticated framework is provided by *quantum mechanics*. In quantum mechanics, both the aspects that we would like to observe (the state of the particle) and the act of observation itself are modelled.

The *state* of the particle is now modelled by an element ψ of a Hilbert space H. Let $(\, , \,)$ denote the inner product on H. We require that states are normalised in such a way that $(\psi, \psi) = 1$. The quantities that we wish to observe, the *observables*, are modelled as self-adjoint elements of the set of bounded linear operators, $B(H)$, on H. The result of a measurement of an observable a is conceived of as an eigenvalue of a. (Typically these are discrete: such discretization of observed values is confirmed experimentally.) The actual values are not defined (thereby avoiding the above difficulty) but the *probability* that a measurement of an observable a in a state ψ will yield the eigenvalue λ is defined. It is defined to be $(\psi, a_\lambda \psi)$, where a_λ denotes orthogonal projection onto the eigenspace of λ. As a result, the expectation value of an observable a in a state ψ is $(\psi, a\psi)$.

Our particular quantum system is to be based on the classical system (G_1, M, α). It is assumed that the properties that we would like to observe correspond to the classical observables. Thus for each $\xi \in g_1$ and each $f \in C^\infty(M)$ we would like to define self-adjoint operators $\hat{\xi}$ and $\hat{f}$. Of these we require the (generalized) *commutation relations*

$$[\hat{f}, \hat{f}'] = 0, \qquad [\hat{\xi}, \hat{\xi}'] = \widehat{[\xi, \xi']}, \qquad [\hat{\xi}, \hat{f}] = \widehat{\alpha_*(f)\xi},$$
$$\forall f, f' \in C^\infty(M), \qquad \forall \xi, \xi' \in g_1.$$

One may ask if there is any $s \in M$ such that $(\psi, \hat{f}\psi) = f(s) \forall f \in C^\infty(M)$. If so, one would say that the particle is at position s. The first commutation relation ensures that (any finite number of) the $\hat{f}$'s are simultaneously diagonalizable. Hence there are approximate simultaneous eigenstates ψ_s of all the $\hat{f}$'s for $f \in C^\infty(M)$. We further require that the corresponding approximate eigenvalues coincide with the classical values $f(s)$. This ensures that in such a state ψ_s the expectation value of $\hat{f}$ is approximately $f(s)$ so that the particle can be said to be near s. The additional requirement means essentially that for all f, $g \in C^\infty(M)$,

$$\hat{f}\hat{g}\psi_s = \hat{f}g(s)\psi_s = f(s)\, g(s)\psi_s = (fg)(s)\psi_s = \widehat{fg}\psi_s, \; \hat{1}\psi_s = \psi_s, \qquad \forall s \in M,$$

i.e., that quantization should preserve the algebra structure of $C^\infty(M)$. Note that a general state ψ is not localised in the manner described for such a ψ_s.

Usually physicists choose some local co-ordinate chart $q = (q^1, q^2, ..., q^n)$, $q^i \in C^\infty(M)$, valid in a suitable open subset $U \subset M$. This asserts that q is a diffeomorphism of U onto some open subset $q(U)$ of $\mathbb{R}^n$. A point $s \in U$ can then typically be found such that $(\psi, \hat{q}^i\psi) = q^i(s) \forall i$. It is the "expected position." However, if $(\psi, \hat{q}\psi)$ lies outside of $q(U)$ then this interpretation breaks down.

Similarly, by linearity, there exists $p \in g_1^*$ for which $(\psi, \hat{\xi}\psi) = \xi(p) \forall \xi \in g_1$. If $\{e_i\}$ denotes a basis of g_1 and $\{f^i\}$ a dual basis, then $p = \sum_i (\psi, \hat{e}_i\psi) f^i$. It is the "expected momentum." The $\hat{e}_i$ are often denoted "$\hat{p}_i$" for this reason.

Of the third commutation relation physicists say "momentum is the generator of the group of translations." The second commutation relation is required for consistency of the Jacobi identity, assuming that α is an effective action. These relations ensure that the element $\sum_{ij} (K^{ij}/2m\hbar)\hat{e}_i\hat{e}_j$ (called the *free particle Hamiltonian*) acting on state $\psi \in H$ correctly generates proper-time evolution approximating as $\hbar \to 0$ to the desired classical motion [9, Sect. 2.2]. Here K is the inverse of the inner product on g_1 that was used in the process of inducing a metric on M in the previous subsection. If g_1 is semisimple K is the inverse Killing form. It is assumed that α also depends on the parameter $\hbar$ such that α_* is $O(\hbar)$.

The *quantization problem* is to actually find suitable operators $\hat{\xi}$ and $\hat{f}$ on a suitable Hilbert space H. These maps $\xi \mapsto \hat{\xi}$, $f \mapsto \hat{f}$ are respectively the *momentum* and *position quantization maps*.

In the algebraic setting $\psi \in V$ an inner product space and $a \in \mathrm{End}_k V$. Hence $(\psi, a\psi) \in k$. The probabilistic interpretation no longer applies, but is in any case redundant since all the information that we were given is contained in such "expectation values." Next, instead of g_1 we shall work with G_1 as explained in the previous subsection. Note that representations of G_1 are precisely the algebra representations of $k[G_1]$. Then the quantization problem for this system is the following; to find an endomorphism algebra B and algebra maps $\hat{}$ into B, such that

$$k[G_1] \xrightarrow{\wedge} B \xleftarrow{\wedge} k(M), \qquad \hat{u}\hat{f}\widehat{u^{-1}} = \widehat{\alpha_u(f)},$$
$$\forall u \in G_1, f \in k(M);\ \alpha_u(f)(s) \equiv f(\alpha_{u^{-1}}(s)).$$

The extra condition says precisely that the position quantization map is a $k[G_1]$-module map where B has the adjoint action due to the momentum quantization map. The quantization problem as stated is therefore precisely the definition of the smash product; $k[G_1] \ltimes_\alpha k(M)$ is the universal solution. The universal property of semidirect products precisely asserts that any solution of the quantization problem is just a linear representation of the smash product algebra. The smash or cross product is recalled in more detail where it is needed in Section 3.

A simple example, quantum mechanics on a group, is provided by the case $M = G_1 = G$ and α the left action. The algebra $k[G] \ltimes_{\mathrm{left}} k(G)$ will be called the *group Weyl algebra* essentially in keeping with the analytic setting when G is a vector space. The latter is extensively studied by operator algebraists; cf. [10, Chap. 5.2]. As an example, the well-known *Schroedinger representation* is that induced by the universal property by the left regular representation of $k[G]$ on $V = \mathrm{Map}(G, k)$ and the representation by pointwise multiplication of $k(G)$. The ordinary Weyl algebra A_n, $n \geqslant 1$, is similar to the group Weyl algebras; it is $U \ltimes k[x_1, ..., x_n]$, where $k[x_1, ..., x_n]$ is the commutative algebra of polynomials in the x_i, and U is the universal enveloping algebra of the Abelian Lie algebra generated by $\partial/\partial x_1, ..., \partial/\partial x_n$ [11].

1.1.3. *Mach's Principle and Hopf Algebra Duality*

The classical and quantum mechanics that we have described is only half the story. Not only is the classical motion of particles determined entirely by the spacetime geometry (modelled above by α), but conversely the spacetime geometry is determined by the particle distribution. This is formulated by Einstein's equation

$$(\text{Ricci Curvature}) - \tfrac{1}{2}(\text{Scalar Curvature})\, g + \Lambda g = T.$$

Here the curvature is that of the spacetime with metric g. T is the *stress-energy tensor* and expresses the particle distribution, their masses, and momenta. Λ is a real parameter called the *cosmological constant.*

This is an example of a *closed system.* This means that reality is not specified a priori but rather characterized as the solution of a circular self-consistency problem. Concerning the motivation of his equation in the ideas of the German philosopher E. Mach, Einstein wrote "it is contrary to the mode of thinking in science to conceive of a thing (the space-time continuum) which acts itself, but which cannot be acted upon ... in this way the series of causes of mechanical phenomena was [in Mach's ideas] closed, in contrast to the mechanics of Newton and Galileo" [12, p. 54]. The mode of thinking referred to is a positivist one put forward in this context by Mach. In modern usage *Mach's principle* is the maxim "for every action there is an equal and opposite *back-reaction.*"

Mach recognized that this principle, which he encountered in the form of Newton's third law, had a very deep origin in the relation between the observer and the observed. Mach's critique of Newtonian conceptions of acceleration and gravity [13, II.v–vii] may be idealized as follows. If there was just one particle in the universe, how would we know if it was accelerating? We need another particle for reference. Thus it is the existence of the second particle that determines the acceleration of the first. Thus we might as well say that it is the very existence of the second particle that causes the acceleration of the first. By reversing roles, an equal and opposite acceleration in induced in the second particle by the first. In attempting to formulate classical mechanics along these lines, Einstein was not able to do away with space or spacetime altogether as Mach had wanted, but by virtue of Einstein's equation spacetime can be viewed as playing only an auxiliary role. Thus by moving along geodesics, one particle maps out or "observes" the spacetime geometry determined by the other particle. Conversely, reversing the roles, the first particle maps out or "observes" the geometry determined by the second. In fact both processes occur at once; the spacetime geometry is determined by all the particles present through T.

We shall now try to extend these ideas to the quantum mechanical setting. The first step is an established slight generalization, *quantum statistical mechanics*, of the concrete operator framework adopted in the previous section. This abstract approach was emphasized by [14, Chap. 17], and developed in [15]. The *observables* are now the self-adjoint elements, a, of an abstract C^* algebra A, the *algebra of observables*. The *states* are now the positive elements, Ψ, of the dual linear space A^*. Positive means $\Psi(a^*a)\geqslant 0$. The pairing $\Psi(a)$ is the "expectation value of a in state Ψ." For convenience it is assumed that A is unital and that states are scaled such that $\Psi(1)=1$.

This includes the previous framework where the algebra of observables was constructed as a subalgebra of $B(H)$ and where $\Psi(a)$ was of the form $(\psi, a\psi)$ for $\psi\in H$. In the case of $B(H)$, all positive linear functionals are of the form

$$\Psi(a)=(\psi_1, a\psi_1)t_1+\cdots+(\psi_n, a\psi_n)t_n, \qquad t_i>0, \sum t_i=1$$

(Gleason's theorem [14, Chap. 17]). Thus a general state should be viewed as describing a statistical combination of an arbitrary finite number of single-particle states ψ_i. Note also that a single abstract quantization can lead to many concrete solutions of the quantization problem by composing with different representations.

In the algebraic case we work with an abstract algebra over k. The $*$ structure and the restrictions defined in terms of it will be neglected. These restrictions are physically important. They guarantee that probabilities lie in the range [0, 1]. They also enable one to establish the precise sense in which elements of the dual linear space A^* are representations of A. Thus for C^* algebras there is a one–one correspondence between states and faithful representations of A as an operator algebra [6, Chap.1]. The set of all states is a locally convex space and its extreme points correspond to irreducibles. The $*$ structure is compatible with the bicrossproduct construction, but does not play a direct role. For this reason it is suppressed.

In the algebraic case without such a $*$ structure the situation is weaker with regard to interpretation.

We now ask, following the ideas of Mach, to maintain symmetry between observer and observed, i.e., between observables and states. In our case the algebra of observables was found to be $k[G_1] \ltimes_\alpha k(M)$. We therefore ask for a similar structure on the dual linear space A^*, i.e., a semidirect coproduct coalgebra structure on A by a dual action. In this case it is natural to ask that these fit together to form a Hopf algebra. This places certain restrictions on α. These can be viewed as equations for α with the dual action playing an auxiliary role. Recall that in our formulation α plays the role of the spacetime metric, so these equations are somewhat comparable to Einstein's equations.

For example, for rectilinear motion in $\mathbb{R}^n$, the quantum algebra of observables (the Weyl algebra) admits no Hopf algebra structure of self-dual type. The Hopf algebra requirement forces a modification of α (Lemma 2.6). This modifies the classical motion. Thus gravity-like forces are literally induced by the observables–states symmetry consideration, in accordance with the philosophical ideas of E. Mach.

Apart from its philosophical value, it is expected that more complex models of closed systems combining quantum mechanics with gravity could be obtained. They would correspond to more complex self-dual algebraic structures. The advantage for physicists is that this approach is highly constrained. It may solve a long-standing problem in theoretical physics known as the "puzzle of the cosmological constant." This refers to the parameter Λ in Einstein's equation. Experimentally it has been measured to be extremely small or zero. But there is no theoretical reason for this, and conventional estimates would predict a very large typical value. It may be that the cosmological constant vanishes on symmetry grounds: we propose the observable–state symmetry as a possible candidate for this. It remains to find more realistic models, preferably a class of ones in which $\Lambda = 0$ when T is computed in a natural state Ψ.

1.1.4. *Remark on "String Theory"*

We have described above the author's approach to algebraic systems combining quantum mechanics and gravity along fairly traditional lines. In recent times a more topical approach to this goal is provided by "string theory." This section indicates briefly how the two approaches might ultimately be related.

In "string theory" the role of spacetime in the above is played by $M = \mathrm{Map}(S^1, M_0)$, where M_0 is spacetime. Trajectories in this space M should be viewed as mapping out tubes in M_0. Unlike the above treatment where the motion was purely along geodesics, these tubes are allowed to interact. For example, two tubes may join. Quantization reduces the study of the system to algebra. In this algebraic setting, there are constraints on the allowed interactions or "couplings." In a recent formulation [16], V denotes a vector space, "the space of couplings," and the constraints take the form of relations satisfied by a "braiding operator" B and a "fusion operator" F from a subspace of $V\otimes V$ to $V\otimes V$. Writing $R\equiv\tau\circ B$, where $\tau\colon V\otimes V\to V\otimes V$ is the twist map, the relations in [16] for B take the form of the "quantum Yang–Baxter equations" for R. These will be described in detail in Section 1.2. The relations for $W\equiv\tau\circ F$, the "pentagon identities" [16], take the form

$$\sum W_{(1)i}W_{(1)j}\otimes W_{(2)i}W_{(1)k}\otimes W_{(2)j}W_{(2)k}=\sum W_{(1)i}\otimes W_{(1)j}W_{(2)i}\otimes W_{(2)j},$$

where $W\equiv\sum W_{(1)}\otimes W_{(2)}$ (formal $\sum$ notation).

It is widely conjectured that "string theories" are part of a larger class of *quantum integrable systems*. This means roughly that they can be completely solved by algebraic means. In many of these systems these "quantum Yang–Baxter equations" also play a central role, and Hopf algebras have been introduced by Drinfel'd as a tool to solve them. This is described in Section 1.2 also. The equations also turn up in Braid theory. The entire situation is reviewed in [17], where references may be found.

One might hope then that Hopf algebras play a central role in the structure of "string theory" too. So far such an underlying Hopf algebra has not been found. However, I would like to make the fresh observation that these "pentagon identities" for an invertible element $W\in A\otimes A$ for an algebra A imply that A is a bialgebra with

$$\Delta a=W^{-1}(1\otimes a)W,\quad \forall a\in A.$$

This observation is borrowed from Kac algebra theory [18, Sect. 3.1.7]. In the shorthand notation $W_{13}=\sum_i W_{(1)i}\otimes 1\otimes W_{(2)i}$, etc., the proof is

$$(\Delta\otimes 1)\,\Delta a=W_{12}^{-1}W_{23}^{-1}(1\otimes 1\otimes a)\,W_{23}W_{12},$$
$$(1\otimes\Delta)\,\Delta a=W_{23}^{-1}W_{13}^{-1}(1\otimes 1\otimes a)\,W_{13}W_{23}.$$

These are equal if $W_{23}W_{12}=W_{12}W_{13}W_{23}$ (our form of the "pentagon identity") because W_{12} commutes with $1\otimes 1\otimes a$. Unfortunately, the F, B, R, W that arise in the "string theory" context are not defined on all of $V\otimes V$ so only a partial bialgebra structure is obtained.

If Hopf algebras do turn out to play a central role in the structure of "string theory," then that role might persist to those aspects of "string theory" that correspond to quantum mechanics of particles in spacetime combined with gravity. Conversely, in this case the interesting interpretation of Hopf algebra duality in the last section might extend to a deeper duality in the "string theory." The structure of "string theory" is an area of active research at the time of writing, and provides a second physical motivation for interest in the bicrossproduct.

1.2. *Non-commutative Geometry and the Yang–Baxter Equations*

Commutative rings or algebras A in general have a geometrical significance according to the standard interpretation of algebraic geometry. They essentially take the form $A \sim k(X)$, where X is the spectrum space of A. This is made precise in the C^* algebra setting by the theorem of Gelfand and Naimark. This asserts that every commutative C^* algebra A is isomorphic to a C^* algebra of the form $C(X)$. Here $C(X)$ denotes the C^* algebra of continuous functions vanishing at infinity on the locally compact space X. X is obtained explicitly as the space of non-trivial morphisms from A to $\mathbb{C}$ with the weak* topology [6, Sect. 1.2.1].

When A is non-commutative there is no longer any underlying space X such that A is its ring or algebra of functions. Nevertheless, it has been found that many ideas of algebraic geometry still make sense in the non-commutative case. One can carry on doing geometry as if there were an underlying space X, called a *non-commutative space* [19]. Thus Hopf algebras can be thought of heuristically as groups in this generalized, possibly non-commutative sense.

A further refinement of this picture has proven fruitful, for example, the *non-commutative torus* [20]. This tries to obtain non-commutative spaces in a natural way as "extensions" or *deformations* of ordinary ones. The non-commutative torus is a semidirect product of the convolution algebra on $\mathbb{Z}$ with the function algebra on S^1, and the action is generated by rotation by a fixed angle (the parameter). When the parameter is zero, the algebra reduces (after a Fourier transform in one variable) to functions on $S^1 \times S^1$. The present paper grew out of the author's attempt to understand [20] from a physical point of view (the picture is given in Section 1.1). One would then hope to use this picture to obtain more examples.

If at the same time as deforming the algebra structure, we are able to define a non-cocommutative coproduct structure then we have a non-commutative space version of a non-Abelian group. Indeed, in a C^* algebra setting the results of Lemma 2.6 can essentially be interpreted as a non-commutative space version of the non-Abelian Lie group of two dimensions.

For ordinary Lie groups, non-commutativity of the group corresponds to the presence of Riemannian curvature for the intrinsic connection. Thus in the general case non-cocommutativity of the coproduct corresponds to curvature of X in the generalized sense. A substantial amount of machinery of "non-commutative geometry" exists to be applied in such a situation. For example, one may build vector bundles (these correspond to finitely generated projective modules) as in [20]. However, a problem in the subject of non-commutative geometry is a general shortage of examples of such non-commutative geometries exhibiting curvature. Thus the significance of our results from this point of view is that they do provide a new class of such examples once the bicrossproduct construction is translated into a Hopf–von Neumann algebra or C^* algebra setting.

In the Hopf algebra setting one may attempt to quantify the degree of non-cocommutativity, i.e., the curvature on X. We remark that such a quantity can arise naturally in the following construction of Drinfel'd (from a very different context).

Some Constructions of Drinfel'd

Drinfel'd considers the following situation [2, Sect. 10]. Suppose that the k-Hopf algebra H has an invertible element $R \in H \otimes H$ such that

$$\tau \circ \Delta h = R(\Delta h) R^{-1}, \qquad \forall h \in H,$$

where $\tau\colon H \otimes H \to H \otimes H$ is the twist map. Suppose further that

$$(\Delta \otimes 1) R = \sum R_{(1)i} \otimes R_{(1)j} \otimes R_{(2)i} R_{(2)j},$$

$$(1 \otimes \Delta) R = \sum R_{(1)i} R_{(1)j} \otimes R_{(2)j} \otimes R_{(2)i},$$

where $R \equiv \sum R_{(1)} \otimes R_{(2)}$ (formal $\sum$ notation). These latter equations assert respectively that $R\colon H^* \to H\colon \phi \mapsto \sum \phi(R_{(1)}) R_{(2)}$ is an algebra map and (in the case when H^* is finite dimensional) an anti-coalgebra map. This is an easy exercise. It will not be needed in the sequel.

In this situation Drinfel'd says that the Hopf algebra is *quasitriangular* and it is easy to show that R necessarily obeys the *Quantum Yang–Baxter Equations*

$$\sum R_{(1)i} R_{(1)j} \otimes R_{(2)i} R_{(1)k} \otimes R_{(2)j} R_{(2)k}$$

$$= \sum R_{(1)j} R_{(1)i} \otimes R_{(1)k} R_{(2)i} \otimes R_{(2)k} R_{(2)j}. \qquad \text{(QYBE)}$$

This is because in this case, $(1 \otimes \tau \circ \Delta) R = R_{(1)i} R_{(1)j} \otimes R_{(2)i} \otimes R_{(2)j}$ while $(1 \otimes R)((1 \otimes \Delta) R)(1 \otimes R^{-1}) = (R_{(1)j} R_{(1)i} \otimes R_{(1)k} R_{(2)i} \otimes R_{(2)k} R_{(2)j})$ $(1 \otimes R^{-1})$ after a relabelling of the formal summation indices. Equality of these two expressions is the QYBE after multiplying by $(1 \otimes R)$ on the right.

The QYBE are usually posed for $R \in \mathrm{End}_k V \otimes \mathrm{End}_k V$, where V is a vector space, and usually written in the compact from $R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$. Here $R_{13} \equiv \sum R_{(1)} \otimes 1 R_{(2)}$, etc. The QYBE arise in a wide variety of contexts. A relevant one was given in Section 1.1.4. The reader is referred to [2, Sect. 11] for a beautiful account, intended for algebraists, of the original context in physics. Many of the solutions known in a physical context can be obtained from a quasitriangular Hopf algebra by composing with a representation in $\mathrm{End}_k V$.

One approach to solving the QYBE is to first solve the linearized version. We work for the moment over $\mathbb{C}$ and suppose that R has an expansion in powers of $\hbar$ of the form $R=1\otimes 1+\hbar\sum r_{(1)}\otimes r_{(2)}+O(\hbar^2)$. Here $\hbar$ is a real parameter and typically this will be an asymptotic expansion with respect to an operator norm. Then the QYBE become the *Classical Yang–Baxter Equations*

$$\sum [r_{(1)i}, r_{(1)j}]\otimes r_{(2)i}\otimes r_{(2)j}+r_{(1)i}\otimes [r_{(2)i}, r_{(1)j}]\otimes r_{(2)j} + r_{(1)i}\otimes r_{(1)j}\otimes [r_{(2)i}, r_{(2)j}]=0 \qquad \text{(CYBE)}$$

This makes sense in $\mathrm{End}_k V\otimes \mathrm{End}_k V\otimes \mathrm{End}_k V$. In fact, since the CYBE only involves commutators it can be solved by first solving abstractly for an element $r\in g\otimes g$, where g is a Lie algebra, and then composing with a representation of g in $\mathrm{End}_k V$.

Analogous to the notion of a quasitriangular Hopf algebra, Drinfel'd introduces the notion of a quasitriangular *Lie bialgebra.* Thus a *Lie coalgebra* is a pair (g, δ), where g is a vector space and δ: $g\to g\otimes g$ obeys axioms dual to those of a Lie algebra. Explicitly, these are [21]

$$\text{image}\,\delta\subset \text{image}(1-\tau), \qquad (1\otimes\delta)\,\delta\xi+\text{cyclic}=0, \qquad \forall\xi\in g.$$

(The second of these will be called the "coJacobi identity.") Here τ is the twist map and "+ cyclic" means to add cyclic permutations in $g\otimes g\otimes g$. A pair (g, δ) in which g is both a Lie algebra and a Lie coalgebra is a Lie bialgebra if $\delta\in Z^1_{\mathrm{ad}}(g, g\otimes g)$ (i.e., a 1-cocycle [22, Chap. VII, Sect. 4]). Here g acts on $g\otimes g$ via the adjoint action. Explicitly, the condition on δ is

$$\delta([\xi, \xi'])-\mathrm{ad}_\xi(\delta(\xi'))+\mathrm{ad}_{\xi'}(\delta(\xi))=0, \qquad \forall\xi, \xi'\in g.$$

Explicitly, the adjoint action on g is $\mathrm{ad}_\xi(\xi')=[\xi, \xi']$ and this extends as a derivation to higher tensor products.

A Lie bialgebra is quasitriangular if δ is the Lie algebra coboundary of an element $r\in g\otimes g$ which obeys the CYBE. We now show, following Drinfel'd, that every solution of the CYBE on Lie algebra g such that r has ad-invariant symmetric part is associated to a quasitriangular Lie bialgebra. Thus suppose that r is of the form $r=r^-+r^+$ and r^+ is ad-invariant. Denote by $C(r)$ the expression on the left hand side of the

CYBE for r above (so the CYBE for r is $C(r)=0$). We first observe that $C(r)=C(r^-)+C(r^+)$. This is because C is quadratic and the cross terms vanish by the symmetries of $r^\pm$ and ad-invariance of r^+. Hence, that r obeys the CYBE is $C(r^-)=-C(r^+)$. As an element of $g\otimes g\otimes g$ the right hand side of this is manifestly ad-invariant (because both the Lie bracket in g and r^+ are ad-invariant). To complete the proof we need the following lemma.

LEMMA 1.1 (Drinfel'd [23]). *Let g be a Lie algebra, $r\in \Lambda^2 g$. Let $C(r)$ denote the left hand side of the CYBE. Let $\delta\colon g\to g\otimes g$ be defined by $\delta=dr$, where r is viewed as an element of $C^0_{\mathrm{ad}}(g, g\otimes g)$, the Lie algebra cohomology complex with values in $g\otimes g$ and the adjoint action. Explicitly, this means*

$$\delta\xi=\sum [\xi, r_{(1)}]\otimes r_{(2)}+r_{(1)}\otimes [\xi, r_{(2)}]=(1-\tau)\sum (r_{(1)}\otimes [\xi, r_{(2)}]),\quad \forall\xi\in g.$$

Then (g,δ) is a Lie bialgebra iff $C(r)$ is ad-invariant.

Proof. From the second expression for δ we see that image$(\delta)\subset$ image$(1-\tau)$. This is the axiom for a Lie coalgebra which, in the finite dimensional case, is the dual of the condition that the bracket on g^* corresponding to δ (i.e., such that $\langle\delta\xi, l\otimes l'\rangle=\langle\xi, [l, l']\rangle$, $\forall l, l'\in g^*$, $\xi\in g$) obeys $[l, l]=0$ for all $l\in g^*$. Since $\delta=dr$, $d^2=0$ implies that $\delta\in Z^1_{\mathrm{ad}}(g, g\otimes g)$ (i.e., a one-cocycle). Hence a pair (g,δ) of this form is a Lie bialgebra iff δ obeys the coJacobi identity (corresponding in the finite dimensional case to the Jacobi identity on g^*). The coJacobi identity is the vanishing of $(1\otimes\delta)\,\delta\xi+$ cyclic permutations in $g\otimes g\otimes g$. But

$$\begin{aligned}(1\otimes\delta)\,\delta\xi+\text{cyclic}&=[\xi, r_{(1)i}]\otimes[r_{(2)i}, r_{(1)j}]\otimes r_{(2)j}\\ &\quad+[\xi, r_{(1)i}]\otimes r_{(1)j}\otimes[r_{(2)i}, r_{(2)j}]\\ &\quad+[[\xi, r_{(2)i}], r_{(1)j}]\otimes r_{(2)j}\otimes r_{(1)i}\\ &\quad+[[\xi, r_{(2)i}], r_{(2)j}]\otimes r_{(1)i}\otimes r_{(1)j}+\text{cyclic}\\ &=[\xi, C(r)_{(1)}]\otimes C(r)_{(2)}\otimes C(r)_{(3)}+\text{cyclic},\end{aligned}$$

where in the first step, the terms in the cyclic average involving ξ in the second and third factors of $g\otimes g\otimes g$ were rotated into the first factor. In the second step the formal summation indices were interchanged in the third term, and we use antisymmetry of r in the third and fourth terms, and the Jacobi identity in g, to obtain the last expression. But since r is antisymmetric, $C(r)$ is already invariant under the permutations, so the last expression is identical to $[\xi, C(r)_{(1)}] \otimes C(r)_{(2)} \otimes C(r)_{(3)} + C(r)_{(1)} \otimes [\xi, C(r)_{(2)}] \otimes C(r)_{(3)} + C(r)_{(1)} \otimes C(r)_{(2)} \otimes [\xi, C(r)_{(3)}] = \mathrm{ad}_\xi(C(r))$. Thus the coJacobi identity is precisely that $C(r)$ be ad-invariant. This completes the lemma. To use the lemma in the present case we define $\delta = dr^-$. Then applying the lemma to r^-, we have that (g, δ) is a Lie bialgebra if r obeys the CYBE and $r = r^- + r^+$ with r^+ ad-invariant. It is quasitriangular because $\delta = dr^- = dr$ by a further use of the ad-invariance of r^+.

In the context of non-commutative geometry, which is the context relevant to the present paper, the linearization r of R plays a role roughly comparable to "Riemannian curvature" on the non-commutative space. This is because we have seen in the context that it was introduced that r measures the linearized extent to which Δ is non-cocommutative. However, because we are working with Hopf algebras, the same deformation idea can also be formulated in a dual language. This is the point of view in which the QYBE actually arises (and Drinfel'd's point of view). Thus if g is a Lie algebra over $\mathbb{C}$ and $U(g)$ its universal enveloping algebra, we can look for a non-cocommutative deformation $U_\hbar(g)$. It should be thought of as the dual of the function algebra of a non-commutative-geometric version of a group with Lie algebra g. In this interpretation the r measures the degree to which the geometry is non-commutative in the sense of non-commutative geometry. Such $U_\hbar(g)$ have been found by mathematical physicists for all simple complex Lie algebras g. Moreover, $U_\hbar(g)$ is non-commutative as an algebra because g is non-commutative as a Lie algebra. In this context, non-commutative non-cocommutative Hopf algebras have been called *quantum groups.*

One basis for the construction of $U_\hbar(g)$ is as follows. First find a quasitriangular structure on g, (g, r). Then try to find a quasitriangular Hopf algebra $(U_\hbar(g), R)$ over $\mathbb{C}[[\hbar]]$ such that $U_\hbar(g)/\hbar U_\hbar(g) = U(g)$ and $U_\hbar(g)$ is a topologically free $\mathbb{C}[[\hbar]]$-module and $(R - 1 \otimes 1)/\hbar \bmod \hbar = r$. This is achieved following the methods of physicists to solve the QYBE, and the result is called the quantized universal enveloping algebra. For completeness, we describe it briefly but without proof. It should be stressed that the non-commutative non-cocommutative Hopf algebras introduced in the present paper are quite different from these quantum groups, and arose independently in a different physical setting, as described in Section 1.1.

Indeed, the geometrical significance of Lie bialgebras was explained by Drinfel'd as follows [23]. If G is a simple simply connected Lie group with real Lie algebra g, then $C^\infty(G)$ has a Poisson bracket suitably respecting the group product if and only if g is a Lie bialgebra [23]. "Poisson bracket" means a skew bilinear map $\{\ ,\ \}: C^\infty(G) \otimes C^\infty(G) \to C^\infty(G)$ such that $\{f,\ \}$ is a derivation for all $f \in C^\infty(G)$, and the Jacobi identity holds. When the Lie bialgebra is a quasitriangular one and the associated solution r of the CYBE is antisymmetric, then explicitly the Poisson bracket structure is $\{f, g\} = r^{\mathrm{L}}(df, dg) - r^{\mathrm{R}}(df, dg)$. Here r^{L}, r^{R} denote r extended to T^2G respectively as a left-invariant and a right-invariant tensor field.

Poisson bracket structures on a phase space G (as a manifold) are the starting point for one approach to classical and quantum mechanics (related to the formulation of Section 1.1). This replaces the "classical observables," functions on G, by an operator algebra, the "quantum algebra of observables" in such a way that $i\hbar\widehat{\{f, g\}} = [\hat{f}, \hat{g}]$ (cf. the *commutation relations* in Section 1.1). Applying this formalism and dualizing is the heuristic derivation of $U_\hbar(g)$, the quantized version of $U(g)$. Formal definitions are in [2]. Note that these are formal constructions to solve the QYBE. The actual physical significance of these Hopf algebras is only slowly emerging. A reference is [24]. The case $U_\hbar(su(2))$ was obtained independently in a dual form by Woronowicz, and the general case was also obtained in another context by M. Jimbo; see [24]. Note that $q = e^\hbar$ is another current notation for the parameter. In the setting of a quantum mechanical system, one would choose physical units such that $\hbar = 1$.

The Examples of $D(g)$ and $D(H)$

Some important examples of Lie bialgebras arise as follows. For every finite dimensional Lie bialgebra g there is a Lie bialgebra $D(g)$. The Lie algebra structure is defined in [23] as the unique Lie algebra structure on $g \oplus g^*$ such that the canonical symmetric bilinear form on $g \oplus g^*$ is ad-invariant. In [2, Sect. 13] it is observed that the canonical element of $g \otimes g^*$ can be viewed as an element r of $(g \oplus g^*) \otimes (g \oplus g^*)$. With respect to this, $D(g)$ is a quasitriangular Lie bialgebra.

To see this, let $\{e_a\}$ be a basis for g and $\{f^a\}$ a dual basis. The canonical r is then explicitly $r = (e_a, 0) \otimes (0, f^a) \in (g \oplus g^*) \otimes (g \oplus g^*)$ (summation over repeated indices understood). The Lie algebra structure $D(g)$ on $g \oplus g^*$ is explicitly defined for (ξ, l), $(\xi', l') \in D(g)$ by

$$\begin{aligned}
&\langle (l'', \xi''), [(\xi, l), (\xi', l')] \rangle \\
&\quad = \langle l'', [\xi, \xi'] \rangle + \langle \delta\xi', l'' \otimes l \rangle - \langle \delta\xi, l'' \otimes l' \rangle + \langle \delta\xi'', l \otimes l' \rangle \\
&\qquad + \langle l', [\xi'', \xi] \rangle - \langle l, [\xi'', \xi'] \rangle, \qquad \forall (l'', \xi'') \in D(g)^*.
\end{aligned}$$

Note that g and g^* are subalgebras. Using these definitions it is easy to compute (by evaluating $C(r)$ on arbitrary elements of $g^* \oplus g$) that $C(r) = 0$, i.e., r obeys the CYBE. In characteristic 2, the lemma above cannot be applied directly, but $\delta = dr$ still works (see Example 4.2). In other characteristics, we write $r = r^- + r^+$, where $r^+ = \frac{1}{2}((e_a, 0) \otimes (0, f^a) +$

$(0, f^a) \otimes (e_a, 0))$, and check that r^+ is ad-invariant. Hence by the above, $(D(g), r)$ is a quasitriangular Lie bialgebra.

Similarly, for every finite dimensional Hopf algebra H there is a quasitriangular Hopf algebra $D(H)$. It is constructed as follows [2, Sect. 13]. Let $H^{*\mathrm{op}}$ denote the Hopf algebra dual with the opposite coproduct. On the set $H \otimes H^{*\mathrm{op}}$ define the coproduct to be the tensor coproduct structure. Define the product as

$$(h \otimes \phi) \cdot (g \otimes \psi) = \sum h_{(2)} g \otimes \phi \langle \psi, (S' h_{(3)})? \, (h_{(1)}) \rangle, \quad \forall h, g \in H, \phi, \psi \in H^{*\mathrm{op}}.$$

Here $\langle \ , \ \rangle$ denotes the pairing of H with $H^{*\mathrm{op}}$ as a linear space and ψ is written as $\langle \psi, ? \rangle$, where "?" denotes an unused argument. Then $\langle \psi, (S' h_{(3)}) \, ? \, (h_{(1)}) \rangle$ is a convenient way to write ψ precomposed with operations of left and right multiplication in H as indicated. S' denotes the skew-antipode of H and $\Delta h = \sum h_{(1)} \otimes h_{(2)}$ the coproduct. The unit and counit are the tensor product unit and counit. There is also a suitable antipode (see Section 4).

$D(H)$ is quasitriangular with respect to R the canonical element of $H \otimes H^{*\mathrm{op}}$ viewed in $H \otimes H^{*\mathrm{op}} \otimes H \otimes H^{*\mathrm{op}}$. To see this, let $\{e_a\}$ be a basis for H and $\{f^a\}$ a dual basis. The canonical element referred to is $R = (e_a \otimes 1) \otimes (1 \otimes f^a)$. To verify $\tau \circ \Delta = R(\Delta) R^{-1}$ we have to show that

$$\begin{aligned}&(h_{(2)} \otimes \phi_{(2)}) \cdot (e_a \otimes 1) \otimes (h_{(1)} \otimes \phi_{(1)}) \cdot (1 \otimes f^a) \\ &\quad = (e_a \otimes 1) \cdot (h_{(1)} \otimes \phi_{(1)}) \otimes (1 \otimes f^a) \cdot (h_{(2)} \otimes \phi_{(2)}).\end{aligned}$$

Multiplying out as defined and evaluating on arbitrary $\phi' \otimes h' \otimes \phi'' \otimes h''$ one obtains $\langle \phi', h''_{(2)} h_{(1)} \rangle \langle \phi, h' h''_{(1)} \rangle \langle \phi'', h_{(2)} \rangle$ for both sides. For the computation one should use the relations between the structures of H and $H^{*\mathrm{op}}$, and the skew-antipode property $h_{(2)} S' h_{(1)} = \varepsilon(h) = (S' h_{(2)}) h_{(1)}$ for all $h \in H$. (Note in particular that the unit in $H^{*\mathrm{op}}$ is given by the counit ε_H in H.) We also have to see that R is invertible. R^{-1} is explicitly given by $R^{-1} = (e_a \otimes 1) \otimes (1 \otimes f^a \circ S)$, as is readily verified by multiplying out RR^{-1} and $R^{-1}R$ and evaluating on elements of $H^{*\mathrm{op}} \otimes H \otimes H^{*\mathrm{op}} \otimes H$. Finally, to verify $(\Delta \otimes 1) R = R_{13} R_{23}$ we have to show that $(e_{a(1)} \otimes 1) \otimes (e_{a(2)} \otimes 1) \otimes (1 \otimes f^a) = (e_a \otimes 1) \otimes (e_b \otimes 1) \otimes (1 \otimes f^a) \cdot (1 \otimes f^b)$. This holds because evaluating on arbitrary $\phi \otimes h \otimes \phi' \otimes h' \otimes \phi'' \otimes h''$ gives both sides to be $\phi''(1) \langle h'', \phi \phi' \rangle \, \varepsilon_H(h) \, \varepsilon_H(h')$. The computation of $(1 \otimes \Delta) R = R_{13} R_{12}$ is similar.

Thus $D(H)$ is quasitriangular. Drinfel'd also defines $D(H)$ as the unique quasitriangular Hopf algebra containing H and $H^{*\mathrm{op}}$ and obeying certain conditions. These examples will play a role in Section 4. This completes the review of constructions of Drinfel'd and non-commutative geometry.

2. Examples of the Form $k[G_1] \bowtie k(G_2)$

In the Introduction we have motivated the following question: When is a semidirect product algebra a Hopf algebra of self-dual type? In the simplest case based on groups, the semidirect construction can be handled explicitly. In this section we state the explicit solution in this case, and give a number of examples. Theorem 2.1 (in the context of algebraic groups) and Theorem 2.3 were also obtained by Takeuchi [1] following work of Singer. The non-trivial examples are new. The abstract Hopf algebra treatment is given in Section 3.

In the remaining sections, unless otherwise stated, k denotes an arbitrary commutative ground field. Let G_1 and G_2 be two finite groups. We consider the situation in which α is an action of group G_1 on the set G_2, and β an action of the group G_2 on the set G_1, i.e.,

$$\begin{aligned}&\forall u, v \in G_1,\ s, t \in G_2;\\ &\alpha_{uv}(s) = \alpha_u(\alpha_v(s)), \qquad \alpha_e(s) = s, \qquad \beta_{st}(u) = \beta_s(\beta_t(u)), \qquad \beta_e(u) = u,\end{aligned} \tag{1}$$

where e denotes a group identity. Note that G_1 (and $k[G_1]$) automatically acts on $k(G_2)$ and we continue to denote this by α; $\alpha_u(f)(s) = f(\alpha_{u^{-1}}(s))$, $\forall u \in G_1$, $s \in G_2$, $f \in k(G_2)$.

Given the action α, the semidirect or cross product algebra can be constructed explicitly as follows. Let K denote the set of k-linear combinations of elements in G_1 and coefficients in $k(G_2)$, $K = \{F = \sum_{u \in G_1} f_u u \mid f_u \in k(G_2)\}$. Define the algebra structure on K by

$$fugv = f\alpha_u(g)uv, \qquad 1_K = 1_{k(G_2)}e, \qquad \forall u, v \in G_1, f, g \in k(G_2).$$

We denote it $k[G_1] \ltimes_\alpha k(G_2)$ (in keeping with standard notation in the theory of operator algebras).

Similarly, the action β allows a dual construction, the semidirect or cross coproduct coalgebra. It can be defined explicitly on the same set K by

$$(\Delta fu)(s, t) = f(st)u \otimes \beta_{s^{-1}}(u), \qquad \varepsilon(fu) = f(e), \qquad \forall u \in G_1, f \in k(G_2).$$

We denote it $k[G_1]^\beta \rtimes k(G_2)$.

THEOREM 2.1. *Let G_1 and G_2 be finite groups acting on each other as sets by α, β. The cross product algebra–cross coproduct coalgebra K is a bialgebra iff $\forall u, v \in G_1$, $s, t \in G_2$*

$$\alpha_u(e) = e, \qquad \alpha_u(st) = \alpha_u(s)\alpha_{\beta_{s^{-1}}(u^{-1})^{-1}}(t) \tag{A}$$

$$\beta_s(e) = e, \qquad \beta_s(uv) = \beta_s(u)\beta_{\alpha_{u^{-1}}(s^{-1})^{-1}}(v). \tag{B}$$

In this case there is an antipode defined by

$$(SF)(s)=\sum_{u\in G_1} f_{\beta_{s^{-1}}(u)^{-1}}(\alpha_{u^{-1}}(s)^{-1})u, \qquad \forall F=\sum_{u\in G_1} f_u u\in K,$$

and we denote the resulting Hopf algebra by $k[G_1]^\beta \bowtie_\alpha k(G_2)$, *the* bicrossproduct Hopf algebra *with structure maps* α, β.

This is straightforward to check explicitly from the definitions. Such an explicit proof also makes it clear that G_1, G_2 need only be monoids (i.e., unital semigroups). For in this case one can work with the anti-actions $\bar\alpha$, $\bar\beta$,

$$\bar\alpha_{uv}(s)=\bar\alpha_v(\bar\alpha_u(s)), \qquad \bar\alpha_e(s)=s, \qquad \bar\beta_{st}(u)=\bar\beta_t(\bar\beta_s(u)), \qquad \bar\beta_e(u)=u \tag{2}$$

acting by bijective maps obeying

$$\bar\alpha_u(st)=\bar\alpha_u(s)\,\bar\alpha_{\bar\beta_s(u)}(t), \qquad \bar\beta_s(uv)=\bar\beta_s(u)\,\bar\beta_{\bar\alpha_u(s)}(v), \qquad \bar\alpha_u(e)=e, \qquad \bar\beta_s(e)=e. \tag{3}$$

In this case $k[G_1]^\beta \bowtie_\alpha k(G_2)$ is a unital and counital k-bialgebra defined analogously to Theorem 2.1. Here $\bar\alpha_u$ plays the role of $\alpha_{u^{-1}}$.

The condition (A) asserts that α is an action of G_1 as a group almost by automorphisms of G_2, as modified by β. The condition (B) asserts that β is an action of G_2 almost by automorphisms of G_1, as modified by α. The antipode S has order 2. Note that only this construction can lead to a self-dual Hopf algebra structure on $k[G_1] \ltimes_\alpha k(G_2)$. In general it is of self-dual type in the following sense.

THEOREM 2.2. *Let G_1, G_2 be finite groups and k a field. Then under the conditions of Theorem* 2.1,

$$(k[G_1]^\beta \bowtie_\alpha k(G_2))^* \cong k[G_2]^\alpha \bowtie_\beta k(G_1).$$

Proof. The dual Hopf algebra structure was defined in the Introduction in this finite dimensional case. The definition does not require any restrictions on k. The proof in our case can be checked explicitly as follows from the fact that $k[G]$ and $k(G)$ are dual for any finite group G. The pairing is given by $\langle f, s\rangle = f(s)$, $\forall f\in k(G)$, $s\in G\subset k[G]$, extended by linearity to all of $k[G]$. Hence, for $\phi, \psi\in k(G_1)$, $s, t\in G_2\subset k[G_2]$ we have $\langle \Delta fu, \phi s\otimes \psi t\rangle = \langle (\Delta fu)(s,t), \phi\otimes\psi\rangle = f(st)\,\phi(u)\,\psi(\beta_{s^{-1}}(u)) = \langle fu, \phi\beta_s(\psi)st\rangle = \langle fu, \phi s\psi t\rangle$. The units, counits, and antipodes are similarly in duality. To check the antipodes in this way, the identities $\alpha_{u^{-1}}(s)^{-1}=\alpha_{\beta_{s^{-1}}(u)^{-1}}(s^{-1})$ and $\beta_{s^{-1}}(u)^{-1}=\beta_{\alpha_{u^{-1}}(s)^{-1}}(u^{-1})$ are useful.

It is evident in Theorem 2.1 that the necessary data $(G_1, G_2, \alpha, \beta)$ are defined at the level of groups. One may therefore expect the same data to lead to an interesting construction at the level of groups. The following observation was also made previously by [1, Sect. 2]. It plays an important role in constructing examples.

THEOREM 2.3. *Let $(G_1, G_2, \alpha, \beta)$ be a quadruple consisting of groups G_1 and G_2 acting on each other as sets, by actions α, β. Then Eqs.* (A), (B) *in the statement of Theorem* 2.1 *hold iff the set $G_1 \times G_2$ with product*

$$(u, s) \cdot (v, t) = (\beta_{t^{-1}}(u^{-1})^{-1} v, s\alpha_u(t))$$

and identity (e, e) is a monoid. In this situation there is an inverse

$$(u, s)^{-1} = (\beta_s(u)^{-1}, \alpha_{u^{-1}}(s^{-1}))$$

and the resulting group structure is denoted $G_{1\beta}\bowtie_\alpha G_2$, the bicrossproduct group.

In this situation $(G_1, G_2, \alpha, \beta)$ is called a *matched pair* in [1, Sect. 2]. Reference [1] also gave an abstract definition of $G_1 \bowtie G_2$ when it exists, as a group containing G_1 and G_2 as subgroups and such that the map $G_1 \times G_2 \to G_1 \bowtie G_2 : (u, s) \mapsto su$ is a bijection (the reversal here is due to our choice of conventions). This will be developed further in Section 3.2 in the abstract Hopf algebra setting.

$G_{1\beta}\bowtie_\alpha G_2$ is not generally related to the k-Hopf algebras of Theorem 2.1 by the standard functors. This is because the Hopf algebras of Theorem 2.1 are neither commutative nor cocommutative. But in the case G_1 Abelian and α trivial or the case G_2 Abelian and β trivial we have respectively (in fact as Hopf algebras)

$$k[G_1]^\beta \rtimes k(G_2) \cong k(G_2 \ltimes_\beta \hat{G}_1), \qquad k[G_1] \ltimes_\alpha k(G_2) \cong k[G_1 \ltimes_\alpha \hat{G}_2]. \quad (4)$$

Here the actions in the respective cases are related by

$$\beta_s(\tau)(u) = \tau(\beta_{s^{-1}}(u)), \ \forall u \in G_1, \ \tau \in \hat{G}_1;$$

$$\alpha_u(\chi)(s) = \chi(\alpha_{u^{-1}}(s)), \ \forall s \in G_2, \ \chi \in \hat{G}_2$$

and we assume that k has characteristic 0. The assertions are easily proved by Fourier transformation in one variable. In particular, the general case of $k[G_1]^\beta \bowtie_\alpha k(G_2)$ may therefore be viewed as the "non-commutative version" of the non-Abelian group $G_2 \ltimes_\beta \hat{G}_1$ (in the sense of the function algebra $k(\ )$ becoming non-commutative). This was motivated in Section 1.2.

2.1. *Examples*

(A) Examples of the data $(G_1, G_2, \alpha, \beta)$ for Theorems 2.1–2.3 are obtained if β is trivial and α an action by automorphisms; similarly if α is trivial and β an action by automorphisms. For a concrete example,

EXAMPLE 2.4. Let $G_1 = G_2 = G$ be finite, $\alpha = \mathrm{Ad}$, and k a field. Then

$$k[G] \ltimes_{\mathrm{Ad}} k(G)$$

is a k-Hopf algebra with the tensor coproduct coalgebra structure. By a dual construction, $\beta = \mathrm{Ad}$ gives

$$k[G] \,{}^{\mathrm{Ad}}\!\rtimes k(G),$$

a k-Hopf algebra with the tensor product algebra structure. It is the dual of $k[G] \ltimes_{\mathrm{Ad}} k(G)$.

(B) More generally let $\alpha \in \hom(G_1, \mathrm{Aut}(G_2))$, $\beta \in \hom(G_2, \mathrm{Aut}(G_1))$ be actions by automorphisms such that

$$(\beta_{s^{-1}}(u)u^{-1})^{-1} \subset \ker \alpha, \qquad (\alpha_{u^{-1}}(s)s^{-1})^{-1} \subset \ker \beta \qquad \forall u \in G_1,\ s \in G_2.$$

Then again (A) and (B) hold. For example, let $G_1 \subset G$ and $G_2 \subset G$ be two subgroups of G a finite group that are mutually normalizing,

$$[G_1, G_2] \subset G_2, \qquad [G_2, G_1] \subset G_1,$$

let α and β be actions by the Adjoint action (conjugation) in G, and suppose that the subgroups are "weakly interacting" in the sense

$$[[G_1, G_2], G_1] = \{e\}, \qquad [[G_1, G_2], G_2] = \{e\}.$$

Then

$$k[G_1] \,{}^{\mathrm{Ad}}\!\bowtie_{\mathrm{Ad}} k(G_2)$$

is a k-Hopf algebra twisted according to the "interaction" of G_1 with G_2 as subgroups of G. Explicitly we assume

$$\alpha_u(s) = usu^{-1} \in G_2, \qquad \beta_s(u) = sus^{-1} \in G_1, \qquad \forall u \in G_1,\ s \in G_2.$$

For a concrete example,

EXAMPLE 2.5. Let $G_1 = G_2 = G$ be a finite group that is nilpotent of class 1. Then

$$k[G] \,{}^{\mathrm{Ad}}\!\bowtie_{\mathrm{Ad}} k(G)$$

is a self-dual Hopf algebra.

It should be possible to generalize this example to G an arbitrary nilpotent group.

(C) This is the general solution of the data $(G_1, G_2, \alpha, \beta)$ in the case $G_1 = \mathbb{R}$, $G_2 = \mathbb{R}$.

LEMMA 2.6. *Let $G_1 = G_2 = \mathbb{R}$ as additive groups. The general twice differentiable solution to* (1), (A), (B) *in a neighborhood of the origin in $\mathbb{R} \times \mathbb{R}$ is the two-parameter family of solutions*

$$\alpha_u(s) = \frac{1}{B} \ln(1 + e^{Au}(e^{Bs} - 1)),$$

$$\beta_s(u) = \frac{1}{A} \ln(1 + e^{Bs}(e^{Au} - 1)), \qquad \forall u \in G_1,\ s \in G_2$$

and $A, B \in \mathbb{R}$. These extend to the domain

$$D_{AB} = \{(u, s) \in \mathbb{R}^2 \mid e^{Au} + e^{Bs} > 1\}.$$

Proof. Equations (1), (A), (B) are to be solved for α, β twice differentiable real functions in the neighborhood of $(u, s) = (0, 0)$. One solution is $\alpha_u(s) = s$ and $\beta_s(u) = u$ (trivial actions). Hence writing $\alpha_u(s) \equiv s + f(-u, s)$, $\beta_s(u) \equiv u + g(-s, u)$ the equations become

$$\begin{aligned} f(u+v, s) &= f(v, s) + f(u, s + f(v, s)), \\ g(s+t, u) &= g(t, u) + g(s, u + g(t, u)) \end{aligned} \tag{5}$$

$$\begin{aligned} f(u, s+t) &= f(u, s) + f(u + g(s, u), t), \\ g(s, u+v) &= g(s, u) + g(s + f(u, s), v) \end{aligned} \tag{6}$$

$$f(u, 0) = f(0, s) = g(0, u) = g(s, 0) = 0. \tag{7}$$

We write

$$f_1(s) = \partial_1 f(0, s), \quad f_2(u) = \partial_2 f(u, 0), \quad g_1(u) = \partial_1 g(0, u), \quad g_2(s) = \partial_2 g(s, 0),$$

where ∂_i are derivatives in the ith argument, and let $\Delta(u, s) = 1 - f_1(s)\, g_1(u)$. Differentiating (5), (6), (7) one finds that either $\Delta(u, s) = 0$ or

$$\frac{f_1(s)(f_2(u) + 1)}{\Delta(u, s)} = \partial_1 f(u, s) = f_1(s + f(u, s)) \tag{8}$$

$$\frac{f_2(u) + g_1(u)\, f_1(s)}{\Delta(u, s)} = \partial_2 f(u, s) = f_2(u + g(s, u)) \tag{9}$$

$$\frac{g_1(u)(g_2(s)+1)}{\Delta(u,s)}=\partial_1\, g(s,u)=g_1(u+g(s,u)) \tag{10}$$

$$\frac{g_2(s)+f_1(s)\, g_1(u)}{\Delta(u,s)}=\partial_2\, g(s,u)=g_2(s+f(u,s)) \tag{11}$$

$$f_1(0)=f_2(0)=g_1(0)=g_2(0)=0.$$

Hence in principle f, g can be determined from the f_1, f_2, g_1, g_2. $\Delta(0,0)=1$ so in a suitable neighborhood of $(0,0)$, $\Delta(u,s)$ can be assumed to be non-zero. Differentiate Eqs. (9) and (11) by $(\partial/\partial u)|_0$ and $(\partial/\partial s)|_0$, respectively, to obtain

$$f_1'(s)=f_1(s)\, g_1'(0)+f_2'(0), \qquad g_1'(u)=g_1(u)\, f_1'(0)+g_2'(0).$$

These have unique solutions of the form

$$f_1(s)=\frac{A}{B}(e^{-sB}-1), \qquad g_1(u)=\frac{B}{A}(e^{-uA}-1), \qquad A, B\in\mathbb{R}.$$

Differentiating Eqs. (8) and (10) by $(\partial/\partial s)|_0$ and $(\partial/\partial u)|_0$, respectively, gives

$$f_2'(u)=f_2(u)\, f_1'(0)+f_1'(0), \qquad g_2'(s)=g_2(s)\, g_1'(0)+g_1'(0).$$

Solving these and inserting in the above gives f, g and hence α, β as shown. It is interesting that the solution extends only as far as the domain in $\mathbb{R}\times\mathbb{R}$ stated. The solution restricts to $(\mathbb{R}, \mathbb{R}_{\geqslant 0})$ and $(\mathbb{R}_{\geqslant 0}, \mathbb{R})$, where $\mathbb{R}_{\geqslant 0}$ is the additive semigroup.

The associated bialgebra corresponding to $(\mathbb{R}, \mathbb{R}_{\geqslant 0}, \alpha, \beta)$ will be described elsewhere in its topological setting (but see remark below Theorem 2.1). In the limit $A\mapsto 0$ we obtain that α is trivial and β an action by automorphisms. Then the associated bialgebra is isomorphic to functions on the non-Abelian (semi) group of the form $\hat{\mathbb{R}}_\beta \rtimes \mathbb{R}_{\geqslant 0}$ cf. Eq. (4). (The restriction to $\mathbb{R}_{\geqslant 0}$ is not essential.) This point of view was motivated in Section 1.2.

On the other hand, in the limit $A\mapsto\infty$ with $A/B=\hbar$ a fixed parameter, we have

$$\alpha_u(s)=\begin{cases}0, & s=0 \text{ else},\\ 0, & s+\hbar u\leqslant 0,\\ s+\hbar u, & s+\hbar u\geqslant 0,\end{cases} \qquad D_{\alpha\alpha}=(\mathbb{R}\times\mathbb{R}_{\geqslant 0})\cup(\mathbb{R}_{\geqslant 0}\times\mathbb{R}).$$

This limit remains an action (i.e., $\alpha_{uv}(s)=\alpha_u(\alpha_v(s))$) if the image of α remains positive. It has the physical interpretation given in Section 1.1

where it corresponds to rectilinear motion in the region $s>0$. The essentially self-dual case (in the sense of Theorem 2.2) is $A=B$ and lies in between these two limits.

The isomorphism ln: $\mathbb{R}_{>0} \cong \mathbb{R}$ can be used to put these solutions in the form

$$\alpha_u(s)=(1+u^A(s^B-1))^{1/B}, \qquad \beta_s(u)=(1+s^B(u^A-1))^{1/A}.$$

According to Theorem 2.3 we also have a group except that the solutions do not extend to all of $\mathbb{R}_{>0}$ as a multiplicative group. Let $u \oplus s = u+s-us$ so that $u \oplus s \geqslant 0$ iff $1/u+1/s \geqslant 1$. Then the semigroup structure for $A, B \geqslant 0$ is

$$\forall (u,s),\ (v,t) \in \mathbb{R}_{>0} \bowtie \mathbb{R}_{\geqslant 1},$$
$$(u,s)\cdot(v,t)=(u(u^A \oplus t^{-B})^{-1/A}\, v,\ s(u^A \oplus t^{-B})^{1/B}\, t).$$

Inverses when they are defined are given by

$$(u,s)^{-1}=(u^{-1}(u^{-A} \oplus s^B)^{-1/A},\ (u^{-A} \oplus s^B)^{1/B} s^{-1}).$$

In this form one may also take $A=B=1$ and $G_1=\mathbb{Q}_{>0}$, $G_2=\mathbb{Q}_{\geqslant 1}$ as a submonoid of the invertibles $\mathbb{Q}^*$. The solution of this section motivates the next example.

(D) The following construction is somewhat complementary to the adjoint actions of example (B).

LEMMA 2.7. *Let G_1 and G_2 be submonoids of A^* the group of invertible elements of a not necessarily commutative unital ring A, such that*

$$1+G_1^{-1}(G_2-1) \subset G_2, \qquad 1+G_2^{-1}(G_1-1) \subset G_1.$$

Then

$$\bar{\alpha}_u(s)=1+u^{-1}(s-1), \qquad \bar{\beta}_s(u)=1+s^{-1}(u-1), \qquad u \in G_1,\ s \in G_2$$

fulfill (2)–(3) *giving the bicrossproduct k-bialgebra*

$$k[G_1] \bowtie k(G_2).$$

This is a k-Hopf algebra if G_1 and G_2 are actually groups.

Proof (Similarly for $\bar{\beta}$). Note that inverses may be taken in A^*. We must check (2)–(3). Thus

$$\bar{\alpha}_{uv}(s)=1+(uv)^{-1}(s-1)=1+v^{-1}((1+u^{-1}(s-1))-1)=\bar{\alpha}_v(\bar{\alpha}_u(s)).$$

$\bar{\alpha}_1(s)=s$ and $\bar{\alpha}_u(1)=1$ because the units are embedded as the unit in A^*. Finally, we need $\forall s, t \in G_2$, $u \in G_1$,

$$\bar{\alpha}_u(st)=1+u^{-1}(st-1)=(1+u^{-1}(s-1))(1+X(t-1)),$$

where $X=\beta_s(u)^{-1}=(1+s^{-1}(u-1))^{-1}$. A sufficient condition for the desired equality, under the assumptions, is therefore $u^{-1}st = u^{-1}s+X(t-1)+u^{-1}(s-1)\,X(t-1)$. A sufficient condition for this is $u^{-1}s=(1+u^{-1}(s-1))X$ (then terms with and without t each equate). But this is valid on multiplying both sides on the left by $(u^{-1}s)^{-1}$ and inserting the expression for X.

For a finite dimensional example let $G_1=G_2=T_1(n, \mathbb{F}_q)$ the group of upper triangular matrices with values in $\mathbb{F}_q$ and 1 on the diagonal. This is a subgroup of the group of invertibles of the unital ring $M(n, \mathbb{F}_q)$. One may check

$$u, v \in T_1(n, \mathbb{F}_q) \Rightarrow 1+u^{-1}(v-1) \in T_1(n, \mathbb{F}_q)$$

so that the lemma applies.

EXAMPLE 2.8. Let $G_1=G_2=T_1(n, \mathbb{F}_q)$ and

$$\alpha_u(v)=\beta_u(v)=1+u(v-1).$$

Then

$$k[T_1(n, \mathbb{F}_q)] \bowtie k(T_1(n, \mathbb{F}_q))$$

is a self-dual k-Hopf algebra.

This example may be compared with the group Weyl algebra, i.e., the semi-direct product by action $\alpha_u(v)=uv$. The construction here gives data $(G_1, G_2, \alpha, \beta)$ for arbitrary fields in place of $\mathbb{F}_q$. It is also interesting to note that $T_1(3, \mathbb{R})$ is the Heisenberg group.

3. ABSTRACT BICROSSPRODUCTS $H_1 \bowtie H_2$

In this section we generalize the constructions of Section 2 to general k-Hopf algebras H_1 and H_2. The first part generalizes Theorem 2.1 and originated as a sequel to the work of [3]. The notation of [3] is adopted. The second part generalizes Theorem 2.3 and the relation with Theorem 2.1. It is more closely related to the work of Takeuchi [1] and constructions of Drinfel'd.

3.1. *Bicrossproducts $H_1{}^{\beta}\bowtie_{\alpha} H_2$ and Factorization Property*

Let $H_1 = H$ and $H_2 = A$ be two k-Hopf algebras. We consider the situation in which A is a right H-module algebra [4, Sect. 7.2; 3, Sect. 2.1] with structure map α and H a left A-comodule coalgebra with structure map β. These will be denoted

$$\alpha: A \otimes H \to A,\ \alpha(a \otimes h) = a.h, \quad \beta: H \to A \otimes H,\ \beta(h) = \sum h^{(\bar{1})} \otimes h^{(\bar{2})},$$
$$\forall a \in A,\ h \in H.$$

Then the assumptions are $\forall a, b \in A,\ g, h \in H$,

$$a.1_H = a, \qquad (a.h).g = a.(hg) \qquad \text{(module action structure)} \tag{12}$$

$$1_A.h = 1_A \varepsilon_H(h), \quad (ab).h = \sum a.h_{(1)} b.h_{(2)} \quad \text{(module algebra)} \tag{13}$$

$$\begin{gathered}\sum \varepsilon_A(h^{(\bar{1})}) \otimes h^{(\bar{2})} = 1_A \otimes h \equiv h,\\ \sum h^{(\bar{1})} \otimes h^{(\bar{2})(\bar{1})} \otimes h^{(\bar{2})(\bar{2})} = \sum h^{(\bar{1})}{}_{(1)} \otimes h^{(\bar{1})}{}_{(2)} \otimes h^{(\bar{2})}\end{gathered} \tag{14}$$

(comodule coaction structure),

$$\begin{gathered}\sum h^{(\bar{1})} \varepsilon_H(h^{(\bar{2})}) = 1_A \varepsilon_H(h),\\ \sum h^{(\bar{1})} \otimes h^{(\bar{2})}{}_{(1)} \otimes h^{(\bar{2})}{}_{(2)} = \sum h_{(1)}{}^{(\bar{1})} h_{(2)}{}^{(\bar{1})} \otimes h_{(1)}{}^{(\bar{2})} \otimes h_{(2)}{}^{(\bar{2})}\end{gathered} \tag{15}$$

(comodule coalgebra). The notions of "module coalgebra" and "comodule algebra" are similar. A module coalgebra would mean here that α respects the coproduct of A.

In this situation the cross or smash product and cross or smash coproduct are both defined. Note that if H is a Hopf algebra, B an algebra, and $\phi \in \mathrm{alg}(H, B)$, then B becomes a right H-module, B_ϕ. Here the action is the adjoint action, ad_ϕ, induced by ϕ [3, Sect. 2.3]. Explicitly it is $\mathrm{ad}_\phi(b \otimes h) = \sum \phi(S_H(h_{(1)}))\, b\phi(h_{(2)})$, $\forall h \in H,\ b \in B$. A similar construction holds dually.

The smash product $H \# A$ may be defined abstractly as an algebra with inclusion maps $j \in \mathrm{alg}(H, H \# A)$ and $i \in \mathrm{alg}_H(A, H \# A_j)$ obeying a certain universal property. Here $H \# A_j$ denotes $H \# A$ as an H-module algebra by the adjoint action induced by j, and alg_H denotes H-module algebra morphisms. The universal property is that if (ϕ, B, ψ) is another system

$$\phi \in \mathrm{alg}(H, B), \qquad \psi \in \mathrm{alg}_H(A, B_\phi),$$

then these maps factor through $(j, H \# A, i)$. Similarly the smash coproduct $H \times A$ may be defined abstractly as a coalgebra with projection maps $p \in \mathrm{colg}^A({}^q H \times A, H)$ and $q \in \mathrm{colg}(H \times A, A)$ and a dual universal property. Here ${}^q H \times A$ denotes $H \times A$ as an A-comodule coalgebra by the coadjoint coaction induced by q, and colg^A denotes A-comodule coalgebra morphisms. The dual universal property is that if (ϕ, B, ψ) is another system

$$\phi \in \mathrm{colg}^A({}^\psi B, H), \qquad \psi \in \mathrm{colg}(B, A)$$

then these maps factor through $(p, H \times A, q)$. Schematically

$$\begin{tikzcd} H \arrow[r, hook, "j"] \arrow[dr, "\phi"'] & H \# A \arrow[d, "\phi \# \psi"] & A \arrow[l, hook', "i"'] \arrow[dl, "\psi"] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} H & H \times A \arrow[l, "p"'] \arrow[r, "q"] & A \\ & B \arrow[ul, "\phi"] \arrow[u, dashed, "\phi \times \psi"] \arrow[ur, "\psi"'] & \end{tikzcd}$$

commute. The smash product and coproduct are each unique up to unique isomorphism but there is a standard construction of each on $H \otimes A$. This will always be used for any explicit computations. Then explicitly $\forall h, g \in H$, $a, b \in A$,

$$(h \otimes a) \cdot_K (g \otimes b) = \sum hg_{(1)} \otimes a.g_{(2)} b,$$

$$\Delta_K h \otimes a = \sum h_{(1)} \otimes h_{(2)}{}^{(\bar{1})} a_{(1)} \otimes h_{(2)}{}^{(\bar{2})} \otimes a_{(2)}$$

$$\varepsilon_K = \varepsilon_H \otimes \varepsilon_A, \qquad 1_K = 1_H \otimes 1_A$$

$$j = 1 \otimes 1_A, \qquad i = 1_H \otimes 1, \qquad p = 1 \otimes \varepsilon_A, \qquad q = \varepsilon_H \otimes 1.$$

Explicitly, the condition for an algebra map $\psi \in \mathrm{alg}(A, B)$ to be an H-module map with respect to the adjoint action due to a $\phi \in \mathrm{alg}(H, B)$ is

$$\psi(a.h) = \sum \phi(S_H(h_{(1)}))\, \psi(a)\, \phi(h_{(2)}), \qquad \forall a \in A,\ h \in H.$$

The induced map is $(\phi \# \psi)(h \otimes a) = \phi(h)\, \psi(a)$. Similarly, the condition for a coalgebra map $\phi \in \mathrm{colg}(B, H)$ to be an A-comodule map with respect to the coadjoint coaction due to a $\psi \in \mathrm{colg}(B, A)$ is

$$\sum \phi(b)^{(\bar{1})} \otimes \phi(b)^{(\bar{2})} = \sum \psi(b_{(1)})\, S_A \psi(b_{(3)}) \otimes \phi(b_{(2)}), \qquad \forall b \in B.$$

The induced map is $(\phi \times \psi)(b) = \sum \phi(b_{(1)}) \otimes \psi(b_{(2)})$. These induced maps are respectively module and comodule maps.

DEFINITION 3.1. Let A, H be Hopf algebras, A a right H-module, and H a left A-comodule. We define $(K(H, A), i, j, p, q)$ to be the explicit smash product algebra and smash coproduct coalgebra on $H \otimes A$. We often denote it simply by K.

We wish to know when K is a Hopf algebra. Note that in choosing to build the smash products and coproducts on the same set, we are also introducing relations between the maps. For example,

$$p \circ i = 1_H \circ \varepsilon_A, \qquad p \circ j = 1, \qquad q \circ i = 1, \qquad q \circ j = 1_A \circ \varepsilon_H.$$

Some other relevant elementary facts can be immediately computed.

LEMMA 3.2. (i) *ε_K an algebra map $\Leftrightarrow p$ an algebra map.*

(ii) *1_K a coalgebra map $\Leftrightarrow i$ a coalgebra map.*

(iii) *β trivial $\Leftrightarrow j$ a coalgebra map.*

(iv) *α trivial $\Leftrightarrow q$ an algebra map.*

This is an appropriate point to mention that a search of the literature revealed another "biproduct" of Hopf algebras, considered in [25]. The setup there is somewhat different, namely A is (by contrast) both an H-module algebra and an H-comodule coalgebra. The author then shows that such "biproduct" bialgebras B are characterized by the bialgebra projection map $B \rightleftarrows H$. By contrast we see that (j, p) or (i, q) constitute such a pair only when β or α are respectively trivial.

THEOREM 3.3. *Let H, A be Hopf algebras and A an H-module algebra by α and H an A-comodule coalgebra by β. The smash product algebra–smash coproduct coalgebra K is a bialgebra iff $\forall a \in A$, $h, g \in H$,*

$$\begin{gathered}\varepsilon_A(a.h) = \varepsilon_A(a)\,\varepsilon_H(h),\\ \sum (a.h)_{(1)} \otimes (a.h)_{(2)} = \sum a_{(1)}.h_{(1)}h_{(2)}{}^{(\bar{1})} \otimes a_{(2)}.h_{(2)}{}^{(\bar{2})}\end{gathered} \tag{A}$$

$$\begin{gathered}\sum 1_H{}^{(\bar{1})} \otimes 1_H{}^{(\bar{2})} = 1_A \otimes 1_H,\\ \sum (hg)^{(\bar{1})} \otimes (hg)^{(\bar{2})} = \sum h^{(\bar{1})}.g_{(1)}\, g_{(2)}{}^{(\bar{1})} \otimes h^{(\bar{2})} g_{(2)}{}^{(\bar{2})}\end{gathered} \tag{B}$$

$$\sum h_{(1)}{}^{(\bar{1})} a.h_{(2)} \otimes h_{(1)}{}^{(\bar{2})} = \sum a.h_{(1)} h_{(2)}{}^{(\bar{1})} \otimes h_{(2)}{}^{(\bar{2})}. \tag{C}$$

In this case there is an antipode

$$S_K(h \otimes a) = \sum (1_H \otimes S_A(h^{(\bar{1})}a)) \cdot_K (S_H(h^{(\bar{2})}) \otimes 1_A)$$

and we call the resulting Hopf algebra, $H^{\beta} \bowtie_{\alpha} A$, a bicrossproduct *of Hopf algebras. If H and A have skew antipodes, S'_H and S'_A, then so does $H^{\beta} \bowtie_{\alpha} A$. Explicitly, it is*

$$S'_K(h\otimes a)=\sum (S'_H(h_{(1)}))^{(\bar{2})}\otimes S'_A(a.S'_H(h_{(2)})(S'_H(h_{(1)}))^{(\bar{1})}).$$

Proof. The conditions involving ε_A and 1_H are Lemma 3.2(i) and (ii). Suppose that K is a bialgebra. Then (omitting "$\sum$")

$$\begin{aligned}\Delta_K(h\otimes a)\cdot_K(g\otimes 1_A)&=\Delta_K(hg_{(1)}\otimes a.g_{(2)})\\&=h_{(1)}g_{(1)}\otimes(h_{(2)}g_{(2)})^{(\bar{1})}(a.g_{(3)})_{(1)}\\&\quad\otimes(h_{(2)}g_{(2)})^{(\bar{2})}\otimes(a.g_{(3)})_{(2)}\end{aligned}$$

equals the product in $K\otimes K$ of $\Delta_K(h\otimes a)$ and $\Delta_K(g\otimes 1_A)$. This is

$$h_{(1)}g_{(1)}\otimes(h_{(2)}{}^{(\bar{1})}a_{(1)}).g_{(2)}g_{(3)}{}^{(\bar{1})}\otimes h_{(2)}{}^{(\bar{2})}g_{(3)}{}^{(\bar{2})}{}_{(1)}\otimes a_{(2)}.g_{(3)}{}^{(\bar{2})}{}_{(2)}.$$

To the two expressions we first apply $\varepsilon_H\otimes 1\otimes 1\otimes 1$. Setting $h=1_H$ and applying $1\otimes\varepsilon_H\otimes 1$ to both sides gives (A) for a, g. Alternatively, setting $a=1_A$ gives (B). The unital and counital parts of (A), (B) are similar. Since these are necessary, we now use (A) and (B) to simplify the above expressions. Thus with $h=1_H$ we obtain

$$\begin{aligned}&g_{(1)}{}^{(\bar{1})}a_{(1)}.g_{(2)}g_{(3)}{}^{(\bar{1})}\otimes g_{(1)}{}^{(\bar{2})}\otimes a_{(2)}.g_{(3)}{}^{(\bar{2})}\\&\quad=a_{(1)}.g_{(1)}g_{(2)}{}^{(\bar{1})}g_{(3)}{}^{(\bar{1})}\otimes g_{(2)}{}^{(\bar{2})}\otimes a_{(2)}.g_{(3)}{}^{(\bar{2})}.\end{aligned}$$

Applying $1\otimes 1\otimes\varepsilon_A$ to both sides gives (C) for a, g.

Conversely, if (A), (B), (C) hold, a similar computation gives that Δ_K is an algebra map. This, along with the desired properties of ε_K and 1_K, means that K is a bialgebra. It is straightforward to check that in this case the map S_K is an antipode. In computing $(h\otimes a)_{(1)}S_K(h\otimes a)_{(2)}$, etc., use associativity of multiplication in K in the factorized expression for S_K. This convenient trick of factorizing the antipode S_K comes from [1]; cf. Example 3.6 below. Finally, suppose that S'_H and S'_A are skew antipodes on H and A. Then

$$\begin{aligned}S_K\circ S'_K(h\otimes a)&=S_H((S'_Hh_{(1)})^{(\bar{2})(\bar{2})}{}_{(2)})\otimes(S_A((S'_Hh_{(1)})^{(\bar{2})(\bar{1})}(S'_A(S'_Hh_{(1)})^{(\bar{1})})\\&\quad\times S'_A(a.S'_Hh_{(2)}))).S_H((S'_Hh_{(1)})^{(\bar{2})(\bar{2})}{}_{(1)})\\&=S_H((S'_Hh_{(1)})^{(\bar{2})}{}_{(2)})\otimes(S_A((S'_Hh_{(1)})^{(\bar{1})}{}_{(2)}(S'_A(S'_Hh_{(1)})^{(\bar{1})}{}_{(1)})\\&\quad\times S'_A(a.S'_Hh_{(2)}))).S_H((S'_Hh_{(1)})^{(\bar{2})}{}_{(1)})\\&=S_H((S'_Hh_{(1)})_{(2)})\otimes a.((S'_Hh_{(2)})\,S_H((S'_Hh_{(1)})_{(1)}))=h\otimes a,\end{aligned}$$

where we use in sequence the definitions of S_K and S'_K, that β is a coaction, the skew antipode property of S'_A, that α is an action, and the skew antipode property of S'_H. The proof that $S'_K \circ S_K = 1$ is similar. This concludes the theorem.

We now give an example. If β is trivial, then (A) says that α respects the coproduct of A. Hence α respects the entire bialgebra structure of A. This situation was studied by Molnar [3]. He defines in this situation that α makes A an H-module *bialgebra*. In this case (and if H is cocommutative) he defines a Hopf algebra with the tensor coproduct coalgebra structure and the smash product algebra structure. It is the natural semidirect product in the category of Hopf algebras. The semidirect coproduct is defined similarly and requires α trivial. Thus the bicrossproduct generalizes the construction of Molnar;

EXAMPLE 3.4. If $\beta = 1_A \otimes 1$ (trivial coaction) then K is Hopf algebra iff A is an H-module bialgebra and H is cocommutative. The antipode is explicitly

$$S_K(h \otimes a) = \sum S_H(h_{(2)}) \otimes (S_A a).S_H(h_{(1)}).$$

In this case K is the semidirect product in the category of Hopf algebras [3].

Similarly, if $\alpha = 1 \otimes \varepsilon_H$ (trivial action) then K is a Hopf algebra iff H is an A-comodule bialgebra and A is commutative. The antipode is explicitly

$$S_K(h \otimes a) = \sum S_H(h^{(\bar{2})}) \otimes S_A(h^{(\bar{1})}a).$$

In this case K is the semidirect coproduct in the category of Hopf algebras [3].

This point of view motivates the following definitions.

DEFINITION 3.5. Let A be an H-module algebra by α and H an A-comodule coalgebra by β.

We say under conditions (A) of Theorem 3.3, i.e., if

$$\varepsilon_A \circ \alpha = \varepsilon_{A \otimes H},$$

$$\Delta_A \circ \alpha = (\cdot_A \otimes 1) \circ (\alpha \otimes 1 \otimes \alpha) \circ (1 \otimes 1 \otimes \tau \otimes 1) \circ (1 \otimes 1 \otimes 1 \otimes \beta) \circ \Delta_{A \otimes H},$$

that α gives A the structure of a right H-module algebra β-coalgebra.

We say under conditions (B), i.e., if

$$\beta(1_H)=1_{A\otimes H},$$

$$\beta\circ\cdot_H=\cdot_{A\otimes H}\circ(\alpha\otimes 1\otimes 1\otimes 1)\circ(1\otimes\tau\otimes 1\otimes 1)\circ(\beta\otimes 1\otimes\beta)\circ(1\otimes\Delta_H),$$

that β gives H the structure of a left A-comodule coalgebra α-algebra.

We say under conditions (C), i.e., if

$$(\cdot_A\otimes 1)\circ(1\otimes\alpha\otimes 1)\circ(\tau\otimes\tau)\circ(1\otimes\beta\otimes 1)\circ(1\otimes\Delta_H)$$
$$=(\cdot_A\otimes 1)\circ(\alpha\otimes\beta)\circ(1\otimes\Delta_H),$$

that (H, A) are compatible.

EXAMPLE 3.6. If H is cocommutative and A is commutative then the compatibility condition (C) is automatic. This case was introduced independently by Singer and used by [1] in the context of algebraic groups. The examples of Section 2, Theorem 2.1, are all of this form.

It remains to investigate the abstract meaning of these conditions (A), (B), (C) separately. The meaning of (C) is deferred to Section 3.2.

DEFINITION 3.7. Let A be an H-module algebra and H an A-comodule coalgebra. The coproduct Δ_K of the smash algebra–coalgebra K *factorizes* over the smash product if it can be induced by suitable maps from H and A by the universal property. Similarly the product $\cdot_K$ *factorizes* over the smash coproduct if it can be induced by suitable maps to H and A,

$$\begin{tikzcd} H \arrow[r, hook, "j"] \arrow[dr] & K \arrow[d, "\Delta_K"] & A \arrow[l, hook', "i"'] \arrow[dl] \\ & K\otimes K & \end{tikzcd} \qquad \begin{tikzcd} H & K \arrow[l, "p"'] \arrow[r, "q"] & A \\ & K\otimes K \arrow[ul] \arrow[u, "\cdot_K"] \arrow[ur] & \end{tikzcd}$$

Similarly for the counit and unit where a short analysis gives that ε_K factorizes iff it is an algebra map and 1_K factorizes iff it is a coalgebra map, so that factorizing is not a strong condition. Moreover, we know precisely what the suitable maps are. Thus let

$$\mu=\Delta_K\circ j=(1\otimes 1\otimes j)\circ(1\otimes\beta)\circ\Delta_H,\quad \nu=q\circ\cdot_K=\cdot_A\circ(\alpha\otimes 1)\circ(q\otimes 1\otimes 1).$$

An easy computation gives

LEMMA 3.8. *Under the conditions of Definition* 3.7 *and* μ, ν *defined above*,

(i) ν *is a coalgebra map* $\Leftrightarrow$ (A).

(ii) μ *is an algebra map* $\Leftrightarrow$ (B).

Proof. For (ii) from the definitions we have

$$\begin{aligned}\mu(h)\,\mu(g) &= (h_{(1)}\otimes h_{(2)}{}^{(\bar{1})})\cdot_K(g_{(1)}\otimes g_{(2)}{}^{(\bar{1})})\otimes(h_{(2)}{}^{(\bar{2})}\otimes 1)\cdot_K(g_{(2)}{}^{(\bar{2})}\otimes 1)\\ &= h_{(1)}g_{(1)}\otimes h_{(2)}{}^{(\bar{1})}.g_{(2)}g_{(3)}{}^{(\bar{1})}\otimes h_{(2)}{}^{(\bar{2})}g_{(3)}{}^{(\bar{2})}\otimes 1,\\ \mu(hg) &= (hg\otimes 1)_{(1)}\otimes(hg\otimes 1)_{(2)}\\ &= h_{(1)}g_{(1)}\otimes(h_{(2)}g_{(2)})^{(\bar{1})}\otimes(h_{(2)}g_{(2)})^{(\bar{2})}\otimes 1.\end{aligned}$$

These are equal if (B). Conversely, apply $\varepsilon_H\otimes 1\otimes 1\otimes\varepsilon_A$ to the expressions for $\mu(hg)$ and $\mu(h)\,\mu(g)$ (and $\mu(1)=1$) to obtain (B). Part (i) follows dually to part (ii). The factorization property characterizes the bicrossproduct construction as follows.

THEOREM 3.9. *Let A be an H-module algebra and H an A-comodule coalgebra and K the smash algebra–coalgebra. The following are equivalent.*

(i) *The product and unit of K factorize.*

(ii) *The coproduct and counit of K factorize.*

(iii) *K is a Hopf algebra (the bicrossproduct).*

Proof. That the coproduct factorizes means $\mu\in\mathrm{alg}(H, K\otimes K)$ and $\Delta_K\circ i\in\mathrm{alg}_H(A, K\otimes K_\mu)$. The first condition is (B) (by Lemma 3.8). In particular, it implies that $1_H{}^{(\bar{1})}\otimes 1_H{}^{(\bar{2})}=1_A\otimes 1_H$ which implies that 1_K is a coalgebra map. Factorization also implies that Δ_K and ε_K are algebra maps. Hence K is a bialgebra. From Theorem 3.3 we know that in this case it is a Hopf algebra.

Conversely, by Theorem 3.3, if K is a bialgebra then (A), (B), (C) hold and by Lemma 3.2, i is a coalgebra map. The second condition for factorization of Δ_K becomes

$$1_H\otimes(a.h)_{(1)}\otimes 1_H\otimes(a.h)_{(2)}=\mu(S_Hh_{(1)})\cdot(1_H\otimes a_{(1)}\otimes 1_H\otimes a_{(2)})\cdot\mu(h_{(2)}).$$

Multiplying out the right hand side in $K\otimes K$ and using property (C) reduces this condition to (A). The proof that (i) and (iii) are equivalent is similar. This completes the proof.

Theorem 3.9 gives the meaning of the requirement that the smash product and coproduct K form a Hopf algebra. It says that $H^\beta\bowtie_\alpha A$, the bicrossproduct, is in some sense "made up" from its two factors H and A. Finally, one might try to define the bicrossproduct more abstractly. One result in this direction is

PROPOSITION 3.10. *Let (H, A) be a pair of Hopf algebras having a* *bicrossproduct* $K = H^{\beta} \bowtie_{\alpha} A$. *If* (K', i', j', p', q') *is any bialgebra and maps such that*

$$(j', K'^{\mathrm{alg}}, i') \text{ and } (p', K'^{\mathrm{colg}}, q')$$

have respectively the smash product algebra and smash coproduct coalgebra properties, and

(i) $p' \circ i' = 1_H \circ \varepsilon_A, \quad p' \circ j' = 1, \quad q' \circ i' = 1, \quad q' \circ j' = 1_A \circ \varepsilon_H$

(ii) p' *and* i' *are bialgebra maps*

(iii) $\varepsilon_K \circ j' = \varepsilon_H$,

$$\sum (j'(h))_{(1)} \otimes (j'(h))_{(2)} = \sum j'(h_{(1)})\, i'(h_{(2)}{}^{(\bar{1})}) \otimes j'(h_{(2)}{}^{(\bar{2})})$$

$$q'(1_K) = 1_A, \qquad q'(mk) = \sum q'(m).p'(k_{(1)})\, q'(k_{(2)}),$$

then as bialgebras $K' \cong K$ *by a unique isomorphism.*

Proof. The hypotheses (which are trivially satisfied for K with the algebra–coalgebra defined at the beginning of the section) are such that $j' \# i'$ induced by K is a coalgebra map, hence equals $p \times q$ induced by K' since it makes the required diagrams commute. One then shows that $p' \times q'$ induced by K is an algebra map hence equals $j \# i$ induced by K' since it makes the required diagrams commute, and that the two bialgebra maps are inverse. The condition (iii) states the extent to which j' and q' are not bialgebra maps.

Similar results obtain if α is a *left* action and β a left coaction. This is the correct situation for duality considerations. For if A is a finite dimensional left H-module and H a finite dimensional left A-comodule then A^* is a left H^*-comodule coalgebra by α^* and H^* is a left A^*-module algebra by β^*. Hence the dual situation to (H, A, α, β) is of the same form $(H', A', \alpha', \beta')$ with $A^* = H'$, $H^* = A'$, $\alpha^* = \beta'$, $\beta^* = \alpha'$. Discussion of duality will be deferred to a sequel, but this simple finite dimensional consideration is useful to keep in mind. It corresponds in the above constructions to the fact that all expressions above, when written in terms of α and β as linear maps, are transformed into each other under the interchange of α with β, products with coproducts, counits with units, H with A, left–right reversal, and reversal of all compositions (for example, the two halves of Definition 3.5). In this sense, the antipode defined in Theorem 3.3 is precisely self-dual:

$$\begin{aligned} S_K = {} & (1 \otimes \alpha) \circ (1 \otimes \tau) \circ (1 \otimes 1 \otimes S_A) \circ (1 \otimes 1 \otimes \cdot_A) \\ & \circ (\Delta_H \otimes 1 \otimes 1) \circ (S_H \otimes 1 \otimes 1) \circ (\tau \otimes 1) \circ (\beta \otimes 1). \end{aligned}$$

3.2. *Double Crossproducts* $H_{1\beta}\bowtie_\alpha H_2$

This section generalizes Theorem 2.3 and is inspired by the paper of [1]. This is made clear in Example 3.14.

DEFINITION 3.11. Two Hopf algebras (H, B) will be said to be *matched* if there exists a Hopf algebra K, inclusions j, i of H, B, respectively, as sub-Hopf algebras, and coalgebra maps p, q to H, B such that

(i) the map $\Phi: H\otimes B\to K$, $\Phi=\cdot_K\circ\tau\circ(j\otimes i)$ is a linear isomorphism,

(ii) the map $\Psi: K\to H\otimes B$, $\Psi=(p\otimes q)\circ\tau\circ\Delta_K$ obeys $\Phi\circ\Psi: i(b)\, j(h)\mapsto j(h)\, i(b)\forall h\in H,\ b\in B$.

This definition subsumes the definition of matched pairs of groups given in [1]. Proposition 3.13 below makes clear the connection with bicrossproducts (and a previous notion of matched pairs of Hopf algebras due to Singer, see [1]).

PROPOSITION 3.12. *(H, B) are a matched pair of Hopf algebras iff B is an H-module coalgebra, denoted α, and H is a B-module coalgebra, denoted β,*

$$\alpha: H\otimes B\to B,\qquad \alpha(h\otimes b)=h.b,\qquad \beta: H\otimes B\to H,\qquad \beta(h\otimes b)=h\bullet b,$$

such that $\forall b, c\in B,\ h, g\in H$

$$h.1_B=\varepsilon_H(h)\,1_B,\qquad h.(bc)=\sum h_{(1)}.b_{(1)}(h_{(2)}\bullet b_{(2)}).c \tag{A}$$

$$1_H\bullet b=1_H\varepsilon_B(b),\qquad (hg)\bullet b=\sum h\bullet(g_{(1)}.b_{(1)})\,g_{(2)}\bullet b_{(2)} \tag{B}$$

$$\sum h_{(1)}\bullet b_{(1)}\otimes h_{(2)}.b_{(2)}=\sum h_{(2)}\bullet b_{(2)}\otimes h_{(1)}.b_{(1)}. \tag{C}$$

In this case, the Hopf algebra K is explicitly realized on the set $H\otimes B$ as

$$(h\otimes b)\cdot_K(g\otimes c)=\sum (h_{(2)}\bullet c_{(2)})\,g\otimes bh_{(1)}.c_{(1)},$$

$$\Delta_K(h\otimes b)=\sum h_{(1)}\otimes b_{(1)}\otimes h_{(2)}\otimes b_{(2)},$$

$$\varepsilon_K(h\otimes b)=\varepsilon_H(h)\,\varepsilon_B(b),\qquad 1_K=1_H\otimes 1_B,$$

$$S_K(h\otimes b)=(S_Hh\otimes 1_B)\cdot_K(1_H\otimes S_Bb)$$

and denoted $H_\beta\bowtie_\alpha B$. The condition (C) *can be stated as*

$$(\Delta_K-\tau\circ\Delta_K)(h\otimes b)\in\ker\beta\otimes\alpha,\qquad \forall h\in H,\ b\in B.$$

If H and B possess skew antipodes S'_H and S'_B, then so does $H_\beta\bowtie_\alpha B$. Explicitly, it is $S'_K(h\otimes b)=(S'_Hh\otimes 1_B)\cdot_K(1_H\otimes S'_Bb)$.

Proof. If (A), (B), (C) hold it is straightforward to check that the definition given does define a Hopf algebra $H_\beta \bowtie_\alpha B$ with the desired properties. (For the antipode, use associativity of multiplication in K. The proof for the skew-antipode, when defined, is similar.) Conversely, if (H, B) are a matched pair then the map Φ means that without loss of generality we can consider the product K concretely on the set $H \otimes B$ with $i(b)\, j(h) = h \otimes b$ (the reversal here is due to our conventions). Then $\Delta_K i(b)\, j(h) = ((i \otimes i) \circ \Delta_B b)((j \otimes j) \circ \Delta_H h)$, which is as given in the proposition. The counit is similar.

The assumption involving Ψ asserts that $(h \otimes 1_B)(1_H \otimes b) = p(h_{(2)} \otimes b_{(2)}) \otimes q(h_{(1)} \otimes b_{(1)})$ for some coalgebra maps p, q. Set $\alpha = q$, $\beta = p$. Then

$$\begin{aligned}(hg)_{(2)} \bullet b_{(2)} \otimes (hg)_{(1)}.b_{(1)} &= j(h)\, j(g)\, i(b)\\ &= h_{(2)} \bullet (g_{(1)}.b_{(1)})_{(2)}\, g_{(2)} \bullet b_{(2)} \otimes h_{(1)}.(g_{(1)}.b_{(1)})_{(1)}.\end{aligned}$$

Applying $1 \otimes \varepsilon_B$ to this yields (B) and applying $\varepsilon_H \otimes 1$ yields that α is an action (the other remaining facts are similar). The proof is modeled on [1, Proposition 2.4], where analogous computations are made for matched pairs of groups.

PROPOSITION 3.13. *Let B be a finite dimensional Hopf algebra. Then (H, B, α, β) is a matched pair of Hopf algebras iff $(H, B^*, \alpha^*, \beta^*)$ have a bicrossproduct. Thus the same data lead to $H^{\beta^*} \bowtie_{\alpha^*} B^*$ and to $H_\beta \bowtie_\alpha B$. Here the actions and coactions are related by*

$$\beta(h \otimes b) = \sum \langle h^{(\bar{1})}, b \rangle h^{(\bar{2})}, \qquad \langle a, \alpha(h \otimes b) \rangle = \langle \alpha^*(a \otimes h), b \rangle,$$

$$\forall h \in H,\ b \in B,\ a \in B^*,$$

where $\beta^*(h) = \sum h^{(\bar{1})} \otimes h^{(\bar{2})}$.

Proof. That an H-module coalgebra structure on B dualizes to an H-module algebra structure on B^* is standard; cf. [4]. Similarly the B-module coalgebra structure on H dualizes to a B^*-comodule coalgebra structure. It remains to check (A), (B), (C). Write $\alpha^*(a \otimes h) \equiv a \blacktriangleleft h$. Then from the definitions,

$$\begin{aligned}&\langle (a \blacktriangleleft h)_{(1)} \otimes (a \blacktriangleleft h)_{(2)}, b \otimes c \rangle\\ &\qquad = \langle a \blacktriangleleft h,\ bc \rangle = \langle a, h.(bc) \rangle,\\ &\langle a_{(1)} \blacktriangleleft h_{(1)} h_{(2)}{}^{(\bar{1})} \otimes a_{(2)} \blacktriangleleft h_{(2)}{}^{(\bar{2})}, b \otimes c \rangle\\ &\qquad = \langle a_{(1)}, h_{(1)}.b_{(1)} \rangle \langle h_{(2)}{}^{(\bar{1})}, b_{(2)} \rangle \langle a_{(2)}, h_{(2)}{}^{(\bar{2})}.c \rangle\\ &\qquad = \langle a, h_{(1)}.b_{(1)}(h_{(2)} \bullet b_{(2)}).c \rangle.\end{aligned}$$

In this way, (A) in Proposition 3.12 is equivalent to (A) in Theorem 3.3 applied to $(H, B^*, \alpha^*, \beta^*)$. The proofs for (B) and (C) are similarly straightforward.

EXAMPLE 3.14 (cf. [1]). Let G_1, G_2 be finite groups, α a left action of G_1 on the set G_2, β a *right* action of G_2 on the set G_1. $(G_1, G_2, \alpha, \beta)$ are a matched pair of finite groups, i.e., $\forall u, v \in G_1$, $s, t \in G_2$,

$$\alpha_u(e) = e, \qquad \beta_s(e) = e, \qquad \alpha_u(st) = \alpha_u(s)\,\alpha_{\beta_s(u)}(t), \qquad \beta_s(uv) = \beta_{\alpha_v(s)}(u)\,\beta_s(v)$$

(this a variant of Section 2, suitable for β a right action) iff these actions extend linearly to actions rendering $(k[G_1], k[G_2])$ a matched pair.

The double crossproduct is $k[G_1]_\beta\bowtie_\alpha k[G_2] = k[G_{1\beta}\bowtie_\alpha G_2]$, where $G_{1\beta}\bowtie_\alpha G_2$ is defined by

$$(u, s)\cdot(v, t) = (\beta_t(u)v, s\alpha_u(t))$$

(this is a variant of that given in Theorem 2.3). The bicrossproduct $k[G_1]^{\alpha^*}\bowtie_{\beta^*} k(G_2)$ is a variant of that given in Section 2 (cf. Theorem 2.1).

Proof. (i) Recall that $k[G_1]$ and $k[G_2]$ are formal linear combinations of elements in G_1 and G_2, respectively. $G_1 \subset k[G_1]$ as the group-like elements so that $\Delta u = u \otimes u$, $\forall u \in G_1$. Similarly $\Delta s = s \otimes s$, $\forall s \in G_2$. On such group-like elements $u, v \in G_1 \subset k[G_1]$ and $s, t \in G_2 \subset k[G_2]$, conditions (A), (B) in Proposition 3.12 reduce precisely to the conditions stated for a matched pair of finite groups. Conversely, these latter conditions extend by linearity to actions obeying the conditions (A) and (B). Condition (C) is automatic because both $k[G_1]$ and $k[G_2]$ are cocommutative.

(ii) When $k[G_1]$ and $k[G_2]$ are matched, the product and coproduct structures for $k[G_1]_\beta\bowtie_\alpha k[G_2]$ given explicitly in Proposition 3.12 reduce in the case of $u \otimes s$, $v \otimes t \in k[G_1]_\beta\bowtie_\alpha k[G_2]$ to

$$(u \otimes s)\cdot(v \otimes t) = \beta_t(u)v \otimes s\alpha_u(t), \qquad \Delta(u \otimes s) = u \otimes s \otimes u \otimes s.$$

Conversely, all the group-like elements are of the form $(u \otimes s)$ for a pair $(u, s) \in G_{1\beta}\bowtie_\alpha G_2$, where $G_{1\beta}\bowtie_\alpha G_2$ is in the form stated. The unit, counit, and antipode can similarly be checked. Hence $k[G_1]_\beta\bowtie_\alpha k[G_2] = k[G_{1\beta}\bowtie_\alpha G_2]$. Note that when β is a right action, the map $\tilde{\beta}_s(u) = \beta_{s^{-1}}(u^{-1})^{-1}$ is a left action. The notions of matched pair of finite groups, etc., recovered here are precisely the notions given in Section 2 applied to $(G_1, G_2, \alpha, \tilde{\beta})$.

(iii) The explicit formula for $\beta^*(h) \equiv \sum h^{(\bar{1})} \otimes h^{(\bar{2})}$ given in Proposition 3.13 is $\langle u^{(\bar{1})}, s\rangle u^{(\bar{2})} = \beta_s(u)$, $\forall u \in G_1 \subset k[G_1]$, $s \in G_2 \subset k[G_2]$, i.e., $\beta^*(u) = \beta.(u) \in k(G_2) \otimes G_1 \subset k(G_2) \otimes k[G_1]$. Here the pairing between $f \in k(G_2)$ and $s \in G_2 \subset k[G_2]$ is the evaluation $f(s)$. Similarly, α^* defined in

Proposition 3.13 is $\alpha_u^*(f)(s)=f(\alpha_u(s))$, $\forall f\in k(G_2)$, $s\in G_2$, $u\in G_1$. The smash product and coproduct structures given in Theorem 3.3 applied to $(k[G_1], k(G_2), \alpha^*, \beta^*)$ are then, $\forall u, v\in G_1\subset k[G_1]$, $s, t\in G_2$, $f, f'\in k(G_2)$,

$$((u\otimes f)\cdot(v\otimes f'))(s)=uv\otimes f(\alpha_v(s))\,f'(s),\quad (\Delta(u\otimes f))(s,t)=u\otimes\beta_s(u)\,f(st).$$

This is a variant of the structure given in Theorem 2.1, suitable for α^* a *right* action of $k[G_1]$ on $k(G_2)$, and β^* a matching left coaction of $k(G_2)$ on $k[G_1]$. This completes the proof of the example.

4. Further Examples Including the Classical Yang–Baxter Equations

(A) The k-Lie algebra version of Theorem 2.3 is

THEOREM 4.1. *Let α be a linear representation of Lie algebra g_1 on finite dimensional g_2 as a vector space, and let β be a linear representation of Lie algebra g_2 on g_1 as a vector space,*

$$\forall\xi,\xi'\in g_1,\ l,l'\in g_2,\quad \alpha_{[\xi,\xi']}=\alpha_\xi\circ\alpha_{\xi'}-\alpha_{\xi'}\circ\alpha_\xi,\quad \beta_{[l,l']}=\beta_l\circ\beta_{l'}-\beta_{l'}\circ\beta_l,$$

then the following are equivalent:

$$\text{(i)}\quad \alpha_\xi([l,l'])=[\alpha_\xi(l),l']+[l,\alpha_\xi(l')]+\alpha_{\beta_{l'}(\xi)}(l)-\alpha_{\beta_l(\xi)}(l')$$
$$\beta_l([\xi,\xi'])=[\beta_l(\xi),\xi']+[\xi,\beta_l(\xi')]+\beta_{\alpha_{\xi'}(l)}(\xi)-\beta_{\alpha_\xi(l)}(\xi').$$

*We say in this situation that α is an action by Lie algebra β-*derivations *and β is an action by Lie algebra α-*derivations.

(ii) *The bracket on $g_1\oplus g_2$ defined by*

$$[(\xi,l),(\xi',l')]=([\xi,\xi']+\beta_l(\xi')-\beta_{l'}(\xi),\ [l,l']+\alpha_\xi(l')-\alpha_{\xi'}(l))$$

is a Lie algebra structure, denoted $g_{1\beta}\bowtie_\alpha g_2$, the bicrossproduct *(or bicross-sum) Lie algebra.*

Proof. Consider the Jacobi identity for the bracket shown in (ii), i.e., the vanishing of

$$\begin{aligned}&[(\xi,l),[(\xi',l'),(\xi'',l'')]]+\text{cyclic}\\&\quad=([(\xi,l),([\xi',\xi'']+\beta_{l'}(\xi'')-\beta_{l''}(\xi'),[l',l'']\\&\qquad+\alpha_{\xi'}(l'')-\alpha_{\xi''}(l'))])+\text{cyclic}\end{aligned}$$

$$\begin{aligned}
&=([\xi,[\xi',\xi'']]+[\xi,\beta_{l'}(\xi'')-\beta_{l''}(\xi')]+\beta_l([\xi',\xi'']+\beta_{l'}(\xi'')-\beta_{l''}(\xi'))\\
&\quad-\beta_{[l',l'']+\alpha_{\xi'}(l'')-\alpha_{\xi}(l')}(\xi),\ldots)+\text{cyclic}\\
&=([\xi,[\xi',\xi'']]+\beta_l([\xi',\xi''])-[\xi',\beta_l(\xi'')]+[\xi'',\beta_l(\xi')]+\beta_{\alpha_{\xi'}(l)}(\xi'')\\
&\quad-\beta_{\alpha_{\xi}(l')}(\xi'')+\beta_{l'}(\beta_{l''}(\xi))-\beta_{l''}(\beta_{l'}(\xi))-\beta_{[l',l'']}(\xi),\ldots)+\text{cyclic.}
\end{aligned}$$

Here "$\cdots$" means the same expression with all occurrences of the symbols α, ξ interchanged with those of the symbols β, l. "+cyclic" means to add the cyclic permutation $(\xi, l)\to(\xi', l')\to(\xi'', l'')\to(\xi, l)$ and its square. Because the expression shown is cyclically averaged, any term may be separately rotated. This was done in the last step to obtain the expression shown. From this it is clear that α and β are actions and obey conditions (i) iff the Jacobi identity holds. (To see the converse direction, consider the special case $[(\xi, 0), [(0, l'), (0, l'')]]+\text{cyclic}=0$ to see that β is an action, and the case $[(0, l), [(\xi', 0), (\xi'', 0)]]+\text{cyclic}=0$ to see that β_l is an α-derivation for each l, etc.) $[(\xi, l), (\xi, l)]=0$ holds identically.

One can show that under favorable circumstances, such an interacting pair (g_1, g_2) does exponentiate to a pair of interacting Lie groups (G_1, G_2) as in Theorem 2.3, and $g_1 \bowtie g_2$ is the Lie algebra of $G_1 \bowtie G_2$; cf. [26].

EXAMPLE 4.2 (Lie Bialgebras and $D(g)$). Let $g_1=g$ and $g_2=g^*$ be finite dimensional Lie algebras. Let $\alpha=\mathrm{ad}^*$ and $\beta=\mathrm{ad}^*$ be mutual coadjoint actions (i.e., $\langle\alpha_\xi(l), \xi'\rangle=\langle l, [\xi', \xi]\rangle$ and $\langle\beta_l(\xi), l'\rangle=\langle\xi, [l', l]\rangle$). Then the following are equivalent:

(i) g (with cobracket from g^*) is a Lie bialgebra as defined in [23].

(ii) (g, g^*) have a bicrossproduct by the mutual coadjoint actions.

In this case, $g_{\mathrm{ad}^*}\bowtie_{\mathrm{ad}^*} g^*=D(g)$ as a Lie algebra. Let $g^{*\mathrm{op}}$ denote g^* with the opposite Lie coalgebra structure. Then $D(g)=g_{\mathrm{ad}^*}\bowtie_{\mathrm{ad}^*} g^{*\mathrm{op}}$ is a Lie bialgebra with the direct sum Lie coalgebra structure.

Proof. Recall that the compatibility condition for the Lie algebra and Lie coalgebra (g, δ) to be a bialgebra is $\delta\in Z^1_{\mathrm{ad}}(g, g\otimes g)$. Explicitly $d\delta=0$ is $\delta([\xi, \xi'])=\mathrm{ad}_\xi(\delta(\xi'))-\mathrm{ad}_{\xi'}(\delta(\xi))\,\forall\xi, \xi'\in g$. In the present case we are given (g, g^*) a pair of finite dimensional Lie algebras. Let δ denote the dual of the Lie algebra structure on g^*, and let α, β be the mutual coadjoint actions as stated (so that $(1\otimes l)(\delta\xi)=\beta_l(\xi)$ for all $l\in g^*$). In terms of these and the notation $\delta\xi\equiv\xi_{(1)}\otimes\xi_{(2)}$, we compute

$$\begin{aligned}
&\langle l\otimes l', \delta([\xi, \xi'])-\mathrm{ad}_\xi(\delta(\xi'))+\mathrm{ad}_{\xi'}(\delta(\xi))\rangle\\
&\quad=\langle l, \beta_{l'}([\xi, \xi'])\rangle-\langle l\otimes l', [\xi, \xi'_{(1)}]\otimes\xi'_{(2)}+\xi'_{(1)}\otimes[\xi, \xi'_{(2)}]\rangle\\
&\qquad+\langle l\otimes l', [\xi', \xi_{(1)}]\otimes\xi_{(2)}+\xi_{(1)}\otimes[\xi', \xi_{(2)}]\rangle\\
&\quad=\langle l, \beta_{l'}([\xi, \xi'])-[\xi, \beta_{l'}(\xi')]+\beta_{\alpha_\xi(l')}(\xi')+[\xi', \beta_{l'}(\xi)]-\beta_{\alpha_{\xi'}(l')}(\xi)\rangle.
\end{aligned}$$

So $d\delta = 0$, i.e., (g, δ) is a bialgebra, iff β is an α-derivation. Also, by self-duality in the axioms, this is true iff g^* is a bialgebra with respect to the dual of the Lie algebra structure on g, which is true iff α is a β-derivation. Thus for pairs (g, g^*) the two equations in Theorem 4.1(i) are equivalent and equivalent to a Lie bialgebra structure.

In this case we compute

$$\begin{aligned}\langle [(\xi, l), (\xi', l')], (l'', \xi'')\rangle &= \langle l'', [\xi, \xi'] + \beta_l(\xi') - \beta_{l'}(\xi)\rangle + \cdots \\ &= \langle l'', [\xi, \xi']\rangle + \langle \delta\xi', l''\otimes l\rangle - \langle \delta\xi, l''\otimes l'\rangle \\ &\quad + \langle \delta\xi'', l\otimes l'\rangle + \langle l', [\xi'', \xi]\rangle - \langle l, [\xi'', \xi']\rangle,\end{aligned}$$

where "$\cdots$" denotes all occurrences of α, ξ interchanged with those of β, l. Comparing this with Section 1.2, we see that $g \bowtie g^* = D(g)$. We saw in Section 1.2 that this has a canonical solution of the CYBE on it with ad-invariant symmetric part, hence by Drinfel'd's work it is itself a quasitriangular Lie bialgebra. The corresponding Lie bracket structure on $(g \bowtie g^*)^*$ is explicitly

$$\begin{aligned}&\langle [(l, \xi), (l', \xi')], (\xi'', l'')\rangle \\ &= \langle [(\xi'', l''), (e_a, 0)] \otimes (0, f^a) \\ &\quad + (e_a, 0) \otimes [(\xi'', l''), (0, f^a)], (l, \xi) \otimes (l', \xi')\rangle \\ &= \langle [\xi'', \xi'] + \beta_{l''}(\xi'), l\rangle - \langle \alpha_{\xi'}(l''), \xi\rangle \\ &\quad - \langle \beta_l(\xi''), l'\rangle + \langle [l'', l] + \alpha_{\xi''}(l), \xi'\rangle \\ &= \langle ([l, l'], -[\xi, \xi']), (\xi'', l'')\rangle,\end{aligned}$$

i.e., as a Lie bialgebra, $D(g) = g \bowtie g^{*\mathrm{op}}$, where $g^{*\mathrm{op}}$ is g^* with the opposite Lie coalgebra structure.

The example shows how the Lie algebra bicrossproduct includes the concept of a finite dimensional Lie bialgebra introduced independently by Drinfel'd and reviewed in Section 1.2. Taking this further, Lemma 1.1 motivates the following observation. It indicates how to obtain the data $(g_1, g_2, \alpha, \beta)$ from data of the form (g_1, g_2, α, r).

LEMMA 4.3. *Let g_1 and g_2 be finite dimensional Lie algebras, $r: g_2 \to g_1$ a linear map, and α an action of g_1 on g_2. Define $\beta: g_2 \otimes g_1 \to g_1$ by*

$$\beta_l(\xi) = r \circ \alpha_\xi(l) + [r(l), \xi],$$

and suppose that α is a β-derivation. In this case β is automatically an α-derivation. Then β is an action iff the expression

$$r([\ ,\ ]) - [r(\ ), r(\ )] \in g_1 \otimes g_2^* \otimes g_2^*$$

is g_1-invariant (under the action defined by the adjoint action on g_1 and the dual of α on g_2^*).

Proof. Explicitly, we suppose the equation for $\alpha_\xi([l, l'])$ in Theorem 4.1(i), where β is the linear map stated. Then $\beta_l([\xi, \xi']) = r \circ \alpha_{[\xi, \xi']}(l) + [r(l), [\xi, \xi']] = [r \circ \alpha_\xi(l), \xi'] + [[r(l), \xi], \xi'] + r \circ \alpha_\xi(\alpha_{\xi'}(l)) + [r \circ \alpha_{\xi'}(l), \xi] - \xi \leftrightarrow \xi' = [\beta_l(\xi), \xi'] + [\xi, \beta_l(\xi')] + \beta_{\alpha_{\xi'}(l)}(\xi) - \beta_{\alpha_\xi(l)}(\xi')$. Hence β_l is automatically an α-derivation. Next, $\beta_{[l,l']}(\xi) = r([\alpha_\xi(l), l']) + r \circ \alpha_{\beta_l(\xi)}(l) - l \leftrightarrow l' + [r([l, l']), \xi]$ while $\beta_l(\beta_{l'}(\xi)) - \beta_{l'}(\beta_l(\xi)) = r \circ \alpha_{\beta_{l'}(\xi)}(l) + [r(l), r \circ \alpha_\xi(l')] + [r(l), [r(l'), \xi]] - l \leftrightarrow l'$. Hence β is an action iff

$$[\xi, r([l, l']) - [r(l), r(l')]] = r([\alpha_\xi(l), l']) - [r \circ \alpha_\xi(l), r(l')] - l \leftrightarrow l',$$

i.e., iff $r([\ ,\ ]) - [r(\ ), r(\ )]$ is invariant. This completes the proof of the lemma.

This lemma plays an important role in [26], where it is generalized to the level of Lie groups [26, Lemma 4.1]. Drinfel'd's construction corresponds to a variant of the lemma in the special case $g_1 = g$, $g_2 = g^*$. In this variant the Lie algebra structure on g_2 is itself constructed from r such that β in the lemma is the coadjoint action. (Explicitly, it is $[l, l'] = \alpha_{r(l)}(l') + \alpha_{r^*(l')}(l)$. Here r is assumed to have a decomposition into self-adjoint and anti-self-adjoint parts with the former ad-invariant.) Then g^* is indeed a Lie algebra (i.e., β is indeed an action) and g a Lie bialgebra, iff $r([\ ,\ ]) - [r(\ ), r(\ )]$ is ad-invariant. In particular, the vanishing of this expression is precisely the CYBE.

Numerous examples of Lie bialgebras are known, associated to solutions of the CYBE. This includes a canonical one for all simple complex Lie algebras g. Let $\{E_\lambda\}$ denote the root vectors of a Weyl basis with ordered root system $\{\lambda\}$. Let $K \in g^* \otimes g^*$ be the Killing form and $K^{-1} \in g \otimes g$ its inverse. The canonical solution is

$$r = \sum_\lambda \frac{E_\lambda \otimes E_{-\lambda} \operatorname{sgn} \lambda}{K(E_\lambda, E_{-\lambda})} + K^{-1}.$$

Some new examples can be obtained by restricting this to real forms [26]. An example is [26, Lemma 2.2].

EXAMPLE 4.4. Let $g_1 = su(2)$. This can be described explicitly as vectors in $\mathbb{R}^3$ with the vector product, $[\xi, \xi'] = \xi \times \xi'$. Let g_2 also be defined on $\mathbb{R}^3$ by the rule $[l, l'] = e_3 \times (l \times l')$, where $e_3 = (0, 0, 1)$. Then $\alpha_\xi(l) = \xi \times l$ and $\beta_l(\xi) = l \times (\xi \times e_3)$ satisfy Theorem 4.1.

If $\xi \in \mathbb{R}^3$ and $l \in \mathbb{R}^3$ are paired by the Euclidean inner product, then the corresponding Lie bialgebra structure on g_1 according to Example 4.2 is $\delta\xi = \xi \otimes e_3 - e_3 \otimes \xi$.

This is generalized to include *all compact simple real Lie algebras* g_1 *in* [26]. The purpose of these examples is not, however, to obtain bicrossproduct groups (these, per se, tend to be not very interesting) but to use the same data to construct new Hopf algebras. We now turn to this.

(B) The above leads to examples of Proposition 3.12.

EXAMPLE 4.5. Let $(g_1, g_2, \alpha, \beta)$ be a pair of interacting Lie algebras. Then (g_1, g_2) have a bicrossproduct $g_{1\beta}\bowtie_\alpha g_2$ iff $(U(g_1), U(g_2))$ is a matched pair of Hopf algebras. In this case $U(g_1)_\beta\bowtie_\alpha U(g_2) = U(g_{1\beta}\bowtie_\alpha g_2)$.

Proof. Note that for the conventions of Section 3.2, $g_{1\beta}\bowtie_\alpha g_2$ should be defined analogously to Theorem 4.1 but with β a right action (the corresponding left action in Theorem 4.1 is $-\beta$). The Hopf algebra actions α, β are then induced by the Lie algebra actions as follows. β extends to a right action of g_2 on $U(g_1)$ as an *α-derivation*, i.e.,

$$\beta_l(\xi\xi') = \beta_l(\xi)\xi' + \xi\beta_l(\xi') + \beta_{\alpha_\xi(l)}(\xi), \qquad \beta_1(\xi) = \xi,\ \beta_l(1) = 0,$$
$$\forall \xi' \in g_1,\ \xi \in U(g_1).$$

This makes sense in $U(g_1)$ because of the conditions analogous to Theorem 4.1(i) and because α is a Lie algebra action. Similarly, α extends to a left action of g_1 on $U(g_2)$ as a β-derivation. Then, by the universal properties of universal enveloping algebras, these extend to actions of $U(g_2)$ on $U(g_1)$ and of $U(g_1)$ on $U(g_2)$ and one can check that they are a matched pair as in Proposition 3.12. The converse and the stated isomorphism are obtained by restricting to the primitive elements.

(C) By analogy with the finite dimensional case, Proposition 3.13, one expects that the same data $(g_1, g_2, \alpha, \beta)$ as in the previous example lead under favorable circumstances to bicrossproducts of the form

$$U(g_1)^{\beta^*}\bowtie_{\alpha^*} k[[g_2]],$$

where $k[[g_2]]$ denotes a *suitable* dual of $U(g_2)$. In the case when β is trivial, α is an action by bialgebra maps. Hence in this case $U(g_2)^\circ$ [4, Chap. 6] is a suitable definition. For in this case the action α dualizes to an action α^* that restricts to $U(g_2)^\circ$. It would appear that the correct definition of the dual for our purposes should be modified according to β. This, and the duality theorem analogous to Theorem 2.2, is deferred for further work.

(D) The Hopf algebra analogue of Example 4.2 is

EXAMPLE 4.6. Let H be a finite dimensional Hopf algebra and S' its skew-antipode. Let $B = H^{*\mathrm{op}}$ denote the dual Hopf algebra to H but with the opposite coalgebra. Let α and β be given, $\forall h \in H$, $b \in B$, $a \in H$, by

$$\langle \alpha(h \otimes b), a \rangle = \sum \langle b, S'(h_{(2)}) a h_{(1)} \rangle, \qquad \beta(h \otimes b) = \sum h_{(2)} \langle b, S'(h_{(3)}) h_{(1)} \rangle.$$

Proof. That (α, β) have the properties needed for Proposition 3.12 can be checked directly. It also follows from Example 4.7 (which will be checked explicitly) via Proposition 3.13. From the explicit formulae stated in Proposition 3.12 the Hopf algebra structure is $(h \otimes b).(g \otimes c) = h_{(4)} \langle c_{(2)}, (S'h_{(5)}) h_{(3)} \rangle \, g \otimes b \langle c_{(1)}, (S'h_{(2)}) \,?\, h_{(1)} \rangle = h_{(4)} g \otimes b \langle c, (S'h_{(5)}) h_{(3)} (S'h_{(2)}) \,?\, h_{(1)} \rangle$. Here ? denotes an unused argument as in Section 1.2. After using the skew-antipode property, this coincides with the algebra structure of Drinfel'd's $D(H)$ given in Section 1.2. The other facts are similar.

Thus we have recovered Drinfel'd's Hopf algebras $D(H)$, introduced in Section 1.2, as examples of Proposition 3.12. There is no restriction on H to be commutative or cocommutative. These Hopf algebras are interesting because they have elements R obeying the Quantum Yang–Baxter Equations as explained in Section 1.2. It is not known if such equations play a role in the general double crossproduct construction.

(E) Finally, applying Proposition 3.13 to the last example leads to new non-commutative, non-cocommutative Hopf algebras for *every* Hopf algebra with skew-antipode. Finiteness is no longer required.

EXAMPLE 4.7. Let H be a Hopf algebra with skew-antipode and $A = H^{\mathrm{op}}$ the Hopf algebra with opposite algebra structure. Let $\alpha = \mathrm{ad}_i$ the adjoint action of H on H^{op} induced by the identity map $i\colon H \to H^{\mathrm{op}}$. Explicitly, this is

$$\alpha(a \otimes h) = \sum i(h_{(1)}) \, a S_A i(h_{(2)}), \qquad \forall h \in H, \ a \in H^{\mathrm{op}}.$$

Here S_A is the antipode on H^{op}. Let $\beta = \mathrm{co}_i$ the coaction of H^{op} on H induced by i. Explicitly, this is

$$\beta(h) = \sum i(h_{(1)}) \, S_A i(h_{(3)}) \otimes h_{(2)}.$$

Then $H^{\mathrm{co}_i} \bowtie_{\mathrm{ad}_i} H^{\mathrm{op}}$ is a bicrossproduct. It has a skew-antipode.

This is an example of Theorem 3.3 with $A = H^{\mathrm{op}}$. The definition of ad_f for $f \in \mathrm{alg}(H, A)$ was reviewed in the introduction of Section 3.1. The definition of co_f for $f\in \mathrm{colg}(H, A)$ is dual [3, Sect. 2.3]. In the present case i is an anti-algebra map. This makes ad_i a right action as the next lemma asserts. It is a variant of similar constructions in [3, Sect. 2.3], so the proof is omitted.

LEMMA 4.8 (cf. [3, Sect. 2.3]). *Let H, A be Hopf algebras. If $f\in \mathrm{antialg}(H, A)$ and if there exists S' a skew-antipode on H, then*

$$\mathrm{ad}_f(a\otimes h)=\sum f(h_{(1)})\, af(S'(h_{(2)}))$$

is a right H-module algebra structure on A.

If $f\in \mathrm{colg}(H, A)$ and S_A is the antipode on A, then

$$\mathrm{co}_f(h)=\sum f(h_{(1)})\, S_A f(h_{(3)})\otimes h_{(2)}$$

is a left A-comodule coalgebra structure on H.

In the present case, we apply the lemma to $f=i\colon H\to A=H^{\mathrm{op}}$, the linear identity map. Note that $S_A=i\circ S'\circ i^{-1}$. Thus to verify the data for Theorem 3.3, we only need to verify (A), (B), (C). For (A) we compute using the properties of i and S' that

$$(a.h)_{(1)}\otimes (a.h)_{(2)}=i(h_{(1)})\, a_{(1)} i(S'h_{(4)})\otimes i(h_{(2)})\, a_{(2)} i(S'h_{(3)}),$$

$$a_{(1)}.h_{(1)} h_{(2)}{}^{(\bar{1})}\otimes a_{(2)}.h_{(2)}{}^{(\bar{2})}=i(h_{(1)})\, a_{(1)} i(S'h_{(2)})\, i(h_{(3)})\, i(S'h_{(6)})$$
$$\otimes\, i(h_{(4)})\, a_{(2)} i(S'h_{(5)}).$$

These expressions are equal since i is an anti-algebra map and S' a skew-antipode. In addition i a coalgebra map implies $\varepsilon_A(a.h)=\varepsilon_A(i(h_{(1)})\, ai(S'h_{(2)}))=\varepsilon_H(h)\,\varepsilon_A(a)$. For (B), we compute

$$(hg)^{(\bar{1})}\otimes (hg)^{(\bar{2})}$$
$$=i(g_{(1)})\, i(h_{(1)})\, i(S'h_{(3)})\, i(S'g_{(3)})\otimes h_{(2)} g_{(2)},$$
$$h^{(\bar{1})}.g_{(1)} g_{(2)}{}^{(\bar{1})}\otimes h^{(\bar{2})} g_{(2)}{}^{(\bar{2})}$$
$$=i(g_{(1)})\, i(h_{(1)})\, i(S'h_{(3)})\, i(S'g_{(2)})\, i(g_{(3)})\, i(S'g_{(5)})\otimes h_{(2)} g_{(4)}.$$

These expressions are equal since i is an anti-algebra map and S' is a skew-antipode. In addition $1_H{}^{(\bar{1})}\otimes 1_H{}^{(\bar{2})}=i(1_H)\, i(S'1_H)\otimes 1_H=1_A\otimes 1_H$. Finally, for (C) we compute

$$h_{(1)}{}^{(\bar{1})}a.h_{(2)}\otimes h_{(1)}{}^{(\bar{2})}=i(h_{(1)})\, i(S'h_{(3)})\, i(h_{(4)})\, ai(S'h_{(5)})\otimes h_{(2)},$$
$$a.h_{(1)} h_{(2)}{}^{(\bar{1})}\otimes h_{(2)}{}^{(\bar{2})}=i(h_{(1)})\, ai(S'h_{(2)})\, i(h_{(3)})\, i(S'h_{(5)})\otimes h_{(4)}.$$

These are equal by the properties of i and the skew-antipode property. This completes the proof of the example.

For every Hopf algebra H with skew-antipode, this construction gives a new Hopf algebra $H^{\mathrm{co}_i} \bowtie_{\mathrm{ad}_i} H^{\mathrm{op}}$, the "mirror product." Since it also possesses a skew-antipode, the construction can be iterated. Although the "mirror double" is not generally self-dual, the construction was inspired by duality symmetry considerations (see the final paragraph of Section 3.1). In the finite dimensional case it is related to Drinfel'd's "quantum double" $D(H)$ by Proposition 3.13 and is non-commutative and non-cocommutative if H is.

From our point of view then, $D(H) = H_{\mathrm{co}_i^*} \bowtie_{\mathrm{ad}_i^*} H^{*\mathrm{op}}$. To see this, we apply Proposition 3.13 with $\alpha, \beta, \alpha^*, \beta^*$ given by $\mathrm{ad}_i^*, \mathrm{co}_i^*, \mathrm{ad}_i, \mathrm{co}_i$, respectively. For $h \in H$, $b \in B = H^{*\mathrm{op}}$, $a \in H^{\mathrm{op}}$ the stated relations are then $\mathrm{co}_i^*(h \otimes b) = \langle h^{(\bar{1})}, b \rangle h^{(\bar{2})} = \langle b, i(h_{(1)}) S_A i(h_{(3)}) \rangle h_{(2)} = \langle b, i((S'h_{(3)}) h_{(1)}) \rangle h_{(2)}$ and $\langle a, \mathrm{ad}_i^*(h \otimes b) \rangle = \langle \mathrm{ad}_i(a \otimes h), b \rangle = \langle i(h_{(1)}) a S_A i(h_{(2)}), b \rangle = \langle i((S'h_{(2)}) i^{-1}(a) h_{(1)}), b \rangle$. Thus ad_i^* and co_i^* coincide with the actions used in Example 4.6, where $H^{*\mathrm{op}}$ was identified as a linear space with H^*. This completes the proof that Example 4.6 is recovered via Proposition 3.13 from Example 4.7.

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