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Solution of the One-Dimensional N-Body Problems with Quadratic and/or Inversely Quadratic Pair Potentials

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The quantum-mechanical problems of N 1-dimensional equal particles of mass m interacting pairwise via quadratic ("harmonical") and/or inversely quadratic ("centrifugal") potentials is solved. In the first case, characterized by the pair potential $\frac{1}{4}m\omega^2(x_i - x_j)^2 + g(x_i - x_j)^{-2}$, $g > -\hbar^2/(4m)$, the complete energy spectrum (in the center-of-mass frame) is given by the formula

$$E = \hbar\omega(\tfrac{1}{2}N)^{\frac{1}{2}}\left[\tfrac{1}{2}(N-1) + \tfrac{1}{2}N(N-1)(a+\tfrac{1}{2}) + \sum_{l=2}^{N} ln_l\right],$$

with $a = \frac{1}{2}(1 + 4mg\hbar^{-2})^{\frac{1}{2}}$. The $N - 1$ quantum numbers n_l are nonnegative integers; each set $\{n_l;\ l = 2, 3, \cdots, N\}$ characterizes uniquely one eigenstate. This energy spectrum can also be written in the form $E_s = \hbar\omega(\frac{1}{2}N)^{\frac{1}{2}}[\frac{1}{2}(N-1) + \frac{1}{2}N(N-1)(a+\frac{1}{2}) + s]$, $s = 0, 2, 3, 4, \cdots$, the multiplicity of the sth level being then given by the number of different sets of $N - 1$ nonnegative integers n_l that are consistent with the condition $s = \sum_{l=2}^{N} ln_l$. These equations are valid independently of the statistics that the particles satisfy, if $g \neq 0$; for $g = 0$, the equations remain valid with $a = \frac{1}{2}$ for Fermi statistics, $a = -\frac{1}{2}$ for Bose statistics. The eigenfunctions corresponding to these energy levels are not obtained explicitly, but they are rather fully characterized. A more general model is similarly solved, in which the N particles are divided in families, with the same quadratic interaction acting between all pairs, but with the inversely quadratic interaction acting only between particles belonging to the same family, with a strength that may be different for different families. The second model, characterized by the pair potential $g(x_i - x_j)^{-2}$, $g > -\hbar^2/(4m)$, contains only scattering states. It is proved that an initial scattering configuration, characterized (in the phase space sector defined by the inequalities $x_i \geq x_{i+1}$, $i = 1, 2, \cdots, N - 1$, to which attention may be restricted without loss of generality) by (initial) momenta p_i, $i = 1, 2, \cdots, N$, goes over into a final configuration characterized uniquely by the (final) momenta p_i', with $p_i' = p_{N+1-i}$. This remarkably simple outcome is a peculiarity of the case with equal particles (i.e., equal masses and equal strengths of all pair potentials).

1. INTRODUCTION

The motivation for a physicist to study 1-dimensional problems is best illustrated by the story of the man who, returning home late at night after an alcoholic evening, was scanning the ground for his key under a lamppost; he knew, to be sure, that he had dropped it somewhere else, but only under the lamppost was there enough light to conduct a proper search. In a more serious vein, a motivation is perceived[1] in the insight that exact solutions, even of oversimplified models, may provide, and in the possibility to assess the reliability of approximation techniques that can be used in more realistic contexts, by first testing them in exactly solvable cases. Moreover for some physical problems a 1-dimensional schematization may indeed be appropriate. And a final argument emphasizes the formal elegance that exactly solvable models often reveal.

There do not exist many examples of N-body problems with pair forces that can be solved exactly, even in one dimension. The case with only quadratic ("harmonical") potentials may be reduced, by a linear reshuffle of the particle variables, to a problem with decoupled oscillators; thus it can easily be solved, both in the classical and quantal cases, and in a space with any number of dimensions. However, just because it can be reduced to a problem with decoupled variables, it is not so interesting as an example of many-body dynamics, though of course it is quite important from other points of view (for instance, to provide a denumerable basic set of eigenstates for the description of the system). The only other known solvable example, first introduced by Berezin, Pochil, and Finkelberg and by McGuire, is that of N equal-mass particles interacting in one dimension via 2-body equal-strength zero-range δ-function potentials.[2] This example is very interesting, especially in the case of attractive forces, when a collection of many-body bound states exist.[3] But the zero-range character of the forces, implying that two particles interact only when their positions coincide, reduces the problem, at least in the classical case, to a sequence of 2-body processes, whose outcome is primarily determined by kinematics alone (energy and momentum conservation imply that in a 1-dimensional 2-body collision between equal-mass particles no new momenta can be produced; the two particles either maintain their initial momenta or exchange them). And indeed, as remarked by McGuire, it is just because this simplification is maintained in the quantal case (for equal particles) that the problem is also solvable in this case.

In this paper we present a rather complete analysis (in the framework of quantum mechanics) of two new 1-dimensional N-body problems that bear some resemblance to the two models mentioned above. The first problem considers N equal particles interacting via pair potentials that are the sum of a quadratic ("harmonical") plus an inversely quadratic ("centrifugal") term:

$$V(x_i - x_j) = \tfrac{1}{4}m\omega^2(x_i - x_j)^2 + g(x_i - x_j)^{-2},\qquad g > -\hbar^2/(4m). \quad (1.1)$$

For $N = 3$ this problem was completely solved recently, i.e., all its eigenvalues and eigenfunctions were explicitly exhibited.[4] Moreover, in the N-body case, a subset (including the ground-state) of the wavefunctions and energy levels were found, and the conjecture was put forward that, irrespective of the statistics that the particle obey (Boltzmann, Bose, or Fermi), the complete energy spectrum for this problem differ from the spectrum of the corresponding problem with $g = 0$ (i.e., with only harmonical forces) and with Fermi statistics, only by the (N-dependent) constant[5]

$$\Delta_F E = \tfrac{1}{2}N(N-1)\hbar\omega(\tfrac{1}{2}N)^{\frac{1}{2}}\tfrac{1}{2}[(1 + 4mg\hbar^{-2})^{\frac{1}{2}} - 1]. \quad (1.2)$$

This conjecture is validated in the present paper by a computation of the complete spectrum. Indeed we prove that, *irrespective of the statistics (Boltzmann, Bose, or Fermi) that the particles obey, the energy spectrum (in the center-of-mass frame) of the 1-dimensional N-body problem with the pair potential (1.1) (with $g \neq 0$) coincides (except for a constant shift of all energy levels) with the energy spectrum of the corresponding problem with only harmonical forces ($g = 0$) and identical particles (bosons or fermions). The shift has the value $\Delta_F E$, Eq. (1.2), relative to the fermion case, and*

$$\Delta_B E = \tfrac{1}{2}N(N-1)\hbar\omega(\tfrac{1}{2}N)^{\frac{1}{2}}\tfrac{1}{2}[(1 + 4mg\hbar^{-2})^{\frac{1}{2}} + 1] \quad (1.3)$$

relative to the boson case. The coincidence of these spectra (after the shift has been taken into account) refers not only to the values of the energy levels, but also to their multiplicities. This result also implies the coincidence [except for the energy shift

$$-\Delta_F E + \Delta_B E = \tfrac{1}{2}N(N-1)\hbar\omega(\tfrac{1}{2}N)^{\frac{1}{2}}]$$

of the spectra of the 1-dimensional N-body systems with only harmonical forces and Fermi or Bose statistics.[6] It should be emphasized that, even though the spectrum of the N-body system under consideration coincides, except for a constant shift, with that of the N-body system with only harmonical forces and Bose or Fermi statistics, the presence of the additional "centrifugal" potential excludes the possibility of reducing the problem to one with decoupled variables by a simple redefinition of the particle variables.

The second model that we study might be regarded as a special case of the first, obtained eliminating from it the harmonical potential, i.e., setting $\omega = 0$. But, in fact, the two models differ qualitatively, because, in the first case, the energy spectrum (in the center-of-mass frame) is discrete, and only localized states are allowed (corresponding to classical orbits that are restricted to a finite phase-space region; in fact, these classical orbits are presumably all closed[7]), while in the latter case the spectrum is continuous and only scattering states occur. The solution of the scattering problem for this model in the 3-body case was evinced by Marchioro[8] from the stationary eigenfunctions given explicitly in Ref. 4. He proved that, both in the classical and the quantal cases, *an initial "ingoing" scattering configuration, characterized (in the sector of configuration space defined by the inequalities $x_i \geq x_{i+1}$, $i = 1, 2, \cdots, N-1$, to which attention may be confined without loss of generality; see below) by the momenta p_i, $i = 1, 2, \cdots, N$, goes eventually over into a uniquely determined "outgoing" configuration characterized by (final) momenta p_i' determined by the simple rule*

$$p_i' = p_{N+1-i}, \quad i = 1, 2, \cdots, N. \quad (1.4)$$

This remarkable outcome (proved by Marchioro for $N = 3$) coincides with that that obtains in the case of infinitely repulsive δ-function potentials. However, while in the δ-function case this result is, at least in the classical case, quite trivial, because, as mentioned above, the zero-range character of the forces reduces the scattering process to a sequence of 2-body encounters whose outcome is determined by kinematics alone,[2] in the present case, owing to the long range character of the interaction, the simple relation (1.4) between the asymptotic momenta obtains after a quite complicated, definitely nontrivial, time evolution. In this paper we prove that, as conjectured by Marchioro,[8] also in the N-body case the remarkable rule of Eq. (1.4) obtains. The proof is done directly in the quantal case; clearly the validity of the result also in the classical case is implied (but a detailed proof through the explicit solution of the equations of motion would be a nontrivial task). Arguments are also given that imply that the simple rule (1.4) is characteristic only of the case with equal particles (i.e., with equal masses and equal coupling constants).

Let us finally emphasize the two features of the models considered in this paper that presumably have more potential for applications, especially for testing approximation techniques: the singular nature at zero range of the inversely quadratic potential, and its long range.

The first model is treated in Sec. 2; in Sec. 3, a generalization of it is discussed, where the N particles are divided into families, and there are equal quadratic potentials acting between all particle pairs but inversely quadratic potentials acting only between pairs belonging to the same family, with coupling constants that can be different for different families. In Sec. 4, we treat the second type of model, characterized by inversely quadratic potentials and only scattering states. Finally, in Sec. 5 we mention some queries that are naturally suggested by the quest for a more complete understanding of these types of models and we discuss the prospects of further generalizations. Some material has been relegated in five Appendices. Although we use some results of previous papers (especially of C) without reporting their proofs, the presentation should be sufficiently selfcontained to be understandable by readers not already familiar with the subject.

2. THE SYSTEM WITH QUADRATIC AND INVERSELY QUADRATIC POTENTIALS

The Hamiltonian of the system under consideration is

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \sum_{i=2}^{N}\sum_{j=1}^{i-1}\{\tfrac{1}{4}m\omega^2(x_i - x_j)^2 + g(x_i - x_j)^{-2}\}. \quad (2.1)$$

Throughout this paper we assume that

$$g > -\hbar^2/(4m) \quad (2.2)$$

to prevent the collapse that a more attractive inversely quadratic potential would cause.[4,9] (In the classical case the condition becomes of course $g > 0$.) Our task is to solve the eigenvalue equation

$$H\psi_s = E_s\psi_s, \quad (2.3)$$

where ψ_s is a translation invariant eigenfunction; in particular, we shall find all the eigenvalues E_s together with their multiplicities.

Hereafter we restrict attention to the sector of configuration space corresponding to a definite ordering of the particles, say

$$x_i \geq x_{i+1}, \quad i = 1, 2, \cdots, N-1. \quad (2.4)$$

In fact, the singular nature of the centrifugal interaction, together with the restriction to one space dimension, forbids any particle from overtaking any other particle. This is reflected in the vanishing not only of the wavefunction, but also of its derivative with respect to any particle coordinate, whenever the coordinates of two particles coincide.[10] As a consequence the extension of the wavefunction to the whole configuration space is achieved by the simple prescription

$$\psi(Px) = \eta_P\psi(x), \quad (2.5)$$

where x indicates the set $\{x_i; i = 1, 2, \cdots, N\}$ of Eq. (2.4), P indicates an arbitrary permutation, η_P equals unity if the particles obey Bose statistics and equals the parity of the permutation if the particles obey Fermi statistics. If the particles are distinguishable (Boltzmann statistics), each wavefunction ψ may correspond to $N!$ different states, each one of these being characterized by a wavefunction vanishing identically for all but one of the $N!$ possible orderings of the particles, and for that one being given by Eq. (2.5). However, rather than considering these $N!$ states as degenerate eigenstates of the same system, it is more appropriate to view them as $N!$ altogether different systems, the difference being enforced by the dynamical superselection rule that prevents particles from crossing over each other. Therefore, hereafter, when counting the multiplicity of the eigenvalues, we shall assume that, irrespective of the original nature of the particles (identical or distinguishable), one and only one state correspond to each different eigenfunction of Eq. (2.3). Hence, there shall be no need to distinguish between the different statistics, since such a distinction affects neither the spectrum nor its multiplicities, nor indeed the wavefunctions in the sector (2.4), but only the prescription to continue them elsewhere. This situation is of course consistent with the remark that, since the singular centrifugal force prevents the particles from crossing over one another, their sequential order can in fact be used to identify them even if they were undistinguishable to begin with.

We now assert that the normalizable solutions of Eq. (2.3) can be cast into the form

$$\psi(x) = z^{a+\frac{1}{2}}\varphi(r)P_k(x), \quad (2.6)$$

where the variables z and r and the constant a are defined as in C, namely

$$z = \prod_{i=2}^{N} \prod_{j=1}^{i-1} (x_i - x_j), \tag{2.7}$$

$$r^2 = \frac{1}{N} \sum_{i=2}^{N} \sum_{j=1}^{i-1} (x_i - x_j)^2, \tag{2.8}$$

$$a = \tfrac{1}{2}(1 + 4mg\hbar^{-2})^{\frac{1}{2}}, \tag{2.9}$$

and $P_k(x)$ is a homogeneous polynomial of degree k in the particle coordinates.[11] These polynomials are also assumed to be translation invariant and to be solutions of the "generalized Laplace equation"

$$\left[\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2(a + \tfrac{1}{2}) \sum_{i=2}^{N} \sum_{j=1}^{i-1} (x_i - x_j)^{-1} \times \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)\right] P_k(x) = 0. \tag{2.10}$$

We indicate hereafter with $g(N, k)$ the number of independent polynomials, homogeneous of degree k and translation invariant, that are solutions of this generalized Laplace equation [a nonpositive value of $g(N, k)$ indicates that no translation-invariant polynomial solution of Eq. (2.10) exists for the corresponding values of N and k]. The properties of these polynomials and the values of the quantities $g(N, k)$ are discussed below and in Appendix C.

To prove our assertion we insert the ansatz (2.6) into Eq. (2.3), and using Eq. (2.10) and the homogeneity of $P_k(x)$ we obtain (see Appendix A for a detailed derivation)

$$-\hbar^2/(2m)\{\varphi'' + [N + 2k - 2 + N(N-1)(a + \tfrac{1}{2})]r^{-1}\varphi'\} + \{\tfrac{1}{4}m\omega^2 N r^2 - E\}\varphi = 0, \tag{2.11}$$

where the primes indicate differentiation. The normalizable solutions of this equation are of course the functions

$$\varphi_{nk}(r) = \exp[-\tfrac{1}{2}(m\omega/\hbar)(\tfrac{1}{2}N)^{\frac{1}{2}} r^2] L_n^b[(m\omega/\hbar)(\tfrac{1}{2}N)^{\frac{1}{2}} r^2], \quad n = 0, 1, 2, \cdots, \tag{2.12}$$

where L_n^b is a Laguerre polynomial[12] and

$$b = k + \tfrac{1}{2}(N - 3) + \tfrac{1}{2}N(N-1)(a + \tfrac{1}{2}). \tag{2.13}$$

The corresponding energy eigenvalues are

$$E_{2n+k} = \hbar\omega(\tfrac{1}{2}N)^{\frac{1}{2}} \times [\tfrac{1}{2}(N-1) + \tfrac{1}{2}N(N-1)(a + \tfrac{1}{2}) + 2n + k], \quad n = 0, 1, 2, \cdots, \quad k = 0, 1, 2, \cdots. \tag{2.14}$$

This formula may be rewritten as follows:

$$E_s = E_s^F + \Delta_F E, \quad s = 0, 1, 2, \cdots, \tag{2.15a}$$
$$= E^B + \Delta_B E, \quad s = 0, 1, 2, \cdots, \tag{2.15b}$$

with

$$E_s^F = \hbar\omega(\tfrac{1}{2}N)^{\frac{1}{2}}[\tfrac{1}{2}(N^2 - 1) + s], \quad s = 0, 1, 2, \cdots, \tag{2.16a}$$

$$E_s^B = \hbar\omega(\tfrac{1}{2}N)^{\frac{1}{2}}[\tfrac{1}{2}(N - 1) + s], \quad s = 0, 1, 2, \cdots, \tag{2.16b}$$

and

$$\Delta_F E = \tfrac{1}{2}N(N-1)\hbar\omega(\tfrac{1}{2}N)^{\frac{1}{2}}(a - \tfrac{1}{2}), \tag{2.17a}$$
$$\Delta_B E = \tfrac{1}{2}N(N-1)\hbar\omega(\tfrac{1}{2}N)^{\frac{1}{2}}(a + \tfrac{1}{2}). \tag{2.17b}$$

As shall be proved below, E_s^F, respectively E_s^B, Eqs. (2.16a) and (2.16b), are the energy levels of the 1-dimensional N-body problem with only oscillator forces ($g = 0$) and with Fermi, respectively Bose, statistics. Moreover, the multiplicity of the energy levels E_s, E_s^F, and E_s^B (for the respective problems) is the same, and it is given by the formula (proved in Appendix C)

$$f(N, s) = \sum_n g(N, s - 2n)\theta(s - 2n)\theta(n). \tag{2.18}$$

Here, and always in the following,

$$\theta(x) = 1, \quad x \geq 0, \qquad = 0, \quad x < 0, \tag{2.19}$$

and all sums run over integral values of the dummy indices. A vanishing value of $f(N, s)$ (as it occurs, for instance, for $s = 1$; see below) indicates that the corresponding energy level is not present. A more detailed discussion of $f(N, s)$ is given in Appendix C; in particular, it is shown that $f(N, s)$ is the number of completely symmetrical polynomials that are homogeneous of degree s and translation invariant, and that it also coincides with the number of different solutions of the equation

$$s = \sum_{l=2}^{N} l n_l, \tag{2.20}$$

where n_l are nonnegative integers. Explicit expressions for $f(N, s)$ for N up to 5 and arbitrary s are also reported in Appendix C, together with a proof of the asymptotic formula

$$\lim_{s \to \infty} [f(N, s)s^{2-N}] = [N!\,(N-2)!]^{-1}. \tag{2.21}$$

While these equations provide explicit information on the multiplicity of the energy level E_s, it is often more useful to use for the energy spectrum the formula

$$E_{\{n_l\}} = \hbar\omega(\tfrac{1}{2}N)^{\frac{1}{2}} \times \left[\tfrac{1}{2}(N-1) + \tfrac{1}{2}N(N-1)(a + \tfrac{1}{2}) + \sum_{l=2}^{N} l n_l\right]. \tag{2.22}$$

Here, each of the $N-1$ integers n_i can take any nonnegative value, and to each set $\{n_i\}$ there corresponds one and only one eigenstate. Thus, this form of the formula for the spectrum automatically takes care of the multiplicity problem; this is implied by a comparison with the previous formula, Eq. (2.15), and by the statement reported above [see Eq. (2.20)].

The polynomials $P_k(x)$ are, by definition, translation invariant and homogeneous of degree k, and they satisfy the generalized Laplace equation (2.10).[13] *The last requirement implies that they are completely symmetrical under the exchange of any two coordinates* x_i, x_j; a formal proof of this most important property is given in Appendix B. Thus, they can be written in the form

$$P_k(x) = S \sum_{\{n_i\}} a_{\{n_i\}} \left[\prod_{i=1}^{N} x_i^{n_i}\right]\left[\prod_{i=1}^{N} \theta(n_i - n_{i-1})\right]\delta_{k,\Sigma}, \tag{2.23}$$

this being the most general form that a completely symmetrical polynomial can take. Here, by definition,

$$\Sigma = \sum_{i=1}^{N} n_i, \tag{2.24}$$

S is the operator that symmetrizes over all exchanges of the coordinates, $\{n_i\}$ indicates a set of the N (nonnegative) integers n_i, and $\sum_{\{n_i\}}$ indicates the sum over all such sets. The sum extends, in fact, only over the sets satisfying the conditions

$$n_i \geq n_{i-1} \geq n_0 \equiv 0, \quad i = 1, 2, \cdots, N, \tag{2.25}$$

$$k = \sum_{i=1}^{N} n_i, \tag{2.26}$$

these restrictions being explicitly enforced by the θ functions and by the Kronecker-δ function; by convention, $n_0 \equiv 0$. The constants $a_{\{n_i\}}$ must, of course, be chosen so that the polynomial of Eq. (2.23) be translation invariant and satisfy the generalized Laplace equation (2.10); for small values of N and k one can easily find in this manner all the polynomials $P_k(x)$. A different way to write the most general completely symmetrical polynomial of N variables that is homogeneous of degree k is reported in Appendix C.

There remains to prove that the spectra of Eqs. (2.16a) and respectively (2.16b) [or, equivalently, Eq. (2.22) with $a = \frac{1}{2}$, respectively $a = -\frac{1}{2}$] correspond to the cases with only oscillator forces ($g = 0$) and Fermi, respectively Bose, statistics. This follows from the remark that, for $g = 0$, the ansatz of Eq. (2.6) may be used, with $a = \frac{1}{2}$ (Fermi statistics) or $a = -\frac{1}{2}$ (Bose statistics), throughout all configuration space—i.e., not only in the sector specified by Eq. (2.4).[14] In the Fermi case, one obtains thereby again the generalized Laplace equation (2.10) (with $a = \frac{1}{2}$), and the fact that only symmetrical polynomials are solutions of this equation is consistent with the requirement that the wavefunction ψ of Eq. (2.6) be completely antisymmetrical [note that z is completely antisymmetrical, while of course r, and therefore also any function $\varphi(r)$, is completely symmetrical]. In the Bose case, the ansatz of Eq. (2.6) with $a = -\frac{1}{2}$ implies that the (homogeneous and translation-invariant) polynomials $P_k(x)$ satisfy the usual Laplace equation, namely Eq. (2.10) with $a = -\frac{1}{2}$; though nonsymmetrical polynomial solutions of this equation do now exist, they are excluded by the symmetry requirement itself, and this guarantees that also the multiplicity for the Bose case coincides with that of the general case (the number of completely symmetrical polynomial solutions, homogeneous of degree k and translation invariant, of the Laplace equation, coincides with the number of polynomial solutions, homogeneous of degree k and translation invariant, of the generalized Laplace equation, since the latter are automatically constrained to be completely symmetrical; see Appendices B and C).

The result we have just proved implies that the (Gibbs) partition function for the 1-dimensional system composed of N identical oscillators coupled by pair harmonical potentials is the same (except for a multiplicative constant) for Bose and Fermi statistics [it is also the same (except for a multiplicative constant) for the system discussed in this paper, having additional pair inverse-square potentials of arbitrary strength (but the same for all pairs)[15]]. This fact does not appear to have been previously noted, possibly because the possibility to handle the many-oscillator problem in higher-dimensional spaces focused attention on the more realistic 3-dimensional case.[16]

3. GENERALIZED MODEL WITH QUADRATIC AND INVERSELY QUADRATIC POTENTIALS

In this section we discuss briefly the generalized problem characterized by the Hamiltonian

$$\begin{aligned} H = &-\frac{\hbar^2}{2m}\sum_{\alpha=1}^{A}\sum_{i=1}^{N_\alpha}\frac{\partial^2}{\partial x_{\alpha i}^2} \\ &+ \tfrac{1}{8}m\omega^2 \sum_{\alpha=1}^{A}\sum_{i=1}^{N_\alpha}\sum_{\beta=1}^{A}\sum_{j=1}^{N_\beta}(x_{\alpha i} - x_{\beta j})^2 \\ &+ \sum_{\alpha=1}^{A} g_\alpha \sum_{i=2}^{N_\alpha}\sum_{j=1}^{i-1}(x_{\alpha i} - x_{\alpha j})^{-2}. \end{aligned} \tag{3.1}$$

This Hamiltonian describes N particles, with

$$N=\sum_{\alpha=1}^{A} N_\alpha, \tag{3.2}$$

divided in A families of N_α particles, $\alpha=1,2,\cdots,A$. All the N particles interact pairwise through the harmonical potential $\frac{1}{4}m\omega^2(x_{\alpha i}-x_{\beta j})^2$, and in addition within each family (but not between different families) the particles interact pairwise through the potential $g_\alpha(x_{\alpha i}-x_{\alpha j})^{-2}$. To avoid 2-body collapse,[4,9] we assume as usual that

$$g_\alpha > -\hbar^2/(4m), \quad \alpha=1,2,\cdots,A. \tag{3.3}$$

The case of the preceding section corresponds to the present one with $A=1$. The proofs of the results reported in this section parallel closely those of the corresponding results of Sec. 2, and are therefore omitted. We assume, whenever relevant, that within each family the particles are identical, satisfying Bose or Fermi statistics.

As in the preceding case, we need solve the eigenvalue problem only in the sector of configuration space characterized by the prescriptions

$$x_{\alpha i} \geq x_{\alpha, i+1}, \quad \alpha=1,2,\cdots,A';\ i=1,2,\cdots,N_\alpha-1, \tag{3.4}$$

where A' is characterized by the condition

$$g_\alpha \neq 0, \quad \alpha=1,2,\cdots,A',$$
$$g_\alpha = 0, \quad \alpha=A'+1, A'+2,\cdots,A;$$

A' may be zero or it may coincide with A. Once the problem is solved within the sector (3.4), the extension of the many-body wavefunctions to the whole configuration space is easily achieved by the obvious generalization of the prescriptions discussed in detail at the beginning of the preceding section.

It is convenient to introduce the quantities

$$z_\alpha=\prod_{i=2}^{N_\alpha}\prod_{j=1}^{i-1}(x_{\alpha i}-x_{\alpha j}), \tag{3.5}$$

$$r^2=\frac{1}{2N}\sum_{\alpha=1}^{A}\sum_{i=1}^{N_\alpha}\sum_{\beta=1}^{A}\sum_{j=1}^{N_\beta}(x_{\alpha i}-x_{\beta j})^2, \tag{3.6}$$

$$a_\alpha=\tfrac{1}{2}(1+4mg_\alpha\hbar^{-2})^{\frac{1}{2}}. \tag{3.7}$$

If $g_\alpha=0$, $a_\alpha=\frac{1}{2}$ for Fermi statistics, $a_\alpha=-\frac{1}{2}$ for Bose statistics.

It can then be shown that the eigenfunctions of the eigenvalue equation

$$H\psi_{nk}=E_{nk}\psi_{nk} \tag{3.8}$$

have the form

$$\psi_{nk}=\left\{\prod_{\alpha=1}^{A}[z_\alpha^{a_\alpha+\frac{1}{2}}]\right\}\exp\left(-\frac{1}{2}\frac{m\omega}{\hbar}(\tfrac{1}{2}N)^{\frac{1}{2}}r^2\right) \times L_n^b\left[\frac{m\omega}{\hbar}(\tfrac{1}{2}N)^{\frac{1}{2}}r^2\right]\cdot P_k(x), \tag{3.9}$$

where L_n^b is a Laguerre polynomial,[12]

$$b=k+\tfrac{1}{2}(N-3)+\sum_{\alpha=1}^{A}\tfrac{1}{2}N_\alpha(N_\alpha-1)(a_\alpha+\tfrac{1}{2}), \tag{3.10}$$

and $P_k(x)$ is a translation-invariant polynomial in the N variables $x_{\alpha i}$, homogeneous of degree k and satisfying the generalized Laplace equation

$$\left[\sum_{\alpha=1}^{A}\sum_{i=1}^{N_\alpha}\frac{\partial^2}{\partial x_{\alpha i}^2}+2\sum_{\alpha=1}^{A}(a_\alpha+\tfrac{1}{2})\sum_{i=2}^{N_\alpha}\sum_{j=1}^{i-1}(x_{\alpha i}-x_{\alpha j})^{-1} \times\left(\frac{\partial}{\partial x_{\alpha i}}-\frac{\partial}{\partial x_{\alpha j}}\right)\right]P_k(x)=0. \tag{3.11}$$

It can be shown that the polynomials $P_k(x)$ are completely symmetrical (invariant) for any coordinate exchange within the same family [provided the corresponding g_α does not vanish, or, if it vanishes, provided the particles satisfy Fermi statistics, so that $a_\alpha=\frac{1}{2}$; if, instead, g_α vanishes and the particles of the αth family satisfy Bose statistics, so that $a_\alpha=-\frac{1}{2}$, only solutions that are completely symmetrical in the variables $x_{\alpha i}$, $i=1,2,\cdots,N_\alpha$, should be considered, even though nonsymmetrical solutions of the generalized Laplace equation (3.11) exist].

The energy eigenvalues are given by the formula

$$E_{nk}=E_{2n+k}=\hbar\omega(\tfrac{1}{2}N)^{\frac{1}{2}}\left[\tfrac{1}{2}(N-1)+\sum_{\alpha=1}^{A}\tfrac{1}{2}N_\alpha(N_\alpha-1)(a_\alpha+\tfrac{1}{2})+2n+k\right]. \tag{3.12}$$

All these equations are straightforward generalizations of the corresponding equations of the preceding section. The analysis of the degeneracy of these energy levels could also be carried out in analogy to the treatment given there and in Appendix C.

4. THE SYSTEM WITH INVERSELY QUADRATIC POTENTIALS

The Hamiltonian of the system under consideration is

$$H=-\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}+g\sum_{i=2}^{N}\sum_{j=1}^{i-1}(x_i-x_j)^{-2}, \tag{4.1}$$

again with the condition (2.2) to prevent 2-body collapse.[4,9] This Hamiltonian has of course no discrete spectrum; it describes only scattering states. The treatment of Sec. 2 and Appendix A implies that the complete set of stationary eigenfunctions of this problem is (in the center-of-mass frame)

$$\psi_{pk} = z^{a+\frac{1}{2}} r^{-b} J_b(pr) P_k(x), \quad k = 0, 1, 2, \cdots, \quad p \geq 0, \tag{4.2}$$

with z, r, a, b, and $P_k(x)$ defined as in Sec. 2 [Eqs. (2.7)–(2.10) and (2.13)], and with p connected to the energy eigenvalue by

$$E = \hbar^2 p^2/(2m). \tag{4.3}$$

The product of r^{-b} times the Bessel function $J_b(pr)$ is the (regular) solution of the differential equation

$$-\frac{\hbar^2}{2m}\left[\frac{d^2}{dr^2} + (1 + 2b)\frac{1}{r}\frac{d}{dr} + p^2\right]\varphi(r) = 0, \tag{4.4}$$

which coincides with Eq. (2.11) when $\omega = 0$ and the energy E is given by Eq. (4.3). Of course, for each value of k there are $g(N, k)$ independent eigenfunctions, corresponding to the $g(N, k)$ independent solutions $P_k(x)$ of Eq. (2.10). We shall indicate them hereafter as $P_{kq}(x)$, using the quantum number q, which takes all integral values from 1 to $g(N, k)$, to label them. It is important to recall that, as proved in Appendix B, all these polynomials $P_{kq}(x)$ are completely symmetrical (invariant) under the exchange of any two coordinates x_i, x_j.

To simplify the discussion we assume hereafter that the particles are distinguishable. Attention need be confined only to the sector of phase space corresponding to a definite ordering of the particles, say that specified by the inequalities (2.4), whose validity is assumed hereafter.

The most general stationary eigenfunction of the Hamiltonian (4.1), corresponding to the eigenvalue (4.3), can be written in the form

$$\psi = z^{a+\frac{1}{2}} \sum_{k=0}^{\infty} \sum_{q=1}^{g(N,k)} c_{kq} r^{-A-k} J_{A+k}(pr) P_{kq}(x), \tag{4.5}$$

where to expose the dependence on the quantum number k we have introduced the convenient constant

$$A = b - k = \tfrac{1}{2}(N - 3) + \tfrac{1}{2}N(N - 1)(a + \tfrac{1}{2}). \tag{4.6}$$

To discuss scattering, we need only the asymptotic behavior of this function when all particles are far apart from each other. Then

$$\psi \sim \psi_{\text{in}} + \psi_{\text{out}}, \tag{4.7}$$

where

$$\psi_{\text{in}} \sim (\tfrac{1}{2}\pi pr)^{-\frac{1}{2}} z^{a+\frac{1}{2}} r^{-A} \times \sum_{k=0}^{\infty} \sum_{q=1}^{g(N,k)} c_{kq} r^{-k} e^{i(A+k+\frac{1}{2})\frac{1}{2}\pi - ipr} P_{kq}(x), \tag{4.8}$$

and

$$\psi_{\text{out}} \sim (\tfrac{1}{2}\pi pr)^{-\frac{1}{2}} z^{a+\frac{1}{2}} r^{-A} \times \sum_{k=0}^{\infty} \sum_{q=1}^{g(N,k)} c_{kq} r^{-k} e^{-i(A+k+\frac{1}{2})\frac{1}{2}\pi + ipr} P_{kq}(x). \tag{4.9}$$

The wavy symbol $\sim$ in these equations and below indicates asymptotic equality, i.e., equality up to corrections of order r^{-2}.

The stationary eigenfunction describing, in the center-of-mass frame, the scattering situation is characterized by the form

$$\psi_{\text{in}} \sim c \exp\left[i \sum_{i=1}^{N} p_i x_i\right], \tag{4.10}$$

with

$$p_i \leq p_{i+1}, \quad i = 1, 2, \cdots, N - 1, \tag{4.11}$$

$$p^2 = \sum_{i=1}^{N} p_i^2, \tag{4.12}$$

and

$$\sum_{i=1}^{N} p_i = 0. \tag{4.13}$$

We now prove that if the constants c_{kq} are chosen so that Eq. (4.8) yields (4.10), then from Eq. (4.9) there also follows

$$\psi_{\text{out}} \sim e^{-iA\pi} c \exp\left[i \sum_{i=1}^{N} p_{N+1-i} x_i\right]. \tag{4.14}$$

In fact, the symmetry and the homogeneity of $P_{kq}(x)$ imply that

$$P_{kq}(-Sx) = e^{-ik\pi} P_{kq}(x), \tag{4.15}$$

where S indicates an arbitrary permutation of the coordinates x_i. We focus attention hereafter on the special permutation T, defined by

$$Tx_i = x_{N+1-i}, \quad i = 1, 2, \cdots, N. \tag{4.16}$$

The distinguishing property of this permutation is that the set $\{-Tx\}$ belongs to the same sector (2.4) of configuration space as the set $\{x\}$, because the inequalities

$$x_i \geq x_{i+1}, \quad i = 1, 2, \cdots, N - 1, \tag{2.4}$$

also imply

$$-x_{N+1-i} \geq -x_{N-i}, \quad i = 1, 2, \cdots, N - 1. \tag{4.17}$$

Now using Eq. (4.15) we can rewrite ψ_{out}, Eq. (4.9), in the form

$$\psi_{\text{out}} \sim e^{-i\pi A} (\tfrac{1}{2}\pi \bar{p} r)^{-\frac{1}{2}} z^{a+\frac{1}{2}} r^{-A} \times \sum_{k=0}^{\infty} \sum_{q=1}^{g(N,k)} c_{kq} r^{-k} e^{i(A+k+\frac{1}{2})\frac{1}{2}\pi - i\bar{p}r} P_{kq}(-Tx), \tag{4.18}$$

where we have set formally

$$\bar{p} = -p. \tag{4.19}$$

Let us re-emphasize that this representation of ψ_{out} applies in the sector (2.4) of configuration space. It may therefore be compared with the representation of ψ_{in}, Eq. (4.8); the comparison yields, through Eq. (4.10), the result

$$\psi_{\text{out}} \sim e^{-i\pi A} c \exp\left[-i \sum_{i=1}^{N} \bar{p}_i T x_i\right], \qquad (4.20)$$

where $\bar{p}_i$ is related to $\bar{p}$ in the same manner as p_i is related to p. Since by dimensional arguments we can assert that

$$p_i = p\alpha_i, \qquad (4.21)$$

where the "angular" variables α_i are independent of p, Eq. (4.19) implies simply

$$\bar{p}_i = -p_i. \qquad (4.22)$$

Substituting this equality and Eq. (4.16) in Eq. (4.20) and changing the dummy index i into $N + 1 - i$ yield Eq. (4.14). QED

Let us now discuss the implications of the result we have just proved. The initial wavefunction (4.10) describes, in the sector (2.4) of configuration space, a (free) state where particle 1 has momentum p_1, particle 2 has momentum p_2, etc., the inequality (4.11) insuring that this is indeed an incoming scattering state, i.e., one where each particle gets less close to every other particle if time runs backward. The final wavefunction (4.14) describes, in the same sector (2.4) of configuration space, a (free) state where particle 1 has momentum p_N, particle 2 has momentum p_{N-1}, etc., the inequality (4.11) insuring that this is indeed an outgoing scattering state, i.e., one where each particle gets farther away from every other particle as time goes on. The result just proven implies that the stationary eigenfunction of the Hamiltonian H, Eq. (4.1), which is identified by the condition that its incoming part coincide with Eq. (4.10), contains only the outgoing wave Eq. (4.14); thus the initial state characterized by particle 1 having momentum p_1, particle 2 having momentum p_2, etc., can go only into the final state characterized by particle 1 having momentum p_N, particle 2 having momentum p_{N-1}, etc. This corresponds to the rule

$$p'_i = p_{N+1-i}, \quad i = 1, 2, \cdots, N, \qquad (4.23)$$

where p_i is the initial momentum of the ith particle and p'_i is the (only allowed value of the) final momentum of the same particle; we already noted above that the inequalities (4.11), satisfied by the initial momenta p_i, imply automatically that the final momenta p'_i given by Eq. (4.23) satisfy the inequalities

$$p'_i \geq p'_{i+1}, \quad i = 1, 2, \cdots, N - 1, \qquad (4.24)$$

which characterize outgoing states. It should perhaps be emphasized that the rule (4.23) is not implied, for $N > 2$, by energy and momentum conservation [Eqs. (4.12) and (4.13), and the corresponding equations with p'_i in place of p_i], although it is of course consistent with these requirements.

The result (4.23) is certainly true, but nontrivial, also in the classical case; note that it is independent of the value of the coupling constant g, that only enters, through Eq. (4.6), in the phase factor $\exp(-i\pi A)$ multiplying Eq. (4.14).

The result (4.23) was already known for the case of infinitely repulsive zero-range δ-function interactions, in which case it is actually quite trivial in the classical case, although a bit less so in the quantal case[2]; that problem corresponds indeed to the limit of the present one as $g \to 0$, $a \to \frac{1}{2}$, since for extremely small g the interaction $g(x_i - x_j)^{-2}$ is effective only at very short interparticle separation $x_i \approx x_j$, where its singular nature continues to prevent the particles from crossing over—namely, it has the same effect as an infinitely repulsive δ-function potential.

Finally we discuss whether the result (4.23) holds only for the case of equal particles, or if it remains true even if the coupling constants g_{ij} multiplying the inversely quadratic potentials acting between the ith and jth particles are not all equal. The comparison with the infinitely repulsive zero-range δ-function case, as discussed above, suggests that this is, at least approximately, the case if all the constants g_{ij} are extremely small, independently from their being equal or different. But the measure of the smallness of the (nondimensionless) coupling constants g_{ij} is nontrivial. In the quantal case the dimensionless constants $2mg_{ij}\hbar^{-2}$ could be considered; but they become infinite in the classical limit, whereas the phenomenon just discussed should continue to be relevant in this limit. On the other hand, in the classical case an examination of the example discussed in Ref. 8 and in Appendix D suggests that the distinguishing parameter between the "small g" and the "not-small g" cases is the dimensionless quantity

$$\eta = \max_{i=1,2,\cdots,N-1}\left(\min_t \{[x_i(t) - x_{i+1}(t)]/r(t)\}\right), \qquad (4.25)$$

where t is the time and $r(t)$ is connected to the coordinates $x_i(t)$ through Eq. (2.8). But this quantity does not depend only on the parameters m and g_{ij} of the system; it depends also on the initial data p_i and a_i, defined by

$$x_i(t) \xrightarrow[t\to-\infty]{} p_i t/m + a_i + O(t^{-1}). \qquad (4.26)$$

Therefore, no analogous quantity exists in the quantal case, since, in this case, we cannot specify both p_i and a_i. One might therefore be inclined to conjecture that the result (4.23) hold in all cases, for it should hold for small coupling constants g_{ij} independently from their being equal, and at the same time it should be independent of their scale. It is instead much more reasonable to conjecture that the result (4.23) does not hold, unless all the coupling constants (and the masses) are equal. In the quantal case, the asymptotic outgoing wavefunction ψ_{out}, corresponding to the asymptotic wavefunction ψ_{in} of Eq. (4.10), has, in general, the form

$$\psi_{\text{out}} \sim c \int dp_1' \cdots dp_N' \delta\left(\sum_{i=1}^{N}[p_i'^2 - p_i^2]\right)\delta\left(\sum_{i=1}^{N}[p_i' - p_i]\right) \times \left[\prod_{i=1}^{N-1}\theta(p_i' - p_{i+1}')\right] \exp\left[i\sum_{i=1}^{N} p_i' x_i\right] S(p_i, p_i'), \tag{4.27}$$

where the δ functions insure energy and momentum conservation, while the θ functions insure that the momenta satisfy the inequalities (4.24) characterizing a final scattering state. The S-matrix function, in the general case, need not contain any additional δ function and, in particular, need not reduce to a product of δ functions; only if all the coupling constants g_{ij} are equal (and all the masses are also equal), i.e., only in the case treated above, the function $S(p_i, p_i')$ (can be computed and) reduces to a product of δ functions[17]

$$S(p_i, p_i') = e^{-i\pi A}\left[\prod_{i=1}^{N} \delta(p_i' - p_{N+1-i})\right] \Big/ \left[\delta\left(\sum_{i=1}^{N}[p_i'^2 - p_i^2]\right)\delta\left(\sum_{i=1}^{N}[p_i' - p_i]\right)\right], \tag{4.28}$$

so that Eq. (4.27) reproduces Eq. (4.14). In addition, one conjectures that a measure of the deviation of the S matrix $S(p_i, p_i')$ for the general case from that of Eq. (4.28) be given by the quantity

$$\epsilon = \max_{i,j,i',j'} |(g_{ij} - g_{i'j'})| \Big/ \sum_{l=2}^{N}\sum_{k=1}^{l-1}|g_{lk}|, \tag{4.29}$$

which is clearly independent of the scale of the coupling constants g_{ij}. Of course, the value of this quantity is relevant to measuring the violation of the rule (4.23) also in the classical case, in addition to the quantity η of Eq. (4.25).

The above conjecture concerning the invalidity of the rule (4.23) in the general case (with different coupling constants g_{ij}) is strongly supported by the study of the 3-body classical case. Indeed in Appendix D it is proved that the rule (4.23) does not hold, at least in some cases, if the coupling constants g_{ij} are different; and the treatment suggests that if the coupling constants are different, the rule (4.23) almost never holds—namely, it is always violated except possibly for some symmetrical set of coupling constants and of initial data, the set of such initial data having, however, presumably a null measure relative to the whole set of possible initial conditions. These statements refer of course to the classical case; obviously they imply that, also in the quantal case, the S matrix does not reduce to the simple form (4.28), unless all the coupling constants, and the masses, be equal.

5. OPEN PROBLEMS

A number of questions are suggested by the results obtained. There is the problem of finding an explicit representation for the eigenfunctions, as it was done in the 3-body case.[4] This problem coincides with that of finding an explicit representation of the (translation-invariant) homogeneous polynomial solutions $P_k(x)$ of the generalized Laplace equations (2.10) and (3.11).[18]

Then there is the problem of solving the classical case, displaying explicitly the time evolution of the particle coordinates, for both types of models. For the first type (Secs. 2 and 3), it should be verified whether the plausible conjecture of periodic motion (already mentioned in the Introduction, and strongly suggested by the degenerate nature of the quantal spectrum) is confirmed.[7] For the second type (Sec. 4), the simple rule (4.23) connecting the initial momenta p_i to the final momenta p_i' should of course be recovered.

In view of the simplicity of the results characteristic of both types of models, it should be possible to reobtain the same conclusions, possibly more simply, by displaying the group-theoretical structure that certainly underlies them.[18] Note that this group-theoretical structure must be a peculiarity of the equal-particle models (see below).

As regards generalizations, the first problem that comes to mind is the extension to models characterized by different coupling constants g_{ij} of the inversely quadratic potential. Is such a generalization of the models of Secs. 2 and 3 still characterized by a completely[19] linear spectrum (in the quantal case, and by periodic motion, in the classical case)? It is plausible to conjecture that this is generally not the case [even though the linear formula

$$E_s = \hbar\omega(\tfrac{1}{2}N)^{\frac{1}{2}}\left[s + \tfrac{1}{2}(N-1) + \sum_{i=2}^{N}\sum_{j=1}^{i-1}(a_{ij} + \tfrac{1}{2})\right], \tag{5.1}$$

with

$$a_{ij} = \tfrac{1}{2}(1 + 4mg_{ij}\hbar^{-2})^{\frac{1}{2}}, \qquad (5.2)$$

is an appealing generalization of Eq. (3.12), also in view of the simple physical interpretation it suggests, according to which the potential $g_{ij}(x_i - x_j)^{-2}$ produces simply an energy shift of the amount $\hbar\omega(\frac{1}{2}N)^{\frac{1}{2}}(a_{ij} \pm \frac{1}{2})$ relative to the case with $g_{ij} = 0$ and with the particles i and j being identical bosons (+ sign) or fermions (− sign)]. This conjecture is validated in Appendix E by a perturbative computation in the 3-body case.[20] For the second type of model (Sec. 4), we have already all but proved (in Sec. 4 and Appendix D) that the rule (4.23) breaks down unless all the coupling constants g_{ij} coincide.

A much more interesting generalization is of course in the direction of two or more space dimensions. It is expected that neither the completely linear nature of the spectrum for the first type of models nor the simple rule (4.23) for the second type of model is maintained, if the restriction to 1-space is dropped.

Another category of models that is somewhat similar to those considered in Secs. 2 and 3 of this paper and whose solution, even only in 1-dimensional space, would be most interesting, is defined by Hamiltonians such as

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \tfrac{1}{4}m\omega^2\sum_{i=2}^{n}\sum_{j=1}^{i-1}(x_i - x_j)^2 + g\sum_{i=2}^{N}\sum_{j=1}^{i-1}(x_i - x_j)^{-2}, \qquad (5.3)$$

with $n < N$. Already for $N = 3$ and $n = 2$ this Hamiltonian describes both a system with a discrete spectrum and one with scattering (between particle 3 and the bound state of particles 1 and 2, with the possibility of anelastic collisions occurring, although of course without break up). These two systems are characterized by (the same Hamiltonian but) different particle orderings, in the former case with particle 3 trapped between particles 1 and 2, in the latter case with particle 3 to the left (or to the right) of both particles 1 and 2.

Finally, we mention the possibility to interpret the results of this paper, and of the preceding ones,[4,8] in the context of the classical theory of wave propagation, say physical optics. In fact, these results imply the existence of a potential for which the fundamental equation of wave propagation

$$[\Delta + p^2 - V]\psi = 0 \qquad (5.4)$$

can be solved exactly. Here, of course, Δ is the Laplace operator (in N dimensions; clearly the case $N = 3$ is the most interesting), and the potential is

$$V = G\sum_{i=2}^{N}\sum_{j=1}^{i-1}(x_i - x_j)^{-2}, \quad G > -\tfrac{1}{2}, \qquad (5.5)$$

where now x_i are the (Cartesian) coordinates of a point in space.[21] This potential is of course not spherically symmetrical, and it is singular on the planes $x_i = x_j$; but it is otherwise acceptable within each of the wedges (sectors) in which the planes $x_i = x_j$ slice the whole space (in particular, it vanishes asymptotically). We leave to the interested reader the task of formulating in the language of physical optics the conclusions of Sec. 4 of this paper (or of Ref. 8). Such a translation is just the converse exercise to that performed by those (for instance, McGuire,[2] and before him Nussenzweig[2]) who invented solvable 1-dimensional problems by reinterpreting results that had been previously obtained in the framework of the theory of electromagnetic wave propagation.

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APPENDIX A

In this appendix we report in detail the steps required to derive Eq. (2.11).

We begin by noting that

$$\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} = \frac{1}{N}\frac{\partial^2}{\partial y_0^2} + \sum_{i=1}^{N-1}\frac{\partial^2}{\partial y_i^2}, \qquad (A1)$$

where the translation-invariant "Jacobi coordinates" are defined by[22]

$$y_i = [i(i+1)]^{-\frac{1}{2}}\Big(ix_{i+1} - \sum_{j=1}^{i}x_j\Big), \quad i = 1, 2, \cdots, N-1, \qquad (A2)$$

while

$$y_0 = \frac{1}{N}\sum_{i=1}^{N}x_i \qquad (A3)$$

is the center-of-mass coordinate. We also note that a transition from the "Cartesian" coordinates y_i, $i = 1, 2, \cdots, N-1$, to the "spherical" coordinates r, Ω_i, $i = 1, 2, \cdots, N-2$, with[23]

$$r^2 = \sum_{i=1}^{N-1} y_i^2, \tag{A4}$$

yields

$$\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} = \frac{\partial^2}{\partial y_0^2} + r^{2-N} \frac{\partial}{\partial r} r^{N-2} \frac{\partial}{\partial r} + r^{-2}\hat{L}, \tag{A5}$$

where the operator $\hat{L}$ acts only on the $N-2$ "angular" coordinates Ω_i.

It is important to note that *the radial coordinate r defined by Eq. (A4) coincides with that introduced previously through Eq. (2.8)*, as may be verified by an explicit computation.

Similarly, we note that

$$\sum_{i=2}^{N} \sum_{j=1}^{i-1} (x_i - x_j)^{-1} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right) = \tfrac{1}{2}N(N-1)\frac{1}{r}\frac{\partial}{\partial r} + r^{-2}\hat{M}, \tag{A6}$$

where the operator $\hat{M}$ acts only on the angular coordinates. This equation follows from the relation

$$\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right) r = (x_i - x_j)/r, \tag{A7}$$

which is implied by the definition (2.8) of r.

Next we note that if $P_k(x)$ is a translation-invariant homogeneous polynomial of degree k in the N x_i coordinates, it is also a homogeneous polynomial of degree k in the $N-1$ y_i coordinates, and, therefore, $r^{-k}P_k(x)$ depends on the $N-2$ angular coordinates Ω_i but is independent of r. Using this remark, and Eqs. (A5) and (A6), we may cast Eq. (2.10) in the form

$$(\hat{L} + 2(a + \tfrac{1}{2})\hat{M}) r^{-k}P_k(x) = -k[k + N - 3 + N(N-1)(a + \tfrac{1}{2})] r^{-k}P_k(x). \tag{A8}$$

Finally, we note the two important relations

$$\sum_{i=1}^{N} \left(\frac{\partial}{\partial x_i} z^{a+\frac{1}{2}}\right)\frac{\partial}{\partial x_i} = (a + \tfrac{1}{2}) z^{a+\frac{1}{2}} \sum_{i=2}^{N}\sum_{j=1}^{i-1} (x_i - x_j)^{-1}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right), \tag{A9}$$

$$\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} z^{a+\frac{1}{2}} = 2mg\hbar^{-2} z^{a+\frac{1}{2}} \sum_{i=2}^{N}\sum_{j=1}^{i-1} (x_i - x_j)^{-2}. \tag{A10}$$

The first one follows directly from the definition of z, Eq. (2.7), which implies

$$\frac{1}{z}\frac{\partial z}{\partial x_i} = \sum_{\substack{j=1 \\ j\neq i}}^{N} (x_i - x_j)^{-1}, \tag{A11}$$

and from the identity

$$\sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N} (x_i - x_j)^{-1}\frac{\partial}{\partial x_i} = \sum_{i=2}^{N}\sum_{j=1}^{i-1} (x_i - x_j)^{-1}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right). \tag{A12}$$

The second equation obtains evaluating the second derivative of $z^{a+\frac{1}{2}}$,

$$\frac{\partial^2}{\partial x_i^2} z^{a+\frac{1}{2}} = z^{a+\frac{1}{2}}\left[(a^2 - \tfrac{1}{4}) z^{-2}\left(\frac{\partial z}{\partial x_i}\right)^2 + (a + \tfrac{1}{2}) z^{-1}\frac{\partial^2 z}{\partial x_i^2}\right], \tag{A13}$$

and then using the definition of a, Eq. (2.9), and the two equations

$$z^{-2}\sum_{i=1}^{N}\left(\frac{\partial z}{\partial x_i}\right)^2 = 2\sum_{i=2}^{N}\sum_{j=1}^{i-1}(x_i - x_j)^{-2}, \tag{A14}$$

$$\sum_{i=1}^{N}\frac{\partial^2 z}{\partial x_i^2} = 0, \tag{A15}$$

which are proved in C.

All the necessary tools having been prepared, we turn now to the derivation of Eq. (2.11). Substituting Eq. (2.6) into Eq. (2.3) and using Eqs. (A9) and (A10), we get

$$\left\{-\frac{\hbar^2}{2m}\left[\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + 2(a + \tfrac{1}{2})\sum_{i=2}^{N}\sum_{j=1}^{i-1}(x_i - x_j)^{-1}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)\right] + \tfrac{1}{4}m\omega^2 N r^2 - E\right\} r^{-k}P_k(x) r^k \varphi(r) = 0, \tag{A16}$$

where we have explicitly displayed the function $r^{-k}P_k(x)$, which depends on the $N-2$ angular variables Ω_i, and the function $r^k\varphi(r)$, which depends only upon the radial coordinate r. Then we use Eqs. (A5), (A6), and (A8), and we get

$$\left[-\frac{\hbar^2}{2m}\left(r^{2-N}\frac{d}{dr} r^{N-2}\frac{d}{dr} + N(N-1)(a + \tfrac{1}{2})\frac{1}{r}\frac{d}{dr} - k[k + N - 3 + N(N-1)(a + \tfrac{1}{2})] r^{-2}\right) + \tfrac{1}{4}m\omega^2 N r^2 - E\right] r^k\varphi(r) = 0. \tag{A17}$$

This equation immediately yields Eq. (2.11). QED

APPENDIX B

In this appendix we prove that any polynomial solution of the generalized Laplace equation

$$L_G P(x) \equiv \left[\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + G \sum_{i=2}^{N}\sum_{j=1}^{i-1} (x_i - x_j)^{-1}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)\right] P(x) = 0 \tag{B1}$$

is completely symmetrical under the exchange of any two coordinates x_i, x_j. The only assumption needed for the proof is that

$$G \neq -2p, \quad p = 0, 1, 2, \cdots. \tag{B2}$$

This condition is of course fulfilled in the case of Eq. (2.10), since $G = 2(a + \frac{1}{2})$ is in fact positive.

Let us assume, *per absurdum*, that $P(x)$ is not symmetrical under the exchange of two coordinates, say x_1 and x_2. Then $P(x)$ and $P_{12}P(x)$, where P_{12} is the operator that interchanges x_1 and x_2, are different, and the polynomial

$$Q(x) = (1 - P_{12})P(x) \tag{B3}$$

does not vanish identically. This polynomial is also a solution of Eq. (B1), since P_{12} commutes with the (completely symmetrical) operator L_G of Eq. (B1); moreover, it is antisymmetrical under the exchange of the coordinates x_1 and x_2, and therefore it can be written in the form

$$Q(x) = (x_1 - x_2)^{2p+1} R(x), \tag{B4}$$

where p is an integer not less than zero and $R(x)$ is a polynomial, symmetrical under the exchange of x_1 and x_2 and not identically vanishing for $x_1 = x_2$:

$$R(x)|_{x_1=x_2} \neq 0. \tag{B5}$$

But now from Eq. (B4) there follows that, in the neighborhood of $x_1 = x_2$,

$$(x_1 - x_2)^{-1}\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right) Q(x) = 2(2p + 1)(x_1 - x_2)^{2p-1} R(x) + O[(x_1 - x_2)^{2p}]. \tag{B6}$$

From Eq. (B4) we also get

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) Q(x) = 4p(2p + 1)(x_1 - x_2)^{2p-1} R(x) + O[(x_1 - x_2)^{2p}]. \tag{B7}$$

Inserting $Q(x)$ in Eq. (B1) and noting that all other terms besides those just computed produce contributions of order $(x_1 - x_2)^{2p}$, we finally obtain

$$2(2p + 1)[2p + G]R(x)(x_1 - x_2)^{2p-1} + O[(x_1 - x_2)^{2p}] = 0. \tag{B8}$$

Dividing this equation by $(x_1 - x_2)^{2p-1}$ and then setting $x_1 = x_2$ yields

$$2(2p + 1)[2p + G]R(x)|_{x_1=x_2} = 0. \tag{B9}$$

This equation, together with Eq. (B2), is in contradiction with Eq. (B5); the proof *per absurdum* is therefore accomplished.

APPENDIX C

In this appendix we discuss some results concerning the multiplicity of the energy levels.

We begin with a proof of Eq. (2.18). Let us assume that two eigenfunctions of the form[24]

$$\psi_{nk} = z^{a+\frac{1}{2}} \exp\left[-\frac{1}{2}\frac{m\omega}{\hbar}(\tfrac{1}{2}N)^{\frac{1}{2}} r^2\right] \times L_n^b\left[\frac{m\omega}{\hbar}(\tfrac{1}{2}N)^{\frac{1}{2}} r^2\right] P_k(x), \tag{C1}$$

say $\psi_{n_1k_1}$ and $\psi_{n_2k_2}$, are mutually orthogonal (and therefore linearly independent) unless $n_1 = n_2$ and $k_1 = k_2$; also, let us recall that by definition $g(N, k)$ is the number of linearly independent eigenfunctions corresponding to an assigned value of the quantum number k. Then the number $f(N, k)$ of linearly independent eigenfunctions corresponding to the quantum number $s = 2n + k$ is the sum of the numbers $g(N, k)$ over all allowed (i.e., nonnegative integral) values of n and k that are consistent with the relation $s = 2n + k$:

$$f(N, s) = \sum_{n,k} g(N, k)\delta_{s,2n+k}\theta(n)\theta(k). \tag{C2}$$

Performing the sum over k by means of the Kronecker delta yields Eq. (2.18). QED

There remains to prove the orthogonality of $\psi_{n_1k_1}$ and $\psi_{n_2k_2}$. This (rather obvious) property is most conveniently proved by rewriting ψ_{nk} in the form

$$\psi_{nk} = R_{nk}(r)\chi_k(\Omega_i), \tag{C1'}$$

where

$$R_{nk}(r) = r^{b-\frac{1}{2}(N-3)} \exp\left[-\frac{1}{2}\frac{m\omega}{\hbar}(\tfrac{1}{2}N)^{\frac{1}{2}} r^2\right] \times L_n^b\left[\frac{m\omega}{\hbar}(\tfrac{1}{2}N)^{\frac{1}{2}} r^2\right], \tag{C3a}$$

$$\chi_k(\Omega_i) = [z/r^{\frac{1}{2}N(N-1)}]^{a+\frac{1}{2}} r^{-k} P_k(x), \tag{C3b}$$

and by noticing that the Hamiltonian operator H of Eq. (2.1) may be rewritten in the separated form

$$H = H_{CM} + H_r + r^{-2}H_\Omega, \tag{C4}$$

with

$$H_{CM} = -\frac{\hbar^2}{2mN}\frac{\partial^2}{\partial y_0^2}, \tag{C5}$$

$$H_r = -\frac{\hbar^2}{2m} r^{2-N}\frac{\partial}{\partial r} r^{N-2}\frac{\partial}{\partial r} + \tfrac{1}{4}Nm\omega^2 r^2, \tag{C6}$$

$$H_\Omega = -\frac{\hbar^2}{2m}\hat{L} + g\sum_{i=2}^{N}\sum_{j=1}^{i-1}[r/(x_i - x_j)]^2. \tag{C7}$$

Here we are using the notation, and some of the results, introduced at the beginning of Appendix A.

Using the separated form of the eigenfunction, Eq. (C1′), and of the Hamiltonian, Eq. (C4), we can recast the eigenvalue equation

$$H\psi_{nk} = E_{nk}\psi_{nk} \tag{C8}$$

into the separated form

$$H_\Omega\chi_k(\Omega_i) = \lambda_k\chi_k(\Omega_i), \tag{C9}$$

$$(H_r + r^{-2}\lambda_k)R_{nk}(r) = E_{nk}R_{nk}(r), \tag{C10}$$

and from Eqs. (C3a) and (C6) we get

$$\lambda_k = [\hbar^2/(2m)][b^2 - \tfrac{1}{4}(N-3)^2] \tag{C11a}$$

$$= [\hbar^2/(2m)][k + \tfrac{1}{2}N(N-1)(a+\tfrac{1}{2})] \times [k + N - 3 + \tfrac{1}{2}N(N-1)(a+\tfrac{1}{2})]. \tag{C11b}$$

Since the operators H_Ω and H_r are Hermitian, Eqs. (C9) and (C11) imply the orthogonality of $\chi_{k_1}(\Omega_i)$ and $\chi_{k_2}(\Omega_i)$ unless $k_1 = k_2$, and Eqs. (C10) and (2.14) imply the orthogonality of R_{n_1k} and R_{n_2k} unless $n_1 = n_2$. The combination of these two results implies the orthogonality of $\psi_{n_1k_1}$ and $\psi_{n_2k_2}$ unless $n_1 = n_2$ and $k_1 = k_2$. QED

[It can be explicitly verified, using the formula

$$\prod_{i=1}^{N} dx_i\,\delta\left(\sum_{i=1}^{N}x_i\right) \propto d\Omega r^{N-2}\,dr, \tag{C12}$$

where $d\Omega$ indicates the differential element for the "angular" coordinates Ω_i, that the orthogonality of R_{n_1k} to R_{n_2k} reproduce the well-known orthogonality relation for Laguerre polynomials.]

We turn now to a discussion of the multiplicity indices $g(N,k)$ and $f(N,k)$, beginning with some definitions. Let $a(N,k)$ be the number of linearly independent (completely symmetrical) polynomials of N variables x_i that are homogeneous of degree k. Let $b(N,k)$ be the number of linearly independent completely symmetrical polynomials of N variables x_i that are homogeneous of degree k and translation invariant [we show presently that $b(N,k)$ coincides with $f(N,k)$]. Let $c(N,k)$ be the number of linearly independent (completely symmetrical) polynomials of N variables x_i that are homogeneous of degree k and that are solutions of a given generalized Laplace equation such as Eq. (2.10). Finally, as in Sec. 2, let $g(N,k)$ be the number of linearly independent (completely symmetrical) polynomials that are homogeneous of degree k, that are translation invariant, and that are solutions of a given generalized Laplace equation such as Eq. (2.10). Then of course

$$b(N,k) = a(N,k) - a(N,k-1), \tag{C13}$$

because the requirement that $P_k(x)$ be translation invariant is equivalent to the condition that the polynomial $\sum_{i=1}^{N}\partial P_k(x)/\partial x_i$ vanish identically, and it corresponds therefore to $a(N,k-1)$ equations, namely as many equations as the number of different monomials that make up the general completely symmetrical polynomial homogeneous of degree $k-1$. An analogous argument yields

$$c(N,k) = a(N,k) - a(N,k-2), \tag{C14}$$

since the application of the (generalized) Laplace operator to a completely symmetrical homogeneous polynomial of degree k yields a completely symmetrical homogeneous polynomial of degree $k-2$. Analogous arguments also yield

$$g(N,k) = b(N,k) - b(N,k-2), \tag{C15a}$$

$$g(N,k) = c(N,k) - c(N,k-1), \tag{C15b}$$

and then either one of these equations yields

$$g(N,k) = a(N,k) - a(N,k-1) - a(N,k-2) + a(N,k-3). \tag{C16}$$

On the other hand, Eqs. (C15a) and (2.18) immediately yield

$$f(N,k) = b(N,k). \tag{C17}$$

An explicit formula for the computation of $a(N,k)$ is

$$a(N,k) = \sum_{\{n_i\}}\delta_{k,\Sigma}\prod_{i=1}^{N}\theta(n_i - n_{i-1}). \tag{C18}$$

Here all symbols are defined as in Eq. (2.23); indeed the validity of this equation is a consequence of the possibility of writing in the form (2.23) the most general polynomial that is homogeneous of degree k and completely symmetrical, for Eq. (C18) is obtained by counting the number of coefficients $a_{\{n_i\}}$ that enter in Eq. (2.23). A general closed expression for this sum is not known; the computation of $a(N,k)$ from this formula is a tedious task already for $N = 3$. Explicit expressions for $a(N, k)$ for N up to 5 and arbitrary k have been obtained by a different procedure.[25] It is based on the possibility of writing the most general completely symmetrical polynomial in the alternative form

$$\sum a_{\{n_l\}}\delta_{k,\Sigma_{l=1}^{N} l n_l} \prod_{l=1}^{N} [S_l^{n_l}\theta(n_l)], \tag{C19}$$

where

$$S_l = \sum_{i=1}^{N} x_i^l, \tag{C20}$$

and the sum extends over all sets of N nonnegative integers n_l that are consistent with the homogeneity condition enforced by the Kronecker δ; note that it can be similarly asserted that the most general completely symmetrical and translation invariant polynomial of N variables, that is homogeneous of degree k, can be written in the form

$$\sum a_{\{n_l\}}\delta_{k,\Sigma_{l=2}^{N} l n_l} \prod_{l=2}^{N} [T_l^{n_l}\theta(n_l)], \tag{C21}$$

where

$$T_l = \sum_{i=1}^{N} (x_i - N^{-1}S_1)^l \tag{C22}$$

(so that obviously $T_1 = 0$ and T_l is translation invariant). The numbei $a(N, k)$ of linearly independent completely symmetrical polynomials of N variables, homogeneous of degree k, coincides with the number of coefficients $a_{\{n_l\}}$ entering in Eq. (C19); thus, it is the number of different sets $\{n_l; l = 1, 2, \cdots, N\}$ of N nonnegative integers that are consistent with the equation

$$k = \sum_{l=1}^{N} l n_l. \tag{C23}$$

Similarly Eq. (C22) implies that the number $b(N, k)$ of linearly independent completely symmetrical polynomials of N variables, homogeneous of degree k and translation invariant, coincides with the number of different sets $\{n_l; l = 2, 3, \cdots, N\}$ of $N - 1$ nonnegative integers that are consistent with the equation

$$k = \sum_{l=2}^{N} l n_l. \tag{C24}$$

But we proved above that this number coincides with the multiplicity index $f(N, k)$. Thus the statement reported in Sec. 2, and used there to obtain Eq. (2.22), is now proved.

The following trick[25] is convenient to evaluate $a(N, k)$. Introduce the generating function

$$A_N(z) = \sum_{k=0}^{\infty} a(N, k) z^k. \tag{C25}$$

The statement just proved implies the formula

$$A_N(z) = \sum_{n_1, n_2, \cdots, n_N = 0}^{\infty} z^{\Sigma_{l=1}^{N} l n_l}. \tag{C26}$$

From this we get

$$A_N(z) = \prod_{l=1}^{N} (1 - z^l)^{-1}. \tag{C27}$$

Separating this expression into partial fractions, and then re-expanding in powers of z and identifying the coefficients with those in Eq. (C25), yield explicit expressions for $a(N, k)$. In this manner, in Ref. 25, the following formulas were obtained:

$$a(2, k) = 1 + [\tfrac{1}{2}k], \tag{C28a}$$

$$a(3, k) = \{\tfrac{1}{12}(k + 2)(k + 4)\}, \tag{C28b}$$

$$a(4, k) = \{\tfrac{1}{144}(k + 2)(k^2 + 13k + 37 + \tfrac{9}{2}(1 + (-)^k))\}, \tag{C28c}$$

$$a(5, k) = [(\tfrac{1}{2880}((k + 1)(k + 2)(k + 3)(k + 24) + 155k^2 + 15k(67 + 3(-)^k))]. \tag{C28d}$$

In these four equations $[x]$ and $\{x\}$ indicate, respectively, the integral part of x and the integer closest to x. From these quantities and the formula

$$f(N, k) = a(n, k) - a(N, k - 1), \tag{C29}$$

which follows from Eqs. (C13) and (C17), one may immediately obtain explicit expressions for $f(N, k)$ for N up to 5 and arbitrary k. Alternatively, one may compute $f(N, k)$ directly in a similar manner, starting from the generating function

$$F_N(z) = \sum_{k=0}^{\infty} f(N, k) z^k \tag{C30a}$$

$$= \prod_{l=2}^{N} (1 - z^l)^{-1}. \tag{C30b}$$

In fact, if $a(N, k)$ is not already known, it is more convenient to compute $f(N, k)$ directly in this manner. Note that Eqs. (C30b) and (C27) imply

$$F_N(z) = (1 - z) A_N(z), \tag{C31}$$

and that this relation is consistent with Eqs. (C30a), (C25), and (C29). Also $g(N, k)$ is more easily obtainable directly from the generating function

$$G_N(z) = \sum_{k=0}^{\infty} g(N, k) z^k \tag{C32a}$$

$$= (1 - z^2) F_N(z) \tag{C32b}$$

$$= \prod_{l=3}^{N} (1 - z^l)^{-1}, \tag{C32c}$$

rather than from Eq. (C16) or

$$g(N, k) = f(N, k) - f(N, k - 2). \tag{C33}$$

Note that these expressions imply that $g(N, k)$ is the number of different solutions of the equation

$$k = \sum_{l=3}^{N} l n_l, \quad \text{(C34)}$$

where n_l are nonnegative integers. Thus, for $N = 3$, $g(3, k)$ is unity if k is a multiple of 3 and vanishes otherwise, a result consistent with the findings of Ref. 4.

Finally we note that the generating function technique is also convenient to obtain the asymptotic behavior of the quantities $a(N, k)$, $f(N, k)$, and $g(N, k)$ at large k. This is done identifying the residue of the pole at $z = 1$,

$$\lim_{z\to 1} [(1 - z)^N A_N(z)] = (N!)^{-1}, \quad \text{(C35a)}$$

$$\lim_{z\to 1} [(1 - z)^{N-1} F_N(z)] = (N!)^{-1}, \quad \text{(C35b)}$$

$$\lim_{z\to 1} [(1 - z)^{N-2} G_N(z)] = 2/N!, \quad \text{(C35c)}$$

with the singular behavior of the power expansions of these functions at $z = 1$. In this manner the asymptotic expression

$$\lim_{k\to\infty} [a(N, k)k^{1-N}] = [N!\,(N - 1)!]^{-1} \quad \text{(C36a)}$$

is obtained in Ref. 25. In an analogous manner one gets

$$\lim_{k\to\infty} [f(N, k)k^{2-N}] = [N!\,(N - 2)!]^{-1}, \quad \text{(C36b)}$$

$$\lim_{k\to\infty} [g(N, k)k^{3-N}] = 2/[N!\,(N - 3)!]. \quad \text{(C36c)}$$

APPENDIX D

In this appendix we study [in the sector (2.4)] the classical 1-dimensional 3-body problem with pair inversely quadratic potentials of unequal strength, and we produce an explicit example that violates the rule

$$p_i' = p_{4-i}, \quad \text{(D1)}$$

where p_i indicates the initial momentum of the ith particle and p_i' its final momentum. This rule is of course enforced if the coupling constants are all equal; an explicit proof in the classical case has been given by Marchioro,[8] whose treatment we follow here.

The Hamiltonian of the problem is

$$H = \frac{1}{2m}\sum_{i=1}^{3} p_i^2 + g_3(x_1 - x_2)^{-2} + g_1(x_2 - x_3)^{-2} + g_2(x_3 - x_1)^{-2}, \quad \text{(D2)}$$

where the coupling constants g_i are all nonnegative. It is convenient to introduce[4,8] the variables r and φ, defined by

$$r\cos\varphi = (x_1 + x_2 - 2x_3)/\sqrt{6}, \quad \text{(D3a)}$$

$$r\sin\varphi = (x_1 - x_2)/\sqrt{2}. \quad \text{(D3b)}$$

Then in the center-of-mass frame the Hamiltonian becomes

$$H = \frac{1}{2m}(p_r^2 + r^{-2}p_\varphi^2) + \tfrac{1}{2}r^{-2}\{g_3(\sin\varphi)^{-2} + g_1[\sin(\varphi + \tfrac{2}{3}\pi)]^{-2} + g_2[\sin(\varphi + \tfrac{4}{3}\pi)]^{-2}\}, \quad \text{(D4)}$$

where p_r and p_φ are the momenta conjugate to r and φ:

$$p_r = m\frac{dr}{dt}, \quad \text{(D5)}$$

$$p_\varphi = mr^2\frac{d\varphi}{dt}. \quad \text{(D6)}$$

The separability of the Hamiltonian (D4) implies the existence of a second constant of the motion B^2, in addition to the energy E:

$$E = \frac{1}{2m}p_r^2 + B^2 r^{-2}, \quad \text{(D7)}$$

$$B^2 = \frac{1}{2m}p_\varphi^2 + \tfrac{1}{2}\{g_3(\sin\varphi)^{-2} + g_1[\sin(\varphi + \tfrac{2}{3}\pi)]^{-2} + g_2[\sin(\varphi + \tfrac{4}{3}\pi)]^{-2}\}. \quad \text{(D8)}$$

From Eqs. (D7) and (D5) we get[8]

$$r(t) = \left(\frac{2E}{m}t^2 + r_{\min}^2\right)^{\frac{1}{2}}, \quad \text{(D9)}$$

where

$$r_{\min} = r(0) = BE^{-\frac{1}{2}}; \quad \text{(D10)}$$

then from Eqs. (D8), (D6), and (D9) we get

$$F[\varphi(t)] = F[\varphi(-\infty)] + \operatorname{arctg}(t/T) + \tfrac{1}{2}\pi, \quad \text{(D11)}$$

where

$$T = r_{\min}(m/2E)^{\frac{1}{2}} = (B/E)(\tfrac{1}{2}m)^{\frac{1}{2}}, \quad \text{(D12)}$$

and

$$F(\varphi') = \int^{\varphi'} d\varphi[1 - \tfrac{1}{2}B^{-2}\{g_3[\sin\varphi]^{-2} + g_1[\sin(\varphi + \tfrac{2}{3}\pi)]^{-2} + g_2[\sin(\varphi + \tfrac{4}{3}\pi)]^{-2}\}]^{-\frac{1}{2}}. \quad \text{(D13)}$$

We are interested in the asymptotic behavior of the particles in the remote past and future, when they are widely separated and move freely. We set by definition

$$x_i(t) \xrightarrow[t\to+\infty]{} p_i't/m + a_i' + O(t^{-1}), \quad \text{(D14)}$$

$$x_i(t) \xrightarrow[t\to-\infty]{} p_i t/m + a_i + O(t^{-1}), \quad \text{(D15)}$$

and since we are working in the center-of-mass frame, we also have

$$\sum_{i=1}^{3} p_i = \sum_{i=1}^{3} p_i' = 0. \tag{D16}$$

Moreover the initial momenta p_i must be consistent with the inequalities

$$p_1 \le p_2 \le p_3 \tag{D17a}$$

characterizing an incoming state (particles approaching each other), while the momenta p_i' shall be consistent with the opposite inequalities

$$p_1' \ge p_2' \ge p_3' \tag{D17b}$$

characterizing an outgoing state. For simplicity, we also choose the origin of the x axis to coincide with the center-of-mass, so that

$$\sum_{i=1}^{3} a_i = \sum_{i=1}^{3} a_i' = 0. \tag{D18}$$

Now from Eq. (D9) we get

$$r(t) \xrightarrow[t\to\pm\infty]{} \pm(2E/m)^{\frac{1}{2}} t + O(t^{-1}); \tag{D19}$$

from this equation and Eq. (D3b) we obtain

$$\varphi(-\infty) = \arcsin\,[\tfrac{1}{2}(p_2 - p_1)(p_1^2 + p_2^2 + p_1p_2)^{-\frac{1}{2}}]. \tag{D20}$$

We have also used the obvious relation

$$E = \frac{1}{2m}\sum_{i=1}^{3} p_i^2 = \frac{1}{m}(p_1^2 + p_2^2 + p_1p_2), \tag{D21}$$

which can be explicitly obtained from Eqs. (D15), (D16), and (D19), and from the relation

$$r^2 = \tfrac{1}{3}[(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2], \tag{D22}$$

which is implied by Eqs. (D3).

But once $\varphi(-\infty)$ is known, $\varphi(+\infty)$ can be evaluated from the implicit equation

$$F[\varphi(+\infty)] = F[\varphi(-\infty)] + \pi, \tag{D23}$$

which is a special case of Eq. (D11); from $\varphi(+\infty)$ the final momenta are determined through

$$p_1' - p_2' = 2(p_1^2 + p_2^2 + p_1p_2)^{\frac{1}{2}} \sin\varphi(+\infty), \tag{D24}$$

$$p_1' + p_2' = \left(\frac{2}{\sqrt{3}}\right)(p_1^2 + p_2^2 + p_1p_2)^{\frac{1}{2}} \cos\varphi(+\infty), \tag{D25}$$

which follow from Eqs. (D14), (D19), and (D3). Note, however, that the implicit function $F[\varphi]$ contains also the constant B^2, which must also be determined from the initial data. This can be done from Eq. (D8), which yields

$$\begin{aligned} B^2 = \frac{1}{2m} p_\varphi^2(-\infty) + \tfrac{1}{2}\{&g_3[\sin\varphi(-\infty)]^{-2} \\ &+ g_1[\sin(\varphi(-\infty) + \tfrac{2}{3}\pi)]^{-2} \\ &+ g_2[\sin(\varphi(-\infty) + \tfrac{4}{3}\pi)]^{-2}\}. \end{aligned} \tag{D26}$$

As for $p_\varphi(-\infty)$, it can be obtained from Eqs. (D6), (D3), (D15), and (D18), and we find

$$p_\varphi(-\infty) = \sqrt{3}(p_1a_2 - p_2a_1). \tag{D27}$$

Note that $p_\varphi(-\infty)$, and therefore also B^2, depend not only on the initial momenta p_i, but also on the constants a_i of Eq. (D15).

All the steps for the computation of the final momenta p_i' from the initial data p_i and a_i are now ready, the relevant equations being (D16), (D20) [to evaluate $\varphi(-\infty)$ from the initial momenta p_i], (D16), (D18), (D27), (D26) (to evaluate the constant B^2 from the initial data p_i and a_i), (D23) [to evaluate $\varphi(+\infty)$ from $\varphi(-\infty)$ and B^2], and finally (D16), (D24), and (D25) [to evaluate the final momenta p_i' from $\varphi(+\infty)$]. The remaining difficulty is the inability to perform explicitly, in the general case, the integral in the definition of $F(\varphi)$, Eq. (D13). In the special case

$$g_1 = g_2, \tag{D28}$$

to which attention is hereafter confined, this integral can be cast into the form

$$\begin{aligned} F(\varphi) = \frac{-1}{2}\int^{\cos^2\varphi} &dx(1 - 4x) \\ &\times (c_1x + c_2x^2 + c_3x^3 + c_4x^4)^{-\frac{1}{2}}, \end{aligned} \tag{D29}$$

where

$$c_1 = 1 - \tfrac{1}{2}(8g_1 + g_3)B^{-2}, \tag{D30a}$$

$$c_2 = -9 + 4(g_3 - g_1)B^{-2}, \tag{D30b}$$

$$c_3 = 24 - 8(g_3 - g_1)B^{-2}, \tag{D30c}$$

$$c_4 = -16. \tag{D30d}$$

If in addition we assume the initial data to be such that

$$c_1 = 0 \tag{D31a}$$

or, equivalently,

$$B^2 = \tfrac{1}{2}(8g_1 + g_3), \tag{D31b}$$

then the integral of Eq. (D29) can be performed and it yields

$$\begin{aligned} F(\varphi) = \frac{-1}{2}\Bigg((9 + \gamma)^{-\frac{1}{2}} &\arcsin\frac{(12 + \gamma)\cos^2\varphi - 9 - \gamma}{\cos^2\varphi\,[\gamma(\gamma + 8)]^{\frac{1}{2}}} \\ &+ \arcsin\frac{-16\cos^2\varphi + 12 + \gamma}{[\gamma(\gamma + 8)]^{\frac{1}{2}}}\Bigg), \end{aligned} \tag{D32}$$

with

$$\gamma = 4(g_1 - g_3)B^{-2} = 8(g_1 - g_3)/(8g_1 + g_3). \tag{D33}$$

We are assuming that γ is positive.

The derivation of $\varphi(+\infty)$ from $\varphi(-\infty)$ through Eqs. (D32) and (D23) still involves the solution of a transcendental equation; it cannot therefore be done explicitly. But the main purpose of this discussion is to prove that the rule (D1) need not hold if the three coupling constants g_i are not all equal. This is easily done by assuming its validity in one specific example, and showing that there results an inconsistency. Thus, we assume

$$p_1 = -p, \quad p_2 = 0, \quad p_3 = p, \tag{D34}$$

and

$$p_1' = p, \quad p_2' = 0, \quad p_3' = -p, \tag{D35}$$

with p positive [so that the inequalities (D17) are satisfied]. The choice (D34) implies, through Eq. (D20),

$$\varphi(-\infty) = \tfrac{1}{6}\pi, \tag{D36}$$

while the choice (D35) yields

$$\varphi(+\infty) = \tfrac{1}{6}\pi = \varphi(-\infty). \tag{D37}$$

[This is consistent with the remark[8] that generally the rule (D1) corresponds to $\varphi(\infty) = \frac{1}{3}\pi - \varphi(-\infty)$].

But the result (D37) is clearly inconsistent with Eqs. (D23) and (D32), at least so long as the arguments of the arcsin functions of Eq. (D32) are in the interval between -1 and 1. [Note that this is never the case if $g_1 = g_3$, i.e., if all coupling constants are equal; in fact, in this case, Eq. (D32) is meaningless, because γ vanishes. Indeed, the correct integration of Eq. (D29) in this case reproduces Marchioro's results,[8] yielding the rule (D1).] When g_1 is larger than g_3, this condition is easily satisfied, for instance, by the simple choice $g_3 = 0$, in which case $\gamma = 1$ and the arguments of the arcsin functions become $-\frac{1}{9}$ and $\frac{1}{3}$, respectively. As for the condition (D31), which we had to assume in order to perform explicitly the integration that led to Eq. (D32), it is easily seen that, through Eqs. (D27), (D34), and (D26), it becomes

$$a_2^2 = m(g_1 - g_3)/p^2,$$

so that it can always be enforced by appropriate choice of the initial constant a_2.

APPENDIX E

In this appendix we disprove the conjecture that the spectrum of the problem of Sec. 2 is completely linear even if the coupling constants of the inversely quadratic interaction are different for different pairs. This we do by displaying an explicit example that violates this conjecture. This is the 3-body model characterized by the Hamiltonian

$$H = H_0 + \epsilon(x_1 - x_2)^{-2}, \tag{E1}$$

where H_0 is the Hamiltonian of Sec. 2, Eq. (2.1), with $N = 3$, and ϵ is a small parameter. The spectrum of this model is then given by the formula[4]

$$E_{nl} = \hbar\omega(\tfrac{3}{2})^{\frac{1}{2}}[2n + B_l(\epsilon) + 1], \tag{E2}$$

where $B_l^2(\epsilon)$ are the eigenvalues of the differential equation[4]

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\varphi^2} + \frac{9g}{2\sin^2 3\varphi} + \frac{\epsilon}{2\sin^2\varphi}\right)F_l(\varphi) = B_l^2(\epsilon)F_l(\varphi), \quad 0 \le \varphi \le \tfrac{1}{3}\pi. \tag{E3}$$

For $\epsilon = 0$, the complete set of eigenfunctions of this equation are[4]

$$F_l(\varphi) = (\sin 3\varphi)^{a+\frac{1}{2}}C_l^{a+\frac{1}{2}}(\cos 3\varphi), \quad l = 0, 1, 2, \cdots, \tag{E4}$$

where C_l^p is a Gegenbauer polynomial,[12] and the corresponding eigenvalues are

$$B_l(0) = 3(l + a + \tfrac{1}{2})(\hbar^2/2m)^{\frac{1}{2}}, \quad l = 0, 1, 2, \cdots. \tag{E5}$$

Here, a is defined by Eq. (2.9).

To first order in ϵ the eigenvalues $B_l^2(\epsilon)$ are given by

$$B_l^2(\epsilon) = B_l^2(0) + \tfrac{1}{2}\epsilon\beta_l, \tag{E6}$$

where

$$\beta_l = \int_0^{\frac{1}{3}\pi} d\varphi \sin^{-2}\varphi F_l^2(\varphi) \Big/ \int_0^{\frac{1}{3}\pi} d\varphi F_l^2(\varphi). \tag{E7}$$

Thus, to first order in ϵ the energy spectrum is

$$E_{nl} = \hbar\omega(\tfrac{3}{2})^{\frac{1}{2}}[2n + 3l + 3a + \tfrac{5}{2} + \tfrac{1}{6}\epsilon\beta_l(l + a + \tfrac{1}{2})^{-1}m\hbar^{-2}]. \tag{E8}$$

Therefore, if the spectrum is to be completely linear, it should be true that

$$\beta_l = \mu(l + a + \tfrac{1}{2}) + \nu(l + a + \tfrac{1}{2})^2, \tag{E9}$$

where μ and ν are numerical constants. This implies the condition

$$6\beta_0 - 8\beta_1 + 3\beta_2 = 0. \tag{E10}$$

A simple example that demonstrates that this equation is not verified obtains considering the special case where $a = \frac{3}{2}$.[26] Then with a little algebra this condition becomes

$$\int_0^{\frac{1}{3}\pi} d\varphi \sin 3\varphi[\sin 15\varphi - \tfrac{5}{9}\sin 9\varphi][\sin 3\varphi/\sin\varphi]^2 = 0, \tag{E11}$$

and it is easily seen that this equality does not hold.

Note that if the perturbing potential had maintained the equality of the coupling constants, i.e., if it had been of the form

$$\epsilon[(x_1 - x_2)^{-2} + (x_2 - x_3)^{-2} + (x_3 - x_1)^{-2}] = \tfrac{1}{2}\epsilon r^{-2} \sin^{-2} 3\varphi, \quad \text{(E12)}$$

the term $(\sin 3\varphi/\sin \varphi)^2$ in the integrand of Eq. (E11) would be missing; then of course this equation would be satisfied. Indeed, in such a case Eq. (E9) holds, with $\nu = 0$ and $\mu = 9/a$.

Although we have for simplicity displayed only a very special example in which the conjecture of complete linearity of the spectrum is violated, it is obvious that, for 1-dimensional models with quadratic and inversely-quadratic forces, this property is the exception rather than the rule, holding only in some special cases such as those considered in Sec. 2 (all particles equal) or in Sec. 3 (all particles equal within each family, the same quadratic interaction between all pairs, no inversely quadratic interaction between different families).

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[1] E. H. Lieb and D. C. Mattis, *Mathematical Physics in One Dimension* (Academic, New York, 1966).

[2] F. A. Berezin, G. P. Pochil, and V. M. Finkelberg, Moscow Univ. Vestnik **1**, 21 (1964) (in Russian); J. B. McGuire, J. Math. Phys. **5**, 622 (1964); E. Brezin and J. Zinn-Justine, Compt. Rend. Acad. Sci. (Paris) **B263**, 670 (1966); C. N. Yang, Phys. Rev. Lett. **19**, 1312 (1967); Phys. Rev. **168**, 1920 (1968). The 1-dimensional 3-body problem with zero-range interactions had been previously solved in some special cases. For a detailed treatment of one such example, and a review of work in this field up to 1960, see H. M. Nussenzweig, Proc. Roy. Soc. (London) **A264**, 408 (1961).

[3] A complete analysis of the S matrix describing all scattering processes between these bound states has been given by Yang (see the second of his papers, Ref. 2 above).

[4] F. Calogero, J. Math. Phys. **10**, 2191 (1969).

[5] F. Calogero, J. Math. Phys. **10**, 2197 (1969), hereafter referred to as C. This paper is marred by several misprints: In the second term in Eq. (2.8), $\partial^2 z/\partial x_i^2$ should appear in place of $\partial z/\partial x_i$; in Eqs. (2.15) and (2.18), $\frac{1}{2}$ should appear in place of m within the square root; in Eq. (2.20), r^2 should appear in place of z^2 in the argument of the exponential; in Eq. (2.22), the plus sign after a should be a minus sign; in the sentence before Eq. (2.4), the word "symmetrical" should be replaced by "antisymmetrical and symmetrical."

[6] The coincidence of the energy levels (except for a constant shift) is essentially trivial, but the coincidence of the multiplicities (that probably is peculiar to the 1-dimensional case) is remarkable (see below).

[7] For $N = 3$, this has been proved by C. Marchioro (unpublished).

[8] C. Marchioro, J. Math. Phys. **11**, 2193 (1970).

[9] See, for instance, L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, New York, 1958), Sec. 35.

[10] Actually, the derivative of the wavefunction with respect to the variable x_i vanishes, when x_i coincide with another variable x_j, only if the inversely quadratic potential is repulsive ($g > 0$); but when multiplied with the wavefunction itself, it vanishes in all cases; and it is this that counts (see below and Ref. 4).

[11] The solutions considered in C correspond to the subset of these solutions characterized by the condition $k = 0$.

[12] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1965); A. Erdélyi, Ed., *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. II.

[13] Nonpolynomial solutions of Eq. (2.10) contain singularities and are therefore excluded by the requirement that the wavefunction (2.6) be physically acceptable (for an explicit illustration in the 3-body case, see Ref. 4). An additional argument to justify the absence of physically acceptable nonpolynomial solutions of Eq. (2.10) is that the polynomial solutions alone constitute a complete set [in the sector (2.4); see below].

[14] The ansatz (2.6), when inserted in the eigenvalue equation (2.3), yields Eqs. (2.10) and (2.11), provided $a^2 = \frac{1}{4}(1 + 4mg\hbar^{-2})$ (see Appendix A). The positive solution of this quadratic equation is the only acceptable one if g does not vanish, but both solutions $a = \pm\frac{1}{2}$ are acceptable (in the whole configuration space) if g vanishes. Acceptability is conditioned by the continuity of ψ^2 and $\psi\psi'$ at $x_i = x_j$, the prime indicating here differentiation relative to x_i or x_j.

[15] While the proportionality of the partition functions implies that these systems are thermodynamically equivalent, differences would show up in observables (such as, for instance, correlation functions) depending not only on the energy spectrum of the system but also on the form of its many-body wavefunction. Incidentally, the partition function is trivially computed from Eq. (2.26); indeed, up to a multiplicative constant, it coincides with the generating function $F_N(z)$ of Appendix C, with $z = \exp(\hbar\omega(\frac{1}{2}N)^{\frac{1}{2}}/KT)$. The (more interesting) problem of the gas composed of N 1-dimensional particles interacting by inversely square potentials has been studied, in the $N \to \infty$ limit with constant density, by C. Marchioro and E. Presutti, Nuovo Cim. Lett. **4**, 488 (1970); and by Bill Sutherland, J. Math. Phys. **11**, 3183 (1970).

[16] R. G. Storer, Phys. Rev. Lett. **24**, 5 (1970).

[17] We use for simplicity a symbolic notation, with δ functions in the denominator, whose significance should be self-evident [see Eq. (4.27)].

[18] Some progress in this direction has been made by A. M. Perelomov (private communication).

[19] Of course the separability of the Hamiltonian into "radial" and "angular" parts (see Appendices C and A), which obtains independently of the equality of the particles, and the simple nature of the harmonical potential, imply that the spectrum depends in all cases linearly upon the "radial" quantum number n.

[20] I owe to A. M. Perelomov the suggestion to test this conjecture by perturbation theory.

[21] Other, possibly more interesting, potentials, for which the basic equation of wave propagation can be solved exactly, are obtained from the above one by going over to the "Jacobi" coordinates (and eliminating the "center-of-mass coordinate" y_0 from the Laplace operator, thereby reducing by one the dimensionality of the space it refers to; see Appendix A).

[22] Different definitions of the Jacobi coordinates could be chosen without affecting the final result. See, e.g., M. Grynberg and Z. Koba, Ann. Phys. (N.Y.) **26**, 418 (1964).

[23] The exact choice of the angular coordinates Ω_i is immaterial, the only important property being that of Eq. (A5) below. A possible choice are the standard polar coordinates; another possible choice are the so called zonal coordinates. See, e.g., P. Appel and J. Kampé de Fériet, *Fonctions Hypergéométriques et Hypersphériques* (Gauthier-Villars, Paris, 1926), Pt. II, Chap. II.

[24] Here and below z, r, a, and b are defined as in Sec. 2, Eqs. (2.7)–(2.9) and (2.13), and $P_k(x)$ is a translation-invariant polynomial in the N variables x_i, homogeneous of degree k, and satisfying the generalized Laplace equation (2.10).

[25] A. M. Perelomov, V. S. Popov, and I. A. Malkin, ITEF preprint 337 [a slightly abridged version of this paper, in which the results reported here have been omitted, has been published: Sov. J. Nucl. Phys. **2**, 533 (1965)].

[26] The first three Gegenbauer polynomials with superscript 2 are

$$C_0^2(x) = 1, \quad C_1^2(x) = x, \quad C_2^2(x) = 1 - 6x^2.$$