Multi-instanton calculus in $N=2$ supersymmetric gauge theory

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The Seiberg-Witten solution of $N=2$ supersymmetric SU(2) gauge theory may be viewed as a prediction for the infinite family of constants $\{\mathcal{F}_n\}$ measuring the n-instanton contribution to the prepotential $\mathcal{F}$. Here we examine the instanton physics directly, in particular the contribution of the general self-dual solution of topological charge n constructed by Atiyah, Drinfeld, Hitchin, and Manin (ADHM). In both the bosonic and supersymmetric cases, we determine both the large- and short-distance behavior of all the fields in this background. This allows us to construct the exact classical interaction between n ADHM (super)instantons mediated by the adjoint Higgs bosons. We calculate the one- and two-instanton contributions to the low-energy Seiberg-Witten effective action and find precise agreement with their predicted values of $\mathcal{F}_1$ and $\mathcal{F}_2$. [S0556-2821(96)03416-9]

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I. INTRODUCTION

A. Instanton series in $N=2$ supersymmetric gauge theory

The last two years have seen remarkable progress in the study of $N=2$ supersymmetric Yang-Mills theories. This progress was initiated by the work of Seiberg and Witten [1], who determined the exact low-energy effective Lagrangian for the gauge group SU(2). Their work, which relies on a novel version of Montonen-Olive duality [2], has subsequently been generalized to include various matter couplings [3] and larger gauge groups [4–8]. In $N=2$ supersymmetric Yang-Mills theory, the low-energy effective Lagrangian is determined in terms of a single object: the prepotential $\mathcal{F}$ [9,10]. The prepotential is a holomorphic function of the $N=2$ chiral superfield [11] Ψ and the dynamically generated scale of the theory Λ. By determining its behavior in the vicinity of its singularities, Seiberg and Witten were able to reconstruct it exactly:

$$\mathcal{F}_{\mathrm{sw}}(\Psi) \equiv \mathcal{F}_{\mathrm{pert}}(\Psi) + \mathcal{F}_{\mathrm{inst}}(\Psi)$$

$$= \frac{i}{2\pi}\,\Psi^2\,\ln\frac{2\Psi^2}{e^3\Lambda^2} - \frac{i}{\pi}\sum_{n=1}^{\infty}\mathcal{F}_n\!\left(\frac{\Lambda}{\Psi}\right)^{4n}\Psi^2, \quad (1.1)$$

where the inverse powers of Ψ are understood in the sense of a Taylor expansion about the vacuum expectation value (VEV) v. That the expansion has this general form had been known for some time [10]; the nth term in the series is renormalization group invariant and has precisely the right transformation properties under the anomalous U(1) symmetry to

be identified with an n-instanton effect. The new information in the Seiberg-Witten solution is the precise numerical value of each of the coefficients $\mathcal{F}_n$; in particular,[1] $\mathcal{F}_1 = 1/2$ and $\mathcal{F}_2 = 5/16$, with the higher $\mathcal{F}_n$'s being easily determined by a recursion relation [13]. This constitutes a set of highly nontrivial predictions for *all* multi-instanton contributions to the low-energy physics in this theory.

In principle, it should be possible to calculate the instanton series without appealing to duality, by directly evaluating the saddle-point contributions to the path integral in the semiclassical limit. Previously, this has only been accomplished in the one-instanton sector [12,14]. In this paper, we examine the role of the complete set of multi-instantons, constructed long ago by Atiyah, Drinfeld, Hitchin, and Manin (ADHM) [15,16]. At a purely formal level, we succeed in recasting the $\{\mathcal{F}_n\}$ as integrals over the $8n$-parameter families of ADHM collective coordinates, together with their superpartners. For $n=1$ and $n=2$, we are able to perform these integrations explicitly and confirm the predictions of Seiberg and Witten. In Ref. [17] we extend our analysis to include $N=2$ matter hypermultiplets, paralleling Ref. [3].

Our motivation is twofold. First, our calculation serves as an independent check on the proposed exact solution and therefore on the realization of electric-magnetic duality on which it relies. A second motivation concerns the old questions about the role of instantons in the strongly coupled vacuum of QCD. The availability of exact information about multi-instanton effects in this theory makes it a promising theoretical laboratory for investigating this issue. Indeed, several important simplifications occur for instanton physics

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in this model. In particular, the fact that the $\mathcal{F}_n$ are independent of the coupling implies a powerful nonrenormalization theorem for perturbation theory in instanton backgrounds. And in addition, while usually an expansion such as Eq. (1.1) makes sense only in weak coupling ($v \gg \Lambda$), the exact solution may be analytically continued all the way into the strong-coupling regime, up to the singular points at which the theory admits a dual description in terms of massless monopoles. This suggests the interesting possibility that the strongly coupled vacuum can be described as a nondilute gas of instantons which undergoes a phase transition near the singular point.

The general approach we adopt, and extend, herein was originally developed by Affleck, Dine, and Seiberg for $N=1$ models [18] and adapted to the $N=2$ case by Seiberg [10]. In this approach, the long-distance physics is studied by focusing on certain chirality-violating antifermion Green's functions, $\langle \bar{\psi}(x_1) \bar{\psi}(x_2) \rangle$ in the $N=1$ theories or $\langle \bar{\psi}(x_1) \bar{\psi}(x_2) \bar{\lambda}(x_3) \bar{\lambda}(x_4) \rangle$ in the $N=2$ theories as dictated by the anomaly structure. These Green's functions are then saturated in saddle-point approximation by their classical values, $\bar{\psi} \rightarrow \bar{\psi}_{\text{cl}}$, etc., obtained by solving the inhomogeneous Dirac equation in the classical instanton background.

Strictly speaking, when the gauge group is spontaneously broken, the instantons are no longer exact solutions of the equations of motion [19]. We follow Affleck *et al.* [20,18] and work instead with the *constrained* instanton (see Sec. III). Despite its approximate nature, this field configuration has universal properties both at short and long distances which yield unambiguous answers for the corresponding asymptotics of the correlation functions. At lowest order in g^2, the short-distance constrained instanton simply coincides with the exact self-dual solution of the unbroken theory. Furthermore, in supersymmetric models, the leading-order short-distance fermions and scalars assume their classical zero-mode values in this self-dual background. In contrast, at large distances, the fields satisfy linearized equations; nevertheless, the overall amplitudes of the long-distance fields are determined by the short-distance behavior [20,18].

When one generalizes to $n>1$, there is an additional set of issues to be confronted. The saddle-point contribution of ADHM multi-instantons to the functional integral for pure bosonic Yang-Mills theory has been studied extensively with inconclusive results.[2] This program was never completed for several reasons. First, it was not possible to calculate the functional determinant for small fluctuations around the general ADHM solution which provides a vital prefactor for the saddle-point exponential. Second, the calculations were plagued with the usual infrared-divergent integrals over instanton sizes which are ubiquitous in theories with classical scale invariance. Finally the parameters which appear in the general ADHM solution are not independent, but obey nonlinear constraints. These constraints have only been solved explicitly for $n \leq 3$ [22]. Fortunately, in the supersymmetric theory at hand, the first two difficulties are simply not present. The nonzero eigenvalues in the functional determi-

nants for small fluctuations cancel precisely between bosonic and fermionic degrees of freedom. In fact, this is just the manifestation at one loop of the nonrenormalization theorem mentioned above. The second problem is eliminated because the VEV acts as an infrared cutoff on the contribution of large instantons. The third problem is harder to avoid, but there are preliminary indications that supersymmetry may provide simplifications in this regard also. Certainly, it would be gratifying if knowledge of the specific instanton series (1.1), in this one particular model, were to lead to fundamental progress in the general theory of multi-instantons.

B. Plan of this paper

This paper is organized as follows. In Sec. II we review both the microscopic SU(2) gauge theory and also, the effective long-distance U(1) theory valid for length scales $\gg 1/gv$. In Sec. III we discuss the coupled Euler-Lagrange equations at both distance scales, as well as the "patching condition" which strictly relates the tail of the constrained instanton to its core [20,18].

The heart of the paper is the study of ADHM multi-instantons, Secs. VI–VIII. However, as essential groundwork, we first present a detailed study of the single superinstanton sector in Secs. IV and V. This subject has been studied extensively [18,23–26]; in particular, Finnell and Pouliot [12] have applied the methods of [18] to the $N=2$ theory, and confirmed the one-instanton coefficient $\mathcal{F}_1 = 1/2$ in Eq. (1.1). Nevertheless, there are two reasons for revisiting this subject in a careful way. The first is curiosity about the instanton methods themselves; we would like to see how to reproduce the full one-instanton sector of Eq. (1.1), not just the four-fermion vertex. Specifically, this means understanding the fermion bilinear contributions to the classical Bose fields. The detailed structure of the superinstanton is the subject of Sec. IV. We then recover, in Sec. V, the one-instanton contribution to *all* the effective Seiberg-Witten vertices that can be probed by saturating m-point functions with m insertions of long-distance classical fields (e.g., the anomalous magnetic moment vertex). Ultimately, however, there is a much more important reason to revisit the one-instanton sector: As we shall make clear, detailed knowledge of these fermion-bilinear contributions provides an *explicit roadmap*[3] for how to solve the analogous equations for $n>1$.

Section VI is a self-contained steepest-descent introduction to the ADHM construction of multi-instantons. In Sec. VII we apply the lessons learned in Secs. IV and V to the more challenging problem of solving the coupled boson-fermion equations of motion in the general ADHM background. Based on our explicit one-instanton expressions, we

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[2]See Ref. [21] for review of progress made in this direction as well as useful set of references.

[3]Technically, the reason that the problem is inherently much more challenging for $n>1$ can be traced to the following. In the one-instanton sector, the full complement of fermion zero modes can be generated by applying Lagrangian symmetries to the Belavin-Polyakov-Schwartz-Tyupin (BPST) instanton [27], specifically the generators of the superconformal group [23–25] (Appendix A). But for $n>1$ these only sweep out a fixed subset of the $8n$-dimensional space of fermionic collective coordinates, so that the methods of Refs. [23–25] are insufficient.

are able to intuit the correct solutions and verify our guess-work *a posteriori*. The principal result of this section, and of the paper as a whole, is the explicit construction of the adjoint Higgs boson in the exact ADHM multi-instanton background and, with it, the exact classical interaction between n instantons, both in the bosonic and in the supersymmetric cases [Eq. (7.32)]. As expected on general grounds from Eq. (1.1), this interaction lifts all but four of the $8n$ fermion zero modes.

Finally, in Sec. VIII we specialize to $n=2$. The main technical hurdle in this section is the construction of the two-instanton measure. In this task we rely heavily on some remarkable results due to Osborn [21] and to Corrigan and collaborators [28,29], which are reviewed as needed. We also supply some essential new ingredients: principally, the calculation of the fermion zero-mode determinant (Appendixes B and C) and the pinpointing of the residual discrete symmetry group that must be modded out in order to identify the physical moduli space of inequivalent field configurations and, thus, properly to normalize the measure (Appendix D). With a convenient resolution of ADHM constraints, the 28-fold integration over the supersymmetric moduli proceeds in a straightforward manner and confirms that $\mathcal{F}_2 = 5/16$.

II. MICROSCOPIC VS EFFECTIVE LAGRANGIANS

In this section we review both the microscopic non-Abelian theory where the instantons live and, also, the effective long-distance Abelian theory that the power-law ''tail'' of the instanton is supposed to reproduce. The particle content of pure $N=2$ supersymmetric SU(2) gauge theory consists, in $N=1$ language, of a gauge multiplet (v_m, λ, D) coupled to a complex chiral matter multiplet (A, ψ, F) which transforms in the adjoint representation of the gauge group. Here v_m is the gauge field, A is the Higgs field, and the Weyl fermions λ and ψ are the gaugino and Higgsino, while D and F are auxiliary fields. In Wess-Bagger notation [30], the component Lagrangian reads[4]

$$\mathcal{L}_{SU(2)} = \mathcal{L}_{gauge} + \mathcal{L}_{chiral}, \qquad (2.1)$$

where

$$\mathcal{L}_{gauge} = tr_2\{-\tfrac{1}{2} v_{mn} v^{mn} - i\bar{\lambda}\bar{\mathcal{D}}\lambda - i\lambda\mathcal{D}\bar{\lambda} + D^2\} \qquad (2.2)$$

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[4]Note on conventions: we use an undertilde as a shorthand for SU(2) matrix notation; thus, $X = \Sigma_{a=1,2,3} X^a \tau^a/2$, where τ^a are Pauli matrices. Letters from the beginning of the alphabet are adjoint SU(2) indices running over 1,2,3, whereas letters from the middle of the alphabet run over 0,1,2,3 (or in Euclidean space 1,2,3,4). Also $v_{mn} = \partial_m v_n - \partial_n v_m - ig[v_m, v_n]$, $\mathcal{D}_{\alpha\dot\alpha} = \mathcal{D}_m \sigma^m_{\alpha\dot\alpha}$, and $\bar{\mathcal{D}}^{\dot\alpha\alpha} = \mathcal{D}^m \bar\sigma_m^{\dot\alpha\alpha}$, where $\mathcal{D}_m X = \partial_m X - ig[v_m, X]$. Wess-Bagger conventions are used throughout [30]: $\chi\zeta = \chi^\alpha \zeta_\alpha$, $\bar\chi\bar\zeta = \bar\chi_{\dot\alpha}\bar\zeta^{\dot\alpha}$, $\chi\sigma^{mn}\zeta = \chi^\alpha \sigma^{mn}{}_\alpha{}^\beta \zeta_\beta$, and $\bar\chi\bar\sigma^{mn}\bar\zeta = \bar\chi_{\dot\alpha}\bar\sigma^{mn\dot\alpha}{}_{\dot\beta}\bar\zeta^{\dot\beta}$. The metric is $\eta_{mn} = \text{diag}(-1,1,1,1)$. Throughout this paper we work in Minkowski space even when dealing with instantons; analytic continuation to Euclidean space poses no problems. Through Sec. V we will usually keep factors of g explicit, unlike Seiberg and Witten who set $g=1$; thus, for instance, their condition for weak coupling, $v \gg \Lambda$, becomes for us $M_W \gg \Lambda$.

and

$$\mathcal{L}_{chiral} = tr_2\{-2\mathcal{D}_m A^\dagger \mathcal{D}^m A - i\bar\psi\bar{\mathcal{D}}\psi - i\psi\mathcal{D}\bar\psi + 2F^\dagger F$$
$$-2gD[A, A^\dagger] + 2\sqrt{2} gi([A^\dagger, \psi]\lambda + \bar\lambda[A, \bar\psi])\}. \qquad (2.3)$$

In addition to superconformal invariance (see Appendix A for a review), $\mathcal{L}_{SU(2)}$ is classically invariant under so-called $SU(2)_R$ rotations of λ and ψ:

$$\begin{pmatrix} \lambda \\ \psi \end{pmatrix} \to \mathbf{M}_R \cdot \begin{pmatrix} \lambda \\ \psi \end{pmatrix}, \quad (\bar\lambda, \bar\psi) \to (\bar\lambda, \bar\psi) \cdot \mathbf{M}_R^{-1}, \quad \mathbf{M}_R \in SU(2). \qquad (2.4)$$

The bosons are $SU(2)_R$ singlets. $N=1$ invariance together with $SU(2)_R$ invariance guarantees $N=2$ invariance, a labor-saving observation which we exploit below.

Eliminating the auxiliary field D from $\mathcal{L}_{SU(2)}$ produces a scalar potential proportional to $tr_2[A, A^\dagger]^2$. The classical vacua of the theory therefore consist of all constant fields A and $A^\dagger$ satisfying $A \propto A^\dagger$. With a gauge transformation we can always align the VEV along a specific direction, for instance, $A = v\tau^3/2$ and $A^\dagger = \bar{v}\tau^3/2$; the gauge-inequivalent vacua are then labeled by the arbitrary complex number v. Importantly, since $N=2$ supersymmetry protects against the generation of a superpotential, this degeneracy of vacua persists at the quantum level as well [10].

For nonzero v the gauge group SU(2) spontaneously breaks down to U(1). The components of the fields that are aligned with the VEV remain massless and are neutral under the unbroken U(1), whereas the remaining charged components acquire a mass $M_W = \sqrt{2} g|v|$. For length scales $x \gg 1/M_W$, the massless modes can be described by an effective Lagrangian $\mathcal{L}_{eff}$ constructed from a single $N=2$ superfield Ψ, or, in more familiar language, from an $N=1$ photon superfield $W^\alpha = (v_m, \lambda, D)$ and chiral superfield $\Phi = (A, \psi, F)$. Underlying $N=2$ supersymmetry forces $\mathcal{L}_{eff}$ to have the form

$$\mathcal{L}_{eff} = \frac{1}{4\pi} \text{Im}\left[\int d^4\theta \mathcal{F}'_{SW}(\Phi)\bar\Phi \right.$$
$$\left. + \int d^2\theta \tfrac{1}{2} \mathcal{F}''_{SW}(\Phi) W^\alpha W_\alpha \right] + \cdots, \qquad (2.5)$$

where the prepotential $\mathcal{F}_{SW}$ was defined in Eq. (1.1) and the dots represent terms of higher order in ''chiral perturbation theory.'' In this paper we will be focusing on the region of moduli space $M_W \gg \Lambda$, where the theory is weakly coupled due to asymptotic freedom and instantons may be studied in the semiclassical approximation. Nevertheless, we ought to mention an especially surprising feature of the exact Seiberg-Witten solution in the opposite regime, namely, that the classical symmetry-restoring point $v=0$ of the effective theory is actually absent from the quantum moduli space.

As with the microscopic theory, it will be useful to write out $\mathcal{L}_{eff}$ in components:

$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi}\,\text{Im}\Bigg[-\mathcal{F}''_{\text{SW}}(A)[\partial_m A^\dagger \partial^m A + i\psi\,\slashed{\partial}\overline{\psi} + i\lambda\,\slashed{\partial}\overline{\lambda}$$

$$+ \tfrac{1}{2}(v^{\text{SD}}_{mn})^2] + \frac{1}{\sqrt{2}}\,\mathcal{F}'''_{\text{SW}}(A)\lambda\,\sigma^{mn}\psi v_{mn}$$

$$+ \tfrac{1}{4}\mathcal{F}''''_{\text{SW}}(A)\psi^2\lambda^2\Bigg] + \mathcal{L}_{\text{aux}}. \qquad (2.6)$$

Here $\mathcal{L}_{\text{aux}}$ assembles all the dependence on the auxiliary fields F, $F^\dagger$, and D:

$$\mathcal{L}_{\text{aux}} = \frac{1}{4\pi}\,\text{Im}\{\mathcal{F}''_{\text{SW}}(A)(F^\dagger F + \tfrac{1}{2}D^2)$$

$$- \tfrac{1}{2}\mathcal{F}'''_{\text{SW}}(A)[F^\dagger(\psi^2 + \lambda^2) - i\sqrt{2}D\psi\lambda]\}. \qquad (2.7)$$

The quantity v^{SD}_{mn} in Eq. (2.6) denotes the self-dual part of the field strength[5] so that $(v^{\text{SD}}_{mn})^2$ comprises both $(v_{mn})^2$ and $iv_{mn}v_{kl}\epsilon^{mnkl}$ terms. For these two terms to have their customary couplings, $\mathcal{F}''_{\text{SW}}$ at the VEV scale, or indeed at any scale μ, must be identified with the complexified coupling $\tau(\mu)$:

$$\mathcal{F}'_{\text{SW}}(\mu) = \tau(\mu) = \frac{4\pi i}{g^2(\mu)} + \frac{\vartheta(\mu)}{2\pi}, \qquad (2.8)$$

where ϑ is the effective theta parameter. Note that in this paper we are using the notation of Ref. [1] for the complexified coupling (2.8). Trivial rescaling connects it with the notation of Ref. [3]: Multiply the right-hand side of Eq. (2.8) by 2, and divide by 2 the right-hand sides of Eqs. (2.5)–(2.7) and (1.1).

We make several comments.

(1) The correspondence between the renormalization-group (RG) invariant dynamical mass scale Λ in the effective U(1) theory and the running coupling $g(\mu)$ in the microscopic SU(2) theory have been carefully examined by Finnell and Pouliot [12] using matching to perturbation theory in the weak-coupling regime. In the Pauli-Villars regularization scheme, which is the natural scheme for doing instanton calculations, the result of [12] is simply

$$\Lambda^4 = \Lambda^4_{\text{PV}} \equiv \mu^4 e^{-8\pi^2/g^2_{\text{PV}}(\mu)}. \qquad (2.9)$$

A different choice of Λ would imply an altered prescription for the $\{\mathcal{F}_n\}$. From now on the Pauli-Villars scheme will be always assumed, and in what follows we suppress the PV subscript.

(2) As always in effective theories, the relationship between the effective fields $\{A, \psi, \dots\}$ and the microscopic fields $\{\underline{A}, \underline{\psi}, \dots\}$ is in no way unique and may be quite complicated.

But for long-distance physics ($x \gg 1/M_W$), we can simply equate the effective fields with the surviving massless components; thus, $A = \tfrac{1}{2}g\,\text{tr}_2\tau^3\underline{A}$, $\psi = \tfrac{1}{2}g\,\text{tr}_2\tau^3\underline{\psi}$, etc., assuming the VEV points in the τ^3 direction. The factor of g compensates for the overall normalization of $\mathcal{L}_{\text{eff}}$ by $1/g^2$ as opposed to $\mathcal{L}_{\text{SU(2)}}$.

(3) Integrating out the auxiliary fields from Eq. (2.7) gives new contributions to the effective vertices, for instance to the $\psi^2\lambda^2$ operator. However, these are down by a relative factor of $[\text{Im}\mathcal{F}_{\text{SW}}(A)'']^{-1} \sim g^2$ compared with existing vertices, term by term in the instanton expansion. For simplicity, we ignore $\mathcal{L}_{\text{aux}}$ in the following and choose to concentrate, at any given order in the instanton expansion, on the leading contributions in g^2. In principle, the contribution of $\mathcal{L}_{\text{aux}}$ can be recaptured with instanton methods by analyzing Feynman graphs in instanton backgrounds.

III. DEFINING EQUATIONS FOR THE CONSTRAINED SUPERINSTANTON

Throughout this paper, we will use the term ''superinstanton'' in a loose way, simply to denote a nontrivial finite-action (exact or approximate) solution to the coupled supersymmetric Euler-Lagrange equations of the theory.[6] For the model at hand, these equations read, in Minkowski space,

$$\mathcal{D}^m \underline{v}_{mn} = -ig([\underline{A}, \mathcal{D}_n\underline{A}^\dagger] + [\underline{A}^\dagger, \mathcal{D}_n\underline{A}])$$

$$- g(\underline{\lambda}\sigma_n\overline{\underline{\lambda}} + \overline{\underline{\lambda}}\,\overline{\sigma}_n\underline{\lambda} + \underline{\psi}\sigma_n\overline{\underline{\psi}} + \overline{\underline{\psi}}\,\overline{\sigma}_n\underline{\psi}), \qquad (3.1a)$$

$$\overline{\mathcal{D}}\underline{\lambda} = \sqrt{2}g[\underline{A}, \overline{\underline{\psi}}], \qquad (3.1b)$$

$$\slashed{\mathcal{D}}\overline{\underline{\lambda}} = \sqrt{2}g[\underline{A}^\dagger, \underline{\psi}], \qquad (3.1c)$$

$$\underline{D} = g[\underline{A}, \underline{A}^\dagger] \qquad (3.1d)$$

for the $N=1$ gauge multiplet, and

$$\mathcal{D}^2\underline{A} = \sqrt{2}ig[\underline{\lambda}, \underline{\psi}] + g[\underline{D}, \underline{A}], \qquad (3.2a)$$

$$\overline{\mathcal{D}}\underline{\psi} = \sqrt{2}g[\overline{\underline{\lambda}}, \underline{A}], \qquad (3.2b)$$

$$\underline{F} = 0 \qquad (3.2c)$$

for the $N=1$ chiral multiplet, and finally

$$\mathcal{D}^2\underline{A}^\dagger = \sqrt{2}ig[\overline{\underline{\lambda}}, \overline{\underline{\psi}}] - g[\underline{D}, \underline{A}^\dagger], \qquad (3.3a)$$

$$\slashed{\mathcal{D}}\overline{\underline{\psi}} = \sqrt{2}g[\underline{\lambda}, \underline{A}^\dagger], \qquad (3.3b)$$

$$\underline{F}^\dagger = 0 \qquad (3.3c)$$

for the $N=1$ antichiral multiplet.

There are actually two philosophically distinct, but mathematically equivalent, ways of viewing the fermion compo-

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[5]In Minkowski space the self-dual and anti-self-dual components of v_{mn} are projected out using $v^{\text{SD}}_{mn} = \tfrac{1}{4}(\eta_{mk}\eta_{nl} - \eta_{ml}\eta_{nk} + i\epsilon_{mnkl})v^{kl}$ and $v^{\text{ASD}}_{mn} = (v^{\text{SD}}_{mn})^*$, where $\epsilon_{0123} = -\epsilon^{0123} = -1$. Also, since $\sigma^{mn} = \tfrac{1}{4}\sigma^{[m}\overline{\sigma}^{n]}$ and $\overline{\sigma}^{mn} = \tfrac{1}{4}\overline{\sigma}^{[m}\sigma^{n]}$ are self-dual and anti-self-dual, respectively, it follows that $\sigma^{mn}{}_\alpha{}^\beta v_{mn} = \sigma^{mn}{}_\alpha{}^\beta v^{\text{SD}}_{mn}$ and $\overline{\sigma}^{mn\dot\alpha}{}_{\dot\beta}v_{mn} = \overline{\sigma}^{mn\dot\alpha}{}_{\dot\beta}v^{\text{ASD}}_{mn}$.

[6]In particular, we are making no claim about the exchangeability, at all length scales simultaneously, of an active supersymmetry transformation on the fields for a passive transformation on the collective coordinates, which property is at the heart the use of the term by Novikov et al. [23].

nents in these equations. In one approach, the classical background configuration is always purely bosonic and the fermions are treated as a particular set of fluctuations. This is the viewpoint that one finds in 't Hooft's original paper [31] and which is implicit in the language used by Affleck, Dine, and Seiberg [18]. In contrast, in the formalism of Novikov *et al.* [23], the fermion modes naturally appear as the superpartners of the classical gauge and Higgs configurations (as we shall see below). In this approach the fermions acquire a geometric meaning and are thought of as facets of the classical solution. This is the viewpoint, and the language, we adopt herein.

Returning to these equations, we recall that for a nonzero VEV a nontrivial solution cannot exist, thanks to Derrick's theorem [19]: For any putative solution, one can lower the action further simply by shrinking the configuration. One famous fix to this problem, due to Affleck [20], is as follows. A new operator, or ''Affleck constraint,'' is introduced into the action by means of a Faddeev-Popov insertion of unity. If this operator is of suitably high dimension, Derrick's theorem is avoided, and the instanton stabilizes at a fixed scale size ρ. The integration over the Faddeev-Popov Lagrange multiplier can then be traded off for the integration over ρ. The now-stable solutions are known as constrained instantons.

Of course, the detailed shape of the constrained (super)instanton depends in a complicated way on one's choice of constraint. But certain important features remain constraint independent.

(1) The short-distance regime $x \ll 1/M_W$. In this regime Eqs. (3.1)–(3.3) can be solved perturbatively in $g^2 \rho^2 |v|^2$; since ultimately the integration over scale size is dominated by $\rho \sim |v|^{-1}$, this is tantamount to perturbation theory in g^2. As the constraints do not enter into these equations until some high order, the first few terms in this expansion are robust. For $\rho \ll x \ll 1/M_W$ the various fields fall off as powers of ρ^2/x^2; again, the first few terms in the expansion in ρ^2/x^2 are constraint independent. Note that none of these inequalities conflict in any way with the requirement for semiclassical physics, $M_W \gg \Lambda$.

(2) The long-distance regime $x \gg 1/M_W$. The long-distance ''tail'' of the instanton reflects the Higgs mechanism. In the model at hand, the superinstanton components perpendicular to the VEV decay as $\exp(-M_W|x|)$. In contrast, the components parallel to the VEV fall off merely as powers of ρ^2/x^2. They are constrained to obey the U(1) reductions of Eqs. (3.1)–(3.3); namely,

$$0 = \partial^m v_{mn}^{(3)} = \bar{\partial}\lambda^{(3)} = \bar{\partial}\bar{\lambda}^{(3)} = D^{(3)} = \partial^2 A^{(3)}$$

$$= \bar{\partial}\psi^{(3)} = F^{(3)} = \partial^2 A^{\dagger(3)} = \partial\bar{\psi}^{(3)} = F^{\dagger(3)}, \qquad (3.4)$$

again assuming that the VEV is aligned in the τ^3 direction. By themselves, these equations allow a distressingly broad range of possible behaviors for the instanton tail; for example, the long-distance Higgs boson can decay as $v + C_{\rm LD}/(x-x_0)^2$ for any constant $C_{\rm LD}$. This is a hallmark of linear equations. Fortunately, Eqs. (3.4) are supplemented by a *patching condition* which ties $C_{\rm LD}$ to the analogous short-distance constant $C_{\rm SD}$ derived in the regime $\rho \ll x \ll 1/M_W$. Briefly, the patching condition states

$$C_{\rm LD} = C_{\rm SD}[1 + O(g^2)], \qquad (3.5)$$

with the specific form of the Affleck constraints only affecting these $O(g^2)$ corrections.

It is precisely because of this patching condition that it is important to study the short-distance properties of the instanton, even if, as here, one is ultimately interested in the tail.[7] Accordingly, in the following section we give an unusually careful analysis of the short-distance superinstanton and are twice rewarded for our efforts. First, this will enable us to rederive *all* of the pieces of $\mathcal{L}_{\rm eff}$ that can be obtained by saturating n-point functions with n insertions of single instantons. And, second, our explicit one-instanton results will allow us later to intuit the general solutions to Euler-Lagrange equations in arbitrary multi-instanton backgrounds.

IV. SINGLE SHORT-DISTANCE SUPERINSTANTON

A. Overview of calculation

In this section and the next, we specialize to the one-instanton sector.[8] In this case the short-distance superinstanton is most easily constructed with the ''sweeping-out'' technique originated by Novikov *et al.* [23] and extended by Fuchs and Schmidt [24,25]. By design, this technique covers the solution space of the coupled Euler-Lagrange equations, to any desired order in $g^2 \rho^2 |v|^2$. Specifically, starting with a ''reference'' configuration $\Psi^{(0)}$ which solves Eqs. (3.1)–(3.3) at a given order in perturbation theory, the superinstantons are then the family of configurations generated by

$$e^{\xi(x)Q} \times e^{\bar{\Theta}_0 \bar{Q}} \times \Psi^{(0)}. \qquad (4.1)$$

$\Psi^{(0)}$ stands for the initial values for all the bosons and fermions in the model. As explained in Appendix A, the product $\xi(x)Q$ encapsulates both Q^α and $\bar{S}_{\dot{\alpha}}$, where Q is the $N=1$ supersymmetry generator and $\bar{S}$ is the fermionic superpartner of the special conformal generator K_μ. The action of $\bar{Q}$, Q, and $\bar{S}$ on all the component fields in $\Psi^{(0)}$ is specified in Eqs. (A4)–(A6).

The calculation that follows has the following features. (1) Since the construction (4.1) is only manifestly $N=1$ invariant,[9] it is necessary also to check SU(2)$_R$ invariance,

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[7] As opposed to the short- and long-distance regimes, very little can be said about the shape of the instanton at length scales $x \sim 1/M_W$. This is the domain where the short- and long-distance behaviors are patched together and the effect of the constraints is unsuppressed. While it underlies the work of Affleck, Dine, and Seiberg and much of the literature on high-energy baryon number violation, so far as we know the patching condition is still at the level of a ''folk theorem'' about nonlinear differential equations; a rigorous proof would start by recasting these as integral equations.

[8] Those readers primarily interested in what we have to say about multi-instantons may skip directly to Sec. VI.

[9] While, in principle, an ($N=2$)-invariant sweeping-out procedure would be more aesthetic for the particular problem at hand, in practice it seems much harder to carry out, the primary reason being that ($N=2$)-invariant Grassmannian exponentials such as $\exp(\bar{\Theta}^{(1)}\bar{Q}^{(1)} + \bar{\Theta}^{(2)}\bar{Q}^{(2)})$ terminate at quartic rather than quadratic order. This technical difficulty, together with the desirability of making contact with the work of Novikov *et al.* and Fuchs and Schmidt, leads us to favor the $N=1$ formalism.

Eq. (2.4). We shall find the following. At leading order, the ψ and λ components of the superinstanton appear in an $SU(2)_R$ symmetric way, but $\bar{\psi}$ and $\bar{\lambda}$ do not; at the next leading order, $SU(2)_R$ invariance of the antifermions is recaptured, but now the fermions fail to enter symmetrically by a small amount; and so forth, order by order in $g^2\rho^2|v|^2$. Only at infinite order is exact $SU(2)_R$ symmetry and hence $N=2$ supersymmetry manifest, assuming that the Affleck constraints themselves respect $N=2$ supersymmetry (which in principle they need not).

(2) This ''leapfrog'' pattern characterizes the Euler-Lagrange equations (3.1)–(3.3) as well. At leading order the $N=1$ matter equations (3.2) and (3.3) are exact, whereas the gauge equations (3.1) are approximate; at next leading order, the latter become exact, while the former are off by a small amount; and so forth.

In practice, we content ourselves with a next-leading-order calculation, where nontrivial checks on the proposed Seiberg-Witten solution can first be made. Our explicit next-leading-order results, as well as our formulas for the components of the antichiral superfields, are beyond the existing literature.

B. Leading-order calculation

Following Fuchs and Schmidt [24], we start with a reference configuration $\Psi^{(0)}$ which is *almost* purely bosonic. Specifically, we take for the gauge field $v_m^{(0)}=v_m^{\rm cl}$ where $v_m^{\rm cl}$ is a BPST instanton [27] having some particular position x_0, scale size ρ, and iso-orientation R_{ab}. We take for the Higgs boson, $A^{(0)}=A_{\rm cl}$ and $A^{\dagger(0)}=A_{\rm cl}^{\dagger}$ where $A_{\rm cl}$ satisfies the massless Klein Gordon equation in the BPST background, $\mathcal{D}^2 A_{\rm cl}=0$, while approaching the VEV $v^a\tau^a/2$ at infinity. All other bosonic and fermionic components are initially set to zero, *except* for ψ. As noted by Fuchs and Schmidt,[10] this component needs to be loaded initially with a supersymmetric zero mode, or else none is generated by the sweeping-out procedure (4.1). Thus we set

$$\psi^{(0)}=\xi'\,\sigma^{mn}v_{mn}^{\rm cl}, \qquad (4.2)$$

where ξ' is a new Grassmannian collective coordinate. This defines our starting point $\Psi^{(0)}$.

Sweeping out as per Eq. (4.1) is a two-step process. First, one applies $\exp(\bar{\Theta}_0\bar{Q})$ to each component field and refers the final expressions to their prescribed values in $\Psi^{(0)}$ [see Eq. (A7) for an example]. With $\Psi^{(0)}$ as specified above and remembering that $\bar{\sigma}^{mn}v_{mn}^{\rm cl}=0$ by self-duality, we find that the *only* field that transforms is the Higgsino, which picks up an admixture of a superconformal zero mode (see Appendix A): $\psi\to\psi^{(0)}-i\sqrt{2}\bar{\Theta}_0\bar{\mathcal{D}}A_{\rm cl}$. Next, one repeats the procedure with $\exp[\xi(x)Q]$. The result of this simple exercise is the

leading-order superinstanton with the following functional form for the gauge, chiral, and antichiral multiplets, respectively:

$$v_m=v_m^{\rm cl}, \quad \lambda=-\xi(x)\sigma^{mn}v_{mn}^{\rm cl}, \quad \bar{\lambda}=D=0, \qquad (4.3a)$$

$$A=A_{\rm cl}+\sqrt{2}\xi'(x)\sigma^{mn}\xi(x)v_{mn}^{\rm cl},$$

$$\psi=\xi'(x)\sigma^{mn}v_{mn}^{\rm cl}, \quad F=0, \qquad (4.3b)$$

$$A^{\dagger}=A_{\rm cl}^{\dagger}, \quad \bar{\psi}=-i\sqrt{2}[\xi(x)\mathcal{D}+2\bar{\eta}]A_{\rm cl}^{\dagger}, \quad F^{\dagger}=0. \qquad (4.3c)$$

The reader can verify that Eqs. (4.3a,b) are equivalent, up to a collective-coordinate-dependent gauge transformation, to the elegant superfield expressions for $W_{\rm cl}^{\alpha}$ and $\Phi_{\rm cl}$, respectively, derived by Novikov *et al.* [23]. For the present purposes, however, Eq. (4.3) has the advantage of commuting with gauge fixing. This will allow for a clean physical interpretation for the quanta that propagate to infinity in the manner of Affleck, Dine, and Seiberg [18].

In arriving at Eq. (4.3), we have used the equations of motion (3.1)–(3.3) to eliminate all quadratic terms in the expansions of the Grassmann exponentials. Furthermore, in order to highlight the manifest $SU(2)_R$ symmetry between λ and ψ, we have traded $\bar{\Theta}_0$ for a linearly related collective coordinate $\bar{\eta}'$, defined in analogy to Eq. (A5),

$$\xi'(x)=\xi'-(x^k-x_0^k)\bar{\eta}'\,\bar{\sigma}_k, \qquad (4.4a)$$

$$\bar{\eta}'=\tfrac{1}{8}i\sqrt{2}gv^a\bar{\eta}_{mn}'^a\bar{\Theta}_0\bar{\sigma}^{mn}, \qquad (4.4b)$$

$$\bar{\Theta}_0=i\sqrt{2}g^{-1}\frac{\bar{v}^a}{|v|^2}\bar{\eta}_{mn}'^a\bar{\eta}'\bar{\sigma}^{mn}. \qquad (4.4c)$$

Here $\bar{\eta}_{mn}'^a$ is short for $R_{ab}\bar{\eta}_{mn}^b$, with $\bar{\eta}_{mn}^b$ a 't Hooft symbol [31]. This definition allows us to translate back and forth between the two distinct-looking representations of the superconformal zero mode (see Appendix A), since

$$i\sqrt{2}\bar{\Theta}_0\bar{\mathcal{D}}A_{\rm cl}=(x_k-x_k^0)\bar{\eta}'\bar{\sigma}^k\sigma^{mn}v_{mn}^{\rm cl}, \qquad (4.5)$$

as the reader can verify in his favorite gauge [e.g., Eq. (4.8) below].

As is clear from Eq. (4.3), $SU(2)_R$ symmetry, originally defined in Eq. (2.4) as an active transformation on the fermions, can be recast as a passive transformation on the Grassmannian collective coordinates:

$$\begin{pmatrix} \xi(x) \\ -\xi'(x) \end{pmatrix}\to\mathbf{M}_R\cdot\begin{pmatrix} \xi(x) \\ -\xi'(x) \end{pmatrix}. \qquad (4.6)$$

Note that the fermion bilinear contribution to A, while breaking the symmetry between A and $A^{\dagger}$, is an $SU(2)_R$ singlet under Eq. (4.6), as it must be.

In contrast, $SU(2)_R$ is *not* respected by the antifermions; indeed, $\bar{\lambda}$ is not even turned on at this order. And while Eqs. (3.2) and (3.3) are exactly satisfied by the leading-order superinstanton (4.3), Eqs. (3.1) are not; instead, what is satisfied are the homogeneous variants of Eqs. (3.1), with the

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[10]This is a minor peculiarity of the $N=1$ approach in the presence of an adjoint Higgs boson, which we could avoid by adopting from the outset an $N=2$ sweeping-out procedure. Our choice of notation ξ' reminds us that this is really the collective coordinate for the second supersymmetry; see Eq. (A1f).

right-hand sides set to zero. Both these drawbacks are rectified at next leading order, as mentioned above, and confirmed in Sec. IV D below.

C. Superinstanton action

Knowledge of the leading-order short-distance superinstanton (4.3) suffices to construct the action, up to $g^2\rho^2|v|^2$ corrections. We shall do so in the way that generalizes most readily to multi-instantons, by expressing the answer as a surface integral. At leading order the only nonvanishing terms are

$$S_{\text{inst}} \equiv S_{\text{cl}} + S_{\text{Higgs}} + S_{\text{Yuk}}$$

$$= \text{tr}_2 \int d^4x \left(-\tfrac{1}{2}\underline{v}_{mn}\underline{v}^{mn} - 2\mathcal{D}_m\underline{A}^\dagger\mathcal{D}^m\underline{A} + 2\sqrt{2}gi[\underline{A}^\dagger,\psi]\underline{\lambda}\right)$$

$$= \frac{8\pi^2}{g^2} - 2\,\text{tr}_2 \int d^3S\,\underline{A}^\dagger\hat{x}_m\mathcal{D}^m\underline{A}, \tag{4.7}$$

where $\hat{x}_m = x_m/\sqrt{|x|^2}$ and S is the 3-sphere at infinity.[11] The last equality follows from an integration by parts together with the Euler-Lagrange equation (3.2a).

The surface integral is most conveniently evaluated in singular gauge, in which[12] $\underline{A}_{\text{cl}} \to v^a\tau^a/2$ so that $\underline{A}^\dagger$ can simply be replaced by $\bar{v}^a\tau^a/2$. Singular gauge is defined by

$$\underline{v}_m^{\text{cl}} = \frac{2}{g}\frac{\rho^2}{x^2(x^2+\rho^2)}\,\bar{\eta}'^a_{ml}x_l\,\frac{\tau^a}{2}, \tag{4.8a}$$

$$\underline{v}_{mn}^{\text{cl}} = \frac{4\rho^2}{g}\frac{1}{x^2(x^2+\rho^2)^2}$$

$$\times(-x^2\bar{\eta}'^a_{mn} + 2x_lx_n\bar{\eta}'^a_{ml} + 2x_lx_m\bar{\eta}'^a_{ln})\frac{\tau^a}{2}, \tag{4.8b}$$

$$\underline{A}_{\text{cl}} = \frac{x^2}{x^2+\rho^2}v^a\frac{\tau^a}{2}, \tag{4.8c}$$

$$\mathcal{D}_l\underline{A}_{\text{cl}} = \frac{2\rho^2}{(x^2+\rho^2)^2}\,x_m v^b\bar{\eta}'^b_{ln}\bar{\eta}'^a_{mn}\frac{\tau^a}{2}. \tag{4.8d}$$

The gradient of the Higgs field (4.3b) including both the classical and fermion-bilinear contributions is easily obtained and gives the well-known result[13]

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[11]Actually up to subleading corrections the radius of the sphere should be taken to be $\gg\rho$, but $\ll 1/M_W$, so that the short-distance formulas (4.3) are applicable.

[12]This is not true in regular gauge, where $\underline{A}_{\text{cl}}$ has nontrivial spatial structure even at infinity.

[13]See Refs. [23–26]. Such expressions for the action are of course the source of the familiar lore that the integration over instanton scale size is typically dominated by ρ^2 on the order of $1/|v|^2$, at least for m-point functions with $m\ll 1/g^2$. Surprisingly, for certain types of supersymmetric correlators, although not the ones considered in this paper, it is actually the *zero*-size instantons which control the physics [23].

$$S_{\text{Higgs}} + S_{\text{Yuk}} = 4\pi^2|v|^2\rho^2\left(1 + 4\sqrt{2}g^{-1}\frac{\bar{v}^a}{|v|^2}\,\bar{\eta}'^a_{mn}\bar{\eta}'\,\bar{\sigma}^{mn}\bar{\eta}\right)$$

$$= 4\pi^2|v|^2\rho^2(1 - 4i\bar{\Theta}_0\bar{\eta}). \tag{4.9}$$

D. Next-leading-order calculation

Returning once again to Eq. (4.3a), we note that the entire right-handed gauge superfield $\overline{W}_{\dot{\alpha}} = (\bar{\lambda}, \underline{D}, \bar{\sigma}^{mn}\underline{v}_{mn}, \overline{\mathcal{D}}\underline{\lambda})$ is identically zero; the $N=1$ gauge components of the superinstanton live exclusively in the left-handed superfield W^a at leading order in $g^2\rho^2|v|^2$. We now refine the superinstanton to next leading order in $g^2\rho^2|v|^2$ and verify that $\overline{W}_{\dot{\alpha}}$ turns on at this order. The first step is to improve the choice of reference configuration $\Psi^{(0)}$. As discussed above, the strategy at this order is to freeze the $N=1$ matter components at their earlier values, but choose gauge components so that the Euler-Lagrange equations (3.1) hold true. The improved initial choices are thus dictated by the equations

$$\mathcal{D}\underline{\lambda}^{(0)} = \sqrt{2}g[\underline{A}_{\text{cl}}^\dagger,\psi^{(0)}] \equiv \sqrt{2}g[\underline{A}_{\text{cl}}^\dagger,\xi'\sigma^{mn}\underline{v}_{mn}^{\text{cl}}] \tag{4.10}$$

and

$$\mathcal{D}^m\underline{v}_{mn}^{(0)} = -2ig[\underline{A}_{\text{cl}}^\dagger,\mathcal{D}_n\underline{A}_{\text{cl}}], \tag{4.11}$$

with $\underline{\lambda}^{(0)}$ and $\underline{D}^{(0)}$ still zero. Repeating the earlier two-step sweeping-out procedure then gives the improved gauge components of the superinstanton, which supersede Eq. (4.3a):

$$\underline{v}_{mn} = \underline{v}_{mn}^{(0)} + i\xi(x)\sigma_{[n}\mathcal{D}_{m]}\bar{\underline{\lambda}}^{(0)} + 4i\underline{\lambda}^{(0)}\bar{\sigma}_{mn}\bar{\eta}$$

$$+ 2ig\xi(x)^2\bar{\underline{\lambda}}^{(0)}\bar{\sigma}_{mn}\bar{\underline{\lambda}}^{(0)}, \tag{4.12a}$$

$$\underline{\lambda} = -\xi(x)\sigma^{mn}\underline{v}_{mn}^{\text{SD}(0)} + i\xi(x)^2\mathcal{D}\bar{\underline{\lambda}}^{(0)}, \tag{4.12b}$$

$$\bar{\underline{\lambda}} = \bar{\underline{\lambda}}^{(0)} + \bar{\sigma}^{mn}\bar{\Theta}_0\underline{v}_{mn}^{\text{ASD}(0)}, \tag{4.12c}$$

$$\underline{D} = -\xi(x)\mathcal{D}(\bar{\underline{\lambda}}^{(0)} + \bar{\sigma}^{mn}\bar{\Theta}_0\underline{v}_{mn}^{\text{ASD}(0)}). \tag{4.12d}$$

We need to verify that the expression (4.12c) for $\bar{\underline{\lambda}}$ is actually related by $SU(2)_R$ symmetry (4.6) to the expression (4.3c) for $\bar{\psi}$, despite appearances. This is easily accomplished, first by decomposing the field strength into self-dual and anti-self-dual components, $\underline{v}_{mn}^{(0)} = \underline{v}_{mn}^{\text{SD}(0)} + \underline{v}_{mn}^{\text{ASD}(0)}$, and next by recasting the Bianchi identity (A3d) as $\mathcal{D}^m\underline{v}_{mn}^{\text{SD}(0)} = \mathcal{D}^m\underline{v}_{mn}^{\text{ASD}(0)}$. This fact, together with Eqs. (4.5) and (A.2c), then allows us to rewrite Eq. (4.11) as

$$\mathcal{D}(\bar{\sigma}^{mn}\bar{\Theta}_0\underline{v}_{mn}^{\text{ASD}(0)}) = 2ig[\underline{A}_{\text{cl}}^\dagger, -\bar{\Theta}_0\overline{\mathcal{D}}\underline{A}_{\text{cl}}]$$

$$= \sqrt{2}g[\underline{A}_{\text{cl}}^\dagger, -(x_k - x_k^0)\bar{\eta}'\bar{\sigma}^k\sigma^{mn}\underline{v}_{mn}^{\text{cl}}]. \tag{4.13}$$

Adding together Eqs. (4.10) and (4.13) yields the following condition for the right-hand side of Eq. (4.12c):

$$\mathcal{D}\bar{\underline{\lambda}} \equiv \mathcal{D}(\bar{\underline{\lambda}}^{(0)} + \bar{\sigma}^{mn}\bar{\Theta}_0\underline{v}_{mn}^{\text{ASD}(0)}) = \sqrt{2}g[\underline{A}_{\text{cl}}^\dagger,\xi'(x)\sigma^{mn}\underline{v}_{mn}^{\text{cl}}], \tag{4.14}$$

where $\xi'(x)$ was defined in Eq. (4.4a). Save for the switch $\xi(x) \to \xi'(x)$, this is *precisely* Eq. (3.3b) for $\bar{\psi}$, from which we conclude

$$\bar{\underline{\lambda}} = -i\sqrt{2}[\xi'(x)\mathcal{D} + 2\bar{\eta}']\underline{A}^\dagger_{\mathrm{cl}},$$

$$\bar{\underline{\lambda}}^{(0)} = \bar{\underline{\lambda}}\big|_{\bar{\eta}'=0} = -i\sqrt{2}\xi'\mathcal{D}\underline{A}^\dagger_{\mathrm{cl}} \qquad (4.15)$$

by analogy with Eq. (4.3c). $SU(2)_R$ symmetry for the anti-fermions is now manifest, as is the validity of the Euler-Lagrange equations (3.1c,d); the reader can verify that Eqs. (3.1a,b) are true as well, up to still higher-order perturbative corrections.

To complete the determination of the short-distance superinstanton through this order, it remains only to specify $\underline{v}_{mn}^{(0)}$. The anti-self-dual part is given by $\bar{\sigma}^{mn}\bar{\Theta}_0 \underline{v}_{mn}^{\mathrm{ASD}(0)} = \bar{\underline{\lambda}}\big|_{\xi'=0}$. The self-dual piece has the general form $\underline{v}_{mn}^{\mathrm{SD}(0)} = \underline{v}_{mn}^{\mathrm{cl}} + \delta\underline{v}_{mn}^{\mathrm{SD}(0)}$; here, $\delta\underline{v}_{mn}^{\mathrm{SD}(0)}$ is a relative correction of order $g^2\rho^2|v|^2$ to the BPST field strength, but as it is only determined up to an admixture of bosonic zero modes, explicit expressions are not particularly illuminating, nor are they needed in what follows.

V. SEIBERG-WITTEN EFFECTIVE ACTION AT THE ONE-INSTANTON LEVEL

We now apply these results to some explicit calculational checks of the Seiberg-Witten effective Lagrangian, Eqs. (1.1) and (2.6). We will evaluate the one-instanton contribution to the family of $(l+4)$-point functions[14]

$$\langle\bar{\lambda}_\alpha(x_1)\bar{\lambda}_\beta(x_2)\bar{\psi}_{\dot\gamma}(x_3)\bar{\psi}_{\dot\delta}(x_4)\,\delta A^\dagger(x_5)\cdots\delta A^\dagger(x_{l+4})\rangle, \tag{5.1}$$

extending slightly the analysis of Finnell and Pouliot [12] (for higher gauge groups, see Ref. [14]). However, we will also evaluate the $(l+3)$-point and $(l+2)$-point functions

$$\langle v_{mn}(x_1)\bar{\lambda}_{\dot\alpha}(x_2)\bar{\psi}_{\dot\beta}(x_3)\,\delta A^\dagger(x_4)\cdots\delta A^\dagger(x_{l+3})\rangle \tag{5.2}$$

and

$$\langle v_{mn}(x_1)v_{kl}(x_2)\,\delta A^\dagger(x_3)\cdots\delta A^\dagger(x_{l+2})\rangle, \tag{5.3}$$

both of which require the extra machinery developed above.

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[14]As is obvious from Eqs. (1.1) and (2.6), these chirality-violating four-fermion Green's functions receive their leading contribution, not from $\mathcal{F}'''_{\mathrm{inst}}$, but from $\mathcal{F}'''_{\mathrm{pert}}$. Diagrammatically, this perturbative contribution sums the one-loop polygon graphs with four massless fermions plus l fluctuating Higgs antibosons on the external legs and massive quanta running around the polygon. Our point of view is that understanding the instanton series is important theoretically, even if in weak coupling the instanton contribution to individual cross sections is negligible. The reason that one examines instanton correlators of antifermions rather than fermions is simply that, while the latter appear at one lower order in perturbation theory in g, they fall off one power more slowly with x. Thus, in the long-distance domain $x \gg 1/M_W$, the fermion Green's functions are actually subdominant and contribute to the higher-derivative corrections to Eq. (2.5) (vice versa in *anti*-instanton backgrounds) [18].

The first task is to extrapolate the relevant long-distance effective U(1) fields from the short-distance singular-gauge superinstanton, which is summarized in Eqs. (4.3b,c), (4.8), (4.12), and (4.15). The simplest field is the Higgs antiboson, which, unlike the Higgs boson is free of fermion bilinears through subleading order:

$$A^\dagger = A^\dagger_{\mathrm{cl}} = \frac{(x-x_0)^2}{(x-x_0)^2+\rho^2}\,\bar{v}^u\frac{\tau^a}{2}$$

$$= \left(1 - \frac{\rho^2}{(x-x_0)^2} + \frac{\rho^4}{(x-x_0)^4} - \cdots\right)\bar{v}^u\frac{\tau^a}{2}. \tag{5.4}$$

This Taylor expansion is strictly valid only in the regime $\rho^2 \ll (x-x_0)^2 \ll 1/M_W^2$. Notice, however, that the first two terms (but not the higher terms) also satisfy the long-distance Euler-Lagrange equations (3.4), which are valid for $(x-x_0)^2 \gg 1/M_W^2$. This illustrates the "patching condition" discussed in Sec. III. Accordingly, up to $O(g^2)$ corrections, we equate the long-distance tail of $A^\dagger$ with the truncated expression $\bar{v} - \bar{v}\rho^2/(x-x_0)^2$. In other words (subtracting off the VEV),

$$\delta A^\dagger(x) = -\frac{\bar{v}\rho^2}{(x-x_0)^2} = -v^{-1}S_{\mathrm{Higgs}}G(x,x_0), \tag{5.5}$$

where $G(x,x_0) = 1/4\pi^2(x-x_0)^2$ is the massless Euclidean propagator.

So now consider each of the three families of Green's functions in turn, starting with Eq. (5.1). The one-instanton measure [31] in the $N=2$ supersymmetric gauge theory has the well-known form

$$\int 2^{10}\pi^6 g^{-8}(\mu\rho)^8 d^4x_0\,\frac{d\rho}{\rho^5}\left(\frac{16\pi^2\mu}{g^2}\right)^{-2}$$

$$\times d^2\xi\,d^2\xi'\left(\frac{32\pi^2\rho^2\mu}{g^2}\right)^{-2}d^2\bar\eta\,d^2\bar\eta'\,e^{-S_{\mathrm{inst}}}. \tag{5.6}$$

The factors in large parentheses contain the norms of the supersymmetric and superconformal fermion zero modes, respectively (see Appendix A). Since S_{inst} does not depend on ξ and ξ', these Grassmann integrations must be saturated by the antifermions in Eq. (5.1). In terms of the spinor propagator $S(x,x_0) \equiv \not{\partial}G(x,x_0)$, the ξ and ξ' components of the antifermions are[15]

$$\bar\psi_{\dot\alpha}(x) = -i\sqrt{2}\xi^\alpha\not{\partial}_{\alpha\dot\alpha}A^\dagger_{\mathrm{cl}}(x) = i\sqrt{2}v^{-1}S_{\mathrm{Higgs}}\xi^\alpha S_{\alpha\dot\alpha}(x,x_0) \tag{5.7}$$

and, likewise,

$$\bar\lambda_{\dot\alpha}(x) = i\sqrt{2}v^{-1}S_{\mathrm{Higgs}}\xi'^\alpha S_{\alpha\dot\alpha}(x,x_0), \tag{5.8}$$

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[15]Recall that the long-distance fields are neutral under the unbroken U(1) so that $\mathcal{D}_n \to \partial_n$. Here and in Eq. (5.5), we choose to express the right-hand sides in terms of S_{Higgs}, in order that these expressions may be immediately promotable to the multi-instanton case, where S_{Higgs} is much more complicated.

as follows from Eqs. (4.3c), (4.15), and (5.5). These expressions, too, satisfy the long-distance equations (3.4) and are therefore dictated by the ''patching condition.'' Integrating out the (lifted) superconformal modes from the measure gives

$$\int d^2\bar\eta \, d^2\bar\eta' \, \exp(-S_{\text{Yuk}}) = -2^9\pi^4 g^{-2}\rho^4\bar v^2. \quad (5.9)$$

The remaining integrations are elementary and yield

$$\int d^4x_0 \epsilon^{\alpha\beta} S_{\alpha\dot\alpha}(x_1,x_0) S_{\beta\dot\beta}(x_2,x_0)$$
$$\times \epsilon^{\gamma\delta} S_{\gamma\dot\gamma}(x_3,x_0) S_{\delta\dot\delta}(x_4,x_0) G(x_5,x_0)\times\cdots$$
$$\times G(x_{l+4},x_0)\frac{(5+l)!\Lambda^4}{16\pi^2 g^2(-v)^{6+l}}, \quad (5.10)$$

with Λ as in Eq. (2.9). This position-space Green's function may be interpreted as coming from an instanton-induced effective local vertex

$$\sum_{l=0}^{\infty}\frac{1}{2!2!l!}\psi^2\lambda^2(\delta A)^l\frac{(5+l)!\Lambda^4}{16\pi^2 g^2(-v)^{6+l}} = \frac{15\Lambda^4}{8\pi^2 g^2}\frac{\psi^2\lambda^2}{A^6}. \quad (5.11)$$

The binomial theorem has been used to reconstitute in the denominator the total Higgs field $A(x_0)=v+\delta A(x_0)$. It is interesting to see how the combinatorics of the collective coordinate integration promotes v^{-6} to A^{-6}, as required by the Seiberg-Witten effective action.

Next we look at the Green's functions (5.2) and (5.3). In order to saturate the ξ and ξ' integrations, we need to extract the piece of v_{mn} bilinear in ξ and ξ'. Here our careful sweeping-out procedure in Sec. IV D pays off: The second term on the right-hand side of Eq. (4.12a) contains just such a piece, which we write as

$$\sqrt{2}v^{-1}S_{\text{Higgs}}\xi\sigma_{[n}\bar\theta\,\partial_{m]}G(x,x_0)\xi', \quad (5.12)$$

using Eqs. (4.15) and (5.8). This expression, too, satisfies, the long-distance equations (3.4). It is also purely anti-self-dual (as pointed out to us by Yung), meaning that it fits into the right-handed superfield $\bar W_{\dot\alpha}$, rather than in W^α where v_{mn}^{cl} itself lives.

We can reexpress Eq. (5.12) in a more illuminating way, as

$$-\sqrt{2}v^{-1}S_{\text{Higgs}}\xi\sigma^{kl}\xi' G_{mn,kl}(x,x_0), \quad (5.13)$$

where $G_{mn,kl}$ is the gauge-invariant propagator of U(1) field strengths:

$$G_{mn,kl}(x,x_0)=(\eta_{nl}\partial_m\partial_k-\eta_{nk}\partial_m\partial_l-\eta_{ml}\partial_n\partial_k$$
$$+\eta_{mk}\partial_n\partial_l)G(x,x_0). \quad (5.14)$$

The families of Green's functions (5.2) and (5.3) then work out to, respectively,

$$\int d^4x_0\sigma^{kl\alpha\beta}G_{mn,kl}(x_1,x_0)S_{\alpha\dot\alpha}(x_2,x_0)$$
$$\times S_{\beta\dot\beta}(x_3,x_0)G(x_4,x_0)\times\cdots$$
$$\times G(x_{l+3},x_0)\frac{(4+l)!\Lambda^4}{16\sqrt{2}\pi^2 g^2(-v)^{5+l}} \quad (5.15)$$

and

$$\int d^4x_0\text{tr}_2\sigma^{pq}\sigma^{rs}\cdot G_{mn,pq}(x_1,x_0)$$
$$\times G_{kl,rs}(x_2,x_0)G(x_3,x_0)\times\cdots$$
$$\times G(x_{l+2},x_0)\frac{(3+l)!\Lambda^4}{32\pi^2 g^2(-v)^{4+l}}. \quad (5.16)$$

These, in turn, correspond to the effective pointlike vertices

$$\sum_{l=0}^{\infty}\frac{1}{l!}\lambda\sigma^{kj}\psi v_{kj}(\delta A)^l\frac{(4+l)!\Lambda^4}{16\sqrt{2}\pi^2 g^2(-v)^{5+l}}$$
$$=-\frac{3\Lambda^4}{2\sqrt{2}\pi^2 g^2}\frac{\lambda\sigma^{kl}\psi v_{kl}}{A^5} \quad (5.17)$$

and

$$\sum_{l=0}^{\infty}\frac{1}{2!l!}\text{tr}_2\sigma^{pq}\sigma^{rs}\cdot v_{pq}v_{rs}(\delta A)^l\frac{(3+l)!\Lambda^4}{32\pi^2 g^2(-v)^{4+l}}$$
$$=\frac{3\Lambda^4}{32\pi^2 g^2}\frac{\text{tr}_2\sigma^{pq}\sigma^{rs}\cdot v_{pq}v_{rs}}{A^4}=-\frac{3\Lambda^4}{16\pi^2 g^2}\frac{(v_{pq}^{\text{SD}})^2}{A^4},$$
$$(5.18)$$

again with the help of the binomial theorem.

Comparing the three effective vertices (5.11), (5.17), and (5.18) to their counterparts in the Seiberg-Witten Lagrangian, Eqs. (1.1) and (2.6), we find agreement if and only if $\mathcal{F}_1=1/2$, confirming their prediction. The remaining vertices in $\mathcal{L}_{\text{eff}}$, namely, the renormalized fermion and Higgs kinetic energies, share the property that (up to an integration by parts) they vanish identically when the leading-order equations of motion are used. This means that they only affect on-shell processes at a higher order in perturbation theory. In instanton language, they presumably map onto propagator corrections in instanton backgrounds, which lie beyond the scope of this paper.

We now apply the lessons learned at the one-instanton level to the study of multi-instantons.

VI. ABC's of ADHM

A. Three Ansätze

In this section we give a physicist's introduction to the ADHM construction of multi-instantons [15,16], which in our idiosyncratic formulation will be seen to rely on three remarkably simple *Ansätze*. (For additional information, see especially Refs. [21,22,28].) We will specialize to the gauge group SU(2) from the outset, and we will also preserve the distinction between dotted and undotted SU(2) indices in or-

der to facilitate supersymmetrization. In fact (in this section only), we shall often exhibit *all* indices (five different types[16]) for maximal clarity.

A heuristic motivation for the ADHM construction is to notice an interesting pattern: Just as a zero-instanton solution of pure SU(2) Yang-Mills theory is by definition a pure gauge,

$$v_n{}^{\dot\alpha}{}_{\dot\beta} = \bar U^{\dot\alpha\alpha}\partial_n U_{\alpha\dot\beta}, \quad \bar U^{\dot\alpha\alpha}U_{\alpha\dot\beta} = \delta^{\dot\alpha}{}_{\dot\beta}, \quad (6.1)$$

so too a one-instanton configuration is expressible as

$$v_n{}^{\dot\alpha}{}_{\dot\beta} = \bar U_\lambda^{\dot\alpha\alpha}\partial_n U_{\lambda\ \alpha\dot\beta}, \quad \bar U_\lambda^{\dot\alpha\alpha}U_{\lambda\ \alpha\dot\beta} = \delta^{\dot\alpha}{}_{\dot\beta}, \quad (6.2)$$

where the new index λ is summed over 0 and 1. For example, the usual singular-gauge instanton (4.8a) follows from the choice

$$U_{0\ \alpha\dot\beta} = \sqrt{\frac{x^2}{x^2+\rho^2}}\ \sigma_{0\ \alpha\dot\beta},$$

$$U_{1\ \alpha\dot\beta} = -\frac{\rho}{x^2}\ \sqrt{\frac{x^2}{x^2+\rho^2}}\ x_{\alpha\dot\alpha}\bar u^{\dot\alpha\beta}\sigma_{0\ \beta\dot\beta}, \quad (6.3)$$

while the regular-gauge instanton follows from

$$U_{0\ \alpha\dot\beta} = -\frac{1}{\sqrt{x^2+\rho^2}}\ u_{\alpha\dot\alpha}\bar x^{\dot\alpha\beta}\sigma_{0\ \beta\dot\beta},$$

$$U_{1\ \alpha\dot\beta} = \sqrt{\frac{\rho^2}{x^2+\rho^2}}\ \sigma_{0\ \alpha\dot\beta}, \quad (6.4)$$

where in quaternionic notation $u_{\alpha\dot\alpha} \equiv u_n\sigma^n_{\alpha\dot\alpha}$ is an iso-orientation matrix in the spin-1/2 representation of SU(2), i.e., satisfying $|u|^2 = 1$. ADHM's first key *Ansatz* for where to search for the n-instanton solutions is, naturally enough, Eq. (6.2) again, where now $\lambda = 0,1,\dots,n$.

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[16]Apart from the SU(2) indices $\alpha,\beta,\dots$ and $\dot\alpha,\dot\beta,\dots$ and Lorentz indices m,n, there are also Greek and Roman ADHM matrix indices $\kappa,\lambda = 0,1,\dots,n$ and $k,l = 1,2,\dots,n$, where n is the winding number (number of instantons). Additional conventions for the remainder of the paper: Each Lorentz vector z_m has associated with it *quaternions* $z_{\alpha\dot\alpha} = z_m\sigma^m_{\alpha\dot\alpha}$ and $\bar z^{\dot\alpha\alpha} = z^m\bar\sigma_m^{\dot\alpha\alpha}$. For quantities with two undotted (likewise two dotted) SU(2) indices, overbarring reverses the sign of the $\sigma_\alpha{}^\beta$ components, but not of the $\delta_\alpha{}^\beta$ components, so that $\bar z_\alpha{}^\beta = \delta_\alpha{}^\beta \mathrm{tr}_2 z - z_\alpha{}^\beta$; if a quantity also carries ADHM indices, then the overbar transposes in these as well (overbarring also conjugates any multiplicative complex phase). In contrast, the superscript T transposes only in the ADHM matrix indices, whereas the symbol tr_2 traces only over the dotted or undotted SU(2) indices. Note the very useful identity $z_{\alpha\dot\alpha}\bar y^{\dot\alpha\beta} + y_{\alpha\dot\alpha}\bar z^{\dot\alpha\beta} = \mathrm{tr}_2\overline{z y}$; more generally, for a quantity $X_\alpha{}^\beta$ that carries ADHM indices, one has $\mathrm{tr}_2 X = X + \bar X^T$. We define $|z|^2 = z\bar z = \bar z z$ for any quantity not having ADHM indices, be it $z_{\alpha\dot\alpha}$, $z_\alpha{}^\beta$, or $z^{\dot\alpha}{}_{\dot\beta}$. Note that $|z_{\alpha\dot\alpha}|^2 = -z_n z^n$ due to the Wess-Bagger metric. Unit-normalized quaternions, $|z|^2 = 1$, are elements of SU(2). Finally note that $z^{\alpha\beta} = z^{\beta\alpha} - \epsilon^{\beta\alpha}\mathrm{tr}_2 z$ whereas $z^{\dot\alpha\dot\beta} = z^{\dot\beta\dot\alpha} + \epsilon^{\dot\beta\dot\alpha}\mathrm{tr}_2 z$.

To find a solution of the Yang-Mills equations, it suffices to establish self-duality of the field strength[17]

$$v_{mn}{}^{\dot\alpha}{}_{\dot\beta} \equiv \partial_{[m}v_{n]}{}^{\dot\alpha}{}_{\dot\beta} + v_{[m}{}^{\dot\alpha}{}_{\dot\gamma}v_{n]}{}^{\dot\gamma}{}_{\dot\beta}$$

$$= \partial_{[m}\bar U_\lambda^{\dot\alpha\alpha}(\delta_{\lambda\kappa}\delta_\alpha{}^\beta - \mathcal{P}_{\lambda\kappa\alpha}{}^\beta)\partial_{n]}U_{\kappa\beta\dot\beta}, \quad (6.5)$$

where

$$\mathcal{P}_{\lambda\kappa\alpha}{}^\beta = U_{\lambda\alpha\dot\alpha}\bar U_\kappa^{\dot\alpha\beta}. \quad (6.6)$$

Note that $\mathcal{P}$, and hence also $1-\mathcal{P}$, is a projection operator, satisfying

$$0 = (1-\mathcal{P})U = \bar U(1-\mathcal{P}), \quad \mathcal{P} = \bar{\mathcal{P}}, \quad \mathcal{P}^2 = \mathcal{P}. \quad (6.7)$$

ADHM's second key *Ansatz* is to assume that $1-\mathcal{P}$ can be factorized as

$$\delta_{\lambda\kappa}\delta_\alpha{}^\beta - \mathcal{P}_{\lambda\kappa\alpha}{}^\beta = \Delta_{\lambda l\ \alpha\dot\alpha} f_{lk}{}^{\dot\alpha}{}_{\dot\beta}\bar\Delta_{k\kappa}^{\dot\beta\beta} \quad (6.8)$$

for some matrices Δ and f, and that Eq. (6.7) is satisfied by virtue of the more basic conditions

$$0 = \bar\Delta_{k\kappa}^{\dot\beta\beta}U_{\kappa\beta\dot\alpha} = \bar U_\lambda^{\dot\beta\alpha}\Delta_{\lambda l\ \alpha\dot\alpha}, \quad f = \bar f. \quad (6.9)$$

In a moment we will solve for $\Delta_{\lambda l}$, but already we know its size: By comparing the dimensions of the nullspaces of $U\bar U$ and $1-\Delta f\bar\Delta$, we conclude that $\Delta_{\lambda l}$ must be a rectangular quaternion-valued matrix of dimension $(n+1)\times n$. We will refer to both $\lambda = 0,\dots,n$ and $l = 1,\dots,n$ as "ADHM indices."

Returning to the issue of self-duality, we can now rewrite Eq. (6.5) as

$$v_{mn}{}^{\dot\alpha}{}_{\dot\beta} = \bar U_\lambda^{\dot\alpha\alpha}\partial_{[m}\Delta_{\lambda l\ \alpha\dot\gamma} f_{lk}{}^{\dot\gamma}{}_{\dot\delta}\ \partial_{n]}\bar\Delta_{k\kappa}^{\dot\delta\beta}U_{\kappa\ \beta\dot\beta}, \quad (6.10)$$

with an integration by parts. ADHM's third and final key *Ansatz* is now easily anticipated: If one can arrange that $\partial_m\Delta_{\lambda l\alpha\dot\alpha} = \text{stuff}\times\sigma_{m\beta\dot\alpha}$ and also that σ_m commutes through f, then the right-hand side of Eq. (6.10) will be of the form stuff$\times\sigma_{mn}\times$stuff, and self-duality is guaranteed (being a built-in property of σ_{mn}). So we postulate, first, that Δ is a linear left-multiplied function of $x_{\beta\dot\alpha} \equiv x^m\sigma_{m\beta\dot\alpha}$,

$$\Delta_{\lambda l\ \alpha\dot\alpha} = a_{\lambda l\ \alpha\dot\alpha} + b_{\lambda l\ \alpha}{}^\beta x_{\beta\dot\alpha}, \quad (6.11)$$

where $a_{\lambda l}$ and $b_{\lambda l}$ are $[(n+1)\times n]$-dimensional matrices of constant quaternions; and second

$$f_{lk}{}^{\dot\alpha}{}_{\dot\beta} = f_{lk}\delta^{\dot\alpha}{}_{\dot\beta}, \quad (6.12)$$

so that it commutes with the Pauli matrices. We then find, as promised,

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[17]The shortest imaginable proof that (anti-) self-dual gauge fields automatically satisfy the Yang-Mills equations of motion uses the Bianchi identity (A3d) $D_m\underline{v}_{mn} \propto \mathcal{D}_m\epsilon^{mnpq}\underline{v}_{pq} = 0$. Henceforth, rather than $\underline{v}_n = v_n{}^a\tau^a/2$ as before, the SU(2) gauge field will be denoted $v_n{}^{\dot\alpha}{}_{\dot\beta}$ and obey $\mathrm{tr}_2 v_n{}^{\dot\alpha}{}_{\dot\beta} \equiv v_n{}^{\dot\alpha}{}_{\dot\alpha} = 0$, and likewise for the other fields. Furthermore, following the ADHM tradition we will work with anti-Hermitian gauge fields and set $g = 1$ in the following.

$$v_{mn}{}^{\dot\alpha}{}_{\dot\beta}=(v_{mn}{}^{\dot\alpha}{}_{\dot\beta})^{\text{dual}}=4\,\overline{U}_\lambda^{\dot\alpha\alpha}b_{\lambda l}{}_\alpha{}^\beta\sigma_{mn}{}_\beta{}^\gamma f_{lk}\overline{b}_{k\kappa\gamma}{}^{\dot\delta}U_{\kappa\,\delta\dot\beta}. \tag{6.13}$$

That the winding number is in fact n is checked in Sec. VII D below.

B. Solving the ADHM equations

At this point no further *Ansätze* are needed; the problem of solving the Yang-Mills equations in the n-instanton sector has been reduced to constructing ADHM matrices f_{lk}, $U_{\lambda\alpha\dot\alpha}$, $b_{\lambda\,l\alpha}{}^\beta$, and $a_{\lambda l\alpha\dot\alpha}$ such that Eqs. (6.6)–(6.9) are a consistent set of equations. Let us solve for each of these four quantities in turn. First, these equations imply that $\mathcal{P}\Delta=0$; therefore, f can be eliminated in terms of Δ, via

$$\overline{\Delta}_{k\kappa}^{\dot\beta\beta}\Delta_{\kappa l\,\beta\dot\alpha}=(f^{-1})_{kl}\delta^{\dot\beta}{}_{\dot\alpha}. \tag{6.14}$$

Of course, this condition only makes sense if the left-hand side really is proportional to $\delta^{\dot\beta}{}_{\dot\alpha}$, which is tantamount to requiring

$$\overline{a}a=(\overline{a}a)^T\propto\delta^{\dot\beta}{}_{\dot\alpha}, \tag{6.15a}$$

$$\overline{b}a=(\overline{b}a)^T, \tag{6.15b}$$

$$\overline{b}b=(\overline{b}b)^T\propto\delta_\alpha{}^\beta, \tag{6.15c}$$

as follows from a Taylor expansion in x (the superscript T denotes a transpose in the ADHM indices only). We will return to these conditions shortly.

Second, let us solve for U_λ, and hence v_n itself, in terms of Δ. Equating the two alternative expressions (6.6) and (6.8) for $\mathcal{P}$ and setting $\kappa=0$ implies, for $\lambda=0$ and $\lambda\neq0$, respectively,

$$|U_0|^2=1-\tfrac{1}{2}f_{lk}\,\text{tr}_2 w_l\overline{w}_k, \quad w_{l\alpha\dot\alpha}=\Delta_{0l\alpha\dot\alpha}, \tag{6.16}$$

and

$$U_\lambda=-\frac{1}{|U_0|^2}\Delta_{\lambda l}f_{lk}\overline{w}_k U_0, \quad \lambda\neq0. \tag{6.17}$$

Crucially, thanks to Eq. (6.14), these solutions are consistent: The expressions (6.6) and (6.8) automatically remain equal even when both κ and λ are nonzero. Note that there is an infinite number of quaternions U_0 satisfying Eq. (6.16), but these are all equivalent up to gauge transformations of v_n. Generalizing the singular-gauge expression (6.3) which served us well in the one-instanton sector, we will specify

$$U_{0\alpha\dot\alpha}=\sigma_{0\alpha\dot\alpha}(1-\tfrac{1}{2}f_{lk}\,\text{tr}_2 w_l\overline{w}_k)^{1/2} \tag{6.18}$$

in what follows, whenever an explicit choice is called for.

Third, let us eliminate the degrees of freedom of $b_{\lambda\,l\alpha}{}^\beta$ entirely from the problem. To this end, it is helpful to catalog the complete set of invariances of the ADHM construction. The usual SU(2) gauge transformations of v_n read

$$U_{\lambda\alpha\dot\alpha}\to U_{\lambda\alpha\dot\beta}\Omega(x)^{\dot\beta}{}_{\dot\alpha}, \quad \Delta\to\Delta, \quad f\to f, \tag{6.19}$$

where $\overline{\Omega}^{\dot\alpha}{}_{\dot\gamma}\Omega^{\dot\gamma}{}_{\dot\beta}=\delta^{\dot\alpha}{}_{\dot\beta}$. For example, the gauge transformation between Eqs. (6.3) and (6.4) is induced by

$$\Omega(x)^{\dot\beta}{}_{\dot\alpha}=-\frac{1}{\sqrt{x^2}}\,\overline{\sigma}_0^{\dot\beta\beta}u_{\beta\dot\gamma}\overline{x}^{\dot\gamma\gamma}\sigma_{0\gamma\dot\alpha}. \tag{6.20}$$

In contrast, consider the two sets of transformations

$$\Delta_{\lambda l\beta\dot\beta}\to\Delta_{\lambda k\beta\dot\beta}B_{kl}, \quad f\to B^{-1}\cdot f\cdot(B^{-1})^T, \quad U\to U, \tag{6.21}$$

and

$$\Delta_{\lambda l\beta\dot\beta}\to\Lambda_{\lambda\kappa\beta}{}^\alpha\Delta_{\kappa l\alpha\dot\beta}, \quad f\to f,$$
$$U_{\lambda\beta\dot\beta}\to\Lambda_{\lambda\kappa\beta}{}^\alpha U_{\kappa\alpha\dot\beta}, \quad \overline{\Lambda}\Lambda=1, \tag{6.22}$$

where B and Λ are independent of x. While these obviously preserve the various ADHM relations and constraints detailed above, they have no effect on the gauge field (6.2) itself; hence, they commute with gauge fixing. We now exploit these two invariances to simplify b, as follows. With an initial Λ whose top row lives in the $\perp$ space of b, b can be brought into the form

$$b=\begin{pmatrix}0\cdots0\\b'\end{pmatrix}.$$

Since the product $\overline{b}'b'$ is a real symmetric *scalar*-valued $n\times n$ matrix as follows from Eq. (6.15c), it can be factored as $O\cdot\mu\cdot O^T$, where O is an orthogonal matrix and μ is a diagonal matrix of nonnegative eigenvalues. So next we let $\Delta\to\Delta B$ as per Eq. (6.21), with $B=O\cdot\mu^{-1/2}$. This ensures that the new b' is unit normalized: $\overline{b}'_{kk'\,\alpha}{}^\beta b'_{k'l\,\beta}{}^\gamma=\delta_{kl}\delta_\alpha{}^\gamma$. Applying a followup unitary transformation (6.22), with

$$\Lambda=\begin{pmatrix}1&0&\cdots&0\\0&&&\\\vdots&&\overline{b}'&\\0&&&\end{pmatrix},$$

we then rotate b into its final canonical form, while fixing our notation for the constant matrix a as follows:

$$b_{\lambda k}=\begin{pmatrix}0&\cdots&0\\\delta_\alpha{}^\beta&\cdots&0\\\vdots&\ddots&\vdots\\0&\cdots&\delta_\alpha{}^\beta\end{pmatrix}, \quad a_{\lambda k}=\begin{pmatrix}w_1&\cdots&w_n\\&a'_{lk}&\end{pmatrix}. \tag{6.23}$$

Here $a'_{lk\,\alpha\dot\alpha}$ is an $n\times n$ matrix of quaternions.

Fourth, we discuss the remaining unknown, $a_{\lambda k}$ itself, which is now constrained *solely* by the two conditions (6.15a, b). Notice that these constraints are invariant under the linear shifts

$$a\to a-bx_0. \tag{6.24}$$

So the degree of freedom of a that is proportional to b itself has a special role to play: that of the *position* of the multi-instanton (in our notation, this is $-x_0$). With the canonical choice for b, Eq. (6.15b) simply means

$$a'=a'^T. \tag{6.25}$$

Equation (6.15a) is more complicated; it defines a set of coupled quadratic constraints on the quaternionic elements of a. For $n=1$, it is automatically satisfied; for $n=2$, it is easily solved [22] (as reviewed in Sec. VIII A below); and for $n=3$, an intricate solution has been constructed in Ref. [22]. However, for $n>3$, no solution of this constraint is known. This is arguably the biggest failing of the ADHM construction and has hampered progress in the study of multi-instantons, particularly as regards the construction of the moduli space integration measure.

Let us pause to count the independent parameters of the quaternion-valued matrix $a_{\lambda k}$. Thanks to Eq. (6.25) it contains $2n(n+3)$ scalar degrees of freedom. Irrespective of the existence of an explicit solution, Eq. (6.15a) then imposes $\frac{3}{2}n(n-1)$ constraints on the upper-triangular traceless quaternionic elements of $\bar{a}a$, leaving $\frac{1}{2}n(n+15)$ degrees of freedom. But for $n>1$ this is still too large a set; by considering the limit of n widely separated distinguishable instantons, we see that the correct number should of course be $8n$ ($4n$ positions, n scale sizes, and $3n$ iso-orientations). In other words, we are still lacking $\frac{1}{2}n(n+15)-8n=\frac{1}{2}n(n-1)$ constraints. To see where these come from, look again at Eqs. (6.21)–(6.23). Even after one fixes the canonical forms (6.23) and (6.18) for b and U_0, there are still x-independent O(n) transformations which act nontrivially on a, namely,

$$\Delta \rightarrow \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & R^T & \\ 0 & & & \end{pmatrix} \cdot \Delta \cdot R, \quad f \rightarrow R^T \cdot f \cdot R,$$

$$U_0 \rightarrow U_0, \quad U_l \rightarrow R_{kl} U_k, \qquad (6.26)$$

where $R \in \text{O}(n)$ and carries no SU(2) indices. Obviously, this is a composite of transformations of the type (6.21) and (6.22), specifically arranged to preserve the canonical form of b and U_0. As such, it too commutes with ordinary gauge fixing of v_m. This O(n) freedom can be used to fix the remaining redundancies in a in any number of ways, for instance, to diagonalize any one of the four σ_m components of the submatrix a' (we use a different prescription in Sec. VIII below).

Since O(n) has $\frac{1}{2}n(n-1)$ generators, we have succeeded, in principle, in reducing the number of independent scalar parameters in $a_{\lambda k}$ to $8n$ continuous degrees of freedom, as expected. But we are still not done: There may yet be a residual *discrete* degeneracy that needs to be modded out [21]. To understand this point, imagine that the elements of a are expressed in some definite way in terms of $8n$ unconstrained parameters, labeled $\{X_i\}$, with $i=1,\ldots,8n$. It may still happen that two different points in the parameter space, $\{X_i\}$ and $\{\widetilde{X}_i\}$, correspond to equivalent field configurations. From Eq. (6.26), we see that this occurs if there exists an $R \in \text{O}(n)$ such that

$$\vec{w}(\{X_i\}) \cdot R = \vec{w}(\{\widetilde{X}_i\}), \quad R^T a'(\{X_i\}) R = a'(\{\widetilde{X}_i\}), \quad (6.27)$$

where $\vec{w}=(w_1,\ldots,w_n)$. In order to obtain the physical moduli space of inequivalent instantons, it is necessary to identify all such points. As we will see below, the degeneracy corresponds to the action of a discrete symmetry group

G_n on the ADHM parameter space and the required identification can be made by taking a quotient in the usual way. The symmetry factor or ''statistical weight'' $\mathcal{S}_n$ of the parametrization counts the number of redundant copies of each configuration. Hence $\mathcal{S}_n$ is given by the number of solutions of Eq. (6.27), which is just the order of the symmetry group G_n. Knowledge of $\mathcal{S}_n$ is necessary in order properly to normalize the integration measure over the moduli [21].

This concludes our overview of the ADHM construction of multi-instantons. The reader may wish to verify that the single-instanton expressions (6.3) and (6.4) follow from this construction for $n=1$. Fortunately, in what follows we will not actually need the explicit expressions for multi-instantons, which are quite cumbersome even for $n=2$. With judicious use of integrations by parts, it will suffice to know only the asymptotic behavior of the configurations as $|x|\rightarrow\infty$. We therefore list for later use the asymptotic behavior of several key ADHM quantities:

$$\Delta \rightarrow bx, \qquad (6.28a)$$

$$f_{kl} \rightarrow \frac{1}{|x|^2}\,\delta_{kl}, \qquad (6.28b)$$

$$U_k \rightarrow -\frac{1}{|x|^2}\,x\bar{w}_k U_0, \qquad (6.28c)$$

$$U_0 \rightarrow \sigma_0, \qquad (6.28d)$$

where Eq. (6.28d) assumes the gauge choice (6.18).

VII. MULTI-INSTANTONS AND SUPER-MULTI-INSTANTONS WITH ADJOINT MATTER

This section contains our principal findings for general winding number n. In Sec. VII A we review the construction of adjoint fermion zero modes in the general ADHM background. Sections VII B–VII D contain new results. In Secs. VII B and VII C we construct the adjoint Higgs field, first in the absence of fermions, then in the presence of a Yukawa source term as in the $N=2$ supersymmetric theory. A common feature of Secs. VII A–VII C is that, with the correct *Ansätze*, the solution to a linear differential equation is mapped onto the solution of a linear finite matrix equation. In Sec. VII D we construct the multi-instanton action in both the bosonic and supersymmetric cases. Finally, in Sec. VII E we discuss the formulation of the multi-instanton measure.

A. Adjoint fermion zero modes

As in the one-instanton case, the classical gaugino and Higgsino fields are the zero modes for $\overline{\mathcal{D}}$ (in contrast, $\mathcal{D}$ is still invertible). The most general such solution was found by Corrigan *et al.* using the method of tensor products [28,29,22]. In this approach, the required adjoint fermion zero modes can be constructed from the simpler expressions for zero modes in the fundamental representation of the gauge group. Explicitly, for the gaugino we have [suppressing the ADHM indices but retaining the SU(2) indices for clarity]

$$(\lambda_\alpha)^{\dot\beta}{}_{\dot\gamma} = \bar{U}^{\dot\beta\gamma}\mathcal{M}_\gamma f\bar{b}U_{\alpha\dot\gamma} - \bar{U}^{\dot\beta}{}_\alpha bf\mathcal{M}^{\gamma T}U_{\gamma\dot\gamma}, \quad (7.1)$$

where $\mathcal{M}_\gamma$ is an $[(n+1)\times n]$-dimensional matrix of constant Grassmann spinors. From Eq. (7.1) we calculate (using some of the tricks of Sec. VII B below)

$$\bar{\mathcal{D}}^{\dot\alpha\alpha}(\lambda_\alpha)^{\dot\beta}{}_{\dot\gamma} = 2\bar{U}^{\dot\beta\alpha}bf(\bar{\Delta}^{\dot\alpha\gamma}\mathcal{M}_\gamma + \mathcal{M}^{\gamma T}\Delta_\gamma{}^{\dot\alpha})f\bar{b}U_{\alpha\dot\gamma}. \quad (7.2)$$

Hence the condition for a gaugino zero mode [28,29] is the following two sets of linear constraints on $\mathcal{M}_\gamma$ which ensure that the right-hand side vanishes [expanding $\Delta(x)$ as $a+bx$]:

$$a^{\dot\alpha\gamma}\mathcal{M}_\gamma = -\mathcal{M}^{\gamma T}a_\gamma{}^{\dot\alpha} \quad (7.3)$$

and

$$\bar{b}_\alpha{}^\gamma\mathcal{M}_\gamma = \mathcal{M}^{\gamma T}b_{\gamma\alpha}. \quad (7.4)$$

Counting the number of degrees of freedom, one finds $2n(n+1)$ individual Grassmann numbers in the matrix $\mathcal{M}_\gamma$, subject to $n(n-1)$ constraints from each of Eqs. (7.3) and (7.4), for a net of $4n$ gaugino zero modes. Since in the $N=2$ theory these are reduplicated in the Higgsino ψ as well (to which we associate the matrix $\mathcal{N}_\gamma$), we have $8n$ linearly independent modes in all, the same as for the bosons. The corresponding entries in the matrices $\mathcal{M}_\gamma$ and $\mathcal{N}_\gamma$ are Grassmann collective coordinates which must be integrated over just like their counterparts $\{\xi,\xi',\bar{\eta},\bar{\eta}'\}$ in the one-instanton sector.

A crucial distinction among these modes should be made between the lifted modes, on which the superinstanton action S_{inst} depends, and the unbroken modes which do not appear in S_{inst}. Thus, in the one-instanton sector, we saw in Sec. IV C that the four superconformal modes $\bar{\eta}_{\dot\alpha}$ and $\bar{\eta}'_{\dot\alpha}$ are lifted, while the four supersymmetric modes ξ_α and ξ'_α are exact. We will soon find that this pattern is extended for general n in the following way. There are always precisely four unlifted modes, corresponding to a single spinor degree of freedom in each of $\mathcal{M}_\gamma$ and $\mathcal{N}_\gamma$: the one proportional to $\xi_{1,2}\sigma^{mn}v_{mn}$, swept out by the exact $N=2$ supersymmetry generators Q_1 and Q_2, respectively [see Eqs. (A1b, f)]. Comparing expressions (6.13) and (7.1), we see that these are the zero modes for which

$$\mathcal{M}_\gamma \propto b, \quad \mathcal{N}_\gamma \propto b, \quad (7.5)$$

in harmony with the global translational mode (6.24). In this instance the constraints (7.3) and (7.4) simply boil down to the bosonic constraints (6.15b, c). The remaining $8n-4$ modes are lifted by the Yukawa interactions. Among the lifted modes are the superconformal modes. In contrast with these, most of the lifted modes do not correspond to (approximate) Lagrangian symmetries.

This mode counting is quite different from the case of $N=1$ supersymmetric Yang-Mills theory coupled to fundamental Higgs fields, where the number of unbroken modes, rather than remaining constant, grows linearly with n [18]. Therefore, in those models, the different winding number sectors do not interfere (e.g., the fermion mass term is a pure

one-instanton effect), in contrast to the $N=2$ model at hand, where correlators receive contributions simultaneously from all n, including $n=0$.

B. Classical adjoint Higgs field

Here we construct the classical adjoint Higgs field A_{cl} in the general multi-instanton background and with the fermions turned off. We assume that the adjoint Higgs field is of the general form

$$iA_{\mathrm{cl}}{}^{\dot\alpha}{}_{\dot\beta} = \bar{U}_\lambda^{\dot\alpha\alpha}\mathcal{A}(x)_{\lambda\kappa}{}_\alpha{}^\beta U_{\kappa\beta\dot\beta}, \quad (7.6)$$

where the $[(n+1)\times(n+1)]$-dimensional matrix $\mathcal{A}_{\lambda\kappa}$ is to be determined. Note that $\mathcal{A}_{\lambda\kappa}$ is not, strictly speaking, a quaternion; rather, it is a quaternion multiplied by a complex phase, namely, that carried by the VEV v. Tracelessness of A_{cl} implies $\bar{\mathcal{A}} = -\mathcal{A}$ in the case of a real VEV; more generally, $\bar{\mathcal{A}} = -\mathcal{A}\cdot(\bar{v}/v)$.

From Eqs. (6.2) and (6.6) one calculates the commutator (dropping indices from now on)

$$[v_n, iA_{\mathrm{cl}}] = -\partial_n\bar{U}\mathcal{P}\mathcal{A}U - \bar{U}\mathcal{A}\mathcal{P}\partial_n U, \quad (7.7)$$

so that, with Eqs. (6.8) and (6.9),

$$\begin{aligned}\mathcal{D}_n iA_{\mathrm{cl}} &\equiv \partial_n iA_{\mathrm{cl}} + [v_n, iA_{\mathrm{cl}}] \\ &= -\bar{U}\partial_n\Delta f\bar{\Delta}\mathcal{A}U - \bar{U}\mathcal{A}\Delta f\partial_n\bar{\Delta}U + \bar{U}\partial_n\mathcal{A}U.\end{aligned} \quad (7.8)$$

This expression can be further differentiated with the help of

$$\partial_n f = -f\partial_n(\bar{\Delta}\Delta)f = -f(\bar{\sigma}_n\bar{b}\Delta + \bar{\Delta}b\sigma_n)f. \quad (7.9)$$

A straightforward calculation gives

$$\begin{aligned}\mathcal{D}^2 iA_{\mathrm{cl}} &= 4\bar{U}\{bf\bar{b}, \mathcal{A}(x)\}U - 4\bar{U}bf\cdot\mathrm{tr}_2\bar{\Delta}\mathcal{A}(x)\Delta\cdot f\bar{b}U \\ &\quad + \bar{U}\partial^2\mathcal{A}(x)U - 2\bar{U}bf\sigma_n\bar{\Delta}\partial^n\mathcal{A}(x)U \\ &\quad - 2\bar{U}\partial^n\mathcal{A}(x)\Delta\bar{\sigma}_n f\bar{b}U.\end{aligned} \quad (7.10)$$

In obtaining this relatively simple answer, we have exploited, in addition to the usual Pauli matrix identity [30] $\sigma^n_{\alpha\dot\alpha}\bar{\sigma}_n^{\dot\beta\beta} = -2\delta_\alpha{}^\beta\delta_{\dot\alpha}{}^{\dot\beta}$, the fact that

$$\sigma^n\bar{\Delta}b\sigma_n = 2(\bar{b}\Delta)^T = 2\bar{b}\Delta. \quad (7.11)$$

The first equality in Eq. (7.11) follows from $\sigma^n\bar{\sigma}_m\sigma_n = 2\sigma_m$ and the second from the ADHM constraints (6.15).

We wish to solve $\mathcal{D}^2A_{\mathrm{cl}} = 0$ together with the VEV boundary conditions

$$iA_{\mathrm{cl}}{}^{\dot\alpha}{}_{\dot\beta} \xrightarrow{|x|\to\infty} \frac{i}{2}v^a\tau^{a\dot\alpha}{}_{\dot\beta}. \quad (7.12)$$

As we stressed in the Introduction, our general strategy in solving this type of ADHM equation is to seek inspiration whenever possible from the one-instanton sector. In that instance we need only compare Eqs. (4.8c) and (6.3) to see the answer. We find that $\mathcal{A}(x)$ is in fact a constant, namely,

$$\mathcal{A}(x) = \begin{pmatrix} \mathcal{A}_{00} & 0 \\ 0 & 0 \end{pmatrix}, \qquad (7.13)$$

where

$$\mathcal{A}_{00\,\alpha}{}^{\beta} = \frac{i}{2}\, v^a \tau^a{}_{\alpha}{}^{\beta}. \qquad (7.14)$$

Extrapolating to multi-instantons, we will guess that $\mathcal{A}(x)$ remains a constant. From the asymptotic behavior (6.28c,d) for U_λ, we then draw two important conclusions. First, we see that the top-left entry $\mathcal{A}_{00}$ must by itself account for the boundary conditions (7.12); thus, Eq. (7.14) continues to hold for multi-instantons as well. Second, if we assume that the *next* leading behavior of A_{cl} goes like $1/|x|^2$ rather than $1/|x|$, again as in the one-instanton sector, then the ($\lambda = 0$, $\kappa \neq 0$) and ($\lambda \neq 0$, $\kappa = 0$) elements of $\mathcal{A}$ should vanish. In short, our *Ansatz* is

$$\mathcal{A}_{\kappa\lambda} = \begin{pmatrix} \mathcal{A}_{00} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathcal{A}'_{kl} & \\ 0 & & & \end{pmatrix}, \quad \partial_n \mathcal{A} = 0. \qquad (7.15)$$

Consequently, Eq. (7.10) collapses to

$$\mathcal{D}^2 i A_{\mathrm{cl}} = 4\bar{U}'\{f,\mathcal{A}'\}U' - 4\bar{U}'f\cdot\mathrm{tr}_2\bar{\Delta}\mathcal{A}\Delta\cdot fU', \qquad (7.16)$$

where U'_l is the truncated vector obtained by lopping off the $\lambda = 0$ component from U_λ.

To make further progress, it is convenient to denote by $\Delta'_{kl}(x) = a'_{kl} + \delta_{kl}\cdot x$ the lower $n \times n$ submatrix of $\Delta_{\kappa l}(x)$. We also define the three $n \times n$ matrices Λ, $\widetilde{W}$, and $W = \mathrm{tr}_2\widetilde{W}$ by their matrix elements

$$\Lambda_{kl} = \bar{w}_k \mathcal{A}_{00} w_l - \bar{w}_l \mathcal{A}_{00} w_k, \quad \widetilde{W}_{kl} = \bar{w}_k w_l,$$

$$W_{kl} = \bar{w}_k w_l + \bar{w}_l w_k. \qquad (7.17)$$

We then rewrite

$$\mathrm{tr}_2\bar{\Delta}\mathcal{A}\Delta = \Lambda + \mathrm{tr}_2\bar{\Delta}'\mathcal{A}'\Delta'$$

$$= \Lambda + \tfrac{1}{2}\,\mathrm{tr}_2([\bar{\Delta}',\mathcal{A}']\Delta' - \bar{\Delta}'[\Delta',\mathcal{A}'] + \{\mathcal{A}',\bar{\Delta}'\Delta'\})$$

$$= \Lambda + \tfrac{1}{2}\mathrm{tr}_2([\bar{\Delta}',\mathcal{A}']\Delta' - \bar{\Delta}'[\Delta',\mathcal{A}']$$

$$- \{\mathcal{A}',\widetilde{W}\} + \{\mathcal{A}',f^{-1}\}), \qquad (7.18)$$

where the final equality follows from Eq. (6.14).

Notice that the last term in Eq. (7.18), namely, $\tfrac{1}{2}\{\mathrm{tr}_2\mathcal{A}', f^{-1}\}$, gives a contribution that is naturally combined with the anticommutator in Eq. (7.16), for a total of $4\bar{U}'\{f,\mathcal{A}' - \tfrac{1}{2}\mathrm{tr}_2\mathcal{A}'\}U'$. With the obvious further *Ansatz*

$$\mathcal{A}'_{kl\,\alpha}{}^{\beta} = \mathcal{A}'_{kl}\delta_\alpha{}^{\beta}, \qquad (7.19)$$

this vanishes, leaving only the terms containing two factors of f:

$$\mathcal{D}^2 i A_{\mathrm{cl}} = -4\bar{U}'f(\Lambda - \tfrac{1}{2}\{\mathcal{A}',W\} + \tfrac{1}{2}\mathrm{tr}_2([\bar{a}',\mathcal{A}']a'$$

$$- \bar{a}'[a',\mathcal{A}']))fU'. \qquad (7.20)$$

Thanks to Eq. (7.19) the $\delta_{kl}\cdot x$ part of $\Delta'_{kl}(x)$ has canceled out of the commutator terms. In fact, with the x dependence gone, the entire $n \times n$ matrix in the outer parentheses in Eq. (7.20) can now sensibly be set to zero. With $a'_{\alpha\dot\alpha} = a'_m \sigma^m_{\alpha\dot\alpha}$ we arrive, finally, at the defining equation for the classical adjoint Higgs field:

$$-[a'^m,[a'_m,\mathcal{A}']] + \tfrac{1}{2}\{\mathcal{A}',W\} = \Lambda. \qquad (7.21)$$

From Eqs. (6.25), (7.15), (7.17) and (7.19), together with the requirement $\bar{\mathcal{A}} = -\mathcal{A}\cdot(\bar{v}/v)$, we see that both $\mathcal{A}'_{kl}$ and Λ_{kl} are antisymmetric, while W_{kl} and a'_{kl} are symmetric; therefore, both sides of this matrix equation are consistent. For the n-instanton sector, Eq. (7.21) defines a set of $\tfrac{1}{2}n(n-1)$ coupled linear inhomogeneous equations for the $\tfrac{1}{2}n(n-1)$ independent entries of $\mathcal{A}'$, in terms of the constrained $[(n+1)\times n]$-dimensional ADHM matrix a. We note for future reference [17] that the $[\tfrac{1}{2}n(n+1)\times\tfrac{1}{2}n(n+1)]$-dimensional linear operator on the space of antisymmetric matrices defined by the left-hand side of Eq. (7.21) is actually self-adjoint on this space, as the reader can easily check in the obvious basis.

For an alternative route to Eq. (7.21), see Appendix C.

C. Adjoint Higgs field in the presence of a Yukawa source

Next we solve the more challenging equation (3.2a), for the adjoint Higgs field in the presence of an adjoint fermion bilinear source term, subject once again to the VEV boundary conditions (7.12). In terms of $\mathcal{M}_\gamma$ and $\mathcal{N}_\gamma$ the right-hand side of Eq. (3.2a) is expanded as

$$-\sqrt{2}i\bar{U}^{\beta\gamma}(\mathcal{N}_\gamma f\bar{b}\mathcal{P}^{\delta\alpha}\mathcal{M}_\alpha f\bar{b} + bf\mathcal{N}^{\alpha T}\mathcal{P}_{\alpha\gamma}bf\mathcal{M}^{\delta T}$$

$$+ \mathcal{N}_\gamma f\bar{b}\cdot\mathrm{tr}_2\mathcal{P}\cdot bf\mathcal{M}^{\delta T} - \delta_\gamma{}^\delta bf\mathcal{N}^{\alpha T}\mathcal{P}_\alpha{}^\beta\mathcal{M}_\beta f\bar{b})U_{\delta\dot\gamma}$$

$$-(\mathcal{M}\rightleftharpoons\mathcal{N}). \qquad (7.22)$$

We can ignore the auxiliary field D at this order, as in the one-instanton case. To solve for the Higgs field, we exploit the linearity of the Klein-Gordon equation and decompose A into $A_{\mathrm{cl}} + A_f$. As constructed in the preceding section, A_{cl} solves the homogeneous equation and soaks up the boundary conditions (7.12). This leaves A_f (the subscript stands for ''fermionic'') to account for the Yukawa source (7.22), while approaching zero as $|x| \to \infty$.

Unlike the one-instanton case of Sec. IV, here we cannot rely on a ''sweeping-out'' procedure to generate A_f automatically. The reason is that for $n > 1$, the superconformal group will only produce a fixed subset of the adjoint fermion zero modes; most of these zero modes are not associated with (approximate) Lagrangian symmetries. Nevertheless, an intelligent first guess for A_f may be intuited once again from the one-instanton sector. There, A_f is the fermion bilinear piece of Eq. (4.3b) proportional to $\xi'(x)\sigma^{mn}\xi(x)v^{\mathrm{cl}}_{mn}$, where $\xi(x)$ and $\xi'(x)$ parametrize the supersymmetric and superconformal modes of, respectively, the gaugino and Higgsino. This expression may be regarded either as the Higgsino collective coordinates dotted into the most general gaugino zero mode or, symmetrically, as the gaugino collective coordinates dotted into the most general Higgsino zero mode. Extrapolating to the n-instanton case at hand and dotting $\mathcal{N}^\alpha$

into the most general gaugino zero mode (7.1), we are led immediately to the symmetric Ansatz

$$i(A_f)^{\dot{\beta}}{}_{\dot{\gamma}} \stackrel{?}{=} \frac{1}{2\sqrt{2}}\, \bar{U}^{\dot{\beta}\alpha}(\mathcal{N}_\alpha f\mathcal{M}^{\beta T} - \mathcal{M}_\alpha f\mathcal{N}^{\beta T})U_{\beta\dot{\gamma}}, \tag{7.23}$$

where the constant in front comes from a careful comparison of normalizations.

In actuality, for $n>1$ this guess for A_f is almost, but not quite, correct, as the reader can check by plugging it into Eq. (7.10). Specifically, it accounts for everything in Eq. (7.22) except the last term, where it gives $\mathcal{P}-1$ rather than $\mathcal{P}$ in the middle of the expression. Taking advantage of the linearity of Eq. (3.2a) for a second time, one further decomposes

$$i(A_f)^{\dot{\beta}}{}_{\dot{\gamma}} = \frac{1}{2\sqrt{2}}\, \bar{U}^{\dot{\beta}\alpha}(\mathcal{N}_\alpha f\mathcal{M}^{\beta T} - \mathcal{M}_\alpha f\mathcal{N}^{\beta T})U_{\beta\dot{\gamma}}$$

$$+ i(A_f')^{\dot{\beta}}{}_{\dot{\gamma}}. \tag{7.24}$$

By design, A_f' accounts for the missing piece of the last term:

$$\mathcal{D}^2(A_f')^{\dot{\beta}}{}_{\dot{\gamma}} = -4i\bar{U}^{\dot{\beta}\alpha}bf\Lambda_f f\bar{b}U_{\alpha\dot{\gamma}}, \tag{7.25}$$

with

$$\Lambda_f = -\Lambda_f^T = -\frac{1}{2\sqrt{2}}(\mathcal{N}^{\beta T}\mathcal{M}_\beta - \mathcal{M}^{\beta T}\mathcal{N}_\beta). \tag{7.26}$$

Fortunately, this equation is easily solved by the methods of Sec. VII B. Making the Ansatz

$$iA_f' = \bar{U} \cdot \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \mathcal{A}_f' & \\ 0 & & \end{pmatrix} \cdot U, \tag{7.27}$$

where

$$\partial_n \mathcal{A}_f' = 0, \quad (\mathcal{A}_f')_{kl}{}_\alpha{}^\beta = (\mathcal{A}_f')_{kl}\delta_\alpha{}^\beta, \quad (\mathcal{A}_f')_{kl} = -(\mathcal{A}_f')_{lk}, \tag{7.28}$$

as before, one regains Eq. (7.20), where now $\mathcal{A}' \to \mathcal{A}_f'$ and $\Lambda \to \Lambda_f$. The solution follows by direct analogy with Eq. (7.21) and is given by the antisymmetric matrix equation

$$-[a'^m, [a_m, \mathcal{A}_f']] + \tfrac{1}{2}\{\mathcal{A}_f', W\} = \Lambda_f. \tag{7.29}$$

Like Eq. (7.21), this defines a set of $\frac{1}{2}n(n-1)$ linear inhomogeneous equations for the $\frac{1}{2}n(n-1)$ independent scalars in $\mathcal{A}_f'$, completing our task.

D. Multi-instanton and super-multi-instanton action

We now construct the supersymmetric multi-instanton action, through order g^0. The pure Yang-Mills contribution is, of course, $S_{\text{cl}} = 8n\pi^2/g^2$. An instant way to derive this fact, or equivalently that the winding number is n, is to think of the field strength (6.13) as a translational vector zero mode, i.e., Eq. (B1) with $C^{(m)} = g^{-1}b\sigma_m$, and then to use the integration formula (B5). (Alternatively one can exploit the interesting identity [32,21] $\text{tr}_2 v_{mn}v^{mn} = \partial^4\ln\det f^{-1}$.)

At the next order in $g^2|v|^2$, one needs to evaluate the surface integral shown in Eq. (4.7). In the long-distance limit, we can simply replace $A^\dagger$ in that expression by $\bar{\mathcal{A}}_{00}$, as in the one-instanton case. With the help of Eqs. (6.28) and (7.8), we calculate the gradient of the bosonic and fermion-bilinear pieces of the Higgs field, respectively:[18]

$$\hat{x}_m\mathcal{D}^m A_{\text{cl}} \xrightarrow{|x|\to\infty} \frac{2}{|x|^3}\left(\mathcal{A}_{00}{}_\alpha{}^\beta\sum_k |w_k|^2 - w_{k\alpha\dot{\alpha}}\mathcal{A}_{kl}'\bar{w}_l^{\dot{\alpha}\beta}\right), \tag{7.30a}$$

$$\hat{x}_m\mathcal{D}^m A_f \xrightarrow{|x|\to\infty} -\frac{1}{|x|^3\sqrt{2}}(\nu_{k\alpha}\mu_k^\beta - \mu_{k\alpha}\nu_k^\beta)$$

$$-\frac{2}{|x|^3}\, w_{k\alpha\dot{\alpha}}(\mathcal{A}_f')_{kl}\bar{w}_l^{\dot{\alpha}\beta}, \tag{7.30b}$$

where μ_k and ν_k denote the top-row elements,

$$\mu_k^\alpha = \mathcal{M}_{0k}^\alpha, \quad \nu_k^\alpha = \mathcal{N}_{0k}^\alpha. \tag{7.31}$$

Substituting Eqs. (7.30) into Eq. (4.7) and performing the traces gives, finally,

$$S_{\text{inst}} \equiv S_{\text{cl}} + S_{\text{Higgs}} + S_{\text{Yuk}}$$

$$= \frac{8n\pi^2}{g^2} + 16\pi^2|\mathcal{A}_{00}|^2\sum_k |w_k|^2$$

$$- 8\pi^2(\bar{w}_l\bar{\mathcal{A}}_{00}w_k - \bar{w}_k\bar{\mathcal{A}}_{00}w_l)\mathcal{A}_{kl}' + 4\sqrt{2}\pi^2\mu_k^\alpha\bar{\mathcal{A}}_{00\alpha}{}^\beta\nu_{k\beta}$$

$$- 8\pi^2(\bar{w}_l\bar{\mathcal{A}}_{00}w_k - \bar{w}_k\bar{\mathcal{A}}_{00}w_l)(\mathcal{A}_f')_{kl}. \tag{7.32}$$

We remind the reader where the various threads of this equation may be located: Eqs. (6.23), (7.1), (7.14), (7.17), (7.21), (7.24), (7.26), (7.29), and (7.31).

Expression (7.32) constitutes the exact classical interaction between an arbitrary number of superinstantons in $N=2$ supersymmetric Yang-Mills theory and is our main result in the general case; eliminating the fermion bilinears gives the analogous result in the purely bosonic version of the theory. Some comments about the parts and the whole are in order.

(1) Beyond its role in the action, the quantity S_{Higgs} also governs the asymptotics of various component fields in the problem. For example, we discover that the one-instanton relation (5.5) between $\delta A^\dagger$ and S_{Higgs} continues to hold in the multi-instanton case, even though S_{Higgs} is much more complicated. On reflection this is no surprise, for Eqs. (5.5) and (7.30a) are essentially equivalent statements. Similarly for the antifermions specifically in the background of the exact supersymmetric mode, which are proportional to $\mathcal{D}A^\dagger$ [see Eqs. (A1 c,i)]; again, the one-instanton expressions (5.7) and

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[18]We can ignore pre- and post-multiplication by $\bar{\sigma}_0$ and σ_0, which merely interchanges dotted and undotted indices. Note also that unlike for the bosons, the order in g of a fermionic contribution to the action is convention dependent; it is a property of Grassmannian integrals that any rescaling of the fermionic action can be compensated by a change in the measure.

(5.8) generalize immediately to $n \geq 1$. We will need these expressions once more in the following section.

(2) As for S_{Yuk}, the reader can check that the only Grassmann modes not lifted by this expression are the exact supersymmetry modes (7.5). Thanks to the constraint (7.4), these modes cancel out of Λ_f and, therefore, do not appear in $\mathcal{A}_f'$.

(3) In Ref. [17] we verify that the sum $S_{\text{Higgs}} + S_{\text{Yuk}}$ as given above is in fact a supersymmetric invariant, as it must be.

E. Multi-instanton measure

To proceed to a complete calculation of the multi-instanton contributions to the prepotential $\mathcal{F}$, we require the measure for integration over the moduli. The collective coordinate integration measure for n ADHM instantons in $N=2$ supersymmetric Yang-Mills theory is obtained in the usual way by changing variables in the functional integral from the fields to the moduli [31,26]. Listing the $8n$ unconstrained bosonic and fermionic coordinates as X_i and ξ_i, respectively, we have

$$\int d\mu_n = \frac{1}{\mathcal{S}_n} \int \left(\prod_{i=1}^{8n} dX_i d\xi_i \right) (J_{\text{Bose}}/J_{\text{Fermi}})^{1/2} \exp(-S_{\text{inst}}),$$
(7.33)

where J_{Bose} (J_{Fermi}) is the Jacobian for the change of variables for the bosonic (fermionic) degrees of freedom. A key simplification in a supersymmetric theory is that there is no additional small-fluctuations 't Hooft determinant to be calculated, as the positive frequency bosonic and fermionic excitations cancel identically. As we discussed above, any unconstrained parametrization of the n-instanton solution will, in general, contain several redundant copies of each field configuration. Hence, to obtain the correct normalization of the measure, we must divide out the relevant symmetry factor $\mathcal{S}_n$.

The fermionic Jacobian J_{Fermi} is simply the determinant of the normalization matrix of the fermion zero modes. In Appendix B, we calculate this matrix in terms of the (constrained) bosonic parameters of the ADHM solution for general n and evaluate its determinant for $n=2$. The bosonic Jacobian has been studied in detail by Osborn [21]. J_{Bose} is much harder to evaluate than J_{Fermi}, because the corresponding variations of the collective coordinates are not simply equal to the physical bosonic zero modes, but differ from them by transformations of the type (6.19)–(6.22). In particular, an explicit expression for J_{Bose} cannot be obtained without first specifying an unconstrained parametrization of the solution. Given any such parametrization, the required measure can then be constructed using the methods developed by Osborn and the corresponding multi-instanton contribution to the prepotential can, in principle, be written as a finite-dimensional integral. Unfortunately, as mentioned above, the constraints have only been solved for $n \leq 3$, and so we cannot make immediate progress in the general case.

It can easily be checked that applying our general results to the case $n=1$ reproduces all the known results for the action, the measure, and the modes in the one-instanton sector. One result we will need below is the correctly normal-

ized measure $d\mu_1$ for a single ADHM instanton. The bosonic and fermionic collective coordinates of the $n=1$ ADHM super-instanton are contained in the three 2×1 matrices of unconstrained parameters:

$$a = \begin{pmatrix} w \\ X \end{pmatrix}, \quad \mathcal{M}_\gamma = \begin{pmatrix} \mu_\gamma \\ M_\gamma \end{pmatrix}, \quad \mathcal{N}_\gamma = \begin{pmatrix} \nu_\gamma \\ N_\gamma \end{pmatrix}.$$
(7.34)

These coordinates are related in a simple way to the parameters of the single instanton appearing in the measure (5.6): $X = x_0$, $|w| = \rho$, $M = 4\xi$, $N = 4\xi'$, $\mu = 4w\bar{\eta}$, and $\nu = 4w\bar{\eta}'$. In these variables the correctly normalized measure (5.6) can be rewritten as

$$\int d\mu_1 = 2^7 \pi^{-4} \Lambda^4 \int d^4X \, d^4w \, d^2M \, d^2N \, d^2\mu \, d^2\nu$$
$$\times \exp(-S_{\text{Higgs}} - S_{\text{Yuk}}). \tag{7.35}$$

In the following section we focus on the case $n=2$.

VIII. THE USES OF $N=2$, $n=2$ SUPERINSTANTONS

A. Collective coordinates, the measure, and the action

We now specialize to the two-instanton sector. The 16 gauge, 8 gaugino, and 8 Higgsino collective coordinates live, respectively, in the following matrices:

$$a = \begin{pmatrix} w_1 & w_2 \\ x_0 + a_3 & a_1 \\ a_1 & x_0 - a_3 \end{pmatrix},$$
(8.1a)

$$\mathcal{M}_\gamma = \begin{pmatrix} \mu_{1\gamma} & \mu_{2\gamma} \\ 4\xi_\gamma + \mathcal{M}_{3\gamma} & \mathcal{M}_{1\gamma} \\ \mathcal{M}_{1\gamma} & 4\xi_\gamma - \mathcal{M}_{3\gamma} \end{pmatrix},$$
(8.1b)

$$\mathcal{N}_\gamma = \begin{pmatrix} \nu_{1\gamma} & \nu_{2\gamma} \\ 4\xi_\gamma' + \mathcal{N}_{3\gamma} & \mathcal{N}_{1\gamma} \\ \mathcal{N}_{1\gamma} & 4\xi_\gamma' - \mathcal{N}_{3\gamma} \end{pmatrix}.$$
(8.1c)

As in the one-instanton case, the modes x_0, ξ_γ, and ξ_γ' represent, respectively, global translations and global $N=2$ supersymmetries Q_1 and Q_2; see Eqs. (6.24) and (7.5). The lower 2×2 subblock of each of these matrices is symmetric, as forced by the constraints (6.25) and (7.4). The remaining constraints (6.15a) and (7.3) may be used to eliminate a_1, $\mathcal{M}_1$, and $\mathcal{N}_1$:

$$a_1 = \frac{1}{4|a_3|^2} a_3(\bar{w}_2 w_1 - \bar{w}_1 w_2 + \Sigma),$$
(8.2)

$$\mathcal{M}_1 = \frac{1}{2|a_3|^2} a_3(2\bar{a}_1 \mathcal{M}_3 + \bar{w}_2 \mu_1 - \bar{w}_1 \mu_2),$$
(8.3)

and

$$\mathcal{N}_1 = \frac{1}{2|a_3|^2} a_3(2\bar{a}_1 \mathcal{N}_3 + \bar{w}_2 \nu_1 - \bar{w}_1 \nu_2).$$
(8.4)

Notice that the ADHM constraints have only determined the

four quaternionic components of a_1 up to a new ''17th collective coordinate'' Σ, which apart from the requirement $\Sigma^{\dot\alpha}{}_{\dot\beta} \propto \delta^{\dot\alpha}{}_{\dot\beta}$ is completely unrestricted. This is precisely consistent with our counting in Sec. VI B; as discussed there, this extra degree of freedom may be eliminated in a variety of ways by invoking the leftover O(2) symmetry (6.26). In the following we will use this O(2) freedom to fix $\Sigma \equiv 0$, which simplifies the algebra enormously. In Appendix D we find that the discrete symmetry group of this parametrization is the dihedral group D_8 and hence the corresponding symmetry factor is $\mathcal{S}_2 = 16$.

Following Sec. VII E, the $N=2$, $n=2$ superinstanton measure is written as

$$\frac{1}{\mathcal{S}_2} \int d^4 x_0 d^4 a_3 d^4 w_1 d^4 w_2 d^2 \xi \, d^2 \mathcal{M}_3$$

$$\times d^2 \mu_1 d^2 \mu_2 d^2 \xi' d^2 \mathcal{N}_3 d^2 \nu_1 d^2 \nu_2$$

$$\times \exp(-S_{\mathrm{inst}})(J_{\mathrm{Bose}}/J_{\mathrm{Fermi}})^{1/2}. \qquad (8.5)$$

It is helpful to have in mind a physical picture of these integration variables [22]. As discussed above, the coordinates x_0, ξ, and ξ' can always be thought of as the location of the center of the two-instanton configuration in superspace. In contrast, the other coordinates can only be given a definite interpretation in a clustering limit where the two-instanton solution is approximately the linear superposition of two single instantons. For the choice $\Sigma \equiv 0$, one such limit is $|a_3| \to \infty$. In this limit the off-diagonal elements in the collective coordinate matrices go to zero and it is straightforward to identify the corresponding one-instanton degrees of freedom. In particular, the combinations $x_{(1)} = -(x_0 + a_3)$ and $x_{(2)} = -(x_0 - a_3)$ can be identified as the centers of two well-separated singular gauge instantons. The scale sizes of these instantons are given by the magnitudes of the quaternions w_1 and w_2, $\rho_{(i)} = |w_i|$ for $i=1,2$, while their SU(2) iso-orientations [see Eq. (6.4) ff] are given by the corresponding unit-normalized quaternions $w_1/\rho_{(1)}$ and $w_2/\rho_{(2)}$.

Next we discuss the integrand of Eq. (8.5). It is convenient to define the four frequently occurring combinations of the bosonic parameters from Eqs. (7.14) and (8.1), as follows:

$$L = |w_1|^2 + |w_2|^2, \qquad (8.6a)$$

$$H = L + 4|a_1|^2 + 4|a_3|^2, \qquad (8.6b)$$

$$\Omega = w_1 \bar{w}_2 - w_2 \bar{w}_1, \qquad (8.6c)$$

$$\omega = \tfrac{1}{2} \mathrm{tr}_2 \Omega \mathcal{A}_{00} = \bar{w}_2 \mathcal{A}_{00} w_1 - \bar{w}_1 \mathcal{A}_{00} w_2, \qquad (8.6d)$$

in terms of which the action (7.32) works out to

$$S_{\mathrm{inst}} \equiv S_{\mathrm{cl}} + S_{\mathrm{Higgs}} + S_{\mathrm{Yuk}}$$

$$= \frac{16\pi^2}{g^2} + 16\pi^2 \left(L|\mathcal{A}_{00}|^2 - \frac{|\omega|^2}{H} \right) + 4\sqrt{2}\,\pi^2 [-\nu_k \bar{\mathcal{A}}_{00} \mu_k$$

$$+ (\bar{\omega}/H)(\mu_1 \nu_2 - \nu_1 \mu_2 + 2\mathcal{M}_3 \mathcal{N}_1 - 2\mathcal{N}_3 \mathcal{M}_1)]. \qquad (8.7)$$

Remarkably, Osborn [21] obtained an explicit expression for the bosonic collective coordinate Jacobian for an arbitrary unconstrained parametrization of the two instanton solution.[19] In our notation, Osborn's result reads

$$J_{\mathrm{Bose}}^{1/2} \propto \frac{H}{|a_3|^4} \left| |a_3|^2 - |a_1|^2 - \frac{1}{8} \frac{d\Sigma}{d\theta} \right|_{\theta=0} \Big|, \qquad (8.8)$$

where the angle θ parametrizes the O(2) symmetry (6.26). As for the fermionic Jacobian, it is given by the determinant of the overlap matrix of the fermion zero modes. We calculate this explicitly in Appendix C and find

$$J_{\mathrm{Fermi}}^{1/2} \propto \frac{H^2}{|a_3|^4}. \qquad (8.9)$$

Putting the factors which occur in the measure together and specializing to $\Sigma \equiv 0$, we obtain

$$\frac{1}{\mathcal{S}_2} (J_{\mathrm{Bose}}/J_{\mathrm{Fermi}})^{1/2} \exp(-S_{\mathrm{cl}}) = C_J \frac{\big||a_3|^2 - |a_1|^2\big|}{H}. \qquad (8.10)$$

The overall constant C_J, in Eq. (8.10) can easily be determined by noting that the Jacobians, being local functionals of the fields, factorize into the product of single-instanton Jacobians in the clustering limit $|a_3| \to \infty$. Taking account of the symmetry factors, this implies the following relationship between the the correctly normalized measures $d\mu_n$ for the cases $n=2$ (with this particular parametrization) and $n=1$:

$$\int d\mu_2 \to \frac{\mathcal{S}_1^2}{\mathcal{S}_2} \int d\mu_1^{(1)} d\mu_1^{(2)} \qquad (8.11)$$

as $|a_3| \to \infty$. Here $d\mu_1^{(1)}$ and $d\mu_1^{(2)}$ are the normalized measures for the collective coordinates of two well-separated single instantons. In Appendix D we note that the symmetry factor for the single-instanton measure is given by $\mathcal{S}_1 = 2$. Hence, using $\mathcal{S}_2 = 16$ and Eq. (7.35), we extract the value

$$C_J = 2^6 \pi^{-8} \Lambda^8, \qquad (8.12)$$

which completes the specification of the two-instanton measure.

---

[19]The measure for the particular parametrization of Jackiw, Nohl, and Rebbi [33] had been reported earlier in Refs. [34,35].

B. Four-fermion Green's functions and a two-instanton check on Seiberg and Witten

In what follows we shall focus on the family of four-fermion Green's functions (5.1) that we examined earlier in the one-instanton sector. The 28-fold integration proceeds stepwise, and more or less painlessly, as follows.

(1) As in the one-instanton case, the four antifermions are needed to saturate the integration over the exact supersymmetric modes $d^2\xi d^2\xi'$ which do not otherwise appear in the integrand. The remaining (lifted) Grassmann modes are then necessarily saturated by pulling down appropriate powers of the action. A straightforward calculation gives, for these lifted modes,

$$\int d^2\mathcal{M}_3 d^2\mu_1 d^2\mu_2 d^2\mathcal{N}_3 d^2\nu_1 d^2\nu_2 \, \exp(-S_{\text{Yuk}})$$
$$= -\left(\frac{16\sqrt{2}\pi^6\bar{\omega}}{|a_3|^2 H|\Omega|}\right)^2 \{[-(\bar{\mathcal{A}}_{00})^2|\Omega|^2 + \tau_1\bar{\omega}^2]^2$$
$$- \tau_2\bar{\omega}^2[(\bar{\mathcal{A}}_{00})^2|\Omega|^2 + \bar{\omega}^2]\}, \quad (8.13)$$

where $\tau_1 = L/H$ and $\tau_2 = (L^2 - |\Omega|^2)/H^2$. Comparing this expression to its one-instanton counterpart (5.9) reflects the order-of-magnitude increase in complexity in the step between the one-instanton and two-instanton sectors. It is convenient to substitute $\bar{\omega}^2 = |\omega|^2(\bar{v}/v)$ and also $(\bar{\mathcal{A}}_{00})^2 = -|\mathcal{A}_{00}|^2(\bar{v}/v) = -\bar{v}^2/4$.

(2) We now insert the $l+4$ fields from Eq. (5.1) or, more specifically, their asymptotics as given in Eqs. (5.5), (5.7), and (5.8), and perform the trivial integration over the exact supersymmetric modes $d^2\xi \, d^2\xi'$. We are left with a purely bosonic integral. The integrand is greatly simplified by the familiar trick of letting $S_{\text{Higgs}} \to \lambda S_{\text{Higgs}}$ in the action and representing the $l+4$ powers of S_{Higgs} coming from these equations as parametric derivatives with respect to λ. [Alternatively, if one is careful to maintain the distinction between v and $\bar{v}$, these field insertions may be obtained from the action (8.7) by differentiation with respect to v.]

(3) Next we change integration variables from the quaternions $\{a_3, w_1, w_2\}$ into the more natural coordinates in the problem, $\{H, L, \Omega\}$. Using Eq. (8.2) and fixing $\Sigma \equiv 0$, we calculate

$$\int_{-\infty}^{\infty} d^4 a_3 \frac{||a_3|^2 - |a_1|^2|}{|a_3|^4} = \pi^2 \int_0^{\infty} |a_3|^2 d|a_3|^2 \frac{||a_3|^2 - |a_1|^2|}{|a_3|^4}$$
$$\to 2\frac{\pi^2}{4}\int_{L+2|\Omega|}^{\infty} dH \quad (8.14)$$

and, likewise,

$$\int_{-\infty}^{\infty} d^4 w_1 d^4 w_2 \to 2\int_0^{\infty} dL \int_{|\Omega|\leqslant L} d^3\Omega \int_{L_-}^{L_+} \pi^2 |w_1|^2 d|w_1|^2$$
$$\times \frac{1}{16|w_1|^2\sqrt{(L_+ - |w_1|^2)(|w_1|^2 - L_-)}}$$
$$= 2\frac{\pi^3}{16}\int_0^{\infty} dL \int_{|\Omega|\leqslant L} d^3\Omega. \quad (8.15)$$

In Eq. (8.14) the numerator and denominator of the integrand are supplied by Eqs. (8.10) and (8.13), respectively; in Eq. (8.15), the (irrelevant) limits of integration are $L_\pm = \frac{1}{2}(L \pm \sqrt{L^2 - |\Omega|^2})$; and in both Eqs. (8.14) and (8.15) the extra overall factors of 2 reflect the fact that each of these changes of variables is a 2-to-1 mapping.

(4) Next we rescale variables,

$$\Omega' = \Omega/L, \quad H' = H/L, \quad \omega' = \frac{1}{2}\text{tr}_2\Omega'\mathcal{A}_{00}, \quad (8.16)$$

and switch to polar coordinates

$$d^3\Omega' \to 2\pi\int_{-1}^{1} d(\cos\theta)\int_0^1 |\Omega'|^2 d|\Omega'|, \quad (8.17)$$

where the polar angle is defined by $|\omega'| = |\Omega'||\mathcal{A}_{00}|\cos\theta = \frac{1}{2}|\Omega'||v|\cos\theta$. With a bookkeeper's eye on the various factors of 2 as well as on the distinction between v and $\bar{v}$, we can then reexpress the Green's function (5.1) as

$$\int d^4 x_0 \int_0^1 d|\Omega'||\Omega'|^6 \int_{-1}^1 d(\cos\theta)\cos^2\theta \int_{1+2|\Omega'|}^{\infty} \frac{dH'}{H'^3} \int_0^{\infty} dL \, L^5 \bar{v}^6 (-v)^{-(4+l)}\Lambda^8\pi^{10}2^6$$

$$\times\left[\left(1 + \frac{\cos^2\theta}{H'}\right)^2 + \frac{1 - |\Omega'|^2}{4H'^2}\sin^2 2\theta\right]\epsilon_{\alpha\beta}S^{\dot\alpha\alpha}(x_1,x_0)S^{\dot\beta\beta}(x_2,x_0)\epsilon_{\gamma\delta}S^{\dot\gamma\gamma}(x_3,x_0)S^{\dot\delta\delta}(x_4,x_0)G(x_5,x_0)\times\cdots$$

$$\times G(x_{l+4},x_0)\left(-\frac{\partial}{\partial\lambda}\right)^{l+4}\bigg|_{\lambda=1}\exp[-4\pi^2\lambda L|v|^2(1 - |\Omega'|^2(\cos^2\theta)/H')]$$

$$= \int d^4 x_0 \epsilon_{\alpha\beta}S^{\dot\alpha\alpha}(x_1,x_0)S^{\dot\beta\beta}(x_2,x_0)\epsilon_{\gamma\delta}S^{\dot\gamma\gamma}(x_3,x_0)S^{\dot\delta\delta}(x_4,x_0)G(x_5,x_0)\times\cdots\times G(x_{l+4},x_0)(9+l)!\frac{\Lambda^8 I_{3D}}{\pi^2 2^6(-v)^{10+l}}.$$

$$(8.18)$$

In obtaining this equality we have performed first the trivial integration over L and then the λ differentiation.

(5) The quantity I_{3D} introduced in Eq. (8.18) stands for the remaining scaleless three-dimensional integral,

$$
I_{3D}=\int_0^1 d|\Omega'||\Omega'|^6\int_{-1}^1 d(\cos\,\theta)\cos^2\,\theta\int_{1+2|\Omega'|}^\infty\frac{dH'}{H'^3}
$$

$$
\times\frac{[1+(\cos^2\,\theta)/H']^2+[(1-|\Omega'|^2)/4H'^2]\sin^2\,2\theta}{[1-|\Omega'|^2(\cos^2\,\theta)/H']^6}.
$$

$$(8.19)$$

It is elementary and gives $1/48$.

(6) As in the one-instanton case, we can view the two-instanton result (8.18) as arising from the following effective local vertex, built out of the total Higgs field $A(x_0)=v+\delta A(x_0)$:

$$
\sum_{l=0}^\infty\frac{1}{2!2!l!}\,\psi^2\lambda^2(\delta A)^l\,\frac{(9+l)!\Lambda^8}{3\cdot 2^{10}\pi^2(-v)^{10+l}}=\frac{945\Lambda^8}{32\pi^2g^6}\,\frac{\psi^2\lambda^2}{A^{10}}.
$$

$$(8.20)$$

We have used the binomial theorem once again and have also restored the explicit g dependence. Identical manipulations may be used for the families of anomalous magnetic moment and field-strength Green's functions, respectively, (5.2) and (5.3), as in Sec. V. Comparing the fraction $945/32$ with the Seiberg-Witten effective Lagrangian, Eqs. (1.1) and (2.6), we deduce for the two-instanton coefficient $\mathcal{F}_2=5/16$, confirming their prediction.

APPENDIX A: SUPERSYMMETRIC AND SUPERCONFORMAL INVARIANCE

In this appendix we review some of the classical invariances of the microscopic Lagrangian (2.1) together with the induced Euler-Lagrange equations (3.1)–(3.3). To begin with, they are invariant under the following $N=2$ supersymmetry transformations, which commute with Wess-Zumino gauge fixing:

$$
\delta v^m=-i\underline{\lambda}\bar{\sigma}^m\xi_1+i\bar{\xi}_1\bar{\sigma}^m\underline{\lambda}-i\bar{\psi}\bar{\sigma}^m\xi_2+i\bar{\xi}_2\bar{\sigma}^m\psi,
$$

$$(A1a)$$

$$
\delta\underline{\lambda}=-\xi_1\sigma^{mn}v_{mn}+i\xi_1\underline{D}+i\sqrt{2}\bar{\xi}_2\bar{\mathcal{D}}\underline{A}-\sqrt{2}\xi_2\underline{F},\quad(A1b)
$$

$$
\delta\bar{\underline{\lambda}}=-\bar{\xi}_1\bar{\sigma}^{mn}v_{mn}-i\bar{\xi}_1\underline{D}+i\sqrt{2}\xi_2\mathcal{D}\underline{A}^\dagger-\sqrt{2}\bar{\xi}_2\underline{F}^\dagger,
$$

$$(A1c)$$

$$
\delta\underline{D}=-\xi_1\mathcal{D}\bar{\underline{\lambda}}+\bar{\xi}_1\bar{\mathcal{D}}\underline{\lambda}-\xi_2\mathcal{D}\bar{\psi}+\bar{\xi}_2\bar{\mathcal{D}}\psi,\qquad(A1d)
$$

$$
\delta\underline{A}=\sqrt{2}\xi_1\psi-\sqrt{2}\xi_2\underline{\lambda},\qquad\qquad(A1e)
$$

$$
\delta\psi=-i\sqrt{2}\bar{\xi}_1\bar{\mathcal{D}}\underline{A}+\sqrt{2}\xi_1\underline{F}-\xi_2\sigma^{mn}v_{mn}+i\xi_2\underline{D},\quad(A1f)
$$

$$
\delta\underline{F}=i\sqrt{2}\bar{\xi}_1\bar{\mathcal{D}}\psi+2ig[\underline{A},\bar{\xi}_1\bar{\underline{\lambda}}]-i\sqrt{2}\bar{\xi}_2\bar{\mathcal{D}}\underline{\lambda}+2ig[\underline{A},\bar{\xi}_2\bar{\psi}],
$$

$$(A1g)$$

$$
\delta\underline{A}^\dagger=\sqrt{2}\bar{\xi}_1\bar{\psi}-\sqrt{2}\bar{\xi}_2\bar{\underline{\lambda}},\qquad\qquad(A1h)
$$

$$
\delta\bar{\psi}=-i\sqrt{2}\xi_1\mathcal{D}\underline{A}^\dagger+\sqrt{2}\bar{\xi}_1\underline{F}^\dagger-\bar{\xi}_2\bar{\sigma}^{mn}v_{mn}-i\bar{\xi}_2\underline{D},
$$

$$(A1i)$$

$$
\delta\underline{F}^\dagger=i\sqrt{2}\xi_1\mathcal{D}\bar{\psi}+2ig[\underline{A}^\dagger,\xi_1\underline{\lambda}]-i\sqrt{2}\xi_2\mathcal{D}\bar{\underline{\lambda}}-2ig[\underline{A}^\dagger,\xi_2\psi].
$$

$$(A1j)$$

For purposes of comparison with Sec. IV, take $\xi_1\rightarrow\xi$ and $\xi_2\rightarrow-\xi'$. Actually, it suffices to restrict our attention to $N=1$ invariance, setting $\xi_2=\bar{\xi}_2=0$ henceforth, as long as we also enforce $SU(2)_R$ invariance as per Eq. (2.4).

A heuristic route to the superconformal group is to pose the question, can these supersymmetry transformations be made *local*, that is to say, $\xi_1\rightarrow\xi(x)$ and $\bar{\xi}_1\rightarrow\bar{\xi}(x)$? The answer involves a lengthy but straightforward calculation, in which one repeatedly exploits not only the standard Wess-Bagger identities, but also the following useful facts about Pauli matrices and covariant derivatives, respectively:

$$
\bar{\sigma}^m\sigma^{kl}=\tfrac{1}{2}(\eta^{ml}\bar{\sigma}^k-\eta^{mk}\bar{\sigma}^l-i\epsilon^{mkln}\bar{\sigma}_n),\qquad(A2a)
$$

$$
\bar{\sigma}^{kl}\bar{\sigma}^m=\tfrac{1}{2}(-\eta^{ml}\bar{\sigma}^k+\eta^{mk}\bar{\sigma}^l-i\epsilon^{mkln}\bar{\sigma}_n),\qquad(A2b)
$$

$$
\sigma^m\bar{\sigma}^{kl}=\tfrac{1}{2}(\eta^{ml}\sigma^k-\eta^{mk}\sigma^l+i\epsilon^{mkln}\sigma_n),\qquad(A2c)
$$

$$
\sigma^{kl}\sigma^m=\tfrac{1}{2}(-\eta^{ml}\sigma^k+\eta^{mk}\sigma^l+i\epsilon^{mkln}\sigma_n),\qquad(A2d)
$$

and

$$
[\mathcal{D}_m,\mathcal{D}_n]\underline{X}=-ig[v_{mn},\underline{X}],\qquad\qquad(A3a)
$$

$$
(\mathcal{D}\bar{\mathcal{D}})_\alpha{}^\beta\underline{X}=-\delta_\alpha{}^\beta\mathcal{D}^2\underline{X}-ig\sigma^{mn}{}_\alpha{}^\beta[v_{mn},\underline{X}],\quad(A3b)
$$

$$
(\bar{\mathcal{D}}\mathcal{D})^{\dot\alpha}{}_{\dot\beta}\underline{X}=-\delta^{\dot\alpha}{}_{\dot\beta}\mathcal{D}^2\underline{X}-ig\bar{\sigma}^{mn\dot\alpha}{}_{\dot\beta}[v_{mn},\underline{X}],\quad(A3c)
$$

$$
\epsilon^{klmn}\mathcal{D}_k v_{lm}=0.\qquad\qquad(A3d)
$$

In addition, one needs the transformation law for v_{mn}, which follows from that for v_m:

$$
\delta v_{mn}=i\xi(x)(\sigma_n\mathcal{D}_m-\sigma_m\mathcal{D}_n)\bar{\underline{\lambda}}+i\bar{\xi}(x)(\bar{\sigma}_n\mathcal{D}_m-\bar{\sigma}_m\mathcal{D}_n)\underline{\lambda}
$$

$$
-i\bar{\underline{\lambda}}(\bar{\sigma}_n\partial_m-\bar{\sigma}_m\partial_n)\xi(x)-i\underline{\lambda}(\sigma_n\partial_m-\sigma_m\partial_n)\bar{\xi}(x).
$$

$$(A4)$$

The results of this exercise are as follows. Focusing first on $\mathcal{L}_{\text{gauge}}$, one finds invariance if and only if

$$\xi(x) = \xi - \overline{\eta}\overline{\sigma}^k x_k, \quad \overline{\xi}(x) = \overline{\xi} + \eta\sigma^k x_k. \tag{A5}$$

The new Grassmann parameters η^α and $\overline{\eta}_{\dot{\alpha}}$ are associated with the superconformal generators S^α and $\overline{S}_{\dot{\alpha}}$, which are the fermionic superpartners of the special conformal generators K_n [37]. As for $\mathcal{L}_{\text{chiral}}$, the substitution (A5) does not *quite* work; one has to add an extra piece to the transformations for ψ and $\overline{\psi}$, proportional to $\eta\underset{\sim}{A}$ and $\overline{\eta}\underset{\sim}{A}^\dagger$, respectively. These extra pieces are just a particular resolution of the operator ordering ambiguity between $\mathcal{D}_m$ and $\xi(x)$; they are present even in the simplest superconformally invariant model, that of a single massless chiral superfield [38]. In sum, $\mathcal{L}_{\text{SU(2)}}$ is invariant under the space-dependent Wess-Zumino gauge-preserving supersymmetric transformations (A5), as follows:

$$\delta\underset{\sim}{v}^m = -i\overline{\underset{\sim}{\lambda}}\overline{\sigma}^m \xi(x) + i\overline{\xi}(x)\overline{\sigma}^m\underset{\sim}{\lambda}, \tag{A6a}$$

$$\delta\underset{\sim}{\lambda} = -\xi(x)\sigma^{mn}\underset{\sim}{v}_{mn} + i\xi(x)\underset{\sim}{D}, \tag{A6b}$$

$$\delta\overline{\underset{\sim}{\lambda}} = -\overline{\xi}(x)\overline{\sigma}^{mn}\underset{\sim}{v}_{mn} - i\overline{\xi}(x)\underset{\sim}{D}, \tag{A6c}$$

$$\delta\underset{\sim}{D} = -\xi(x)\mathcal{D}\overline{\underset{\sim}{\lambda}} + \overline{\xi}(x)\overline{\mathcal{D}}\underset{\sim}{\lambda}, \tag{A6d}$$

$$\delta\underset{\sim}{A} = \sqrt{2}\xi(x)\psi, \tag{A6e}$$

$$\delta\psi = -i\sqrt{2}\overline{\xi}(x)\overline{\mathcal{D}}\underset{\sim}{A} + \sqrt{2}\xi(x)\underset{\sim}{F} + 2\sqrt{2}i\,\eta\underset{\sim}{A}, \tag{A6f}$$

$$\delta\underset{\sim}{F} = i\sqrt{2}\overline{\xi}(x)\overline{\mathcal{D}}\psi + 2ig[\underset{\sim}{A}, \overline{\xi}(x)\overline{\underset{\sim}{\lambda}}], \tag{A6g}$$

$$\delta\underset{\sim}{A}^\dagger = \sqrt{2}\overline{\xi}(x)\overline{\psi}, \tag{A6h}$$

$$\delta\overline{\psi} = -i\sqrt{2}\xi(x)\mathcal{D}\underset{\sim}{A}^\dagger + \sqrt{2}\overline{\xi}(x)\underset{\sim}{F}^\dagger - 2\sqrt{2}i\,\overline{\eta}\underset{\sim}{A}^\dagger, \tag{A6i}$$

$$\delta\underset{\sim}{F}^\dagger = i\sqrt{2}\xi(x)\mathcal{D}\overline{\psi} + 2ig[\underset{\sim}{A}^\dagger, \xi(x)\underset{\sim}{\lambda}]. \tag{A6j}$$

Of course, given these *infinitesimal* transformations, one automatically knows the *finite* transformations as well since exponential series in Grassmannians terminate. For instance, $\exp(\overline{\xi}\overline{Q}) = 1 + \overline{\xi}\overline{Q} + \frac{1}{2}(\overline{\xi}\overline{Q})^2$, so that

$$\exp(\overline{\xi}\overline{Q}) \times \underset{\sim}{A}^\dagger = \underset{\sim}{A}^\dagger + \sqrt{2}\overline{\xi}\overline{\psi} + \frac{1}{2}\sqrt{2}\overline{\xi}(\sqrt{2}\overline{\xi}\underset{\sim}{F}^\dagger); \tag{A7}$$

for our purposes, the fields on the right are then replaced by the initial ''reference'' choice of configuration. In practice, for superinstantons, the quadratic terms often vanish by virtue of the equations of motion.

Finally, it is easily checked using Eqs. (A2) and (A3) that these transformations automatically generate from the instanton the two different types of Weyl zero modes of $\overline{\mathcal{D}}$: the supersymmetric modes $\xi\sigma^{mn}v_{mn}^{\text{cl}}$ and the superconformal modes which, for $n=1$, appear both as $x_k\overline{\eta}\overline{\sigma}^k\sigma^{mn}v_{mn}^{\text{cl}}$ and as $\overline{\xi}\overline{\mathcal{D}}\underset{\sim}{A}_{\text{cl}}$, the relationship between them being given by Eqs. (4.4) and (4.5).

APPENDIX B: ADJOINT FERMION ZERO-MODE JACOBIAN IN THE TWO-INSTANTON SECTOR

Here we derive the expression (8.9) for the adjoint fermion zero-mode Jacobian J_{Fermi}. Rather than work directly with the adjoint spinor zero modes (7.1), it is equivalent but somewhat more convenient to calculate, instead, the overlap matrix for the closely related adjoint *vector* zero modes Z_n. These are given by [21]

$$Z_n = \overline{U}C\overline{\sigma}_n f\overline{b}U - \overline{U}bf\sigma_n\overline{C}U, \tag{B1}$$

where C is an $[(n+1)\times n]$-dimensional matrix of constant quaternions. The defining equations for background-gauge vector zero modes, namely,

$$\mathcal{D}_{[n}Z_{m]} = (\mathcal{D}_{[n}Z_{m]})^{\text{dual}}, \quad \mathcal{D}_n Z^n = 0, \tag{B2}$$

ensure that, when they are viewed as infinitesimal transformations on the gauge field, the field strength remains self-dual. The first of these conditions leads to $\overline{\Delta}C - (\overline{\Delta}C)^T = -\overline{C}\Delta + (\overline{C}\Delta)^T$, while the second works out to $\overline{\Delta}C - (\overline{\Delta}C)^T = \overline{C}\Delta - (\overline{C}\Delta)^T$; combining these gives $\overline{\Delta}C = (\overline{\Delta}C)^T$ or, equivalently,

$$\overline{a}C = (\overline{a}C)^T, \quad \overline{b}C = (\overline{b}C)^T. \tag{B3}$$

These are the precise analogs of the linear constraints on the fermion zero modes, Eqs. (7.3) and (7.4); indeed, the fermion zero modes can be linearly obtained from the vector zero modes by folding in a spin matrix times a spinor.

That a calculation of the overlap matrix between different Z_{rn} is at all tractable is thanks to a remarkable identity due to Corrigan:[20]

$$\text{tr}_2 Z_{rn}Z_s^n = \frac{1}{2}\partial_n\partial^n\text{Tr}\,\overline{C}_r(\mathcal{P}+1)C_s f, \tag{B4}$$

where the capitalized Tr means a trace over both SU(2) and ADHM indices. The inner product is then

$$\langle Z_r|Z_s\rangle \equiv -\int d^4x\,\text{tr}_2\,Z_{rn}Z_s^n$$

$$= -\frac{1}{2}\int d^4x\,\partial_n\partial^n\,\text{Tr}\,\overline{C}_r(\mathcal{P}+1)C_s f$$

$$= -\pi^2\lim_{r\to\infty} r^3\,\text{Tr}\,\overline{C}_r(\mathcal{P}_\infty+1)C_s f'(r)$$

$$= 2\pi^2\,\text{Tr}\,\overline{C}_r(\mathcal{P}_\infty+1)C_s. \tag{B5}$$

Here

$$\mathcal{P}_\infty + 1 \equiv \lim_{r\to\infty}(2 - \Delta f\overline{\Delta}) = 2 - b\overline{b} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{B6}$$

specializing at last to the two-instanton case. In obtaining these expressions we have used the asymptotic forms

---

[20]The index r labels the different zero modes. This identity is quoted in Osborn as his Eq. (3.17), but its proof is apparently not to be found in the literature; we supply one in Appendix C.

(6.28a,b). Also, we have disregarded the term where the normal derivative hits $\mathcal{P}$ rather than f, as this costs two powers of r.

We parametrize the quaternion matrix C as

$$C=\begin{pmatrix} m_1 & m_2 \\ M_3 & M_1 \\ M_1 & -M_3 \end{pmatrix}, \tag{B7}$$

omitting the translational modes $\propto b$ since they do not mix with the others. Thus

$$\operatorname{Tr}\bar{C}(\mathcal{P}_\infty+1)C \propto |m_1|^2+|m_2|^2+|M_3|^2+|M_1|^2. \tag{B8}$$

Thanks to the constraints, M_1 can be eliminated, just like its fermionic counterpart $\mathcal{M}_1$ in Eq. (8.3). It is convenient, first, to define the rotated quaternionic variables $\widetilde{m}_1$ and $\widetilde{m}_2$, via

$$m_1=\frac{w_2\bar{a}_1\widetilde{m}_1}{\sqrt{|w_2|^2|a_1|^2}}, \quad m_2=\frac{w_1\bar{a}_1\widetilde{m}_2}{\sqrt{|w_1|^2|a_1|^2}}. \tag{B9}$$

Since $|m_1|^2+|m_2|^2=|\widetilde{m}_1|^2+|\widetilde{m}_2|^2$, this change of variables does not affect the determinant. The constraint on M_1 is then resolved as

$$M_1=\frac{a_3}{2|a_3|^2}(2\bar{a}_1M_3+\bar{w}_2m_1-\bar{w}_1m_2)$$

$$=\frac{a_3\bar{a}_1}{2|a_3|^2\sqrt{|a_1|^2}}(2\sqrt{|a_1|^2}M_3+\sqrt{|w_2|^2}\widetilde{m}_1-\sqrt{|w_1|^2}\widetilde{m}_2). \tag{B10}$$

From Eqs. (B8) and (B10), it is obvious that the matrix whose determinant we want can be written as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}+Q\cdot Q^T, \tag{B11}$$

where

$$Q^T=\frac{1}{2\sqrt{|a_3|^2}}(2\sqrt{|a_1|^2},\sqrt{|w_2|^2},-\sqrt{|w_1|^2}). \tag{B12}$$

This matrix has two eigenvalues equaling unity (spanning the $\perp$ space of Q) and a third eigenvalue, hence the determinant itself, equaling

$$1+|Q|^2=\frac{H}{4|a_3|^2}, \tag{B13}$$

with H as in Eq. (8.6). Actually, this is the determinant for *each* of the four decoupled [thanks to (B9)] σ_n components of the quaternion, so that

$$J_{\text{Fermi}}\propto\frac{H^4}{|a_3|^8}, \tag{B14}$$

confirming Eq. (8.9).

APPENDIX C: VECTOR ZERO-MODE IDENTITIES

1. Alternative route to the classical adjoint Higgs field

One important example of a vector zero mode [see Eqs. (B1), (B2), and (A3a)] is $\mathcal{D}_nA_{\text{cl}}$. Here we confirm that our solution for A_{cl} does indeed have this property. In performing this check, we are giving in effect an alternative route to the construction of A_{cl}.

We focus on the right-hand side of Eq. (7.8) and once again posit that $\mathcal{A}$ is a constant $[(n+1)\times(n+1)]$-dimensional matrix of the form shown in Eq. (7.15), in particular that $\mathcal{A}'$ satisfies the condition (7.19). We can then reexpress

$$\mathcal{A}\Delta=\begin{pmatrix} \mathcal{A}_{00}w_1 & \cdots & \mathcal{A}_{00}w_n \\ & \mathcal{A}',a' & \end{pmatrix}$$

$$+\begin{pmatrix} 0 & \Uparrow \\ \vdots & bx \\ 0 & \Downarrow \end{pmatrix}\begin{pmatrix} 0 & \cdots & 0 \\ & \mathcal{A}' & \end{pmatrix}$$

$$\equiv\begin{pmatrix} \mathcal{A}_{00}w_1-w_k\mathcal{A}'_{k1} & \cdots & \mathcal{A}_{00}w_n-w_k\mathcal{A}'_{kn} \\ & [\mathcal{A}',a'] & \end{pmatrix}, \tag{C1}$$

with a' as in Eq. (6.23). In the final rewrite we have exploited the fact that the matrix product is to be left multiplied by $\bar{U}_\lambda$ and have therefore used Eq. (6.9) to eliminate the x dependence. We can now check that this final matrix satisfies the two conditions (B3) for a vector zero-mode matrix C. The latter is automatically satisfied when $\mathcal{A}'$ obeys Eq. (7.19) (and is therefore necessarily antisymmetric in ADHM indices). Less obviously, the former is equivalent to Eq. (7.21); verifying this claim is entirely straightforward, once one rewrites Eq. (6.15a) in terms of the submatrix a'.

By dotting the vector zero mode $\mathcal{D}_nA_{\text{cl}}$ with $\bar{\theta}_0\bar{\sigma}^n$, we of course construct the fermionic zero modes (4.5), which are among the lifted fermionic modes in the problem.

2. Proof of Corrigan's inner product formula

Next, we supply a proof of Corrigan's remarkable identity, Eq. (B4). First, we expand the left-hand side, using Eqs. (B1), (6.6), and (7.11):

$$\operatorname{tr}_2Z_{rm}Z_s^m=2\operatorname{Tr}[(\bar{C}_s\mathcal{P}C_r+\bar{C}_r\mathcal{P}C_s)f\bar{b}\cdot\operatorname{tr}_2\mathcal{P}\cdot bf$$

$$+\bar{C}_r\mathcal{P}bf(\mathcal{P}C_s)^Tbf+\mathcal{P}C_rf\bar{b}(\bar{C}_s\mathcal{P})^Tf\bar{b}]. \tag{C2}$$

In order to differentiate the right-hand side of Eq. (B4) twice, one needs, in addition to Eq. (7.9), the following useful facts:

$$\partial_m\partial^mf=4f\bar{b}\cdot\operatorname{tr}_2\mathcal{P}\cdot bf, \tag{C3a}$$

$$\partial^m\mathcal{P}=-\Delta f\bar{b}\bar{\sigma}^m\mathcal{P}-\mathcal{P}b\sigma^mf\bar{\Delta}, \tag{C3b}$$

$$\partial_m\partial^m\mathcal{P}=4\{\mathcal{P},bf\bar{b}\}-4\Delta f\bar{b}\cdot\operatorname{tr}_2\mathcal{P}\cdot bf\bar{\Delta}. \tag{C3c}$$

So the right-hand side of Eq. (B4) becomes

$$\mathrm{Tr}[2\bar{C}_r\{\mathcal{P},bf\bar{b}\}C_sf+2\bar{C}_r(\mathcal{P}+1)C_sf\bar{b}\cdot\mathrm{tr}_2\mathcal{P}\cdot bf$$

$$-2\bar{C}_r\Delta f\bar{b}\cdot\mathrm{tr}_2\mathcal{P}\cdot bf\bar{\Delta}C_sf+\bar{C}_r\Delta f\bar{b}\bar{\sigma}^m\mathcal{P}C_sf\partial_m(\bar{\Delta}\Delta)f$$

$$+\bar{C}_r\mathcal{P}b\sigma^mf\bar{\Delta}C_sf\partial_m(\bar{\Delta}\Delta)f]. \tag{C4}$$

Comparing Eqs. (C2) and (C4), it is clear that we need to eliminate all explicit factors of Δ and $\bar{\Delta}$ in favor of $\mathcal{P}$, using Eq. (6.8). In fact, the third, fourth, and fifth terms in Eq. (C4) can individually be rewritten in this manner, yielding, respectively,

$$2\mathrm{Tr}[\bar{C}_r(\mathcal{P}-1)C_sf\bar{b}\cdot\mathrm{tr}_2\mathcal{P}\cdot bf+C_sf\bar{b}(\bar{C}_s\mathcal{P})^Tf\bar{b}(\mathcal{P}-1)$$

$$+(\mathcal{P}C_r)^Tbf\bar{C}_s(\mathcal{P}-1)bf]. \tag{C5}$$

In proving this last step, it is helpful to use $\bar{C}\Delta=-\bar{\Delta}C+\mathrm{tr}_2\bar{C}\Delta$ as follows from Eqs. (B3). Now the ''-1'' pieces in the last two terms in (C5) cancel the anticommutator piece in (C4); by inspection, the surviving terms establish the claimed equality with (C2). Q.E.D.

APPENDIX D: RESIDUAL DISCRETE SYMMETRIES IN THE ADHM MEASURE

The purpose of this appendix is to determine the relevant discrete symmetry groups G_n and the corresponding symmetry factors $\mathcal{S}_n$ for the case $n=1$ and the particular parametrization of the $n=2$ solution described in the text.

After fixing the canonical form (6.23), the collective coordinates of the ADHM multi-instanton live in the vector $\vec{w}=(w_1,\ldots,w_n)$ and the $n\times n$ submatrix a'. To solve the constraint we need to specify $\vec{w}$ and a' in terms of a set of unconstrained parameters $\{X_i\}$, $i=1,2,\ldots,8n$. The residual redundancy is then obtained by finding all matrices $R\in O(n)$ satisfying the condition (6.27) discussed in the text. In general, the solutions of this condition will form a discrete subgroup of $O(n)$, the symmetry group G_n. The number of such solutions is the symmetry factor $\mathcal{S}_n$. Notice that the symmetry group and symmetry factor depend not only on n but also on the details of the particular parametrization $\{X_i\}$ chosen.

1. $n=1$

In the ADHM construction, the single-instanton solution is parametrized as in Eq. (7.34); in this case, the residual transformations are simply generated by $R=\pm1$. Hence the symmetry group is just Z_2 and $\mathcal{S}_1=2$.

2. $n=2$

Here we have

$$\vec{w}=(w_1,w_2), \quad a'=\begin{pmatrix} x_0+a_3 & a_1 \\ a_1 & x_0-a_3 \end{pmatrix}. \tag{D1}$$

The constraint is solved by eliminating the parameter a_1 via the relation (8.2). We complete the parametrization by specifying $\Sigma\equiv0$ so that

$$a_1=\frac{1}{4|a_3|^2}a_3(\bar{w}_2w_1-\bar{w}_1w_2). \tag{D2}$$

Now let us solve the condition (6.27). This is equivalent to finding the matrices R such that the transformed collective coordinates still obey the condition (D2). By definition, any matrix $R\in O(2)$ must have $\det R=\pm1$. If $\det R=+1$, R is just a rotation matrix,

$$R=R_\theta=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad 0\leq\theta<2\pi, \tag{D3}$$

and the transformed parameters from Eq. (6.26) may be rewritten as

$$(w_1^\theta,w_2^\theta)=(w_1,w_2)R_\theta, \quad (a_3^\theta,a_1^\theta)=(a_3,a_1)R_{2\theta}, \quad x_0^\theta=x_0. \tag{D4}$$

Hence the equation we have to solve is

$$a_1^\theta=\frac{1}{4|a_3^\theta|^2}a_3^\theta(\bar{w}_2^\theta w_1^\theta-\bar{w}_1^\theta w_2^\theta). \tag{D5}$$

Since $\bar{w}_2^\theta w_1^\theta-\bar{w}_1^\theta w_2^\theta=\bar{w}_2w_1-\bar{w}_1w_2$, this simplifies to

$$0=\bar{a}_3^\theta a_1^\theta-\bar{a}_3a_1$$

$$=\tfrac{1}{2}(|a_3|^2-|a_1|^2)\sin4\theta-(\bar{a}_3a_1+\bar{a}_1a_3)\sin^22\theta. \tag{D6}$$

Noting that the last term vanishes by virtue of Eq. (D2) and excepting the special case $|a_1|=|a_3|$ discussed below, we see that the solutions with $\det R=+1$ are given by $\sin4\theta=0$, i.e., $\theta=\theta_k=(k-1)\pi/4$, with $k=1,2,\ldots,8$. Our analysis is completed by considering the $O(2)$ matrices with $\det R=-1$. These can be written as the product of a rotation R_θ and a reflection $R_r=\sigma_3$. The analysis proceeds as before and yields the same values of θ. Hence we have 16 solutions of Eq. (6.27) in all:

$$R=R_{\theta_k}, \quad R=R_rR_{\theta_k}. \tag{D7}$$

It follows that $\mathcal{S}_2=16$. Because R_τ does not commute with the R_{θ_k}, G_2 is the dihedral group D_8, the symmetry group of a regular octagon under reflections and rotations.

Some particular symmetries of the two-instanton solution are the following.

(1) The transformation generated by $R_rR_{3\pi/2}$ permutes the labels 1 and 2. In the clustering limit this has the effect of interchanging the two well-separated instantons.

(2) The matrix $R_rR_{3\pi/4}$ has the effect of interchanging the parameters a_1 and a_3. As described in the text, the parameter a_3 has the interpretation of (half) the distance between two well-separated instantons in the clustering limit $|a_3|\to\infty$. The discrete symmetry means that the region of the parameter space $|a_3|<|a_1|$ is equivalent to the region $|a_3|>|a_1|$. Hence the limit $|a_3|\to0$ is also a clustering limit. The fixed points of the discrete symmetry at $|a_1|=|a_3|$ are the central points in the moduli space at which the two instantons are coincident. Reasoning by analogy with the Jackiw-Nohl-Rebbi parametrization discussed by Osborn (see Sec. 4 of Ref. [21]), we expect that at these central points the solution degenerates to one of lower topological charge $n=1$.

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