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Fermions and vortex solutions in Abelian and non-Abelian gauge theories

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The interaction of fermions with an extended vortex solution of the Higgs model is investigated. It is found that this interaction has a long-range inverse-square tail. It is caused by the coupling of the fermion angular momentum with the vortex gauge field itself. The fermion-vortex bound states present at the threshold and the fermion-vortex scattering are studied. The scattering phase shifts and the Jost functions are obtained for large and small fermion momenta as well as the low-energy cross section which diverges at zero momentum. The quantum field theory in the one-vortex sectors is developed. It is found that, in the presence of fermions, a vortex with an even (odd) number of flux quanta has a half-integer (integer) fermionic number. It follows that a two-quantum vortex is stable. Finally, the stable vortex solution of an SU(2) Higgs model is investigated. The appropriate ansatz for the field is given and radial equations are discussed. It is shown that the interaction of a vortex with any nonsinglet particle has a long-range inverse-square tail.

I. INTRODUCTION

It has been known for a long time that certain nonlinear classical field theories possess exact wave solutions of finite energy. They are interesting in particle physics because they represent new states in the corresponding quantum field theory.[1,2,3] One of the first models of this type was developed by Nielsen and Olesen.[4] They pointed out, in analogy with the Ginzburg-Landau theory of superconductivity, that the Higgs Lagrangian has classical vortex solutions which behave classically as Nambu strings in the strong-coupling limit. The vortex can be considered as an infinitely long object in three spatial dimensions or as a two-dimensional finitely extended solution. Otherwise one must place sources and sinks of magnetic flux to get a vortex line of finite length in three-dimensional space.[5,6]

In this paper we study the interaction of fermions with a vortex in three-dimensional space-time. The fermions are coupled to the gauge field in the usual minimal way.

Jackiw and Rebbi have studied fermions in field theories with bosonic extended solutions.[7] They show the connection between the quantized Dirac field and the solutions of the Dirac equation in the external field given by the bosonic solutions.

Thus, as a first step to study the quantum mechanics of the one-vortex sectors, we investigate the Dirac equation in the extended gauge field of an n-quantum vortex. It can be pointed out that for small coupling the fermion-vortex interaction can be shown to be of order one.

In that way, we find that for long distances ($\rho \gg \mu^{-1}$, where μ is the vector-meson mass), the fermion-vortex interaction has an inverse-square tail. Because the magnetic field of the vortex decreases exponentially (like $e^{-\mu\rho}$) for long distances, that long-range interaction is not originated by the Lorentz force on the fermion. Nor is it the Aharonov-Bohm effect[8] because this effect is absent when the confined magnetic flux is an integer multiple of $2\pi/e$, like in our case.

One finds that the $1/\rho^2$ long-range fermion-vortex interaction is a modification of the centrifugal barrier produced by the minimal gauge coupling of the angular momentum with the gauge field. At long distances, the polar component of the vector field has a nonzero value proportional to the topological charge of the vortex ($-n/e$). The topological charge ensures the stability of the classical solution.[9] The minimal gauge coupling performed on the polar component of the momentum reads

$$p_\varphi \to p_\varphi - eA_\varphi(\rho) \underset{\mu\rho \gg 1}{\simeq} p_\varphi + n\,. \tag{1.1}$$

This shows explicitly that the centrifugal barrier of the particle is modified in an amount proportional to the topological charge of the vortex. It is clear that, for all charged particles of any spin, a $1/\rho^2$ interaction will be present for $\rho \gg \mu^{-1}$.

At small distances from the vortex line, we find that the fermion-vortex interaction is of the magnetic-moment type. In other words, for small distances the fermion angular momentum couples with the magnetic field, whereas for long distances it interacts with the gauge field itself.

After separating angular and spin dependence in Sec. II, the Dirac equation is reduced to a pair of uncoupled second-order differential equations of the Schrödinger type. The effective potentials that appear in these Schrödinger-type equations are very useful in getting information on the fermion-vortex interaction.

The Dirac equation on the vortex field is exactly soluble at the threshold. All solutions with energy equal to plus or minus the fermion mass (m) are expressed by quadratures in terms of the radial function of the vortex gauge field [Eqs. (2.19)–(2.20)]. There are no bound states with nonzero binding energy.[10] We find that normalizable fermion solutions are present, for an n-quantum vortex, with an energy $\omega = m\,\mathrm{sgn}(n)$ if

$$\tfrac{1}{2} \leq J\,\mathrm{sgn}(n) \leq |n| - \tfrac{3}{2}\,. \qquad (1.2)$$

Here J is the total fermion angular momentum. It can take any half-integer value (positive or negative). We see from the preceding equation that for positive (negative) n the vortex can bind only fermions (antifermions). There are $|n|-1$ bound states at the threshold, each in a different partial wave. In particular, a vortex with one unit of flux does not bind fermions.

The presence or absence of bound states at the threshold can be related to the strength of the centrifugal barrier modified by the topological charge and to the sign of the magnetic-moment interaction. We find that bound states are present when the magnetic-moment interaction is attractive at short distances, and the modified centrifugal barrier is repulsive enough. In this way it keeps the fermion near the center of the vortex line.

In Sec. III, we study the scattering of fermions by a vortex line located at the origin in a partial-wave analysis. In this context, we investigate the radial scattering solutions of the decoupled Schrödinger-type equations mentioned above. We consider two limiting cases: high and low fermion momenta $k^2 \gg \mu^2$ and $k^2 \ll \mu^2$. For the high-momentum behavior, we find that the scattering phase shift vanishes like (μ/k) for $k \to \infty$. This is related to the regular behavior of the fermion-vortex interaction at short distances. In the low-energy case there are more interesting results because of the long-range nature of the interaction.

Then, the phase shifts are found for all values of m and J for low energies [Eqs. (3.13)–(3.16)]. The phase shifts at zero energy are nonzero [we normalize them so that $\delta_J(\infty)=0$] but Levinson's theorem[11] does not hold because of the long-range tail of the interaction. It is interesting to note that, for odd n, every phase shift attains its unitarity limit at the threshold. This phenomenon produces a Dirac δ behavior on the forward direction in the scattering amplitude at $\omega = m$.

We find that the fermion-vortex cross section diverges for low momenta, like $k^{-1}\ln^{-2}k$ [Eqs. (3.22)], isotropically. This is due to the zero-energy resonance in the wave $J = n - \frac{1}{2}$, clearly produced by the modified centrifugal barrier. The study of the Dirac equation in the vortex field carried out in Secs. II and III is a step previous to treating the quantum fields in the one-vortex sectors of the theory. This is done in Sec. IV within the collective-coordinate method.[3] In this context the fermionic field is expanded in the eigenfunctions of the Dirac equation previously considered. The lower-energy state in each n-quantum sector is the vortex state with a mass M_n of order e^{-2}. If $n \geq 2$ ($n \leq -2$) the next state is the fermion-vortex (antifermion-vortex) bound state with a mass equal to $M_n + m$, at first approximation. Then there is the continuum of the fermion-vortex scattering states.

Although the theory is invariant under charge conjugation ($\mathfrak{C}$), each vortex sector is not. As it is clear, an n-quantum sector goes into the $(-n)$-quantum sector under $\mathfrak{C}$.

We find that the fermionic number of a vortex is nonzero when fermions are present. For an n-quantum vortex [Eqs. (4.20) and (4.21)]

$$N_F(n\text{-quantum vortex}) = \frac{1-|n|}{2}\,\mathrm{sgn}(n)\,. \qquad (1.3)$$

Then a vortex with an even number of quanta has a half-integer fermionic number. Because vortices with an odd number of quanta have an integer fermionic number, it follows, from fermionic and topological number conservation laws, that a vortex with $|n|=2$ is stable.

As in Refs. 7 and 12, we find quantum states in the theory with quantum numbers which are a fraction of those of the fields explicit in the Lagrangian.

Finally, we discuss the Dirac-field matrix elements between vortex and fermion-vortex states. We show that the fermion-vortex form factor has singularities at correct physical thresholds, in the case when both bosons have equal masses.

Finally, in Sec. V, we consider vortex solutions of a non-Abelian gauge theory. They exist, as in the Abelian case, if the symmetry breaking is maximum. In other words, if the vacuum is invariant only under group elements which are mapped into the unit matrix of the adjoint representation.[6] We work in detail the SU(2) case only, where there is only one topologically stable vortex solution.[6] We give the appropriate isospin and angular dependence of the fields [Eqs. (5.7) and (5.8)] which separate the equations of motion. The solutions of the radial equations [Eq. (5.9)] admit asymptotic solutions and short-distance ones, not very different from those of the Abelian vortex. As in the Abelian case, the energy density is confined in a disk with a diameter of a few boson Compton wavelengths.

Subsequently, we study isospin $-\frac{1}{2}$ fermions coupled gauge-invariantly to the gauge and Higgs fields. Because the non-Abelian vortex solution is invariant under space plus isospin rotations, the fermions acquire integer angular momentum as in the monopole field.[12,13] We find a long-distance fermion-vortex interaction, as in the Abelian case, because the fermion angular momentum couples to the gauge field through

$$p_\varphi \to p_\varphi - eT^a A_\varphi^a(\rho) \underset{\rho\to\infty}{\simeq} p_\varphi - T^3 . \tag{1.4}$$

This last equation shows that all nonsinglet particles will interact with the vortex through a long-range force due to the modification of the centrifugal barrier. The interaction strength is proportional to the isospin of the particle.

In the case when the fermions are coupled to the gauge field only, we find that there are neither fermion-vortex bound states at the threshold nor with nonzero binding energy. The modified centrifugal barrier has not enough strength to bind isospin $-\frac{1}{2}$ fermions.

It can be pointed out that for monopole-type solutions in three-dimensional space[13,14] and for the pseudoparticle solution in four-dimensional space,[15] the covariant components of the gauge field in the angular directions are nonzero at spatial infinity. Thus, long-range $1/r^2$ interactions will be present between any nonsinglet particle and a monopole in four-dimensional space-time. The same thing is true around a pseudoparticle solution in five-dimensional space-time which binds fermions at the threshold[16] as the vortex solutions do in three-dimensional space-time.

II. FERMIONS AND ABELIAN VORTICES: LONG-RANGE AND SHORT-RANGE INTERACTIONS

We consider the Abelian Ginzburg-Landau-Higgs model coupled to spin-$\frac{1}{2}$ particles in three-dimensional space-time, with the Lagrangian density

$$\mathcal{L}(x) = -\tfrac{1}{4} F_{\mu\nu}{}^2 + |(\partial_\mu + ieA_\mu)\Phi|^2 + \tfrac{1}{2}\mu_S{}^2|\Phi|^2$$
$$-\tfrac{1}{2}\lambda|\Phi|^4 + \bar\psi(i\not\partial - e\not A - m)\psi , \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. At the tree level the masses of the scalar and vector bosons are μ_S and $\mu = e\mu_S/\sqrt{\lambda}$. We consider $\mu_S \geq \mu$ but $\lambda = O(e^2)$, which corresponds to type-II superconductivity.[17]

As it is known, in the absence of fermions this model has classical vortex solutions.[4,17] For these vortex solutions, the total magnetic flux through the xy plane is quantized in units of $2\pi/e$. An n-quantum solution, located around the origin, has the following form:

$$\Phi(\vec\rho) = \frac{\mu}{\sqrt{2\lambda}} F_n\left(\mu\rho, \frac{e^2}{\lambda}\right) e^{-in\varphi} , \tag{2.2}$$

$$\vec A(\vec\rho) = \frac{\hat e_\varphi}{\rho}\left[n - \mathcal{K}_n\left(\mu\rho, \frac{e^2}{\lambda}\right)\right] , \tag{2.3}$$

$$H(\rho) = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = -\frac{1}{e\rho}\frac{d\mathcal{K}_n}{d\rho} . \tag{2.4}$$

The classical field solution is of order $1/e$ and gives the leading approximation to the quantum theory of the bosonic fields for weak coupling.[1,2,3]

Now we will consider the leading approximation for fermionic fields when they are coupled to the Abelian Higgs model.

One can easily prove that the Dirac equation in the external classical field given by the vortex has solutions of order e^0. More precisely, the solutions of

$$(i\not\partial - e\not A_{\rm cl} - m)\psi(x) = 0 \tag{2.5}$$

(where $A^\mu_{\rm cl}$ stands for the classical vortex field) are self-consistent solutions of the coupled set of Euler-Lagrange equations of the model, for small coupling. One checks immediately that the fermionic current

$$e\bar\psi\gamma_\mu\psi = O(e)$$

is a higher-order correction to the bosonic one

$$e^2|\Phi|^2 A_\mu = O(1/e) .$$

Now we turn to the study of the solutions of the Dirac equation in the gauge field of a vortex (always in three-dimensional space-time). We use the following two-by-two representation for the Dirac matrices

$$\gamma^0 = \sigma_z, \quad \gamma^1 = i\sigma_y, \quad \gamma^2 = -i\sigma_x , \tag{2.6}$$

where σ's are the Pauli matrices.

We will seek stationary solutions of our Dirac equation and also eigenstates of the total angular momentum

$$\psi_{kJ}(t,\vec\rho) = \frac{e^{i(J\varphi-\omega t)}}{\sqrt{\rho}}\begin{pmatrix}\varphi_+^J(k,\rho)e^{-i\varphi/2}\\ \varphi_-^J(k,\rho)e^{+i\varphi/2}\end{pmatrix} , \tag{2.7}$$

where $k = +(\omega^2 - m^2)^{1/2}$ and J can take any half-integer value (positive or negative). The radial functions verify the following equations:

$$\left[\frac{d}{d\rho} \pm \frac{J - n + \mathcal{K}_n(\rho)}{\rho}\right]\varphi_\mp^J(k,\rho) = i(\omega \mp m)\varphi_\pm^J(k,\rho) . \tag{2.8}$$

These equations differ from the free Dirac ones, only in the terms $\pm[\mathcal{K}_n(\rho) - n]/\rho$ in the left-hand side. From this fact and the boundary condition of the classical gauge field at infinity $[\mathcal{K}_n(\infty) = 0]$ we get a first result on the fermion-vortex interaction. We see that for long distances ($\rho \gg \mu^{-1}$) the Dirac equation in the field of the vortex is like a free one, but with a factor $(J-n)$ instead of J. In other words, at long distances from the vortex, the fermion "sees" the interaction as a modification in the strength of the centrifugal barrier.

The system of radial Dirac equations [Eq. (2.8)] can be decoupled into two Schrödinger-type equations for the functions $\varphi_\pm^J(k,\rho)$

$$\left[\frac{d^2}{d\rho^2}+k^2-U_{nJ}^\pm(\rho)\right]\varphi_\pm^J(k,\rho)=0\,. \tag{2.9}$$

Here the functions

$$U_{nJ}^\pm(\rho)=\pm\frac{1}{\rho}\frac{d\mathcal{H}_n}{d\rho}+\frac{[J-n+\mathcal{H}_n(\rho)](J-n+\mathcal{H}_n\mp 1)}{\rho^2} \tag{2.10}$$

play the role of effective potentials. In the case of free fermions, one would find only a centrifugal term for the effective potentials

$$U_{0J}^\pm(\rho)=\frac{J(J\mp 1)}{\rho^2}\,. \tag{2.11}$$

We also consider an interaction potential which only takes into account the fermion-vortex interaction

$$\begin{aligned}V_{nJ}^\pm(\rho)&=U_{nJ}^\pm(\rho)-U_{0J}^\pm(\rho)\\&=\pm\frac{1}{\rho}\frac{d\mathcal{H}_n}{d\rho}-\frac{(n-\mathcal{H}_n)(2J+\mathcal{H}_n-n\mp 1)}{\rho^2}\,.\end{aligned} \tag{2.12}$$

In Figs. 1 and 2 we plot this interaction potential for $\lambda=e^2$, $n=1$, and $J=+\frac{1}{2}$ and $J=-\frac{1}{2}$, respectively, from the solution given in Ref. 18.

It can be pointed out that the Schrödinger-type equation (2.9) for the upper component is formally identical to a radial (three-dimensional) Schrödinger equation with a half-integer orbital angular momentum $l=J-1$ ($l=-J$) if $J\geq\frac{1}{2}$ ($J\leq-\frac{1}{2}$). The same thing is true for the lower-component equation with $l=J$ ($l=-J-1$) if $J\geq\frac{1}{2}$ ($J\leq-\frac{1}{2}$).

From the boundary value of the gauge field vortex solution at long distance [Eq. (A2)] one obtains, for the asymptotic behavior of the effective potential,

$$U_{nJ}^\pm(\rho)=\frac{(J-n)(J-n\pm 1)}{\rho^2}+O(e^{-\mu\rho})\,. \tag{2.13}$$

Comparison of this formula with the effective potential for a free fermion [Eq. (2.11)] explicitly shows the centrifugal barrier modification produced by the gauge field of a vortex line. This implies that the interaction potential $V_{nJ}^\pm(\rho)$ decreases like ρ^{-2} for $\rho\gg\mu^{-1}$.

At short distances from the center of the vortex line, the fermion-vortex interaction is regular. The interaction potential, for small values of ρ, can be written as

$$\begin{aligned}V_{nJ}^\pm(\rho)=&-e(J\pm\tfrac{1}{2})H(0)\\&+\mu^2\{(d_1^n\mu\rho)^2+d_2^n[2J\pm(2n+1)](\mu\rho)^{2|n|}\}\\&+O((\mu\rho)^{2|n|+2})\,,\end{aligned} \tag{2.14}$$

where we have used the short-distance behavior of the classical vortex solution [Eqs. (A1) and (2.4)]. One sees clearly that the fermion-vortex interaction near the vortex center is entirely due to the magnetic moment.

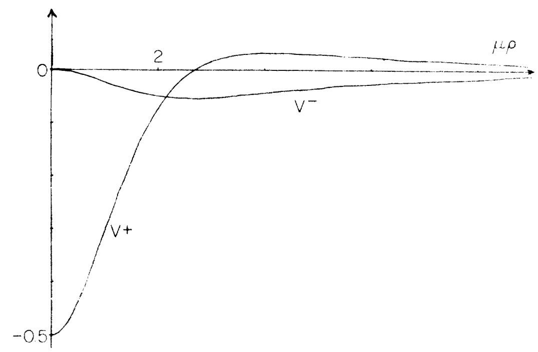

FIG 1. Interaction potentials $V_{1,1/2}^+$ and $V_{1,1/2}^-$ between a fermion and a one-quantum vortex in the wave $J=\frac{1}{2}$. The potential has been obtained from the Abelian vortex solution given in Ref. 18 for $\lambda=e^2$ and it is plotted in units of μ^2.

The second term in Eq. (2.20) can be recast in a more familiar form by using the well-known fact that the canonical momentum of a charged particle in a magnetic field is the sum of its mechanical one plus eA_μ. In our case only the azimuthal canonical momentum differs from the mechanical one. It is given by

$$P_\varphi=p_\varphi+eA_\varphi=p_\varphi+n-\mathcal{H}_n(\rho)\,, \tag{2.15}$$

where p_φ is the mechanical momentum (which is not conserved).

Although our wave function [Eq. (2.7)] is not an eigenstate of

$$\hat{p}_\varphi=\frac{1}{i}\frac{\partial}{\partial\varphi}\,,$$

their upper and lower components are indeed eigenfunctions of it with $(J\mp\frac{1}{2})$ as eigenvalues. Then

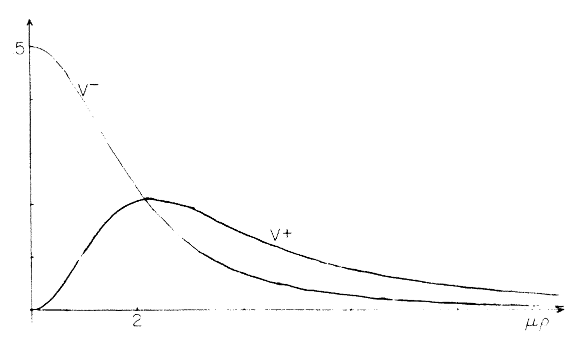

FIG. 2. Same as Fig. 1, but for the wave $J=-\frac{1}{2}$.

$$p_\varphi = J - n + \mathcal{K}_n(\rho) \mp \tfrac{1}{2}\,. \tag{2.16}$$

Taking into account this and the expression for the magnetic field [Eq. (2.4)], the effective potential can be written as

$$U^{\pm}_{nJ}(\rho) = \mp\, eH(\rho) + \frac{p_\varphi{}^2 - \frac{1}{4}}{\rho^2}\,. \tag{2.17}$$

The second term of Eq. (2.17) has the form of the ordinary centrifugal barrier. Then, one can interpret the long-range $1/\rho^2$ interaction as coming from the "minimal gauge coupling" performed on the centrifugal barrier.

Let us now consider the radial solutions of the Dirac equation in the vortex field. We normalize the regular wave functions such that, at small distances,

$$\begin{aligned}\varphi^J_{\pm}(k,\rho) &= \rho^{|J|}\left[1 - \frac{(k\rho)^2}{2(1+2|J|)} + O(\rho^4)\right],\\ \varphi^J_{\mp}(k,\rho) &= \rho^{1+|J|}\,\frac{i(\omega \mp m)}{1+2|J|}\,[1 + O(\rho^2)]\,,\end{aligned} \tag{2.18}$$

where $\pm J > \frac{1}{2}$.

The regular solutions for $\omega = \pm m$ are given by

$$\begin{aligned}\varphi^J_{\pm}(0,\rho) &= \left[\rho^{J-n}\,\frac{f_n(\rho)}{L_n}\right]^{\pm 1},\\ \varphi^J_{\mp}(0,\rho) &= 0 \quad \text{if } \pm J \geq \tfrac{1}{2}\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}\varphi^J_{\pm}(0,\rho) &= 2m[iL_n\rho^{J-n}f_n(\rho)]^{\pm 1}\\ &\quad\times \int_0^\rho \frac{d\rho'}{[\rho'^{J-n}f_n(\rho')]^{\pm 2}}\,,\\ \varphi^J_{\mp}(0,\rho) &= \left[\frac{L_n}{\rho^{J-n}f_n(\rho)}\right]^{\pm 1} \quad \text{for } \mp J \geq \tfrac{1}{2}\,.\end{aligned} \tag{2.20}$$

Here

$$f_n(\rho) \equiv \exp\left[-\int_\rho^\infty \frac{d\rho'}{\rho'}\,\mathcal{K}_n(\rho')\right]. \tag{2.21}$$

Of course, the set of negative-energy solutions can be obtained by charge-conjugation of the set of positive-energy solutions.

All the solutions at the threshold given in Eqs. (2.19)–(2.20) are regular in the sense that they are locally square-integrable. Among them, those that decrease sufficiently fast for $\rho \to \infty$ are also bound states. The asymptotic behavior of the functions $\varphi^{\pm}_J(0,\rho)$ can easily be obtained from Eqs. (2.19)–(2.20) and the Appendix. We therefore find for $n \geq 1$ ($n \leq -1$) that the positive- (negative-) frequency radial function given by Eq. (2.19) is normalizable if

$$\tfrac{1}{2} \leq J\,\mathrm{sgn}(n) \leq |n| - \tfrac{3}{2}\,. \tag{2.22}$$

The wave functions corresponding to $J\,\mathrm{sgn}(n) \leq -\frac{1}{2}$ [Eq. (2.20)] are not normalizable.

Thus, vortices with positive (negative) topological number n bind only fermions (antifermions) at the threshold. The total number of bound states is $|n| - 1$ in both cases. For $|n| = 1$ there are no normalizable solutions. If $n = 2$ ($n = -2$) there is only one at $\omega = m$ ($\omega = -m$) with $J = \frac{1}{2}$ ($J = -\frac{1}{2}$).

The condition (2.22) that ensures the presence of bound states can be physically understood by analyzing the strength of the modified centrifugal barrier and the magnetic-moment interactions (at short distances).

The centrifugal barrier, modified by the direct coupling with the vector potential [Eq. (2.13)] can be written as

$$U^{\pm}_{nJ}(\rho) = \frac{(J - n \mp \frac{1}{2})^2 - \frac{1}{4}}{\rho^2} + O(e^{-\mu\rho})\,. \tag{2.23}$$

If there are bound states at the threshold, for the case $n \geq 1$, we find from Eq. (2.22)

$$(J - n + \tfrac{1}{2})^2 - \tfrac{1}{4} \geq \tfrac{3}{4} \quad \text{and} \quad (J - n - \tfrac{1}{2})^2 - \tfrac{1}{4} \geq \tfrac{7}{4}\,. \tag{2.24}$$

For $n \leq -1$ one reaches similar conclusions.

We conclude then that bound states at the threshold are present if the magnetic-moment interaction is attractive and if the modified centrifugal barrier is repulsive enough in order to keep the fermion near the center of the vortex line.

We wish to make a final remark about the gauge invariance of our preceding analysis. It is clear that the particular form of the effective fermion-vortex potential [Eq. (2.10)] depends on the gauge chosen. However, the long-range ρ^{-2} interaction will be present in all gauges because $A_\varphi(\rho = \infty, \varphi)$ cannot vanish identically after any regular gauge transformation.

III. SCATTERING OF FERMIONS BY STATIC VORTEX

In the small-coupling limit, the scattering solutions of the Dirac equation in the gauge field of a vortex [Eq. (2.5)] describe the quantum scattering of fermions by a vortex line located at the origin. This follows from their relation with the Dirac-field matrix elements between a vortex state and a vortex-plus-one-fermion state (Sec. IV). Recoil effects from the vortex will be neglected, which is consistent with its large mass $[O(1/e^2)]$.

We consider positive-energy fermions inciding along the x axis with momentum k. Then the asymptotic behavior of the wave function must be

$$\psi_k(\vec{\rho},t)=\begin{pmatrix}1\\ \left(\frac{\omega-m}{\omega+m}\right)^{1/2}\end{pmatrix}e^{i(kx-\omega t)}+\begin{pmatrix}e^{-i\varphi/2}\\ \left(\frac{\omega-m}{\omega+m}\right)^{1/2}e^{i\varphi/2}\end{pmatrix}\frac{f(\varphi)}{\sqrt{\rho}}e^{i(k\rho-\omega t)}, \tag{3.1}$$

where $f(\varphi)$ are the scattering amplitude and $\omega=+(k^2+m^2)^{1/2}$. It can be pointed out that in two-dimensional space the spin is not an independent degree of freedom for a fermion. The quantum state of the fermion can be completely specified by its momentum and sign of the energy.

It is convenient to expand the scattering solution into partial waves

$$\psi_k(\vec{\rho},t)=\left(\frac{2}{i\pi k\rho}\right)^{1/2}\sum_{\substack{J=-\infty\\(J,\text{ half integer})}}^{+\infty} i^J e^{i[J\varphi-\omega t+\delta_J(k)]}\begin{pmatrix}(-1)^{(J-1/2)\theta(-J)}e^{-i\varphi/2}\mu_+^J(k,\rho)\\ i\left(\frac{\omega-m}{\omega+m}\right)^{1/2}(-1)^{(J+1/2)\theta(-J-1)}e^{+i\varphi/2}\mu_-^J(k,\rho)\end{pmatrix}. \tag{3.2}$$

Here $\mu_\pm^J(k,\rho)$ are radial solutions of Eq. (2.9), regular at the origin and with asymptotic behavior

$$\mu_\pm^J(k,\rho)\underset{\rho\to\infty}{\sim}\sin\{k\rho-[\,|J|-\theta(\pm J)]\tfrac{1}{2}\pi+\delta_J(k)\}. \tag{3.3}$$

The scattering amplitude can also be expanded in partial waves

$$f(\varphi)=\frac{1}{(2\pi ik)^{1/2}}\sum_{\substack{J=-\infty\\(J,\text{ half integer})}}^{+\infty} e^{iJ\varphi}(e^{2i\delta_J(k)}-1). \tag{3.4}$$

The scattering solutions $\mu_\pm^J(k,\rho)$ are of course proportional to the regular solutions $\varphi_\pm^J(k,\rho)$ defined in Sec. II [Eq. (2.18)]. The proportionality factor can be related with the Jost functions which are defined as the following Wronskians[11]:

$$\mathfrak{F}_J(k)\equiv\frac{i^{1/2-|J-1/2|}}{(|2J-1|)!!}k^{|J-1/2|-1/2}\times W[f_J(k,\rho),\varphi_+^J(k,\rho)].$$

Here $f_J(k,\rho)$ is the Jost solution (irregular) of the radial equation (2.9). It verifies

$$\lim_{\rho\to\infty}e^{-ik\rho}f_J(k,\rho)=1.$$

Thus, we can write the following relations:

$$\mu_+^J(k,\rho)=\frac{k^{|J-1/2|+1/2}}{|\mathfrak{F}_J(k)|(2|J-\frac{1}{2}|)!!}\varphi_+^J(k,\rho). \tag{3.5}$$

Here

$$(2l+1)!!\equiv\frac{2^{l+1}}{\sqrt{\pi}}\Gamma(l+\tfrac{1}{2}),$$

$$\mathfrak{F}_J(k)=|\mathfrak{F}_J(k)|e^{-i\delta_J(k)}. \tag{3.6}$$

We have considered the radial solutions for two limiting cases: $k^2\gg\mu^2$ and $k^2\ll\mu^2$. We recall that the order of magnitude of the fermion-vortex interaction is given by $V_{nJ}\sim\mu^2$.

In the high-energy region the phase shifts can be obtained by the Born approximation, which in our two-spatial dimension case gives

$$\sin\delta_J^{\rm Born}(k)=-\frac{\pi}{2}\int_0^\infty\rho\,d\rho[J_{J-1/2}(k\rho)]^2V_{nJ}^+(\rho). \tag{3.7}$$

This is a convergent integral because of the properties of the interaction potential. For $k^2\to\infty$ and J fixed, one can approximate[11] this expression by the simpler one

$$\delta_J^{\rm Born}(k)\simeq-\frac{1}{2k}\int_0^\infty V_{nJ}^+(\rho)d\rho=\frac{\mu}{k}\left[\frac{1}{2}\int_0^\infty\mathcal{H}_n(\alpha)d\alpha-(J-\tfrac{3}{2})\int_0^\infty\frac{d\mathcal{H}_n}{d\alpha}\frac{d\alpha}{\alpha}\right]. \tag{3.8}$$

We conclude that, for high energies, the phases shifts tend to zero like μ/k.

Let us now study the behavior for small energies. In order to obtain the phase shifts and Jost functions for small k, we will consider the regular radial solutions first. They will be obtained as a Born series around the exact solutions.

The knowledge of the regular solutions in powers of k^2 is not sufficient to get the phase shifts. However, we know that for large distances ($\mu\rho\gg1$) the effective potential can be approximated by the modified centrifugal barrier [Eq. (2.13)]. Then the radial solutions can be approximated by the solution of the following equation:

$$\left[\frac{d^2}{d\rho^2}+k^2-\frac{(J-n)(J-n-1)}{\rho^2}\right]\mu_{\rm as}^J(k,\rho)=0, \tag{3.9}$$

which is exactly solvable in terms of Bessel functions. This function $\mu_{\rm as}^J(k,\rho)$ differs from $\mu_+^J(k,\rho)$ in quantities of order $e^{-\mu\rho}$ for large distances.

The final step to obtain the phase shifts and Jost functions consists in matching the regular solutions for $\mu\rho \gg 1$ and (fixed) small k with the functions $\mu_{as}^{J}(k,\rho)$ for small k and fixed p.[19]

Let us now return to consider the regular solutions. The radial differential equation (2.9) plus the boundary conditions are equivalent to the following integral equation (for $J \geq \frac{1}{2}$):

$$\varphi^J(k,p) = \rho^{J-n}\frac{f_n(\rho)}{L_n} - k^2\int_0^\rho g_J(\rho,\rho')\varphi^J(k,\rho')d\rho' , \tag{3.10}$$

where $\theta(\rho-\rho')g_J(\rho,\rho')$ is the Green's function of Eq. (2.9) at $k^2=0$. It can be written as

$$g_J(\rho,\rho') = \frac{f_n(\rho)f_n(\rho')}{(\rho\rho')^{n-J}}\int_{\rho'}^{\rho}\frac{d\rho''}{[\rho''^{J-n}f_n(\rho'')]^2} . \tag{3.11}$$

By iterating Eq. (3.10), we get

$$\varphi^J(k,\rho) = \frac{\rho^{J-n}f_n(\rho)}{L_n}\left\{1 - k^2\int_0^\rho d\rho'\left[\frac{f_n(\rho')}{\rho'^{n-J}}\right]^2\int_{\rho'}^{\rho}\frac{d\rho''}{[\rho''^{J-n}f_n(\rho'')]^2} + O(k^4)\right\} . \tag{3.12}$$

By partial integration from Eq. (3.12), one can easily get the large-distance behavior ($\mu\rho \gg 1$) of the regular radial solutions at fixed (small) k^2, taking into account the Appendix and Eq. (2.21). Then we can use Eq. (3.5) to relate the behavior for $\mu\rho \gg 1$ of these regular solutions with the solutions $\mu_{as}^{J}(k,\rho)$ taken at small k and fixed ρ.

After some algebra, we reach the following results for the phase shifts:

$$\delta_J(k) = -\frac{n\pi}{2} + \frac{\pi k^{2(n-J)+1}2^{2(J-n)}}{\Gamma(n+\frac{1}{2}-J)\Gamma(n+\frac{3}{2}-J)}\int_0^\infty \frac{\mathcal{K}_n(\rho)d\rho}{[\rho^{J-n}f_n(\rho)]^2}[1+O(k^2)] \quad \text{for } J \leq -\tfrac{1}{2} , \tag{3.13}$$

$$\delta_J(k) = \frac{\pi}{2}(2J+1-n) - \frac{\pi k^{2(n-J)-1}2^{2(J-n)+1}[1+O(k^2)]}{\Gamma(n+\frac{1}{2}-J)\Gamma(n-\frac{1}{2}-J)\int_0^\infty [\rho^{J-n}f_n(\rho)]^2\mathcal{K}_n d\rho} \quad \text{for } \tfrac{1}{2} \leq J \leq n-\tfrac{3}{2} \text{ if } n \geq 2 , \tag{3.14}$$

$$\delta_{n-1/2}(k) = \frac{n\pi}{2} - \frac{(\pi/2)[1+O(k^2\ln k)]}{\ln(2/k) - C - 2\int_0^\infty(\ln\rho/\rho)f_n(\rho)^2\mathcal{K}_n(\rho)d\rho} , \tag{3.15}$$

$$\delta_J(k) = \frac{n\pi}{2} - \frac{\pi k^{2(J-n)+1}\int_0^\infty[\rho^{J-n}f_n(\rho)]^2\mathcal{K}_n(\rho)d\rho[1+O(k^2)]}{2^{2(J-n)}\Gamma(J-n+\frac{1}{2})\Gamma(J-n+\frac{3}{2})} \tag{3.16}$$

for $J \geq n+\frac{1}{2}$. Here C stands for the Euler-Mascheroni constant. We have taken the normalization

$$\delta_J(\infty) = 0 . \tag{3.17}$$

In the process of the derivation of Eqs. (3.13)–(3.16) we have used the relation between the phase shift at $k=0$ and the small-energy behavior of the Jost function

$$\delta_J(0) = \tfrac{1}{2}\pi q \quad \text{if } \mathcal{F}_J(k) \simeq k^q(\ln k)^{q'} \text{ for } k \to 0 .$$

The modulus of the Jost function has a simple physical interpretation.[20] It gives the ratio between the free wave function and the wave function in the presence of the interaction at the origin, i.e., the center of the vortex. For this reason

$$A_J \equiv \frac{1}{|\mathcal{F}_J(k)|^2}$$

is called the enhancement factor. We find for the enhancement factor at low fermion momenta

$$A_J(k) = \begin{cases} k^{2n}, & J \leq -\frac{1}{2} \\ k^{2(n-2J-1)}, & \frac{1}{2} \leq J \leq n-\frac{3}{2} \\ k^{-2n}\ln^{-2}(k/\Lambda_n), & J = n-\frac{1}{2} \\ k^{-2n}, & J \geq n+\frac{1}{2} . \end{cases} \tag{3.18}$$

We see that the probability of finding a low-energy fermion near the center of the vortex is reduced for $J<(n-1)/2$ $[A_J(0)=0]$ and enhanced for $J>(n-1)/2$ $[A_J(0)=\infty]$.

From the results given in Eqs. (3.13)–(3.16) we can get some conclusions on the fermion-vortex scattering at low fermion momenta. For $J \neq n-\frac{1}{2}$, we can resume the low-momentum behavior of the S matrix as

$$e^{2i\delta_J(k)} = (-1)^n + D_J^n\left(\frac{e^2}{\lambda}\right)\left(\frac{k}{\mu}\right)^{2|n-J-\text{sgn}(J)/2|} \times [1+O(k^2)] , \tag{3.19}$$

where the coefficients $D_J^n(e^2/\lambda)$ can be easily obtained from Eqs. (3.13)–(3.16). For the case $J=n-\frac{1}{2}$, it follows from Eq. (3.15) that

$$e^{2i\delta_{n-1/2}(k)} = (-1)^n\left[1 - \frac{i\pi}{\ln(\Lambda_n/k) + O(k^2\ln k)}\right] , \tag{3.20}$$

where

$$\Lambda_n = 2\mu\exp\left[-2\int_0^\infty \frac{\ln\alpha}{\alpha}f_n(\alpha)^2\mathcal{K}_n(\alpha)d\alpha - C\right] . \tag{3.21}$$

It must be pointed out that for odd n all phase shifts attain the unitarity maximum at $k=0$. This produces a δ-function behavior in the forward scattering amplitude. In other words, we have a zero-energy resonance in all partial waves.

For $\varphi\neq 0$ the scattering for small k is dominated by the $J=n-\frac{1}{2}$ wave. From Eq. (3.19) we see that all other partial amplitudes decrease faster for $k^2\to 0$. More precisely, the absolute value of the difference $J-n$ determines how fast the phase shift attains its value at $k=0$.

Finally we find that the cross section diverges when the fermion momentum goes to zero as

$$\frac{d\sigma}{d\varphi}=|f(\varphi)|^2=\frac{\pi}{2k\ln^2(k/\Lambda_n)}[1+O(k^2\ln k)]\ . \tag{3.22}$$

IV. FIELD-THEORETICAL ASPECTS

In this section we consider the quantum fields in the one-vortex sectors of the theory when fermions are present.

The Dirac field in a one-vortex sector can be appropriately treated by introducing a collective coordinate[3] $\vec{R}$ associated with the vortex position and its canonical conjugate $\vec{P}$, which corresponds to the vortex momentum.

Then, the fermonic field can be expanded in the eigenfunctions of the Dirac equation in the classical field of the n-quantum vortex solution. This set of eigenfunctions contains a discrete part (if $|n|\geq 2$) formed by the bound states at the threshold and a continuum part corresponding to the scattering solutions.

Thus we write the Dirac field[3,7] in the n-quantum vortex sector as

$$\psi(\vec{\rho},t)=\Xi_n(\vec{\rho}-\vec{R}(t),t)+\hat{\psi}_n(\vec{\rho}-\vec{R}(t),t)\ . \tag{4.1}$$

Here the first term contains the discrete eigenfunctions and the second one corresponds to the continuum states. Explicitly,

$$\Xi_n(\vec{x})=\sum_{J=1/2}^{n-3/2}a_{nJ}\xi_{nJ}(\vec{x})\quad\text{for } n\geq 2\ , \tag{4.2}$$

$$\Xi_n(\vec{x})=\sum_{J=1/2}^{|n|-3/2}a^\dagger_{nJ}\eta_{nJ}(\vec{x})\quad\text{for } n\leq -2\ , \tag{4.3}$$

$$\Xi_{\pm 1}\equiv 0\ .$$

Here the normalized bound-state eigenfunctions are

$$\xi_{nJ}(\vec{x})=\frac{f_n(\rho)}{N_{nJ}\,\rho^{n-J+1/2}}\begin{pmatrix}1\\0\end{pmatrix}e^{i(J-1/2)\varphi}\quad\text{for } n\geq 2 \tag{4.4}$$

and

$$\eta_{nJ}(\vec{x})=C\,\xi^*_{-n,\,J}(\vec{x})\quad\text{for } n\leq -2 \tag{4.5}$$

with $C=\gamma^0\gamma^1$ and where we have used Eq. (2.19). N_{nJ} is a normalization constant. We recall that ξ_{nJ} (η_{nJ}) are positive- (negative-) energy eigenfunctions. Thus, the coefficient of ξ_{nJ} in Eq. (4.2) is an annihilation operator and the coefficient of η_{nJ} in Eq. (4.3) is a creation one. These operators annihilate and create fermion-vortex (antifermion-vortex) bound states at the threshold for $n\geq 2$ ($n\leq -2$). They verify the relations

$$\{a_{nJ},a^\dagger_{nJ'}\}=\delta_{JJ'},\quad \{a_{nJ},a_{nJ'}\}=\{a^\dagger_{nJ},a^\dagger_{nJ'}\}=0\ . \tag{4.6}$$

Then, for $n\geq 2$,

$$\{\Xi^\dagger_n(\vec{x}),\Xi_n(\vec{y})\}=\sum_{J=1/2}^{n-3/2}\xi^\dagger_{nJ}(\vec{x})\xi_{nJ}(\vec{y})\ , \tag{4.7}$$

and we have an analogous relation for $n\leq -2$.

The second piece of the Dirac field in the n-quantum vortex sector [Eq. (4.1)] verifies

$$\int d^2x\,\hat{\psi}_n(\vec{x},t)\Xi_n(\vec{x},t)=0\ .$$

It can be expanded in the scattering solutions of Dirac equation (2.5)

$$\hat{\psi}_n(\vec{x})=\int\frac{d^2k}{(2\pi)^2}[b_n(\vec{k})\psi^{(+)}_{n,\vec{k}}(\vec{x})+d^\dagger_n(\vec{k})\psi^{(-)}_{n,\vec{k}}(\vec{x})]\ . \tag{4.8}$$

The operators $b_n(\vec{k})$ [$d_n(\vec{k})$] correspond to annihilation [creation] of fermion-vortex [antifermion-vortex] scattering states. These operators verify, as usual

$$\{b_n(\vec{k}),b^\dagger_n(\vec{k}')\}=\{d_n(\vec{k}),d^\dagger_n(\vec{k}')\}=\delta(\vec{k}-\vec{k}')\ ; \tag{4.9}$$

all other anticommutators vanish. The anticommutation relations (4.6) and (4.9) together with the completeness of the eigenfunctions of the Dirac equation ensure that the canonical equal-time anticommutation relations for the Dirac field hold. The wave function $\psi^{(+)}_{n\vec{k}}$ in Eq. (4.8) corresponds to a scattering solution of Dirac equation (2.5) for a fermion with incident momentum $\vec{k}$. The $\psi^{(-)}_{n\vec{k}}$ are negative-energy scattering solutions of Dirac equation (2.7) with incident fermion momentum $(-\vec{k})$.

The quantum field theory is invariant under charge conjugation. Under this discrete symmetry

$$\begin{aligned}&\mathcal{C}\psi(x)\mathcal{C}^{-1}=C\psi(x)^*\ ,\\&\mathcal{C}A_\mu(x)\mathcal{C}^{-1}=-A_\mu(x)\ .\end{aligned} \tag{4.10}$$

Because n goes in $(-n)$ under $\mathcal{C}$, the n-quantum vortex sector must go into the $(-n)$-quantum sector. Then, it must be true that

$$\begin{aligned}&\mathcal{C}a_{nJ}\mathcal{C}^{-1}=a_{-nJ}\ ,\\&\mathcal{C}b_n(\vec{k})\mathcal{C}^{-1}=d_{-n}(\vec{k})\ ,\\&\mathcal{C}d_n(\vec{k})\mathcal{C}^{-1}=b_{-n}(\vec{k})\ .\end{aligned} \tag{4.11}$$

Now let us consider the charge-conjugation properties of the fermion-vortex states. An n-quantum vortex-fermion bound state, with angular momentum J, can be written as

$$|n\vec{P}J\rangle = a_{nJ}^{\dagger}|n\vec{P}\rangle . \tag{4.12}$$

Here $|n\vec{P}\rangle$ stands for an n-quantum vortex state of momentum $\vec{P}$. Then by charge conjugation, an n-quantum vortex-fermion bound state becomes a $(-n)$-quantum vortex-antifermion bound state

$$\mathcal{C}|n\vec{P}J\rangle = a_{-nJ}^{\dagger}\mathcal{C}|n\vec{P}\rangle = a_{-nJ}^{\dagger}|-n,\vec{P}\rangle = |-n,\vec{P},J\rangle . \tag{4.13}$$

One can easily calculate the energy of the fermion-vortex and antifermion-vortex bound states at leading order. The fermionic part of the Hamiltonian can be written as

$$H_F(t) = \int d^2x \colon\ \psi^{\dagger}(\vec{x})[-\vec{\sigma}\cdot(i\vec{\nabla}+e\vec{A})+m\sigma_z]\psi(\vec{x})\colon . \tag{4.14}$$

Thus, the fermion-vortex or antifermion-vortex bound-state energy is given at leading order by

$$\begin{aligned} E_{PJm} &= \langle n\vec{P}J|H|n\vec{P}J\rangle \\ &= \langle n\vec{P}|a_{nJ}Ha_{nJ}^{\dagger}|n\vec{P}\rangle \\ &= E_n(P) + \{a_{nJ},[H_F,a_{nJ}^{\dagger}]\}, \end{aligned} \tag{4.15}$$

where H is the total Hamiltonian and $E_n(P)$ the energy of an n-quantum vortex with momentum $\vec{P}$ at leading order. The anticommutator in the preceding equation can easily be evaluated using Eqs. (4.1)–(4.8) and the Dirac equation for the solutions at the threshold. One gets the same result for a positive as well as for a negative n

$$E_{PJn} = E_n(P) + m \tag{4.16}$$

as expected. This energy value will be corrected by higher-order contributions. It is not clear whether that shift will be equal to the quantum corrections to the vortex mass. In other words, higher-order quantum corrections perhaps may split the fermion-vortex bound state from the fermion-vortex threshold.

Fermionic number is a conserved magnitude in the quantum theory considered here. Thus, it is interesting to know what its value is for the quantum states $|\vec{P}n\rangle$ and $|\vec{P}nJ\rangle$.

As it is well known, if we agree to assign zero fermion number to the vacuum state, the fermionic vacuum operator can be written as

$$N_F = \tfrac{1}{2}\int d^2x\,[\psi^{\dagger}(\vec{x})\psi(\vec{x}) - \psi(\vec{x})\psi^{\dagger}(\vec{x})] . \tag{4.17}$$

Then, the fermionic number value for an n-quantum vortex state is given by

$$\begin{aligned} \langle n\vec{P}|N_F|n\vec{P}\rangle &= -\tfrac{1}{2}\int d^2x\,\langle n\vec{P}|\Xi_n(\vec{x})\Xi_n^{\dagger}(\vec{x})|n\vec{P}\rangle \\ &= -\tfrac{1}{2}\int d^2x \sum_{J=1/2}^{n-3/2}\xi_{nJ}^{\dagger}(\vec{x})\xi_{nJ}(\vec{x}) \\ &= \frac{1-n}{2}\quad \text{if } n \geq 1 . \end{aligned} \tag{4.18}$$

Here we have used the relation

$$\int d^2x\,\langle n\vec{P}|\hat{\psi}^{\dagger}(\vec{x})\hat{\psi}(\vec{x})|n\vec{P}\rangle = \int d^2x\,\langle n\vec{P}|\hat{\psi}(\vec{x})\hat{\psi}^{\dagger}(\vec{x})|n\vec{P}\rangle . \tag{4.19}$$

In a similar way, we find for a fermion-vortex bound state

$$\begin{aligned} \langle n\vec{P}J|N_F|n\vec{P}J\rangle &= \tfrac{1}{2}\int d^2x\,\langle n\vec{P}|a_{nJ}(\Xi_n^{\dagger}\Xi_n - \Xi_n\Xi_n^{\dagger})a_{nJ}^{\dagger}|n\vec{P}\rangle \\ &= (3-n)/2 , \end{aligned} \tag{4.20}$$

where Eq. (4.19) has also been used. For the fermion-vortex bound states we find that N_F is one plus the result for a vortex, as expected.

The fermionic number for vortices with negative n as well as for antifermion-vortex bound states can easily be obtained by charge conjugation of the results found for positive n. In a compact notation

$$N_F(n\text{-quantum vortex}) = \tfrac{1}{2}[\operatorname{sgn}(n) - n] . \tag{4.21}$$

For a fermion-vortex or antifermion-vortex bound state, one must simply add $+\operatorname{sgn}(n)$ to this expression.

It is now clear that fermion number conservation forbids the decay of a two-quantum vortex state ($n_F = -\frac{1}{2}$) into two vortices, each with one unit of flux ($n_F = 0$). This decay would be allowed by the topological conservation law and also by energy conservation in the particular case $\lambda = e^2$.[18] By analogous arguments the vortex state with $n = -2$ must be stable.

According to the topological and fermionic number conservation laws, a vortex with $2k+2$ units of flux (k = positive integer) could decay into a two-quantum vortex, k antifermions, and $2k$ one-quantum vortices. In a similar way, a $(2k+1)$-quantum vortex could decay into $(2k+1)$ one-quantum vortices and k antifermions.

The Dirac field matrix elements between vortex states and vortex-fermion bound states can easily be compared using the collective-coordinate expressions for the fermionic fields given by Eqs. (4.1)–(4.3). Then, at leading order in e^2 and λ

$$\langle n\vec{P}'|\psi(\vec{x})|n\vec{P}\vec{J}\rangle = \int d^2x'\,e^{i(\vec{p}-\vec{p}')\cdot\vec{x}'} \times \xi_{nJ}(\vec{x}-\vec{x}') , \tag{4.22}$$

where $\frac{1}{2} \leq J \leq n - \frac{3}{2}$. One can also easily obtain one-fermion matrix elements at leading order:

$$\langle n\vec{P}'|\psi(\vec{x})|n\vec{P},\vec{k}\rangle = \int d^2x' e^{i(\vec{p}-\vec{p}')\cdot\vec{x}'} \times \psi_{n\vec{k}}^{(+)}(\vec{x}-\vec{x}'), \qquad (4.23)$$

where $|n\vec{P},\vec{k}\rangle$ stands for a fermion-vortex scattering state, where $\vec{k}$ is the incident fermion momentum.

Taking into account expression (4.4) the matrix element given by Eq. (4.22) can be written as

$$\langle n\vec{P}|\psi(\vec{x})|n\vec{P}J\rangle = e^{-i\vec{q}\cdot\vec{x}} e^{i(J-1/2)(\beta+\pi/2)} \times F_J(|\vec{q}|)\begin{pmatrix}1\\0\end{pmatrix}. \qquad (4.24)$$

Here $\vec{q}=\vec{p}-\vec{p}'$, β is the polar angle of $\vec{q}$, and

$$F_J(q) \equiv \int_0^\infty J_{J-1/2}(q\rho) f_n(\rho)\rho^{J-n+1/2} d\rho\,. \qquad (4.25)$$

The behavior of the function $f_n(\rho)$ for large ρ determines the position of the singularities of the fermion-vortex form factor as a function of $t=-\vec{q}^{\,2}$. This behavior is known for the particular case $\lambda=e^2$ (Ref. 18):

$$f_n(\rho) = 1 + \sum_{s=1}^\infty c_s \frac{e^{-s\rho}}{\rho^{P_s}}[1 + d_s/\rho + O(1/\rho^2)]\,. \qquad (4.26)$$

Here c_s, d_s, P_s are numerical coefficients. This asymptotic expansion for f_n and the large argument behavior of the Bessel functions shows that the fermion-vortex form factors have singularities when $t=(s\mu)^2$, $s=1,2,3,\ldots$.

Finally it can be pointed out that if the fermions are massless, the n-quantum vortex state and the corresponding fermion-vortex bound state become degenerate. In this particular case, one can interpret the vortex states in the presence of fermions *à la* Jackiw-Rebbi,[7] that is, as a doublet of degenerate states with different fermionic numbers. The fermionic numbers differ by one unit between them. However, one can still consider the two states like in the massive fermion case, that is, as a vortex and a fermion-vortex state of the same mass. Both interpretations seem plausible up to this level.

V. VORTEX LINES IN NON-ABELIAN GAUGE THEORIES

Let us consider the topological stability of vortex solutions of a classical gauge theory with symmetry group G and spontaneous symmetry breaking, via a Higgs field, such that the vacuum is invariant under a subgroup H of G. Then, in order to have topologically stable static solutions in two space dimensions, $\pi_1(G/H)$ must have more than one element.[9] That is, there must be homotopically inequivalent mappings from the quotient group G/H to the circumference at $\rho=\infty$ in ordinary space. In the Abelian case[4,9] $G=\mathrm{U}(1)$, $H=\phi$, then $\pi_1(G/H)=\pi_1(\mathrm{U}(1))=Z$ (the set of integers). There is, in this case, an infinite number of topologically stable vortices labeled by the integer n. For $n=0$ we have the vacuum.

By analogy with the Abelian case, for a non-Abelian gauge field coupled to a Higgs field in the adjoint representation, we suppose that the spontaneous symmetry breaking is maximum. By maximum we understand that the vacuum is invariant only under the unit matrix in the adjoint representation. Then it will be invariant only under elements of G that are mapped onto the unit matrix of the adjoint representation. If $G=\mathrm{SU}(n)$ there are only n elements with such a property. In the fundamental representation they have the following form:

$$I_n e^{2\pi i k/n}, \quad k=0,1,\ldots,n-1, \qquad (5.1)$$

where I_n stands for the $n\times n$ unit matrix. In this case $\pi_1(\mathrm{SU}(n)/H)=Z_n$ and we have $(n-1)$ topologically stable vortex solutions, besides the vacuum solution.[6]

In SU(2) gauge theory a triplet of Higgs fields is clearly not sufficient to get the maximum symmetry breaking. If one introduces two Higgs fields, say $\vec{\phi}$ and $\vec{\chi}$,[4] such that for the vacuum solution $\vec{\phi}_0$ is not parallel to $\vec{\chi}_0$ in isospace, we have $\pi_1(\mathrm{SU}(2)/H)=Z_2$. In other words, we have one topologically stable vortex solution.[6]

We consider the more general SU(2)-invariant and renormalizable (in four-dimensional space-time) Lagrangian density

$$\mathcal{L}(x) = -\tfrac{1}{4}(\vec{F}_{\mu\gamma})^2 + \tfrac{1}{2}(\vec{D}_\mu\vec{\phi})^2 + \tfrac{1}{2}(\vec{D}_\mu\vec{\chi})^2 + \tfrac{1}{2}(\mu_1^2\vec{\phi}^2 + \mu_2^2\vec{\chi}^2) - \tfrac{1}{4}[\lambda_1(\vec{\phi}^2)^2 + \lambda_2(\vec{\chi}^2)^2] - \tfrac{1}{2}\beta(\vec{\phi}\cdot\vec{\chi})^2 - \tfrac{1}{2}\gamma\vec{\phi}^2\vec{\chi}^2\,. \qquad (5.2)$$

Here

$$\begin{aligned} \vec{F}_{\mu\nu} &= \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu + e\vec{A}_\mu\times\vec{A}_\nu\,,\\ \vec{D}_\mu\vec{\phi} &= \partial_\mu\vec{\phi} + e\vec{A}_\mu\times\vec{\phi}\,, \end{aligned} \qquad (5.3)$$

and $\vec{\phi}, \vec{\chi}, \vec{A}_\mu$ stand for isovector fields.

In order to get the maximum symmetry breaking, the parameters of the Lagrangian must verify

$$\lambda_2\left(\frac{\mu_1}{\mu_2}\right)^2 > \gamma + \beta < \lambda_1\left(\frac{\mu_2}{\mu_1}\right)^2, \quad \lambda_1>0,\ \lambda_2>0\,, \qquad (5.4)$$

which also ensures that the energy is bounded from below. One finds for the minimum energy classical configuration

$$\bar{\phi}_0\cdot\bar{\chi}_0=0\,, \tag{5.5}$$

$$\phi_0=|\bar{\phi}_0|=\left(\frac{\mu_1^2\lambda_2-\mu_2^2\gamma}{\lambda_1\lambda_2-\gamma^2}\right)^{1/2},$$
$$\chi_0=|\bar{\chi}_0|=\left(\frac{\mu_2^2\lambda_1-\mu_1^2\gamma}{\lambda_1\lambda_2-\gamma^2}\right)^{1/2}. \tag{5.6}$$

Let us now consider the vortex solution. The appropriate ansatz for the angular and isospin dependence of the fields is

$$\bar{\phi}(\vec{\rho})=f(\rho)\begin{pmatrix}\cos\varphi\\ \sin\varphi\\ 0\end{pmatrix},\quad \bar{\chi}(\vec{\rho})=g(\rho)\begin{pmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix}, \tag{5.7}$$

$$\vec{A}_a(\vec{\rho})=\hat{e}_\varphi\,\frac{1-H(\rho)}{e\rho}\delta_{a3},\quad A_0=0\,. \tag{5.8}$$

The boundary conditions are

$$f(\infty)=\phi_0,\quad g(\infty)=\chi_0,\quad H(\infty)=0,\quad H(0)=1\,.$$

The angular and isospin dependence given by Eqs. (5.7) can be obtained by imposing to the Higgs fields at large distances the following condition:

$$\left[\frac{1}{i}\frac{\partial}{\partial\varphi}+\frac{\sigma_3}{2},\Phi(\varphi)\right]=0\,,$$

where

$$\Phi(\varphi)=\tfrac{1}{2}\sum_{a=1}^{3}\sigma_a\phi_a(\vec{\rho})\Big|_{\rho=\infty}=\tfrac{1}{2}\bar{\sigma}\cdot\bar{\phi}(\vec{\rho})\big|_{\rho=\infty}\,,$$

in analogy with the monopole-type solutions.[7,8] Also, the ansatz for the gauge fields follows from the requirement

$$\bar{D}\bar{\phi}\,|_{\rho\to\infty}=\bar{D}\bar{\chi}\,|_{\rho\to\infty}=0\,.$$

The equations of motion of the fields are separated by the ansatz given by Eqs. (5.7), (5.8). After some algebra one gets

$$\frac{d^2f}{d\rho^2}+\frac{1}{\rho}\frac{df}{d\rho}+\frac{H(H-2)}{\rho^2}f+\mu_1^2f-\lambda_1f^3-\gamma fg^2=0\,,$$
$$\frac{d^2g}{d\rho^2}+\frac{1}{\rho}\frac{dg}{d\rho}+\frac{H(H-2)}{\rho^2}g+\mu_2^2g-\lambda_2g^3-\gamma gf^2=0\,, \tag{5.9}$$
$$\frac{d^2H}{d\rho^2}-\frac{1}{\rho}\frac{dH}{d\rho}-e^2H(f^2+g^2)=0\,.$$

From the preceding equation we get that a regular solution must verify

$$f(\rho)\sim\rho,\quad g(\rho)\sim\rho,\quad H(\rho)-1\sim\rho^2 \tag{5.10}$$

for small distances. That is, both Higgs fields have simple zeros at the center of the vortex line. As like in the Abelian case, we can write the asymptotic solutions

$$H(\rho)=Z\,\mu\rho K_1(\mu\rho)[1+O(e^{-\mu\rho})]\,,$$
$$f(\rho)=\phi_0[1+O(e^{-\mu_1\rho})]\,, \tag{5.11}$$
$$g(\rho)=\chi_0[1+O(e^{-\mu_1\rho})]\,,$$

where $\mu\equiv e(\phi_0^2+\chi_0^2)^{1/2}$ is the vector-boson mass.

The preceding relations allow us to make a qualitative picture of the field behavior: the gauge field tensor decreasing with a characteristic length μ^{-1} and the Higgs fields $\bar{\phi}$ and $\bar{\chi}$ increasing with characteristic lengths $(\mu_1)^{-1}$ and $(\mu_2)^{-1}$ from zero at the origin to their vacuum values at infinity, respectively.

Let us now consider fermions coupled to the bosonic fields considered up to now. For simplicity, we consider isospin-$\frac{1}{2}$ fermions. We couple them to the bosonic fields through the Lagrangian density

$$\mathcal{L}_F(x)=\bar{\psi}(i\partial\!\!\!/-\tfrac{1}{2}ie\tau_a A\!\!\!/_a-m)\psi(x)+\bar{\psi}\tau_a\psi(g_1\phi_a+g_2\chi_a)\,, \tag{5.12}$$

where τ_a are Pauli matrices acting on the isospin indices of the Dirac field.

As in the Abelian case, the c-number Dirac equation in the external potential given by the bosonic fields of the vortex is related to the small-coupling regime of the Dirac quantum field. Thus, we look for the solutions of

$$[i\partial\!\!\!/-m-\tfrac{1}{2}ie\tau_a A\!\!\!/_a+(g_1\phi_a+g_2\chi_a)\tau_a]\psi(x)=0\,, \tag{5.13}$$

where the $A\!\!\!/_a$, ϕ_a, and χ_a correspond to the classical vortex solution.

Because the non-Abelian vortex solution is invariant under space-plus-isospin rotations and not under either separately, the conserved angular momentum is given by

$$\hat{K}=\frac{1}{i}\frac{\partial}{\partial\varphi}+\tfrac{1}{2}\sigma_z+\tfrac{1}{2}\tau_3\,. \tag{5.14}$$

Here $\sigma_z/2$ and $\tau_3/2$ are the spin and isospin generators of rotations around the z axis (ordinary space) and 3 axis (isospace), respectively. As in the monopole field,[12,13] the angular momentum consists of the sum of an orbital part plus a spin part plus an isospin part.

We seek for stationary solutions of the Dirac equation in the field of the non-Abelian vortex [Eq. (5.13)] which are eigenstates of angular momentum. That is

$$\psi(\vec{\rho},t)=\frac{e^{i(K\varphi-\omega t)}}{\sqrt{\rho}}\begin{bmatrix}\varphi_1^+(\rho)e^{-i\varphi}\\ \varphi_1^-(\rho)\\ \varphi_2^+(\rho)\\ \varphi_2^-(\rho)e^{i\varphi}\end{bmatrix}. \tag{5.15}$$

Here (1.2) are the isospin indices whereas ± are the spin ones. K, the eigenvalue of the operator (5.14), can take any integer value. The spin $\frac{1}{2}$ coupled with the isospin $\frac{1}{2}$ and the orbital angular momentum have produced an integer total angular momentum. The angular momentum stored in the non-Abelian vortex field configuration, in the presence of an isospinor particle, contributes to K.[12]

The Dirac equation for the radial wave functions can be written as

$$\left[\frac{d}{d\rho} \pm \frac{K-\frac{1}{2}-h(\rho)/2}{\rho}\right]\varphi_1^{\mp}(\rho) = i(\omega \mp m)\varphi_1^{\pm}(\rho) \pm W(\rho)\varphi_2^{\pm}(\rho), \quad (5.16)$$

$$\left[\frac{d}{d\rho} \pm \frac{K+\frac{1}{2}+h(\rho)/2}{\rho}\right]\varphi_2^{\mp}(\rho) = i(\omega \mp m)\varphi_2^{\pm}(\rho) \mp W^*(\rho)\varphi_1^{\pm}(\rho), \quad (5.17)$$

where

$$W(\rho) \equiv g_2 g(\rho) + i g_1 f(\rho)$$

and

$$h(\rho) \equiv 1 - H(\rho).$$

Let us consider the case when the Higgs fields are decoupled, i.e., $g_1 = g_2 = 0$. In this situation, we see that for large distances ($\mu\rho \gg 1$) the Dirac radial equations [Eqs. (5.16), (5.17)] are like free ones but with the centrifugal terms modified by the replacement $K \to K - T_3$. In our case $T_3 = \pm\frac{1}{2}$, but it is clear that in general the fermion-vortex interaction at large distances will consist of a simple modification of the centrifugal barrier according to the isospin of the fermion.

This long-range interaction can be interpreted, as in the Abelian case, as a consequence of the minimal gauge coupling of the azimuthal canonical momentum. This long-range interaction will clearly be present between any particle of nonzero isospin and the non-Abelian vortex line.

The study of the radial Dirac equations (5.16), (5.17) is greatly simplified for $g_1 = g_2 = 0$. In this case the isospin is a good quantum number and the four-equation system decouples into two systems of two equations. Each of them can be obtained from the Abelian radial Dirac equation (2.7) by the following replacements:

$$\begin{aligned} &n \to \pm\tfrac{1}{2}, \\ &J \to K \mp \tfrac{1}{2}, \\ &\mathcal{H}_n(\rho) \to \pm H(\rho)/2. \end{aligned} \quad (5.18)$$

We conclude that there are no normalizable wave functions for $k^2 < 0$, nor for $k^2 = 0$ because the conditions for the existence of bound states at the threshold [Eq. (2.22)] are not satisfied after the replacements (5.18).

It must be pointed out, that fermion-vortex bound states at the threshold could be present if the fermion isospin was higher than one. This will increase the strength of the long-range fermion-vortex interaction.

Note added in proof. In a recent paper, E. B. Bogomol'nyi hasshown [Yad. Fiz. 24, 861 (1976) (Sov. J. Nucl. Phys. 24, 449 (1977))] that Abelian vortices with flux $n_1 |n| \geq 2$ are unstable for $\mu_s > \mu$ (in the absence of fermions). This could indicate a breaking of fermion number conservation if this instability survives when fermions are coupled to the vortices.

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APPENDIX

It can be shown that the radial functions on the vortex solution [Eqs. (2.2) and (2.3)] have the following small-distance behavior:

$$\begin{aligned} \mathcal{H}_n(\rho) &= n - d_1^n\left(\frac{e^2}{\lambda}\right)(\mu\rho)^2 + d_2^n\left(\frac{e^2}{\lambda}\right)(\mu\rho)^{2+2|n|}[1 + O(\mu\rho)^2], \\ F_n(\rho) &= L_n\left(\frac{e^2}{\lambda}\right)(\mu\rho)^{|n|}[1 + O(\mu\rho)^2]. \end{aligned} \quad (A1)$$

For long distances we find

$$\begin{aligned} \mathcal{H}_n(\rho) &= Z_n(e^2/\lambda)\mu\rho K_1(\mu\rho)[1 + O(e^{-\mu\rho})], \\ F_n(\rho) &= 1 + O(e^{-\mu_s\rho}). \end{aligned} \quad (A2)$$

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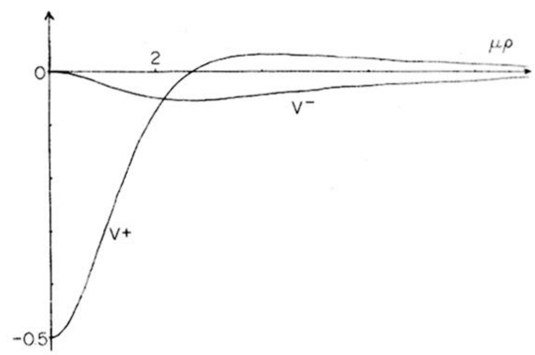

FIG 1. Interaction potentials $V^+_{1,1/2}$ and $V^-_{1,1/2}$ between a fermion and a one-quantum vortex in the wave $J=\frac{1}{2}$. The potential has been obtained from the Abelian vortex solution given in Ref. 18 for $\lambda=e^2$ and it is plotted in units of μ^2.

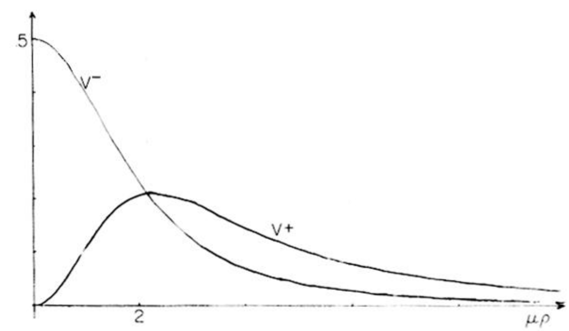

FIG. 2. Same as Fig. 1, but for the wave $J = -\frac{1}{2}$.