FERMION SOLUTIONS IN THE SUPERSYMMETRIC CP^{n-1} MODEL

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Received 8 January 1981

We find all normalizable fermion solutions of the supersymmetric CP^{n-1} model in the background of a general bosonic solution of the equations of motion. The index theorem is seen to be satisfied in a non-minimal way.

The supersymmetric CP^{n-1} model [1] in two euclidean space–time dimensions constitutes an interesting example of an interacting theory of bosons and fermions, where various features like $1/n$ expansion and topological structure can be studied in an explicit fashion. Here we will be interested in investigating the problem of finding solutions to the field equations of motion.

For the case of the purely bosonic CP^{n-1} model this problem has been solved completely [2,3]. It can be shown that all stationary points of the lagrangian

$$\mathcal{L} = 2(|D_+z|^2 + |D_-z|^2), \tag{1}$$

where z is an n-component complex field (with $|z|^2 = 1$) and the covariant derivatives $D_\pm = \partial_\pm - \bar{z}\partial_+ z$ with $x_\pm = x_1 \pm \mathrm{i}x_2$, can be given explicitly in terms of holomorphic (polynomial) vectors $f(x_+)$ by

$$z = \hat{z}^k/|\hat{z}^k|, \tag{2}$$

where

$$\hat{z}^k = P_+^k f, \quad k = 0, ..., n-1, \tag{3}$$

and the operator P_+ is defined by

$$P_+ g = \partial_+ g - (\bar{g}\partial_+ g/|g|^2)g,$$

For $k = 0$ and $k = n-1$ one recovers the usual instantons and anti-instantons.

The interpretation [4] of the solutions with $0 < k < n-1$ is found to be that of a non-interacting unstable mixture of instantons and anti-instantons. One can express the action S_k and topological charge Q_k of the kth solution by

$$S_k = 2\pi(\alpha_{k+1} + \alpha_k), \tag{4}$$

$$Q_k = \alpha_{k+1} - \alpha_k, \tag{5}$$

where α_i, $i = 0, ..., n$, are non-negative integers (with $\alpha_0 = \alpha_n = 0$) which can be evaluated in terms of a given f.

The generalization of this solution structure to the supersymmetric CP^{n-1} which has a lagrangian

$$\mathcal{L} = \overline{D_\mu z}\, D_\mu z - \mathrm{i}\bar{\psi}\not{D}\psi + \tfrac{1}{4}[(\bar{\psi}\psi)^2 + (\bar{\psi}\gamma_5\psi)^2 - (\bar{\psi}\gamma_\mu\psi)^2], \tag{6}$$

where the fermion field ψ fulfils

$$\bar{z}\cdot\psi = 0, \tag{7}$$

does not seem to be straightforward. At this stage we cannot solve the full nonlinear equations of motion and will therefore focus our attention on the simpler problem of a fermion in the fixed background of a bosonic solution.

The supersymmetric external field Dirac equation is

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$$\not{D}\psi - \bar{z}\cdot\not{D}\psi z = 0, \tag{8}$$

with the constraint (7).

Decomposing ψ in eigenstates of γ_5

$$\psi = \binom{1}{-i}\psi^+ + \binom{1}{i}\psi^-, \tag{9}$$

one gets the equivalent equations

$$D_\pm \psi^\pm = \lambda_\pm z, \tag{10}$$

with the constraints

$$\bar{z}\cdot\psi^\pm = 0, \tag{11}$$

where $\lambda_\pm$ are some functions of x_+ and x_-.

The solution of eqs. (10) and (11) is known for the case of the background field z being an instanton or anti-instanton solution [1].

If we let $\xi_\pm$ denote the number of independent $\psi^\pm$ solutions which are normalizable on the sphere, i.e.

$$\int d^2x(1+x^2)^{-1}|\psi^\pm|^2 < \infty, \tag{12}$$

then it turns out that ξ_+ is zero for an instanton background and ξ_- is zero for an anti-instanton background. In these cases the Atiyah–Singer index theorem

$$\xi_+ - \xi_- = -nQ, \tag{13}$$

where Q is the topological charge of the background field, is seen to be satisfied in a minimal way.

It was found in ref. [4] how a special background field corresponding to a $k = 1$ solution in the CP^2 model gave rise to fermion solutions with both ξ_+ and ξ_- different from zero. The existence of such solutions are to be expected on general grounds [5].

We will now proceed to find the general solution of (10) and (11) for a general background field (2) and (3). Consider first the positive helicity equation. A ψ^+ field orthogonal to $\hat{z}^k$ can be written in general as

$$\psi^+ = |\hat{z}^k| \sum_{i\neq k} \frac{\overline{\hat{z}^i}\cdot g^+}{|\hat{z}^i|^2}\hat{z}^i, \tag{14}$$

for some vector g^+ since the system $\hat{z}^k$, $k = 0, ..., n-1$, is orthogonal. A factor $|\hat{z}^k|$ has been extracted because the relation $D_+|\hat{z}^k| = 0$ allows us to write (10) as

$$|\hat{z}^k|\partial_+ \sum_{i\neq k} \frac{\overline{\hat{z}^i}\cdot g^+}{|\hat{z}^i|^2}\hat{z}^i = \lambda_+ \hat{z}^k/|\hat{z}^k|. \tag{15}$$

Using the relations

$$\partial_+ \hat{z}^k/|\hat{z}^k|^2 = |\hat{z}^k|^{-2}\hat{z}^{k+1}$$

and

$$\partial_- \hat{z}^{k+1} = -|\hat{z}^{k+1}|^2|\hat{z}^k|^{-2}\hat{z}^k$$

it follows that g^+ and $\hat{z}^{k+1}\cdot g^+$ can be functions of x_- only. For $|x| \sim \infty$ one has $|\hat{z}^k| \sim |x|^{Q_k}$ and $Q_k > Q_{k+1}$. The normalization condition on ψ^+ (12) thus enforces $\hat{z}^{k+1}\cdot g^+ = 0$ and it also follows that $\overline{\hat{z}^i}\cdot g^+ = 0$ for $i = k+1, ..., n-1$. We therefore find the general form of ψ^+ fulfilling (10) to be

$$\psi^+ = |\hat{z}^k| \sum_{i=0}^{k-1} \frac{\overline{\hat{z}^i}\cdot g^+}{|\hat{z}^i|^2}\hat{z}^i, \quad g^+ = g^+(x_-). \tag{16}$$

The normalization condition is, however, not automatically satisfied and it remains to determine the g^+ for which this is the case.

Using similar arguments and the property $D_-|\hat{z}^k|^{-1} = 0$ one can also find the general solution for ψ^- as

$$\psi^- = \frac{1}{|\hat{z}^k|} \sum_{i=k+1}^{n-1} \frac{\overline{\hat{z}^i}\cdot g^-}{|\hat{z}^i|^2}\hat{z}^i, \quad g^- = g^-(x_+). \tag{17}$$

We will now try to determine the numbers $\xi_\pm$, i.e. find how many independent $\psi^\pm$ which are allowed by the freedom in the (polynomial) $g^\pm$ and the normalization constraint.

For the norms of $\psi^\pm$ we have

$$|\psi^+|^2 = |\hat{z}^k|^2 \sum_{i=0}^{k-1} \frac{|\overline{\hat{z}^i}\cdot g^+|^2}{|\hat{z}^i|^2}, \tag{18}$$

$$|\psi^-|^2 = \frac{1}{|\hat{z}^k|^2} \sum_{i=k+1}^{n-1} \frac{|\overline{\hat{z}^i}\cdot g^-|^2}{|\hat{z}^i|^2}. \tag{19}$$

Consider first the case of ψ^+. The functions $\overline{\hat{z}^i}\cdot g^+$, $i = 0, ..., k-1$, are independent because of the orthogonality of the system $\{\hat{z}^i\}$. It now follows that the maximal degree in $|x|$ for $|x| \sim \infty$,

$$\deg \overline{\hat{z}^i}\cdot g^+ = Q_i - Q_k - 1.$$

For $i = 0$, $\overline{\hat{z}^i}\cdot g^+$ is a polynomial in x_- and therefore contains $Q_0 - Q_k$ independent parameters. To count the number of independent parameters for $i \neq 0$ in $\overline{\hat{z}^i}\cdot g^+$ one may for example consider how many independent derivatives with respect to ∂_+ and ∂_- there exist. Now

$$\partial_+ \overline{\hat{z}^i}\cdot g^+ = -|\hat{z}^i|^2 |\hat{z}^{i-1}|^{-2} \overline{\hat{z}^{i-1}}\cdot g^+ ,$$

which is an already counted set of variables. Thus only ∂_- derivatives will give independent quantities and since the degree at infinity was $Q_i - Q_k - 1$ there will be precisely $Q_i - Q_k$ independent parameters.

The number of independent ψ^+ solutions is therefore given by

$$\xi_+ = \sum_{i=0}^{k-1} (Q_i - Q_k) = \alpha_k - kQ_k . \tag{20}$$

Considering the case of ψ^- one finds that

$$\deg (\overline{\hat{z}^i}/|\hat{z}^i|^2)\cdot g^- = Q_k - Q_i - 1 ,$$

$i = k+1, ..., n-1$. For $i = n-1$, $(\overline{\hat{z}^i}/|\hat{z}^i|^2)\cdot g^-$ is a polynomial in x_+ and therefore has $Q_k - Q_{n-1}$ independent parameters. A similar reasoning as for ψ^+ shows that the number of independent parameters is $Q_k - Q_i$. Thus

$$\xi_- = \sum_{i=k+1}^{n-1} (Q_k - Q_i) = \alpha_{k+1} + (n-k-1)Q_k . \tag{21}$$

It is possible to derive the parameter counting results (20) and (21) in a more explicit and detailed manner, but the one described above seems to be the intuitively simplest one.

One can now evaluate

$$\xi_+ - \xi_- = \alpha_k - \alpha_{k+1} - (n-1)Q_k = -nQ_k ,$$

and thus the index theorem (13) is fulfilled.

To be explicit one may consider the example of a polynomial f of degree β with no accidental linear dependencies for which case [3] one has $Q_k = \beta - 2k$, $k = 0, ..., n-2$. Inserting this in (20) one gets $\xi_+ = k(k+1)$. The special fermion solutions studied in ref. [4] correspond to $k = 1$ and $\xi_+ = 2$ in agreement with the result found.

Instead of introducing the fermion field ψ in a supersymmetric way one could simply couple it to the bosonic z field of the CP^{n-1} model in a minimal way. Then the Dirac equation would read

$$\not{D}\psi = 0 , \tag{22}$$

without any constraint on ψ.

Decomposing again in positive and negative helicity modes $\psi^\pm$ one finds

$$D_\pm \psi^\pm = 0 . \tag{23}$$

The general solutions of these equations are simply

$$\psi^+ = |\hat{z}^k| g^+(x_-), \quad \psi^- = |\hat{z}^k|^{-1} g^-(x_+). \tag{24}$$

It is easy to see that if the topological charge of the background field is positive, i.e. $Q_k > 0$ then $\xi_+ = 0$ and $\xi_- = nQ_k$ and if $Q_k < 0$ then $\xi_+ = -nQ_k$ and $\xi_- = 0$. The index theorem (13) is fulfilled in a minimal way. In this case the values of $\xi_\pm$ are sensitive only to the total topological charge of the background field and not to the particular instanton–anti-instanton content of the latter. We thus see that it is the constraint (7) which introduces a dependence on the instanton–anti-instanton mixture of the background field.

Let us finally remark that for a CP^{n-1} solution z which is an embedding of a solution of a lower dimensional model (as for example a one-instanton solution in CP^{n-1}, $n \geqslant 3$) the corresponding fermion solutions can be found simply using the above considerations. In this case the system $\{\hat{z}^i\}$ is truncated, $i = 0, ..., m < n-1$. For the projection of a fermion field on the subspace $\{\hat{z}^0, ..., \hat{z}^m\}$ the discussion proceeds as in the case of the supersymmetric Dirac equation. For the projection on the orthogonal subspace the discussion of the simple Dirac equation applies. In this way one can easily evaluate ξ_+ and ξ_- and verify the index theorem.

One of us (W.J.Z.) would like to thank Yale University, Los Alamos Scientific Laboratory and Lawrence Berkeley Laboratory for their kind hospitality during the summer 1980 where part of the work reported here was performed, as well as the first UK Institute for theoretical high energy physics. One of us (A.M.D.) would also like to thank the CERN Theory Division for hospitality.

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