(Table I). It follows that the over-all gain is limited to $\sim Gl_c$ or 10^1-10^2 for picosecond 10-GW pump pulses. This conclusion is supported by our failure to observe nonfilamentary stimulated Raman emission at 10-GW-cm^{-2} pump intensity over a distance of 10-30 cm.

It appears that picosecond light pulse excitation of Raman scattering may provide a valuable technique for probing molecular orientation dynamics in liquids by the use of pulse times on the scale of the molecular orientation time.

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METHOD FOR SOLVING THE KORTEWEG-deVRIES EQUATION*

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A method for solving the initial-value problem of the Korteweg-deVries equation is presented which is applicable to initial data that approach a constant sufficiently rapidly as $|x|\to\infty$. The method can be used to predict exactly the "solitons," or solitary waves, which emerge from arbitrary initial conditions. Solutions that describe any finite number of solitons in interaction can be expressed in closed form.

For a large class of physical systems, nonlinear and dispersive processes compete while dissipation is negligible. In particular, the Korteweg-deVries (KdV) equation,

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1}$$

(subscripts x and t denoting partial differentiations), has been shown to describe the asymptotic development of small- but finite-amplitude shallow-water waves,[1] hydromagnetic waves in a cold plasma,[2] ion-acoustic waves,[3] and acoustic waves in an anharmonic crystal.[4]

The quantities u, x, and t can be rescaled to produce any desired coefficients for the terms of Eq. (1). The present choice is convenient for this paper. Note that u is reversed in sign from previous work since the coefficient of the second term is negative. Further, the KdV equation is Galilean-invariant so that $u(x-6Vt, t)-V$ forms a one-parameter family of solutions.

Previous numerical computations,[5] as well as more recent ones,[6] indicate that for large t the solution of an initial-value problem consists of a finite train of "solitons," or solitary waves, traveling to the right, and an oscillatory train or "tail" spreading to the left. The solitons exhibit a remarkable stability in that their identity is preserved through nonlinear interactions. This property of solitons, which was discovered numerically[5] and justifies the name suggestive of particles, has been proved by Lax[7] for two of them, and can be demonstrated for any number using the solution described below.

We now sketch a general method of solution that can be used to establish these results rigorously. It is applicable to initial data that approach a constant sufficiently rapidly as $|x| \to \infty$. The Galilean invariance described above permits us to set this constant equal to zero.

First consider the differential equation[8]

$$\psi_{xx}-(u-\lambda)\psi=0, \tag{2}$$

where $u(x,t)$ is a solution of Eq. (1), so that $\psi(x,t)$ and $\lambda(t)$ depend parametrically on t. Solving Eq. (2) for u and inserting the result in Eq. (1) yields

$$\lambda_t\psi^2+[\psi Q_x-\psi_x Q]_x=0, \tag{3}$$

with

$$Q\equiv\psi_t+\psi_{xxx}-3(u+\lambda)\psi_x, \tag{4}$$

for the time development of the solutions of Eq. (2). If ψ vanishes as $|x| \to \infty$, the second term of Eq. (3) vanishes on integration over the interval $(-\infty,\infty)$. Therefore $\lambda_t=0$, i.e., the discrete eigenvalues of Eq. (2) are constant when u evolves according to the KdV equation.

Dropping the first term, we can integrate Eq. (3) twice to yield

$$\psi_t+\psi_{xxx}-3(u+\lambda)\psi_x=C\psi+D\varphi. \tag{5}$$

Here $C(t)$ and $D(t)$ are the constants of integration, and φ is a solution of Eq. (2) that is linearly independent of ψ. Thus $\varphi\equiv\psi\int^x dx/\psi^2$.

It is now straightforward to compute the evolution of ψ in regions where u vanishes, and, in particular, asymptotically for $|x| \to \infty$. For a (time-independent) discrete eigenvalue λ_n <0, $D=0$ because the corresponding ψ_n satisfies Eq. (5) and vanishes exponentially as $|x| \to \infty$, and $C=0$ because we are assuming the normalization $\int\psi_n^2dx=1$. Then inserting

$$\psi_n\approx c_n(t)\exp(-\kappa_n x) \text{ for } x\to\infty, \tag{6}$$

with $\kappa_n=(-\lambda_n)^{1/2}>0$ into Eq. (5), we find

$$c_n(t)=c_n(0)\exp(4\kappa_n{}^3t). \tag{7}$$

The analogous coefficients for large negative x decay exponentially in time.

For $\lambda=k^2>0$, a solution of Eq. (2) for large $|x|$ is a linear combination of $\exp(\pm ikx)$. We impose on ψ the boundary conditions

$$\psi\approx\exp(-ikx)+b\exp(ikx),\quad x\to\infty, \tag{8}$$

$$\psi\approx a\exp(-ikx),\quad x\to-\infty. \tag{9}$$

In the frequent interpretation of Eq. (2) as describing the normal modes of a wave equation, the coefficients of unity in Eq. (8) and (implied) zero in Eq. (9) indicate prescribed steady radiation arriving from $+\infty$ only. The coefficients of transmission $a(k,t)$ and reflection $b(k,t)$ can be shown to satisfy $|a|^2+|b|^2=1$.

The spectrum for $\lambda>0$ is continuous and we may choose λ constant, so that Eq. (5) is again valid. Inserting Eqs. (8) and (9) into Eq. (5) and equating the coefficients of the two independent solutions at $+\infty$ and at $-\infty$, we find $D=0$, $C=4ik^3$, and two equations which integrate trivially to yield

$$a(k,t)=a(k,0), \tag{10}$$

$$b(k,t)=b(k,0)\exp(8ik^3t). \tag{11}$$

This information on the development of ψ is sufficient to reconstruct u for any value of time! Specifically, given the reflection coefficient $b(k)$ and the κ_n and c_n, let $K(x,y)$ for $y\geq x$ be the solution of the Gel'fand-Levitan equation,[9,10]

$$K(x,y)+B(x+y)+\int_x^\infty K(x,z)B(y+z)dz=0, \tag{12}$$

with

$$B(\xi)\equiv\frac{1}{2\pi}\int_{-\infty}^\infty b(k)\exp(ik\xi)dk+\sum_n c_n{}^2\exp(\kappa_n\xi). \tag{13}$$

Then

$$u(x,t)=2(d/dx)K(x,x). \tag{14}$$

The evolution of $u(x,t)$ is obtained from the explicit dependence on time of $b(k)$ and the c_n given by Eqs. (11) and (7). [In all these formulas the signs of x, y, and z have been reversed from Kay's[10] usage, thus the reference end in Eqs. (6), (8), and (9) is $+\infty$.] Note that $K(x, x)$ as determined by Eq. (12) is independent of values of $B(\xi)$ for $\xi < 2x$.

A number of results can be established by further elaboration of this method, which we mention without going into details.

When u represents a single soliton, there is perfect transmission $[b(k) \equiv 0]$ and exactly one discrete eigenvalue $\lambda_1 = \frac{1}{2}u_{\min}$. More generally, Kay and Moses[11] have given the general solution of Eq. (12) with $b(k) \equiv 0$ in closed form in terms of exponentials. This includes all cases where u decomposes exactly into solitons.

It is more difficult to find exact solutions when $b(k)$ does not vanish. The time dependence of $b(k)$ indicates a strong phase mixing in the integral of Eq. (13) as $t \to \infty$ for positive ξ. The behavior for negative ξ is more complicated since the integrand then has points of stationary phase. This is reflected (in computer studies) by the "tail" moving toward the left.

Since the c_n grow exponentially, as long as there is at least one of them $B(\xi)$ can be approximated by the summation when Eq. (12) is to be solved for $x > 0$ and $t \to \infty$. The solution then reduces to that found by Kay and Moses[11] described above. Thus the magnitude, velocity, and position of each soliton can be found in the limit of large time. Furthermore, the solitons for large negative time can be found from the usual version[10] of Eq. (12) where the reference end is $-\infty$.

A fuller treatment together with other applications and generalizations will be published subsequently.

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FREQUENCY BROADENING IN LIQUIDS BY A SHORT LIGHT PULSE*

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Several authors have reported the observations of frequency broadening in filaments which were produced by the self-focusing of a Q-switched laser in liquids.[1-3] This broadening has been attributed to the generation of new frequency components through an intensity-dependent refractive index and stimulated Rayleigh scattering.[1-4] Theories of the frequency broadening in an optical pulse by an intensity-dependent refractive index have been given in connection with the pulse distortion.[5,6] But comparison of experiment with theory has been difficult, because the broadening is usually irregular and the observation of the spectrum in filaments is obscured by the strong background. We report here the observation of the frequency broadening in a filament with short duration time, under such experimental conditions that the intensity of the stimulated Raman emission in the filament is much less than that of the laser. The structure of the frequency spectrum shows a pattern which can be explained by phase modulation through the intensity-dependent refractive index.