PHYSICS REPORTS

Physics Reports 371 (2002) 231–459

www.elsevier.com/locate/physrep

The calculus of many instantons

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Accepted 1 June 2002
editor: A. Schwimmer

Abstract

We describe the modern formalism, ideas and applications of the instanton calculus for gauge theories with, and without, supersymmetry. Particular emphasis is put on developing a formalism that can deal with any number of instantons. This necessitates a thorough review of the ADHM construction of instantons with arbitrary charge and an in-depth analysis of the resulting moduli space of solutions. We review the construction of the ADHM moduli space as a hyper-Kähler quotient. We show how the functional integral in the semi-classical approximation reduces to an integral over the instanton moduli space in each instanton sector and how the resulting matrix partition function involves various geometrical quantities on the instanton moduli space: volume form, connection, curvature, isometries, etc. One important conclusion is that this partition function is the dimensional reduction of a higher-dimensional gauged linear sigma model which naturally leads us to describe the relation of the instanton calculus to D-branes in string theory. Along the way we describe powerful applications of the calculus of many instantons to supersymmetric gauge theories including (i) the gluino condensate puzzle in $\mathcal{N}=1$ theories (ii) Seiberg–Witten theory in $\mathcal{N}=2$ theories; and (iii) the AdS/CFT correspondence in $\mathcal{N}=2$ and 4 theories. Finally, we briefly review the modifications of the instanton calculus for a gauge theory defined on a non-commutative spacetime and we also describe a new method for calculating instanton processes using a form of localization on the instanton moduli space.
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PACS: 11.15.−q; 11.30.Pb

Contents

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PII: S0370-1573(02)00301-0

1. Introduction

Yang–Mills instantons [1] have provided an enduring interest for a generation of physicists and mathematicians. On the physics side, instanton configurations give the leading non-perturbative contributions to the functional integral in the semi-classical approximation [2]. After the initial disappointment that instantons were not going to provide a simple explanation of quark confinement, they have proved interesting and useful both in phenomenological models of QCD (see, for example [3–5]) and for describing exact non-perturbative phenomena in supersymmetric gauge theories (as we describe in this review). On the mathematical side, instantons lie at the heart of some important recent developments in topology and, in particular, of Donaldson's construction of topological invariants of four manifolds (see [6] and references therein).

The main applications of instantons we will consider in this review are to supersymmetric gauge theories. Recent developments have provided an extraordinary web of exact non-perturbative results for these theories, making them an ideal field-theoretical laboratory in which to test our ideas about strongly coupled gauge dynamics. Highlights include Seiberg duality for $\mathcal{N}=1$ theories, the Seiberg–Witten solution of $\mathcal{N}=2$ supersymmetric Yang–Mills and the AdS/CFT correspondence. Typically supersymmetry constrains the form of quantum corrections allowing exact results to be obtained for some special quantities. Despite these constraints, supersymmetric gauge theories still exhibit many interesting physical phenomena including quark confinement and chiral symmetry breaking. Yang–Mills instantons have played an important rôle in some of these developments which we will review in the following.

Instantons in supersymmetric gauge theories were studied in detail by several groups in the 1980s. This early work focused on the contribution of a single instanton in theories with $\mathcal{N}=1$ supersymmetry. Impressive results were obtained for the vacuum structure of these theories, including exact formulae for condensates of chiral operators. Although precise numerical answers were obtained, interest in these was limited as there was nothing with which to compare them. [1] The new developments motivate the extension of the instanton calculus to theories with extended supersymmetry and, in particular, to instantons of arbitrary topological charge. This is the main goal of the first part of this review. Developments such as Seiberg–Witten theory and the AdS/CFT correspondence also yield precise predictions for instanton effects, which can be checked explicitly using the methods we will develop below. These applications, which are described in the second half of the review are important for two reasons. Firstly, they provide as a quantitative test of conjectural dualities which underlie the recent progress in supersymmetric gauge theories. Secondly, they also increase our confidence in weak-coupling instanton calculations and some of the technology which they require such as Wick rotation to Euclidean space, constrained instantons, etc.

A semi-classical evaluation of the path integral requires us to find the complete set of finite-action configurations which minimize the Euclidean action. In pure Yang–Mills theory, this was accomplished many years ago in pioneering work by Atiyah, Drinfeld, Hitchin and Manin (ADHM) [7]. In particular, these authors found the complete set of self-dual gauge fields of arbitrary topological charge k. Their construction, which works for arbitrary $\mathrm{SU}(N)$, $\mathrm{SO}(N)$ or $\mathrm{Sp}(N)$ gauge groups (but not for the exceptional groups), reduces the self-dual Yang–Mills equation to a set of non-linear algebraic equations (the "ADHM constraints") constraining a matrix of parameters (the "ADHM data"). After modding out a residual symmetry group, each solution of the ADHM constraints defines a gauge-inequivalent, k-instanton configuration. The space of such solutions, which we denote $\mathfrak{M}_k$, is also known as the k-instanton moduli space.

The space $\mathfrak{M}_k$, has many remarkable properties which will play a central rôle in our story. Firstly, apart from having some isolated singularities, it is a Riemannian manifold with a natural metric. The

[1] Note, however, that several puzzles emerged on comparing numerical values for condensates obtained using different approaches to instanton calculations. These will be re-examined in Section 4.

singularities are relatively mild, being of conical type, and have a natural interpretation as points where instantons shrink to zero size. In addition, the moduli space carries families of inequivalent complex structures and, when endowed with its natural metric, defines a hyper-Kähler manifold. This property is manifest in the ADHM construction, which is actually an example of a general method for constructing hyper-Kähler spaces known as the *hyper-Kähler quotient*. This viewpoint turns out to be very useful and we will find that many of the structures which emerge in the semi-classical analysis of the path integral have a nice geometrical interpretations in terms of the hyper-Kähler quotient.

Applying the semi-classical approximation to a Green's function requires us to replace the fields by their ADHM values and then integrate over the moduli space $\mathfrak{M}_k$. Constructing the appropriate measure for this integration, both in pure gauge theory and its supersymmetric generalizations, is the main problem we address in the first part of the review. The main obstacle to overcome is the fact the ADHM constraints cannot generally be solved except for small values of k. Hence, we cannot find an explicit unconstrained parameterization of the moduli space. However, we will be able to find explicit formulae for the measure in terms of the ADHM data together with Lagrange multiplier fields which impose the ADHM equations as δ-function constraints. Under certain circumstances we will be able to evaluate the moduli space integrals even for general k. For instance at large-N the ADHM constraints can be solved, at least on a certain generic region of the moduli space, and this means that the instanton calculus becomes tractable for arbitrary k. In the final section we shall briefly explain important new developments that allow one to make progress even at finite N when one is on a Higgs or Coulomb branch.

The plan of this report is as follows. As a prelude we briefly review the basic philosophy of instanton calculations. Section 2 provides a solid grounding of the ADHM construction of instantons in SU(N) gauge theory.[2] Firstly, we describe some general features of instantons. This begins with the notion of zero modes in the instanton background, collective coordinates and the moduli space of instantons $\mathfrak{M}_k$. Very general arguments then show that $\mathfrak{M}_k$ is a hyper-Kähler manifold with a natural Hermitian metric. After this, we describe all aspects of the ADHM construction of $\mathfrak{M}_k$, putting particular emphasis on the hyper-Kähler quotient description. Most of the material in Section 2 is necessarily very mathematical. A reader primarily interested in physical applications of instantons can skip most of Sections 2.3.1, 2.4.1, 2.5 and 2.6.

Section 3 shows how the semi-classical limit of the functional integral reduces to an integral over $\mathfrak{M}_k$ with a volume form which is derived from the Hermitian metric constructed previously. However, in the setting of pure gauge theory, integrating out all the non-zero modes of the quadratic fluctuations around the instanton yields a non-trivial determinant factor which accounts for the one-loop effects in perturbation theory in the instanton background. We briefly describe, using results reviewed in [8], how the determinants may be evaluated in the ADHM background, although the resulting expressions are rather unsatisfactory (involving spacetime integrals). This latter point will not overly concern us, because in the supersymmetric applications, the non-trivial parts of the determinant factors cancel between bosons and fermions.

Section 4 describes how the instanton calculus is generalized in a supersymmetric gauge theory. This is done in the context of $\mathcal{N}=1,2$ and 4 supersymmetric theories with no additional matter fields. (Adding matter fields is considered separately in Section 6.3.) First of all, we explain how to construct the Euclidean version of the theory by Wick rotation from Minkowski space and how this

[2] The ADHM construction for the other classical groups is described separately in Section 6.

inevitably leads to a theory where the fermion action is not real. This is a fact of life in the supersymmetric instanton calculus, just as it is when fermionic theories are investigated on the lattice, but does not lead to any inconsistencies or pathologies. In an instanton background, there are fermion zero modes and their associated Grassmann collective coordinates. We show how these are superpartners of the bosonic collective coordinates and so in a supersymmetric gauge theory the moduli space $\mathfrak{M}_k$ is, itself, supersymmetrized. We show how this fits in with the hyper-Kähler quotient approach. In this section, we explain in detail the concept of the super-instanton. In particular, in the $\mathcal{N}=4$ theory, the appropriate supersymmetric instanton solution is only an approximate solution of the equations-of-motion, but one which captures the leading-order semi-classical behaviour of the functional integral. We also explain in this section how the addition of VEVs for the scalar fields (for the $\mathcal{N}=2$ and 4 theories) leads to Affleck's notion of a constrained instanton. Our point of view is that constrained instantons are—in a sense—only a mild modification of the conventional instanton: some of the collective coordinates cease to be exact moduli and the whole effect can be described by the turning on a potential on $\mathfrak{M}_k$. We call this approximate solution a "quasi-instanton". Section 5 describes the semi-classical approximation in a supersymmetric gauge theory. In all cases we can formulate the leading-order semi-classical approximation to the functional integral as an integral over the supersymmetric version of $\mathfrak{M}_k$.

Some generalizations, and other important miscellany, are collected in Section 6. In particular, we explain how the ADHM constraints can be solved on generic orbit of the gauge group when $N \geqslant 2k$; how to extend the ADHM construction to gauge groups $\mathrm{SO}(N)$ and $\mathrm{Sp}(N)$; how to add matter fields transforming in the fundamental representation; the effect of adding masses which break various amounts of supersymmetry; and how to define the notion of the instanton partition function.

Sections 7–9 each describe an application of the multi-instanton calculus. In Section 7 we review the two distinct approaches to instanton calculus in $\mathcal{N}=1$ theories developed by different groups in the 1980s. One approach follows closely the methodology developed in the first part of the report, working with constrained instantons in a weakly coupled Higgs phase. As we review, this approach yields explicit agreement with the web of exact results which predict a precise numerical value for the gluino condensate in $\mathcal{N}=1$ supersymmetric Yang–Mills. The second, more controversial approach, attempts to evaluate the gluino condensate directly in the strongly coupled confining phase. We cast further doubt on this method, by showing explicitly that the resulting formulae violate the clustering property, a general axiom of quantum field theory, and therefore cannot yield the exact answer. This is just as one might have expected: instantons are a semi-classical phenomenon and as such are not expected to yield quantitatively exact results in a strongly coupled phase.

We then go on in Section 8 to consider instanton contributions to the low-energy effective action of $\mathcal{N}=2$ gauge theories on their Coulomb branch. These calculations are very important because they can be compared with a completely different approach based on the celebrated theory of Seiberg and Witten. Perfect agreement is found providing strong evidence in favour of the instanton calculus—including all this entails like the imaginary time formalism and the resulting saddle-point approximation of the functional integral—but also in favour of Seiberg and Witten's ingenious theory. Several minor modifications of the original Seiberg–Witten solution for $\mathcal{N}=2$ theories with matter are also described. These involve ambiguities in the definition of parameters appearing in the exact solution.

In Section 9 we consider conformal, or finite, gauge theories with $\mathcal{N}=2$ or 4 supersymmetry in their non-abelian Coulomb phase. Here, we show how various instanton effects for arbitrary instanton charge can be evaluated exactly in the limit of large N. The large-N instanton calculus that we develop provides substantial evidence in favour of the remarkable AdS/CFT correspondence which relates the gauge theories to ten-dimensional superstring theory on a particular background. In particular we will describe how, even at weak coupling, the large-N instanton calculus probes the background ten-dimensional spacetime geometry of the string theory directly.

In Section 10 we describe how instantons can be embedded in higher-dimensional theories as brane-like solitons. We show in detail, as expected from general principles, the resulting collective coordinate dynamics involves a certain σ-model on the brane world volume with the moduli space of instantons as the target space. This provides a useful way to understand the relationship of instantons to D-branes in superstring theory, a subject we go on to describe. Studying instantons in the context of string theory provides a powerful way to derive many aspects of the instanton calculus developed in the previous section and in the process removes much of the cloak of mystery surrounding the ADHM construction.

Finally, in Section 11 we describe two recent developments in the instanton calculus. Firstly, how the instanton is modified when the underlying gauge theory is defined on a non-commutative space. It turns out that the instanton moduli space is modified in a particularly natural way. Secondly, we describe a new calculational technique which promises to make instanton effects tractable even at finite N and for all instanton charge. The key idea is that in the presence of scalar field vacuum expectation values (VEVs), a potential develops on the instanton moduli space and the collective coordinate integrals localize around the critical points of this potential.

Appendix A details our conventions for spinors in different spacetime dimensions. Appendix B includes an introduction to hyper-Kähler geometry. Appendix C reviews some useful identities for calculations involving the ADHM data.

There are many other excellent reviews of instanton physics including [4,9–13]; however, these reviews concentrate on the single instanton.

1.1. The philosophy of instanton calculations

Before we begin in earnest, it is perhaps useful to remind the reader of the basic philosophy of instanton calculations. In quantum field theory, all information about the physical observables (i.e. the spectrum and the S-matrix) can be obtained by calculating the correlation functions of appropriate operators. These correlation functions are defined by the Feynman path integral. The most convenient formulation is one where the correlation functions and the path integral are analytically continued to Euclidean spacetime. The path integral was first introduced as a formal generating functional for perturbation theory. However, thanks to the work of Wilson, it is widely believed that the Euclidean path integral actually provides a first-principles non-perturbative definition of quantum field theory. Implementing this definition in practice involves replacing continuous spacetime by a finite number of points and leads to the subject of lattice field theory which is beyond the scope of this article. Nevertheless, the success of this viewpoint inspires us to take even the *continuum* path integral seriously.

In continuum quantum field theory we are generally limited to calculating at weak coupling. In four dimensions, the only exceptions are certain supersymmetric gauge theories which we will discuss further below. If the continuum path integral makes any sense at all, then the very least one should expect is that it should yield sensible calculable answers at weak coupling. As mentioned above, the Feynman rules of ordinary weak-coupling perturbation theory can easily be derived from the path integral. However, even at weak coupling, the path integral contains much more information, including effects which are non-perturbative in the coupling constant.

In the simplest cases, the generating functional in Euclidean spacetime takes the schematic form

$$Z[J] = \int [\mathrm{d}\Phi] \exp\left(-\frac{1}{g^2} S[\Phi] + \int \mathrm{d}^4x\, J\Phi \right) , \tag{1.1}$$

where Φ denotes some set of fields with sources J and $S[\Phi]$ is the Euclidean action which is real and bounded below. The gauge theories we consider below are characterized (classically) by a single dimensionless coupling, denoted g^2, and one may always re-scale the fields so that the coupling appears in front of the action as indicated in (1.1). Of course, in the quantum theory, we will certainly need to modify our discussion to account for the running of the coupling but we will, for the moment, postpone this discussion.

The basic idea of the semi-classical approximation [3] is that, for small g^2, the path integral is dominated by the configurations of lowest Euclidean action and we may proceed by expanding around these configurations. The simplest such configurations are the perturbative vacua of the theory (i.e. minima of the classical potential) and the corresponding expansion is just the ordinary loop expansion. However the Euclidean action may have other minima with finite action. Such configurations are known as instantons and the same logic dictates that we should also expand in fluctuations around them. For an instanton of finite Euclidean action $S_{\rm cl}$, the leading semi-classical contribution goes like $\exp(-S_{\rm cl}/g^2)$ and expanding in fluctuation leads to corrections which are suppressed by further powers of g^2.

As we will see, there are many extra complications and technical difficulties in carrying out this program in practice. As mentioned above, for asymptotically free theories of direct physical interest the coupling g^2 runs as a function of the energy and becomes large in the infra-red at the dynamical scale Λ. In this case, the fluctuations of the fields in the path integral become large and neither the instanton nor any other classical field configuration determines the physics. It seems therefore that instantons have little to tell us directly about these theories. On the other hand, there are two types of theory we will meet where instanton methods are directly applicable. Firstly, and most straightforwardly, there are theories like $\mathcal{N}=4$ supersymmetric Yang–Mills theory where the β-function vanishes and the coupling does not run. In this case we may set $g^2 \ll 1$ and safely apply semi-classical methods. The second type of theory is asymptotically free but also contains scalar fields which can acquire a VEV spontaneously breaking the gauge group. The Higgs mechanism can then cut off the running of the coupling in the IR. If the mass scale of the VEV is much larger than the dynamical scale Λ, the coupling is small at all length scales and semi-classical reasoning is valid.

In the case of a non-zero scalar VEV, we will also have to contend with problems related to the fact that instantons are no longer true minima of the action. In fact, both types of theory described above require a more complicated application of the semi-classical method than simply expanding

[3] The term "semi-classical" originates in the observation that g^2 appears in the exponent of (1.1) in exactly the same position as Planck's constant $\hbar$ would if we restored physical units. Thus the $g^2 \to 0$ limit is identical to the $\hbar \to 0$ limit.

around exact solutions of the equations-of-motion. We will frequently solve the classical equations only approximately order by order in g^2. However, the basic philosophy remains the same: as long as the theory is genuinely weakly coupled, an appropriate semi-classical approximation to the path integral should be reliable and we can use it to evaluate the leading non-perturbative corrections to observables. In the following sections we will develop the necessary technology to accomplish this goal.

2. Instantons in pure gauge theory

In this section, we describe instanton solutions of pure gauge theory, i.e. without any additional fields present. To begin with we introduce the central concept of the moduli space of instantons and describe how this space has a lot structure: it is a hyper-Kähler manifold with singularities. In Section 2.4 we describe the ADHM construction of arbitrary instanton solutions and then explain how the "ADHM construction" can be viewed as a particular example of the hyper-Kähler quotient construction. This point of view is necessarily rather mathematical and some of the background required is reviewed in Appendix B.

2.1. Some basic facts

We start with pure SU(N) gauge theory described by a Euclidean space action

$$S[A] = -\frac{1}{2}\int d^4x\, \mathrm{tr}_N\, F_{mn}^2 + i\theta k\ , \tag{2.1}$$

where the field strength $F_{mn} = \partial_m A_n - \partial_n A_m + g[A_m, A_n]$ and the topological charge is

$$k = -\frac{g^2}{16\pi^2}\int d^4x\, \mathrm{tr}_N\, F_{mn}{}^*F_{mn} \in \mathbb{Z} \tag{2.2}$$

with ${}^*F_{mn} = \frac{1}{2}\varepsilon_{mnkl}F_{kl}$. In these conventions, characteristic of the mathematical instanton literature, the gauge field A_m is anti-Hermitian and so the covariant derivative is $D_m = \partial_m + gA_m$.

Instantons are the finite action solutions of the classical equations-of-motion which consequently satisfy a first-order equation. From the inequality

$$\int d^4x\, \mathrm{tr}_N(F_{mn} \pm {}^*F_{mn})^2 \leqslant 0\ , \tag{2.3}$$

one establishes a lower bound on the real part of the action:

$$-\frac{1}{2}\int d^4x\, \mathrm{tr}_N\, F_{mn}^2 \geqslant \frac{8\pi^2}{g^2}|k|\ , \tag{2.4}$$

with equality when the gauge field satisfies the self-dual, for $k > 0$, or anti-self-dual, for $k < 0$, Yang–Mills equations:

$${}^*F_{mn} \equiv \tfrac{1}{2}\varepsilon_{mnkl}F_{kl} = \pm F_{mn}\ . \tag{2.5}$$

In our convention, instantons are the self-dual solutions and carry positive topological (instanton) charge $k > 0$. In contrast, anti-instantons satisfy the anti-self-dual Yang–Mills equations and carry negative topological charge $k < 0$. Their action is

$$S = \frac{8\pi^2|k|}{g^2} + \mathrm{i}k\theta = \begin{cases} -2\pi\mathrm{i}k\tau, & k > 0 \ , \\ -2\mathrm{i}\pi k\tau^*, & k < 0 \ , \end{cases} \tag{2.6}$$

where we have defined the complex coupling

$$\tau = \frac{4\pi\mathrm{i}}{g^2} + \frac{\theta}{2\pi} \ . \tag{2.7}$$

In this review we will discuss exclusively instantons, rather than anti-instantons. The fundamental problem then, and the main subject of Section 2.4, is to find all the solutions to (2.5) with a + sign. When discussing the instanton calculus it is very convenient to introduce a *quaternionic* notation for four-dimensional Euclidean spacetime. The covering group of the SO(4) Euclidean Lorentz group in four dimensions is $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. A four-vector x_n can be rewritten as a $(\mathbf{2},\mathbf{2})$ of this group with components $x_{\alpha\dot\alpha}$ (or $\bar{x}^{\dot\alpha\alpha}$). Here, $\alpha, \dot\alpha = 1,2$ are spinor indices of $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$, respectively. The explicit relation between the two bases is

$$x_{\alpha\dot\alpha} = x_n\sigma_{n\alpha\dot\alpha}, \qquad \bar{x}^{\dot\alpha\alpha} = x_n\bar\sigma_n^{\dot\alpha\alpha} \ , \tag{2.8}$$

where $\sigma_{n\alpha\dot\alpha}$ are four 2×2 matrices $\sigma_n = (\mathrm{i}\vec\tau, 1_{[2]\times[2]})$ (τ^c, $c = 1$–3, are the three Pauli matrices). In addition we define the Hermitian conjugate matrices $\bar\sigma_n \equiv \sigma_n^\dagger = (-\mathrm{i}\vec\tau, 1_{[2]\times[2]})$ with components $\bar\sigma_n^{\dot\alpha\alpha}$.[4] As explicit 2×2-dimensional matrices

$$x_{\alpha\dot\alpha} = \begin{pmatrix} \mathrm{i}x_3 + x_4 & \mathrm{i}x_1 + x_2 \\ \mathrm{i}x_1 - x_2 & -\mathrm{i}x_3 + x_4 \end{pmatrix}, \qquad \bar{x}^{\dot\alpha\alpha} = \begin{pmatrix} -\mathrm{i}x_3 + x_4 & -\mathrm{i}x_1 - x_2 \\ -\mathrm{i}x_1 + x_2 & \mathrm{i}x_3 + x_4 \end{pmatrix} \ . \tag{2.9}$$

Notice that derivatives are defined in the same way:

$$\partial_{\alpha\dot\alpha} = \sigma_{n\alpha\dot\alpha}\partial_n, \qquad \bar\partial^{\dot\alpha\alpha} = \bar\sigma_n^{\dot\alpha\alpha}\partial_n \ . \tag{2.10}$$

But note with this definition $\partial_{\alpha\dot\alpha} \neq \partial/\partial x_{\alpha\dot\alpha}$.

We now introduce the Lorentz generators,

$$\sigma_{mn} = \tfrac14(\sigma_m\bar\sigma_n - \sigma_n\bar\sigma_m), \qquad \bar\sigma_{mn} = \tfrac14(\bar\sigma_m\sigma_n - \bar\sigma_n\sigma_m) \ , \tag{2.11}$$

which are, respectively, self-dual and anti-self-dual:

$$\sigma_{mn} = \tfrac12\varepsilon_{mnkl}\sigma_{kl}, \qquad \bar\sigma_{mn} = -\tfrac12\varepsilon_{mnkl}\bar\sigma_{kl} \ . \tag{2.12}$$

In terms of these, the explicit one-instanton solution of the SU(2) gauge theory in regular gauge, known as the BPST instanton [1], is

$$A_n = g^{-1}\frac{2(x-X)_m\sigma_{mn}}{(x-X)^2 + \rho^2} \ . \tag{2.13}$$

[4] Notice that indices are raised and lowered with the ε-tensor defined in Appendix A: $\bar{x}^{\dot\alpha\alpha} = \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}x_{\beta\dot\beta}$.

Note the identification SU(2) gauge indices and those of $SU(2)_L$ introduced above, reflecting the fact that the instanton configuration breaks the product of these groups down to a diagonal subgroup. The instanton gauge field above depends on five parameters: one scale-size ρ and the four-vector instanton position X_m. Performing global SU(2) gauge rotations of the right-hand side of (2.13) the total number of free parameters of a single-instanton solution becomes eight. This is our first exposure to instanton collective coordinates. We will see that in general a k-instanton solution in the SU(N) theory will contain $4kN$ collective coordinates (for the SU(2) theory and $k=1$ we have $4kN=8$). It is straightforward to see that the corresponding field strength

$$F_{mn} = g^{-1}\,\frac{4\rho^2\sigma_{mn}}{((x-X)^2+\rho^2)^2} \tag{2.14}$$

is self-dual. The anti-instanton is obtained from expressions above via a substitution $\sigma_{mn} \to \bar{\sigma}_{mn}$.

Note that BPST instanton (2.13) is a non-singular expression which falls off at large distances as x^{-1}. This slow fall off would make it difficult to construct square-integrable quantities involving instanton gauge fields (see [2] for more detail). An elegant and straightforward resolution of this technical problem is to gauge transform regular instanton (2.13) with a singular gauge transformation $U(x) = \bar{\sigma}_m(x-X)_m/|x-X|$. The resulting expression for the instanton in singular gauge is

$$A_n = g^{-1}\,\frac{2\rho^2(x-X)_m\bar{\sigma}_{mn}}{(x-X)^2((x-X)^2+\rho^2)}\ . \tag{2.15}$$

Note in singular gauge the SU(2) gauge indices are identified with $SU(2)_R$ indices in contrast to the expression in regular gauge. This expression falls off as x^{-3} at large distances which improves the convergence of various integrals. [5] Hence, from now on we will always assume that all the multi-instanton solutions we are dealing with are written in singular gauge. The price to pay for this is the apparent singularity of solution (2.15) at the instanton centre $x_m = X_m$. However, this singularity is not a problem, since it is, by construction, removable by a gauge transformation. A rigorous way of dealing with a singular gauge transformation is to introduce it on a punctured Euclidean space with the singular point(s) being removed. Then the singular-gauge instanton remains regular on the punctured space. Note that the punctures contribute to the boundary, hence the integrals of total derivatives, such as instanton charge (2.2), will receive contributions from these punctures.

2.2. Collective coordinates and moduli space

One of the key concepts associated to instantons (and more generally to solitons) is the idea of a *moduli space of solutions*. For instantons, this is the space of inequivalent solutions of the self-dual Yang–Mills equations (2.5). Since we are dealing with a gauge theory, the word "inequivalent" requires some clarification. For present purposes, it is most convenient to think of "inequivalent" as being equivalence up to *local* gauge transformations. So, for instance, solutions differing by *global* gauge transformation are deemed to be inequivalent to each other. The reason for the distinction between local and global gauge transformations is discussed in [9]. In the present context the main

[5] The fast fall off of the singular gauge is also required for applying the LSZ reduction formulae to various Green functions in the instanton background.

point is simply that the usual covariant gauge fixing condition does not fix global gauge transformations and hence we must still integrate over the corresponding orbits in the path integral. If one is calculating a correlation function of gauge-invariant operators, this integral simply leads to an additional factor of the volume of the gauge group.

It will turn out that the moduli space of instantons has a lot of mathematical structure, but for the moment, there are two properties that are paramount. Firstly, since finite action classical solutions of gauge theories on four-dimensional Euclidean space are classified by the topological (instanton) charge (2.2), the moduli space must contain distinct components describing the inequivalent solutions for each topological charge k. When the gauge group is $\mathrm{SU}(N)$ (or $\mathrm{U}(N)$ since the abelian factor makes no difference to the instanton solutions on a commutative Euclidean space) we will denote the moduli space of instantons with topological charge k as $\mathfrak{M}_k$. The second property that these moduli spaces have is that they are manifolds, a fact which is not a priori obvious. Strictly speaking, as we shall see later, they have conical-type singularities (which occur physically when instantons shrink to zero size) and so we will use the term manifold in a slightly looser sense to encompass spaces with these kinds of features. We shall also see that $\mathfrak{M}_k$ has even more structure: it is a complex manifold of a very particular type known as a hyper-Kähler manifold.

Since the moduli space is a manifold, albeit with singularities, we can introduce local coordinates to label its points. The coordinates on the moduli space label various *collective* properties of the gauge field and are therefore called *collective coordinates*. So the gauge fields of the instanton $A_n(x;X)$ depend not only on the coordinates on $\mathbb{R}^4$, x_n, but also on a set of collective coordinates that we denote X^μ, $\mu=1,2,\ldots,\dim\mathfrak{M}_k$. Some of the collective coordinates have an obvious physical interpretation; for instance, if we have a given instanton solution, then since the solution is localized in $\mathbb{R}^4$ it has a definable notion of centre. We can obviously translate the whole configuration in $\mathbb{R}^4$ and so there must be collective coordinates that specify the position of the centre which we denote as X_n, $n=1$–4. Notice that, by symmetry, the gauge fields can only depend on these coordinates through the difference $x_n - X_n$. To reflect this, the moduli space $\mathfrak{M}_k$ is a product

$$\mathfrak{M}_k = \mathbb{R}^4 \times \widehat{\mathfrak{M}}_k \ . \tag{2.16}$$

The component $\widehat{\mathfrak{M}}_k$, with the centre factored out, is known as the *centred moduli space*.

Notice that the collective coordinates X_n arise because, by its very nature, a given instanton solution breaks the translational symmetry of the theory. Another way to say this is that the subspace spanned by the collective coordinates X_n around a given point in the moduli space can be generated by acting on the instanton solution with the group elements of the broken symmetry. In this case the symmetry in question are translations in spacetime. So if $T_X = -X_n\partial/\partial x_n$ are the generators of translations, then

$$A_n(x;X,\ldots) = \mathrm{e}^{T_X} A_n(x;0,\ldots) = A_n(x-X;0,\ldots) \ . \tag{2.17}$$

This is all quite trivial for the translational collective coordinates; however, it illustrates a general principle: there are collective coordinates associated to all the symmetries of the gauge theory that are broken by a given instanton solution. However, not all symmetries of the gauge theory give inequivalent collective coordinates and not all collective coordinates correspond to broken symmetries: indeed typically the majority do not. Another way to phrase this is that the symmetries of the classical equations-of-motion are realized as symmetries of the moduli space and different symmetries may sweep out the same subspace of the moduli space while some symmetries may act trivially on (leave invariant) the moduli space. Finally, there may be some directions in the moduli space, and typically the majority, that are not related to any symmetry.

We now describe the symmetries of our theory. First of all, we have spacetime symmetries including the Poincaré symmetry of four-dimensional Euclidean space. However, classical gauge theory is actually invariant under the larger group that includes conformal transformations. In all, this group has 15 generators which includes four translations, six rotations, four special conformal transformations generated by

$$\delta x_n = 2x_n(x\cdot\varepsilon) - \varepsilon_n x^2 \ , \tag{2.18}$$

where ε_n is an infinitesimal four-vector, and dilatations $\delta x_n = \alpha x_n$. In quaternionic language where the vector x_n is represented as the 2×2 matrix $x_{\alpha\dot\alpha}$ in (2.9), the action of the whole conformal group can be written elegantly as

$$x \to x' = (Ax+B)(Cx+D)^{-1} \ , \quad \det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = 1 \ , \tag{2.19}$$

where A, B, C and D are quaternions. Notice that there are 15 variables in transformation (2.19), matching the dimension of the conformal group. In addition to these spacetime symmetries, there are global gauge transformations in our chosen gauge group $\mathrm{SU}(N)$. Some of these symmetries will, like translations, be broken by an instanton solution, and consequently be realized non-trivially on the moduli space.

Another key concept that is related to the idea of a collective coordinate is the notion of a *zero mode*. Suppose that $A_n(x)$ is an instanton solution. Consider some small fluctuation $A_n(x) + \delta A_n(x)$ around this solution which is also a solution of the self-dual Yang–Mills equations (2.5). To linear order

$$\mathscr{D}_m \delta A_n - \mathscr{D}_n \delta A_m = \varepsilon_{mnkl} \mathscr{D}_k \delta A_l \ . \tag{2.20}$$

There are actually three independent equations here, manifested by writing them in quaternionic form:

$$\vec\tau^{\dot\alpha}{}_{\dot\beta}\bar{\mathscr{D}}^{\dot\beta\alpha}\delta A_{\alpha\dot\alpha} = 0 \ , \tag{2.21}$$

where $\vec\tau$ are the three Pauli matrices and we define the covariant derivatives in the quaternionic basis as

$$\mathscr{D\!\!\!/} \equiv \sigma_n \mathscr{D}_n, \quad \bar{\mathscr{D\!\!\!/}} \equiv \bar\sigma_n \mathscr{D}_n \ . \tag{2.22}$$

We must also fix the gauge in some way to weed out the variations that are just local gauge transformations rather than genuine physical variations of the instanton solution. This is conveniently achieved by demanding that all fluctuations, including the zero modes, are orthogonal to gauge transformations. Here, orthogonality is defined in a functional sense by means of the inner product on adjoint-valued (anti-Hermitian) variations

$$\langle \delta A_n, \delta' A_n \rangle = -2\int \mathrm{d}^4x\, \mathrm{tr}_N\, \delta A_n(x)\delta' A_n(x) \ . \tag{2.23}$$

Implicitly, we are discussing fluctuations which are square integrable with respect to this inner product. So if δA_n is orthogonal to a gauge transformation, $\int d^4x\,\mathrm{tr}_N\,\mathscr{D}_n\Omega\delta A_n=0$, then by integrating by parts

$$\mathscr{D}_n\delta A_n=0\ . \tag{2.24}$$

Again it is quite instructive to write this in quaternionic form:

$$\bar{\mathscr{D}}^{\dot\alpha\alpha}\delta A_{\alpha\dot\alpha}=0\ . \tag{2.25}$$

The conditions for a zero mode, (2.21) and (2.25), can then be written as a single quaternion equation:

$$\bar{\mathscr{D}}^{\dot\alpha\alpha}\delta A_{\alpha\dot\beta}=0\ . \tag{2.26}$$

We recognize this as the covariant Weyl equation for a Weyl spinor $\psi_\alpha=\delta A_{\alpha\dot\beta}$ in the instanton background. Due to the free $\dot\beta$ index each gauge zero mode actually corresponds to two independent solutions of the Weyl equation.

The fluctuations $\delta A_n(x)$ satisfying (2.20) and (2.24), or (2.26), are then "zero modes" in the sense that they represent physical fluctuations in field space which do not change the value of the action. Non-zero mode fluctuations necessarily increase the action of the instanton solution. To quadratic order around the instanton solution, one finds

$$\begin{aligned}S[A]&=-2\pi\mathrm{i}k\tau+\int d^4x\,\mathrm{tr}_N\,\delta A_m(\mathscr{D}^2\delta_{mn}+2gF_{mn})\delta A_n+\cdots\\&=-2\pi ik\tau-\frac{1}{2}\int d^4x\,\mathrm{tr}_N\,\delta A^{\dot\alpha\alpha}\Delta^{(+)}{}_\alpha{}^\beta\delta A_{\beta\dot\alpha}+\cdots\ ,\end{aligned} \tag{2.27}$$

where we have defined the fluctuation operator $\Delta^{(+)}$, and, for future use, a companion $\Delta^{(-)}$:

$$\Delta^{(+)}\equiv-\mathscr{D}\bar{\mathscr{D}}=-1_{[2]\times[2]}\mathscr{D}^2-gF_{mn}\sigma_{mn}\ , \tag{2.28a}$$

$$\Delta^{(-)}\equiv-\bar{\mathscr{D}}\mathscr{D}=-1_{[2]\times[2]}\mathscr{D}^2-gF_{mn}\bar\sigma_{mn}\ . \tag{2.28b}$$

The equality between the two expressions in (2.27) is established by using the fact that σ_{mn} is a projector onto self-dual tensors (A.9). In addition, in an instanton (as opposed to an anti-instanton) background, the second term in $\Delta^{(-)}$ vanishes; hence

$$\Delta^{(-)}=-1_{[2]\times[2]}\mathscr{D}^2\ . \tag{2.29}$$

The two operators $\Delta^{(\pm)}$ play an important rôle in governing the behaviour of fluctuations around the instanton in the semi-classical approximation. An important result is that in an instanton (self-dual) background $\Delta^{(+)}$ has (normalizable) zero modes, whereas, by virtue of (2.29), $\Delta^{(-)}$ is a positive semi-definite operator and therefore has no normalizable zero modes.

The zero modes are associated to the collective coordinates in the following way. Consider the space of solutions $A_n(x;X)$ where x_n are the Euclidean spacetime coordinates and X^μ are the collective coordinates. The derivative of the gauge field with respect to a collective coordinate, $\partial A_n/\partial X^\mu$, is guaranteed to satisfy the zero mode equation (2.20): in other words it is a potential zero mode. In order to ensure that it is a genuine zero mode, we have to satisfy gauge condition (2.24). The way to achieve this is to notice that (2.20) is trivially satisfied by a gauge transformation $\delta A_n = D_n\Omega$, for some function Ω in the Lie algebra of the gauge group. We can then consider a linear combination of the derivative by the collective coordinate and a compensating gauge transformation:

$$\delta_\mu A_n(x;X) \equiv \frac{\partial A_n(x;X)}{\partial X^\mu} - \mathscr{D}_n\Omega_\mu(x) . \tag{2.30}$$

The parameter of the gauge transformation $\Omega_\mu(x)$ is chosen in order that (2.24) is satisfied by $\delta_\mu A_n$; hence

$$\mathscr{D}_n\left(\frac{\partial A_n}{\partial X^\mu}\right) = \mathscr{D}^2\Omega_\mu . \tag{2.31}$$

The quantity $\delta_\mu A_n$ is then the "genuine" zero mode associated to X^μ. For the case of instantons, as long as we work in a special gauge known as *singular gauge* (as in Eq. (2.15)), there are no subtleties associated with normalizability and $\delta_\mu A_n$ are all square integrable. This rather convenient choice of gauge will be described in Section 2.4.3. Subsequently in Section 2.5 we will show in singular gauge that $\delta_\mu A_n$ is $\mathcal{O}(x^{-3})$ for large x.

We have seen that all collective coordinates are associated to zero modes. However, when gauge fields couple to other fields, it can happen that instanton zero modes are no longer associated to collective coordinates because they can fail to integrate from solutions of the linearized problem to solutions of the full coupled equations. Often we find that zero modes of the linearized problem are lifted by higher-order interactions or by external interactions (for instance when scalar fields have VEVs). In these circumstances it is still useful to introduce the notion of *quasi-collective coordinates* which are lifted by a non-trivial effective action. The quasi-collective coordinates give rise to the notion of a *quasi-instanton* which only satisfies the classical equation-of-motion up to a certain order in the coupling.

2.3. General properties of the moduli space of instantons

The moduli space of instanton solutions plays a central rôle in our story, so it is important to describe in some detail how one arrives at a description of it. The story is necessarily rather mathematical, but the final answer, known as the ADHM construction, after Atiyah, Drinfeld, Hitchin and Manin [7] is a great mathematical achievement which we will review in Section 2.4. The ADHM moduli space has a lot of structure that can be deduced from general considerations. For example, the dimension of $\mathfrak{M}_k$ can be obtained by simply counting the number of zero modes at a point in the moduli space using the Atiyah–Singer Index Theorem [14] (for applications to instantons see [15]). The result is that $\mathfrak{M}_k$ has real dimension $4kN$.

2.3.1. The moduli space as a complex manifold

We have argued that the moduli space $\mathfrak{M}_k$ is a space with dimension $4kN$. In fact it is a Riemannian manifold endowed with a natural metric defined as the functional inner product of the zero modes (in singular gauge),[6]

$$g_{\mu\nu}(X) = -2g^2 \int d^4x \, \mathrm{tr}_N \, \delta_\mu A_n(x;X)\delta_\nu A_n(x;X) \ . \tag{2.32}$$

This metric plays an important rôle in the theory since, as we establish in Section 3, it defines the volume form on $\mathfrak{M}_k$ that arises from changing variables in the path integral from the gauge field to the collective coordinates. However, we shall find that various other quantities that are derived from the metric, like the connection and curvature, also have an important rôle to play in the instanton calculus.

In fact there is more structure to the moduli space than simply the existence of a metric. It turns out that it is also a complex manifold of a very particular kind, namely a hyper-Kähler space (with singularities). A short review of some relevant aspects of such spaces is provided in Appendix B. Fundamentally, these spaces admit three linearly independent complex structures $\boldsymbol{I}^{(c)}$, $c = 1$–3, that satisfy the algebra

$$\boldsymbol{I}^{(c)}\boldsymbol{I}^{(d)} = -\delta^{cd} + \varepsilon^{cde}\boldsymbol{I}^{(e)} \ . \tag{2.33}$$

Often we will represent them as a three-vector $\vec{\boldsymbol{I}}$. The key idea for constructing the triplet of complex structures on $\mathfrak{M}_k$ arises after noticing that Euclidean spacetime is itself hyper-Kähler. The three complex structures of $\mathbb{R}^4$ can be chosen so that $I^c_{mn} = -\bar{\eta}^c_{mn}$, where $\bar{\eta}^c_{mn}$ is a 't Hooft η-symbol defined in Appendix A. In the quaternion basis $x_{\dot{\alpha}\alpha}$

$$(\vec{\boldsymbol{I}} \cdot x)_{\alpha\dot{\alpha}} = \mathrm{i}x_{\alpha\dot{\beta}}\vec{\tau}^{\dot{\beta}}{}_{\dot{\alpha}}, \quad (\vec{\boldsymbol{I}} \cdot \bar{x})^{\dot{\alpha}\alpha} = -\mathrm{i}\vec{\tau}^{\dot{\alpha}}{}_{\dot{\beta}}\bar{x}^{\dot{\beta}\alpha} \ . \tag{2.34}$$

These descend to give the three complex structures on $\mathfrak{M}_k$ in the following way. Notice that in the zero mode equation (2.26) $\dot{\beta}$ is a free index. This means that if $\delta_\mu A_{\alpha\dot{\alpha}}$ is a zero mode then so is $\delta_\mu A_{\alpha\dot{\beta}}G^{\dot{\beta}}{}_{\dot{\alpha}}$, for any constant matrix G. In particular, if $\delta_\mu A_{\alpha\dot{\alpha}}$ is a zero mode then so is $(\vec{\boldsymbol{I}} \cdot \delta_\mu A)_{\alpha\dot{\alpha}} = \mathrm{i}\delta_\mu A_{\alpha\dot{\beta}}\vec{\tau}^{\dot{\beta}}{}_{\dot{\alpha}}$. Since the zero modes form a complete set, there must exist $\vec{\boldsymbol{I}}^\nu{}_\mu$ such that

$$(\vec{\boldsymbol{I}} \cdot \delta_\mu A)_{\alpha\dot{\alpha}} = \delta_\nu A_{\alpha\dot{\alpha}}\vec{\boldsymbol{I}}^\nu{}_\mu \ , \tag{2.35}$$

from which it follows that the triplet $\vec{\boldsymbol{I}}^\nu{}_\mu$ satisfies algebra (2.33).

At this stage, $\vec{\boldsymbol{I}}^\mu{}_\nu$ are *almost* complex structures because we have not proved that they are integrable (B.1). Rather than prove this directly we will follow the analysis of Maciocia [16] and construct a *hyper-Kähler potential* for $\mathfrak{M}_k$. This proves that $\mathfrak{M}_k$ is not only hyper-Kähler but also it is a rather special kind for which each complex structure shares the same Kähler potential (called the hyper-Kähler potential). The expression for the potential is [16]

$$\chi = -\frac{g^2}{4} \int d^4x \, x^2 \, \mathrm{tr}_N \, F^2_{mn} \ . \tag{2.36}$$

In order to prove this, we pick out one of the complex structures $\boldsymbol{I}^{(c)}$ of $\mathbb{R}^4$. We will choose holomorphic coordinates $(z^i, \bar{z}^i)$, $i=1,2$, with respect to this particular complex structure. For example,

[6] The factor of g^2 is inserted here so that the metric is independent of the coupling.

choosing $\boldsymbol{I}^{(3)}$ we have $z^1 = \mathrm{i}x^3 + x^4$ and $z^2 = \mathrm{i}x^1 - x^2$. The complex structure is associated, via (2.35), with a complex structure on $\mathfrak{M}_k$. We can then choose a set of matching holomorphic coordinates on $\mathfrak{M}_k$ $(Z^{\mathrm{i}}, \bar{Z}^{\mathrm{i}})$, $\mathrm{i} = 1, \ldots, \frac{1}{2}\dim\mathfrak{M}_k$ for which the complex structure on the moduli space is

$$\boldsymbol{I}^{(c)} = \begin{pmatrix} \mathrm{i}\delta^{\mathrm{i}}{}_{\mathrm{j}} & 0 \\ 0 & -\mathrm{i}\delta^{\mathrm{i}}{}_{\mathrm{j}} \end{pmatrix} . \tag{2.37}$$

Then, from (2.35) it follows that

$$\frac{\partial A_{\alpha\dot\beta}}{\partial Z^{\mathrm{i}}}\tau^{(c)\dot\beta}{}_{\dot\alpha} = \frac{\partial A_{\alpha\dot\alpha}}{\partial Z^{\mathrm{i}}} , \quad \frac{\partial A_{\alpha\dot\beta}}{\partial \bar{Z}^{\mathrm{i}}}\tau^{(c)\dot\beta}{}_{\dot\alpha} = -\frac{\partial A_{\alpha\dot\alpha}}{\partial \bar{Z}^{\mathrm{i}}} . \tag{2.38}$$

For example, for $\boldsymbol{I}^{(3)}$ this implies

$$\frac{\partial A_{\alpha 2}}{\partial Z^{\mathrm{i}}} = 0, \quad \frac{\partial A_{\alpha 1}}{\partial \bar{Z}^{\mathrm{i}}} = 0 . \tag{2.39}$$

Furthermore, variation equation (2.20) implies that the derivatives above automatically satisfy the background gauge condition (2.25). Consequently

$$\delta_{\mathrm{i}} A_n \equiv \frac{\partial A_n}{\partial Z^{\mathrm{i}}}, \quad \bar\delta_{\mathrm{i}} A_n \equiv \frac{\partial A_n}{\partial \bar{Z}^{\mathrm{i}}} \tag{2.40}$$

are zero modes directly without the need for a compensating gauge transformation (2.30). This fact turns out to be crucial. Furthermore,

$$\frac{\partial^2 A_n}{\partial \bar{Z}^{\mathrm{j}} \partial Z^{\mathrm{i}}} = 0 , \tag{2.41}$$

as is clear from (2.39) for the case of $\boldsymbol{I}^{(3)}$.

By explicit calculation, using the zero mode condition (2.26) and (2.41), one finds

$$\frac{\partial^2}{\partial \bar{Z}^{\mathrm{j}} \partial Z^{\mathrm{i}}} \operatorname{tr}_N F_{mn}^2 = \Box \operatorname{tr}_N \delta_{\mathrm{i}} A_n \bar\delta_{\mathrm{j}} A_n - 2\partial_m \partial_n \operatorname{tr}_N \delta_{\mathrm{i}} A_m \bar\delta_{\mathrm{j}} A_n . \tag{2.42}$$

Using this in (2.36) and integrating by parts twice, discarding the surface terms since the zero modes decay as $\mathcal{O}(x^{-3})$, we have

$$\frac{\partial^2 \chi}{\partial \bar{Z}^{\mathrm{j}} \partial Z^{\mathrm{i}}} = -2g^2 \int \mathrm{d}^4 x \operatorname{tr}_N \delta_{\mathrm{i}} A_n \, \bar\delta_{\mathrm{j}} A_n . \tag{2.43}$$

By comparing with (2.32), we see that the above expression is a component of the metric on the space of zero modes:

$$g(X) = \frac{\partial^2 \chi}{\partial Z^{\mathrm{j}} \partial Z^{\mathrm{i}}} \, \mathrm{d}Z^{\mathrm{i}} \, \mathrm{d}\bar{Z}^{\mathrm{j}} . \tag{2.44}$$

This proves that χ is the Kähler potential for the complex structure $\boldsymbol{I}^{(c)}$ for $c=3$. However, χ manifestly does not depend on the choice of the index $c=1$–3 and so it is a hyper-Kähler potential, and by implication $\mathfrak{M}_k$ is a hyper-Kähler space.

2.4. The ADHM construction of instantons

In this section we describe the construction of instantons due to ADHM [7] and how this leads to a description of $\mathfrak{M}_k$ for which the hyper-Kähler property is manifest. This remarkable construction of ADHM was originally discussed in Refs. [17–19]. Here we follow, with minor modifications, the SU(N) formalism of Refs. [20,21]. Our approach here is to describe the ADHM construction as an ansatz for producing instanton solutions and we direct the reader to the references above for the more mathematical technicalities.

The basic object in the ADHM construction is the $(N+2k)\times 2k$ complex-valued matrix $\Delta_{\lambda i\dot\alpha}$ which is taken to be linear in the spacetime variable x_n:

$$\Delta_{\lambda i\dot\alpha}(x) = a_{\lambda i\dot\alpha} + b^{\alpha}_{\lambda i}x_{\alpha\dot\alpha}, \quad \bar\Delta^{\dot\alpha\lambda}_{i}(x) = \bar a^{\dot\alpha\lambda}_{i} + \bar x^{\dot\alpha\alpha}\bar b^{\lambda}_{i\alpha} . \tag{2.45}$$

Here, we have introduced "ADHM indices" $\lambda, \mu,\ldots=1,\ldots,N+2k$ and "instanton indices" $i,j,\ldots=1,\ldots,k$ and used the quaternionic representation of x_n as in (2.8) and (2.9). By definition the conjugate is [7]

$$\bar\Delta^{\dot\alpha\lambda}_{i} \equiv (\Delta_{\lambda i\dot\alpha})^* . \tag{2.46}$$

We will soon verify by direct calculation that k is the instanton charge of the solution. As discussed below, the complex-valued constant matrices a and b in (2.45) constitute a (highly over-complete) set of collective coordinates on $\mathfrak{M}_k$.

Generically, the null space of the Hermitian conjugate matrix $\bar\Delta(x)$ is N-dimensional, as it has N fewer rows than columns. The basis vectors for this null space can be assembled into an $(N+2k)\times N$-dimensional complex-valued matrix $U_{\lambda u}(x)$, $u=1,\ldots,N$:

$$\bar\Delta^{\dot\alpha\lambda}_{i}U_{\lambda u} = 0 = \bar U^{\lambda}_{u}\Delta_{\lambda i\dot\alpha} , \tag{2.47}$$

where U is orthonormalized according to

$$\bar U^{\lambda}_{u}U_{\lambda v} = \delta_{uv} . \tag{2.48}$$

The construction requires a non-degeneracy condition: the maps $\Delta_{\dot\alpha}(x)$: $\mathbb{C}^k \rightarrow \mathbb{C}^{N+2k}$ must be injective while the maps $\bar\Delta^{\dot\alpha}(x)$: $\mathbb{C}^{N+2k} \rightarrow \mathbb{C}^k$ must be surjective. Having said this, we will see in Section 2.6 that points where the non-degeneracy conditions break down have an interesting physical interpretation.

[7] Throughout this, and other section, an over-bar means Hermitian conjugation: $\bar\Delta\equiv\Delta^\dagger$. Notice as usual we have to distinguish between upper and lower spinor indices, and hence also for the ADHM indices, but not for the other types of index. Frequently, where confusion cannot arise, we do not indicate all the indices. Often only the spinor indices need to be labelled explicitly.

In turn, the classical ADHM gauge field A_n is constructed from U as follows. Note first that for the special case $k=0$, the anti-symmetric gauge configuration A_n defined by

$$(A_n)_{uv} = g^{-1}\bar{U}^{\lambda}_{u}\partial_n U_{\lambda v} \tag{2.49}$$

is "pure gauge" so that it automatically solves the self-duality equation (2.5) in the vacuum sector. The ADHM ansatz is that Eq. (2.49) continues to give a solution to Eq. (2.5), even for non-zero k. As we shall see, this requires the additional condition

$$\bar{\Delta}^{\dot\alpha\lambda}_{i}\Delta_{\lambda j\dot\beta} = \delta^{\dot\alpha}{}_{\dot\beta}(f^{-1})_{ij} \ , \tag{2.50}$$

where f is an arbitrary x-dependent $k\times k$-dimensional Hermitian matrix. Note that the existence of the inverse f^{-1} is guaranteed by the non-degeneracy condition.

To check the validity of the ADHM ansatz, we first observe that Eq. (2.50) combined with the null-space condition (2.47) implies the completeness relation

$$\mathscr{P}_{\lambda}{}^{\mu} \equiv U_{\lambda u}\bar{U}^{\mu}_{u} = \delta^{\mu}_{\lambda} - \Delta_{\lambda i\dot\alpha}f_{ij}\bar{\Delta}^{\dot\alpha\mu}_{j} \ . \tag{2.51}$$

Note that $\mathscr{P}$, as defined, is actually a projection operator; the fact that one can write $\mathscr{P}$ in these two equivalent ways turns out to be a useful trick in ADHM algebra, used pervasively throughout the instanton calculus. With the above relations the expression for the field strength F_{mn} may be massaged as follows:

$$\begin{aligned} F_{mn} &\equiv \partial_m A_n - \partial_n A_m + g[A_m, A_n] = g^{-1}\partial_{[m}(\bar{U}\partial_{n]}U) + g^{-1}(\bar{U}\partial_{[m}U)(\bar{U}\partial_{n]}U) \\ &= g^{-1}\partial_{[m}\bar{U}(1-U\bar{U})\partial_{n]}U = g^{-1}\partial_{[m}\bar{U}\Delta f\bar{\Delta}\partial_{n]}U \\ &= g^{-1}\bar{U}\partial_{[m}\Delta f\partial_{n]}\bar{\Delta}U = g^{-1}\bar{U}b\sigma_{[m}\bar{\sigma}_{n]}f\bar{b}U \\ &= 4g^{-1}\bar{U}b\sigma_{mn}f\bar{b}U \ . \end{aligned} \tag{2.52}$$

Self-duality of the field strength then follows automatically from the self-duality property of the tensor σ_{mn} (2.12).

The instanton number of the ADHM configuration can be calculated directly using a remarkable identity of Osborn [8] for $\mathrm{tr}_N F^2$ in the ADHM background proved in Appendix C:

$$-\frac{g^2}{16\pi^2}\int d^4x\,\mathrm{tr}_N F^2_{mn} = \frac{1}{16\pi^2}\int d^4x\,\Box^2\,\mathrm{tr}_k \log f \ . \tag{2.53}$$

Now from (2.50) one can deduce the asymptotic form for $f(x)$ for large x, $f(x)\overset{x\to\infty}{\to}x^{-2}1_{[k]\times[k]}$, and therefore the right-hand side is equal to k as claimed.

Let us analyse factorization condition (2.50) in more detail. Noting that $f_{ij}(x)$ is arbitrary, one extracts three x-independent conditions on a and b:

$$\bar{a}^{\dot\alpha\lambda}_{i}a_{\lambda j\dot\beta} = (\tfrac{1}{2}\bar{a}a)_{ij}\delta^{\dot\alpha}{}_{\dot\beta} \ , \tag{2.54a}$$

$$\bar a_i^{\dot\alpha\lambda} b_{\lambda j}^{\beta} = \bar b_i^{\beta\lambda} a_{\lambda j}^{\dot\alpha} \ , \tag{2.54b}$$

$$\bar b_{\alpha i}^{\lambda} b_{\lambda j}^{\beta} = (\tfrac{1}{2}\bar b b)_{ij}\delta_\alpha{}^\beta \ . \tag{2.54c}$$

These three conditions are generally known as the "ADHM constraints" [18,19].[8] They define a set of coupled quadratic conditions on the matrix elements of a, $\bar a$ b and $\bar b$. Note that (2.54b) and (2.54c) can be combined in the useful form

$$\bar\Delta^{\dot\alpha} b^\alpha = \bar b^\alpha \Delta^{\dot\alpha} \ . \tag{2.55}$$

The fact that the ADHM construction involves non-linear constraints presents considerable difficulties for practical applications. However, it turns out, as we shall in Section 6.1, that the ADHM constraints can be resolved in a simple way, at least generically, when $N \geqslant 2k$.

The elements of the matrices a and b comprise the collective coordinates for the k-instanton gauge configuration. Clearly the number of independent such elements grows as k^2, even after accounting for constraints (2.54a)–(2.54c). In contrast, the number of physical collective coordinates should equal $4kN$ which scales linearly with k. It follows that a and b constitute a highly redundant set. Much of this redundancy can be eliminated by noting that the ADHM construction is unaffected by x-independent transformations of the form

$$\Delta \to \Lambda\Delta\Upsilon^{-1}, \quad U \to \Lambda U, \quad f \to \Upsilon f \Upsilon^\dagger \ , \tag{2.56}$$

provided $\Lambda \in \mathrm{U}(N+2k)$ and $\Upsilon \in \mathrm{Gl}(k,\mathbb{C})$. Exploiting these symmetries, one can choose a representation in which b assumes a simple canonical form [18]. Decomposing the index $\lambda = u + \mathrm{i}\alpha$

$$b_{\lambda j}^{\beta} = b_{(u+\mathrm{i}\alpha)j}^{\beta} = \begin{pmatrix} 0 \\ \delta_\alpha{}^\beta \delta_{ij} \end{pmatrix}, \quad \bar b_{\beta j}^{\lambda} = \bar b_{\beta j}^{(u+\mathrm{i}\alpha)} = (0 \quad \delta_\beta{}^\alpha \delta_{ji}) \ . \tag{2.57}$$

The remaining variables all reside in a and we will split them up in a way that mirrors the canonical form for b:

$$a_{\lambda j\dot\alpha} = a_{(u+\mathrm{i}\alpha)j\dot\alpha} = \begin{pmatrix} w_{uj\dot\alpha} \\ (a'_{\alpha\dot\alpha})_{ij} \end{pmatrix}, \quad \bar a_j^{\dot\alpha\lambda} = \bar a_j^{\dot\alpha(u+\mathrm{i}\alpha)} = (\bar w_{ju}^{\dot\alpha} \quad (\bar a'^{\dot\alpha\alpha})_{ji}) \ . \tag{2.58}$$

With b in canonical form (2.57), the third ADHM constraint of (2.54c) is satisfied automatically, while the remaining constraints (2.54a) and (2.54b) boil down to the $k \times k$ matrix equations

$$\vec\tau^{\dot\alpha}{}_{\dot\beta}(\bar a^{\dot\beta} a_{\dot\alpha}) = 0 \ , \tag{2.59a}$$

$$(a'_n)^\dagger = a'_n \ . \tag{2.59b}$$

In (2.59a) there are three separate equations since we have contracted $\bar a^{\dot\beta} a_{\dot\alpha}$ with any of the three Pauli matrices, while in Eq. (2.59b) we have decomposed $a'_{\alpha\dot\alpha}$ and $\bar a'^{\dot\alpha\alpha}$ in our usual quaternionic

[8] We should warn the reader that we use the term "ADHM constraints" in a rather more restricted sense after a canonical choice has been made below.

basis of spin matrices (2.8):

$$a'_{\alpha\dot{\alpha}} = a'_n \sigma_{n\alpha\dot{\alpha}}, \quad \bar{a}'^{\dot{\alpha}\alpha} = a'_n \bar{\sigma}_n^{\dot{\alpha}\alpha} . \tag{2.60}$$

In canonical form (2.57) we also have the useful identity

$$\bar{b}_\alpha b^\beta = \delta_\alpha{}^\beta 1_{[k]\times[k]} \tag{2.61}$$

and the ADHM matrix f takes the form

$$f = 2(\bar{w}^{\dot{\alpha}} w_{\dot{\alpha}} + (a'_n + x_n 1_{[k]\times[k]})^2)^{-1} . \tag{2.62}$$

Note that the canonical form for b (2.57) is preserved by a residual $\mathrm{U}(k)$ subgroup of the $\mathrm{U}(N+2k) \times \mathrm{Gl}(k, \mathbb{C})$ symmetry group (2.56):

$$\Lambda = \begin{pmatrix} 1_{[N]\times[N]} & 0 \\ 0 & \Xi 1_{[2]\times[2]} \end{pmatrix}, \quad \Upsilon = \Xi, \quad \Xi \in \mathrm{U}(k) . \tag{2.63}$$

These residual transformations act non-trivially on the remaining variables:

$$w_{ui\dot{\alpha}} \to w_{\dot{\alpha}} \Xi, \quad a'_n \to \Xi^\dagger a'_n \Xi . \tag{2.64}$$

Henceforth, we shall use almost exclusively the streamlined version of the ADHM construction obtained by fixing b as in (2.57). The basic variables will be $a_{\dot{\alpha}} = \{w_{\dot{\alpha}}, a'_n\}$ where we will automatically assume (2.59b) so that the four $k \times k$ matrices a'_n are defined from the outset to be Hermitian. The remaining constraints (2.59a)

$$\vec{\tau}^{\dot{\alpha}}{}_{\dot{\beta}} \bar{a}^{\dot{\beta}} a_{\dot{\alpha}} \equiv \vec{\tau}^{\dot{\alpha}}{}_{\dot{\beta}} (\bar{w}^{\dot{\beta}} w_{\dot{\alpha}} + a'^{\dot{\beta}\alpha} a'_{\alpha\dot{\alpha}}) = 0 \tag{2.65}$$

will be called the "ADHM constraints".

2.4.1. The ADHM construction as a hyper-Kähler quotient

It follows from the ADHM construction of the solutions of the self-dual Yang–Mills equations that the moduli space $\mathfrak{M}_k$ is identified with the variables $a_{\dot{\alpha}}$ subject to ADHM constraints (2.65) quotiented by the residual symmetry group $\mathrm{U}(k)$ (2.64). Subsequently it was realized that the ADHM construction is an example of a more general construction known as the hyper-Kähler quotient [22]. This construction is reviewed in Appendix B and provides a way of constructing a new hyper-Kähler space $\mathfrak{M}$ starting from a "mother" hyper-Kähler space $\tilde{\mathfrak{M}}$ with suitable isometries. In the case of the ADHM construction, the mother space $\tilde{\mathfrak{M}}$ is simply the Euclidean space $\mathbb{R}^{4k(k+N)}$ with coordinates $a_{\dot{\alpha}}$ and metric

$$\tilde{g} = 4\pi^2 (2 \mathrm{d}\bar{w}^{\dot{\alpha}}_{iu} \mathrm{d}w_{ui\dot{\alpha}} + \mathrm{d}(\mathrm{a}'^{\dot{\alpha}\alpha})_{ij} \, \mathrm{d}(a'_{\alpha\dot{\alpha}})_{ji}) \equiv 8\pi^2 \operatorname{tr}_k (\mathrm{d}\bar{w}^{\dot{\alpha}} \, \mathrm{d}w_{\dot{\alpha}} + \mathrm{d}a'_n \, \mathrm{d}a'_n) . \tag{2.66}$$

The normalization factor here will be justified in Section 2.5. Flat space with dimension a multiple of four is trivially hyper-Kähler as described in Appendix B. The tangent space of a hyper-Kähler space of real dimension $4n$, admits a distinguished $\mathrm{Sp}(n) \times \mathrm{SU}(2)$ basis of tangent vectors. For flat space $\mathbb{R}^{4k(N+k)}$, this symplectic structure is realized by a set of coordinates $z^{\tilde{\mathrm{i}}\dot{\alpha}}$, $\tilde{\mathrm{i}} = 1, \ldots, 2k(N+k)$ and $\dot{\alpha} = 1, 2$, in terms of which the flat metric is

$$\tilde{g} = \tilde{\Omega}_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}} \varepsilon_{\dot{\alpha}\dot{\beta}} \, \mathrm{d}z^{\tilde{\mathrm{i}}\dot{\alpha}} \, \mathrm{d}z^{\tilde{\mathrm{j}}\dot{\beta}} \ , \tag{2.67}$$

where $\tilde{\Omega}$ is a symplectic matrix. In the present case, where $\tilde{\mathfrak{M}} = \mathbb{R}^{4k(N+k)}$ is identified with $a_{\dot{\alpha}}$, the coordinates and matrix $\tilde{\Omega}$ are

$$z^{\tilde{\mathrm{i}}\dot{\alpha}} = \begin{pmatrix} \bar{w}^{\dot{\alpha}}_{iu} \\ (\bar{a}'^{\dot{\alpha}1})_{ij} \\ \varepsilon^{\dot{\alpha}\dot{\beta}} w_{ui\dot{\beta}} \\ \varepsilon^{\dot{\alpha}\dot{\beta}} (a'_{1\dot{\beta}})_{ij} \end{pmatrix}, \quad \tilde{\Omega}_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}} = 4\pi^2 \begin{pmatrix} 0 & 0 & 1_{[kN]\times[kN]} & 0 \\ 0 & 0 & 0 & 1_{[k^2]\times[k^2]} \\ -1_{[kN]\times[kN]} & 0 & 0 & \\ 0 & -1_{[k^2]\times[k^2]} & 0 & 0 \end{pmatrix} , \tag{2.68}$$

where the Roman indices $\tilde{\mathrm{i}}, \tilde{\mathrm{j}}, \ldots$ each run over the set of $2k(N+k)$ composite indices $\{iu, ij, ui, ij\}$.

The ADHM construction is the hyper-Kähler quotient of $\tilde{\mathfrak{M}} = \mathbb{R}^{4k(N+k)}$ by the isometry group $\mathrm{U}(k)$ which acts on the variables $a_{\dot{\alpha}}$ as in (2.64). This action defines a set of tri-holomorphic Killing vector fields on $\tilde{\mathfrak{M}}$:

$$X_r = \mathrm{i} T^r_{ij} \bar{w}^{\dot{\alpha}}_{ju} \frac{\partial}{\partial \bar{w}^{\dot{\alpha}}_{iu}} - \mathrm{i} T^r_{ji} w_{uj\dot{\alpha}} \frac{\partial}{\partial w_{ui\dot{\alpha}}} + \mathrm{i}[T^r, a'_n]_{ij} \frac{\partial}{\partial (a'_n)_{ij}} \ , \tag{2.69}$$

where T^r is a generator of $\mathrm{U}(k)$ in the fundamental representation.[9]

There are two main parts to the quotient construction. Firstly, one restricts to the *level set* $\mathfrak{N} \subset \tilde{\mathfrak{M}}$, defined by the vanishing of the moment maps associated to the isometries, in this case the $\mathrm{U}(k)$ vector fields (2.69). The expressions for the moment maps are given in (B.39). In the present case, using (2.68) and (2.69), the moment maps as

$$\mathrm{i}\vec{\mu}^{X_r} = 4\pi^2 \vec{\tau}^{\dot{\beta}}{}_{\dot{\alpha}} \, \mathrm{tr}_k \, (T^r \bar{a}^{\dot{\alpha}} a_{\dot{\beta}}) - \vec{\zeta}^r \ . \tag{2.70}$$

The central elements $\vec{\zeta}^r$ can, in this case, take values in the Lie algebra of the $\mathrm{U}(1)$ factor of the gauge group. One notices immediately that ADHM constraints (2.65) are precisely the conditions $\vec{\mu}^{X_r} = 0$ (with vanishing central element). In other words, the ADHM constraints explicitly implement the first part of the quotient construction. The second part of the quotient construction involves an ordinary quotient of $\mathfrak{N}$ by the $\mathrm{U}(k)$ action (which is guaranteed to fix $\mathfrak{N}$). But this quotient, as we have seen, is also an essential ingredient of the ADHM construction.

The conclusion is that the ADHM construction realizes the instanton moduli space $\mathfrak{M}_k$ as a hyper-Kähler quotient of flat space by a $\mathrm{U}(k)$ group of isometries (with vanishing central elements

[9] We will choose a normalization $\mathrm{tr}_k \, T^r T^s = \delta^{rs}$.

$\vec{\zeta}^r = 0$). Later in Section 11.1, we shall describe the physical interpretation of taking a non-vanishing $\vec{\zeta}^r$. The dimension of the quotient space $\mathfrak{M}_k$ is

$$\dim \mathfrak{M}_k = \dim \tilde{\mathfrak{M}} - 4 \dim \mathrm{U}(k) = 4kN \tag{2.71}$$

as anticipated by the Index Theorem.

The importance of the hyper-Kähler quotient construction is that geometric properties of $\mathfrak{M}_k$ are inherited from the mother space $\tilde{\mathfrak{M}}$ in a rather straightforward way (as described in Appendix B). For example of particular importance will be the metric on $\mathfrak{M}_k$ and we will now describe in some detail how to construct it. First of all, we focus on $T\mathfrak{N}$, the tangent space of the level set. Locally, as described in Appendix B, this is the subspace of $T\tilde{\mathfrak{M}}$ orthogonal to the $3k^2$ vectors $\tilde{\boldsymbol{I}}^{(c)}X_r$, where $\tilde{\boldsymbol{I}}^{(c)}$ are the three independent complex structures of the mother space $\tilde{\mathfrak{M}}$. Then we have the following decomposition

$$T\mathfrak{N} = \mathscr{V} \oplus \mathscr{H} , \tag{2.72}$$

where the *horizontal* subspace $\mathscr{H}$ is the subspace orthogonal to the vectors X_r and the *vertical* subspace $\mathscr{V}$ is the orthogonal complement. The tangent space of $\mathfrak{M}_k$ is identified with the quotient $T\mathfrak{N}/\mathscr{V}$. This means that each $X \in T\mathfrak{M}$ has a unique lift to $\mathscr{H}$, which, by a slight abuse of notation, we denote by the same letter. The metric on the quotient $g(X, Y)$ is then identified with the metric on $\tilde{\mathfrak{M}}$, $\tilde{g}(X, Y)$, evaluated on the lifts to $\mathscr{H}$.

What we would like to show is that the metric on $\mathfrak{M}_k$ inherited from the quotient construction is equal to the metric on $\mathfrak{M}_k$ that arises from the functional inner product of zero modes (2.32). Rather than implement the procedure that we describe above explicitly, it is more straightforward to use the fact that the hyper-Kähler spaces that we are considering are of a special class which admit a hyper-Kähler potential. We have already determined in (2.36) the form of this potential for the metric arising from the inner product of zero modes. Is this equal to the hyper-Kähler potential arising from the quotient construction? First of all, the mother space $\tilde{\mathfrak{M}}$ trivially admits such a potential with

$$\tilde{\chi} = \tilde{\Omega}_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}}\varepsilon_{\dot{\alpha}\dot{\beta}} z^{\tilde{\mathrm{i}}\dot{\alpha}} z^{\tilde{\mathrm{j}}\dot{\beta}} \equiv 8\pi^2 \operatorname{tr}_k(\bar{w}^{\dot{\alpha}} w_{\dot{\alpha}} + a'_n a'_n) . \tag{2.73}$$

The hyper-Kähler potential on the quotient space $\mathfrak{M}_k$ is then simply obtained by finding a parameterization of the ADHM variables in terms of the coordinates $\{X^\mu\}$ on $\mathfrak{M}_k$. In other words $z^{\tilde{\mathrm{i}}\dot{\alpha}}(X)$, or $a_{\dot{\alpha}}(X)$, which solves the ADHM constraints and for some choice of gauge slice for the U(k) action on the level set $\mathfrak{N}$. The hyper-Kähler potential for $\mathfrak{M}_k$ is then obtained by restriction

$$\chi(X) = \tilde{\chi}(z^{\tilde{\mathrm{i}}\dot{\alpha}}(X)) = 8\pi^2 \operatorname{tr}_k(\bar{w}^{\dot{\alpha}}(X) w_{\dot{\alpha}}(X) + a'_n(X) a'_n(X)) . \tag{2.74}$$

We now evaluate (2.36) using Osborn's formula (C.6) for $\operatorname{tr}_N F_{mn}^2$, giving

$$\chi = \frac{g^2}{4} \int d^4x\, x^2 \,\square^2 \operatorname{tr}_k \log f . \tag{2.75}$$

Applying Stokes' Theorem along with the ADHM form for the matrix f (2.62) and (2.45), one finds precisely (2.74).

2.4.2. Symmetries and the moduli space

We now show how the symmetries of the classical gauge theory broken by an instanton solution are realized on the ADHM moduli space. Consider, first of all, the action of the conformal group on the instanton solutions. The action of the conformal group is given most elegantly in the quaternion basis (2.19). So acting on the ADHM variable $\Delta(x)$ we find

$$\Delta(x'; a, b) = \Delta(x; aD + bB, aC + bA)(Cx + D)^{-1} . \tag{2.76}$$

Notice that since the gauge field only depends on $\bar{U}$ and U, defined by (2.47), the factor of $(Cx + D)^{-1}$ on the right is redundant. Hence, the action of the conformal group on the ADHM variables is

$$a \to aD + bB, \quad b \to aC + bA . \tag{2.77}$$

To get the transformation on our canonical basis (2.57) and (2.58) we have to perform a compensating transformation of form (2.56) in order to return b to its canonical form (2.57). So there exist transformations of form (2.56), with Λ and Υ dependent on a and the element of the conformal group, that takes

$$\Lambda(aC + bA)\Upsilon^{-1} = b , \tag{2.78}$$

where b assumes its canonical form (2.57). The resulting action of the conformal group on the ADHM variable a is then

$$a \to \Lambda(aD + bB)\Upsilon^{-1} . \tag{2.79}$$

For example, translations act on the ADHM coordinates in the following way. From

$$\Delta(x + \varepsilon; a, b) = \Delta(x; a + b\varepsilon, b) , \tag{2.80}$$

we deduce

$$a'_n \to a'_n + \varepsilon_n 1_{[k]\times[k]}, \quad w_{\dot\alpha} \to w_{\dot\alpha} . \tag{2.81}$$

This allows us to identify the coordinates of the centre of the instanton with the component of a'_n proportional to the identity matrix:

$$X_n = -k^{-1}\,\mathrm{tr}_k\, a'_n . \tag{2.82}$$

Note that these centre-of-mass coordinates do not appear in ADHM constraints (2.65) reflecting the fact that the moduli space is product (2.16).

Global gauge transformations act on the gauge indices $u, v, \ldots$, so only on the quantities $w_{ui\dot\alpha}$. Generically, $w_{ui\dot\alpha}$ constitute a set of $2k$ complex N-vectors. Consequently if $N \leqslant 2k$ then all global gauge transformations generically act non-trivially on the ADHM variables, while if $N > 2k$ then there is a non-trivial subgroup that leaves the instanton fixed. This is the *stability group* of the instanton. In order to identify it, we follow Bernard's description of the one instanton moduli space [23] and embed the k instanton solution in an SU($2k$)-dimensional subgroup of the gauge group. This involves choosing a suitable gauge transformation that puts the $N \times 2k$ matrix w, with elements

$w_{ui\dot\alpha}$, into upper-triangular form:

$$\begin{pmatrix} w_{11} & \cdots & w_{12} \\ \vdots & \ddots & \vdots \\ w_{N1} & \cdots & w_{N,2k} \end{pmatrix} = \mathscr{U}\cdot\begin{pmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1,2k} \\ 0 & \xi_{22} & \cdots & \xi_{2,2k} \\ & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \xi_{2k,2k} \\ & & & 0 \\ & & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} . \tag{2.83}$$

The ξ_{ab}, $a,b=1,\dots,2k$, are complex except for the diagonal elements ξ_{aa} which we can choose to be real. Note that the group elements $\mathscr{U}$ in the $\mathrm{SU}(N-2k)$ subgroup in the lower $(N-2k)\times(N-2k)$ corner leave ξ invariant. Therefore, at least generically, $\mathscr{U}$ is valued in the coset

$$\mathscr{U}\in\frac{\mathrm{SU}(N)}{\mathrm{SU}(N-2k)} . \tag{2.84}$$

There are $4kN$ independent real parameters in w on the left-hand side of (2.83), matching $4k(N-k)$ in $\mathscr{U}$ and $4k^2$ in ξ, on the right-hand side.

With parameterization (2.83), the instanton solution has the form

$$A_n=\mathscr{U}^\dagger\begin{pmatrix} (A_n^{\mathrm{inst}})_{[2k]\times[2k]} & 0_{[2k]\times[N-2k]} \\ 0_{[N-2k]\times[2k]} & 0_{[N-2k]\times[N-2k]} \end{pmatrix}\mathscr{U} , \tag{2.85}$$

where A_n^{inst} is the k-instanton solution lying, generically, in $\mathrm{SU}(2k)\subset\mathrm{SU}(N)$. Generically, the stability group of the instanton solution consists of the $\mathrm{SU}(N-2k)$ in the denominator of (2.84) along an additional U(1) transformation generated by

$$\lambda=\begin{pmatrix} 1_{[2k]\times[2k]} & 0_{[2k]\times[N-2k]} \\ 0_{[N-2k]\times[2k]} & -\dfrac{2k}{N-2k}\,1_{[N-2k]\times[N-2k]} \end{pmatrix} . \tag{2.86}$$

The stability group, therefore, consists of elements $e^{i\lambda\theta}g$, with $g\in\mathrm{SU}(N-2k)$. Notice that this is not exactly $\mathrm{SU}(N-2k)\times\mathrm{U}(1)$ because elements $e^{i\lambda\theta_1}$ and $e^{i\lambda\theta_2}$ with $\theta_1-\theta_2=\pi$ differ by an element of the centre of $\mathrm{SU}(N-2k)$. More precisely, therefore, the stability group is $\mathrm{S}(\mathrm{U}(N-2k)\times\mathrm{U}(1))\subset\mathrm{SU}(N)$. To summarize, the non-trivial global gauge transformations acting on a generic instanton solution are

$$\mathrm{SU}(N)\quad\text{for}\quad N\leqslant 2k,\qquad \frac{\mathrm{SU}(N)}{\mathrm{S}(\mathrm{U}(N-2k)\times\mathrm{U}(1))}\quad\text{for } N>2k . \tag{2.87}$$

For $N > 2k$, we will loosely refer to $\mathscr{U}$, taking values in (2.84), as the "gauge orientation" of the instanton, even though the true gauge orientation involves quotienting by the additional U(1) described above. Notice, the additional U(1) can be identified with the U(1) subgroup of the U(k) auxiliary group.

2.4.3. Singular gauge, one instanton, the dilute limit and asymptotics

Let us determine the gauge field A_n more explicitly. This entails solving for U, and hence A_n itself via (2.49), in terms of Δ. It is convenient to make the decomposition:

$$U_{\lambda v} = U_{(u+i\alpha)v} = \begin{pmatrix} V_{uv} \\ (U'_\alpha)_{iv} \end{pmatrix}, \quad \Delta_{\lambda j\dot\alpha} = \Delta_{(u+i\alpha)j\dot\alpha} = \begin{pmatrix} w_{uj\dot\alpha} \\ (\Delta'_{\alpha\dot\alpha})_{ij} \end{pmatrix} . \tag{2.88}$$

Then from completeness condition (2.51) one finds

$$V = 1_{[N]\times[N]} - w_{\dot\alpha} f \bar w^{\dot\alpha} . \tag{2.89}$$

For any V that solves this equation, one can find another by right-multiplying it by an x-dependent U(N) matrix. A specific choice of V corresponds to fixing the spacetime gauge. The "singular gauges" correspond to taking any one of the 2^N choices of matrix square roots:

$$V = (1_{[N]\times[N]} - w_{\dot\alpha} f \bar w^{\dot\alpha})^{1/2} . \tag{2.90}$$

Next, U' in (2.88) is determined in terms of V via

$$U' = -\Delta'_{\dot\alpha} f \bar w^{\dot\alpha} \bar V^{-1} \tag{2.91}$$

which likewise follows from (2.51). Eqs. (2.90) and (2.91) determine U in (2.88), and hence the gauge field A_n via (2.49).

We now show how the well-known expression (2.15) of 't Hooft for the one instanton solution in SU(2) (in singular gauge) is reproduced by the ADHM construction. Adopting canonical form (2.57), we set the instanton number $k = 1$, thus dropping the instanton, i, j, indices. Eq. (2.59b) then says that a'_n is a real four-vector which from the last section we will identify as minus the centre $-X_n$ of the instanton, as per (2.82),

$$a'_n \equiv -X_n \in \mathbb{R}^4 . \tag{2.92}$$

ADHM constraint (2.65) collapses to

$$\bar w^{\dot\alpha}_u w_{u\dot\beta} = \rho^2 \delta^{\dot\alpha}{}_{\dot\beta} . \tag{2.93}$$

The parameter ρ will soon be identified with the instanton scale size. Constraint (2.93) can be explicitly solved by taking

$$w = \rho\mathscr{U} \begin{pmatrix} 1_{[2]\times[2]} \\ 0_{[N-2]\times[2]} \end{pmatrix} , \tag{2.94}$$

which is decomposition (2.83) for one instanton and identifies $\mathcal{U} \in \mathrm{SU}(N)$ as the gauge orientation of the instanton. For one instanton, the ADHM quantity f is a scalar; from (2.62),

$$f = \frac{1}{(x-X)^2 + \rho^2} \ . \tag{2.95}$$

Then, from (2.90) and (2.91),

$$V = 1_{[N]\times[N]} + \frac{1}{\rho^2}\left(\sqrt{\frac{(x-X)^2}{(x-X)^2+\rho^2}} - 1\right) w_{\dot\alpha}\bar w^{\dot\alpha} \ ,$$

$$U' = -\frac{(x-X)_{\alpha\dot\alpha}\bar w^{\dot\alpha}}{|x-X|\sqrt{(x-X)^2+\rho^2}} \ . \tag{2.96}$$

Using (2.49), one finds the expression for the gauge potential in singular gauge

$$A_n = g^{-1}\,\frac{2w_{\dot\alpha}(x-X)_m\bar\sigma_{mn}{}^{\dot\alpha}{}_{\dot\beta}\bar w^{\dot\beta}}{(x-X)^2((x-X)^2+\rho^2)} \ . \tag{2.97}$$

Comparing with the SU(2) solution (2.15), this form of the solution manifests the fact that a single SU(N) instanton is described by taking the SU(2) instanton solution and embedding it in SU(2) $\subset$ SU(N). In this case the three SU(2) generators are

$$T^c_{uv} = \rho^{-2} w_{u\dot\alpha}\tau^{c\dot\alpha}{}_{\dot\beta}\bar w^{\dot\beta}_v, \quad c = 1\text{–}3 \ . \tag{2.98}$$

The fact that these generators satisfy the SU(2) algebra is guaranteed by ADHM constraints (2.93). With explicit solution (2.94)

$$A_n = \mathcal{U}\begin{pmatrix} A_n^{\mathrm{SU}(2)} & 0 \\ 0 & 0 \end{pmatrix}\mathcal{U}^\dagger \ , \tag{2.99}$$

which manifests the fact that $\mathcal{U}$ is the gauge orientation of the instanton taking values in the coset $\mathrm{SU}(N)/\mathrm{S}(\mathrm{U}(N-2)\times\mathrm{U}(1))$.

In general a multi-instanton configuration cannot be thought of as a combination of single instantons. However, there are asymptotic regions of $\mathfrak{M}_k$ where the solutions can be identified as being composed of well-separated single instantons. Here, we consider the *completely clustered limit* in which the k-instanton configuration looks like k well-separated single instantons. Up to the action of the auxiliary U(k) symmetry, the completely clustered limit is the region of moduli space where the differences between the diagonal elements of a'_n are much greater than the off-diagonal elements (in a sense that we make precise below). We can then identify $X^i_n \equiv -(a'_n)_{ii}$ as the centres of each of the k single instantons. To be more specific, it is useful to fix the U(k) symmetry by setting to zero the off-diagonal components of a'_n that are generated by U(k) adjoint action on the diagonal matrix $\mathrm{diag}(-X^1_n,\ldots,-X^k_n)$. The "gauge choice" amounts to taking

$$(a'_n)_{ij}(X^i - X^j)_n = 0 \tag{2.100}$$

and we will denote the a' so constrained by $\tilde{a}'$. This leaves the diagonal symmetry $\mathrm{U}(1)^k$ which will be identified with the auxiliary symmetry of each of the k single instantons. In the complete clustering limit, the terms $(\tilde{a}'^{\dot{\alpha}\alpha})_{ik}(\tilde{a}'_{\alpha\dot{\beta}})_{kj}$, $k \neq i,j$, can be ignored in ADHM constraints (2.65). In this case, these off-diagonal constraints are linear in $(\tilde{a}'_{\alpha\dot{\alpha}})_{ij}$, for $i \neq j$,

$$(\bar{X}^i - \bar{X}^j)^{\dot{\alpha}\alpha}(\tilde{a}'_{\alpha\dot{\beta}})_{ij} + \bar{w}^{\dot{\alpha}}_{iu}w_{uj\dot{\beta}} \propto \delta^{\dot{\alpha}}{}_{\dot{\beta}} \tag{2.101}$$

and can be solved, although we will not require explicit expressions for the solutions. The diagonal components of ADHM constraints (2.65) in this limit are then simply

$$\bar{w}^{\dot{\alpha}}_{iu}w_{ui\dot{\beta}} = \rho_i^2\delta^{\dot{\alpha}}{}_{\dot{\beta}} \tag{2.102}$$

(no sum on i) for arbitrary ρ_i. Constraint (2.102) is then the ADHM constraint of a single instanton and we can therefore identify ρ_i with the scale size of the ith instanton. Each instanton is associated with a particular SU(2) embedding of SU(N) defined by the generators

$$(T_i^c)_{uv} = \rho_i^{-2}w_{ui\dot{\alpha}}\tau^{c\dot{\alpha}}{}_{\dot{\beta}}\bar{w}^{\dot{\beta}}_{iv}, \quad c = 1\text{–}3 , \tag{2.103}$$

with no sum on i. One can show that the completely clustered limit is valid when, for each $i \neq j$,

$$(X^i - X^j)^2 \gg \rho_i\rho_j \operatorname{tr}_N(T_i^cT_j^c) . \tag{2.104}$$

In other words, the separation between the instantons must be much greater than the product of the scale sizes times a trace over generators which measures the overlap of the SU(2) embeddings of each of the instantons.

It will be important for many of the applications of the instanton calculus to describe the asymptotic fall off of the fields from the centre of the instanton. The nature of the fall off depends on the gauge used which for us means singular gauge as described above. From the previous formulae of this section for arbitrary instanton charge k, we find the leading large-x asymptotic behaviour of several key ADHM quantities, in singular gauge (2.90):

$$\Delta_{\dot{\alpha}} \to b^\alpha x_{\alpha\dot{\alpha}}, \quad f_{ij} \to \frac{1}{x^2}\delta_{ij}, \quad U' \to -\frac{x_{\alpha\dot{\alpha}}}{x^2}\bar{w}^{\dot{\alpha}}, \quad V \to 1_{[N]\times[N]} \tag{2.105}$$

and in addition, for the gauge field

$$A_n \to g^{-1}\frac{x_m}{x^4}w_{\dot{\alpha}}\bar{\sigma}_{mn}{}^{\dot{\alpha}}{}_{\dot{\beta}}\bar{w}^{\dot{\beta}} . \tag{2.106}$$

2.5. Zero modes and the metric on $\mathfrak{M}_k$

The ADHM construction yields an explicit solution for the self-dual gauge field in terms of the (over-complete) ADHM collective coordinates $a_{\dot{\alpha}}$. We now show how one can find explicit expressions for the zero modes. In Section 2.2, we showed, in the quaternion basis, that zero modes $\delta A_{\alpha\dot{\alpha}}$ are solutions of the covariant Weyl equation $\bar{\mathcal{D}}^{\dot{\alpha}\alpha}\delta A_{\alpha\dot{\beta}} = 0$. In Appendix C (Eq. (C.11)) we verify that the following linear functions

$$\Lambda_\alpha(C) \overset{\text{def}}{=} \bar{U}Cf\bar{b}_\alpha U - \bar{U}b_\alpha f\bar{C}U , \tag{2.107}$$

for a constant $(N+2k)\times k$ matrix $C_{\lambda i}$, are a solution of the covariant Weyl equation in the background of the instanton,

$$\bar{\mathscr{D}}^{\dot\alpha\alpha}\Lambda_\alpha(C)=0\ , \tag{2.108}$$

as long as C satisfies the constraints

$$\bar{C}_{i\lambda}a_{\lambda j\dot\alpha}=-\bar{a}_{i\dot\alpha\lambda}C_{\lambda j}\ , \tag{2.109a}$$

$$\bar{C}_{i\lambda}b^\alpha_{\lambda j}=\bar{b}^\alpha_{i\lambda}C_{\lambda j}\ . \tag{2.109b}$$

There are $2k(N+2k)$ real degrees of freedom in C subject to $4k^2$ constraints (2.109a)–(2.109b). Hence, there are $2kN$ independent solutions to the Weyl equation as anticipated by the Index Theorem.

The question is how these solutions are related to the derivatives of the gauge field with respect to the collective coordinates? If X^μ is an arbitrary collective coordinate, then we prove in Appendix C (Eq. (C.16)) that

$$\frac{\partial A_n}{\partial X^\mu}=-g^{-1}\mathscr{D}_n\left(\frac{\partial\bar{U}}{\partial X^\mu}U\right)+g^{-1}\bar{U}\frac{\partial a}{\partial X^\mu}f\bar{\sigma}_nbU-g^{-1}\bar{U}b\sigma_nf\frac{\partial\bar{a}}{\partial X^\mu}U\ . \tag{2.110}$$

In the quaternion basis we recognize the second and third terms as being $2g^{-1}\Lambda_\alpha(C_{\dot\alpha})$ with $C_{\dot\alpha}=\partial a_{\dot\alpha}/\partial X^\mu$:

$$\frac{\partial A_{\alpha\dot\alpha}}{\partial X^\mu}=-g^{-1}\mathscr{D}_{\alpha\dot\alpha}\left(\frac{\partial\bar{U}}{\partial X^\mu}U\right)+2g^{-1}\Lambda_\alpha(\partial a_{\dot\alpha}/\partial X^\mu)\ . \tag{2.111}$$

The first term in (2.111), as its form suggests, is precisely the compensating gauge transformation (2.30) needed in order to force the zero mode into background gauge:

$$\Omega_\mu=-g^{-1}\frac{\partial\bar{U}}{\partial X^\mu}U\ . \tag{2.112}$$

Hence the expression for the zero modes is given explicitly by the linear function defined in (2.107)

$$\delta_\mu A_{\alpha\dot\alpha}=2g^{-1}\Lambda_\alpha(\partial a_{\dot\alpha}/\partial X^\mu)\equiv 2g^{-1}\bar{U}\left(\frac{\partial a_{\dot\alpha}}{\partial X^\mu}f\bar{b}_\alpha-b_\alpha f\frac{\partial\bar{a}_{\dot\alpha}}{\partial X^\mu}\right)U\ . \tag{2.113}$$

The fact that $C_{\dot\alpha}=\partial a_{\dot\alpha}/\partial X^\mu$ satisfies constraints (2.109a) and (2.109b) is partially taken care of by taking the X^μ derivative of ADHM constraints (2.54a) and (2.54b). From the hyper-Kähler quotient perspective, described in Section 2.4.1 and Appendix B, the conditions that arise from taking the X^μ derivative of (2.54a) are equivalent to the requirement that $\partial a_{\dot\alpha}/\partial X^\mu$ is orthogonal to the $3k^2$ vectors $\tilde{\boldsymbol{I}}^{(c)}X_r$, i.e. $\partial a_{\dot\alpha}/\partial X^\mu$ lies in the tangent space $T\mathfrak{N}\subset T\tilde{\mathfrak{M}}$. However, this leaves k^2 additional constraints on $\partial a_{\dot\alpha}/\partial X^\mu$ which have not been accounted for; namely,

$$\frac{\partial\bar{a}^{\dot\alpha}}{\partial X^\mu}a_{\dot\alpha}-\bar{a}^{\dot\alpha}\frac{\partial a_{\dot\alpha}}{\partial X^\mu}=0\ . \tag{2.114}$$

From the hyper-Kähler quotient point of view, these extra constraints simply require that the vector $\partial a_{\dot\alpha}/\partial X^\mu$ is orthogonal to the tri-holomorphic Killings vectors X_r, (2.69), which generate the $\mathrm{U}(k)$

isometries of $\tilde{\mathfrak{M}}$. This means that it lies in the horizontal subspace $\mathscr{H} \subset T\mathfrak{N}$. The horizontal subspace is identified with the tangent space $T\mathfrak{M}_k$ by a unique lifting procedure. Hence, the zero modes are naturally associated to tangent vectors to $\mathfrak{M}_k$.

It is remarkable that one can calculate the explicit functional inner product of the zero modes in singular gauge, a result attributed in Ref. [8] to Corrigan [24]. The proof, reviewed in Appendix C (Eq. (C.20)), was first published in [25], for gauge group Sp(1), and extended to gauge group SU(N) in [21]. The identity can then be written in terms of the Weyl spinor quantities $\Lambda(C)$ as [10]

$$\int d^4x\, \mathrm{tr}_N\, \Lambda(C)\Lambda(C') = -\frac{\pi^2}{2}\, \mathrm{tr}_k[\bar{C}(\mathscr{P}_\infty + 1)C' - \bar{C}'(\mathscr{P}_\infty + 1)C]\ . \tag{2.115}$$

Here

$$\mathscr{P}_\infty = \lim_{x\to\infty} \mathscr{P} = 1 - b\bar{b} = \begin{pmatrix} 1_{[N]\times[N]} & 0_{[N]\times[2k]} \\ 0_{[2k]\times[N]} & 0_{[2k]\times[2k]} \end{pmatrix}\ . \tag{2.116}$$

From this identity, and expression (2.113), we can deduce the expression for the metric on the space of collective coordinates (2.32)

$$g_{\mu\nu}(X) = 2\pi^2\, \mathrm{tr}_k \left(\frac{\partial \bar{a}^{\dot\alpha}}{\partial X^\mu}(\mathscr{P}_\infty + 1)\frac{\partial a_{\dot\alpha}}{\partial X^\nu} + \frac{\partial \bar{a}^{\dot\alpha}}{\partial X^\nu}(\mathscr{P}_\infty + 1)\frac{\partial a_{\dot\alpha}}{\partial X^\mu} \right)\ . \tag{2.117}$$

Since, $\partial a_{\dot\alpha}/\partial X^\mu$ are the components of a vector in $\mathscr{H}$ (due to constraints (2.109a) and (2.109b)), the metric arising from the functional inner product of zero modes (in singular gauge) (2.117) is identical to the metric on $\mathfrak{M}_k$ induced by the hyper-Kähler quotient construction. This is completely in accord with the argument in Section 2.4.1 involving the hyper-Kähler potential.

Now that we have found the explicit form of the zero modes and compensating gauge transformations in singular gauge, one can easily verify that for large x

$$\delta_\mu A_m \sim \mathscr{O}(x^{-3}), \quad \frac{\partial A_m}{\partial X^\mu} \sim \mathscr{O}(x^{-3}), \quad \Omega_\mu \sim \mathscr{O}(x^{-2})\ . \tag{2.118}$$

These formulae follow from explicit expressions (2.110) and (2.112) and asymptotic formulae in (2.105).

2.6. Singularities and small instantons

We have previously remarked that the instanton moduli space $\mathfrak{M}_k$ fails to be a smooth manifold due to certain singularities. In this section we will explore these interesting features in more detail. Before we embark on this analysis, it is important to emphasize that these singularities are *not* evidence of any pathology in the instanton calculus since the integrals over the $\mathfrak{M}_k$ appearing in the semi-classical approximation of the functional integral are perfectly well defined in the vicinity of these singularities.

[10] The relative minus sign in the second term compared to Eq. (2.61) in [21] is due to the fact that in that reference we have written the identity for the fermion zero modes and so $\mathscr{M}$ and $\mathscr{N}$ are Grassmann collective coordinates, whereas here C and C' are c-number-valued.

Before we talk about the singularities specifically, let us develop a picture of the geometry of $\mathfrak{M}_k$. In fact it will be more convenient, for the most part, to consider the centred moduli space $\hat{\mathfrak{M}}_k$ defined in (2.16). As we have shown, the ADHM construction of $\mathfrak{M}_k$, or $\hat{\mathfrak{M}}_k$, is an example of a hyper-Kähler quotient based on the quotient group $\mathrm{U}(k)$ and starting from flat space. Since the quotient group has an abelian factor, the quotient construction can, in general, involve the parameters ζ^c of (2.70) taking values in the $u(1)$ subalgebra of the Lie algebra $\mathrm{U}(1) \subset \mathrm{U}(k)$. However, the ADHM construction requires $\zeta^c = 0$ and this has some interesting consequences that we now elaborate.[11] Firstly, the quotient space admits a *dilatation* generated by a *homothetic Killing vector* which we denote by κ.[12] To see this, let us note that the mother space $\tilde{\mathfrak{M}} = \mathbb{R}^{4n}$ trivially admits a dilatation generated by the homothetic Killing vector $\tilde{\kappa}$ with components

$$\tilde{\kappa}^{\tilde{\mathrm{i}}\dot{\alpha}} = z^{\tilde{\mathrm{i}}\dot{\alpha}} . \tag{2.119}$$

In addition κ is *hypersurface orthogonal*, i.e.

$$\tilde{\kappa}^{\tilde{\mathrm{i}}\dot{\alpha}} = \partial^{\tilde{\mathrm{i}}\dot{\alpha}}\tilde{\chi} , \tag{2.120}$$

for some function $\tilde{\chi}$, which we identify with hyper-Kähler potential (2.73). When the central terms vanish, $\zeta^c = 0$, it is easy to see that $\tilde{\kappa}$ preserves the level set $\mathfrak{N}$ and hence is the lift to $\mathscr{H}$ of a (hypersurface orthogonal) homothetic Killing vector κ on the quotient space. The dilatation arises as a consequence of the classical conformal invariance of the gauge theory described in Section 2.4.2. The existence of the dilatation implies that the metric on the quotient $\mathfrak{M}_k$ has the form of a cone [26] or more specifically a *hyper-Kähler cone*. At this point it is convenient to talk about the centred moduli space $\hat{\mathfrak{M}}_k$. Everything we have said above is equally applicable to $\hat{\mathfrak{M}}_k$ which has hyper-Kähler potential $\hat{\chi}$ equal to χ in (2.74) with the trace components of a'_n set to zero. The fact that $\hat{\mathfrak{M}}_k$ is a cone means that $\hat{\mathfrak{M}}_k \simeq \mathbb{R}^+ \times \mathfrak{F}$ with a metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}\hat{\chi}^2}{4\hat{\chi}} + \hat{\chi}\,\mathrm{d}s^2_{\mathfrak{F}} . \tag{2.121}$$

Since $\mathfrak{F}$ is not a sphere, $\hat{\mathfrak{M}}_k$ has a conical singularity at $\hat{\chi} = 0$ of which we will have more to say below.

The quotient space also inherits the obvious SU(2) isometries (acting in the obvious way on the $\dot{\alpha}$ indices) from the mother space. These isometries on $\tilde{\mathfrak{M}}$ are generated by three Killing vectors with components

$$\tilde{n}^{(c)\tilde{\mathrm{i}}\dot{\alpha}} = \tilde{\boldsymbol{I}}^{(c)\dot{\alpha}}{}_{\dot{\beta}} z^{\tilde{\mathrm{i}}\dot{\beta}} \equiv \mathrm{i}\tau^{c\dot{\alpha}}{}_{\dot{\beta}} z^{\tilde{\mathrm{i}}\dot{\beta}} . \tag{2.122}$$

As with the dilatation above, it is precisely when the central terms ζ^c vanish that these symmetries are inherited by the quotient space $\mathfrak{M}_k$. To see this one simply has to verify that the action preserves the level set $\mu^c = 0$.[13] Notice that the four vectors κ and $n^{(c)}$ are mutually orthogonal.

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[11] The case $\zeta^c \neq 0$ corresponds to the ADHM construction in the gauge theory defined on a non-commutative spacetime as described in Section 11.1.

[12] A homothetic Killing vector is a conformal Killing vector, so $\mathscr{L}_\kappa g = \phi g$, for which ϕ is a constant.

[13] It is worth pointing out that the action is *not* tri-holomorphic because the SU(2) rotates the complex structures.

The existence of the dilatation and SU(2) isometries together implies that the centred moduli space is locally a product $\hat{\mathfrak{M}}_k = [\mathbb{R}^4/\mathbb{Z}_2] \times \mathfrak{Q}$ for some $4kN-8$-dimensional space $\mathfrak{Q}$.[14] In fact, when $\hat{\mathfrak{M}}_k$ is described as cone (2.121) the space $\mathfrak{F}$ is a tri-Sasakian manifold, which due to the structure of SU(2) isometries described above, has the form of a non-trivial fibration of SO(3) over $\mathfrak{Q}$. One can further show that $\mathfrak{Q}$ is a *quaternionic Kähler space* (see [27,28] for more details and other references).

Having established the interesting cone geometry of $\mathfrak{M}_k$ we now turn more specifically to its singularities. It is not hard to pin down the reason for the singularities. The ADHM construction requires us to quotient the level set $\mathfrak{N}$ by the $\mathrm{U}(k)$ auxiliary symmetry group and this procedure will introduce orbifold singularities at points in $\mathfrak{N}$ where $\mathrm{U}(k)$ does not act freely, i.e. where some subgroup of $\mathrm{U}(k)$ leaves $a_{\dot{\alpha}}$, a solution to (2.65), fixed. Geometrically, the volume of the gauge orbit through any point in $\mathfrak{N}$ is proportional to the determinant of the matrix of inner products of the $\mathrm{U}(k)$ tri-holomorphic Killing vectors $\{X_r\}$:

$$|\mathrm{det}_{k^2}\, \tilde{g}(X_r, X_s)|^{1/2} \ . \tag{2.123}$$

The $k^2 \times k^2$ matrix with elements $\tilde{g}(X_r, X_s)$ plays a ubiquitous rôle in the instanton calculus and we introduce the notation

$$\tilde{g}(X_r, X_s) \equiv \boldsymbol{L}_{rs} \overset{\text{def}}{=} 8\pi^2\, \mathrm{tr}_k(T^r \boldsymbol{L} T^s) \ , \tag{2.124}$$

where the generators T^r of $\mathrm{U}(k)$ were defined previously in Section 2.4.1. Here, using (2.66) and (2.69), $\boldsymbol{L}$ is an operator on $k \times k$ Hermitian matrices of the form

$$\begin{aligned} \boldsymbol{L} \cdot \Omega &\overset{\text{def}}{=} \tfrac{1}{2}\{\bar{w}^{\dot{\alpha}} w_{\dot{\alpha}}, \Omega\} + \tfrac{1}{2}\, \bar{a}'^{\dot{\alpha}\alpha} a'_{\alpha\dot{\alpha}} \Omega - \bar{a}'^{\dot{\alpha}\alpha} \Omega a'_{\alpha\dot{\alpha}} + \tfrac{1}{2}\, \Omega \bar{a}'^{\dot{\alpha}\alpha} a'_{\alpha\dot{\alpha}} \\ &= \tfrac{1}{2}\{\bar{w}^{\dot{\alpha}} w_{\dot{\alpha}}, \Omega\} + [a'_n, [a'_n, \Omega]] \ . \end{aligned} \tag{2.125}$$

At a point where $\mathrm{U}(k)$ does not act freely, the operator $\boldsymbol{L}$ develops one, or more, null eigenvectors. The relevant effect can be seen already at the one instanton level. In this case, the ADHM constraints are more explicitly (2.93). The auxiliary group U(1) acts by phase rotation, $w_{\dot{\alpha}} \to e^{i\phi} w_{\dot{\alpha}}$, but does not act freely when $w_{\dot{\alpha}} = 0$ which, from Section 2.4.3, is the point at which the instanton has zero scale size ρ. This is precisely in accord with our description of the centred moduli space $\hat{\mathfrak{M}}_k$ above as a cone over the tri-Sasakian space $\mathfrak{F}$. For the single instanton the apex of the cone is the point $\rho = 0$. At the one-instanton level we can be more explicit. In (2.87) we have identified the action of global gauge transformations on $\mathfrak{M}_k$. In fact at the one-instanton level all gauge orbits are equivalent to the generic one (2.87) and therefore the whole centred moduli space $\hat{\mathfrak{M}}_1$ is simply a cone over the gauge orbit:

$$\hat{\mathfrak{M}}_1 \simeq \mathbb{R}^+ \times \frac{\mathrm{SU}(N)}{\mathrm{S}(\mathrm{U}(N-2) \times \mathrm{U}(1))} \ , \tag{2.126}$$

where the variable along the cone is $\hat{\chi} \propto \rho^2$. Since, at the one-instanton level, there are no additional singularities in $\hat{\mathfrak{M}}_k$, the corresponding $\mathfrak{F}$ is an example of a homogeneous tri-Sasakian space [27].

[14] The $\mathbb{Z}_2$ quotient (acting by inversion) arises because the centre of SU(2) lies in the $\mathrm{U}(k)$ quotient group.

As discussed above, the centred moduli spaces $\widehat{\mathfrak{M}}_k$ for instanton charge $k>1$ also have the structure of a cone over a tri-Sasakian space. The apex of the cone in this case corresponds to the point where $w_{\dot\alpha}=0$ and a'_n is proportional to the identity. We will shortly interpret this as the point in the moduli space where all the instantons have shrunk to zero size and all lie coincident in $\mathbb{R}^4$. This is the point of maximal degeneracy where the whole of $\mathrm{U}(k)$ is fixed. However, there are other singularities of smaller co-dimension in the moduli space reflecting the fact that the tri-Sasakian space $\mathfrak{F}$ in this case is not homogeneous and has singularities of its own. These can be uncovered in the following way. In Section 2.4.3, we have described how multi-instanton configurations in various asymptotic regions of the moduli space can be identified with clusters of smaller numbers of instantons. In particular, when $w_{ui\dot\alpha}\to 0$ for some fixed $i=l$, the ADHM constraints imply that the elements $(a'_n)_{il},(a'_n)_{li}\to 0$, for $i\neq l$. In the limit, the subgroup $\mathrm{U}(1)\subset\mathrm{U}(k)$ corresponding to

$$w_{ui\dot\alpha}\to w_{ui\dot\alpha}e^{i\delta_{il}\phi},\quad (a'_n)_{ij}\to e^{i(\delta_{jl}-\delta_{il})\phi}(a'_n)_{ij} \tag{2.127}$$

does not act freely. This is a limit in the moduli space, where the k-instanton configuration looks like a smooth $(k-1)$-instanton configuration along with a single instanton that has shrunk to zero size. The point-like instanton still has a position $X_n^l=-(a'_n)_{ll}$; hence, in the limit $\mathfrak{M}_k\to\mathfrak{M}_{k-1}\times\mathbb{R}^4$. This process can continue. There are regions where some subgroup $\mathrm{U}(1)^r\subset\mathrm{U}(k)$ does not act freely, which corresponds to a $(k-r)$-instanton configuration along with r point-like instantons. In this limit the moduli space is of the form

$$\mathfrak{M}_k\to\mathfrak{M}_{k-r}\times\mathrm{Sym}^r\,\mathbb{R}^4\ . \tag{2.128}$$

Here, $\mathrm{Sym}^r\,\mathbb{R}^4$ is the symmetric product of r points in $\mathbb{R}^4$.[15] Additional singularities arise when the point-like instantons come together at the same spacetime point. This describes a situation where a non-abelian subgroup of $\mathrm{U}(k)$ does not act freely. The maximally degenerate situation is when the whole of $\mathrm{U}(k)$ is fixed. This occurs at the apex of the hyper-Kähler cone and describes a configuration where all the instantons have shrunk to zero size at the same point in $\mathbb{R}^4$.

In a certain mathematical sense, the moduli space $\mathfrak{M}_k$ excludes the regions with point-like instantons. It then has a natural compactification by including the regions with point-like instantons at the boundary [29] (see also [6]). However, in the semi-classical approximation of the functional integral, the singularities of the instanton moduli space do not lead to a divergence of the collective coordinate integral and—for all practical purposes—we need not distinguish between $\mathfrak{M}_k$ and its compactification.

3. The collective coordinate integral

In this section, we describe how instantons contribute to the functional integral of the field theory in the semi-classical limit. To start with in Section 3.1 we consider the problem of expanding around an instanton solution in the functional integral. We follow the approach of Bernard [23] suitably generalized in an obvious way to $k>1$ (see also the thorough treatment in Osborn's review [8]).

[15] This is the product $(\mathbb{R}^4)^r$ modded out by the group of permutations on r objects arising from the subgroup of the Weyl group of $\mathrm{U}(k)$ that permutes the r labels $\{i\}$ for which $w_{ui\dot\alpha}\to 0$.

In particular, following [23], we take advantage of certain simplifications that occur in singular gauge arising from the fast fall off of the gauge potential. The goal of this section is to show that the leading-order semi-classical approximation involves an integral over the collective coordinates with a measure defined by the inner product of the corresponding zero modes along with a ratio of determinants arising from the fluctuations around the instanton. In Section 3.2 we explain how a concrete expression for the integration measure can be obtained using the hyper-Kähler quotient perspective, although this was not originally how the measure was found in Refs. [30,31]. We go on in Section 3.3 to review the successes and failures of the old instanton literature regarding the fluctuation determinants in the general k instanton background. We do not develop this subject any further because the ratio of fluctuations in a supersymmetric theory is a trivial collective coordinate independent constant factor.

3.1. From the functional to the collective coordinate integral

The semi-classical approximation is a saddle-point method and, as such, in order to find the leading-order behaviour of the functional integral, we need to consider the fluctuations around the instanton solution. To this end, we expand

$$A_n(x) = A_n(x;X) + \delta A_n(x;X)\ , \tag{3.1}$$

where $A_n(x;X)$ is the instanton solution as a function of the collective coordinates and $\delta A_n(x;X)$ are the fluctuations which are chosen to satisfy the background gauge condition (2.24) [16]

$$\mathscr{D}_n \delta A_n = 0\ . \tag{3.2}$$

Notice that the fluctuations depend implicitly on the collective coordinates. The action of the theory to quadratic order in the fluctuations is (2.27)

$$S = -2\pi i k\tau - \frac{1}{2}\int d^4x\, \mathrm{tr}_N\, \delta\bar{A}^{\dot\alpha\alpha} \varDelta^{(+)}{}_\alpha{}^\beta \delta A_{\beta\dot\alpha} + \cdots\ . \tag{3.3}$$

We must also include the gauge-fixing term involving the ghost fields

$$S_{\rm gf} = 2\int d^4x\, \mathrm{tr}_N\, b\mathscr{D}^2 c\ . \tag{3.4}$$

The fluctuations can be expanded in terms of the eigenfunctions of the operator $\varDelta^{(+)}$ which can be split into the zero modes, already extensively investigated in Section 2.5, and the non-zero modes:

$$\delta A_n = \sum_\mu \xi^\mu \delta_\mu A_n + \tilde{A}_n\ . \tag{3.5}$$

Here, the non-zero mode fluctuations $\tilde{A}_n$ are functionally orthogonal to the zero modes. In the functional integral we now separate out the integrals over the zero and non-zero modes:

$$\int [dA_n] = g^{-4kN}\int \left\{\sqrt{\det g(X)}\prod_\mu \frac{d\xi^\mu}{\sqrt{2\pi}}\right\}[d\tilde{A}_n]\ . \tag{3.6}$$

[16] In the following all covariant derivatives are defined with respect to the instanton solution $A_n(x;X)$.

Here, $g(X)$ is the metric on the zero modes defined in (2.32). The term in braces is the integral over the zero mode subspace. The factors of g in front of the measure arise from the fact that we included a factor of g^2 in the definition of metric (2.32).

The non-zero mode fluctuations $\tilde{A}_n$ and the ghosts can now be integrated out, producing the usual determinant factors:

$$\int [\mathrm{d}\tilde{A}_n][\mathrm{d}b][\mathrm{d}c]\exp\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{tr}_N\{\tilde{\bar{A}}^{\dot{\alpha}\alpha}\Delta^{(+)}{}_\alpha{}^\beta\tilde{A}_{\beta\dot{\alpha}} - 4b\mathscr{D}^2c\} = \frac{\det(-\mathscr{D}^2)}{\det'\Delta^{(+)}}\ . \tag{3.7}$$

As is conventional, the prime on the determinant indicates that the operator $\Delta^{(+)}$ has zero modes and these must be excluded in the product over eigenvalues that defines the determinant. So the leading-order expression for the functional integral in the charge-k sector is

$$\frac{\mathrm{e}^{2\pi\mathrm{i}k\tau}}{g^{4kN}}\int\left\{\sqrt{\det g(X)}\prod_\mu\frac{\mathrm{d}\xi^\mu}{\sqrt{2\pi}}\right\}\frac{\det(-\mathscr{D}^2)}{\det'\Delta^{(+)}}\ . \tag{3.8}$$

At this stage, (3.8) is only schematic because in order to define the determinants rigorously, we must regularize the theory in some way. We shall consider this problem in Section 3.3.

In the final part of this section, we explain how the integrals over the expansion coefficients of the zero modes $\{\xi^\mu\}$ may be traded for integrals over the collective coordinates $\{X^\mu\}$. This change of variables is facilitated by the well-known trick of inserting an expression for unity in the guise of

$$1 \equiv \int\prod_\mu \mathrm{d}X^\mu\left|\det\frac{\partial f_\nu}{\partial X^\mu}\right|\prod_\nu\delta(f_\nu(X))\ . \tag{3.9}$$

A judicious choice for the function $f_\nu(X)$ is the inner product of fluctuation (3.1) with the zero mode $\delta_\nu A_n$:

$$f_\nu(X) = -2g^2\int \mathrm{d}^4x\,\mathrm{tr}_N\,\delta A_n\delta_\nu A_n = \sum_\mu \xi^\mu g_{\mu\nu}(X)\ , \tag{3.10}$$

where we used (3.5) and the fact that the zero and non-zero modes are functionally orthogonal, along with the definition of the metric $g_{\mu\nu}(X)$ as the inner product of zero modes (2.32).

Now we compute the derivative

$$\frac{\partial f_\nu}{\partial X^\mu} = -2g^2\int \mathrm{d}^4x\,\mathrm{tr}_N\left\{\frac{\partial\delta A_n}{\partial X^\mu}\,\delta_\nu A_n + \delta A_n\,\frac{\partial\delta_\nu A_n}{\partial X^\mu}\right\}\ . \tag{3.11}$$

Since the total field $A_n(x)$ in Eq. (3.1) does not depend on the collective coordinates, we can replace δA_n by $-A_n(x;X)$ in the first term. Using the fact that

$$\int \mathrm{d}^4x\,\mathrm{tr}_N\,\mathscr{D}_n\Omega_\mu\delta_\nu A_n = -\int \mathrm{d}^4x\,\mathrm{tr}_N\,\Omega_\mu\mathscr{D}_n\delta_\nu A_n = 0\ , \tag{3.12}$$

where the surface term vanishes due to the asymptotic form of Ω_μ and $\delta_\nu A_n$ (see Section 2.5), and the last equality follows from (2.24), we have

$$-2g^2\int \mathrm{d}^4x\,\mathrm{tr}_N\,\frac{\partial A_n(x;X)}{\partial X^\mu}\,\delta_\nu A_n = g_{\mu\nu}(X)\ . \tag{3.13}$$

The insertion of unity is then

$$1 \equiv \int \prod_\mu \mathrm{d}X^\mu \left|\det\left(g_{\mu\nu}(X) - 2g^2 \int \mathrm{d}^4x\, \mathrm{tr}_N\, \delta A_n \frac{\partial \delta_\nu A_n}{\partial X^\mu}\right)\right| \prod_\nu \delta\left(\sum_\mu \xi^\mu g_{\mu\nu}(X)\right) . \tag{3.14}$$

Plugging this into (3.8), we perform the integrals over the expansion coefficients using the δ-functions. Since $g_{\mu\nu}(X)$ is invertible, the δ-functions enforce $\xi^\mu = 0$. Consequently in the second term of the determinant we can replace δA_n by the non-zero mode part of the fluctuation $\tilde{A}_n$. This term is higher order in g and may be dropped to leading order. Finally, we have to leading order in the charge-k sector

$$\int [\mathrm{d}A_n][\mathrm{d}b][\mathrm{d}c]\mathrm{e}^{-S[A,b,c]}\bigg|_{\text{charge-}k} \overset{g\to 0}{\to} \frac{\mathrm{e}^{2\pi\mathrm{i}k\tau}}{g^{4kN}} \int_{\mathfrak{M}_k} \boldsymbol{\omega}\, \frac{\det(-\mathcal{D}^2)}{\det' \Delta^{(+)}} . \tag{3.15}$$

Notice that the resulting form of the collective coordinate integral is proportional to the canonical volume form on the moduli space $\mathfrak{M}_k$ associated to the metric $g_{\mu\nu}(X)$:

$$\int_{\mathfrak{M}_k} \boldsymbol{\omega} \equiv \int \sqrt{\det g(X)} \prod_\mu \frac{\mathrm{d}X^\mu}{\sqrt{2\pi}} , \tag{3.16}$$

along with a non-trivial function on $\mathfrak{M}_k$ equal to the determinants of the operators governing the Gaussian fluctuations of the gauge field and ghosts in the instanton background (3.7).

Finally, when calculating a correlation function $\langle \mathcal{O}_1(x^{(1)}) \cdots \mathcal{O}_n(x^{(n)})\rangle$ in the semi-classical limit, the fields insertions $\mathcal{O}_i(x^{(i)})$ are replaced by their values in the instanton background and so, to leading semi-classical order, become functions of the collective coordinates.

3.2. The volume form on the instanton moduli space

We have shown that the leading-order semi-classical approximation of the functional integral involves an integral over instanton moduli space with a measure associated to the natural metric inherited from the inner product of zero modes (2.32). At leading-order in the saddle-point expansion, the integrand includes determinants of fluctuations (3.15), but in the present section we focus on the volume form.

The most direct way to obtain a workable expression for the volume form on the instanton moduli space is to use the fact that this space can be described as a hyper-Kähler quotient of flat space. Actually this is not how the volume form was originally found in Refs. [20,30,31]. In these references, the constraints of supersymmetry, the Index Theorem and other consistency conditions (most notably "clustering" in the dilute instanton gas limit) were used to write down the unique expression for the volume form for a supersymmetric gauge theory. Then, decoupling the fermions and scalars, by giving them large masses, an expression for the volume form on $\mathfrak{M}_k$ was derived. Here, we will not follow this route, rather we shall take as our central theme the hyper-Kähler quotient construction (an approach also adopted in [32]). Since the quotient space $\mathfrak{M}_k$ inherits a metric from the mother space $\tilde{\mathfrak{M}}$ it also inherits a volume form. Recall from Section 2.4.1 and Appendix B that there are two parts to the hyper-Kähler quotient. Firstly, one restricts to the level sets $\mathfrak{N} \subset \tilde{\mathfrak{M}}$ defined by the vanishing of the moment maps $\vec{\mu} = 0$. Then one performs a quotient by the group G finally giving $\mathfrak{M}_k = \mathfrak{N}/G$. An expression for the volume form on $\mathfrak{M}_k$ is be obtained from that of $\tilde{\mathfrak{M}}$ by imposing the vanishing of the moment maps by explicit δ-functions with the appropriate Jacobian. One then divides by the volume of the orbits of the G-symmetry. This leads in general to expression (B.33). An expression for the volume form on $\mathfrak{M}_k$ is then obtained by gauge fixing the G-symmetry. However, it turns out to be more convenient not to fix the symmetry. This is because the expressions for all the fields in an instanton background are G-invariant.

For the ADHM construction the mother space is $\tilde{\mathfrak{M}} = \mathbb{R}^{4k(N+k)}$ parameterized by $a_{\dot\alpha}$ with metric (2.66) and G is $\mathrm{U}(k)$ acting in (2.64). The moment maps are ADHM constraints (2.59a). The $\mathrm{U}(k)$ action defines a set of vector fields on $\tilde{\mathfrak{M}}$, X_r, $r = 1, \ldots, k^2$, given in (2.69). The inner product of these vector fields is given by (2.124) which defines the ADHM operator $\boldsymbol{L}$ which appears in the formula for the volume form. The volume form of the ADHM moduli space in its $\mathrm{U}(k)$-unfixed form is, from (B.40),

$$\int_{\mathfrak{M}_k} \omega = \frac{C_k}{\mathrm{Vol}\,\mathrm{U}(k)} \int \mathrm{d}^{4k(N+k)}a \left|\det{}_{k^2} \boldsymbol{L}\right| \prod_{r=1}^{k^2} \prod_{c=1}^{3} \delta\big(\tfrac{1}{2}\,\mathrm{tr}_k\, T^r(\tau^{c\dot\alpha}{}_{\dot\beta}\bar{a}^{\dot\beta} a_{\dot\alpha})\big)\ . \tag{3.17}$$

In the above, we have defined

$$\int \mathrm{d}^{4k(N+k)}a \equiv \int \prod_{n=1}^{4} \prod_{r=1}^{k^2} \mathrm{d}(a'_n)^r \prod_{i=1}^{k} \prod_{u=1}^{N} \prod_{\dot\alpha=1}^{2} \mathrm{d}\bar{w}^{\dot\alpha}_{iu}\, \mathrm{d}w_{\dot\alpha ui}\ , \tag{3.18}$$

where the integral over the $k \times k$ matrices a'_n and the arguments of the ADHM constraints are defined with respect to the generators of $\mathrm{U}(k)$ in the fundamental representation, normalized so that $\mathrm{tr}_k\, T^r T^s = \delta^{rs}$. The volume of the $\mathrm{U}(k)$ is the constant

$$\mathrm{Vol}\,\mathrm{U}(k) = \frac{2^k \pi^{k(k+1)/2}}{\prod_{i=1}^{k-1} \mathrm{i}!}\ . \tag{3.19}$$

The normalization factor C_k can be determined by taking into account the normalization of metric (2.66), giving

$$C_k = 2^{-k(k-1)/2}(2\pi)^{2kN}\ . \tag{3.20}$$

Our expression for the volume form on the instanton moduli space can be compared with the one-instanton expressions in the literature [2,23]. For $k = 1$ the ADHM are resolved as described in Section 2.4.3. In fact the necessary change of variables from $w_{u\dot\alpha}$ to the scale size ρ and gauge orientation $\mathcal{U}$ is described in Section 6.1. In particular, the (non-supersymmetric) one-instanton measure is given by (6.29) with $\mathcal{N} = 0$, $k = 1$ and with $W^0 \rho^2$ and $a'_n = -X_n$. In order to compare with the literature we integrate over the gauge orientation $\mathcal{U}$ which is conveniently normalized so that $\int \mathrm{d}\mathcal{U} = 1$. This leaves

$$\int_{\mathfrak{M}_1} \omega = \frac{2^{4N+2}\pi^{4N-2}}{(N-1)!(N-2)!} \int \mathrm{d}^4X\, \mathrm{d}\rho\, \rho^{4N-5}\ , \tag{3.21}$$

which agrees with the expression for gauge group $\mathrm{SU}(N)$ derived in [23].

3.2.1. Clustering

The relative normalization constants C_k, for different k, can be checked by using the clustering property of the instanton measure. This requires in certain regions of moduli space where a k instanton configuration can be interpreted as well separated k_1 and k_2 instanton configurations, $\mathfrak{M}_k$ is approximately $\mathfrak{M}_{k_1} \times \mathfrak{M}_{k_2}$, and so the volume form factorizes as

$$\int_{\mathfrak{M}_k} \boldsymbol{\omega} \to \frac{k_1!k_2!}{k!} \int_{\mathfrak{M}_{k_1}} \boldsymbol{\omega} \times \int_{\mathfrak{M}_{k_2}} \boldsymbol{\omega} . \tag{3.22}$$

We can determine the overall normalization of the volume form by going to the completely clustered limit defined in Section 2.4.3. In this limit, we can describe the positions of the single instanton by the diagonal elements of a'_n, $(a'_n)_{ii} = -X^i_n$. Part of the U(k) symmetry can be fixed by setting to zero the off-diagonal elements of a'_n that are generated by U(k) adjoint action on the diagonal matrix diag$(-X^1_n, \dots, -X^k_n)$; as in Section 2.4.3 we will denote the remaining off-diagonal elements as $\tilde{a}'_{ij}$. In the measure, this gauge fixing involves a Jacobian factor: [17]

$$\frac{1}{\mathrm{Vol}\,\mathrm{U}(k)} \int d^{4k^2} a' \to \frac{2^k(k-1)/2}{[\mathrm{Vol}\,\mathrm{U}(1)]^k} \frac{1}{k!} \int \prod_{i=1}^{k} d^4 X^i \left\{ \prod_{\substack{i,j=1\\ i\neq j}}^{k} d^3 \tilde{a}'_{ij} |X^i - X^j| \right\} . \tag{3.23}$$

The off-diagonal ADHM constraints are linear in $\tilde{a}'_{ij}$ (2.101) and the corresponding δ-functions in (3.17) can be used to integrate out completely the $3k(k-1)$ variables $\tilde{a}'_n$. These integrals produce a factor $\prod_{i\neq j}(X^i - X^j)^{-3}$. To complete the analysis we note that in the complete clustering limit

$$\det \boldsymbol{L} = \prod_{i=1}^{k} 2\rho_i^2 \prod_{i\neq j} (X^i - X^j)^2 + \cdots , \tag{3.24}$$

where $2\rho_i^2 \equiv \boldsymbol{L}_i$, the $\boldsymbol{L}$-operator of each individual instanton.

Putting everything together one finds in the complete clustering limit, with the normalization constant given in (3.20),

$$\int_{\mathfrak{M}_k} \boldsymbol{\omega} \to \frac{1}{k!} \underbrace{\int_{\mathfrak{M}_1} \boldsymbol{\omega} \times \cdots \times \int_{\mathfrak{M}_1} \boldsymbol{\omega}}_{k \text{ times}} , \tag{3.25}$$

as required.

3.3. Fluctuation determinants in the instanton background

Determinants of fluctuation operators in the background of a single instanton were first evaluated by 't Hooft in the classic paper [2]. In this section, we sketch how the determinants of the fluctuation

[17] The following formula can easily be derived from the well-known Jacobian that arises from changing variables from the elements of a $k \times k$ Hermitian matrix X (defined with respect to our basis T^r) to its eigenvalues X^i: $\int dX = (2\pi)^{k(k-1)/2} / \prod_{i=1}^k \mathrm{i}! \int dX_i \prod_{i<j} (X^i - X^j)^2$. See for example [33].

operators in a *general* ADHM instanton background can be found, although we omit many of the technical parts of the calculation, principally because in the supersymmetric case the determinants over non-zero modes cancel between the bosonic and fermionic fields [34]. A complete review of the old instanton literature pertaining to the fluctuation determinants may be found in Osborn's review [8]. Ideally, one would want to express the results in terms of geometric quantities on the instanton moduli space. This was never achieved and the final answers are quite implicit, involving spacetime integrals.

The first thing we need to do is to provide properly UV-regularized definitions of the determinants. We will do this by introducing Pauli–Villars regulator fields with large masses μ_i and alternating "metric" e_i, such that

$$\sum_{i=1}^{v} e_i = -1, \quad \sum_{i=1}^{v} e_i \mu_i^{2p} = 0, \quad p = 1, \ldots, v-1 \ . \tag{3.26}$$

We will also define

$$\log \mu = -\sum_{i=1}^{v} e_i \log \mu_i \ . \tag{3.27}$$

For a consistent regularization the number of regulator fields v must exceed three. The regularized determinants are

$$\log \det{}' \Delta^{(+)} = \mathrm{Tr} \left\{ \log(\Delta^{(+)} + \mathscr{P}_0) + \sum_{i=1}^{v} e_i \log(\Delta^{(+)} + \mu_i^2) \right\} \ , \tag{3.28a}$$

$$\log \det \Delta^{(-)} = \mathrm{Tr} \left\{ \log \Delta^{(-)} + \sum_{i=1}^{v} e_i \log(\Delta^{(-)} + \mu_i^2) \right\} \ . \tag{3.28b}$$

In (3.28a), $\mathscr{P}_0$ is the projector onto the zero mode subspace of $\Delta^{(+)}$. Since the non-zero eigenvalues of $\Delta^{(\pm)}$ are identical, we can extract a very simple expression for the ratio

$$\frac{\det{}' \Delta^{(+)}}{\det \Delta^{(-)}} = \exp\left(2kN \sum_{i=1}^{v} e_i \mu_i^2 \right) = \mu^{-4kN} \ . \tag{3.29}$$

Here, $2kN$ is the number of zero modes of $\Delta^{(+)}$ and μ is the overall Pauli–Villars mass scale (3.27). Using this relation and the fact that $\det \Delta^{(-)} = \{\det(-\mathscr{D}^2)\}^2$, we can express the fluctuation determinants in (3.15) in terms of the determinant of the covariant Laplacian:

$$\frac{\det(-\mathscr{D}^2)}{\det{}' \Delta^{(+)}} = \mu^{4kN} \det(-\mathscr{D}^2)^{-1} \ . \tag{3.30}$$

We have succeeded in reducing the problem to that of the fluctuation determinant of a scalar field transforming in the adjoint representation of the gauge group. By pooling together results from the old instanton literature, we can find an expression for this determinant. Firstly, one can construct the determinant for a scalar field transforming in the fundamental representation of SU(N) following

Refs. [8,35,36]. The determinant for an adjoint-valued field is then related to this by a remarkable formula of Jack [37].

So we begin with a brief sketch of how one determines the fluctuation determinant of a scalar field transforming in the fundamental representation. The key idea involves considering the variation of $\log\det(-\mathscr{D}^2)_{\rm fund.}$ by the collective coordinates. [18] The resulting formula established in [8,35,36] is

$$\frac{\partial}{\partial X^\mu}\log\det(-\mathscr{D}^2)_{\rm fund.}=\frac{1}{6\pi^2}\int d^4x\,{\rm tr}_N\,\delta_\mu A_n J_n\ , \tag{3.31}$$

where J_n is the conserved current

$$J_n=\bar U\sigma_{n\alpha\dot\alpha}b^\alpha f\bar\Delta^{\dot\alpha}b^\beta f\bar b_\beta U-\bar U b^\beta f\bar b_\beta\Delta_{\dot\alpha}f\bar b_\alpha\bar\sigma_n^{\dot\alpha\alpha}U \tag{3.32}$$

and $\delta_\mu A_n$ is the zero mode associated to X^μ. By integrating this expression one can extract the ratio of the determinant in the instanton background to the determinant in the vacuum. The final expression established in [35] is expressed solely in terms of the ADHM matrix f:

$$\log\left[\frac{\det(-\mathscr{D}^2)}{\det(-\mathscr{D}_0^2)}\right]_{\rm fund.}=\frac{k}{6}\log\mu+p_k+\frac{1}{48\pi^2}\int d^4x(I_1(x)+I_2(x))\ , \tag{3.33}$$

where $\mathscr{D}_0$ is the covariant derivative in the vacuum ($A_n=0$), p_k is a constant which will be determined shortly and the integrands are

$$I_1(x)={\rm tr}_k(f\partial_n f^{-1}f\partial_n f^{-1}f\partial_m f^{-1}f\partial_m f^{-1}-20f^2)+\frac{4k}{(1+x^2)^2}\ , \tag{3.34a}$$

$$I_2(x)=\int_0^1 dt\,\varepsilon_{mnkl}\,{\rm tr}_k(\tilde f\partial_t\tilde f^{-1}\tilde f\partial_m\tilde f^{-1}\tilde f\partial_n\tilde f^{-1}\tilde f\partial_k\tilde f^{-1}\tilde f\partial_l\tilde f^{-1})\ . \tag{3.34b}$$

Here, t is an auxiliary variable and $\tilde f(x,t)$ is the $k\times k$-dimensional matrix derived from $f(x)$:

$$\tilde f^{-1}(x,t)=tf^{-1}(x)+(1-t)(1+x^2)1_{[k]\times[k]}\ . \tag{3.35}$$

In order to fix the constant p_k we can appeal to a clustering argument. In the complete clustering limit the $k\times k$ matrix f is diagonal:

$$f={\rm diag}\left(\ldots,\frac{1}{(x-X^i)^2+\rho_i^2},\ldots\right)\ , \tag{3.36}$$

where ρ_i and X_n^i are the scale size and position of the ith instanton, respectively. In this limit, the integrals in (3.33) can easily be evaluated. (Actually the integral of I_2 vanishes.) One finds

$$\log\left[\frac{\det(-\mathscr{D}^2)}{\det(-\mathscr{D}_0^2)}\right]_{\rm fund.}=\sum_{i=1}^k\left(\tfrac16\log(\mu\rho_i)-\tfrac{5}{18}\right)+p_k\ . \tag{3.37}$$

[18] Here, the subscript reminds us that the scalar field transforms in the fundamental representation of SU(N).

Comparing with the one-instanton expression for this determinant [2] fixes the constant to be

$$p_k = k(\alpha(\tfrac{1}{2}) + \tfrac{5}{18}) = k(-2\zeta'(-1) - \tfrac{1}{6}\log 2 + \tfrac{15}{72}) \ . \tag{3.38}$$

Here, $\alpha(\frac{1}{2})$ is a constant defined in [2].

The second part of the problem involves relating the fluctuation determinant associated to the fundamental representation of the gauge group to that of the adjoint representation. The way this is achieved [37] relies on the tensor product ADHM formalism developed in [18]. The analysis provides the following explicit relation for the fluctuation determinant of the Laplace operator for an adjoint-valued field in terms of one for a fundamental-valued field:

$$\log\frac{\det(-\mathscr{D}^2)}{\det(-\mathscr{D}_0^2)} = 2N\log\left[\frac{\det(-\mathscr{D}^2)}{\det(-\mathscr{D}_0^2)}\right]_{\text{fund.}} + \log\det{}_{k^2}\boldsymbol{L} - \frac{1}{16\pi^2}\int d^4x \log\det{}_k f\,\square^2\log\det{}_k f + q_k \ . \tag{3.39}$$

Here, $\boldsymbol{L}$ is the ubiquitous ADHM operator on $k\times k$ matrices that we defined in (2.125). As above, we will determine the constant q_k by a clustering argument. In the complete clustering limit, we find

$$\log\frac{\det(-\mathscr{D}^2)}{\det(-\mathscr{D}_0^2)} = 2N\sum_{i=1}^{k}(\tfrac{1}{6}\log(\mu\rho_i) + \alpha(\tfrac{1}{2})) + k(\log 2 - \tfrac{5}{6}) + q_k \ . \tag{3.40}$$

Comparing this with the one-instanton expression [2], we find

$$q_k = k(\alpha(1) - 4\alpha(\tfrac{1}{2}) - \log 2 + \tfrac{5}{6}) = \tfrac{5}{9}k \ . \tag{3.41}$$

4. Instantons in supersymmetric gauge theories

In this section, we consider instanton configurations in gauge theories with supersymmetry. There is an intimate relation between instantons and supersymmetry which can be traced to the fact that, in any supersymmetric gauge theory, self-dual gauge fields are invariant under precisely half the supersymmetry generators. As we will see, this is manifest in the ADHM construction, which has a very natural supersymmetric generalization. Specifically we consider the minimal theories in four dimensions with gauge group SU(N) and $\mathcal{N}=1$, 2 and 4 supersymmetry. Like the purely bosonic gauge theory considered in previous sections, the $\mathcal{N}=1$ and 2 theories are asymptotically free and the relevance of instantons in the quantum theory is not immediately obvious. However, we will begin by focusing on the classical aspects of these configurations.

Before searching for supersymmetric instantons, there is the worrisome issue of supersymmetry in Euclidean space to discuss. The nub of the issue is the following: Weyl spinors in $D=4$ Minkowski space are in a “real” representation of the covering group of the Lorentz group $\overline{\mathrm{SO}}(3,1)$.[19] Consequently the minimal spinor in $D=4$ is a Majorana spinor, which we can think of

[19] Our conventions for spinors are described in Appendix A. In particular in Minkowski space our conventions are those of Wess and Bagger.

as two Weyl spinors λ and $\bar{\lambda}$ subject to the reality condition

$$\bar{\lambda}^{\dot{\alpha}} = (\lambda_\alpha)^\dagger \quad (\dot{\alpha} = \alpha) \; . \tag{4.1}$$

This ensures, for instance, that the canonical fermion kinetic term is real. On the contrary, in $D = 4$ Euclidean space, Weyl spinors are in "pseudo-real" representations of $\overline{\mathrm{SO}}(4)$ and one cannot impose reality condition (4.1). On the contrary, $\overline{\mathrm{SO}}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$, so the spinor indices α and $\dot{\alpha}$ refer to each of the SU(2)'s and they are not mixed under complex conjugation. In Euclidean space there is no notion of a Majorana spinor and, apparently, no Euclidean version of the theory with a real action. This problem may be by-passed in theories with extended supersymmetry as the Weyl spinors in these theories may be combined in pairs to form Dirac spinors which do have Euclidean counterparts. However, the problem is unavoidable for theories with $\mathcal{N} = 1$ supersymmetry.

There are several alternative approaches to this issue favoured by different workers in the field. The most conservative approach as described in [38] (see Appendix A of this reference) is to abandon the idea of constructing a Euclidean quantum field theory with $\mathcal{N} = 1$ supersymmetric as unnecessary. In particular, our main interest should be calculating Green's functions in the Minkowski space theory. As in standard perturbative calculations, these Green's functions are most conveniently calculated by analytic continuation of the corresponding Minkowski space path integral to Euclidean spacetime. This is effected by the standard Wick rotation of the time coordinate $x_0 \to -\mathrm{i}x_4$. The path-integral exponent is then $-S_E = \mathrm{i}S_M$. The resulting path integral can then be evaluated in the saddle-point approximation by expanding in fluctuations around the minima of S_E. This procedure yields finite and well-defined answers for the original Minkowski space Green's functions. In this context, the fact that the fermionic part of S_E is not real is inconsequential: it does not affect the convergence of the integrals. Some authors argue that reality of the action is not, in any case, the appropriate condition for Euclidean space theories. Rather we should impose a modified condition known as "reflection positivity" which is characteristic of fermionic actions on a spacetime lattice.

In the following we will adopt the conservative viewpoint described above. However, there are other approaches to instantons which divorce themselves from the Minkowski space theory and seek to define the supersymmetric theory directly in Euclidean space [13,39]. This follows the work of several authors who have shown that it is actually possible to define Euclidean versions of the theories with both extended supersymmetry and real actions [40–44]. These theories have the potentially undesirable feature of a non-compact R-symmetry group and a scalar field with a negative kinetic term. The philosophy is therefore slightly different from our viewpoint—and the relation with the original Minkowski space theory is now rather obscure—but the resulting calculations are essentially identical to those described below.

In the bulk of this section, we will discuss the $\mathcal{N} = 1$, 2 and 4 theories in a unified formalism. In the remainder of this introductory section we will introduce some of the key points which arise, starting with the $\mathcal{N} = 1$ theory which contains the gauge field and a single species of Weyl fermion λ_α in the adjoint representation of the gauge group. Setting the fermion fields to zero to start with, the ADHM instanton configuration trivially solves the equations-of-motion. However, we must now consider the fluctuations of the fermions around this solution. These are governed by the following covariant Dirac equations:

$$\bar{\mathcal{D}}\!\!\!/\,\lambda^A = 0 \; , \tag{4.2a}$$

$$\mathcal{D}\!\!\!/\,\bar{\lambda}_A = 0 \; , \tag{4.2b}$$

where the covariant derivatives are taken in the adjoint representation and evaluated in the instanton background of topological charge k. As already discussed in Section 2.3, a standard application of the Atiyah–Singer Index Theorem shows that Eq. (4.2a) has $2kN$ linearly independent (normalizable) solutions. These were identified explicitly in Section 2.5 as $\Lambda_\alpha(C_{\rm i})$, where ${\rm i} = 1,\ldots,2kN$ labels the solutions of constraints (2.109a) and (2.109b). On the contrary, (4.2b) has no non-trivial solutions in an instanton background.

These solutions are known as the *fermion zero modes* of the instanton. There are two distinct—but ultimately equivalent—approaches to treating these modes. One way of incorporating these modes is to treat them perturbatively as fluctuations around the ordinary bosonic instanton. This expansion is perfectly consistent, but is hard to carry out in practice beyond the lowest orders. An alternative approach, and the one we will adopt here, originates in the work of Novikov et al. [38]. In this approach, we interpret the fermionic zero modes as corresponding to a degeneracy of classical solutions just like zero modes of the gauge field we met in the previous section. More generally, the standard semi-classical reasoning suggests that we should look for finite-action solutions of the full coupled equations-of-motion of the supersymmetric theory. We will call these configurations *super-instantons*. In the present case they are easy to find: the gauge field takes ADHM value, while the fermions solve Eqs. (4.2a) and (4.2b). Thus the right-handed fermion $\bar\lambda_{\dot\alpha}$ is zero; this ensures that the fermions do not modify the Yang–Mills equation for the gauge field. Meanwhile, the left-handed fermion λ_α is general linear superposition of the normalizable zero modes:

$$\lambda_\alpha = \sum_{i=1}^{2kN} \psi^{\rm i} \Lambda_\alpha(C_{\rm i}) \ . \tag{4.3}$$

As λ_α is a fermionic field, the $2kN$ coefficients $\psi^{\rm i}$ are Grassmann variables. In fact (4.3) can be written more compactly as $\lambda_\alpha = \Lambda_\alpha(\mathscr{M})$, where the Grassmann quantities $\mathscr{M} = \psi^{\rm i} C_{\rm i}$ themselves satisfy constraints (2.109a) and (2.109b). We will call these constraints the "fermionic ADHM constraints" since they will turn out to be the Grassmann superpartners of the "bosonic" ADHM constraints (2.65). The solutions to these constraints will then be parameterized as $\mathscr{M}(\psi^{\rm i})$.

The Grassmann variables $\psi^{\rm i}$ are the fermionic analogues of the collective coordinates which parameterize the general ADHM solution and we will henceforth refer to them as *Grassmann collective coordinates*. While the bosonic collective coordinates X^μ have the interpretation as intrinsic coordinates on the moduli space $\mathfrak{M}_k$, their Grassmann counterparts correspond to intrinsic *symplectic tangent vectors* on this manifold (see Appendix B). Like the bosonic collective coordinates they parameterize degenerate minima of the action and we must integrate over them. We will work out the exact integration measure in the following but the important qualitative features follow from the basic rules of Grassmann integration,

$$\int d\psi^{\rm i} = 0, \quad \int d\psi^{\rm i}\, \psi^{\rm j} = \delta^{\rm ij} \ . \tag{4.4}$$

We see that to obtain a non-zero answer, each Grassmann integration in the measure must be saturated by a single power of the integration variable. This leads to simple selection rule for fermionic Green's functions in the k instanton background. Specifically Green's function $\langle \lambda(x_1)\lambda(x_2)\cdots\lambda(x_l)\rangle$ will vanish unless $l = 2kN$. This counting can be understood by noting that $\mathcal{N} = 1$ supersymmetric Yang–Mills theory has an anomalous abelian R-symmetry under which the left- and right-handed fermions have charges ± 1. The exact anomaly in the R-symmetry current j^R_m is determined at one loop as

$$\partial_m j^R_m = \frac{Ng^2}{8\pi^2} \operatorname{tr}_N F^*_{mn} F_{mn} \ . \tag{4.5}$$

Integrating this equation over spacetime we find that conservation of the corresponding U(1) charge is violated by $2kN$ units in the background of topological charge k. This agrees with the selection rule described above. The symmetry means that observables of definite R-charge receive corrections from a single topological sector if at all. As we will see below, much more interesting behaviour is possible in theories without an anomalous abelian R-symmetry.

In the previous section, we saw that a subset of the bosonic collective coordinates can be understood in terms of the action of symmetries of the theory on the instanton solution. The same is true for the Grassmann collective coordinates. As mentioned above, any self-dual configuration of the gauge field is invariant under half the generators of the supersymmetry algebra. On the other hand, the other half of the generators act non-trivially on the solution generating fermion zero modes. The $\mathcal{N}=1$ supersymmetry algebra has four supercharges and yields two zero modes. In fact, classical supersymmetric gauge theories in four dimensions are invariant under a larger superconformal algebra which, in the $\mathcal{N}=1$ case, includes four additional fermionic generators, two of which act non-trivially on the instanton. Broken symmetries thus yield a total of four fermion zero modes. The argument which guarantees the existence of these zero modes is very robust (it is essentially Goldstone's theorem) and holds at all orders in the full quantum theory. As above, the actual number of zero modes is $2kN$ which equals four in the minimal case $k=1$, $N=2$. Thus, the situation for fermionic zero modes matches nicely with the corresponding counting of bosonic zero modes given in the previous section. Specifically, the bosonic and fermionic zero modes of a single instanton of gauge group SU(2) are all associated with the action of broken symmetry generators. For higher instanton number and/or larger gauge group there are additional zero modes which do not correspond to broken symmetries.

Theories with extended supersymmetry, or more generally $\mathcal{N}=1$ theories with additional matter, necessarily contain scalar fields. As usual, the scalars can acquire VEVs which spontaneously break the gauge symmetry. This possibility is particularly important for applications of instanton calculus to theories with asymptotic freedom. As usual such theories are characterized by logarithmic running of the coupling which introduces a dynamical scale Λ. In the absence of scalar VEVs, the running coupling becomes large in the IR at mass scales of order Λ and we do not expect semi-classical methods to work. However, if we introduce a scalar VEV which breaks the gauge group at some scale v then the effective coupling will not run below this scale. In particular, if the scale of the VEV is much greater than Λ, the running coupling is frozen before it has a chance to become large and the theory is weakly coupled at all length scales. In these circumstances we can expect a semi-classical analysis of the path integral to be reliable.

Introducing scalar fields, with or without VEVs, affects the instanton calculus in several important ways. In addition to the gauge couplings of the scalars, supersymmetric Lagrangians necessarily contain Yukawa couplings between the scalars and fermions. As above, we are looking for a super-instanton which solves the full equations-of-motion of the theory. A promising starting point is the configuration described above where the gauge field takes its ADHM value and the fermions are a general linear combination of the zero modes. The scalars themselves satisfy a covariant Laplace equation in the gauge-field background with a fermion bilinear source term which comes from the Yukawa coupling and, as we review below, this equation can be solved explicitly. The solution exhibits a new feature of the super-instanton: bosonic fields can have pieces which are bilinear (or of higher even power) in the Grassmann collective coordinates. A potentially worrying feature is that a complex scalar field ϕ typically acquires a non-zero Grassmann bilinear part, while its complex conjugate $\phi^\dagger$ does not. There is no inconsistency here, as we can illustrate by considering the following toy integral:

$$\mathscr{I} = \int d^2\phi \, e^{-|\phi|^2+\phi A+\phi^* B} \ , \tag{4.6}$$

where A and B are quadratic expressions in Grassmann variables and we suppose $A \neq B^*$. Obviously, one should expand in the Grassmann composites A and B, since the series terminates because there are only a finite number of Grassmann variables, and then do the ϕ integral:

$$\mathscr{I} = \sum_{m,n=0}^{P} \frac{1}{m!n!} A^m B^n \int d^2\phi \, \phi^m \phi^{*n} e^{-|\phi|^2} = \pi \sum_{m=0}^{P} \frac{1}{m!}(AB)^m = \pi e^{AB} \ . \tag{4.7}$$

However, the same result can be obtained by solving the "equations-of-motion" of ϕ and ϕ^* and shifting the integration variables by the solution. The solutions are

$$\phi = B, \quad \phi^* = A \ , \tag{4.8}$$

which, since by hypothesis $A \neq B^*$, violate the reality condition on ϕ. However,

$$\mathscr{I} = \int d^2\phi \, e^{-(\phi-B)(\phi^*-A)+AB} = e^{AB} \int d^2\phi \, e^{-|\phi|^2} = \pi e^{AB} \tag{4.9}$$

reproducing (4.7). In this way solving the equation-of-motion in the instanton background will involve Grassmann composite terms which violate the reality conditions of the fields, however, as we have seen with the toy integral above, it is a convenient book-keeping device.

Even in the absence of scalar VEVs, the configuration described above is not the end of the story because it does not necessarily solve the full coupled equations-of-motion. In particular, the non-zero Grassmann bilinear part of the scalar field can modify the equations-of-motion for the fermions and even for the gauge field itself, invalidating our starting ansatz for these fields. In general this modification will be non-trivial unless it is forbidden by the symmetries of the theory. For theories with an abelian R-symmetry, such as the $\mathcal{N}=2$ theory without scalar VEVs, the new terms in the equations are zero and our candidate super-instanton actually solves the full equations-of-motion. However, in one of the most important cases, that of $\mathcal{N}=4$ supersymmetric Yang–Mills, there is no such symmetry and the modification is unavoidable. Even in this case the situation is not as hopeless as it might appear because the offending couplings are each suppressed by powers of g^2 at weak coupling. Fortunately, for our stated purpose of calculating the leading semi-classical contributions to Green's functions, it suffices to solve the equations-of-motion perturbatively to some order in g^2. The resulting configuration differs from being an exact solution by a power of g^2 and hence we will refer to it as a *quasi-instanton*. The question of whether a corresponding *exact* solution exists is an interesting one. As we are expanding in Grassmann bilinears, the perturbation series must truncate at some finite order and one might imagine that the final result should be an exact solution. However, at least for the $\mathcal{N}=4$ theory we will argue that this is not the case. In fact, we will exhibit an obstruction to solving the equations beyond next-to-leading order in this case. In any case, the utility of proceeding beyond the first few orders in the classical equations is unclear, because one must also take into account quantum corrections which modify the equations themselves at the same order in g^2.

An important property of the quasi-instanton can be exhibited by calculating its action. The $\mathcal{N}=4$ theory has four species of Weyl fermions which lie in the fundamental representation of an $SU(4)$ R-symmetry. The corresponding Grassmann collective coordinates are $\psi^{\mathrm{i}A}$ where, as in the $\mathcal{N}=1$ theory discussed above, $\mathrm{i}=1,\ldots,2kN$ and $A=1$–4 is the R-symmetry index. In the absence of scalar VEVs, the action of our quasi-instanton of topological charge k is

$$\tilde{S}=\frac{8\pi^2 k}{g^2}-\mathrm{i}k\theta+\frac{1}{96}\,\varepsilon_{ABCD}R_{\mathrm{ijkl}}(X)\psi^{\mathrm{i}A}\psi^{\mathrm{j}B}\psi^{\mathrm{k}C}\psi^{\mathrm{l}D}\ . \tag{4.10}$$

Here, $R(X)$ is the symplectic curvature of the hyper-Kähler quotient space $\mathfrak{M}_k$, which depends explicitly on the chosen point in the moduli space. The action depends explicitly on both the bosonic and fermionic collective coordinates reflecting the fact that we are not dealing with an exact solution of the equations-of-motion.

From the point of view of the semi-classical approximation, expression (4.10) can be thought of as an "effective action" for the quasi-zero modes, to order g^0 in the coupling, which results from integrating out the remaining modes of non-zero frequency. As we will see in the next section, the exponential of this action is an essential ingredient in the collective coordinate integration measure. The effective action for the instanton collective coordinates is closely related to the idea of a world-volume effective action for solitons and other extended object in higher dimensions. This point of view will be developed substantially in Section 10.

The dependence of action (4.10) on the Grassmann collective coordinates means that the corresponding fermionic modes are not exact zero modes. These modes are *lifted* at order g^0 in the semi-classical expansion. The zero modes which are generated by the action of fermionic symmetry generators on the instanton are an important exception. The same symmetries lead to zero eigenvalues of the curvature tensor which mean that action (4.10) is independent of the corresponding Grassmann collective coordinates. The associated zero modes remain unlifted to all orders in g^2. The lifting of fermion zero modes means that the selection rules determining which Green's functions can receive instanton corrections are much less restrictive than those of the $\mathcal{N}=1$ theory described above. Typically this is also related to the absence of an anomalous abelian R-symmetry. For example, in the $\mathcal{N}=4$ theory with zero scalar VEVs, the total number of exact fermion zero modes is 16 (two supersymmetric and two superconformal modes for each of the four species of Weyl fermion). This number does not depend on the topological charge and, for example, 16-fermion correlators receive an infinite series of corrections from all numbers of instantons.

Finally we turn to the case where scalar fields develop non-zero VEVs. In this case, it is well known that there is no non-trivial instanton solution of the coupled equations-of-motion for the gauge field and scalar. Indeed, the existence of such a solution is forbidden by Derrick's theorem. The problem is best illustrated by proceeding naively in the SU(2) theory. If we solve the covariant Laplace equation for an adjoint scalar ϕ with VEV $\mathrm{diag}(\phi^0,-\phi^0)$ in the background of a single

SU(2) instanton of scale size ρ and substitute the solution back into the action, the result is

$$S=\frac{8\pi^2}{g^2}-\mathrm{i}\theta+4\pi^2\rho^2(\phi^0)^2\ . \tag{4.11}$$

The instanton action depends explicitly on the scale size and may be lowered continuously to zero by shrinking the instanton. However, it is useful to note that the non-trivial term in action (4.11) is down by a power of g^2 relative to the constant.[20] Thus our candidate configuration is actually a quasi-instanton in the same sense as in our discussion of the $\mathcal{N}=4$ theory above. In either case, our aim is to determine the leading semi-classical behaviour of Green's functions and it is legitimate to solve the saddle-point conditions order by order in g^2.

In the case of non-zero scalar VEVs, this approach was first developed by Affleck who referred to the corresponding field configurations as *constrained instantons*. The formalism of constrained instantons is rather technical. However, as we explain below, a large part of it can actually be understood in terms of a modified instanton action like that given in (4.11). As mentioned above, any quasi-instanton will have an action which depends explicitly on the collective coordinates. In the case of the $\mathcal{N}=4$ theory without scalar VEVs, the resulting expression, given as (4.10), is determined in terms of the symplectic curvature on $\mathfrak{M}_k$. One of our main results is that the effect of introducing scalar VEVs is simply to introduce an appropriate potential on the moduli space. Furthermore this potential has a nice interpretation as the norm squared of a tri-holomorphic Killing vector on $\mathfrak{M}_k$ (for related results in the context of dyons see [45,46]).

The remainder of this section is organized as follows. In Section 4.1, we present the Minkowski space action, equations-of-motion and supersymmetric transformations of the minimal gauge theories with $\mathcal{N}=1$, 2 and 4 supersymmetries in four dimensions. We discuss the analytic continuation of the theory to Euclidean spacetime. In Section 4.2 we construct the super-instanton at the first non-trivial order in g^2. This necessitates solving the adjoint Dirac equation (4.2b) for the left-handed fermion in the general ADHM background. The general solution involves a matrix of Grassmann variables constrained by a linear equation which generalizes the ADHM constraint equations of the previous section. We introduce intrinsic Grassmann collective coordinates and identify them as symplectic tangent vectors on $\mathfrak{M}_k$. We discuss the action of supersymmetry on the collective coordinates.

In Section 4.3, we consider the construction of the super-instanton beyond linear order. This requires solving the covariant Laplace equation for the adjoint scalar in the general ADHM background with an appropriate fermion bilinear source term. We show that this yields an exact super-instanton for the $\mathcal{N}=2$ theory (without scalar VEVs), but only a quasi-instanton, in the sense described above, in the $\mathcal{N}=4$ case. We demonstrate an obstruction to the existence of an exact solution in this case. In Section 4.4, we review the necessary aspects of the constrained instanton formalism. Finally, in Section 4.5 we explain how the supersymmetry of the field theory is inherited by the collective coordinates.

In the next section, we derive the instanton measure in supersymmetric theories. We introduce the *instanton effective action* for these theories and discuss the consequent lifting of fermion

[20] Strictly speaking, this is true as long as we treat $\rho\phi^0$ as order g^0. As we must integrate over all values of ρ this assumption needs to be justified. In the cases of interest it is not hard to show that larger values of $\rho\phi^0$ are exponentially suppressed and the naive scaling is correct.

zero modes. Then we derive an explicit formulae for the appropriate supersymmetric volume form on $\mathfrak{M}_k$.

4.1. Action, supersymmetry and equations-of-motion

We start by defining theories in four-dimensional Minkowski space with $\mathcal{N}=1,2$ and 4 supersymmetry. In the interests of brevity we will develop a unified notion that allows us to deal with all these cases together. To this end, we introduce the fermionic partners of the gauge field λ^A and $\bar\lambda_A$. Here, $A=1,\ldots,\mathcal{N}$ is an R-symmetry index of the supersymmetry. Since we are working—at least initially—in Minkowski space, these spinors are subject to the reality conditions

$$(\lambda^A_\alpha)^\dagger=\bar\lambda_{\dot\alpha A},\quad (\bar\lambda^{\dot\alpha}_A)^\dagger=\lambda^{\alpha A}\quad (\alpha=\dot\alpha)\ . \tag{4.12}$$

In addition, for the theories with extended supersymmetry there are real scalar fields ϕ_a, $a=1,\ldots,2(\mathcal{N}-1)$. The Minkowski space action is [21]

$$S^{\rm Mink}=\int d^4x\,{\rm tr}_N\Big\{\frac12 F^2_{mn}+\frac{{\rm i}\theta g^2}{16\pi^2}F^*_{mn}F^{mn}+2{\rm i}\mathscr{D}_n\bar\lambda_A\bar\sigma^n\lambda^A-\mathscr{D}^n\phi_a\mathscr{D}_n\phi_a$$
$$+g\bar\lambda_A\Sigma^{AB}_a[\phi_a,\bar\lambda_B]+g\lambda^A\bar\Sigma_{aAB}[\phi_a,\lambda^B]+\frac12 g^2[\phi_a,\phi_b]^2\Big\}\ . \tag{4.13}$$

The terms involving the scalar fields are, of course, absent in the $\mathcal{N}=1$ theory. The Σ-matrices are associated to the SU(2) and SU(4) R-symmetry group of the $\mathcal{N}=2$ and 4 theories, respectively. For $\mathcal{N}=2$ we take

$$\Sigma^{AB}_a=\varepsilon^{AB}({\rm i},1),\quad \bar\Sigma_{aAB}=\varepsilon_{AB}(-{\rm i},1)\ . \tag{4.14}$$

In this case the indices $A,B,\ldots=1,2$ are spinor indices of the SU(2) subgroup of the U(1) $\times$ SU(2) R-symmetry group. In this case, we can raise and lower the indices using the ε-tensor in the usual way following the conventions of [47]. For the $\mathcal{N}=4$ case

$$\Sigma_a=(\eta^3,{\rm i}\bar\eta^3,\eta^2,{\rm i}\bar\eta^2,\eta^1,{\rm i}\bar\eta^1)\ ,$$
$$\bar\Sigma_a=(-\eta^3,{\rm i}\bar\eta^3,-\eta^2,{\rm i}\bar\eta^2,-\eta^1,{\rm i}\bar\eta^1)\ , \tag{4.15}$$

where η^c, $\bar\eta^c$, $c=1$–3, are 't Hooft's η-symbols defined in Appendix A.

Theory (4.13) is invariant under the on-shell supersymmetry transformations

$$\delta A_n=-\xi^A\sigma_n\bar\lambda_A-\bar\xi_A\bar\sigma_n\lambda^A\ , \tag{4.16a}$$

$$\delta\lambda^A=-{\rm i}\sigma^{mn}\xi^AF_{mn}-{\rm i}g\Sigma_{ab}{}^A{}_B\xi^B[\phi_a,\phi_b]+\Sigma^{AB}_a\sigma^n\bar\xi_B\mathscr{D}_n\phi_a\ , \tag{4.16b}$$

[21] We remind the reader that our gauge field is anti-Hermitian rather than Hermitian, otherwise our conventions in Minkowski space are those of Wess and Bagger [47].

$$\delta\bar{\lambda}_A = -\mathrm{i}\bar{\sigma}^{mn}\bar{\xi}_A F_{mn} - \mathrm{i}g\bar{\Sigma}_{abA}{}^B\bar{\xi}_B[\phi_a,\phi_b] + \bar{\Sigma}_{aAB}\bar{\sigma}^n\xi^B\mathscr{D}_n\phi_a \ , \tag{4.16c}$$

$$\delta\phi_a = \mathrm{i}\xi^A\bar{\Sigma}_{aAB}\lambda^B + \mathrm{i}\bar{\xi}_A\Sigma_a^{AB}\bar{\lambda}_B \ . \tag{4.16d}$$

In the above,

$$\sigma^{mn} = \tfrac{1}{4}(\sigma^m\bar{\sigma}^n - \sigma^n\bar{\sigma}^m), \quad \bar{\sigma}^{mn} = \tfrac{1}{4}(\bar{\sigma}^m\sigma^n - \bar{\sigma}^n\sigma^m) \tag{4.17}$$

and

$$\Sigma_{ab} = \tfrac{1}{4}(\Sigma_a\bar{\Sigma}_b - \Sigma_b\bar{\Sigma}_a), \quad \bar{\Sigma}_{ab} = \tfrac{1}{4}(\bar{\Sigma}_a\Sigma_b - \bar{\Sigma}_b\Sigma_a) \ . \tag{4.18}$$

In order to construct instanton solutions, we now Wick rotate to Euclidean space. Vector quantities in Minkowski space $a^n = (a^0, \vec{a})$, with $n = 0$–3, become $a_n = (\vec{a}, \mathrm{i}a^0)$, with $n = 1$–4, in Euclidean space. The Euclidean action is then $-\mathrm{i}$ times the Minkowski space action. The exception to this is that we define the Euclidean σ-matrices as in (2.8) and (2.9). So in Minkowski space $\sigma^n = (-1, \vec{\tau})$ and $\bar{\sigma}^n = (-1, -\vec{\tau})$, whereas in Euclidean space $\sigma_n = (\mathrm{i}\vec{\tau}, 1)$ and $\bar{\sigma}_n = (-\mathrm{i}\vec{\tau}, 1)$. Operationally, this means that when Wick rotating from Minkowski space to Euclidean space we should actually replace the Minkowski space σ-matrices by $-\mathrm{i}$ times the Euclidean space σ-matrices. As usual we treat λ_α and $\bar{\lambda}^{\dot{\alpha}}$ as independent spinors, i.e. independent integration variables in the functional integral. The Euclidean space action is

$$\begin{aligned} S = \int \mathrm{d}^4x\,\mathrm{tr}_N \Bigg\{ & -\tfrac{1}{2}F_{mn}^2 - \frac{\mathrm{i}\theta g^2}{16\pi^2}F_{mn}^* F_{mn} - 2\mathscr{D}_n\bar{\lambda}_A\bar{\sigma}_n\lambda^A + \mathscr{D}_n\phi_a\mathscr{D}_n\phi_a \\ & - g\bar{\lambda}_A\Sigma_a^{AB}[\phi_a, \bar{\lambda}_B] - g\lambda^A\bar{\Sigma}_{aAB}[\phi_a, \lambda^B] - \tfrac{1}{2}g^2[\phi_a,\phi_b]^2 \Bigg\} \ . \end{aligned} \tag{4.19}$$

As discussed above, the fact that the fermionic terms in this action are not real will not concern us further. For the case with $\mathscr{N}=2$ supersymmetry, we can recover the more conventional presentation of the theory by defining a complex scalar field

$$\phi = \phi_1 - \mathrm{i}\phi_2, \quad \phi^\dagger = \phi_1 + \mathrm{i}\phi_2 \tag{4.20}$$

and spinors $\lambda \equiv \lambda^1$ and $\psi \equiv \lambda^2$. The fields $\Phi = \{\phi/\sqrt{2}, \psi\}$ form a chiral multiplet and $V = \{A_m, \lambda\}$ a vector multiplet of $\mathscr{N}=1$ supersymmetry. In terms of these variables, the Euclidean space action of the $\mathscr{N}=2$ theory (4.19) is

$$\begin{aligned} S_{\mathscr{N}=2} = \int \mathrm{d}^4x\,\mathrm{tr}_N \Bigg\{ & -\frac{1}{2}F_{mn}^2 - \frac{\mathrm{i}\theta g^2}{16\pi^2}F_{mn}^* F_{mn} - 2\mathscr{D}_n\bar{\lambda}\bar{\sigma}_n\lambda - 2\mathscr{D}_n\bar{\psi}\bar{\sigma}_n\psi + \mathscr{D}_n\phi^\dagger\mathscr{D}_n\phi \\ & + 2\mathrm{i}g\bar{\psi}[\phi, \bar{\lambda}] + 2\mathrm{i}g[\phi^\dagger, \lambda]\psi + \frac{1}{4}g^2[\phi, \phi^\dagger]^2 \Bigg\} \ . \end{aligned} \tag{4.21}$$

In the following, we prefer the presentation of the theory in (4.19) since this will allow us to deal with the theories with different numbers of supersymmetries in a unified way.

The equations-of-motion following from (4.19) are

$$\mathscr{D}_m F_{nm} = 2g[\phi_a, \mathscr{D}_n\phi_a] + 2g\bar\sigma_n\{\lambda^A, \bar\lambda_A\} , \tag{4.22a}$$

$$\bar{\mathscr{D}}\!\!\!/\,\lambda^A = g\Sigma_a^{AB}[\phi_a, \bar\lambda_B] , \tag{4.22b}$$

$$\mathscr{D}\!\!\!/\,\bar\lambda_A = g\bar\Sigma_{aAB}[\phi_a, \lambda^B] , \tag{4.22c}$$

$$\mathscr{D}^2\phi_a = g^2[\phi_b, [\phi_b, \phi_a]] + g\bar\Sigma_{aAB}\lambda^A\lambda^B + g\Sigma_a^{AB}\bar\lambda_A\bar\lambda_B . \tag{4.22d}$$

The supersymmetry transformations in Euclidean space are given by (4.16a)–(4.16d) by replacing the sigma matrices with $-\mathrm{i}$ times their Euclidean space versions and by replacing Minkowski space inner products with Euclidean ones:

$$\delta A_n = \mathrm{i}\xi^A\sigma_n\bar\lambda_A + \mathrm{i}\bar\xi_A\bar\sigma_n\lambda^A , \tag{4.23a}$$

$$\delta\lambda^A = \mathrm{i}\sigma_{mn}\xi^A F_{mn} - \mathrm{i}g\Sigma_{ab}{}^A{}_B\xi^B[\phi_a, \phi_b] - \mathrm{i}\Sigma_a^{AB}\mathscr{D}\!\!\!/\,\phi_a\bar\xi_B , \tag{4.23b}$$

$$\delta\bar\lambda_A = \mathrm{i}\bar\sigma_{mn}\bar\xi_A F_{mn} - \mathrm{i}g\bar\Sigma_{abA}{}^B\bar\xi_B[\phi_a, \phi_b] - \mathrm{i}\bar\Sigma_{aAB}\mathscr{D}\!\!\!/\,\phi_a\xi^B , \tag{4.23c}$$

$$\delta\phi_a = \mathrm{i}\xi^A\bar\Sigma_{aAB}\lambda^B + \mathrm{i}\bar\xi_A\Sigma_a^{AB}\bar\lambda_B . \tag{4.23d}$$

4.2. The super-instanton at linear order

We will now attempt to find *super-instanton* configurations which solve the full coupled equations-of-motion (4.22a)–(4.22d). First notice that the original instanton solution of the pure gauge theory (2.49) is a solution of the full equations-of-motion when all other fields are set to zero. In fact, we can use $A_m(x;X)$ as a starting point to find the more general solutions where the fermion and scalar fields are non-vanishing. As explained in the introduction to this section we will proceed perturbatively order by order in the coupling. In this connection note the explicit powers of g appearing on the right-hand side of Eqs. (4.22a)–(4.22d).

The first step, following [38], is to expand to linear order in the fields around the bosonic instanton solution. To the next order, we must therefore solve the covariant Weyl equations

$$\bar{\mathscr{D}}\!\!\!/\,\lambda^A = 0 , \tag{4.24a}$$

$$\mathscr{D}\!\!\!/\,\bar\lambda_A = 0 \tag{4.24b}$$

for the fermions, and the covariant Laplace equation

$$\mathscr{D}^2\phi_a = 0 \tag{4.25}$$

for the scalars. It then remains to be seen whether the original instanton solution needs to be modified due the source term on the right-hand side of (4.22a).

A key result follows from the fact that $\mathscr{D}\!\!\!/$ has no zero modes in an instanton (rather than anti-instanton) background. Consequently, the solution to (4.24b) is $\bar\lambda_A = 0$. To prove this, (4.24b) implies $\bar{\mathscr{D}}\!\!\!/\,\mathscr{D}\!\!\!/\,\bar{\lambda}_A = 0$. But we can expand the product of operators using the definition of $\Delta^{(-)}$ in (2.28b), so

$$\bar{\mathscr{D}}\!\!\!/\,\mathscr{D}\!\!\!/\,\bar{\lambda}_A \equiv -\Delta^{(-)}\bar{\lambda}_A = \mathscr{D}^2\bar{\lambda}_A + \bar{\sigma}_{mn}F_{mn}\bar{\lambda}_A \tag{4.26}$$

and use the fact that in an instanton background F_{mn} is self-dual. Since $\bar{\sigma}_{mn}$ is a projector onto the anti-self-dual part (2.12), we have

$$\bar{\mathscr{D}}\!\!\!/\,\mathscr{D}\!\!\!/\,\bar{\lambda}_A = \mathscr{D}^2\bar{\lambda}_A = 0 \ . \tag{4.27}$$

But $\mathscr{D}^2 \equiv \mathscr{D}_n\mathscr{D}_n$ is a positive operator. This means that the only normalizable solution to $\mathscr{D}^2\bar{\lambda}_A = 0$ is $\bar{\lambda}_A = 0$ and consequently there are no zero modes for the anti-chiral fermions. The discussion above also implies, for vanishing VEVs at least, that the scalar fields also vanish at linear order.

On the contrary, for λ^A a similar maneuver to (4.26) gives

$$\mathscr{D}\!\!\!/\,\bar{\mathscr{D}}\!\!\!/\,\lambda^A \equiv -\Delta^{(+)}\lambda^A = \mathscr{D}^2\lambda^A + F_{mn}\sigma_{mn}\lambda^A = 0 \ . \tag{4.28}$$

In this case the second term does not vanish and so there is no reason for λ^A to vanish. In fact, we should have anticipated this since, as discussed in Section 2.5, the Atiyah–Singer index theorem dictates that the operator $\bar{\mathscr{D}}\!\!\!/$ has $2kN$ normalizable zero modes in the k instanton background for gauge group SU(N).

4.2.1. Adjoint fermion zero modes

In this section, we consider in more detail the adjoint-valued fermion zero modes in the background of the bosonic instanton solution; in other words, the solutions of the covariant Weyl equation (4.24a) in the ADHM background. Due to the linearity of the equation we can consider a single Weyl fermion λ with $\bar{\mathscr{D}}\!\!\!/\,\lambda = 0$. Fortunately, no additional work need be done since we have already solved this equation in Section 2.5 in the context of the gauge-field zero modes. We can immediately write down the solutions in terms of the linear functions defined in (2.107):

$$\lambda_\alpha = g^{-1/2}\Lambda_\alpha(\mathscr{M}) \equiv g^{-1/2}(\bar{U}\mathscr{M}f\bar{b}_\alpha U - \bar{U}b_\alpha f\bar{\mathscr{M}}U) \ . \tag{4.29}$$

(The unconventional power of $g^{-1/2}$ in the definition reflects the true g-scaling of the fermion zero modes, as will emerge in due course.) One difference from the gauge zero modes follows from the fact that λ is a Grassmann quantity; hence, $\mathscr{M}_{\lambda i}$ and $\bar{\mathscr{M}}_i^{\lambda}$ are constant $(N+2k)\times k$ and $k\times(N+2k)$ matrices of *Grassmann* collective coordinates, respectively, which replace the c-number quantities $C_{\dot\alpha}$ and $\bar{C}_{\dot\alpha}$ in (2.107). In addition, as indicated, the Grassmann collective coordinates do *not* carry the Weyl spinor index $\dot\alpha$.

In order for (4.29) to be a solution of (4.24a), the Grassmann collective coordinates must satisfy the constraints [18] (C.14)

$$\bar{\Delta}^{\dot\alpha}\mathscr{M} + \bar{\mathscr{M}}\Delta^{\dot\alpha} = 0 \ . \tag{4.30}$$

Expanding $\bar{\Delta}$ and Δ as in (2.45) and writing all the indices explicitly, this becomes

$$\bar{\mathscr{M}}_i^{\lambda}a_{\lambda j\dot\alpha} = -\bar{a}_{i\dot\alpha}^{\lambda}\mathscr{M}_{\lambda j} \ , \tag{4.31a}$$

$$\bar{\mathscr{M}}_i^{\lambda}b_{\lambda j}^{\alpha} = \bar{b}_i^{\alpha\lambda}\mathscr{M}_{\lambda j} \ . \tag{4.31b}$$

In a formal sense discussed in Ref. [20], these fermionic constraints are the "spin-$\frac{1}{2}$" superpartners of the original "spin-1" ADHM constraints (2.54a) and (2.54b), respectively. Note further that (4.31b) is easily solved when b is in canonical form (2.57). With the ADHM index decomposition $\lambda = u + \mathrm{i}\alpha$, we set

$$\mathcal{M}_{\lambda j} \equiv \mathcal{M}_{(u+i\alpha)j} = \begin{pmatrix} \mu_{uj} \\ (\mathcal{M}'_\alpha)_{ij} \end{pmatrix}, \quad \bar{\mathcal{M}}^\lambda_j \equiv \bar{\mathcal{M}}_{j(u+i\alpha)} = (\bar{\mu}_{ju} (\bar{\mathcal{M}}'^\alpha)_{ji}) \ . \tag{4.32}$$

Eq. (4.31b) then collapses to

$$\bar{\mathcal{M}}'_\alpha = \mathcal{M}'_\alpha \tag{4.33}$$

which allows us to eliminate $\bar{\mathcal{M}}'$ in favor of $\mathcal{M}'$. In terms of the variables μ, $\bar{\mu}$ and $\mathcal{M}'_\alpha$, the "fermionic ADHM constraints" (4.31a) are

$$\bar{\mathcal{M}} a_{\dot\alpha} + \bar{a}_{\dot\alpha} \mathcal{M} \equiv \bar{\mu} w_{\dot\alpha} + \bar{w}_{\dot\alpha} \mu + [\mathcal{M}^\alpha, a'_{\alpha\dot\alpha}] = 0 \ . \tag{4.34}$$

The quantities $\mathcal{M} = \{\mu, \bar{\mu}, \mathcal{M}'_\alpha\}$, subject to (4.34), are the Grassmann-valued partners of the ADHM variables $a_{\dot\alpha} = \{w_{\dot\alpha}, a'_n\}$. Later in Section 4.5 we shall see that they are, indeed, related by supersymmetry.

We can now count the number of independent fermion zero modes. There are $2k(N+k)$ independent Grassmann variables $\mathcal{M} = \{\mu, \bar{\mu}, \mathcal{M}'\}$, subject to $2k^2$ constraints (4.34). Hence, there are $2kN$ independent zero modes, in agreement with the Index Theorem.

4.2.2. Grassmann collective coordinates and the hyper-Kähler quotient construction

It is interesting to establish the relationship between these Grassmann collective coordinates and the hyper-Kähler quotient construction. A hyper-Kähler space of dimension $4n$ admits a preferred $\mathrm{SU}(2) \times \mathrm{Sp}(n)$ basis of tangent vectors. In particular, this leads to the notion of a *symplectic tangent vector* (B.30). With the relation between the ADHM data $a_{\dot\alpha}$ and the symplectic variables $z^{\tilde{\mathrm{i}}\dot\alpha}$ in (2.68), we can see that the fermionic ADHM constraints (4.34) are precisely the condition that the symplectic tangent vector to the mother space,

$$\mathcal{M}^{\tilde{\mathrm{i}}} = \begin{pmatrix} \bar{\mu}_{iu} \\ (\mathcal{M}'^1)_{ij} \\ \mu_{ui} \\ (\mathcal{M}'_1)_{ij} \end{pmatrix} , \tag{4.35}$$

is a symplectic tangent vector to the hyper-Kähler quotient space $\mathfrak{M}_k$. Here, $\tilde{\mathrm{i}}$ is the an index that, as earlier in Section 2.4.1, runs over $\{iu, ij, ui, ij\}$. In order to prove this we must show that the inner product

$$\mathcal{M}^{\tilde{\mathrm{i}}} \tilde{\Omega}_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}} X_r^{\tilde{\mathrm{j}}\dot\alpha} = -4\mathrm{i}\pi^2 \operatorname{tr}_k T^r (\bar{\mu} w^{\dot\alpha} + \bar{w}^{\dot\alpha} \mu + [\bar{a}'^{\dot\alpha\alpha}, \mathcal{M}'_\alpha]) \tag{4.36}$$

vanishes, where X_r are the vectors, defined in (2.69), that generate the $\mathrm{U}(k)$ action on $\tilde{\mathfrak{M}}$. It is easy to see that this condition is equivalent to the fermionic ADHM constraints (4.34). To summarize, we have shown that the Grassmann collective coordinates are Grassmann-valued symplectic tangent vectors to the instanton moduli space $\mathfrak{M}_k$.

The functional inner product of the fermion zero modes can be calculated using the same integral formula that we used to establish the functional inner product of the gauge zero modes in Section 2.5 and Appendix C (Eq. (C.20)). For two such zero modes

$$\int d^4x\, \mathrm{tr}_N\, \Lambda(\mathcal{M})\Lambda(\mathcal{N}) = -\frac{\pi^2}{2}\,\mathrm{tr}_k[\bar{\mathcal{M}}(\mathcal{P}_\infty + 1)\mathcal{N} + \bar{\mathcal{N}}(\mathcal{P}_\infty + 1)\mathcal{M}] \ . \tag{4.37}$$

Notice that the extra minus sign in (4.37) relative to (2.115) arises because of the Grassmann-valued nature of the collective coordinates. This is precisely the inner product of symplectic tangent vectors on $\tilde{\mathfrak{M}}$:

$$\tilde{\Omega}(\mathcal{M},\mathcal{N}) = -4\int d^4x\, \mathrm{tr}_N\, \Lambda(\mathcal{M})\Lambda(\mathcal{N}) \ . \tag{4.38}$$

Just as in the c-number sector, where we have $\{X^\mu\}$ as intrinsic coordinates on $\mathfrak{M}_k$, we can define intrinsic Grassmann-valued symplectic tangent vectors on $\mathfrak{M}_k$. To do this we solve the fermionic ADHM constraints $\mathcal{M}=\mathcal{M}(\psi,X)$. Since the fermionic ADHM constraints are linear in the Grassmann variables, the parameterization $\mathcal{M}(\psi,X)$ will be linear in the intrinsic Grassmann coordinates ψ^{i}, $\mathrm{i} = 1,\ldots,2kN$. In much the same way that the metric on $\mathfrak{M}_k$ is induced by that on $\tilde{\mathfrak{M}}$, the inner product of Grassmann-valued symplectic tangent vectors on $\tilde{\mathfrak{M}}$, (4.38), then induces a similar inner product on $\mathfrak{M}_k$. So for a pair of symplectic tangent vectors ψ and θ:

$$\Omega_{\mathrm{ij}}(X)\psi^{\mathrm{i}}\theta^{\mathrm{j}} = \tilde{\Omega}(\mathcal{M}(\psi,X),\mathcal{M}(\theta,X)) \ . \tag{4.39}$$

Here, $\Omega_{\mathrm{ij}}(X)$ is the symplectic matrix which appears in the expression for the metric on $\mathfrak{M}_k$ in (B.9).

4.2.3. Supersymmetric and superconformal zero modes

In Section 2.4.2, we described how those global symmetries of the classical equations-of-motion which act non-trivially on the instanton are represented on the moduli space. The symmetries described there—global gauge and conformal—will also have an action on the Grassmann collective coordinates. In addition, we have supersymmetry, enhanced to the superconformal group, that acts as symmetries of the classical equations-of-motion. These will act on the supersymmetrized moduli space [20,38,48].

A special set of the fermion zero modes can be identified with the action of supersymmetry transformations on the bosonic instanton solution. As is evident from (4.23b) and (4.23c), supersymmetry transformations on the purely bosonic instanton turn on the fermionic fields:

$$\lambda^A = \mathrm{i}\sigma_{mn}\xi^A F_{mn} \ , \tag{4.40a}$$

$$\bar{\lambda}_A = \mathrm{i}\bar{\sigma}_{mn}\bar{\xi}_A F_{mn} \ . \tag{4.40b}$$

Actually, to be more precise, in the bosonic instanton background $\bar{\sigma}_{mn}F_{mn} = 0$, so the anti-chiral fermions are not turned on. In particular, this means that the bosonic instanton is invariant under half the supersymmetries, namely the anti-chiral ones $\bar{\xi}_A$. Correspondingly the chiral supersymmetry transformations generate fermion zero modes. Using the expression for the field strength and fermion zero modes in the ADHM instanton background, Eqs. (2.52) and(4.29), respectively, we find

$$\lambda^A_\alpha = 4\mathrm{i}g^{-1/2}(\sigma_{mn}\xi^A)_\alpha \bar{U}b\sigma_{mn}\bar{b}fU$$
$$= -4\mathrm{i}g^{-1}\bar{U}(b\xi^A f\bar{b}_\alpha - b_\alpha f\xi^A\bar{b})U \equiv g^{-1/2}\Lambda_\alpha(-4ib\xi^A) \; . \tag{4.41}$$

Here, we have re-scaled

$$\xi^A \to g^{1/2}\xi^A \tag{4.42}$$

so that the following equations do not have g dependence. Consequently, the chiral supersymmetry transformations generate fermionic zero modes with the Grassmann collective coordinates

$$\mathscr{M}^A_{\lambda i} = -4\mathrm{i}\xi^A_\alpha b^\alpha_{\lambda i}, \quad \bar{\mathscr{M}}^{\lambda A}_i = -4\mathrm{i}\xi^{\alpha A}\bar{b}^\lambda_{\alpha i} \; . \tag{4.43}$$

These privileged fermion zero modes are known as the "supersymmetric zero modes". There are obviously two such modes for each species of fermion, hence the total number of supersymmetric modes is equal to half the number of component supercharges of the gauge theory, i.e. $2\mathscr{N}$.

In addition to these supersymmetric zero modes, there are fermionic zero modes corresponding to broken superconformal invariance. These transformations can be obtained by generalizing supersymmetry transformations (4.23a)–(4.23d) by making the parameters ξ^A local in a particular way:

$$\xi^A_\alpha(x) = \xi^A_\alpha - x_{\alpha\dot\alpha}\bar\eta^{\dot\alpha A}, \quad \bar\xi^\alpha_A(x) = \bar\xi^\alpha_A + \eta_{\dot\alpha A}\bar{x}^{\dot\alpha\alpha} \; . \tag{4.44}$$

This defines a basis of both supersymmetric and superconformal transformations generated by $\{\xi_A, \bar\xi^A\}$ and $\{\eta^A, \bar\eta_A\}$, respectively. The bosonic instanton breaks half the superconformal transformations, namely those generated by $\bar\eta^A$. These transformations generate fermion zero modes in much the same way as (4.41):

$$\lambda^A_\alpha = -4\mathrm{i}g^{-1/2}(\sigma_{mn}x\bar\eta^A)_\alpha\bar{U}b\sigma_{mn}\bar{b}fU = 4\mathrm{i}g^{-1/2}\bar{U}(bx\bar\eta^A f\bar{b}_\alpha - b_\alpha f\bar\eta^A\bar{x}\bar{b})U$$
$$= -4\mathrm{i}g^{-1/2}\bar{U}(a\bar\eta^A f\bar{b}_\alpha - b_\alpha f\bar\eta^A\bar{a})U \equiv g^{-1/2}\Lambda_\alpha(-4a\mathrm{i}\bar\eta^A) \; , \tag{4.45}$$

where in the second to last equality, we used ADHM relations (2.47) which imply $\bar{U}bx = -\bar{U}a$ and $\bar{x}\bar{b}U = -\bar{a}U$. Consequently, the associated Grassmann collective coordinates are

$$\mathscr{M}_{\lambda i} = -4\mathrm{i}a_{\lambda i\dot\alpha}\bar\eta^{\dot\alpha}, \quad \bar{\mathscr{M}}^\lambda_i = -4\mathrm{i}\bar\eta_{\dot\alpha}\bar{a}^{\dot\alpha\lambda}_i \; . \tag{4.46}$$

As for the supersymmetric modes, there are two such modes for each species of fermion, hence the total number of "superconformal modes" is equal again to $2\mathscr{N}$.

4.3. Going beyond linear order: the quasi-instanton

In order to evaluate correlation functions at their first non-vanishing order in the semi-classical expansion, we will find it necessary to go beyond the linear order analysis of the previous subsection. The systematics of the semi-classical expansion can be deduced from equations-of-motion (4.22a)–(4.22d) and the fact that the fermion zero modes λ^A are order $\mathscr{O}(g^{-1/2})$ and the gauge field is $\mathcal{O}(g^{-1})$. Schematically, the various fields have the following g-expansions:

$$\begin{aligned}
A_m &= g^{-1}A_m^{(0)} + gA_m^{(1)} + g^3A_m^{(2)} + \cdots ,\\
\lambda^A &= g^{-1/2}\lambda^{(0)A} + g^{3/2}\lambda^{(1)A} + g^{7/2}\lambda^{(2)A} + \cdots ,\\
\bar\lambda_A &= g^{1/2}\bar\lambda_A^{(0)} + g^{5/2}\bar\lambda_A^{(1)} + g^{9/2}\bar\lambda_A^{(2)} + \cdots ,\\
\phi_a &= g^0\phi_a^{(0)} + g^2\phi_a^{(1)} + g^4\phi_a^{(2)} + \cdots .
\end{aligned} \tag{4.47}$$

In the above,

$$A_m^{(0)} = \bar U \partial_m U, \qquad \lambda^{(0)A} = \Lambda(\mathcal{M}^A) . \tag{4.48}$$

Note in the $\mathcal{N}=1$ theory there are no scalar fields, hence the leading-order instanton solution,

$$A_m = g^{-1}A_m^{(0)}, \qquad \lambda^A = g^{-1/2}\lambda^{(0)A}, \qquad \bar\lambda_A = 0 , \tag{4.49}$$

is exact. We will also see shortly that in the absence of scalar VEVs the leading-order instanton configuration of the $\mathcal{N}=2$ theory,

$$A_m = g^{-1}A_m^{(0)}, \qquad \lambda^A = g^{-1/2}\lambda^{(0)A}, \qquad \bar\lambda_A = 0, \qquad \phi_a = g^0\phi_a^{(0)} , \tag{4.50}$$

is again an exact solution. This will no longer be the case when VEVs are turned on. The $\mathcal{N}=4$ case is more subtle, and we will treat it carefully.

When the expansions are substituted into the action, the latter is

$$S = \frac{8\pi^2 k}{g^2} + \mathrm{i}k\theta + \tilde S g^0 + \mathcal{O}(g^2) , \tag{4.51}$$

where we have defined

$$\tilde S = \int \mathrm{d}^4x\, \mathrm{tr}_N\{\mathcal{D}_n\phi_a^{(0)}\mathcal{D}_n\phi_a^{(0)} - \lambda^{(0)A}\bar\Sigma_{aAB}[\phi_a^{(0)}, \lambda^{(0)B}]\} , \tag{4.52}$$

which will play an important role in what follows. So if we wish to work to leading order in g, we only need to solve the equations-of-motion to order g^0.[22] From (4.51), it follows that we need the expression for the scalar field to order g^0. This is given by the solution of equation-of-motion (4.22d) with only the source bi-linear in the fermion zero modes included:

$$\mathcal{D}^2\phi_a = g\bar\Sigma_{aAB}\lambda^A\lambda^B . \tag{4.53}$$

Since λ^A is $\mathcal{O}(g^{-1/2})$ the solution is $\mathcal{O}(g^0)$. The solution to (4.53) is found in Appendix C (Eq. (C.25)) with arbitrary VEV. Setting the VEV to zero, the solution has the form [21,25]

$$\phi_a = -\frac{1}{4}\bar\Sigma_{aAB}\bar U\mathcal{M}^A f\bar{\mathcal{M}}^B U + \bar U \begin{pmatrix} 0_{[N]\times[N]} & 0_{[N]\times[2k]} \\ 0_{[2k]\times[N]} & \varphi_a 1_{[2]\times[2]} \end{pmatrix} U , \tag{4.54}$$

[22] The higher-order effects are, in any case, mixed non-trivially with the quantum effects from the fluctuations as we shall see in Section 5. The fluctuations actually also contribute determinant factors at $\mathcal{O}(g^0)$. However, since our theory is supersymmetric this is just a constant.

where the $k \times k$ matrices φ_a are

$$\varphi_a = \tfrac{1}{4}\,\bar{\Sigma}_{aAB}\boldsymbol{L}^{-1}(\bar{\mathcal{M}}^A\mathcal{M}^B)\ . \tag{4.55}$$

For the case of $\mathcal{N}=2$, one can see that (4.54) implies that the holomorphic field ϕ is non-trivial, while $\phi^\dagger$ remains zero. In other words, we can see that the lack of a reality condition on the fermions means that the scalar field in the instanton background also violates its reality condition. The same is true in the $\mathcal{N}=4$ theory. However, this violation of the reality condition only occurs if terms are quadratic in the Grassmann collective coordinates and so will not affect the convergence of the integrals over the fluctuations of the scalar field (which will still satisfy the usual reality condition). The fact that the reality properties of fields are violated in the instanton background by polynomial factors in the Grassmann collective coordinates will be a constant feature of the supersymmetric instanton calculus.

To evaluate correlation functions at leading semi-classical order, it turns out that we need go no further in iterating the equations-of-motion. However, it is instructive to go one step further by solving for the anti-chiral fermions at order $g^{1/2}$. At this order it appears from (4.22c) that the source term for the anti-chiral fermion turns on. However, for the $\mathcal{N}=2$ theories there is a major simplification compared with the $\mathcal{N}=4$ theories. From (4.53) we can see that only the components of ϕ_a which are non-trivial are of the form $\phi_a=\phi_{AB}\bar{\Sigma}_{aAB}$. But we recall from (4.14) for the $\mathcal{N}=2$ theories

$$\bar{\Sigma}_{aAB}=\varepsilon_{AB}\bar{\Sigma}_a, \qquad \bar{\Sigma}_a=(-\mathrm{i},1)\ . \tag{4.56}$$

Consequently

$$\mathcal{N}=2: \qquad \bar{\Sigma}_{aAB}\bar{\Sigma}_{aCD}=\varepsilon_{AB}\varepsilon_{CD}\bar{\Sigma}_a\bar{\Sigma}_a=0 \tag{4.57}$$

and so the source term for the anti-chiral fermions vanishes and the anti-chiral fermions remain zero. This should be contrasted with the situation in the $\mathcal{N}=4$ theories where one can show from the definition (4.15)

$$\mathcal{N}=4: \qquad \bar{\Sigma}_{aAB}\bar{\Sigma}_{aCD}=2\varepsilon_{ABCD}\ . \tag{4.58}$$

Hence, in these theories the source for $\bar{\lambda}_A$ is non-vanishing and the anti-chiral fermions are non-trivial.

Another way to see the difference between the $\mathcal{N}=2$ and 4 theories is to note that the former has an abelian factor in their R-symmetry group under which the fields have the following charges:

$$q(A_n)=0, \qquad q(\lambda)=1, \qquad q(\bar{\lambda})=-1, \qquad q(\phi)=2, \qquad q(\phi^\dagger)=-2\ . \tag{4.59}$$

In the instanton background, the charge 2 component ϕ is non-trivial, however, it is $\phi^\dagger$, the charge -2 component, that couples in the source for $\bar{\lambda}$. In the $\mathcal{N}=4$ case there is no abelian R-symmetry to provide a similar selection rule of this kind. The existence of the abelian R-symmetry in the $\mathcal{N}=2$ theory means that no other sources are turned on and so there is fully fledged supersymmetric generalization of the bosonic instanton solution which depends on $4kN$ Grassmann collective coordinates. The solution has A_n, λ^A and ϕ_a all non-trivial, where A_n is the original gauge field, λ^A are the zero modes of the Weyl equation and ϕ_a is solution (4.54). It is easy to verify that this "supersymmetric instanton" is degenerate with the original bosonic instanton and so has action $-2\pi\mathrm{i}\tau$. In this case, therefore, the instanton effective action $\tilde{S}$ in (4.51) vanishes, as do all terms higher order in g.

The picture in the $\mathcal{N}=4$ theory is much more subtle. Naïvely one would think that a supersymmetric instanton with $8kN$ Grassmann collective coordinates exists also in this case, albeit that it is much harder to find. We will argue that this is not the case. Remarkably, we will find that most of the Grassmann collective coordinates, exactly the $8kN-16$ of them, are not true collective coordinates in the sense of parameterizing a space of solutions of the classical equations-of-motion. In view of this, we shall call them *quasi-collective coordinates* and associate to them a *quasi-instanton*. They correspond to zero modes of the linearized system that are lifted by interactions. As a consequence, the action of the theory evaluated on instanton solution (4.51) actually depends non-trivially on the quasi-collective coordinate modes. The only "genuine" collective coordinates are those protected by symmetries of the theory. For the $\mathcal{N}=4$ theory these number 16 in total, corresponding to the supersymmetry and superconformal generators which, as explained in Section 4.2.3, act non-trivially on the instanton. These zero modes are protected by symmetries and cannot be lifted by interactions. Hence the exact instanton solution of the equations-of-motion of the $\mathcal{N}=4$ theory contains only 16 Grassmann collective coordinates. The expressions for the multiplet of fields of this solution can be found by acting on the bosonic instanton with a series of supersymmetry transformations (4.23a)–(4.23d), with the x-dependent Grassmann parameters ξ^A in (4.44) and with the re-scaling (4.42). Iterating this "sweeping-out" procedure to fourth order yields

$$\begin{aligned}
\lambda^A &= \mathrm{i}g^{1/2}\sigma_{mn}\xi^A(x)F_{mn}\ ,\\
\phi_a &= -\tfrac{1}{2}\,g\bar\Sigma_{aAB}\xi^A(x)\sigma_{mn}\xi^B(x)F_{mn}\ ,\\
\bar\lambda_A &= \tfrac{1}{3}\,\mathrm{i}g^{3/2}\varepsilon_{ABCD}(\bar{\mathscr{D}}\!\!\!/ F_{mn})\xi^B(x)[\xi^C(x)\sigma_{mn}\xi^D(x)]\ ,\\
A_m &= A_m - \tfrac{1}{6}\,g^2\varepsilon_{ABCD}[\xi^A(x)\sigma_{mn}\xi^B(x)][\xi^C(x)\sigma_{kl}\xi^D(x)]\mathscr{D}_nF_{kl}\ .
\end{aligned}\tag{4.60}$$

However, this exact instanton solution is not the most convenient starting point for the semi-classical approximation to the functional integral. Integrating over the quadratic fluctuations in the background of this exact solution, one would have to introduce the additional $8kN-16$ fermion zero modes of the Dirac operator. These modes will couple to the scalar field fluctuation and in order to lift them one would have to re-sum tree-level perturbative contributions in the instanton background. A much faster and more elegant way of addressing this problem is to modify the instanton background by including in it from the beginning all $8kN$ fermion zero modes. Thus, as in Ref. [21], we will always work with the quasi-instanton configuration in the $\mathcal{N}=4$ theory which is not an exact solution, but does not require perturbation theory to lift the quasi-zero modes.

The subtlety of the quasi-instanton of the $\mathcal{N}=4$ theory appears when we try to solve for the anti-chiral fermions (4.22c). In Appendix C (Eq. (C.34)), we show that the right-hand side of (4.22c) can be decomposed as

$$g\bar\Sigma_{aAB}[\phi_a,\lambda^B] = g^{1/2}(\bar{\mathscr{D}}\!\!\!/\bar\psi_A + \Lambda(\mathcal{N}_A))\ .\tag{4.61}$$

Here, $\mathcal{N}_A$ is a Grassmann odd function of the collective coordinates which satisfies the fermionic ADHM constraints (4.31a) and (4.31b) whose form is written down in Appendix C, Eq. (C.44).[23]

[23] Actually Appendix C considers the more general case when the scalar fields have VEVs.

This means that the second term in (4.61) is a zero mode of $\mathcal{D}\!\!\!/$. The expression for ψ_A in (4.61) is also given in Appendix C, Eq. (C.38). Note that the right-hand side of (4.61) has a component in the kernel of $\mathcal{D}\!\!\!/$. However this kernel is not in the image of $\mathcal{D}\!\!\!/$, in other words there is no $\bar\lambda_A$ such that $\mathcal{D}\!\!\!/\bar\lambda_A = g^{1/2}(\mathcal{D}\!\!\!/\bar\psi_A + \Xi_A)$. Hence it is impossible to solve this equation-of-motion. The best that can be done is to set

$$\bar\lambda_A = g^{1/2}\bar\psi_A\ , \tag{4.62}$$

which gives the leading order $\mathcal{O}(g^{1/2})$ expression for the anti-chiral fermion in the instanton background. In particular, from the expression for $\bar\psi_A$ (C.38), this solution contains a cubic dependence on the Grassmann collective coordinates.

We shall see in Section 5 that the failure to solve the anti-chiral fermion equation-of-motion is a symptom of the fact that, all but the 16 Grassmann collective coordinates associated to broken supersymmetric and superconformal invariance, are lifted by interactions and so are only *quasi*-collective coordinates. The leading-order effect is captured by the order g^0 term in action (4.52). Even though an exact supersymmetric instanton does not exist in the $\mathcal{N}=4$ case, dependent on all $4kN$ Grassmann collective coordinates, the approximate solution $\{A_m, \lambda^A, \phi_a\}$, the "quasi-instanton", is sufficient to capture semi-classical contributions to the leading order in g. The expression for the anti-chiral fermion is then only needed when one considers correlation functions with explicit insertions of $\bar\lambda_A$.

4.4. Scalar VEVs and constrained instantons

In this section, we examine instantons in cases where the scalar fields have non-vanishing VEVs. From the outset, we emphasized that instantons are a semi-classical phenomenon and therefore are only expected to describe the physics of these theories in a weakly coupled phase. The $\mathcal{N}=4$ theory has a weakly coupled regime, obtained simply by taking g—which does not run—to be arbitrarily small. In this case the theory is in a non-abelian Coulomb phase and the semi-classical approximation is reliable. However, the pure $\mathcal{N}=1$ and 2 theories (with vanishing VEVs in the latter) are strongly interacting. These theories can be rendered weakly coupled by breaking the gauge symmetry via the Higgs mechanism. Either the gauge symmetry is broken to an abelian subgroup, yielding a Coulomb phase, or it is broken completely, yielding a Higgs phase. In the $\mathcal{N}=1$ theory this can only be achieved by adding matter fields, a subject that we will pursue in Sections 6.3 and 7. The $\mathcal{N}=2$, like the $\mathcal{N}=4$, theory has adjoint-valued scalar fields, ϕ_a, which can develop a VEV driving the theory into an abelian Coulomb phase. By taking the VEV to be large, the theory is weakly coupled and semi-classical methods can be rigorously justified. As mentioned in the introduction to this section, scalar VEVs have an unfortunate side effect on instantons: strictly speaking they no longer exist! The way to resurrect them and make sense of the theory in the Coulomb or Higgs phase was worked out some time ago by Affleck [49] (see also [50]). Instantons are replaced by "constrained instantons" in a rather technically demanding formalism. But it turns out that the constrained instanton formalism is a paper tiger: working to lowest order in the semi-classical expansion with constrained instantons involves only a relatively mild generalization of the instanton calculus and, moreover, one which has a nice geometrical interpretation in the moduli space picture. Roughly speaking, instantons are no longer true minima of the action and a potential develops on $\mathfrak{M}_k$: instantons now have non-trivial action as well as entropy. In this sense constrained instanton are examples of the more general notion of a quasi-instanton that we have already seen in the $\mathcal{N}=4$ theory.

4.4.1. Constrained instantons on the Coulomb branch

The $\mathcal{N}=2$ and 4 theories contain two and six real adjoint-valued scalar fields, respectively. The classical potentials of these theories have flat directions along which these scalars develop VEVs breaking the gauge group to its maximal abelian subgroup by the adjoint Higgs mechanism. In the case of gauge group SU(N) the unbroken subgroup is $\mathrm{U}(1)^{N-1}$. Without loss of generality, we can simultaneously diagonalize all the components of the VEVs and we label the elements

$$\phi_a^0 = \mathrm{diag}((\phi_a^0)_1, (\phi_a^0)_2, \ldots, (\phi_a^0)_N) , \tag{4.63}$$

with $\sum_{u=1}^{N} (\phi_a^0)_u = 0$ for tracelessness. Here, and in the following, the superscript "0" on a scalar field denotes a VEV. Up to Weyl transformations, these VEVs parameterize a moduli space of inequivalent vacua known as the Coulomb branch.

After the Higgs mechanism has done its work, the diagonal components of all the fields remain massless whereas the off-diagonal components gain masses $|(\phi_a^0)_u - (\phi_a^0)_v|$. The $\mathcal{N}=2$ theory is asymptotically free with dynamical scale Λ. The running of the coupling is cut-off in the IR at a scale set by the masses of the off-diagonal components. Thus, as long as we choose VEVs such that $|(\phi_a^0)_u - (\phi_a^0)_v| \gg \Lambda$, for all $u \neq v$, the theory is weakly coupled at all length scales and semi-classical (instanton) methods should be reliable. In contrast, the coupling constant of the $\mathcal{N}=4$ theory does not run and we may achieve weak coupling simply by setting $g^2 \ll 1$, either on the Coulomb branch or at the conformal point where the non-abelian gauge symmetry is restored.

As we have mentioned in the introduction to this section, in the background of scalar VEVs, instantons are no longer exact solutions of the equations-of-motion due to Derrick's Theorem [51]. The action can always be lowered by shrinking an instanton, so the size of the configuration cannot be a genuine modulus when the VEVs are turned on. The way to implement the semi-classical approximation around instanton quasi-solutions in theories with symmetry breaking was developed by Affleck [49,50]. For simplicity, we review this approach in the context of a single BPST instanton of scale size ρ. The basic idea is to introduce a new operator, or "Affleck constraint", into the action by means of a Faddeev–Popov insertion of unity. If this operator is of suitably high dimension, Derrick's Theorem is avoided and instantons stabilize at a fixed scale size ρ. The integration over the Faddeev–Popov Lagrange multiplier can then be traded off for the integration over ρ. The now-stable solutions are known as *constrained* instantons. Of course, the detailed shape of the constrained instanton depends in a complicated way on one's choice of constraint, but certain important features remain constraint independent, namely:

(i) *The short-distance regime*, $x \ll 1/(g\phi^0)$.[24] In this regime the equations-of-motion can be solved perturbatively in $g^2\rho^2\phi^0$; since ultimately, as seen ex post facto in Section 5.2, the integration over scale size is dominated by $\rho \lesssim 1/\phi^0$ this is tantamount to perturbation theory in g. As the constraints do not enter into these equations until some high order, the first few terms in this expansion are robust. In particular, to leading order in the semi-classical approximation, the gauge fields and

[24] Here, ϕ^0 is the characteristic scale of the VEVs.

fermions are equal to their BPST expressions while the scalar fields undergo a minor modification to take account of the VEV.

(ii) *The long-distance regime*, $x \gg 1/(g\phi^0)$. The long-distance "tail" of the instanton reflects the Higgs mechanism. In the model at hand, the instanton component fields which gain a mass via the Higgs mechanism decay exponentially. In contrast, the diagonal components of the fields fall off as powers of ρ^2/x^2. It is an important assumption of the constrained instanton method that to leading order in the semi-classical approximation, the long-range behaviour of these massless fields is simply an extrapolation of the BPST core.[25] So for the massless components, the BPST form of the solution is all that one needs to discuss the instanton on all length scales (at leading order). However, even for the massive fields the BPST form often suffices, even though it is not correct at large distance. The reason is that the error made in using the BPST form rather than the actual exponential fall off is higher order in the coupling.

The most important conclusion of the constrained instanton method is that to leading order, small constrained instantons are well approximated by ordinary BPST instantons. But since these are the ones that are favoured by the now size-sensitive action of the constrained instanton, little error is made by replacing the constrained instanton by a conventional instanton. We will apply the same reasoning to the case of arbitrary topological charge to find that the core of the required constrained instanton is more or less the ADHM instanton that we have constructed: the gauge field assumes its ADHM form, (2.49), and the fermions λ^A are the linear combination of zero modes of $\not{\mathcal{D}}$, (4.29). The scalar field continues to obey (4.53) but now the boundary condition on ϕ_a is that it must approach the VEV ϕ_a^0 at large distance from the instanton. The general solution with VEV [25] is derived in Appendix C (Eq. (C.25))

$$\phi_a = -\frac{1}{4}\bar{\Sigma}_{aAB}\bar{U}\mathcal{M}^A f \bar{\mathcal{M}}^B U + \bar{U}\begin{pmatrix} \phi_a^0 & 0 \\ 0 & \varphi_a 1_{[2]\times[2]} \end{pmatrix} U \ , \tag{4.64}$$

with

$$\varphi_a = \boldsymbol{L}^{-1}(\tfrac{1}{4}\bar{\Sigma}_{aAB}\bar{\mathcal{M}}^A\mathcal{M}^B + \bar{w}^{\dot{\alpha}}\phi_a^0 w_{\dot{\alpha}}) \ . \tag{4.65}$$

That this has the requisite boundary condition (4.63) can be verified using the asymptotic formulae in Section 2.4.3.

As long as we work to leading order in g, the Affleck constraint does not explicitly appear and furthermore there is no need to iterate the instanton solution any further. Just as in the $\mathcal{N}=4$ theory with zero VEV, the quasi-instanton solution $\{A_m, \lambda^A, \phi_a\}$ is sufficient to capture the leading-order semi-classical approximation of the functional integral, as we describe in Section 5.

4.5. Collective coordinate supersymmetry

In Section 4.2.3, we saw that chiral supersymmetry transformations on the bosonic instanton generate fermion zero modes: these are the supersymmetries that are broken by the purely bosonic

[25] In principle, this simple patching of the short- and long-distance behaviour is modified at higher order in a way that is dependent on the precise form of the Affleck constraint.

instanton. Once, the fermion zero modes are turned on, the other half of the supersymmetry generators, the anti-chiral ones which left unbroken the bosonic instanton, now act non-trivially on the super-instanton. It turns out, as we uncover in this section, that the supersymmetry transformations on the super-instanton can be traded for supersymmetry transformations on the collective coordinates themselves. In fact the unbroken supersymmetries of the ADHM background are linearly realized on the bosonic and fermionic collective coordinates. This is an example of a general feature of BPS saturated solitons and corresponding higher-dimensional extended objects (i.e. branes). In all of these cases the unbroken supersymmetries are linearly realized in the world-volume theory. Invariance under these symmetries is an important constraint which must be satisfied by the super-instanton measure we will construct in the next section.

To find the transformations, we need to consider the supersymmetry variation of the gauge field A_m and fermions λ^A in the background of the non-linear supersymmetric instanton solution. By the latter, we mean the solution with A_m and λ^A equal to their ADHM forms, (2.49) and (4.29), and with the scalar fields given by (4.64). In the $\mathcal{N}=4$ case, or on the Coulomb branch of the $\mathcal{N}=2$ theory, this quasi-instanton configuration, as we have already described in Sections 4.3 and 4.4, is only an approximate solution to the equations-of-motion, but, nevertheless, provides a convenient way of capturing the leading-order semi-classical contribution to the functional integral. As a symptom of the non-exactness of the instanton solution in these cases, supersymmetry transformations on the fields, as well as transforming the collective coordinates, also turn on components of the fields at a higher order in the semi-classical expansion. We have already seen this phenomenon as the "sweeping-out" procedure that led to the expressions for ϕ_a, $\bar{\lambda}_A$ and A_m in (4.60).

We start by considering the supersymmetry variation of gauge field (4.23a). Since the anti-chiral fermions vanish, we have

$$\delta A_{\alpha\dot\alpha} = 2\mathrm{i}g^{-1/2}\bar{\xi}_{\dot\alpha A}\lambda^A_\alpha \equiv 2g^{-1}\Lambda_\alpha(\mathrm{i}\bar{\xi}_{\dot\alpha A}\mathcal{M}^A) \ . \tag{4.66}$$

Here, for convenience we have re-scaled

$$\bar{\xi}_A \to g^{-1/2}\bar{\xi}_A \ . \tag{4.67}$$

Comparing this with (2.110), the variation of the gauge field up to a local gauge transformation, under a variation of the c-number collective coordinates, we deduce the simple rule

$$\delta a_{\dot\alpha} = \mathrm{i}\bar{\xi}_{\dot\alpha A}\mathcal{M}^A, \qquad \delta\bar{a}^{\dot\alpha} = \mathrm{i}\bar{\xi}^{\dot\alpha}_A\bar{\mathcal{M}}^A \ . \tag{4.68}$$

This leaves the transformations of the Grassmann collective coordinates which are deduced by considering the variation of fermions (4.23b). We have already shown how the first term lifts to the collective coordinates in Section 4.2.3 but for consistency, and contrary to (4.67), we should re-scale

$$\xi^A \to g^{1/2}\xi^A \ . \tag{4.69}$$

The second term involves an expression which is higher order in the VEVs and Grassmann collective coordinates of $\mathcal{O}(g^{3/2})$. In other words this term does not contribute to $\delta\lambda^{(0)A}$ but rather the next term $\delta\lambda^{(1)A}$ in expansion (4.47). This is an example of the sweeping-out procedure turning on a higher-order term in the semi-classical expansion of a field. Note with vanishing VEVs and $\mathcal{N}<4$ supersymmetry, this term vanishes. The final term is $\mathcal{O}(g^{-1/2})$ and in Appendix C (Eq. (C.45)), we show how it lifts to a variation of the Grassmann collective coordinates. Putting the two variations, from the first and third terms, together, we have

$$\delta\mathscr{M}^A = -4\mathrm{i}\xi^A_\alpha b^\alpha + 2\mathrm{i}\Sigma^{AB}_a \mathscr{C}_{a\dot\alpha}\bar\xi^{\dot\alpha}_B, \qquad \delta\bar{\mathscr{M}}^A = -4\mathrm{i}\xi^{\alpha A}\bar b_\alpha + 2\mathrm{i}\Sigma^{AB}_a \bar\xi_{\dot\alpha B}\bar{\mathscr{C}}^{\dot\alpha}_a \ , \tag{4.70}$$

where

$$\mathscr{C}_{a\dot\alpha} = \begin{pmatrix} \phi^0_a & 0 \\ 0 & \varphi_a \end{pmatrix} a_{\dot\alpha} - a_{\dot\alpha}\varphi_a, \qquad \bar{\mathscr{C}}^{\dot\alpha}_a = \bar a^{\dot\alpha} \begin{pmatrix} \phi^0_a & 0 \\ 0 & \varphi_a \end{pmatrix} - \varphi_a \bar a^{\dot\alpha} \ , \tag{4.71}$$

where φ_a was defined in (4.65) in the context of the scalar field equation. Notice that with re-scaling (4.67) and (4.69), all g-dependence drops out of (4.70).

The anti-chiral supersymmetry transformations also turn on the anti-chiral fermions at $\mathcal{O}(g^{1/2})$:

$$\delta\bar\lambda_A = -\mathrm{i}g^{1/2}\Sigma^B_{abA}\bar\xi_B[\phi_a, \phi_b] \ . \tag{4.72}$$

Again, this is a symptom of the fact that the quasi-instanton is not an exact solution to the equation-of-motion.[26] Nevertheless, the supersymmetry transformations on the collective coordinates that we have derived in (4.68) and (4.70) will turn out to be symmetries of the leading-order semi-classical approximation of the functional integral.

5. The supersymmetric collective coordinate integral

This section is a companion to the previous one and hence we consider how the supersymmetric instanton is used to implement the semi-classical approximation of the functional integral in the context of a supersymmetric gauge theory. As we have already seen, there are extra subtleties in the supersymmetric case arising from the existence of quasi-zero modes. In this section we show how the procedure that we adopted to solve the equations-of-motion is precisely in accord with an expansion in g^2. First of all, in Section 5.1 we describe how the semi-classical approximation of the functional integral leads to a supersymmetrized version of the collective coordinate integral in which the Grassmann collective coordinates are also integrated over. We show that, even though the non-trivial parts of the fluctuation determinants cancel between bosonic and fermionic fluctuations, there is still, in general, a non-trivial integrand which arises as a consequence of quasi-zero modes. In fact the integrand involves the exponential of the "instanton effective action", a suitably supersymmetrized potential on the instanton moduli space. In the case of constrained instantons, the instanton effective action penalizes large instantons but, even when the VEVs vanish it is a non-trivial function of the Grassmann collective coordinates in the $\mathcal{N}=4$ theory. The instanton effective action is constructed and analysed in Section 5.2. In Section 5.3 we find an expression for the supersymmetric volume form on the instanton moduli space. As in the pure gauge theory, the hyper-Kähler quotient construction plays a central role here. Finally, in Section 5.3.1, we show how the supersymmetry of the underlying field theory is manifested on the collective coordinates.

[26] Except for $\mathcal{N} < 4$, with zero VEVs, in which case (4.72) vanishes.

5.1. The supersymmetric collective coordinate measure

In order to implement the semi-classical approximation we must expand around the instanton. In this section, we will initially consider the case where the VEVs vanish, so avoiding the complications of the constrained instanton. The gauge field is expanded as in (3.1) and (3.5) where we recall that the fluctuations are split into the zero mode piece and the component orthogonal to the zero modes, the latter being denoted $\tilde{A}_n$. The fermions are treated in a similar way: we separate out the zero modes of $\bar{\mathscr{D}}\!\!\!/$ and write the chiral fermion as

$$\lambda^A(x) = g^{-1/2}\lambda^{(0)A}(x; X, \psi^A) + \tilde{\lambda}^A(x; X, \psi) \ . \tag{5.1}$$

The $2kN$ Grassmann collective coordinates, for each species, are denoted by ψ^A. Since the anti-chiral fermions and scalar field have no zero modes in an instanton background we continue to denote them as $\bar{\lambda}_A$ and ϕ_a.

The functional integral over the fermions can be factorized into integrals over the Grassmann collective coordinates ψ^A and the non-zero mode fluctuations; schematically[27]

$$\int \prod_{A=1}^{\mathcal{N}} [\mathrm{d}\lambda^A][\mathrm{d}\bar{\lambda}_A] = g^{kN\mathcal{N}} \int \prod_{A=1}^{\mathcal{N}} \left\{ \frac{\prod_{\mathrm{i}=1}^{2kN} \mathrm{d}\psi^{\mathrm{i}A}}{\mathrm{Pfaff}\,\frac{1}{2}\,\Omega(X)} [\mathrm{d}\tilde{\lambda}^A][\mathrm{d}\bar{\lambda}_A] \right\} \ . \tag{5.2}$$

Here, $\Omega(X)$ is the anti-symmetric $2kN \times 2kN$ matrix defined by the functional inner product of the zero modes (4.39):

$$\int \mathrm{d}^4x\, \mathrm{tr}_N\, \lambda^{(0)}(x; X, \psi^A)\lambda^{(0)}(x; X, \psi^B) = -\tfrac{1}{4}\,\delta^{AB}\Omega_{\mathrm{ij}}(X)\psi^{\mathrm{i}A}\psi^{\mathrm{j}B} \tag{5.3}$$

and the Pfaffian ensures that the integral is invariant under re-parameterizations of the Grassmann collective coordinates.

We now substitute the expansion of the fields into the action of theory and obtain

$$S[g^{-1}A_m^{(0)} + \tilde{A}_m, g^{-1/2}\lambda^{(0)A} + \tilde{\lambda}^A, \bar{\lambda}^A, \phi_a] = -2\pi\mathrm{i}k\tau + S_{\mathrm{kin}} + S_{\mathrm{int}} \ . \tag{5.4}$$

Here, the second term denotes the kinetic terms for the non-zero mode fluctuations:[28]

$$S_{\mathrm{kin}} = \int \mathrm{d}^4x\, \mathrm{tr}_N \{ -\tfrac{1}{2}\,\tilde{\bar{A}}^{\dot{\alpha}\alpha} \Delta^{(+)}{}_{\alpha}{}^{\beta} \tilde{A}_{\beta\dot{\alpha}} - 2\mathscr{D}_n\bar{\lambda}_A\bar{\sigma}_n\tilde{\lambda}^A + \mathscr{D}_n\phi_a\mathscr{D}_n\phi_a \} \tag{5.5}$$

and the third term includes the interactions between the zero and non-zero modes:

$$S_{\mathrm{int}} = \int \mathrm{d}^4x\, \mathrm{tr}_N \{ -\lambda^{(0)A}\bar{\Sigma}_{aAB}[\phi_a, \lambda^{(0)B}] - 2g^{1/2}[\tilde{A}_n, \bar{\lambda}_A]\bar{\sigma}_n\lambda^{(0)A}$$
$$- 2g^{1/2}\lambda^{(0)A}\bar{\Sigma}_{aAB}[\phi_a, \tilde{\lambda}^B] + \cdots \} \ . \tag{5.6}$$

The remaining terms, whose presence is indicated by the ellipsis, are higher order in g. We have also not written down any terms involving the zero-mode piece of the gauge field because, as

[27] The factor of $g^{kN\mathcal{N}}$ arises from the factor of $g^{-1/2}$ included in the definition of each fermion zero modes in (4.29).

[28] In the following, as previously, all covariant derivatives are defined with respect to the bosonic instanton solution (2.49).

we saw in Section 3.1, once the integrals of the expansion parameters of the zero modes, the ξ^μ in (3.5), are traded for integrals over the collective coordinates X^μ, at leading order the ξ^μ are set to zero.

We can now integrate over the non-zero modes $\{\tilde{A}_m, \tilde{\lambda}^A, \bar{\tilde{\lambda}}^A, \phi_a\}$, and ghosts. This defines a kind of effective action on the collective coordinates $S_{\rm eff}$, which we call the *instanton effective action*:

$$e^{-S_{\rm eff}} \stackrel{\rm def}{=} e^{2\pi i k\tau} \int [d\tilde{A}]\,[db]\,[dc]\,[d\tilde{\lambda}]\,[d\bar{\tilde{\lambda}}]\,[d\phi]\, e^{-S_{\rm kin}-S_{\rm int}-S_{\rm gh}} \; . \tag{5.7}$$

The instanton effective action is only non-constant in the $\mathcal{N}=4$ theory (or more generally in the $\mathcal{N}=1$ and 2 theories with non-vanishing VEVs) when the supersymmetric instanton is not an exact solution to the equations-of-motion. We can think of (5.5) and (5.6) as specifying a set of Feynman rules which determines $S_{\rm eff}$ perturbatively in g. Only the non-zero mode pieces of the fields actually propagate and, in particular, the zero modes of the fermions $\lambda^{(0)A}$ are non-propagating and act as sources. When we integrate out the fluctuations, the second and third terms in (5.6) will not contribute at the leading g^0 order since they are order $g^{1/2}$; only the first term which is linear in the fluctuation of the scalar field is relevant. We can include the leading-order effect in an efficient way via a shift of the scalar field by $\phi_a^{(0)}$, the solution to the equation

$$\mathscr{D}^2\phi_a^{(0)} = \bar{\Sigma}_{aAB}\lambda^{(0)A}\lambda^{(0)B} \; . \tag{5.8}$$

This equation is identical to one which we solved in Section 4.3 and yields an expression for ϕ_a which is bi-linear in the Grassmann collective coordinates. So when working to leading order in g, it is convenient to think of the background configuration of the instanton as being the multiplet $\{A_m, \lambda^A, \phi_a\}$, defined in (2.49), (4.29) and (4.64). Notice that as we explained in the introduction to Section 4, the solution for the scalar field has terms bilinear in the Grassmann collective coordinates which violates the reality condition on the field: $\phi_a^\dagger$ is no longer the Hermitian conjugate of ϕ_a. However, as we explained, there is no inconsistency.

Once we have shifted the scalar field, the fluctuations can be integrated out. At leading order this yields the usual determinant factors. The gauge field and ghosts have already been dealt with in Section 3.1, while for the fermions we obtain

$$\mathrm{Pfaff}'\begin{pmatrix} 0 & \not{\mathscr{D}} \\ \not{\mathscr{D}} & 0 \end{pmatrix} \stackrel{\rm def}{=} \left|\det{}'\begin{pmatrix} 0 & \not{\mathscr{D}} \\ \not{\mathscr{D}} & 0 \end{pmatrix}^2\right|^{1/4} = |\det{}'\,\Delta^{(+)}\det\Delta^{(-)}|^{1/4} \; , \tag{5.9}$$

for each flavour of fermion and where $\Delta^{(\pm)}$ are defined in (2.28a) and (2.28b). In (5.9), the prime, as usual, indicates that $\Delta^{(+)}$ has zero modes and the determinant must be taken over the subspace orthogonal to the zero mode space. Finally, the integrals over the scalar field fluctuation $\tilde{\phi}_a$ simply give an additional factor of $[\det(-\mathscr{D}^2)]^{1-\mathcal{N}}$. Putting the determinant factors together we have

$$\left|\frac{\det(-\mathscr{D}^2)}{\det{}'\,\Delta^{(+)}}\right| |\det{}'\,\Delta^{(+)}|^{\mathcal{N}/4}|\det\Delta^{(-)}|^{\mathcal{N}/4}|\det(-\mathscr{D}^2)|^{1-\mathcal{N}} = \left|\frac{\det{}'\,\Delta^{(+)}}{\det\Delta^{(-)}}\right|^{\mathcal{N}/4-1} \; , \tag{5.10}$$

using (2.29) valid in an instanton background.

There are two leading order g^0 contributions to $S_{\rm eff}$. The first contribution arises from evaluating the action of the theory on the multiplet $\{A_m, \lambda^A, \phi_a\}$. This is simply the quantity $\tilde{S}$ defined in (4.52).

The second contribution comes from determinants (5.10). Putting these together we have

$$S_{\rm eff} = -2\pi i k\tau + \{\tilde{S} + (\mathcal{N}/4 - 1)\log[\det{}' \Delta^{(+)}/\det \Delta^{(-)}]\}g^0 + \mathcal{O}(g^2) \ . \tag{5.11}$$

As we described in Section 3.3, the ratio of determinants in (5.11) is simply a power of the Pauli–Villars mass scale:

$$\frac{\det{}' \Delta^{(+)}}{\det \Delta^{(-)}} = \mu^{-4Nk} \ . \tag{5.12}$$

This cancelling of the determinants, up to a constant factor, is perhaps the biggest simplification that occurs for instantons in a supersymmetric gauge theory [34]. Recall that $\tilde{S}$ is an expression quartic in the Grassmann collective coordinates. In the $\mathcal{N}=1$ theory, there are no scalar fields, so this term cannot appear. In the $\mathcal{N}=2$ theory with zero VEVs, this term also vanishes because the multiplet $\{A_m, \lambda^A, \phi_a\}$ is an exact solution to the equations-of-motion. With zero VEVs, it is only in the $\mathcal{N}=4$ theory that this contribution to $\tilde{S}$ is non-vanishing as we shall find by explicit evaluation in Section 5.2.

Including the bosonic parts of the functional integral as described in Section 3.1, the final expression for the leading-order semi-classical approximation of the functional integral in the charge-k sector is

$$\left(\frac{\mu}{g}\right)^{kN(4-\mathcal{N})} e^{2\pi i k\tau} \int \frac{\sqrt{\det g(X)}}{[\mathrm{Pfaff}\ \frac{1}{2}\ \Omega]^{\mathcal{N}}} \left\{\prod_{\mu=1}^{4kN} \frac{dX^\mu}{\sqrt{2\pi}} \prod_{A=1}^{\mathcal{N}} \prod_{i=1}^{2kN} d\psi^{iA}\right\} e^{-\tilde{S}(X,\psi)} \ . \tag{5.13}$$

So the final expression for the leading-order semi-classical approximation of the functional integral in the k-instanton sector is simply a multiple of the supersymmetrized volume form on the instanton moduli space $\mathfrak{M}_k$ with an integrand involving the instanton effective action $\tilde{S}$:

$$\int [dA]\,[d\lambda]\,[d\phi]\,[db]\,[dc]e^{-S}\Big|_{\text{charge-}k} \overset{g\to 0}{\to} \left(\frac{\mu}{g}\right)^{kN(4-\mathcal{N})} e^{2\pi i k\tau} \int_{\mathfrak{M}_k} \boldsymbol{\omega}^{(\mathcal{N})} e^{-\tilde{S}} \ , \tag{5.14}$$

where we have defined an $\mathcal{N}$ supersymmetric volume form on $\mathfrak{M}_k$:

$$\int_{\mathfrak{M}_k} \boldsymbol{\omega}^{(\mathcal{N})} \overset{\rm def}{=} \int \frac{\sqrt{\det g(X)}}{[\mathrm{Pfaff}\ \frac{1}{2}\ \Omega(X)]^{\mathcal{N}}} \prod_{\mu=1}^{4kN} \frac{dX^\mu}{\sqrt{2\pi}} \prod_{A=1}^{\mathcal{N}} \prod_{i=1}^{2kN} d\psi^{iA} \ . \tag{5.15}$$

We will refer to the quantity

$$\mathscr{Z}_k^{(\mathcal{N})} = \int_{\mathfrak{M}_k} \boldsymbol{\omega}^{(\mathcal{N})} e^{-\tilde{S}} \tag{5.16}$$

as the *instanton partition function* since it has the form of a partition function of a zero-dimensional field theory. Later in Section 10 we will see that for $\mathcal{N}>1$ it can be viewed as the dimensional reduction of the partition function of a higher-dimensional field theory. Specifically for $\mathcal{N}=2$, respectively $\mathcal{N}=4$, the field theory is a two-dimensional, respectively four-dimensional, σ-model with $\mathfrak{M}_k$ as the target space. This point of view leads very naturally to the relation of the instanton calculus to the dynamics of D-branes in string theory described in Section 10.3.

In the pure $\mathcal{N}=1$ and 2 theories, the pre-factors of the collective coordinates measures (5.14) can be related to the renormalization group invariant Λ-parameters. The point is that the coupling g must run with the Pauli–Villars mass scale in such a way that the combination in front of the measure is a renormalization group invariant. This defines the Λ-parameters in the Pauli–Villars scheme:

$$\Lambda^{3N}_{\mathcal{N}=1} = \mu^{3N} g(\mu)^{-2N} \mathrm{e}^{-8\pi^2/g(\mu)^2+\mathrm{i}\theta}, \qquad \Lambda^{2N}_{\mathcal{N}=2} = \mu^{2N} \mathrm{e}^{-8\pi^2/g(\mu)^2+\mathrm{i}\theta} . \tag{5.17}$$

(Notice that the powers of g in (5.14) and (5.17) do not match. The additional powers of g come from the insertions when one calculates correlations functions.) On the contrary, the $\mathcal{N}=4$ theory is finite, the coupling does not run and as a consequence integration measure (5.14) is independent of the Pauli–Villars mass scale μ.

5.2. The instanton effective action

In this section, we evaluate the leading-order contribution to the instanton effective action $\tilde{S}$ in (4.52). When the scalar fields have a non-trivial VEV, the semi-classical approximation proceeds via Affleck's constraint method as explained in Section 4.4. In this case, the scale sizes of instantons cease to be true moduli and the instanton effective action will be a non-trivial function on $\mathfrak{M}_k$ which breaks superconformal invariance.[29] However, the now *quasi*-collective coordinates are still to be integrated over in the semi-classical approximation of the functional integral. Furthermore, the Affleck constraint, at leading order, does not explicitly appear and further discussion of it is unnecessary. To leading order, the net effect of introducing the VEV is to change the boundary condition on ϕ_a, as indicated in (4.64) and this will feed into the instanton effective action $\tilde{S}$ in a way we now calculate.

With an integration by parts together with the scalar equation-of-motion, Eq. (4.52) may be re-cast as[30]

$$\tilde{S} = \int \mathrm{d}^4x \{\partial_n \,\mathrm{tr}_N(\phi_a \mathscr{D}_n \phi_a) - \tfrac{1}{2}\, g \,\mathrm{tr}_N\, \lambda^A \bar{\Sigma}_{aAB}[\phi_a, \lambda^B]\} . \tag{5.18}$$

The first term, being a total derivative may be converted to a surface integral over the sphere at infinity in spacetime. Since it is gauge invariant we can evaluate it in any convenient gauge. In particular, in singular gauge defined in Section 4.3, Eqs. (2.105) and (4.64) imply, in the limit of large x,

$$\frac{x_n}{x}\mathscr{D}_n\phi_a \overset{x\to\infty}{\to} \frac{1}{x^3}(\tfrac{1}{2}\bar{\Sigma}_{aAB}\mu^A\bar{\mu}^B + w_{\dot{\alpha}}\bar{w}^{\dot{\alpha}}\phi^0_a + \phi^0_a w_{\dot{\alpha}}\bar{w}^{\dot{\alpha}} - 2w_{\dot{\alpha}}\varphi_a\bar{w}^{\dot{\alpha}}) . \tag{5.19}$$

Hence, the first term in (5.18) is

$$4\pi^2\,\mathrm{tr}_k[\tfrac{1}{4}\bar{\Sigma}_{aAB}\bar{\mu}^A\phi^0_a\mu^B + \bar{w}^{\dot{\alpha}}\phi^0_a\phi^0_a w_{\dot{\alpha}} - \bar{w}^{\dot{\alpha}}\phi^0_a w_{\dot{\alpha}}\varphi_a] . \tag{5.20}$$

[29] In the $\mathcal{N}=4$ case, this is in addition to the non-trivial instanton effective action that is present even when the VEVs vanish.

[30] In the following $\{A_m, \lambda^A, \phi_a\}$ take ADHM expressions (2.49), (4.29) and (4.64).

The second term can be evaluated by using identity (C.34) in Appendix C:

$$\bar{\Sigma}_{aAB}[\phi_a, \Lambda(\mathcal{M}^B)] = \bar{\mathcal{D}}\bar{\psi}_A + \Lambda(\mathcal{N}_A) . \tag{5.21}$$

Then

$$\begin{aligned}
&-\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{tr}_N\,\Lambda(\mathcal{M}^A)(\bar{\mathcal{D}}\bar{\psi}_A + \Lambda(\mathcal{N}_A)) \\
&\quad = -\frac{1}{2}\int \mathrm{d}^4x(\partial_n\,\mathrm{tr}_N\,\Lambda(\mathcal{M}^A)\sigma_n\bar{\psi}_A + \mathrm{tr}_N\,\Lambda(\mathcal{M}^A)\Lambda(\mathcal{N}_A)) .
\end{aligned} \tag{5.22}$$

In the first term, we have used the fact $\bar{\mathcal{D}}\lambda^A = 0$ to pull the derivative outside the trace. One can verify that for large x, $\bar{\psi} \sim x^{-2}$ and $\lambda \sim x^{-3}$; hence, the first term on the right-hand side of (5.22) gives a vanishing contribution at infinity and may be dropped. The second term can be evaluated using the inner-product formula (4.37):

$$\begin{aligned}
-\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{tr}_N\,\Lambda(\mathcal{M}^A)\Lambda(\mathcal{N}_A) &= -\frac{\pi^2}{4}\,\mathrm{tr}_k[\bar{\mathcal{M}}^A(\mathcal{P}_\infty + 1)\mathcal{N}_A + \bar{\mathcal{N}}_A(\mathcal{P}_\infty + 1)\mathcal{M}^A] \\
&= \pi^2\bar{\Sigma}_{aAB}\,\mathrm{tr}_k[\bar{\mu}^A\phi_a^0\mu^B - \bar{\mathcal{M}}^A\mathcal{M}^B\varphi_a] ,
\end{aligned} \tag{5.23}$$

where we used the expression for $\mathcal{N}_A$ in (C.44) Appendix C:

$$\mathcal{N}_A = -\bar{\Sigma}_{aAB}\left\{\begin{pmatrix}\phi_a^0 & 0 \\ 0 & \varphi_a\end{pmatrix}\mathcal{M}^B - \mathcal{M}^B\varphi_a\right\} + 2\begin{pmatrix}0 & 0 \\ 0 & \mathcal{G}_A^{\dot{\alpha}}\end{pmatrix}a_{\dot{\alpha}} - 2a_{\dot{\alpha}}\mathcal{G}^{\dot{\alpha}} . \tag{5.24}$$

Here, $\mathcal{G}^{\dot{\alpha}A}$ is a quantity needed to ensure that $\mathcal{N}_A$ satisfies the fermionic ADHM constraints (4.34), but which, as one can show by explicit substitution, does not contribute to (5.23).

Assembling all the pieces, the leading-order contribution to the instanton effective action is

$$\begin{aligned}
\tilde{S} &= 4\pi^2\,\mathrm{tr}_k\{\tfrac{1}{2}\bar{\Sigma}_{aAB}\bar{\mu}^A\phi_a^0\mu^B + \bar{w}^{\dot{\alpha}}\phi_a^0\phi_a^0 w_{\dot{\alpha}} - \varphi_a\boldsymbol{L}\varphi_a\} \\
&= 4\pi^2\,\mathrm{tr}_k\{\tfrac{1}{2}\bar{\Sigma}_{aAB}\bar{\mu}^A\phi_a^0\mu^B + \bar{w}^{\dot{\alpha}}\phi_a^0\phi_a^0 w_{\dot{\alpha}} \\
&\quad - (\tfrac{1}{4}\bar{\Sigma}_{aAB}\bar{\mathcal{M}}^A\mathcal{M}^B + \bar{w}^{\dot{\alpha}}\phi_a^0 w_{\dot{\alpha}})\boldsymbol{L}^{-1}(\tfrac{1}{4}\bar{\Sigma}_{aCD}\bar{\mathcal{M}}^C\mathcal{M}^D + \bar{w}^{\dot{\beta}}\phi_a^0 w_{\dot{\beta}})\} .
\end{aligned} \tag{5.25}$$

One can verify explicitly that the expression above is invariant under supersymmetry transformations (4.68) and (4.70). Notice in the $\mathcal{N}=2$ theory when the VEVs vanish, $\tilde{S}=0$. However, in the $\mathcal{N}=4$ theory with vanishing VEVs, $\tilde{S}$ remains a non-trivial expression quartic in the Grassmann collective coordinates:

$$\tilde{S} = -\frac{\pi^2}{2}\varepsilon_{ABCD}\,\mathrm{tr}_k(\bar{\mathcal{M}}^A\mathcal{M}^B\boldsymbol{L}^{-1}\bar{\mathcal{M}}^C\mathcal{M}^D) . \tag{5.26}$$

This reflects the fact that in the $\mathcal{N}=4$ theory, the super-instanton is not an exact solution of the equations-of-motion.

5.2.1. Geometric interpretation

The terms in instanton effective (5.25) have an elegant interpretation in terms of the geometry of $\mathfrak{M}_k$. To start with, the quartic coupling is precisely the coupling of the Grassmann-valued symplectic tangent vectors $\mathscr{M}^A$ to the symplectic curvature of $\mathfrak{M}_k$:

$$\tilde{S} = \frac{1}{96}\,\varepsilon_{ABCD}R(\mathscr{M}^A,\mathscr{M}^B,\mathscr{M}^C,\mathscr{M}^D) = \frac{1}{96}\,\varepsilon_{ABCD}R_{\mathrm{ijkl}}\psi^{\mathrm{i}A}\psi^{\mathrm{j}B}\psi^{\mathrm{k}C}\psi^{\mathrm{l}D}\ . \tag{5.27}$$

The second expression is written in terms of the intrinsic coordinates ψ^A. Here, R is the symplectic curvature of the hyper-Kähler quotient space $\mathfrak{M}_k$. It is a totally symmetric tensor in the $\mathrm{Sp}(n)$ indices. In Appendix B we derive a general formula (B.48) for the symplectic curvature of a hyper-Kähler quotient in terms of the curvature and connection of the mother space $\tilde{\mathfrak{M}}$. Using relation (4.35) and the explicit representation of the generators of the $\mathrm{U}(k)$ symmetry in Eq. (2.69), we find

$$\mathscr{M}^{\tilde{\mathrm{i}}A}(\tilde{\Omega}T^r)_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}}\mathscr{M}^{\tilde{\mathrm{j}}B} \equiv -4\pi^2\mathrm{i}\,\mathrm{tr}_k\,T^r(\bar{\mathscr{M}}^A\mathscr{M}^B - \bar{\mathscr{M}}^B\mathscr{M}^A)\ . \tag{5.28}$$

Hence, using the formula for symplectic curvature (B.48) (in terms of the ADHM variables $\mathscr{M}^{\tilde{\mathrm{i}}A} = \mathscr{M}^{\tilde{\mathrm{i}}}(\psi^A, X)$) and the definition of $\boldsymbol{L}_{rs}$ in Eq. (2.124), we have

$$\begin{aligned}
&\frac{1}{96}\,\varepsilon_{ABCD}R_{\mathrm{ijkl}}\psi^{\mathrm{i}A}\psi^{\mathrm{j}B}\psi^{\mathrm{k}C}\psi^{\mathrm{l}D}\\
&\quad= \frac{1}{16}\,\varepsilon_{ABCD}\sum_{rs}[\mathscr{M}^{\tilde{\mathrm{i}}A}(\tilde{\Omega}T^r)_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}}\mathscr{M}^{\tilde{\mathrm{j}}B}]\boldsymbol{L}^{-1}_{rs}[\mathscr{M}^{\tilde{\mathrm{k}}C}(\tilde{\Omega}T^s)_{\tilde{\mathrm{k}}\tilde{\mathrm{l}}}\mathscr{M}^{\tilde{\mathrm{l}}D}]\\
&\quad= -\,\frac{\pi^2}{2}\,\varepsilon_{ABCD}\,\mathrm{tr}_k(\bar{\mathscr{M}}^A\mathscr{M}^B\boldsymbol{L}^{-1}\bar{\mathscr{M}}^C\mathscr{M}^D)\ ,
\end{aligned} \tag{5.29}$$

which is precisely (5.26).

The pieces of the instanton effective action which do not depend on the Grassmann collective coordinates also have a very concrete geometrical interpretation in terms of the hyper-Kähler quotient construction. The first observation is that the $\mathrm{SU}(N)$ gauge symmetry of the theory acts as a group of isometries on the hyper-Kähler quotient space. Firstly, consider the $\mathrm{SU}(N)$ isometries generated by the Killing vectors

$$\tilde{V} = \mathrm{i}\bar{w}^{\dot{\alpha}}_{iu}\frac{\partial}{\partial\bar{w}^{\dot{\alpha}}_{iv}} - \mathrm{i}w_{ui\dot{\alpha}}\frac{\partial}{\partial w_{iv\dot{\alpha}}} \tag{5.30}$$

on the mother space $\tilde{\mathfrak{M}}$. This group action descend to act as isometries on the quotient $\mathfrak{M}_k$. To see this, it suffices to notice that the moment maps are invariant: $\tilde{V}\mu^{X_r} = 0$. In fact the group action is also holomorphic with respect to each of the three independent complex structures, $\mathscr{L}_{\tilde{V}}\tilde{\boldsymbol{I}}^{(c)} = 0$, a property that can be easily shown to be inherited by the action on the quotient. Consequently the vector fields generated by the group action are *tri-holomorphic*.

Now consider a set of such isometries associated to $\mathrm{U}(1)^{N-1} \subset \mathrm{U}(N)$ picked out by the VEVs of the scalar field:

$$\tilde{V}_a = \mathrm{i}\bar{w}^{\dot{\alpha}}_{iu}(\phi^0_a)_u\frac{\partial}{\partial\bar{w}^{\dot{\alpha}}_{iu}} - \mathrm{i}(\phi^0_a)_u w_{ui\dot{\alpha}}\frac{\partial}{\partial w_{iu\dot{\alpha}}}\ . \tag{5.31}$$

This action, as described above is tri-holomorphic on the quotient space $\mathfrak{M}_k$. The lifts of the Killing vectors on the quotient $\mathfrak{M}_k$, V_a, are equal to projection of the $\tilde{V}_a$ to $\mathscr{H} \subset T\mathfrak{N}$. The bosonic parts of action (5.25) are then equal to a sum over the inner products of the vectors V_a:

$$4\pi^2 \operatorname{tr}_k[\bar{w}^{\dot\alpha}\phi_a^0\phi_a^0 w_{\dot\alpha} - \bar{w}^{\dot\alpha}\phi_a^0 w_{\dot\alpha}\boldsymbol{L}^{-1}\bar{w}^{\dot\alpha}\phi_a^0 w_{\dot\alpha}] = \tfrac{1}{2}\, g_{\mu\nu}(X)V_a^\mu V_a^\nu \ , \tag{5.32}$$

where g is the metric on the quotient space. As explained in Section 2.4.1 (and more fully in Appendix B) the metric on the quotient space $g(X,Y)$ is equal to $\tilde{g}(X,Y)$, the metric on the mother space evaluated on the lifts of X and Y. Hence,

$$g(V_a,V_a) = \tilde{g}(V_a,V_a) = \tilde{g}(\tilde{V}_a,\tilde{V}_a) - \tilde{g}(\tilde{V}_a^{\perp},\tilde{V}_a^{\perp}) \ , \tag{5.33}$$

where $\tilde{V}_a^{\perp}$ is the projection of $\tilde{V}_a$ to the vertical subspace $\mathscr{V}$. The $\mathrm{U}(1)^{N-1}$ group action on $\tilde{\mathfrak{M}}$ is tri-holomorphic, hence $\tilde{g}(\tilde{V}_a,\boldsymbol{I}^{(c)}X_r)=0$, so that $\tilde{V}_a$ already lies in the tangent space of the level set $T\mathfrak{N} \subset T\tilde{\mathfrak{M}}$. Since the vectors X_r, $r=1,\ldots,k^2$, form a basis for the vertical subspace,

$$\tilde{V}_a^{\perp} = \sum_{rs} X_r \boldsymbol{L}_{rs}^{-1}\tilde{g}(X_s,\tilde{V}_a) \tag{5.34}$$

and so

$$g(V_a,V_a) = \tilde{g}(\tilde{V}_a,\tilde{V}_a) - \sum_{rs}\tilde{g}(\tilde{V}_a,X_r)\boldsymbol{L}_{rs}^{-1}\tilde{g}(X_s,\tilde{V}_a) \ . \tag{5.35}$$

The matrix of inner products $\boldsymbol{L}_{rs}$ is given in (2.124). By explicit calculation we have

$$\tilde{g}(\tilde{V}_a,\tilde{V}_a) = 8\pi^2 \operatorname{tr}_k(\bar{w}^{\dot\alpha}\phi_a^0\phi_a^0 w_{\dot\alpha}), \qquad \tilde{g}(\tilde{V}_a,X_r) = 8\pi^2 \operatorname{tr}_k(T^r\bar{w}^{\dot\alpha}\phi_a^0 w_{\dot\alpha}) \ . \tag{5.36}$$

Substituting these expressions in (5.35) gives (5.32).

The remaining terms in (5.25) involving the Grassmann collective coordinates can also be given a geometric interpretation. First of all, we recall from Section 4.2.2 that, from a geometric perspective, the Grassmann collective coordinates are Grassmann-valued symplectic tangent vectors to the instanton moduli space. The action of the $\mathrm{U}(1)^{N-1}$ symmetry on the Grassmann collective coordinates generated by the vector fields $\tilde{V}_a$ in (5.31) is simply

$$\delta_a\mu_{ui}^A = \mathrm{i}(\phi_a^0)_u\mu_{ui}^A, \qquad \delta_a\bar{\mu}_{iu}^A = \mathrm{i}\bar{\mu}_{iu}^A(\phi_a^0)_u, \qquad \delta_a\mathscr{M}'^{A}_{ij\alpha} = 0 \ , \tag{5.37}$$

which defines $\delta_a\mathscr{M}^A$. This variation can be written in a completely geometric way using the (flat) connection on the mother space:

$$\delta_a\mathscr{M}^{\tilde{\mathrm{i}}A} = (\tilde{\nabla}_{\tilde{\mathrm{j}}\dot\alpha}\tilde{V}_a^{\tilde{\mathrm{i}}\dot\alpha})\mathscr{M}^{\tilde{\mathrm{j}}A} \ . \tag{5.38}$$

We now show that the Grassmann terms in the instanton effective action (5.25) can be interpreted in terms of the intrinsic geometry of $\mathfrak{M}_k$ as

$$2\pi^2\bar{\Sigma}_{aAB}\operatorname{tr}_k\{\bar{\mu}^A\phi_a^0\mu^B - \bar{w}^{\dot\alpha}\phi_a^0 w_{\dot\alpha}\boldsymbol{L}^{-1}\bar{\mathscr{M}}^A\mathscr{M}^B\} = \frac{\mathrm{i}}{4}\bar{\Sigma}_{aAB}\Omega_{\mathrm{ij}}(X)\psi^{\mathrm{i}A}(\nabla_{\mathrm{k}\dot\alpha}V_a^{\mathrm{j}\dot\alpha})\psi^{\mathrm{k}B} \ . \tag{5.39}$$

This follows by proving

$$\Omega_{\mathrm{ij}}(X)\psi^{\mathrm{i}A}(\nabla_{\mathrm{k}\dot\alpha}V_a^{\mathrm{j}\dot\alpha})\psi^{\mathrm{k}B} = \tilde{\Omega}_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}}\mathscr{M}^{\tilde{\mathrm{i}}A}(\tilde{\nabla}_{\tilde{\mathrm{k}}\dot\alpha}(\tilde{V}_a - \tilde{V}_a^{\perp})^{\tilde{\mathrm{j}}\dot\alpha})\mathscr{M}^{\tilde{\mathrm{k}}B} \tag{5.40}$$

and then, from (2.68), (2.69), (4.35) and (5.37),

$$\bar{\Sigma}_{aAB}\tilde{\Omega}_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}}\mathscr{M}^{\tilde{\mathrm{i}}A}(\tilde{\nabla}_{\tilde{\mathrm{k}}\dot{\alpha}}\tilde{V}_a^{\tilde{\mathrm{j}}\dot{\alpha}})\mathscr{M}^{\tilde{\mathrm{k}}B} = -8\mathrm{i}\pi^2\bar{\Sigma}_{aAB}\,\mathrm{tr}_k\,\bar{\mu}^A\phi_a^0\mu^B\ ,$$

$$\bar{\Sigma}_{aAB}\tilde{\Omega}_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}}\mathscr{M}^{\tilde{\mathrm{i}}A}(\tilde{\nabla}_{\tilde{\mathrm{k}}\dot{\alpha}}\tilde{V}_a^{\perp\tilde{\mathrm{j}}\dot{\alpha}})\mathscr{M}^{\tilde{\mathrm{k}}B} = -8\mathrm{i}\pi^2\bar{\Sigma}_{aAB}\,\mathrm{tr}_k\,\bar{w}^{\dot{\alpha}}\phi_a^0 w_{\dot{\alpha}}\boldsymbol{L}^{-1}\bar{\mathscr{M}}^A\mathscr{M}^B\ . \tag{5.41}$$

Putting this together with (5.36), and using (2.124), we have proved (5.39).

To summarize, the instanton effective action (5.25) can be written in an elegant way involving only the intrinsic geometry of $\mathfrak{M}_k$:

$$\tilde{S} = \frac{1}{2}\left\{g_{\mu\nu}(X)V_a^\mu V_a^\nu + \frac{\mathrm{i}}{2}\bar{\Sigma}_{aAB}\Omega_{\mathrm{ij}}(X)\psi^{\mathrm{i}A}(\nabla_{\mathrm{k}\dot{\alpha}}V_a^{\mathrm{j}\dot{\alpha}})\psi^{\mathrm{k}B} + \frac{1}{48}\varepsilon_{ABCD}R_{\mathrm{ijkl}}\psi^{\mathrm{i}A}\psi^{\mathrm{j}B}\psi^{\mathrm{k}C}\psi^{\mathrm{l}D}\right\}\ . \tag{5.42}$$

We will see in Section 10 that this effective action can be obtained from a non-trivial dimensional reduction of a σ-model in two and six dimensions, for $\mathscr{N}=2$ and 4, respectively, with the instanton moduli space $\mathfrak{M}_k$ as target.

5.2.2. The size of a constrained instanton

Having derived the leading-order expression for the instanton effective action, (5.25) or (5.42), we can, ex post facto, justify the main assumption of the constrained instanton method that there is an effective cut-off on large instanton sizes. First of all, since the bosonic part of the instanton effective action is simply (5.32), we can identify the minima of $\tilde{S}$ as fixed points of the $\mathrm{U}(1)^{N-1}$ action on the instanton moduli space $\mathfrak{M}_k$. Since the construction of $\mathfrak{M}_k$ involves a quotient by $\mathrm{U}(k)$, the fixed-point condition is

$$(\phi_a^0)_u w_{ui\dot{\alpha}} = w_{uj\dot{\alpha}}(\chi_a)_{ji}, \qquad [a'_n, \chi_a] = 0\ , \tag{5.43}$$

in addition to ADHM constraints (2.65). Here, χ_a are $k\times k$ Hermitian matrices acting as infinitesimal compensating transformation in the auxiliary group $\mathrm{U}(k)$ which can depend on the VEVs ϕ_a^0. For generic values of the VEVs, the solutions of Eqs. (5.43) are as follows. At least until we impose the ADHM constraints, each fixed-point set is associated to the partition

$$k \to k_1 + k_2 + \cdots + k_N\ , \tag{5.44}$$

up to the $\mathrm{U}(k)$ auxiliary symmetry. Each $i\in\{1,2,\ldots,k\}$ is then associated to a given u by a map u_i as follows:

$$\Big\{\underbrace{1,2,\ldots,k_1}_{u=1},\underbrace{k_1+1,\ldots,k_1+k_2}_{u=2},\ldots,$$

$$\ldots,\underbrace{k_1+\cdots+k_{u-1}+1,\ldots,k_1+\cdots+k_u}_{u},\ldots,\ldots,\underbrace{k_1+\cdots+k_{N-1}+1,\ldots,k}_{u=N}\Big\}\ . \tag{5.45}$$

For a given partition the variables have a block-diagonal form

$$(\chi_a)_{ij} = -(\phi_a^0)_{u_i}\delta_{ij}, \qquad w_{ui\dot{\alpha}} \propto \delta_{uu_i}, \qquad (a'_n)_{ij} \propto \delta_{u_iu_j}\ . \tag{5.46}$$

Now we impose ADHM constraints (2.65). In the uth block, the constraints are of the form of a set of ADHM constraints for k_u instantons for which $N=1$. In fact, the N blocks each correspond to the N U(1) subgroups of the gauge group picked out by the VEVs.[31] Taking the trace of the ADHM constraints within the block removes the a'_n dependent terms to leave equations of the form

$$\vec{\tau}^{\dot\alpha}{}_{\dot\beta} \sum_{i=k_{u-1}+1}^{k_u} \bar{w}^{\dot\beta}_{iu} w_{ui\dot\alpha} = 0 \ . \tag{5.47}$$

However, the solution to these equations is $w_{ui\dot\alpha}=0$. Therefore the structure of the partitions collapses to leave, up to the U(k) symmetry, a'_n and χ_a diagonal:

$$w_{ui\dot\alpha} = 0, \qquad a'_n = -\mathrm{diag}(X^1_n, \ldots, X^k_n) \ . \tag{5.48}$$

Taking the solution above fixes all of the auxiliary symmetry apart from permutations; hence, the fixed-point space is simply $\mathrm{Sym}^k\,\mathbb{R}^4$, the symmetric product of k points in $\mathbb{R}^4$. So the fixed-point set describes a configuration where all the instantons have shrunk down to zero size. It is easy to verify that the gauge potential on this singular subspace is pure gauge. As the sizes of the instantons grow, the effective action favours instantons with sizes up to a scale $\sim 1/\phi^0$, where ϕ^0 is the characteristic scale of the VEVs, after which they are exponentially suppressed. Therefore, constrained instantons have a natural cut-off on their scale size.

5.2.3. The lifting of zero modes

Of paramount importance for applications is the process of lifting fermion zero modes encoded in the instanton effective action. First of all, consider the case with vanishing VEVs. In this case, the only genuinely weakly coupled scenario is the $\mathcal{N}=4$ theory. As we have emphasized in Section 4.3, only the eight supersymmetric and eight superconformal zero modes are exact, the remaining $8kN-16$ are lifted by interactions and consequently we expect the instanton effective action $\tilde{S}$ to depend on all the Grassmann collective coordinates except (4.43) and (4.46). It is easily verified from (5.26) (using the fermionic ADHM constraints (4.34)) that the latter 16 variables decouple from $\tilde{S}$ as expected. There is another way to phrase this result. The decoupling of the supersymmetric and superconformal Grassmann collective coordinates implies that the symplectic curvature of $\mathfrak{M}_k$ admits four null eigenvectors:

$$R_{\mathrm{ijkl}}\ell^{\mathrm{i}} = 0 \ . \tag{5.49}$$

The fact that the symplectic curvature has four null eigenvectors implies that the holonomy group of $\mathfrak{M}_k$ is reduced from Sp(kN) to Sp($kN-2$) (or for $\hat{\mathfrak{M}}_k$ from Sp($kN-1$) to Sp($kN-2$)) [28,52].

When VEVs are turned on, it is easy to see that the instanton effective action now lifts the superconformal zero modes via the first term in (5.25), as one expects since conformal invariance is broken. Of course, the supersymmetric zero modes remain unlifted because the introduction of VEVs does not break supersymmetry.

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[31] Actually this statement is not quite correct because we are taking a gauge group SU(N). However, the ADHM construction actually yields a U(N) gauge potential, where the abelian part of the gauge group is pure gauge: in this sense the statement is correct.

5.3. The supersymmetric volume form on $\mathfrak{M}_k$

In this section, we show how to construct a volume form on the space of collective coordinates of a supersymmetric instanton, generalizing result (3.17) in the pure gauge theory. We shall adopt an approach based on the hyper-Kähler quotient construction, but we shall find a result that is identical with the original approach of Refs. [20,30,31] which relied on various consistency conditions, principally supersymmetry and clustering, in order to construct the volume form.

We have already seen how the Grassmann collective coordinates arise in the context of the hyper-Kähler quotient construction. The quantities $\{\mu, \bar{\mu}, \mathcal{M}'_\alpha\}$ can be arranged as in (4.35) to get an n-vector $\mathcal{M}^{\tilde{\mathrm{i}}}$ ($n = 2k(N+k)$). The fermionic ADHM constraints are precisely the conditions for $\mathcal{M}^{\tilde{\mathrm{i}}}$ to be a symplectic tangent vector to the hyper-Kähler quotient space. Using this fact it is straightforward to write down the integration measure for the Grassmann collective coordinates. As for the volume form itself, the integration measure on the quotient $\mathfrak{M}_k$ is induced from that on the mother space $\tilde{\mathfrak{M}}$. The covariant expression for this latter quantity is

$$\int \frac{\prod_{\tilde{\mathrm{i}}=1}^{2n} \mathrm{d}\mathcal{M}^{\tilde{\mathrm{i}}}}{\mathrm{Pfaff}\,\tilde{\Omega}} \,, \tag{5.50}$$

where $\tilde{\Omega}_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}}$ is the anti-symmetric symplectic tensor on mother space (2.68). We restrict to the quotient space by inserting explicit Grassmann-valued δ-functions to impose the symplectic tangent vector condition (B.30). These δ-functions must be accompanied by a suitable Jacobian giving

$$\int \frac{\prod_{\tilde{\mathrm{i}}=1}^{2n} \mathrm{d}\mathcal{M}^{\tilde{\mathrm{i}}}}{\mathrm{Pfaff}\,\tilde{\Omega}} \frac{1}{J_f} \prod_{r=1}^{\dim G} \prod_{\dot{\alpha}=1}^{2} \delta(\mathcal{M}^{\tilde{\mathrm{i}}} \tilde{\Omega}_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}} X_r^{\tilde{\mathrm{j}}\dot{\alpha}}) \,. \tag{5.51}$$

The Jacobian J_f is related to the determinant of $\boldsymbol{L}$, (2.124), the matrix of inner products of the normal vectors to the quotient space:[32]

$$J_f = |\det{}_{k^2} \boldsymbol{L}| \,. \tag{5.52}$$

For the particular quotient that yields the ADHM construction $\tilde{\mathfrak{M}}$ is flat, and Pfaff $\tilde{\Omega}$ is simply a numerical factor and we can write the measure for each species of fermion as

$$\int \mathrm{d}^{2k(N+k)}\mathcal{M} |\det{}_{k^2} \boldsymbol{L}|^{-1} \prod_{r=1}^{k^2} \prod_{\dot{\alpha}=1}^{2} \delta(\mathrm{tr}_k\, T^r(\bar{\mathcal{M}} a_{\dot{\alpha}} + \bar{a}_{\dot{\alpha}} \mathcal{M})) \,, \tag{5.53}$$

where[33]

$$\int \mathrm{d}^{2k(N+k)}\mathcal{M} \stackrel{\mathrm{def}}{=} \int \prod_{r=1}^{k^2} \mathrm{d}^2(\mathcal{M})^r \prod_{i=1}^{k} \prod_{u=1}^{N} \mathrm{d}\bar{\mu}_{iu}\, \mathrm{d}\mu_{ui} \,. \tag{5.54}$$

[32] The following can be deduced from Eq. (5.79).

[33] Our convention for integrating a two-component spinor ψ_α is $\int \mathrm{d}^2\psi \equiv \int \mathrm{d}\psi_1\, \mathrm{d}\psi_2$.

Putting this together with volume form for $\mathfrak{M}_k$, (3.17), the collective coordinate volume form for an arbitrary supersymmetric theory is

$$\int_{\mathfrak{M}_k} \boldsymbol{\omega}^{(\mathcal{N})} = \frac{C_k^{(\mathcal{N})}}{\mathrm{Vol}\,\mathrm{U}(k)} \int \mathrm{d}^{4k(N+k)}a \prod_{A=1}^{\mathcal{N}} \mathrm{d}^{2k(N+k)}\mathcal{M}^A |\det{}_{k^2}\boldsymbol{L}|^{1-\mathcal{N}}$$
$$\times \prod_{r=1}^{k^2}\left\{\prod_{c=1}^{3}\delta\left(\tfrac{1}{2}\,\mathrm{tr}_k\,T^r(\tau^{c\dot\alpha}{}_{\dot\beta}\bar a^{\dot\beta}a_{\dot\alpha})\right)\prod_{A=1}^{\mathcal{N}}\prod_{\dot\alpha=1}^{2}\delta(\mathrm{tr}_k\,T^r(\bar{\mathcal{M}}^A a_{\dot\alpha}+\bar a_{\dot\alpha}\mathcal{M}^A))\right\}\,. \tag{5.55}$$

The normalization factor $C_k^{(\mathcal{N})}$ can be determined by a careful analysis of the inner products of the zero modes:

$$C_k^{(\mathcal{N})} = 2^{-k(k-1)/2+kN(2-\mathcal{N})}\pi^{2kN(1-\mathcal{N})}\,. \tag{5.56}$$

An independent check of the normalization constants $C_k^{(\mathcal{N})}$ can be achieved, as in Section 3.2.1, by invoking the clustering property of the instanton integration measure. In order to apply this argument we have to consider how the Grassmann integrals behave in the complete clustering limit. In this limit, the off-diagonal components of the fermionic ADHM constraints are dominated by the term

$$(\bar X^i - \bar X^j)^{\dot\alpha\alpha}\mathcal{M}'_\alpha + \cdots\,. \tag{5.57}$$

We can therefore use the off-diagonal constraints to saturate the $(\mathcal{M}'_\alpha)_{ij}$, $i\neq j$, integrals. This yields a factor of

$$\prod_{i\neq j}(X^i - X^j)^2 \tag{5.58}$$

for each species of fermion. However, each species of fermion is accompanied by a factor of $|\det_{k^2}\boldsymbol{L}|^{-1}$ which clusters as (3.24). Hence, what remains in the complete clustering limit are the diagonal fermionic ADHM constraints which to leading order are the fermionic ADHM of the individual instantons. Hence, taking into account the clustering of the purely bosonic parts of the volume form, as in Section 3.2.1, we find that the supersymmetric measures cluster consistently.

5.3.1. Supersymmetry

In this section, we will verify that the volume form equation (5.55) is invariant under the supersymmetry transformations acting on the collective coordinates that we established in Section 4.5.

To begin with, consider the supersymmetry variations of the bosonic and fermionic ADHM constraints, (2.65) and (4.34), since these appear as the argument of the δ-functions in (5.55). First, the c-number ADHM constraints (2.65):

$$\delta(\vec\tau^{\dot\beta}{}_{\dot\alpha}\bar a^{\dot\alpha}a_{\dot\beta}) = \vec\tau^{\dot\beta}{}_{\dot\alpha}(-\mathrm{i}\bar\xi^{\dot\alpha}_A\bar{\mathcal{M}}^A a_{\dot\beta} + \mathrm{i}\bar\xi_{\dot\beta A}\bar a^{\dot\alpha}\mathcal{M}^A) = -\mathrm{i}\vec\tau^{\dot\beta}{}_{\dot\alpha}\xi^{\dot\alpha}_A(\bar{\mathcal{M}}^A a_{\dot\beta} + \bar a_{\dot\beta}\mathcal{M}^A)\,. \tag{5.59}$$

In other words, the bosonic ADHM constraints transform into a linear combination of the fermionic ADHM constraints (4.34). Now the fermionic ADHM constraints themselves:

$$\delta(\bar{\mathcal{M}}^A a_{\dot\alpha} + \bar a_{\dot\alpha}\mathcal{M}^A) = 4\mathrm{i}\xi^A_\alpha(b^\alpha a_{\dot\alpha} - \bar a_{\dot\alpha}b^\alpha) + \mathrm{i}\bar\xi_{\dot\alpha B}(\bar{\mathcal{M}}^A\mathcal{M}^B - \bar{\mathcal{M}}^B\mathcal{M}^A) + 2\mathrm{i}\bar\xi_{\dot\beta B}\Sigma_a^{AB}(\bar{\mathcal{C}}^{\dot\beta}_a a_{\dot\alpha} - \bar a_{\dot\alpha}\mathcal{C}^{\dot\beta}_a)$$
$$= \mathrm{i}\bar\xi_{\dot\alpha B}(\bar{\mathcal{M}}^A\mathcal{M}^B - \bar{\mathcal{M}}^B\mathcal{M}^A - 2\Sigma_a^{AB}\bar w^{\dot\alpha}\phi^0_a w_{\dot\alpha} + 2\Sigma_a^{AB}\boldsymbol{L}\varphi_a)\,. \tag{5.60}$$

To prove this, we used (2.54b) to show that first bracket on the right-hand side of (5.60) vanishes. Going from the first line to the second line involved using the bosonic ADHM constraints (2.65) along with the definition of $\boldsymbol{L}$ in Eq. (2.125). The final expression then vanishes by virtue of the definition of φ_a in Eq. (4.65) along with a Σ-matrix identity (A.20).

The fact that the variation of the bosonic (fermionic) ADHM constraints involves the fermionic (bosonic) ADHM constraints looks promising because it means that the variation of the product of δ-functions in (5.55) vanishes to linear order. For the $\mathcal{N}=1$ measure, this is sufficient to prove supersymmetric invariance. The reason is that the transformation of $a_{\dot\alpha}$ involves $\mathcal{M}$ but that of $\mathcal{M}$ is a constant. Hence, the super-Jacobian for the transformation of $\{a_{\dot\alpha}, \mathcal{M}\}$ vanishes at linear order. Since there are no other contributions to consider, the $\mathcal{N}=1$ measure is a supersymmetric invariant.

However, the story is more involved with extended supersymmetry. The reason is that, contrary to the $\mathcal{N}=1$ case, the variation of the Grassmann collective coordinates (4.70) involves the Grassmann collective coordinates through the dependence of $\mathscr{C}_a^{\dot\alpha}$ and $\bar{\mathscr{C}}_a^{\dot\alpha}$ on φ_a, (4.71), which in turn depends on $\mathcal{M}^A$ via (4.65). This means that at linear order there is non-trivial super-Jacobian for the transformation of $\{a_{\dot\alpha}, \mathcal{M}^A\}$. In order to prove supersymmetric invariance, this super-Jacobian must cancel the transformation of the remaining factor $|\det_{k^2}\boldsymbol{L}|^{1-\mathcal{N}}$ that we have not, hitherto, considered.

Rather than evaluate the variations of these two quantities, and then show that they cancel, we will proceed in a more indirect fashion. The idea is to remove the dependence of $\delta\mathcal{M}^A$ on the Grassmann collective coordinates by introducing some auxiliary variables. The obvious candidate is φ_a itself. In order to implement this idea, we need to find out how φ_a transforms. We can answer this by considering the variation of definition (4.65). Writing this as

$$\boldsymbol{L}\cdot\varphi_a = \tfrac{1}{4}\,\bar\Sigma_{aAB}\bar{\mathcal{M}}^A\mathcal{M}^B + \bar w^{\dot\alpha}\phi_a^0 w_{\dot\alpha}\ , \tag{5.61}$$

the variation of the left-hand side is the sum of $\boldsymbol{L}\cdot\delta\varphi_a$ and

$$\begin{aligned}(\delta\boldsymbol{L})\cdot\varphi_a &= \frac{\mathrm{i}}{2}\,\bar\xi_A^{\dot\alpha}(\{\bar\mu^A w_{\dot\alpha} - \bar w_{\dot\alpha}\mu^A + \mathcal{M}'^{\alpha A}a'_{\alpha\dot\alpha} + a'_{\alpha\dot\alpha}\mathcal{M}'^{A}, \varphi_a\} - 2\mathcal{M}'^{\alpha A}\varphi_a a'_{\alpha\dot\alpha} - 2a'_{\alpha\dot\alpha}\varphi_a\mathcal{M}'^{\alpha A})\\ &= \mathrm{i}\bar\xi_A^{\dot\alpha}(\bar\mu^A w_{\dot\alpha}\varphi_a + [\mathcal{M}'^{\alpha A}a'_{\alpha\dot\alpha}, \varphi_a] - \varphi_a\bar w_{\dot\alpha}\mu^A + [\varphi_a, a'_{\alpha\dot\alpha}\mathcal{M}'^{\alpha A}])\ ,\end{aligned} \tag{5.62}$$

using the definition of $\boldsymbol{L}$ in (2.125). The last equality follows by using the fermionic ADHM constraints (4.34). For the variation of the right-hand side of (5.61), one finds

$$\frac{\mathrm{i}}{2}\,\bar\Sigma_{aBC}\Sigma_b^{CA}\bar\xi_A^{\dot\alpha}(\bar{\mathscr{C}}_{b\dot\alpha}\mathcal{M}^B - \bar{\mathcal{M}}^B\mathscr{C}_{b\dot\alpha}) + \mathrm{i}\bar\xi_A^{\dot\alpha}(\bar\mu^A\phi_a^0 w_{\dot\alpha} - \bar w_{\dot\alpha}\phi_a^0\mu^A)\ . \tag{5.63}$$

Taking the difference of (5.62) and (5.63), one deduces the variation of φ_a:

$$\boldsymbol{L}\cdot\delta\varphi_a = \mathrm{i}\{\tfrac{1}{2}\,\bar\Sigma_{aBC}\Sigma_b^{CA} - \delta_{ab}\delta_B^A\}\bar\xi_A^{\dot\alpha}\mathscr{F}_{b\dot\alpha}^B\ , \tag{5.64}$$

where we have defined

$$\mathscr{F}_{b\dot\alpha}^B = (\bar{\mathscr{C}}_{b\dot\alpha}\mathcal{M}^B - \bar{\mathcal{M}}^B\mathscr{C}_{b\dot\alpha})\ . \tag{5.65}$$

This looks disappointing because the variation of φ_a seems to depend on itself via $\mathscr{C}_{b\dot\alpha}$ and $\bar{\mathscr{C}}_{b\dot\alpha}$. However, notice that the variation of the Grassmann collective coordinates (4.70) actually depends not on φ_a directly but, more precisely, on the combination $\Sigma_a^{AB}\varphi_a$. From (5.64), we have

$$\boldsymbol{L}\cdot\delta(\Sigma_a^{AB}\varphi_a)=\mathrm{i}\bar{\xi}^{\dot\alpha}_C(-\Sigma_a^{BC}\mathscr{F}^A_{a\dot\alpha}+\Sigma_a^{AC}\mathscr{F}^B_{a\dot\alpha}-\Sigma_a^{AB}\mathscr{F}^C_{a\dot\alpha})\ . \tag{5.66}$$

Now for $\mathscr{N}=2$ supersymmetry, there is only a single independent quantity $\Sigma_a^{AB}\varphi_a$ which we can take to be $\varphi\equiv\Sigma_a^{12}\varphi_a=\mathrm{i}\varphi_1+\varphi_2$. In this case the right-hand side of (5.66) vanishes identically; hence

$$\delta\varphi=0\ . \tag{5.67}$$

We now consider the extended set of variables $\{a_{\dot\alpha},\mathscr{M}^A,\varphi\}$, where the latter is subject to its own "ADHM constraint" following from its definition (4.65):

$$\boldsymbol{L}\cdot\varphi=-\tfrac{1}{2}(\bar{\mathscr{M}}^1\mathscr{M}^2-\bar{\mathscr{M}}^2\mathscr{M}^1)+\bar{w}^{\dot\alpha}\phi^0w_{\dot\alpha}\ , \tag{5.68}$$

where $\phi^0\equiv\Sigma_a^{12}\phi_a^0$. The $\mathscr{N}=2$ measure (5.55) can then be written in a suggestive way by replacing the factor of $|\det_{k^2}\boldsymbol{L}|^{-1}$ with an integral over the auxiliary variable φ with an explicit δ-function which imposes the new ADHM constraint (5.68):

$$|\det{}_{k^2}\boldsymbol{L}|^{-1}=\int d^{k^2}\varphi\prod_{r=1}^{k^2}\delta(\mathrm{tr}_k\,T^r(\boldsymbol{L}\cdot\varphi+\tfrac{1}{2}\bar{\mathscr{M}}^1\mathscr{M}^2-\tfrac{1}{2}\bar{\mathscr{M}}^2\mathscr{M}^1-\bar{w}^{\dot\alpha}\phi^0w_{\dot\alpha}))\ . \tag{5.69}$$

It is then easy to see that, when re-written using the above identity, the new form of the measure is supersymmetric. This follows because the variations of $\{a_{\dot\alpha},\mathscr{M}^A,\varphi\}$ are either off-diagonal or vanish; hence, the super-Jacobian vanishes to linear order. To complete the proof, the variations of the ADHM constraints, (2.65), (4.34) and (5.68), are also either off-diagonal or vanish,[34] and so the variation of the product of δ-functions also vanishes to linear order.

For $\mathscr{N}=4$ supersymmetry, we have to go back to (5.66) and follow the same logic. To make things simpler, and ultimately without loss of generality, we can focus on a particular supersymmetry variation, say $\bar{\xi}^{\dot\alpha}_1$. In that case, the variation of the Grassmann collective coordinates (4.70) involves the three quantities $\varphi^{A1}\equiv\Sigma_a^{A1}\varphi$, $A=2$–4. In that case, from (5.66), we have

$$\boldsymbol{L}\cdot\delta\varphi^{A1}=\mathrm{i}\bar{\xi}^{\dot\alpha}_1(-\Sigma_a^{11}\mathscr{F}^A_{a\dot\alpha}+\Sigma_a^{A1}\mathscr{F}^1_{a\dot\alpha}-\Sigma_a^{A1}F^1_{a\dot\alpha})=0\ . \tag{5.70}$$

The new auxiliary variables φ^{A1} are subject to their own "ADHM constraints":

$$\boldsymbol{L}\cdot\varphi^{A1}=\tfrac{1}{2}(\bar{\mathscr{M}}^1\mathscr{M}^A-\bar{\mathscr{M}}^A\mathscr{M}^1)+\Sigma_a^{A1}\bar{w}^{\dot\alpha}\phi_a^0w_{\dot\alpha}\ . \tag{5.71}$$

So for this particular supersymmetry variation, we consider the multiplet of variables $\{a_{\dot\alpha},\mathscr{M}^A,\varphi^{A1}\}$. The $\mathscr{N}=4$ collective coordinate integral (5.55) can then be re-cast by introducing integrals over the auxiliary variables φ^{A1} along with explicit δ-functions to impose (5.71):

$$|\det{}_{k^2}\boldsymbol{L}|^{-3}=\int\prod_{A=2}^{4}\left\{d^{k^2}\varphi^{A1}\prod_{r=1}^{k^2}\delta(\mathrm{tr}_k\,T^r(\boldsymbol{L}\cdot\varphi^{A1}-\tfrac{1}{2}\bar{\mathscr{M}}^1\mathscr{M}^A+\tfrac{1}{2}\bar{\mathscr{M}}^A\mathscr{M}^1-\Sigma_a^{A1}\bar{w}^{\dot\alpha}\phi^0w_{\dot\alpha}))\right\}\ . \tag{5.72}$$

[34] Note that the variation of (5.68) vanishes since we imposed it to find the variation of φ_a above.

It is then straightforward to prove, following the same logic as for $\mathcal{N}=2$, supersymmetric invariance. Firstly, the variations of $\{a_{\dot{\alpha}}, \mathcal{M}^A, \varphi^{A1}\}$ are either off-diagonal or vanish; hence, the super-Jacobian vanishes to linear order. In addition, the variations of ADHM constraints (2.65), (4.34) and (5.71) are also off-diagonal or zero. Consequently, the variation of the product of δ-functions also vanishes to linear order. The new twist in the $\mathcal{N}=4$ case is, for each supersymmetry variation $\bar{\xi}_A$, we must use a different set of three auxiliary variables $\Sigma_a^{AB}\varphi_a$, $B\neq A$, to prove supersymmetric invariance. Invariance of measure (5.55) under a general supersymmetry variation then follows by linearity.

5.4. From $\mathcal{N}=4$ to 0 via decoupling

An interesting consistency check on our collective coordinate integrals, Eq. (5.14) along with (5.55), which relates the expressions for different numbers of supersymmetries follows from renormalization group decoupling. The idea is to take one of the supersymmetric theories and add mass terms for some of the fields in such a way that for large masses the massive fields can be "integrated out" and the theory flows in the infra-red to a theory with fewer supersymmetries. This procedure, when implemented at the level of the semi-classical approximation, provides a way to relate the instanton integration measure for different numbers of supersymmetries.

We begin, with the $\mathcal{N}=2$ theory, and give a mass to one of the two $\mathcal{N}=1$ chiral multiplets. Adding such mass terms is discussed in Section 6.4. For a single flavour of fermion, the leading-order effect, is to introduce (6.89)

$$\tilde{S}_{\rm mass} = m\int d^4x\,{\rm tr}_N\,\lambda^2 = -\frac{m\pi^2}{g}\,{\rm tr}_k\,\bar{\mathcal{M}}(\mathcal{P}_\infty+1)\mathcal{M} \equiv -\frac{m}{4g}\,\tilde{\Omega}_{\tilde{\rm i}\tilde{\rm j}}\mathcal{M}^{\tilde{\rm i}}\mathcal{M}^{\tilde{\rm j}} \tag{5.73}$$

into the instanton effective action, where the inner product of two Grassmann symplectic tangent vectors was defined in (4.38).

The integral over the Grassmann collective coordinates $\mathcal{M}$ is written down in (5.53). It is convenient to re-write the argument of the fermionic ADHM constraints using (4.36) and then including the mass term, we have

$$\mathcal{I} = |{\rm det}_{k^2}\,\boldsymbol{L}|^{-1}\int d^{2k(N+k)}\mathcal{M}\prod_{r=1}^{k^2}\prod_{\dot{\alpha}=1}^{2}\delta\left(\frac{1}{4\pi^2}\,\mathcal{M}^{\tilde{\rm i}}\tilde{\Omega}_{\tilde{\rm i}\tilde{\rm j}}X_r^{\tilde{\rm j}\dot{\alpha}}\right)\exp\left(\frac{m}{4g}\,\tilde{\Omega}_{\tilde{\rm i}\tilde{\rm j}}\mathcal{M}^{\tilde{\rm i}}\mathcal{M}^{\tilde{\rm j}}\right)\ . \tag{5.74}$$

In order to evaluate the integral it is useful to decompose

$$\mathcal{M}^{\tilde{\rm i}} = \sigma_{r\dot{\alpha}}X_r^{\tilde{\rm i}\dot{\alpha}} + \mathcal{M}^{\perp}\ , \tag{5.75}$$

where $(\mathcal{M}^{\perp})^{\tilde{\rm i}}\tilde{\Omega}_{\tilde{\rm i}\tilde{\rm j}}X_r^{\tilde{\rm j}\dot{\alpha}}=0$ for all $r=1,\ldots,k$ and $\dot{\alpha}=1,2$. The quantity $\mathcal{M}^{\perp}$ is the projection of $\mathcal{M}$ which does not appear in the arguments of the δ-functions and its integrals must be saturated by bringing down powers of the mass term. Hence with decomposition (5.75), integral (5.74) factorizes as

$$\mathcal{I} = |{\rm det}_{k^2}\,\boldsymbol{L}|^{-1}\,\frac{\int\prod_{r=1}^{k^2}\prod_{\dot{\alpha}=1}^{2}\,d\sigma_{r\dot{\alpha}}\,\delta\left(\frac{1}{4\pi^2}\,\sigma_{s\dot{\beta}}X_s^{\tilde{\rm i}\dot{\beta}}\tilde{\Omega}_{\tilde{\rm i}\tilde{\rm j}}X_r^{\tilde{\rm j}\dot{\alpha}}\right)}{\left|{\rm det}_{2k^2}\,\frac{1}{4\pi^2}\,X_r^{\tilde{\rm i}\dot{\alpha}}\tilde{\Omega}_{\tilde{\rm i}\tilde{\rm j}}X_s^{\tilde{\rm j}\dot{\beta}}\right|^{1/2}}$$

$$\times\int d^{2kN}\mathcal{M}^{\perp}\exp\left(\frac{m}{8g}\,\tilde{\Omega}_{\tilde{\rm i}\tilde{\rm j}}\mathcal{M}^{\perp\tilde{\rm i}}\mathcal{M}^{\perp\tilde{\rm j}}\right)\ . \tag{5.76}$$

The determinant in the denominator is the Jacobian for transforming to the variables $\sigma_{r\dot\alpha}$. The Grassmann integrals can now be done to give

$$\mathscr{I} = |\det_{k^2} \boldsymbol{L}|^{-1} \left| \det_{2k^2} \frac{1}{4\pi^2} X_r^{\tilde{\mathrm{i}}\dot\alpha} \tilde\Omega_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}} X_s^{\tilde{\mathrm{j}}\dot\beta} \right|^{1/2} \left(\frac{2m\pi^2}{g} \right)^{kN} . \tag{5.77}$$

The second determinant factor can be related to a more familiar quantity by noting the following. From Appendix B, we established that on the level set of the hyper-Kähler quotient, the U(k)-vectors X_r satisfy $\tilde g(X_r, \tilde{\boldsymbol{I}}^{(c)} X_s) = 0$. In terms of components

$$\varepsilon_{\dot\alpha\dot\beta} X_r^{\tilde{\mathrm{i}}\dot\alpha} \tilde\Omega_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}} X_s^{\tilde{\mathrm{j}}\dot\gamma} \vec\tau^{\dot\beta}{}_{\dot\gamma} = 0 \quad \text{implies} \quad X_r^{\tilde{\mathrm{i}}\dot\alpha} \tilde\Omega_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}} X_s^{\tilde{\mathrm{j}}\dot\beta} \propto \varepsilon^{\dot\alpha\dot\beta} . \tag{5.78}$$

Hence, from the definition of $\boldsymbol{L}$ in (2.124) and the explicit relation $\tilde g(X_r, X_s) = \varepsilon_{\dot\alpha\dot\beta} X_r^{\tilde{\mathrm{i}}\dot\alpha} \tilde\Omega_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}} X_s^{\tilde{\mathrm{j}}\dot\beta}$, we have

$$\left| \det_{2k^2} \frac{1}{4\pi^2} X_r^{\tilde{\mathrm{i}}\dot\alpha} \tilde\Omega_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}} X_s^{\tilde{\mathrm{j}}\dot\beta} \right| = \left| \det_{k^2} \frac{1}{8\pi^2} \frac{1}{2} \varepsilon_{\dot\alpha\dot\beta} X_r^{\tilde{\mathrm{i}}\dot\alpha} \tilde\Omega_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}} X_s^{\tilde{\mathrm{j}}\dot\beta} \right|^2 \equiv |\det_{k^2} \boldsymbol{L}|^2 . \tag{5.79}$$

Consequently, the two determinants in (5.77) cancel to leave a purely numerical factor

$$\mathscr{I} = \left(\frac{2m\pi^2}{g} \right)^{kN} . \tag{5.80}$$

Using this expression for integral (5.74), we find that after decoupling the $\mathscr{N} = 2$ collective coordinate integral gives the $\mathscr{N} = 1$ collective coordinate integral with the following relation between the Pauli–Villars mass scales:

$$\mu_{\mathscr{N}=1}^{3N} = m^N \mu_{\mathscr{N}=2}^{2N} . \tag{5.81}$$

Alternatively, we can phrase the result in terms of the renormalization group Λ-parameters defined in (5.17):

$$\Lambda_{\mathscr{N}=1} = (g^2 m)^N \Lambda_{\mathscr{N}=2} . \tag{5.82}$$

Therefore, the decoupling limit is $g^2 m \to \infty$ and $\Lambda_{\mathscr{N}=2} \to 0$ in such a way that $\Lambda_{\mathscr{N}=1}$ is fixed.

Now we turn to the relation between the $\mathscr{N} = 4$ and 2 theory. We start with the $\mathscr{N} = 4$ theory and add an $\mathscr{N} = 2$ preserving mass term for a pair of $\mathscr{N} = 1$ chiral multiplets. This results in contribution (6.89) to the instanton effective action:

$$\tilde S_{\text{mass}} = m \int \mathrm{d}^4 x\, \mathrm{tr}_N(\lambda^3\lambda^3 + \lambda^4\lambda^4) = -\frac{m\pi^2}{g} \mathrm{tr}_k[\bar{\mathscr{M}}^3(\mathscr{P}_\infty + 1)\mathscr{M}^3 + \bar{\mathscr{M}}^4(\mathscr{P}_\infty + 1)\mathscr{M}^4] . \tag{5.83}$$

We can now integrate out the two flavours of Grassmann collective coordinates $\mathscr{M}^3$ and $\mathscr{M}^4$ as above to verify that the collective coordinate integrals are related if

$$\mu_{\mathscr{N}=2}^{2N} = m^{2N} . \tag{5.84}$$

This means that

$$\Lambda_{\mathscr{N}=2} = m^{2N} \mathrm{e}^{-8\pi^2/g^2 + \mathrm{i}\theta} \tag{5.85}$$

and so the decoupling limit is $m \to \infty$ and $g^2 \to 0$, in such a way that $\Lambda_{\mathscr{N}=2}$ is fixed.

We remark that (5.82) and (5.85) are consistent with the standard prescriptions in the literature for the renormalization group matching of a low- and a high-energy theory [53,54]. The absence of numerical factors on the right-hand side of these relations reflects the absence of threshold corrections in the Pauli–Villars scheme.

Finally, we can take the $\mathcal{N}=1$ theory and decouple the gluino to flow at low energies to the non-supersymmetric theory. In this limit, the $\mathcal{N}=1$ collective coordinate integral gives the extrapolation of (5.55) to $\mathcal{N}=0$ with the relation

$$\mu_{\mathcal{N}=0}^{4N} = m^N \mu_{\mathcal{N}=1}^{3N} \ . \tag{5.86}$$

However, in the non-supersymmetric theory there is also a non-trivial fluctuation determinant described in Section 3.3 which is not, of course, reproduced.

6. Generalizations and miscellany

In this section, we describe some important generalizations of the instanton calculus and other results. In Section 6.1, we describe the extent to which the ADHM constraints can be solved when $N \geqslant 2k$. This will be useful in applications where a large-N limit is involved, as in Sections 7 and 9. Then, in Section 6.2, we explain how to generalize the instanton calculus to theories with gauge groups $\mathrm{Sp}(N)$ and $\mathrm{SO}(N)$. This is useful in the application to $\mathcal{N}=2$ theories described in Section 8 involving gauge group SU(2) because it turns out that the instanton calculus of Sp(1) ($\simeq$ SU(2)) is actually more economical and convenient than the SU(2)-as-an-example-of-SU(N) formalism developed previously. Moving on, in Section 6.3, we describe how to include matter fields in the calculus. This is important for the applications in Sections 7 and 10.1. The question of how to modify the instanton calculus when fields have masses is considered in Section 6.4. Finally in Section 6.5 we consider the instanton partition function in more detail. In particular we show how it can be "linearized" by introducing various auxiliary variables including Lagrange multipliers for the bosonic and fermionic ADHM constraints. We also show how the integrals over the overall position coordinate and its superpartners can be separated out, leading to the notion of the centred instanton partition function.

6.1. Solving the ADHM constraints for $N \geqslant 2k$

It turns out that when $N \geqslant 2k$, the ADHM constraints can be solved in a certain generic region of the moduli space by a change of variables [21]. The idea is to introduce the gauge-invariant U(N)-invariant bi-linears

$$(W^{\dot\alpha}{}_{\dot\beta})_{ij} = \bar w_{iu}^{\dot\alpha} w_{uj\dot\beta} \tag{6.1}$$

which can be incorporated into four $k\times k$ matrices

$$W^0 = \mathrm{tr}_2\, W, \quad W^c = \mathrm{tr}_2(\tau^c W) \ . \tag{6.2}$$

When $N \geqslant 2k$, these variables are independent and the change of variables from the $4kN$ real variables $\{w_{\dot\alpha}, \bar w^{\dot\alpha}\}$ to the bi-linear variables $\{W^0, W^c\}$ and the gauge orientation $\mathscr{U}$ (defined in Section 2.4.2) is invertible, at least when the gauge orbit is generic as in Eq. (2.87). In changing variables, one must also, of course, specify the integration domain for the new variables. In particular, since the W variables are the inner products of vectors, they will be constrained by various triangle inequalities.[35]

The reason why this change of variables is useful is because ADHM constraints (2.65) are linear in W^c; indeed they can be re-cast as

$$W^c = -a'_m a'_n \, \mathrm{tr}_2(\tau^c \bar{\sigma}_m \sigma_n) \, . \tag{6.3}$$

The final description of the subspace of the moduli space on a generic orbit of the gauge group for $N \geqslant 2k$ is then in terms of the five $k \times k$ Hermitian matrices, W^0 and a'_n, in addition to $\mathcal{U}$. However, as long as one is in a phase where the gauge symmetry is not broken, then physical quantities are gauge invariant and one can readily integrate over the gauge orientation $\mathcal{U}$.

In order to exploit this linearizing change of variables, we must determine how the instanton integration measure transforms in going from $\{w_{\dot\alpha}, \bar{w}^{\dot\alpha}\}$ to the bi-linears $\{W^0, W^c\}$ in Eq. (6.2) and the gauge orientation $\mathcal{U}$. In what follows it will be useful to think of $w_{\dot\alpha}$ as an $N \times K$-dimensional matrix w, where $K = 2k$, and to introduce the composite index $a \equiv i\dot\alpha = 1, \ldots, K$, so that the elements of w are w_{ua}. A suitable $\mathrm{SU}(N)$ gauge transformation $\mathcal{U}$ puts the matrix w into upper-triangular form, as in (2.83). The ξ_{ab} are complex except for the diagonal elements ξ_{aa} which we can choose to be real. While a priori the ξ_{ab} are only defined for $1 \leqslant a \leqslant b \leqslant K$, it is convenient to extend them to $1 \leqslant b < a \leqslant K$ as well, by defining $\xi_{ab} = (\xi_{ba})^*$. In terms of these extended variables, the gauge-invariant bi-linear W defined in (6.1) is the matrix

$$W = w^\dagger w = \begin{pmatrix} \xi_{11} & 0 & \cdots & 0 \\ \xi_{21} & \xi_{22} & & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \xi_{K1} & \xi_{K2} & \cdots & \xi_{KK} \end{pmatrix} \begin{pmatrix} \xi_{11} & \xi_{21} & \cdots & \xi_{K1} \\ 0 & \xi_{22} & \cdots & \xi_{K2} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \xi_{KK} \end{pmatrix} \, . \tag{6.4}$$

Note that there are as many real degrees of freedom in the $\{W_{ab}\}$ as in the $\{\xi_{ab}\}$. From Eq. (6.4) it follows that

$$\det_K W = \left(\prod_{a=1}^{K} \xi_{aa} \right)^2 \, . \tag{6.5}$$

When calculating the Jacobian, it is useful to pass through an intermediate change of variables involving ξ rather than the W. The Jacobian for this is

$$\int \mathrm{d}^{K^2} W = 2^K \int \mathrm{d}^{K^2} \xi \prod_{a=1}^{K} \xi_{aa}^{2K-2a+1} \, . \tag{6.6}$$

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[35] Fortunately, these technicalities will not be relevant in our applications at large-N. In this case, steepest-descent methods apply and the saddle-point values of the W's obtained in Sections 7 and 9 satisfy all such triangle inequalities by inspection. In addition, we will show that on the saddle points the k instantons inhabit k commuting $\mathrm{SU}(2)$ subgroups of the gauge group. Hence, manifestly, they lie on a generic orbit of gauge group (2.87).

This can be proved by induction. For $K = 1$ one has simply $W_{11} = \xi_{11}^2$. From Eq. (6.4) we can also easily relate the Jacobian for K to that for $K-1$; one finds

$$\left.\frac{\partial(\{W_{ab}\})}{\partial(\{\xi_{ab}\})}\right|_K = \frac{\partial W_{KK}}{\partial \xi_{KK}}\left(\prod_{a=1}^{K-1}\frac{\partial W_{aK}}{\partial \xi_{aK}}\frac{\partial W_{Ka}}{\partial \xi_{Ka}}\right)\left.\frac{\partial(\{W_{ab}\})}{\partial(\{\xi_{ab}\})}\right|_{K-1}$$

$$= 2\xi_{KK}\left(\prod_{a=1}^{K-1}\xi_{aa}^2\right)\left.\frac{\partial(\{W_{ab}\})}{\partial(\{\xi_{ab}\})}\right|_{K-1} . \tag{6.7}$$

Result (6.6) follows by induction.

Next, we calculate the Jacobian for the change of variables from w to $\{\xi, \mathcal{U}\}$. To this end, define $u^a \equiv \mathcal{U}_{ua}$ to be the ath column of $\mathcal{U}$, $a = 1, \ldots, K$. Since $\mathcal{U} \subset \mathrm{SU}(N)$

$$(u^\dagger)^a \cdot u^b = \delta^{ab} . \tag{6.8}$$

The K N-vectors u^a then provide the well-known parameterization of coset (2.84) as a product of spheres [55]. To see this, u^1 is a unit vector in an N-dimensional complex space and consequently parameterizes S^{2N-1}. The second vector u^2 is also a unit vector, but one which is orthogonal to u^1, and consequently parameterizes S^{2N-3}. Continuing this chain of argument, we see that the vectors $\{u^a\}$ parameterize the product of spheres

$$\frac{\mathrm{SU}(N)}{\mathrm{SU}(N-K)} \simeq S^{2N-1} \times S^{2N-3} \times \cdots \times S^{2N-2K+1} . \tag{6.9}$$

From (2.83) we can read off the expansion of the elements of w in terms of the elements of the vectors u^a:

$$w_{ua} = \sum_{b=1}^{a} \xi_{ba} u_u^b . \tag{6.10}$$

The Jacobian can be determined through the following iterative process. First we start with $w_{u1} = \xi_{11} u_u^1$, whose measure can be written in polar coordinates as

$$\int \prod_{u=1}^{N} \mathrm{d}w_{u1}\,\mathrm{d}w_{u1}^* = 2^N \int \xi_{11}^{2N-1}\,\mathrm{d}\xi_{11}\,\mathrm{d}^{2N-1}\hat{\Omega}_1 . \tag{6.11}$$

Here $\mathrm{d}^{2N-1}\hat{\Omega}_1$ is the usual measure for the solid angles on S^{2N-1} parameterized by u^1. Continuing the process on the next vector $w_{u2} = \xi_{12} u_u^1 + \xi_{22} u_u^2$, we have

$$\int \prod_{u=1}^{N} \mathrm{d}w_{u2}\,\mathrm{d}w_{u2}^* = 2^{N-1} \int \mathrm{d}\xi_{12}\,\mathrm{d}\xi_{12}^*\,\xi_{22}^{2N-3}\,\mathrm{d}\xi_{22}\,\mathrm{d}^{2N-3}\hat{\Omega}_2 . \tag{6.12}$$

In general

$$\int \prod_{u=1}^{N} \mathrm{d}w_{ua}\,\mathrm{d}w_{ua}^* = 2^{N-a-1} \int \left\{\prod_{b=1}^{a-1} \mathrm{d}\xi_{ba}\,\mathrm{d}\xi_{ba}^*\right\} \xi_{aa}^{2N-2a+1}\,\mathrm{d}\xi_{aa}\,\mathrm{d}^{2N-2a+1}\hat{\Omega}_a , \tag{6.13}$$

where $\hat{\Omega}_a$ is parameterized by u^a. Hence

$$\int \mathrm{d}^{2KN} w = 2^{NK-K(K-1)/2} \int \left\{ \prod_{a=1}^{K} \xi_{aa}^{2N-2a+1}\, \mathrm{d}\xi_{aa}\, \mathrm{d}^{2N-2a+1}\hat{\Omega}_a \right\} \left\{ \prod_{a<b} \mathrm{d}\xi_{ab}\, \mathrm{d}\xi^*_{ab} \right\} . \tag{6.14}$$

Using (6.5) and (6.6), we obtain

$$\int \mathrm{d}^{2KN} w = 2^{NK-K(K+1)/2} \int |\mathrm{det}_K\, W|^{N-K}\, \mathrm{d}^{K^2} W \left\{ \prod_{a=1}^{K} \mathrm{d}^{2N-2a+1}\hat{\Omega}_a \right\} . \tag{6.15}$$

Re-introducing $k = K/2$, we have

$$\int \mathrm{d}^{2kN} w\, \mathrm{d}^{2kN} \bar{w} = A_k \int |\mathrm{det}_{2k}\, W|^{N-2k}\, \mathrm{d}^{k^2} W^0 \prod_{c=1,2,3} \mathrm{d}^{k^2} W^c\, \mathrm{d}^{4k(N-k)}\mathscr{U} . \tag{6.16}$$

The integral over the $2k \times 2k$ matrix W has been written as four separate integrals over the $k \times k$ matrices W^0 and W^c, defined in (6.2), with respect to the basis $\{T^r\}$, defined in Section 3.2. [36] For convenience we have defined a unit normalized measure on the coset space:

$$\int \mathrm{d}^{4k(N-k)}\mathscr{U} \stackrel{\mathrm{def}}{=} \frac{1}{\prod_{a=1}^{2k} \mathrm{Vol}\, S^{2(N-a)+1}} \int \prod_{a=1}^{2k} \mathrm{d}^{2N-2a+1}\hat{\Omega}_a \tag{6.17}$$

and this fixes the normalization constant to be

$$A_k = 2^{2kN-4k^2-k} \prod_{a=1}^{2k} \mathrm{Vol}\, S^{2(N-a)+1} = \frac{2^{2kN-4k^2+k}\pi^{2kN-2k^2+k}}{\prod_{a=1}^{2k}(N-a)!} . \tag{6.18}$$

In expression (3.17) for volume form on $\mathfrak{M}_k$, the δ-functions imposing the ADHM constraints simply soak up the integrals over W^c (giving rise to the numerical factor of 2^{3k^2} from the $\frac{1}{2}$'s in the arguments of the δ-functions) to leave

$$\int_{\mathfrak{M}_k} \boldsymbol{\omega} = \frac{2^{3k^2} A_k C_k}{\mathrm{Vol}\, \mathrm{U}(k)} \int \mathrm{d}^{4k^2} a'\, \mathrm{d}^{k^2} W^0\, \mathrm{d}^{4k(N-k)}\mathscr{U} |\mathrm{det}_{k^2} \boldsymbol{L}|\, |\mathrm{det}_{2k}\, W|^{N-2k} . \tag{6.19}$$

The simplifications for $N \geqslant 2k$ also extend to the Grassmann sector. The trick is to find a change of variables in the Grassmann sector which mirrors that which we have just described for the c-number collective coordinates. To this end, let us identify the superpartners of the collective coordinates associated to global gauge transformations on the instanton solution. Infinitesimally, the latter are the subset of δw which preserve the gauge-invariant variables W, i.e. which satisfy

$$\bar{w}^{\dot{\alpha}}_{iu}\delta w_{uj\dot{\beta}} + \delta\bar{w}^{\dot{\alpha}}_{iu} w_{uj\dot{\beta}} = 0 . \tag{6.20}$$

Under supersymmetry transformation (4.68) one has

$$\delta w_{ui\dot{\alpha}} = \mathrm{i}\bar{\xi}_{\dot{\alpha}A}\mu^A_{ui}, \qquad \delta\bar{w}^{\dot{\alpha}}_{iu} = -\mathrm{i}\bar{\mu}^A_{iu}\bar{\xi}^{\dot{\alpha}}_A . \tag{6.21}$$

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[36] This accounts for an additional factor of 2^{-2k^2}.

Inserting Eq. (6.21) into Eq. (6.20) produces the gauge-invariant conditions

$$\bar{\xi}_{\dot\beta A}\bar{w}^{\dot\alpha}_{iu}\mu^A_{uj}+\bar{\xi}^{\dot\alpha}_{A}\bar{\mu}^A_{iu}w_{uj\dot\beta}=0 \tag{6.22}$$

or equivalently,

$$\bar{w}^{\dot\alpha}_{iu}\mu^A_{uj}=0 \quad \text{and} \quad \bar{\mu}^A_{iu}w_{uj\dot\alpha}=0\ . \tag{6.23}$$

To satisfy these constraints, it is convenient to decompose μ^A as follows:

$$\mu^A_{iu}=w_{uj\dot\alpha}(\zeta^{\dot\alpha A})_{ji}+v^A_{iu}, \qquad \bar{\mu}^A_{iu}=(\bar{\zeta}^A_{\dot\alpha})_{ij}\bar{w}^{\dot\alpha}_{ju}+\bar{v}^A_{iu}\ , \tag{6.24}$$

where v^A lies in the orthogonal subspace to w:

$$\bar{w}^{\dot\alpha}_{iu}v^A_{uj}=0, \qquad \bar{v}^A_{iu}w_{uj\dot\alpha}=0\ . \tag{6.25}$$

The superpartners of the bosonic coset coordinates $\mathscr{U}$ are then precisely the variables $\{v^A,\bar{v}^A\}$.[37] Notice in the case of a single instanton, the variables $\bar{\zeta}^A=\zeta^A$ are precisely the Grassmann collective coordinates associated to superconformal transformations.

We now turn to the Grassmann part of instanton measure (5.55). As explained above, the superpartners of the global gauge collective coordinates $\mathscr{U}$ are the Grassmann variables $\{v^A,\bar{v}^A\}$ defined in Eqs. (6.24) and (6.25). Since these coordinates are orthogonal to the $\bar{w}$ and w vectors, respectively, it is easy to see from (4.31a) that they do not appear in the fermionic ADHM constraints. The Jacobian for the change of variables from the original Grassmann coordinates $\{\mu^A,\bar{\mu}^A\}$ to $\{\zeta^A,\bar{\zeta}^A,v^A,\bar{v}^A\}$, is, for each value of A,[38]

$$\frac{\partial(\{\mu^A,\bar{\mu}^A\})}{\partial(\{\zeta^A,\bar{\zeta}^A,v^A,\bar{v}^A\})}=|\det_{2k}W|^{-k}\ . \tag{6.26}$$

As in the bosonic sector, the change of variables allows us to integrate out the Grassmann-valued δ-functions in measure (5.55) trivially. The arguments of the δ-functions are the fermionic ADHM constraints which, in terms of the new variables, are (4.31a)

$$\bar{\zeta}^A_{\dot\beta}W^{\dot\beta}{}_{\dot\alpha}+W_{\dot\alpha\dot\beta}\zeta^{\dot\beta A}+[\mathscr{M}'^{\alpha A},a'_{\alpha\dot\alpha}]=0\ . \tag{6.27}$$

These equations can be used to eliminate the $2k^2$ variables $\bar{\zeta}^A_{\dot\alpha}$, for each A. The relevant integral is simply

$$\int \mathrm{d}^{2k^2}\bar{\zeta}^A\prod_{r=1}^{k^2}\prod_{\dot\alpha=1}^{2}\delta(\mathrm{tr}_k\,T^r(\bar{\zeta}^A_{\dot\beta}W^{\dot\beta}{}_{\dot\alpha}+W_{\dot\alpha\dot\beta}\zeta^{\dot\beta A}+[\mathscr{M}'^{\alpha A},a'_{\alpha\dot\alpha}]))=|\det_{2k}W|^{k}\ . \tag{6.28}$$

Notice that the factor on the right-hand side conveniently cancels the Jacobian of (6.26).

[37] It is worth mentioning that although the coset coordinates correspond to bosonic zero modes which are generated by Lagrangian symmetries, this is not true of their Grassmann partners.

[38] We define the integrals over the $k\times k$ matrices $\zeta^{\dot\alpha A}$ and $\bar{\zeta}^{\dot\alpha A}$ with respect to the Hermitian basis T^r introduced in Section 4.1.

Table 1
Gauge and associated auxiliary groups

G	SU(N)	Sp(N)	SO(N)
$H(k)$	U(k)	O(k)	Sp(k)
N'	N	$2N$	N
k'	k	k	$2k$

The expression for the supersymmetric volume form on the instanton moduli space in the case $N \geqslant 2k$ is

$$\int_{\mathfrak{M}_k} \boldsymbol{\omega}^{(\mathcal{N})} = \frac{2^{3k^2} C_k^{(\mathcal{N})} A_k}{\mathrm{Vol}\, \mathrm{U}(k)} \int \mathrm{d}^{4k^2} a' \, \mathrm{d}^{k^2} W^0 \, \mathrm{d}^{4k(N-k)} \mathscr{U} \prod_{A=1}^{\mathcal{N}} \{ \mathrm{d}^{k(N-2k)} v^A \, \mathrm{d}^{k(N-2k)} \bar{v}^A \, \mathrm{d}^{2k^2} \zeta^A \, \mathrm{d}^{2k^2} \mathscr{M}'^A \}$$
$$\times \quad |\mathrm{det}_{k^2} \boldsymbol{L}|^{1-\mathcal{N}} |\mathrm{det}_{2k} W|^{N-2k} \ . \qquad (6.29)$$

This expression for the measure is the starting point for two applications of the multi-instanton calculus at large N that we describe in Sections 8 and 9.

6.2. The ADHM construction for $Sp(N)$ and $SO(N)$

The ADHM formalism [7] for constructing instanton solutions was adapted for dealing with any of the classical gauge groups in the early instanton literature. The method adopted was to consider the construction for one of the series of classical groups, e.g. symplectic groups in Ref. [56] and orthogonal groups in Ref. [57], and then embed the other two series in this series. Our approach will be no different, although we will start from the unitary series. The advantage of this is that all the previous formulae that we have established in the SU(N) case can easily be adapted to describe the other gauge groups.

In order to construct instanton solutions for gauge theories with Sp(N) and SO(N) gauge groups, we use the embeddings [39]

$$\mathrm{Sp}(N) \subset \mathrm{SU}(2N), \qquad \mathrm{SO}(N) \subset \mathrm{SU}(N) \qquad (6.30)$$

to extract the ADHM formalism for these groups in terms of the SU(N) ADHM construction. The surprising feature of the resulting formalism is that the auxiliary group, U(k) in the SU(N) case and denoted generally as $H(k)$ at instanton number k, is *not* in the same series as the gauge group G. Table 1 shows the auxiliary groups and defines the quantities N' and k' allowing us to present a unified treatment of Sp(N) and SO(N).

To describe the other classical groups we start with the theory with gauge group SU(N') at instanton number k'. Instanton solutions in the Sp(N) and SO(N) theories follow by simply imposing certain reality conditions on the ADHM construction of the SU(N') theory which ensures that the gauge potential lies in the appropriate $sp(N)$ and $so(N)$ subalgebra of $su(N')$. In order to deal with

[39] For the orthogonal groups we restrict $N \geqslant 4$.

both the Sp(N) and SO(N) cases at the same time it is useful to define the notion of a generalized transpose operation denoted t which acts either on gauge or instanton indices. Specifically, on Sp(n) group indices t acts as a symplectic transpose, i.e. on a column vector v, $v^{\rm t} = v^{\rm T}J^{\rm T}$, where J is the $2n \times 2n$ symplectic matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{6.31}$$

while on O(n) group indices t is a conventional transpose t $\equiv$ T. The adjoint representations of both groups are Hermitian t-anti-symmetric matrices and have dimensions $n(2n+1)$ and $n(n-1)/2$, respectively. Hermitian t-symmetric matrices correspond to the *anti-symmetric* representation of Sp(n), with dimension $n(2n-1)$ and the symmetric representation of SO(n), with dimensions $n(n+1)/2$.

The additional reality conditions on the ADHM variables are

$$\bar{w}^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}(w_{\dot\beta})^{\rm t}, \qquad (a'_{\alpha\dot\alpha})^{\rm t} = a'_{\alpha\dot\alpha} \ . \tag{6.32}$$

These reality conditions are only preserved by the subgroup $H(k) \subset {\rm U}(k')$ of the auxiliary symmetry group of the SU(N') theory. The matrices a'_n are Hermitian and, by (6.32), t-symmetric, i.e. real symmetric in the case of auxiliary group O(k), and symplectic anti-symmetric in the case of auxiliary group Sp(k). It is easy to verify that ADHM constraints (2.65) themselves are anti-Hermitian t-anti-symmetric, in other words $H(k)$ adjoint-valued. It is straightforward to show that these reality conditions are precisely what is required to render gauge field (2.49) t-anti-symmetric, in other words to restrict it to an $sp(N)$ and $so(N)$ subalgebra of $su(N')$, respectively.

The Grassmann collective coordinates are subject to a similar set of reality conditions:

$$\bar\mu = \mu^t, \qquad (\mathscr{M}'_\alpha)^t = \mathscr{M}'_\alpha \ . \tag{6.33}$$

So $\mathscr{M}'_\alpha$ is t-symmetric. The fermionic ADHM constraints (4.34) are, like their bosonic counterparts, t-anti-symmetric. [40]

We can now count the number of c-number and Grassmann collective coordinates. For both Sp(N) and SO(N) at instanton number k, there are $4kN$ real independent w variables, taking into account the reality conditions. The number of a'_n variables is $4\times k(k+1)/2$ and $4\times k(2k-1)$, for Sp(N) and SO(N), respectively. The physical moduli space is then the space of these variables modulo the three $H(k)$-valued ADHM constraints (2.65) and auxiliary $H(k)$ symmetry. Hence the dimension of the physical moduli space is $4k(N+1)$ and $4k(N-2)$, for Sp(N) and SO(N), respectively. This agrees with the counting via the Index Theorem. The counting of the Grassmann sector of the physical moduli space goes as follows. There are $2kN$ real degrees of freedom in μ and $2\times k(k+1)/2$ and $2\times k(2k-1)$, in $\mathscr{M}'_\alpha$, for Sp(N) and SO(N), respectively. The ADHM constraints then impose $2\times k(k-1)/2$ and $2\times k(2k+1)$ conditions, for Sp(N) and SO(N), respectively. Hence there are $2k(N+1)$ and $2k(N-2)$ real physical Grassmann collective coordinates for Sp(N) and SO(N), respectively. Again this agrees with the counting via the Index Theorem.

As an example, consider the Sp(1) theory. Since Sp(1) $\simeq$ SU(2), this should have the same content as the SU(2) ADHM construction that we have described in Section 2.4; however, we will

[40] In proving this it is useful to notice that $(w^{\rm t}_{\dot\alpha})^{\rm t} = -w_{\dot\alpha}$ and $(\mu^{\rm t})^{\rm t} = -\mu$.

find that the Sp(1) description is more economical in that there are fewer ADHM variables subject to fewer constraints for a given instanton number. This makes the Sp(1) formalism particularly attractive in certain applications (for instance in Section 8). In this case, reality conditions (6.32) are explicitly

$$w^*_{ui\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta} J_{uv} w_{ui\dot\beta}, \qquad (a'_n)_{ij} = (a'_n)_{ji} \ . \tag{6.34}$$

Given that a'_n are Hermitian, the second condition implies that a'_n are real symmetric $k\times k$ matrices. We now write the gauge group indices as $\alpha=1,2$ rather than $u=1,2$. In this case the first condition becomes

$$w^*_{i\alpha\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta} w_{i\beta\dot\beta} \ . \tag{6.35}$$

This means that w_i are quaternions of form (2.9), so $w_{i\alpha\dot\alpha} = w_{in}\sigma_{n\alpha\dot\alpha}$ for real w_{in}. The fact that the gauge indices are also labelled by α means that the ADHM variable $a_{\dot\alpha}$ can be written as a quaternion:

$$a_{\alpha\dot\alpha} = \begin{pmatrix} w_{\alpha\dot\alpha} \\ a'_{\alpha\dot\alpha} \end{pmatrix} . \tag{6.36}$$

At instanton number k, there are $2k(k+3)$ variables $a_{\alpha\dot\alpha}$ subject to $3k(k-1)/2$ ADHM constraints and $k(k-1)/2$ symmetries to give the dimension of $\mathfrak{M}_k$ as $8k$. This compares with the SU(2) description, where there are $4k(k+2)$ variables $a_{\dot\alpha}$ subject to $3k^2$ ADHM constraints and k^2 symmetries. Clearly the Sp(1) formalism is more economical. For example, at the one-instanton level there are no ADHM constraints in the Sp(1) formalism compared with three in the SU(2) formalism.

In general all the formulae that we have established in the instanton calculus carry through to the Sp(N) and SO(N) cases without change. The only difference is the ADHM variables are subject to reality conditions (6.32) and (6.33). In particular, the collective coordinate integration measure and the instanton effective action have the same form established in Section 5. Only the normalization constant for the integral over the instanton moduli space $C^{(\mathcal{N})}$ is changed.

6.3. Matter fields and the ADHM construction

In this section we consider various aspects of the instanton calculus in supersymmetric theories with matter fields. We will only consider matter in the fundamental representation of the gauge group (although the ADHM formalism can be extended to any higher representation by the tensor product formalism developed in [18]). There are two distinct applications: to $\mathcal{N}=1$ theories on the Higgs branch and $\mathcal{N}=2$ theories on the Coulomb branch.

6.3.1. $\mathcal{N}=1$ theories on the Higgs branch

Let us start by considering a theory with the $\mathcal{N}=1$ vector multiplet coupled to a single fundamental chiral multiplet $Q=(q,\chi)$, where q is the scalar (Higgs) field and χ is the Weyl fermion partner (Higgsino). The Euclidean space action for the matter fields (with no superpotential) is

$$S_{\text{matter}} = \int d^4x \{\mathscr{D}_n q^\dagger \mathscr{D}_n q - \mathscr{D}_n\bar\chi\bar\sigma_n\chi - \sqrt{2}ig\bar\chi\bar\lambda q + \sqrt{2}igq^\dagger\lambda\chi + \tfrac14 g^2(q^\dagger q)^2\} \ . \tag{6.37}$$

On the Higgs branch, the scalar field will have an arbitrary VEV q^0 and the instantons become constrained. As previously we can capture the leading-order behaviour in the semi-classical limit by an appropriate approximate instanton solution. As on the Coulomb branch of the $\mathcal{N}=2$ theories discussed in Section 4, the anti-chiral fermions are zero for the approximate instanton: $\bar\lambda=\bar\chi=0$. To leading order, therefore, the fundamental-valued chiral fermion χ satisfies the source free Weyl equation in the ADHM instanton background

$$\bar{\mathscr{D}}\chi = 0\ . \tag{6.38}$$

The fundamental fermion zero modes were originally constructed in [18]. In our language, they read

$$\chi_\alpha = g^{-1/2}\bar U b_\alpha f\mathscr{K}\ , \tag{6.39}$$

where $\mathscr{K}_i$ are k new Grassmann collective coordinates. In the case of a fundamental fermion, there are no analogues of the fermionic ADHM constraints for the new Grassmann coordinates $\mathscr{K}$. So there are k independent zero modes which agrees with counting via the Index Theorem. The proof that (6.39) satisfies (6.38) is a straightforward exercise in ADHM algebra, using (2.51), (C.19) and (C.2a):

$$\begin{aligned}\bar{\mathscr{D}}^{\dot\alpha\alpha}\chi_\alpha &= g^{-1/2}\bar\sigma_n^{\dot\alpha\alpha}(\partial_n\bar U\Delta_{\dot\beta}f\bar\Delta^{\dot\beta}b_\alpha f+\bar U b_\alpha\partial_n f)\mathscr{K}\\ &= g^{-1/2}\bar\sigma_n^{\dot\alpha\alpha}\bar U(-b_\alpha f\bar\Delta^{\dot\beta}b^\beta\sigma_{n\beta\dot\beta}f-b^\beta\sigma_{n\beta\dot\beta}f\bar\Delta^{\dot\beta}b_\alpha f)\mathscr{K}=0\ .\end{aligned} \tag{6.40}$$

On the other hand, to leading order the Higgs field q satisfies an inhomogeneous covariant Laplace equation

$$\mathscr{D}^2 q=\sqrt{2}\mathrm{i}g\lambda\chi \tag{6.41}$$

together with the VEV boundary conditions

$$q\overset{x\to\infty}{=}q^0\ , \tag{6.42}$$

where $q^0=(q^0_1,\dots,q^0_N)$ denotes the fundamental VEV. The right-hand side of Eq. (6.41) is the product of classical configurations (4.29) and (6.39), respectively. The general solution to Eqs. (6.41) and (6.42) is

$$q=\bar U\begin{pmatrix} q^0_{[N]}\\ 0_{[2k]}\end{pmatrix}-\frac{\mathrm{i}}{2\sqrt2}\bar U\mathscr{M}f\mathscr{K}\ . \tag{6.43}$$

The proof is another straightforward exercise in ADHM algebra (very similar to that in Appendix C, Eq. (C.25)). Firstly, similar to (C.3b),

$$\mathscr{D}^2(\bar U\mathscr{J})=-4\bar U b^\alpha f\bar b_\alpha\mathscr{J}+\bar U\partial^2\mathscr{J}-2\bar U b^\alpha f\sigma_{n\alpha\dot\alpha}\bar\Delta^{\dot\alpha}\partial_n\mathscr{J}\ . \tag{6.44}$$

Now take $\mathscr{J}=-\mathrm{i}/4\mathscr{M}f\mathscr{K}$ and compare with

$$g\lambda\chi=\bar U(\mathscr{M}f\bar b^\alpha-b^\alpha f\bar{\mathscr{M}})\mathscr{P}b_\alpha f\mathscr{K}\ . \tag{6.45}$$

Using differentiation formulae (C.2b) and (C.2a) along with $\bar\Delta_{\dot\alpha}\mathscr{M}=-\bar{\mathscr{M}}\Delta_{\dot\alpha}$ and $\bar b_\alpha\mathscr{M}=\bar{\mathscr{M}}b_\alpha$, one finds that the second term in (6.44) matches the first term in (6.45), while the sum of the first and third terms in (6.44) matches the second term in (6.45).

As usual in the supersymmetric instanton calculus, $q^\dagger$ ceases to be the conjugate of q in the instanton background, since it satisfies $\mathscr{D}^2 q^\dagger = 0$ to leading order. Consequently

$$q^\dagger = ((q^0)^\dagger_{[N]} \quad 0_{[2k]})U \ . \tag{6.46}$$

Notice that $q^\dagger$ fails to be the conjugate of q due to the presence of terms bi-linear in the Grassmann collective coordinates in the latter. This is a—by now familiar—symptom of working in Euclidean space.

To capture the leading-order behaviour we have to take the approximate instanton solution, including the matter field solutions discussed above, and substitute them into the Euclidean action. The gauge field term in the component Lagrangian yields $-2\pi i\tau$ as always. Following the method of Refs. [48,58], the two other relevant terms of the action, namely the Higgs kinetic term and the Yukawa interaction involving the chiral fermions, are turned into a surface term with an integration by parts in the former together with Euler–Lagrange equation (6.41) for the fundamental scalar:

$$\tilde{S} = \int d^4x\{\mathscr{D}_n q^\dagger \mathscr{D}_n q + i\sqrt{2}q^\dagger\lambda\chi\} = \int d^4x\{\partial_n(q^\dagger\mathscr{D}_n q) + q^\dagger(-\mathscr{D}^2 q + \sqrt{2}ig\lambda\chi)\} \ . \tag{6.47}$$

Using Stokes' Theorem, the contribution to the action may then be extracted from the asymptotic fall-off at infinity

$$\frac{x_n}{x}\mathscr{D}_n q \overset{x\to\infty}{\to} \frac{1}{2x^3}\left(w_{\dot\alpha}\bar{w}^{\dot\alpha}q^0 + \frac{i}{\sqrt{2}}\mu\mathscr{K}\right) \tag{6.48}$$

and hence to lowest order the instanton effective action is

$$\tilde{S} = \pi^2\left\{(q^0)^\dagger w_{\dot\alpha}\bar{w}^{\dot\alpha}q^0 + \frac{i}{\sqrt{2}}(q^0)^\dagger\mu\mathscr{K}\right\} \ . \tag{6.49}$$

This k-instanton formula, although written in ADHM collective coordinates, is nevertheless easily compared with the one-instanton expression for the action with SU(2) gauge symmetry found in Ref. [38]. The first term in parentheses is equivalent to $\sum_i |q^0|^2\rho_i^2$, summed over the k different instantons, where ρ_i is the scale size of the ith instanton. Also the second term in parentheses is the fermion bi-linear necessary to promote this ρ_i^2 to the supersymmetric invariant scale size constructed in [38]. Independent of one's choice of collective coordinates, the presence of the VEV in the instanton effective action (6.49) gives a natural cut-off to the integrations over instanton scale sizes [2], leading to an infra-red-safe application of instanton calculus.

The expressions given above may be immediately extended to phenomenologically more interesting models with N_F fundamental flavours of Dirac fermions (known as a hypermultiplet). In this case the gauge multiplet is minimally coupled to $2N_F$ chiral superfields Q_f and $\tilde{Q}_f$, $1 \leqslant f \leqslant N_F$, where Q_f transforms in the $\boldsymbol{N}$ and $\tilde{Q}_f$ in the $\bar{\boldsymbol{N}}$ representation of the gauge group. We will take Q ($\tilde{Q}$) to be an $N \times N_F$ ($N_F \times N$), but for greater clarity we will often write the flavour indices explicitly. The action for the matter fields is

$$\begin{aligned} S_{\text{matter}} = \int d^4x \Big\{ &\mathscr{D}_n q^\dagger \mathscr{D}_n q + \mathscr{D}_n\tilde{q}\mathscr{D}_n\tilde{q}^\dagger - \mathscr{D}_n\bar{\chi}\bar{\sigma}_n\chi + \tilde{\chi}\sigma_n\mathscr{D}_n\bar{\tilde{\chi}} \\ &-\sqrt{2}ig\bar{\chi}\bar{\lambda}q + i\sqrt{2}q^\dagger\lambda\chi + \sqrt{2}ig\tilde{q}\bar{\lambda}\bar{\tilde{\chi}} - \sqrt{2}ig\tilde{\chi}\lambda\tilde{q}^\dagger + \tfrac{1}{4}g^2(q^\dagger q - \tilde{q}\tilde{q}^\dagger)^2 \Big\} \ . \end{aligned} \tag{6.50}$$

The classical moduli space of the theory in the Higgs phase is given by [59,60]

$$q^0_{uf} = \begin{pmatrix} v_1 & 0 & \ldots & 0 \\ 0 & v_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & v_f \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}, \qquad \tilde q^0_{fu} = \begin{pmatrix} \tilde v_1 & 0 & \ldots & 0 & \ldots & 0 \\ 0 & \tilde v_2 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \tilde v_f & \ldots & 0 \end{pmatrix} . \tag{6.51}$$

The VEV matrices in Eq. (6.51) correspond to the cases $N_F \leqslant N$. The cases $N_F > N$ are similar except that the VEV matrices have extra rows of zeroes rather than columns or vice versa. These VEVs are not all independent; the D-flatness condition requires that for each value of f,

$$|v_f|^2 = \begin{cases} |\tilde v_f|^2 + a^2, & N_F \geqslant N \ , \\ |\tilde v_f|^2 & N_F < N \ , \end{cases} \tag{6.52}$$

where a^2 is an arbitrary constant, independent of the colour index u.

Now Eqs. (6.39) and (6.41) generalize to [41]

$$\chi_{\alpha f} = g^{-1/2} \bar U b_\alpha f \mathscr{K}_f, \qquad \tilde\chi_{f\alpha} = g^{-1/2} \tilde{\mathscr{K}}_f f \bar b_\alpha U \tag{6.53}$$

and

$$q_f = \bar U \begin{pmatrix} q^0_f \\ 0 \end{pmatrix} - \frac{\mathrm{i}}{2\sqrt2} \bar U \mathscr{M} f \mathscr{K}_f, \qquad \tilde q_f = (\tilde q^0_f \quad 0) U + \frac{\mathrm{i}}{2\sqrt2} \tilde{\mathscr{K}}_f f \bar{\mathscr{M}} U \ , \tag{6.54}$$

respectively. The leading contribution to the instanton effective action can be worked out in a completely analogous way to (6.49), yielding

$$\tilde S = \pi^2 \sum_{f=1}^{N_F} \left\{ q^{0\dagger}_f w_{\dot\alpha} \bar w^{\dot\alpha} q^0_f + \frac{\mathrm{i}}{\sqrt2} q^{0\dagger}_f \mu \mathscr{K}_f + \tilde q^0_f w_{\dot\alpha} \bar w^{\dot\alpha} \tilde q^{0\dagger} - \frac{\mathrm{i}}{\sqrt2} \tilde{\mathscr{K}}_f \bar\mu \tilde q^{0\dagger}_f \right\} . \tag{6.55}$$

The $\mathscr{N}=1$ supersymmetry transformation properties of the ADHM variables were constructed in Section 4.5. To check the invariance of expression (6.55), it is necessary as well to derive the transformation properties for the Grassmann collective coordinates $\mathscr{K}$ and $\tilde{\mathscr{K}}$ associated with the fundamental fermions. As with the other collective coordinates, this may be straightforwardly accomplished by equating "active" and "passive" supersymmetry transformations on the Higgsinos χ and $\tilde\chi$. In this way one obtains

$$\delta\mathscr{K}_f = -2\sqrt2 \bar\xi_{\dot\alpha} \bar w^{\dot\alpha} q^0_f, \qquad \delta\tilde{\mathscr{K}}_f = -2\sqrt2 \tilde q^0_f w_{\dot\alpha} \bar\xi^{\dot\alpha} . \tag{6.56}$$

[41] Here, and in the following, one should not confuse the flavour subscript f with the ADHM quantity f.

It is now easily checked that action (6.55) is invariant under supersymmetry transformations (4.68), (4.70) and (6.56).

It remains to determine how one integrates over the new Grassmann collective coordinates in the semi-classical approximation of the functional integral. As for the adjoint-valued fields, the non-zero mode fluctuation determinants cancel, up to a power of the Pauli–Villars mass scale μ^{-2kN_F}. The measure for integrating over the matter zero modes follows from the inner-product formula

$$\int \mathrm{d}^4x \sum_{f=1}^{N_F} \tilde{\chi}_f \chi_f = g^{-1} \int \mathrm{d}^4x \sum_{f=1}^{N_F} \tilde{\mathscr{K}}_f f \bar{b}_\alpha \mathscr{P} b^\alpha \mathscr{K}_f = -\frac{1}{4g} \int \mathrm{d}^4x \sum_{f=1}^{N_F} \tilde{\mathscr{K}}_f \Box f \mathscr{K}_f$$

$$= \frac{\pi^2}{g} \sum_{f=1}^{N_F} \tilde{\mathscr{K}}_f \mathscr{K}_f \ , \tag{6.57}$$

employing (C.2b). The integration measure for the matter Grassmann collective coordinates is then

$$\pi^{-2kN_F} \mu^{-kN_F} g^{kN_F} \int \mathrm{d}^{kN_F} \mathscr{K} \, \mathrm{d}^{kN_F} \tilde{\mathscr{K}} \ . \tag{6.58}$$

For convenience, we define the combination

$$\int_{\mathfrak{M}_k} \boldsymbol{\omega}^{(\mathscr{N},N_F)} = \pi^{-2kN_F} \int_{\mathfrak{M}_k} \boldsymbol{\omega}^{(\mathscr{N})} \cdot \mathrm{d}^{kN_F} \mathscr{K} \, \mathrm{d}^{kN_F} \tilde{\mathscr{K}} \tag{6.59}$$

and then a generalization of the instanton partition function (5.16)

$$\mathscr{Z}_k^{(\mathscr{N},N_F)} = \int_{\mathfrak{M}_k} \boldsymbol{\omega}^{(\mathscr{N},N_F)} \mathrm{e}^{-\tilde{S}} \tag{6.60}$$

to include the matter fields.

We will see that the $\mathscr{N}=1$ action (6.55) possesses two simplifying properties that the $\mathscr{N}=2$ action, to be discussed below, does not. First, Eq. (6.55) has the form of a disconnected sum of k single instantons; with the choice of these ADHM coordinates there is no interaction between them. Second, the only gaugino modes that are lifted (i.e. that appear in the action) are those associated with the top elements μ and $\bar{\mu}$ of the collective coordinate matrices $\mathscr{M}$ and $\bar{\mathscr{M}}$. This leaves $\mathscr{O}(k)$ unlifted gaugino modes after one implements fermionic constraints (4.34). This counting contrasts sharply with the $\mathscr{N}=2$ theories in which the number of unlifted modes is independent of the winding number k. Saturating each of these unlifted modes with an anti-Higgsino as per Affleck et al. [50] one sees that unlike the $\mathscr{N}=2$ theory, here the sectors of different topological number cannot interfere with one another, since the corresponding Green's functions are distinguished by different (anti-)fermion content.

6.3.2. $\mathscr{N}=2$ theories on the Coulomb branch

Now we consider $\mathscr{N}=2$ theories with N_F flavours of fundamental hypermultiplets. Each such hypermultiplet comprises a pair of $\mathscr{N}=1$ chiral multiplets, Q_f and $\tilde{Q}_f$, with the same conventions for component fields as in the $\mathscr{N}=1$ case discussed in the last section. In $\mathscr{N}=1$ language, these matter fields couple to the gauge multiplet via a superpotential,

$$W = \sqrt{2} g \sum_{f=1}^{N_F} \tilde{Q}_f \Phi Q_f \ . \tag{6.61}$$

In component form, the matter fields have the Euclidean action

$$S_{\rm matter} = \int d^4x\{\mathscr{D}_n q^\dagger \mathscr{D}_n q + \mathscr{D}_n \tilde{q} \mathscr{D}_n \tilde{q}^\dagger - \mathscr{D}_n \bar{\chi}\bar{\sigma}_n \chi + \tilde{\chi}\sigma_n \mathscr{D}_n \bar{\tilde{\chi}}$$
$$-\sqrt{2}ig\bar{\chi}\bar{\lambda}q + i\sqrt{2}q^\dagger \lambda\chi + \sqrt{2}ig\tilde{q}\bar{\lambda}\bar{\tilde{\chi}} - \sqrt{2}ig\tilde{\chi}\lambda\tilde{q}^\dagger - \sqrt{2}g\tilde{\chi}\psi q - \sqrt{2}g\tilde{q}\psi\chi$$
$$-g\tilde{\chi}\phi\chi - \sqrt{2}gq^\dagger\bar{\psi}\bar{\tilde{\chi}} - \sqrt{2}g\bar{\chi}\bar{\psi}\tilde{q}^\dagger - g\bar{\chi}\phi^\dagger\bar{\tilde{\chi}}\} + S_{\rm scalar} \;, \tag{6.62}$$

where $S_{\rm scalar}$ are the interaction terms between the scalar fields whose explicit form we do not need. In what follows, we will restrict our attention to the Coulomb branch of the $\mathcal{N}=2$ theory where the hypermultiplet squarks do not acquire VEVs. The classical component fields χ_f, $\tilde{\chi}_f$, q_f and $\tilde{q}_f$ are still given by Eqs. (6.53)–(6.54), except that on the Coulomb branch the first terms on the right-hand sides of Eq. (6.54) are zero. The adjoint-valued fermions have their usual ADHM form $\lambda = g^{-1/2}\Lambda(\mathscr{M}^1)$ and $\psi = g^{-1/2}\Lambda(\mathscr{M}^2)$. To leading order, the scalar field ϕ satisfies the same equation-of-motion as in the pure $\mathcal{N}=2$ gauge theory and so the solution follows from Eqs. (4.64) and (4.65) (along with definition of ϕ in (4.20)): [42]

$$\phi = \frac{i}{2}\bar{U}\mathscr{M}^A f \bar{\mathscr{M}}_A U + \bar{U}\begin{pmatrix}\phi^0 & 0\\ 0 & \varphi 1_{[2]\times[2]}\end{pmatrix}U \;, \tag{6.63}$$

where

$$\varphi = \boldsymbol{L}^{-1}\left(-\frac{i}{2}\bar{\mathscr{M}}^A\mathscr{M}_A + \bar{w}^{\dot{\alpha}}\phi^0 w_{\dot{\alpha}}\right) \;. \tag{6.64}$$

On the other hand, the anti-holomorphic component now satisfies the inhomogeneous equation

$$\mathscr{D}^2\phi^\dagger = -g\chi\tilde{\chi} \;. \tag{6.65}$$

The solution of this equation is readily shown to be

$$\phi^\dagger = \bar{U}\begin{pmatrix}\phi^{0\dagger} & 0\\ 0 & \varphi^\dagger 1_{[2]\times[2]}\end{pmatrix}U \;, \tag{6.66}$$

where

$$\varphi^\dagger = \boldsymbol{L}^{-1}\left(-\frac{1}{4}\sum_{f=1}^{N_F}\mathscr{K}_f\tilde{\mathscr{K}}_f + \bar{w}^{\dot{\alpha}}\phi^{0\dagger}w_{\dot{\alpha}}\right) \;. \tag{6.67}$$

Now we consider the matter fields themselves. The new feature in the $\mathcal{N}=2$ theory is that the fields $q^\dagger$ and $\tilde{q}^\dagger$ now satisfy the non-trivial equations [43]

$$\mathscr{D}^2 q^\dagger = -\sqrt{2}g\tilde{\chi}\psi, \qquad \mathscr{D}^2\tilde{q}^\dagger = -\sqrt{2}g\psi\chi \tag{6.68}$$

[42] In the following when discussing the $\mathcal{N}=2$ theory, we use the ε tensor to raise and lower SU(2) R-symmetry spinor indices in the usual way following the conventions of [47].

[43] As usual, to leading order where we set the anti-chiral fermions to zero and ignore the potential terms for the scalar fields.

with solutions

$$q_f^\dagger = \frac{1}{2\sqrt{2}}\,\tilde{\mathscr{K}}_f f \bar{\mathscr{M}}^2 U, \qquad \tilde{q}_f^\dagger = \frac{1}{2\sqrt{2}}\,\bar{U}\mathscr{M}^2 f \mathscr{K}_f\ . \tag{6.69}$$

As usual, only the kinetic terms of the scalar fields and the Yukawa interactions involving the fermions (rather than the anti-chiral fermions) contribute at leading order to the instanton effective action:

$$\tilde{S} = \int \mathrm{d}^4x\{\mathscr{D}_n\phi^\dagger\mathscr{D}_n\phi + \mathscr{D}_n q^\dagger\mathscr{D}_n q + \mathscr{D}_n\tilde{q}\mathscr{D}_n\tilde{q}^\dagger + 2\mathrm{i}g[\phi^\dagger,\lambda]\psi$$
$$+\sqrt{2}\mathrm{i}gq^\dagger\lambda\chi - \sqrt{2}\mathrm{i}g\tilde{\chi}\lambda\tilde{q}^\dagger - \sqrt{2}g\tilde{\chi}\psi q - \sqrt{2}g\tilde{q}\psi\chi - g\tilde{\chi}\phi\chi\}\ . \tag{6.70}$$

This can be simplified by integrating the kinetic terms by parts and using the equations-of-motion for the scalar fields:

$$\tilde{S} = \int \mathrm{d}^4x\{\partial_n(\phi^\dagger\mathscr{D}_n\phi) + \partial_n(q^\dagger\mathscr{D}_n q) + \partial_n(\tilde{q}\mathscr{D}_n\tilde{q}^\dagger) - g\tilde{\chi}(\sqrt{2}\mathrm{i}\lambda\tilde{q}^\dagger + \sqrt{2}\psi q + \phi\chi)\}\ . \tag{6.71}$$

The first three terms are converted into surface terms on a large sphere at infinity. Only ϕ has a VEV, so only the first term is none vanishing. Using

$$\frac{x_n}{x}\mathscr{D}_n\phi \overset{x\to\infty}{\to} \frac{1}{x^3}\{2\mathrm{i}\mu^A\bar{\mu}_A + w_{\dot\alpha}\bar{w}^{\dot\alpha}\phi^0 + \phi^0 w_{\dot\alpha}\bar{w}^{\dot\alpha} - 2w_{\dot\alpha}\varphi\bar{w}^{\dot\alpha}\} \tag{6.72}$$

we find the contribution to $\tilde{S}$:

$$4\pi^2\,\mathrm{tr}_k\left\{-\frac{\mathrm{i}}{2}\,\bar{\mu}^A\phi^{0\dagger}\mu_A + \bar{w}^{\dot\alpha}|\phi^0|^2 w_{\dot\alpha} - \varphi\bar{w}^{\dot\alpha}\phi^{0\dagger}w_{\dot\alpha}\right\}\ . \tag{6.73}$$

The contribution from the remaining Yukawa interactions can be evaluated in a similar way to the manipulations required to evaluate the instanton effective action in Section 5.2 (see also Appendix C, Eq. (C.34) and [61]). The strategy is to write

$$\sqrt{2}\mathrm{i}\lambda\tilde{q}^\dagger + \sqrt{2}\psi q + \phi\chi = g^{-1/2}(\mathscr{D}\bar{\Upsilon} + \Theta)\ , \tag{6.74}$$

where Θ is a fundamental zero mode, $\bar{\mathscr{D}}\Theta = 0$. We now verify that the solution for $\bar{\Upsilon}$ and Θ is

$$\bar{\Upsilon}_f^{\dot\alpha} = -\frac{\mathrm{i}}{4}\,\bar{U}\mathscr{M}^A f\bar{\Delta}^{\dot\alpha}\mathscr{M}_A f\mathscr{K}_f + \bar{U}\begin{pmatrix}\phi^0 & 0\\ 0 & \varphi\end{pmatrix}\Delta^{\dot\alpha} f\mathscr{K}_f \tag{6.75}$$

and

$$\Theta_f = \bar{U}b_\alpha f\varphi\mathscr{K}_f\ . \tag{6.76}$$

First of all, using (2.51), the left-hand side of (6.74) is

$$g^{1/2}(\sqrt{2}\mathrm{i}\lambda\tilde{q}^\dagger + \sqrt{2}\psi q + \phi\chi) = \frac{\mathrm{i}}{2}\,\bar{U}\{\mathscr{M}^A f b_\alpha\mathscr{P}\mathscr{M}_A - b_\alpha f\bar{\mathscr{M}}^A\mathscr{P}\mathscr{M}_A$$
$$+\mathscr{M}^A f\bar{\mathscr{M}}_A\mathscr{P}b_\alpha\}f\mathscr{K}_f + \bar{U}\begin{pmatrix}\phi^0 & 0\\ 0 & \varphi\end{pmatrix}\mathscr{P}b_\alpha f\mathscr{K}_f\ . \tag{6.77}$$

Then using the formula

$$\mathscr{D}_{\alpha\dot\alpha}(\bar U\mathscr{J}) = \bar U\partial_{\alpha\dot\alpha}\mathscr{J} - 2\bar U b_\alpha f\bar\Delta_{\dot\alpha}\mathscr{J} \ , \tag{6.78}$$

we have

$$\begin{aligned}\mathscr{D}\!\!\!/_{\alpha\dot\alpha}\bar\Upsilon^{\dot\alpha} &= \frac{\mathrm{i}}{2}\,\varepsilon_{AB}\bar U\{b_\alpha f\bar{\mathscr{M}}^A(1-\mathscr{P})\mathscr{M}_A + \mathscr{M}^A f\bar b_\alpha\mathscr{P}\mathscr{M}_A + \mathscr{M}^A f\bar{\mathscr{M}}_A\mathscr{P}b_\alpha\}f\mathscr{K}_f \\ &\quad + \bar U\left\{\begin{pmatrix}\phi^0 & 0\\ 0 & \varphi\end{pmatrix}(\mathscr{P}-1)b_\alpha + b_\alpha f\bar\Delta^{\dot\alpha}\begin{pmatrix}\phi^0 & 0\\ 0 & \varphi\end{pmatrix}\Delta_{\dot\alpha}\right\}f\mathscr{K}_f \ .\end{aligned} \tag{6.79}$$

In the last term we can now use identity (C.32), along with the expression for φ in (6.64):

$$\bar\Delta^{\dot\alpha}\begin{pmatrix}\phi^0 & 0\\ 0 & \varphi\end{pmatrix}\Delta_{\dot\alpha} = -\frac{\mathrm{i}}{2}\,\bar{\mathscr{M}}^A\mathscr{M}_A + \{\varphi, f^{-1}\} \ . \tag{6.80}$$

This simplifies (6.79)

$$\begin{aligned}\mathscr{D}\!\!\!/_{\alpha\dot\alpha}\bar\Upsilon^{\dot\alpha} &= \frac{\mathrm{i}}{2}\,\bar U\{-b_\alpha f\bar{\mathscr{M}}^A\mathscr{P}\mathscr{M}_A + \mathscr{M}^A f\bar b_\alpha\mathscr{P}\mathscr{M}_A + \mathscr{M}^A f\bar{\mathscr{M}}_A\mathscr{P}b_\alpha\}f\mathscr{K}_f \\ &\quad + \bar U\begin{pmatrix}\phi^0 & 0\\ 0 & \varphi\end{pmatrix}\mathscr{P}b_\alpha f\mathscr{K}_f + \bar U b_\alpha f\varphi\mathscr{K}_f \ .\end{aligned} \tag{6.81}$$

Now one can see that apart from the last term, which is the fundamental zero mode Θ, this is precisely (6.77). So the contribution to $\tilde S$ from the Yukawa terms is given by

$$-\,g^{1/2}\int d^4x\,\tilde\chi\Theta \ , \tag{6.82}$$

which can be evaluated using inner-product formula (6.57):

$$-\,\pi^2\sum_{f=1}^{N_F}\tilde{\mathscr{K}}_f\varphi\mathscr{K}_f = \frac{\pi^2}{g^2}\,\mathrm{tr}_k\left[\left(\sum_{f=1}^{N_F}\mathscr{K}_f\tilde{\mathscr{K}}_f\right)\boldsymbol{L}^{-1}\left(-\frac{\mathrm{i}}{2}\,\bar{\mathscr{M}}^A\mathscr{M}_A + \bar w^{\dot\alpha}\phi^0 w_{\dot\alpha}\right)\right] \ . \tag{6.83}$$

Summing the result with (6.73) gives the leading-order expression for the instanton effective action:

$$\begin{aligned}\tilde S = 4\pi^2\,\mathrm{tr}_k&\left\{-\frac{\mathrm{i}}{2}\,\bar\mu^A\phi^{0\dagger}\mu_A + \bar w^{\dot\alpha}|\phi^0|^2 w_{\dot\alpha}\right. \\ &\left. + \left(\frac{1}{4}\sum_{f=1}^{N_F}\mathscr{K}_f\tilde{\mathscr{K}}_f - \bar w^{\dot\alpha}\phi^{0\dagger}w_{\dot\alpha}\right)\boldsymbol{L}^{-1}\left(-\frac{\mathrm{i}}{2}\bar{\mathscr{M}}^A\mathscr{M}_A + \bar w^{\dot\alpha}\phi^0 w_{\dot\alpha}\right)\right\} \ .\end{aligned} \tag{6.84}$$

As with the $\mathscr{N}=1$ action (6.55), one can check that this expression is a supersymmetric invariant. Notice on the Coulomb branch, Eq. (6.56) collapses to

$$\delta\mathscr{K}_f = \delta\tilde{\mathscr{K}}_f = 0 \ . \tag{6.85}$$

We can also add $\mathcal{N}=2$ preserving masses for the hypermultiplets as explained in Section 6.4. Finally, the collective coordinate integration measure for the Grassmann collective coordinates is given by the same expression, Eq. (6.58), as in the $\mathcal{N}=1$ theory.

6.4. Masses

In certain circumstances one wants to add masses for some of the fields. Here we assess the affects on the instanton calculus. It turns out that the effect will be simple to incorporate when working to leading order in the semi-classical approximation.

The interesting cases involve $\mathcal{N}=1$ preserving mass terms in either $\mathcal{N}=2$ or 4 theories. The simplest case consists of a mass term for a fundamental hypermultiplet transforming in the $(\boldsymbol{N},\bar{\boldsymbol{N}})$ as described in Section 6.3. When Wick rotated to Euclidean space, the conventional mass term in components is

$$S_{\text{mass}} = \int d^4x\{m\tilde{\chi}\chi + m^*\bar{\chi}\bar{\tilde{\chi}} + 2|m|^2q^\dagger q + 2|m|^2\tilde{q}\tilde{q}^\dagger\} \ . \tag{6.86}$$

In principle, these mass terms contribute new terms to the equations-of-motion which will affect the instanton solutions themselves. In applications one is typically interested in quantities that are known to have holomorphic dependence on the mass. We then expect that the semi-classical approximation respects this dependence.[44] Hence for a holomorphic quantity, we can treat m and m^* as independent variables and then set $m^*=0$. The mass terms then only affect the equations-of-motion of the anti-chiral fermions $\bar{\chi}$ and $\bar{\tilde{\chi}}$ and the supersymmetric instanton remains unaffected to leading order. Hence, we simply have to evaluate the first term in Eq. (6.86) in the background of the supersymmetric instanton and add the resulting expression to the instanton effective action $\tilde{S}$. The contribution can be determined using inner-product formula (6.57). For N_F such hypermultiplets

$$\tilde{S}_{\text{mass}} = \frac{\pi^2}{g}\sum_{f=1}^{N_F} m_f\tilde{\mathcal{K}}_f\mathcal{K}_f \ . \tag{6.87}$$

Now we turn to masses for adjoint chiral superfields. As discussed above only the mass term for the chiral fermions is relevant. In the $\mathcal{N}=4$ theory, the most general mass term for the chiral fermions is of the form

$$S_{\text{mass}} = \int d^4x\, m_{AB}\,\text{tr}_N\,\lambda^A\lambda^B \ , \tag{6.88}$$

where m_{AB} is a symmetric matrix which in a suitable basis is diagonal $m_{AB}=\text{diag}(m_1,m_2,m_3,m_4)$. In the general case, all the supersymmetry is broken, while if one, respectively two, of the eigenvalues vanishes then the coupling arises from mass terms that preserve $\mathcal{N}=1$, respectively $\mathcal{N}=2$, supersymmetry. To find the effect on the instanton effective action we substitute the ADHM form

[44] It should be possible to prove this by showing that the relevant Ward identity is respected. Such a proof would follow the lines of Section 7.1 which considers a different Ward identity.

$\lambda^A = g^{-1/2}\Lambda(\mathcal{M}^A)$ and evaluate the integral using Corrigan's inner-product formula (4.37):

$$\tilde{S}_{\rm mass} = -\frac{m_{AB}\pi^2}{g}\,{\rm tr}_k\,\bar{\mathcal{M}}^A(\mathcal{P}_\infty + 1)\mathcal{M}^B = -\frac{m_{AB}\pi^2}{g}\,{\rm tr}_k[2\bar{\mu}^A\mu^B + \mathcal{M}'^{\alpha A}\mathcal{M}'^B_{\alpha}]\ . \qquad (6.89)$$

6.5. The instanton partition function

For many applications, and to anticipate later conceptual developments, it is useful to re-formulate the collective coordinate integral, or instanton partition function (5.16), by introducing some auxiliary variables which have the effect of "linearizing" the integral in a sense to be described. To start with, we introduce Lagrange multipliers for the bosonic and fermionic ADHM δ-function constraints and also, for $\mathcal{N} > 1$, other additional auxiliary variables are introduced. In Section 10 we will see for $\mathcal{N} > 1$ this linearized form of the collective coordinate integral has an important relation to higher-dimensional field theories. This point of view will also be the starting point for understanding the relation of the instanton calculus to D-branes in string theory as described in Section 10.3.

In the linearized formalism one introduces auxiliary variables in the form of: χ_a, a $2(\mathcal{N}-1)$-vector of Hermitian $k\times k$ matrices; $\vec{D}$, a 3-vector of Hermitian $k\times k$ matrices; and $k\times k$ matrices of Grassmann superpartners $\bar{\psi}^{\dot{\alpha}}_A$, $A = 1,\ldots,\mathcal{N}$. The instanton partition function (5.16) can be written as

$$\mathcal{Z}^{(\mathcal{N})}_k = \frac{2^{2(2-\mathcal{N})k^2}\pi^{(2-3\mathcal{N})k^2}C^{(\mathcal{N})}_k}{{\rm Vol}\,{\rm U}(k)}\int {\rm d}^{4k(N+k)}a\,{\rm d}^{3k^2}D\,{\rm d}^{2(\mathcal{N}-1)k^2}\chi\prod_{A=1}^{\mathcal{N}}{\rm d}^{2k(N+k)}\mathcal{M}^A\,{\rm d}^{2k^2}\bar{\psi}_A{\rm e}^{-\tilde{S}}\ , \qquad (6.90)$$

where the instanton effective action is

$$\tilde{S} = 4\pi^2\,{\rm tr}_k\{\chi_a\boldsymbol{L}\chi_a + \tfrac{1}{2}\bar{\Sigma}_{aAB}\mathcal{M}^A\mathcal{M}^B\chi_a\} + \tilde{S}_{\rm L.m.}\ , \qquad (6.91)$$

where

$$\tilde{S}_{\rm L.m.} = -4{\rm i}\pi^2\,{\rm tr}_k\{\bar{\psi}^{\dot{\alpha}}_A(\bar{\mathcal{M}}^A a_{\dot{\alpha}} + \bar{a}_{\dot{\alpha}}\mathcal{M}^A) + \vec{D}\cdot\vec{\tau}^{c\dot{\alpha}}{}_{\dot{\beta}}\bar{a}^{\dot{\beta}}a_{\dot{\alpha}}\}\ . \qquad (6.92)$$

The previous form of the collective coordinate integral (5.55) is recovered by integrating out the auxiliary variables $\{\chi_a, \vec{D}, \bar{\psi}^{\dot{\alpha}}_A\}$. Specifically, integrating out the Lagrange multipliers $\vec{D}$ and $\bar{\psi}_A$ yields the δ-functions in (5.55) imposing the ADHM constraints and their Grassmann analogues, (2.65) and (4.34). The Gaussian integrals over χ_a yield the appropriate power of $\det_{k^2}\boldsymbol{L}$ in (5.55) as well as the quadrilinear couplings of Grassmann collective coordinates of the $\mathcal{N}=4$ theory (5.26).

One of the advantages of the linearized form of the partition function is that it is straightforward to incorporate scalar VEVs on the Coulomb branch in the $\mathcal{N}=2$ and 4 theories. One simply generalizes the following couplings in the instanton effective action $\tilde{S}$:

$$w_{\dot{\alpha}}\chi_a \to w_{\dot{\alpha}}\chi_a + \phi^0_a w_{\dot{\alpha}}, \qquad \chi_a\bar{w}^{\dot{\alpha}} \to \chi_a\bar{w}^{\dot{\alpha}} + \bar{w}^{\dot{\alpha}}\phi^0_a\ ,$$

$$\mu^A\chi_a \to \mu^A\chi_a + \phi^0_a\mu^A, \qquad \chi_a\bar{\mu}^A \to \chi_a\bar{\mu}^A + \bar{\mu}^A\phi^0_a\ . \qquad (6.93)$$

The effective instanton action is now

$$\tilde{S} = 4\pi^2 \operatorname{tr}_k\{|w_{\dot\alpha}\chi_a + \phi^0_a w_{\dot\alpha}|^2 - [\chi_a, a'_n]^2 + \tfrac{1}{2}\bar\Sigma_{aAB}\bar\mu^A(\mu^B\chi_a + \phi^0_a\mu^B) + \tfrac{1}{2}\bar\Sigma_{aAB}\mathscr{M}'^A\mathscr{M}'^B\chi_a\} + \tilde{S}_{\text{L.m.}}\,, \tag{6.94}$$

where we used the definition of $\boldsymbol{L}$ in (2.125). The supersymmetry transformations on the linearized system are

$$\delta a'_{\alpha\dot\alpha} = \mathrm{i}\bar\xi_{\dot\alpha A}\mathscr{M}'^A_\alpha, \qquad \delta\mathscr{M}'^A_\alpha = -2\mathrm{i}\Sigma_a^{AB}\bar\xi^{\dot\alpha}_B[a'_{\alpha\dot\alpha}, \chi_a]\,, \tag{6.95a}$$

$$\delta w_{\dot\alpha} = \mathrm{i}\bar\xi_{\dot\alpha A}\mu^A, \qquad \delta\mu^A = -2\mathrm{i}\Sigma_a^{AB}\bar\xi^{\dot\alpha}_B(w_{\dot\alpha}\chi_a + \phi^0_a w_{\dot\alpha})\,, \tag{6.95b}$$

$$\delta\chi_a = -\Sigma_a^{AB}\bar\xi_{\dot\alpha A}\bar\psi^{\dot\alpha}_B, \qquad \delta\bar\psi^{\dot\alpha}_A = 2\bar\Sigma_{abA}{}^B[\chi_a, \chi_b]\bar\xi^{\dot\alpha}_B - \mathrm{i}\vec{D}\cdot\vec\tau^{\dot\alpha}{}_{\dot\beta}\bar\xi^{\dot\beta}_A\,, \tag{6.95c}$$

$$\delta\vec{D} = -\mathrm{i}\vec\tau^{\dot\alpha}{}_{\dot\beta}\Sigma_a^{AB}\bar\xi_{\dot\alpha B}[\bar\psi^{\dot\beta}_A, \chi_a]\,. \tag{6.95d}$$

On integrating out the auxiliary variables $\{\chi_a, \vec{D}, \bar\psi_A\}$, these transformations reduce to those constructed in Section 6.5.

In the $\mathcal{N}=2$ theory it is rather simple to incorporate fundamental hypermultiplets. The instanton effective action, generalizing (6.94), is

$$\tilde{S} = 4\pi^2 \operatorname{tr}_k\left\{|w_{\dot\alpha}\chi_a + \phi^0_a w_{\dot\alpha}|^2 - [\chi_a, a'_n]^2 + \frac{\mathrm{i}}{2}\bar\mu^A(\mu_A\chi^\dagger + \phi^{0\dagger}\mu_A) + \frac{\mathrm{i}}{2}\mathscr{M}'^A\mathscr{M}'_A\chi^\dagger + \frac{1}{4}\sum_{f=1}^{N_F}\mathscr{K}_f\tilde{\mathscr{K}}_f(\chi - g^{-1}m_f)\right\} + \tilde{S}_{\text{L.m.}}\,, \tag{6.96}$$

where we have defined

$$\phi^0 = \phi^0_1 - \mathrm{i}\phi^0_2, \qquad \chi = \chi_1 - \mathrm{i}\chi_2 \tag{6.97}$$

and we have allowed for arbitrary hypermultiplet masses (6.87). Integrating out χ_a gives the instanton effective action (6.84). Supersymmetry transformations (6.95a)–(6.95d) are then augmented with (6.85).

In applications of the instanton calculus it is useful to define the notion of the "centred instanton partition function". This is defined in terms of an integral over the centred instanton moduli space $\hat{\mathfrak{M}}_k$, (2.16), where the overall position coordinates and their superpartners, the Grassmann collective coordinates for the supersymmetries broken by the bosonic instanton solution (see Section 4.2.3),

$$X_n = -k^{-1}\operatorname{tr}_k a'_n, \qquad \xi^A = \frac{\mathrm{i}}{4}k^{-1}\operatorname{tr}_k\mathscr{M}'^A\,, \tag{6.98}$$

have been factored off. First of all, from the expression for metric (2.66) we have

$$ds^2_{\mathfrak{M}_k} = 8\pi^2 k\, dX_n\, dX_n + ds^2_{\hat{\mathfrak{M}}_k}\,. \tag{6.99}$$

Therefore taking account of the normalization of the measure in the non-supersymmetric theory (3.16), we have

$$\int_{\mathfrak{M}_k} \boldsymbol{\omega} = \int (4\pi k)^2\, \mathrm{d}^4 X \cdot \int_{\hat{\mathfrak{M}}_k} \boldsymbol{\omega}\ . \tag{6.100}$$

Now we consider the supersymmetric integral. The inner product of the supersymmetric zero modes (4.43) is from (4.37)

$$\int \mathrm{d}^4 x\, \mathrm{tr}_N\, \Lambda(-4\mathrm{i}\xi^A 1_{[k]\times[k]})\Lambda(-4\mathrm{i}\xi^B 1_{[k]\times[k]}) = 16\pi^2 k\, \xi^A \xi^B\ . \tag{6.101}$$

This means

$$\int_{\mathfrak{M}_k} \boldsymbol{\omega}^{(\mathcal{N},N_F)} = \int (4\pi k)^2\, \mathrm{d}^4 X \prod_{A=1}^{\mathcal{N}} (32\pi^2 k)^{-1}\, \mathrm{d}^2 \xi^A \cdot \int_{\hat{\mathfrak{M}}_k} \boldsymbol{\omega}^{(\mathcal{N},N_F)}\ . \tag{6.102}$$

Since the instanton effective action is always independent of X_n and ξ^A, we can define the *centred instanton partition function*, generalizing (6.60) as

$$\hat{\mathscr{Z}}_k^{(\mathcal{N},N_F)} = \int_{\hat{\mathfrak{M}}_k} \boldsymbol{\omega}^{(\mathcal{N},N_F)} \mathrm{e}^{-\tilde{S}}\ . \tag{6.103}$$

7. The gluino condensate in $\mathcal{N}=1$ theories

Pure $\mathcal{N}=1$ gauge theory is in some respects the simplest supersymmetric gauge theory; however, a puzzle arose in the mid-1980s over the numerical value of the gluino condensate. In this section, we will describe how the puzzle—described in detail below—can be resolved using the calculus of many instantons and, in particular, by exploiting the simplifications that occur in the large-N limit.

It is well established that $\mathcal{N}=1$ supersymmetric gauge theory has N physically equivalent vacua, as dictated by the Witten Index, which differ by the phase of the gluino condensate. By dimensional analysis one expects

$$\left\langle \frac{g^2}{16\pi^2}\, \mathrm{tr}_N\, \lambda^2 \right\rangle = c\Lambda^3 \mathrm{e}^{2\pi\mathrm{i}u/N}, \quad u = 1,\ldots,N\ , \tag{7.1}$$

where Λ is the dynamical scale in the theory, while c is a numerical constant. In particular, by a powerful supersymmetric non-renormalization theorem there can be no perturbative corrections to form (7.1) since this would be a series in g and hence would not be holomorphic in Λ.[45] This suggests that if c can be calculated to leading semi-classical order, then it will be exact. Remarkably, there are two approaches in the literature for calculating the gluino condensate each superficially appearing to be exact but which differ in their predictions of the constant c. This disagreement is especially vexing in the light of the fact that *both* involve the use of a single supersymmetric instanton. The first, we shall call the "strong-coupling instanton" (SCI) approach [10,38,62–64], while the second, we call the "weak-coupling instanton" (WCI) approach [50,64–66].

[45] In this context, Λ is a complex quantity carrying with it the phase $\mathrm{e}^{\mathrm{i}\theta/3N}$.

In the SCI approach, one calculates the instanton contributions to the correlation functions of $g^2\,\mathrm{tr}_N\,\lambda^2$ directly in the strongly coupled confining phase of the theory. Strictly speaking, it is therefore not a semi-classical calculation even though it uses instantons (ultimately we will conclude that this is why the SCI approach is flawed). In an instanton background, each insertion of $g^2\,\mathrm{tr}_N\,\lambda^2$ is quadratic in the Grassmann collective coordinates and since the charge-k instanton has $2kN$ fermion zero modes, the latter can only contribute to the kN-point function.[46] Furthermore, by a Ward identity, which we prove in Section 7.1 is respected in the instanton approximation, the kN-point correlation function is independent of the kN spacetime insertion points $\{x^{(i)}\}$. The one-instanton calculation of the N-point function, reviewed in Section 7.2.1, can be done exactly and yields the result

$$\left\langle \frac{g^2}{16\pi^2}\,\mathrm{tr}_N\,\lambda^2(x^{(1)})\times\cdots\times\frac{g^2}{16\pi^2}\,\mathrm{tr}_N\,\lambda^2(x^{(N)})\right\rangle=\frac{2^N}{(N-1)!(3N-1)}\,\Lambda^{3N}\ . \tag{7.2}$$

In order to extract $\langle g^2\,\mathrm{tr}_N\,\lambda^2\rangle$ from correlator (7.2), one then invokes cluster decomposition: taking $|x^{(i)}-x^{(j)}|\gg\Lambda^{-1}$, and remembering the independence of the correlator on $\{x^{(i)}\}$, one simply replaces the left-hand side of Eq. (7.2) by $\langle(g^2/16\pi^2)\mathrm{tr}_N\,\lambda^2\rangle^N$. Taking the Nth root, the net result, based on this one-instanton calculation, reads

$$\left\langle \frac{g^2}{16\pi^2}\,\mathrm{tr}_N\,\lambda^2\right\rangle_{\mathrm{SCI}}=\frac{2}{[(N-1)!(3N-1)]^{1/N}}\,\Lambda^3 e^{2\pi i u/N}\ , \tag{7.3}$$

where $u=1,\dots,N$ indexes the N vacua of the SU(N) theory, and reflects the ambiguity in taking the Nth root. In retrospect—as argued in Refs. [10,63]—the reason why the naive instanton calculation of $\langle(g^2/16\pi^2)\mathrm{tr}_N\,\lambda^2$ gives zero is that the N vacua are being averaged over with equal weight in the instanton approximation and the phases cancel.[47]

In contrast, in the WCI approach, one modifies the pure gauge theory by adding matter superfields in such a way that the gauge symmetry is either entirely broken, or only an abelian subgroup remains, after the Higgs mechanism. In this way the theory can be rendered weakly coupled in the infra-red and semi-classical approaches should be reliable. Here, we choose the former option by adding $N_F=N-1$ matter hypermultiplets in the fundamental representation of the gauge group. In the Higgs phase, all the n-point functions of $g^2\,\mathrm{tr}_N\,\lambda^2$ receive contributions from constrained instantons. In particular, based on fermion zero-mode counting, k-instantons now contribute uniquely to the k-point function. We will argue in Section 7.1 that, just as for the SCI approach, the supersymmetric Ward identity guaranteeing independence of the result on the insertion points is respected in the instanton approximation. In particular, the one-instanton contribution to the one-point function can be performed exactly, as reviewed in Section 7.2.2. The non-renormalization theorems of $\mathcal{N}=1$ supersymmetric gauge theory then permit the analytic continuation of the answer into the confining phase by decoupling the extraneous matter fields. This is achieved by giving them a mass m, and taking the joint limit $m\to\infty$ and $\Lambda_{(N_F)}\to 0$ in the manner dictated by renormalization group

[46] This can also be deduced from the relation between Λ and the running coupling (5.17) $\Lambda^{3N}\sim e^{2\pi i\tau(\mu)}$.

[47] The same argument implies that only the kN-point functions are non-vanishing, matching the selection rule arising from counting fermion zero modes in the k-instanton background.

decoupling. Matching the exact one-instanton calculation onto the effective low-energy theory without matter gives

$$\left\langle \frac{g^2}{16\pi^2} \mathrm{tr}_N\, \lambda^2 \right\rangle_{\mathrm{WCI}} = \Lambda^3 \mathrm{e}^{2\pi \mathrm{i} u/N} . \tag{7.4}$$

Note that the renormalization group decoupling procedure forces the low-energy theory into one of the N degenerate vacua. The puzzle now reveals itself as the mismatch between (7.3) and (7.4).

In this section, following Ref. [67], we review this old controversy, using the many-instanton calculus that we have developed in previous sections. In particular, we shall calculate the k-instanton contribution to the kN-point correlator, in the SCI approach, and the k-point function, in the WCI approach, in the large-N limit where the solution of the ADHM constraints described in Section 6.1 in conjunction with saddle-point methods simplifies the instanton calculus. In a nutshell, our results cast serious doubt on the validity of the SCI calculations of the condensate. Specifically, we will demonstrate that in the SCI approach cluster decomposition is violated at leading order in $1/N$:

$$\left\langle \frac{g^2}{16\pi^2} \mathrm{tr}_N\, \lambda^2(x^{(1)}) \times \cdots \times \frac{g^2}{16\pi^2} \mathrm{tr}_N\, \lambda^2(x^{(kN)}) \right\rangle_{\mathrm{SCI}} \overset{N\to\infty}{\neq} \left\langle \frac{g^2}{16\pi^2} \mathrm{tr}_N\, \lambda^2 \right\rangle_{\mathrm{SCI}}^{k} . \tag{7.5}$$

On the contrary, we will present a new calculation showing that the WCI approach is perfectly consistent with clustering at leading order in $1/N$. But we shall go further and argue that the WCI approach is also consistent with clustering for finite N. The important implications of this observation are as follows. Since cluster decomposition is an essential requirement of quantum field theories (with very mild assumptions that are certainly met by supersymmetric Yang–Mills), the exact quantum correlators must have this property. The fact that cluster decomposition is violated by the instanton-saturated SCI correlators means that—contrary to claims in the literature—the SCI approximation is only giving *part* of the full answer. Since the SCI correlators obey supersymmetric perturbative non-renormalization theorems [65], it necessarily follows that additional *non*-perturbative configurations must be contributing to the correlators. On the contrary, we will argue that the WCI approach, which is a genuine semi-classical technique, is consistent with cluster decomposition. We believe that this conclusion is perfectly natural and simply underlines the fact that instantons are a semi-classical phenomenon and should only be trusted at weak coupling. We should add that there are other non-instanton approaches to calculating the gluino condensate, for instance, via softly broken $\mathcal{N}=2$ gauge theory [67] solved using the theory of Seiberg and Witten [68], or from monopoles acting as instantons in the three-dimensional gauge theory that arises after compactification on a circle [69,70]. These alternative approaches all agree with the WCI answer.

7.1. A supersymmetric Ward identity

A fundamental property of correlation functions involving insertions of lowest components of gauge invariant chiral superfields, of which $\mathrm{tr}_N\, \lambda^2$ is an example, is that they are independent of the insertion points [38,62]. It is of paramount importance for our subsequent arguments that this property is preserved within both the SCI and WCI approaches. To start with, let us briefly review the field theoretic argument. For any gauge-invariant chiral superfield $\Phi = A(x) + \sqrt{2}\theta\psi(x) + \cdots$

one can show that

$$\frac{\partial}{\partial x_n} A(x) = \frac{\mathrm{i}}{4}\,\bar\sigma_n^{\dot\alpha\alpha}\bar\delta_{\dot\alpha}\psi_\alpha(x)\ . \tag{7.6}$$

Here, we have defined $\bar\delta_{\dot\alpha}$ via the supersymmetry variation $\delta = \xi^\alpha\delta_\alpha + \bar\xi_{\dot\alpha}\bar\delta^{\dot\alpha}$. In particular,

$$\frac{\partial}{\partial x_n}\,\mathrm{tr}_N\,\lambda^2(x) = \bar\delta_{\dot\alpha}(\bar\sigma_m^{\dot\alpha\alpha}\,\mathrm{tr}_N\,F_{mn}\lambda_\alpha)\ . \tag{7.7}$$

Now consider the derivative of a correlator of lowest-component fields with respect to one of the insertion points:

$$\frac{\partial}{\partial x_n^{(l)}}\langle A_1(x^{(1)})\times\cdots\times A_p(x^{(p)})\rangle = \frac{\mathrm{i}}{4}\,\bar\sigma_n^{\dot\alpha\alpha}\langle A_1(x^{(1)})\cdots\bar\delta_{\dot\alpha}\psi_\alpha(x^{(l)})\cdots A_p(x^{(p)})\rangle\ . \tag{7.8}$$

The supersymmetry variation can then be commuted through the operators to the left and right, since for a lowest component of a gauge invariant chiral multiplet $\bar\delta_{\dot\alpha}A_j(x)=0$. Furthermore, the vacuum is a supersymmetry invariant, so the right-hand side of (7.8) vanishes and the correlation function is independent of the insertion point $\{x^{(l)}\}$.

The question which we now address is whether the supersymmetric Ward identity described above is respected in either the SCI or WCI approaches? In an $\mathcal{N}=1$ supersymmetric theory, the supersymmetry transformation $\bar\delta_{\dot\alpha}$ lifts to the collective coordinates as (4.68) and (4.70): $\bar\delta^{\dot\alpha}a_{\dot\beta} = \mathrm{i}\delta^{\dot\alpha}{}_{\dot\beta}\mathcal{M}$ and $\bar\delta^{\dot\alpha}\mathcal{M}=0$. Using (7.7), this means

$$\frac{\partial}{\partial x_n}\,\mathrm{tr}_N\,\lambda^2(x) = \mathrm{i}\left(\mu_{ui}\frac{\partial}{\partial w_{ui\dot\alpha}} + \bar\mu_{iu}\frac{\partial}{\partial\bar w_{iu\dot\alpha}} + \mathcal{M}'_{ij\alpha}\frac{\partial}{\partial a'_{ij\alpha\dot\alpha}}\right)\sigma_{m\alpha\dot\alpha}\,\mathrm{tr}_N\,F_{mn}\lambda^\alpha \tag{7.9}$$

and, in addition, $\bar\delta^{\dot\alpha}\,\mathrm{tr}_N\,\lambda^2=0$. In the instanton background, therefore, the $x^{(l)}$ derivative of the multi-point correlator of $g^2\,\mathrm{tr}_N\,\lambda^2$ is equal to an integral over $\mathfrak{M}_k$ of a total derivative. We can use Stokes' Theorem to write this as an integral over the boundary of $\mathfrak{M}_k$. The only possible contributions can come from the large sphere at infinity or the small spheres surrounding one of the insertion points. To judge whether the contributions are non-vanishing, we need to determine the asymptotic behaviour of the integrand.

First of all, consider the SCI instanton approach, for which the relevant collective coordinate measure is (5.55) with $\mathcal{N}=1$. Consider the contribution from the sphere of large radius R. By this we mean where the ADHM variables scale as

$$\mathrm{tr}_k(\bar w^{\dot\alpha}w_{\dot\alpha} + a'_n a'_n) \overset{R\to\infty}{\sim} R^2\ . \tag{7.10}$$

The relevant asymptotic behaviour we need is

$$\mathrm{tr}_N\,\lambda^2 \sim R^{-4}, \qquad \mathrm{tr}_N\,F_{mn}\lambda \sim R^{-4}\ , \tag{7.11}$$

so, collectively, the kN operator insertions scale as R^{-4kN}. To complete the analysis we have to determine the scaling of the supersymmetric volume form (5.55). There are two sources of R dependence: firstly there are $4k(N+k)-1$ integrals over the c-number collective coordinates on the boundary, giving $R^{4k(N+k)-1}$, and secondly the δ-functions for the bosonic and fermionic ADHM constraints scale as R^{-4k^2}.[48] Overall, therefore, the volume form on the boundary scales as R^{4kN-1}. Putting this together with the scaling of the operator insertions, we see that the contribution from the large sphere scales as $R^{4kN-1}\times R^{-4kN}=1/R$ and therefore vanishes as $R\to\infty$. Now consider the behaviour on a small sphere of radius R around one of the insertion points $x^{(j)}$:

$$\mathrm{tr}_k(\bar{w}^{\dot\alpha}w_{\dot\alpha}+(a'_n+x_n^{(j)}1_{[k]\times[k]})(a'_n+x_n^{(j)}1_{[k]\times[k]}))\overset{R\to 0}{\sim}R^2\ . \tag{7.12}$$

On this sphere, the insertion at $x^{(j)}$ scales as R^{-4}, while the other insertions at other points remain finite.[49] The collective coordinate integral over this boundary scales as R^{4kN-1}. Therefore, as long as we avoid making all the insertions at the same point, there is a vanishing contribution as $R\to 0$. Since there are no contributions from the boundaries, we conclude that the integral vanishes and the supersymmetric Ward identity is respected within the SCI approach.

Now we turn to the same considerations in the WCI approach. In this case, k instantons contribute to the k-point function rather than the kN-point function. To leading order in the semi-classical expansion, even though the instantons are now constrained, the insertions take their ADHM form and so satisfy (7.9). Hence, we use the same logic to write the $x^{(l)}$-derivative of the k-point function with respect to one of the insertion points as a total derivative on $\mathfrak{M}_k$. However, there are some new subtleties. Firstly, since we are on the Higgs branch, there is a non-trivial instanton effective action (6.55) characteristic of a constrained instanton calculation. Since it is a supersymmetric invariant though, we can pull $\bar\delta^{\dot\alpha}$ past $\mathrm{e}^{-\tilde S}$, as well as the other insertions, in order to apply Stokes' Theorem. However, the instanton effective action (6.55) will modify the asymptotic behaviour on the large sphere of radius R. The bosonic terms in $\tilde S$ effectively prevent $4k(N-1)$ of the variables $\{w_{ui\dot\alpha},\bar{w}^{\dot\alpha}_{iu}\}$ from becoming large on the boundary manifesting the fact that the instantons are constrained and there is a cut-off on their size. Taking this into account, the measure now scales effectively as R^{4k-1} on the sphere, rather than the R^{4kN-1} of the SCI approach. Each of the insertions (which are equal to their ADHM expressions to leading order) still scales as (7.1) and so the insertions, together, scale as R^{4k}. Overall, measure and insertions scale as $1/R$ and so there is a vanishing contribution as $R\to\infty$. Likewise, it is straightforward to show that there are no contributions from around the insertion points themselves. The point is that in this case the instantons are small and so the constraining plays no role and the analysis is identical to the SCI case above. Consequently the k-point correlations functions are independent of the insertions points in the WCI approach.

7.2. One-instanton calculations of the gluino condensate

In the following subsections, we review the one-instanton SCI and WCI calculations of the gluino condensate.

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[48] The $3k^2$ bosonic ADHM constraints are quadratic in the bosonic ADHM variables and so scale as R^{-6k^2}, while the $2k^2$ fermionic ADHM constraints are linear in the bosonic variables and so scale as R^{2k^2}, giving R^{-4k^2}.

[49] Note that this argument is only valid if the insertion points are distinct. We shall find out by explicit calculation that ambiguities can arise if too many insertions are made at the same point.

7.2.1. Strong coupling

The strong-coupling one-instanton calculation was done originally for gauge group SU(2) in [38] and then extended to the SU(N) theories in [63] (see also the very comprehensive review articles [10,12]).

To begin with, the supersymmetric volume form on the instanton moduli space is given in (5.55). However, since at the one-instanton level $N \geqslant 2k \equiv 2$, we find it more convenient to use the expression given in (6.29) where the ADHM δ-functions have been resolved: [50]

$$\int_{\mathfrak{M}_1} \boldsymbol{\omega}^{(1)} = \frac{2^{3N}\pi^{2N-2}}{(N-1)!(N-2)!}\int \rho^{4N-8}\,\mathrm{d}^4X\,\mathrm{d}\rho^2\,\mathrm{d}^{4(N-1)}\mathscr{U}\,\mathrm{d}^2\mathscr{M}'\,\mathrm{d}^2\zeta\,\mathrm{d}^{(N-2)}v\,\mathrm{d}^{(N-2)}\bar{v}\ . \tag{7.13}$$

Here, $\mathrm{d}^{4(N-1)}\mathscr{U}$ is the unit normalized ($\int \mathrm{d}^{4(N-1)}\mathscr{U} = 1$) volume form for the gauge orientation of the instanton. We have identified $X_n = -a'_n$, the position of the instanton, $\rho = \sqrt{W^0/2}$, the scale size, and $\{\mathscr{M}', \zeta\}$ as the Grassmann collective coordinates associated to broken supersymmetric and superconformal invariance. In particular, recall from Section 6.1, the definitions

$$\mu = w_{\dot\alpha}\zeta^{\dot\alpha} + v, \qquad \bar\mu = \zeta_{\dot\alpha}\bar{w}^{\dot\alpha} + \bar{v}\ . \tag{7.14}$$

By explicit evaluation, using (4.29) and the one-instanton formulae of Section 4.3, the gluino insertions in the one-instanton background are

$$g^2\,\mathrm{tr}_N\,\lambda^2(x) = \frac{4g\rho^2\bar{v}v}{((x-X)^2+\rho^2)^3} + \frac{6g\rho^4(\mathscr{M}'^{\alpha} + \zeta_{\dot\alpha}(\bar{x}-\bar{X})^{\dot\alpha\alpha})(\mathscr{M}'_{\alpha} - (x-X)_{\alpha\dot\alpha}\zeta^{\dot\alpha})}{((x-X)^2+\rho^2)^4}\ . \tag{7.15}$$

We then insert $\prod_{i=1}^{N}(g^2/16\pi^2)\mathrm{tr}_N\,\lambda^2(x^{(i)})$ into the collective coordinate integral. Since in the confining phase there is no symmetry breaking and the insertions are gauge invariant, there is no dependence on the gauge orientation of the instanton and one can simply integrate over the associated collective coordinates $\int \mathrm{d}^{4(N-1)}\mathscr{U} = 1$. Next, let us carry out the Grassmann integrations. Obviously, the ζ and $\mathscr{M}'$ Grassmann integrals have to be saturated from the insertions at two points $\{x^{(i)}, x^{(j)}\}$ chosen from amongst the N. After integrating over ζ and $\mathscr{M}'$, the contribution from this pair is

$$\frac{36\rho^8\,(x^{(i)}-x^{(j)})^2}{((x^{(i)}-X)^2+\rho^2)^4((x^{(j)}-X)^2+\rho^2)^4}\ . \tag{7.16}$$

Now we take advantage of the fact that the N-point function is independent of the $x^{(i)}$, to choose the insertion points for maximum simplicity of the algebra. The simplest conceivable such choice, $x^{(i)} = 0$ for all i, turns out to give an ill-defined answer of the form "$0 \times \infty$" (the zero coming from the Grassmann integrations follows from Eq. (7.16), and the infinity from divergences in the ρ^2 integration due to coincident poles). In order to sidestep this ambiguity, one chooses instead

$$x^{(1)} = \cdots = x^{(N-1)} = 0, \qquad x^{(N)} = x\ . \tag{7.17}$$

This choice is the simplest which gives a well-defined answer with no "$0 \times \infty$" ambiguity. More ambitiously, one can still perform the calculation even if all the insertion points are taken to be

[50] Note for one instanton the solution of the ADHM constraints presented in Section 6.1 is generic.

arbitrary [10,63]; however, we find it convenient to take the minimal resolution provided by (7.17). Due to the $(x^{(i)}-x^{(j)})^2$ factor in Eq. (7.16), it follows that the pair of insertions $\{x^{(i)},x^{(j)}\}$ responsible for the $\{\mathscr{M}',\zeta\}$ integrations must include the point $x^{(N)}=x$; there are $N-1$ possible such pairs, giving

$$\frac{36(N-1)\rho^8x^2}{((x-X)^2+\rho^2)^4(X^2+\rho^2)^4} \tag{7.18}$$

for these contributions. The remaining Grassmann integrations over $\{\nu,\bar{\nu}\}$ are saturated by the $N-1$ insertions at $x^{(i)}=0$, and give

$$(N-2)!\left[\frac{4\rho^2}{(X^2+\rho^2)^3}\right]^{N-2} . \tag{7.19}$$

Combining the denominators in Eqs. (7.18) and (7.19) with a Feynman parameter α,

$$\begin{aligned}&\frac{1}{(X^2+\rho^2)^{3N-2}}\,\frac{1}{((x-X)^2+\rho^2)^4}\\ &\quad=\frac{(3N+1)!}{3!\,(3N-3)!}\int_0^1 \mathrm{d}\alpha\,\frac{\alpha^3(1-\alpha)^{3N-3}}{((\alpha x-X)^2+\alpha(1-\alpha)x^2+\rho^2)^{3N+2}}\end{aligned} \tag{7.20}$$

and performing the X integrals yields

$$\begin{aligned}&\left\langle\frac{g^2}{16\pi^2}\,\mathrm{tr}_N\,\lambda^2(x^{(1)})\times\cdots\times\frac{g^2}{16\pi^2}\,\mathrm{tr}_N\,\lambda^2(x^{(N)})\right\rangle\\ &\quad=\frac{3(3N-2)2^{N-1}\mu^{3N}\mathrm{e}^{2\pi i\tau}}{g^{2N}(N-2)!}\,x^2\int_0^1\mathrm{d}\alpha\int_0^\infty \mathrm{d}\rho^2\,(\rho^2)^{2N-4}\,\frac{\rho^{6N-4}\alpha^3(1-\alpha)^{3N-3}}{(\rho^2+\alpha(1-\alpha)x^2)^{3N}}\\ &\quad=\frac{3(3N-2)\,2^{N-1}\mu^{3N}\mathrm{e}^{2\pi i\tau}}{g^{2N}(N-2)!}\int_0^1\mathrm{d}\alpha\,\alpha^2(1-\alpha)^{3N-4}\\ &\quad=\frac{2^N\,\Lambda^{3N}}{(N-1)!(3N-1)} ,\end{aligned} \tag{7.21}$$

in agreement with Eqs. (7.2) and (7.3). In the final expression, we have used (5.17) to remove any dependence on the running coupling in favour of the Λ-parameter.

7.2.2. Weak coupling

Next, let us review the weak-coupling instanton calculation of the gluino condensate. These kinds of calculation were originally done in [50,64–66] and reviewed in [12]. The strategy here is to add sufficient matter fields in order to completely break the gauge group by the Higgs mechanism. For matter fields transforming in the fundamental representation, this means we need to add $N_F=N-1$ hypermultiplets, that is $N-1$ chiral multiplets in both the $\boldsymbol{N}$ and $\bar{\boldsymbol{N}}$ representations. On the Higgs branch, the scalar fields of the chiral multiplets q_f and $\tilde{q}_f$, $f=1,\dots,N_F$, gain a VEV. For the case $N_F=N-1$, we can choose the VEVs as in (6.51). For large values of the VEVs, the theory is weakly coupled and semi-classical methods should be reliable. In contrast with the SCI calculation, in the weakly coupled Higgs phase $\langle g^2\,\mathrm{tr}_N\,\lambda^2\rangle$ receives a non-zero contribution directly at the one-instanton level. Decoupling the extraneous matter and matching to the low-energy pure gauge theory is then accomplished using standard renormalization group prescriptions. What is absolutely crucial is that, due to holomorphy, the value of the gluino condensate is not renormalized as the matter is decoupled and in this way a result calculated at weak coupling can yield a result also valid at strong coupling.

We now calculate the instanton contribution to the one-point function $\langle(g^2/16\pi^2)\mathrm{tr}_N\,\lambda^2\rangle$ in the theory with $N_F=N-1$. The instanton calculus in theories with fundamental matter fields is described in Section 6.3. Since the scalar fields in the matter sector have VEVs, the calculation involves constrained instantons. As we have seen, to leading order (which in the present context is exact) the effect of constraining the instantons appears at the level of the instanton effective action which becomes a non-trivial function of the collective coordinates as is evident in (6.55).

The existence of the N_F fundamental hypermultiplets leads to new Grassmann collective coordinates $\{\mathscr{K},\tilde{\mathscr{K}}\}$ as described in Section 6.3. The new feature of the semi-classical limit is integrals over these new collective coordinates (Eq. (6.58)). To leading order in the semi-classical approximation, the instanton measure is obtained by amalgamating (5.14), for $\mathcal{N}=1$, with (6.58):

$$\mu^{k(3N-N_F)}g^{k(N_F-3N)}\mathrm{e}^{2\pi ik\tau}\int_{\mathfrak{M}_k}\omega^{(\mathcal{N}=1,N_F)}\mathrm{e}^{-\tilde{S}}\ , \tag{7.22}$$

where the volume form $\omega^{(\mathcal{N},N_F)}$ is defined in (6.59).

In order to calculate the gluino condensate, we must insert into (7.22) (with $N_F=N-1$) the one-instanton expression for $(g^2/16\pi^2)\mathrm{tr}_N\,\lambda^2$. Note that in contrast to the SCI approach, in this WCI calculation a single instanton contributes directly to the one-point function. However, since the matter fields have non-vanishing VEVs, the instantons are constrained as described in Section 4.4. This seems to preclude an evaluation of the condensate because we do not actually know the exact profile of λ in the constrained instanton background. However, we do know the profile of λ in the core of the instanton where it is well approximated by ADHM form (4.29). Now suppose we simply use the ADHM expression for $(g^2/16\pi^2)\mathrm{tr}_N\,\lambda^2$ (written in (7.15)). Have we any right to expect this to be a good approximation? At first it appears not because the ADHM profile behaves differently in the tail of the instanton where it has a power-law fall off rather than the exponential fall off of the constrained instanton. However, this would only be a problem for large instantons which are in any case suppressed by the instanton effective action (7.27). Consequently the error made in substituting the ADHM profile for the insertion will be higher order in g and as long as we are only after the leading-order expression, the ADHM profile will suffice. Here is the rub: we know that the condensate cannot receive any corrections in g by holomorphy and so the result we obtain will therefore be exact.

The insertion $(g^2/16\pi^2)\mathrm{tr}_N\,\lambda^2$ in the background of one instanton is given in (7.15) and is quadratic in Grassmann collective coordinates. Since in the measure (7.22) includes an integral over the two Grassmann collective coordinates associated to broken supersymmetry transformations $\mathscr{M}'$ and the instanton effective action (7.27) is independent of them, the insertion must saturate these integrals:

$$\int \mathrm{d}^2\mathscr{M}'\,\mathrm{tr}_N\,\lambda^2(x)=g^{-1}\,\frac{6\rho^4}{((x-X)^2+\rho^2)^4}\ . \tag{7.23}$$

The instanton effective action is likewise independent of the position of the instanton X_n and so the integrals over these collective coordinates give

$$\int d^4X \, \frac{1}{((x-X)^2+\rho^2)^4} = \frac{\pi^2}{6\rho^4} \ . \tag{7.24}$$

What remains is then a supersymmetrized integral over the centred moduli space $\widehat{\mathfrak{M}}_1$. In fact, the gluino condensate in the case $N_F = N-1$ can be written simply in terms of the centred instanton partition function (6.103) as

$$\left\langle \frac{g^2}{16\pi^2} \operatorname{tr}_N \lambda^2 \right\rangle = \tfrac{1}{2}\,\mu^{2N+1} g^{-2N} e^{2\pi i\tau} \hat{\mathscr{Z}}_1^{(\mathcal{N}=1,N_F=N-1)} \ . \tag{7.25}$$

At the one-instanton level, since $N \geqslant 2k \equiv 2$, we can use results from Section 6.1 to resolve the bosonic ADHM constraints. When this has been done the centred one-instanton partition function is

$$\begin{aligned}\hat{\mathscr{Z}}_1^{(\mathcal{N}=1,N_F=N-1)} &= \frac{2^{3N-3}\pi^{2N-2-2N_F}}{(N-1)!(N-2)!} \int \rho^{4N-8} \, d\rho^2 \, d^{4(N-1)}\mathscr{U} \\ &\quad \times d^N\bar{\mu} \, d^N\mu \, d^{N_F}\mathscr{K} \, d^{N_F}\tilde{\mathscr{K}} \prod_{\dot{\alpha}=1}^{2} \delta(\bar{w}_{\dot{\alpha}}\mu + \bar{\mu}w_{\dot{\alpha}}) e^{-\tilde{S}} \ ,\end{aligned} \tag{7.26}$$

where the instanton effective action is (6.55) (with the choice of VEVs (6.51))

$$\tilde{S} = 2\pi^2\rho^2 \sum_{f=1}^{N-1} |v_f|^2(|\mathscr{U}_{f1}|^2 + |\mathscr{U}_{f2}|^2) + \frac{i\pi^2}{\sqrt{2}} \sum_{f=1}^{N-1} (v_f^* \mathscr{K}_f \mu_f - \tilde{v}_f^* \bar{\mu}_f \tilde{\mathscr{K}}_f) \ . \tag{7.27}$$

Here, we have used the one-instanton form for $w_{\dot{\alpha}}$ in terms of the gauge orientation $\mathscr{U}$ in Eq. (2.94). Notice that the form of the measure that we will use, (7.26), is rather schizophrenic, since we have chosen to resolve the bosonic ADHM constraints as in Section 6.1, but not the fermionic ones, as indicated by the remaining Grassmann δ-functions in (7.26).

We now evaluate the partition function explicitly. First of all, the $\{\mathscr{K}_f, \tilde{\mathscr{K}}_f\}$ and $\{\mu_u, \bar{\mu}_u\}$, $u \neq N$, integrals are saturated by pulling down powers of the Yukawa coupling terms in the instanton effective action (7.27). This yields a constant term

$$\left(\frac{\pi^2}{\sqrt{2}}\right)^{2(N-1)} \prod_{f=1}^{N-1} (v_f \tilde{v}_f)^* \ . \tag{7.28}$$

The integrals over the two remaining Grassmann variables $\{\mu_N, \bar{\mu}_N\}$ are saturated by the Grassmann δ-functions:

$$\int d\mu_N \, d\bar{\mu}_N \prod_{\dot{\alpha}=1}^{2} \delta(\bar{w}_{N\dot{\alpha}}\mu_N + \bar{\mu}_N w_{N\dot{\alpha}}) = \rho^2(|\mathscr{U}_{N1}|^2 + |\mathscr{U}_{N2}|^2) \ . \tag{7.29}$$

Putting all these factors together, we have the remaining bosonic integrals to perform

$$\hat{\mathscr{Z}}_1^{(\mathscr{N}=1,N_F=N-1)} = \frac{\pi^{4N-8}2^{2N-1}}{(N-1)!(N-2)!}\prod_{f=1}^{N-1} v_f^*\tilde{v}_f^* \int d^4X\, d\rho^2\, d^{4(N-1)}\mathscr{U}$$
$$\times \rho^{4N-6}(|\mathscr{U}_{N1}|^2 + |\mathscr{U}_{N2}|^2)\exp\left[-2\pi^2\rho^2\sum_{f=1}^{N-1}|v_f|^2(|\mathscr{U}_{f1}|^2+|\mathscr{U}_{f2}|^2)\right] . \quad (7.30)$$

The integral over the scale size is

$$\int_0^\infty d\rho^2\,\rho^{4N-6}\exp\left[-2\pi^2\rho^2\sum_{f=1}^{N-1}v_f^2(|\mathscr{U}_{f1}|^2+|\mathscr{U}_{f2}|^2)\right]$$
$$= \frac{(2N-3)!}{(2\pi^2\sum_{f=1}^{N-1} v_f^2(|\mathscr{U}_{f1}|^2+|\mathscr{U}_{f2}|^2))^{2N-2}} . \quad (7.31)$$

This leaves the remaining integral over the gauge orientation $\mathscr{U}$ which can be done by using the formulae in Ref. [71]: [51]

$$\int d^{4(N-1)}\mathscr{U}\,\frac{|\mathscr{U}_{N1}|^2+|\mathscr{U}_{N2}|^2}{[\sum_{f=1}^{N-1} v_f^2(|\mathscr{U}_{f1}|^2+|\mathscr{U}_{f2}|^2)]^{2N-2}} = \frac{(N-1)!(N-2)!}{(2N-3)!}\prod_{f=1}^{N-1}|v_f|^{-4} . \quad (7.32)$$

Collecting the results, we have

$$\hat{\mathscr{Z}}_1^{(\mathscr{N}=1,N_F=N-1)} = 2\prod_{f=1}^{N-1}(v_f\tilde{v}_f)^{-1} \quad (7.33)$$

and hence

$$\left\langle \frac{g^2}{16\pi^2}\,\mathrm{tr}_N\,\lambda^2\right\rangle = \mu^{2N+1}g^{-2N}e^{2\pi i\tau}\prod_{f=1}^{N-1}(v_f\tilde{v}_f)^{-1} . \quad (7.34)$$

Result (7.34) is expressed in terms of bare quantities and involves the Pauli–Villars mass scale μ. We need to re-express it in terms of renormalized quantities. Firstly, the bare VEVs v_f and $\tilde{v}_f$ should be replaced by $Z_f^{1/2}v_f^{\text{ren.}}$ and $Z_f^{1/2}\tilde{v}_f^{\text{ren.}}$, respectively, where Z_f is the usual multiplicative wavefunction renormalization factor. From now on we shall drop the "ren." superscripts on the renormalized VEVs with the understanding that, henceforth, all quantities are understood to be renormalized. The coupling constant g and multiplicative renormalization factors Z_f run with the Pauli–Villars mass scale μ in such a way that the remaining factors are equal to a certain power of the strong coupling

[51] Alternatively, making the choice $v_f = \tilde{v}_f \equiv v$ yields an elementary integral.

scale of the theory in the Pauli–Villars scheme:

$$\Lambda_{(N-1)}^{2N+1} = \mathrm{e}^{-8\pi^2/g(\mu)^2+\mathrm{i}\theta}\mu^{2N+1}g(\mu)^{-2N}\prod_{f=1}^{N-1} Z_f(\mu)^{-1} \,, \tag{7.35}$$

generalizing the $N_F=0$ relation (5.17). Note that due to holomorphy this relation must be exact. By differentiating with respect to μ this yields an exact expression for the β-function. It is an example of the more general relation for arbitrary N_F

$$\Lambda_{(N_F)}^{3N-N_F} = \mathrm{e}^{-8\pi^2/g(\mu)^2+\mathrm{i}\theta}\mu^{3N-N_F}g(\mu)^{-2N}\prod_{f=1}^{N_F} Z_f(\mu)^{-1} \,, \tag{7.36}$$

which upon differentiation leads to the famous Novikov–Shifman–Vainstein–Zakharov β-function of supersymmetric QCD [12,72]:

$$\mu\frac{\partial\alpha}{\partial\mu} = -\frac{\alpha^2}{2\pi}\frac{3N-\sum_{f=1}^{N_F}(1-\gamma_f)}{1-\dfrac{N\alpha}{2\pi}}, \qquad \alpha\equiv\frac{g^2}{4\pi}, \qquad \gamma_f\equiv-\frac{\mathrm{d}\ln Z_f}{\mathrm{d}\ln\mu} \,. \tag{7.37}$$

The final result for the gluino condensate is

$$\left\langle\frac{g^2}{16\pi^2}\mathrm{tr}_N\,\lambda^2\right\rangle = \Lambda_{(N-1)}^{2N+1}\prod_{f=1}^{N-1}(v_f\tilde{v}_f)^{-1} \,. \tag{7.38}$$

We still have to relate the value of gluino condensate (7.38) in the Higgs phase to that in the confining phase. In other words, we have to track the value of the gluino condensate as we decouple the matter fields. In order to achieve this in a controlled way, it is useful to re-interpret (7.38) in terms of a coupling in the low-energy effective superpotential W_{eff} for the matter fields in the Higgs phase. In order to determine the superpotential we exploit the general functional identity (see for example [73]):

$$\left\langle\frac{g^2}{16\pi^2}\mathrm{tr}_N\,\lambda^2\right\rangle = b_0^{-1}\Lambda\frac{\partial}{\partial\Lambda}\langle W_{\mathrm{eff}}\rangle \,. \tag{7.39}$$

Here, Λ is the appropriate strong coupling scale and b_0 is the first coefficient of the β-function, equal to $3N-N_F=2N+1$ in the present context. Since, the expression for the superpotential must be invariant under gauge and global symmetries, it can be uniquely determined:

$$W_{\mathrm{eff}} = \frac{\Lambda_{(N-1)}^{2N+1}}{\det\tilde{Q}Q} \,. \tag{7.40}$$

This is the famous Affleck–Dine–Seiberg (ADS) superpotential for $\mathcal{N}=1$ supersymmetric SU(N) QCD. (See Ref. [50] for the SU(2) case and Ref. [74] for the SU(N) calculation.) Note we have arrived at the ADS superpotential by a somewhat different route to Refs. [50,74]. In those references, the form of the superpotential was deduced by first noticing that after the Higgs mechanism, there is a combination of the matter fields which remains classically massless. The ADS superpotential then implies that these classically massless fields receive a non-perturbative mass through a one-instanton effect. This mass, and hence the form of the superpotential itself, can be deduced by calculating the long-distance behaviour of the two-point function of the massless component of the anti-Higgsinos $\bar{\chi}$ and $\bar{\tilde{\chi}}$. The computation involves using a single constrained instanton but to leading order we can replace the massless component of $\bar{\chi}$ and $\bar{\tilde{\chi}}$ by their value in the ADHM instanton background.[52] The net result of this calculation is entirely consistent with our approach via the gluino condensate.

As it stands, the ADS superpotential implies that the vacuum runs away to infinity. One way to stabilize the theory is to add a small mass term. Using the ubiquitous holomorphy argument, the mass cannot get renormalized and will appear directly in the effective superpotential:

$$W_{\rm eff} = \frac{\Lambda_{(N-1)}^{2N+1}}{\det \tilde{Q}Q} + \sum_{f=1}^{N-1} m_f \tilde{Q}_f Q_f \ . \tag{7.41}$$

Superpotential (7.41) now has well-defined critical points at

$$m_f v_f \tilde{v}_f = \frac{\Lambda_{(N-1)}^{2N+1}}{\prod_{f'=1}^{N-1} v_{f'}\tilde{v}_{f'}} \ . \tag{7.42}$$

It is easy to see that there are N solutions where $v_f\tilde{v}_f$ differ by the Nth roots of unity. At these critical points the values of the gluino condensate obtained by using (7.39) are

$$\left\langle \frac{g^2}{16\pi^2} {\rm tr}_N\, \lambda^2 \right\rangle = \left(\Lambda_{(N-1)}^{2N+1} \prod_{f=1}^{N-1} m_f \right)^{1/N} \ , \tag{7.43}$$

which exhibits the N-fold degeneracy explicitly via the choice of the Nth root.

Result (7.43) is justified in the weakly coupled Higgs phase where the VEVs are large (or masses small) compared with $\Lambda_{(N-1)}$. However the powerful Ward identities of supersymmetric theory allow us to extrapolate the result into the regime of small VEV (large masses) and strong coupling. The point is that since ${\rm tr}_N\, \lambda^2$ is gauge-invariant and the lowest component of a chiral superfield, its VEV must be a holomorphic expression in the coupling constants of the theory. In the present context this means $\Lambda_{(N-1)}$ and the masses. This implies that result (7.43) cannot be subject to any perturbative corrections since a power series in g would translate into an expression which could not be holomorphic in $\Lambda_{(N-1)}$. Hence, (7.43) must also be valid for small VEVs, or, correspondingly from (7.42), large masses. This is precisely the limit in which the matter fields are decoupled and the theory should flow into the pure $\mathcal{N}=1$ supersymmetric gauge theory. In particular the N supersymmetric vacua that we see at weak coupling are continuously connected with the N supersymmetric vacua of the pure gauge theory. In order to track the value of the gluino condensate from weak to strong coupling we simply have to match the Λ-parameters of the two theories. The correct renormalization group matching in this case is [54,75]

$$\Lambda^{3N} = \Lambda_{(N-1)}^{2N+1} \prod_{f=1}^{N-1} m_f \ . \tag{7.44}$$

[52] Notice that this involves iterating the constrained instanton to the required order where the anti-chiral fermions are non-zero.

Finally putting (7.43) together with (7.44), we deduce

$$\left\langle \frac{g^2}{16\pi^2}\,\mathrm{tr}_N\,\lambda^2 \right\rangle = \Lambda^3 e^{2\pi i u/N} \tag{7.45}$$

for $u = 1,\ldots,N$ which is the expression in (7.4).

7.3. Multi-instanton calculations of the gluino condensate

In this section, we calculate the contributions of arbitrary numbers of instantons to the gluino condensate in both the SCI and WCI approaches at large N where saddle-point methods are available to simplify the instanton calculus.

7.3.1. Strong coupling

In Section 6.1, we derived a form for the multi-instanton volume form in a supersymmetric gauge theory with $N \geqslant 2k$. In the confining phase there is no spontaneous symmetry breaking and all insertions are gauge invariant, hence, we can immediately integrate over the gauge orientation: $\int d^{4k(N-k)}\mathscr{U} = 1$. This gauge-invariant form for the multi-instanton volume form is now particularly useful for taking a large-N limit.

In order to calculate the kN-point correlation function, we must insert into the measure the expression for $\mathrm{tr}_N\,\lambda^2$ evaluated in the k ADHM instanton background. At the multi-instanton level we find it useful to use identity (C.22) to express the insertions as

$$\mathrm{tr}_N\,\lambda^2 = \frac{1}{4g}\Box\;\mathrm{tr}_k\,\bar{\mathscr{M}}(\mathscr{P}+1)\mathscr{M}f\ . \tag{7.46}$$

As proved earlier, the correlation function is expected to be independent of the insertion points and so, as in the one-instanton sector in Section 7.1, the $\{x^{(l)}\}$ can therefore be chosen for maximum simplicity of the algebra. The simplest conceivable choice, $x^{(l)} = 0$ for all l, results in an ill-defined answer of the form "$0\times\infty$" (the zero coming from unsaturated Grassmann integrations, and the infinity from divergences in the bosonic integrations due to coincident poles); we have already noted this fact in the one-instanton sector in Section 2.1. The simplest choice of the $\{x^{(l)}\}$ that avoids this problem, generalizing (7.17), is

$$\begin{aligned} &x^{(1)} = \cdots = x^{(kN-k^2)} = 0\ ,\\ &x^{(kN-k^2+1)} = \cdots = x^{(kN)} \equiv x \end{aligned} \tag{7.47}$$

which we adopt for the remainder of this section. [53]

In the large-N limit, there is a large preponderance of insertions (7.47) at $x^{(l)}=0$, and the resulting factor of $(\mathrm{tr}_N\,\lambda^2(0))^{k(N-k)}$, taken together with the Jacobian factor $|\det_{2k}W|^{N-2k}$ from measure (6.29),

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[53] As a non-trivial check on the Ward identity, we have also numerically integrated the large-N correlator for insertions other than Eq. (7.47), and verified the constancy of the answer presented below.

dominates the integral and can be treated in a saddle-point approximation. Below we will carry out this saddle-point evaluation in full detail, but we can already quite easily understand the source of the linear dependence on k in our final result. The argument goes as follows:

(i) Let us imagine carrying out all the Grassmann integrations in the problem. The remaining large-N integrand will then have the form $\exp(-N\Gamma + \mathcal{O}(\log N))$ where Γ might be termed the "effective large-N instanton action". The large-N saddle points are then the stationary points of Γ with respect to the c-number collective coordinates. By Lorentz symmetry, Γ can only depend on the four $k \times k$ matrices a'_n through even powers of a'_n. (This is because the bulk of the insertions have been chosen to be at $x^{(l)} = 0$; otherwise one could form the Lorentz scalar $x_n a'_n$ and so have odd powers of a'_n.) It follows that the ansatz

$$a'_n = 0 , \quad n = 1\text{–}4 , \tag{7.48a}$$

$$W^c = 0 , \quad c = 1\text{–}3 \tag{7.48b}$$

is automatically a stationary point of Γ with respect to these collective coordinates. (Note that (7.48b) follows automatically from (7.48a) by virtue of ADHM constraints (6.3).) It will actually turn out that, once one assumes these saddle-point values, Γ is independent of the remaining collective coordinate matrix W^0; furthermore we will verify that this saddle point is actually a minimum of Γ.

(ii) Having anticipated saddle point (7.48a) and (7.48b) using these elementary symmetry considerations, let us back up to a stage in the analysis prior to the Grassmann integration, and proceed a little more carefully. Evaluating the insertions $\mathrm{tr}_N\, \lambda^2(x^{(l)})$ on this saddle point, one easily verifies that the ζ modes vanish when $x^{(l)} = 0$; consequently, the ζ integrations must be saturated entirely from the k^2 insertions at $x^{(l)} = x$. This leaves the $\mathcal{M}'$, v and $\bar{v}$ integrations to be saturated purely from the insertions at $x^{(l)} = 0$. Moreover, because $\mathcal{M}'$ carries a Weyl spinor index α whereas v and $\bar{v}$ do not, the $\mathrm{tr}_N\, \lambda^2(0)$ insertions depend on these Grassmann coordinates only through bi-linears of the form $\bar{v} \times v$ or $\mathcal{M}' \times \mathcal{M}'$; there are no cross terms.

(iii) Performing all the Grassmann integrations then automatically generates a combinatoric factor

$$(k^2)!\,(k^2)!\,(kN - 2k^2)! \begin{pmatrix} kN - k^2 \\ k^2 \end{pmatrix} . \tag{7.49}$$

Here the first three factors account for the indistinguishable bi-linear insertions of the ζ, $\mathcal{M}'$, and $\{v, \bar{v}\}$ modes, respectively, while the final factor counts the ways of selecting the k^2 bi-linears in $\mathcal{M}'$ from the $kN - k^2$ insertions at $x^{(i)} = 0$. Multiplying these combinatoric factors together, as well as the normalization constants from the instanton measure, and taking the (kN)th root yields, in the large-N limit,

$$[A_k (C_1^{(1)})^k\, (k^2)!\, (kN - k^2)!]^{1/kN} \overset{N\to\infty}{\to} 2^3 \pi^2 \mathrm{e} N^{-1} k \Lambda^3 + \mathcal{O}(N^{-2}) . \tag{7.50}$$

Remarkably, apart from a factor of four, this back-of-the-envelope analysis precisely accounts for the leading term in $1/N$ in the final answer, Eq. (7.63). Note that most of the remaining contributions to the saddle-point analysis, which involve a specific convergent bosonic integral derived below, as well as the factor $2^{3k^2}/\mathrm{Vol}\,\mathrm{U}(k)$ from Eq. (6.29), reduce to unity when the (kN)th root is taken in the large-N limit; the missing factor of four will simply come from the leading saddle-point evaluation of the bosonic integrand.

Here are the details of the large-N calculation of the kN-point correlation function. First, we find it convenient to partially fix the auxiliary U(k) symmetry by taking a basis where W^0 is diagonal:

$$W^0 = \mathrm{diag}(2\rho_1^2, \ldots, 2\rho_k^2) \ . \tag{7.51}$$

As the notation implies, in the dilute instanton gas limit ρ_i can be identified with the scale size of the ith instanton in the k-instanton sector (see Section 4.2). The appropriate gauge fixing involves a Jacobian:

$$\frac{1}{\mathrm{Vol}\,\mathrm{U}(k)} \int \mathrm{d}^{k^2} W^0 \ \rightarrow \ \frac{2^{3k(k-1)/2}\pi^{-k}}{k!} \int_0^\infty \prod_{i=1}^{k} \mathrm{d}\rho_i^2 \prod_{i<j=1}^{k} (\rho_i^2 - \rho_j^2)^2 \ . \tag{7.52}$$

For $k = 1$ one has, of course, $\int \mathrm{d}W^0 \rightarrow 2\int_0^\infty \mathrm{d}\rho^2$.

Now let us consider the Grassmann integrations, beginning with the ζ variables. We assume the saddle-point conditions (7.48a) and (7.48b), in which case

$$\Delta = \begin{pmatrix} w \\ x \cdot 1_{[k]\times[k]} \end{pmatrix} , \qquad f = \mathrm{diag}\left(\frac{1}{x^2+\rho_1^2}, \ldots, \frac{1}{x^2+\rho_k^2} \right) \tag{7.53}$$

and from Eq. (7.46),

$$\mathrm{tr}_N\, \lambda^2(x) = \sum_{i,j=1}^{k} (\zeta_{\dot\alpha})_{ij} (\zeta^{\dot\alpha})_{ji}\, F_{ij}(x) + \cdots \ , \tag{7.54}$$

where

$$F_{ij}(x) = \frac{1}{4g} \Box \frac{x^4}{(x^2+\rho_i^2)(x^2+\rho_j^2)} \tag{7.55}$$

and the terms omitted in Eq. (7.54) represent dependence on the other Grassmann collective coordinates $\{\mathscr{M}', v, \bar v\}$. It is obvious from Eq. (7.55) that $F_{ij}(0) = 0$, so the ζ must be entirely saturated from the k^2 insertions at $x^{(i)} = x$ as claimed above. Performing the ζ integrations then yields

$$(k^2)! \prod_{i,j=1}^{k} F_{ij}(x) \ . \tag{7.56}$$

Next we consider the insertions at $x^{(l)} = 0$. Focusing on the $\mathscr{M}'$ modes first, one finds from Eq. (7.46)

$$\mathrm{tr}_N\, \lambda^2(0) = \frac{2}{g} \sum_{i,j=1}^{k} (\mathscr{M}'^{\alpha})_{ij} (\mathscr{M}'_{\alpha})_{ji}\, (\rho_i^{-4} + \rho_j^{-4} + \rho_i^{-2}\rho_j^{-2}) + \cdots \ , \tag{7.57}$$

omitting the $v \times \bar{v}$ terms. Hence the $\mathscr{M}'$ integrations yield

$$\frac{2^{k^2}}{g^{k^2}}\begin{pmatrix} kN-k^2 \\ k^2 \end{pmatrix}(k^2)!\prod_{i,j=1}^{k}(\rho_i^{-4}+\rho_j^{-4}+\rho_i^{-2}\rho_j^{-2}) , \tag{7.58}$$

where the combinatoric factors in (7.58) (as well as in (7.56)) have been explained previously.[54]

Finally we turn to the $\{v,\bar{v}\}$ integrations. Since (unlike ζ and $\mathscr{M}'$) the number of v and $\bar{v}$ variables grows with N as $kN-2k^2$, it does not suffice merely to plug in the saddle-point values (7.48a) and (7.48b) and (7.53). One must also calculate the Gaussian determinant about the saddle point, which provides an $\mathcal{O}(N^0)$ multiplicative contribution to the answer. Accordingly we expand about (7.48a) and (7.48b) to quadratic order in the a'_n. From (7.46), the $v\times\bar{v}$ contribution to $\mathrm{tr}_N\,\lambda^2(0)$ has the form

$$-\frac{1}{2g}\,\mathrm{tr}_k\,\bar{v}v\Box f\bigg|_{x=0}=\frac{2}{g}\,\mathrm{tr}_k\,\bar{v}vf\bar{b}_\alpha\mathscr{P}b^\alpha f\bigg|_{x=0} \tag{7.59}$$

as follows from differentiation formula (C.2b). Performing the $\{v,\bar{v}\}$ integrations therefore gives

$$\begin{aligned}&(kN-2k^2)!\exp\{(N-2k)\mathrm{tr}_k\log(2f\bar{b}_\alpha\mathscr{P}b^\alpha f)|_{x=0}\}\\ &\quad=(kN-2k^2)!\exp\Bigg\{(N-2k)\\ &\qquad\times\left(\log\det{}_k\,16(W^0)^{-2}-\frac{3}{2}\sum_{i,j=1}^{k}(a'_n)_{ij}(a'_n)_{ji}(\rho_i^{-2}+\rho_j^{-2})+\mathcal{O}(a'_n)^4\right)\Bigg\} .\end{aligned} \tag{7.60}$$

The negative sign in front of the quadratic term in a'_n confirms that our saddle point (7.48a), (7.48b) is in fact a minimum of the action. Combining this expression with the measure factor in Eq. (6.29), namely

$$|\det{}_{2k}W|^{N-2k}=\exp((N-2k)\log\det{}_{2k}W)=\exp\{(N-2k)(\log\det{}_k(\tfrac{1}{2}W^0)^2+\mathcal{O}(a'_n)^4)\} , \tag{7.61}$$

and performing the Gaussian integrations over a'_n, yields

$$2^{2k(N-2k)}(kN-2k^2)!\prod_{i,j=1}^{k}\left(\frac{2\pi}{3N(\rho_i^{-2}+\rho_j^{-2})}\right)^2+\cdots , \tag{7.62}$$

where the omitted terms are suppressed by powers of $1/N$.

[54] One can easily check that these large-N formulae are consistent with the explicit one-instanton calculation presented in Section 2.1 which is exact in N.

Finally one combines Eqs. (6.29), (3.23), (7.56), (7.58) and (7.62) (and the definition of the Λ-parameter in (5.17)) to obtain the leading-order result for the correlator:

$$\left\langle \frac{g^2}{16\pi^2}\,\mathrm{tr}_N\,\lambda^2(x^{(1)}) \times \cdots \times \frac{g^2}{16\pi^2}\,\mathrm{tr}_N\,\lambda^2(x^{(kN)}) \right\rangle$$

$$\underset{\rightarrow}{N\rightarrow\infty}\ \frac{2^{kN+k^2-k+1/2}\pi^{-k+1/2}\mathrm{e}^{kN}(k^2)!k^{kN-k^2+1/2}\Lambda^{3kN}}{3^{2k^2}N^{kN+k^2-1/2}k!}\,\mathscr{I}_k\ , \tag{7.63}$$

where $\mathscr{I}_k$ is the convergent integral,

$$\mathscr{I}_k = \int_0^\infty \prod_{i=1}^k \mathrm{d}\rho_i^2 \prod_{\substack{i,j\\ i\neq j}}^k |\rho_i^2-\rho_j^2| \prod_{i,j=1}^k F_{ij}(x)(1-(\rho_j/\rho_i+\rho_i/\rho_j)^{-2})\ . \tag{7.64}$$

Note that $\mathscr{I}_k$ is independent of x as a simple re-scaling argument confirms. For the simple case $k=1$, $\mathscr{I}_1=\frac{3}{2}$ and expression (7.63) agrees—as it must—with the large-N limit of the one-instanton result in Eq. (7.21).

7.3.2. Weak coupling

In this section, we turn to multi-instanton effects in the WCI approach. We apply the same large-N saddle-point methods to extract the large-N behaviour of the k-instanton contribution to the k-point of $(g^2/16\pi^2)\mathrm{tr}_N\,\lambda^2$. In order to make the calculation more manageable, we will choose the VEVs from the outset to be $v_1=\cdots=v_{N-1}=\tilde{v}_1=\cdots=\tilde{v}_{N-1}\equiv v$. In this case, the bosonic part of the instanton effective action (6.55) can be written as

$$\tilde{S}_\mathrm{b}=\pi^2v^2\,\mathrm{tr}_{2k}(W-\xi^\dagger uu^\dagger\xi)\ . \tag{7.65}$$

Here, W and ξ are the $2k\times 2k$ matrix defined in (6.1) and (2.83), respectively, and the complex $2k$ vector $u_{i\dot\alpha}$ is defined in terms of the gauge orientation via

$$u=(\mathscr{U}_{1N},\ldots,\mathscr{U}_{2k,N})\ . \tag{7.66}$$

Notice that the correlation function only depends on the gauge orientation through the vector u in the instanton effective action (7.65). So we need an expression for the reduced measure $\int \mathrm{d}\mathscr{U} = \int \mathrm{d}^{2k}u\,f(u)$. We now pause to evaluate the function $f(u)$. The complex $2k$-vector u is composed of the first $2k$ components of the complex unit N-vector $U=(\mathscr{U}_{1N},\ldots,\mathscr{U}_{NN})$. Let $\tilde{u}=(\mathscr{U}_{2k+1,N},\ldots,\mathscr{U}_{NN})$ be the complementary complex $(N-2k)$-vector: $U=u+\tilde{u}$. It is clear that the measure on the unit vector U inherited from the group measure $\int\mathrm{d}\mathscr{U}$ is proportional to

$$\int \mathrm{d}^{2N}U\,\delta(|U|^2-1)=\int \mathrm{d}^{4k}u\,\mathrm{d}^{2(N-2k)}\tilde{u}\,\delta(|u|^2+|\tilde{u}|^2-1)\ . \tag{7.67}$$

The measure on u is obtained by integrating over $\tilde{u}$ giving (up to constant)

$$\int_{|u|\leqslant 1}\mathrm{d}^{4k}u(1-|u|^2)^{N-2k-1}\ . \tag{7.68}$$

Actually (7.68) is still too general for our needs. It will be sufficient for our purposes to know the measure on the k quantities $z_i = \bar{u}_i^{\dot\alpha} u_{i\dot\alpha}$ (with no sum on i). The measure on the $\{z_i\}$ follows easily from (7.68). Finally, taking into account normalization (6.17), we have the required expression for the reduced measure of the gauge orientation:

$$\int d^{4k(N-k)}\mathscr{U} = \frac{\Gamma(N)}{2\Gamma(N-2k)} \int_0^1 \left\{ \prod_{i=1}^{k} dz_i\, z_i \right\} \left(1 - \sum_{i=1}^{k} z_i \right)^{N-2k-1} , \tag{7.69}$$

with the constraint $\sum_{i=1}^{k} z_i \leqslant 1$.

We now have all the ingredients needed to start a large-N analysis. In this limit, we will evaluate the collective coordinate integral by a method of steepest descent. As usual in such a saddle-point analysis, one exponentiates any terms in the collective coordinate integral raised to the power N, and includes them, along with the bosonic terms in the instanton effective action, in a large-N effective action. There are two contributions besides the instanton effective action (7.65). The first, is the factor of $|\det_{2k} W|^N$ in measure (6.29) and the second the factor of $(1-|u|^2)^N$ in the measure for gauge orientation (7.68). Placing all these contributions up in exponent defines a saddle-point action:

$$\Gamma = -N \log \det_{2k} \left(\frac{1}{2} W^0 1_{[2]\times[2]} - 2a'_m a'_n \bar\sigma_{mn} \right) + \pi^2 v^2 \operatorname{tr}_{2k}(W - \bar\xi u \bar u \xi) - N \log(1-|u|^2) . \tag{7.70}$$

Now we find the extrema of Γ with respect to the independent variables a'_n, W^0 and $u_{\dot\alpha}$.

Rather than write down the saddle-point equations and proceed to solve them, we will simply write down an ansatz for the solution and then show ex post facto from a fluctuation analysis that it is indeed an extremum. Up to the auxiliary U(k) symmetry, our ansatz is

$$W^0 = \frac{N}{\pi^2 v^2} 1_{[k]\times[k]} , \tag{7.71a}$$

$$a'_n = -\operatorname{diag}(X_n^1, \ldots, X_n^k) , \tag{7.71b}$$

$$u_{i\dot\alpha} = 0 . \tag{7.71c}$$

These values imply

$$\xi = \sqrt{\frac{N}{2\pi^2 v^2}} 1_{[2k]\times[2k]}, \qquad W = \frac{N}{2\pi^2 v^2} 1_{[2k]\times[2k]} . \tag{7.72}$$

The solution has a simple physical interpretation. It represents k instantons in the dilute-gas limit with positions X_n^i, all with the same scale size $\sqrt{N/(2\pi^2 v^2)}$ inhabiting k mutually commuting SU(2) subgroups of the gauge group [55] orthogonal to the gauge orientation of the VEV (since $u_{i\dot\alpha} = 0$). The fact that the instantons live in commuting subgroups of the gauge group implies that they are non-interacting and the solution is dilute-gas like.

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[55] The generators of the SU(2) embeddings of the k instantons in the dilute limit are given in (2.10). The fact that the commutator of these generators for different instantons vanishes follows from the fact that the solution above has $W^c = 0$.

We now turn to an analysis of the fluctuations around solution (7.71a)–(7.71c). The explicit expression for the expansion of action around the saddle point to quadratic order is

$$\Gamma^{(2)} = 2kN(1+\log(2\pi^2 v^2/N)) + \frac{(2\pi^2 v^2)^2}{4N}\sum_{i,j=1}^{k}\delta W^0_{ij}\delta W^0_{ji} + \frac{(2\pi^2 v^2)^2}{N}\sum_{\substack{i,j=1\\ i\neq j}}^{k}(X^i - X^j)^2 (a'_n)_{ij}(a'_n)_{ji} + \frac{N}{4}\sum_{i=1}^{k} z_i^2 \ , \tag{7.73}$$

where we have removed the cross terms between fluctuations involving δW^0 with u and a'_n with u by appropriate shifts in δW^0 and a'_n. At this point, we find it convenient to fix the U(k) symmetry by making the "gauge choice" (2.100) for the fluctuations a'_n and then denote the remaining fluctuations $\tilde{a}'_n$.

To leading order in the $1/N$ expansion, we substitute the saddle-point expressions for variables (7.71a) and (7.71b) into the insertions. Due to the dilute-gas nature of the saddle-point solution, at leading order, each insertion $(g^2/16\pi^2)\,\mathrm{tr}_N\,\lambda^2$ will be a sum over k one-instanton expressions involving the one-instanton collective coordinates $\{\rho_i, X^i_n, u_{i\dot\alpha}, \mathscr{M}'_{ii}, \mu_{ui}, \bar\mu_{ui}\}$ for a fixed $i\in\{1,\ldots,k\}$. In particular, the only dependence on the Grassmann collective coordinates $\mathscr{M}'_{ii}$ is through the insertions because at leading order they decouple from the fermionic ADHM constraints. Therefore using the one-instanton expression (7.15), we can replace each insertion with

$$\frac{g^2}{16\pi^2}\,\mathrm{tr}_N\,\lambda^2(x) \to \sum_{i=1}^{k}\frac{2^{-5}3\pi^{-6}v^{-4}gN^2}{((x-X^i)^2 + N/(2\pi^2 v^2))^4}(\mathscr{M}'^{\alpha})_{ii}(\mathscr{M}'_{\alpha})_{ii} \ . \tag{7.74}$$

Even at this intermediate stage in the calculation we can draw an important conclusion. Since, to leading order, the insertions do not involve any "off-diagonal" collective coordinates which communicate between the individual instantons of the saddle-point solution, the result for the correlation function is destined to cluster correctly. Nevertheless, it is interesting to show how this happens explicitly.

As we have noted, the leading-order expressions for the insertions are independent of the fluctuations δW^0_{ij} and $\tilde{a}'_n$; hence, we can immediately integrate them out:

$$\int d^{k^2}\delta W^0\, d^{3k(k-1)}\tilde{a}' \exp\left\{-\frac{(2\pi^2 v^2)^2}{4N}\sum_{i,j=1}^{k}\delta W^0_{ij}\delta W^0_{ji} - \frac{(2\pi^2 v^2)^2}{N}\sum_{\substack{i,j=1\\ i\neq j}}^{k}(X^i - X^j)^2 (a'_n)_{ij}(a'_n)_{ji}\right\}$$

$$= \left(\frac{N\pi}{\pi^4 v^4}\right)^{k^2/2}\left(\frac{N\pi}{4\pi^4 v^4}\right)^{4k(k-1)/2}\prod_{\substack{i,j=1\\ i\neq j}}^{k}(X^i - X^j)^{-3} \ . \tag{7.75}$$

We have succeeded in integrating out all the off-diagonal c-number collective coordinates. Now we shall deal with the Grassmann sector. The off-diagonal elements of $\mathscr{M}'_\alpha$ do not appear in the instanton effective action (6.55) and, at leading order in $1/N$, in insertions (7.74); hence, the integrals over these quantities must be saturated by the fermionic ADHM constraints. Since at the saddle point a'_n is diagonal, the integrals are

$$\int \prod_{\substack{i,j=1\\ i\neq j}}^{k} \left\{ \mathrm{d}^2\mathscr{M}'_{ij} \prod_{\dot\alpha=1}^{2} \delta((\bar X^i - \bar X^j)^{\dot\alpha\alpha}(\mathscr{M}'_\alpha)_{ij}) \right\} = \prod_{\substack{i,j=1\\ i\neq j}}^{k} (X^i - X^j)^2 \; . \tag{7.76}$$

This conveniently cancels against similar factors in (3.23) and (7.75).

The integrals that remain are over the positions of the instantons X_n^i, the gauge orientation coordinates z_i and the Grassmann collective coordinates $\{\mu_{iu}, \bar\mu_{iu}, (\mathscr{M}'_\alpha)_{ii}, \mathscr{K}, \tilde{\mathscr{K}}\}$. The integrals over the remaining Grassmann collective coordinates proceed as for the exact one-instanton calculation in Section 7.2.2: the $\{\mathscr{K}, \tilde{\mathscr{K}}\}$ and $\{\mu_{ui}, \bar\mu_{iu}\}$ integrals are saturated by bringing down terms from the instanton effective action (6.55), as in (7.28) for each i. The integrals over the remaining variables $\{\mu_{Ni}, \bar\mu_{iN}\}$ are then saturated by the diagonal fermionic ADHM constraints, as in (7.29), for each i.

Putting all the pieces together, we have the leading order $1/N$ contribution:

$$\begin{aligned} &\left\langle \frac{g^2}{16\pi^2}\,\mathrm{tr}_N\,\lambda^2(x^{(1)}) \times \cdots \times \frac{g^2}{16\pi^2}\,\mathrm{tr}_N\,\lambda^2(x^{(k)}) \right\rangle \\ &\quad \overset{N\to\infty}{\longrightarrow} \frac{3^k \mu^{k(2N+1)} g^{-2kN} \mathrm{e}^{2\pi \mathrm{i} k\tau} v^{-2kN} N^{3k/2}}{2^k \pi^{9k/2} k!} \\ &\quad \times \sum_{\substack{\text{perms}\\ \{l_i\}\text{ of }\{1,\dots,k\}}} \prod_{i=1}^{k} \left\{ \int \mathrm{d}^4X^i \, \frac{1}{((x^{(l_i)} - X^i)^2 + N/(2\pi^2 v^2))^4} \int_0^\infty \mathrm{d}z_i\, z_i^2 \mathrm{e}^{-Nz_i^2/4} \right\} \; . \end{aligned} \tag{7.77}$$

Performing the remaining integrals

$$\left\langle \frac{g^2}{16\pi^2}\,\mathrm{tr}_N\,\lambda^2(x^{(1)}) \times \cdots \times \frac{g^2}{16\pi^2}\,\mathrm{tr}_N\lambda^2(x^{(k)}) \right\rangle \overset{N\to\infty}{\longrightarrow} (\mu^{2N+1} g^{2N} \mathrm{e}^{2\pi \mathrm{i}\tau} v^{-2(N-1)})^k \; . \tag{7.78}$$

After renormalizing the VEV and matching to the pure gauge theory in the way described in the end of Section 7.2.2, our leading-order large-N result is

$$\left\langle \frac{g^2}{16\pi^2}\,\mathrm{tr}_N\,\lambda^2(x^{(1)}) \times \cdots \times \frac{g^2}{16\pi^2}\,\mathrm{tr}_N\lambda^2(x^{(k)}) \right\rangle \overset{N\to\infty}{\longrightarrow} \Lambda^{3k} \; . \tag{7.79}$$

As noted above, clustering is satisfied at large N.

7.4. Clustering in instanton calculations

Taking the SCI calculations in Sections 7.2.1 and 7.3.1, compare the kNth root of (7.63) with the large-N limit of the Nth root of the one-instanton result (7.21):

$$\frac{2\mathrm{e}}{N} k\Lambda^3 + \mathcal{O}(N^{-2}) \quad \text{versus} \; \frac{2\mathrm{e}}{N} \Lambda^3 + \mathcal{O}(N^{-2}) \; . \tag{7.80}$$

The implication is that the clustering property of the field theory is not respected by approximating the functional integral with an integral over instantons in the confining phase. In Ref. [67], it was further shown that the existence of the Shifman–Kovner vacuum designed to reconcile the SCI and WCI at the one-instanton level, cannot account for mismatch (7.80) for general k. We interpret the result to mean that instantons in the confining phase by themselves cannot account for all of the gluino condensate and that additional non-perturbative configurations must necessarily contribute. This conclusion is also supported by numerical calculations of the two-instanton contribution to the four-point function for gauge group SU(2) which were performed in [67].

In contrast, the WCI approach is consistent with clustering, at least at large N: compare (7.45) with (7.79). But we can argue further that consistency with clustering is guaranteed in the WCI approach, even at finite N. The argument proceeds as follows. Suppose one calculates the k-instanton contribution to the k-point function in the regime where $|x^{(i)}-x^{(j)}|\gg g/v$, where v is the characteristic scale of the VEVs. In other words, the separation between the insertions is much greater than the effective cut-off on the instanton scale size imposed by the instanton effective action (6.55). Since each insertion can saturate the integrals over two Grassmann collective coordinates, the dominant contribution to the integral comes from a region where one of the k instantons lies in the vicinity of each of the k insertion points. Choosing the insertion points in this way selects a region of $\mathfrak{M}_k$ in which the k instanton configuration is completely clustered, as described in Section 2.4.3. To leading order we can ignore any interactions between them. The collective coordinate integration measure then clusters as in (3.25). The factor of $1/k!$ is then cancelled by the $k!$ ways of pairing the k instantons with the k insertion points. Hence, to leading order, for these large separations we have

$$\left\langle \frac{g^2}{16\pi^2}\,\mathrm{tr}_N\,\lambda^2(x^{(1)})\times\cdots\times\frac{g^2}{16\pi^2}\,\mathrm{tr}_N\lambda^2(x^{(k)})_{k\text{-inst}}\right\rangle = \left\langle \frac{g^2}{16\pi^2}\,\mathrm{tr}_N\,\lambda^2\right\rangle^k_{1\text{-inst}} . \tag{7.81}$$

But we have proved in Section 7.1 that the correlation functions are independent of the insertion points and so (7.81) is an exact statement.

Now that we have established that the SCI approach is inconsistent with clustering, it is appropriate for us to make some comments regarding this methodology. The usual justification put forward for this method can be paraphrased as follows [10,38,63]. The correlation functions of $\mathrm{tr}_N\,\lambda^2$ are independent of the insertion points and so we can consider the configuration where the insertions are very close compared with the scale of strong coupling effects Λ. In this case only small instanton configurations should contribute to the integral over the instanton moduli space. If this is true then we should be able in invoke asymptotic freedom, namely the fact that the integral over the instanton moduli space involves the instanton action factor $\exp(-8\pi^2/g^2)$, where g runs with the characteristic scale size of the instanton, to argue that the calculation is reliable for insertions which are arbitrarily close. The fact that the correlation functions are independent of the insertions then allows us to continue to large separations. Unfortunately this argument is potentially flawed because, as is clear from (7.21), the integral over the instanton scale size is *not* cut-off exponentially by the separations between the insertions but only by a power law and so large instantons are not adequately suppressed. This behaviour should at least question the use of asymptotic freedom as an argument for exactness.

Although we have only chosen to discuss the gluino condensate, the analysis can be generalized to any correlation function involving the lowest components of gauge-invariant chiral superfields. For example, in the theories with fundamental hypermultiplets, the composite $\tilde{q}_f q_{f'}$ is gauge invariant and the lowest component of a chiral superfield. So correlation functions of this insertion, along with $g^2\,\mathrm{tr}_N\,\lambda^2$ will, by the Ward identity, be independent of the insertion points and holomorphic in the couplings. The SCI versus WCI mismatch that we have witnessed for the gluino condensate extends to these correlation functions and the same conclusion applies: only the WCI approach is reliable. The mismatch is particularly striking in the theory with $N_F = N-1$ hypermultiplets. In this case the correlation function

$$\mathscr{G} = \left\langle \frac{g^2}{16\pi^2}\,\mathrm{tr}_N\,\lambda^2(x^{(1)})\tilde{q}_{f_1}q_{f_1'}(x^{(2)}) \times \cdots \times \tilde{q}_{f_{N-1}}q_{f_{N-1}'}(x^{(N)}) \right\rangle \tag{7.82}$$

receives a contribution from a single instanton in both the SCI and WCI approaches. In the SCI methodology one takes the Higgs VEV to be zero in the bare theory so that the instantons are not constrained. VEVs for the scalar fields then arise from the instanton effects directly. On the contrary, in the WCI methodology the Higgs VEV is included in the bare theory and the instantons are constrained. The SCI approach yields the result [10]

$$\mathscr{G}_{\mathrm{SCI}} = \frac{\Lambda_{(N-1)}^{2N+1}}{N!}\,\varepsilon_{\{f_l\},\{f_l'\}}\ , \tag{7.83}$$

where $\varepsilon_{\{f_l\},\{f_l'\}}$ equals 1 if all the f_l are distinct and $\{f_l\} \equiv \{f_l'\}$, otherwise it is 0. The WCI result is very easy to establish given the calculation of $\langle (g^2/16\pi^2)\lambda^2\rangle$ in Section 7.2.2. This is because the insertions of the matter fields can be replaced by their VEVs $\tilde{q}_f q_{f'} \to \tilde{v}_f v_f \delta_{ff'}$. Multiplying (7.34) by the product of VEVs one finds

$$\mathscr{G}_{\mathrm{WCI}} = \Lambda_{(N-1)}^{2N+1} \prod_{l=1}^{N-1} \delta_{f_l f_l'}\ . \tag{7.84}$$

What makes the mismatch between (7.83) and (7.84) particularly striking is that the integral over the collective coordinates that yields the WCI result is formally just the limit of the SCI integral with the VEVs set to zero: $v_f = \tilde{v}_f = 0$ as emphasized in the context of gauge group SU(2) in Ref. [76]. So the collective coordinate integral is discontinuous at this point. These considerations show that the SCI approach yields an incorrect result even when applied in a weakly coupled Higgs phase.

8. On the Coulomb branch of $\mathscr{N} = 2$ gauge theories

Theories with $\mathscr{N} = 2$ supersymmetry have an adjoint-valued scalar field and consequently a Coulomb branch where the gauge symmetry is broken to the maximal abelian subgroup. For large values of the VEVs, the theory is weakly coupled and constrained instanton methods should be reliable. Moreover certain holomorphic quantities, and in particular the *prepotential* described below, are protected against quantum corrections beyond one loop and leading-order semi-classical methods should be exact. On the Coulomb branch of an $\mathscr{N} = 2$ theory, there is a completely different approach to calculating the prepotential based on the theory of Seiberg and Witten [68,77]. This remarkable theory predicts in a rather implicit way the exact form of the prepotential which can then be expanded in the semi-classical regime yielding the sum of a tree-level and one-loop contribution plus a series of instanton terms of arbitrary charge. In general closed-form expressions for higher instanton numbers are not known; however, what has been established are recursion relations which give the higher instanton coefficients, in terms of those of lower instanton number. The situation gives us an unprecedented laboratory for testing semi-classical instanton methods and all the underlying conceptual baggage like the imaginary-time formalism, Euclidean path integral, etc. Actually the result cuts both ways: not only can we quantitatively test semi-classical instanton methods in gauge theory, but, in addition, we can subject the ingenious theory of Seiberg–Witten to stringent tests. [56]

After introducing the notion of the low-energy effective action and the prepotential, we then go on to show how the instanton contribution at each instanton number is proportional to the centred instanton partition function introduced in Section 6.5. We then go on to describe the explicit evaluation of this partition function for the particular case of gauge group SU(2) with N_F flavours of hypermultiplets in the fundamental representation for instanton number $k=1$ and 2. This calculation relies on the fact that the gauge group SU(2) is also Sp(1) and the ADHM construction based on Sp(N), for $N=1$, is more economical than that for SU(N), for $N=2$. We then present the one-instanton calculation for general unitary groups. The results we obtain are in precise agreement with Seiberg–Witten theory for any number of flavours. Nevertheless, the cases with $N_F=2N$ are rather special since there are non-trivial dictionary issues to resolve.

We shall conclude this overture with a brief guide to literature which studied instanton effects in $\mathcal{N}=2$ gauge theories. First-instanton tests of pure $\mathcal{N}=2$ supersymmetric SU(2) theories were performed in [54] at a one-instanton level and in [25] at a two-instanton level. Two-instanton contributions to the prepotential in SU(2) theories with N_F fundamental hypermultiplets were calculated in [48,61,78] and the general expression for the k-instanton contribution to the prepotential as an integral over the ADHM moduli space was derived in [48]. The relation of Matone [80] between the prepotential and the condensate u_2 in SU(2) was tested at a two-instanton level in [81] and derived to all orders in instantons in [82]. SU(N) gauge theories with and without matter fields were studied at one-instanton level in [20,83,84]. In all these cases exact results of Seiberg–Witten and their generalizations [85–90] for the low-energy effective action were reproduced exactly for $N_F<2N$. The case of $N_F=2N$ was considered in [20,91] where it was pointed out that the mismatch between instanton calculations and the proposed exact solutions for $N_F=2N$ arises due to a finite (perturbative and non-perturbative) renormalization of the coupling constant of the low-energy effective theory. This effect has to be incorporated into exact solutions which then agree with instanton results. For a recent careful treatment of these issues see [92]. In addition to these effects, explicit instanton calculations are also necessary in order to fix the dictionary between the quantum moduli used in constructing exact solutions and the gauge-invariant VEVs u_n, see [20,58,61,84,91–94] for more detail. A completely new technique for evaluating the instanton contributions to the prepotential has been pioneered in [95,96] leading to the first calculations of instanton effects for all instanton number (beyond the large-N calculations reported in Sections 7 and 9) and the first

[56] We take it as unreasonably perverse that the instanton calculus and Seiberg–Witten theory could both be wrong while being consistent with one another. In any case, there are other tests of Seiberg–Witten theory based, for example, on the spectrum of BPS states [79].

test of Seiberg–Witten theory to all orders in the instanton expansion. We will briefly describe this new formalism in Section 9.2 where we also mention the extraordinary recent work of Nekrasov [179] which calculates the instanton contributions to the prepotential to all order in the instanton charge.

8.1. Seiberg–Witten theory and the prepotential

Theories with $\mathcal{N}=2$ supersymmetry and gauge group SU(N) on the Coulomb branch have an $N-1$ complex dimensional vacuum moduli space parameterized by the VEVs (4.63) for the complex scalar field ϕ. At a generic point on this classical moduli space the gauge group is broken to its maximal abelian subgroup $\mathrm{U}(1)^{N-1} \subset \mathrm{SU}(N)$ and the theory is in a Coulomb phase. In these circumstances it is possible to describe the long-distance behaviour of the theory in terms of a low-energy effective action. This can be written in terms of abelian $\mathcal{N}=1$ superfields [57] $W_{\alpha u}=(A_{mu},\lambda_u)$ and $\Phi_u=(\phi_u,\psi_u)$:

$$S_{\mathrm{eff}} = \frac{1}{4\pi}\int \mathrm{d}^4x\, \mathrm{Im}\left\{\frac{1}{2}\,\tau_{uv}(\Phi)W_u^{\alpha}W_{\alpha v}|_{\theta^2} + \Phi_{\mathrm{D}u}(\Phi)\Phi_u^{\dagger}|_{\theta^2\bar\theta^2}\right\}\ , \tag{8.1}$$

where the VEVs of the scalar components of Φ_u are identified with $g\phi_u^0/\sqrt{2}$ of the field of the microscopic theory. This low-energy effective action is uniquely determined by the holomorphic prepotential $\mathscr{F}(\Phi)$ [68] as

$$\Phi_{\mathrm{D}u} \equiv \frac{\partial\mathscr{F}}{\partial\Phi_u}, \qquad \tau_{uv} \equiv \frac{\partial^2\mathscr{F}}{\partial\Phi_u\partial\Phi_v}\ . \tag{8.2}$$

Eq. (8.1) implies that the matrix of complexified coupling constants of the low-energy $\mathrm{U}(1)^{N-1}$ theory has components τ_{uv} that depend on the scalar VEVs.

The electric–magnetic duality of the $\mathrm{U}(1)^{N-1}$ theory uncovered by Seiberg and Witten [68,77] identifies $\Phi_{\mathrm{D}u}$ with the magnetic dual of the original (electric) matter superfield Φ_u. The vacuum expectation values of the scalar component of $\Phi_{\mathrm{D}u}$ are $g\phi_{\mathrm{D}u}^0/\sqrt{2}$, with $\sum_{u=1}^N \phi_{\mathrm{D}u}^0=0$, and these provide an alternative parameterization of the vacuum moduli space. Both parameterizations, $\{\phi_u^0\}$ and $\{\phi_{\mathrm{D}u}^0\}$, are not valid globally on the moduli space. Such a global parameterization is provided [68,77] by the gauge-invariant condensates $\{u_n\}$ $n=2,\dots,N$:

$$u_n = 2^{-n/2}g^n\langle \mathrm{tr}_N\,\phi^n\rangle\ . \tag{8.3}$$

On a patch of the moduli space where $\{\phi_u^0\}$ are a local coordinate system we have $\phi_u^0=\phi_u^0(u_n)$, while on the patch where the dual variables $\{\phi_{\mathrm{D}u}^0\}$ are a local coordinate system we have $\phi_{\mathrm{D}u}^0=\phi_{\mathrm{D}u}^0(u_n)$. The analysis of Seiberg and Witten [68,77], and its generalizations, explicitly determine these functions in terms of an auxiliary Riemann surface defined by a family of hyperelliptic curves.

The effective action specifies the long-range behaviour, compared with the scale $1/(g\phi^0)$, of a series of correlation functions. In particular, we will focus on a four-point anti-chiral fermion correlator

[57] For each of these superfields we have the traceless condition $\sum_{u=1}^N W_{\alpha u}=\sum_{u=1}^N \Phi_u=0$.

which involves the fourth derivative of the prepotential. In Euclidean space

$$\langle \bar\lambda^{\dot\alpha}_{u_1}(x^{(1)})\bar\lambda^{\dot\beta}_{u_2}(x^{(2)})\bar\psi^{\dot\gamma}_{u_3}(x^{(3)})\bar\psi^{\dot\delta}_{u_4}(x^{(4)})\rangle = \frac{1}{2\pi \mathrm{i}}\,\frac{\partial^4\mathscr{F}}{\partial\phi^0_{u_1}\partial\phi^0_{u_2}\partial\phi^0_{u_3}\partial\phi^0_{u_4}}\int d^4X\,\bar{\mathrm{S}}^{\dot\alpha\alpha}(x^{(1)},X)\bar{\mathrm{S}}^{\dot\beta}{}_\beta(x^{(2)},X)\bar{\mathrm{S}}^{\dot\gamma\gamma}(x^{(3)},X)\bar{\mathrm{S}}^{\dot\delta}{}_\gamma(x^{(4)},X)\ , \tag{8.4}$$

where $\bar{\mathrm{S}}(x,X)$ is the free anti-Weyl spinor propagator,

$$\bar{\mathrm{S}}(x,X) = \frac{1}{4\pi^2}\,\bar{\partial}\,\frac{1}{(x-X)^2}\ . \tag{8.5}$$

The restrictions imposed by holomorphy, renormalization group invariance and the anomaly imply that the prepotential has a weak coupling expansion consisting of perturbative piece that is one-loop exact and a sum over instanton contributions which are exact to leading order in the semi-classical expansion. For $SU(N)$ with $N_F < 2N$ fundamental hypermultiplets

$$\mathscr{F} = \mathscr{F}_{\text{pert}} + \frac{1}{2\pi \mathrm{i}}\sum_{k=1}^{\infty}\mathscr{F}_k\Lambda^{k(2N-N_F)}\ . \tag{8.6}$$

In the special case $N_F = 2N$, for which the β-function vanishes, the perturbative part is purely classical:

$$\mathscr{F}|_{N_F=2N} = \frac{g^2}{2}\,\tau\sum_{u=1}^{N}(\phi^0_u)^2 + \frac{1}{2\pi \mathrm{i}}\sum_{k=1}^{\infty}\mathscr{F}_k e^{2\pi \mathrm{i}k\tau}\ . \tag{8.7}$$

In general, the coefficients $\mathscr{F}_k$ depend on the VEVs and hypermultiplet masses.

The theory of Seiberg and Witten, extended from the original setting for $SU(2)$ [68], to $SU(N)$, $N > 2$, in Refs. [85–89], determines the exact form of the prepotential and in particular the k-instanton coefficients $\mathscr{F}_k$. The construction involves an auxiliary algebraic curve, or Riemann surface, the *Seiberg–Witten curve*.[58] Once the curve has been identified, the coefficients can, in principle, be extracted. In particular, explicit expression for low instanton number have been found in this way [83,90]. For arbitrary instanton number, explicit expressions are hard to obtain; however, recursion relations have been established that relate instanton contributions at charge k to those of lower charge [80,97–100]. These recursion relations are only valid in the asymptotically free theories with $N_F < 2N$. The finite theory with $N_F = 2N$ is rather special since there are various re-definitions of the physical quantities that have to be taken into account; see Ref. [92].

Here, we use the notation of [90] to write down the $k = 1$ and 2 contributions for $N_F < 2N$:

$$\mathscr{F}_1 = \sum_{u=1}^{N} S_u(\phi^0_u)\ , \tag{8.8a}$$

$$\mathscr{F}_2 = \sum_{\substack{u,v=1\\(u\neq v)}}^{N}\frac{S_u(\phi^0_u)S_v(\phi^0_v)}{(\phi^0_u-\phi^0_v)^2} + \frac{1}{4}\sum_{u=1}^{N}S_u(\phi^0_u)\,\frac{\partial^2 S_u(\phi^0_u)}{\partial(\phi^0_u)^2}\ , \tag{8.8b}$$

[58] In a nutshell, the matrix of coupling τ_{uv} is identified with the period matrix of the curve in a suitable basis.

where we have used the function

$$S_u(x) \equiv \prod_{\substack{v=1\\(\neq u)}}^{N} \frac{1}{(x-\phi_v^0)^2} \prod_{f=1}^{N_F} (m_f + x) \ . \tag{8.9}$$

8.2. Extracting the prepotential from instantons

There are two distinct—but ultimately equivalent—ways of determining the instanton contributions to the prepotential. The first involves calculating the leading semi-classical contribution to Green's functions whose long-distance limit can be matched with the low-energy effective action (8.1). Here, we will concentrate on the four-point anti-chiral fermion correlator (8.4) which involves the fourth derivative of the prepotential. However, other correlators which we do not consider explicitly determine the second derivative of the prepotential in an analogous way. The second approach involves calculating the instanton contribution to the condensate u_2 defined in (8.3) and then relating this to the prepotential by use of a renormalization group equation which yields the derivative of the prepotential with respect to $\log \Lambda$ (for $N_F < 2N$) [80,101–103]. Alternatively, we will take the view that the instanton calculation of the condensate provides an independent way to establish the renormalization group relation.

For the first approach we turn our focus on the instanton contributions to the four-point anti-chiral fermion correlator (8.4). The first point to make is that since there are non-trivial scalar VEVs, the instantons are constrained as described in Section 4.4.1. It will transpire that the integral over the four-vector X in (8.4) arises in the instanton context from the integral over the centre of instanton (2.82). Consequently, in order to extract the long-distance behaviour of the correlator, we should insert the corresponding behaviour of the anti-chiral fermions $\bar\lambda_A = (\bar\lambda, \bar\psi)$ far from the core of the constrained instanton. Recall from our discussion in Section 4.4.1 that in the tail of the constrained instanton, fields decay exponentially except for the components which remain massless after the Higgs mechanism. These are the fields valued in the $\mathrm{U}(1)^{N-1}$ subgroup picked out by the VEV ϕ^0; in other words, the diagonal elements of the $N \times N$ matrices $\bar\lambda_A$. It is precisely these components of the anti-chiral fermions that we need in order to calculate the long-distance behaviour of the correlator. This is fortunate because, as we argued in Section 4.4.1, to leading order in the semi-classical expansion these massless components are simply given by their ADHM expressions. Moreover, we will only need the components of the anti-chiral fermions that depend on the four supersymmetric Grassmann collective coordinates ξ^A (4.43), since these are the only Grassmann variables whose integrals are not saturated by the instanton effective action (6.84). It will turn out that each insertion of $\bar\lambda_A$ is then linear in ξ^B and therefore the insertion of the four anti-chiral fermions precisely saturates the integrals over the four Grassmann variables associated to the unlifted supersymmetric zero modes. In principle, we can extract the terms that we need from the solution for the anti-chiral fermions in (4.62) (with $\bar\psi_A$ given in (C.38) and (C.39a)–(C.39c)). However, it is more straightforward to use the sweeping-out procedure described in Section 4.3. The supersymmetry transformation (4.23c) (re-scaled by $g^{1/2}$) in the super-instanton background gives the required dependence on ξ^A:

$$\bar\lambda_A = -\mathrm{i} g^{1/2} \bar\Sigma_{aAB} \bar{\mathscr{D}} \phi_a \xi^B = -g^{1/2} \bar{\mathscr{D}} \phi^\dagger \xi_A \ , \tag{8.10}$$

where in the latter expression we used relations (4.20). The expression for the anti-holomorphic component of the scalar field is given in (6.66) (it is important to remember that in an instanton background $\phi^\dagger$ is *not* the conjugate of ϕ). We then expand the diagonal components of $\bar{\mathscr{D}}\phi^\dagger$ (these are the massless components) for large distances from the instanton core using the asymptotic formulae in Section 4.3:

$$(\bar{\mathscr{D}}\phi^\dagger)_{uu} = -\bar{\partial}\,\frac{1}{(x-X)^2}\,w_{u\dot{\alpha}}\left\{\phi_u^{0\dagger}1_{[k]\times[k]} + \boldsymbol{L}^{-1}\left(\bar{w}^{\dot{\beta}}\phi^{0\dagger}w_{\dot{\beta}} - \frac{1}{4}\sum_{f=1}^{N_F}\mathscr{K}_f\tilde{\mathscr{K}}_f\right)\right\}\bar{w}_u^{\dot{\alpha}}\ . \tag{8.11}$$

Plugging this into (8.10), and using the form of the instanton effective action (6.84), we are able to establish the following simple relation between the long-distance behaviour of the anti-chiral fermion and the instanton effective action: [59]

$$\bar{\lambda}_{Au}^{\dot{\alpha}} = 2\sqrt{g}\,\bar{\mathrm{S}}^{\dot{\alpha}\alpha}(x,X)\varepsilon_{AB}\xi_\alpha^A\,\frac{\partial\tilde{S}}{\partial\phi_u^0} + \cdots\ . \tag{8.12}$$

The ellipsis represents terms depending on other Grassmann collective coordinate which will not be required to determine the long-range behaviour of the correlator in question.

The k-instanton contribution to correlation function (8.4) is evaluated in the usual way by making insertions of the long-distance component (8.12) into the collective coordinate integral:

$$\begin{aligned}&\langle\bar{\lambda}_{u_1}^{\dot{\alpha}}(x_1)\bar{\lambda}_{u_2}^{\dot{\beta}}(x_2)\bar{\psi}_{u_3}^{\dot{\gamma}}(x_3)\bar{\psi}_{u_4}^{\dot{\delta}}(x_4)\rangle_k\\ &\quad=\left(\frac{\mu}{g}\right)^{k(2N-N_F)}\mathrm{e}^{2\pi ik\tau}\int_{\mathfrak{M}_k}\boldsymbol{\omega}^{(\mathscr{N}=2,N_F)}\,\mathrm{e}^{-\tilde{S}}\,\bar{\lambda}_{u_1}^{\dot{\alpha}}(x^{(1)})\bar{\lambda}_{u_2}^{\dot{\beta}}(x^{(2)})\bar{\psi}_{u_3}^{\dot{\gamma}}(x^{(3)})\bar{\psi}_{u_4}^{\dot{\delta}}(x^{(4)})\ .\end{aligned} \tag{8.13}$$

Here, $\boldsymbol{\omega}^{(\mathscr{N}=2,N_F)}$ is the supersymmetric volume form on the instanton moduli space in a theory with N_F hypermultiplets defined in (5.55) and (6.59). The instanton effective action $\tilde{S}$ is precisely (6.84) along with the hypermultiplet mass term (6.87). Following [48,82], we substitute expression (8.12) into the right-hand side and perform the integrals over the four supersymmetric Grassmann variables ξ^A. This leaves

$$\begin{aligned}&\langle\bar{\lambda}^{\dot{\alpha}}(x^{(1)})\bar{\lambda}^{\dot{\beta}}(x^{(2)})\bar{\psi}^{\dot{\gamma}}(x^{(3)})\bar{\psi}^{\dot{\delta}}(x_4)\rangle_k\\ &\quad=\frac{1}{4\pi^2}\,g^2\left(\frac{\mu}{g}\right)^{k(2N-N_F)}\mathrm{e}^{2\pi ik\tau}\,\frac{\partial^4}{\partial\phi_{u_1}^0\partial\phi_{u_2}^0\partial\phi_{u_3}^0\partial\phi_{u_4}^0}\int_{\hat{\mathfrak{M}}_k}\boldsymbol{\omega}^{(\mathscr{N}=2,N_F)}\mathrm{e}^{-\tilde{S}}\\ &\qquad\times\int \mathrm{d}^4X\,\bar{\mathrm{S}}^{\dot{\alpha}\alpha}(x^{(1)},X)\bar{\mathrm{S}}^{\dot{\beta}}{}_{\alpha}(x^{(2)},X)\bar{\mathrm{S}}^{\dot{\gamma}\gamma}(x^{(3)},X)\bar{\mathrm{S}}^{\dot{\delta}}{}_{\gamma}(x^{(4)},X)\ .\end{aligned} \tag{8.14}$$

The integral over the centred moduli space $\hat{\mathfrak{M}}_k$ is precisely the *centred instanton partition function* $\hat{\mathscr{Z}}_k^{(\mathscr{N}=2,N_F)}$ defined in (6.103). Note in (8.14) the linearity of $\tilde{S}$ in ϕ_u^0, apparent in (6.84), has been used to pull the ϕ_u^0 derivatives outside the collective coordinate integral.

[59] Here, $\bar{\lambda}_u \equiv (\bar{\lambda})_{uu}$, the diagonal elements of $\bar{\lambda}$.

Comparing semi-classical expression (8.14) with its exact counterpart (8.4), we deduce the following expression for the k-instanton expansion coefficient of the prepotential in (8.6):

$$\mathscr{F}_k = g^{-k(2N-N_F)+2}\,\hat{\mathscr{Y}}_k^{(\mathscr{N}=2,N_F)} \ . \tag{8.15}$$

One might have thought that we could add to this relation any function whose fourth derivative with respect to the VEVs vanishes. Actually, by considering other correlators, one can establish (8.15) up to functions whose second derivative with respect to the VEVs vanishes. However, since there are no possible linear functions of the VEVs, (8.15) must be true up to an undetermined constant which does not affect the physics because the low-energy effective action only depends on derivatives of $\mathscr{F}$. Note that in order to write the instanton contributions in form (8.6) we have substituted the expression for the Λ-parameter of the theory in terms of the running coupling:

$$\Lambda_{(N_F)}^{2N-N_F} \equiv \mu^{2N-N_F}\,\mathrm{e}^{2\pi\mathrm{i}\tau} \ . \tag{8.16}$$

For the special case when $N_f = 2N$, where the β-function vanishes, the factor $\Lambda_{(N_F)}^{2N-N_F}$ should be replaced by $q = \mathrm{e}^{2\mathrm{i}\pi\tau}$.

The other way to calculate the instanton expansion of the prepotential is to use the renormalization group equation for the prepotential first established by Matone in the context of pure SU(2) theory [80] and then generalized to SU(N) in Refs. [101–103]. Rather than write down the relevant equation we shall proceed to prove a version of the renormalization group equation using the instanton calculus, generalizing the SU(2) calculation of [82]. One may then check that our instanton version is consistent with those in the literature cited above. Our relation will also be valid in the finite theory where $N_F = 2N$. The approach is rather different from that just follows. Instead of calculating the instanton contribution to the long-range behaviour of a correlator, one calculates the instanton contributions to the condensate u_2.

To begin, we establish the form of the insertion $\mathrm{tr}_N\,\phi^2$ in the constrained instanton background. Just as in the calculation of the gluino condensate in the $\mathscr{N} = 1$ theories in the Higgs phase described in Sections 7.2.2 and 7.3.2, to leading order we may replace the insertion by its value in the unconstrained ADHM instanton background. This is because any error incurred is necessarily of a higher order in g. Just as for the four-point anti-chiral fermion correlator (8.4), the insertion must saturate the integrals over the four supersymmetric Grassmann variables ξ^A. Since, as is evident from (4.64), the scalar field is quadratic in the Grassmann collective coordinates, so the composite $\mathrm{tr}_N\,\phi^2$ is indeed quartic in the Grassmann collective coordinates. The dependence on ξ^A may be obtained by the sweeping-out procedure using supersymmetry transformation (4.23d):

$$\phi_a = -\tfrac{1}{2}\,g\bar{\Sigma}_{aAB}\xi^A\sigma_{mn}\xi^B F_{mn} + \cdots \ . \tag{8.17}$$

Therefore, using (4.20), $\phi = \phi_1 - \mathrm{i}\phi_2$, and the identity

$$\xi\sigma_{mn}\eta\xi\sigma_{pq}\eta = -\tfrac{1}{8}\,(\xi\xi)(\eta\eta) \ , \tag{8.18}$$

for arbitrary spinors ξ and η, we have

$$\tfrac{1}{2}\,g^2\,\mathrm{tr}_N\,\phi^2(x) = \tfrac{1}{2}\,g^4(\xi^1\xi^1)(\xi^2\xi^2)\,\mathrm{tr}_N\,F_{mn}^2(x) + \cdots \ . \tag{8.19}$$

We now insert (8.19) into the collective coordinate integral. Separating out the variables $\{X, \xi^A\}$ as in (6.102) one can then trivially integrate over the Grassmann variables ξ^A. The integral over the four-vector X is also trivial because by translational symmetry it may be traded for an integral over the insertion point x yielding $\int d^4x\, \mathrm{tr}_N\, F_{mn}^2(x)$ which is the integral that gives instanton charge (2.2). Hence, the k-instanton contribution to the condensate is

$$u_2|_k = k\Lambda_{(N_F)}^{k(2N-N_F)} g^{-k(2N-N_F)+2} \hat{\mathscr{Z}}_k^{(\mathcal{N}=2,N_F)} . \tag{8.20}$$

Note that the instanton contribution to the condensate u_2 is proportional to the centred instanton partition function just as for expression for gluino condensate (7.25) in the $\mathcal{N}=1$ theory. Of course in the $\mathcal{N}=2$ theory the condensate receives contributions from all instanton numbers, whereas one can easily show that in the $\mathcal{N}=1$ theory the centred instanton partition function vanishes for $k>1$.

Comparing (8.20) with our formula for prepotential (8.24) we obtain a version of the renormalization group equation in the form of Ref. [82]. The result is most easy to state in terms of the k-instanton contribution:

$$u_2|_k = k\Lambda^{k(2N-N_F)} \mathscr{F}_k . \tag{8.21}$$

This expression relates only the non-perturbative contributions to both quantities. The rest, however, is easy to determine via direct perturbative calculation. In addition to instantons, the prepotential receives only one-loop perturbative contributions, while u_2 receives no perturbative corrections to its classical value. It is remarkable that we were able to derive (8.21) using instanton calculus, but without actually having to integrate over the instanton moduli space. Note that the result is equally valid in the finite theory $N_F = 2N$ on replacing $\Lambda^{2N-N_F} \to e^{2\pi i\tau}$ and it is also valid for arbitrary hypermultiplet masses.

At this point we find it useful to perform various re-scalings of the variables by powers of the coupling g in order that there are is no explicit g dependence in $\hat{\mathscr{Z}}_k$. The required re-scalings are

$$\phi^0 \to g^{-1}\phi^0, \quad a_{\dot\alpha} \to g a_{\dot\alpha}, \quad \mathscr{M}^A \to g^{1/2}\mathscr{M}^A, \quad \mathscr{K} \to g^{1/2}\mathscr{K}, \quad \tilde{\mathscr{K}} \to g^{1/2}\tilde{\mathscr{K}} . \tag{8.22}$$

Under these re-scalings, the centred instanton partition function scales as

$$\hat{\mathscr{Z}}_k^{(\mathcal{N}=2,N_F)} \to g^{k(2N-N_F)+2} \hat{\mathscr{Z}}_k^{(\mathcal{N}=2,N_F)} . \tag{8.23}$$

In particular there is no g dependence in the re-scaled instanton effective action and the re-scaled version of (8.15) is simply

$$\mathscr{F}_k = \hat{\mathscr{Z}}_k^{(\mathcal{N}=2,N_F)} , \tag{8.24}$$

where all g dependence has disappeared (as it should).

8.3. *Gauge group* SU(2)

In this section, we specialize to gauge group SU(2) and compute the one- and two-instanton contributions to the prepotential in order to compare with the exact theory of Seiberg and Witten. For SU(2), there is a single VEV $\phi^0 \equiv \phi_1^0/2 = -\phi_2^0/2$. The predictions for the one- and two-instanton coefficients are obtained from (8.8a) by setting $N=2$.

Since the gauge group SU(2) is isomorphic to Sp(1), one can proceed with either of the two formalisms; however, as described in Section 6.2, the ADHM construction for Sp(1) is more economical in the sense that for a given k there are fewer ADHM variables and constraints compared with the SU(2) formalism. Hence, the Sp(1) formalism is better suited for a direct calculation of the prepotential.

Almost all of the formulae that we established in the SU(N) instanton calculus case carry over to Sp(1) by simply imposing the extra restrictions (6.32) and (6.33) on $a_{\dot\alpha}$ and $\mathscr{M}^A$, respectively. Recall that once we have replaced the gauge index $u=1,2$ by a spinor index $\alpha=1,2$, the ADHM variables $w_{\dot\alpha ui}$ become quaternions $w_{\alpha\dot\alpha i}$. From (6.36)

$$a=\begin{pmatrix} w \\ a' \end{pmatrix}\ , \tag{8.25}$$

where w is a k-vector of quaternions and a' is a $k\times k$ symmetric matrix whose elements are quaternions. In (8.25) and in much of the following the quaternion indices are understood. Products of quaternions are defined as 2×2 matrix multiplication. We also define the quaternion inner product

$$x\cdot y=\tfrac{1}{4}\,\mathrm{tr}_2(\bar x y+\bar y x),\qquad |x|^2\equiv x\cdot x=x_nx_n\ . \tag{8.26}$$

For the Grassmann collective coordinates

$$\mathscr{M}^A=\begin{pmatrix} \mu^A \\ \mathscr{M}'^A \end{pmatrix} \tag{8.27}$$

for $A=1,2$, where μ^A and $\mathscr{M}'^A$ are Weyl spinor-valued k-vectors and $k\times k$ symmetric matrices, respectively.

The collective coordinate integral will involve the volume form on the instanton moduli space which is given by an obvious translation of the general SU(N) formula (5.55):

$$\begin{aligned}\int_{\mathfrak{M}_k}\boldsymbol{\omega}^{(\mathscr{N}=2)}=\frac{C_k^{(\mathscr{N}=2)}}{\mathrm{Vol}\,\mathrm{O}(k)}&\int\prod_{i=1}^{k}\mathrm{d}^4w_i\prod_{i\leqslant j}\mathrm{d}^4a'_{ij}\left\{\prod_{A=1}^{2}\prod_{i=1}^{k}\mathrm{d}^2\mu_i^A\prod_{i\leqslant j=1}^{k}\mathrm{d}^2\mathscr{M}'^A_{ij}\right\}\\ &\times|\det\boldsymbol{L}|^{-1}\prod_{i<j=1}^{k}\left\{\prod_{c=1}^{3}\delta\left(\frac{1}{2}\,\mathrm{tr}_k\,\tau^{c\dot\alpha}{}_{\dot\beta}((\bar a^{\dot\beta}a_{\dot\alpha})_{ij}-(\bar a^{\dot\beta}a_{\dot\alpha})_{ji})\right)\right.\\ &\left.\prod_{A=1}^{2}\prod_{\dot\alpha=1}^{2}\delta((\bar{\mathscr{M}}^Aa_{\dot\alpha})_{ij}-(\bar{\mathscr{M}}^Aa_{\dot\alpha})_{ji})\right\}\ .\end{aligned} \tag{8.28}$$

In this expression, integrals over a quaternion w are defined as $\int\mathrm{d}^4w\equiv\int\mathrm{d}w_1\,\mathrm{d}w_2\,\mathrm{d}w_3\,\mathrm{d}w_4$. Note in the one-instanton sector the δ-functions are absent (unlike the SU(2) formalism). In this formalism $\boldsymbol{L}$ is now a linear operator on the space of $k\times k$ anti-symmetric matrices but still defined as in (2.125). Finally, the normalization constant is fixed by using clustering and comparison with the one-instanton collective coordinate integral in [2]:

$$C_k^{(\mathscr{N}=2)}=2^{5k-k^2}\pi^{-4k}\ . \tag{8.29}$$

The leading-order expression for the instanton effective action of the $\mathscr{N}=2$ theory with N_F fundamental hypermultiplets can be read of the expression in the SU(N) theory (6.84). In addition, we add mass terms (6.87). The collective coordinate integral also includes integrals over the hypermultiplet collective coordinates $\{\mathscr{K},\tilde{\mathscr{K}}\}$ in (6.58). Consider the complete expression for the collective coordinate integral in the chiral limit, $m_f=0$. In this limit, for fixed flavour index f, Grassmann measure (6.58) is obviously even or odd under the discrete symmetry

$$\mathscr{K}_{if} \leftrightarrow \tilde{\mathscr{K}}_{fi} , \tag{8.30}$$

depending on whether k itself is even or odd. On the other hand, the instanton effective action (6.84) (without the mass terms) is always even under this symmetry. Therefore, for $N_F>0$, only the even-instanton sectors $k=0,2,\ldots$ can contribute in the chiral limit [60] (recall that when $N_F=0$, all instanton numbers contribute). This selection rule was already noted by Seiberg and Witten in Section 3 of [77], so it is satisfying to see it arising naturally in the instanton calculus. Of course it is violated once the masses are non-zero, since $\mathscr{K}_{if}\tilde{\mathscr{K}}_{fi}$ is odd under the symmetry.

8.3.1. One instanton

Recall that in order to calculate instanton contributions to the prepotential, we have to calculate the centred instanton partition function (8.24). We begin with the one-instanton case. In ADHM language, the bosonic and fermionic parameters of a single $\mathcal{N}=2$ super-instanton are contained in three 2×1 matrices of unconstrained parameters:

$$a=\begin{pmatrix} w \\ -X \end{pmatrix}, \quad \mathscr{M}^A=\begin{pmatrix} \mu^A \\ -4\mathrm{i}\xi^A \end{pmatrix} . \tag{8.31}$$

In addition, there are $2N_F$ Grassmann variables $\{\mathscr{K}_f,\tilde{\mathscr{K}}_f\}$ which parameterize the fundamental zero modes. The centred-instanton volume form is extracted from (8.28)

$$\int_{\hat{\mathfrak{M}}_1} \omega^{(\mathcal{N}=2,N_F)} = \frac{2^3}{\pi^{4+2N_F}} \int \mathrm{d}^4 w \prod_{A=1}^{2} \mathrm{d}^2\mu^A \prod_{f=1}^{N_F} \mathrm{d}\mathscr{K}_f\, \mathrm{d}\tilde{\mathscr{K}}_f , \tag{8.32}$$

where the instanton effective action for $k=1$ is easily deduced from (6.84) and (6.87):

$$\tilde{S} = 8\pi^2|w|^2|\phi^0|^2 - 2\mathrm{i}\pi^2\mu^A\phi^{0\dagger}\mu_A + \pi^2\sum_{f=1}^{N_F} m_f\mathscr{K}_f\tilde{\mathscr{K}}_f . \tag{8.33}$$

Notice that the only dependence on $\mathscr{K}_f$ and $\tilde{\mathscr{K}}_f$ comes exclusively from the mass term at the one-instanton level. The corresponding Grassmann integrations are easily saturated by bringing down appropriate powers of this term from the exponent. As expected from discrete symmetry (8.30), the result is non-zero only when all the m_f are non-vanishing. The remaining integrals are easily performed, yielding

$$\hat{\mathscr{Z}}_1^{(\mathcal{N}=2,N_F)}|_{N=2} = \frac{2}{(\phi^0)^2}\prod_{f=1}^{N_F} m_f . \tag{8.34}$$

[60] The absence of a one-instanton contribution is quite easy to see since $\{\mathscr{K}_f,\tilde{\mathscr{K}}_f\}$ completely decouple from the instanton effective action (6.84) and consequently their integrals remain unsaturated.

Using (8.24), this gives $\mathscr{F}_1$ which is in agreement with (8.8a) for $N_F < 4$, up to irrelevant VEV-independent constants for the cases $N_F = 2, 3$.

8.3.2. Two instantons

Next we turn to the more calculationally intensive two-instanton contribution. The parameters of the $k = 2$ ADHM super-instanton are contained in the following 3×2 matrices:

$$a = \begin{pmatrix} w_1 & w_2 \\ -X + a_3 & a_1 \\ a_1 & -X - a_3 \end{pmatrix}, \qquad \mathscr{M}^A = \begin{pmatrix} \mu_1^A & \mu_2^A \\ -4\mathrm{i}\xi^A + \mathscr{M}'^A_3 & \mathscr{M}'^A_1 \\ \mathscr{M}'^A_1 & -4\mathrm{i}\xi^A - \mathscr{M}'^A_3 \end{pmatrix}. \tag{8.35}$$

Each element of a is a quaternion and of $\mathscr{M}^A$ a Weyl spinor. In addition, there are now $4N_F$ fundamental zero modes parameterized by the Grassmann numbers $\mathscr{K}_{if}$ and $\tilde{\mathscr{K}}_{fi}$. We also define the following frequently occurring combinations of these collective coordinates:

$$\begin{aligned} &L = |w_1|^2 + |w_2|^2, \quad H = |w_1|^2 + |w_2|^2 + 4|a_1|^2 + 4|a_3|^2 \,, \\ &\Omega = w_1\bar{w}_2 - w_2\bar{w}_1, \quad \omega = \tfrac{1}{2}\,\phi^0\,\mathrm{tr}_2(\bar{w}_2\tau^3 w_1 - \bar{w}_1\tau^3 w_2)\,, \\ &Y = -\mu_1^A\mu_{2A} - 2\mathscr{M}'^A_3\mathscr{M}'_{1A}, \quad Z = \sum_{f=1}^{N_F}(\mathscr{K}_{1f}\tilde{\mathscr{K}}_{f2} - \mathscr{K}_{2f}\tilde{\mathscr{K}}_{f1})\,. \end{aligned} \tag{8.36}$$

For $k = 2$, $\boldsymbol{L}$ is just multiplication by the quantity H. In terms of these variables, the instanton effective action (6.84) (with mass term (6.87) and re-scalings (8.22)) is written as

$$\begin{aligned} \tilde{S} = {}& 8\pi^2 L|\phi^0|^2 - 2\pi^2\mathrm{i}\phi^{0\dagger}(\mu_1^A\mu_{1A} - \mu_2^A\mu_{2A}) \\ & - \frac{8\pi^2}{H}\left(\bar{\omega} - \frac{1}{8}Z\right)\left(\omega - \frac{\mathrm{i}}{2}Y\right) + \pi^2\sum_{f=1}^{N_F} m_f(\mathscr{K}_{1f}\tilde{\mathscr{K}}_{f1} + \mathscr{K}_{2f}\tilde{\mathscr{K}}_{f2})\,. \end{aligned} \tag{8.37}$$

We can now proceed to evaluate the centred instanton partition function. To start with, we can explicitly solve the bosonic and fermionic ADHM constraints. This is conveniently done by eliminating the off-diagonal elements a_1 and $\mathscr{M}'^A_1$ as follows:

$$a_1 = \frac{1}{4|a_3|^2}\,a_3(\bar{w}_2 w_1 - \bar{w}_1 w_2), \quad \mathscr{M}'^A_1 = \frac{1}{2|a_3|^2}\,a_3(2\bar{a}_1\mathscr{M}'^A_3 + \bar{w}_2\mu_1^A - \bar{w}_1\mu_2^A)\,. \tag{8.38}$$

We can now explicitly integrate out the δ-functions in the expression for the centred $k=2$ instanton volume form (8.28):

$$\begin{aligned} \int_{\hat{\mathfrak{M}}_k} \boldsymbol{\omega}^{(\mathscr{N}=2)} = \frac{C_2^{(\mathscr{N}=2)}}{2^{10}} \int \mathrm{d}^4 a'_1 \mathrm{d}^4 w_1\, \mathrm{d}^4 w_2 \left\{ \prod_{A=1}^{2} \mathrm{d}^2\mathscr{M}'^A_3\, \mathrm{d}^2\mu_1^A\, \mathrm{d}^2\mu_2^A \right\} \\ \frac{||a_3|^2 - |a_1|^2|}{H}\,. \end{aligned} \tag{8.39}$$

The bosonic parts of the two-instanton collective coordinate integral were originally derived in Refs. [8,104,105] by directly changing variables in the path integral.

We now have a series of integrals to perform. First of all, performing the Grassmann integrals over the parameters of the adjoint zero modes gives [48]

$$\int \left\{ \prod_{A=1}^{2} d^2 \mathscr{M}_3'^A \, d^2\mu_1^A \, d^2\mu_2^A \right\} \exp\left\{ 2\pi^2 i \left(\phi^{0\dagger}(\mu_1^A \mu_{1A} - \mu_2^A \mu_{2A}) + \frac{Y}{H}\left(\bar{\omega} - \frac{1}{8} Z\right)\right)\right\}$$

$$= - \left(\frac{16\pi^6(\bar{\omega} - \frac{1}{8}Z)}{|a_3|^2 H} \right)^2 \left\{ 2^{-8}(\phi^{0\dagger})^4 |\Omega|^2 + \frac{L}{8H}(\phi^0)^{\dagger 2}(\bar{\omega} - \tfrac{1}{8}Z)\bar{\omega} \right.$$

$$\left. + \tfrac{1}{4} H^{-2}(\bar{\omega} - \tfrac{1}{8}Z)^2 (\tfrac{1}{4}(\phi^{0\dagger})^2 (L^2 - |\Omega|^2) + \bar{\omega}^2) \right\} . \tag{8.40}$$

This is the generalization to $N_F > 0$ of the Yukawa determinant given in Eq. (8.13) of [25]. The next step is to integrate over the Grassmann collective coordinates $\{\mathscr{K}, \tilde{\mathscr{K}}\}$ using the identity

$$\int d^{2N_F}\mathscr{K} \, d^{2N_F}\tilde{\mathscr{K}} \, G(Z) \exp\left(-\pi^2 \sum_{f=1}^{N_F} m_f (\mathscr{K}_{1f}\tilde{\mathscr{K}}_{f1} + \mathscr{K}_{2f}\tilde{\mathscr{K}}_{f2}) \right)$$

$$= \sum_{l=0}^{N_F} \frac{M_{N_F-l}^{(N_F)}}{\pi^{4l}} \left. \frac{\partial^{2l} G}{\partial Z^{2l}} \right|_{Z=0} , \tag{8.41}$$

where

$$M_l^{(N_F)} \stackrel{\rm def}{=} \sum_{f_1 < f_2 < \cdots < f_l = 1}^{N_F} m_{f_1}^2 m_{f_2}^2 \cdots m_{f_l}^2 \quad (M_0^{(N_F)} = 1, \; M_l^{(N_F)} = 0, \; l < 0) . \tag{8.42}$$

Finally we turn to the remaining integration over the bosonic moduli. Following [25], it is convenient to change variables in the bosonic measure from $\{a_3, w_1, w_2\}$ to the new set $\{H, L, \Omega\}$ defined in (8.36). The relevant Jacobians are

$$\int_{-\infty}^{\infty} d^4 a_3 \, \frac{||a_3|^2 - |a_1|^2|}{|a_3|^4} \to \frac{\pi^2}{2} \int_{L+2|\Omega|}^{\infty} dH , \tag{8.43a}$$

$$\int_{-\infty}^{\infty} d^4 w_1 \, d^4 w_2 \to \frac{\pi^3}{8} \int_0^{\infty} dL \int_{|\Omega| \leqslant L} d^3\Omega . \tag{8.43b}$$

The numerator and denominator on the left-hand side of (8.43a) are supplied by (8.39) and (8.40), respectively. In addition, we introduce re-scaled variables $\Omega = L\Omega'$, $H = LH'$, and $\omega = L\omega'$. The integral over L is now trivial. Finally we switch to spherical polar coordinates,

$$\int d^3\Omega' \to 2\pi \int_{-1}^{1} d(\cos\theta) \int_0^1 |\Omega'|^2 \, d|\Omega'| , \tag{8.44}$$

where the polar angle is defined by $|\omega'| = \frac{1}{2}|\Omega'||\phi^0|\cos\theta$. This leaves an ordinary three-dimensional integral over the remaining variables H, $\cos\theta$ and $|\Omega'|$ which is the precise analogue of Eq. (8.19) in [25]. Performing this integral with the help of a standard symbolic manipulation routine gives results which can be summed up by the formula

$$\mathscr{F}_k|_{N=2,N_F} = \frac{5}{(\phi^0)^6} M_{N_F}^{(N_F)} - \frac{3}{4(\phi^0)^4} M_{N_F-1}^{(N_F)} \frac{1}{16(\phi^0)^2} M_{N_F-2}^{(N_F)} - \frac{5}{2^6 3^3} M_{N_F-3}^{(N_F)} + \frac{7(\phi^0)^2}{2^8 3^5} M_{N_F-4}^{(N_F)} , \tag{8.45}$$

where the coefficients $M_l^{(N_F)}$ are defined in (8.42). For $N_F < 4$ these expressions are identical to the predictions from the Seiberg–Witten curve (8.8b), up to a physically unimportant additive constant in the case $N_F = 3$.

The situation in the $N_F = 4$ theory is rather more subtle. To our knowledge the two-instanton prediction from Seiberg–Witten theory with arbitrary hypermultiplet masses has not been determined in the literature. However, in the massless case one expects that the prepotential is classically exact: in other words given by the first term in (8.7). Result (8.45) for $N_F = 4$ with vanishing masses

$$\mathscr{F}_2|_{N_F=4,m_f=0} = \frac{7(\phi^0)^2}{2^8 3^5} \tag{8.46}$$

is obviously in contradiction to this. The resolution of this discrepancy is explained in Refs. [48,91]. The point is that the Seiberg–Witten curve is parameterized by an effective coupling τ_{eff} rather than by the microscopic coupling τ. The two definitions differ by an infinite series of even-charge instanton corrections:

$$\tau_{\text{eff}} = \tau + \sum_{k=2,4,\ldots} c_k e^{2\pi i k\tau} . \tag{8.47}$$

The two-instanton computation above shows that

$$c_2 = \frac{1}{i\pi} \frac{7}{2^7 3^5} . \tag{8.48}$$

8.4. One-instanton prepotential in SU(N)

In this section we perform, following [20], the explicit evaluation of the centred instanton partition function for $k=1$ in a theory with arbitrary gauge group SU(N). It is best to start from the linearized formulation described in Section 6.5. The instanton effective action for $k=1$ is, from (6.96) (with factors of g removed by the re-scalings (8.22)),

$$\tilde{S} = 4\pi^2 \left\{ |w_{u\dot\alpha}\chi + \phi^0 w_{u\dot\alpha}|^2 + \frac{i}{2}\bar\mu_u^A(\mu_{uA}\chi^* + \phi_u^{0*}\mu_{uA}) + \frac{1}{4}\sum_{f=1}^{N_F} \mathscr{K}_f \tilde{\mathscr{K}}_f(\chi - m_f) \right\} + \tilde{S}_{\text{L.m.}} , \tag{8.49}$$

where the Lagrange multiplier terms for the ADHM constraints are

$$\tilde{S}_{\mathrm{L.m.}} = -4\mathrm{i}\pi^2\{\bar{\psi}^{\dot\alpha}_A(\bar{\mu}^A_u w_{u\dot\alpha} + \bar{w}_{u\dot\alpha}\mu^A_u) + \vec{D}\cdot\vec{\tau}^{\dot\alpha}{}_{\dot\beta}\bar{w}^{\dot\beta}_u w_{u\dot\alpha}\} \ . \tag{8.50}$$

Note that the quantity χ for $k=1$ is just a complex variable rather than being a matrix. The linear shifts

$$\mu^A_u \to \mu^A_u - \frac{2w_{u\dot\alpha}}{\alpha^*_u}\,\bar{\psi}^{\dot\alpha A}, \qquad \bar{\mu}^A_u \to \bar{\mu}^A_u + \frac{2\bar{w}_{u\dot\alpha}}{\alpha^*_u}\,\bar{\psi}^{\dot\alpha A} \ , \tag{8.51}$$

eliminate the linear terms of these variables in the action. The Grassmann integrals over $\{\mu^A_u, \bar{\mu}^A_u\}$ then bring down the factors

$$\prod_{u=1}^{N}(2\pi^2\alpha^*_u)^2 \ , \tag{8.52}$$

where we have defined

$$\alpha_u = \chi + \phi^0_u, \quad \alpha^* = \chi^* + \phi^{0*}_u \ . \tag{8.53}$$

The Grassmann integrals over the matter field coordinates $\{\mathscr{K}_f, \tilde{\mathscr{K}}_f\}$ are simply evaluated:

$$\int \mathrm{d}^{N_F}\mathscr{K}\,\mathrm{d}^{N_F}\tilde{\mathscr{K}}\exp\left(-\pi^2\sum_{f=1}^{N_F}\mathscr{K}_f\tilde{\mathscr{K}}_f(\chi - m_f)\right) = \pi^{2N_F}\prod_{f=1}^{N}(m_f - \chi) \ . \tag{8.54}$$

The integrals over $\{w_{u\dot\alpha}, \bar{w}^{\dot\alpha}_u\}$ are Gaussian and are accomplished using the identity

$$\int \mathrm{d}^{2N}w\,\mathrm{d}^{2N}\bar{w}\exp(-4\pi^2 A_u\bar{w}^{\dot\alpha}_u w_{u\dot\alpha} + 4\mathrm{i}\pi^2\vec{B}_u\cdot\vec{\tau}^{\dot\alpha}{}_{\dot\beta}\bar{w}^{\dot\beta}_u w_{u\dot\alpha}) = (2\pi)^{-2N}\prod_{u=1}^{N}\frac{1}{A_u^2 + \vec{B}_u^2} \ . \tag{8.55}$$

All that remains are integrals over the auxiliary variables $\{\chi, \vec{D}, \bar{\psi}_A\}$:

$$\hat{\mathscr{Z}}_k^{(\mathscr{N}=2,N_F)} = \frac{1}{(2\pi)^3}\int \mathrm{d}^2\chi\,\mathrm{d}^3D\prod_{A=1}^{2}\mathrm{d}^2\bar{\psi}_A\prod_{u=1}^{N}\frac{\alpha^{*2}_u}{|\alpha_u|^4 + (\vec{D}+\vec{\Xi}_u)^2}\prod_{f=1}^{N_F}(m_f - \chi) \ , \tag{8.56}$$

where $\vec{\Xi}_u$ is the Grassmann bilinear

$$\vec{\Xi}_u = (\alpha^*_u)^{-1}\bar{\psi}^A_{\dot\alpha}\vec{\tau}^{\dot\alpha}{}_{\dot\beta}\bar{\psi}^{\dot\beta}_A \ . \tag{8.57}$$

The integrals over $\bar{\psi}_A$ must be saturated with two insertions of Ξ:

$$\int\prod_{A=1}^{2}\mathrm{d}^2\bar{\psi}_A\,\Xi^c_u\Xi^d_v = -8\,\frac{\delta^{cd}}{\alpha^*_u\alpha^*_v} \ , \tag{8.58}$$

leading to the identity

$$\int\prod_{A=1}^{2}\mathrm{d}^2\bar{\psi}_A\,F(\Xi) = -4\sum_{u,v=1}^{N}\frac{1}{\alpha^*_u\alpha^*_v}\left.\frac{\partial^2 F(\Xi)}{\partial\Xi^c_u\partial\Xi^c_v}\right|_{\Xi=0} = -\frac{4}{\vec{D}^2}\frac{\partial^2 F(\Xi=0)}{\partial\chi^{*2}} \ , \tag{8.59}$$

where, in our case,

$$F(\Xi) = \prod_{u=1}^{N} \frac{\alpha_u^{*2}}{|\alpha_u|^4 + (\vec{D} + \vec{\Xi}_u)^2} \ . \tag{8.60}$$

We are now in a position to integrate out the Lagrange multipliers $\vec{D}$. The non-trivial part of the integral is easily performed by a standard contour integration in the variable $|\vec{D}| \equiv \sqrt{\vec{D} \cdot \vec{D}}$ extended to run from $-\infty$ to $+\infty$:

$$\int \frac{\mathrm{d}^3 D}{\vec{D}^2} \prod_{u=1}^{N} \frac{\alpha_u^{*2}}{|\alpha_u|^4 + \vec{D}^2} = 2\pi^2 \sum_{u=1}^{N} \frac{\alpha_u^*}{\alpha_u} \prod_{\substack{v=1 \\ (\neq u)}}^{N} \frac{\alpha_v^{*2}}{|\alpha_v|^4 - |\alpha_u|^4} \ . \tag{8.61}$$

Finally, it only remains to integrate over χ:

$$\hat{\mathscr{Z}}_k^{(\mathcal{N}=2, N_F)} = -\pi^{-1} \int \mathrm{d}^2\chi \, \frac{\partial^2}{\partial \chi^{*2}} f_1(\chi, \chi^*) f_2(\chi) \ , \tag{8.62}$$

where we have defined the two quantities

$$f_1(\chi, \chi^*) = \sum_{u=1}^{N} \frac{\alpha_u^*}{\alpha_u} \prod_{\substack{v=1 \\ (\neq u)}}^{N} \frac{\alpha_v^{*2}}{|\alpha_v|^4 - |\alpha_u|^4}, \quad f_2(\chi) = \prod_{f=1}^{N_F} (m_f - \chi) \ . \tag{8.63}$$

The resulting integral over the χ-plane can be evaluated by Stoke's Theorem. There are two kinds of boundary to consider: on the sphere at infinity and around the singularities of the integrand. First we shall consider the singularities. Contrary to appearances the integrand is completely regular at $|\alpha_u|^2 - |\alpha_v|^2$ due to the cancellation between the uth and vth terms in the sum. However, there are N singularities at $\alpha_u = 0$, i.e. $\chi = -\phi_u^0$, following from the fact that for $z = x + \mathrm{i}y$

$$\frac{\partial}{\partial z^*} \frac{1}{z} = \pi \delta(x) \delta(y) \ . \tag{8.64}$$

Introducing polar coordinates in the vicinity of the point $\chi = -\phi_u^0$, $\alpha_u = r\mathrm{e}^{\mathrm{i}\theta}$, we take a boundary in the form of a small circle of radius $r \to 0$. The resulting contribution is then

$$\lim_{r \to 0} \frac{1}{4\pi} \left(2 + r \frac{\partial}{\partial r} \right) \int_0^{2\pi} \mathrm{d}\theta \, \mathrm{e}^{2\mathrm{i}\theta} f_1(r, \theta) f_2(r\mathrm{e}^{\mathrm{i}\theta}) \ . \tag{8.65}$$

Applying this formula in the vicinity of $\chi = -\phi_u^0$ we find a contribution to the centred instanton partition function of

$$\prod_{\substack{v=1 \\ (\neq u)}}^{N} \frac{1}{(\phi_u^0 - \phi_v^0)^2} \prod_{f=1}^{N_F} (m_f + \phi_u^0) \ . \tag{8.66}$$

We can determine the contribution from the sphere at infinity by once again introducing polar coordinates. Then in a similar fashion to (8.65) the contribution is

$$-\lim_{r \to \infty} \frac{1}{4\pi} \left(2 + r \frac{\partial}{\partial r} \right) \int_0^{2\pi} \mathrm{d}\theta \, \mathrm{e}^{2\mathrm{i}\theta} f_1(r, \theta) f_2(r\mathrm{e}^{\mathrm{i}\theta}) \ . \tag{8.67}$$

It is easy to establish the following asymptotic forms for large r:

$$f_1(r,\theta) \sim \frac{e^{-2iN\theta}}{r^{2(N-1)}}, \quad f_2(re^{i\theta}) \sim e^{iN_F\theta} r^{N_F} . \tag{8.68}$$

This means that the contribution from the sphere at infinity is only non-vanishing when $N_F \geqslant 2(N-1)$, i.e. in the three cases $N_F = 2N-2$, $2N-1$, $2N$. Taking into account the selection rule arising from the integration over θ, there are only two relevant terms in the asymptotic expansion of $f_1(r,\theta)$, namely

$$f_1(r,\theta) \sim \alpha_1 \frac{e^{-2iN\theta}}{r^{2(N-1)}} + \alpha_2 \frac{e^{-2i(N+1)\theta}}{r^{2N}} \sum_{u=1}^{N} (\phi_u^0)^2 , \tag{8.69}$$

where we have defined the two constants

$$\alpha_1 = 2^{3-2N} \begin{pmatrix} 2N-3 \\ N-1 \end{pmatrix}, \quad \alpha_2 = 2^{-2N} \begin{pmatrix} 2N \\ N-1 \end{pmatrix} . \tag{8.70}$$

All other terms are either sub-leading in $1/r$ or have the wrong θ dependence (taking account of the fact that f_2 is a polynomial in $e^{i\theta}$ along with the factor of $e^{2i\theta}$ in (8.67)). Hence, the contribution to the centred instanton partition function from the sphere at infinity is

$$\mathscr{S}_{k=1}^{(N_F)} = -\begin{cases} 0, & N_F < 2N-2 , \\ \alpha_1, & N_F = 2N-2 , \\ \alpha_1 \displaystyle\sum_{f=1}^{N} m_f, & N_F = 2N-1 , \\ \alpha_1 \displaystyle\sum_{\substack{f,f'=1 \\ (f<f')}}^{N_F} m_f m_{f'} + \alpha_2 \sum_{u=1}^{N} (\phi_u^0)^2, & N_F = 2N . \end{cases} \tag{8.71}$$

All-in-all, the $k=1$ contribution to the centred partition function, and hence the coefficient $\mathscr{F}_1$ of the prepotential, is the sum of (8.66) and (8.71):

$$\mathscr{F}_1 \equiv \hat{\mathscr{Z}}_k^{(\mathcal{N}=2,N_F)} = \sum_{u=1}^{N} \prod_{\substack{v=1 \\ (\neq u)}}^{N} \frac{1}{(\phi_v^0 - \phi_u^0)^2} \prod_{f=1}^{N_F} (m_f + \phi_u^0) + \mathscr{S}_1^{(N_F)} . \tag{8.72}$$

This calculation illustrates the complexity of integrating over the instanton moduli space even at the one-instanton level. However, if one follows the method in detail, an interesting intuitive picture emerges. First recall that the final integral over χ was evaluated by using Stoke's Theorem: there are only contributions from an isolated set of N points along with a contribution from the sphere at infinity. We now show that these contributions arise at the critical points of the instanton effective action (8.49). The latter correspond to the vanishing

$$w_{\dot\alpha}\chi + \phi^0 w_{\dot\alpha} = 0 . \tag{8.73}$$

So there is a branch consisting of N solutions, labelled by u, where

$$\chi = -\phi_u^0, \quad w_{v\dot{\alpha}} \propto \delta_{uv} , \tag{8.74}$$

i.e. precisely at the positions of the isolated contributions to the instanton effective action. However, it is apparent that these are critical points of $\tilde{S}$ only on a branch where the instanton has non-zero scale size: $\rho^2 = \frac{1}{2}\bar{w}^{\dot{\alpha}} w_{\dot{\alpha}} > 0$. There is another branch of solutions when $\rho = 0$, i.e. $w_{\dot{\alpha}} = 0$ where physically the instanton has shrunk to zero size. This second branch is associated to the contribution from the integral over the sphere at infinity in χ-space. To see this note that when χ is eliminated via its "equation-of-motion" one has $\chi \sim \rho^{-1}$: so $\rho \to 0$ does indeed correspond to the circle at infinity in χ-space. These observations are intended as anecdotal evidence in favour of some kind of localization on the moduli space of instantons. This is the subject that we will pursue in earnest in Section 11.2.

9. Conformal gauge theories at large N

Instanton calculations are only reliable at weak coupling which can be achieved in a Higgs or Coulomb phase, giving rise to the applications reported in Sections 7 and 10.1. However, theories can be weakly coupled without invoking the Higgs mechanism if they are finite, or conformal. In this case the gauge coupling does not run with scale and weak coupling prevails for small g. The two main examples that we discuss here are the $\mathcal{N}=4$ theory and the $\mathcal{N}=2$ theory with $N_F=2N$ hypermultiplets in the fundamental representation of the gauge group. We shall see that these two examples have some interesting features in common and, in particular, the calculations of instanton effects in both cases considerably simplify in the large-N limit (see Refs. [21,106,107]). The most striking feature of this limit is that the instanton measure concentrates on a subspace of the moduli space, which, in the $\mathcal{N}=4$ case, is essentially $AdS_5 \times S^5$. Of course this phenomenon is directly related to the AdS/CFT correspondence, which we discuss in Section 9.3. The approach of Refs. [21,106,107] that we will describe has been generalized to other finite gauge theories. Hollowood et al. [108] considered the $\mathcal{N}=4$ theory with gauge groups $\mathrm{Sp}(N)$ and $\mathrm{SO}(N)$; finite $\mathcal{N}=2$ gauge theories with gauge group $\mathrm{Sp}(N)$ were considered in [109,110]; and finite $\mathcal{N}=2$ theories with product gauge group $\mathrm{SU}(N)^k$ and hypermultiplets in bi-fundamental representations of the gauge group—the "quiver" models—were considered in [111].

The main common feature of finite gauge theories is the fact, discussed in Section 4.3, that the supersymmetric instanton with the full set of fermion zero modes turned on is *not* an exact solution of the equations-of-motion. Put another way, most of the fermion zero modes that appear at linear order around the instanton background are lifted by Yukawa interactions as evidenced by the leading-order expression for the instanton effective action (5.25) or (6.84) (with $\phi_a^0=0$ in the present context). In both cases, this effective action involves a quadrilinear coupling of the Grassmann collective coordinates. Of course, in both cases, the supersymmetric and superconformal zero modes are protected by symmetries and remain unlifted by the instanton action. The pattern of lifting dictated by the Grassmann quadrilinear implies that instantons of all charges k will contribute to a class of correlation functions whose insertions evaluated in the instanton background are responsible for saturating the integrals over a finite number of unlifted modes. In both cases, the number of unlifted fermion modes is 16. This is directly the number of supersymmetric and superconformal modes in the $\mathcal{N}=4$ theory while in the $\mathcal{N}=2$ theory it includes the supersymmetric and superconformal modes—numbering eight—along with eight additional fundamental fermion zero modes.

The aim of this section is to show how instanton contributions to these kinds of correlation functions can be calculated in the large-N limit. They will turn out to have remarkable properties. The only dependence on the instanton charge appears in an overall numerical pre-factor:

$$\langle \mathcal{O}_1(x^{(1)}) \times \cdots \times \mathcal{O}_n(x^{(n)})\rangle_{\text{inst.}} \overset{N\to\infty}{\to} \left(\sum_{k=1}^{\infty} k^n c_k \mathrm{e}^{2\pi\mathrm{i}k\tau}\right) f(x^{(1)},\ldots,x^{(n)}) \tag{9.1}$$

for some function f and coefficients c_k independent of the correlator in question. At first sight, limit (9.1) looks absurd. How could it be that the only dependence on the instanton number k is via an overall multiplicative factor? The reason for the simple dependence on k rests on certain very special properties of the large-N limit of the instanton calculus. Intuitively what happens in the large-N limit is that integral over the k-instanton moduli space is dominated by configurations of k single instantons occupying k commuting SU(2) subgroups of the gauge group which are therefore totally non-interacting. In this sense the dominant configurations are dilute-gas like. But, contrary to the dilute gas, the dominant configuration also has all the instantons lying at the same spacetime point X_n and having the same scale size ρ. This accounts for the fact that the functional dependence on the insertion points is the same for all instanton number. We will show that the k-dependent numerical factors c_k are related to interesting matrix integrals, namely, the partition functions of dimensionally reduced gauge theories.

In addition, when taking the large-N limit, it is convenient to introduce auxiliary bosonic collective coordinates to bi-linearize the Grassmann quadrilinear in the instanton effective action. In the large-N limit, these additional variables are confined to a sphere: S^5 for $\mathcal{N}=4$ theory [21] and S^1 for $\mathcal{N}=2$ [107]. The appearance of these auxiliary collective coordinate is especially interesting in the light of the AdS/CFT correspondence [112,113] relating the $\mathcal{N}=4$ theory to Type IIB string theory on $AdS_5\times S^5$ as discussed in Section 9.3.

9.1. The collective coordinate integrals at large N

We begin our analysis by establishing an expression for the instanton partition function in the large-N limit. There are two relevant cases: $\mathcal{N}=4$ and 2 with $N_F=2N$ hypermultiplets. The expressions for the collective coordinate integral (5.14) are obtained from (5.55), with $\tilde{S}$ equal to (5.25), in the $\mathcal{N}=4$ theory and (6.84), for the $\mathcal{N}=2$ case. [61] Since we are interested in pursuing a large-N limit, we can use the results described in Section 6.1 to resolve the ADHM constraints and therefore use the explicit version for the supersymmetric volume form in (6.29). Note that the requirement $N\geqslant 2k$ will certainly be met in the large-N limit (for fixed k). Since we are working in a non-abelian Coulomb phase, all the VEV vanish and the gauge symmetry remains unbroken. In this case, we can immediately integrate over the gauge orientation of the instanton: $\int \mathrm{d}^{4k(N-k)}\mathcal{U}=1$.

The key to taking a large-N limit is to bi-linearize the Grassmann quadrilinear effective action (5.26) or (6.84), by introducing some auxiliary variables. This kind of transformation is a well-known tool for analysing the large-N limit of field theories with four-fermion interactions, like the Gross–Neveu and Thirring models [114]. In fact we have already introduced the necessary auxiliary variables

[61] The VEVs are to set to zero in both cases to preserve conformal invariance in the present applications.

in the form of the $k\times k$ matrices χ_a in Section 6.5. In particular, the linearized instanton effective actions (6.91) and (6.96) involve only Grassmann bilinears.

In the $\mathcal{N}=2$ theory the identity we use is

$$|\mathrm{det}_{k^2}\,\boldsymbol{L}|^{-1}\exp\left\{\frac{\mathrm{i}\pi^2}{2}\sum_{f=1}^{N_F}\mathrm{tr}_k\,\mathcal{K}_f\tilde{\mathcal{K}}_f\boldsymbol{L}^{-1}\bar{\mathcal{M}}^A\mathcal{M}_A\right\}$$
$$=2^{2k^2}\pi^{k^2}\int\mathrm{d}^{2k^2}\chi\exp\left\{-4\pi^2\left(\mathrm{tr}_k\,\chi_a\boldsymbol{L}\chi_a\right.\right.$$
$$\left.\left.+\frac{\mathrm{i}}{2}\,\mathrm{tr}_k\,\bar{\mathcal{M}}^A\mathcal{M}_A\chi^\dagger+\frac{1}{4}\sum_{f=1}^{N_F}\mathrm{tr}_k\,\mathcal{K}_f\tilde{\mathcal{K}}_f\chi\right)\right\}\ , \tag{9.2}$$

whilst in the $\mathcal{N}=4$ case

$$|\mathrm{det}_{k^2}\,\boldsymbol{L}|^{-3}\exp\left\{\frac{\pi^2}{2}\,\varepsilon_{ABCD}\,\mathrm{tr}_k(\bar{\mathcal{M}}^A\mathcal{M}^B\boldsymbol{L}^{-1}\bar{\mathcal{M}}^C\mathcal{M}^D)\right\}$$
$$=2^{6k^2}\pi^{3k^2}\int\mathrm{d}^{6k^2}\chi\exp\{-4\pi^2(\mathrm{tr}_k\,\chi_a\boldsymbol{L}\chi_a+\tfrac{1}{2}\,\bar{\Sigma}_{aAB}\,\mathrm{tr}_k\,\bar{\mathcal{M}}^A\mathcal{M}^B\chi_a)\}\ . \tag{9.3}$$

Note, in both cases, that the appropriate factor of $|\mathrm{det}_{k^2}\,\boldsymbol{L}|$ is already present in the supersymmetric volume form on $\mathfrak{M}_k$ (5.55) (or (6.29)).

9.1.1. *The $\mathcal{N}=4$ case*

Before we proceed, and with the large-N limit in mind, it is useful to re-scale

$$\chi_a\to\frac{\sqrt{N}}{2\pi}\,\chi_a\ . \tag{9.4}$$

It is also convenient to define

$$\chi_{AB}=\tfrac{1}{\sqrt{8}}\,\bar{\Sigma}_{aAB}\chi_a,\quad \chi_a=-\tfrac{1}{\sqrt{2}}\,\Sigma_a^{AB}\chi_{AB} \tag{9.5}$$

which satisfies a pseudo-reality condition following from the Hermiticity of χ_a:

$$\chi_{AB}^\dagger=\tfrac{1}{2}\,\varepsilon^{ABCD}\chi_{CD}\ , \tag{9.6}$$

where $\dagger$ only acts on the instanton indices rather than the SU(4) R-symmetry indices.

After integrating over the gauge orientation and using identity (9.3), the instanton partition function is

$$\mathcal{Z}_k^{(\mathcal{N}=4)}=\frac{2^{3k^2}N^{3k^2}C_k^{(\mathcal{N}=4)}A_k}{\pi^{3k^2}\,\mathrm{Vol}\,\mathrm{U}(k)}\int\mathrm{d}^{4k^2}a'\,\mathrm{d}^{k^2}W^0\,\mathrm{d}^{6k^2}\chi\prod_{A=1}^{4}\{\mathrm{d}^{k(N-2k)}v^A\,\mathrm{d}^{k(N-2k)}\bar{v}^A\,\mathrm{d}^{2k^2}\zeta^A\,\mathrm{d}^{2k^2}\mathcal{M}'^A\}$$
$$\times|\mathrm{det}_{2k}\,W|^{N-2k}\,\mathrm{e}^{-N\,\mathrm{tr}_k\,\chi_a\boldsymbol{L}\chi_a+\sqrt{8N}\pi\,\mathrm{tr}_k\,\bar{\mathcal{M}}^A\mathcal{M}^B\chi_{AB}}\ . \tag{9.7}$$

In this expression, and in the following, W is the $2k \times 2k$ matrix defined in (6.1), where ADHM constraints (6.3) are imposed; hence,

$$W = \tfrac{1}{2} W^0 1_{[2]\times[2]} - 2a'_m a'_n \bar{\sigma}_{mn} \ . \tag{9.8}$$

The next stage in the large-N programme is to integrate out the Grassmann variables $\{\bar{v}^A, v^A\}$ by pulling down powers of the second term in the exponential in (9.3):

$$\int \left\{ \prod_{A=1}^{4} \mathrm{d}^{k(N-2k)} v^A \, \mathrm{d}^{k(N-2k)} \bar{v}^A \right\} \exp \sqrt{8N}\pi \, \mathrm{tr}_k \, \bar{v}^A v^B \chi_{AB} = (8N\pi^2)^{2k(N-2k)} |\mathrm{det}_{4k}\, \chi|^{N-2k} \ , \tag{9.9}$$

where the determinant is over the $4k \times 4k$-dimensional matrix with elements $(\chi_{AB})_{ij}$. One might think that it is rather premature to integrate out the Grassmann variables $\{\bar{v}^A, v^A\}$ since they could be saturated by insertions made into the functional integral. This is possible, but we shall find later a very simple prescription for including this kind of dependence when working at large N.

After integrating out the subset of Grassmann variables as above, we can collect together all the non-constant terms in the collective coordinate integral which are raised to the power N as $\exp -N\Gamma$. The large-N "effective action" is [62]

$$\Gamma = -\log \mathrm{det}_{2k} W - \log \mathrm{det}_{4k}\, \chi + \mathrm{tr}_k\, \chi_a \boldsymbol{L} \chi_a \ . \tag{9.10}$$

This expression involves the $11k^2$ bosonic variables comprising the 11 independent $k \times k$ Hermitian matrices W^0, a'_n and χ_a. The remaining components W^c, $c = 1$–3, are eliminated in favour of the a'_n via ADHM constraints (6.3). The action is invariant under U(k) symmetry (2.64) which acts by adjoint action on all the variables.

With N factored out of the exponent, the measure is in a form which is amenable to a saddle-point treatment as $N \to \infty$. The critical points of Γ satisfy the equations

$$\varepsilon^{ABCD} (\boldsymbol{L} \cdot \chi_{AB}) \chi_{CE} = \tfrac{1}{2} \delta^D_E 1_{[k]\times[k]} \ , \tag{9.11a}$$

$$\chi_a \chi_a = \tfrac{1}{2} (W^{-1})^0 \ , \tag{9.11b}$$

$$[\chi_a, [\chi_a, a'_n]] = \mathrm{tr}_2 (\tau^c \bar{\sigma}_{nm}) [(W^{-1})^c, a'_m] \ . \tag{9.11c}$$

These are obtained by varying Γ with respect to the matrix elements of χ, W^0 and a'_n, respectively, and rewriting "log det" as "tr log". We have defined the $k \times k$ matrices

$$(W^{-1})^0 = \mathrm{tr}_2\, W^{-1}, \quad (W^{-1})^c = \mathrm{tr}_2 (\tau^c W^{-1}) \ . \tag{9.12}$$

The general solution to these coupled saddle-point equations is easily found. It has the simple property that all the quantities are diagonal in instanton indices:

$$W^0 = \mathrm{diag}(2\rho_1^2, \ldots, 2\rho_k^2) \ , \tag{9.13a}$$

[62] In the following we translate back and forth as convenient between the anti-symmetric tensor representation χ_{AB} and the $\mathrm{SO}(6)_R$ vector representation χ_a, $a = 1, \ldots, 6$ (9.5).

$$\chi_a = \mathrm{diag}(\rho_1^{-1}\hat{\Omega}_a^1, \ldots, \rho_k^{-1}\hat{\Omega}_a^k) , \tag{9.13b}$$

$$a'_n = \mathrm{diag}(-X_n^1, \ldots, -X_n^k) , \tag{9.13c}$$

up to adjoint action by the U(k) auxiliary symmetry. For quantities $\hat{\Omega}_a^i$ parameterize k unit six-vectors, or points in S^5,

$$\hat{\Omega}_a^i \hat{\Omega}_a^i = 1 \quad (\text{no sum on } i) , \tag{9.14}$$

where the radius of the ith S^5 factor is ρ_i^{-1}.

A simple picture of this leading-order saddle-point solution emerges: it can be thought of as k independent copies of a large-N one-instanton saddle-point solution, where the ith large-N instanton is parameterized by the triple $\{X_n^i, \rho_i, \hat{\Omega}_a^i\}$. Additional insight into this solution emerges from considering the SU(2) generators $(T_i^c)_{uv}$ describing the embedding of the ith instanton inside SU(N). From Eqs. (2.103), (6.2) and (9.13a) one derives the commutation relations

$$[T_i^a, T_j^b] = 2\mathrm{i}\delta_{ij}\varepsilon_{abc}T_i^c \tag{9.15}$$

so that at the saddle point, thanks to the Kronecker-δ, the k individual instantons lie in k mutually commuting SU(2) subgroups. Actually this feature follows intuitively from large-N statistics alone,[63] and has nothing to do with either the existence of supersymmetry or with the details of the ADHM construction. Another important property is that effective action (9.10) evaluated on these saddle-point solutions is zero; hence there is no exponential dependence on N in the final result. Finally we should make the technical point that, thanks to the diagonal structure of these solutions, they are automatically consistent with the triangle inequalities on the bosonic bi-linear W discussed in the paragraph following Eq. (2.84); hence we never need to specify more explicitly the integration limits on the W variables.

The saddle-point solution depends on the moduli of each of the k large-N instantons. However, not all these $10k$ parameters are the genuine moduli, or flat directions of Γ, because, as we shall see, they are "lifted" by terms beyond quadratic order in the expansion of Γ. In fact the only genuine moduli are the 10 overall large-N coordinates. The correct large-N behaviour is obtained [21] by expanding Γ around the *maximally degenerate* saddle-point solution

$$W^0 = 2\rho^2 1_{[k]\times[k]}, \quad \chi_a = \rho^{-1}\hat{\Omega}_a 1_{[k]\times[k]}, \quad a'_n = -X_n 1_{[k]\times[k]} , \tag{9.16}$$

which corresponds to the k large-N instantons living at a common point $\{X_n, \hat{\Omega}_a, \rho\}$. (From ADHM constraint (6.3) it follows that the remaining components of W vanish: $W^c = 0$ for $c = 1$–3.) This degenerate solution, unlike Eqs. (9.13a)–(9.13c), is invariant under the residual U(k). With the instantons sitting on top of one another, it looks like the complete opposite of the dilute instanton gas limit; however the instantons still live in k mutually commuting SU(2) subgroups of SU(N) as per Eq. (9.15), which is a dilute-gas-like feature.

In order to expand about this special solution, one first needs to factor out the integrals over the parameters $\{X_n, \rho, \hat{\Omega}_a\}$. This is done in the following way: for each $k \times k$ matrix, we introduce

[63] Consider the analogous problem of k randomly oriented vectors in $\mathbb{R}^N$ in the limit $N \to \infty$; clearly the dot products of these vectors tend to zero simply due to statistics.

a basis of traceless Hermitian matrices $\hat{T}^r$, $r = 1, \ldots, k^2 - 1$, normalized by $\mathrm{tr}_k\, \hat{T}^r \hat{T}^s = \delta^{rs}$. For each $k \times k$ matrix v we separate out the "scalar" component v_0 by taking

$$v = v_0 1_{[k]\times[k]} + \hat{v}^r \hat{T}^r \ . \tag{9.17}$$

The change of variables from the $\{T^r\}$ basis used in (9.7) to $\{1_{[k]\times[k]}, \hat{T}^r\}$ involves a Jacobian

$$\int \mathrm{d}^{k^2} v = k^{\pm 1/2} \int \mathrm{d}v_0\, \mathrm{d}^{k^2-1} \hat{v} \ , \tag{9.18}$$

where ± 1 refers to c-number and Grassmann quantities, respectively. For the moment we continue to focus on the bosonic variables, which are decomposed as follows:

$$a'_n = -X_n 1_{[k]\times[k]} + \hat{a}'_n \ , \tag{9.19a}$$

$$\chi_a = \rho^{-1} \hat{\Omega}_a 1_{[k]\times[k]} + \hat{\chi}_a \ . \tag{9.19b}$$

By definition the traceless matrix variables $\hat{a}'_n$ and $\hat{\chi}_a$ are the fluctuating fields.

Inserting Eqs. (9.19a) and (9.19b) into Eq. (9.10) and Taylor expanding is a tedious though straightforward exercise. It is necessary to expand to fourth order in the fluctuating fields around the solution parameterized by the 10 exact moduli. The expansion of the determinant terms in (9.10) is facilitated by first writing "log det" as "tr log" and then expanding the logarithm:

$$\begin{aligned} \mathrm{tr}_{2k}\, W = {} & 2k \log \rho^2 + \frac{1}{\rho^2} \mathrm{tr}_k(\delta W^0) - \frac{1}{4\rho^4} \mathrm{tr}_k(\delta W^0)^2 \\ & + \frac{1}{12\rho^6} \mathrm{tr}_k(\delta W^0)^3 - \frac{1}{32\rho^8} \mathrm{tr}_k(\delta W^0)^4 + \frac{1}{2\rho^4} \mathrm{tr}_k[\hat{a}'_n, \hat{a}'_m]^2 + \cdots \end{aligned} \tag{9.20}$$

and

$$\begin{aligned} \mathrm{tr}_{4k} \log \chi = {} & -2k \log(8\rho^2) - 2^5 \rho^2\, \mathrm{tr}_{4k}(\hat{\Omega}^* \hat{\chi})^2 + \frac{2^9 \rho^3}{3} \mathrm{tr}_{4k}(\hat{\Omega}^* \hat{\chi})^3 \\ & - 2^{10} \rho^4\, \mathrm{tr}_{4k}(\hat{\Omega}^* \hat{\chi})^4 + \cdots \ . \end{aligned} \tag{9.21}$$

In these expansions we have dropped fifth- and higher-order terms in the fluctuating fields. Here we are anticipating the fact that these terms are not needed at leading order in $1/N$. To see this we can re-scale the integration variables in a standard way which shows that the fluctuations around the maximally degenerate saddle point are of order $N^{-1/4}$. The higher-order terms in the exponent therefore yield subleading contributions in the $1/N$ expansion. In particular, this is true for the diagonal components of $\hat{a}'_n$ and $\hat{\chi}_a$ which correspond to the moduli of the generic saddle point solution discussed earlier in this section. This shows that our large-N expansion around the maximally degenerate saddle point is self-consistent. In Eq. (9.21), and in subsequent equations, we move back and forth as convenient between the six-vector and the anti-symmetric tensor representations of $\hat{\Omega}$ and χ using Eq. (9.5). In particular, the SO(6) orthonormality condition $\hat{\Omega} \cdot \hat{\Omega} = 1$ becomes, in 4×4 matrix language,

$$\hat{\Omega}\hat{\Omega}^* = -\tfrac{1}{8} 1_{[4]\times[4]} \quad \text{or } \hat{\Omega}^{-1} = -8\hat{\Omega}^* \ , \tag{9.22}$$

which has been implemented in Eq. (9.21).

Next we need a systematic method for re-expressing the traces over $4k\times 4k$ matrices in Eq. (9.21) as traces over $k\times k$ matrices. We will exploit the following "moves": [64]

$$\hat{\Omega}^*\hat{\chi} = -\hat{\chi}^\dagger\hat{\Omega} - \tfrac{1}{4}(\hat{\Omega}\cdot\hat{\chi})1_{[4]\times[4]}\ , \tag{9.23a}$$

$$\mathrm{tr}_4\, E^\dagger F = \mathrm{tr}_4\, F^\dagger E = -\tfrac{1}{2}(E\cdot F)\ , \tag{9.23b}$$

$$\mathrm{tr}_4\, E^\dagger F G^\dagger H = \tfrac{1}{16}(E_aF_aG_bH_b - E_aF_bG_aH_b + E_aF_bG_bH_a)\ . \tag{9.23c}$$

On the left-hand sides of Eqs. (9.23b) and (9.23c), the $4k\times 4k$ matrices $\{E,F,G,H\}$ are assumed to be antisymmetric in $\mathrm{SU}(4)_R$ indices and subject to usual conditions (9.6) and (9.5); identity (9.23a) follows from a double application of Eq. (9.6). Using Eqs. (9.22)–(9.23c) in an iterative fashion, it is then easy to derive the following trace identities:

$$\begin{aligned}\mathrm{tr}_{4k}(\hat{\Omega}^*\hat{\chi})^2 &= -\mathrm{tr}_{4k}\,\hat{\chi}^\dagger\hat{\Omega}\hat{\Omega}^*\hat{\chi} - \tfrac{1}{4}\,\mathrm{tr}_k(\hat{\Omega}\cdot\hat{\chi}\,\mathrm{tr}_4(\hat{\Omega}^*\hat{\chi}))\\ &= \frac{1}{2^3}\,\mathrm{tr}_k(\hat{\Omega}\cdot\hat{\chi})^2 - \frac{1}{2^4}\,\mathrm{tr}_k\,\hat{\chi}\cdot\hat{\chi}\ ,\end{aligned} \tag{9.24a}$$

$$\begin{aligned}\mathrm{tr}_{4k}(\hat{\Omega}^*\hat{\chi})^3 &= \tfrac{1}{8}\,\mathrm{tr}_{4k}(\hat{\chi}^\dagger\hat{\chi}\hat{\Omega}^*\hat{\chi}) + \tfrac{1}{64}\,\mathrm{tr}_k\,(\hat{\Omega}\cdot\hat{\chi})^2\hat{\chi}\cdot\hat{\chi} - \tfrac{1}{32}\,\mathrm{tr}_k(\hat{\Omega}\cdot\hat{\chi})^3\\ &= -\frac{1}{2^5}\,\mathrm{tr}_k(\hat{\Omega}\cdot\hat{\chi})^3 + \frac{3}{2^7}\,\mathrm{tr}_k\,\hat{\chi}\cdot\hat{\chi}\hat{\Omega}\cdot\hat{\chi}\ ,\end{aligned} \tag{9.24b}$$

$$\begin{aligned}\mathrm{tr}_{4k}(\hat{\Omega}^*\hat{\chi})^4 &= \mathrm{tr}_{4k}(\tfrac{1}{64}\,\hat{\chi}^\dagger\hat{\chi}\hat{\chi}^\dagger\hat{\chi} - \tfrac{1}{32}\,\hat{\chi}^\dagger\hat{\chi}(\hat{\Omega}\cdot\hat{\chi})\hat{\Omega}^*\hat{\chi}\\ &\quad - \tfrac{1}{32}(\hat{\Omega}\cdot\hat{\chi})\hat{\Omega}^*\hat{\chi}\hat{\chi}^\dagger\hat{\chi} + \tfrac{1}{16}\,(\hat{\Omega}\cdot\hat{\chi})\hat{\Omega}^*\hat{\chi}(\hat{\Omega}\cdot\hat{\chi})\hat{\Omega}^*\hat{\chi})\\ &= \frac{1}{2^7}\,\mathrm{tr}_k(\hat{\Omega}\cdot\hat{\chi})^4 - \frac{1}{2^7}\,\mathrm{tr}_k(\hat{\Omega}\cdot\hat{\chi})^2\hat{\chi}\cdot\hat{\chi} + \frac{1}{2^9}\,\mathrm{tr}_k(\hat{\chi}\cdot\hat{\chi})^2 - \frac{1}{2^{10}}\,\mathrm{tr}_k\hat{\chi}_a\hat{\chi}_b\hat{\chi}_a\hat{\chi}_b\ .\end{aligned} \tag{9.24c}$$

As before, on the left-hand side of these formulae, $4k\times 4k$ matrix multiplication is implied, whereas on the right-hand side, all SO(6) indices are saturated in standard six-vector inner products, leaving the traces over $k\times k$ matrices.

From Eqs. (9.10), (9.20), (9.21), and (9.24a)–(9.24c), one obtains for the bosonic part of the expansion of the action Γ:

$$\Gamma_{\mathrm{b}} = \Gamma^{(2)} + \Gamma^{(3)} + \Gamma^{(4)} + \cdots\ , \tag{9.25}$$

where the quadratic, cubic and quartic actions are now given entirely as k-dimensional (rather than $2k$- or $4k$-dimensional) traces:

$$\Gamma^{(2)} = \mathrm{tr}_k\,\varphi^2, \quad \varphi = 2\rho\hat{\Omega}\cdot\hat{\chi} + \frac{1}{2\rho^2}\,\delta W^0\ , \tag{9.26a}$$

[64] In the following, we should emphasize that † only acts on instanton indices, as per reality condition (9.6), and *not* on SU(4) matrix indices.

$$\begin{aligned}\Gamma^{(3)} &= -\frac{1}{12\rho^6}\,\mathrm{tr}_k(\delta W^0)^3 + 4\rho^3\,\mathrm{tr}_k\,\hat{\Omega}\cdot\hat{\chi}\hat{\chi}\cdot\hat{\chi} - \frac{16\rho^3}{3}\,\mathrm{tr}_k(\hat{\Omega}\cdot\hat{\chi})^3 + \mathrm{tr}_k\,\delta W^0\hat{\chi}\cdot\hat{\chi} \\ &= 2\rho^2\,\mathrm{tr}_k\,\varphi(\hat{\chi}\cdot\hat{\chi} - 4(\hat{\Omega}\cdot\hat{\chi})^2) + \cdots\ , \end{aligned} \tag{9.26b}$$

$$\begin{aligned}\Gamma^{(4)} &= -\frac{1}{2\rho^4}\,\mathrm{tr}_k[\hat{a}'_n, \hat{a}'_m]^2 + \frac{1}{32\rho^8}\,\mathrm{tr}_k(\delta W^0)^4 - \mathrm{tr}_k[\hat{\chi}_a, \hat{a}'_n][\hat{\chi}_a, \hat{a}'_n] + 8\rho^4\,\mathrm{tr}_k(\hat{\Omega}\cdot\hat{\chi})^4 \\ &\quad - 8\rho^4\,\mathrm{tr}_k(\hat{\Omega}\cdot\hat{\chi})^2\hat{\chi}\cdot\hat{\chi} + 2\rho^4\,\mathrm{tr}_k(\hat{\chi}\cdot\hat{\chi})^2 - \rho^4\,\mathrm{tr}_k\,\hat{\chi}_a\hat{\chi}_b\hat{\chi}_a\hat{\chi}_b \\ &= -\frac{1}{2\rho^4}\,\mathrm{tr}_k[\hat{a}'_n, \hat{a}'_m]^2 - 8\rho^4\,\mathrm{tr}_k(\hat{\Omega}\cdot\hat{\chi})^2\hat{\chi}\cdot\hat{\chi} + 2\rho^4\,\mathrm{tr}_k(\hat{\chi}\cdot\hat{\chi})^2 \\ &\quad + 16\rho^4\,\mathrm{tr}_k(\hat{\Omega}\cdot\hat{\chi})^4 - \rho^4\,\mathrm{tr}_k\,\hat{\chi}_a\hat{\chi}_b\hat{\chi}_a\hat{\chi}_b - \mathrm{tr}_k[\hat{\chi}_a, \hat{a}'_n][\hat{\chi}_a, \hat{a}'_n] + \cdots\ . \end{aligned} \tag{9.26c}$$

Notice that only k^2 fluctuations, denoted φ, are actually lifted at quadratic order. This, in turn, implies that certain terms in $\Gamma^{(3)}$ and $\Gamma^{(4)}$ are subleading, and can be omitted. Specifically, the omitted terms in the final rewrites in Eqs. (9.26b) and (9.26c) contain, respectively, two or more, and one or more, factors of the quadratically lifted φ modes, and consequently are suppressed in $1/N$ (as a simple re-scaling argument again confirms).

Now let us perform the elementary Gaussian integration over the φ's. Changing integration variables in Eq. (9.7) from $d^{k^2}W^0$ to $d^{k^2}\varphi$ using Eq. (9.26a), one finds

$$\int d^{k^2}W^0\, e^{-N(\Gamma^{(2)}+\Gamma^{(3)})} = \left(\frac{4\pi\rho^4}{N}\right)^{k^2/2} e^{-N\Gamma^{(4)\prime}}\ , \tag{9.27}$$

where the new induced quartic coupling reads

$$\Gamma^{(4)\prime} = -\rho^4\,\mathrm{tr}_k(\hat{\chi}\cdot\hat{\chi} - 4(\hat{\Omega}\cdot\hat{\chi})^2)^2\ . \tag{9.28}$$

Combining $\Gamma^{(4)\prime}$ with the original quartic coupling (9.26c) gives for the effective bosonic small-fluctuations action

$$\Gamma_{\mathrm{b}} = -\frac{1}{2}\,\mathrm{tr}_k(\rho^{-4}[\hat{a}'_n, \hat{a}'_m]^2 + 2[\hat{\chi}_a, \hat{a}'_n]^2 + \rho^4[\hat{\chi}_a, \hat{\chi}_b]^2)\ . \tag{9.29}$$

Remarkably, all dependence on the unit vector $\hat{\Omega}_a$ has cancelled out.

Note that apart from the absence of derivative terms, expression (9.29) looks like a Yang–Mills field strength for the gauge group SU(k)! We can make this explicit by introducing a 10-dimensional vector field A_M,

$$A_M = N^{1/4}(\rho^{-1}\hat{a}'_n, \rho\hat{\chi}_a), \quad M = 1, \ldots, 10\ , \tag{9.30}$$

in terms of which

$$N\Gamma_{\mathrm{b}}(A_M) = -\frac{1}{2}\,\mathrm{tr}_k[A_M, A_N]^2\ . \tag{9.31}$$

We recognize this as precisely the action of 10-dimensional SU(k) gauge theory, reduced to 0 dimensions, i.e. with all derivatives set to zero.

Now let us turn to the Grassmann collective coordinates. Since $\mathcal{N}=4$ supersymmetry in four dimensions descends from $\mathcal{N}=1$ supersymmetry in 10 dimensions, and since all our saddle-point manipulations commute with supersymmetry, we expect to find the $\mathcal{N}=1$ supersymmetric completion of the 10-dimensional dimensionally reduced action (9.31), namely

$$N\Gamma_{\rm f}(A_M,\Psi) = -{\rm i}\,{\rm tr}_k\,\bar{\Psi}\Gamma_M[A_M,\Psi]\ , \tag{9.32}$$

where Ψ is a 10-dimensional Majorana–Weyl spinor, and Γ_M is an element of the 10-dimensional Clifford algebra. To see how this comes about, we first separate out from the fermionic collective coordinates the exact zero modes, in analogy to Eqs. (9.19a) and (9.19b):

$$\mathcal{M}'^{A}_{\alpha} = -4{\rm i}\xi^A_\alpha 1_{[k]\times[k]} - 4{\rm i}a'_{\alpha\dot\alpha}\bar\eta^{\dot\alpha A} + \hat{\mathcal{M}}'^{A}_{\alpha}\ , \tag{9.33a}$$

$$\zeta^{\dot\alpha A} = -4{\rm i}\bar\eta^{\dot\alpha A}1_{[k]\times[k]} + \hat\zeta^{\dot\alpha A}\ . \tag{9.33b}$$

Here, ξ^A_α and $\bar\eta^{\dot\alpha A}$ are the supersymmetric and superconformal fermion modes (4.43)–(4.46). Expanding the remaining part of the fermion coupling in the exponent of (9.3) around special solution (9.16) and using relations (6.3) and (9.26a), we find

$$\begin{aligned} N\Gamma_{\rm f} = (8\pi^2N)^{1/2}\,{\rm tr}_k[(\varphi - 2\rho(\hat\Omega\cdot\hat\chi))\rho\hat\Omega_{AB}\hat\zeta^{\dot\alpha A}\hat\zeta^{B}_{\dot\alpha} + \rho^{-1}\hat\Omega_{AB}[\hat a'_{\alpha\dot\alpha},\hat{\mathcal{M}}'^{\alpha A}]\hat\zeta^{\dot\alpha B} \\ +\hat\chi_{AB}(\rho^2\hat\zeta^{\dot\alpha A}\hat\zeta^{B}_{\dot\alpha} + \hat{\mathcal{M}}'^{\alpha A}\hat{\mathcal{M}}'^{B}_{\alpha})]\ . \end{aligned} \tag{9.34}$$

If we now define the 10-dimensional Majorana–Weyl fermion field Ψ,

$$\Psi = \sqrt{\frac{\pi}{2}}N^{1/8}\left[\rho^{-1/2}\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\hat{\mathcal{M}}'^{A}_{\alpha} + \rho^{1/2}\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\hat\zeta^{\dot\alpha A}\right]\ , \tag{9.35}$$

and the Γ_M matrices according to Eq. (9.36), we do in fact recover the simple form (9.32). In moving from Eq. (9.34) to Eq. (9.32) we have dropped the term depending on φ; since φ is a quadratically lifted bosonic mode its contribution is suppressed in large N compared to the other couplings in Eq. (9.34), as a simple re-scaling argument confirms. The representation of the SO(10) Γ-matrices is constructed via the decomposition SO(10) $\supset$ SO(6) $\times$ SO(4) in the following way. Firstly, We define an SO(6) rotation matrix R, $RR^{\rm T}=1$, such that $\hat\Omega'_a = R_{ab}\hat\Omega_b$ lies entirely along, say, the first direction, i.e. $\hat\Omega'_a \propto \delta_{a1}$. In the new basis, we have a representation of the SO(6) Clifford algebra $\Gamma'_a = R_{ab}\Gamma_b$. In the rotated basis, we can construct a representation of the SO(10) Clifford algebra as follows:

$$\Gamma'_M = \{\Gamma'_1\otimes\gamma_n, \Gamma'_a\otimes(\delta_{a1}\gamma_5 + (1-\delta_{a1})1_{[4]\times[4]})\}\ , \tag{9.36}$$

where $n=1,\ldots,4$ and $a=1,\ldots,6$. The representation of the SO(10) Clifford algebra that we need is then found by undoing the rotation on the six-dimensional subspace:

$$\Gamma_M = \{\Gamma'_1\otimes\gamma_n, (R^{-1})_{ab}\Gamma'_b\otimes(\delta_{b1}\gamma_5 + (1-\delta_{b1})1_{[4]\times[4]})\}\ . \tag{9.37}$$

One can then verify that (9.35) is Majorana–Weyl with respect to this basis. Note that unlike the bosonic sector, the $\hat{\Omega}_a$ dependence of the fermionic action does not actually disappear; it is simply subsumed into the representation of the SO(10) Γ-matrices.

Finally our effective large-N k-instanton partition function has the form

$$\mathscr{Z}_k^{(\mathcal{N}=4)} \overset{N\to\infty}{\rightarrow} \frac{\sqrt{N}}{k^3 2^{17k^2/2-k/2+25}\pi^{9k^2/2+9}\,\mathrm{Vol}\,\mathrm{U}(k)} \int \rho^{-5}\,\mathrm{d}\rho\,\mathrm{d}^4X\,\mathrm{d}^5\hat{\Omega} \times \prod_{A=1}^{4} \mathrm{d}^2\xi^A\,\mathrm{d}^2\bar{\eta}^A \cdot \hat{\mathscr{Z}}_{\mathrm{SU}(k)}^{(d=10)} , \tag{9.38}$$

where $\hat{\mathscr{Z}}_{\mathrm{SU}(k)}^{(d=10)}$ is the partition function of an $\mathcal{N}=1$ supersymmetric SU(k) gauge theory in 10 dimensions dimensionally reduced to zero dimensions:

$$\hat{\mathscr{Z}}_{\mathrm{SU}(k)}^{(d=10)} = \int_{\mathrm{SU}(k)} \mathrm{d}^{10}A\,\mathrm{d}^{16}\Psi\,\mathrm{e}^{-S(A_\mu,\Psi)} ,$$

$$S(A_M,\Psi) = N(\Gamma_\mathrm{b}+\Gamma_\mathrm{f}) = -\frac{1}{2}\mathrm{tr}_k[A_M,A_N]^2 - \mathrm{i}\,\mathrm{tr}_k\,\bar{\Psi}\Gamma_M[A_M,\Psi] . \tag{9.39}$$

Note that the rest of the measure, up to numerical factors, is independent of the instanton number k. When integrating expressions which are independent of the SU(k) degrees of freedom, $\hat{\mathscr{Z}}_{\mathrm{SU}(k)}^{(d=10)}$ is simply an overall constant factor. A calculation of Ref. [115] revealed that $\hat{\mathscr{Z}}_{\mathrm{SU}(k)}^{(d=10)}$ is proportional to $\sum_{d|k} d^{-2}$, a sum over the positive integer divisors d of k. However, the constant of proportionality was fixed definitively in Ref. [116] to give [65]

$$\hat{\mathscr{Z}}_{\mathrm{SU}(k)}^{(d=10)} = \frac{2^{17k^2/2+k/2-9}\pi^{5k^2+k/2-7/2}k^{-1/2}}{\prod_{i=1}^{k-1} \mathrm{i}!} \sum_{d|k} \frac{1}{d^2} . \tag{9.40}$$

In summary, the effective large-N, semi-classically leading-order, collective coordinate measure has the following simple form [21]:

$$\mathrm{e}^{2\pi\mathrm{i}k\tau}\mathscr{Z}_k^{(\mathcal{N}=4)} \overset{N\to\infty}{\rightarrow} \frac{\sqrt{N}}{2^{33}\pi^{27/2}} k^{-7/2}\mathrm{e}^{2\pi\mathrm{i}k\tau}\left\{\sum_{d|k}\frac{1}{d^2}\right\} \int \frac{\mathrm{d}^4X\,\mathrm{d}\rho}{\rho^5}\,\mathrm{d}^5\,\hat{\Omega} \prod_{A=1}^{4} \mathrm{d}^2\xi^A\,\mathrm{d}^2\bar{\eta}^A . \tag{9.41}$$

9.1.2. The $\mathcal{N}=2$ case

We now follow essentially the same steps to deduce a form for the large-N collective coordinates integral in the finite $\mathcal{N}=2$ theory following Ref. [107]. In view of the large-N limit to come, we first re-scale

$$\chi \to \frac{\sqrt{N}}{2\pi}\chi . \tag{9.42}$$

[65] In comparing to Ref. [116], it is important to note that our convention for the normalization of the generators is $\mathrm{tr}_k\,\hat{T}^r\hat{T}^s = \delta^{rs}$, rather than $\frac{1}{2}\delta^{rs}$ in Ref. [116].

Performing the integration over the v^A's and $\bar{v}^A$'s gives

$$\int \prod_{A=1,2} \mathrm{d}^{k(N-2k)} v^A \, \mathrm{d}^{k(N-2k)} \bar{v}^A \exp[-\sqrt{N}\pi \, \mathrm{tr}_k \, \chi^\dagger \bar{v}^A v_A] = (N\pi^2)^{k(N-2k)} |\det{}_k \, \chi^\dagger|^{2(N-2k)} \; . \tag{9.43}$$

Similarly, integrating out the matter Grassmann collective coordinates

$$\int \mathrm{d}^{2kN} \mathscr{K} \, \mathrm{d}^{2kN} \tilde{\mathscr{K}} \exp\left[\sqrt{\frac{N}{4}}\,\pi \sum_{f=1}^{N_F=2N} \mathrm{tr}_k \, \mathscr{K}_f \tilde{\mathscr{K}}_f \chi\right] = \left(\frac{N\pi^2}{4}\right)^{kN} |\det{}_k \, \chi|^{2N} \; . \tag{9.44}$$

Once again, we collect all the terms to the power N to give the large-N "effective action"

$$\Gamma = -\log \det{}_{2k} W - 2 \log \det{}_k \, \chi\chi^\dagger + \mathrm{tr}_k \, \chi_a \boldsymbol{L} \chi_a \; . \tag{9.45}$$

We now turn to the solution of the large-N saddle-point equations. These are the coupled Euler–Lagrange equations that come from extremizing Γ with respect to $\{a'_n, W^0, \chi_a\}$. The analysis of these coupled equations is virtually identical to the $\mathcal{N}=4$ case in Section 9.1.1; hence we suppress the calculational details. In particular, the dominant configuration is the maximally degenerate solution (9.16) where now $\hat{\Omega}_a$ is a unit 2-vector. It is convenient to parameterize $\hat{\Omega}_a$ by the phase angle ϕ:

$$\hat{\Omega}_1 + \mathrm{i}\hat{\Omega}_2 = \mathrm{e}^{\mathrm{i}\phi} \; . \tag{9.46}$$

The next stage of the analysis is to expand the effective action in the fluctuations out to sufficient order to ensure the convergence of the integrals over the fluctuations. Fortunately, the analysis need not be repeated because our effective action (9.45) is, up to a constant, simply the $\mathcal{N}=4$ effective action (9.10) with the replacements $\chi_{12} \to \chi/\sqrt{8}$, $\chi_{34} \to \chi^\dagger/\sqrt{8}$ and all other components of χ_{AB} set to zero. The fluctuations in W^0 are integrated out at Gaussian order to leave integrals over the remaining fluctuations in $\hat{a}'_n$ and $\hat{\chi}_a$, the traceless parts, that are lifted at quartic order; the terms beyond quartic order are then formally suppressed by (fractional) powers of $1/N$, and may be dropped. The "action" for these quartic fluctuations is precisely the action for six-dimensional Yang–Mills theory, as in (9.30) and (9.31) but where the indices $M = 1, \ldots, 6$.

Not surprisingly the coupling to the Grassmann collective coordinates completes the six-dimensional theory to a supersymmetric gauge theory in six dimensions dimensionally reduced to 0 dimensions. The six-dimensional theory has eight independent supersymmetries and is therefore an $\mathcal{N}=(1,0)$ theory in six dimensions. The eight-component fermion field of the $\mathcal{N}=(1,0)$ theory has components

$$\Psi = \sqrt{\frac{\pi}{2}} N^{1/8} \mathrm{e}^{-\mathrm{i}\phi/2} (\rho^{-1/2} \hat{\mathscr{M}}'^{A}_{\alpha}, \rho^{1/2} \hat{\zeta}^{\dot{\alpha}A}) \; . \tag{9.47}$$

Following the same steps as in the $\mathcal{N}=4$ theory, the effective large-N k-instanton measure, for the leading-order semi-classical approximation of the functional integral, has the form

$$\mathrm{e}^{2\pi\mathrm{i}k\tau} \mathscr{Z}_k^{(\mathcal{N}=2, N_F=2N)} \overset{N\to\infty}{\to} \frac{N^{1/2} \mathrm{e}^{2\pi\mathrm{i}k\tau}}{k 2^{9k^2/2 - k/2 + 12} \pi^{5k^2/2+4} \, \mathrm{Vol}\, \mathrm{U}(k)} \times \int \rho^{-5} \mathrm{d}\rho \, \mathrm{d}^4 X \, \mathrm{d}\phi \, \mathrm{e}^{4\mathrm{i}\phi} \prod_{A=1}^{2} \mathrm{d}^2 \xi^A \mathrm{d}^2 \bar{\eta}^A \cdot \hat{\mathscr{Z}}^{(d=6)}_{\mathrm{SU}(k)} \; , \tag{9.48}$$

where $\hat{\mathscr{Z}}_k$ is the partition function of the $\mathcal{N}=(1,0)$ supersymmetric SU(k) gauge theory in six dimensions dimensionally reduced to zero dimensions:

$$\hat{\mathscr{Z}}^{(d=6)}_{\mathrm{SU}(k)} = \int_{\mathrm{SU}(k)} \mathrm{d}^6 A\, \mathrm{d}^8 \Psi\, \mathrm{e}^{-S(A_\mu,\Psi)}\ ,$$

$$S(A_M,\Psi) = N(\Gamma_{\mathrm{b}} + \Gamma_{\mathrm{f}}) = -\frac{1}{2}\,\mathrm{tr}_k [A_M, A_N]^2 - \mathrm{i}\,\mathrm{tr}_k\, \bar{\Psi}\Gamma_M[A_M,\Psi]\ . \tag{9.49}$$

When integrating expressions which are independent of the SU(k) degrees of freedom, $\hat{\mathscr{Z}}^{(d=6)}_{\mathrm{SU}(k)}$ is simply an overall constant factor that was evaluated in [116]. In our notation [66]

$$\hat{\mathscr{Z}}^{(d=6)}_{\mathrm{SU}(k)} = (\sqrt{2\pi})^{6(k^2-1)}(\sqrt{2})^{(8-6)(k^2-1)}\,\frac{2^{k(k+1)/2}\pi^{(k-1)/2}}{2\sqrt{k}\prod_{i=1}^{k-1} \mathrm{i}!}\,\frac{1}{k^2}\ . \tag{9.50}$$

In summary, the effective large-N collective coordinate measure has the following simple form [107]:

$$\mathrm{e}^{2\pi \mathrm{i} k\tau}\mathscr{Z}_k^{(\mathcal{N}=2,N_F=2N)} \overset{N\to\infty}{\rightarrow} \frac{\sqrt{N}}{2^{17}\pi^{15/2}}\,k^{-7/2}\mathrm{e}^{2\pi\mathrm{i}k\tau} \times \int \frac{\mathrm{d}^4 X\,\mathrm{d}\rho}{\rho^5}\,\mathrm{d}\phi\,\mathrm{e}^{4\mathrm{i}\phi}\prod_{A=1}^{2}\mathrm{d}^2\xi^A\,\mathrm{d}^2\bar{\eta}^A\ . \tag{9.51}$$

This has a remarkable similarity to the form of the $\mathcal{N}=4$ measure (9.41). Apart from the differences in the overall numerical factors, the integral over S^5 is replaced by S^1 and the $\mathcal{N}=4$ measure involves, in addition, the factor $\sum_{d|k} d^{-2}$, the sum over the integer divisors of k. Notice that the $\sqrt{N}$ dependence and factor of $k^{-7/2}$ is the same in both cases.

The appearance of the $\mathrm{e}^{4\mathrm{i}\phi}$ phase in (9.51) implies a selection rule in order that correlation functions are non-vanishing. This is a relic of fermion zero mode counting. From (9.2), we see each insertion of an adjoint fermion Grassmann collective coordinate (other than those associated to broken supersymmetry and superconformal invariance which drop out from the couplings in (9.2)) implies an insertion of $\mathrm{e}^{\mathrm{i}\phi/2}$ and each insertion of a fundamental fermion Grassmann collective coordinate implies an insertion of $\mathrm{e}^{-\mathrm{i}\phi/2}$. The selection rule implies that the difference between the latter and former is 8. Therefore the simplest kind of non-vanishing correlation functions would involve insertions that saturate the eight integrals over ξ^A and $\bar{\eta}^A$ and, in addition, integrals over eight of the fundamental fermion Grassmann collective coordinates $\{\mathscr{K},\tilde{\mathscr{K}}\}$.

9.2. Large-N correlation functions

Having established the form for the collective coordinate integral in the large-N limit, we can now calculate correlation functions of various composite operators. We are primarily interested in those

[66] We have written the result in a way which allows an easy comparison with [116]. The factors of $\sqrt{2\pi}$ and $\sqrt{2}$ arise, respectively, from the difference in the definition of the bosonic integrals and the normalization of the generators: we have $\mathrm{tr}_k\, T^r T^s = \delta^{rs}$ rather than $\frac{1}{2}\delta^{rs}$. The remaining factors are the result of [116].

correlators that receive contributions from all instanton numbers. From the form of integrals (9.38) and (9.48), we can now identify those as the ones for which the leading-order (in $1/N$) expression for the insertion is independent of the SU(k) variables. Hence the only non-trivial dependence is on $\{X_n, \rho, \hat{\Omega}_a\}$ and $\{\xi^A, \bar{\eta}^A\}$.[67] In this case, we can use the reduced form of integrals (9.41) and (9.51). For these special correlation functions the only dependence on the instanton charge will be through an overall multiplicative factor as advertised in (9.1).

Before considering specific insertions, we can make some useful general statements. At leading order in $1/N$, we can replace each operator insertion with its classical k-instanton saddle-point value. Since the saddle-point solution (9.16) is relatively simple, this observation greatly streamlines the form of the operator insertions. We will restrict our attention to operators consisting of a single trace on the gauge group index of a product of adjoint scalars, fermions and field strengths. Each of these three adjoint quantities is of the type $\bar{U}\boldsymbol{X}U$ where U is the ADHM quantity defined in (2.47) and $\boldsymbol{X}$ is some matrix of ADHM variables; consequently the insertions have the form

$$\mathcal{O}(x) = \mathrm{tr}_N[\bar{U}\boldsymbol{X}_1 U\bar{U}\boldsymbol{X}_2 U\cdots\bar{U}\boldsymbol{X}_p U] = \mathrm{tr}_{N+2k}[\mathcal{P}\boldsymbol{X}_1\mathcal{P}\boldsymbol{X}_2\mathcal{P}\cdots\mathcal{P}\boldsymbol{X}_p]\ , \tag{9.52}$$

where $\mathcal{P}=U\bar{U}$ is projection operator (2.51). It is easily checked that at the saddle point, the bosonic ADHM quantities f, a'_n, $\boldsymbol{L}$ and $\mathcal{P}$ collapse to

$$\begin{aligned} &f \to \frac{1}{(x-X)^2+\rho^2}\,1_{[k]\times[k]}, \quad a'_n \to -X_n 1_{[k]\times[k]}, \quad \boldsymbol{L} \to 2\rho^2\ , \\ &\mathcal{P} = \frac{1}{(x-X)^2+\rho^2}\begin{pmatrix} ((x-X)^2+\rho^2)1_{[N]\times[N]} - w_{\dot\alpha}\bar{w}^{\dot\alpha} & -w_{\dot\alpha}(\bar{x}-\bar{X})^{\dot\alpha\alpha} \\ -(x-X)_{\alpha\dot\alpha}\bar{w}^{\dot\alpha} & \rho^2 1_{[2k]\times[2k]} \end{pmatrix}\ . \end{aligned} \tag{9.53}$$

One can verify that deviations from these saddle-point values are suppressed by powers of $N^{-1/4}$.

The analogous replacement prescription for the fermionic ADHM quantities is, in general, somewhat trickier. Recall expansions (9.33a) for the Grassmann collective coordinates. In analogy with bosonic quantities (9.16)–(9.53), it is useful to think of the unlifted variables ξ^A and $\bar{\eta}^A$ themselves as arising from a saddle-point evaluation:

$$\mathcal{M}'^{A}_{\alpha} \to -4\mathrm{i}\xi^A_\alpha 1_{[k]\times[k]}, \qquad \zeta^{\dot\alpha A} \to -4\mathrm{i}\bar{\eta}^{\dot\alpha A}1_{[k]\times[k]}\ . \tag{9.54}$$

Indeed, scaling (9.35) implies that the remaining variables denoted $\hat{\mathcal{M}}'^A$ and $\hat{\zeta}^A$ in (9.33a) are subleading compared to ξ^A and $\bar{\eta}^A$ by a factor of $N^{-1/8}$. There remain the modes v^A and $\bar{v}^A$, which are distinct from the others in that they carry an SU(N) index u. From their coupling to χ_a in Eq. (9.9), one sees that each $\bar{v}^A v^B$ pair in an insertion, for a fixed, unsummed value of the index u, costs a factor of $N^{1/2}$; however, summing on u (as required by gauge invariance) then turns this $N^{-1/2}$ suppression into an $N^{1/2}$ enhancement. In other words, v^A and $\bar{v}^A$ factors in the insertions should each be thought of as being enhanced by $N^{1/4}$. The large-N rule of thumb for choosing Grassmann collective coordinates in correlators is now clear: the insertions should saturate as many

[67] We will describe how dependence on $\hat{\Omega}_a$ in the insertion arises in due course.

of the $\{v^A, \bar{v}^A\}$ integrals as possible, with the proviso that the integrals over the $\{\xi^A, \bar{\eta}^A\}$ must be saturated.

Notice in order to consider insertions of $\{v^A, \bar{v}^A\}$ and $\{\mathscr{K}, \tilde{\mathscr{K}}\}$ we must return to the collective coordinates before these variables have been integrated out. We take the $\mathcal{N}=4$ case first and show, to leading order in $1/N$, the gauge-invariant combination $\bar{v}^A v^B$ in the effective integration measure (9.41) is replaced by

$$\bar{v}^A v^B \rightarrow -\frac{\rho N^{1/2}}{\pi} \Sigma_a^{AB} \hat{\Omega}_a 1_{[k]\times[k]} , \tag{9.55}$$

where the $N^{1/2}$ dependence has already been noted. To this end, consider a general insertion with a string of such combinations $\bar{v}^{A_1} v^{B_1} \otimes \cdots \otimes \bar{v}^{A_p} v^{B_p}$. We must insert this expression into the measure *before* the v integrals have been performed. Then performing the v integrals as in (9.9) in the presence of the insertion leads to a modified expression involving factors of χ^{-1} which can be derived by considering

$$\begin{aligned} &\frac{\partial}{\partial \chi^{\mathrm{t}}_{A_1B_1}} \otimes \cdots \otimes \frac{\partial}{\partial \chi^{\mathrm{t}}_{A_pB_p}} |\mathrm{det}_{4k}\, \chi|^{N-2k} \\ &\quad = N^p |\mathrm{det}_{4k}\, \chi|^{N-2k} (\chi^{-1})_{B_1A_1} \otimes \cdots \otimes (\chi^{-1})_{B_pA_p} + \cdots , \end{aligned} \tag{9.56}$$

where t (transpose) acts on instanton indices and the ellipses represent terms of lower order in $1/N$. This shows that, after performing the v integrals, a term of the form $\bar{v}^A v^B$ is replaced by $-(\sqrt{N}/\sqrt{8}\pi)(\chi^{-1})_{BA}$, to leading order. Now we replace χ with its saddle-point value (9.16) and use (9.5) to give (9.55). The replacement elucidates the mysterious appearance of the variables $\hat{\Omega}_a$.

In the $\mathcal{N}=2$ theory, examination of Eq. (9.2) a similar replacement

$$\bar{v}^A v^B \rightarrow \frac{g\rho\sqrt{2\mathrm{i}N}}{\pi} \varepsilon^{AB} \mathrm{e}^{\mathrm{i}\phi} 1_{[k]\times[k]} . \tag{9.57}$$

This leaves the matter Grassmann collective coordinates $\{\mathscr{K}, \tilde{\mathscr{K}}\}$. Again after examining (9.2), for each flavour $f = 1, \ldots, 2N$, we have the replacement

$$\mathscr{K}_f \tilde{\mathscr{K}}_f \rightarrow \frac{\rho\sqrt{2\mathrm{i}}}{\pi\sqrt{N}} \mathrm{e}^{-\mathrm{i}\phi}\, 1_{[k]\times[k]} . \tag{9.58}$$

We now consider some examples. First of all, in the $\mathcal{N}=4$ theory there are a class of correlation functions of type described above motivated by the AdS/CFT correspondence (discussed in Section 9.3). The simplest involves the 16-point correlation function of the fermionic composite operator

$$\Lambda^A = g^2\, \sigma_{mn}\, \mathrm{tr}_N\, F_{mn}\, \lambda^A . \tag{9.59}$$

Since, to leading order in g, each insertion is linear in Grassmann collective coordinates, we only need the dependence on the collective coordinates $\{\xi^A, \bar{\eta}^A\}$, since the other Grassmann coordinates must be lifted by the instanton effective action. In this case to find the dependence, we can use the supersymmetric sweeping-out procedure defined in Section 6.3 with the x-dependent variation parameter (4.44) giving [68]

$$\Lambda^A = \mathrm{i}g^{5/2}\sigma_{mn}\sigma_{kl}\xi^A(x)\,\mathrm{tr}_N\,F_{mn}F_{kl} + \cdots = -\frac{\mathrm{i}}{4}\,g^{5/2}\xi^A(x)\,\mathrm{tr}_N\,F_{mn}^2 + \cdots\ . \tag{9.60}$$

The ellipses represent the dependence on the other Grassmann collective coordinates that are not needed. Next we can use identity (C.6) and finally insert the saddle-point expression for f in Eq. (9.53):

$$\Lambda^A = 24\mathrm{i}kg^{1/2}\xi^A(x)\,\frac{\rho^4}{((x-X)^2+\rho^2)^4} + \cdots\ . \tag{9.61}$$

Then using Eqs. (9.41) and (9.61), the large-N k-instanton contribution to the 16-point correlator is precisely [21]

$$\langle \Lambda^{A_1}_{\alpha_1}(x^{(1)}) \times \cdots \times \Lambda^{A_{16}}_{\alpha_{16}}(x^{(16)})\rangle_{\text{inst}} \overset{N\to\infty}{\to} \frac{3^{16}2^{15}g^8\sqrt{N}}{\pi^{27/2}}\left\{\sum_{k=1}^{\infty} k^{25/2}\mathrm{e}^{2\pi\mathrm{i}k\tau}\sum_{d|k}\frac{1}{d^2}\right\}$$
$$\times \int \frac{\mathrm{d}^4X\,\mathrm{d}\rho}{\rho^5}\prod_{A=1}^{4}\mathrm{d}^2\xi^A\,\mathrm{d}^2\bar{\eta}^A\prod_{l=1}^{16}\frac{\xi^{A_l}_{\alpha_l}(x^{(l)})\rho^4}{((x^{(l)}-X)^2+\rho^2)^4}\ . \tag{9.62}$$

For application to the AdS/CFT correspondence it is unnecessary to perform the remaining integrals.

Correlator (9.62) has the distinguishing property that the functional dependence on the insertion points given by the integral term can be reproduced by a single-instanton calculation with gauge group SU(2). The reason is clear: only the dependence of the insertions on the supersymmetric and superconformal Grassmann coordinates was required. In particular the saddle-point evaluation of the relevant part of insertion (9.61) is identical to the single instanton in SU(2). So at large N, the dependence on k and N only appears in an overall multiplicative factor. In fact there is a whole family of related correlators considered in Ref. [117] for which the same property holds and a calculation for a single instanton in SU(2) is sufficient to obtain the functional dependence on the insertion points. We have now explained the puzzle of why the analysis of Ref. [117] involving a single instanton in SU(2) was able to capture effects that ultimately, via the AdS/CFT correspondence, should have been valid at large-N only.

There are more general correlation functions involving insertions with non-trivial SO(6) R-transformation properties for which a calculation in SU(2) would not suffice. For example consider the insertion of the composite field

$$\mathcal{O}_{a_1\cdots a_p}(x) = \mathrm{tr}_N\,[\phi_{a_1}\cdots\phi_{a_p}F_{mn}^2]\ , \tag{9.63}$$

whose form is motivated by the AdS/CFT correspondence. We now consider this evaluated at the saddle point. In general, $\mathcal{O}$ depends on the Grassmann variables $\{\mu^A,\bar{\mu}^A\}$, as well as the supersymmetric and superconformal variables. The resulting expressions are rather cumbersome, so we will assume that the integrals over the supersymmetric and superconformal variables are saturated by other insertions, for example, by 16 insertions of Λ^A. In this case we only need the dependence

[68] The extra factor of $g^{1/2}$ follows from re-scaling (4.42).

of $\mathcal{O}$ on the $\{v^A, \bar{v}^A\}$:

$$\mathcal{O}_{a_1\cdots a_p}(x) \sim \frac{\rho^4}{((x-X)^2+\rho^2)^{p+4}}\,\bar{\Sigma}_{a_1A_1B_1}\cdots\bar{\Sigma}_{a_pA_pB_p}\,\mathrm{tr}_k[\bar{v}^{A_1}v^{B_1}\cdots\bar{v}^{A_p}v^{B_p}]\ . \tag{9.64}$$

At leading order in $1/N$ we need use replacement (9.55) to obtain

$$\mathcal{O}_{a_1\cdots a_p}(x) \sim kN^{p/2}\frac{\rho^{4+p}}{((x-X)^2+\rho^2)^{p+4}}\,\hat{\Omega}_{a_1}\cdots\hat{\Omega}_{a_p}\ . \tag{9.65}$$

In this case, we see that the insertion depends explicitly on the S^5 coordinate $\hat{\Omega}_a$.

In the $\mathcal{N}=2$ theory, a simple correlator involves eight insertions of Λ^A, as in (9.59) where now $A=1,2$, and 4 insertions of the gauge-invariant operator

$$Q_f(x) = g^2\,\mathrm{tr}_N\,\chi_f\,\tilde{\chi}_f\ , \tag{9.66}$$

which is quadratic in the matter Grassmann collective coordinates. Evaluating this on the saddle-point solution using expression (6.53) along with the leading-order replacement (9.58), we have

$$Q_f(x) = \frac{gk\sqrt{8\mathrm{i}}}{\pi\sqrt{N}}\,\frac{\rho^2}{((x-X)^2+\rho^2)^3}\,\mathrm{e}^{-\mathrm{i}\phi}+\cdots\ . \tag{9.67}$$

Notice that the four insertions of Q_f involve a factor of $\mathrm{e}^{-4\mathrm{i}\phi}$ which cancels the factor of $\mathrm{e}^{4\mathrm{i}\phi}$ in Eq. (9.51), so the integral over ϕ yields the constant 2π. Hence,

$$\begin{aligned}
&\langle\Lambda^{A_1}_{\alpha_1}(x^{(1)})\times\cdots\times\Lambda^{A_8}_{\alpha_8}(x^{(8)})Q_{f_1}(x^{(9)})\times\cdots\times Q_{f_4}(x^{(12)})\rangle_{\mathrm{inst}}\\
&\times\ \overset{N\to\infty}{\to}\ \frac{3^8 2^{14} g^8}{\pi^{21/2}N^{3/2}}\left\{\sum_{k=1}^{\infty}k^{17/2}\mathrm{e}^{2\pi\mathrm{i}k\tau}\right\}\\
&\times\ \int\frac{\mathrm{d}^4X\,\mathrm{d}\rho}{\rho^5}\prod_{A=1}^{2}\mathrm{d}^2\xi^A\,\mathrm{d}^2\bar{\eta}^A\prod_{l=1}^{8}\frac{\xi^{A_l}_{\alpha_l}(x^{(l)})\rho^4}{((x^{(l)}-X)^2+\rho^2)^4}\prod_{l=1}^{4}\frac{\rho^2}{((x^{(l+8)}-X)^2+\rho^2)^3}\ .
\end{aligned} \tag{9.68}$$

9.3. Instantons and the AdS/CFT correspondence

The formalism that we have used to calculate the instanton contributions to certain correlation functions at leading order in $1/N$ is particularly interesting in the light of the AdS/CFT correspondence. The basic example of the AdS/CFT correspondence, due to Maldacena [112] (see also the comprehensive review [113]), states that the $\mathcal{N}=4$ supersymmetric gauge theory is equivalent to Type IIB string theory on an $AdS_5\times S^5$ background. In particular, the gauge coupling g and vacuum angle θ of the four-dimensional theory are given in terms of the string parameters by

$$g=\sqrt{4\pi g_{\mathrm{st}}}=\sqrt{4\pi\mathrm{e}^{\phi}},\quad \theta=2\pi C^{(0)}\ . \tag{9.69}$$

Here, g_{st} is the string coupling while ϕ and $C^{(0)}$ are the expectation values of the dilaton and Ramond–Ramond scalar, respectively, of Type IIB string theory. Also N appears explicitly, through the relation

$$\frac{L^2}{\alpha'} = \sqrt{g^2 N}\ , \tag{9.70}$$

where $(\alpha')^{-1}$ is the string tension and L is the radius of both the AdS_5 and S^5 factors of the background.

9.3.1. The instanton collective coordinate integral

Even before we consider application of the AdS/CFT correspondence to correlation functions, and how it applies to instantons, we can already see in the form of the effective large-N collective coordinate integral (9.41) a remarkable relation to $AdS_5 \times S^5$. The c-number integrals are precisely the volume form on this space, where the AdS_5 part has the metric

$$\mathrm{d}s^2 = \rho^{-2}(\mathrm{d}X_n^2 + \mathrm{d}\rho^2) \tag{9.71}$$

so that ρ is the radial variable. This fact alone strongly suggests that Yang–Mills instantons are identified in the large-N limit with D-instantons in Type IIB string theory. Some relevant aspects of D-branes in Type II string theory are summarized in Section 10.3. In particular, the contribution of k D-instantons to low-energy correlators is governed by the $\mathrm{U}(k)$ matrix integral (10.52). For the most part, the $\mathrm{SU}(k)$ part of the integral can be factored off as an overall numerical constant; hence,

$$\mathscr{Z}_k \sim \frac{\mathrm{e}^{-2\pi k(\mathrm{e}^{-\phi}+\mathrm{i}C^{(0)})}}{\mathrm{Vol}\,\mathrm{U}(k)} \int \mathrm{d}^{10}X\,\mathrm{d}^{16}\Theta \cdot \hat{\mathscr{Z}}_k\ , \tag{9.72}$$

where $\hat{\mathscr{Z}}_k$ is the same $\mathrm{SU}(k)$ matrix integral that we defined in (9.49) and X_M and Θ are the abelian components of the fields. This D-instanton collective coordinate integral is appropriate to the case of flat 10-dimensional space. However, the results of Banks and Green [118] imply that since $AdS_5 \times S^5$ is conformally flat, a similar expression should apply to this background with the bosonic abelian integrals replaced by the appropriate volume form of $AdS_5 \times S^5$. Being careful with the pre-factor, one has the D-instanton collective coordinate integral [21,117]

$$\mathscr{Z}_k|_{AdS_5\times S^5} \sim (\alpha')^{-1}(k\mathrm{e}^{\phi})^{-7/2}\,\mathrm{e}^{-2\pi k(\mathrm{e}^{-\phi}+\mathrm{i}C^{(0)})} \sum_{d|k} \frac{1}{d^2} \int \frac{\mathrm{d}\rho\,\mathrm{d}^4X}{\rho^5}\,\mathrm{d}^5\hat{\Omega}\,\mathrm{d}^{16}\Theta\ . \tag{9.73}$$

Using the relation between the couplings of the string theory and the gauge theory in (9.69) this is precisely the leading-order collective coordinate integral in the large-N limit of the $\mathscr{N}=4$ instanton calculus (9.41). [69] This equivalence of the large-N Yang–Mills instanton measure and the D-instanton measure on $AdS_5 \times S^5$ is rather stunning evidence for the AdS/CFT correspondence. It is very satisfying that instantons at large N seem to probe the $AdS_5 \times S^5$ geometry directly. Although, as discussed below, this relation implies the existence of an as yet unproved non-renormalization theorem for a certain class of correlation functions (like those discussed in Section 9.2) protecting them

[69] There appears to be a mismatch of g^8; however, this is due to our normalization of the Grassmann collective coordinates. Re-scaling the supersymmetric and superconformal collective coordinates, ξ^A and $\bar{\eta}^A$, by $g^{1/2}$ produces the missing factor of g^8.

against perturbative corrections in g^2N. In what follows, we shall make this correspondence between large-N instanton and D-instanton effects even more convincing by considering these correlation functions.

In Section 10.3 we will establish a relation between Yang–Mills instantons and D-instantons which is apparently rather different from that we have just described. We will show in Section 10.3 how k D-instantons in the presence of N D3-branes are precisely identified with k Yang–Mills instantons in $\mathrm{U}(N)$ gauge theory describing the collective dynamics of the D3-branes. This is true in the limit $\alpha' \to 0$ for fixed g_{st} (in other words fixed coupling g on the D3-branes) in which bulk supergravity modes decouple from the world-volume theory of the D3-branes. The question is how this description in terms of D-instantons moving in the background of D3-branes relates to the description of D-instantons moving in the $AdS_5 \times S^5$ background (but with no D3-branes) established above? The answer is interesting because it illuminates some basic features of the AdS/CFT correspondence. In the limit of large N (with g^2N large and small g) the background of D3-branes is replaced by its near-horizon geometry, namely $AdS_5 \times S^5$. In a certain sense the D3-branes disappear to be replaced by non-trivial geometry. We can see this happening explicitly with the D-instanton/D3-brane system. The presence of the D3-branes in the D-instanton matrix theory is signalled by the fundamental hypermultiplet variables $\{w_{\dot\alpha}, \mu^A\}$, and their conjugates, describing open strings stretched between the D-instantons and D3-branes. At large N, these degrees of freedom can be integrated out in the way described in Section 9.1 to yield an effective collective coordinate integral (9.38) which exhibits the $AdS_5 \times S^5$ geometry explicitly. So the analogue of taking the near-horizon geometry is the process of integrating out the degrees of freedom of open strings stretched between the D-instantons and D3-branes and taking the large-N limit. The remaining puzzle is that this should be done at weak coupling, i.e. small g^2N, whereas the dual supergravity region of the AdS/CFT correspondence is valid at large g^2N. Yet again, this strongly suggests that some non-renormalization theorem in g^2N is a work: something we shall comment on in the next section.

9.3.2. Correlation functions

A more precise statement of the AdS/CFT correspondence is presented in Refs. [119,120]. The $\mathcal{N}=4$ gauge theory is to be thought of living on the four-dimensional boundary of AdS_5. In particular, each chiral primary operator $\mathcal{O}$ in the boundary conformal field theory is identified with a particular Kaluza–Klein mode of the supergravity fields which we denote as $\Phi_{\mathcal{O}}$. In general, it is not known how to solve string theory on an AdS background. However, the AdS/CFT correspondence is still useful because in a certain limit we can approximate the full string theory by its supergravity low-energy limit. This requires weak coupling (small g) but, in addition, the length scale L must be large compared with the string length scale $\sqrt{\alpha'}$. This latter requirement is met when the 't Hooft coupling g^2N of the gauge theory is large and conventional perturbation theory breaks down. In this limit [119,120], the generating function for the correlation functions of $\mathcal{O}$ is then given in terms of the supergravity action $S_{\mathrm{IIB}}[\phi_{\mathcal{O}}]$ according to

$$\left\langle \exp \int \mathrm{d}^4x\, J_{\mathcal{O}}(x)\mathcal{O}(x) \right\rangle = \exp - S_{\mathrm{IIB}}[\Phi_{\mathcal{O}}; J] \; . \tag{9.74}$$

The IIB action on the right-hand side of the equation is evaluated on a configuration which solves the classical field equations subject to the condition $\Phi_{\mathcal{O}}(x)=J_{\mathcal{O}}(x)$ on the four-dimensional boundary.

In most applications considered so far, relation (9.74) has primarily been applied at the level of classical supergravity, which corresponds to $N \to \infty$, with g^2N fixed and large, in the boundary theory [119–132]. However, the full equivalence of IIB superstrings on $AdS_5 \times S^5$ and $\mathcal{N}=4$ supersymmetric Yang–Mills theory conjectured in [112] suggests that (9.74) should hold more generally, with quantum and stringy corrections to the classical supergravity action corresponding to g^2 and $1/(g^2N)$ corrections in the $\mathcal{N}=4$ theory, respectively. The particular comparison that concerns us here is between Yang–Mills instanton contributions to the correlators of $\mathcal{O}$ generated by the left-hand side of (9.74) and D-instanton corrections to the IIB effective action on the right-hand side.

Let us describe these D-instanton effects in more detail. Before we look at some specific correlators, let us consider in more detail the effects of D-instantons in the string theory [118] (closely following the more detailed treatment in [117]). In [133], Green and Gutperle conjectured an exact form for certain non-perturbative (in $g_{\rm st}$) corrections to certain terms in the Type IIB supergravity effective action. In the present application, where the string theory is compactified on $AdS_5 \times S^5$, it is important for the overall consistency of the Banks–Green prediction that the non-perturbative terms in the effective action do not alter the $AdS_5 \times S^5$ background, since the latter is conformally flat [118]. In particular, at leading order beyond the Einstein–Hilbert term in the derivative expansion, the IIB effective action is expected to contain a totally anti-symmetric 16-dilatino effective vertex of the form [134,135]

$$(\alpha')^{-1} \int \mathrm{d}^{10}x \sqrt{\det g}\, \mathrm{e}^{-\phi/2} f_{16}(\tau,\bar\tau)\, \Lambda^{16} + \text{h.c}\ . \tag{9.75}$$

Here Λ is a complex chiral SO(9, 1) spinor, and f_{16} is a certain weight $(12,-12)$ modular form under $(2,\mathbb{Z})$. At the same order in the derivative expansion there are other terms related to (9.75) by supersymmetry and involving other modular forms [134–136] and, in particular, $f_n(\tau,\bar\tau)$, with $n=4$ and 8. The modular symmetry is precisely S-duality of Type IIB superstring, and although this does not completely determine the modular forms f_n, for the $n=4$ term, Green and Gutperle [133] made the following conjecture, later proved in Ref. [137] and generalized to $n\neq 4$ in [135,136]:

$$f_n(\tau,\bar\tau) = (\mathrm{Im}\,\tau)^{3/2} \sum_{(p,q)\neq(0,0)} (p+q\bar\tau)^{n-11/2}(p+q\tau)^{-n+5/2}\ . \tag{9.76}$$

These rather arcane expressions turn out to have the right modular properties, i.e. weight $(n-4,-n+4)$, and also have very suggestive weak-coupling expansions [133–135,138]:

$$\mathrm{e}^{-\phi/2} f_n = 32\pi^2\zeta(3) g^{-4} - \frac{2\pi^2}{3(9-2n)(7-2n)} + \sum_{k=1}^{\infty} \mathcal{G}_{k,n}\ , \tag{9.77}$$

where

$$\begin{aligned}\mathcal{G}_{k,n} = {}& \left(\frac{8\pi^2 k}{g^2}\right)^{n-7/2} \left\{\sum_{d|k} \frac{1}{d^2}\right\} \left[\mathrm{e}^{-(8\pi^2/g^2 - \mathrm{i}\theta)k} \sum_{j=0}^{\infty} c_{4-n,j-n+4} \left(\frac{g^2}{8\pi^2 k}\right)^{j}\right.\\ &\left. + \mathrm{e}^{-(8\pi^2/g^2+\mathrm{i}\theta)k} \sum_{j=0}^{\infty} c_{n-4,j+n-4} \left(\frac{g^2}{8\pi^2 k}\right)^{j+2n-8}\right]\end{aligned} \tag{9.78}$$

and the numerical coefficients are

$$c_{n,r} = \frac{(-1)^n\sqrt{8\pi}\,\Gamma(3/2)\Gamma(r-1/2)}{2^r\Gamma(r-n+1)\Gamma(n+3/2)\Gamma(-r-1/2)} \, . \tag{9.79}$$

As previously, the summation over d in (9.78) runs over the integral divisors of k. Notice that, having taken into account the conjectured correspondence (9.69) to the couplings of four-dimensional Yang–Mills theory, series (9.77) has the structure of a semi-classical expansion: the first two terms correspond to the tree and one-loop pieces, while the sum on k is interpretable as a sum on Yang–Mills instanton number, the first and second terms in the square bracket being instantons and anti-instantons, respectively (as dictated by the θ dependence). Each of these terms includes a perturbative expansion around the instantons, although notice that the leading-order anti-instanton contributions are suppressed by a factor of g^{4n-16} (so not suppressed for $n=4$) relative to the leading-order instanton contributions.

Let us focus on the leading semi-classical contributions to the f_n; by this we mean, for each value of the topological number k, the leading-order contribution in g^2. For f_{16} and f_8 the leading semi-classical contributions come from instantons only and have the form

$$e^{-\phi/2} f_n|_{k\text{-instanton}} = \text{const}\left(\frac{k}{g^2}\right)^{n-7/2} e^{2\pi i k\tau}\sum_{d|k}\frac{1}{d^2} \, , \tag{9.80}$$

neglecting g^2 corrections. For the special case of f_4 there is an identical anti-instanton contribution with $i\tau \to -i\bar{\tau}$. Comparing with (9.73), we see that the terms in the square bracket in (9.78), which are non-perturbative in the string coupling, are interpreted as being due to D-instantons.

From effective vertex (9.75) one can construct Green's functions $G_{16}(x^{(1)},\dots,x^{(16)})$ for 16 dilatinos $\Lambda(x^{(l)})$, $1 \leqslant l \leqslant 16$, which live on the boundary of AdS_5:

$$\begin{aligned} G_{16} &= \langle \Lambda(x^{(1)}) \times \cdots \times \Lambda(x^{(16)})\rangle \sim (\alpha')^{-1} e^{-\phi/2} f_{16}\, t_{16} \\ &\quad \times \int \frac{d^4X\, d\rho}{\rho^5}\prod_{l=1}^{16} K^F_{7/2}(X,\rho;x^{(l)},0) \end{aligned} \tag{9.81}$$

suppressing spinor indices. Here $K^F_{7/2}$ is the bulk-to-boundary propagator for a spin-$\frac{1}{2}$ Dirac fermion of mass $m = -\frac{3}{2}L^{-1}$ and scaling dimension $\Delta = \frac{7}{2}$ [119,120,123,139]:

$$K^F_{7/2}(X,\rho;x,0) = K_4(X,\rho;x,0)\,(\rho^{1/2}\gamma_5 - \rho^{-1/2}(x-X)_n\gamma^n) \tag{9.82}$$

with

$$K_4(X,\rho;x,0) = \frac{\rho^4}{(\rho^2+(x-X)^2)^4} \, . \tag{9.83}$$

In these expressions the $x^{(l)}$ are four-dimensional spacetime coordinates for the boundary of AdS_5 while ρ is the fifth radial coordinate. The quantity t_{16} in Eq. (9.81) is (in the notation of Ref. [117]) a 16-index anti-symmetric invariant tensor which enforces Fermi statistics and ensures, inter alia, that precisely 8 factors of $\rho^{1/2}\gamma_5$ and 8 factors of $\rho^{-1/2}\gamma^n$ are picked out in the product over $K^F_{7/2}$. It is not difficult to see that the form of (9.81) matches precisely the large-N instanton contribution to the 16-point correlator in gauge theory (9.62). In particular, the dilatino corresponds, on the field theory side, to the fermionic composite operator defined in (9.59) [117]. This shows that not only does the form of the D-instanton collective coordinate integral match the large-N instanton collective coordinate integral, but, in addition, the various bulk-to-boundary propagators on the supergravity side arise in the field theory as operator insertions evaluated in the large-N instanton background. In the case of G_{16}, and related correlators, because of the nature of the saddle-point solution, the insertions have the form of k times the same insertion in the theory with gauge group SU(2) at the one-instanton level (precisely the case considered in [117]).

Of course this equivalence between D-instanton and Yang–Mills instanton effects in correlation functions extends to the other cases, G_8 and G_4, related by supersymmetry. Note that there is no explicit dependence in these expressions on the coordinates on S^5; in particular, the propagator does not depend on them. This is because G_{16}, G_8 and G_4 are correlators of operators whose supergravity associates are constant on S^5. Generalizations involving non-trivial dependence on S^5 are possible [21].

As emphasized above, the comparison between the Yang–Mills and supergravity descriptions can be quantitative if and only if there exists a non-renormalization theorem that allows one to relate the small g^2N to the large g^2N behaviour of chiral correlators such as G_n, as has been suggested in Ref. [140]. In the absence of such a theorem the best one can hope for is that qualitative features of the agreement persist beyond leading order while the exact numerical factor in each instanton sector does not, in analogy with the mismatch in the numerical pre-factor between weak- and strong-coupling results for black-hole entropy [120]. In our view, however, our results provide strong evidence in favour of such a non-renormalization theorem for the correlators G_n, for the following reason. Consider the planar diagram corrections to the leading semi-classical (i.e. $g^2N \to 0$) result for, say, G_{16}, Eq. (9.62). In principle, these would not only modify the above result by an infinite series in g^2N, but also, at each order in this expansion, and independently for each value of k, they would produce a different function of spacetime. The fact that the leading semi-classical form for G_{16} that we obtain is not only k-independent, but already reproduces the spacetime dependence of the D-instanton/supergravity prediction exactly, strongly suggests that such diagrammatic corrections (planar and otherwise) must vanish. (Note there are necessary subleading corrections, both in $1/N$ and in g^2, to our leading semi-classical results.) By evaluating the possible form of non-perturbative corrections on the string theory side of the correspondence, it has been argued that the non-renormalization theorem is rather natural [141]; however, there is still no purely field theoretic proof.

It is interesting to generalize the relation of large-N instantons in other conformal gauge theories to the D-instantons via the more general AdS/CFT correspondences. This has been done for the finite $\mathcal{N}=2$ theories with product gauge group $\mathrm{SU}(N)^k$ in [111], the $\mathcal{N}=4\mathrm{Sp}(N)$ and $\mathrm{SO}(N)$ theories in [111] and for the finite $\mathcal{N}=2$ $\mathrm{Sp}(N)$ theories in [109,110].

The finite $\mathcal{N}=2$ theory with gauge group $\mathrm{SU}(N)$ and $2N$ fundamental hypermultiplets has not yet been discussed in this respect. However, it is not difficult to find a set-up which can be used to discuss this case. Here we sketch the details. The idea is to consider the finite $\mathcal{N}=2$ theory with product gauge group $\mathrm{SU}(N)\times\mathrm{SU}(N)$ and matter in bi-fundamental representations of the gauge group. This theory can be realized, as explained in Section 10.2.5, on the world-volume of N D3-branes lying transverse to the orbifold $\mathbb{R}^2 \times (\mathbb{R}^4/\mathbb{Z}_2)$, where $\mathbb{Z}_2$ acts by inversion. The near-horizon geometry of the D3-branes is $AdS_5 \times S^5/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on S^5, realized as $x_a^2 = 1$ by taking $x^3, x^4, x^5, x^6 \to -x^3, -x^4, -x^5, -x^6$, leaving x^1 and x^2 fixed. So the AdS/CFT correspondence in this case involves Type IIB string theory on the $AdS_5 \times S^5/\mathbb{Z}_2$ background. Notice that the $\mathbb{Z}_2$ leaves an $S^1 \subset S^5$ fixed and so in this case the dual geometry is not smooth. Now consider instanton effects in one of the SU(N) gauge group factors. At leading order in such an instanton background, the adjoint-valued fields of the other gauge remain zero and so the instanton effects in question are identical to the SU(N) theory with $2N$ hypermultiplets (the latter arising from the bi-fundamental hypermultiplets of the original theory). On the string theory side, these instanton effects of charge k will be related to k fractional D-instantons. These fractional D-instantons are to be thought of as D-strings which are wrapped around the non-trivial S^2 of the smooth resolution of the orbifold $\mathbb{R}^4/\mathbb{Z}_2$ in the limit that the S^2 cycle collapses. In the near-horizon geometry, these fractional instantons are consequently stuck at the orbifold singularity of $S^1 \subset S^5/\mathbb{Z}_2$. So the fractional D-instanton collective coordinate integral involves an integral over the six-dimensional space $AdS_5 \times S^1$ rather than 10-dimensions. Furthermore, this measure includes, in analogy with (9.72), a partition function of six-dimensional SU(k) gauge theory dimensionally reduced to zero dimensions. This is precisely what is seen in the large-N instanton collective coordinate integral (9.51). It would be interesting to extend the analysis to particular correlation functions in order to place the relationship on the same footing as the $\mathcal{N} = 4$ case.

10. Instantons as solitons in higher dimensions and string theory

Instantons are classical solutions of Euclidean gauge theories in four-dimensional spacetime with finite action. They are consequently localized in *spacetime* rather than space. However, it will turn out to be rewarding to think about instantons in gauge theories in dimensions greater than four. In these other contexts, if the higher-dimensional theories are defined in Minkowski space then the instanton solutions have an interpretation as solitons. For example, consider five-dimensional gauge theories. We can easily take an instanton solution of four-dimensional gauge theory and embed it in the five-dimensional theory, by identifying the coordinates x_n, $n = 1$–4, on which the instanton solution depends, with the Euclidean subspace of the five-dimensional Minkowski space [70] coordinates y^N, N=0–4, with $y_n \equiv x_n$, n=1–4. The resulting solution of the five-dimensional Yang–Mills equations is then independent of the time coordinate y^0. The solution, which had finite action in four dimensions, now has finite energy in five dimensions and so is interpreted as a lump localized in space but evolving in time. In other words, the instanton is a particle-like soliton in five dimensions. Adding an extra dimension yields a much richer system: instantons in four-dimensional gauge theories have no dynamics, but when lifted to five dimensions they can move in the four-dimensional space and have complicated dynamics.

We can continue this process of embedding instantons in gauge theories of even higher dimension. In six dimensions, the embedded instanton solutions are independent of two coordinates and so, thinking of one of them as a time direction, means the solution is extended in a single space direction; in other words, it is a string-like "defect". In seven dimensions, one gets a membrane and

[70] In Minkowski space we choose a metric $\eta_{MN} = \mathrm{diag}(-1, 1, \ldots, 1)$.

so on. Let us introduce some modern terminology, arising originally in the context of supergravity and string theory, which affords us a certain brevity in talking about these more exotic possibilities. Consider gauge theories in D-dimensional Minkowski space. An instanton embedded in this theory will be independent of $D-4$ coordinates (one being the time). This means that the solution will be extended in $D-5$ spatial dimensions. Such a configuration is called a "$(D-5)$-brane". The case with $D=4$, describes a solution which is localized in both space and time dimensions and this is precisely the situation for the original instanton of four-dimensional gauge theory. This case is therefore a "(-1)-brane". For $D=5$, the solution is localized in space, but not time, and so the instanton lump-like soliton of five-dimensional gauge theory is a 0-brane. It may seem strange to think about instanton solutions in higher-dimensional gauge theories but one of the major lessons of string theory and its generalizations is that it is compulsory to think in arbitrary numbers of dimensions and big advantages ensue.

We will show how the instanton moduli space plays an important role in describing the dynamics of instanton branes in higher-dimensional gauge theories. In Section 10.1.1 we begin by discussing the case of a pure gauge theory. Supersymmetric generalizations will be considered in Section 10.2. It is worth pointing out that much of the analysis for instantons is very closely related to the problem of describing the semi-classical behaviour of monopoles in supersymmetric gauge theories described in Refs. [142–144], and references therein. In particular, the description of instanton lumps in five-dimensional gauge theory with either 4 or 8 supercharges (corresponding to $\mathcal{N}=2$ and 4 supersymmetry in four dimensions) is closely related to the description of monopoles in four-dimensional gauge theories, with $\mathcal{N}=2$ and 4 supersymmetry, respectively, since both involve a quantum mechanical σ-model on the appropriate moduli space. In both cases, the instanton and monopole moduli space are both hyper-Kähler spaces and so the structure of the σ-models is identical. The main difference is that the σ-model in the instanton case admits a straightforward linear realization via the ADHM hyper-Kähler quotient construction, while in the monopole case the analogous quotient construction based on Nahm's equations is somewhat more complicated.

After we have described the dynamics of instanton branes, we will, in Section 10.3 describe how such objects appear naturally in string theory. In this context they correspond to D-branes dissolved within other higher-dimensional D-branes. We will show how solving for the low-energy dynamics of these configurations of D-branes actually leads directly to the instanton calculus. Not only is the ADHM construction obtained directly, but also the leading-order expression for the collective coordinate integral for the $\mathcal{N}=4$ and 2 supersymmetric theories. We also briefly describe how the actual profile of the instanton can be obtained by using a suitable probe brane.

10.1. Non-supersymmetric instanton branes

We begin with the case of pure gauge theory. The key idea, which originated with Manton [145] and was elaborated by Ward in the context of a $2+1$-dimensional model [146], is that the dynamics of slowly moving solitons can be approximated by assuming that the evolution is adiabatic in the moduli space of classical solutions. In many cases involving lumps, the time evolution of the system is governed by the geodesic motion on the moduli space with respect to the Levi–Civita connection induced by the metric that arises as the inner product of zero modes (2.32). The intuitive idea that lies behind this approximation is that, for sufficiently low velocities, the non-zero modes of the fields are only very weakly excited and therefore it is consistent to ignore these modes to leading order.

In our case we start with $D > 4$-dimensional gauge theory in Minkowski space with coordinates y^M, $M = 0, 1, \ldots, D-1$, having the conventional action [71]

$$S = \frac{1}{2g_D^2} \int \mathrm{d}^D y \, \mathrm{Tr}_N \, F_{MN} F^{MN} \ . \tag{10.1}$$

Here, g_D is the gauge coupling in D dimensions which has dimensions of $[L]^{(D-4)/2}$.

We now consider how to embed an instanton in the D-dimensional theory. To this end, we decompose D-dimensional Minkowski space into $D-4$-dimensional Minkowski space and four-dimensional Euclidean space, by taking $y^M = (\xi^a, x_m)$, $a = 0, \ldots, p$, where $p = D-5$, and $m = 1$–4. We now embed the instanton solution in the subspace x_m by taking

$$A_M(y) = (0, A_m(x; X)) \ , \tag{10.2}$$

where $A_m(x; X)$ is the instanton solution constructed in Section 2 (with coupling g set to 1). Recall that X^μ denote the collective coordinates of $\mathfrak{M}_k$. This ansatz obviously satisfies the equations-of-motion of the D-dimensional theory and represents an object extended in $p = D-5$ space dimensions, i.e. a p-brane.

10.1.1. The moduli space approximation

Manton's moduli space approximation amounts to modelling the dynamics of the p-brane by allowing dependence on the transverse coordinates ξ^a (so including time ξ^0) to enter implicitly through the collective coordinates. Of course, the original classical solution with ξ^a-dependent collective coordinates $A_N(0, x; X(\xi))$ does *not* satisfy the classical equations-of-motion. The idea is that for sufficiently slowly varying $X^\mu(\xi)$ it is *almost* a solution. The point can be illustrated with the position coordinates of the centre of the instanton X_n. In this case we know the exact solution for motion in time $t \equiv \xi^0$ with constant velocity because we can Lorentz-boost the static solution. The boosted solution only approximates $A_N(0, x; X_n = v_n t)$ if the velocity $|v| \ll 1$.

The moduli space approximation is an expansion in powers of ξ^a derivatives. In order to find the leading-order effective dynamics we have to substitute the instanton solution with ξ^a-dependent collective coordinates into the action. This then yields an effective action for the collective coordinates now interpreted as fields on the $p+1$-dimensional world volume of the p-brane: $X^\mu(\xi)$. In order that the effective action is at least quadratic order in the ξ^a derivatives, we must ensure that the equations-of-motion are satisfied to linear order in the ξ^a derivatives. In components (A^a, A_n), the equations-of-motion, $\mathscr{D}^N F_{MN} = 0$, are

$$\mathscr{D}_m F_{mn} + \mathscr{D}^a(\partial_a A_n - \mathscr{D}_n A_a) = 0 \ , \tag{10.3a}$$

$$\mathscr{D}_n(\mathscr{D}_n A_a - \partial_a A_n) + \mathscr{D}^b F_{ab} = 0 \ . \tag{10.3b}$$

Now we substitute in the ξ-dependent instanton solution $A_n = A_n(x; X(\xi))$. To linear order in ξ^a we can ignore the second term in (10.3a) and the second term in (10.3b); this leaves

$$\left(\frac{\partial A_n}{\partial X^\mu} \, \partial_a X^\mu - \mathscr{D}_n A_a \right) = 0 \tag{10.4}$$

[71] In this and following sections, we will choose a normalization for the fields where the coupling constant appears outside the action. So the field strength, for instance, has no factor of g in front of the commutator: $F_{MN} = \partial_M A_N - \partial_N A_M + [A_M, A_N]$.

for A_a. Notice that this equation is very similar to the background gauge condition for an instanton zero mode (2.31). Indeed, if we take [72]

$$A_a = \Omega_\mu \partial_a X^\mu \ , \tag{10.5}$$

where Ω_μ is the compensating gauge transformation associated to the collective coordinate X^μ, then (10.4) is satisfied by virtue of (2.31). Since A_a is linear in $\partial_a X^\mu$, it is then trivial to see that the ansatz

$$A_N = (\Omega_\mu \partial_a X^\mu(\xi), A_n(x; X(\xi))) \tag{10.6}$$

satisfies equations-of-motion (10.3a) and (10.3b) to linear order in ξ^a-derivatives. In making this ansatz, we have fixed the gauge symmetry of the theory in a rather unconventional way. We are working in the gauge where $A_0 = \Omega_\mu \partial_0 X^\mu$. Of course we could always return to a more familiar gauge, say $A_0 = 0$, by performing a gauge transformation on the ansatz.

Since the ξ^a-dependent expression (10.6) is only a solution of the equations-of-motion to linear order in ξ^a derivatives, it will now contribute non-trivially to the action at quadratic order:

$$\begin{aligned} S^{(2)} &= \frac{1}{g_D^2} \int \mathrm{d}^{p+1}\xi \, \mathrm{d}^4 x \, \mathrm{tr}_N \, \delta_\mu A_n(x; X(\xi)) \partial^a X^\mu(\xi) \, \delta_\nu A_n(x; X(\xi)) \partial_a X^\nu(\xi) \\ &= -\frac{1}{2g_D^2} \int \mathrm{d}^{p+1}\xi \, g_{\mu\nu}(X) \partial^a X^\mu \partial_a X^\nu \ , \end{aligned} \tag{10.7}$$

where $g_{\mu\nu}(X)$ is the metric tensor on the instanton moduli space defined in (2.32).

The effective collective coordinate dynamics embodied in (10.7) has the form of a σ-model in $p+1$ dimensions whose target space is the moduli space $\mathfrak{M}_k$. The classical collective coordinate dynamics follows from the equation-of-motion

$$\partial^a \partial_a X^\mu + \tfrac{1}{2} g^{\mu\nu} \frac{\partial g_{\rho\sigma}}{\partial X^\nu} \partial^a X^\rho \partial_a X^\sigma = 0 \ . \tag{10.8}$$

For the case with $D=5$, where the instanton is a lump in five dimensions, this is simply the equation for geodesics in $\mathfrak{M}_k$.

10.2. Supersymmetric instanton branes

In this section, we will investigate how the description of instanton branes extends to the supersymmetric theories. Necessarily the theories must have extended supersymmetry from the four-dimensional perspective, because the $\mathcal{N}=1$ theory cannot be obtained by dimensional reduction from higher dimensions. Since theories with the same number of supercharges are related by dimensional reduction, we lose no loss of generality by considering the theories in their maximal dimension: $D=10$ for the theory having 16 supersymmetries ($\mathcal{N}=4$ in $D=4$) and $D=6$ for the theory having 8 supersymmetries ($\mathcal{N}=2$ in $D=4$). In the last section, we have shown how the

[72] The following discussion and its supersymmetric generalization is motivated by the treatment of the analogous monopole problem in Refs. [142,144].

moduli space approximation involves a $p+1=D-4$-dimensional σ-model with $\mathfrak{M}_k$ as target. In the supersymmetric case, we expect the moduli space approximation to lead to a supersymmetric σ-model with the appropriate number of supersymmetries. Since an instanton breaks half the supersymmetries of the parent theory (see Section 4.2.3), we expect this σ-model to manifest half the supersymmetries, i.e. have 8 supercharges, in the $p=5$ case, and 4 supercharges in the $p=1$ case. Furthermore, the supersymmetry in both cases is chiral, being $\mathcal{N}=(0,1)$ in six spacetime dimensions and $\mathcal{N}=(0,4)$ in two spacetime dimensions, respectively. It is a well-established fact that σ-models with the relevant kinds of supersymmetry only exist in these dimensions if the target space is hyper-kähler [147–149]. In some respects, the very fact that instantons can be embedded as branes in the higher-dimensional supersymmetric gauge theories "proves" that the instanton moduli space $\mathfrak{M}_k$ has to be hyper-Kähler.

What is interesting about these σ-models is that on Wick rotation and complete dimensional reduction to zero dimensions their partition functions reproduce the leading-order semi-classical expression for the collective coordinate integral of the instanton calculus. This includes both the volume form on the instanton moduli space as well as the non-trivial instanton effective action. In fact the σ-models have both a non-linear realization, where the target space is directly $\mathfrak{M}_k$, and also, importantly, a *linear* realization. In this second formulation there is an auxiliary supermultiplet involving a non-dynamical $\mathrm{U}(k)$ gauge field. It turns out that this linear formulation is directly related to the ADHM construction, where the bosonic and fermionic ADHM constraints arise from integrating out Lagrange multiplier fields. However, there is more. When one dimensionally reduces these σ-models there are certain kinds of deformation which can be added in the form of very special potentials [143,150,151]. These potentials naturally arise in the instanton calculus when one moves onto the Coulomb branch of the original theory and correspond to the adjoint-valued VEVs described in Section 2.5. In the two-dimensional σ-model example, we also have the freedom to add other fields to the σ-model which correspond in the $\mathcal{N}=2$ instanton calculus to the Grassmann collective coordinates of hypermultiplet matter fields. We will subsequently show how these σ-models arise very naturally in the context of D-branes in string theory.

10.2.1. Action, supersymmetry and equations-of-motion

The actions of the $D=6$ and 10 theories with 8 and 16 supercharges, respectively, can both be written as

$$S=\frac{1}{g_D^2}\int d^D y\,\mathrm{tr}_N\left\{\frac{1}{2}F_{MN}F^{MN}-\mathrm{i}\bar{\Psi}\Gamma^M\mathscr{D}_M\Psi\right\}\ . \tag{10.9}$$

The theory is invariant under the supersymmetry transformations

$$\delta A_N=-\bar{\Xi}\,\Gamma_N\Psi\ , \tag{10.10a}$$

$$\delta\Psi=\mathrm{i}\Gamma^{MN}\Xi F_{MN}\ . \tag{10.10b}$$

As in the purely bosonic case, $y^M=(\xi^a,x_m)$, where ξ^a, $a=0,1,\ldots,p$, are $p+1=D-4$-dimensional Minkowski space coordinates and x_m, $m=1$–4, are four-dimensional Euclidean coordinates.

The spectrum of fields in both cases consists of a gauge field $A_N=(A_a,A_n)$, with $a=0,1$ in $D=6$ and $a=0,1,\ldots,5$ in $D=10$, and the minimal spinor Ψ. (Our conventions for spinors are described in Appendix A.) In $D = 6$ this is a Weyl, while in $D = 10$ it is a Majorana–Weyl, spinor. In both cases, it is convenient to represent the Γ-matrices using a tensor product notation that reflects the subgroups of the Lorentz group, $\mathrm{SO}(4) \times \mathrm{SO}(1,1)$ and $\mathrm{SO}(4) \times \mathrm{SO}(5,1)$, respectively:

$$\Gamma_N = \{\Gamma_a \otimes \gamma_5, 1 \otimes \gamma_n\} , \tag{10.11}$$

where γ_n are the SO(4) γ-matrices, (A.7), and Γ_a are the SO(1,1) and SO(5,1) Γ-matrices, in the two cases, respectively. In both cases we write

$$\Gamma_a = \begin{pmatrix} 0 & \Sigma_a \\ \bar{\Sigma}_a & 0 \end{pmatrix} , \tag{10.12}$$

where, for SO(1,1), $\Sigma_a = (1,1)$ and $\bar{\Sigma}_a = (-1,1)$ and for SO(5,1) the Σ-matrices are defined in (A.19).

In both $D = 6$ and 10, in the tensor product notation (10.11), a Weyl spinor can be written

$$\Psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \lambda_\alpha + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \bar{\lambda}^{\dot\alpha} , \tag{10.13}$$

where the α and $\dot\alpha$ are four-dimensional Euclidean space spinor indices. In the $D = 6$ case, Ψ is a pseudo-real spinor and $\lambda_{\dot\alpha}$ and $\bar{\lambda}^{\dot\alpha}$ are independent complex quantities. It is convenient, in this case, to introduce the notion of a two-component symplectic real spinor (see Appendix A). In terms of the component spinors, we define $\lambda^A_{\dot\alpha}$ and $\bar{\lambda}^{\dot\alpha}_A$, $A = 1,2$, via

$$\lambda^1_\alpha = \tfrac{1}{\sqrt{2}}\varepsilon_{\alpha\beta}(\lambda_\beta)^\dagger, \qquad \lambda^2_\alpha = \tfrac{1}{\sqrt{2}}\lambda_\alpha, \qquad \bar{\lambda}^{\dot\alpha}_1 = \tfrac{1}{\sqrt{2}}\bar{\lambda}^{\dot\alpha}, \qquad \bar{\lambda}^{\dot\alpha}_2 = -\tfrac{1}{\sqrt{2}}\varepsilon^{\dot\alpha\dot\beta}(\bar{\lambda}^{\dot\beta})^\dagger . \tag{10.14}$$

The reason for choosing these definitions will emerge shortly. To accompany the two-component spinors we also define the Σ-matrices

$$\Sigma^{AB}_a = \Sigma_a \varepsilon^{AB}, \qquad \bar{\Sigma}_{aAB} = \bar{\Sigma}_a \varepsilon_{AB} . \tag{10.15}$$

By virtue of (10.14), the two-component spinors satisfy the pseudo-reality conditions

$$(\lambda^A)^\dagger = \bar{\Sigma}^0_{AB}\lambda^B, \qquad (\bar{\lambda}_A)^\dagger = \Sigma^{0AB}\bar{\lambda}_B . \tag{10.16}$$

In the $D = 10$ case, Ψ is subjected to a Majorana spinor condition, which means in terms of the component spinors, λ^A and $\bar{\lambda}_A$, $A = 1$–4, that

$$(\lambda^A)^\dagger = \bar{\Sigma}^0_{AB}\lambda^B, \qquad (\bar{\lambda}_A)^\dagger = \Sigma^{0AB}\bar{\lambda}_B . \tag{10.17}$$

Notice that this is identical in notion to the reality condition for the symplectic real spinor in $D = 6$ case (10.16) (and explains our choice of definition).

In terms of the spinors λ and $\bar{\lambda}$, action (10.9) is

$$S = \frac{1}{g_D^2} \int \mathrm{d}^D y \, \mathrm{tr}_N \{ \tfrac{1}{2} F_{MN}F^{MN} + 2\mathscr{D}_n\bar{\lambda}_A\bar{\sigma}_n\lambda^A - \mathrm{i}\bar{\lambda}_A\Sigma^{aAB}\mathscr{D}_a\bar{\lambda}_B - \mathrm{i}\lambda^A\bar{\Sigma}^a_{AB}\mathscr{D}_a\lambda^B \} . \tag{10.18}$$

Our definitions have been chosen so that for ξ-independent field configurations, the Lagrangian density is identical to *minus* that in (4.13) with the relation $A_a = \mathrm{i}\phi_a$ and with the metric on the a-indices changed from a Euclidean to Minkowski signature. The subgroups of the Lorentz group, SO(1,1) and SO(5,1), respectively, are identified, after the signature change, with part of the R-symmetry group of the four-dimensional theory, namely SO(2) and SO(6), respectively. The other difference is the reality condition on the spinors: compare (10.16) or (10.17) with the situation in the four-dimensional context where the spinors λ^A and $\bar{\lambda}_A$ are independently real.

The equations-of-motion which follow form (10.18) are

$$\mathscr{D}_m F_{mn} + \mathscr{D}^a F_{an} = 2\bar{\sigma}_n\{\lambda^A, \bar{\lambda}_A\} , \tag{10.19a}$$

$$\bar{\not{\mathscr{D}}}\lambda^A = -\mathrm{i}\Sigma^{aAB}\mathscr{D}_a\bar{\lambda}_B , \tag{10.19b}$$

$$\not{\mathscr{D}}\bar{\lambda}_A = -\mathrm{i}\bar{\Sigma}^a_{AB}\mathscr{D}_a\lambda^B , \tag{10.19c}$$

$$\mathscr{D}_n F_{na} + \mathscr{D}^b F_{ab} = \mathrm{i}\bar{\Sigma}_{aAB}\lambda^A\lambda^B + \mathrm{i}\Sigma_a^{AB}\bar{\lambda}_A\bar{\lambda}_B . \tag{10.19d}$$

10.2.2. The moduli space approximation

For the case of pure gauge theory, the effective moduli space dynamics involved an expansion in ξ_a derivatives. When there are fermion fields involved, we also allow the associated Grassmann collective coordinates to depend on ξ^a. The moduli space approximation is then an expansion in $n = n_\partial + \frac{1}{2}n_f$: the number of ξ^a derivatives plus half the number of Grassmann collective coordinates. The lowest non-trivial terms in the effective action are those terms of order $n = 2$ and in order to derive them we must solve the equations-of-motions up to order $n = 1$.[73]

The equations to order $n=0$ are solved, as in Section 10.1, by embedding the instanton solution as $A_N = (0, A_n(x; X(\xi)))$. At the next order, $n = \frac{1}{2}$, the fermions λ^A satisfy the covariant Weyl equation (4.24a) with the solution $\lambda^A = \Lambda(\mathscr{M}^A)$, where $\mathscr{M}^A$ are the Grassmann collective coordinates. In order to extend the notion of the moduli space approximation to the supersymmetric theories, we should also allow the Grassmann collective coordinates $\mathscr{M}^A$ to depend on ξ^a.

At order $n = 1$, we have the following equation (generalizing (10.4)):

$$\mathscr{D}_n\left(\frac{\partial A_n}{\partial X^\mu}\,\partial_a X^\mu - \mathscr{D}_n A_a\right) = -\mathrm{i}\bar{\Sigma}_{aAB}\lambda^A\lambda^B . \tag{10.20}$$

As in the case of pure gauge theory, we can use the background gauge condition on the zero modes of A_n to solve these equations for A_a. The new ingredient is the presence of the source bi-linear in fermion zero modes on the right-hand side. However, as the equation is linear in A_a we can take the linear combination

$$A_a(x; X(\xi), \mathscr{M}^A(\xi)) = \Omega_\mu(x; X(\xi))\partial_a X^\mu(\xi) + \mathrm{i}\phi_a(x; X(\xi), \mathscr{M}^A(\xi)) , \tag{10.21}$$

where the Hermitian field ϕ_a satisfies

$$\mathscr{D}^2\phi_a = \bar{\Sigma}_{aAB}\lambda^A\lambda^B . \tag{10.22}$$

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[73] Note that the possible cross-terms between fields of lower order, $n = 0$ and $1/2$, and fields of higher order, $n = 3/2$ and 2, which potentially could contribute to the effective action at orders $\leqslant 2$, actually vanish by the equations-of-motion.

Fortunately, we have already solved the covariant Laplace equation in the ADHM background of a bi-fermion source (see Section 6.3 and Appendix C, Eq. (C.25)).

10.2.3. The effective action

We now substitute our solution into the action of the theory and extract the terms of order $n=2$. At this order, the terms which contribute to the effective action are

$$S^{(2)} = \frac{1}{g_D^2}\int d^D y\,\mathrm{tr}_N\,\{F_{na}F^{na} - i\lambda^A\bar{\Sigma}^a_{AB}\mathscr{D}_a\lambda^B\}\ . \tag{10.23}$$

We now evaluate this expression. First of all, we have

$$F_{na} = \mathscr{D}_n A_a - \partial_a A_n = i\mathscr{D}_n\phi_a - \delta_\mu A_n \partial_a X^\mu \tag{10.24}$$

and, writing $\lambda^B = \Lambda(\mathscr{M}^B)$,

$$\begin{aligned}\lambda^A\bar{\Sigma}^a_{AB}\mathscr{D}_a\lambda^B &= \lambda^{\alpha A}\bar{\Sigma}^a_{AB}\left\{\Lambda_\alpha(\partial_a\mathscr{M}^B) + \frac{\partial\lambda^B_\alpha}{\partial X^\mu}\partial_a X^\mu - \Lambda_\alpha\left(\frac{\partial\mathscr{M}^B}{\partial X^\mu}\right)\partial_a X^\mu + [A_a,\lambda^B]\right\}\\ &= \lambda^{\alpha A}\bar{\Sigma}^a_{AB}\left\{\Lambda_\alpha(\partial_a\mathscr{M}^B) + \left(\frac{\partial\lambda^B_\alpha}{\partial X^\mu} + [\Omega_\mu,\lambda^B_\alpha]\right.\right.\\ &\qquad\left.\left. -\Lambda_\alpha\left(\frac{\partial\mathscr{M}^B}{\partial X^\mu}\right)\right)\partial_a X^\mu + i[\phi_a,\lambda^B_\alpha]\right\}\ .\end{aligned} \tag{10.25}$$

Consequently, using gauge condition (2.24), action (10.23) becomes

$$\begin{aligned}S^{(2)} = \frac{1}{g_D^2}\int d^D y\,\mathrm{tr}_N\ &\{\delta_\mu A_n\partial^a X^\mu\delta_\nu A_n\partial_a X^\nu - i\lambda^A\bar{\Sigma}^a_{AB}\Lambda(\partial_a\mathscr{M}^B)\\ &-i\lambda^A\bar{\Sigma}^a_{AB}\left(\frac{\partial\lambda^B}{\partial X^\mu} + [\Omega_\mu,\lambda^B] - \Lambda\left(\frac{\partial\mathscr{M}^B}{\partial X^\mu}\right)\right)\partial_a X^\mu\\ &-\mathscr{D}_n\phi^a\mathscr{D}_n\phi_a + \lambda^A\bar{\Sigma}^a_{AB}[\phi_a,\lambda^B]\}\ .\end{aligned} \tag{10.26}$$

The first two terms can be evaluated using inner-product formulae (2.32) and (4.37). The third term can be evaluated using identity (C.51) established in Appendix C:

$$\frac{\partial\lambda^A}{\partial X^\mu} + [\Omega_\mu,\lambda^A] = \not{D}\bar{\varrho}^A_\mu + \Lambda\left(\frac{\partial\mathscr{M}^A}{\partial X^\mu}\right)\ , \tag{10.27}$$

where

$$\bar{\varrho}^{\dot\alpha A}_\mu = \frac{1}{4}\bar{U}\,\frac{\partial a^{\dot\alpha}}{\partial X^\mu}\,f\bar{\mathscr{M}}^A U\ . \tag{10.28}$$

Therefore the third term in (10.26) involves the integral

$$\int \mathrm{d}^4x\,\mathrm{tr}_N\,\lambda^A \not{\mathcal{D}}\bar{\varrho}^B_\mu\ . \tag{10.29}$$

We can now use the fact that $\bar{\not{\mathcal{D}}}\lambda^A = 0$ to write this as

$$\int \mathrm{d}^4x\,\partial_{\alpha\dot\alpha}(\mathrm{tr}_N\,\lambda^{\alpha A}\bar{\varrho}^{\dot\alpha B})\ . \tag{10.30}$$

Using the asymptotic formulae in Section 2.4.3, one sees that the surface term at infinity vanishes and so there is no contribution from the third term in (10.26). The final two terms (10.26) can be evaluated following an identical analysis to the construction of the instanton effective action in Section 5.2. Notice that this term vanishes in the $p=1$ case.[74]

Putting everything together, we have

$$S^{(2)} = \frac{1}{g_D^2}\int \mathrm{d}^{p+1}\xi\,\mathrm{tr}_k\left\{-\frac{1}{2}g_{\mu\nu}(X)\partial^a X^\mu\partial_a X^\nu - 2\mathrm{i}\pi^2\bar\Sigma^a_{AB}\bar\mu^A\partial_a\mu^B - \mathrm{i}\pi^2\bar\Sigma^a_{AB}\mathcal{M}'^A\partial_a\mathcal{M}'^B\right.$$
$$\left.+\frac{1}{2}\pi^2\varepsilon_{ABCD}\bar{\mathcal{M}}^A\mathcal{M}^B\boldsymbol{L}^{-1}\bar{\mathcal{M}}^C\mathcal{M}^D\right\}\ . \tag{10.31}$$

This expression is somewhat schizophrenic because the bosonic fields X^μ are the intrinsic coordinates while the fermionic fields $\mathcal{M}^A$ are the ADHM variables, subject to the fermionic ADHM constraints. In order to unify things we can proceed in two alternative ways, either writing everything in terms of quantities intrinsic to $\mathfrak{M}_k$ or in terms of the ADHM variables with the ADHM constraints explicitly imposed. Both viewpoints are worth developing.

First we shall consider the intrinsic expression. In Section 4.2.2, we introduced $\psi^{\mathrm{i}A}$, $\mathrm{i}=1,\dots,2kN$, the intrinsic Grassmann-valued symplectic tangent vectors to $\mathfrak{M}_k$. In (10.31), the terms quadratic in the fermions are written in terms of the intrinsic objects ψ^A as

$$\mathrm{i}\pi^2\bar\Sigma^a_{AB}(2\bar\mu^A\partial_a\mu^B + \mathcal{M}'^A\partial_a\mathcal{M}'^B) = 4\bar\Sigma^a_{AB}\tilde\Omega\left(\mathcal{M}(\psi^A,X),\mathcal{M}(\partial_a\psi^B,X) + \frac{\partial\mathcal{M}^B}{\partial X^\mu}\partial_a X^\mu\right) \tag{10.32}$$

using the linearity of $\mathcal{M}^A$ on ψ^A. The first term can be written in terms of the intrinsic symplectic matrix $\Omega_{ij}(X)$ defined in (4.39). The relevant geometrical quantity corresponding to the second term is the symplectic spin connection on $\mathfrak{M}_k$. In Appendix B we explain in the context of the hyper-Kähler quotient construction how the connection ∇ on the quotient space is inherited from the connection $\tilde\nabla$ on the mother space $\tilde{\mathfrak{M}}$ by orthogonal projection to $\mathscr{H}$. In a completely analogous way the spin connection for symplectic tangent vectors on $\mathfrak{M}_k$ is obtained from that of $\tilde{\mathfrak{M}}$ by projection via the fermionic ADHM constraints. So since $\mathcal{M}^A$ are subject to the fermionic ADHM constraints

$$\tilde\Omega(\mathcal{M}^A,\nabla_X\mathcal{M}^B) = \tilde\Omega(\mathcal{M}^A,\tilde\nabla_X\mathcal{M}^B)\ . \tag{10.33}$$

[74] Note the fact that we are in Minkowski space does not change the result since $(\bar\Sigma^a_{AB}\bar\Sigma_{aCD})_{\mathrm{Mink}} = (\bar\Sigma_{aAB}\bar\Sigma_{aCD})_{\mathrm{Eucl}} = 2\varepsilon_{ABCD}$.

Then since $\tilde{\mathfrak{M}}$ is flat, we can identify the second term in (10.32) with

$$4\bar{\Sigma}^a_{AB}\Omega_{\rm ij}(X)\psi^{{\rm i}A}\,\omega_{\mu}{}^{\rm j}_{\rm k}\partial_a X^\mu \psi^{{\rm k}B}\ , \tag{10.34}$$

where $\omega_{\mu}{}^{\rm i}_{\rm j}$ are the elements of the symplectic spin connection.

The final term, as we have explained in Section 5.2, involves a coupling to the symplectic curvature of $\mathfrak{M}_k$. Writing all the terms using in the intrinsic variables X^μ and $\psi^{{\rm i}A}$, we have

$$S^{(2)} = -\frac{1}{2g_D^2}\int {\rm d}^{p+1}\xi\,\Big\{g_{\mu\nu}(X)\partial^a X^\mu \partial_a X^\mu + \frac{\rm i}{2}\bar{\Sigma}_{aAB}\Omega_{\rm ij}(X)\psi^{{\rm i}A}(\partial_a\delta^{\rm j}_{\rm k} + \omega_{\mu}{}^{\rm j}_{\rm k}\partial_a X^\mu)\psi^{{\rm k}B}$$
$$+\frac{1}{48}R_{\rm ijkl}(X)\varepsilon_{ABCD}\psi^{{\rm i}A}\psi^{{\rm j}B}\psi^{{\rm k}C}\psi^{{\rm l}D}\Big\}\ . \tag{10.35}$$

In both cases $p = 1, 5$, this is the action of a non-linear σ-model with $\mathfrak{M}_k$ as target with, as we shall show in Section 10.2.4, $\mathcal{N} = (0,4)$ or $(0,1)$ supersymmetry, respectively. For references on two-dimensional $\mathcal{N} = (0,4)$ theories see [148,149], while for six-dimensional $\mathcal{N} = (0,1)$ theories see [153,154] and references therein.

We now follow the opposite philosophy and attempt to write effective action (10.31) entirely in terms of the ADHM variables $a_{\dot\alpha}$ and $\mathcal{M}^A$. In this form, the bosonic, as well as the fermionic, ADHM constraints are now implicit. We claim that the correct expression is

$$S^{(2)} = \frac{\pi^2}{g_D^2}\int {\rm d}^{p+1}\xi\,{\rm tr}_k\{-4\partial^a\bar{w}^{\dot\alpha}\partial_a w_{\dot\alpha} - 4\partial^a a'_n\partial_a a'_n - 2{\rm i}\bar{\Sigma}^a_{AB}\bar{\mu}^A\partial_a\mu^B - {\rm i}\bar{\Sigma}^a_{AB}\mathcal{M}'^{\alpha A}\partial_a\mathcal{M}'^{\,B}_{\alpha}$$
$$+\tfrac{1}{4}[\bar{\Sigma}^a_{AB}\bar{\mathcal{M}}^A\mathcal{M}^B + 2{\rm i}(\bar{a}^{\dot\alpha}\partial^a a_{\dot\alpha} - \partial^a\bar{a}^{\dot\alpha}a_{\dot\alpha})]\boldsymbol{L}^{-1}[\bar{\Sigma}_{aCD}\bar{\mathcal{M}}^C\mathcal{M}^D + 2{\rm i}(\bar{a}^{\dot\beta}\partial_a a_{\dot\beta} - \partial_a\bar{a}^{\dot\beta}a_{\dot\beta})]\}\ . \tag{10.36}$$

We must show that this reduces to (10.31) when we substitute $a_{\dot\alpha} = a_{\dot\alpha}(X)$. Firstly, we have to recall expression (2.117) for the metric on the quotient space in terms of the ADHM variables $a_{\dot\alpha}(X)$. This accounts for the first two terms in (10.36). Most of the fermionic terms in (10.36) are already present in (10.31) in particular the fermion quadrilinear term is equal to that in (10.31) by the Σ-matrix identity (A.20). We seem to have the additional terms dependent on $\bar{a}^{\dot\alpha}\partial^a a_{\dot\alpha} - \partial^a\bar{a}^{\dot\alpha}a_{\dot\alpha}$; however this vanishes when we substitute $a_{\dot\alpha}(X^\mu)$ since

$$\bar{a}^{\dot\alpha}\,\partial_a a_{\dot\alpha} - \partial_a\bar{a}^{\dot\alpha}\,a_{\dot\alpha} = \left(\bar{a}^{\dot\alpha}\,\frac{\partial a_{\dot\alpha}}{\partial X^\mu} - \frac{\partial\bar{a}^{\dot\alpha}}{\partial X^\mu}\,a_{\dot\alpha}\right)\partial_a X^\mu = 0\ , \tag{10.37}$$

by virtue of constraint (2.114).

Form (10.36) is motivated by the fact that one can introduce a non-dynamical U(k) (Hermitian) gauge field χ_a coupled to the fields via the covariant derivatives

$$\mathscr{D}_a w_{\dot\alpha} = \partial_a w_{\dot\alpha} - {\rm i}w_{\dot\alpha}\chi_a,\quad \mathscr{D}_a\bar{w}^{\dot\alpha} = \partial_a\bar{w}^{\dot\alpha} + {\rm i}\chi_a\bar{w}^{\dot\alpha},\quad \mathscr{D}_a a'_n = \partial_a a'_n + {\rm i}[\chi_a, a'_n]\ ,$$
$$\mathscr{D}_a\mu^A = \partial_a\mu^A - {\rm i}\mu^A\chi_a,\quad \mathscr{D}_a\bar{\mu}^A = \partial_a\bar{\mu}^A + {\rm i}\chi_a\bar{\mu}^A,\quad \mathscr{D}_a\mathcal{M}'^A = \partial_a\mathcal{M}'^A + {\rm i}[\chi_a, \mathcal{M}'^A]\ . \tag{10.38}$$

Action (10.36) can then be written in the form of a gauged linear σ-model

$$S^{(2)} = -\frac{4\pi^2}{g_D^2}\int d^{p+1}\xi\,\mathrm{tr}_k\Big\{\mathscr{D}^a\bar{w}^{\dot\alpha}\mathscr{D}_a w_{\dot\alpha} + \mathscr{D}^a a'_n\mathscr{D}_a a'_n + \frac{\mathrm{i}}{2}\bar\Sigma^a_{AB}\bar\mu^A\mathscr{D}_a\mu^B + \frac{\mathrm{i}}{4}\bar\Sigma^a_{AB}\mathscr{M}'^A\mathscr{D}_a\mathscr{M}'^B\Big\}\ . \tag{10.39}$$

Remember that the bosonic and fermionic ADHM constraints have to be imposed. This can be achieved explicitly by introducing, as in Section 6.5, bosonic and fermionic Lagrange multipliers in the form of $k\times k$ Hermitian matrix fields, $\vec{D}$ and $\bar\psi^{\dot\alpha}_A$, respectively:

$$S_{\mathrm{L.m.}} = \frac{4\pi^2\mathrm{i}}{g_D^2}\int d^{p+1}\xi\,\mathrm{tr}_k\{\vec{D}\cdot\vec{\tau}^{\dot\alpha}{}_{\dot\beta}\bar{a}^{\dot\beta}a_{\dot\alpha} + \bar\psi^{\dot\alpha}_A(\bar{\mathscr{M}}^A a_{\dot\alpha} + \bar{a}_{\dot\alpha}\mathscr{M}^A)\}\ . \tag{10.40}$$

The non-dynamical gauge field along with the Lagrange multipliers then forms a vector multiplet $\{\chi_a,\vec{D},\bar\psi_A\}$ of the supersymmetry appropriate to the $p=1$ and 5 cases.

10.2.4. Supersymmetry

Just as the supersymmetry of the parent theory is inherited by the collective coordinates of instanton, we expect the same to be true for the instanton branes. The only difference now is that the collective coordinates are now fields and so the supersymmetry transformations will involve ξ^a derivatives.

From (10.10a) and (10.10b), one finds

$$\delta A_m = \mathrm{i}\xi^A\sigma_m\bar\lambda_A + \mathrm{i}\bar\xi_A\bar\sigma_m\lambda^A\ , \tag{10.41a}$$

$$\delta\lambda^A = \mathrm{i}\Sigma^{abA}{}_B\xi^B F_{ab} + \mathrm{i}\sigma_{mn}\xi^A F_{mn} + \Sigma^{aAB}\sigma_m\bar\xi_B F_{am}\ . \tag{10.41b}$$

These transformations are closely related to (4.23a) and(4.23b), respectively. (Indeed, by removing ξ^a derivatives and replacing A_a by $\mathrm{i}\phi_a$ they are identical.) Now we let A_m and λ^A take their ADHM values with ξ^a-dependent collective coordinates and A_a as in (10.21). The variations lift to variations of the collective coordinates in an almost identical way to those derived in Section 6.5. The only difference is an extra contribution from the final term in the variation of λ^A. Using (10.21), this term is

$$-\mathrm{i}\Sigma^{aAB}D\!\!\!/\phi_a\bar\xi_B + \Sigma^{aAB}(\partial_a A_n - \Omega_\mu\partial_a X^\mu)\sigma_n\bar\xi^A\ . \tag{10.42}$$

The first term, here, is identical to the final term in (4.23b) and so is already accounted by the analysis in Section 4.5. It is the second term which is new. Since

$$\partial_a A_n - \Omega_\mu\partial_a X^\mu = \frac{\partial A_n}{\partial X^\mu}\,\partial_a X^\mu - \Omega_\mu\partial_a X^\mu \equiv \delta_\mu A_n\ , \tag{10.43}$$

we can use the explicit expression for zero modes (2.113) to write the final term as

$$2\Sigma^{aAB}\bar\xi^{\dot\alpha}_B\Lambda\left(\frac{\partial a_{\dot\alpha}}{\partial X^\mu}\right)\partial_a X^\mu\ . \tag{10.44}$$

So there is an extra ξ^a-derivative term, relative to (4.70), in the transformation of the Grassmann collective coordinates:

$$\delta\mathscr{M}^A = -4\mathrm{i}\xi^A_\alpha b^\alpha + 2\mathrm{i}\Sigma^{aAB}\mathscr{C}_{a\dot\alpha}\bar\xi^{\dot\alpha}_B + 2\Sigma^{aAB}\partial_a a_{\dot\alpha}\bar\xi^{\dot\alpha}_B\ ,$$
$$\delta\bar{\mathscr{M}}^A = -4\mathrm{i}\xi^{\alpha A}\bar b_\alpha + 2\mathrm{i}\Sigma^{aAB}\bar\xi_{\dot\alpha B}\bar{\mathscr{C}}^{\dot\alpha}_a + 2\Sigma^{aAB}\partial_a\bar a^{\dot\alpha}\bar\xi_{\dot\alpha B}\ . \qquad (10.45)$$

These transformations are symmetries of effective action (10.31). In counting the number of supersymmetries, we do not include the trivial shifts of the fermions generated by ξ^A and this means that the effective theory of the instanton branes has half the number of supersymmetries of the parent theory: so 8 for the six-dimensional ($p=5$) theory and 4 for the two-dimensional ($p=1$) theory. In both cases the supersymmetries are all anti-chiral and hence, using the usual convention, are denoted as $\mathscr{N}=(0,1)$ and $(0,4)$, for $p=5$ and 1, respectively.

10.2.5. Relation to the instanton calculus

The instanton calculus for $\mathscr{N}=2$ and 4 supersymmetry can be obtained as a particular limit of the σ-model described above. This limit involves performing a Wick rotation of the $p+1$-dimensional world volume of the brane and then a dimensional reduction to zero dimensions. In this limit, the field theory reduces to a matrix theory and the only part of the "dynamics" that remains, in the case with $p=5$ (8 supercharges) is the four-fermion coupling in (10.35). This four-fermion coupling then reproduces the quadrilinear coupling of the Grassmann collective coordinates described in Section 5.2. On top of this, the collective coordinate integral of the instanton calculus is obtained directly from the Wick-rotated partition function of the dimensionally reduced σ-model. This is most easily seen in the gauged linear σ-model description of Eq. (10.39). On dimensional reduction, the partition function of the σ simply gives the instanton partition in the linearized version constructed in Section 6.5. One can see that the auxiliary variables $\{\chi_a,\vec D,\bar\psi_A\}$ introduced in Section 6.5 arise from the dimensional reduction of the fields of the vector multiplet of the σ-model.

It is interesting to ask whether the constrained instanton formalism describing the Coulomb branch of the $\mathscr{N}=2$ and 4 theories can be re-produced in this way. To answer this question, we use the fact that for our six-dimensional σ-model with 8 supercharges, relevant to the $\mathscr{N}=4$ case, it is possible when dimensionally reducing to add a potential which does not break supersymmetry [150]. This potential term has the form of the inner product of a tri-holomorphic Killing vector field on the hyper-Kähler target space. In fact the most general type of potential for an 8 supercharge σ-model in $6-l$ dimensions involves the sum of the inner products of l *commuting* tri-holomorphic Killing vector fields [143,151]. At the level of the non-linear σ-model, the introduction of this potential can be described by non-trivial dimensional reduction of the Scherk–Schwarz kind [152]. At the level of the linear σ-model we can describe it as gauging an additional $\mathrm{U}(1)^l$ symmetry, over and above $\mathrm{U}(k)$, corresponding to the action of the tri-holomorphic Killing vector fields, and then giving VEVs to the l components of the six-dimensional gauge field χ_a that lie in the dimensionally reduced directions.

In the present setting of the instanton calculus, $\mathfrak{M}_k$ admits an $\mathrm{SU}(N)$ tri-holomorphic action corresponding to global gauge transformation in the original $\mathrm{SU}(N)$ gauge theory. The action of these transformations on the bosonic ADHM variables is described in Section 2.4.3. Under global gauge transformations a'_n and $\mathscr{M}'^A$ are invariant, but

$$w_{\dot\alpha}\to\mathscr{U}w_{\dot\alpha},\qquad \mu^A\to\mathscr{U}\mu^A\ . \qquad (10.46)$$

Commuting actions can be obtained by considering global gauge transformations in the Cartan subgroup $\mathrm{U}(1)^{N-1} \subset \mathrm{SU}(N)$ generated by diagonal matrices. We now gauge this symmetry and reduce to zero dimensions. This amounts to replacing the covariant derivatives in (10.39) by

$$\mathscr{D}_a w_{\dot\alpha} \to -\mathrm{i} w_{\dot\alpha}\chi_a - \mathrm{i}\phi^0_a w_{\dot\alpha}, \quad \mathscr{D}_a \bar w^{\dot\alpha} \to \mathrm{i}\chi_a \bar w^{\dot\alpha} + \mathrm{i}\bar w^{\dot\alpha}\phi^0_a, \quad \mathscr{D}_a a'_n \to -\mathrm{i}[a'_n, \chi_a]\ ,$$
$$\mathscr{D}_a \mu^A \to -\mathrm{i}\mu^A\chi_a - \mathrm{i}\phi^0_a\mu^A, \quad \mathscr{D}_a \bar\mu^A \to \mathrm{i}\chi_a\bar\mu^A + \mathrm{i}\bar\mu^A\phi^0_a, \quad \mathscr{D}_a \mathscr{M}'^A \to -\mathrm{i}[\mathscr{M}'^A, \chi_a]\ , \tag{10.47}$$

where ϕ^0_a are six diagonal $N \times N$ matrices. It is easy to see that after integrating out the gauge field χ_a one is left with the instanton effective action describing constrained instantons on the Coulomb branch of $\mathscr{N}=4$ gauge theory (6.94).

In a similar way, the constrained instanton calculus of the $\mathscr{N}=2$ theory can be obtained by gauging the same $\mathrm{U}(1)^{N-1}$ symmetry and dimensionally reducing to zero dimensions the two-dimensional σ-model with $\mathscr{N}=(0,4)$ supersymmetry. In this case, there is another kind of generalization which plays a role in the instanton calculus. Since the $\mathscr{N}=(0,4)$ supersymmetry is purely chiral, it is possible to have fermions of the opposite chirality which are singlets under the supersymmetry. This is precisely what is required to describe the instanton calculus when the $\mathscr{N}=2$ gauge theory involves additional matter hypermultiplets. As described in Section 6.3.2, instantons have additional fermion zero modes and there are new Grassmann collective coordinates $\{\mathscr{K}_f, \tilde{\mathscr{K}}_f\}$. Consequently the σ-model describing the collective coordinate dynamics will have additional fermionic fields describing the fluctuations of the hypermultiplet Grassmann collective coordinates. These are incorporated in the two-dimensional σ-model dynamics in precisely the same way as in description of monopole dynamics in an $\mathscr{N}=2$ theory with hypermultiplets described in [144] (although in this reference the σ-model is in one dimension, i.e. is a quantum mechanical system).

10.3. Instantons and string theory

The most remarkable and unexpected development of the instanton calculus has come with the realization that the ADHM formalism arises naturally in the context of string theory. The point is that supersymmetric instanton branes, as previously described in Section 10.2, arise when D-branes are "absorbed", in a way to be made precise, on other D-branes. In certain respects, the string theory point of view provides an "explanation" for the rather mysterious ADHM construction in the sense that the ADHM variables, constraints and the internal $\mathrm{U}(k)$ symmetry have a simple interpretation in terms of a conventional (but auxiliary) gauge theory with gauge group $\mathrm{U}(k)$ (see Refs. [155–157]). Moreover, the connection is very explicit: not only can the ADHM gauge potential be derived [158] but the σ-model describing the dynamics of instanton branes is obtained in a direct way [21]. Both are obtained in a certain decoupling limit where stringy effects can be neglected. In this way one "derives" the leading-order semi-classical expression for the collective coordinate integral of the original gauge theory. Some aspects of the relation between Yang–Mills and D-instantons, particularly for $k=1$, are also discussed in Refs. [159,160].

10.3.1. The $\mathscr{N}=4$ instanton calculus

We begin by describing how the $\mathscr{N}=4$ instanton calculus can be recovered from string theory. The basic idea is that the $\mathscr{N}=4$ theory with gauge group $\mathrm{U}(N)$ arises as the collective dynamics of N D3-branes in Type IIB string theory and, according to [156,156], D-instantons located on the D3-branes are equivalent to Yang–Mills instantons in the collective coordinate world-volume gauge theory. More generally, Dp-branes located on a stack of D$(p+4)$-branes appear as instanton p-branes in the world-volume theory of the higher-dimensional branes. So our attention will focus on a general system of k Dp-branes and N D$(p+4)$-branes.

We begin by briefly reviewing some basic facts about D-branes in Type II string theory [161,162]. There are two distinct ways to think of a Dp-brane. Firstly, it can be viewed as a brane-like soliton of Type II supergravity in 10 dimensions, the low-energy limit of the string theory. In this respect, we expect that in an appropriate limit its behaviour should be captured by a generalization of Manton's moduli space dynamics: in this case some $(p+1)$-dimensional field theory on the world volume. Secondly, a Dp-brane can be described as a p-dimensional hyperplane on which open strings can end. The connection between these two points of view is that the massless states in the open string spectrum in the presence of the Dp-brane give rise to massless fields which propagate on the $p+1$-dimensional world volume of the D-brane and these massless modes are identified with the collective coordinates of the moduli space approximation.

Specifically, the massless modes or collective coordinates of a single Dp-brane come from a simple dimensional reduction to $p+1$ dimensions of a U(1) vector multiplet of $\mathcal{N}=1$ supersymmetry in 10 dimensions. This supersymmetric theory, which has 16 supercharges, consequently describes the moduli space dynamics of the D-brane. The action is (10.9) with $D=10$ dimensionally reduced to $p+1$ dimensions. After dimensional reduction, the 10-dimensional gauge field A_N, $N=0,1,2,\dots 9$, yields a $p+1$-dimensional gauge field A_n, $n=0,\dots,p$ and $9-p$ scalars ϕ_a, $a=1,\dots,9-p$. In the collective coordinate interpretation, the scalars specify the location of the D-brane in the $9-p$ dimensions transverse to its world volume. The $d=10$ multiplet also includes a Majorana–Weyl fermion Ψ. The 16 independent components of Ψ correspond to the 16 fermion zero modes of the D-brane. This number reflects the fact that the D-brane is a BPS configuration which breaks half of the 32 supersymmetries of Type II theory.

Remarkably, the collective dynamics of a system of N parallel Dp-branes is obtained by simply "non-abelianizing" the U(1) gauge group thereby replacing it with U(N). At low energies, the adjoint-valued scalar fields can acquire VEVs breaking the gauge group to $\mathrm{U}(1)^N$ describing a configuration of Dp-branes separated in the transverse directions. However, when two or more D-branes coincide, additional states corresponding to open strings stretched between the two branes become massless leading to enhanced gauge symmetry [163]. In the maximal case, where all N Dp-branes coincide, the unbroken gauge group is U(N). The low-energy effective action for the world-volume theory can be obtained from dimensional reduction of 10-dimensional super-Yang–Mills theory (10.9) (with $D=10$) with gauge group U(N). Dimensional reduction to $p+1$ dimensions proceeds by setting all spacetime derivatives in the reduced directions to zero. As in the case of a single Dp-brane, the 10-dimensional gauge field yields a $p+1$-dimensional, but now U(N)-valued, gauge field and $9-p$ real adjoint scalars. Configurations with some or all of the D-branes separated in spacetime correspond to the Coulomb branch of the world-volume gauge theory. In terms of string theory parameters, the Yang–Mills coupling constant in $p+1$ dimensions is identified as

$$g_{p+1}^2 = 2(2\pi)^{p-2} g_{\mathrm{st}} \alpha'^{(p-3)/2} , \tag{10.48}$$

where g_{st} is the string coupling constant and α' is the inverse string tension.

In the case of N parallel D3-branes, the low-energy effective theory on the brane world volume is four-dimensional $\mathcal{N}=4$ supersymmetric Yang–Mills theory with gauge group $U(N)$ and coupling constant $g^2=4\pi g_{\rm st}$. In this case we can also introduce a world-volume vacuum angle $\theta=2\pi C^{(0)}$, where $C^{(0)}$ is the VEV of the Ramond–Ramond scalar of the IIB theory. In general, the four-dimensional fields which propagate on the brane also have couplings to the 10-dimensional graviton and the other bulk closed string modes. To decouple the four-dimensional theory from these bulk modes it is necessary to take the limit $\alpha' \to 0$ with g (i.e. $g_{\rm st}$) held fixed and small. This gives a weakly coupled gauge theory on the D3-brane.

A D-instanton (or D(-1)-brane) corresponds to the extreme case where the dimensional reduction is complete and the world volume is a single point in Euclidean spacetime.[75] Correspondingly, rather than having finite mass or tension, a single D-instanton has finite action

$$2\pi(g_{\rm st}^{-1} - {\rm i}C^{(0)}) \equiv -2\pi{\rm i}\tau \ , \tag{10.49}$$

using relations (10.48). Hence, a D-instanton carries the same action as a Yang–Mills instanton in the world-volume gauge theory of N D3-branes described above. From our discussion above, the collective coordinates of a charge-k D-instanton correspond to a $U(k)$ vector multiplet of 10-dimensional $\mathcal{N}=1$ supersymmetric gauge theory dimensionally reduced to zero spacetime dimensions. As we dimensionally reduce to zero dimensions, the fields become both c-number and Grassmann matrix degrees of freedom. In addition to a constant part equal to $-2\pi{\rm i}k\tau$, the action of a charge-k D-instanton also depends on the collective coordinates via the (Wick-rotated) dimensional reduction of (10.9) (with $D=10$):

$$S = -\frac{1}{2g_0^2}\,{\rm tr}_k\,[A_M, A_N]^2 + \frac{\rm i}{g_0^2}\,{\rm tr}_k\,\bar{\Psi}\Gamma_M[A_M, \Psi] \ . \tag{10.50}$$

In addition to the manifest SO(10) symmetry under 10-dimensional rotations, the action is trivially invariant under translations of the form $A_M \to A_M + x_M 1_{[k]\times[k]}$. Hence $k^{-1}\,{\rm tr}_k\,A_M$, which corresponds to the abelian factor of the $U(k)$ gauge group, is identified with the position of the centre of the charge k D-instanton in $\mathbb{R}^{10}$.

Action (10.50) inherits supersymmetries (10.10) and (10.10b) along with linear shifts in the Grassmann collective coordinates:

$$\begin{aligned} \delta A_M &= -\bar{\Xi}\Gamma_M\Psi \ , \\ \delta\Psi &= {\rm i}\Gamma_{MN}\Xi[A_M, A_N] + 1_{[k]\times[k]}\,\varepsilon \ . \end{aligned} \tag{10.51}$$

The 16 components of the Majorana–Weyl SO(10) spinor ϵ correspond to the 16 zero modes of the D-instanton configuration generated by the action of the $D=10$ supercharges. Like the bosonic translation modes, these modes live in the abelian factor of the corresponding $U(k)$ field, $k^{-1}{\rm tr}_k\,\Psi$. In contrast, the Majorana–Weyl spinor Ξ parameterizes the 16 supersymmetries left unbroken by the D-instanton.

In ordinary field theory, as we have seen, instantons yield non-perturbative corrections to correlation functions via their saddle-point contribution to the Euclidean path integral. In the semi-classical

[75] So before dimensional reduction, we must Wick rotate the $D=10$ action (10.9) to Euclidean space.

limit, the path integral in each topological sector reduces to an ordinary integral over the instanton moduli space (5.14). The extent to which similar ideas apply to D-instantons is less clear, in part because string theory lacks a second-quantized formalism analogous to the path integral. Despite this, there is considerable evidence that D-instanton contributions to string theory amplitudes also reduce to integrals over collective coordinates at weak string coupling [133]. In this case the relevant collective coordinates are the components of the 10-dimensional $U(k)$ gauge field A_M and their superpartners Ψ which in the dimensional reduction end up as matrices. According to Green and Gutperle [133,138,164,165], the charge-k D-instanton contributions to the low-energy correlators of the IIB theory are consequently governed by the partition function

$$\mathscr{Z}_k = \frac{e^{-2\pi k(e^{-\phi}+iC^{(0)})}}{\mathrm{Vol}\,U(k)} \int d^{10}A\, d^{16}\Psi\, e^{-S} \ . \tag{10.52}$$

This partition function can be thought of as the collective coordinate integration measure for k D-instantons. In particular, the leading-order semi-classical contribution of k D-instantons to the correlators of the low-energy supergravity fields is obtained by inserting into (10.52) the classical value of each field. The collective coordinate action (10.50) does not depend on the $U(1)$ components of the fields A_M and Ψ. Hence, to obtain a non-zero answer, the inserted fields must, at the very least, involve a product of at least 16 fermions to saturate the corresponding Grassmann integrations. As in field theory instanton calculations, the resulting amplitudes can be interpreted in terms of an instanton-induced vertex in the low-energy effective action. The spacetime position of the D-instanton, $X_M = k^{-1}\,\mathrm{tr}_k\, A_N$, is interpreted as the location of the vertex. In particular, the work of Green and Gutperle [133] has focused on a term of the form R^4 in the IIB effective action (here R is the 10-dimensional curvature tensor) and its supersymmetric completion.

So far we have only considered D-instantons in the IIB theory in a flat 10-dimensional background and in the absence of other branes. In order to make contact with four-dimensional gauge theory, we need to understand how these ideas apply to D-instantons in the presence of D3-branes. In particular, we wish to determine how the D-instanton collective coordinate integral (10.52) is modified by the introduction of N parallel D3-branes. Conversely, in the absence of D-instantons, the theory on the four-dimensional world volume of the D3-branes is $\mathcal{N}=4$ Yang–Mills with gauge group $U(N)$. Hence a related question is how the D-instantons appear from the point of view of the four-dimensional world-volume theory of the D3-branes. In fact, the brane configuration considered here is a special case of a system which has been studied intensively involving k Dp-branes in the presence of N D$(p+4)$-branes, with all branes parallel. As we will review below, the lower-dimensional branes corresponds to Yang–Mills instantons in the world-volume gauge theory of the higher-dimensional branes [156]. We begin by reviewing the maximal case $p=5$, which was first considered (in the context of Type I string theory) by Witten [155]. The cases with $p<5$ then follow by straightforward dimensional reduction.

We start by considering a theory of k parallel D5-branes (in Type IIB string theory) in isolation. As above, the world-volume theory is obtained by dimensional reduction of 10-dimensional $\mathcal{N}=1$ Yang–Mills theory with gauge group $U(k)$. The resulting theory in six-dimensions has two Weyl supercharges of opposite chirality and, therefore, conventionally has $\mathcal{N}=(1,1)$ supersymmetry. [76]

[76] Some convenient facts about six-dimensional supersymmetry are reviewed in [166, p. 67 in particular].

After dimensional reduction, the SO(10) Lorentz group of the Minkowski theory in 10 dimensions is broken to $H = \mathrm{SO}(5,1) \times \mathrm{SO}(4)$. The SO(5,1) factor is the Lorentz group of the six-dimensional theory while the SO(4) is an R-symmetry. The 10-dimensional gauge field A_M splits up into an adjoint scalar in the vector representation of SO(4) and a six-dimensional gauge field. Explicitly we set

$$A_M = \mathrm{i}\left(\chi_a, \frac{1}{2\pi\alpha'} a'_n\right), \quad a = 1,\dots,6, \quad n = 1,\dots,4 . \tag{10.53}$$

Since, in our conventions, A_M is anti-Hermitian, both a'_n and χ_a are Hermitian. The factor of $(2\pi\alpha')^{-1}$ has been inserted so that a'_n will subsequently be identified with the quantity of the same name in the instanton calculus.

In order to describe the fermion content of the theory, consider the covering group of H, $\bar{H} = \mathrm{SU}(4) \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. We introduce spinor indices $A = 1$–4 and $\alpha, \dot\alpha = 1,2$ corresponding to each factor. As mentioned above, a 10-dimensional Majorana–Weyl spinor is decomposed under $\bar{H}$ as

$$\mathbf{16} \to (\mathbf{4},\mathbf{2},\mathbf{1}) \oplus (\bar{\mathbf{4}},\mathbf{1},\mathbf{2}) , \tag{10.54}$$

so it contains two Weyl spinors of opposite chirality in six dimensions. With the representation of the 10-dimensional Clifford algebra as in (10.11), a Majorana–Weyl fermion, as in (10.13), can be written as

$$\Psi = \frac{1}{4\pi\alpha'} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathcal{M}'^{A}_{\alpha} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \bar\psi^{\dot\alpha}_{A} , \tag{10.55}$$

Altogether, the fields $(\chi_a, a'_n, \mathcal{M}'^{A}_{\alpha}, \bar\psi^{\dot\alpha}_{A})$ form a vector multiplet of $\mathcal{N} = (1,1)$ supersymmetry in six dimensions. In terms of an $\mathcal{N} = (0,1)$ subalgebra, the $\mathcal{N} = (1,1)$ vector multiplet splits up into an $\mathcal{N} = (0,1)$ vector multiplet containing $\{\chi_a, \bar\psi^{\dot\alpha}_{A}\}$ and an adjoint hypermultiplet containing $\{a'_n, \mathcal{M}'^{A}_{\alpha}\}$. The action of the $\mathcal{N} = (1,1)$ theory is deduced from (10.18) (with $g_D \to g_6$):

$$S = \frac{1}{g_6^2}\left(S_{\text{gauge}} + \frac{1}{4\pi^2\alpha'^2} S^{(\text{a})}_{\text{matter}}\right) \equiv \frac{4\pi^2}{g_{p+5}^2}\left(4\pi^2\alpha'^2 S_{\text{gauge}} + S^{(\text{a})}_{\text{matter}}\right) , \tag{10.56}$$

(in the present case $p = 5$) where

$$S_{\text{gauge}} = \int \mathrm{d}^6\xi\, \mathrm{tr}_k\{\tfrac{1}{2} F_{ab}^2 - \mathrm{i}\Sigma^{aAB}\bar\psi_A \mathcal{D}_a \bar\psi_B + \tfrac{1}{2} D_{mn}^2\} \tag{10.57}$$

and

$$S^{(\text{a})}_{\text{matter}} = \int \mathrm{d}^6\xi\, \mathrm{tr}_k \left\{ \mathcal{D}^a a'_n \mathcal{D}_a a'_n - \frac{\mathrm{i}}{4} \bar\Sigma^a_{AB} \mathcal{M}^A \mathcal{D}_a \mathcal{M}'^B - \mathrm{i}[\mathcal{M}'^{\alpha A}, a'_{\alpha\dot\alpha}]\bar\psi^{\dot\alpha}_{A} + \mathrm{i}\vec{D}\cdot\vec\tau^{\dot\alpha}{}_{\dot\beta}\bar{a}'^{\dot\beta\alpha} a'_{\alpha\dot\alpha} \right\} . \tag{10.58}$$

Here, the covariant derivatives are for adjoint-valued fields: $\mathcal{D}_a a'_n = \partial_a a'_n + \mathrm{i}[\chi_a, a'_n]$, etc. For later convenience we have introduced a real anti-self dual auxiliary field for the vector multiplet,

$D_{mn} = -^*D_{mn}$), transforming in the adjoint of $SU(2)_R$. Since D_{mn} is anti-self-dual we can write

$$D_{mn} = \mathrm{tr}_2(\vec{\tau}\bar{\sigma}_{mn}) \cdot \vec{D} , \tag{10.59}$$

which defines the three-vector $\vec{D}$. We have also used the relation $g^2_{p+5} = (2\pi)^4\alpha'^2 g^2_{p+1}$.

Following [156], the next step is to introduce N D9-branes of Type IIB theory whose world-volume fills the 10-dimensional spacetime.[77] These will give rise on dimensional reduction to the N D3-branes in the $p = -1$ case on which we will eventually focus. The world-volume theory of the D9-branes in isolation (i.e. in the absence of the D5-branes) is simply 10-dimensional $U(N)$ $\mathcal{N} = 1$ supersymmetric gauge theory. As explained by Douglas [156], the effective action for this system contains a coupling between the two-form field strength F of the world-volume gauge field and a six-form field $C^{(6)}$ which comes from the Ramond–Ramond sector of Type IIB theory. This coupling has the form

$$\int C^{(6)} \wedge F \wedge F , \tag{10.60}$$

where the integration is over the 10-dimensional world volume of the D9-branes. The same six-form field $C^{(6)}$ also couples minimally to the Ramond–Ramond charge carried by D5-branes. Hence a configuration of the $U(N)$ gauge fields with non-zero second Chern class, $F \wedge F$, acts as a source for D5-brane charge. More concretely, if the D9-brane gauge field is chosen to be independent of six of the world-volume dimensions and an ordinary Yang–Mills instanton is embedded in the remaining four dimensions, then the resulting configuration has the same charge density as a single D5-brane. Both objects are also BPS saturated and therefore they also have the same tension. Further confirmation of the identification of D5-branes on a D9-brane as instantons was found in [156] where the gauge-field background due to a Type I D5-brane was shown to be self-dual via its coupling to the world volume of a D1-brane probe.

As described above, D5-branes appear as BPS-saturated instanton configurations on the D9-brane which break half of the supersymmetries of the world-volume theory. Conversely, the presence of D9-branes also breaks half of the supersymmetries of the D5-brane world-volume theory described by action (10.56). Specifically, the $\mathcal{N}=(1,1)$ supersymmetry of the six-dimensional theory is broken down to the $\mathcal{N} = (0,1)$ subalgebra described above Eq. (10.56). To explain how this happens we recall that open strings stretched between branes give rise to fields which propagate on the D-brane world volume. So far we have only included the adjoint representation fields which arise from strings stretching between pairs of D5-branes. As our configuration now includes both D5- and D9-branes there is the additional possibility of states corresponding to strings with one end on each of the two different types of brane. As the D5- and D9-brane ends of the string carry $U(k)$ and $U(N)$ Chan–Paton indices respectively, the resulting states transform in the $(\boldsymbol{k},\boldsymbol{N})$ representation of $U(k) \times U(N)$.

In fact, the additional degrees of freedom transform as hypermultiplets of $\mathcal{N}=(0,1)$ supersymmetry in six dimensions [155]. As these hypermultiplets cannot be combined to form multiplets of $\mathcal{N}=(1,1)$ supersymmetry, the residual supersymmetry of the six-dimensional theory is $\mathcal{N} = (0,1)$ as claimed

[77] In fact a IIB background with non-vanishing D9-brane charge suffers from inconsistencies at the quantum level. This is not relevant here because the D9-branes in question are just a starting point for a classical dimensional reduction.

Table 2
Transformation properties of the fields

	U(k)	U(N)	SU(4)	SU(2)$_L$	SU(2)$_R$
χ	**adj**	**1**	**6**	**1**	**1**
$\bar{\psi}$	**adj**	**1**	$\bar{\mathbf{4}}$	**1**	**2**
D	**adj**	**1**	**1**	**1**	**3**
a'	**adj**	**1**	**1**	**2**	**2**
$\mathscr{M}'$	**adj**	**1**	**4**	**2**	**1**
w	$\boldsymbol{k}$	$\boldsymbol{N}$	**1**	**1**	**2**
$\bar{w}$	$\bar{\boldsymbol{k}}$	$\bar{\boldsymbol{N}}$	**1**	**1**	**2**
μ	$\boldsymbol{k}$	$\boldsymbol{N}$	**4**	**1**	**1**
$\bar{\mu}$	$\bar{\boldsymbol{k}}$	$\bar{\boldsymbol{N}}$	**4**	**1**	**1**

above. Each hypermultiplet contains two complex scalars $w_{ui\dot{\alpha}}$, $\dot{\alpha}=1,2$. Here, as previously, i and u are fundamental representation indices of U(k) and U(N) respectively. The fact that hypermultiplet scalars transform as doublets of the SU(2) R-symmetry is familiar from $\mathcal{N}=2$ theories in four dimensions. Each hypermultiplet also contains a pair of complex Weyl spinors, μ^A_{ui} and $\bar{\mu}^A_{iu}$. The six-dimensional action for the hypermultiplets can be deduced from action (10.58) of the $\{a'_n, \mathscr{M}^A\}$ hypermultiplet:

$$\frac{4\pi^2}{g^2_{p+5}} S^{(\mathrm{f})}_{\mathrm{matter}} = \frac{4\pi^2}{g^2_{p+5}} \int d^6\xi \,\mathrm{tr}_k \left\{ -\mathscr{D}^a \bar{w}^{\dot{\alpha}} \mathscr{D}_a w_{\dot{\alpha}} - \frac{\mathrm{i}}{2} \bar{\Sigma}^a_{AB} \bar{\mu}^A \mathscr{D}_a \mu^B \right.$$
$$\left. - \mathrm{i}(\bar{\mu}^A w_{\dot{\alpha}} + \bar{w}_{\dot{\alpha}} \mu^A) \bar{\psi}^{\dot{\alpha}}_A + \mathrm{i}\vec{D} \cdot \vec{\tau}^{\dot{\alpha}}{}_{\dot{\beta}} \bar{w}^{\dot{\beta}} w_{\dot{\alpha}} \right\} . \tag{10.61}$$

The scalar and fermion kinetic terms in the above action include the fundamental representation covariant derivative, $\mathscr{D}_a w = \partial_a w - \mathrm{i} w \chi_a$, etc. The remaining two terms in (10.61) are the fundamental representation versions of the Yukawa coupling and D-terms which appear in (10.58). The complete action of the six-dimensional theory is then the amalgam of (10.57), (10.58) and (10.61):

$$S = \frac{4\pi^2}{g^2_{p+5}} (4\pi^2 \alpha'^2 S_{\mathrm{gauge}} + S^{(\mathrm{a})}_{\mathrm{matter}} + S^{(\mathrm{f})}_{\mathrm{matter}}) . \tag{10.62}$$

The various fields of the six-dimensional theory and their transformation properties under U(k) $\times$ U(N) $\times \bar{H}$ are shown in Table 2.

The $\mathcal{N}=(0,1)$ supersymmetry transformations for the theory can be deduced from the supersymmetry transformations of 10-dimensional Yang–Mills theory in Eqs. (10.10a) and (10.10b). The $\mathcal{N}=(0,1)$ supersymmetry of the six-dimensional action (10.57), (10.58) and (10.61) is then obtained as the subalgebra of this 10-dimensional $\mathcal{N}=1$ supersymmetry by taking

$$\Xi = -\mathrm{i} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \bar{\xi}^{\dot{\alpha}}_A . \tag{10.63}$$

This yields the transformations

$$\delta a'_{\alpha\dot\alpha} = \mathrm{i}\bar\xi_{\dot\alpha A}\mathcal{M}'^{A}_{\alpha} \qquad \delta\mathcal{M}'^{A}_{\alpha} = 2\Sigma^{aAB}\bar\xi^{\dot\alpha}_{B}\mathcal{D}_a a'_{\alpha\dot\alpha}\ , \tag{10.64a}$$

$$\delta\chi_a = \mathrm{i}\Sigma^{AB}_{a}\bar\xi_A\bar\psi_B, \qquad \delta\bar\psi_A = \bar\Sigma^{ab}{}_{A}{}^{B}F_{ab}\bar\xi_B - \mathrm{i}\bar\sigma_{mn}D_{mn}\bar\xi_A\ . \tag{10.64b}$$

The transformations act on the fundamental hypermultiplets $\{w_{\dot\alpha},\mu^A\}$ in an analogous way to the action on the adjoint hypermultiplet $\{a'_n,\mathcal{M}'^A\}$:

$$\delta w_{\dot\alpha} = \mathrm{i}\bar\xi_{\dot\alpha A}\mu^A, \qquad \delta\mu^A = 2\Sigma^{aAB}\bar\xi^{\dot\alpha}_{B}\mathcal{D}_a w_{\dot\alpha}\ . \tag{10.65}$$

The reader will have noticed that we have chosen our notation so that each hypermultiplet field has a counterpart, denoted by the same letter, in the $\mathcal{N}=4$ instanton calculus. In particular the various indices on these fields correspond with those on the corresponding ADHM variable. The physical reason for this correspondence is simple: the six-dimensional fields are the collective coordinates of the D5-branes. As the D5-branes are equivalent to Yang–Mills instantons, the vacuum moduli space of the U(k) gauge theory on the D5-branes should coincide with the $\mathcal{N}=4$ supersymmetric k-instanton moduli space described by the ADHM construction. As the only scalar fields in the six-dimensional theory lie in $\mathcal{N}=(0,1)$ hypermultiplets, the relevant vacuum moduli space is conventionally referred to as a Higgs branch. Precisely how the proposed equivalence arises was explained in [155]. The Higgs branch is described by the vanishing of the D-term. The D-term equation-of-motion which follows from varying the action is

$$\alpha'^2\vec D = \frac{\mathrm{i}}{16\pi^2}\vec\tau^{\dot\alpha}{}_{\dot\beta}(\bar w^{\dot\beta}w_{\dot\alpha} + \bar a'^{\dot\beta\alpha}a'_{\alpha\dot\alpha})\ . \tag{10.66}$$

So the condition for a supersymmetric vacuum, $\vec D=0$, is equivalent to ADHM constraints (2.65). Since global U(k) gauge transformations relate equivalent vacua, we see that the Higgs branch of theory yields precisely the ADHM construction of the instanton moduli space $\mathfrak{M}_k$ in the form of a hyper-Kähler quotient. This is a particular example of the more general fact that the Higgs branch of a gauge theory with eight supercharges is a hyper-Kähler quotient (for example see Refs. [147,167,168]). The general construction involves a gauge group G and hypermultiplets transforming in some representation of G. The canonical kinetic terms of the hypermultiplet scalars $z^{\mathrm{i}\dot\alpha}$ define the metric on mother space (B.35) and the gauge group action defines the vector fields X_r in (B.37). The auxiliary fields $\vec D$ are moment maps (B.39). Setting $\vec D=0$ and dividing by global gauge transformations gives the hyper-Kähler quotient construction. [78]

At a generic point on the Higgs branch, the U(k) gauge symmetry is completely broken and at low energies the vector multiplet can be integrated out leading to a six-dimensional $\mathcal{N}=(0,1)$ supersymmetric σ-model with $\mathfrak{M}_k$ as target space. At leading order, the low-energy effective action is obtained by simply removing $S_{\rm gauge}$, the kinetic terms of the vector multiplet, from action (10.62). In this limit, the vector multiplet fields $\{\chi_a,\vec D,\bar\psi_A\}$ become auxiliary and the resulting theory is precisely the linear σ-model describing the moduli space dynamics of instanton branes in 10-dimensional $\mathcal{N}=1$ gauge theory constructed in Section 10.2. Note in this low-energy limit, $\vec D$ and $\bar\psi_A$ become Lagrange multipliers (10.40) for the c-number and Grassmann ADHM constraints, (2.65) and (4.34),

[78] Note the ζ^c in (B.39) are precisely Fayet–Illiopolos terms for any abelian factors of G.

respectively. Moreover, χ_a is the auxiliary U(k) gauge field in linear action (10.39) (also, up to a re-scaling, the auxiliary variables that we used to bi-linearize the action in Section 10). At higher energies, or for small VEVs, where non-abelian subgroups are restored, the stringy corrections involving the kinetic terms for the vector multiplet become increasingly important and the simple moduli space picture breaks down.

Starting from the configuration of D5- and D9-branes described above, the general case of parallel Dp- and D($p+4$)-branes with $p<5$ can be obtained by dimensional reduction on the brane world volumes. In this case the vacuum structure of the gauge theory describing the brane configuration is richer. The reason is that on dimensional reduction $5-p$ of the components of the gauge field $\chi_{\hat{a}}$, $\hat{a}=p+1,\dots,5$, become adjoint scalars and can develop VEVs. These scalars describe the freedom for the Dp-branes to move off into the $5-p$ dimensions transverse to both the Dp- and D($p+4$)-branes. In addition to the ADHM constraints (2.65), the classical equations for a supersymmetric vacuum now include

$$[\chi_{\hat{a}},\chi_{\hat{b}}]=w_{\dot{\alpha}}\chi_{\hat{a}}=[a'_n,\chi_{\hat{a}}]=0\ . \tag{10.67}$$

The classical vacuum moduli space consists of distinct branches. First of all, there is a Coulomb branch where $\chi_{\hat{a}}$ and a'_n are diagonal with $w_{\dot{\alpha}}=0$ and on which the gauge group is broken to U$(1)^k$. This corresponds to situation where the Dp-branes are located (generically) in the bulk away from the D($p+4$)-branes. The diagonal elements of $\chi_{\hat{a}}$ give the positions of the k Dp-brane transverse to the D($p+4$)-branes, while the diagonal elements of a'_n give the positions of the Dp-branes along the world volume of the D($p+4$)-branes. When a given element $(\chi_a)_{ii}$ goes to zero, the corresponding Dp-brane touches the world volume of the D3-branes. In this case, it is clear from (10.67) that it is then possible for $w_{ui\dot{\alpha}}$ (for the given value of i) to become non-vanishing. The Dp-brane is then "absorbed" onto the D($p+4$)-branes. On this new branch of the vacuum moduli space the Dp-brane becomes a genuine Yang–Mills instanton with a non-zero scale size $\rho_i^2=\frac{1}{2}\bar{w}^{\dot{\alpha}}_{iu}w_{ui\dot{\alpha}}$ (no sum on i). Other kinds of "mixed" branches arise when more of the k Dp-branes are absorbed into the D($p+4$)-branes. The Higgs branch described the situation where all the Dp-branes have been absorbed into the D($p+4$)-branes and so $\chi_{\hat{a}}=0$. Notice that the vacuum moduli space on the Higgs branch with $p<5$ continues to be the instanton moduli space $\mathfrak{M}_k$. In the reverse sense, when an instanton shrinks down to zero size, i.e. becomes an ideal instanton in the gauge theory on the world volume of the D($p+4$)-branes, it becomes a Dp-brane which can move off into the bulk. These transitions move through the points where the different branches of the vacuum moduli space touch. As long as $p>1$, so that the world-volume theory on the Dp-branes is more than two dimensional, we expect the qualitative picture of distinct branches to be valid, at least semi-classically. However, the points at which the different branches touch correspond to situations where a non-abelian subgroup of the gauge group is restored. The theory here will be strongly coupled and so we expect the moduli space description to break down. These are regimes where we cannot ignore the kinetic terms of the vector multiplet and stringy corrections are expected to be important. The situations with $p\leqslant 1$ are rather different since there can be no symmetry breaking in this case. However, the picture of the low-energy dynamics being described by σ-models on the appropriate moduli spaces is still expected to be valid.

In particular the case of D0/D4-branes in Type IIA theory has been studied extensively because of its application as a light-cone matrix model of the $\mathcal{N}=(2,0)$ theory in six dimensions [169]. In this case, the D0-branes correspond to solitons in the $4+1$-dimensional gauge theory on the D4-brane world volume. These solitons are just four-dimensional Yang–Mills instantons thought of as static finite-energy configurations in five dimensions. At weak coupling, we expect the dynamics of these solitons to be correctly described by supersymmetric quantum mechanics on the instanton moduli space. Hence, it is not surprising that precisely this quantum mechanical system is obtained in [169] by dimensionally reducing the six-dimensional $\mathcal{N}=(0,1)$ theory described above down to a single time dimension. On adding another world-volume dimension, we obtain the D1/D5 system discussed by Witten in [170]. The D1-branes are now strings in a six-dimensional Yang–Mills theory and, in the decoupling limit, their world-sheet dynamics is described by a two-dimensional $\mathcal{N}=(4,4)$ non-linear σ-model with the ADHM moduli space as the target manifold. This σ-model has kinetic terms for the coordinates on the target and their superpartners and as usual the supersymmetric completion of the action involves a four-fermion term which couples to the Riemann tensor of the target. If we reduce this action back to one dimension by discarding spatial derivatives we obtain the quantum mechanics of [169].

In order to arrive at a description of D-instantons in the presence of D3-branes we must dimensionally reduce the world-volume theory of the D5-branes all the way to zero dimensions. However, in order to recover the instanton calculus, we must first perform a Wick rotation in the world volume of the D5-branes. Vector quantities, including the Σ-matrices, in Minkowski space $v^a=(v^0,\vec{v})$, $a=0,\dots,5$, become $v_a=(\vec{v},\mathrm{i}v^0)$, $a=1,\dots,6$, and the Euclidean action is $S^{\rm Eucl}=-\mathrm{i}S^{\rm Mink}$. After dimensional reduction, the symmetry group $\mathrm{U}(k)\times\mathrm{U}(N)\times\bar{H}$ of the six-dimensional system now has a simple interpretation: $\mathrm{U}(k)$ is the auxiliary symmetry of the ADHM construction, $\mathrm{U}(N)$ is the gauge group of the D3-brane theory, SU(4) is the R-symmetry group of the $\mathcal{N}=4$ supersymmetry algebra and $\overline{\mathrm{SO}}(4)=\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ is the four-dimensional Lorentz group.

We will now write down the collective-coordinate integral which determines the leading semi-classical contribution of k D-instantons to correlation functions of the low-energy fields of the IIB theory in the presence of N D3-branes. From the above discussion, the appropriate generalization of (10.52) is obtained by Wick rotating and then dimensionally reducing the partition function of the six-dimensional theory. The resulting matrix theory has a partition function [79]

$$\mathscr{Z}_k=\frac{1}{\mathrm{Vol}\,\mathrm{U}(k)}\int \mathrm{d}^6\chi\,\mathrm{d}^8\lambda\,\mathrm{d}^3D\,\mathrm{d}^4a'\,\mathrm{d}^8\mathscr{M}'\,\mathrm{d}^2w\,\mathrm{d}^2\bar{w}\,\mathrm{d}^4\mu\,\mathrm{d}^4\bar{\mu}\,\mathrm{e}^{-S}\ , \tag{10.68}$$

where the action is deduced from (10.57), (10.58) and (10.61) reduced to zero dimensions:

$$S=\frac{4\pi^2}{g^2}(4\pi^2\alpha'^2S_G+S_K+S_D)\ , \tag{10.69}$$

where $g\equiv g_4$ and

$$S_G=\mathrm{tr}_k\{\tfrac12\,[\chi_a,\chi_b]^2-\Sigma_a^{AB}\bar{\psi}_A[\chi_a,\bar{\psi}_B]-\tfrac12\,D_{mn}^2\}\ , \tag{10.70a}$$

[79] In this section, we shall not keep a careful track of the overall normalization of the partition function which, of course, is important for calculating physical quantities.

$$S_K = \mathrm{tr}_k\{-[\chi_a, a'_n]^2 + \chi_a \bar{w}^{\dot\alpha} w_{\dot\alpha}\chi_a - \tfrac{1}{4}\,\bar\Sigma_{aAB}\mathscr{M}'^{\alpha A}[\chi_a, \mathscr{M}'^{B}_{\alpha}] + \tfrac{1}{2}\,\bar\Sigma_{aAB}\bar\mu^A\mu^B\chi_a\}\ , \tag{10.70b}$$

$$S_D = \mathrm{tr}_k\{-\mathrm{i}\vec{D}\cdot\vec{\tau}^{\dot\alpha}\quad \dot\beta(\bar{w}^{\dot\beta} w_{\dot\alpha} + \bar{a}'^{\dot\beta\alpha} a'_{\alpha\dot\alpha}) + \mathrm{i}(\bar\mu^A w_{\dot\alpha} + \bar{w}_{\dot\alpha}\mu^A + [\mathscr{M}'^A, a'_{\alpha\dot\alpha}])\bar\psi^{\dot\alpha}_A\}\ . \tag{10.70c}$$

Note that S_G arises from the dimensional reduction of S_{gauge} and $S_K + S_D$ comes from the dimensional reduction of $S^{(\mathrm{a})}_{\text{matter}} + S^{(\mathrm{f})}_{\text{matter}}$. Specifically, S_K contains the six-dimensional gauge couplings of the hypermultiplets while S_D contains the Yukawa couplings and D-terms.

Semi-classical correlation functions of the light fields can be calculated by replacing each field with its value in the D-instanton background and performing the collective coordinate integrations with measure (10.68). In the case of the low-energy gauge fields on the D3-brane, the relevant classical configuration is simply the charge-k Yang–Mills instanton specified by the ADHM data which appears explicitly in action (10.70a)–(10.70c). Note that the collective coordinate integral (10.68) depends explicitly on the string length scale α' only via the zero-dimensional remnant of the kinetic terms of the vector multiplet. As a consequence, correlation functions which include fields inserted at distinct spacetime points $x^{(i)}$ and $x^{(j)}$ will have a non-trivial expansion in powers of $\sqrt{\alpha'}/|x^{(i)} - x^{(j)}|$. In order to decouple the world-volume gauge theory from gravity, we must take the limit $\alpha' \to 0$ keeping the four-dimensional coupling g fixed. In this limit, the coupling of $S_K + S_D$ is held fixed while the remnant of the kinetic term of the vector multiplet, S_G, decouples from the action. In this limit, the collective coordinate integral (10.68) becomes

$$\mathscr{Z}_k = \frac{1}{\mathrm{Vol}\,\mathrm{U}(k)} \int \mathrm{d}^6\chi\, \mathrm{d}^8\lambda\, \mathrm{d}^3 D\, \mathrm{d}^4 a'\, \mathrm{d}^8\mathscr{M}'\, \mathrm{d}^4 w\, \mathrm{d}^4\mu\, \mathrm{d}^4\bar\mu\, \exp(-S_K - S_D)\ . \tag{10.71}$$

We can now make contact with the $\mathscr{N} = 4$ instanton calculus. Note that S_K can be written more compactly as

$$S_K = \frac{4\pi^2}{g^2}\,\mathrm{tr}_k\,\chi_a \boldsymbol{L}\chi_a + \frac{2\pi^2}{g^2}\,\bar\Sigma_{aAB}\,\mathrm{tr}_k\,\bar{\mathscr{M}}^A\mathscr{M}^B\chi_a\ . \tag{10.72}$$

We now recognize partition function (10.71) as being the instanton partition function in its linearized form (6.90) with $S_K + S_D$ being the instanton effective action $\tilde{S}$ in (6.91).

We have therefore recovered the leading-order semi-classical expression for the collective coordinate integral for instantons in the $\mathscr{N} = 4$ supersymmetric theory. In addition, the action $S_K + S_D$ is invariant under eight supercharges which are inherited from the $\mathscr{N} = (0, 1)$ theory in six dimensions. One can check that after Wick rotation and dimensional reduction, supersymmetry transformations (10.64a) and (10.65) (up to appropriate re-scaling by g) match the collective coordinate supersymmetries of the $\mathscr{N} = 4$ ADHM instanton calculus written down in (4.68) and (4.70), or (6.95a)–(6.95d). There are also eight additional fermionic symmetries, corresponding to ξ^A in (4.70), which only appear after taking the decoupling limit $\alpha' \to 0$. These correspond to half of the superconformal transformations of the $\mathscr{N} = 4$ theory which leave the instanton invariant.

One can ask how constrained instantons appear in this context. Constrained instantons arise when the $\mathscr{N} = 4$ theory is on its Coulomb branch realized by separating the N D3-branes in the six-dimensional transverse space. This obviously changes the lengths of string stretched between the D-instantons and D3-branes and has the effect of adding mass terms for the fundamental hypermultiplet fields. At the level of the matrix theory action (10.47) the relevant effect can be introduced by replacements (6.93). Taking the decoupling limit and integrating out the vector multiplet one recovers the instanton effective action (5.25) for $\mathcal{N}=4$ constrained instantons.

10.3.2. Probing the stringy instanton

We have seen in the preceding section that many of the features of the instanton calculus are reproduced by considering the dynamics of Dp-branes lying inside D($p+4$)-branes. It is interesting to ask whether the actual ADHM form for gauge potential (2.49) can be obtained in this string theory context. The answer is affirmative once an appropriate probe is identified. The concept of a probe which is able to "feel" the instanton background was first described by Witten [158] in the context of Type I string theory. The idea was generalized to Type II theories needed for the present discussion by Douglas in [156]. The appropriate probe turns out to be a D-brane as well. In the context of the D5/D9-brane system described in the last section, the probe is a single D1-brane (or D-string) whose world volume lies parallel to the other branes. The D-string "feels" the D5/D9-brane background since the fields of the D-string world-sheet theory include a subset that arise from open strings stretched between the D-string and the other higher-dimensional branes. The whole configuration is like a Russian doll: the D5/D9 and D1/D5 sub-systems are both examples of our general Dp/D($p+4$) system described in Section 10.3.1. In the composite system, the ADHM variables appear both as fields of the D5-branes and as couplings of the world-sheet theory of the D-string. The conditions for the resulting world-sheet theory to be $\mathcal{N}=(0,4)$ supersymmetric are precisely ADHM constraints (2.65).

In more detail, we now analyse the dynamics of a D-string in the background of k D5-branes and N D9-branes. The system breaks the 10-dimensional Lorentz group to $\mathrm{SO}(1,1)\times\mathrm{SO}(4)_1\times\mathrm{SO}(4)_2$. Here, $\mathrm{SO}(4)_1$ describes the directions transverse to the D5-branes denoted by vector index $n=1$–4. From the earlier discussion, $\overline{\mathrm{SO}}(4)_1\simeq\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$, where the latter correspond to the spinor indices α and $\dot\alpha$. The other factors $\mathrm{SO}(1,1)\times\mathrm{SO}(4)_2$ are a subgroup $\mathrm{SO}(5,1)$, the Lorentz group of the D5-branes' world volume. We will denote the corresponding spinor indices of $\overline{\mathrm{SO}}(4)_2\simeq\mathrm{SU}(2)_A\times\mathrm{SU}(2)_B$ by δ and $\dot\delta$, respectively. Under the decomposition $\mathrm{SO}(5,1)\supset\mathrm{SO}(1,1)\times\mathrm{SO}(4)_2$

$$\mathbf{4}\to(\mathbf{2},\mathbf{1})_1+(\mathbf{1},\mathbf{2})_{-1},\qquad \bar{\mathbf{4}}\to(\mathbf{2},\mathbf{1})_{-1}+(\mathbf{1},\mathbf{2})_1,\qquad \mathbf{6}\to(\mathbf{2},\mathbf{2})_0+(\mathbf{1},\mathbf{1})_2+(\mathbf{1},\mathbf{1})_{-2}\ . \tag{10.73}$$

Here, the subscripts indicate the SO(1, 1) chirality.

Let us ignore, for the moment, the presence of the D9-branes. The D-string/D5-brane system is an example of the Dp/D($p+4$)-brane system studied in the last section. Recall that the D-string world-sheet theory can best be derived as the dimensional reduction of the D5/D9-brane system, with one D5-brane and k D9-branes. Obviously, we will have to use a different notation to describe the fields of the D-string theory as well as re-assigning indices appropriately. The action of the theory is the dimensional reduction to two dimensions of (10.62) with the following replacements. Firstly, there is a U(1) gauge field and fermions

$$\chi_a\to(A_\pm,x_n),\qquad \bar\psi^{\dot\alpha}_A\to(\bar\zeta^{\dot\delta}_\alpha,\bar\zeta^{\dot\delta}_{\dot\alpha})\ , \tag{10.74}$$

where $A_\pm$ are the light-cone components of the two-dimensional abelian gauge field. Note that x_n denotes, in this context, a field in the world-sheet theory rather than a spacetime coordinate. Since the world-sheet theory is abelian the adjoint hypermultiplet previously denoted $\{a'_n,\mathcal{M}'^A_\alpha\}$ decouples and we can safely ignore it. Finally, the fundamental hypermultiplets describing string stretched between the D-string and D5-branes are described by the replacements

$$w_{ui\dot\alpha} \to \phi_{i\dot\delta}, \qquad \mu^A \to (\theta_i^\alpha, \theta_i^{\dot\alpha}), \qquad \bar w_{iu}^{\dot\alpha} \to \bar\phi_i^{\dot\delta}, \qquad \bar\mu^A \to (\bar\theta_i^\alpha, \bar\theta_i^{\dot\alpha}) \;. \tag{10.75}$$

The world-sheet theory describing the D-string in the presence of the D5-branes has conventional kinetic terms along with a potential [80]

$$\mathscr{L}_{\rm pot} = -\bar\phi_i^{\dot\delta} \phi_{i\dot\delta} x_n x_n + \sum_{c=1}^{3} (\tau^{c\dot\delta}{}_{\dot\varepsilon} \bar\phi_i^{\dot\varepsilon} \phi_{i\dot\delta})^2 \;. \tag{10.76}$$

Here, the second term comes from the D-terms which can be viewed as the ADHM constraint for the D-string viewed as a single instanton inside the U(k) gauge theory on the D5-branes. There are also Yukawa couplings

$$\mathscr{L}_{\rm Yuk} = \bar\theta_i^\alpha x_{\alpha\dot\alpha} \theta_i^{\dot\alpha} + \bar\theta_{i\dot\alpha} \bar x^{\dot\alpha\alpha} \theta_{i\alpha} + (\bar\zeta_\alpha^{\dot\delta} \bar\theta_i^\alpha + \bar\zeta_{\dot\alpha}^{\dot\delta} \bar\theta_i^{\dot\alpha}) \phi_{i\dot\delta} + (\bar\zeta_\alpha^{\dot\delta} \theta_i^\alpha + \bar\zeta_{\dot\alpha}^{\dot\delta} \theta_i^{\dot\alpha}) \bar\phi_{i\dot\delta} \;. \tag{10.77}$$

The first two terms are the fermionic ADHM constraints for the Grassmann collective coordinates of the single U(k) instanton.

At the moment we have assumed all the D5-branes are coincident. In reality, the D5-branes will be separated (in addition to being "thickened out") in a way described by the ADHM matrices a_n'. By translational symmetry, the effect can be introduced into the world-sheet theory by replacing x_n by the matrix quantity

$$x_n \to x_n 1_{[k]\times[k]} + a_n' \;. \tag{10.78}$$

In this way the ADHM variables a_n' appear as couplings in the world-sheet theory.

Now we must consider the effect of the D9-branes. First of all, there are new fields in the world-sheet theory describing open strings stretched between the D-string and the D9-branes. Since there are no additional moduli the new fields are only fermionic and are denoted ε_u and $\bar\varepsilon_u$. The second effect of the D9-branes is that in the configuration we are interested in, they absorb the D5-branes and the latter thicken out in a way parameterized by the ADHM variables $w_{\dot\alpha}$. These variables will, like a_n', appear as couplings in the world-sheet theory. Taking into account the D9-branes leads to a modified potential which has a very suggestive form:

$$\mathscr{L}_{\rm pot} = -\bar\phi_i^{\dot\delta} \bar\Delta_i^{\dot\alpha\lambda} \Delta_{\lambda j\dot\alpha} \phi_{j\dot\delta} + \sum_{c=1}^{3} (\tau^{c\dot\delta}{}_{\dot\varepsilon} \bar\phi_i^{\dot\varepsilon} \phi_{i\dot\delta})^2 \;. \tag{10.79}$$

Here, Δ and $\bar\Delta$ are the ADHM quantities introduced in Section 2.4 (with canonical choice (2.57)) but where now x_n is viewed as a field rather than as a spacetime coordinate. Note (10.79) subsumes (10.76). The Yukawa couplings have a similar suggestive form:

$$\mathscr{L}_{\rm Yuk} = \bar\Upsilon^\lambda \Delta_{\lambda i\dot\alpha} \theta_i^{\dot\alpha} + \bar\theta_{i\dot\alpha} \bar\Delta_i^{\dot\alpha\lambda} \Upsilon_\lambda + (\bar\zeta_\alpha^{\dot\delta} \bar\theta_i^\alpha + \bar\zeta_{\dot\alpha}^{\dot\delta} \bar\theta_i^{\dot\alpha}) \phi_{i\dot\delta} + (\bar\zeta_\alpha^{\dot\delta} \theta_i^\alpha + \bar\zeta_{\dot\alpha}^{\dot\delta} \theta_i^{\dot\alpha}) \bar\phi_{i\dot\delta} \;, \tag{10.80}$$

[80] In the following expressions we do not keep careful track of the coupling constants and normalizations.

where we have defined the composites[81]

$$\Upsilon_\lambda = \begin{pmatrix} \varepsilon_u \\ \theta_{i\alpha} \end{pmatrix}, \qquad \bar{\Upsilon}^\lambda = (\bar{\varepsilon}_u \quad \bar{\theta}^\alpha_i) \; . \tag{10.81}$$

Note (10.80) subsumes (10.77).

As is often the case, the condition that the world-sheet theory is $\mathcal{N}=(0,4)$ invariant is identical to the equations-of-motion in the target space: in this case the D-flatness condition in the D5-brane theory. In other words, as argued in [158], the conditions for extended $\mathcal{N}=(0,4)$ supersymmetry are the ADHM constraints (2.65) on $\{w_{\dot\alpha}, a'_n\}$ viewed as coupling in the world-sheet theory.

For generic ADHM data, potential (10.79) gives a mass to the fields ϕ. These fields can then be integrated out leading to a (gauged) σ-model. At the classical level, we can simply set $\phi=0$. The fields x_n are massless and the resulting σ-model involves the flat metric on $\mathbb{R}^4$. The fermionic sector is more interesting. The first two Yukawa couplings in (10.80) give masses to $\theta^{\dot\alpha}_i$, $\bar{\theta}_{i\dot\alpha}$ and a $2k$-dimensional subspace of both Υ_λ and $\bar{\Upsilon}^\lambda$. What we are interested in are the remaining massless fermionic degrees of freedom. In order to identify these modes we need bases for the null spaces of Δ and $\bar{\Delta}$. But these are provided by the ADHM quantities $\bar{U}$ and U, respectively, as is apparent in (2.47). Hence, the massless modes $\{\rho_u, \bar{\rho}_u\}$ are picked out by

$$\Upsilon_\lambda = U_{\lambda u}\rho_u, \qquad \bar{\Upsilon}^\lambda = \bar{\rho}_u \bar{U}^\lambda_u \; . \tag{10.82}$$

The kinetic term for the massless modes then follows by substitution into the kinetic term for Υ:

$$\bar{\Upsilon}^\lambda \partial_- \Upsilon_\lambda \to \bar{\rho}_u(\delta_{uv}\partial_- + (\partial_- x_n)(A_n)_{uv})\rho_v \tag{10.83}$$

with

$$(A_n)_{uv}(x) = \bar{U}^\lambda_u \frac{\partial U_{\lambda v}}{\partial x_n} \; . \tag{10.84}$$

This is precisely the ADHM expression for gauge potential (2.49) as a function of the field x_n (with $g \to 1$). Of course the expression that we have derived for the gauge potential is only valid in the classical limit and there will be stringy corrections to the instanton profile on a scale set by $\sqrt{\alpha'}$.

10.3.3. The $\mathcal{N}=2$ instanton calculus

There are a number of ways to obtain four-dimensional $\mathcal{N}=2$ theories as the collective dynamics of branes in string theory. The first observation is that the $\mathcal{N}=2$ theory has eight supercharges and may be obtained from the dimensional reduction of a six-dimensional theory with $\mathcal{N}=(0,1)$ supersymmetry. Just as in the case of the $\mathcal{N}=4$ theory it is useful to realize the $\mathcal{N}=2$ theory in its maximal dimension. In this case, an instanton will be a 1-brane soliton as described in Section 10.2.

One way to realize the six-dimensional theory is to consider the collective dynamics of D5-branes embedded in a spacetime where the four transverse dimensions are orbifold $\mathbb{R}^4/\mathbb{Z}_2$. The collective dynamics of the D5-branes can be deduced in the following way [171–173]. First of all, suppose we start with a set of $\tilde{N}$ parallel D5-branes in flat 10-dimensional Minkowski space. From

[81] Recall, from Section 2.4, λ is the ADHM composite index $u+\mathrm{i}\alpha$.

the discussion in Section 10.3.1, the world-volume theory is simply the dimensional reduction to six of $\mathcal{N}=1$ supersymmetric $\mathrm{U}(\tilde{N})$ gauge theory in 10 dimensions. This theory has 16 supercharges, i.e. has $\mathcal{N}=(1,1)$ supersymmetry in six dimensions. The theory has an $\mathrm{SU}(2)_A\times\mathrm{SU}(2)_B$ R-symmetry which occurs because of the SO(4) rotational symmetry of the transverse space. Now we replace the transverse space with the orbifold $\mathbb{R}^4/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ is chosen to act as the centre of $\mathrm{SU}(2)_A$. The resulting world-volume theory of the D5-branes is deduced by a process of projection in the following way. Notice that the $\sigma_R=\mathbb{Z}_2$ act on the fields via their R-symmetry indices. The next part of the procedure involves embedding the $\mathbb{Z}_2$ group action as $\sigma_{\mathrm{U}(\tilde{N})}$ in the gauge group. There are different ways to do this. However, up to conjugation we can take

$$\sigma_{\mathrm{U}(\tilde{N})}=\begin{pmatrix} 1_{[N]\times[N]} & 0 \\ 0 & -1_{[M]\times[M]}\end{pmatrix} \tag{10.85}$$

with $\tilde{N}=N+M$. The resulting theory is then obtained by taking the fields and action of the original $\mathcal{N}=(1,1)$ theory and projecting out by hand all the fields which are not invariant under the simultaneous transformation by $\sigma_R\sigma_{\mathrm{U}(N)}$. The resulting theory has gauge group $\mathrm{U}(N)\times\mathrm{U}(M)$, the supersymmetry is reduced to $\mathcal{N}=(0,1)$ and there are two hypermultiplets in the bi-fundamental representation of the gauge group, i.e. each having fields in the $(\boldsymbol{N},\bar{\boldsymbol{M}})+(\bar{\boldsymbol{N}},\boldsymbol{M})$.

Now we consider instantons. In the six-dimensional theory the instantons correspond to D1-branes lying inside the D5-brane, the same situation as in Section 10.3.1, but now lying transverse to the orbifold rather than $\mathbb{R}^4$. The collective dynamics of the D1-branes in this configuration can be deduced from the Dp/D($p+4$)-brane system discussed in Section 10.3.1 (with $\tilde{N}$ D($p+4$)-branes) and then performing the same kind of projection on the D1-brane world-sheet theory that we did for the D5-brane world-volume theory above. Firstly, one embeds $\mathbb{Z}_2$ both in the R-symmetry group and in the gauge group, but now the gauge group pertains to the D1-branes. Firstly, consider the R-symmetry. In terms of the variables of the instanton calculus, the $\mathbb{Z}_2$ is embedded as σ_R in the group $\overline{\mathrm{SO}}(6)\simeq\mathrm{SU}(4)$ that arose from the world volume Lorentz symmetry of the D5-branes in the D5/D9-brane system. On dimensional reduction to the D1/D5-brane system, $\overline{\mathrm{SO}}(6)$ is broken to $\mathrm{SO}(1,1)\times\mathrm{SU}(2)_A\times\mathrm{SU}(2)_B$ and σ_R acts as the centre of $\mathrm{SU}(2)_A$. For instance, χ_a which transforms in the vector of SO(6), breaks up into

$$\sigma_R(\chi_a)=\begin{cases}\chi_a, & a=0,1,\\ -\chi_a, & a=2,3,4,5\ .\end{cases} \tag{10.86}$$

Here, the invariant components are the two-dimensional U(k) gauge field on the D1-brane world sheet. For spinor quantities like $\bar{\psi}_A$ and $\mathcal{M}'^A$, we have

$$\sigma_R(\bar{\psi}_A)=\begin{cases}\bar{\psi}^A, & A=1,2,\\ -\bar{\psi}_A, & A=3,4,\end{cases}\qquad \sigma_R(\mathcal{M}'^A)=\begin{cases}\mathcal{M}'^A, & A=1,2,\\ -\mathcal{M}'^A, & A=3,4,\end{cases} \tag{10.87}$$

and similarly for μ^A and $\bar{\mu}^A$. The embedding $\mathbb{Z}_2$ in the U(k) gauge group of the D1-brane world-volume theory determines the instanton charge of the configuration. For instance

$$\sigma_{\mathrm{U}(k)}=\begin{pmatrix} 1_{[k_1]\times[k_1]} & 0 \\ 0 & -1_{[k_2]\times[k_2]}\end{pmatrix} \tag{10.88}$$

with $k = k_1 + k_2$ describes an instanton configuration with charges k_1 and k_2 with respect to the $\mathrm{U}(N)$ and $\mathrm{U}(M)$ factors of the gauge group. In order to have an instanton which lives solely in the $\mathrm{U}(N)$ factor we must set $k_2 = 0$ so that $k_1 \equiv k$ and in this case $\sigma_{\mathrm{U}(k)} = 1_{[k]\times[k]}$. From the point of view of this instanton configuration, the $\mathrm{U}(M)$ gauge group plays the role of a spectator since, to leading order, the $\mathrm{U}(M)$-adjoint fields are zero and the $\mathrm{U}(M)$ symmetry is effectively a global symmetry. Consequently on dimensional reduction of the D5-branes to four dimensions, the resulting theory is effectively an $\mathcal{N} = 2$ theory with $N_F = 2M$ fundamental hypermultiplets.[82]

The world-volume theory on the D1-branes is then obtained from the action of the D5-branes in the D5/D9-brane system, that is (10.56), dimensionally reduced to two dimensions and by removing by hand any fields that are not invariant under a simultaneous transformation $\sigma_R\sigma_{\mathrm{U}(k)}\sigma_{\mathrm{U}(\tilde{N})}$. The latter transformation has to be included because the fundamental hypermultiplets carry $\mathrm{U}(\tilde{N})$ gauge indices. In our case, recall that $\sigma_{\mathrm{U}(k)} = 1_{[k]\times[k]}$. The fields that remain are the following. Firstly, from (10.86), only the components χ_a, $a = 0, 1$ remain. This is a $\mathrm{U}(k)$ gauge field in two dimensions. For the other adjoint-valued fields, the bosonic quantities a'_n survive, since these are invariant under σ_R. For the fermions, only the components $\bar{\psi}_A$ and $\mathcal{M}'^A$, with $A = 1, 2$, remain. The situation is slightly more subtle for the fundamental hypermultiplets. Each index $\tilde{u}$ can now run over $N + M$ values which split into two set: $u = 1, \ldots, N$ and $u' = 1, \ldots, M$. Then, for example,

$$\sigma_{\mathrm{U}(\tilde{N})}(w_{ui\dot{\alpha}}) = w_{ui\dot{\alpha}}, \qquad \sigma_{\mathrm{U}(\tilde{N})}(w_{u'i\dot{\alpha}}) = -w_{u'i\dot{\alpha}} \ , \tag{10.89}$$

and similarly for the other hypermultiplet fields. Therefore, the fields $w_{ui\dot{\alpha}}$ survive along with μ^A_{ui} and $\bar{\mu}^A_{iu}$, for $A = 1, 2$. However, in addition the fermionic fields $\mu^A_{u'i}$ and $\mu^A_{iu'}$, with $A = 3, 4$, are odd with respect to both σ_R and $\sigma_{\mathrm{U}(\tilde{N})}$ and so survive.

We now show that on dimensional reduction to zero dimensions and in the decoupling limit, the partition function of the resulting matrix theory is the leading-order expression for the collective coordinate integral of the $\mathcal{N}=2$ theory with $N_F=2M$ hypermultiplets. The details are almost identical to the $\mathcal{N}=4$ case described in Section 10.3.1. In particular, as previously, in the decoupling limit, the $\vec{D}$ and $\bar{\psi}_A$ $(A = 1, 2)$ become Lagrange multipliers for the bosonic and fermionic ADHM constraints. The main difference is that the dimensionally reduced (Wick-rotated) action (10.72) is subject to the projection described above yielding[83]

$$S_K = \frac{4\pi^2}{g^2}\left\{\mathrm{tr}_k\,\chi_a\boldsymbol{L}\chi_a - \frac{1}{2}(\chi_1 - \mathrm{i}\chi_2)(\bar{\mathcal{M}}^1_u\mathcal{M}^2_u - \bar{\mathcal{M}}^2_u\mathcal{M}^1_u) - \frac{1}{2}(\chi_1 + \mathrm{i}\chi_2)(\bar{\mathcal{M}}^3_{u'}\mathcal{M}^4_{u'} - \bar{\mathcal{M}}^4_{u'}\mathcal{M}^3_{u'})\right\} \ . \tag{10.90}$$

In this expression a sum over $a = 1, 2$ is implied. The integral over χ_a yields the factor of $|\det_{k^2}\boldsymbol{L}|^{-1}$ in (5.55) for $\mathcal{N} = 2$, while the action becomes

$$\tilde{S} = -\frac{\pi^2}{g^2}\,\mathrm{tr}_k(\bar{\mathcal{M}}^1_u\mathcal{M}^2_u - \bar{\mathcal{M}}^2_u\mathcal{M}^2_u)\boldsymbol{L}^{-1}(\bar{\mathcal{M}}^3_{u'}\mathcal{M}^4_{u'} - \bar{\mathcal{M}}^4_{u'}\mathcal{M}^3_{u'}) \ . \tag{10.91}$$

[82] Note only N_F even theories can be obtained in this way.
[83] It is useful to recall that in Euclidean space $\bar{\Sigma}_1 = \eta^3$ and $\bar{\Sigma}_2 = \mathrm{i}\bar{\eta}^3$.

This is precisely equal to (6.84), after a suitable re-scaling by g, with the VEVs set to zero and with the relations

$$\mathcal{K}_f = (\bar{\mathcal{M}}^3_{u'}, \bar{\mathcal{M}}^4_{u'}), \qquad \tilde{\mathcal{K}}_f = \mathrm{i}(\mathcal{M}^3_{u'}, -\mathcal{M}^4_{u'}) , \tag{10.92}$$

where $f = 1,\ldots,N_F = 2M$. Just as in the $\mathcal{N}=4$ case, the instanton effective action on the Coulomb branch of the $\mathcal{N}=2$ theory can easily be obtained by separating the D5-branes in the two dimensions transverse to their world-volume orthogonal to the orbifold.

10.3.4. Mass couplings and soft supersymmetry breaking

In Section 6.3, we described how the instanton calculus was modified when mass terms were added to the field theory breaking supersymmetry successively from $\mathcal{N}=4$, through $\mathcal{N}=2$ and 1 to $\mathcal{N}=0$. Using the response of the instanton to field theory masses, we were able to relate in Section 5.4 the collective coordinate integrals with different numbers of supersymmetries by a process of decoupling and renormalization group matching. In this section, we describe how the effects of mass terms may be realized in the brane description.

First of all, let us establish at the level of the $\mathcal{N}=4$ field theory, the form of the mass deformation. It is useful at this stage to introduce the language of $\mathcal{N}=1$ superfields. The $\mathcal{N}=4$ theory consists of an $\mathcal{N}=1$ vector multiplet along with three adjoint-valued chiral multiplets $\Phi_{\mathrm{i}} = \{\varphi_{\mathrm{i}}/\sqrt{2}, \lambda_{\mathrm{i}}\}$, $\mathrm{i} = 1$–3. The relation between these fields and the fields that we introduced in Section 4.1 can be chosen as

$$\varphi_1 = \mathrm{i}\phi_5 + \phi_6, \qquad \varphi_2 = \mathrm{i}\phi_3 + \phi_4, \qquad \varphi_3 = \mathrm{i}\phi_1 + \phi_2 \tag{10.93}$$

and

$$\lambda_{\mathrm{i}} = \lambda^A, \qquad \bar{\lambda}_{\mathrm{i}} = \bar{\lambda}_A \qquad A = \mathrm{i} = 1\text{–}3 . \tag{10.94}$$

With this choice the vector multiplet contains $\{A_m, \lambda \equiv \lambda^4\}$.

In terms of $\mathcal{N}=1$ superfields, the action of the $\mathcal{N}=4$ theory (in Minkowski space) can be written as

$$S = \frac{1}{g^2}\int \mathrm{d}^4x\, \mathrm{tr}_N \left\{ \frac{g^2}{8\pi}\,\mathrm{Im}\,\tau W^\alpha W_\alpha|_{\theta^2} + 2\Phi^\dagger_{\mathrm{i}} \mathrm{e}^V \Phi_{\mathrm{i}}|_{\theta^2\bar\theta^2} \right.$$
$$\left. + \frac{2}{3}\,\varepsilon_{\mathrm{ijk}}\Phi_{\mathrm{i}}\Phi_{\mathrm{j}}\Phi_{\mathrm{k}}|_{\theta^2} + \frac{2}{3}\,\varepsilon_{\mathrm{ijk}}\Phi^\dagger_{\mathrm{i}}\Phi^\dagger_{\mathrm{j}}\Phi^\dagger_{\mathrm{k}}|_{\bar\theta^2} \right\} . \tag{10.95}$$

The most general mass deformation that preserves $\mathcal{N}=1$ supersymmetry can be obtained by adding

$$S_{\mathrm{mass}} = \frac{1}{g^2}\int \mathrm{d}x^4\, \mathrm{tr}_N \{ m_{\mathrm{i}}\Phi^2_{\mathrm{i}}|_{\theta^2} + m^*_{\mathrm{i}}\Phi^{\dagger 2}_{\mathrm{i}}|_{\bar\theta^2} \} . \tag{10.96}$$

In terms of component fields (in Euclidean space)

$$S_{\mathrm{mass}} = \frac{1}{g^2}\int \mathrm{d}^4x\, \mathrm{tr}_N \{ m_{\mathrm{i}}\varepsilon_{\mathrm{ijk}}\varphi_{\mathrm{i}}\varphi^\dagger_{\mathrm{j}}\varphi^\dagger_{\mathrm{k}} - m^*_{\mathrm{i}}\varepsilon_{\mathrm{ijk}}\varphi^\dagger_{\mathrm{i}}\varphi_{\mathrm{j}}\varphi_{\mathrm{k}} + |m_{\mathrm{i}}|^2|\varphi_{\mathrm{i}}|^2 + m_{\mathrm{i}}\lambda_{\mathrm{i}}\lambda_{\mathrm{i}} + m^*_{\mathrm{i}}\bar\lambda_{\mathrm{i}}\bar\lambda_{\mathrm{i}} \} . \tag{10.97}$$

It is interesting to establish the SO(6) R-symmetry properties of the mass terms above. In general a mass term for the fermions can be written as

$$\frac{1}{g^2}\int d^4x\,\mathrm{tr}_N\{m_{AB}\lambda^A\lambda^B + m^{AB}\bar\lambda_A\bar\lambda_B\}\ , \tag{10.98}$$

where the symmetric matrix m_{AB} ($m^{AB}\equiv m^*_{AB}$) transforms in the $\mathbf{\bar{10}}$ ($\mathbf{10}$) of SO(6). The mass matrix can be chosen to be diagonal: $m_{AB}=\mathrm{diag}(m_1,m_2,m_3,m_4)$. In order to preserve at least $\mathcal{N}=1$ supersymmetry, at least one of the mass eigenvalues must vanish. Choosing $m_4=0$ gives the fermion mass terms in (10.97). The mass matrix can also be written as a anti-symmetric rank-3 tensor T_{abc} of SO(6). The relation between the two bases is provided by the Σ-matrices:

$$T_{abc}\sim m_{AB}(\Sigma_{[a}\bar\Sigma_b\Sigma_{c]})^{AB} + m^{AB}(\bar\Sigma_{[a}\Sigma_b\bar\Sigma_{c]})_{AB}\ . \tag{10.99}$$

With some choice of normalization, the tensor T is associated to the following cubic coupling of the scalar fields:

$$T_{abc}\,\mathrm{tr}_N\,\phi_a\phi_b\phi_c = \varepsilon_{\mathrm{ijk}}\,\mathrm{tr}_N\{m_{\mathrm{i}}\varphi_{\mathrm{i}}\varphi^\dagger_{\mathrm{j}}\varphi^\dagger_{\mathrm{k}} - m^*_{\mathrm{i}}\varphi^\dagger_{\mathrm{i}}\varphi_{\mathrm{j}}\varphi_{\mathrm{k}} + m_4\varphi_{\mathrm{i}}\varphi_{\mathrm{j}}\varphi_{\mathrm{k}} - m^*_4\varphi^\dagger_{\mathrm{i}}\varphi^\dagger_{\mathrm{j}}\varphi^\dagger_{\mathrm{k}}\}\ . \tag{10.100}$$

When $m_4=0$, this is precisely the bosonic part linear in the masses of the $\mathcal{N}=1$ preserving deformation in (10.97).

The question before us is how to introduce the mass deformation when the $\mathcal{N}=4$ theory is realized as the collective dynamics of N coincident D3-branes in Type IIB string theory? A Dp-brane carries charge which couples directly to the Ramond–Ramond $p+1$-form potential $C^{(p+1)}$. However, Dp-branes also couple to other background fields in the string theory. The most general couplings of the Ramond–Ramond potentials to a collection of Dp-branes occur through the Chern–Simons action whose form, in the case of multiple branes, was established by Myers [174]. In particular, D3-branes can carry D5-brane dipole moment which is induced by a coupling to the $C^{(6)}$ potential of the form [84]

$$\frac{1}{g^2}\int d^4x\,\mathrm{tr}_N\{\phi_a\phi_b C^{(6)}_{0123ab}(\phi)\}\ . \tag{10.101}$$

Consider the case where the associated seven-form field strength $F^{(7)}=dC^{(6)}$ is constant. Then the coupling above is simply equal to

$$\frac{1}{3g^2}\int d^4x\,F^{(7)}_{0123abc}\,\mathrm{tr}_N\,\phi_a\phi_b\phi_c = \frac{1}{3g^2}\int d^4x(*_6F^{(3)})_{abc}\,\mathrm{tr}_N\,\phi_a\phi_b\phi_c\ , \tag{10.102}$$

using $F^{(7)}{=}{*}F^{(3)}$. Here, $*_6$ is Hodge duality in the six-dimensional transverse space. Comparison with mass deformation (10.97) shows that the background Ramond–Ramond field produces the bosonic terms linear in the masses if

$$(*_6F^{(3)})_{abc} = 6T_{abc} \tag{10.103}$$

and T is the tensor defined in (10.99).

[84] In the following 0123 refers to the world-volume directions.

In this way we have reproduced the bosonic mass coupling (10.97) using a suitable background Ramond–Ramond potential. However, as one would expect due to supersymmetry, the background field also couples to the fermions. The analogous example of D0-branes coupling to $C^{(3)}$ has been investigated in [175]. By using T-duality we can extract the analogous coupling for D3-branes. The result is most easily written by recalling that the fermionic fields on D3-branes can be obtained by a dimensional reduction of a 10-dimensional Majorana–Weyl spinor. Using the representation of the Clifford algebra as in (10.11) and the decomposition of Ψ in terms of $\{\lambda^A, \bar{\lambda}_A\}$ in (10.13), the coupling to the background $C^{(6)}$ potential can be written in the form

$$\frac{1}{g^2}\int d^4x\, F^{(7)}_{0123abc}\,\mathrm{tr}_N\, \bar{\Psi}\Gamma_a\Gamma_b\Gamma_c\Psi\ . \tag{10.104}$$

Using (10.99) and (10.103) this can be written as

$$\frac{1}{g^2}\int d^4x\,\mathrm{tr}_N\{m_{AB}\lambda^A\lambda^B + m^{AB}\bar{\lambda}_A\bar{\lambda}_B\}\ , \tag{10.105}$$

precisely the fermion mass coupling in (10.97).

Now that we have established how to introduce the mass coupling in the world-volume theory of the D3-branes, we can now consider the effect of the same background field in the D-instanton theory. Recall that the relevant effect for the D3-branes could be described as a D5-brane dipole moment coupling to $C^{(6)}$. But non-trivial $C^{(6)}$ also implies a non-trivial $C^{(2)}$ background since $dC^{(2)} = {*dC^{(6)}}$. A D-instanton will carry D1-dipole moment and so a coupling to the $C^{(2)}$ background field. The couplings can be deduced from the D3–$C^{(6)}$ coupling established above by successive T-dualities reducing the D3-branes to D-instantons. The leads to the following couplings in the D-instanton action:

$$\frac{1}{3g_0^2}F^{(3)}_{abc}\,\mathrm{tr}_k\,\chi_a\chi_b\chi_c + \frac{1}{g_0^2}F^{(3)}_{abc}\,\mathrm{tr}_k\,\bar{\Psi}\Gamma_a\Gamma_b\Gamma_c\Psi\ , \tag{10.106}$$

where Ψ is the 16-component fermion defined in (10.55) and Γ_a are components of the 10-dimensional Γ-matrices (10.11) in the six-dimensional space transverse to the D3-branes. The field strength can be deduced by taking the dual of (10.103):

$$F^{(3)} = 6\tilde{T}\ , \tag{10.107}$$

where $\tilde{T}$ is equal to the form of T with m_{AB} replaced by $-m_{AB}$. Couplings (10.106) are then simply

$$\frac{\pi^2}{g^2}\,\mathrm{tr}_k\,\{-m_{AB}\mathcal{M}'^A\mathcal{M}'^B + (4\pi\alpha')^2(2\tilde{T}_{abc}\chi_a\chi_b\chi_c + m^{AB}\bar{\psi}_A\bar{\psi}_B)\}\ . \tag{10.108}$$

Notice that the couplings involving the vector multiplet $\{\chi_a, \bar{\psi}_A\}$ do not survive in the decoupling limit $\alpha' \to 0$ (fixed g). In this limit, the only surviving term is the mass coupling for $\mathcal{M}'^A$ which is identical to that in (6.89) (after the re-scaling $\mathcal{M}^A \to g^{1/2}\mathcal{M}^A$). In order to completely reproduce (6.89), the hypermultiplet fermions $\{\mu^A, \bar{\mu}^A\}$, arising from open string stretched between the

D-instantons and D3-branes must also couple to $C^{(2)}$ in an analogous way dictated by their SO(6) transformation properties.

11. Further directions

What we have said in the preceding sections is far from the end of the story of the calculus of many instantons. There are two main closely related developments that we turn to in this final section. The first involves the fate of instantons in a rather esoteric generalization of the underlying gauge theory, or rather on the spacetime on which it is defined. The idea is to define the theory on a non-commutative version of $\mathbb{R}^4$. This non-commutative space is characterized by the fact that the spacetime position coordinates x_m no longer commute. The motivation for considering such a generalization comes partly from string theory. We have already described how gauge theories can arise as the collective dynamics of D-branes in string theory and turning on certain background fields can make the resulting world volume non-commutative. What is remarkable is that the non-commutativity of spacetime does little violence to instantons. In particular the moduli space is still a conventional commutative space. In fact it is a very simple deformation of $\mathfrak{M}_k$ obtained by taking non-zero central terms for the $\mathrm{U}(1) \subset \mathrm{U}(k)$ factor in the hyper-Kähler quotient construction. In particular, the deformed space, which we denote $\mathfrak{M}_k^{(\zeta)}$, is still hyper-Kähler and rather remarkably it is a smooth resolution of the original space: instantons can no longer shrink down to zero size due to the non-commutativity. In addition, instantons now become non-trivial in theories with an abelian gauge group like QED.

The second development, described in Section 11.2, is a new way to calculate instanton effects beyond a single instanton, or two for SU(2), when scalar fields have VEVs. The approach is based on a kind of localization on the moduli space of instantons and has its genesis in calculations of the gluino condensate in $\mathcal{N}=1$ theories in the Higgs phase [65,66] (as described in Section 7). Later, the idea was suggested as a potential way to deal with the k-instanton contribution to the prepotential of $\mathcal{N}=2$ theories in [177,178]. In both these situations, as we have seen, one must use the constrained instanton formalism of Affleck. To leading order in the semi-classical approximation, we have seen that the effect of the VEVs is to turn on a non-trivial instanton effective action which acts as a potential on the instanton moduli space. Specifically in the context of the calculation of the prepotential in $\mathcal{N}=2$ theories, we will show that the resulting integrals over the instanton moduli space have the remarkable property that they localize around the critical points of the instanton effective action. This is guaranteed by the existence of a nilpotent fermionic symmetry, or BRST operator [176]. In fact the critical points correspond to configurations where all the instantons have shrunk down to zero size. The singular nature of this configuration may be regularized by making the spacetime non-commutative since this removes the singularities of the instanton moduli space. Using this technology we will be able to rather swiftly, following [95], re-derive the one-instanton contribution to the prepotential in $\mathrm{SU}(N)$ that we calculated in Section 8.4 and then extend the method to two instantons. This technique of localization has for the first time enabled a calculation of instanton effects to all orders in the instanton charge for finite N [96]. The relevant theory is the $\mathcal{N}=4$ theory softly broken to $\mathcal{N}=2$ by giving mass m to two of the three adjoint-valued chiral multiplets and the physical quantity that can be calculated to all orders in the instanton expansion is the $\mathcal{O}(m^4)$ terms in the prepotential.

As this review reaches its final editing stage, the localization idea has been pursued to its logical conclusion in a remarkable paper of Nekrasov [179]. In this paper, Nekrasov provides a closed form expression for the k instanton contribution to the prepotential for the $\mathcal{N}=2$ gauge theory with $\mathrm{SU}(N)$ gauge group with fundamental hypermultiplets. We will make some further comments about this important work at the end of this section.

11.1. Non-commutative gauge theories and instantons

It is not at all obvious but the rather bizarre generalization of gauge theories defined on a non-commutative background spacetime has a rather pleasant and simple effect on instantons. Non-commutativity is also of special interest in the framework of string theory and D-branes [180–183] (see Ref. [184] for a review). Here we will briefly examine non-commutativity from the instanton perspective.

We have already seen in Section 6 that the moduli space $\mathfrak{M}_k$ of instantons in ordinary commutative gauge theories fails to be a smooth manifold due to conical singularities arising when the auxiliary symmetry group $\mathrm{U}(k)$ does not act freely. Physically these are points where individual instantons shrink to zero size, i.e. $w_{ui\dot\alpha}=0$ for a given $i\in\{1,\dots,k\}$. There is a natural way to resolve, or blow up, the singularities of $\mathfrak{M}_k\to\mathfrak{M}_k^{(\zeta)}$ whilst preserving the hyper-Kähler structure as described by Nakajima [185,186]. The important observation is that the quotient group $\mathrm{U}(k)$ has an abelian factor and, as we explain in Appendix B, one has the freedom to add to each of the three moment maps a constant term in the Lie algebra of any abelian factor. In the context of the ADHM construction this freedom involves modifying ADHM constraints (2.59a) by adding a term proportional to the identity $k\times k$ matrix to the right-hand side as in (2.70):

$$\vec\tau^{\dot\alpha}{}_{\dot\beta}\bar a^{\dot\beta}a_{\dot\alpha}=\vec\zeta_{(+)}1_{[k]\times[k]}\ . \tag{11.1}$$

Here, $\vec\zeta_{(+)}$ is a three-vector of constants. Now consider the clustering limit in which the ith instanton is well separated from the remainder. In this limit, the effective ADHM constraints of the single instanton are modified from (2.102) to

$$\bar w_{iu}^{\dot\alpha}w_{ui\dot\beta}=\rho_i^2\delta^{\dot\alpha}{}_{\dot\beta}+\tfrac12\,\vec\tau^{\dot\alpha}{}_{\dot\beta}\cdot\vec\zeta_{(+)}\ , \tag{11.2}$$

where we recall that ρ_i is a measure of the size of the instanton. Without loss of generality, suppose $\vec\zeta_{(+)}\propto(0,0,1)$, then it is easy to see that

$$\rho_i^2\geqslant\tfrac12\,|\vec\zeta_{(+)}|\ . \tag{11.3}$$

Therefore in the presence of the central term, an instanton can no longer shrink to zero size and the singularity in the instanton moduli space corresponding to that process is smoothed out. For instance, the $\mathrm{U}(1)\subset\mathrm{U}(k)$ corresponding to phase rotation of $w_{ui\dot\alpha}$ for fixed i, no longer has a fixed point. In fact, one can show that the whole of the auxiliary group $\mathrm{U}(k)$ no longer has any fixed points on $\mathfrak{M}_k^{(\zeta)}$.

The question is what physical effect can introduce the central term in the ADHM constraints? Remarkably, we will see that precisely this smoothed instanton moduli space $\mathfrak{M}_k^{(\zeta)}$, defined by (11.1), arises in the ADHM construction of instantons in gauge theories formulated on a spacetime with non-commuting coordinates [187]:

$$[x_m, x_n] = \mathrm{i}\theta_{mn} \ . \tag{11.4}$$

The three-vector $\vec{\zeta}_{(+)}$ is precisely the anti-self-dual projection of the non-commutativity parameter θ_{mn}. We also define the self-dual projection which appears in the ADHM constraints of anti-instantons:

$$\zeta^c_{(+)} \equiv \bar{\eta}^c_{mn}\theta_{mn}, \qquad \zeta^c_{(-)} \equiv \eta^c_{mn}\theta_{mn}, \qquad c = 1\text{–}3 \ , \tag{11.5}$$

where η^c_{mn} and $\bar{\eta}^c_{mn}$ are 't Hooft η-symbols (see Appendix A).

It is quite remarkable that non-commutative instantons are non-trivial even for an abelian U(1) gauge group. In this case, the deformed instanton moduli space $\mathfrak{M}_k^{(\zeta)}$ is simply a resolution of the space $\mathrm{Sym}^k\,\mathbb{R}^4$.[85] In fact, the case of an abelian gauge group, considered by Nekrasov and Schwarz in Ref. [187], provided the first examples of explicit non-commutative instanton configurations. Various aspects of the ADHM construction on non-commutative spaces were further developed and clarified in Refs. [188–194]. Explicit examples of instanton solutions in U(N) gauge theories with space–space as well as space–time non-commutativity were constructed and analysed in [193]. In general, for $|\vec{\zeta}| > 0$, the moduli spaces of these solutions contain no singularities. However, for semi-classical functional integral applications, it is important to ensure that the instanton gauge field itself is also non-singular—or to be more precise is gauge equivalent to a non-singular configuration—for all values of the argument [193]. The supersymmetric collective-coordinate measure for non-commutative instantons was constructed and the one-instanton partition function calculated in Ref. [176]. Based on these results, instanton contributions to the prepotential of the non-commutative $\mathcal{N} = 2$ pure gauge theory were determined in [195] and found to be equivalent to the corresponding commutative contributions confirming the hypothesis that the non-commutative version of the theory is described by the same Seiberg–Witten curve [195,196].

11.1.1. ADHM construction on non-commutative $\mathbb{R}^4$

We will work in flat Euclidean spacetime $\mathbb{R}^4$ with non-commutative coordinates x_m which satisfy the commutation relations (11.4) where θ_{mn} is an anti-symmetric real constant matrix. Using Euclidean spacetime rotations, θ_{mn} can be always brought to the form

$$\theta_{mn} = \begin{pmatrix} 0 & \theta_{12} & 0 & 0 \\ -\theta_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_{34} \\ 0 & 0 & -\theta_{34} & 0 \end{pmatrix} \ . \tag{11.6}$$

In terms of complex coordinates

$$\begin{aligned} z_1 &= x_2 + \mathrm{i}x_1, \quad \bar{z}_1 = x_2 - \mathrm{i}x_1 \ , \\ z_2 &= x_4 + \mathrm{i}x_3, \quad \bar{z}_2 = x_4 - \mathrm{i}x_3 \ , \end{aligned} \tag{11.7}$$

[85] This is the k-fold symmetric product of $\mathbb{R}^4$ which arises from solving ADHM constraints (2.65) for $N = 1$. The solution is $w_{i\dot{\alpha}} = 0$ and $a'_n = -\mathrm{diag}(X^1_n, \ldots, X^k_n)$, $X^i \in \mathbb{R}^4$, which fixes all the U(k) symmetry apart from permutations of the positions X^i_n. Modding out by the permutations gives the k-fold symmetric product of $\mathbb{R}^4$. This space is singular whenever two, or more, of the point-like instantons come together, since then the group of permutations does not act freely.

commutation relations (11.4) take the form

$$[z_1,\bar z_1]=-2\theta_{12},\quad [z_i,z_j]=0\ ,$$
$$[z_2,\bar z_2]=-2\theta_{34},\quad [z_i,\bar z_{j\neq i}]=0\ , \tag{11.8}$$

where $i,j=1,2$. Besides the usual commutative case, there are two important cases to consider:

1. When either θ_{12} or θ_{34} vanishes, the matrix θ_{mn} is of rank two. This case corresponds to the direct product of the ordinary commutative two-dimensional space with the non-commutative two-dimensional space, $\mathbb{R}^2_{\rm NC}\times\mathbb{R}^2$. For definiteness we set here $\theta_{34}=0$ and introduce the notation $\theta_{12}\equiv-\zeta/2$ in such a way that

$$[z_1,\bar z_1]=-\zeta,\quad [z_2,\bar z_2]=0,\quad [z_i,z_j]=0\ . \tag{11.9}$$

This situation describes theories defined on a background with non-commutative space but commutative time.

2. A rank-four matrix θ_{mn} (with $\theta_{12}\neq0$ and $\theta_{34}\neq0$) generates the non-commutative Euclidean spacetime $\mathbb{R}^4_{\rm NC}=\mathbb{R}^2_{\rm NC}\times\mathbb{R}^2_{\rm NC}$. The corresponding world-volume gauge theory has non-commutative (Euclidean) time. Since both components of θ are non-vanishing, they can be made equal, $\theta_{12}=\theta_{34}\equiv-\zeta/4$, with appropriate re-scalings of the four coordinates x_m and, if necessary, a parity transformation. Eqs. (11.8) become

$$[z_i,\bar z_j]=-\frac{\zeta}{2}\,\delta_{ij},\quad [z_i,z_j]=0\ . \tag{11.10}$$

In fact, the condition $\theta_{12}=\pm\theta_{34}$ corresponds to (anti-)self-duality: $\frac12\,\varepsilon^{mnkl}\theta_{kl}=\pm\theta_{mn}$.

A Hilbert space representation for non-commutative geometry (11.9) or (11.10) can be easily constructed by using complex variables (11.7) and realizing z and $\bar z$ as creation and annihilation operators in the Fock space for simple harmonic oscillators (SHO). The fields in a non-commutative gauge theory are described by functions of $z_1,\bar z_1,z_2,\bar z_2$. In the case of $\mathbb{R}^2_{\rm NC}\times\mathbb{R}^2$, the arguments z_2 and $\bar z_2$ are ordinary c-number coordinates, while z_1 and $\bar z_1$ are the creation and annihilation operators of a single SHO:

$$z_1|n\rangle=\sqrt{\zeta}\sqrt{n+1}|n\rangle,\quad \bar z_1|n\rangle=\sqrt{\zeta}\sqrt{n}|n-1\rangle\ . \tag{11.11}$$

The non-commutative spacetime $\mathbb{R}^4_{\rm NC}=\mathbb{R}^2_{\rm NC}\times\mathbb{R}^2_{\rm NC}$ requires two oscillators. The SHO Fock space $\mathscr{H}$ is spanned by the basis $|n_1,n_2\rangle$ with $n_1,n_2\geqslant0$:

$$z_1|n_1,n_2\rangle=\sqrt{\frac{\zeta}{2}}\sqrt{n_1+1}|n_1+1,n_2\rangle,\quad z_2|n_1,n_2\rangle=\sqrt{\frac{\zeta}{2}}\sqrt{n_2+1}|n_1,n_2+1\rangle\ ,$$
$$\bar z_1|n_1,n_2\rangle=\sqrt{\frac{\zeta}{2}}\sqrt{n_1}|n_1-1,n_2\rangle,\quad \bar z_2|n_1,n_2\rangle=\sqrt{\frac{\zeta}{2}}\sqrt{n_2}|n_1,n_2-1\rangle\ . \tag{11.12}$$

The integral on $\mathbb{R}^4_{\rm NC}$ is defined by the operator trace:

$$\int d^4x\ *=(2\pi)^2\sqrt{\det\theta}\,{\rm Tr}\ *=\left(\frac{\zeta\pi}{2}\right)^2{\rm Tr}\ *\ . \tag{11.13}$$

The ADHM construction of the instanton now proceeds exactly as in Section 4.2 but one has to keep an eye on the ordering of operators. We assume canonical form (2.57) and (2.58) for the ADHM matrices and analyse the requirements imposed by factorization condition (2.50). They amount to the modified ADHM constraints (cf. (2.65))

$$\vec{\tau}^{\dot\alpha}{}_{\dot\beta}\bar{a}^{\dot\beta}a_{\dot\alpha} = \vec{\zeta}_{(+)}1_{[k]\times[k]}\ . \tag{11.14}$$

The three conditions (11.14) are the modified ADHM constraints for the instanton. When $\vec{\zeta}_{(+)}=0$ Eqs. (11.14) give the standard commutative ADHM constraints (2.59a). When non-commutativity is present, the constraints are modified by the anti-self-dual component of θ. Thus, the ADHM constraints for the instanton in a self-dual-θ background on $\mathbb{R}^4_{\rm NC}$ are equal to those of commutative $\mathbb{R}^4$. However, the constraints for the instanton in non-commutative space $\mathbb{R}^2_{\rm NC}\times\mathbb{R}^2$ are always modified since in this case θ cannot be self-dual. The constraints are

$$\tau^{c\,\dot\alpha}{}_{\dot\beta}\bar{a}^{\dot\beta}a_{\dot\alpha} = \delta^{c3}\zeta 1_{[k]\times[k]}\ . \tag{11.15}$$

The constraints for an anti-instanton follow from solving the same factorization condition (2.50) with the matrix $\Delta=a+b\bar{x}$. In this case the ADHM constraints are modified by the self-dual component of θ:

$$\vec{\tau}^{\dot\alpha}{}_{\dot\beta}\,\bar{a}^{\dot\beta}a_{\dot\alpha} = \vec{\zeta}_{(-)}1_{[k]\times[k]}\ . \tag{11.16}$$

The explicit one-instanton and one-anti-instanton solutions for U(N) gauge theory were constructed in [193] for space–space, $\mathbb{R}^2_{\rm NC}\times\mathbb{R}^2$, and spacetime, $\mathbb{R}^4_{\rm NC}$, non-commutativity by resolving the corresponding (modified) ADHM constraints and solving completeness relation (2.51). As always, one has to distinguish between two types of singularities: the singularities of the instanton field $A_m(x)$ as a function of the argument, and the singularities arising for certain values of the collective coordinates interpreted as singularities on the moduli space. Instanton configurations can be determined in singular or regular gauges; in singular gauge all the singularities of instanton configurations as functions of x_m are simply gauge artifacts and can be gauged away.

Instantons with space–space non-commutativity arise from (11.15) where $\zeta\neq 0$ and their moduli space contains no singularities. Instantons with spacetime non-commutativity contain no singularities on the moduli space unless the self-duality of θ coincides with the self-duality of the instanton field strength such that ADHM constraints (11.14) and (11.16) collapse to (2.59a).

11.1.2. The prepotential of non-commutative $\mathcal{N}=2$ gauge theory

We can now consider the $\mathcal{N}=2$ supersymmetric U(N) gauge theory formulated in non-commutative space. On the Coulomb branch, the gauge group is broken to $\mathrm{U}(1)_{\rm c}\times\mathrm{U}(1)^{N-1}$, where $\mathrm{U}(1)_{\rm c}$ is the overall U(1) factor of the non-commutative U(N) gauge group which decouples in the infra-red due to the infra-red/ultra-violet mixing as explained in Ref. [195]. Our goal is to determine the low-energy dynamics of the $\mathrm{U}(1)^{N-1}$ factor.

In a similar way to the ordinary commutative case reviewed in Section 7, the corresponding low-energy effective action is determined by the Seiberg–Witten prepotential $\mathcal{F}$ as explained in Refs. [195,196]. In particular, note there are no commutative "star-products" in the low-energy effective action since we are concerned only with the leading-order terms in the derivative expansion of the effective action. As in the commutative theory, instantons and anti-instantons contribute to various correlation functions. By considering the instanton contributions to these correlation functions, one can relate the instanton coefficients of the prepotential to the centred instanton partition function. We shall assume that re-scalings (8.22) have been performed and so the relation is identical to (8.24). Although we did not consider the anti-instanton contributions in Section 8, one can easily derive an anti-instanton version of (8.24) which involves the complex conjugate of the prepotential. Summarizing, we have for $k > 0$

$$\mathscr{F}_k = \hat{\mathscr{Z}}_{+k}^{(\mathcal{N}=2,N_F)}, \qquad \mathscr{F}_k^* = \hat{\mathscr{Z}}_{-k}^{(\mathcal{N}=2,N_F)} . \tag{11.17}$$

Here, $\hat{\mathscr{Z}}_{\pm k}^{(\mathcal{N}=2,N_F)}$ are the instanton and anti-instanton centred instanton partition functions of the non-commutative theory and defined as integrals over the moduli spaces $\hat{\mathfrak{M}}_{\pm k}^{(\zeta)}$:

$$\hat{\mathscr{Z}}_{\pm k}^{(\mathcal{N}=2,N_F)} = \int_{\hat{\mathfrak{M}}_{\pm k}^{(\zeta)}} \boldsymbol{\omega}^{(\mathcal{N}=2,N_F)} \mathrm{e}^{-\tilde{S}} , \tag{11.18}$$

where, the supersymmetric volume form over the resolved centred instanton moduli space is explicitly

$$\begin{aligned}\int_{\hat{\mathfrak{M}}_{\pm k}^{(\zeta)}} \boldsymbol{\omega}^{(\mathcal{N})} = 2^{\mathcal{N}-2}\pi^{2(\mathcal{N}-1)} \frac{C_k^{(\mathcal{N})}}{\mathrm{Vol}\,\mathrm{U}(k)} \int \mathrm{d}^{4k(N+k)-4}\hat{a} \prod_{A=1}^{\mathcal{N}} \mathrm{d}^{2k(N+k)-2}\hat{\mathscr{M}}^A |\mathrm{det}_{k^2} \boldsymbol{L}|^{1-\mathcal{N}} \\ \times \prod_{r=1}^{k^2} \Bigg\{ \prod_{c=1}^{3} \delta\big(\tfrac{1}{2}\,\mathrm{tr}_k\, T^r(\tau^{c\dot{\alpha}}{}_{\dot{\beta}} \bar{a}^{\dot{\beta}} a_{\dot{\alpha}} - \zeta^c_{(\pm)} 1_{[k]\times[k]})\big) \prod_{A=1}^{\mathcal{N}} \prod_{\dot{\alpha}=1}^{2} \\ \times\, \delta(\mathrm{tr}_k\, T^r(\bar{\mathscr{M}}^A a_{\dot{\alpha}} + \bar{a}_{\dot{\alpha}} \mathscr{M}^A)) \Bigg\} .\end{aligned} \tag{11.19}$$

The expression is identical to the expression for the supersymmetric volume from of $\hat{\mathfrak{M}}_k$ defined in Section 6.5 apart from the fact that the bosonic ADHM constraints are modified appropriately to include the central terms.

From (11.19) we conclude that the centred instanton partition function can only depend on $\vec{\zeta}_{(+)}$ while the centred anti-instanton function can only depend on $\vec{\zeta}_{(-)}$. Given (11.17), the aforementioned dependences are very restrictive: the prepotential cannot depend on $\vec{\zeta}$ and therefore should be identical to that in the commutative theory. This was the hypothesis that was made in Ref. [195] for the theory with no hypermultiplets. In the next section, we will develop a localization formalism in which we can prove rigorously that the centred (anti-)instanton partition function cannot depend smoothly on $\vec{\zeta}_{(\pm)}$. There are possible discontinuities when $\vec{\zeta}_{(\pm)} = 0$, since at this point the (anti-)instanton moduli space becomes singular and the localization formalism breaks down. However, we can already test this hypothesis at the one-instanton level by re-doing the calculation of Section 8.4 in the context of the non-commutative theory. It is simple to establish the modifications that arise from including the non-commutativity parameter $\vec{\zeta}$. The instanton effective action (8.49) has the additional coupling

$$4\pi^2 \mathrm{i}\vec{D}\cdot\vec{\zeta}_{(+)} . \tag{11.20}$$

This modifies the $\vec{D}$ integral in (8.61)

$$\int \frac{d^3D}{\vec{D}^2}\, e^{-4\pi^2 i\vec{D}\cdot\vec{\zeta}_{(+)}} \prod_{u=1}^{N} \frac{\alpha_u^{*2}}{|\alpha_u|^4+\vec{D}^2}\ . \tag{11.21}$$

The angular integrals over $\vec{D}=(|\vec{D}|,\theta,\phi)$ are now non-trivial:

$$\int d(\cos\theta)\, d\theta\, e^{-4\pi^2 i\vec{D}\cdot\vec{\zeta}_{(+)}} = \frac{\sin(4\pi^2|\vec{D}|\,|\vec{\zeta}_{(+)}|)}{\pi|\vec{D}|\,|\vec{\zeta}_{(+)}|}\ . \tag{11.22}$$

We then extend the integral over $|\vec{D}|$ from $-\infty$ to $+\infty$ and perform it by standard contour integration yielding

$$\frac{1}{2|\vec{\zeta}_{(+)}|}\left(\prod_{\substack{u=1\\(\neq u)}}^{N}\frac{1}{\alpha_u^2} - \sum_{u=1}^{N}\frac{1}{\alpha_u^2}\, e^{-4\pi^2|\vec{\zeta}_{(+)}|\alpha_u^2}\prod_{\substack{v=1\\(\neq u)}}^{N}\frac{\alpha_v^{*2}}{|\alpha_v|^4-|\alpha_u|^4}\right)\ , \tag{11.23}$$

compared with the right-hand side of (8.61). The behaviour of the modified integrand in the vicinity of one of the N singularities $\chi=-\phi_u^0$, i.e. $\alpha_u=0$, is

$$2\pi^2\,\frac{\alpha_u^*}{\alpha_u}\prod_{\substack{v=1\\(\neq u)}}^{N}\frac{\alpha_v^{*2}}{|\alpha_v|^4-|\alpha_u|^4}+\cdots \tag{11.24}$$

and so unmodified from the commutative calculation. The contribution to the centred instanton partition function from the N singularities is therefore unchanged from (8.66). However, the contribution from the sphere at infinity in χ-space is modified since in the non-commutative case the asymptotic behaviour of the integrand is changed. In fact contrary to (8.68) we now have

$$f_1(r,\theta)\sim\frac{e^{-2iN\theta}}{r^{2N}} \tag{11.25}$$

and consequently the contribution from the sphere at infinity vanishes because either the integrand $f_1(r,\theta)f_2(re^{i\theta})$ falls off too fast, for $N_F<2N$, or due to a vanishing angular integral, for $N_F=2N$. Hence, in the non-commutative theory the result for the one-instanton coefficient of the prepotential is

$$\mathscr{F}_1^{\rm nc}=\sum_{u=1}^{N}\prod_{\substack{v=1\\(\neq u)}}^{N}\frac{1}{(\phi_v^0-\phi_u^0)^2}\prod_{f=1}^{N_F}(m_f+\phi_u^0)\ , \tag{11.26}$$

compared with (8.72) in the commutative case.

Notice that the additional contribution denoted $\mathscr{S}_1^{(N_F)}$ which we identified as arising form the vicinity of the singularity on the instanton moduli space, is missing from the non-commutative result, as might have been expected since the singularity of $\widehat{\mathfrak{M}}_1$ has been resolved. At the one-instanton level, we see that the commutative and non-commutative results are equivalent up to an unimportant constant for $N_F<2N$. This is entirely consistent with the hypothesis that commutative and non-commutative theories have the same SU(N) low-energy effective action. However, the finite theory with $N_F = 2N$ does have the apparently physically significant difference due to the term proportional to $\sum_{u=1}^{N} (\phi_u^0)^2$ that arises in commutative theory (8.72) but which is missing from (11.26). This difference, however, is not physically relevant since it can be explained by a re-definition of the coupling between the commutative and non-commutative theories. This follows from the expression for the prepotential in (8.7). If the coupling in the non-commutative theory is related to that in the commutative theory by a series of instanton couplings

$$\tau^{\rm nc} = \tau + \sum_{k=1}^{\infty} c_k e^{2\pi i k\tau} , \tag{11.27}$$

then SU(N) part of the low energy of the non-commutative theory is equal to that of the commutative theory. At the one-instanton level we have

$$c_1 = \frac{\alpha_2}{\pi i} \tag{11.28}$$

and α_2 is the coefficient in (8.70). Matching the non-commutative and commutative theories requires for higher instanton number a relation of the form

$$\mathscr{F}_k^{\rm nc} = \mathscr{F}_k - 2\pi i d_k \sum_{u=1}^{N} (\phi_u^0)^2 , \tag{11.29}$$

up to irrelevant (mass-dependent) constants, where d_k are determined in terms of the c_k. It is difficult to prove this relation rigorously. However, in the next section we prove that $\mathscr{F}_k^{\rm nc}$ cannot depend smoothly on $\vec{\zeta}_{(+)}$, hence, the difference $\mathscr{F}_k^{\rm nc} - \mathscr{F}_k$ cannot depend on $\vec{\zeta}_{(+)}$. Then assuming that the contribution to the difference arises from the sphere at infinity in χ-space,[86] generalizing the situation at the one-instanton level, the result can be argued to be polynomial in the masses and VEVs. Given that the prepotential has mass dimension 2, this leaves (11.29) as the only possible VEV dependence.

It should not have escaped the readers notice that something unexpected happens in the finite theory when the non-commutativity parameters are chosen so that either $\vec{\zeta}_{(+)} = 0$ or $\vec{\zeta}_{(-)} = 0$. In the former case, the instantons are those of the non-commutative theory, described by the smooth moduli space $\mathfrak{M}_1^{(\zeta)}$, while the anti-instantons are conventional commutative ones, described by the singular space $\mathfrak{M}_1$. In this case, even at the one-instanton level (11.17) is violated since the anti-instantons receive the contribution $\mathscr{S}_1^{((N_F=2N)}$ from the singularity on the instanton moduli space, while the anti-instantons do not; hence

$$\hat{\mathscr{L}}_{+1}^{(\mathscr{N}=2,N_F)} \neq (\hat{\mathscr{L}}_{-1}^{(\mathscr{N}=2,N_F)})^* . \tag{11.30}$$

In addition, a re-definition of form (11.27) cannot reconcile the commutative and non-commutative theories. It is not clear how one should describe what is really happening at these points in the non-commutativity parameter space and whether the resulting behaviour of the low-energy effective action is physically acceptable.

[86] This should follow from the localization technology developed in the next section.

11.2. Calculating the prepotential by localization

The major applications of the many instanton calculus that we have hitherto reported (the two-instanton contribution to the SU(2) prepotential in Section 8.3.1 excepted) have involved a large-N limit. Clearly one would like techniques to calculate instanton effects for any N and k. The problem is obvious: the integrals over the instanton moduli space are just too complicated to be done by brute force beyond instanton number one (or two in the special case of gauge group SU(2)). Even the integral over the one-instanton moduli space required to obtain the first-instanton coefficient of the prepotential with SU(N) gauge group, undertaken in Section 8.4 was far from elementary.

However there is hope: there is interesting mathematical structure underlying the integrals over the instanton moduli space that define the instanton partition function and hence the instanton coefficients of the prepotential. They are related to integrals that give topological invariants; for example, in the $\mathcal{N}=4$ case the integral is precisely the Gauss–Bonnet–Chern integral that—at least on a compact manifold—gives the Euler characteristic. [87] When the theory is on the Coulomb branch instantons are constrained and there is a non-trivial potential on the instanton moduli space caused by the non-trivial instanton effective action. This acts as a Morse potential and the integrals localize around various subspaces of the instanton moduli space around which the quadratic approximation is exact. This kind of localization is a generalization of the Duistermaat–Heckman Integration Formula that arises in various supersymmetric contexts (see, for example, [197–201] and references therein). The obvious problem to using this theory is the fact that the instanton moduli space is non-compact and has singularities when instantons shrink to zero size. The problem with the singularities can be alleviated by considering the resolution of the instanton moduli space $\mathfrak{M}_k^{(\zeta)}$ relevant to a non-commutative theory as described in Section 11.1. It will turn out that the integrals over the instanton moduli space in the non-commutative version of the theory on the Higgs or Coulomb branch are amenable to a form of localization. This will allow us to calculate the two-instanton contribution to the prepotential in an SU(N) (or rather U(N) since in the non-commutative version the additional abelian factor is inevitably included) theory. In fact the localization method greatly simplifies the integrals even at the $k>2$ level and we expect that further progress may be possible.

In retrospect it should not be a surprise that the instanton partition function localizes in the non-commutative theory. We will show that the subspaces of $\hat{\mathfrak{M}}_k^{(\zeta)}$ on which the instanton partition function localizes correspond to products of (non-commutative) U(1) moduli spaces $\hat{\mathfrak{M}}_k^{(\zeta)}|_{N=1}$. In fact the localization submanifolds are associated to each partition of $k\to k_1+\cdots+k_N$, $k_u\geqslant 0$, and have the form

$$\hat{\mathfrak{M}}_k^{(\zeta)}\to\bigcup_{\substack{\text{partitions}\\ k_1,\ldots,k_N}}\frac{\mathfrak{M}_{k_1}^{(\zeta)}|_{N=1}\times\cdots\times\mathfrak{M}_{k_N}^{(\zeta)}|_{N=1}}{\mathbb{R}^4}\ ,\tag{11.31}$$

where the quotient is by the overall centre of the instanton. This is a manifestation of the fact that in the non-commutative theory there are exact—no longer constrained—instanton solutions on the Coulomb branch. The reason is that the non-commutativity prevents instantons shrinking to zero size and there are now stable solutions of finite size and the consequences of Derrick's theorem are

[87] We do not discuss the mass-deformed $\mathcal{N}=4$ to 2 theory in this review; see [176].

avoided. The fact that these solutions are exact even in the presence of a VEV follows because the non-trivial fields only take values in the $\mathrm{U}(1)^N$ subgroup of the $\mathrm{U}(N)$ gauge group and so these fields commute with the VEVs. The partitions correspond to the instanton charge k_u in each of the N U(1) subgroups. It is then an issue as how one can relate the prepotential in the non-commutative theory to that in the commutative theory. We have already argued in Section 11.1.2 that they must essentially be the same. Our calculations at the one- and two-instanton level confirm this hypothesis, up to an interesting re-parameterization of the coupling in the theory with $N_F = 2N$.

Before proceeding with the more convenient formalism (from the point of view of performing actual computations) based on the linearized instanton partition function constructed in Section 6.5, it is useful to consider the problem in terms of the intrinsic geometry of $\mathfrak{M}_k$. We first note in the $\mathcal{N}=2$ theory, the number of Grassmann collective coordinates (we consider only the case $N_F=0$ for the moment) equals the dimension of $\mathfrak{M}_k$; namely $4kN$. We have already seen in Section 4.2.2 that the Grassmann collective coordinates ψ^A, for each species A, can be thought of as the components of Grassmann-valued symplectic tangent vectors to $\mathfrak{M}_k$. In the $\mathcal{N}=2$ case, there is also an isomorphism between them and the symplectic basis of one forms

$$\psi^{\mathrm{i}A} \leftrightarrow h^{\mathrm{i}\dot\alpha}, \qquad \dot\alpha \equiv A \ . \tag{11.32}$$

Here, the one forms $h^{\mathrm{i}\dot\alpha}$ manifest the $\mathrm{Sp}(n)\times\mathrm{SU}(2)$ structure of the (co-)tangent space of $T\mathfrak{M}_k$ and are related to the inverse Vielbeins via $h^{\mathrm{i}\dot\alpha} = h^{\mathrm{i}\dot\alpha}_\mu\,\mathrm{d}X^\mu$ (see Appendix B). Note that the isomorphism (11.32) preserves the structure of differential calculus: both Grassmann variables and one forms anti-commute. In addition, the $\mathcal{N}=2$ supersymmetric volume form on $\mathfrak{M}_k$ (5.15) can be re-interpreted in terms of differential forms:

$$\begin{aligned}\int_{\mathfrak{M}_k} \omega^{(\mathcal{N}=2)}\mathscr{F}(X,\psi^{\mathrm{i}A}) &= \pi^{-2kN}\int_{\mathfrak{M}_k}\left\{\bigwedge_{\mathrm{i}=1}^{2kN}\bigwedge_{\dot\alpha=1}^{2} h^{\mathrm{i}\dot\alpha}\right\}\left\{\prod_{\mathrm{i}=1}^{2kN}\prod_{A=1}^{2}\frac{\partial}{\partial\psi^{\mathrm{i}A}}\right\}\mathscr{F}(X,\psi^{\mathrm{i}A})\\ &= \pi^{-2kN}\int_{\mathfrak{M}_k}\mathscr{F}(X,h^{\mathrm{i}\dot\alpha}) \quad (\dot\alpha = A)\ ,\end{aligned} \tag{11.33}$$

where the function $\mathscr{F}$ is expanded in wedge powers of the forms $h^{\mathrm{i}\dot\alpha}$ until a top form is obtained which can then be integrated.

Using the isomorphism between the Grassmann collective coordinates and one forms, the instanton effective action (5.42) for $\mathcal{N}=2$ (so the final curvature coupling is absent) can be written succinctly as the inhomogeneous differential form

$$\tilde S = -\tfrac{1}{4}\,\mathrm{d}_V(V^\dagger_\mu\,\mathrm{d}X^\mu)\ . \tag{11.34}$$

Here, V and $V^\dagger$ are the (anti-)holomorphic components (defined as in (4.20)) of the (commuting) tri-holomorphic vector fields on $\mathfrak{M}_k$ denoted V_a, $a=1,2$, introduced in Section 5.2.1, associated to the VEVs. The other quantity in (11.34) is an *equivariant* exterior derivative [88]

$$\mathrm{d}_V \equiv d - 2\iota_V\ . \tag{11.35}$$

[88] Here, ι_V implies contraction with the vect or V.

One can easily verify that

$$d_V^2 = 2\mathscr{L}_V \ , \tag{11.36}$$

the Lie derivative with respect to V. Hence, d_V is nilpotent on $SU(N)$-invariant differential forms. As noted in [205], the centred instanton partition function can then be written in form notation as

$$\hat{\mathscr{Z}}_k^{(\mathcal{N}=2)} = \pi^{-2kN} \int_{\hat{\mathfrak{M}}_k} \exp(-\tfrac{1}{4}\, d_V(V_\mu^\dagger\, dX^\mu)) \ , \tag{11.37}$$

which means that the terms must be pulled down from the instanton effective action in order to make a top form on the centred moduli space $\hat{\mathfrak{M}}_k$ (of degree $4kN-4$).

Final form (11.37) is precisely the kind that can localize. To see this consider the more general integral

$$\hat{\mathscr{Z}}_k^{(\mathcal{N}=2)}(\lambda) = \pi^{-2kN} \int_{\hat{\mathfrak{M}}_k} \exp(-\lambda^{-1}\, d_V(V_\mu^\dagger\, dX^\mu)) \ . \tag{11.38}$$

We then have

$$\frac{\partial \hat{\mathscr{Z}}_k^{(\mathcal{N}=2)}(\lambda)}{\partial \lambda} = \pi^{-2kN} \int_{\hat{\mathfrak{M}}_k} \lambda^{-2}\, d_V\{V_\mu^\dagger\, dX^\mu \exp(-\lambda^{-1}\, d_V(V_\mu^\dagger\, dX^\mu))\} \tag{11.39}$$

using the fact that d_V^2 is nilpotent on $SU(N)$-invariant quantities. But since the volume form is $SU(N)$ invariant, the integral—under favourable conditions—vanishes and so $\hat{\mathscr{Z}}_k^{(\mathcal{N}=2)}(\lambda)$ is independent of λ. Hence, it can be evaluated in the limit $\lambda \to 0$ where the integral is dominated by the saddle-point approximation around the critical points of the vector field V_a. In the present case, the fixed-point set has been identified in Section 5.2.2 with the configurations where all the instantons have shrunk to zero size. The potential problem is that the theory of localization is most easy to apply to situations involving compact spaces without boundary. In the case at hand, the instanton moduli space is obviously not compact and has conical singularities when instantons shrink to zero size. As we have already intimated, the way to alleviate the problems caused by the singularities is to consider the analogous problem in the theory defined on a non-commutative spacetime. As explained in Section 11.1, the deformed instanton moduli space $\mathfrak{M}_k^{(\zeta)}$ is then a smooth resolution of $\mathfrak{M}_k$: instantons can no longer shrink to zero size and the corresponding canonical singularities are smoothed over. In this case, as we shall find below, the technique of localization can be used to considerably simplify the computation of the centred instanton partition function. In particular the critical-point sets become smooth manifolds of a very suggestive form.

The problem with formulating localization in terms of the intrinsic geometry of the instanton moduli space is that, due to the presence of the ADHM constraints, we do not have an explicit description of the intrinsic geometry. It will prove much more convenient to work in terms of the linearized formalism described in Section 6.5. It will also prove easier to formulate the theory of localization in terms of Grassmann variables instead of differential forms. The expression for the centred instanton partition is (6.103).[89] In the language of Grassmann variables, the translation

[89] We also perform re-scaling (8.22) so that there are no factors of g in the instanton effective action (6.96).

of the covariant derivative is a fermionic symmetry $\mathcal{Q}$, or BRST operator, which is nilpotent—at least up to symmetries. The fermionic symmetry that we need to prove the localization properties of the integrals is defined by picking out a particular supersymmetry transformation. These latter symmetries, acting on the variables of the linearized formulation, are given in (10.64b) and (6.85). From the supersymmetry transformations we can define corresponding supercharges via $\delta = \xi_{\dot\alpha A} Q^{\dot\alpha A}$. The fermionic symmetry that has the appropriate properties is then defined as the combination [90]

$$\mathcal{Q} = \varepsilon_{\dot\alpha A} Q^{\dot\alpha A} \ . \tag{11.40}$$

Notice that the definition of $\mathcal{Q}$ mixes up spacetime and R-symmetry indices as is characteristic of topological twisting. The action of $\mathcal{Q}$ on the variables is

$$\mathcal{Q} w_{\dot\alpha} = \mathrm{i}\varepsilon_{\dot\alpha A}\mu^A, \quad \mathcal{Q}\mu^A = -2\varepsilon^{\dot\alpha A}(w_{\dot\alpha}\chi + \phi^0 w_{\dot\alpha}) \ , \tag{11.41a}$$

$$\mathcal{Q} a'_{\alpha\dot\alpha} = \mathrm{i}\varepsilon_{\dot\alpha A}\mathcal{M}'^{A}_{\alpha}, \quad \mathcal{Q}\mathcal{M}'^{A}_{\alpha} = -2\varepsilon^{\dot\alpha A}[a'_{\alpha\dot\alpha}, \chi] \ , \tag{11.41b}$$

$$\mathcal{Q}\chi = 0, \quad \mathcal{Q}\chi^\dagger = 2\mathrm{i}\delta^{A}{}_{\dot\alpha}\bar\psi^{\dot\alpha}_A \ , \tag{11.41c}$$

$$\mathcal{Q}\bar\psi^{\dot\alpha}_A = \tfrac{1}{2}\,\delta^{\dot\alpha}{}_A[\chi^\dagger, \chi] - \mathrm{i}\vec D\cdot\vec\tau^{\dot\alpha}{}_{\dot\beta}\delta^{\dot\beta}{}_A, \quad \mathcal{Q}\vec D = \delta_{\dot\alpha}{}^A\vec\tau^{\dot\alpha}{}_{\dot\beta}[\bar\psi^{\dot\beta}_A, \chi] \ , \tag{11.41d}$$

$$\mathcal{Q}\mathcal{K} = \mathcal{Q}\tilde{\mathcal{K}} = 0 \ . \tag{11.41e}$$

It is straightforward to show that $\mathcal{Q}$ is nilpotent up to an infinitesimal $\mathrm{U}(k)\times\mathrm{SU}(N)$ transformation generated by χ and ϕ. For example,

$$\mathcal{Q}^2 w_{\dot\alpha} = 2\mathrm{i}(w_{\dot\alpha}\chi + \phi^0 w_{\dot\alpha}) \ . \tag{11.42}$$

In terms of $\mathcal{Q}$, the instanton effective action (6.96) assumes the form

$$\tilde S = \mathcal{Q}\,\Xi + \Gamma \ , \tag{11.43}$$

with

$$\Xi = 4\pi^2\,\mathrm{tr}_k\{\tfrac{1}{2}\,\varepsilon_{\dot\alpha A}\bar w^{\dot\alpha}(\mu^A\chi^\dagger + \phi^{0\dagger}\mu^A) + \tfrac{1}{4}\,\varepsilon_{\dot\alpha A}\bar a'^{\dot\alpha\alpha}[\mathcal{M}'^{A}_{\alpha}, \chi^\dagger] + \delta^{A}{}_{\dot\alpha}\bar\psi^{\dot\beta}_A(\bar a^{\dot\alpha}a_{\dot\beta} - \tfrac{1}{2}\,\vec\zeta_{(+)}\cdot\vec\tau^{\dot\alpha}{}_{\dot\beta})\} \tag{11.44}$$

[90] Note that a BRST-type operator was first constructed in the context of the $\mathcal{N}=2$ instanton calculus in Refs. [202–204]. (In particular, the latter reference is most closely related to the approach that we adopt in this paper.) In particular, these references emphasize the relation with the topologically twisted version of the original gauge theory. However, these references did not go on to use the existence of $\mathcal{Q}$ to develop a calculational technique based on localization. A nilpotent fermionic symmetry was also constructed in the context of the $\mathcal{N}=4$ instanton calculus in Ref. [176] where localization was first proposed as a method to calculate, in this case, the $\mathcal{N}=4$, instanton partition function. It was then shown in [202] that the $\mathcal{Q}$ operator in the $\mathcal{N}=2$ theory could be obtained by the orbifolding procedure described in Section 10.3.3 in the present article. Some recent papers [205] have also considered the $\mathcal{Q}$-operator and the $\mathcal{N}=2$ instanton calculus, although they use the equivalent language of differential forms described above. These references then go some way towards interpreting $\mathcal{F}_k$ as a topological intersection number.

and

$$\Gamma = -\pi^2 \sum_{f=1}^{N_F} \mathrm{tr}_k((m_f - \chi)\mathscr{K}_f \tilde{\mathscr{K}}_f) . \tag{11.45}$$

Note that $\mathscr{Q}\Gamma = 0$ so that the instanton effective action is "$\mathscr{Q}$-closed": $\mathscr{Q}\tilde{S} = 0$. Notice in (11.44), to anticipate what follows, we have allowed for a non-trivial non-commutativity parameter $\vec{\zeta}_{(+)}$.

Notice that both the action $\tilde{S}$ and the integration measure $\boldsymbol{\omega}^{(\mathscr{N}=2,N_F)}$ are $\mathscr{Q}$-invariant. The localization argument proceeds as before. Consider the general integral

$$\hat{\mathscr{Z}}_k^{(\mathscr{N}=2,N_F)}(\lambda) = \int_{\hat{\mathfrak{M}}_k} \boldsymbol{\omega}^{(\mathscr{N}=2,N_F)} \exp(-\lambda^{-1}\mathscr{Q}\,\Xi - \Gamma) . \tag{11.46}$$

We then have

$$\frac{\partial \hat{\mathscr{Z}}_k^{(\mathscr{N}=2,N_F)}(\lambda)}{\partial\lambda} = \lambda^{-2} \int_{\hat{\mathfrak{M}}_k} \boldsymbol{\omega}^{(\mathscr{N}=2,N_F)}\, \mathscr{Q}\{\Xi \exp(-\lambda^{-1}\mathscr{Q}\,\Xi - \Gamma)\} , \tag{11.47}$$

using the fact that $\mathscr{Q}^2\,\Xi = \mathscr{Q}\Gamma = 0$. Since the volume form is $\mathscr{Q}$-invariant, the right-hand side of (11.47) vanishes. Consequently, $\hat{\mathscr{Z}}_k^{(\mathscr{N}=2,N_F)}(\lambda)$ is independent of λ and can, therefore, be evaluated in the limit $\lambda \to 0$ where the integral is dominated by the critical points of $\mathscr{Q}\,\Xi$. Since the result is independent of λ, under favourable circumstances—which will be shown to hold in the present application—the Gaussian approximation is exact.

Note that the quantity Ξ depends on $\vec{\zeta}_{(+)}$ and the anti-holomorphic components of the VEV. Hence, the derivative of the instanton effective action with respect to either $\vec{\zeta}_{(+)}$ or $\phi^{0\dagger}$ is $\mathscr{Q}$-exact and so the instanton partition function cannot depend on these parameters. On the contrary, the holomorphic components of the VEVs enter through the action of $\mathscr{Q}$ itself, while the hypermultiplet masses enter through Γ, and so the centred instanton partition function *can* depend on these parameters. In particular, from what we have said above, we have proved that the centred instanton partition function cannot depend smoothly on the non-commutativity parameter $\vec{\zeta}_{(+)}$. Of course, there will be a discontinuity when $\vec{\zeta}_{(+)} = 0$ when singularities appear on the instanton moduli space and the localization argument breaks down.

Following the logic of localization we should investigate the critical points of $\mathscr{Q}\,\Xi$. For the moment, we suppose $\vec{\zeta}_{(+)} = 0$. The terms to minimize are, from (10.39),

$$|w_{\dot\alpha}\chi_a + \phi^0 w_{\dot\alpha}|^2 - [\chi_a, a'_n]^2 . \tag{11.48}$$

Notice that this is positive semi-definite and the critical points are simply the zeros. Hence

$$w_{\dot\alpha}\chi_a + \phi_a w_{\dot\alpha} = [\chi_a, a'_n] = 0 . \tag{11.49}$$

Notice that these equations are identical to those in (5.43). Hence we can identify the critical points of $Q\Xi$ which are the fixed points of the tri-holomorphic vector fields generated by the VEVs. The fixed-point set corresponds to the singular subspace $\mathrm{Sym}^k\,\mathbb{R}^4$ where all the instantons have shrunk down to zero size and the resulting gauge potential is pure gauge.

In order to avoid the singular nature of point-like instantons, Mollowood [95] suggested a regularization based on the smooth resolution of the instanton moduli space first described in purely geometrical terms without reference to the gauge theory by Nakajima in Ref. [185]. Of course, as discussed in Section 11.1, subsequently it was realized that this smooth resolution of the instanton moduli space $\mathfrak{M}_k^{(\zeta)}$ arises naturally when the theory is defined on a non-commutative spacetime [187]. What happens is that the existence of the central term in ADHM constraints (11.1) prevents instantons shrinking to zero size. The solutions are associated to partitions (5.44) and have form (5.46). The ADHM constraints within the uth block are now those of a non-commutative gauge theory with a U(1) gauge group. Taking the trace of the ADHM constraints within the block we have

$$\vec{\tau}^{\dot\alpha}{}_{\dot\beta} \sum_{i=k_{u-1}+1}^{k_u} \bar{w}_{iu}^{\dot\beta} w_{ui\dot\alpha} = k_u \vec{\zeta}_{(+)} \; . \tag{11.50}$$

Now, in contrast to (5.47), in the presence of the central term, $w_{ui\dot\alpha}$ must be non-trivial. The critical-point set associated to partition (5.44) is consequently a product of non-commutative U(1) instanton moduli spaces:

$$\frac{\mathfrak{M}_{k_1}^{(\zeta)}|_{N=1} \times \cdots \times \mathfrak{M}_{k_N}^{(\zeta)}|_{N=1}}{\mathbb{R}^4} \; . \tag{11.51}$$

Recall from the discussion in Section 11.1 that $\mathfrak{M}_k^{(\zeta)}|_{N=1}$ is a smooth resolution of the symmetric product $\mathrm{Sym}^k\,\mathbb{R}^4$.

In fact in the non-commutative theory, the consequences of Derrick's Theorem are avoided and there are now exact non-singular solutions to the equations-of-motion even in the presence of VEVs. These exact solutions were called "topicons" in [96] since they have a contribution to the action which is localized and, unlike an instanton, they have no size modulus. Actually there are N flavours of topicon, associated to the N block in (5.44), obtained by embedding the spacetime non-commutative U(1) instanton solutions in each of N unbroken abelian factors of the gauge group. So the following picture emerges. For instanton charge k, the exact instanton solutions come as a disjoint union of spaces associated to inequivalent partitions (5.44) where each k_u corresponds to each of N U(1) subgroups picked out by the VEV. Hence, the space of exact solutions, or "moduli space of topicons", lying within the larger instanton moduli space is of the form

$$\hat{\mathfrak{M}}_k \overset{\text{resolve}}{\to} \hat{\mathfrak{M}}_k^{(\zeta)} \overset{\text{exact}}{\supset} \hat{\mathfrak{M}}_k^{(\zeta)}|_{\text{topicon}} = \bigcup_{\substack{\text{partitions}\\ k_1+\cdots+k_N}} \frac{\mathfrak{M}_{k_1}^{(\zeta)}|_{N=1} \times \cdots \times \mathfrak{M}_{k_N}^{(\zeta)}|_{N=1}}{\mathbb{R}^4} \; . \tag{11.52}$$

11.2.1. One instanton

We will now begin using localization to evaluate the one-instanton contribution to the prepotential of the $\mathcal{N}=2$ supersymmetric SU(N) gauge theory with N_F hypermultiplets. The instanton effective action (6.96) has N critical points, labelled by $v \in \{1,2,\ldots,N\}$, at which (5.46)

$$\chi_a = -(\phi_a^0)_v, \qquad w_{u\dot\alpha} \propto \delta_{uv} \; . \tag{11.53}$$

Note that $a'_n = 0$ in the one-instanton sector. Without loss of generality, we choose our non-commutativity parameters

$$\zeta^1_{(+)} = \zeta^2_{(+)} = 0, \qquad \zeta^3_{(+)} \equiv \zeta > 0 . \tag{11.54}$$

In this case ADHM constraints (11.1) are solved on the critical submanifold with

$$w_{u\dot\alpha} = \sqrt{\zeta}\,\mathrm{e}^{\mathrm{i}\theta}\delta_{uv}\delta_{\dot\alpha 1} \tag{11.55}$$

for an arbitrary phase angle θ. The integrals over $w_{v\dot\alpha}$ are then partially annulled by the three δ-functions in (3.17) that impose the ADHM constraints, leaving a trivial integral over the phase angle θ:

$$\int d^2 w_v\, d^2\bar w_v \prod_{c=1}^{3} \delta\big(\tfrac12\,\tau^{c\dot\alpha}{}_{\dot\beta}(\bar w_v^{\dot\beta} w_{v\dot\alpha} - \zeta\delta^{c3})\big) = 8\pi\zeta^{-1} . \tag{11.56}$$

Correspondingly, the δ-functions for the Grassmann ADHM constraints saturate the integrals over $\{\mu^A_v, \bar\mu^A_v\}$:

$$\int d\mu^A_v\, d\bar\mu^A_v \prod_{\dot\alpha=1}^{2} \delta(\bar w_{v\dot\alpha}\mu^A_v + w_{v\dot\alpha}\bar\mu^A_v) = \zeta \tag{11.57}$$

for each $A = 1,2$. The remaining variables, $\{w_{u\dot\alpha}, \mu^A_u, \bar\mu^A_u\}$, $u \neq v$, as well as $\{\mathscr{K}_{if}, \tilde{\mathscr{K}}_{fi}\}$, are all treated as Gaussian fluctuations around the critical point. To this order, the instanton effective action (6.96) is

$$\tilde S = 4\pi^2 \left\{ \zeta\chi_a^2 + \sum_{\substack{u=1\\(\neq v)}}^{N} \left((\phi_a)^2_{vu}|w_{u\dot\alpha}|^2 + \frac{\mathrm{i}}{2}\,\phi^{0\dagger}_{vu}\bar\mu^A_u\mu_{uA}\right) - \frac14 \sum_{f=1}^{N_F} (m_f + \phi^0_v)\mathscr{K}_f\tilde{\mathscr{K}}_f \right\} + \cdots , \tag{11.58}$$

where $(\phi^0_a)_{uv} \equiv (\phi^0_a)_u - (\phi^0_a)_v$. The integrals are easily done. Note that the integral over χ_a yields a factor of ζ^{-1} which cancels against the factors of ζ arising from (11.56) and (11.57), so the final result is, as expected, independent of ζ. Summing over the N critical-point sets gives the centred one-instanton partition function

$$\mathscr{F}^{\rm nc}_1 \equiv \hat{\mathscr{Z}}_1^{(\mathscr{N}=2,N_F)} = \sum_{v=1}^{N} \left\{ \prod_{\substack{u=1\\(\neq v)}}^{N} \frac{1}{(\phi^0_v - \phi^0_u)^2} \prod_{f=1}^{N_F} (m_f + \phi^0_v) \right\} . \tag{11.59}$$

Note that the resulting expression is holomorphic in the VEVs and independent of ζ as required. The result should be compared with the brute-force calculation of the one-instanton contribution in (8.72). The expression is entirely consistent with (11.59) for $N_F < 2N$.[91] We can also compare (11.59) with the predictions of Seiberg–Witten theory written down in (8.8a). The expressions agree for

[91] The constant factor $\mathscr{S}_1^{(N_F)}$ for $N_F = 2N-2$ and $2N-1$ does not affect the low-energy effective action which depends only on derivatives of the prepotential with respect to the VEVs.

all for $N_F < 2N$. Recall that the Seiberg–Witten predictions as written are only valid for $N_F < 2N$; however, our expression (11.59) is simply the obvious extrapolation of (8.8a) to $N_F = 2N$.

The case with $N_F = 2N$, which corresponds to the finite theory, is rather special. Note that the expression in the non-commutative theory misses the contribution $\mathscr{S}_1^{(N_F=2N)}$ of (8.72) which is quadratic in the VEVs. We have already identified this term as arising from the singularity of the one-instanton moduli space $\hat{\mathfrak{M}}_1$. However, we have already argued at the end of Section 11.1.2 that the mismatch is not physically relevant since it can be accommodated by a non-trivial mapping between the couplings of the commutative and non-commutative theories.

11.2.2. Two instantons

We now turn to the situation for $k = 2$. There are two kinds critical point corresponding to two topicons of the same, or of different, flavour, respectively. We now evaluate these two contributions separately. As in the one-instanton sector we choose the non-commutativity parameters as in (11.54).

11.2.2.1. Two topicons of different flavour. For two topicons of the different flavour, the ADHM constraints are solved with

$$w_{ui\dot\alpha} = \sqrt{\zeta}e^{i\theta_i}\delta_{uu_i}\delta_{\dot\alpha 1}, \qquad a'_n = \begin{pmatrix} Y_n & 0 \\ 0 & -Y_n \end{pmatrix}. \tag{11.60}$$

The two phase angles θ_i, $i = 1, 2$, are not genuine moduli since they can be separately rotated by transformations in the U(2) auxiliary group. The variables Y_n are the genuine moduli representing the relative positions of the two topicons. The corresponding solution of the fermionic ADHM constraints (4.34) on the critical-point set is

$$\mu^A = \bar\mu^A = 0, \qquad \mathscr{M}'^A_\alpha = \begin{pmatrix} \rho^A_\alpha & 0 \\ 0 & -\rho^A_\alpha \end{pmatrix}, \tag{11.61}$$

where ρ^A_α are the superpartners of Y_n. Note that in this case the critical-point set is non-compact since the separation $Y_n \in \mathbb{R}^4$. The issue of whether the integral over Y_n converges is rather delicate. By explicit calculation we shall find that the integral is indeed convergent.

We now proceed to evaluate the contribution to the centred instanton partition function from the critical-point set. First we expand around critical values (11.60) and (11.61). It is convenient to use the following notation for the fluctuations

$$\delta a'_n = \begin{pmatrix} 0 & Z_n \\ Z_n^* & 0 \end{pmatrix}, \qquad \delta\mathscr{M}'^A_\alpha = \begin{pmatrix} 0 & \sigma^\alpha_\alpha \\ \varepsilon^A_\alpha & 0 \end{pmatrix} \tag{11.62}$$

and to make the shift

$$\chi_a \to \chi_a - \begin{pmatrix} (\phi^0_a)_{u_1} & 0 \\ 0 & (\phi^0_a)_{u_2} \end{pmatrix}, \tag{11.63}$$

so that $\chi_a = 0$ on the critical submanifold. We then integrate over the Lagrange multipliers $\vec{D}$ and $\bar\psi^{\dot\alpha}_A$ which impose ADHM constraints (2.65) and (4.34). The diagonal components of the constraints (in i,j indices) are the ADHM constraints of the two single U(1) instantons. The off-diagonal components vanish on the critical-point set and must therefore be expanded to linear order in the fluctuations. For the bosonic variables we have

$$\sqrt{\zeta}\mathrm{e}^{-\mathrm{i}\theta_1}(w_{u_12})_2+\sqrt{\zeta}\mathrm{e}^{\mathrm{i}\theta_2}(w_{u_21})_2^*+4\mathrm{i}\bar\eta^1_{mn}Y_mZ_n=0\ , \tag{11.64a}$$

$$-\mathrm{i}\sqrt{\zeta}\mathrm{e}^{-\mathrm{i}\theta_1}(w_{u_12})_2+\mathrm{i}\sqrt{\zeta}\mathrm{e}^{\mathrm{i}\theta_2}(w_{u_21})_2^*+4\mathrm{i}\bar\eta^2_{mn}Y_mZ_n=0\ , \tag{11.64b}$$

$$\sqrt{\zeta}\mathrm{e}^{-\mathrm{i}\theta_1}(w_{u_12})_1+\sqrt{\zeta}\mathrm{e}^{\mathrm{i}\theta_2}(w_{u_21})_1^*+4\mathrm{i}\bar\eta^3_{mn}Y_mZ_n=0\ , \tag{11.64c}$$

where $\bar\eta^c_{mn}=\frac{1}{2\mathrm{i}}\,\mathrm{tr}_2(\tau^c\bar\sigma_m\sigma_n)$ are 't Hooft's η-symbols defined in Appendix A. Similarly in the Grassmann sector

$$\sqrt{\zeta}\mathrm{e}^{\mathrm{i}\theta_2}\bar\mu^A_{1u_2}+2(\rho^{\alpha A}Z_{\alpha 1}-\sigma^{\alpha A}Y_{\alpha 1})=0\ , \tag{11.65a}$$

$$\sqrt{\zeta}\mathrm{e}^{-\mathrm{i}\theta_1}\mu^A_{u_12}+2(\rho^{\alpha A}Z_{\alpha 2}-\sigma^{\alpha A}Y_{\alpha 2})=0\ , \tag{11.65b}$$

$$\sqrt{\zeta}\mathrm{e}^{\mathrm{i}\theta_1}\bar\mu^A_{2u_1}+2(\varepsilon^{\alpha A}Y_{\alpha 1}-\rho^{\alpha A}Z^*_{\alpha 1})=0\ , \tag{11.65c}$$

$$\sqrt{\zeta}\mathrm{e}^{-\mathrm{i}\theta_2}\mu^A_{u_21}+2(\varepsilon^{\alpha A}Y_{\alpha 2}-\rho^{\alpha A}Z^*_{\alpha 2})=0\ , \tag{11.65d}$$

where $Y_{\alpha\dot\alpha}=Y_n\sigma_{\alpha\dot\alpha n}$, etc. These equations correspond to a set of linear relations between the fluctuations. It is convenient to define

$$(w_{u_12})_1=\mathrm{e}^{\mathrm{i}\theta_1}(\xi+\lambda),\qquad (w_{u_21})_1^*=\mathrm{e}^{-\mathrm{i}\theta_2}(-\xi+\lambda)\ , \tag{11.66}$$

so that the fluctuation ξ drops out from (11.64c). We can use (11.64a)–(11.64c) to solve for $(w_{u_12})_2$, $(w_{u_21})_2$ and λ, and (11.65a)–(11.65d) to solve for $\mu^A_{1u_2}$, $\mu^A_{2u_1}$, $\bar\mu^A_{u_12}$ and $\bar\mu^A_{u_21}$. We then use the U(2) symmetry to fix (i) the fluctuation Z_n to be orthogonal to Y_n, $Z_nY_n=0$ and (ii) $\theta_i=0$. The Jacobian for the first part of this gauge fixing is

$$\frac{1}{\mathrm{Vol\,U(2)}}\int d^{12}a'\to\frac{16}{\pi^2}\int d^4Y\,d^3Z\,d^3Z^*\,Y^2\ . \tag{11.67}$$

Now we turn to expanding the instanton effective action (6.96). First let us consider the bosonic pieces. To Gaussian order around the critical point

$$\tilde S_\mathrm{b}=\tilde S_\mathrm{b}^{(1)}+\tilde S_\mathrm{b}^{(2)}+\cdots\ , \tag{11.68}$$

where

$$\begin{aligned}\frac{1}{4\pi^2}\tilde S_\mathrm{b}^{(1)}&=\zeta((\chi_a)_{11}^2+(\chi_a)_{22}^2)+8Y^2|(\chi_a)_{12}|^2+2|(\phi_a^0)_{u_1u_2}\xi\\&\quad+\sqrt{\zeta}(\chi_a)_{12}|^2+2(\phi^0_a)^2_{u_1u_2}(1+4\zeta^{-1}Y^2)|Z|^2\end{aligned} \tag{11.69}$$

and

$$\frac{1}{4\pi^2}\tilde{S}_{\rm b}^{(2)} = \sum_{i=1}^{2}\sum_{\substack{u=1\\(\neq u_1,u_2)}}^{N}(\phi_a^0)_{uu_i}^2|w_{ui\dot\alpha}|^2 \ . \tag{11.70}$$

In order to simplify the integration over the fluctuations, it is convenient to shift

$$\xi \to \xi - \sqrt{\zeta}\frac{(\phi_a^0)_{u_1u_2}(\chi_a)_{12}}{(\phi_a)_{u_1u_2}^2} \tag{11.71}$$

and define the orthogonal decomposition

$$\chi_a = \chi_a^{\|} + \chi_a^{\perp}, \qquad \chi_a^{\perp}(\phi_a^0)_{u_1u_2} = 0 \ . \tag{11.72}$$

After having done this, (11.69) becomes

$$\begin{aligned}\frac{1}{4\pi^2}\tilde{S}_{\rm b}^{(1)} &= \zeta((\chi_a)_{11}^2 + (\chi_a)_{22}^2) + 2\zeta(1+4\zeta^{-1}Y^2)|(\chi_a^{\perp})_{12}|^2 + 8Y^2|(\chi_a^{\|})_{12}|^2\\ &\quad + 2(\phi_a^0)_{u_1u_2}^2(|\xi|^2 + (1+4\zeta^{-1}Y^2)|Z|^2) \ .\end{aligned} \tag{11.73}$$

To Gaussian order, the Grassmann parts of the instanton effective action (6.96) are

$$\tilde{S}_{\rm f} = \tilde{S}_{\rm f}^{(1)} + \tilde{S}_{\rm f}^{(2)} + \cdots \ , \tag{11.74}$$

where

$$\begin{aligned}\frac{1}{4\pi^2}\tilde{S}_{\rm f}^{(1)} &= -\frac{\rm i}{2}\phi_{u_1u_2}^{0\dagger}(1+4\zeta^{-1}Y^2)\sigma^{\alpha A}\varepsilon_{\alpha A} + {\rm i}\rho^{\alpha A}(2\zeta^{-1}(\phi_{u_1u_2}^0)^\dagger Z_{\alpha\dot\alpha}\bar{Y}^{\dot\alpha\beta} + \chi_{12}^\dagger\delta_\alpha{}^\beta)\varepsilon_{\beta A}\\ &\quad + {\rm i}\sigma^{\alpha A}(2\zeta^{-1}(\phi_{u_1u_2}^0)^\dagger Y_{\alpha\dot\alpha}\bar{Z}^{*\dot\alpha\beta} - \chi_{21}^\dagger\delta_\alpha{}^\beta)\rho_{\beta A} - 2{\rm i}\zeta^{-1}\phi_{u_1u_2}^{0\dagger}\rho^{\alpha A}Z_{\alpha\dot\alpha}\bar{Z}^{*\dot\alpha\beta}\rho_{\beta A}\end{aligned} \tag{11.75}$$

and

$$\frac{1}{4\pi^2}\tilde{S}_{\rm f}^{(2)} = \frac{\rm i}{2}\sum_{i=1}^{2}\sum_{\substack{u=1\\(\neq u_1,u_2)}}^{N}\phi_{uu_i}^{0\dagger}\bar\mu_{iu}^A\mu_{uiA} - \frac{1}{4}\sum_{i=1}^{2}\sum_{f=1}^{N_F}(m_f + \phi_{u_i}^0)\mathscr{K}_{if}\tilde{\mathscr{K}}_{fi} \ . \tag{11.76}$$

By shifting the fluctuations σ^A and ε^A by the appropriate amounts of ρ^A, we can complete the square yielding

$$\begin{aligned}\frac{1}{4\pi^2}\tilde{S}_{\rm f}^{(1)} &= -\frac{\rm i}{2}(\phi_{u_1u_2}^0)^\dagger(1+4\zeta^{-1}Y^2)\sigma^{\alpha A}\varepsilon_{\alpha A} - 2{\rm i}\zeta^{-1}(1+4\zeta^{-1}Y^2)^{-1}\rho^{\alpha A}(\phi_{u_1u_2}^{0\dagger}Z_{\alpha\dot\alpha}\bar{Z}^{*\dot\alpha\beta}\\ &\quad + 2\chi_{12}^\dagger Y_{\alpha\dot\alpha}\bar{Z}^{*\dot\alpha\beta} + 2\chi_{21}^\dagger Z_{\alpha\dot\alpha}\bar{Y}^{\dot\alpha\beta})\rho_{\beta A} \ .\end{aligned} \tag{11.77}$$

Before we proceed, let us remind ourselves that only the variables Y_n and ρ_α^A are facets of the critical-point set, the remaining variables are all fluctuations. The contribution to the centred instanton partition function from the critical-point set is then proportional to

$$\int d^4Y\, d\xi\, d\xi^*\, d^3Z\, d^3Z^*\, d^8\chi_a\, d^4\rho\, d^4\sigma\, d^4\varepsilon$$

$$\times\prod_{i=1}^{2}\left\{\prod_{\substack{u=1\\(\neq u_1,u_2)}}^{N} d^2w_{ui}\, d^2\bar{w}_{iu}\, d^2\mu_{ui}\, d^2\bar{\mu}_{iu}\prod_{f=1}^{N_F} d\mathscr{K}_{if}\, d\tilde{\mathscr{K}}_{fi}\right\} Y^2\exp(-\tilde{S}_{\rm b}^{(1)}-\tilde{S}_{\rm b}^{(2)}-\tilde{S}_{\rm f}^{(1)}-\tilde{S}_{\rm f}^{(2)})\ . \tag{11.78}$$

The integrals over the Grassmann variables $\{\sigma^A_\alpha, \varepsilon^A_\alpha, \rho^A_\alpha\}$ are saturated by pulling down terms from $S_{\rm f}^{(1)}$ yielding the factors

$$(\phi^{0\dagger}_{u_1u_2})^4\zeta^{-2}(1+4\zeta^{-1}Y^2)^2(4Y^2(\chi^\dagger_{21}Z-\chi^\dagger_{12}Z^*)^2+(\phi^{0\dagger}_{u_1u_2})^2(Z^2Z^{*2}-(Z\cdot Z^*)^2))\ . \tag{11.79}$$

The integrals over the remaining Grassmann variables $\{\mu^A_{ui}, \bar{\mu}^A_{iu}, \mathscr{K}_{if}, \tilde{\mathscr{K}}_{fi}\}$, $u\neq u_1, u_2$, are saturated by pulling down terms from $S_{\rm f}^{(2)}$ giving rise to

$$\prod_{i=1}^{2}\prod_{\substack{u=1\\(\neq u_1,u_2)}}^{N}(\phi^\dagger_{uu_i})^2\prod_{f=1}^{N_F}(m_f+\phi_{u_1})(m_f+\phi_{u_2})\ . \tag{11.80}$$

The $\{Z, \xi, \chi_a\}$ integrals are

$$\int d\xi\, d\xi^*\, d^3Z\, d^3Z^*\, d^8\chi_a\,(4Y^2(\chi^\dagger_{21}Z-\chi^\dagger_{12}Z^*)^2+2(\phi^0_{u_1u_2})^{\dagger 2}(Z^2Z^{*2}-(Z\cdot Z^*)^2))\, e^{-S^{(1)}}\ , \tag{11.81}$$

which yields the non-trivial factor

$$\frac{(\phi^{0\dagger}_{u_1u_2})^2}{\zeta^3(\phi^0_a)^{12}_{u_1u_2}Y^2(1+4\zeta^{-1}Y^2)^6} \tag{11.82}$$

while those over $w_{ui\dot\alpha}$, $u\neq u_1, u_2$, give a factor

$$\prod_{i=1}^{2}\prod_{\substack{u=1\\(\neq u_1,u_2)}}^{N}\frac{1}{(\phi^0_a)^4_{uu_i}}\ . \tag{11.83}$$

Finally, all that remains is to integrate over the relative position of the instantons. As promised the integral is convergent:

$$\int d^4Y\,\frac{\zeta^2}{(\zeta+4Y^2)^4}=\frac{\pi^2}{96}\ . \tag{11.84}$$

Putting all the pieces together with the correct numerical factors gives the final contribution of the critical-point set to the centred instanton partition function

$$\frac{2}{(\phi^0_{u_1u_2})^6}\prod_{i=1}^{2}\prod_{\substack{u=1\\(\neq u_1,u_2)}}^{N}\frac{1}{(\phi^0_{uu_i})^2}\prod_{f=1}^{N_F}(m_f+\phi^0_{u_1})(m_f+\phi^0_{u_2})\ . \tag{11.85}$$

Notice that the result is holomorphic in the VEVs as required. Summing over the $\frac{1}{2}N(N-1)$ critical points for topicons of different flavours, we have the contribution

$$\sum_{\substack{u,v=1\\(u\neq v)}}^{N} \frac{S_u(\phi_u^0)S_v(\phi_v^0)}{(\phi_u^0-\phi_v^0)^2}\ , \tag{11.86}$$

where we have written the answer in terms of the functions

$$S_u(x) \equiv \prod_{\substack{v=1\\(\neq u)}}^{N} \frac{1}{(x-\phi_v^0)^2}\prod_{f=1}^{N_F}(m_f+x) \tag{11.87}$$

defined in [90].

11.2.2.2. Two topicons of the same flavour. There are N critical points of this type, describing two topicons of the same flavour: $u_1 = u_2 \equiv v \in \{1,\ldots,N\}$. On the critical submanifold, $\{w_{vi\dot\alpha}, a'_n\}$ and $\{\mu^A_{vi}, \bar\mu^A_{iv}, \mathcal{M}'^A_\alpha\}$ satisfy the ADHM constraints, (11.1) and (4.34), respectively, of two instantons in a non-commutative U(1) theory. The remaining variables all vanish and are treated as fluctuations around the critical-point set.

As previously, it is convenient to shift the auxiliary variable χ_a by its critical-point value:

$$\chi_a \to \chi_a - (\phi_a^0)_v 1_{[2]\times[2]}\ . \tag{11.88}$$

We now expand in the fluctuations $\{w_{ui\dot\alpha}, \mu^A_{ui}, \bar\mu^A_{iu}\}$, for $u\neq v$, as well as $\{\mathcal{K}_{if}, \tilde{\mathcal{K}}_{fi}\}$. Since all the components of the ADHM constraints are non-trivial at leading order, the fluctuations decouple from the δ-functions in (3.17) and (5.53) which impose the constraints. The fluctuation integrals only involve the integrand $\exp -\tilde S$, where $\tilde S$ is expanded to Gaussian order around the critical-point set. However, it is important, as we shall see below, to leave χ_a arbitrary rather than set it to its critical-point value, namely, $\chi_a = 0$, after shift (11.88). The fluctuation integrals produce the non-trivial factor

$$\begin{aligned}&\prod_{\substack{u=1\\(\neq v)}}^{N} \frac{1}{(\det_2(\chi+\phi_{uv}^0 1_{[2]\times[2]}))^2}\prod_{f=1}^{N_F}\det_2((m_f+\sqrt2\phi_v^0)1_{[2]\times[2]}-\chi)\\ &\quad = S_v(\phi_v^0-\lambda_1)S_v(\phi_v^0-\lambda_2)\ .\end{aligned} \tag{11.89}$$

Here, λ_i, $i=1,2$, are the eigenvalues of the 2×2 matrix χ and $S_u(x)$ was defined in (11.87).

The remaining integrals involve the supersymmetric volume integral on $\hat{\mathfrak{M}}_2^{(\zeta)}|_{N=1}$, into which we insert integrand (11.89) which depends non-trivially on χ. Now by itself $\int_{\hat{\mathfrak{M}}_2^{(\zeta)}|_{N=1}} \omega^{(\mathcal{N}=2)} = 0$. This is clear from the linearized form of the instanton effective action (6.96) with $N=1$ and $N_F=0$: integrals over the Grassmann collective coordinates pull down two elements of the matrix $\chi^\dagger$ from the action and since there are no compensating factors of χ the resulting integrals over the phases of the elements of χ will integrate to zero. This is why we left χ arbitrary in (11.89) since after expanding in powers of the eigenvalues λ_i, it is potentially the quadratic terms that will give a non-zero result when inserted into $\int_{\hat{\mathfrak{M}}_2^{(\zeta)}|_{N=1}} \boldsymbol{\omega}^{(\mathcal{N}=2)}$. To quadratic order (11.89) is

$$\frac{1}{2}S_v(\phi_v^0)\frac{\partial^2 S_v(\phi_v^0)}{\partial(\phi_v^0)^2}(\lambda_1^2+\lambda_2^2)+\frac{\partial S_v(\phi_v^0)}{\partial\phi_v^0}\frac{\partial S_v(\phi_v)}{\partial\phi_v^0}\lambda_1\lambda_2 \ . \tag{11.90}$$

So the contribution from this critical-point set is of the form:

$$\mathscr{I}_1 S_v(\phi_v^0)\frac{\partial^2 S_v(\phi_v^0)}{\partial(\phi_v^0)^2}+\mathscr{I}_2\frac{\partial S_v(\phi_v^0)}{\partial\phi_v^0}\frac{\partial S_v(\phi_v^0)}{\partial\phi_v^0} \ , \tag{11.91}$$

where the VEV-independent constants $\mathscr{I}_{1,2}$ are given by the following integrals:

$$\mathscr{I}_1=\frac{1}{2}\int_{\hat{\mathfrak{M}}_2^{(\zeta)}|_{N=1}}\boldsymbol{\omega}^{(\mathcal{N}=2)}(\lambda_1^2+\lambda_2^2)\equiv\int_{\hat{\mathfrak{M}}_2^{(\zeta)}|_{N=1}}\boldsymbol{\omega}^{(\mathcal{N}=2)}\left(\frac{1}{2}(\mathrm{tr}_2\,\chi)^2-\det{}_2\,\chi\right) \ ,$$

$$\mathscr{I}_2=\int_{\hat{\mathfrak{M}}_2^{(\zeta)}|_{N=1}}\boldsymbol{\omega}^{(\mathcal{N}=2)}\lambda_1\lambda_2\equiv\int_{\hat{\mathfrak{M}}_2^{(\zeta)}|_{N=1}}\boldsymbol{\omega}^{(\mathcal{N}=2)}\det{}_2\,\chi \ . \tag{11.92}$$

We remark that (11.91) is holomorphic in the VEVs as required.

The moduli space $\hat{\mathfrak{M}}_2^{(\zeta)}|_{N=1}$ is the Eguchi–Hanson space [189], a well-known four-dimensional hyper-Kähler space [206]. So after all the Grassmann variables and χ_a have been integrated out, we can write $\mathscr{I}_{1,2}$ as integrals over the Eguchi–Hanson space of a suitable integrand. The following results were proved in the Appendix of [95]:

$$\mathscr{I}_1=\tfrac{1}{4} \quad \mathscr{I}_2=0 \ . \tag{11.93}$$

Hence the final result for the contributions from the N critical points of this type to the centred instanton partition function is

$$\frac{1}{4}\sum_{u=1}^{N}S_u(\phi_u)\frac{\partial^2 S_u(\phi_u)}{\partial(\phi_u^0)^2} \ . \tag{11.94}$$

Finally, summing (11.94) and (11.86) we have the centred two-instanton partition function

$$\mathscr{F}_2^{\mathrm{nc}}\equiv\hat{\mathscr{Z}}_2=\sum_{\substack{u,v=1\\(u\neq v)}}^{N}\frac{S_u(\phi_u^0)S_v(\phi_v^0)}{(\phi_u^0-\phi_v^0)^2}+\frac{1}{4}\sum_{u=1}^{N}S_u(\phi_u^0)\frac{\partial^2 S_u(\phi_u^0)}{\partial(\phi_u^0)^2} \ . \tag{11.95}$$

For $N<2N_F$ it is astonishing to find exact agreement with the prediction from Seiberg–Witten theory (8.8b). As in the one-instanton sector, our expression (11.95) with $N_F{=}2N$ is simply the extrapolation of formula (8.8b). We can also compare (11.95) with the explicit brute-force integration over the instanton moduli space for gauge group SU(2) that we reviewed in Section 8.3.1. The agreement is exact for $N_F<4$. As in the one-instanton sector, the case with $N_F=4$ is rather special since we expect the non-trivial matching of the coupling constants of the commutative and non-commutative theories displayed in (11.27). At the two-instanton level this matching implies

$$\mathscr{F}_2=\mathscr{F}_2^{\mathrm{nc}}+2\pi\mathrm{i}c_1\mathscr{F}_1^{\mathrm{nc}}+\mathrm{i}\pi c_2\sum_{u=1}^{N}(\phi_u^0)^2 \ , \tag{11.96}$$

modulo constants. Since we have already determined c_1 in (11.28) this matching condition is non-trivial even for gauge group SU(2) since the coefficients depend non-trivially on the masses and the VEVs. Using (11.59), (11.95) (for $N=2$) and (8.45), all with $N_F=4$, one finds complete consistency with

$$c_2|_{\mathrm{SU}(2)} = \frac{1}{2^3\pi\mathrm{i}}\left(1+\frac{7}{2^4 3^5}-\frac{13}{2^4}\right) . \tag{11.97}$$

The question is whether the localization technique can be extended beyond two instantons? In fact it has already in the context of the $\mathcal{N}=4$ theory softly broken to $\mathcal{N}=2$ by mass terms as described in Ref. [96]. In this reference the leading-order terms in the mass expansion of the prepotential were calculated to all orders in the instanton charge and were shown to agree with the predictions of Seiberg–Witten theory.

In fact, as we briefly mentioned at the start of this section, very recently Nekrasov has used the localization idea to find the contribution of arbitrary numbers of instanton to the prepotential in an SU(N) gauge theory with fundamental matter [179]. The philosophy is to calculate an appropriate physical quantity that is related to the prepotential but which also breaks Lorentz invariance in a specific way. At the level of the instanton calculus, the breaking of Lorentz invariance leads to an additional potential on the moduli space associated to the isometries generated by the two Cartan elements of $\mathrm{SU}(2)_L\times \mathrm{SU}(2)_R$. Upon localization, the instantons are now drawn together to the origin in spacetime as well as shrinking in size. Once again, non-commutativity regulates the singularity and what is left is a series of isolated critical points and the non-compactness problem encountered above is avoided. The contribution around each critical point is then calculated by using the techniques of [115,207]. In many respects, the resulting picture is most beautifully appreciated in five-dimensional gauge theory compactified on a circle, where the instantons appear as solitons [208] and the prepotential is related to equivariant index theory on the instanton moduli space. Seiberg–Witten theory is then recovered in the four-dimensional limit.

Acknowledgements

We would like to thank Massimo Bianchi, Chong-Sun Chu, Ed Corrigan, Francesco Fucito, Michael Green, Prem Kumar, Werner Nahm, Hugh Osborn, Misha Shifman, Gabriele Travaglini, Arkady Vainshtein and Stefan Vandoren for a wide variety of discussions on the instanton calculus and related subjects.

Appendix A. Spinors in diverse dimensions

In this appendix we define our conventions for spinors in various dimensions. The treatment will be geared towards the applications needed in the main text.

The Minkowski space metric will be chosen to be $\eta_{MN}=\mathrm{diag}(-1,1,\ldots,1)$. The D-dimensional Minkowski space Clifford algebra is

$$\{\Gamma_M,\Gamma_N\}=2\eta_{MN} \tag{A.1}$$

with $M, N = 0, \ldots, D-1$. The D-dimensional Euclidean space Clifford algebra is

$$\{\Gamma_M, \Gamma_N\} = 2\delta_{MN} \tag{A.2}$$

with $M, N = 1, \ldots, D$.

We now define representations of the Clifford algebra in several (even) dimensions. In all our representations the additional element of the Clifford algebra is

$$\Gamma_{D+1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{A.3}$$

so that Weyl spinors are of the form

$$\begin{pmatrix} \lambda \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ \bar{\psi} \end{pmatrix} . \tag{A.4}$$

For $D = 2$, we take

$$\Gamma_1 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{A.5}$$

in Euclidean space and

$$\Gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{A.6}$$

in Minkowski space. Weyl spinors are complex and real, respectively.

In four-dimensional Euclidean space we take

$$\gamma_n = \begin{pmatrix} 0 & -\mathrm{i}\sigma_n \\ \mathrm{i}\bar{\sigma}_n & 0 \end{pmatrix}, \quad n = 1\text{–}4 , \tag{A.7}$$

where $\sigma_n = (\mathrm{i}\vec{\tau}, 1)$ and $\bar{\sigma}_n = (-\mathrm{i}\vec{\tau}, 1)$ are the Euclidean space σ-matrices.[92] We will also use the quantities

$$\sigma_{mn} = \tfrac{1}{4}(\sigma_m\bar{\sigma}_n - \sigma_n\bar{\sigma}_m) ,$$
$$\bar{\sigma}_{mn} = \tfrac{1}{4}(\bar{\sigma}_m\sigma_n - \bar{\sigma}_n\sigma_m) . \tag{A.8}$$

Importantly, σ_{mn} is self-dual while $\bar{\sigma}_{mn}$ is anti-self-dual:

$$\sigma_{mn} = \tfrac{1}{2}\varepsilon_{mnkl}\sigma_{kl}, \quad \bar{\sigma}_{mn} = -\tfrac{1}{2}\varepsilon_{mnkl}\bar{\sigma}_{kl} . \tag{A.9}$$

[92] Here, $\vec{\tau}$ are the usual Pauli matrices.

They can be expressed as

$$\sigma_{mn} = \tfrac{1}{2}\, \mathrm{i}\eta^c_{mn}\tau^c, \quad \bar\sigma_{mn} = \tfrac{1}{2}\, \mathrm{i}\bar\eta^c_{mn}\tau^c \ , \tag{A.10}$$

in terms of 't Hooft's eta symbols [2] η^c_{mn} and $\bar\eta^c_{mn}$, $a = 1$–3, defined in (A.18).

In Minkowski space, we take

$$\gamma_n = \begin{pmatrix} 0 & \sigma_n \\ -\bar\sigma_n & 0 \end{pmatrix}, \quad n = 0\text{–}3 \ , \tag{A.11}$$

where we take the Minkowski space σ-matrices $\sigma^n = (-1, \vec\tau)$ and $\bar\sigma^n = (-1, -\vec\tau)$ to agree with the notation of Wess and Bagger [47].

In both Euclidean and Minkowski spaces, we follow the usual convention of writing two-component Weyl spinors as ψ_α, $\alpha = 1,2$, and $\bar\psi^{\dot\alpha}$, $\dot\alpha = 1,2$. A Dirac spinor is

$$\psi = \begin{pmatrix} \psi_\alpha \\ \bar\psi^{\dot\alpha}. \end{pmatrix} \ . \tag{A.12}$$

In Euclidean space, the Weyl spinors are pseudo-real and so ψ_α and $\bar\psi^{\dot\alpha}$ are independent quantities. In Minkowski space, the Weyl spinors are complex and one can define a four-component Majorana spinor of form (A.12) with

$$\bar\psi^{\dot\alpha} = (\psi_\alpha)^\dagger, \quad \dot\alpha = \alpha \ . \tag{A.13}$$

In this notation, the σ-matrices have indices $\sigma_{n\alpha\dot\alpha}$ and $\bar\sigma_n^{\dot\alpha\alpha}$. In both Minkowski and Euclidean spaces, the indices are raised and lowered with the anti-symmetric tensor ε:

$$\varepsilon_{21} = \varepsilon^{12} = 1, \quad \varepsilon_{12} = \varepsilon^{21} = -1, \quad \varepsilon_{11} = \varepsilon_{22} = 0 \ . \tag{A.14}$$

In addition, we use the summation conventions:

$$\psi\phi \equiv \psi^\alpha\phi_\alpha, \quad \bar\psi\bar\phi = \bar\psi_{\dot\alpha}\bar\phi^{\dot\alpha} \ . \tag{A.15}$$

In six-dimensional Euclidean space we take

$$\Gamma_a = \begin{pmatrix} 0 & \Sigma_a \\ \bar\Sigma_a & 0 \end{pmatrix} \ , \tag{A.16}$$

where the Σ-matrices are defined as

$$\begin{aligned} \Sigma_a &= (\eta^3, \mathrm{i}\bar\eta^3, \eta^2, \mathrm{i}\bar\eta^2, \eta^1, \mathrm{i}\bar\eta^1) \ , \\ \bar\Sigma_a &= (-\eta^3, \mathrm{i}\bar\eta^3, -\eta^2, \mathrm{i}\bar\eta^2, -\eta^1, \mathrm{i}\bar\eta^1) \ , \end{aligned} \tag{A.17}$$

and in their turn, the 4×4-dimensional matrices η^c and $\bar\eta^c$, $c = 1$–3, are 't Hooft's eta symbols [2]:

$$\begin{aligned} &\bar\eta^c_{AB} = \eta^c_{AB} = \varepsilon_{cAB}, \quad A, B \in \{1,2,3\} \ , \\ &\bar\eta^c_{4A} = \eta^c_{A4} = \delta_{cA} \ , \\ &\eta^c_{AB} = -\eta^c_{BA}, \quad \bar\eta^c_{AB} = -\bar\eta^c_{BA}. \end{aligned} \tag{A.18}$$

In six-dimensional Minkowski space, we can take (A.16) with

$$\Sigma_a = (\mathrm{i}\eta^3, \mathrm{i}\bar{\eta}^3, \eta^2, \mathrm{i}\eta^2, \eta^1, \mathrm{i}\bar{\eta}^1) ,$$
$$\bar{\Sigma}_a = (-\mathrm{i}\eta^3, \mathrm{i}\bar{\eta}^3, -\eta^2, \mathrm{i}\bar{\eta}^2, -\eta^1, \mathrm{i}\bar{\eta}^1) , \tag{A.19}$$

with $a = 0$–5.

The following properties of the Σ-matrices are valid both in Euclidean and Minkowski spaces:

$$\Sigma_{aAB}\Sigma_{aCD} = \bar{\Sigma}_{aAB}\bar{\Sigma}_{aCD} = 2\varepsilon_{ABCD} ,$$
$$\Sigma_{aAB}\bar{\Sigma}_{aCD} = -2\delta_{AC}\delta_{BD} + 2\delta_{AD}\delta_{BC} . \tag{A.20}$$

A positive (negative) chirality Weyl spinor is written ψ^A (ψ_A), $A = 1$–4. For consistency the Σ-matrices have indices Σ_a^{AB} and $\bar{\Sigma}_{aAB}$. In Euclidean space, the Weyl spinors are complex while in Minkowski space they are pseudo-real. In the latter case when there are an even number of such spinors we can introduce the concept of a "symplectic-real spinor" ψ_i^A, $i = 1,\ldots,2p$, satisfying the pseudo-reality condition

$$\psi_i^A = \mathrm{i}\Omega_{ij}\bar{\eta}^1_{AB}(\psi_j^B)^* , \tag{A.21}$$

where Ω_{ij} is a (symplectic) matrix with $\Omega\Omega^* = -1_{[2p]\times[2p]}$.

We can easily build up a representation of Clifford algebras in higher dimensions from those in lower dimensions. Suppose, we have Clifford algebras in p and q dimensions with Euclidean space generators $\Gamma_n^{(p)}$, $n = 1,\ldots,p$, and $\Gamma_a^{(q)}$, $a = 1,\ldots,q$. A representation of the Clifford algebra in $p+q$ dimensions can then be constructed by taking the products

$$\Gamma_N = \{\Gamma_n^{(p)} \otimes 1, \Gamma_{p+1}^{(p)} \otimes \Gamma_a^{(q)}\}, \quad N = 1,\ldots,p+q . \tag{A.22}$$

Appendix B. Complex geometry and the quotient construction

Our intention in this appendix is to introduce some basic properties of complex manifolds [93] and describe the "quotient construction".

A complex manifold $\mathfrak{M}$ admits a complex structure $\boldsymbol{I}$, a linear map of the tangent space to itself, satisfying $\boldsymbol{I}^2 = -1$, which is integrable. This latter property means that the *torsion*, or Neijenhuis tensor, vanishes, so for any two vectors X and Y

$$[\boldsymbol{I}X, \boldsymbol{I}Y] - [X, Y] - \boldsymbol{I}[X, \boldsymbol{I}Y] - \boldsymbol{I}[\boldsymbol{I}X, Y] = 0 . \tag{B.1}$$

Necessarily $\mathfrak{M}$ must be of even dimension, $2n$. A *Hermitian metric* g on $\mathfrak{M}$ is invariant under $\boldsymbol{I}$, so

$$g(\boldsymbol{I}X, \boldsymbol{I}Y) = g(X, Y) \tag{B.2}$$

[93] We will use the term "manifold" loosely to encompass spaces which are generically smooth but may have certain singularities.

for any two tangent vectors X and Y. The *fundamental 2-form* ω is defined by

$$\omega(X,Y) = g(\boldsymbol{I}X,Y) \; . \tag{B.3}$$

(Notice that ω is anti-symmetric as a consequence of (B.2) and $\boldsymbol{I}^2 = -1$.) We will denote (real) local coordinates on $\mathfrak{M}$ by x^μ, $\mu = 1,\dots,2n$; however, a complex manifold always admits local holomorphic coordinates $(z^{\rm i},\bar z^{\rm i})$, ${\rm i} = 1,\dots,n$, for which

$$\boldsymbol{I} = \begin{pmatrix} {\rm i}\delta^{\rm i}_{\rm j} & 0 \\ 0 & -{\rm i}\delta^{\rm i}_{\rm j} \end{pmatrix} \; . \tag{B.4}$$

In this basis, the Hermitian metric and fundamental two-form are

$$g = g_{\rm ij}\,{\rm d}z^{\rm i}\,{\rm d}\bar z^{\rm j}, \quad \omega = {\rm i}g_{\rm ij}\,{\rm d}z^{\rm i}\wedge{\rm d}\bar z^{\rm j} \; . \tag{B.5}$$

A Kähler manifold is one for which the fundamental two-form is closed, in which case the latter is called the *Kähler form*. The closure of the Kähler form implies (and follows from) the complex structure $\boldsymbol{I}$ is covariantly constant

$$\nabla_\mu \boldsymbol{I} = 0 \; , \tag{B.6}$$

with respect to the Levi–Civita connection associated to g. In addition, in holomorphic coordinates, the metric is given in terms of a *Kähler potential* K by

$$g_{\rm ij} = \frac{\partial}{\partial z^{\rm i}}\frac{\partial}{\partial \bar z^{\rm j}} K \; . \tag{B.7}$$

Another way to characterize a Kähler manifold is via its holonomy group. Generically, on a manifold of dimension $2n$ this would be $O(2n)$; however, since parallel transport with the connection ∇_μ respects the holomorphic structure (does not mix holomorphic and anti-holomorphic indices) the holonomy of a Kähler manifold is contained in the subgroup $U(n) \subset O(2n)$.

We now proceed to define a hyper-Kähler manifold. Such a space $\mathfrak{M}$ admits three independent complex structures $\boldsymbol{I}^{(c)}$, $c = 1$–3 (which we often represent as a three-vector $\vec{\boldsymbol{I}}$), satisfying the algebra

$$\boldsymbol{I}^{(c)}\boldsymbol{I}^{(d)} = -\delta^{cd} + \varepsilon^{cde}\boldsymbol{I}^{(e)} \; , \tag{B.8}$$

which are covariantly constant with respect to the Levi–Civita connection of a metric g which is Hermitian (B.2) with respect to each of the $\vec{\boldsymbol{I}}$. There are three Kähler forms $\vec\omega$ each related to the metric as in (B.3). The space $\mathfrak{M}$ is necessarily $4n$ dimensional. As before, we will take x^μ, $\mu = 1,\dots,4n$, to be a set of real local coordinates for $\mathfrak{M}$. There are holomorphic coordinates with respect to each of the complex structures, but generally there are no local coordinates which simultaneously represent the action of all the $\vec{\boldsymbol{I}}$; consequently we will have to work in terms of non-coordinate bases. The tangent space $T\mathfrak{M}$ of a hyper-Kähler manifold admits the following $Sp(n) \times SU(2)$ description. There is a vielbein $h^\mu_{{\rm i}\dot\alpha}$ and an inverse $h^{{\rm i}\dot\alpha}_\mu$. The composite index ${\rm i}\dot\alpha$, ${\rm i} = 1,\dots,2n$ and $\dot\alpha = 1,2$, manifests the fact that the vielbein picks out an $Sp(n)\times SU(2)$ structure in $T\mathfrak{M}$.[94] The metric takes the form

$$g = h^{{\rm i}\dot\alpha}{}_\mu h^{{\rm j}\dot\beta}{}_\nu \Omega_{\rm ij}\varepsilon_{\dot\alpha\dot\beta}\,{\rm d}x^\mu\,{\rm d}x^\nu \; . \tag{B.9}$$

[94] The group $Sp(n)$, which has rank n, is often denoted as $USp(2n)$.

Here, $\Omega_{\rm ij}$, $\rm i=1,\ldots,2\mathit{n}$, is an $\mathrm{Sp}(n)$ two-form. The index $\dot\alpha=1,2$ is an SU(2) spinor index which can be raised and lowered in the usual way by the ε tensor associated to SU(2) spinors (see Appendix A). The complex structures act on the SU(2) indices of the vielbeins in a simple way:

$$(\boldsymbol{I}^{(c)}\cdot h)^{\rm i\dot\alpha}{}_{\mu}=-{\rm i}\tau^{c\dot\alpha}{}_{\dot\beta}h^{\rm i\dot\beta}{}_{\mu}\ , \tag{B.10}$$

where τ^c are the three Pauli matrices. Note that the three complex structures can be rotated as a three-vector under action of the SU(2), although this action is not generally an isometry of $\mathfrak{M}$. The three Kähler forms corresponding to the three complex structures are

$$\vec\omega={\rm i}\Omega_{\rm ij}\varepsilon_{\dot\alpha\dot\beta}\vec\tau^{\dot\alpha}_{\dot\gamma}h^{\rm i\dot\beta}{}_{\mu}h^{\rm j\dot\gamma}{}_{\nu}\,{\rm d}x^{\mu}\wedge{\rm d}x^{\nu}\ . \tag{B.11}$$

On a hyper-Kähler manifold, since the tangent space admits an $\mathrm{Sp}(n)\times\mathrm{SU}(2)$ structure, one can define the notion of *symplectic tangent vectors*. These are quantities, like $\mathscr{M}^{\rm i}$, which carry $\mathrm{Sp}(n)$ indices only and which have a symplectic inner product

$$\Omega(\mathscr{M},\mathscr{N})=\Omega_{\rm ij}\mathscr{M}^{\rm i}\mathscr{N}^{\rm j}\ . \tag{B.12}$$

For a hyper-Kähler manifold, the SU(2) part of the spin connection $\omega_{\mu}{}^{\rm i\dot\alpha}{}_{\rm j\dot\beta}$ (not to be confused with the Kähler forms) vanishes. This means

$$\omega_{\mu}{}^{\rm i\dot\alpha}{}_{\rm j\dot\beta}=\omega_{\mu}{}^{\rm i}{}_{\rm j}\delta^{\dot\alpha}_{\dot\beta}\ . \tag{B.13}$$

The tensor $\Omega_{\rm ij}$ is covariantly constant with respect to the $\mathrm{Sp}(n)$ part of the connection. Similarly, the only non-vanishing components of the Riemann tensor are

$$R_{\rm i\dot\alpha j\dot\beta k\dot\gamma l\dot\delta}=R_{\rm ijkl}\varepsilon_{\dot\alpha\dot\beta}\varepsilon_{\dot\gamma\dot\delta}\ . \tag{B.14}$$

This is a symptom of the fact that the holonomy group of a hyper-Kähler manifold is restricted to $\mathrm{Sp}(n)\subset\mathrm{U}(2n)\subset\mathrm{O}(4n)$. The $\mathrm{Sp}(n)$ curvature $R_{\rm ijkl}$ is a totally symmetric tensor in all its indices.

After the brief introduction to complex geometry and, in particular, hyper-Kähler spaces, let us now turn to a way of constructing such spaces known as the *quotient construction*. We begin, initially, with the simpler Kähler case. The quotient construction is a way of constructing a Kähler manifold $\mathfrak{M}$ in terms of some larger-dimensional Kähler manifold, the "mother" space, $\tilde{\mathfrak{M}}$ which admits some group of isometries G preserving both the metric and complex structure. We will be interested in the case when G is some compact Lie group and we will denote the Hermitian generators T^r, $r=1,\ldots,\dim G$.[95] Each generator T^r of G defines a vector field X_r over $\tilde{\mathfrak{M}}$ in the usual way. Since the group action preserves the metric and complex structure, the Lie derivatives $\mathscr{L}_{X_r}\tilde{\boldsymbol{I}}=\mathscr{L}_{X_r}\tilde g=0$. This implies that X_r is a *holomorphic Killing vector*. The two conditions imply a third, namely, $\mathscr{L}_{X_r}\tilde\omega=0$. Since

$$\mathscr{L}_{X_r}\tilde\omega=\imath_{X_r}{\rm d}\tilde\omega+{\rm d}(\imath_{X_r}\tilde\omega)\equiv{\rm d}(\imath_{X_r}\tilde\omega)\ , \tag{B.15}$$

[95] We will take an inner product on the Lie algebra of G with $\mathrm{Tr}\,T^rT^s=\delta^{rs}$ in some matrix representation.

there exits a *Hamiltonian* function μ^{X_r} where $\mathrm{d}\mu^{X_r} = \iota_{X_r}\tilde{\omega}$.[96] In fact this only defines μ^{X_r} up to a constant which can be fixed, up to the abelian factors in G, by requiring that μ^X is *equivariant*: $X\mu^Y = \mu^{[X,Y]}$. In this case, the Hamiltonians are known as *moment maps*. As T^r varies in the Lie algebra of G, we can define the function μ from $\tilde{\mathfrak{M}}$ to the Lie algebra

$$\mu = \sum_{r=1}^{\dim G} \mu^{X_r} T^r \ . \tag{B.16}$$

The Kähler quotient is then

$$\mathfrak{M} = \mu^{-1}(0)/G \ , \tag{B.17}$$

in other words the quotient of the subspace $\mathfrak{N} \subset \tilde{\mathfrak{M}}$ on which the moment map vanishes, the so-called *level set*, by the group G. The daughter $\mathfrak{M}$ is of dimension

$$\dim \mathfrak{M} = \dim \tilde{\mathfrak{M}} - 2 \dim G \tag{B.18}$$

and inherits a complex structure, metric and connection from $\tilde{\mathfrak{M}}$. In order for the quotient to be well defined, we require that G, at least generically, acts freely on $\tilde{\mathfrak{M}}$. There may be points at which the action fails to be free, in which case there will be orbifold-type singularities in the quotient. Since

$$\tilde{g}(\tilde{\boldsymbol{I}}X_r, X_s) = \tilde{\omega}(X_r, X_s) = -\langle X_r, \mathrm{d}\mu^{X_s}\rangle = -\mu^{[X_r,X_s]} = 0 \ , \tag{B.19}$$

using the equivariance property, a basis for the vectors in the tangent space normal to the level set is given by $\tilde{\boldsymbol{I}}X_r$, $r = 1, \ldots, \dim G$.

Before we discuss the induced metric, complex structure and connection, let us say a little more about the G-quotient part of the construction. It is useful to think of $\mathfrak{N}$ as a principal G-bundle over the quotient $\mathfrak{M} = \mathfrak{N}/G$. Hence there is a projection $p : \mathfrak{N} \to \mathfrak{M}$. The G-action picks out a subspace of $\mathscr{V} \subset T\mathfrak{N}$ called the *vertical* space spanned by the vector fields X_r. The tangent space $T\mathfrak{M}$ is then identified with the quotient vector space $T\mathfrak{N}/\mathscr{V}$. Let the *horizontal* space, $\mathscr{H} \subset T\mathfrak{N}$ be the subspace of vectors orthogonal to $\mathscr{V}$, i.e. $\tilde{g}(X, X_r) = 0$ for all r, so that

$$T\mathfrak{N} = \mathscr{H} \oplus \mathscr{V} \ . \tag{B.20}$$

A tangent vector $X \in T\mathfrak{M}$ has a unique horizontal lift to $\mathscr{H}$ (which by a slight abuse of notation we denote with the same letter). From these definitions it follows that a vector $X \in \mathscr{H} \subset T\tilde{\mathfrak{M}}$ satisfies

$$\tilde{g}(X, \tilde{\boldsymbol{I}}X_r) = \tilde{g}(X, X_r) = 0 \ . \tag{B.21}$$

The complex structure $\tilde{\boldsymbol{I}}$ in the mother space naturally induces one in the daughter. To see this it is enough to notice from (B.21) that $\tilde{\boldsymbol{I}}$ preserves $\mathscr{H}$. Consequently, the lift of $\boldsymbol{I}$ is simply equal to $\tilde{\boldsymbol{I}}$ acting on $\mathscr{H}$ and integrability condition (B.1) is automatically satisfied. The induced Riemannian metric on $\mathfrak{M}$ is obtained in a analogous way. For two vectors $X, Y \in T\mathfrak{M}$ (with lifts to $\mathscr{H}$ denoted by the same letter), $g(X, Y) = \tilde{g}(X, Y)$. Finally a Levi–Civita connection $\tilde{\nabla}$ on $\tilde{\mathfrak{M}}$ induces such a connection ∇ on the daughter in the following way. First of all, the projection of the connection $\tilde{\nabla}$

[96] This reasoning assumes that the first cohomology group $H^1(\tilde{\mathfrak{M}}, \mathbb{R})$ is trivial.

to the tangent space $T\mathfrak{N} \subset T\tilde{\mathfrak{M}}$ defines a connection on $\mathfrak{N}$. It is easy to prove that the connection on $\mathfrak{N}$ is of Levi–Civita type. This is then identified with the pull-back of the Levi–Civita connection ∇ on $\mathfrak{M}$ to $\mathfrak{N}$, under the projection p.

Of particular importance for the instanton calculus is the volume form induced on the daughter space (in the hyper-Kähler case discussed below). It will be convenient for applications to the instanton calculus to define a volume form which, using physicists' language, is not gauge fixed. This means a G-invariant volume form $\boldsymbol{\omega}$ on the level set $\mathfrak{N}$ divided by the volume of the G-orbit at that point. A bona-fide volume form on $\mathfrak{M}$ could then be obtained by a gauge-fixing procedure, but we shall not necessarily need to perform this operation. From what we have said above, it is straightforward to see that the volume form we are after is

$$\int_{\mathfrak{M}} \boldsymbol{\omega} \stackrel{\text{def}}{=} \int_{\tilde{\mathfrak{M}}} \tilde{\boldsymbol{\omega}} \frac{J(x)}{\mathrm{Vol}_G(x)} \prod_{r=1}^{\dim G} \delta(\mu^{X_r}) . \tag{B.22}$$

Here, $\tilde{\boldsymbol{\omega}}$ is the canonical volume form on $\mathfrak{M}$, $\mathrm{Vol}_G(x)$ is the volume of the G-orbit through a point $x \in \mathfrak{M}$ and $J(x)$ is a Jacobian factor that arises when the integral over $\tilde{\mathfrak{M}}$ is restricted to $\mathfrak{N}$ by the explicit δ-functions. Geometrically, $J(x)$ is the square root of the determinant of the matrix of inner products of the normal vectors to the level set. Since a basis of such vectors is provided by $\tilde{\boldsymbol{I}}X_r$, the Jacobian is simply

$$J(x) = |\det \boldsymbol{L}|^{1/2} , \tag{B.23}$$

where $\boldsymbol{L}$ is the $\dim G \times \dim G$ matrix of inner products with elements

$$\boldsymbol{L}_{rs} \equiv \tilde{g}(\tilde{\boldsymbol{I}}X_r, \tilde{\boldsymbol{I}}X_s) = \tilde{g}(X_r, X_s) . \tag{B.24}$$

Since G generically acts freely on $\tilde{\mathfrak{M}}$, each orbit is, up to a scale factor, a copy of G itself. Hence, the volume of the orbit through a point on the level set is

$$\mathrm{Vol}_G(x) = |\det \boldsymbol{L}|^{1/2}\, \mathrm{Vol}\, G . \tag{B.25}$$

Here, $\mathrm{Vol}\, G$ is a constant, the volume of the group in some canonical normalization. Notice that the factors of the determinant of $\boldsymbol{L}$ cancel to leave

$$\int_{\mathfrak{M}} \boldsymbol{\omega} = \frac{1}{\mathrm{Vol}\, G} \int_{\tilde{\mathfrak{M}}} \tilde{\boldsymbol{\omega}} \prod_{r=1}^{\dim G} \delta(\mu^{X_r}) . \tag{B.26}$$

The hyper-Kähler quotient construction is an obvious generalization of the Kähler quotient construction described above. One starts with a hyper-Kähler space $\tilde{\mathfrak{M}}$ admitting a group action G which preserves the metric and three complex structures. The isometries correspond to vector fields which are *tri-holomorphic Killing vectors*, that is (i) holomorphic with respect to each of the three complex structures $\mathscr{L}_X \tilde{\boldsymbol{I}}^{(c)} = 0$ and (ii) preserving the metric $\mathscr{L}_X \tilde{g} = 0$. Associated to each complex structure and Kähler form there is a moment map, defined as in the Kähler case above, which we can assemble into the triplet $\vec{\mu}$. In an identical way to the Kähler case, one can prove that the quotient

$$\mathfrak{M} = \vec{\mu}^{-1}(0)/G \tag{B.27}$$

is a hyper-Kähler space of dimension $\dim\tilde{\mathfrak{M}} - 4\dim G$. In the hyper-Kähler case, a basis of vectors normal to the level set is provided by the $3\dim G$ vectors $\tilde{\boldsymbol{I}}^{(c)}X_r$. As in the Kähler case, a natural metric $g(X,Y)$ is induced on $\mathfrak{M}$ by taking $\tilde{g}(X,Y)$, with X and Y (denoted by the same symbol) being the unique lifts of X and Y to $\mathscr{H}$. The local condition for a vector $X \in T\tilde{\mathfrak{M}}$ to be in $\mathscr{H}$ is

$$\tilde{g}(X, \tilde{\boldsymbol{I}}^{(c)}X_r) = \tilde{g}(X, X_r) = 0\ , \tag{B.28}$$

generalizing (B.21). In the $\mathrm{SU}(2)\times\mathrm{Sp}(n)$ basis, this condition is $X^{\tilde{\mathrm{i}}\dot{\alpha}}\tilde{\Omega}_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}}\varepsilon_{\dot{\alpha}\dot{\beta}}X_r^{\tilde{\mathrm{j}}\dot{\gamma}}\tau^{\dot{\beta}}_{\dot{\gamma}} = X^{\tilde{\mathrm{i}}\dot{\alpha}}\tilde{\Omega}_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}}\varepsilon_{\dot{\alpha}\dot{\beta}}X_r^{\tilde{\mathrm{j}}\dot{\beta}} = 0$, or equivalently

$$X^{\tilde{\mathrm{i}}\dot{\alpha}}\tilde{\Omega}_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}}X_r^{j\dot{\beta}} = 0\ . \tag{B.29}$$

The fact that the SU(2) indices of the tangent vector are un-summed, allows us to identify symplectic tangent vectors to the quotient space. Their lifts are simply symplectic tangent vectors of $\tilde{\mathfrak{M}}$ subject to projection (B.29):

$$\mathscr{M}^{\tilde{\mathrm{i}}}\tilde{\Omega}_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}}X_r^{\tilde{\mathrm{j}}\dot{\beta}} = 0\ . \tag{B.30}$$

Of particular importance to the instanton calculus is the volume form on the quotient space, generalizing expression (B.26) in the Kähler case. The Jacobian factor is, as before, the determinant of the matrix of inner products of the basis vectors $\tilde{\boldsymbol{I}}^{(c)}X_r$ normal to the level set. We have

$$\tilde{g}(\tilde{\boldsymbol{I}}^{(c)}X_r, \tilde{\boldsymbol{I}}^{(d)}X_s) = \delta^{cd}\tilde{g}(X_r,X_s) - \varepsilon^{cde}\tilde{g}(X_r, \tilde{\boldsymbol{I}}^{(e)}X_s) = \begin{cases} \tilde{g}(X_r,X_s), & c=d\ ,\\ 0, & c\neq d\ ,\end{cases} \tag{B.31}$$

where we have used (B.19). This means that

$$J(x) = |\det \boldsymbol{L}|^{3/2}\ , \tag{B.32}$$

where the matrix $\boldsymbol{L}$ is defined in (B.24). As previously the volume of the gauge orbit through a point of the level set is (B.25). Hence the G-invariant volume form is

$$\int_{\mathfrak{M}} \boldsymbol{\omega} = \frac{1}{\mathrm{Vol}\,G}\int_{\tilde{\mathfrak{M}}} \tilde{\boldsymbol{\omega}}|\det\boldsymbol{L}|\prod_{r=1}^{\dim G}\prod_{c=1}^{3}\delta(\mu^{(c)X_r})\ . \tag{B.33}$$

Note the factor of $|\det\boldsymbol{L}|$ as compared with the case of Kähler quotient (B.26).

As an example of hyper-Kähler quotient construction which is directly relevant to the ADHM construction, let us start from flat Euclidean space $\tilde{\mathfrak{M}} = \mathbb{R}^{4n}$. In this case there are local coordinates $z^{\tilde{\mathrm{i}}\dot{\alpha}}$ for which the one forms $h^{\tilde{\mathrm{i}}\dot{\alpha}} = \mathrm{d}z^{\tilde{\mathrm{i}}\dot{\alpha}}$. The symplectic matrix is simply

$$\tilde{\Omega} = \begin{pmatrix} 0 & 1_{[n]\times[n]} \\ -1_{[n]\times[n]} & 0 \end{pmatrix}\ . \tag{B.34}$$

and the flat metric is

$$\tilde{g} = \tilde{\Omega}_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}}\varepsilon_{\dot{\alpha}\dot{\beta}}\,\mathrm{d}z^{\tilde{\mathrm{i}}\dot{\alpha}}\,\mathrm{d}z^{\tilde{\mathrm{j}}\dot{\beta}}\ . \tag{B.35}$$

In this case, there exists a hyper-Kähler potential

$$\chi = \tilde{\Omega}_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}}\varepsilon_{\dot\alpha\dot\beta} z^{\tilde{\mathrm{i}}\dot\alpha} z^{\tilde{\mathrm{j}}\dot\beta} \ . \tag{B.36}$$

We now consider the hyper-Kähler quotient of $\mathbb{R}^{4n}$ by some compact group G of tri-holomorphic isometries. The most general isometries of this type are generated by the vector fields

$$X_r = \mathrm{i} T^r_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}} z^{\tilde{\mathrm{j}}\dot\alpha} \frac{\partial}{\partial z^{\tilde{\mathrm{i}}\dot\alpha}} \ , \tag{B.37}$$

where T^r generators in some $2n$-dimensional representation of G which satisfies

$$(\tilde{\Omega} T^r)^{\mathrm{t}} = \tilde{\Omega} T^r \ . \tag{B.38}$$

The moment maps are

$$\vec{\mu}^{X_r} = -\frac{\mathrm{i}}{2} z^{\tilde{\mathrm{i}}\dot\alpha} (\tilde{\Omega} T^r)_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}} z^{\tilde{\mathrm{j}}\dot\beta} \varepsilon_{\dot\alpha\dot\gamma} \vec{\tau}^{\dot\gamma}{}_{\dot\beta} - \vec{\zeta}^r \ , \tag{B.39}$$

where $\sum \vec{\zeta}^r T^r$ are arbitrary constant elements in the abelian component of the Lie algebra of G.

The (gauge un-fixed) volume form on the quotient follows from general formula (B.33):

$$\int_{\mathfrak{M}} \boldsymbol{\omega} = \frac{1}{\mathrm{Vol}\, G} \int \left\{ \prod_{\dot\alpha=1,2} \prod_{\tilde{\mathrm{i}}=1}^{2n} \mathrm{d} z^{\tilde{\mathrm{i}}\dot\alpha} \right\} |\det \boldsymbol{L}| \prod_{r=1}^{\dim G} \prod_{c=1}^{3} \delta \left(\frac{\mathrm{i}}{2} z^{\tilde{\mathrm{i}}\dot\alpha} (\tilde{\Omega} T^r)_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}} z^{\tilde{\mathrm{j}}\dot\beta} \varepsilon_{\dot\alpha\dot\gamma} \tau^{c\dot\gamma}{}_{\dot\beta} - \zeta^{rc} \right) \ , \tag{B.40}$$

where, as previously,

$$\boldsymbol{L}_{rs} = \tilde{g}(X_r, X_s) = -\tilde{\Omega}_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}} \varepsilon_{\dot\alpha\dot\beta} T^r_{\tilde{\mathrm{i}}\tilde{\mathrm{k}}} z^{\tilde{\mathrm{k}}\dot\alpha} T^s_{\tilde{\mathrm{j}}\tilde{\mathrm{l}}} z^{\tilde{\mathrm{l}}\dot\beta} \ . \tag{B.41}$$

For this example, we now discuss more concretely the connection on the quotient and find an explicit expression for the Riemann tensor. First we shall consider the connection. As we have explained previously, the prescription is simple: lift tangent vectors to $\mathfrak{M}$ to $\mathscr{H}$ (denoted by the same symbol). The Levi–Civita connection is then obtained by projection to $\mathscr{H}$ from that on $\tilde{\mathfrak{M}}$. Locally, the tangent space of the horizontal subspace consists of vectors X that are orthogonal to the $4 \dim G$ vectors $\{X_r, \tilde{\boldsymbol{I}}^{(c)} X_r\}$, conditions (B.28); hence, for two vectors $X, Y \in T\mathfrak{M} (\simeq \mathscr{H})$

$$\nabla_X Y = (\tilde{\nabla}_X Y)^{\|} = \tilde{\nabla}_X Y - (\tilde{\nabla}_X Y)^{\perp} \ , \tag{B.42}$$

where

$$(\tilde{\nabla}_X Y)^{\perp} = \sum_{rs} \boldsymbol{L}^{-1}_{rs} \left\{ \tilde{g}(\tilde{\nabla}_X Y, X_s) X_r + \sum_{c=1}^{3} g(\tilde{\nabla}_X Y, \tilde{\boldsymbol{I}}^{(c)} X_s) \tilde{\boldsymbol{I}}^{(c)} X_r \right\} \ . \tag{B.43}$$

We can make the inner product more explicit by using $(\tilde{\nabla}_X Y)^{i\dot\alpha} = X^{\tilde{\mathrm{j}}\dot\beta} \partial Y^{\tilde{\mathrm{i}}\dot\alpha} / \partial z^{\tilde{\mathrm{j}}\dot\beta}$ and the expression for X_r in (B.37)

$$\tilde{g}(\tilde{\nabla}_X Y, X_r) = -\mathrm{i} X^{\tilde{\mathrm{i}}\dot\alpha} (\tilde{\Omega} T^r)_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}} Y^{\tilde{\mathrm{j}}\dot\beta} \varepsilon_{\dot\beta\dot\alpha} \ ,$$

$$\tilde{g}(\tilde{\nabla}_X Y, \tilde{\boldsymbol{I}}^{(c)} X_r) = -X^{\tilde{\mathrm{i}}\dot\alpha} (\tilde{\Omega} T^r)_{\tilde{\mathrm{i}}\tilde{\mathrm{j}}} Y^{\tilde{\mathrm{j}}\dot\beta} \varepsilon_{\dot\beta\dot\gamma} \tau^{c\dot\gamma}{}_{\dot\alpha} \ . \tag{B.44}$$

In deriving these expressions we used the facts $\tilde{g}(Y, X_r) = \tilde{g}(Y, \boldsymbol{I}^{(c)} X_r) = 0$ and, from (B.38), $\tilde{\Omega} T^r$ is a symmetric matrix.

The Riemann tensor of $\mathfrak{M}$ can be expressed using the standard formula in terms of the lifts $W, X, Y, Z \in \mathscr{H}$:

$$
\begin{aligned}
R(X, Y, W, Z) &= \tilde{g}(W, \tilde{\nabla}_X (\tilde{\nabla}_Y Z)^{\|} - \tilde{\nabla}_Y (\tilde{\nabla}_X Z)^{\|} - \tilde{\nabla}_{[X,Y]^{\|}} Z) \\
&= \tilde{g}(W, \tilde{\nabla}_X (\tilde{\nabla}_Y Z)^{\|} - \tilde{\nabla}_Y (\tilde{\nabla}_X Z)^{\|} - \tilde{\nabla}_Z (\tilde{\nabla}_X Y)^{\|} \\
&\quad + \tilde{\nabla}_Z (\tilde{\nabla}_Y X)^{\|} - [Z, [X, Y]]) \ .
\end{aligned} \tag{B.45}
$$

Notice that some of the projections are unnecessary here, since $W \in \mathscr{H}$ and $[Z, K] \in \mathscr{V}$ for any $K \in \mathscr{V}$. Using the fact that $\tilde{g}(W, \tilde{\nabla}_X (\tilde{\nabla}_Y Z)^{\perp}) = -\tilde{g}(\tilde{\nabla}_X W, (\tilde{\nabla}_Y Z)^{\perp})$, we can write (B.45) as

$$
\begin{aligned}
R(X, Y, W, Z) &= \tilde{R}(X, Y, W, Z) + \tilde{g}((\tilde{\nabla}_X Z)^{\perp}, (\tilde{\nabla}_Y W)^{\perp}) - \tilde{g}((\tilde{\nabla}_X W)^{\perp}, (\tilde{\nabla}_Y Z)^{\perp}) \\
&\quad - \tilde{g}((\tilde{\nabla}_X Y)^{\perp}, (\tilde{\nabla}_W Z)^{\perp}) + \tilde{g}((\tilde{\nabla}_Y X)^{\perp}, (\tilde{\nabla}_W Z)^{\perp}) \ ,
\end{aligned} \tag{B.46}
$$

where $\tilde{R}$ is the Riemann tensor of $\tilde{\mathfrak{M}}$. In the example relevant to the ADHM construction this vanishes since the mother space is flat.

We now extract the components of the Riemann tensor in the $z^{\tilde{i}\dot{\alpha}}$ basis by choosing $X = \partial/\partial z^{\tilde{i}\dot{\alpha}}$, $Y = \partial/\partial z^{\tilde{j}\dot{\beta}}$, $W = \partial/\partial z^{\tilde{k}\dot{\gamma}}$ and $Z = \partial/\partial z^{\tilde{l}\dot{\delta}}$. One finds

$$
\begin{aligned}
&R_{\tilde{i}\dot{\alpha}, \tilde{j}\dot{\beta}, \tilde{k}\dot{\gamma}, \tilde{l}\dot{\delta}} \\
&\quad = 2\varepsilon_{\dot{\alpha}\dot{\beta}} \varepsilon_{\dot{\gamma}\dot{\delta}} \sum_{rs} \{ (\tilde{\Omega} T^r)_{\tilde{i}\tilde{j}} \boldsymbol{L}_{rs}^{-1} (\tilde{\Omega} T^s)_{\tilde{k}\tilde{l}} + (\tilde{\Omega} T^r)_{\tilde{i}\tilde{l}} \boldsymbol{L}_{rs}^{-1} (\tilde{\Omega} T^s)_{\tilde{j}\tilde{k}} + (\tilde{\Omega} T^r)_{\tilde{i}\tilde{k}} \boldsymbol{L}_{rs}^{-1} (\tilde{\Omega} T^s)_{\tilde{i}\tilde{l}} \} \ .
\end{aligned} \tag{B.47}
$$

This form reflects the decomposition of the Riemann tensor in (B.14) and so we can extract the symplectic curvature of the quotient:

$$
R_{\tilde{i}\tilde{j}\tilde{k}\tilde{l}} = 2 \sum_{rs} \{ (\tilde{\Omega} T^r)_{\tilde{i}\tilde{j}} \boldsymbol{L}_{rs}^{-1} (\tilde{\Omega} T^s)_{\tilde{k}\tilde{l}} + (\tilde{\Omega} T^r)_{\tilde{i}\tilde{l}} \boldsymbol{L}_{rs}^{-1} (\tilde{\Omega} T^s)_{\tilde{j}\tilde{k}} + (\tilde{\Omega} T^r)_{\tilde{i}\tilde{k}} \boldsymbol{L}_{rs}^{-1} (\tilde{\Omega} T^s)_{\tilde{j}\tilde{l}} \} \ . \tag{B.48}
$$

Since, from Eq. (B.38), $\tilde{\Omega} T^r$ is a symmetric matrix, it is apparent that the symplectic curvature is totally symmetric in all its indices.

Appendix C. ADHM algebra

In this appendix, we collect together most of the pieces of ADHM algebra that we need in the text. Performing ADHM algebra is rather an art, requiring a good deal of chicanery and experience. The ADHM constraints and identities (2.50), (2.55), (2.61), (4.30) and (4.31b)

$$
\begin{aligned}
&\bar{\Delta}^{\dot{\alpha}} \Delta_{\dot{\beta}} = \delta^{\dot{\alpha}}{}_{\dot{\beta}} f^{-1}, \quad \bar{\Delta}^{\dot{\alpha}} \bar{b}^{\alpha} = \bar{b}^{\alpha} \Delta^{\dot{\alpha}}, \quad \bar{b}_{\alpha} b^{\beta} = \delta_{\alpha}{}^{\beta} 1_{[k]\times[k]} \ , \\
&\bar{\Delta}^{\dot{\alpha}} \mathscr{M} + \bar{\mathscr{M}} \Delta^{\dot{\alpha}} = 0, \quad \bar{\mathscr{M}} b^{\alpha} = \bar{b}^{\alpha} \mathscr{M}
\end{aligned} \tag{C.1}
$$

are used pervasively. The following differentiation formulae which are proved using definitions (2.50) and (2.51), are particularly useful:

$$\partial_n f = -f\partial_n(\tfrac{1}{2}\bar{\Delta}^{\dot\alpha}\Delta_{\dot\alpha})f = \begin{cases} -f\bar\sigma_n^{\dot\alpha\alpha}\bar b_\alpha \Delta_{\dot\alpha} f\ , \\ -f\bar\Delta^{\dot\alpha} b^\alpha \sigma_{n\alpha\dot\alpha} f\ , \end{cases} \tag{C.2a}$$

$$\Box f = -4f\bar b_\alpha \mathscr{P} b^\alpha f\ , \tag{C.2b}$$

$$\partial_n \mathscr{P} = -\Delta_{\dot\alpha} f\bar\sigma_n^{\dot\alpha\alpha}\bar b_\alpha \mathscr{P} - \mathscr{P} b^\alpha \sigma_{n\alpha\dot\alpha} f\bar\Delta^{\dot\alpha}\ , \tag{C.2c}$$

$$\Box \mathscr{P} = -4\{\mathscr{P}, b^\alpha f\bar b_\alpha\} + 4\Delta_{\dot\alpha} f\bar b_\alpha \mathscr{P} b^\alpha f\bar\Delta^{\dot\alpha}\ . \tag{C.2d}$$

To complete this compendium of differentiation formulae, using the expression for gauge potential (2.49) and ADHM identity (2.47) along with (C.1), for any $\mathscr{J}(x)$, we have

$$\begin{aligned} \mathscr{D}_n(\bar U\mathscr{J}U) &= \partial_n(\bar U\mathscr{J}U) + [A_n, \bar U\mathscr{J}U] \\ &= \bar U\partial_n\mathscr{J}U - \bar U b^\alpha\sigma_{n\alpha\dot\alpha} f\bar\Delta^{\dot\alpha}\mathscr{J}U - \bar U\mathscr{J}\Delta_{\dot\alpha} f\bar\sigma_n^{\dot\alpha\alpha}\bar b_\alpha U\ , \end{aligned} \tag{C.3a}$$

$$\begin{aligned} \mathscr{D}^2(\bar U\mathscr{J}U) &= -4\bar U\{b^\alpha f\bar b_\alpha, \mathscr{J}\}U + 4\bar U b^\alpha f\bar\Delta^{\dot\alpha}\mathscr{J}\Delta_{\dot\alpha} f\bar b_\alpha U \\ &\quad + \bar U\partial^2\mathscr{J}U - 2\bar U b^\alpha f\sigma_{n\alpha\dot\alpha}\bar\Delta^{\dot\alpha}\partial_n\mathscr{J}U - 2\bar U\partial_n\mathscr{J}\Delta_{\dot\alpha}\bar\sigma_n^{\dot\alpha\alpha} f\bar b_\alpha U\ . \end{aligned} \tag{C.3b}$$

Finally, there are various other tricks that we will explain in situ; however, there are particularly useful ones involving quantities of the form

$$\mathscr{J} = \begin{pmatrix} B_{[N]\times[N]} & 0_{[N]\times[2k]} \\ 0_{[2k]\times[N]} & C_{[k]\times[k]} 1_{[2]\times[2]} \end{pmatrix}\ . \tag{C.4}$$

The pair of identities is

$$\bar\Delta^{\dot\alpha}\mathscr{J}U = (\bar a^{\dot\alpha}\mathscr{J} - Ca_{\dot\alpha})U, \quad \bar U\mathscr{J}\Delta_{\dot\alpha} = \bar U(\mathscr{J}a_{\dot\alpha} - a_{\dot\alpha}C)\ . \tag{C.5}$$

The first of these is proved by expanding $\bar\Delta = \bar a + \bar x\bar b$, as in (2.45). Then with $\bar b$ assuming canonical form (2.57) we can commute $\bar x\bar b$ through $\mathscr{J}$, pick out the component C, and then use (2.47) to re-write $\bar x\bar bU = -\bar aU$. The other identity is proved in a similar way.

Note that where possible we suppress indices. However, in most situations the spinor indices α and $\dot\alpha$ need to be written explicitly because they are often not contracted in an obvious way.

Osborn's formula

This identity reads [8]

$$\mathrm{tr}_N\, F_{mn}^2 = -g^{-2}\Box^2\, \mathrm{tr}_k \log f\ . \tag{C.6}$$

We start by expanding out the left-hand side using the ADHM form for field strength (2.52):

$$\mathrm{tr}_N\, F_{mn}^2 = -16g^{-2}\, \mathrm{tr}_k(\bar b_\alpha\mathscr{P}b^\beta f\bar b^\alpha\mathscr{P}b_\alpha f + \bar b_\alpha\mathscr{P}b^\alpha f\bar b_\beta\mathscr{P}b^\beta f)\ . \tag{C.7}$$

Now consider the right-hand side. Firstly,

$$\Box \mathrm{tr}_k \log f = \mathrm{tr}_k(f^{-1}\Box f - f^{-1}\partial_n f f^{-1}\partial_n f)$$
$$= -2\mathrm{tr}_k(\bar{b}_\alpha \mathscr{P} b^\alpha f + 2f) , \tag{C.8}$$

using differentiation formulae (C.2a) and (C.2b) along with the expression for $\mathscr{P}$ in (2.51). Then employing differentiation formulae (C.2a)–(C.2d) once again

$$\begin{aligned}\Box^2 \mathrm{tr}_k \log f = 8\mathrm{tr}_k\,(&\bar{b}_\alpha\{\mathscr{P}, b^\beta f \bar{b}_\beta\}b^\alpha f - \bar{b}_\alpha \Delta_{\dot\alpha} f \bar{b}_\beta \mathscr{P} b^\beta f \bar{\Delta}^{\dot\alpha} b^\alpha f \\ &- \bar{b}_\alpha \Delta_{\dot\alpha} f \bar{b}_\beta \mathscr{P} b^\alpha f \bar{\Delta}^{\dot\alpha} b^\beta f - \bar{b}_\alpha \mathscr{P} b^\beta f \bar{\Delta}^{\dot\alpha} b^\alpha f \bar{b}_\beta \Delta_{\dot\alpha} f \\ &+ \bar{b}_+ \alpha \mathscr{P} b^\alpha f \bar{b}_\beta \mathscr{P} b^\beta f + 2f\bar{b}_\alpha \mathscr{P} b^\alpha f) .\end{aligned} \tag{C.9}$$

Then using ADHM identity (2.55), the definition of $\mathscr{P}$ in (2.51) along with (2.61), this becomes

$$\Box^2 \mathrm{tr}_k \log f = 16\mathrm{tr}_k(\bar{b}_\alpha \mathscr{P} b^\beta f \bar{b}^\alpha \mathscr{P} b_\alpha f + \bar{b}_\alpha \mathscr{P} b^\alpha f \bar{b}_\beta \mathscr{P} b^\beta f) , \tag{C.10}$$

precisely $-g^2$ times (C.7).

Zero modes: We prove that the quantity

$$\Lambda_\alpha(C) \equiv \bar{U} C f \bar{b}_\alpha U - \bar{U} b_\alpha f \bar{C} U \tag{C.11}$$

satisfies the zero mode condition

$$\bar{\mathscr{D}}^{\dot\alpha\alpha}\Lambda_\alpha(C) = 0 . \tag{C.12}$$

From Eqs. (C.11), (C.2a) and (C.3a) we calculate

$$\bar{D}^{\dot\alpha\alpha}\Lambda_\alpha(C) = 2\bar{U} b^\alpha f(\bar{\Delta}^{\dot\alpha} C + \bar{C}\Delta^{\dot\alpha}) f \bar{b}_\alpha U . \tag{C.13}$$

Hence the condition for a zero mode is

$$\bar{\Delta}^{\dot\alpha} C + \bar{C}\Delta^{\dot\alpha} = 0 . \tag{C.14}$$

Expanding $\Delta(x)$ as in (2.45), we have

$$\bar{C}^\lambda_i a_{\lambda j \dot\alpha} = -\bar{a}^\lambda_{i\dot\alpha} C_{\lambda j} ,$$
$$\bar{C}^\lambda_i b^\alpha_{\lambda j} = \bar{b}^{\alpha\lambda}_i C_{\lambda j} . \tag{C.15}$$

Derivative of the gauge field by a collective coordinate: We now prove that

$$g\,\frac{\partial A_n}{\partial X^\mu} = -\mathscr{D}_n\left(\frac{\partial \bar{U}}{\partial X^\mu} U\right) + \bar{U}\,\frac{\partial a}{\partial X^\mu} f \bar{\sigma}_n \bar{b} U - \bar{U} b \sigma_n f\,\frac{\partial \bar{a}}{\partial X^\mu} U . \tag{C.16}$$

Firstly, using $\bar{U}U = 1$,

$$g\,\frac{\partial A_n}{\partial X^\mu} = \frac{\partial \bar{U}}{\partial X^\mu}\partial_n U + \bar{U}\partial_n \frac{\partial U}{\partial X^\mu} = \frac{\partial \bar{U}}{\partial X^\mu}\,\partial_n U + \partial_n\left(\bar{U}\frac{\partial U}{\partial X^\mu}\right) - (\partial_n \bar{U})\,\frac{\partial U}{\partial X^\mu} . \tag{C.17}$$

Next we insert $1 \equiv U\bar{U} + \Delta f \bar{\Delta}$ into the middle of the first and third terms, and then use the fact that $A_n = g^{-1}\bar{U}\partial_n U$, to arrive at

$$g\,\frac{\partial A_n}{\partial X^\mu} = -D_n\left(\frac{\partial \bar{U}}{\partial X^\mu}U\right) + \frac{\partial \bar{U}}{\partial X^\mu}\Delta f\bar{\Delta}\partial_n U - (\partial_n\bar{U})\Delta f\bar{\Delta}\,\frac{\partial U}{\partial X^\mu}\ . \tag{C.18}$$

From ADHM identity (2.47), we have

$$\bar{\Delta}\partial_n U = -(\partial_n\bar{\Delta})U = -\bar{\sigma}_n\bar{b}U, \quad (\partial_n\bar{U})\Delta = -\bar{U}\partial_n\Delta = -\bar{U}b\sigma_n\ ,$$
$$\bar{\Delta}\frac{\partial U}{\partial X} = -\frac{\partial\bar{\Delta}}{\partial X}U = -\frac{\partial\bar{a}}{\partial X}U, \quad \frac{\partial\bar{U}}{\partial X}\Delta = -\bar{U}\,\frac{\partial\Delta}{\partial X} = -\bar{U}\,\frac{\partial a}{\partial X}\ , \tag{C.19}$$

from which (C.16) follows.

Corrigan's inner-product formula: The expression to be proved reads

$$\int \mathrm{d}^4x\,\mathrm{tr}_N\,\Lambda(C)\Lambda(C') = -\frac{\pi^2}{2}\,\mathrm{tr}_k[\bar{C}(\mathscr{P}_\infty + 1)C' - \bar{C}'(\mathscr{P}_\infty + 1)C]\ . \tag{C.20}$$

Here

$$\mathscr{P}_\infty = \lim_{x\to\infty}\mathscr{P} = 1 - b\bar{b} = \begin{pmatrix} 1_{[N]\times[N]} & 0_{[N]\times[2k]} \\ 0_{[2k]\times[N]} & 0_{[2k]\times[2k]}\end{pmatrix}\ , \tag{C.21}$$

as per Eqs. (2.51) and (2.105).

The strategy of the proof is to show that the integrand is actually a total derivative,

$$\mathrm{tr}_N\,\Lambda(C)\Lambda(C') = \tfrac{1}{8}\Box\,\mathrm{tr}_k[\bar{C}(\mathscr{P}+1)C'f - \bar{C}'(\mathscr{P}+1)Cf]\ , \tag{C.22}$$

after which Eq. (C.20) follows from Stokes' Theorem, together with the asymptotic formulae of Section 2.4.3. To verify this, let us first write out the left-hand side of Eq. (C.22):[97]

$$\mathrm{tr}[(\bar{C}'\mathscr{P}C - \bar{C}\mathscr{P}C')f\bar{b}_\alpha\mathscr{P}b^\alpha f - \bar{C}\mathscr{P}b^\alpha f\bar{C}'\mathscr{P}b_\alpha f - \mathscr{P}Cf\bar{b}_\alpha\mathscr{P}C'f\bar{b}^\alpha]\ . \tag{C.23}$$

We have used the cyclicity of the trace, together with definition (2.15) for the projector $\mathscr{P}$. Turning to the right-hand side of Eq. (C.22), one calculates

$$\begin{aligned}
\tfrac{1}{8}\Box\,\mathrm{tr}[\bar{C}(\mathscr{P}+1)C'f] &= \tfrac{1}{4}\mathrm{tr}[-2\bar{C}\{\mathscr{P}, b^\alpha f\bar{b}_\alpha\}C'f + 2\bar{C}\Delta_{\dot{\alpha}}f\bar{b}_\alpha\mathscr{P}b^\alpha f\bar{\Delta}^{\dot{\alpha}}C'f \\
&\quad - 2\bar{C}(\mathscr{P}+1)C'f\bar{b}_\alpha\mathscr{P}b^\alpha f + \bar{C}\Delta_{\dot{\alpha}}f\bar{\sigma}^{n\dot{\alpha}\alpha}\bar{b}_\alpha\mathscr{P}C'\partial_n f \\
&\quad - \bar{C}\mathscr{P}b^\alpha\sigma_{n\alpha\dot{\alpha}}f\bar{\Delta}^{\dot{\alpha}}C'\partial_n f] \\
&= \tfrac{1}{2}\,\mathrm{tr}[Cf\bar{b}_\alpha\mathscr{P}b^\alpha\bar{C}'(\mathscr{P}-1) - \bar{C}(\mathscr{P}+1)C'f\bar{b}_\alpha\mathscr{P}b^\alpha f \\
&\quad + Cf\bar{b}^\alpha\mathscr{P}C'f\bar{b}_\alpha\mathscr{P} + \bar{C}\mathscr{P}b_\alpha f\bar{C}'\mathscr{P}b^\alpha f] \\
&= \tfrac{1}{2}\,\mathrm{tr}_N\Lambda(C)\Lambda(C') - \tfrac{1}{2}\,\mathrm{tr}[(\bar{C}C' + \bar{C}'C)f\bar{b}_\alpha\mathscr{P}b^\alpha f]\ .
\end{aligned} \tag{C.24}$$

[97] Here, and in the following, the trace on the right-hand side is either over instanton or ADHM indices, depending on the context.

Here the expressions on the right-hand sides follow from differentiation formulae (C.2a)–(C.2d). We have also invoked relations (4.31a), (4.31b) and (2.51) and, once again, cyclicity under the trace. From the final rewrite in Eq. (C.24), desired result (C.22) follows by inspection upon anti-symmetrization in C and C'.

Covariant Laplace equation with bi-fermion source: We now prove that for an adjoint-valued scalar field the solution of the covariant Laplace equation with a bi-fermion source,

$$\mathscr{D}^2\phi = \Lambda(C)\Lambda(C') \tag{C.25}$$

and boundary condition $\lim_{x\to\infty}\phi(x) = \phi^0$, is

$$\phi = -\frac{1}{4}\,\bar{U}Cf\bar{C}'U + \bar{U}\begin{pmatrix}\phi^0 & 0\\ 0 & \varphi 1_{[2]\times[2]}\end{pmatrix}U\ , \tag{C.26}$$

where

$$\varphi = \boldsymbol{L}^{-1}(\bar{w}^{\dot\alpha}\phi^0 w_{\dot\alpha} + \tfrac{1}{4}\,\bar{C}C') \tag{C.27}$$

is a $k\times k$ matrix.

First of all, using asymptotic formulae (2.105) it is easy to see that (C.26) has the correct boundary condition at infinity. Next we employ differentiation formula (C.3b) with $\mathscr{J} = \frac{1}{4}\,Cf\bar{C}'$ and compare with

$$\begin{aligned}\Lambda(C)\Lambda(C') = &-\bar{U}Cf\bar{b}_\alpha\mathscr{P}C'f\bar{b}^\alpha U + \bar{U}Cf\bar{b}_\alpha\mathscr{P}b^\alpha f\bar{C}'U\\ &+\bar{U}b_\alpha f\bar{C}\mathscr{P}C'f\bar{b}^\alpha U - \bar{U}b_\alpha f\bar{C}\mathscr{P}b^\alpha f\bar{C}'U\ .\end{aligned} \tag{C.28}$$

Using differentiation formula (C.3b) we find that the third term in (C.3d) matches the second term in (C.28). Then writing $\mathscr{P} = 1 - \Delta_{\dot\alpha}f\bar{\Delta}^{\dot\alpha}$ in the first and fourth terms of (C.28) and using Eqs. (C.14) and (C.15), $\bar{\Delta}_{\dot\alpha}C = -\bar{C}\Delta_{\dot\alpha}$ and $\bar{b}_\alpha C = \bar{C}b_\alpha$, and differentiation formula (C.2a), we find that these terms match the first, fourth and fifth terms in (C.3b). This leaves the third term in (C.28) which matches the second term in (C.3b) apart from the fact that $\mathscr{P}$ is replaced by $\mathscr{P}-1$. Hence

$$\mathscr{D}^2(\tfrac{1}{4}\,\bar{U}Cf\bar{C}'U) = \Lambda(C)\Lambda(C') - \bar{U}b^\alpha f\bar{C}C'f\bar{b}_\alpha U\ . \tag{C.29}$$

Now consider (C.3b) with

$$\mathscr{J} = \begin{pmatrix}\phi^0 & 0\\ 0 & \varphi 1_{[2]\times[2]}\end{pmatrix}\ ,\quad \partial_n\varphi = 0\ . \tag{C.30}$$

We find

$$\mathscr{D}^2(\bar{U}\mathscr{J}U) = 4\bar{U}b^\alpha\left\{-\{f,\varphi\} + f\bar{\Delta}^{\dot\alpha}\begin{pmatrix}\phi^0 & 0\\ 0 & \varphi\end{pmatrix}\Delta_{\dot\alpha}f\right\}\bar{b}_\alpha U\ . \tag{C.31}$$

Then

$$\bar{\Delta}^{\dot\alpha}\begin{pmatrix}\phi^0 & 0\\ 0 & \varphi\end{pmatrix}\Delta_{\dot\alpha} = \bar{w}^{\dot\alpha}\phi^0 w_{\dot\alpha} - \boldsymbol{L}\varphi + \{\varphi, f^{-1}\}\ , \tag{C.32}$$

where $\boldsymbol{L}$ is defined in (2.125). Putting this together with (C.29) we have solved (C.25) if

$$\boldsymbol{L}\varphi = \bar{w}^{\dot\alpha}\phi_0 w_{\dot\alpha} + \tfrac{1}{4}\,\bar{C}C' \ . \tag{C.33}$$

Anti-fermion source: We now prove that

$$\bar\Sigma_{aAB}[\phi_a, \Lambda(\mathcal{M}^B)] = \mathcal{D}_{\alpha\dot\alpha}\bar\psi^{\dot\alpha}_A + \Lambda(\mathcal{N}_A) \ . \tag{C.34}$$

Here, ϕ_a is the solution to (C.25) given in (C.26). The collective coordinate matrix $\mathcal{N}_A$ (which will be seen to depend in a non-trivial way on the original collective coordinates $\{a, \mathcal{M}^A\}$) is subject to the usual zero mode conditions (4.31a), (4.31b).

We now solve (C.34) for $\bar\psi^{\dot\alpha A}$ and $\mathcal{N}_A$. First of all, from (4.64) and (A.20)

$$\bar\Sigma_{aAB}\phi_a = -\tfrac{1}{2}\,\varepsilon_{ABCD}\bar{U}\mathcal{M}^C f\bar{\mathcal{M}}^D U + \bar\Sigma_{aAB}\bar{U}\begin{pmatrix}\phi^0_a & 0\\ 0 & \varphi_a 1_{[2]\times[2]}\end{pmatrix}U \ , \tag{C.35}$$

where φ_a is defined in (4.65),

$$\varphi_a = \boldsymbol{L}^{-1}(\tfrac{1}{4}\,\bar\Sigma_{aAB}\bar{\mathcal{M}}^A\mathcal{M}^B + \bar{w}^{\dot\alpha}\phi^0_a w_{\dot\alpha}) \ . \tag{C.36}$$

As usual in ADHM calculus, some educated guesswork is required. To this end we expand the left-hand side of (C.34), using Eqs. (4.29) and (C.35):

$$\begin{aligned}\bar\Sigma_{aAB}[\phi_a, \Lambda(\mathcal{M}^B)] = {}& \tfrac{1}{2}\,\varepsilon_{ABCD}\,\bar{U}\{\mathcal{M}^B f\bar{b}_\alpha\mathcal{P}\mathcal{M}^C f\bar{\mathcal{M}}^D - b_\alpha f\bar{\mathcal{M}}^B\mathcal{P}\mathcal{M}^C f\bar{\mathcal{M}}^D\\ &-\mathcal{M}^C f\bar{\mathcal{M}}^D\mathcal{P}\mathcal{M}^B f\bar{b}_\alpha + \mathcal{M}^C f\bar{\mathcal{M}}^D\mathcal{P}b_\alpha f\bar{\mathcal{M}}^B\}U\\ &+\bar\Sigma_{aAB}\bar{U}\left\{-\mathcal{M}^B f\bar{b}_\alpha\mathcal{P}\begin{pmatrix}\phi^0_a & 0\\ 0 & \varphi_a\end{pmatrix} + b_\alpha f\bar{\mathcal{M}}^B\mathcal{P}\begin{pmatrix}\phi^0_a & 0\\ 0 & \varphi_a\end{pmatrix}\right.\\ &\left.+\begin{pmatrix}\phi^0_a & 0\\ 0 & \varphi_a\end{pmatrix}\mathcal{P}\mathcal{M}f\bar{b}_\alpha - \begin{pmatrix}\phi^0_a & 0\\ 0 & \varphi_a\end{pmatrix}\mathcal{P}b_\alpha f\bar{\mathcal{M}}^B\right\}U \ .\end{aligned} \tag{C.37}$$

Here $\mathcal{P}$ is the projection operator defined in Eq. (2.51) above. Since $\partial_n\Delta = b\sigma_n$, a comparison of Eqs. (C.37) and (C.3a) motivates the ansatz:

$$\bar\psi_A = \bar\psi^{(1)}_A + \bar\psi^{(2)}_A + \bar\psi^{(3)}_A \ , \tag{C.38}$$

where

$$\bar\psi^{(1)}_{\dot\alpha A} = -\tfrac{1}{4}\,\varepsilon_{ABCD}\bar{U}\mathcal{M}^B f\bar\Delta_{\dot\alpha}\mathcal{M}^C f\bar{\mathcal{M}}^D U \ , \tag{C.39a}$$

$$\bar\psi^{(2)}_{\dot\alpha A} = \frac{1}{2}\,\bar\Sigma_{aAB}\bar{U}\left\{\mathcal{M}^B f\bar\Delta_{\dot\alpha}\begin{pmatrix}\phi^0_a & 0\\ 0 & \varphi_a\end{pmatrix} + \begin{pmatrix}\phi^0_a & 0\\ 0 & \varphi_a\end{pmatrix}\Delta_{\dot\alpha}f\bar{\mathcal{M}}^B\right\}U \ , \tag{C.39b}$$

$$\bar\psi^{(3)}_{\dot\alpha A} = \bar{U}\begin{pmatrix}0 & 0\\ 0 & \mathcal{G}_{\dot\alpha A}\end{pmatrix}U, \quad \bar{\mathcal{G}}_{\dot\alpha A} = -\mathcal{G}_{\dot\alpha A}, \quad \partial_n\mathcal{G}_{\dot\alpha A} = 0 \ . \tag{C.39c}$$

We expect $\bar{\psi}_A^{(1)}$ to account (more or less) for the first four terms on the right-hand side of Eq. (C.37), and $\bar{\psi}_A^{(2)}$ to account (more or less) for the final four. The presence of $\bar{\psi}_A^{(3)}$, while less obviously motivated at this stage, will be needed to ensure that the quantity $\mathscr{N}_A$ defined in Eq. (C.34) obeys zero-mode constraints (4.31a), (4.31b).

By an explicit calculation using Eqs. (C.2a), (4.31a), (4.31b), (C.3a) and (C.39a), one finds

$$\mathscr{D}_{\alpha\dot\alpha}\bar{\psi}_A^{(1)\dot\alpha} = \tfrac{1}{2}\,\varepsilon_{ABCD}\bar{U}\{\,\mathscr{M}^B f\bar{b}_\alpha \mathscr{P}\mathscr{M}^C f\bar{\mathscr{M}}^D - b_\alpha f\bar{\mathscr{M}}^B(\mathscr{P}-1)\mathscr{M}^C f\bar{\mathscr{M}}^D + \mathscr{M}^C f\bar{\mathscr{M}}^D\mathscr{P}b_\alpha f\bar{\mathscr{M}}^B - \mathscr{M}^C f\bar{\mathscr{M}}^D(\mathscr{P}-1)\mathscr{M}^B f\bar{b}_\alpha\}U\ . \tag{C.40}$$

Except for the "−1" in the second and fourth terms, this reproduces the first four terms of Eq. (C.37), as expected. Similarly one calculates

$$\begin{aligned}\mathscr{D}_{\alpha\dot\alpha}\bar{\psi}_A^{(2)\dot\alpha} &= -\bar{\Sigma}_{aAB}\bar{U}\left\{\mathscr{M}^B f\bar{b}_\alpha(\mathscr{P}+1)\begin{pmatrix}\phi_a^0 & 0\\ 0 & \varphi_a\end{pmatrix} - b_\alpha f\bar{\mathscr{M}}^B(\mathscr{P}-1)\begin{pmatrix}\phi_a^0 & 0\\ 0 & \varphi_a\end{pmatrix}\right.\\
&\quad + \begin{pmatrix}\phi_a^0 & 0\\ 0 & \varphi_a\end{pmatrix}(\mathscr{P}+1)b_\alpha f\bar{\mathscr{M}}^B - \begin{pmatrix}\phi_a^0 & 0\\ 0 & \varphi_a\end{pmatrix}(\mathscr{P}-1)\mathscr{M}^B f\bar{b}_\alpha\\
&\quad \left. - b_\alpha f\bar{\Delta}^{\dot\alpha}\begin{pmatrix}\phi_a^0 & 0\\ 0 & \varphi_a\end{pmatrix}\Delta_{\dot\alpha} f\bar{\mathscr{M}}^B + \mathscr{M}^B f\bar{\Delta}^{\dot\alpha}\begin{pmatrix}\phi_a^0 & 0\\ 0 & \varphi_a\end{pmatrix}\Delta_a D f\bar{b}_\alpha\right\}U\\
&= \bar{\Sigma}_{aAB}\bar{U}\left\{\mathscr{M}^B f\bar{b}_\alpha\mathscr{P}\begin{pmatrix}\phi_a^0 & 0\\ 0 & \varphi_a\end{pmatrix} - b_\alpha f\bar{\mathscr{M}}^B\mathscr{P}\begin{pmatrix}\phi_a^0 & 0\\ 0 & \varphi_a\end{pmatrix}\right.\\
&\quad + \begin{pmatrix}\phi_a^0 & 0\\ 0 & \varphi_a\end{pmatrix}\mathscr{P}b_\alpha f\bar{\mathscr{M}}^B - \begin{pmatrix}\phi_a^0 & 0\\ 0 & \varphi_a\end{pmatrix}\mathscr{M}^B f b_\alpha\\
&\quad \left. + b_\alpha f\left[\bar{\mathscr{M}}^B\begin{pmatrix}\phi_a^0 & 0\\ 0 & \varphi_a\end{pmatrix} - \varphi_a\bar{\mathscr{M}}^B\right] + \left[\begin{pmatrix}\phi_a & 0\\ 0 & \varphi_a\end{pmatrix}\mathscr{M}^B - \mathscr{M}^B\varphi_a\right]f\bar{b}_\alpha\right\}\\
&\quad - \tfrac{1}{2}\,\varepsilon_{ABCD}U\{b_\alpha f\bar{\mathscr{M}}^B\mathscr{M}^C f\bar{\mathscr{M}}^D + \mathscr{M}^C f\bar{\mathscr{M}}^D\mathscr{M}^B f\bar{b}_\alpha\}U\end{aligned} \tag{C.41}$$

so that the last four lines of Eq. (C.37) are accounted for, as well as the "−1" terms in (C.40). Here the final equality follows from commutator identity (C.32):

$$\begin{aligned}\bar{\Delta}^{\dot\alpha}\begin{pmatrix}\phi_a^0 & 0\\ 0 & \varphi_a\end{pmatrix}\Delta_{\dot\alpha} &= \bar{w}^{\dot\alpha}\phi_a^0 w_{\dot\alpha} - \boldsymbol{L}\cdot\varphi_a + \{\varphi_a, f^{-1}\}\\
&= -\tfrac{1}{4}\,\bar{\Sigma}_{aAB}\bar{\mathscr{M}}^A\mathscr{M}^B + \{\varphi_a, f^{-1}\}\end{aligned} \tag{C.42}$$

implied by Eqs. (2.50), (2.125) and (4.65). Finally,

$$\mathscr{D}_{\alpha\dot\alpha}\bar\psi_A^{(3)\dot\alpha} = 2\bar U\left\{b_\alpha f\bar\Delta_{\dot\alpha}\begin{pmatrix}0 & 0\\ 0 & \mathscr{G}_A^{\dot\alpha}\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & \mathscr{G}_A^{\dot\alpha}\end{pmatrix}\Delta_{\dot\alpha} f\bar b_\alpha\right\}U$$

$$= 2\bar U\left\{b_\alpha f\left[\bar a_{\dot\alpha}\begin{pmatrix}0 & 0\\ 0 & \mathscr{G}_A^{\dot\alpha}\end{pmatrix} - \mathscr{G}_A^{\dot\alpha}\bar a_{\dot\alpha}\right] + \left[\begin{pmatrix}0 & 0\\ 0 & \mathscr{G}_A^{\dot\alpha}\end{pmatrix}a_{\dot\alpha} - a_{\dot\alpha}\mathscr{G}_A^{\dot\alpha}\right] f\bar b_\alpha\right\}U\ , \qquad \text{(C.43)}$$

where to obtain the final equality we used the moves summarized in Eq. (C.5).

Next we sum expressions (C.40), (C.41) and (C.43), and compare to the right-hand side of Eq. (C.37). By inspection, we confirm ansätze (C.38), where the Grassmann zero mode matrix has the form

$$\mathscr{N}_A = -\bar\Sigma_{aAB}\left[\begin{pmatrix}\phi_a^0 & 0\\ 0 & \varphi_a\end{pmatrix}\mathscr{M}^B - \mathscr{M}^B\varphi_a\right] + 2\begin{pmatrix}0 & 0\\ 0 & \mathscr{G}_A^{\dot\alpha}\end{pmatrix}a_{\dot\alpha} - 2a_{\dot\alpha}\mathscr{G}_A^{\dot\alpha}\ . \qquad \text{(C.44)}$$

Up till this point we have yet to solve for $\mathscr{G}_A^{\dot\alpha}$. This is accomplished by inserting $\mathscr{N}_A$ into fermionic constraints (4.31a) and (4.31b). One finds that Eq. (4.31b) is satisfied automatically by expression (C.44). In contrast, Eq. (4.31a) amounts to $2k^2$ independent real linear constraints, which is precisely the number required to fix the anti-Hermitian $k\times k$ matrices $\mathscr{G}_A^{\dot\alpha}$ completely (the explicit form for $\mathscr{G}_A^{\dot\alpha}$ is not required).

Supersymmetry transformations of the fermion zero modes: From the supersymmetry transformation of the fermion zero modes

$$\delta\Lambda(\mathscr{M}^A) = -\mathrm{i}\Sigma_a^{AB}(\not{\mathscr{D}}\phi_a)\bar\xi_B\ , \qquad \text{(C.45)}$$

and expression (4.64) for ϕ_a in the instanton background, we now extract the transformation of the Grassmann collective coordinates

$$\delta\mathscr{M}^A = 2\mathrm{i}\Sigma_a^{AB}\mathscr{C}_{a\dot\alpha}\bar\xi_B^{\dot\alpha},\quad \delta\bar{\mathscr{M}}^A = 2\mathrm{i}\Sigma_a^{AB}\xi_{\dot\alpha B}\bar{\mathscr{C}}_a^{\dot\alpha}\ , \qquad \text{(C.46)}$$

where

$$\mathscr{C}_{a\dot\alpha} = \begin{pmatrix}\phi_a^0 & 0\\ 0 & \varphi_a\end{pmatrix}a_{\dot\alpha} - a_{\dot\alpha}\varphi_a,\quad \bar{\mathscr{C}}_a^{\dot\alpha} = \bar a^{\dot\alpha}\begin{pmatrix}\phi_a^0 & 0\\ 0 & \varphi_a\end{pmatrix} - \varphi_a\bar a^{\dot\alpha}\ . \qquad \text{(C.47)}$$

The proof begins by expressing $\delta\lambda$ in terms of a variation of the c-number collective coordinates, as in (4.68), and a variation of the Grassmann collective coordinates that must be determined. Up to a gauge transformation, one finds

$$\delta\Lambda_\alpha(\mathscr{M}^A) = \Lambda_\alpha(\delta\mathscr{M}'^A) + \mathrm{i}\xi_{\dot\alpha B}\bar U\{-\mathscr{M}^B f\bar\Delta^{\dot\alpha}\mathscr{M}^A f\bar b_\alpha - \mathscr{M} f\bar b_\alpha\Delta^{\dot\alpha} f\bar{\mathscr{M}}^B + \mathscr{M}^B f\bar\Delta^{\dot\alpha} b_\alpha f\bar{\mathscr{M}}^A$$

$$+ b_\alpha f\bar{\mathscr{M}}^A\Delta^{\dot\alpha} f\bar{\mathscr{M}}^B + \mathscr{M}^A f\bar{\mathscr{M}}^B\Delta^{\dot\alpha} f\bar b_\alpha + b_\alpha f\bar\Delta^{\dot\alpha}\mathscr{M}^B f\bar{\mathscr{M}}^A\}U\ . \qquad \text{(C.48)}$$

In order to derive this we used

$$\delta a_{\dot\alpha} = \mathrm{i}\xi_{\dot\alpha A}\mathscr{M}^A,\quad \delta\bar a_{\dot\alpha} = \mathrm{i}\bar\xi_{\dot\alpha A}\bar{\mathscr{M}}^A\ ,$$

$$\delta\bar U=-(\bar U\delta U)\bar U-\bar U\delta a_{\dot\alpha}f\bar\Delta^{\dot\alpha},\quad \delta U=U(\bar U\delta U)-\Delta_{\dot\alpha}f\delta\bar a^{\dot\alpha}U\ ,$$

$$\delta f=-\tfrac12 f(\delta\bar a^{\dot\alpha}\Delta_{\dot\alpha}+\bar\Delta^{\dot\alpha}\delta a_{\dot\alpha})f\ . \tag{C.49}$$

Notice that only the last term in (C.48) depends on the variation of the Grassmann collective coordinates.

From the supersymmetry transformation rule, the right-hand side of (C.48) must be equated with the right-hand side of (C.45) where ϕ_a assumes its value in the instanton background as in (4.64). Using derivative identity (C.3a) and properties of the Σ-matrices, one finds

$$\begin{aligned}-\,\mathrm{i}\Sigma^{AB}_a\not{\mathcal D}\phi_a\bar\xi_B=-\mathrm{i}\bar\xi_{\dot\alpha B}\bar U\Bigg\{&\mathcal M^A f\bar\Delta^{\dot\alpha}b_\alpha f\bar{\mathcal M}^B+b_\alpha f\bar\Delta^{\dot\alpha}\mathcal M^A f\bar{\mathcal M}^B+\mathcal M^A f\bar{\mathcal M}^B\Delta^{\dot\alpha}f\bar b_\alpha\\ &-\mathcal M^B f\bar\Delta^{\dot\alpha}b_\alpha f\bar{\mathcal M}^A-b_\alpha f\bar\Delta^{\dot\alpha}\mathcal M^B f\bar{\mathcal M}^A-\mathcal M^B f\bar{\mathcal M}^A\Delta^{\dot\alpha}f\bar b_\alpha\\ &-2\mathrm{i}\Sigma^{AB}_a b_\alpha f\bar\Delta^{\dot\alpha}\begin{pmatrix}\phi^0_a&0\\0&\varphi_a\end{pmatrix}-2\mathrm{i}\Sigma^{AB}_a\begin{pmatrix}\phi^0_a&0\\0&\varphi_a\end{pmatrix}\Delta^{\dot\alpha}f\bar b_\alpha\Bigg\}U\ .\end{aligned} \tag{C.50}$$

One can verify using the fermionic ADHM constraints (4.34) that the first six terms in the above are equal to the first six terms in (C.48). This means that the variation $\Lambda_\alpha(\delta\mathcal M^A)$ is then equated with the final two terms in (C.50). These terms are not quite in the right form due to the presence of the x-dependent Δ and $\bar\Delta$ terms. However, this can easily be removed by using the tricks (C.5). When this has been done, one extracts the variations of the Grassmann collective coordinates given in (4.70).

Variation of the fermion zero modes: We now prove that under a variation by a collective coordinate X^μ

$$\frac{\partial\Lambda(\mathcal M)}{\partial X^\mu}+[\Omega_\mu,\Lambda(\mathcal M)]=\not{\mathcal D}\bar\varrho_\mu+\Lambda(\partial\mathcal M/\partial X^\mu)\ , \tag{C.51}$$

where

$$\bar\varrho^{\dot\alpha}_\mu=\frac14\,\bar U\,\frac{\partial a^{\dot\alpha}}{\partial X^\mu}f\bar{\mathcal M}U\ . \tag{C.52}$$

Using (4.29), (2.112) and (C.19), the left-hand side of (C.51) is

$$\begin{aligned}\bar U\Bigg\{&\frac{\partial a_{\dot\alpha}}{\partial X^\mu}f\bar\Delta^{\dot\alpha}(b_\alpha f\bar{\mathcal M}-\mathcal M f\bar b_\alpha)+(b_\alpha f\bar{\mathcal M}-\mathcal M f\bar b_\alpha)\Delta_{\dot\alpha}f\frac{\partial\bar a^{\dot\alpha}}{\partial X^\mu}+\frac{\partial\mathcal M}{\partial X^\mu}f\bar b_\alpha-b_\alpha f\frac{\partial\bar{\mathcal M}}{\partial X^\mu}\\ &+\frac12\mathcal M f\left(\frac{\partial\bar a^{\dot\alpha}}{\partial X^\mu}\Delta_{\dot\alpha}+\bar\Delta^{\dot\alpha}\frac{\partial a_{\dot\alpha}}{\partial X^\mu}\right)f\bar b_\alpha-\frac12 b_\alpha f\left(\frac{\partial\bar a^{\dot\alpha}}{\partial X^\mu}\Delta_{\dot\alpha}+\bar\Delta^{\dot\alpha}\frac{\partial a_{\dot\alpha}}{\partial X^\mu}\right)f\bar{\mathcal M}\Bigg\}U\ .\end{aligned} \tag{C.53}$$

Using (C.3a), the right-hand side of (C.51) is

$$-\bar{U}\left\{\frac{\partial a^{\dot{\alpha}}}{\partial X^{\mu}}f\bar{b}_{\alpha}\Delta_{\dot{\alpha}}f\bar{\mathscr{M}}+\mathscr{M}f\bar{b}_{\alpha}\Delta_{\dot{\alpha}}f\frac{\partial\bar{a}^{\dot{\alpha}}}{\partial X^{\mu}}+b_{\alpha}f\bar{\Delta}_{\dot{\alpha}}\frac{\partial a^{\dot{\alpha}}}{\partial X^{\mu}}f\bar{\mathscr{M}}+\frac{\partial\mathscr{M}}{\partial X^{\mu}}f\bar{b}_{\alpha}-b_{\alpha}f\frac{\partial\bar{\mathscr{M}}}{\partial X^{\mu}}\right.$$
$$\left.+\frac{\partial a^{\dot{\alpha}}}{\partial X^{\mu}}f\bar{\mathscr{M}}\Delta_{\dot{\alpha}}f\bar{b}_{\alpha}+b_{\alpha}f\bar{\Delta}_{\dot{\alpha}}\mathscr{M}f\frac{\partial\bar{a}^{\dot{\alpha}}}{\partial X^{m}u}+\mathscr{M}f\frac{\partial\bar{a}^{\dot{\alpha}}}{\partial X^{m}u}\Delta_{\dot{\alpha}}f\bar{b}_{\alpha}\right\}U\ . \tag{C.54}$$

The difference of (C.53) and (C.54) is

$$\frac{1}{2}\bar{U}\left\{\mathscr{M}f\left(\frac{\partial\bar{a}^{\dot{\alpha}}}{\partial X^{\mu}}\Delta_{\dot{\alpha}}+\bar{\Delta}_{\dot{\alpha}}\frac{\partial a^{\dot{\alpha}}}{\partial X^{\mu}}\right)f\bar{b}_{\alpha}-b_{\alpha}f\left(\frac{\partial\bar{a}^{\dot{\alpha}}}{\partial X^{\mu}}\Delta_{\dot{\alpha}}+\bar{\Delta}_{\dot{\alpha}}\frac{\partial a^{\dot{\alpha}}}{\partial X^{\mu}}\right)f\bar{\mathscr{M}}\right\}U\ . \tag{C.55}$$

But this vanishes by virtue of constraints (2.109a) satisfied by $C_{\dot{\alpha}}=\partial a_{\dot{\alpha}}/\partial X^{\mu}$.

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