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# Path integral for gauge theories with fermions

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The Atiyah-Singer index theorem indicates that a naive unitary transformation of basis vectors for fermions interacting with gauge fields is not allowed in general. On the basis of this observation, it was previously shown that the path-integral measure of a gauge-invariant fermion theory is transformed nontrivially under the chiral transformation, and thus leads to a simple derivation of "anomalous" chiral Ward-Takahashi identities. We here clarify some of the technical aspects associated with the discussion. It is shown that the Jacobian factor in the path-integral measure, which corresponds to the Adler-Bell-Jackiw anomaly, is independent of any smooth regularization procedure of large eigenvalues of $\not{D}$ in Euclidean theory; this property holds in any even-dimensional space-time and also for the gravitational anomaly. The appearance of the anomaly and its connection with the index theorem are thus related to the fact that the primary importance is attached to the Lorentz-covariant "energy" operator $\not{D}$ and that $\not{D}$ and $\gamma_5$ do not commute. The abnormal behavior of the path-integral measure at the zero-frequency sector in the presence of instantons and its connection with spontaneous symmetry breaking is also clarified. We comment on several other problems associated with the anomaly and on the Pauli-Villars regularization method.

## I. INTRODUCTION

As one of the important implications of the Atiyah-Singer index theorem,[1] we have shown elsewhere[2] that the Euclidean path-integral measure for gauge theories with fermions is not invariant under the chiral transformation, and it gives rise to an extra phase factor corresponding to the Adler-Bell-Jackiw anomaly.[3-5] The derivation of "anomalous" chiral Ward-Takahashi (WT) identities by means of the variational derivative is thus made consistent in the path-integral formalism.[6,7] The basic observation involved is that the transformation of basis vectors in the functional space from the "Heisenberg picture" to the "interaction picture" does not lead to a unitary transformation of the chirality index associated with the functional measure, and thus leads to the anomalous behavior of WT identities in perturbation theory.

In the present paper, we first clarify several technical aspects associated with the discussion, and then show that many of the known properties of the anomaly are reproduced in a compact manner in the present approach. It is shown that the summation of a conditionally convergent series appearing in the index factor gives rise to a correct result of anomaly for *any* smooth regularization of large eigenvalues in Euclidean theory. We then comment on the consistency of the Pauli-Villars regularization scheme with our analysis, although the measure becomes invariant under the chiral transformation if one includes the regulator ghost field into the functional measure. The appearance of the $\theta$ vacuum and its connection with the spontaneous symmetry breaking in the presence of instantons is also clarified. We finally comment on several other problems such as the ordinary anomaly cancellation mechanism in the case of parity-violating coupling of Yang-Mills fields to fermions, and the gravitational anomaly.

## II. ANOMALOUS CHIRAL WARD-TAKAHASHI IDENTITIES

### Path-integral formalism

We recapitulate the basic aspects of the path-integral approach[2] to anomalous WT identities. We start with the SU($n$) Yang-Mills field coupled to fermions,

$$\mathcal{L} = \bar{\psi} i\gamma^\alpha D_\alpha \psi - m\bar{\psi}\psi + \frac{1}{2g^2}\,\mathrm{Tr}\,F^{\mu\nu}F_{\mu\nu}\,, \tag{2.1}$$

suitably continued to Euclidean space-time. After the Wick rotation $x^0 \to -ix^4$ and $A_0 \to iA_4$, the operator $\not{D} \equiv \gamma^\alpha D_\alpha \equiv \gamma^\alpha(\partial_\alpha + A_\alpha)$ becomes a Hermitian operator

$$\begin{aligned}\not{D} &= i\gamma^0 D_4 + \gamma^k D_k \\ &\equiv \gamma^4 D_4 + \gamma^k D_k\,,\end{aligned} \tag{2.2}$$

where our $\gamma$-matrix convention follows that of Bjorken and Drell[8]: $\gamma^0$ is Hermitian and $\gamma^k$ ($k=1,2,3$) are anti-Hermitian. The *Hermitian* $\gamma_5$ matrix is defined by[8]

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \equiv \gamma^4\gamma^1\gamma^2\gamma^3 = -\gamma^1\gamma^2\gamma^3\gamma^4\,. \tag{2.3}$$

After the Wick rotation, the metric becomes $g_{\mu\nu} = (-1,-1,-1,-1)$.

We consider the fermions in the $n$-dimensional representation of SU($n$) in (2.1), and

 

$$\begin{aligned} iA_\mu &\equiv gA_\mu^a(x)T^a\,, \\ [T^a, T^b] &= if^{abc}T^c\,, \\ \mathrm{Tr}(T^aT^b) &= \tfrac{1}{2}\delta^{ab}\,, \\ F_{\mu\nu} &\equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\,. \end{aligned} \tag{2.4}$$

To define the path integral precisely, we change the basis from the "$x$ representation" to the "$n$ representation" by[7]

$$\begin{aligned} \psi(x) &\equiv \sum_n a_n\varphi_n(x) = \sum_n a_n\langle x|n\rangle\,, \\ \bar\psi(x) &\equiv \sum_n \varphi_n(x)^\dagger \bar b_n = \sum_n \langle n|x\rangle \bar b_n \end{aligned} \tag{2.5}$$

in terms of the complete set of eigenfunctions of the Hermitian operator (2.3),

$$\begin{aligned} \not{D}\varphi_n(x) &= \lambda_n\varphi_n(x)\,, \\ \int d^4x\,\varphi_n(x)^\dagger\varphi_m(x) &= \delta_{n,m}\,, \end{aligned} \tag{2.6}$$

where, for simplicity, all the eigenvalues are treated as if they are discrete.[9] The coefficients $a_n$ and $\bar b_n$ are independent elements of the Grassmann algebra in the classical level. The transformation (2.5) is formally unitary and we have the path-integral measure

$$\begin{aligned} d\mu &\equiv \prod_x [\mathfrak{D}A_\mu(x)]\mathfrak{D}\bar\psi(x)\mathfrak{D}\psi(x) \\ &= \prod_x [\mathfrak{D}A_\mu(x)]\prod_n d\bar b_n \prod_m da_m \end{aligned} \tag{2.7}$$

by choosing the arbitrary normalization factor to be unity. Here and in the following we include the Faddeev-Popov factor into $[\mathfrak{D}A_\mu]$ whenever necessary; for example,

$$[\mathfrak{D}A_\mu(x)] = \mathfrak{D}A_\mu(x)\delta(\partial^\mu A_\mu^\omega(x))\det\left[\frac{\delta}{\delta\omega}\,\partial^\mu A_\mu^\omega(x)\right].$$

Under the *local* chiral transformation

$$\begin{aligned} \psi(x) &\to \psi'(x) \equiv e^{i\alpha(x)\gamma_5}\psi(x)\,, \\ \bar\psi(x) &\to \bar\psi'(x) \equiv \bar\psi(x)e^{i\alpha(x)\gamma_5} \end{aligned} \tag{2.8}$$

the Lagrangian (2.1) is transformed for infinitesimal $\alpha(x)$ as

$$\begin{aligned} \mathcal{L}(x) \to \mathcal{L}(x) &- \partial_\mu\alpha(x)\bar\psi(x)\gamma^\mu\gamma_5\psi(x) \\ &- 2mi\alpha(x)\bar\psi(x)\gamma_5\psi(x)\,. \end{aligned} \tag{2.9}$$

The coefficients in (2.5) are then transformed as

$$\psi'(x) \equiv \sum_n a'_n\varphi_n(x) = \sum_n a_n e^{i\alpha(x)\gamma_5}\varphi_n(x)$$

or

$$\begin{aligned} a'_m &= \sum_n \int d^4x\,\varphi_m(x)^\dagger e^{i\alpha(x)\gamma_5}\varphi_n(x)a_n \\ &\equiv \sum_n C_{m,n}a_n\,. \end{aligned} \tag{2.10}$$

Thus

$$\prod_m da'_m = [\det C_{m,n}]^{-1}\prod_n da_n\,, \tag{2.11}$$

where the inverse of the determinant appears due to the fact that the translation-invariant integral measure over the elements of the Grassmann algebra is defined as the *left-derivative*[6] $da_n \equiv \partial/\partial a_n$. The translation-invariant measure ensures that the Feynman path-integral formula satisfies Schwinger's action principle.

The Jacobian factor in (2.11) is evaluated for infinitesimal $\alpha(x)$ as

$$\begin{aligned} [\det C_{m,n}]^{-1} &= \det\left[\delta_{m,n} + i\int \alpha(x)\varphi_m(x)^\dagger\gamma_5\varphi_n(x)dx\right]^{-1} \\ &= \exp\left[-i\sum_n \int dx\,\alpha(x)\varphi_n(x)^\dagger\gamma_5\varphi_n(x)\right] \\ &\equiv \exp\left[-i\int dx\,\alpha(x)A(x)\right]\,, \end{aligned} \tag{2.12}$$

where

$$A(x) \equiv \sum_n \varphi_n(x)^\dagger\gamma_5\varphi_n(x)\,. \tag{2.13}$$

The Jacobian for $\mathfrak{D}\bar\psi$ gives rise to an identical factor, and thus leading to

$$d\mu \to d\mu\,\exp\left[-2i\int dx\,\alpha(x)A(x)\right]\,, \tag{2.14}$$

$A(x)$ in (2.13) corresponds to our *primary definition* of the "anomaly," and it is an ill-defined conditionally convergent quantity. We may evaluate it by regularizing the large eigenvalues (i.e., $|\lambda_n| \lesssim M$) and changing the basis vectors to "plane waves" as

$$\begin{aligned} A(x) &= \lim_{M\to\infty}\left(\sum_n \varphi_n(x)^\dagger\gamma_5 e^{-(\lambda_n/M)^2}\varphi_n(x)\right) \\ &= \lim_{M\to\infty}\left(\sum_n \varphi_n(x)^\dagger\gamma_5 e^{-(\not{D}/M)^2}\varphi_n(x)\right) \\ &= \lim_{M\to\infty}\mathrm{Tr}\int\frac{d^4k}{(2\pi)^4}\gamma_5 e^{-ikx}e^{-(\not{D}/M)^2}e^{ikx} \\ &= \lim_{M\to\infty}\mathrm{Tr}\int\frac{d^4k}{(2\pi)^4}\gamma_5\exp\left(\frac{-1}{2M^2}\{2[ik_\mu + D_\mu(x)]^2 + [\gamma^\mu,\gamma^\nu]F_{\mu\nu}(x)\}\right) \\ &= \lim_{M\to\infty}\mathrm{Tr}\gamma_5\{[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\}^2\left(\frac{1}{2M^2}\right)^2\frac{1}{2!}\int\frac{d^4k}{(2\pi)^4}e^{-k^\mu k_\mu/M^2} \\ &= \frac{1}{2}\left(\frac{-1}{8\pi^2}\right)\mathrm{Tr}\,{}^*F^{\mu\nu}F_{\mu\nu}(x)\,. \end{aligned} \tag{2.15}$$

Namely,

$$A(x) = \frac{1}{2}\left(\frac{-1}{8\pi^2}\right) \mathrm{Tr}\, {}^*F^{\mu\nu}F_{\mu\nu}(x) \tag{2.16}$$

with ${}^*F^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ ($\epsilon^{1230} \equiv \epsilon^{1234} = 1$). The trace in (2.15) runs over the space of $\gamma$ matrices and internal SU($n$) indices, and the trace in (2.16) over internal SU($n$) indices.

The generating functional of complete Green's functions in Euclidean space is given by

$$Z(\eta, \bar{\eta}, J_\mu) \equiv \frac{1}{N} \int d\mu \exp\left( \int [\mathcal{L} + \bar{\eta}\psi + \bar{\psi}\eta - J^\mu A_\mu] dx \right), \tag{2.17}$$

with $Z(0,0,0) = 1$. The WT identities are then collectively represented by the variational derivative[7]

$$\frac{\delta}{\delta\alpha(x)} Z(\eta, \bar{\eta}, J_\mu)\bigg|_{\alpha=0} \equiv 0 . \tag{2.18}$$

By combining the contributions from (2.8), (2.9), and (2.14) with (2.17), we obtain the WT identity, e.g.,

$$\begin{aligned} \partial_\mu \langle [j_5^\mu(x)\psi(y)\bar{\psi}(z)]_+\rangle = 2mi\langle [j_5(x)\psi(y)\bar{\psi}(z)]_+\rangle - i\delta(x-y)\langle [\gamma_5\psi(y)\bar{\psi}(z)]_+\rangle \\ - i\delta(x-z)\langle [\psi(y)\bar{\psi}(z)\gamma_5]_+\rangle - (i/8\pi^2)\langle [\mathrm{Tr}\, {}^*F^{\mu\nu}F_{\mu\nu}(x)\psi(y)\bar{\psi}(z)]_+\rangle . \end{aligned} \tag{2.19}$$

The Minkowski version is obtained by a Wick rotation which removes the imaginary factor $i$ from the last three terms in (2.19), and one recovers the familiar result.[5] [The result for the Abelian theory is obtained by just discarding the trace operation in (2.19).] The gauge-invariant currents are defined from (2.9) as

$$\begin{aligned} j_5^\mu(x) &\equiv \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x) , \\ j_5(x) &\equiv \bar{\psi}(x)\gamma_5\psi(x) . \end{aligned} \tag{2.20}$$

### Regularization independence of the "anomaly" factor

The evaluation of (2.15) is the transformation of basis vectors from $\varphi_n(x)$ in (2.6) to plane waves $e^{ikx}$ to evaluate $\mathrm{Tr}\gamma_5 e^{-(\not{D}/M)^2}$, and it can be compactly written as

$$\begin{aligned} A(x) &= \lim_{M\to\infty} \left( \sum_n \langle n|x\rangle \gamma_5 e^{-(\not{D}/M)^2} \langle x|n\rangle \right) \\ &= \lim_{M\to\infty} \mathrm{Tr} \int \frac{d^4k}{(2\pi)^4} \langle k|x\rangle \gamma_5 e^{-(\not{D}/M)^2} \langle x|k\rangle . \end{aligned} \tag{2.21}$$

This procedure may be regarded as the extraction of the gauge field dependence of $A(x)$ by using the plane waves which have no gauge field dependence by themselves.

We next show that $A(x)$ in (2.16) is independent of any smooth regularization of large eigenvalues. We make the following replacement in (2.15):

$$\exp[-(\lambda_n/M)^2] \to f[(\lambda_n/M)^2] , \tag{2.22}$$

where $f(z)$ is any *smooth* function which rapidly approaches zero at $z = \infty$:

$$f(\infty) = f'(\infty) = f''(\infty) = \cdots = 0$$

and

$$f(0) = 1 . \tag{2.23}$$

The calculation in (2.15) is now replaced by

$$\begin{aligned} A(x) &= \lim_{M\to\infty} \mathrm{Tr}\gamma_5\{[\gamma^\mu, \gamma^\nu]F_{\mu\nu}\}^2 \frac{1}{4M^4}\frac{1}{2!} \\ &\quad \times \int \frac{d^4k}{(2\pi)^4} f''((k^\mu/M)^2) \end{aligned} \tag{2.24a}$$

$$= -\mathrm{Tr}\{{}^*F^{\mu\nu}F_{\mu\nu}\} \int \frac{d^4k}{(2\pi)^4} f''(k^2) \tag{2.24b}$$

after the rescaling $k^\mu \to Mk^\mu$; all other terms either vanish due to the trace operation over the $\gamma$ matrix or contain the factor $(1/M^2)^l$, $l > 0$. We note that any finite-frequency sector $k^2 \equiv -k^\mu k_\mu \le L =$ constant $<\infty$ in (2.24a) gives rise to a vanishing contribution to $A(x)$ after the rescaling (2.24b). The integral in (2.24b) is evaluated by noting $d^4k = \pi^2 k^2 dk^2$, namely,

$$\begin{aligned} \int \frac{d^4k}{(2\pi)^4} f''(k^2) &= \frac{1}{16\pi^2}\int_0^\infty dk^2 k^2 f''(k^2) \\ &= \frac{-1}{16\pi^2}\int_0^\infty dk^2 f'(k^2) = \frac{-1}{16\pi^2} f(k^2)\bigg|_0^\infty \\ &= \frac{1}{16\pi^2} \end{aligned} \tag{2.25}$$

by (2.23). Incidentally, this property holds in any even-dimensional Euclidean space, for example, the two-dimensional space.[10] This regularization independence of the chiral anomaly allowed us to define the chiral WT identity (2.19) *without* using the detailed form of propagators.[11] A related property for the gravitational anomaly will be commented on later.

This regularization independence shows that the summation in (2.13) is not divergent but rather conditionally convergent (a situation somewhat similar to $+1-1+1-1+\cdots$). Any prescription of summation starting from small eigenvalues of $\not{D}$ (i.e., small in their absolute values) always gives rise to an identical result.[11] This rather subtle aspect is also related to the fact that $\gamma_5$ and $\not{D}$ in (2.2) cannot be simultaneously diagonalized.[12] When one diagonalizes the "energy" operator $\not{D}$, the chirality asymmetry appears and the trace of the operator $\gamma_5$ exhibits the anomalous behavior (2.15), contrary to the naive expectation from $\mathrm{Tr}\gamma_5=0$. (The $\gamma_5$ operator is four-dimensional in the naive sense, but it actually becomes $4\times\infty$ dimensional in the functional space spanned by $\varphi_n(x)$ due to $[\gamma_5,\not{D}]\neq 0$.)

### Connection with the Atiyah-Singer theorem

From (2.13) and (2.16), one formally obtains[8,9]

$$n_+-n_-=\nu\,, \tag{2.26}$$

where $n_\pm$ stand for the number of zero-eigenvalue solutions in (2.6) with positive and negative chirality, respectively, and $\nu$ the Pontryagin index

$$\nu\equiv-\frac{1}{16\pi^2}\mathrm{Tr}\int {}^*F^{\mu\nu}F_{\mu\nu}\,dx\,. \tag{2.27}$$

The left-hand side of (2.26) follows from the term-by-term integration in (2.13) by noting that $\gamma_5$ and $\not{D}$ anticommute. The relation (2.26) corresponds to the Atiyah-Singer theorem[1] when one considers the relation in compact space $S^4$ after the stereographic projection of (2.6) and (2.27) originally defined in Euclidean space $R^4$. Jackiw and Rebbi[13] examined the relation (2.26) in Euclidean space $R^4$ by explicit calculations, and they suggest that the relation (2.26) holds for those gauge fields which rapidly approach the pure gauge configuration at $|x|=\infty$ so that the Euclidean action is finite and the index (2.27) is well-defined. For those well-behaved gauge fields, the relation (2.16) is regarded as a local version of the index theorem. In the path integral, all the quantities appearing in the integrand are regarded as *classical* fields. The path integral therefore provides a convenient means to relate the semiclassical (local) index theorem (2.16) to the quantum-mechanical WT identity (2.19).

The important implication of the index theorem on the path integral is that a *naive* unitary transformation of basis vectors for fermions belonging to different (local) indices, for example, a transformation from the "standard" basis (2.6) to "plane waves,"

$$\begin{gathered}\not{\partial}\,\xi_k(x)=\lambda_k\xi_k(x)\,,\\ \int \xi_k(x)^\dagger\xi_l(x)dx=\delta_{k,l}\,,\end{gathered} \tag{2.28}$$

is not allowed for gauge theory. (For simplicity, we here treat all the eigenvalues as if they are discrete.[9]) Under this change of basis vectors, the path-integral measure is formally invariant:

$$\begin{aligned}\prod_n da_n d\bar{b}_n&=\{\det[\langle\xi_k|\varphi_m\rangle]\det[\langle\varphi_n|\xi_l\rangle]\}^{-1}\\&\quad\times\prod_n da'_n d\bar{b}'_n\\&=\prod_n da'_n d\bar{b}'_n\,.\end{aligned} \tag{2.29}$$

The Jacobian factor associated with the chiral transformation (2.12), however, leads to

$$\begin{aligned}\sum_k \xi_k(x)^\dagger\gamma_5\xi_k(x)&=\sum_{k,m,n}\langle\xi_k|\varphi_n\rangle\langle\varphi_m|\xi_k\rangle\\&\quad\times\varphi_n(x)^\dagger\gamma_5\varphi_m(x)\\&=\sum_n\varphi_n(x)^\dagger\gamma_5\varphi_n(x)\equiv A(x)\,.\end{aligned} \tag{2.30}$$

The left-hand side of (2.30) is independent of the gauge field $A_\mu$, whereas the right-hand side functionally depends on $A_\mu$.

As is seen in (2.15), the chirality asymmetry (2.26) appears in the well-defined zero-eigenvalue sector for the standard basis, whereas the asymmetry is transferred to the sector of infinite frequencies for plane waves as was noted in (2.24). This gives rise to the failure of the naive unitary transformation (2.30). In fact, any sensible evaluation of the left-hand side of (2.30) gives a vanishing result in local as well as in integrated forms.

The nonunitary transformation (2.30) therefore induces an "index defect," and the missing anomaly term in WT identities when the remaining parts of the WT identities are *correctly* evaluated by taking the effects of gauge fields into account in interaction picture perturbation theory.

The present consideration suggests that the anomalous behavior of chiral WT identities in perturbation theory should be traced to the failure of the naive unitary transformation of basis vectors rather than to the failure of naive WT identities. If the naive unitary transformation fails, the most natural choice of basis vectors which preserves the characteristic properties (such as the gauge invariance) of the gauge theory is to utilize the standard basis (2.6), and then the "anomalous" WT identity holds as an identity. This also corresponds to the customary procedure in quantum theory[5] where the primary importance is attached to the Lorentz-covariant "energy" operator $\not{D}$ over $\gamma_5$.

### Pauli-Villars regularization

In the ordinary discussion of chiral WT identities, the Pauli-Villars regularization is often utilized.[5] This regularization scheme preserves the gauge invariance but breaks the chiral symmetry by the large mass term of the regulator field.

We here briefly comment on this regularization scheme from the viewpoint of the path-integral measure. We consider

$$\mathcal{L} = \bar{\psi} i \not{D}\psi - m\bar{\psi}\psi + \bar{\Psi} i \not{D}\Psi - M\bar{\Psi}\Psi + \frac{1}{2g^2}\operatorname{Tr} F^{\mu\nu}F_{\mu\nu} \tag{2.31}$$

instead of (2.1). Although the regulator field $\Psi$ thus introduced, which obeys the Bose statistics, does not regulate all the diagrams, it regulates fermion loop diagrams.

The path-integral measure (2.7) is now replaced by

$$d\tilde{\mu} \equiv \prod_x [\mathfrak{D}A_\mu(x)] \prod_n da_n d\bar{b}_n \prod_m d\alpha_m d\bar{\beta}_m\,, \tag{2.32}$$

with

$$\Psi(x) = \sum_n \alpha_n \varphi_n(x)\,, \qquad \bar{\Psi}(x) = \sum_n \bar{\beta}_n \varphi_n(x)^\dagger\,, \tag{2.33}$$

where $\varphi_n(x)$ is defined in (2.6), and $\alpha_n$ and $\bar{\beta}_n$ are *ordinary numbers*. Owing to the Bose statistics for $\Psi$, the measure in (2.32) now becomes invariant under the *simultaneous* transformations (2.8) and

$$\Psi(x) \to e^{i\alpha(x)\gamma_5}\Psi(x)\,, \qquad \bar{\Psi}(x) \to \bar{\Psi}(x)e^{i\alpha(x)\gamma_5}\,. \tag{2.34}$$

The Lagrangian (2.31) is now transformed as

$$\mathcal{L} \to \mathcal{L} - \partial_\mu\alpha(x) j_5^\mu(x) - 2mi\alpha(x) j_5(x) - \partial_\mu\alpha(x)\bar{\Psi}\gamma^\mu\gamma_5\Psi - 2Mi\alpha(x)\bar{\Psi}\gamma_5\Psi\,. \tag{2.35}$$

Thus the WT identity (2.19) is now replaced by

$$\begin{aligned}\partial_\mu\langle[j_5^\mu(x)\psi(y)\bar{\psi}(z)]_+\rangle = {}& 2mi\langle[j_5(x)\psi(y)\bar{\psi}(z)]_+\rangle \\ & - i\delta(x-y)\langle[\gamma_5\psi(y)\bar{\psi}(z)]_+\rangle \\ & - i\delta(x-z)\langle[\psi(y)\bar{\psi}(z)\gamma_5]_+\rangle \\ & - \partial_\mu\langle[\bar{\Psi}(x)\gamma^\mu\gamma_5\Psi(x)\psi(y)\bar{\psi}(z)]_+\rangle \\ & + 2Mi\langle[\bar{\Psi}(x)\gamma_5\Psi(x)\psi(y)\bar{\psi}(z)]_+\rangle\,.\end{aligned} \tag{2.36}$$

The ordinary WT identity is obtained if

$$\begin{aligned}&\partial_\mu\langle[\bar{\Psi}(x)\gamma^\mu\gamma_5\Psi(x)\psi(y)\bar{\psi}(z)]_+\rangle \\ &\quad - 2Mi\langle[\bar{\Psi}(x)\gamma_5\Psi(x)\psi(y)\bar{\psi}(z)]_+\rangle \\ &\qquad = \frac{i}{8\pi^2}\langle[\operatorname{Tr}{}^*F^{\mu\nu}F_{\mu\nu}(x)\psi(y)\bar{\psi}(z)]_+\rangle\,,\end{aligned} \tag{2.37}$$

which should be proved in this regularization scheme; (2.37) is implied in our analysis by the fact that the regulator field gave rise to the extra Jacobian factor to make the measure (2.32) invariant under the chiral transformation. If one takes the limit $M\to\infty$ in (2.36) first, it may also be regarded as one of the specific choices of $f(z)$ in (2.22).

The ordinary analysis on the basis of the Pauli-Villars regularization is thus perfectly consistent with our analysis. In fact, the Pauli-Villars regularization may be regarded as an ideal perturbative realization of the path-integral analysis of the anomaly presented so far: the Green's function

$$\partial_\mu\langle[j_5^\mu(x)\psi(y)\bar{\psi}(z)]_+\rangle \tag{2.38}$$

in (2.19) defined by *naive* perturbative calculations does not satisfy the well-defined WT identity, as one has to diagonalize $\not{D}$ and "regulate" its large eigenvalues to obtain the well-defined anomaly factor. When one considers the combination

$$\langle[j_5^\mu(x)\psi(y)\bar{\psi}(z)]_+\rangle + \langle[\bar{\Psi}(x)\gamma^\mu\gamma_5\Psi(x)\psi(y)\bar{\psi}(z)]_+\rangle \tag{2.39}$$

in (2.36), the contributions from large eigenvalues of $\not{D}$ cancel each other thanks to the Bose statistics of $\Psi$ and we have

$$\langle[j_5^\mu(x)\psi(y)\bar{\psi}(z)]_+\rangle_{\text{reg}} + \langle[\bar{\Psi}(x)\gamma^\mu\gamma_5\Psi(x)\psi(y)\bar{\psi}(z)]_+\rangle_{\text{reg}}\,. \tag{2.40}$$

The regulated quantity

$$\langle[\bar{\Psi}(x)\gamma^\mu\gamma_5\Psi(x)\psi(y)\bar{\psi}(z)]_+\rangle_{\text{reg}} \tag{2.41}$$

in the gauge-invariant calculation goes to zero for $M\to\infty$, and

$$\partial_\mu\langle[j_5^\mu(x)\psi(y)\bar{\psi}(z)]_+\rangle_{\text{reg}} \tag{2.42}$$

in this limit satisfies the well-defined WT identity with the anomaly factor provided by the mass term

$$2Mi\langle[\bar{\Psi}(x)\gamma_5\Psi(x)\psi(y)\bar{\psi}(z)]_+\rangle \tag{2.43}$$

for $M\to\infty$. The regulator field $\Psi$ thus serves to better define the perturbative calculation without altering the physics.

### Adler-Bardeen theorem

Our analysis so far is on the level of formal manipulation, and it does not tell much about the graphical proof of the so-called Adler-Bardeen theorem,[14] which states that the WT identity (2.19), for example, holds up to all orders in perturbation theory and that each term in (2.19) becomes *separately* finite for the vanishing momentum transfer. Our derivation of (2.19) by the formal variational derivative, which gave rise to all other well-tested WT identities in the past, may, however, make the Adler-Bardeen theorem more plausible. A characteristic property of a renormalizable theory is that the *form* of the Lagrangian is not altered by higher-order corrections. Our derivation of WT identities depends solely on this invariant form of the Lagrangian, and it is tempting to regard (2.19), for example, as a *bare* form of the WT identity, although this does not necessarily imply that all the terms in (2.19) separately represent finite Green's functions. [A suitable choice of the composite operator, for example, $\bar{\psi}(\overleftarrow{\not{D}}\gamma_5 - \gamma_5\overrightarrow{\not{D}})\psi$ in place of $\partial_\mu j_5^\mu$, may make the renormalization property of the local WT identity more transparent.] We need a better understanding of the regularization before we establish (2.19) in the graphical calculation and express the WT identity in terms of finite quantities for non-Abelian gauge theories.[15]

## III. THE $\theta$ VACUUM

### The $\theta$ factor and chirality selection rule

The discovery of the instanton solution[16] in non-Abelian gauge theories and its interpretation as tunneling phenomena in field theory gave rise to an important notion of the $\theta$ vacuum.[17] In the path integral, the sum over all the field configurations is assumed at the starting point, and thus the sum over "all" the topological configurations naturally appears. In gauge theory, one sums over all the gauge inequivalent set of field configurations; the definition of different field configurations thus crucially depends on the definition of allowed set of gauge transformations. In the following, we classify the field configurations by means of the "localized" gauge transformations, which approach unity at space-time infinity. This restriction of the allowed set of gauge transformations is necessary to render a well-defined meaning to the topological index for gauge fields.[17] [In this respect, we have implicitly two kinds of boundary conditions in mind: The first is the true (or strong) boundary condition on the path-integral domain at space-time infinity where all the gauge fields are assumed to approach a pure gauge transformation, and the second (weak) one corresponds to the ordinary Lehmann-Symanzik-Zimmermann (LSZ)-type condition, which is incorporated by a *limiting* form of Green's functions generated via *localized* source functions.]

In the presence of instantons, the measure (2.14) splits into a sum of terms for a *global* parameter $\alpha$ (by assuming the smooth transition[9] from local to global $\alpha$)

$$\sum_\nu d\mu_{(\nu)} e^{-2i\alpha\nu} \tag{3.1}$$

or for the $N$-flavor case in quantum chromodynamics

$$\sum_\nu d\mu_{(\nu)} e^{-2iN\alpha\nu} \tag{3.2}$$

under a chiral U(1) transformation, and the measure is invariant under any chiral flavor SU($N$) transformation, as can be seen by diagonalizing the traceless SU($N$) generators. Here $\nu$ stands for the Pontryagin index (2.27), and it corresponds to the number of instantons in the "dilute-gas" approximation.[17]

For the chiral-invariant action, (3.2) gives rise to the ordinary chirality selection rule as is specified by the index theorem[8] (2.26) with $\nu$ on the right-hand side replaced by $N\nu$, and with the identification $\theta \equiv -2N\alpha$ we have

$$\sum_\nu d\mu_{(\nu)} e^{i\nu\theta}, \tag{3.3}$$

which coincides with the ordinary prescription of the $\theta$ vacuum.[17] Our derivation of (2.19) suggests that the WT identity holds without modification in the $\theta$ vacuum. In general, the parameter $\theta$ has a definite physical meaning only relative to the chirality phase of the fermion mass term. It should be noted that not only the gauge field but also the fermion variables in (3.3) depend on the index $\nu$ (at least in the WKB approximation) in our approach. [In the path integral with an infinite number of degrees of freedom, however, one has to integrate over rather singular bosonic field configurations (or at least over suitable limiting configurations of regular fields) for which the Euclidean action in general diverges[7] and the index theorem in the *proper sense* (2.26) has not been checked in Euclidean space $R^4$. As a result, the validity of the chirality selection rule stated above beyond the semiclassical WKB approximation is not known at present in Euclidean space $R^4$. In comparison, the local index theorem (2.16) and the local WT identity (2.19) are more general as they can be established without referring to the global topological structure of field variables.]

The $\theta$ factor for pure gauge fields (or for scalar fields coupled to gauge fields) does not arise in the present manner. This reflects the fact that all the gauge-invariant operators are automatically chiral invariant for those theories, and there exists a superselection rule for the $\theta$ parameter. The vacuum structure of the scalar gauge theory with spontaneous symmetry breaking such as the Higgs-Kibble model, however, becomes more involved due to the fact that the Kronecker index associated with the scalar field correlates with the Pontryagin index of the gauge field if one imposes a finite Euclidean action.[18]

### Spontaneous symmetry breaking

As is well-known, the anomaly term provides an explicit chiral breaking term for the gauge-invariant current (2.20) in the presence of instantons.[17] The effects of instantons and the $\theta$ vacuum may, however, be regarded as a kind of "spontaneous" symmetry breaking. We now briefly comment on this point. We first define the "conserved" current by

$$J_5^\mu(x) \equiv j_5^\mu(x) + iS^\mu(x)\,, \tag{3.4}$$

with

$$S^\mu(x) \equiv (N/8\pi^2)\epsilon^{\mu\alpha\beta\gamma}\operatorname{Tr}\left[F_{\alpha\beta}A_\gamma - \tfrac{2}{3}A_\alpha A_\beta A_\gamma\right]. \tag{3.5}$$

By using $J_5^\mu$, one can *pretend* as if (2.9) and (2.14) (for $m=0$) were replaced by

$$\mathcal{L} \to \mathcal{L} - \partial_\mu\alpha(x)J_5^\mu(x)\,,$$
$$\sum_\nu d\mu_{(\nu)} \to \sum_\nu d\mu_{(\nu)}\exp\left\{i\int dx\,\partial_\mu[\alpha(x)S^\mu(x)]\right\}, \tag{3.6}$$

which give the same expression for (2.17) in the form of the generating functional.

For *local* $\alpha(x)$, on which WT identities are based, the functional measure as defined in (3.6) is invariant as the surface terms vanish. [Incidentally, this partial-integration procedure corresponds to the definition of $T^*$product in the functional formalism.] For *global* $\alpha$, however, the measure is transformed nontrivially in the presence of instantons. Namely, the zero-frequency sector of the integral measure behaves abnormally, thus leading to the mismatch between the local WT identity and the global symmetry relation of the theory. This is the typical situation of spontaneous symmetry breaking and it generally gives rise to a Nambu-Goldstone pole for the gauge-*variant* current $J_5^\mu$ (3.4) if the quantum corrections to the instanton are suitably included. The chiral property of the gauge theory thus exhibits the behavior similar to the spontaneous breakdown of other symmetry transformations if one uses the conserved current (3.4).

In the ordinary treatment of general continuous symmetry, this abnormal behavior for the global transformation is produced by the wave functional which specifies the boundary condition. In the present case, the boundary condition is already incorporated in the definition of the path-integral domain.

## IV. OTHER RELATED PROBLEMS

### Parity-violating gauge coupling

When one considers a gauge model for weak interactions,[19] the parity-violating gauge coupling appears. We here comment on the treatment of such a case by considering the simple model

$$\mathcal{L} = \bar\psi_L(x)i\not{D}\psi_L(x) + \frac{1}{2g^2}\operatorname{Tr}F^{\mu\nu}F_{\mu\nu} \tag{4.1}$$

in place of (2.1), with $\psi_L(x) \equiv [(1-\gamma_5)/2]\psi(x)$.

By noting that $\gamma_5\varphi_n(x)$ belongs to the eigenvalue $-\lambda_n$ if $\varphi_n(x)$ has an eigenvalue $\lambda_n$ in (2.6), we define[9]

$$\begin{aligned}\varphi_n^L(x) &\equiv \left(\frac{1-\gamma_5}{\sqrt{2}}\right)\varphi_n(x) \quad \text{if}\ \lambda_n>0\\ &\equiv \left(\frac{1-\gamma_5}{2}\right)\varphi_n(x) \quad \text{if}\ \lambda_n=0\,,\end{aligned} \tag{4.2}$$

$$\begin{aligned}\varphi_n^R(x) &\equiv \left(\frac{1+\gamma_5}{\sqrt{2}}\right)\varphi_n(x) \quad \text{if}\ \lambda_n>0\\ &\equiv \left(\frac{1+\gamma_5}{2}\right)\varphi_n(x) \quad \text{if}\ \lambda_n=0\,.\end{aligned} \tag{4.3}$$

Note that $\varphi_n(x)$ with $\lambda_n=0$ can be chosen to be the eigenvector of $\gamma_5$. The basis vectors $\varphi_n^L$ and $\varphi_n^R$ thus defined form a complete orthonormal set. We then expand

$$\psi_L(x) \equiv \sum_{\lambda_n\geq 0} a_n\varphi_n^L(x)\,, \tag{4.4}$$

$$\bar\psi_L(x) \equiv \sum_{\lambda_n\geq 0} \bar b_n\varphi_n^R(x)^\dagger\,. \tag{4.5}$$

Under the local chiral transformation

$$\begin{aligned}\psi_L(x) &\to e^{-i\alpha(x)}\psi_L(x)\,,\\ \bar\psi_L(x) &\to \bar\psi_L(x)e^{i\alpha(x)}\,,\end{aligned} \tag{4.6}$$

the Lagrangian (4.1) changes:

$$\mathcal{L} \to \mathcal{L} + \partial_\mu\alpha(x)\bar\psi_L(x)\gamma^\mu\psi_L(x)\,. \tag{4.7}$$

The path-integral measure then transforms as

$$d\mu \equiv \prod_x [\mathfrak{D} A_\mu(x)] \prod_n da_n \prod_n d\bar{b}_n$$

$$\to d\mu \exp\left\{ i \int \alpha(x) \sum_{\lambda_n \geq 0} [\varphi_n^L(x)^\dagger \varphi_n^L(x) - \varphi_n^R(x)^\dagger \varphi_n^R(x)]\, dx \right\}$$

$$= d\mu \exp\left\{ -i \int \alpha(x) \sum_{\text{all}\,\lambda_n} \varphi_n(x)^\dagger \gamma_5 \varphi_n(x) dx \right\}$$

$$= d\mu \exp\left\{ -i \int \alpha(x) A(x) dx \right\} \qquad (4.8)$$

by using (2.13). Namely, the phase factor becomes one-half of the previous example (2.14). The current defined from (4.7) [in accordance with the phase convention in (2.9) and (2.20)],

$$j^\mu(x) \equiv -\bar{\psi}_L(x)\gamma^\mu \psi_L(x) = -\bar{\psi}(x)\gamma^\mu \left(\frac{1-\gamma_5}{2}\right)\psi(x), \qquad (4.9)$$

thus satisfies a WT identity similar to (2.19) with an anomaly factor which is *one-half* of the previous example. It should be noted that we still sum the series in (4.8) according to the eigenvalue of $\not{D}$, although we transformed the eigenvectors to those of $\gamma_5$ in (4.2) and (4.3). This prescription of summation according to the eigenvalue of $\not{D}$ is essential to obtain a well-defined anomaly factor.

By considering a general chiral transformation

$$\psi_L(x) \to e^{i\alpha(x)\gamma_5 T^a} \psi_L(x) = e^{-i\alpha(x) T^a} \psi_L(x) \qquad (4.10)$$

with $T^a$ the generator of the gauge transformation, the *covariant* derivative of the current

$$j_a^\mu(x) \equiv -\bar{\psi}_L(x)\gamma^\mu T^a \psi_L(x) \qquad (4.11)$$

has an anomaly factor:

$$A^a(x) \equiv \sum_n \varphi_n(x)^\dagger \gamma_5 T^a \varphi_n(x) = \frac{1}{2}\left(\frac{-1}{8\pi^2}\right) \mathrm{Tr}\{T^a {}^*F^{\mu\nu} F_{\mu\nu}\}. \qquad (4.12)$$

This relation (4.12) gives rise to the well-known rule for the anomaly[20]; for SU(2) Yang-Mills fields, for example, (4.12) vanishes.

For a general gauge model such as the Weinberg-Salam model, the ordinary result follows if one uses basis vectors which diagonalize covariant derivative operators by taking the index theorem as a guiding principle.

### Anomaly for Einstein gravity

The anomaly for the gravitational field was first discussed by Kimura,[21] and it has been later clarified by many authors.[22,23] For the sake of completeness, we here briefly comment on the gravitational anomaly.[24] As the renormalization of the Einstein gravity is not well understood yet, we consider only the case of the external gravitational field.

We start with the fermionic part of the Lagrangian

$$\mathcal{L} = h(x)(\bar{\psi} i \gamma^\mu D_\mu \psi - m\bar{\psi}\psi) \qquad (4.13)$$

with

$$\gamma^\mu(x) \equiv h_a^\mu(x)\gamma^a, \quad \{\gamma^a, \gamma^b\} = 2G^{ab},$$
$$D_\mu \equiv \partial_\mu - \tfrac{1}{2} i A_{\mu,mn}(x) S^{mn}, \quad S^{mn} \equiv \tfrac{1}{4} i [\gamma^m, \gamma^n], \qquad (4.14)$$
$$\{\gamma^\mu, \gamma^\nu\} = 2h_a^\mu h_b^\nu G^{ab} \equiv 2g^{\mu\nu}(x), \quad h(x) \equiv \det(h_\mu^a).$$

The metric for the local Lorentz frame is $G^{ab} = (1, -1, -1, -1)$. After a suitable Wick rotation in the *local Lorentz frame*, e.g., $h_0^\mu \to i h_4^\mu$, $h \to -ih$, and $\gamma^0 \to -i\gamma^4$ with $\gamma_5 \equiv \gamma^4\gamma^1\gamma^2\gamma^3$ as before [the Euclidean metric is $G^{ab} = (-1, -1, -1, -1)$], we consider a complete set of bases belonging to the *Hermitian* operator $\not{D} \equiv \gamma^\mu D_\mu$:

$$\not{D}\varphi_n(x) = \lambda_n \varphi_n(x), \qquad \int h(x)\varphi_n(x)^\dagger \varphi_l(x) d^4x = \delta_{n,l}. \qquad (4.15)$$

The discussion from now on parallels the previous one. We obtain the chiral transformation law

$$d\mu \to d\mu \exp\left[-2i \int \alpha(x) A(x) dx\right] \qquad (4.16)$$

with

$$A(x) = \sum_n h(x)\varphi_n(x)^\dagger \gamma_5 \varphi_n(x) \qquad (4.17)$$

which can be evaluated as usual:

$$A(x) = \lim_{M\to\infty} \sum_n h\varphi_n(x)^\dagger \gamma_5 e^{-(\lambda_n/M)^2} \varphi_n(x)$$
$$= \lim_{M\to\infty} \lim_{x'\to x} \mathrm{Tr}\gamma_5 h e^{-(\not{D}/M)^2} \sum_n \varphi_n(x)\varphi_n(x')^\dagger$$
$$= \lim_{M\to\infty} \lim_{x'\to x} \mathrm{Tr}\gamma_5 h \exp\left[-\left(D^\mu D_\mu + \frac{R}{4}\right)/M^2\right] \frac{1}{h(x)} \times \delta(x-x'), \qquad (4.18)$$

where the limit $x' \to x$ is inserted to specify that the differential operator acts on $x$, and $R$ is the scalar curvature. The covariant derivative $D_\mu$ from now on stands in general for the full covariant derivative with the Christoffel affinity included. We represent the invariant $\delta$ function by

$$\frac{1}{h(x)}\,\delta(x-x')\equiv\frac{1}{h(x)}\int\frac{d^4k}{(2\pi)^4}e^{ik_\mu D^\mu\sigma(x,x')}\,, \quad (4.19)$$

with the geodesic biscalar [21,25] $\sigma(x,x')$ [a generalization of $\frac{1}{2}(x-x')^2$ in the flat space] defined by

$$\begin{aligned}\sigma(x,x')&=\tfrac{1}{2}g^{\mu\nu}(x)D_\mu\sigma(x,x')D_\nu\sigma(x,x')\\&=\tfrac{1}{2}g^{\mu'\nu'}(x')D'_\mu\sigma(x,x')D'_\nu\sigma(x,x')\end{aligned} \quad (4.20)$$

and $\sigma(x,x)=0$. It can be confirmed that (4.19) defines a biscalar function, and it reduces to the left-hand side of (4.19) after the integration over $k_\mu$ by noting

$$\lim_{x'\to x} D_\mu D^\nu\sigma(x,x')=g^{\nu}_{\ \mu}\,. \quad (4.21)$$

In (4.19) $k_\mu$ is just a numerical parameter, and we regard (4.19) as a biscalar function formed of $h^a_\mu(x)$ and $\partial^\mu\sigma(x,x')$. By inserting (4.19) into (4.18), we obtain

$$\begin{aligned}A(x)&=\lim_{M\to\infty}\lim_{x'\to x}\int\frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\gamma_5\exp\{-[(i\Delta^\mu+D^\mu)(i\Delta_\mu+D_\mu)+\tfrac{1}{4}R]/M^2\}\\&=\lim_{M\to\infty}\lim_{x'\to x}M^4\int\frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\gamma_5e^{\Delta^\mu\Delta_\mu}\big(e^{-\Delta^\mu\Delta_\mu}\exp\{-[(i\Delta^\mu+D^\mu/M)(i\Delta_\mu+D_\mu/M)+R/(4M^2)]\}\big)\end{aligned} \quad (4.22)$$

after the rescaling $k_\mu\to Mk_\mu$, with $\Delta^\mu(x,x')\equiv k_\alpha D^\mu D^\alpha\sigma(x,x')$. By expanding the quantity inside the bold parentheses in powers of $1/M$ in the last expression of (4.22) by imitating the Dyson expansion, we obtain

$$\begin{aligned}A(x)&=\mathrm{Tr}\gamma_5\int\frac{d^4k}{(2\pi)^4}\,e^{k^\mu k_\mu}\{\tfrac{1}{2}D^\mu D_\mu D^\alpha D_\alpha+\tfrac{2}{3}k_\mu k_\nu k_\alpha k_\beta D^\mu D^\nu D^\alpha D^\beta+\tfrac{2}{3}k_\alpha k_\beta[D^\alpha D^\mu D_\mu D^\beta+D^\mu D_\mu D^\alpha D^\beta+D^\alpha D^\beta D^\mu D_\mu]\}\\&=\frac{h(x)}{192\pi^2}\,\mathrm{Tr}\{\gamma_5[D^\mu,D^\nu][D_\mu,D_\nu]\}\\&=-\frac{1}{384\pi^2}\,\tfrac{1}{2}\epsilon^{\alpha\beta\lambda\rho}R^{\mu\nu}{}_{\alpha\beta}R_{\mu\nu\lambda\rho}\,,\end{aligned} \quad (4.23)$$

where

$$\epsilon^{\alpha\beta\lambda\rho}\equiv h\epsilon^{abcd}h^\alpha_a h^\beta_b h^\lambda_c h^\rho_d \quad (4.24)$$

and $R_{\mu\nu\alpha\beta}$ is the Riemann-Christoffel tensor ($\epsilon^{abcd}$ is normalized as $\epsilon^{1230}=\epsilon^{1234}=1$).

By combining $A(x)$ with the variation of the Lagrangian under the chiral transformation, we obtain the relation

$$\partial_\mu\langle j^\mu_5(x)\rangle=2mi\langle j_5(x)\rangle-\frac{i}{384\pi^2}\,\epsilon^{\alpha\beta\lambda\rho}R^{\mu\nu}{}_{\alpha\beta}R_{\mu\nu\lambda\rho}\,, \quad (4.25)$$

with

$$\begin{aligned}j^{\ \mu}_5(x)&\equiv h(x)\,\bar\psi(x)\gamma^\mu\gamma_5\psi(x)\,,\\j_5(x)&\equiv h(x)\,\bar\psi(x)\gamma_5\,\psi(x)\end{aligned} \quad (4.26)$$

thus recovering the familiar result[21~24] in the background gravitational field.

We emphasize that the anomaly factor (4.23) does not depend on the detailed properties of $\sigma(x,x')$ except for the basic requirement (4.21). Moreover, only the integration over the parameter $k_\mu$ in the form

$$\int d^4k f''(k^2)\,,\quad\int d^4k\,k^2f'''(k^2)\,,\quad\int d^4k\,(k^2)^2f^{(\mathrm{iv})}(k^2) \quad (4.27)$$

contributes to the anomaly factor (4.23) if one uses the notation in (2.24). These integrals (4.27) are independent of the detailed form of $f(z)$ as before. Also, we note that the unit bispinor[21,23] $I(x,x')$ is not required in our calculation (4.22), as the result is automatically gauge invariant after the symmetric integration over $k_\mu$. The regularization independence of the gravitational anomaly is confirmed, for example, by choosing a suitable local coordinate so that

$$h^\mu_a(x)=\delta^\mu_a\,,\quad\partial_\nu h_{a\mu}(x)=0\,. \quad (4.28)$$

A direct contact with the perturbative calculation[21,22] is achieved if one replaces the right-hand side of (4.19) by

$$\lim_{x'\to x}\frac{1}{h(x)}\int\frac{d^4k}{(2\pi)^4}\,e^{ik_\mu(x^\mu-x'^\mu)} \quad (4.29)$$

which corresponds to the use of "plane waves," and the simultaneous replacement of $(\not D/M)^2$ by $(h\not D/M)^2$ in the exponential factor in (4.18). We still obtain the same anomaly factor as before.[26]

The anomaly for the supergravity can be similarly treated[27,24] and the Faddeev-Popov ghost fields contribute to the chiral anomaly, a situation similar to the Pauli-Villars regularization

(2.36). The property as an identity appears very clearly in this example: If one exponentiates the Faddeev-Popov determinant by using the auxiliary fields, both the axial-vector current and the anomaly factor receive the contribution from the Faddeev-Popov fields. If one does not exponentiate, neither of them receive the contribution from the Faddeev-Popov determinant.

## V. CONCLUSION

The present investigation shows that the change of basis vectors in the functional space should be clearly recognized when one formulates the perturbation theory in the path-integral formalism. This change of basis vectors is apparent in the operator formalism when one transforms the Heisenberg (or Schrödinger) picture to the interaction picture, and it is realized by a formal unitary transformation. The corresponding change of basis vectors in the path-integral formalism, however, does not lead to a unitary transformation of the "chirality index" associated with the functional measure. As a result, the missing chirality index leads to the anomalous behavior of chiral WT identities when one calculates other parts of WT identities *correctly* in the interaction picture. The nonunitary nature of the transformation must also exist in the operator formalism, but it appears in a more clear-cut way in the path-integral formalism with the aid of the semiclassical index theorem. We also note that the Wick-rotated Euclidean theory, when looked at in momentum space, renders the time-ordered product at the coincident-time limit more manageable and thus allows us to see the basic mechanism of the anomaly.

In quantum theory, it is customary to attach the primary importance to the Lorentz-covariant "energy" operator[28] $\not{D}$. The fact that $\not{D}$ and $\gamma_5$ do not commute provides an intuitive way of seeing the origin of the anomalous behavior, although it does not necessarily imply the appearance of the anomaly.[12]

We finally note that WT identities are based on the *local* symmetry transformation specified by the parameter $\alpha(x)$, and the anomaly factor always appears in the combination $\int \alpha(x)A(x)dx$ in (2.14). As a result, they are not sensitive to the precise boundary condition at space-time infinity, and they hold irrespective whether the symmetry is spontaneously broken or not. As was noted in Sec. III, the spontaneous symmetry breaking is characterized by the mismatch between the local WT identity and the *global* symmetry property of the theory (or the ground state). A general treatment of the spontaneous symmetry breaking, in particular the dynamical symmetry breaking, requires a more careful treatment of the boundary condition in the path-integral formalism. A difficult aspect of the dynamical symmetry breaking resides in the fact that the field variables appearing in the path integral and the effective field variables after the spontaneous symmetry breaking, which are generally accompanied by Nambu-Goldstone bosons, are not quite identical. In comparison, the "effective" field theory such as the Higgs model (and to a certain extent the instanton phenomenon) allows a basically WKB-type treatment, and it can be readily incorporated into the path-integral formalism by imposing a suitable constraint on the functional domain.

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