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INSTANTON-BASED VACUUM FROM THE FEYNMAN VARIATIONAL PRINCIPLE

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We investigate the possibility that the most essential non-perturbative gauge field configurations in QCD are of an instanton–anti-instanton type Since these are not exact solutions of classical equations, we use a modification of the Feynman variational principle in order to calculate in a mathematically consistent way the contribution of given trial gauge field configurations (with small oscillations around them) to the QCD partition function We observe an effective repulsion of instantons which removes the usual infrared problems of the instanton calculus The repulsion leads to the stabilization of the pseudoparticle "liquid" at distances being a factor of ~ 3 larger than their average sizes We calculate the upper bound for the vacuum energy, the non-perturbative gluon condensate $\langle F_{\mu\nu}^2 \rangle$, the topological susceptibility, etc in terms of renormalization-invariant combinations We find our numerical results reasonable from the phenomenological point of view

The variational principle permits one to improve the upper bound for the vacuum energy by varying the profile function of instantons We find that the power fall-off of instantons is replaced at large distances by a Yukawa-type one Therefore, a mass gap for glueballs is naturally obtained at the classical level

1. Introduction

Instantons – the classical tunnelling trajectories in imaginary (euclidean) time with unity topological charge were discovered in 1975 [1] and since then have become very popular. A good reason for this is that the hypothesis of the importance of the instanton-type fluctuations in the vacuum qualitatively explains many features of the strong-interaction theory, QCD Among these are a solution of the U_1 problem [2], an understanding of the chiral symmetry breaking [3, 4], a basis for the bag model [3]. Also, instantons lead to the non-zero non-perturbative gluon condensate $\langle F_{\mu\nu}^a F_{\mu\nu}^a \rangle$, $\langle F^3 \rangle$, etc [5] Recently from the analysis of a large number of phenomenological consequences new arguments were acquired in favour of the essentiality of instanton configurations in the vacuum [6]

Unfortunately, the most straightforward way of understanding the instanton vacuum, i e. to consider a dilute gas of instantons and anti-instantons (hereafter $I\bar{I}$), meets with severe infrared troubles [3], since the integrals over the sizes ρ of free I's diverge at large ρ. If one tries to cut the ρ integrations "by hand", one runs into a

contradiction with a general theorem arising from the renormalization properties of the theory [7].

A way out of the difficulties of the dilute gas is sometimes seen in taking into account all the exact solutions of classical Yang-Mills equations with arbitrary large topological charges (the so-called multi-instanton solutions). To this end, a very interesting paper [8] is usually referred to, where this program was performed in the two-dimensional σ-model However, we feel that such an approach has little to do with the physical vacuum of QCD Indeed, since there is no CP violation in strong interactions, the ground state of QCD has the quasimomentum ϑ equal to zero This means that in QCD we deal with a phase where there is an equal number (up to thermodynamical fluctuations $\sim \sqrt{N}$) of field configurations with topological charge $\pm n$, i e I's and $\bar{\text{I}}$'s But such configurations were not accounted for in ref [8] since the configurations of the $\text{I}\bar{\text{I}}$ type are not exact solutions of the classical equations A way out of this shortcoming of ref. [8] can be found in ref [9] where $\text{I}\bar{\text{I}}$ configurations of the σ-model were also taken into account, as well as their interactions

Thus, from physical considerations it is clear that the main contribution to the partition function should be given not by exact classical solutions, but by approximate solutions of the $\text{I}\bar{\text{I}}$ type This should not surprise us since the approximate solutions may have a larger entropy (= statistical weight) than the exact solutions which correspond to the local minima of the action Anyhow, one is interested in maximizing the partition function of a theory, and $\text{I}\bar{\text{I}}$ configurations may well give a larger contribution to it than those of multi-instantons

At this point we are, unfortunately, led off the well-trodden path of the quasi-classical approach. Putting it in a more general way, the problem is to calculate the contribution to a functional integral of a given trial field configuration (which is not, generally speaking, a local minimum of the action) together with the small fluctuations around it We feel that the adequate way to formulate (and to solve) this problem is to use a modification of the Feynman variational principle [10] Let us briefly outline it

Let a field theory in the euclidean formulation be given by a partition function

$$Z = \int D\phi \, \mathrm{e}^{-S[\phi]} \tag{1 1}$$

We want to calculate the contribution to Z of a given field configuration $\bar{\phi}$ (together with its vicinity) To do that we introduce an approximate action $S_1[\phi]$ such that $\bar{\phi}$ is a local minimum of S_1 and $S_1[\bar{\phi}] = S[\bar{\phi}]$ Hopefully, the approximate partition function

$$Z_1 = \int D\phi \, \mathrm{e}^{-S_1[\phi]} \tag{1 2}$$

is calculable. We have then

$$Z = \frac{\int D\phi\, e^{-(S[\phi]-S_1[\phi])} e^{-S_1[\phi]}}{Z_1} Z_1$$

$$\geqslant Z_1 \exp(-\langle S - S_1 \rangle), \tag{1 3}$$

where

$$\langle S - S_1 \rangle = \frac{\int D\phi\, (S[\phi] - S_1[\phi]) e^{-S_1[\phi]}}{Z_1} \tag{1.4}$$

The r.h.s. of the inequality (1 3) depends on the trial field configuration $\bar{\phi}$ Therefore, varying the r.h.s of (1.3) with respect to $\bar{\phi}$ one finds field configurations which are most important in saturating the partition function Z Simultaneously, one finds an upper bound for the vacuum energy density ϑ_{00}, since with appropriate boundary conditions Z is a matrix element of $\exp(-\vartheta_{00} VT)$, as the euclidean time $T \to \infty$. It should be noted that the quantity

$$\langle (S - S_1)^2 \rangle_{\text{con}} = \langle (S - S_1)^2 \rangle - \langle S - S_1 \rangle^2 \tag{1 5}$$

gives an estimate for the error of this calculation.

In another work [11] we have performed the above program for some quantum-mechanical problems where tunnelling effects are essential, and obtained a very good accuracy for Z and some other physical quantities. However, passing from quantum mechanics to field theory one meets the following problems in using the variational principle. (i) Z must be normalized, (ii) Z must be regularized, (iii) the appropriate action S_1 should neither spoil the known renormalization properties of a theory nor the gauge invariance (for gauge theories). We shall turn to these problems in the next section.

The purpose of this work is to calculate the contribution to the functional integral Z in QCD (without quarks) of gauge field configurations of the $I\bar{I}$ type To be specific we use the following trial function for the variational principle.

$$\bar{A}^a_\mu(x, \gamma_i) = \sum_{i=0}^{N_+} A^{(+)a}_\mu(x - x_{0i}, \rho_i, O_i) + \sum_{j=0}^{N_-} A^{(-)a}_\mu(x - x_{0j}, \rho_j, O_j), \tag{1.6}$$

where

$$A^{(+)a}_\mu = 2O^{aa'}_i \bar{\eta}_{a'\mu\nu} \frac{(x - x_{0i})_\nu}{(x - x_{0i})^2} f\left(\frac{(x - x_{0i})^2}{\rho_i^2}\right) \tag{1 7}$$

is the field of an instanton with the center at x_{0i}, size ρ_i and colour orientation given by an orthogonal matrix O_i, in the case of an $\bar{\mathrm{I}}$ replace $\bar{\eta}_{a\mu\nu} \to \eta_{a\mu\nu}$, η being the 't Hooft symbols [2]. The set of $4N_c$ parameters characterizing one (anti)instanton (for the SU(N_c) colour group) we denote by γ_i

The $\mathrm{I}\bar{\mathrm{I}}$ superposition (1 6) is meaningful only if individual I's are taken in the singular gauge, with the profile function

$$f\left(\frac{x^2}{\rho^2}\right) = f_0\left(\frac{x^2}{\rho^2}\right) \equiv \frac{1}{1 + x^2/\rho^2} \tag{1 8}$$

An individual I (1.7) with the profile function (1 8) satisfies the Yang-Mills equations and possesses a unit topological charge $\int F^a_{\mu\nu} \tilde{F}^a_{\mu\nu} \mathrm{d}^4x/32\pi^2 = 1$. In the larger part of this paper we will restrict ourselves to instantons, $f = f_0$ However, the variational principle permits one to go further and to calculate the contribution of (1 6) to Z as a functional of a general trial profile function $f(x^2)$ This is performed in sect 6 of the paper The only restriction we impose on $f(x^2)$ is $f(0) = 1$, which is the condition for unity topological charge and for the action to be finite

Our ansatz (1 6) corresponds thus to a hypothesis that the most important trajectories in the path integral of QCD (important from the viewpoint of maximizing the partition function) are tunnelling transitions between neighbour minima of the Yang-Mills potential energy but the concrete forms $f(x^2)$ of the transitions are not necessarily that of I and $\bar{\mathrm{I}}$

The best trial function $f(x^2)$ is determined by an interplay of two trends the minimization of individual actions and the minimization of pseudoparticles' interaction in the medium Let us note that a variation with respect to $f(x^2)$ compensates effectively the crudity of our ansatz (1 6). We derive a differential equation for $f(x^2)$ where both trends can be seen. As it should be expected, $f(x^2)$ follows $f_0(x^2)$ at $x^2 \lesssim \rho^2$ and differs from it only at large values of x^2 where instantons begin to overlap and the effects of the medium become essential It turns out that asymptotically $f(x^2) \sim \exp(-|x|)$ instead of (1 8) Therefore, the correlation functions in such a vacuum will have at large distances Yukawa-type behaviour, giving rise to a mass gap – at least at the classical level We make an estimate of the glueball mass in terms of the fundamental quantities of the theory, see sect 6

In the limiting case when instantons are very dilute, i e when the average distances between them are much larger than their average size, the classical action, defined as $(1/4g^2)\int (F^a_{\mu\nu})^2 \mathrm{d}^4x$ becomes just a sum of $N_+ + N_-$ free instanton actions, each equal to $8\pi^2/g^2 \equiv \beta$ In this case one can integrate, following 't Hooft [2], over the small fluctuations around *each* instanton and write down the contribution of the zero modes in the instanton field as integrals over collective coordinates, x_{0i}, ρ_i and O_i Thus, in the dilute case one obtains the following well-known

approximate expression for the functional integral (divided by the functional integral with no background field) [3]:

$$Z = \frac{1}{N_+!N_-!} \prod_{i=1}^{N_+ + N_-} \int d^4x_{0i} \left(d\rho_i/\rho_i^5\right) dO_i C_{N_c} \beta^{2N_c}(\rho_i) e^{-\beta(\rho_i)}, \qquad (1\ 9)$$

where C_{N_c} is a numerical constant calculated in refs. [2,12], $\beta(\rho_i)$ is the inverse running coupling constant depending on the size of the instanton,

$$\beta(\rho) = b \ln(1/\rho\Lambda), \qquad b = \tfrac{11}{3} N_c. \qquad (1\ 10)$$

Unfortunately, the expression (1.9) is senseless because of the divergence of the ρ integrations at large ρ. However, as the sizes of instantons grow, one can no longer neglect the interactions of individual instantons.

There are two sources of inter-instanton interactions which modify eq. (1.9). First, the classical action is not just the sum of free instanton actions. One can define the classical potential u_{int} by subtracting the sum of free actions from the total one

$$\exp(-\beta u_{\text{int}}) \equiv \exp\left[-\frac{\beta}{32\pi^2} \int d^4x \left(F_{\mu\nu}^2 - \sum_i F_{\mu\nu}^2(i)\right)\right], \qquad \beta = \frac{8\pi^2}{g^2} \qquad (1\ 11)$$

We shall call this part of the interaction classical.

Second, the quantum corrections do not, generally speaking, factorize into a product of functional determinants calculated in the background field of individual instantons. In particular, the zero modes are no longer zero, and the idea of writing down their contribution as integrals over collective coordinates seems to be undermined. However, one can still write the functional integral in the form (1.9), keeping in mind that a correction due to the non-factorization of the determinant should be taken into account. We shall call this part of the effective instanton interaction quantum.

We would like to emphasize that any quantum interaction, being written in the exponential form of the classical interaction (1.11), does not contain the β factor in the exponent. Now, what is β? β, the inverse effective charge or the inverse effective temperature (in the language of statistical mechanics) is a slowly varying logarithmic function of the characteristic size of instantons. If the system stabilizes, one can estimate β, according to eq. (1.10), as $\beta \simeq b \ln(1/\bar{\rho}\Lambda)$, where $\bar{\rho}$ is the average size of instantons, Λ is the usual charge-parametrization parameter. Naturally, $\bar{\rho}$ and, consequently, $\beta(\bar{\rho})$ should be determined in a self-consistent manner at the end of the calculations. Theoretically, we deal with a true strong-interaction case: strictly speaking, all numbers are of the order of unity, $\beta(\bar{\rho})$ among them. However, we find an unexpected gift: it turns out that numerically $\beta(\bar{\rho}) \simeq 12$ for SU_2 and $\beta(\bar{\rho}) \simeq 15$ for SU_3. Therefore, for the larger part of our calculations we shall assume that $\beta(\bar{\rho})$

is a large parameter that stands in the exponent, so that the quantum interactions can be neglected Finally, we shall see that the self-consistency equations have indeed a solution leading to $\beta(\bar{\rho}) \gg 1$ and thus we shall justify the procedure

Naturally, these speculations become meaningful only if the system stabilizes itself at some $\bar{\rho} \ll \Lambda^{-1}$ solely due to the classical part of the interaction Meanwhile, as is well known [3], the classical interaction of I's has a "dipole–dipole" tail at large distances This interaction depends on the relative orientation of I and $\bar{\mathrm{I}}$ in the colour and the configuration spaces. On the average it is zero but there always exist such relative orientations which lead to the attraction of I and $\bar{\mathrm{I}}$ Hence one could think that the system cannot be stable with only classical interactions being taken into account These were the lines of thought that led Callan, Dashen and Gross [3] to speculate on the probable phase transition of the instanton gas to another phase

One of the main results of this paper is that the interaction potential between I and $\bar{\mathrm{I}}$ becomes actually *repulsive* already at distances twice or thrice larger than the size of the instantons (sect 3) We feel that we have deduced a sort of dimensionless hard-core interaction proposed some time ago by Jevicki [13] and investigated in detail by Ilgenfritz and Mueller-Preussker [14] However, in contrast to these authors, we do not introduce the hard core "by hand" – its concrete form follows from the calculation of the action of the trial function (1 6)

One can argue here that the repulsion we observe is an artifact of the concrete trial function (1 6), it is, certainly One could choose another trial configuration (also reduced to the sum of I's in the dilute limit) leading to a classical attraction In this case, however, one would observe the repulsion at the quantum level (we have investigated these effects in some quantum mechanical models) The physics would be the same, only numerical constants could differ somewhat, depending on the concrete ansatz It is a great advantage of the variational approach that this subtle problem can be managed in a straightforward way given a trial ansatz, one has to take the induced interaction of I's at face value, calculate the partition function and, finally, maximize it with respect to the free parameters (functions) of the chosen ansatz

Having at hand the microscopic interaction of instantons, the next question is, what is the phase of the system? What one can say for certain is that it is not in the dilute gas phase (see sect 5)

A more delicate question is, since the interaction depends essentially on the orientation and has a minimum at some distance, and since β, the inverse temperature, is presumably large, – why then do not the instantons form an ordered phase, i e a crystal, thus spontaneously violating colour and/or Lorentz symmetry?

Clearly, this intriguing question is but a quantitative one one should calculate the free energy in an ordered and in a disordered phase and compare the two We calculate the free energy of an instanton crystal in sect 4 and find that the ordered phase is not competitive with the disordered one, at least for small groups SU(N_c) However, for large N_c the question is still open

Therefore, at least for colour groups SU_2, SU_3 one is left with the disordered phase which is not a dilute gas – that is, with a liquid*

In dealing with the statistical mechanics of the instanton liquid we have elaborated several different approximation schemes, all of them producing very similar numbers Therefore, in this paper we present the most simple but reliable approach based again on a variational principle (sect 5)

We find numerical values for some basic quantities of the theory. the vacuum energy density (counted from that of perturbation theory), the non-perturbative gluon condensate $\langle F_{\mu\nu}^2 \rangle$, the mean square of the topological charge (= topological susceptibility), the average sizes of I's and the average distance between them All these quantities are found as renormalization-invariant combinations of charge and the cut-off Moreover, we check a low-energy theorem [7] following from the renormalizability An important manifestation of the relative diluteness of the medium is the average potential between the I's – it is a few percent of the free action of one I It means that the overlap of I's is not too strong and that the main assumptions acquire an a posteriori justification.

Actually, since we use a variational principle, we can state that we have found only the lower bound for, say, the gluon condensate Taking $\Lambda_{\text{Pauli-Villars}} \approx 200$ MeV (corresponding to $\Lambda_{\text{lattice}} \approx 6\,4$ MeV) we observe that our lower bound practically coincides with the value of $\langle F_{\mu\nu}^2 \rangle$ known from phenomenology [15] If one is bold enough to take the numbers at face value, one is led to believe that the widely separated instanton-type configurations are indeed the dominant non-perturbative vacuum fluctuations which lower the energy density as compared to the perturbative one [6] Then one can go further and work out the physical consequences of this striking inhomogeneous vacuum

2. Variational principle

We will start by rewriting identically the path integral of the Yang-Mills theory in terms suitable for evaluating the contribution of a given background field $\bar{A}_\mu(x, \gamma_t)$ with small oscillations of quantum field $B_\mu(x)$ around it Let us parametrize a general gauge field $A_\mu(x)$ as

$$A_\mu(x) = \left(\bar{A}_\mu(x, \gamma_t) + B_\mu(x)\right)^{\Omega(x)}, \tag{2 1}$$

where A_μ^Ω is a gauge-transformed field

$$A_\mu^\Omega = SA_\mu S^+ - \imath \partial_\mu S S^+, \qquad S = \exp(\imath \Omega) \tag{2.2}$$

* We note that recently Shuryak strongly advocated the liquid phase for instantons with a small packing fraction on phenomenological grounds [6] One of us (D D) takes the opportunity to thank E V Shuryak for enlightening discussions which triggered off this work

According to (2.1), any $A_\mu(x)$ is parametrized by $B_\mu(x)$ and $\Omega(x)$ and by a set of p parameters γ_i characterizing the chosen background field. Naturally, that is not unique, and we can impose extra conditions. We shall demand that $B_\mu(x)$ satisfies

$$D_\mu(\bar{A}(x,\gamma_i))B_\mu(x)=0, \tag{2.3}$$

$$\int \psi_\mu^{(k)}(x,\gamma_i)B_\mu(x)\,d^4x=0, \qquad k=1,2,\ \ldots,p, \tag{2.4}$$

where $\psi_\mu^{(k)}$ is a set of p orthonormalized functions, to be determined below. The aim of the restrictions (2.3), (2.4) is to forbid the variation of the quantum field B_μ in the "dangerous" directions connected with pure gauge transformations and with zero (or approximately zero) modes in the background field $\bar{A}_\mu(x,\gamma_i)$.

Introducing a unity à la Faddeev-Popov

$$1=\int DA_\mu\,DB_\mu\,D\Omega\prod_{i=1}^{p}d\gamma_i\,\delta\left(A_\mu-\left(\bar{A}_\mu+B_\mu\right)^{\Omega}\right)\delta\left(D_\mu(\bar{A})B_\mu\right)$$

$$\times\prod_{k=1}^{p}\delta\left(\int d^4x\,\psi_\mu^{(k)}B_\mu\right)\Phi[A_\mu], \tag{2.5}$$

which, in fact, is the definition of the gauge-invariant functional $\Phi[A^\Omega]=\Phi[A]$, one can rewrite the Yang-Mills functional integral as

$$Z=\int DA_\mu\exp\left(-\frac{1}{g^2}S[A]\right)$$

$$=\int D\Omega\,DB_\mu\prod d\gamma_i\,\delta\left(D_\mu(\bar{A})B_\mu\right)\prod\delta\left(\int d^4x\,\psi_\mu^{(k)}B_\mu\right)$$

$$\times\Phi[\bar{A}+B]\exp\left(-\frac{1}{g^2}S[\bar{A}+B]\right), \tag{2.6}$$

where the gauge invariance of Φ and S has been used. The integral $\int D\Omega$ gives the total volume of the gauge group and will be omitted.

Let us calculate

$$\left(\Phi[\bar{A}(\gamma)+B]\right)^{-1}=\int D\Omega'\,DB'_\mu\prod d\gamma'_i\,\delta\left(\bar{A}_\mu(\gamma)+B_\mu-\left(\bar{A}_\mu(\gamma')+B'_\mu\right)^{\Omega'}\right)$$

$$\times\delta\left(D_\mu(\bar{A}(\gamma')B'_\mu\right)\prod\delta\left(\int d^4x\,\psi_\mu^{(k)}(\gamma')B'_\mu\right) \tag{2.7}$$

These integrals are easily calculated owing to the δ-functions which impose

$$B'_\mu = B_\mu, \qquad \gamma'_i = \gamma_i, \qquad \Omega' = 0. \tag{2.8}$$

To find Φ it is sufficient to expand the arguments of the δ-functions with respect to the deviations of $B'_\mu, \Omega', \gamma'_i$ from their values (2.8), and to calculate the integrals.

The argument of the first δ-function is

$$-\sum_i \frac{\partial \bar{A}_\mu}{\partial \gamma_i}(\gamma'_i - \gamma_i) + B_\mu - B'_\mu - D_\mu(\bar{A}(\gamma) + B)\Omega'. \tag{2.9}$$

We use it to perform the integration over B'_μ and obtain

$$\Phi^{-1}[\bar{A}(\gamma) + B]$$

$$= \int D\Omega' \prod \mathrm{d}\gamma'_i \prod \delta\left[\int \mathrm{d}^4x\, \psi^{(k)}_\mu(x, \gamma')\left(B_\mu - \sum_i \frac{\partial \bar{A}_\mu}{\partial \gamma_i}(\gamma'_i - \gamma_i) - D_\mu(\bar{A}(\gamma) + B)\Omega\right)\right]$$

$$\times \delta\left[D_\mu(\bar{A}(\gamma'))\left(B_\mu - \sum_i \frac{\partial \bar{A}_\mu}{\partial \gamma_i}(\gamma'_i - \gamma_i) - D_\mu(\bar{A}(\gamma) + B)\Omega'\right)\right]. \tag{2.10}$$

The expansion of the argument of the last δ-function with respect to $\gamma' - \gamma$ is

$$-D_\mu(\bar{A}(\gamma))D_\mu(\bar{A}(\gamma) + B)\Omega' - \sum_i D_\mu(\bar{A}(\gamma) + B)\frac{\partial \bar{A}_\mu}{\partial \gamma_i}(\gamma_i - \gamma'_i)$$

Integrating over Ω' we get

$$\Phi^{-1}[\bar{A}(\gamma) + B] = \det{}^{-1}[D_\mu(\bar{A}(\gamma))D_\mu(\bar{A}(\gamma) + B)]$$

$$\times \int \prod \mathrm{d}\gamma'_i \prod \delta\left[\int \mathrm{d}^4x \sum_i \left(\frac{\partial \psi^{(k)}_\mu}{\partial \gamma_i} B_\mu - \psi^{(k)}_\mu P_{\mu\nu} \frac{\partial \bar{A}_\nu}{\partial \gamma_i}\right)(\gamma'_i - \gamma_i),\right. \tag{2.11}$$

where $P_{\mu\nu}$ is a transverse projection operator

$$P_{\mu\nu} = \delta_{\mu\nu} - D_\mu(\bar{A} + B)[D_\lambda(\bar{A})D_\lambda(\bar{A} + B)]^{-1}D_\nu(\bar{A} + B) \tag{2.12}$$

One has now to integrate over γ' and substitute (2.11) into (2.6). The result is

$$Z = \int \prod \mathrm{d}\gamma_l \int DB_\mu \prod \delta\left(\int \mathrm{d}^4x\, \psi_\mu^{(l)} B_\mu \right) \det{}_{(kl)} \int \mathrm{d}^4x \left(\psi_\mu^{(k)} P_{\mu\nu} \frac{\partial \bar{A}_\nu}{\partial \gamma_l} - \frac{\partial \psi_\mu^{(k)}}{\partial \gamma_l} B_\mu \right)$$

$$\times \det\left[D_\mu(\bar{A}) D_\mu(\bar{A}+B) \right] \exp\left[-\frac{1}{g^2} \left\{ S[\bar{A}+B] + \frac{1}{2\alpha^2} \int \left(D_\mu(\bar{A}) B_\mu \right)^2 \mathrm{d}^4x \right\} \right] \tag{2.13}$$

Note that we have smeared the condition (2.3) introducing a general background gauge labelled by α. In what follows, we will use the Feynman background gauge with $\alpha = 1$.

Eq. (2.13) is an *exact* expression for the Yang-Mills partition function. Now we are in a position to calculate, according to the variational principle, the contribution to Z of a given trial configuration $\bar{A}_\mu(x, \gamma_l)$.

Let us expand the action $S[\bar{A}+B]$ together with the gauge-fixing term in powers of the quantum field B_μ.

$$S[\bar{A}+B] + \mathrm{GFT} = S[\bar{A}] + \int \mathrm{d}^4x\, D_\mu(\bar{A}) F_{\mu\nu}(\bar{A}) B_\nu$$

$$+ \tfrac{1}{2} \int \mathrm{d}^4x\, B_\mu W_{\mu\nu}(\bar{A}) B_\nu + \mathrm{O}(B^3) + \mathrm{O}(B^4), \tag{2.14}$$

where

$$W_{\mu\nu}(\bar{A}) = -D^2(\bar{A}) \delta_{\mu\nu} + 2\imath F_{\mu\nu}(\bar{A}) \tag{2.15}$$

Had we started to work with this action, our aim would not be achieved. Indeed, since a general trial function $\bar{A}(x, \gamma_l)$ does not satisfy the equation of motion, the linear term in (2.14) would necessarily force us to perform a shift in the B_μ integration and come to some true minimum of the action. As a result, we would calculate not the contribution of the vicinity of a given trial function, but of a shifted one. Therefore, to achieve our goal we have to throw away the linear term in (2.14)! Let us denote the obtained action by $S_1[\bar{A}, B]$ and the corresponding partition function by $Z_1[\bar{A}]$. Note that this $Z_1[\bar{A}]$ is a gauge-invariant functional of the trial function, since everything in (2.13) depends on $\bar{A}$ only through the field strength $F_{\mu\nu}(\bar{A})$ and the covariant derivative $D_\mu(\bar{A})$.

Rewriting formulae leading to (1.3) one gets

$$Z \geqslant Z_1[\bar{A}], \tag{2.16}$$

since in this case

$$\langle S - S_1 \rangle = \left\langle \int D_\mu(\bar{A})\, F_{\mu\nu}(\bar{A})\, B_\nu \, \mathrm{d}^4 x \right\rangle = 0 .$$

Both sides of (2.16) should be normalized and regularized. A natural normalization is achieved by dividing (2.16) by a free partition function Z_{PT} with the background field $\bar{A}$ switched off.

$$Z_{\mathrm{PT}} = \int DB_\mu \det\left[\partial_\mu D_\mu(B)\right] \exp\left[-\frac{1}{g^2}\left\{ S[B] + \tfrac{1}{2}\int \mathrm{d}^4 x \left(\partial_\mu B_\mu\right)^2 \right\}\right] \qquad (2\ 17)$$

Evidently, it does not spoil the inequality (2.16).

As for the ultraviolet regularization, the problem may seem to be more subtle. It is evident, however, that, say, with the lattice regularization the inequality (2.16) is not spoiled Meanwhile, all regularization schemes are physically equivalent when calculating the r h.s of (2.16) Therefore, one can use other schemes as well a cut-off in the eigenvalues of the quadratic form (2.15), or a cut-off in the proper time representation for the Green functions of the B_μ field, etc We check the scheme-invariance explicitly in sect. 5.

After dividing (2.16) by the perturbation theory partition function (2 17) and regularizing, the r h s of (2.16) becomes a well-defined series in the coupling constant The first term in it is a 1-loop contribution. To find it, it is sufficient to neglect the $O(B^3)$ and $O(B^4)$ terms in the action and to neglect B_μ in the determinants entering (2.13). It is convenient to choose the zero-modes fixing functions $\psi_\mu^{(k)}$ satisfying the background gauge condition, $D_\mu(\bar{A})\psi_\mu^{(k)} = 0$ One has then

$$\left(\frac{Z}{Z_{\mathrm{PT}}}\right)_{\mathrm{reg}} \geqslant \left(\frac{Z_1[\bar{A}]}{Z_{\mathrm{PT}}}\right)_{\mathrm{reg}}$$

$$= \int DB_\mu \prod \mathrm{d}\gamma_l \prod \delta\left(\int \psi_\mu^{(k)} B_\mu \, \mathrm{d}^4 x\right)$$

$$\times \exp\left(-\frac{1}{g^2}\left\{ S[\bar{A}] + \tfrac{1}{2}\int \mathrm{d}^4 x\, B_\mu W_{\mu\nu} B_\nu \right\}\right) \det{}_{(kl)} \int \mathrm{d}^4 x\, \psi_\mu^{(k)} \frac{\partial \bar{A}_\mu}{\partial \gamma_l}$$

$$\times \det D^2(\bar{A}) \left(\int DB_\mu \exp\left[-\frac{1}{2g^2}\int B_\mu(-\partial^2) B_\mu \, \mathrm{d}^4 x \right] \det \partial^2 \right)^{-1} + O(g^2)$$

(2 18)

The 1-loop term can be rewritten as

$$\int \prod \mathrm{d}\gamma_t \det{}_{(kl)} \int \mathrm{d}^4x\, \psi_\mu^{(k)} \frac{\partial \bar{A}_\mu}{\partial \gamma_l} \left(\frac{\det' W_{\mu\nu}(\bar{A})}{\det - \partial^2 \delta_{\mu\nu}} \right)_{\mathrm{reg}}^{-1/2} \left(\frac{\det - D^2(\bar{A})}{\det - \partial^2} \right)_{\mathrm{reg}}$$

$$\times \exp\left(- \frac{1}{g^2(M)} S[\bar{A}] \right), \tag{2 19}$$

where $g^2(M)$ is the (bare) coupling constant determined at some cut-off M It implies that in calculating the regularized determinants one has to introduce an explicit cut-off M One of the most straightforward ways to do that is to divide the ratios of the determinants in (2.19) by the same ratios where the operators are shifted by a regulator mass M^2 This is the Pauli-Villars regularization scheme [2]

To calculate $\det' W_{\mu\nu}$, one is to recall that $W_{\mu\nu}(\bar{A})$ has zero or even negative modes connected with the collective coordinates γ_l of the trial function $\bar{A}(x, \gamma_l)$, but that the appropriate directions in the functional space are restricted by p δ-functions in (2 18) It is convenient to introduce, instead of $W_{\mu\nu}$, an integral operator [16a]

$$K_{\mu\nu}(x, y) = \delta^{(4)}(x - y) W_{\mu\nu} - \sum_{k=1}^{p} \psi_\mu^{(k)}(x) \otimes W_{\lambda\nu} \psi_\lambda^{(k)}(y)$$

$$- \sum_{k=1}^{p} W_{\mu\lambda} \psi_\lambda^{(k)}(x) \otimes \psi_\nu^{(k)}(y)$$

$$+ \sum_{k,l=1}^{p} \int \mathrm{d}^4 z\, \psi_\mu^{(k)} W_{\mu\nu} \psi_\nu^{(l)}(z) \psi_\mu^{(k)}(x) \otimes \psi_\nu^{(l)}(y), \tag{2 20}$$

with

$$K_{\mu\nu} B_\nu(x) \equiv \int K_{\mu\nu}(x, y) B_\nu(y)\, \mathrm{d}^4 y$$

The operator $K_{\mu\nu}$ has p zero modes $\psi_\mu^{(k)}(x)$ At the same time $B_\mu K_{\mu\nu} B_\nu = B_\mu W_{\mu\nu} B_\nu$ owing to the δ-functions in (2 18)

We can now define $\det' W_{\mu\nu}$ entering (2 19) as (for details see ref [16a])

$$\left(\frac{\det' W_{\mu\nu}}{\det - \partial^2 \delta_{\mu\nu}} \right)_{\mathrm{reg}}^{-1/2} = \left(\frac{M}{g(M)\sqrt{2\pi}} \right)^p$$

$$\times \exp\left[\tfrac{1}{2} \int_0^\infty \frac{\mathrm{d}t}{t} \left(\mathrm{Sp}\, \mathrm{e}^{K_{\mu\nu} t} - p - \mathrm{Sp}\, \mathrm{e}^{\partial^2 \delta_{\mu\nu} t} \right) \left(1 - \mathrm{e}^{-M^2 t} \right) \right] \tag{2 21}$$

The first factor here emerges from integrating over the p zero modes, the second is the contribution of the non-zero modes of $K_{\mu\nu}$.

The ghost operator $-D^2(A)$ is positive definite and its regularized determinant can be written as

$$\left(\frac{\det - D^2}{\det - \partial^2}\right)_{\text{reg}} = \exp\left[-\int_0^\infty \frac{dt}{t} \operatorname{Sp}\left(e^{D^2 t} - e^{\partial^2 t}\right)\left(1 - e^{-M^2 t}\right)\right] \tag{2.22}$$

In papers [16a, b] a very simple method is proposed of evaluating the functional determinants (2 21), (2 22) The method is based on a marriage of a quasiclassical expansion for the heat kernels taking into account zero or approximately zero eigenvalues of the operators involved In the case of an instanton background field we have checked the accuracy of the method to be of the order of a few percent We hope, therefore, that it would also be of some use in the case when the instanton is replaced by a more general configuration (1 7), and also to estimate the quantum corrections to the instantons' interactions due to the non-factorization of individual determinants

However, as emphasized in the introduction, we expect the classical interactions to be much more significant than the quantum ones (some estimates that we have made confirm these expectations) Therefore, in the case of a medium with widely separated instantons we will neglect the non-factorization and use the known results [2, 12] for the instanton's determinants

Another problem is the renormalization of $\beta(M)u_{\text{int}}$ that is present in the $S[\bar{A}]/g^2(M)$ term in the exponent of eq. (2 18) (for the definition of u_{int} see eq (1 11)) Using the method of ref [16] one can show that the determinants (2 21) and (2 22) renormalize this quantity, as well as the free actions, so that it is replaced by $\beta(\bar{\rho})u_{\text{int}}$, where $\beta(\bar{\rho}) = b \ln(1/a\Lambda\bar{\rho})$. The constant $a = O(1)$ can be, in principle, found, it depends on the regularization scheme, but $\beta(\bar{\rho})$ is scheme-independent To write down this fact explicitly, we choose for convenience

$$\beta(\bar{\rho}) = b \ln\left(C_{N_c}^{1/b}\Lambda\bar{\rho}\right)^{-1}, \tag{2 23}$$

where C_{N_c} is given below (eq (2.26)).

We thus rewrite (2.18) as

$$\left(\frac{Z}{Z_{\text{PT}}}\right)_{\text{reg}} \geqslant \frac{1}{N_+!N_-!}\int \prod_{i=1}^{N_+ + N_-} d^4x_i\, d\rho_i\, dO_i\, d(\rho_i)\, e^{-\beta(\bar{\rho})u_{\text{int}}(\gamma_i)} + O\left(\frac{1}{\beta(\bar{\rho})}\right), \tag{2 24}$$

where $d(\rho)$ is the 1-instanton density.

$$d(\rho) = \frac{1}{\rho^5}\beta(M)^{2N_c} e^{-\beta(\rho)}, \qquad \beta(\rho) = b \ln\left(C_{N_c}^{1/b}\Lambda\rho\right)^{-1}, \tag{2 25}$$

with

$$C_{N_c} \simeq \frac{4.66 \exp(-1.68 N_c)}{\pi^2 (N_c-1)!(N_c-2)!} \qquad \text{(Pauli-Villars scheme)} \qquad (2.26)$$

For other regularization schemes replace $C_{N_c} \to C'_{N_c} = C_{N_c}(\Lambda_{PV}/\Lambda')^b$ The renormalization of the pre-exponential factor $\beta(M)^{2N_c}$ arises only at the 2-loop level

The correction $O(1/\beta(\bar\rho))$ in (2.24) arises from several sources First, they come from the quantum corrections to u_{int}, and may be estimated along the lines of ref [16] Second, they arise when 2-loop corrections are being taken into account The most important of those are due to the 2-loop renormalization of the 1-instanton density $d(\rho)$ (see sect 5)

Strictly speaking, the $O(1/\beta(\bar\rho))$ corrections undermine somewhat the use of the inequality (2.24) Numerically, however, it turns out that $\beta(\bar\rho) \approx 15$ (for SU_3), so that (2.24) appears to give a good estimate for the vacuum energy density arising from $I\bar I$ configurations

3. Instanton interaction

In this section we study the interaction of one instanton and one (anti)instanton that arises at the classical level

Let us consider the most general configuration of I and $\bar I$ in a singular gauge

$$A_\mu^a = A_\mu^{(1)a} + A_\mu^{(2)a}, \qquad A_\mu^{(1)a} = 2\bar\eta_{a\mu\kappa}\frac{y_\kappa \rho_1^2}{y^2(y^2+\rho_1^2)},$$

$$A_\mu^{(2)a} = 2O^{aa'}\eta_{a'\mu\lambda}\frac{z_\lambda \rho_2^2}{z^2(z^2+\rho_2^2)}, \qquad y = x - x_1, \qquad z = x - x_2 \qquad (3.1)$$

Here $x_{1,2}$ are the positions and $\rho_{1,2}$ are the dimensions of instanton (1) and anti-instanton (2) (for two instantons replace $\eta_{a\mu\lambda} \to \bar\eta_{a'\mu\lambda}$) One can always choose the orientation of one of the instantons to be standard, the orientation of the second is given by a general orthogonal matrix O^{ab} (We work now with the SU_2 group, the generalization to $SU(N_c)$ is straightforward and will be performed later on)

The field strength can be written as a sum of three terms where the first two are just the strengths of isolated instantons and the third is due to the non-linear nature of the theory

$$F_{\mu\nu}(A^1 + A^2) = F_{\mu\nu}(1) + F_{\mu\nu}(2) + F_{\mu\nu}(1,2)$$

with

$$F^a_{\mu\nu}(1) = -\frac{8\rho_1^2}{\left(y^2+\rho_1^2\right)^2}\left(\tfrac{1}{2}\bar\eta_{a\mu\nu} + \bar\eta_{a\nu\rho}\frac{y_\mu y_\rho}{y^2} - \bar\eta_{a\mu\rho}\frac{y_\nu y_\rho}{y^2}\right),$$

$$F^a_{\mu\nu}(2) = -\frac{8\rho_2^2}{\left(z^2+\rho_2^2\right)^2}O^{aa'}\left(\tfrac{1}{2}\eta_{a'\mu\nu} + \eta_{a'\nu\rho}\frac{z_\mu z_\rho}{z^2} - \eta_{a'\mu\rho}\frac{z_\nu z_\rho}{z^2}\right),$$

$$F^a_{\mu\nu}(1,2) = \frac{4\varepsilon^{abc}y_\kappa z_\lambda \rho_1^2\rho_2^2}{y^2z^2\left(y^2+\rho_1^2\right)\left(z^2+\rho_2^2\right)}\left(\bar\eta_{b\mu\kappa}O^{cc'}\eta_{c'\nu\lambda} + \bar\eta_{c\nu\kappa}O^{bb'}\eta_{b'\mu\lambda}\right) \qquad (3\,2)$$

By definition (see eq. (1 11)) we have

$$u_{\rm int}(\rho_1,\rho_2,x_1-x_2=R,O_{12}) = \frac{1}{32\pi^2}\int \mathrm{d}^4x\left(2F_1^aF_2^a + 2F_1^aF_{12}^a + 2F_2^aF_{12}^a + F_{12}^{a2}\right) \qquad (3\,3)$$

Consider first the most complicated last term in (3.3). Explicitly,

$$\frac{1}{32\pi^2}\int F_{12}^2\mathrm{d}^4x = \frac{\rho_1^4\rho_2^4}{\pi^2}\varepsilon^{abc}\varepsilon^{ade}\left(\bar\eta_{b\mu\kappa}O^{cc'}\eta_{c'\nu\lambda}\right)\left(\bar\eta_{d\mu\rho}O^{ee'}\eta_{e'\nu\sigma} + \bar\eta_{e\nu\rho}O^{dd'}\eta_{d'\mu\sigma}\right)$$

$$\times\int \mathrm{d}^4x\,\frac{y_\kappa y_\rho z_\lambda z_\sigma}{y^4\left(y^2+\rho_1^2\right)^2z^4\left(z^2+\rho_2^2\right)^2}\,. \qquad (3\,4)$$

Let us investigate eq. (3.4) in two limiting cases. (i) $\rho_{1,2} \gg R$ and (ii) $\rho_{1,2} \ll R$. In the first case one obtains for the integral

$$\tfrac{1}{24}\pi^2\left(\delta_{\kappa\lambda}\delta_{\rho\sigma} + \delta_{\kappa\rho}\delta_{\lambda\sigma} + \delta_{\kappa\sigma}\delta_{\lambda\rho}\right)\ln\frac{\rho_{1,2}^2}{R^2}\,.$$

Contracting this with the 't Hooft's η symbols, one gets

$$u_{\rm int} \underset{R\to 0}{=} \begin{cases} 4\ln\dfrac{\rho^2}{R^2} & (\mathrm{I}\bar{\mathrm{I}}) \\ \left(3+(\mathrm{Tr}\,O)^2\right)\ln\dfrac{\rho^2}{R^2} & (\mathrm{II}\ \text{ or }\ \bar{\mathrm{I}}\bar{\mathrm{I}}) \end{cases} \qquad (3.5)$$

We see that when the distances between I and $\bar{\mathrm{I}}$ are small comparable to their sizes, there is a strong repulsive potential between them, independent of their relative

orientation. In the case of two instantons the repulsion depends on the orientation, being averaged, it gives the same quantity as in the case of $I\bar{I}$

In the opposite case (ii) $\rho_{1,2} \ll R$ the integral in eq (3.4) is equal to

$$\tfrac{1}{4}\pi^2\left(\frac{1}{\rho_1^2}\frac{R_\lambda R_\sigma}{R^8}\delta_{\kappa\rho}+\frac{1}{\rho_2^2}\frac{R_\kappa R_\rho}{R^8}\delta_{\lambda\sigma}\right)$$

Contracting this with the η symbols, we obtain

$$u_{\mathrm{int}}\underset{R\to\infty}{=}\begin{cases}\dfrac{\rho_1^2\rho_2^2(\rho_1^2+\rho_2^2)}{2R^6}(9-\bar{\eta}O\eta RR/R^2) & (I\bar{I})\\ \dfrac{\rho_1^2\rho_2^2(\rho_1^2+\rho_2^2)}{2R^6}(9-\mathrm{Tr}\,O) & (II),\end{cases} \tag{3.6}$$

where we used an abbreviation $\bar{\eta}O\eta RR/R^2 \equiv \bar{\eta}_{a\mu\kappa}O^{ab}\eta_{b\mu\lambda}R_\kappa R_\lambda/R^2$ Again there is a repulsion independent of the orientation, and a relatively small orientation-dependent piece

The $\bar{\eta}O\eta RR/R^2$ structure is identical to that of the famous "dipole–dipole" interaction [3] but has a R^{-6} behaviour To find out where the R^{-4} dipole–dipole interaction is, let us consider the first three terms in eq (3.3) Since the r.h.s. of eq (3.3) has the form of a contraction integral, it is convenient to write it down with the help of Fourier-transformed quantities. For instance,

$$\frac{1}{32\pi^2}\int 2F_1F_2\,\mathrm{d}^4x = 64\pi^2\rho_1^2\rho_2^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-i(k,R)}F(k\rho_1)F(k\rho_2)$$

$$\times\left(\tfrac{1}{2}\bar{\eta}_{a\mu\nu}+\bar{\eta}_{a\nu\rho}\frac{k_\mu k_\rho}{k^2}-\bar{\eta}_{a\mu\rho}\frac{k_\nu k_\rho}{k^2}\right)O^{aa'}\left(\tfrac{1}{2}\eta_{a'\mu\nu}+2\eta_{a'\nu\sigma}\frac{k_\mu k_\sigma}{k^2}\right), \tag{3.7}$$

where

$$F(z)=\frac{8}{z^4}-\frac{4}{z^2}K_2(z)-\frac{2}{z}K_1(z)-\tfrac{1}{2}K_0(z)=\tfrac{1}{8}+\tfrac{1}{24}z^2\ln\tfrac{1}{2}z+$$

Contracting the two parentheses containing the η symbols, one obtains that for $I\bar{I}$ this term is identically zero, and for II one obtains

$$\frac{1}{32\pi^2}\int 2F_1F_2\,\mathrm{d}^4x=\tfrac{4}{3}\mathrm{Tr}\,O\frac{\rho_1^2\rho_2^2(\rho_1^2+\rho_2^2)}{R^6}+\mathrm{O}(R^{-8}) \tag{3.8}$$

Similarly, the general expression for the F_1F_{12} term is

$$\frac{1}{32\pi^2}\int 2F_1F_{12}\,\mathrm{d}^4x = -\frac{4}{\pi^2}\rho_1^4\rho_2^2\bar{\eta}_{a\mu\kappa}O^{aa'}\eta_{a'\mu\lambda}\int \mathrm{d}^4x\,\frac{y_\kappa z_\lambda}{y^2\left(y^2+\rho_1^2\right)^3 z^2\left(z^2+\rho_2^2\right)}$$

$$= -\frac{4}{\pi^2}\rho_1^4\rho_2^2\left(\bar{\eta}O\eta\right)_{\kappa\lambda}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-\imath(k,R)}$$

$$\times\frac{\partial}{\partial k_\kappa}\Phi_1(k,\rho_1)\frac{\partial}{\partial k_\lambda}\Phi_2(k,\rho_2), \tag{3 9}$$

where

$$\Phi_1(k,\rho_1)=\frac{4\pi^2}{k^2\rho_1^6}\left[1-\tfrac{1}{8}k^3\rho_1^3K_3(k\rho_1)\right],\qquad \Phi_2(k,\rho_2)=\frac{4\pi^2}{k^2\rho_2^2}\left[1-k\rho_2K_1(k\rho_2)\right]$$

At $R\to\infty$ one needs only the few first terms in the expansion of the functions $\Phi_{1,2}$ giving

$$\frac{1}{32\pi^2}\int 2F_1F_{12}\,\mathrm{d}^4x=\begin{cases}\left(\bar{\eta}O\eta RR/R^2\right)\left(\dfrac{2\rho_1^2\rho_2^2}{R^4}-\dfrac{4\rho_1^4\rho_2^2+3\rho_2^4\rho_1^2}{R^6}+O\left(R^{-8}\right)\right) & (\mathrm{I\bar{I}})\\ -\left(\frac{4}{3}\rho_1^4\rho_2^2+\rho_2^4\rho_1^2\right)\dfrac{\mathrm{Tr}\,O}{R^6}+O\left(R^{-8}\right) & (\mathrm{II})\end{cases} \tag{3 10}$$

The term F_2F_{12} is immediately obtained from this one by the substitution $\rho_1\leftrightarrow\rho_2$ Combining expressions (3.6), (3.8), (3.10) we get finally the expansion of u_{int} at large distances.

$$u_{\mathrm{int}}^{\mathrm{I\bar{I}}}=\left(\bar{\eta}O\eta RR/R^2\right)\left[\frac{4\rho_1^2\rho_2^2}{R^4}-\frac{15\rho_1^2\rho_2^2\left(\rho_1^2+\rho_2^2\right)}{2R^6}\right]+\frac{9\rho_1^2\rho_2^2\left(\rho_1^2+\rho_2^2\right)}{2R^6}+O\left(R^{-8}\right), \tag{3 11}$$

$$u_{\mathrm{int}}^{\mathrm{II}}=(9+\mathrm{Tr}\,O)\frac{\rho_1^2\rho_2^2\left(\rho_1^2+\rho_2^2\right)}{2R^6}+O\left(R^{-8}\right) \tag{3 12}$$

It is easy to check that, when passing from SU_2 to $SU(N_c)$, the only change is that one has to replace the number 9 in eqs. (3 11) (3.12) by $\frac{27}{2}(N_c/(N_c^2-1))$.

Let us discuss these equations We see that both in the $\mathrm{I\bar{I}}$ and II cases there is a universal repulsion which starts as ρ^6/R^6 and at small distances transforms into a

logarithmic core (see eq (3 5)). However, in the $I\bar{I}$ case an R^{-4} dipole–dipole interaction prevails at large distances (see the first term in eq (3 11)) It can be of either sign since

$$-3 \leqslant \bar{\eta} O \eta R R / R^2 \leqslant 1 \tag{3 13}$$

Averaging over the relative orientation O^{ab} of I and $\bar{I}$ (or over their relative position in space R_λ) we get zero

In general, one can easily get an expression for $u_{\rm int}$ being averaged over the relative orientation of $I\bar{I}$ It is clear that only the F_{12}^2 piece is non-zero after averaging. Using the relation

$$\overline{O^{aa'} O^{bb'}} = \frac{1}{N_c^2 - 1} \delta^{ab} \delta^{a\,b'}, \tag{3 14}$$

one gets

$$\overline{u_{\rm int}^{I\bar{I},I\bar{I}}} = \frac{N_c}{N_c^2 - 1} \frac{1}{\pi^2} \int d^4x \, \frac{\left[7y^2 z^2 - (yz)^2\right] \rho_1^4 \rho_2^4}{y^4 \left(y^2 + \rho_1^2\right)^2 z^4 \left(z^2 + \rho_2^2\right)^2} \tag{3 15}$$

This expression is positive definite demonstrating that being averaged over the relative orientation, $u_{\rm int}$ is repulsive at any distance In the limiting cases $R \ll \rho_{1,2}$ and $R \gg \rho_{1,2}$ one gets the averaged eqs (3.5) and (3 6), respectively, with the appropriate generalization to the SU(N_c) group

A quantity of interest is also $u_{\rm int}$ integrated over all distances R between instantons The integration over d^4R is trivial owing to the contraction form of $u_{\rm int}$ Being integrated, the $F_1 F_2$, $F_1 F_{12}$ and $F_2 F_{12}$ pieces are zero, and one gets

$$\int d^4R \, u_{\rm int}^{I\bar{I},I\bar{I}} = \gamma^2 \rho_1^2 \rho_2^2, \qquad \gamma^2 = \frac{27}{4} \frac{N_c}{N_c^2 - 1} \pi^2 \tag{3 16}$$

Since this quantity does not depend on the orientation, it can be also obtained by integrating eq (3 15) over d^4R

Let us return to the expansion of $u_{\rm int}$ at large distances (3 11), and let us orientate I and $\bar{I}$ in such a way that at very large distances we would have a maximal attraction $\bar{\eta} O \eta R R / R^2 = -3$ Putting for simplicity $\rho_1 = \rho_2 = \rho$ we can rewrite (3.11) as

$$u_{\rm int}^{I\bar{I}} = -12 \frac{\rho^4}{R^4} + \left(45 + \frac{27 N_c}{2\left(N_c^2 - 1\right)}\right) \frac{\rho^6}{R^6} + O\left(\frac{\rho^8}{R^8}\right) \tag{3 17}$$

This function has a minimum at

$$\left\langle \frac{R^2}{\rho^2} \right\rangle = \frac{9}{16} \frac{10(N_c^2-1)+3N_c}{N_c^2-1} = \begin{cases} 6\,75, & N_c = 2 \\ 6.26, & N_c = 3 \\ 5\,63, & N_c = \infty \end{cases} \tag{3 18}$$

The minimal value of u_{int} is rather small

$$u_{\min}^{\text{II}} = \begin{cases} -0\,088, & N_c = 2 \\ -0\,102, & N_c = 3 \\ -0\,126, & N_c = \infty \end{cases} \tag{3 19}$$

Note that the minimum is achieved at large values of R^2/ρ^2, meaning that in finding its position we had the right to restrict ourselves only by the R^{-4} and R^{-6} terms in (3 17) Indeed, we have checked that the R^{-8} correction from the F_1F_{12} and F_2F_{12} pieces is

$$(\bar{\eta}O\eta RR/R^2)\frac{1}{R^8}\left[32\left(1-\ln\frac{R^2}{\rho_1\rho_2}\right)\rho_1^4\rho_2^4 + 10(\rho_1^2\rho_2^6+\rho_2^2\rho_1^6)\right]$$

This term shifts the position of the minimum to still larger R^2/ρ^2, however the shift itself is not larger than about 10%

Let us summarize. We see that even if we take the relative orientation that leads to a maximal possible attraction at large distances due to the R^{-4} dipole–dipole interaction, the interaction force changes for the repulsion already at $R^2 \sim 6\rho^2$ (see eq. (3 18)). We consider the coefficient 6 to be a large number Evidently, for the instanton medium this number gives an estimate for the ratio of the mean square distance between instantons to their mean square size In the disordered phase, however, it is natural to average out the orientation-dependent forces and to obtain a repulsion at all distances given by eq. (3.15) At large R^2/ρ^2 which, as we see, is the only domain of interest, eq. (3.15) reads

$$\overline{u_{\text{int}}^{\text{II, II}}} \underset{R\to\infty}{=} \frac{\gamma^2}{\pi^2}\frac{\rho_1^2\rho_2^2(\rho_1^2+\rho_2^2)}{R^6}, \qquad \gamma^2 = \frac{27}{4}\frac{N_c}{N_c^2-1}\pi^2 \tag{3 20}$$

Thus, we have justified the statement made in the introduction that the trial function (1 6) leads to an effective repulsion of instantons

It is very important that the repulsion starts at $R^2 \gg \rho^2$ Qualitatively, it leads to a stabilization of the system at a stage when the overlap of I's is yet very feeble. Technically, it leads to a significant simplification in the statistical mechanics of the system In particular, one can safely neglect the many-body forces, since the packing fraction parameter appears to be very small.

4. Why not a crystal?

We have shown in the previous section that the $I\bar{I}$ interaction is of a molecular type it has a repulsive core at small distances and a dipole–dipole R^{-4} tail at large distances which can be made attractive by choosing the relative orientation of I and $\bar{I}$ (Note that II or $\bar{I}\bar{I}$ are always repulsive – see eq (3 12)) Therefore, a question arises why do not instantons form a NaCl-type crystal with pseudoparticles sitting in the minima of the potential, given by eqs (3 18), (3 19)? If such an ordered phase had a smaller free energy than a disordered phase (or, speaking more accurately, a smaller thermodynamical potential, since the total number of I and $\bar{I}$ is not fixed but should be found self-consistently from the extremum condition), spontaneous violation both of Lorentz and colour symmetry would occur*

In principle, one has to build all possible types of crystals and investigate them, one by one Here we just confine ourselves to a simplest cubic crystal with alternating I's and $\bar{I}$'s at the sites of the lattice

It occurs that for the SU_2 group one can construct a $d = 4$ cubic lattice in which a maximal gain is achieved at all links The colour orientations of all I's and $\bar{I}$'s are fixed up to a global rotation The construction is reported in ref [17]

In the case of the $SU(N_c)$ group ($N_c > 2$) the symmetry breaking pattern is not fixed completely by the demand for a maximal gain in energy at all the links We did not investigate this interesting problem in detail, it seems, however, that an $SU(N_c - 2) \times U(1)$ subgroup of the $SU(N_c)$ group remains unbroken In what follows we stick to the SU_2 case

Having at hand the organization of the lattice presented in ref [17], one can easily calculate the free energy of the system Let the number of I's be fixed and equal to $\frac{1}{2}N$ (the number of $\bar{I}$'s is the same up to the crystal's defects) Then the length of the lattice link is equal to $(V/N)^{1/4} = R_0$ The equilibrium size of instantons is then found from the minimum of the potential (3 17) and is $\rho_0^2 = \frac{4}{27} R_0^2$ (we neglect the interactions with the non-nearest neighbours, the correction due to them can be seen to be of the order of 10%) Now, let us shift the position of a particular I from its equilibrium state by r_μ, its size by $\delta\rho^2$, and its orientation by ω Let us define the effective potential of the chosen I as a sum of the u_{int} with the nearest 8 neighbours, divided by 2 It is not difficult to find the quadratic terms in the shifts for this effective potential

$$u_{\text{eff}} \simeq 4u_{\min} + 0\ 62\omega^2 + 0\ 35\frac{(\delta\rho^2)^2}{\rho_0^4} + 0\ 26\frac{r_\mu^2}{\rho_0^2}, \tag{4 1}$$

where $u_{\min}$ is the value of the two-body potential in the minimum ($u_{\min} = -0\ 088$

* In principle, it is possible that an intermediate situation takes place the medium is polarized but not ordered in the configuration space Then only colour symmetry would be violated

for SU_2) In order to find the partition function of the canonical ensemble, one has to integrate over $8(N_+ + N_-)$ degrees of freedom Since it is a classical system it is sufficient to integrate over the individual shifts of instantons from their equilibrium state, as if they were independent, with the Gibbs factor given by eq (4 1) Therefore, we have for the partition function of a crystal

$$Y = \left\{ C_2 \beta(\rho_0)^4 \int \mathrm{d}^4 r \mathrm{e}^{-\beta(\rho_0)0\,26(r^2/\rho_0^2)} \frac{1}{\pi^2} \int \frac{\mathrm{d}^3\omega}{\sqrt{1-\omega^2}} \mathrm{e}^{-\beta(\rho_0)0\,62\omega^2} \right.$$

$$\left. \times \int \frac{\mathrm{d}\rho}{\rho^5} \mathrm{e}^{-\beta(\rho(1+4u_{\min})} \mathrm{e}^{-\beta(\rho_0)0\,35(\rho^2-\rho_0^2)^2/\rho_0^4} \right\}^N \tag{4 2}$$

Assuming $\beta(\rho_0)$ to be a large number, we can calculate the Gauss-type integrals which yield

$$Y = \left[C_2 \pi^2 \left(\frac{1}{0\,26} \right)^2 \left(\frac{1}{0.62} \right)^{3/2} \frac{1}{2} \left(\frac{1}{0\,35} \right)^{1/2} (\rho_0 \Lambda)^{b(1+4u_{\min})} \right]^N \tag{4 3}$$

Note the cancellation of the factor β^4 coming from zero modes Denoting $\lambda \equiv 1 + 4u_{\min} = 0\,65$ and recalling that $\rho_0^4 \Lambda^4 = (\frac{4}{27})^2 V \Lambda^4 / N$ and that $C_2 \simeq 0\,16/\pi^2$, $b = \frac{22}{3}$, we obtain

$$Y = \left[0.044 \left(\frac{V\Lambda^4}{N} \right)^{b\lambda/4} \right]^N = \exp\left(-\tfrac{1}{4} b\lambda N \ln \frac{N}{V\Lambda^4} + N \ln 0\,044 \right) \tag{4 4}$$

Before passing from the canonical to the grand canonical ensemble by maximizing Z in N, let us note that eq (4 4) leads to the following relation for the dispersion of the number of particles.

$$\langle N^2 \rangle - \langle N \rangle^2 = \frac{4}{b\lambda} \langle N \rangle \tag{4 5}$$

We note further that in the crystal phase the average action is

$$\left\langle \frac{1}{32\pi^2} \int \mathrm{d}^4 x F_{\mu\nu}^2(x) \right\rangle = \langle N \rangle (1 + 4u_{\min}) = \langle N \rangle \lambda . \tag{4 6}$$

Combining (4.5) and (4 6) we find

$$\left\langle \int \mathrm{d}^4 x F_{\mu\nu}^2(x)/32\pi^2, \int \mathrm{d}^4 y F_{\mu\nu}^2(y)/32\pi^2 \right\rangle_{\mathrm{con}} = \frac{4}{b} \left\langle \int \mathrm{d}^4 x F_{\mu\nu}^2/32\pi^2 \right\rangle \tag{4 7}$$

This, however, is the well-known low-energy theorem [16] which follows from the renormalizability of the theory!

Differentiating (4 4) with respect to N, we find

$$\langle N \rangle = V\Lambda^4 0\ 027,$$

hence the gluon condensate

$$\langle F_{\mu\nu}^2/32\pi^2 \rangle = \lambda \times 0\ 027 \times \Lambda^4 = (0\ 36\Lambda)^4, \qquad \Lambda = \Lambda_{\rm PV} \tag{4 8}$$

The grand partition function is

$$Y = \exp\left(\tfrac{1}{4} b\lambda \langle N \rangle\right) = \exp\left(\tfrac{1}{4} bV \langle F_{\mu\nu}^2/32\pi^2 \rangle\right) \tag{4 9}$$

This form of the partition function is of a more general nature However, we note that the magnitude of $\langle F_{\mu\nu}^2 \rangle$, as given by eq (4 8), is very small (we shall obtain a much larger quantity assuming a disordered phase – see the next section) The characteristic inverse temperature β also appears to be rather small – of the order of unity It means that the Gauss-type integrals in (4 2) are not too sharp Therefore, the picture we started with, viz, that I's are strongly localized in space and in orientation, is not adequate. On the contrary, the numbers obtained seem to indicate that all degrees of freedom are significantly smeared out, i.e the system prefers a disordered phase

5. A theory of instanton liquid

In this section we calculate the partition function Y of the instanton liquid given by the r h s. of (2 24)

$$Y = \frac{1}{N_+!N_-!} \int \prod_{i=1}^{N_+ + N_-} \mathrm{d}^4 x_i \, \mathrm{d}\rho_i \, \mathrm{d}O_i \, \mathrm{d}(\rho_i) \exp\left[-\beta(\bar{\rho}) \sum_{i<j} u_{\rm int}(x_i - x_j, \rho_i, \rho_j, O_{ij}) \right] \tag{5 1}$$

Here we restrict ourselves to the two-body $u_{\rm int}$ studied in sect 3 $N_\pm$ are the numbers of I's and $\bar{\rm I}$'s. At the first stage we will fix $N_\pm$, find Y and finally maximize Y in $N_\pm$

To calculate (5.1) is a rather difficult statistical mechanical problem in itself In order to make sure that we solve it accurately enough, we elaborated several different approximation methods a modification of Kirkwood superposition approximation, a modification of the virial expansion, and others. We obtained remarkably close values for the free energy in all the cases. Therefore, in this paper

we present the most simple (but trust-worthy) approximation based again on a variational principle To be more precise, we replace the integrand in (5 1) by a product of 1-particle densities $\mu_{\pm}(\rho)$ and then find the best $\mu_{\pm}(\rho)$ from the variational principle.

Let us rewrite Y as (γ_i stand for the set of all collective coordinates of an $I(\bar{I})$).

$$Y = \frac{1}{N_+!N_-!}\int \prod \mathrm{d}\gamma_i \, \mathrm{e}^{-E(\gamma)}, \qquad E(\gamma) = \beta U_{\mathrm{int}}(\gamma) - \sum \ln d(\rho_i), \qquad \beta \equiv \beta(\bar{\rho}) \tag{5 2}$$

Define

$$E_1(\gamma) = -\sum_{i=1}^{N_+} \ln \mu_+(\rho_i) - \sum_{j=1}^{N_-} \ln \mu_-(\rho_j), \tag{5 3}$$

$$Y_1 = \frac{1}{N_+!N_-!}\int \prod \mathrm{d}\gamma_i \, \mathrm{e}^{-E_1(\gamma)} = \frac{1}{N_+!N_-!} V^{N_+ + N_-} (\mu_{+0})^{N_+} (\mu_{-0})^{N_-},$$

$$\mu_{\pm 0} = \int_0^\infty \mathrm{d}\rho \, \mu_{\pm}(\rho). \tag{5 4}$$

The variational principle says

$$Y \geqslant Y_1 \exp(-\langle E - E_1 \rangle) \tag{5 5}$$

In the case under consideration one has

$$\langle E - E_1 \rangle = \frac{1}{Y_1}\frac{1}{N_+!N_-!}\int \prod \mathrm{d}\gamma_i \left[\beta U_{\mathrm{int}} - \sum \ln d(\rho_i) + \sum \ln \mu(\rho_i)\right] \exp \sum \ln \mu(\rho_i)$$

$$= \sum_{\varepsilon = \pm} \frac{N_\varepsilon}{\mu_{\varepsilon 0}} \int \mathrm{d}\rho \, \mu_\varepsilon(\rho) \ln \frac{\mu_\varepsilon(\rho)}{d(\rho)} + \frac{\beta}{2V^2} \sum_{\varepsilon, \varepsilon' = \pm} \frac{N_\varepsilon N_{\varepsilon'}}{\mu_{\varepsilon 0}\mu_{\varepsilon' 0}}$$

$$\times \int \mathrm{d}\gamma_1 \mathrm{d}\gamma_2 \, u_{\varepsilon\varepsilon'}(\gamma_1, \gamma_2) \mu_\varepsilon(\rho_1) \mu_{\varepsilon'}(\rho_2) \tag{5 6}$$

The last integrals are easily calculated since, according to (3.16),

$$\int \mathrm{d}O_1 \mathrm{d}O_2 \mathrm{d}^4x_1 \mathrm{d}^4x_2 \, u_{\mathrm{int}}^{\mathrm{II,II}}(x_1 - x_2, \rho_1, \rho_2, O_{12}) = V\gamma^2 \rho_1^2 \rho_2^2,$$

$$\gamma^2 = \frac{27 N_c \pi^2}{4(N_c^2 - 1)} \tag{5 7}$$

Hence, the last term in (5 6) takes the form

$$\frac{\beta\gamma^2}{2V}\sum_{\varepsilon,\varepsilon'=\pm} N_\varepsilon N_{\varepsilon'}\overline{\rho_\varepsilon^2}\,\overline{\rho_{\varepsilon'}^2}\,, \qquad \overline{\rho_\pm^2}=\int \mathrm{d}\rho\,\rho^2\mu_\pm(\rho)/\mu_{\pm 0} \tag{5 8}$$

Putting eqs (5 6) and (5 4) into (5 5) and varying its r h s with respect to $\mu_\pm(\rho)$, one finds that the best $\mu_\pm(\rho)$ is

$$\mu_\pm(\rho)=C_\pm d(\rho)\exp\left[-\frac{\beta\gamma^2}{V}\left(N_+\overline{\rho_+^2}+N_-\overline{\rho_-^2}\right)\rho^2\right] \tag{5 9}$$

We see that owing to the interaction, the free instanton density $d(\rho)$ is modified by a gaussian cut-off, the coefficient γ^2 being connected with the averaged u_{int} (see eq (5 7)). Note that the ρ cut-off is also controlled by the density of I's, $N_\pm/V$

The r h s of (5 5) does not depend on the coefficients $C_\pm$ of eq (5 8), and one can put $C_\pm=1$ Therefore, $\mu_+(\rho)=\mu_-(\rho)$ and hence $\overline{\rho_+^2}=\overline{\rho_-^2}$ Introducing the total number of pseudoparticles $N=N_++N_-$ and recalling the definition of $\overline{\rho^2}$ (5 8), we deduce a relation between the average size and the density N/V.

$$\left(\overline{\rho^2}\right)^2=\frac{\nu}{\beta\gamma^2 N/V}\,, \qquad \nu=\tfrac{1}{2}(b-4)\,, \qquad b=\tfrac{11}{3}N_c \tag{5 10}$$

We can, then, rewrite the 1-particle density (5 9) as

$$\mu(\rho)=C_{N_c}\tilde{\beta}^{2N_c}\Lambda^b\rho^{b-5}\exp\left(-\nu\rho^2/\overline{\rho^2}\right), \tag{5 11}$$

where the 1-loop expression (2.24) for d(p) has been used, $\tilde{\beta}=\beta(M)$ It is straightforward now to calculate the r h s. of (5 5). We obtain

$$Y\geqslant\exp\left\{-\left(\tfrac{1}{2}\nu+1\right)N\left(\ln\frac{N}{V\Lambda^4}-1\right)+N\ln\left[\Gamma(\nu)C_{N_c}\tilde{\beta}^{2N}\left(\beta\gamma^2\nu\right)^{-\nu/2}\right]\right\} \tag{5 12}$$

Now we must maximize this expression with respect to N We find

$$\langle N\rangle=V\Lambda^4\left[\Gamma(\nu)C_{N_c}\tilde{\beta}^{2N_c}\left(\beta(\bar{\rho})\gamma^2\nu\right)^{-\nu/2}\right]^{2/\nu+2} \tag{5 13}$$

Substituting this in (5 12) we obtain the maximal value of the partition function

$$Y=\exp\left[\left(\tfrac{1}{2}\nu+1\right)\langle N\rangle\right] \tag{5 14}$$

It is interesting to compare our results with those of ref [14] where a dimensionless hard core in the form $\theta(R_{12}^4-a'\rho_1^2\rho_2^2)$ was introduced for the II and $\bar{\mathrm{I}}\bar{\mathrm{I}}$

interactions Comparing eqs (5 10)–(5.14) with eqs. (3.14), (3 15) of ref [14] we see that these equations coincide when $a' = 8\beta\gamma^2/b\pi^2$ However, we disagree with the authors of ref. [14] on the role of dipole–dipole corrections According to our estimates based on the virial expansion, they increase the value of $\langle N \rangle$ not more than by 20%, and not by a factor of 2–3 as in ref [14] (We take the opportunity to thank M. Ilgenfritz and M. Mueller-Preussker for a very valuable discussion)

Let us now find the average action (i.e. the gluon condensate) By definition

$$\left\langle \int d^4x F_{\mu\nu}^2/32\pi^2 \right\rangle = \langle N \rangle + \langle U_{\text{int}} \rangle . \tag{5 15}$$

The second term here can be easily found from the definition (5.1) as

$$\langle U_{\text{int}} \rangle = -\frac{\partial}{\partial\beta} \ln Y = \frac{\nu}{2\beta} \langle N \rangle . \tag{5.16}$$

In the last equation we have a differential explicit (5.14) with respect to β (note that the factor $\tilde{\beta}^{2N_c}$ has another origin and should not be differentiated). We see that the deviation of the total action from that of free instantons is a $O(1/\beta)$ correction and can be neglected. Therefore, we can rewrite (5 14) as

$$Y = \exp\left(\tfrac{1}{4} bV \langle F_{\mu\nu}^2/32\pi^2 \rangle\right) . \tag{5.17}$$

The partition function of a liquid written in this form coincides with that of a crystal (see eq. (4 9)) This fact is not accidental but is associated with the trace anomaly [15] Indeed, the Yang-Mills partition function that we are estimating is related to the vacuum energy density ϑ_{00} by

$$\frac{Z}{Z_{\text{PT}}} = \exp\left[- V^{(4)} \left(\vartheta_{00} - \vartheta_{00}^{\text{PT}} \right) \right]$$

Both in the cubic crystal and in the disordered liquid one has $\vartheta_{00} = \frac{1}{4}\vartheta_{\mu\mu}$ This last quantity is equal, through a trace anomaly, to $(\beta(g)/8g^3)F_{\mu\nu}^2$, where $\beta(g)$ is the Gell-Mann–Low function. In the 1-loop order that reduces to $-\frac{1}{4}b(F_{\mu\nu}^2/32\pi^2)$ One then has

$$\frac{Z}{Z_{\text{PT}}} = \exp\left(\tfrac{1}{4} bV \langle F_{\mu\nu}^2/32\pi^2 \rangle_{\text{NP}}\right) , \tag{5 18}$$

where $\langle F_{\mu\nu}^2 \rangle_{\text{NP}}$ is the gluon field v e v which is due to the non-perturbative fluctuations (= gluon condensate [5, 15]) The fact that we obtained the partition function for the instanton media in the form (5 18) demonstrates that we have not violated the renormalization properties of the theory in the one-loop order Simultaneously, our result (5.13) can be interpreted as a lower bound for the gluon

condensate, or as an upper bound for the vacuum energy counted from the perturbative one

Another important manifestation of the renormalization properties of the theory is the low-energy theorem (4 7). We have already checked it for the crystal phase (see eq (4 5)). For the liquid phase one can also easily calculate, using (5.12), the dispersion of the number of particles and get

$$\langle N^2\rangle - \langle N\rangle^2 = \frac{4}{b}\langle N\rangle + O(1/\beta) \tag{5 19}$$

Substituting (5.15) we arrive at the low-energy theorem (4 7) Note that had we cut the ρ integrals "by hands", we would come to a dilute gas with a Poisson distribution in N leading to the dispersion law

$$\langle N^2\rangle - \langle N\rangle^2 = \langle N\rangle$$

This obviously violates the above low-energy theorem [7] We thus learn that the interaction of instantons is crucial to maintain the renormalization properties of the theory – however "dilute" the instanton medium may turn out to be

A very important quantity characterizing the Yang-Mills vacuum is the so-called topological susceptibility

$$\langle Q_t^2\rangle = \left\langle \left(\int d^4x F_{\mu\nu}\tilde{F}_{\mu\nu}/32\pi^2 \right)^2 \right\rangle = - \left. \frac{\partial^2 \ln Z(\vartheta)}{\partial \vartheta^2} \right|_{\vartheta=0}, \tag{5 20}$$

where ϑ is the instanton angle which one introduces by adding the ϑ term to the action

$$\beta(M)\int \frac{d^4x}{32\pi^2} F_{\mu\nu}^2 \to \beta(M)\int \frac{d^4x}{32\pi^2} F_{\mu\nu}^2 - i\vartheta \int \frac{d^4x}{32\pi^2} F_{\mu\nu}\tilde{F}_{\mu\nu}$$

The partition function Y is then multiplied by $\exp[i\vartheta(N_+ - N_-)]$, and one should vary (5 12) with respect to N_+, N_- separately One easily finds, instead of (5 14)

$$Y(\vartheta) = \exp\left[\left(\tfrac{1}{2}\nu + 1\right)\langle N\rangle \left(1 - \frac{\vartheta^2}{\nu+2} + O(\vartheta^4)\right)\right], \tag{5 21}$$

where $\langle N\rangle$ is given, as before, by eq. (5 13). Hence

$$\langle Q_t^2\rangle = - \left. \frac{\partial^2 \ln Y(\vartheta)}{\partial \vartheta A^2} \right|_{\vartheta=0} = \langle (N_+ - N_-)^2\rangle = \langle N\rangle \simeq V\langle F_{\mu\nu}^2/32\pi^2\rangle \tag{5 22}$$

This is a remarkable *dynamical* relation which one can compare with experiment. From the η' meson physics one learns [18]

$$\frac{1}{V}\langle Q_t^2\rangle = (180\text{–}190\ \text{MeV})^4 \tag{5 23}$$

The gluon condensate of the real world has been found from QCD sum rules to be $(\sim 195\ \text{MeV})^4$ [5, 15]. In a pure Yang-Mills world we are dealing with, this quantity may be larger by a factor of 2–3 [15] One then has

$$\langle F_{\mu\nu}^2/32\pi^2\rangle = (230\text{–}260\ \text{MeV})^4. \tag{5 24}$$

We find the closeness of these two fundamental quantities to be strong evidence in favour of the instanton vacuum (Varying the profile function of I's (sect. 6) we find that the gluon condensate should be a factor of 2 larger than $\langle Q_t^2\rangle$, which is more in accordance with (5.23) and (5.24).)

To find the absolute values of physical quantities one has to use eq. (5.13), which is actually a self-consistency equation, since $\beta(\bar\rho)$ entering (5 13) is itself, through eq. (5 10), a function of the density N/V. We remind the reader that $\beta(\bar\rho) = b\ln(C_{N_c}^{1/b}\Lambda\bar\rho)^{-1}$. Unfortunately, a numerical constant of the order of unity can enter the argument of the log here Also, the $\tilde\beta^{2N_c}$ factor is not, strictly speaking, renormalized at the 1-loop level. All this makes the possibility of obtaining reliable numerical values from the 1-loop considerations rather obscure*

However, we can improve our calculations considerably by using the 2-loop expression for the 1-instanton density [7] instead of eq. (2.25):

$$d(\rho) = \frac{1}{\rho^5} C_{N_c}\beta_I(\rho)^{2N_c}\exp\left\{-\beta_{II}(\rho) + \left(2N_c - \frac{b'}{2b}\right)\frac{b'}{2b}\ln\frac{\beta_I(\rho)}{\beta_I(\rho)} + O(1/\beta_I)\right\}, \tag{5.25}$$

where $\beta_{II}(\rho)$ is now given by a 2-loop expression

$$\beta_{II}(\rho) = \beta_I(\rho) + \frac{b'}{2b}\ln\frac{2}{b}\beta_I(\rho), \qquad \beta_I(\rho) = b\ln(\Lambda\rho)^{-1},$$

$$b = \tfrac{11}{3}N_c, \qquad b' = \tfrac{34}{3}N_c^2. \tag{5 26}$$

* We note that the authors of ref [14] chose (without compelling reasons) $\tilde\beta = b\ln(\Lambda\bar\rho)^{-1}$ with Λ of the lattice regularization scheme This procedure increases the value of $\langle N\rangle$ and, hence, of the condensate by a factor of ~ 400 as compared to the 2-loop calculation (see below)

The correction $O(1/\beta_I)$ in eq. (5 25) can be found if the exact 2-loop calculations in the instanton background field are performed, and the 3-loop Gell-Mann–Low function is known, it depends on the regularization scheme

We note that in the liquid phase the use of a 2-loop density (5 25) is parametrically the most important correction of the $O(1/\beta)$ type in the variational principle (2 24) Indeed, the quantum interaction of instantons is actually reduced to the change of the numerical constant in the argument of $\beta(\bar\rho)$ that stands before the classical u_{int} As seen from eq (5 13), this $\beta(\bar\rho)$ has the same status as $\tilde\beta^{2N_c}$ coming from the zero modes of 1 instanton However, the precise argument of β^{2N_c}, as seen from eq (5 25), is fixed only from 3-loop calculations It is clear, therefore, that the use of a 2-loop expression (5 26) for $\beta(\rho)$ in the exponent of (5 25) is parametrically more important when $\beta(\rho)$ is large Note that this would not be true for the crystal phase

The instanton density (and hence the gluon condensate) can be found by a slight modification of eq (5 13).

$$\langle F_{\mu\nu}^2/32\pi^2\rangle = \frac{\langle N\rangle}{V} = \frac{1}{(\overline{\rho^2})^2}\left\{\Gamma(\nu)C_{N_c}(\beta\gamma^2\nu)^{-\nu/2}\beta_I^{2N_c}\right.$$

$$\left.\times\exp\left[-\beta_{II}+\left(2N_c-\frac{b'}{2b}\right)\frac{b'}{2b}\frac{\ln\beta_I}{\beta_I}\right]\right\}^{2/(\nu+2)}, \quad (5\ 27)$$

where

$$\beta = \tfrac{1}{2}b\ln\left(C_{N_c}^{2/b}\Lambda^2\overline{\rho^2}\right)^{-1}, \qquad \beta_I = \tfrac{1}{2}b\ln\left(\Lambda^2\overline{\rho^2}\right)^{-1}, \qquad \beta_{II} = \beta_I + \frac{b'}{2b}\ln\left(\frac{2}{b}\beta_I\right)$$

(5 28)

The connection between $\overline{\rho^2}$ and $\langle N\rangle$ is given by eq (5 10)

We note that this set of eqs is not, generally speaking, invariant under the change of regularization schemes Any particular scheme is labelled by a certain value of Λ Changing Λ, we have also to change the quantity C_{N_c}, so that $C_{N_c}\Lambda^b = \text{const}$ Therefore, β is scheme-invariant, but $\beta_{I,II}$ are not The reason why eq (5 27) is not scheme-invariant can be traced to eq (5 25) the omitted terms $O(1/\beta_I)$ depend on the particular regularization However, in practical terms, we have checked that changing a scheme corresponding to $\Lambda = 100$ MeV to one corresponding to $\Lambda = 0\ 1$ MeV, the value of the gluon condensate changes only by 15%

Choosing the Pauli-Villars scheme we find for the SU_3 group

$$\frac{1}{V}\langle Q_t^2\rangle \simeq \langle F_{\mu\nu}^2/32\pi^2\rangle \simeq \frac{\langle N\rangle}{V} \simeq (0\ 65\Lambda_{PV})^4,$$

$$\sqrt{\overline{\rho^2}} \simeq (2\ 1\Lambda_{PV})^{-1}, \qquad \frac{\overline{R}}{\bar\rho} \simeq 3\ 2, \qquad \beta \simeq 15 \qquad (5\ 29)$$

For the SU_2 we have

$$\frac{1}{V}\langle Q_t^2\rangle \simeq \langle F_{\mu\nu}^2/32\pi^2\rangle \simeq \frac{\langle N\rangle}{V} \simeq (0.7\Lambda_{PV})^4,$$

$$\sqrt{\overline{\rho^2}} \simeq (3\Lambda_{PV})^{-1}, \qquad \frac{\overline{R}}{\bar{\rho}} \simeq 4.2, \qquad \beta \simeq 12. \tag{5 30}$$

Note the diluteness of the medium obtained ($\overline{R}/\bar{\rho} \gg 1$) and the small effective coupling constant $\beta = \beta(\bar{\rho}) \gg 1$. This gives some justification to the previously made assumptions the neglect of many-body interactions and of the non-factorization of the determinants From eq (5.16) we learn that $\langle U_{int}\rangle$ is less than 12% of the free instanton action, which is another manifestation of the diluteness However $\beta\langle U_{int}\rangle \sim 1$, this is as it should be, since it is the actual condition for the interactions to come into play and to stabilize the medium

To confront (5 29) with the phenomenological value of $\langle Q_t^2\rangle$ (eq (5 23)) one needs $\Lambda_{PV} \sim 300$ MeV($\Lambda_{lattice} \sim 10$ MeV) which may seem to be too large a value. We note, however, that (i) at the present stage the value of Λ is not determined accurately enough either in perturbative or lattice QCD, (ii) in taking the "best" profile function (sect. 6) we need a smaller value of Λ, (iii) it should anyhow be remembered that we are calculating the lower bound for the condensate If Λ proves to be much smaller than we need here, we obtain a reason why instantons have little to do with the formation of the gluon condensate, and one has to think of other non-perturbative configurations

Our last comment concerns the large N_c behaviour of the system. It can be easily studied from eqs. (5.10), (5.27) and (5.28) In the first approximation all quantities exhibit a natural behaviour. $\langle Q_t^2\rangle, \langle F_{\mu\nu}^2\rangle, \beta \propto N_c, \bar{\rho} \propto N_c^0$ However, there are logarithmic corrections to these formulae. As a result, the ratio $\overline{R}/\bar{\rho}$ decreases with the growth of N_c, and at $N_c \sim 20$–30 it becomes equal to the value corresponding to the minimum of the potential (see eq. (3.18)) We think that at this point there is a phase transition from a liquid to a crystal phase, with a spontaneous violation of colour and/or Lorentz symmetry Therefore, an approach to QCD based on the consideration of the $N_c \to \infty$ limit needs, to our mind, an additional study

6. The best form of tunnelling trajectory

In this section we find the partition function of the instanton liquid as a functional of the instanton's profile function $f(x^2/\rho^2)$ (see eqs (1.6), (1 7)) For the sake of brevity we shall call the instanton-like object with unit topological charge fremon – from free energy minimization.

As seen from eqs (5.13), (5.14), the dependence of Y on f enters through (i) the quantity $\nu \equiv \frac{1}{2}(bs - 4)$, where s is the action of one fremon normalized to unity in

the instanton case, (ii) the value of the one-fremon determinant represented by C_{N_f}, (iii) the quantity γ associated with the interaction of individual fremons (see eq (5 7))

In the quasiclassical approach to quantum mechanics the action is the only thing to be minimized, and one comes to the classical tunnelling trajectory In QCD the trace of this circumstance can be seen from the fact that the dependence of (5 13) on ν is much more rapid than on other quantities However, one given instanton does not exist in QCD because of the divergency of the ρ integrations which are effectively cut only due to the interactions and the entropy effects of the medium These effects are accumulated in the quantity γ It is important that the action s is sensitive to the region where the fremon's field f is large, whereas γ is sensitive to the tail of f Therefore, we expect that the best form of the tunnelling trajectory would follow that of the instanton in the region where f is of the order of unity, and would deviate from the instanton only at large distances where an individual fremon begins to feel the effect of its neighbours, and the effects of the medium become crucial

Let us discuss the rôle of C_{N_f} Basically, it is defined by the determinants in (2 19) To calculate functional determinants in a more or less arbitrary background field seems to be a hopeless problem Nevertheless, they can be estimated using the approximation method of ref [16] and expressed through invariants such as $\int d^4x f^{abc} F^a_{\alpha\beta} F^b_{\beta\gamma} F^c_{\gamma\alpha}$, etc It is clear that, similar to the action, C_{N_f} is determined by the region where the trial function f is large, and depends weakly on the edge of the fremon Since the dependence of (5 13) on C_N is much weaker than on ν, one could think of neglecting the dependence of C_{N_f} on f and of using the instanton value for C_{N_f}

However, it should be taken into account that the action s is scale-invariant it is the same for $f(x^2/\rho^2)$ and for $f(\alpha^2 x^2/\rho^2)$ The breaking of scale invariance is due to quantum corrections and is, therefore, associated with C_{N_f} It is C_N that determines the actual size of the fremon Evidently, the profile function $f(\alpha^2 x^2/\rho^2)$ has the size ρ/α, not ρ The quantity C_{N_f} transforms in a way that maintains this property $C_{N_f} \to C_{N_f} \alpha^{-b}$

Therefore, one can indeed neglect the f dependence of C_{N_f} provided the fremon size is fixed by some subsidiary condition For example, one can define the size ρ as follows

$$f(x^2)|_{x^2=\rho^2} = \tfrac{1}{2} \tag{6 1}$$

(as it is in the case of instantons) We expect that the fremon's form deviates from that of the instanton only at the distances $x \sim \bar{R} \gg \bar{\rho}$, $\bar{R} = (V/\langle N \rangle)^{1/4}$ Therefore, the condition (6 1) (or any similar condition at $x \sim \bar{\rho} \ll \bar{R}$) does not influence the behaviour of the tail of $f(x^2)$. However, it means also that the tail is a universal

function of the distance and does not depend on the size ρ of a given fremon. For this reason one should use a trial function $f(x^2, \rho^2)$ rather than $f(x^2/\rho^2)$

Let us now find s and γ. The action is

$$s \equiv \int \frac{d^4x}{32\pi^2} F_{\mu\nu}^2 = 3\int_0^\infty \frac{dx^2}{x^2}\left\{\left[x^2 \frac{d}{dx^2} f(x^2, \rho^2)\right]^2 + f^2(1-f)^2\right\} \geqslant 1. \quad (6\,2)$$

Generally speaking, it is now a function of ρ, so that the 1-loop 1-fremon density should be written as

$$d(\rho) = \frac{1}{\rho^5} C_{N_c} \tilde{\beta}^{2N_c} (\rho\Lambda)^{b_1(\rho)} \quad (6\,3)$$

Calculating the partition function Y as in sect. 5 we arrive at eqs. (5 6) which should now be changed Indeed, we have ($\mu(\rho)$ being the variable 1-fremon density)

$$\frac{\beta N^2}{2V^2\mu_0^2}\int dO_1\, dO_2\, d\rho_1\, d\rho_2\, d^4x_1\, d^4x_2\, \mu(\rho_1)\mu(\rho_2) u_{12}(x_1 - x_2, \rho_1, \rho_2, O_{12})$$

$$= \frac{\beta N^2}{2V}\left(\overline{\gamma(\rho)\rho^2}\right)^2,$$

where (cf eq (5 8))

$$\gamma(\rho)\rho^2 = \gamma_0 \int_0^\infty dx^2 f(x^2, \rho^2), \qquad \overline{\gamma(\rho)\rho^2} = \int d\rho\, \gamma(\rho)\rho^2\mu(\rho)/\mu_0,$$

$$\gamma_0^2 = \frac{27N_c\pi^2}{4(N_c^2 - 1)}, \qquad \mu_0 = \int d\rho\, \mu(\rho). \quad (6\,4)$$

From the variational principle (5 5) we obtain

$$\ln Y \geqslant -N \ln\frac{N}{2V} + N - \frac{N}{\mu_0}\int d\rho\, \mu(\rho) \ln\frac{\mu(\rho)}{d(\rho)} - \frac{\beta N^2}{2V}\left(\overline{\gamma(\rho)\rho^2}\right)^2. \quad (6\,5)$$

This expression is to be varied in $\mu(\rho)$, in $f(x^2, \rho^2)$ and in the total number of fremons and anti-fremons N

The best $\mu(\rho)$ is (cf eq. (5 9))

$$\mu(\rho) = d(\rho)\exp\left[-\gamma(\rho)\rho^2\overline{\gamma(\rho)\rho^2}\beta N/V\right] \quad (6\,6)$$

The best profile function $f(x^2, \rho^2)$ is determined by a differential equation

$$-\left(x^2\frac{\mathrm{d}}{\mathrm{d}x^2}\right)^2 f + f - 3f^2 + 2f^3 + \tfrac{1}{3}\gamma_0\frac{N}{V}\frac{\beta(\bar\rho)}{\beta(\rho)}\overline{\gamma(\rho)\rho^2}x^2 f + \frac{1}{6\beta(\rho)}\frac{\delta C_{\gamma}}{\delta f} = 0 \tag{6 7}$$

As discussed above, we neglect the last term, implying that $f(x^2, \rho^2)$ is normalized by the condition (6 1) Also the condition for the finiteness of the action, $f(0, \rho^2) = 1$, should be imposed

The term next to last in eq (6 7) describes effectively the influence of the media on one fremon At small x^2 we can neglect this term and obtain the usual equation for the minimum of the action Its solution is the BPST instanton, $f(x^2, \rho^2) = f_0 = \rho^2(x^2 + \rho^2)^{-1}$, $x^2 \lesssim \rho^2$

However, the $x^2 f$ term radically changes the behaviour of f at large $x^2 > \rho^2$ Indeed, we find

$$f(x^2, \rho^2) \sim \exp(-\sqrt{\alpha^2 x^2})(x^2)^{1,4},$$

$$\alpha^2 = \tfrac{4}{3}\gamma_0\frac{N}{V}\frac{\beta(\bar\rho)}{\beta(\rho)}\overline{\gamma(\rho)\rho^2} \tag{6 8}$$

To find α, it is necessary to solve a system of equations on f, $\gamma(\rho)$, $\mu(\rho)$ and N/V For a quick estimate it is sufficient to use the values found in the instanton case

$$\overline{\gamma(\rho)\rho^2} \simeq \gamma_0\overline{\rho^2} \approx \gamma_0(620\ \mathrm{MeV})^{-2}, \qquad \frac{N}{V} \simeq \langle F_{\mu\nu}^2/32\pi^2\rangle \simeq \langle Q_1^2\rangle \simeq (190\ \mathrm{MeV})^4,$$

$$\beta(\rho) \simeq \beta(\bar\rho) \simeq 15 \tag{6 9}$$

Then $\alpha \simeq 350$ MeV Naturally, we now obtain a better upper bound for the vacuum energy density or a better lower bound for the gluon condensate The change, however, is not too large the gluon condensate increases by a factor of 1 5–2 as compared to the instanton case Therefore, the estimates (6 9) seem to be reasonable

The change in the fall-off of $f(x^2)$ leads to dramatic effects Let us consider a correlation function $\langle F_{\mu\nu}^2(x), F_{\mu\nu}^2(0)\rangle$ In perturbation theory it behaves as $|x|^{-8}$ – a manifestation of massless gluons. Meanwhile, the contribution of one fremon to this quantity is

$$\langle F_{\mu\nu}^2(x) F_{\mu\nu}^2(0)\rangle$$

$$\sim \int \mathrm{d}^4 y \int \mathrm{d}\rho\,\mu(\rho)\left\{\frac{f^2(x-y)[1-f(x-y)]^2}{(x-y)^4} + \left[\frac{\mathrm{d}}{\mathrm{d}(x-y)^2}f(x-y)\right]^2\right\}$$

$$\times\left\{\frac{f^2(y)[1-f(y)]^2}{y^4} + \left[\frac{\mathrm{d}}{\mathrm{d}y^2}f(y)\right]^2\right\} \tag{6 10}$$

In the instanton case ($f = f_0$) we again obtain the $|x|^{-8}$ behaviour, whereas for fremons we get at $|x| \to \infty \cdot \exp(-2\alpha|x|)/|x|^3$. Taken naively, it gives an estimate for the 0^{++} glueball mass,

$$m(0^{++}) = 2\alpha \simeq 700 \text{ MeV},$$

which coincides remarkably with the lattice measurements [19]. It should be noted, however, that eq. (6.10) is only one particular contribution to the correlation function Other contributions with quantum fields B_μ and with more fremons, should be, generally speaking, also taken into account and will be considered elsewhere.

7. Conclusions

We have elaborated a modification of the Feynman variational principle for gauge theories It allows one to estimate the contribution of given trial gauge configurations to the Yang-Mills partition function In this work we studied a trial configuration which is a sum of instantons and anti-instantons We observed an effective repulsion of pseudoparticles leading to the stabilization of the medium at average distances being a factor of 3 larger than average sizes. There seem to be no more infrared problems remaining in the theory The effective charge $g^2/8\pi^2$ does not grow above the value of $\sim \frac{1}{15}$ However, the $N_c \to \infty$ limit of the theory may be in trouble since at $N_c \sim 20$–30 the $I\bar{I}$ system seems to prefer a crystal phase.

One can impose a weak and slowly varying external field $F_{\mu\nu}^{\text{ext}}$ on our trial vacuum The effective action can be easily calculated to be $(5/36g^2(\bar{\rho}))\int d^4x\,(F_{\mu\nu}^{\text{ext}})^2$ It shows that our system is stable under small perturbations. At the moment we have little to say on whether it is stable under large perturbations, i.e. whether we have found the true vacuum of QCD If Λ_{QCD} is somewhat larger than the commonly used value we have the numerical argument that we deal with the vacuum.

Our final comment concerns the problem of glueball masses The lattice measurements [19] give ~ 700 MeV for the lightest glueball On the other hand, the value of the gluon condensate (5 24) shows that the average distance between non-perturbative gauge field fluctuations is $\sim (200 \text{ MeV})^{-1}$ If the appearance of a mass gap in the theory is a sort of a collective effect (as it is in the case of the 3-dimensional compact electrodynamics where the mass gap is obtained as a result of the Debye screening of monopoles [20]) then the glueball mass would be much smaller than 200 MeV The only way we can at present reconcile these two numbers is to suggest that the mass gap is achieved owing to a *single* typical gluon fluctuation It is, therefore, very promising that we obtained an exponential fall-off for the trial function (see (6.8)) and not, say, a power or a Gaussian one To find out whether it does indeed lead to the required spectrum and to confinement needs further work

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