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Covariant and Consistent Anomalies in Even-Dimensional Chiral Gauge Theories

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Abstract. This paper is devoted to various aspects of anomalies in even-dimensional chiral gauge theories. The difference between the covariant and consistent anomalies is carefully explained in terms of their different origins. The consistent current is defined in terms of a gauge-variant effective action constructed from the covariant current. An alternative scheme is set up where the covariant anomaly is unaltered but the consistent anomaly vanishes because the effective action is gauge-invariant. A discussion of theories with vector and axial currents separately gauged is included: here, apart from the covariant anomalies, two different ways of constructing gauge-variant effective actions are possible, giving rise to different structures of the consistent anomalies.

1. Introduction

It has been known for some time now that theories in which chiral symmetries are gauged possess anomalies in the gauge currents. According to the equations of motion these currents seem to be covariantly conserved, but quantum effects destroy this property as can be seen by doing 1-loop calculations. Different authors have calculated the "anomaly", i.e., the covariant divergence of the chiral current, by different methods – ranging from the diagrammatic to the nonperturbative – and results have not always agreed. These differences have been resolved by saying that different currents are involved in these calculations. The original expression derived by Bardeen [1] satisfies the so-called Wess-Zumino condition [2] and is, following Bardeen and Zumino [3], referred to as the consistent anomaly. It is not gauge covariant and cannot be supposed to stem from a covariant current. A covariant anomaly was obtained by Fujikawa [4], for instance. This anomaly does not satisfy the Wess-Zumino condition. However, there is no contradiction in that. Indeed, Bardeen and Zumino [3] have shown that an expression for the difference between the currents giving rise to the consistent anomaly and the covariant one can be explicitly obtained if it is assumed to be a local polynomial in the gauge fields and their derivatives.

In this paper (see also [5]), we present a derivation of the two anomalies in arbitrary even dimensions from a dynamical starting point. We emphasize that the expressions are not new, but we obtain them in a way which highlights the differences in the origin and meaning of the two anomalies. In brief, the covariant anomaly is obtained as the covariant divergence of a covariantly regularized current. The consistent anomaly is then obtained by constructing an effective action in terms of the covariant current.

The covariant and consistent anomalies have been calculated by various authors before us [4, 6, 7]. As far as the covariant anomaly is concerned, our derivation is essentially the same as in [4] except that we take a differential approach instead of considering infinitesimal gauge transformations. Our approach to the consistent anomaly is, however, new. Most authors [6, 8–11] regularize the formal effective action and derive the consistent anomaly therefrom. In doing so, they obtain the standard Bardeen anomaly, plus some normal-parity terms which are then removed by redefinition of the action, just as in Bardeen's original work [1]. We construct the effective action from the covariant current, which figures in the derivation of the covariant anomaly. The current defined in terms of this effective action differs from the covariant current by an amount related to the anomaly of that current, so that the covariant anomaly is used in the calculation of the consistent anomaly. We emphasize that only the minimal Bardeen expression appears, i.e. there is no normal-parity term.

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Special mention must be made of the work of Zumino and collaborators [12, 3]. Their approach is algebraic rather than dynamic: the consistent anomaly is obtained as a solution of the Wess-Zumino condition while the covariant current is obtained from the consistent one by the addition of a local piece chosen to give the correct gauge transformation behavior. Needless to say, the normalization is not obtained in this approach and the more general issue of uniqueness not resolved.

Leutwyler, like us, has calculated both the anomalies [7]. However, his work is in a sense a hybrid one. He seeks to calculate the consistent anomaly by starting from the fermion determinant and finds the anomaly of a covariant current. This is then converted algebraically to the consistent anomaly by imposing integrability. We feel that our direct calculation of the covariant anomaly and the derivation therefrom of the consistent one through an effective action is conceptually cleaner.

Before constructing the gauge-variant effective action, we also discuss a gauge invariant version [14]. The consistent anomaly in this scheme vanishes, but we show that the covariant anomaly, which we regard as more fundamental, is unchanged. Thus two different consistent anomalies – one trivial and one non-trivial – correspond to the same covariant anomaly.

The final aim of this paper is to discuss theories in which both the vector and the axial currents – or equivalently, both the left- and the right-handed currents – are gauged. These theories have often appeared in the literature, but dynamical calculations [1, 6, 8, 13] have been restricted to 4 dimensions; moreover, these calculations have yielded normal parity terms in addition to the minimal consistent anomaly, as in the case of the simpler chiral gauge theories. An algebraic treatment [15] has been given for higher dimensions, but this is open to the same criticism as in the simpler case. We obtain the covariant and consistent anomalies dynamically in arbitrary even dimensions. There are two inequivalent but equally natural ways of regularizing the effective action: one in which the left- and right-handed currents are independent and another where vector-gauge-invariance is achieved. Both are discussed.

The plan of the rest of the paper is as follows. Section 2 discusses covariant and consistent currents and anomalies in a general way without going into methods of calculation. This material is not claimed to be new: it is introduced partly to set up the notation and partly for pedagogical purposes. Section 3, devoted to the covariant current and the gauge-invariant effective action, is also not wholly new, but there is some novelty of perspective. The current is constructed explicitly in the path-integral framework. Its covariant divergence yields the covariant anomaly when Fujikawa's [4] regularization is used. The gauge-invariant effective action is discussed here. Section 4 is concerned with the consistent anomaly. A gauge-variant effective action is represented in terms of the covariant current. It is shown that the consistent current differs from the covariant current by an expression related to the covariant anomaly. Using the expression derived earlier for the covariant anomaly, this difference and finally the consistent anomaly are calculated. Section 5 presents expressions for different anomalies in theories with both left- and right-handed chiral gauge invariance. There is a concluding discussion in Sect. 6.

2. General Ideas about Covariant and Consistent Anomalies in Chiral Gauge Theories

We shall be concerned with the theory described by the Lagrangian density

$$\mathscr{L}=\bar{\psi}\left(\not\partial+\mathrm{i}T^a \not A^a\frac{1-\gamma_{D+1}}{2}\right)\psi\equiv\bar{\psi}\mathscr{D}\psi \tag{2.1}$$

in a euclidean D-dimensional spacetime (D even). The fields A_μ^a may be provisionally regarded as external. The matrices γ_μ are antihermitian, while γ_{D+1} is hermitian. The Dirac spinors $\psi, \bar{\psi}$ are understood to transform under some representation of a Lie group G and T^a are the hermitian generators of the Lie algebra of G in the corresponding representation. As is well known, $\mathscr{L}$ is invariant under the chiral gauge transformations

$$\psi(x)\to \mathrm{e}^{\mathrm{i}\alpha(x)\frac{1-\gamma_{D+1}}{2}}\psi(x),\qquad \bar{\psi}(x)\to\bar{\psi}(x)\mathrm{e}^{-\mathrm{i}\alpha(x)\frac{1+\gamma_{D+1}}{2}} \tag{2.2a}$$

$$A_\mu(x)\to\exp(\mathrm{i}\alpha(x))\left\{A_\mu(x)+\frac{1}{\mathrm{i}}\,\partial_\mu\right\}\exp(-\mathrm{i}\alpha(x)) \tag{2.2b}$$

In writing these transformations, we have followed the usual convention

$$A_\mu\equiv A_\mu^a T^a,\qquad \alpha\equiv\alpha^a T^a \tag{2.3}$$

Note that (2.2b) implies that

$$\mathscr{D}\to\exp\left(\mathrm{i}\alpha\frac{1+\gamma_{D+1}}{2}\right)\mathscr{D}\exp\left(-\mathrm{i}\alpha\frac{1-\gamma_{D+1}}{2}\right) \tag{2.4}$$

which is not a similarity transformation at all. Similarly

$$\mathscr{D}^+\to\exp\left(\mathrm{i}\alpha\frac{1-\gamma_{D+1}}{2}\right)\mathscr{D}^+\exp\left(-\mathrm{i}\alpha\frac{1+\gamma_{D+1}}{2}\right) \tag{2.5}$$

However $\mathscr{D}^+\mathscr{D}$ and $\mathscr{D}\mathscr{D}^+$ transform unitarily:

$$\mathscr{D}^+\mathscr{D}\to\exp\left(\mathrm{i}\alpha\frac{1-\gamma_{D+1}}{2}\right)\mathscr{D}^+\mathscr{D}\exp\left(-\mathrm{i}\alpha\frac{1-\gamma_{D+1}}{2}\right) \tag{2.6a}$$

$$\mathscr{D}\mathscr{D}^+\to\exp\left(\mathrm{i}\alpha\frac{1+\gamma_{D+1}}{2}\right)\mathscr{D}\mathscr{D}^+\exp\left(-\mathrm{i}\alpha\frac{1+\gamma_{D+1}}{2}\right) \tag{2.6b}$$

Our object of interest is the chiral gauge current formally written as $\bar\psi T^a\gamma_\mu\frac{1-\gamma_{D+1}}{2}\psi$. Using (2.2a), one can show that it transforms covariantly, at least at the formal level:

$$\left(\bar\psi T^a\gamma_\mu\frac{1-\gamma_{D+1}}{2}\psi\right)\to\mathrm{e}^{\mathrm{i}\alpha}\left(\bar\psi T^a\gamma_\mu\frac{1-\gamma_{D+1}}{2}\psi\right)\mathrm{e}^{-\mathrm{i}\alpha} \tag{2.7}$$

The equations of motion corresponding to (2.1) also lead to the formal covariant conservation of the current:

$$\partial_\mu\left(\bar\psi T^a\gamma_\mu\frac{1-\gamma_{D+1}}{2}\psi\right)$$
$$-f^{abc}A^b_\mu\left(\bar\psi T^c\gamma_\mu\frac{1-\gamma_{D+1}}{2}\psi\right)=0 \tag{2.8}$$

These formal properties may get modified by quantum effects. Thus one writes

$$\partial_\mu\left\langle\bar\psi T^a\gamma_\mu\frac{1-\gamma_{D+1}}{2}\psi\right\rangle$$
$$-f^{abc}A^b_\mu\left\langle\bar\psi T^c\gamma_\mu\frac{1-\gamma_{D+1}}{2}\psi\right\rangle=G^a \tag{2.9}$$

with the average understood to be taken over the fermionic degrees of freedom. G^a is called the anomaly. It turns out that different ways of defining the averaged current lead to different expressions for G^a, so that each G^a should be recognized to correspond to one specific definition of the current.

If the averaged current is defined by a gauge invariant regularization procedure, it transforms as in (2.7) and is said to be a covariant current. Its anomaly is called the covariant anomaly; and in fact this anomaly itself transforms covariantly.

A common way of looking at the averaged current is to regard it as the functional derivative of an effective action. Formally,

$$\left\langle\bar\psi T^a\gamma_\mu\frac{1-\gamma_{D+1}}{2}\psi\right\rangle$$
$$=\frac{1}{\mathrm{i}}\frac{\delta}{\delta A^a_\mu}\ln\int(\mathrm{d}\psi\,\mathrm{d}\bar\psi)\exp\int\mathrm{d}^Dx(\bar\psi\mathscr{D}\psi) \tag{2.10}$$

This representation is formal because the functional integration needs to be more specifically defined. If it is assumed to be gauge invariant – as it formally is – one can show that the current (2.10) is covariantly conserved (see Eq. (2.16) below). This current is also gauge-covariant. However there exist representations of the functional integral which are not gauge-invariant. The current defined by (2.10) then does not transform covariantly. However it satisfies the integrability condition

$$\frac{\delta\left\langle\bar\psi(x)T^a\gamma_\mu\frac{1-\gamma_{D+1}}{2}\psi(x)\right\rangle}{\delta A^b_\nu(x')}$$
$$=\frac{\delta\left\langle\bar\psi(x')T^b\gamma_\nu\frac{1-\gamma_{D+1}}{2}\psi(x')\right\rangle}{\delta A^a_\mu(x)} \tag{2.11}$$

A consequence of this is that the anomaly of this current satisfies what is known as the Wess-Zumino condition [2]. To state this condition, it is necessary to introduce the operators

$$L^a(x)=\partial_\mu\frac{\delta}{\delta A^a_\mu(x)}-f^{abc}A^b_\mu(x)\frac{\delta}{\delta A^c_\mu(x)} \tag{2.12}$$

which generate gauge transformations in the sense that under a gauge transformation with infinitesimal α, an arbitrary functional f of A transforms as

$$f[A]\to f[A]+\int\mathrm{d}^Dx\,\alpha^a(x)L^a(x)f[A] \tag{2.13}$$

These operators generate the gauge algebra

$$[L^a(x),L^b(x')]=f^{abc}\delta^{(D)}(x-x')L^c(x) \tag{2.14}$$

If W is an arbitrary functional of A, we can define a current J_W in analogy with (2.10) by

$$J^a_{W\mu}(x)=\frac{1}{\mathrm{i}}\frac{\delta}{\delta A^a_\mu(x)}\ln W \tag{2.15}$$

The covariant divergence or anomaly of this current is then

$$G^a_W(x)=\frac{1}{\mathrm{i}}L^a(x)\ln W \tag{2.16}$$

which, by virtue of (2.14), is seen to obey the relation

$$L^a(x)G^b_W(x')-L^b(x')G^a_W(x)=f^{abc}\delta^{(D)}(x-x')G^c_W(x) \tag{2.17}$$

This is the Wess-Zumino consistency condition and is satisfied by the anomaly of the current (2.10), which is just a special case of (2.15). This anomaly has therefore been named the consistent anomaly and the current (2.10) the consistent current.

We have already stated that the consistent current is not in general gauge-covariant, except when the effective action is gauge-invariantly regularized and the consistent anomaly vanishes (note that a zero anomaly is consistent with the Wess-Zumino condition). Conversely, covariant currents are in general not integrable, i.e., cannot be written in the form (2.15); hence they do not in general satisfy the Wess-Zumino condition. We shall in fact show that it is only when a current of one type (covariant or consistent) is anomaly-free that it has the characteristics of the other type as well.

First we consider the consistent current (2.15) and apply $L^b(x')$ on it:

$$L^b(x')J^a_{W\mu}(x)=\frac{1}{i}\frac{\delta}{\delta A^a_\mu(x)}L^b(x')\ln W + \frac{1}{i}\left[L^b(x'),\frac{\delta}{\delta A^a_\mu(x)}\right]\ln W \tag{2.18}$$

The commutator is

$$\left[L^b(x'),\frac{\delta}{\delta A^a_\mu(x)}\right]=-f^{abc}\delta^{(D)}(x-x')\frac{\delta}{\delta A^c_\mu(x)} \tag{2.19}$$

so that

$$L^b(x')J^a_{W\mu}(x)=-f^{abc}\delta^{(D)}(x-x')J^c_{W\mu}(x)+\frac{\delta G^b_W(x')}{\delta A^a_\mu(x)} \tag{2.20}$$

where we have used the definition (2.16). Thus we observe that the current (2.15) transforms covariantly, modulo a piece equal to the functional derivative of the anomaly. The anomaly itself is thus a measure of the non-covariance [3].

Next we consider a current which transforms covariantly:

$$L^b(x')J^a_\mu(x)=-f^{abc}\delta^{(D)}(x-x')J^c_\mu(x) \tag{2.21}$$

Taking the covariant divergence of both sides of this equation, we find

$$\begin{aligned}&L^b(x')\{\partial_\mu J^a_\mu(x)-f^{adc}A^d_\mu(x)J^c_\mu(x)\}\\&\quad+f^{adc}[L^b(x'),A^d_\mu(x)]J^c_\mu(x)\\&=-f^{abc}\partial_\mu\{\delta^{(D)}(x-x')J^c_\mu(x)\}\\&\quad+f^{adc}f^{cbe}\delta^{(D)}(x-x')A^d_\mu J^e_\mu.\end{aligned} \tag{2.22}$$

The commutator is easily calculated and the equation simplifies to

$$L^b(x')G^a(x)=-f^{abc}\delta^{(D)}(x-x')G^c(x) \tag{2.23}$$

where G^a is the (covariant) anomaly of the current J^a_μ. We can put this equation in the more symmetric form

$$L^a(x)G^b(x')-L^b(x')G^a(x)=2f^{abc}\delta^{(D)}(x-x')G^c(x) \tag{2.24}$$

This Equation differs from the Wess-Zumino condition through the extra factor of 2 on the right hand side. We may say that the covariant anomaly satisfies the Wess-Zumino condition modulo a term proportional to the anomaly itself: consistency of the covariant anomaly is again achieved when the anomaly vanishes [13].

3. Gauge-Invariant Effective Action and Covariant Current

The fermion effective action is not unambiguously defined until it is specified how the (Grassmann) functional integral is to be evaluated. The standard attitude (Sect. 4) is that the anomaly destroys its formal gauge-invariance, so that the consistent anomaly measured by the effect of L on the effective action is nonvanishing. In this section we wish to construct a gauge-invariant effective action. This is an alternative way of giving meaning to the ill-defined fermion integral. One may think that a gauge-invariant effective action means an anomaly-free theory. The consistent anomaly indeed vanishes. But then it is not the only anomaly in the theory. We shall show that the standard expression for the covariant anomaly can also be derived from this alternative effective action. Thus the covariant anomaly appears to be the fundamental one.

Formally,

$$W=\int(d\psi\, d\bar\psi)\exp\int d^Dx\,\bar\psi\mathscr{D}\psi \tag{3.1}$$

The point is that the integration over $\psi,\bar\psi$ needs to be defined properly. We think of it as an integration over components of $\psi,\bar\psi$ as expanded in some suitable bases. One can take any bases; the ones we choose here depend on the gauge-fields A. To be specific, we expand ψ in eigenfunctions of $\mathscr{D}^+\mathscr{D}$ and $\bar\psi$ in those of $\mathscr{D}\mathscr{D}^+$. It should be noted that $\mathscr{D}^+$ is not equal to $\mathscr{D}$ and does not commute with it. But $\mathscr{D}^+\mathscr{D}$ and $\mathscr{D}\mathscr{D}^+$ are hermitian and in fact positive semidefinite operators. The choice of their eigenfunctions in building the bases for expansions leads to a gauge invariant effective action, as we shall presently see.

We write

$$\psi(x)=\sum_n a_n\phi_n(x),\qquad \bar\psi(x)=\sum_n \bar b_n\varphi^+_n(x) \tag{3.2}$$

where

$$\mathscr{D}^+\mathscr{D}\phi_n=\lambda^2_n\phi_n,\qquad \mathscr{D}\mathscr{D}^+\varphi_n=\lambda^2_n\varphi_n \tag{3.3}$$

Here the eigenfunctions are taken to be discrete, which presumes an infrared regularization. Note that the eigenvalues of $\mathscr{D}^+\mathscr{D}$ are the same as those of

$\mathscr{D}\mathscr{D}^+$, while the eigenfunctions are related by

$$\mathscr{D}\phi_n=\lambda_n\varphi_n, \quad \mathscr{D}^+\varphi_n=\lambda_n\phi_n \tag{3.4}$$

if they are suitably normalized. The numbers λ_n^2 being real and nonnegative, λ_n too can be taken to be real and nonnegative. Note finally that the coefficients a_n, $\bar{b}_n$ are Grassmann variables.

We now write

$$\tilde{W}=(\prod_n \int \mathrm{d}a_n \int \mathrm{d}\bar{b}_n)\exp\int \mathrm{d}^D x\bar{\psi}\mathscr{D}\psi \tag{3.5}$$

This is certainly a reasonable interpretation of (3.1). Fujikawa [13] has used expressions similar to (3.5) but he has put in an extra factor $(\det\mathscr{D}/\det\mathscr{D}^+)^{1/2}$ – see (3.7) below. We observe that even without any extra factor, $\tilde{W}$ can be taken to define the functional integral.

To show that $\tilde{W}$ is gauge invariant, we evaluate it explicitly. Note that

$$\begin{aligned}\int \mathrm{d}^D x\bar{\psi}\mathscr{D}\psi &= \sum_{m,n}\bar{b}_m a_n \int \mathrm{d}^D x\varphi_m^+\mathscr{D}\phi_n\\ &=\sum_{m,n}\bar{b}_m a_n\lambda_n\int \mathrm{d}^D x\varphi_m^+\varphi_n\\ &=\sum_n \bar{b}_n a_n\lambda_n\end{aligned} \tag{3.6}$$

The Grassmann integrals in (3.5) can be done explicitly and one finds

$$\tilde{W}=\prod_n \lambda_n \tag{3.7}$$

This can also be written as [14] $(\det\mathscr{D}\mathscr{D}^+)^{1/2}$. Since under gauge transformations $\mathscr{D}^+\mathscr{D}$ and $\mathscr{D}\mathscr{D}^+$ change by similarity transformations (see 2.6), the eigenvalues λ_n^2 are gauge-invariant, hence $\tilde{W}$ is so too. It follows that the consistent anomaly $G^a_{\tilde{W}}$ is zero (see 2.16). There is no conflict with the Wess-Zumino condition, which is trivially satisfied.

The covariant anomaly will now be derived in this functional integral approach. Unlike Fujikawa [4], we construct the current explicitly and avoid calculating Jacobians for infinitesimal transformations. Our effective action $\tilde{W}$ is not the standard one, but any extra factors in the measure cancel out in the current:

$$\begin{aligned}&J^a_\mu(x)\\ &=\frac{(\prod_n\int \mathrm{d}a_n\int \mathrm{d}\bar{b}_n)\bar{\psi}(x)T^a\gamma_\mu(1-\gamma_{D+1})/2\psi(x)\exp\int\bar{\psi}\mathscr{D}\psi}{(\prod_n\int \mathrm{d}a_n\int \mathrm{d}\bar{b}_n)\exp\int\bar{\psi}\mathscr{D}\psi}\\ &=\sum_{l,m}\varphi_l^+(x)T^a\gamma_\mu(1-\gamma_{D+1})/2\phi_m(x)\\ &\cdot\frac{(\prod_n\int \mathrm{d}a_n\int \mathrm{d}\bar{b}_n)\bar{b}_l a_m\exp\sum_k\bar{b}_k a_k\lambda_k}{(\prod_n\int \mathrm{d}a_n\int \mathrm{d}\bar{b}_n)\exp\sum_k\bar{b}_k a_k\lambda_k}\\ &=\sum_l\frac{1}{\lambda_l}\varphi_l^+(x)T^a\gamma_\mu(1-\gamma_{D+1})/2\phi_l(x)\end{aligned} \tag{3.8}$$

This can be regularized by suitably suppressing the contributions from terms corresponding to large eigenvalues λ_l^2. Such regularization is gauge-invariant because each λ_l^2 is.

The covariant divergence of the vacuum current is

$$\begin{aligned}&\partial_\mu J^a_\mu - f^{abc}A^b_\mu J^c_\mu\\ &=\sum_l\frac{1}{\lambda_l}\{\varphi_l^+\overleftarrow{\partial\!\!\!/}T^a(1-\gamma_{D+1})/2\phi_l+\varphi_l^+\partial\!\!\!/ T^a(1-\gamma_{D+1})/2\phi_l\\ &\quad -\mathrm{i}\varphi_l^+ A\!\!\!/ T^a(1-\gamma_{D+1})/2\phi_l+\mathrm{i}\varphi_l^+T^a A\!\!\!/(1-\gamma_{D+1})/2\phi_l\\ &=\sum_l\frac{1}{\lambda_l}\{-(\mathscr{D}^+\varphi_l)^+T^a(1-\gamma_{D+1})/2\phi_l\\ &\quad+\varphi_l^+(1+\gamma_{D+1})/2\mathscr{D}\phi_l\}\\ &=\sum_l\{\varphi_l^+T^a(1+\gamma_{D+1})/2\varphi_l-\phi_l^+T^a(1-\gamma_{D+1})/2\phi_l\}\end{aligned} \tag{3.9}$$

Employing Fujikawa's regularization [4], we write this as

$$\begin{aligned}G^a&=\operatorname*{lt}_{M\to\infty}\operatorname{tr}\int\frac{\mathrm{d}^D k}{(2\pi)^D}\left\{T^a\frac{1+\gamma_{D+1}}{2}\mathrm{e}^{\mathrm{i}k\cdot x}\mathrm{e}^{-\frac{\mathscr{D}\mathscr{D}^+}{M^2}}\mathrm{e}^{-\mathrm{i}k\cdot x}\right.\\ &\quad\left.-T^a\frac{1-\gamma_{D+1}}{2}\mathrm{e}^{\mathrm{i}k\cdot x}\mathrm{e}^{-\frac{\mathscr{D}^+\mathscr{D}}{M^2}}\mathrm{e}^{-\mathrm{i}k\cdot x}\right\}\\ &=\frac{(-)^{[D/4]}}{(4\pi)^{D/2}(D/2)!}\varepsilon_{\mu_1\nu_1\cdot\mu_{D/2}\nu_{D/2}}\operatorname{tr}(T^aF_{\mu_1\nu_1}\cdot F_{\mu_{D/2}\nu_{D/2}})\end{aligned} \tag{3.10}$$

where $[D/4]$ is the smallest integer $\geqq D/4$. The M-limit and the spinor trace have been explicitly carried out in this result, which is finite and local.

4. Gauge-Variant Effective Action and Minimal Consistent Anomaly

The effective action $\ln\int(\mathrm{d}\psi\,\mathrm{d}\bar{\psi})\exp\int \mathrm{d}^D x(\bar{\psi}\mathscr{D}\psi)$ is formally gauge invariant. The effect of regularization depends very much on what regularization is used. In the functional integral approach, one may say that the effective action depends on the way in which the functional integral is evaluated. As we have seen in the previous section, it is in fact possible to preserve gauge invariance. But first we give a treatment of the effective action which is more in keeping with the perturbative approach of – say – Bardeen [1]. We do not explicitly consider the functional integration involved; instead, we construct the effective action indirectly by means of some formal relations.

So far we have avoided mention of the gauge coupling constant, which may be understood to be absorbed in the gauge fields. Now we introduce a parameter g and define

$$\mathscr{D}_g = \not{\partial} + \mathrm{i} g T^a \not{A}^a \frac{1-\gamma_{D+1}}{2} \tag{4.1}$$

When $g=1$, $\mathscr{D}_g$ reduces to $\mathscr{D}$. We also write

$$W_g = \int (\mathrm{d}\psi\, \mathrm{d}\bar{\psi}) \exp \int \mathrm{d}^D x (\bar{\psi} \mathscr{D}_g \psi) \tag{4.2}$$

Since W_g depends on A only through gA

$$\frac{\partial}{\partial g} \ln W_g = \int \mathrm{d}^D x A_\mu^a(x) \frac{\delta \ln W_g}{\delta g A_\mu^a(x)}, \tag{4.3}$$

so that

$$\ln W = \int_0^1 \mathrm{d}g \int \mathrm{d}^D x A_\mu^a(x) \frac{\delta \ln W_g}{\delta g A_\mu^a(x)} \tag{4.4}$$

Here we have dropped a term corresponding to the free theory. Making use of (2.10), we rewrite (4.4) as

$$\ln W = \mathrm{i} \int_0^1 \mathrm{d}g \int \mathrm{d}^D x A_\mu^a(x) J_\mu^{ag}(x), \tag{4.5}$$

where the superscript g indicates that this is the coupling constant to be used in the construction of the current. (4.5) is purely formal: neither the effective action nor the current is properly defined. But we regard it as *defining* the effective action in terms of a current, which therefore has to be defined first. The legitimacy of using a non-integrable current will be explained below (after (4.9)).

The current defined by the effective action $\ln W$ is (see (2.15))

$$\begin{aligned} J_{W\mu}^a(x) &= \int_0^1 \mathrm{d}g J_\mu^{ag}(x) + \int_0^1 \mathrm{d}g \int \mathrm{d}^D x' A_\nu^b(x') \frac{\delta J_\nu^{bg}(x')}{\delta A_\mu^a(x)} \\ &= J_\mu^a(x) - \int_0^1 \mathrm{d}g\, g \frac{\partial J_\mu^{ag}(x)}{\partial g} \\ &\quad + \int_0^1 \mathrm{d}g \int \mathrm{d}^D x' A_\nu^b(x') \frac{\delta J_\nu^{bg}(x')}{\delta A_\mu^a(x)} \end{aligned} \tag{4.6}$$

where we have integrated by parts in g. Now J_μ^g depends on g only through gA, hence

$$\frac{\partial J_\mu^{ag}(x)}{\partial g} = \int \mathrm{d}^D x' A_\nu^b(x') \frac{\delta J_\mu^{ag}(x)}{\delta g A_\nu^b(x')} \tag{4.7}$$

It follows that

$$J_{W\mu}^a(x) = J_\mu^a(x) + \int_0^1 \mathrm{d}g \int \mathrm{d}^D x' A_\nu^b(x') \cdot \left\{ \frac{\delta J_\nu^{bg}(x')}{\delta A_\mu^a(x)} - \frac{\delta J_\mu^{ag}(x)}{\delta A_\nu^b(x')} \right\} \tag{4.8}$$

Thus this current differs from the original current by a term involving the functional curl of the latter. If the original current is integrable, this curl vanishes and the two currents are identical. In general, however, the curl will be nonzero. The current $J_{W\mu}^a(x)$ will however be integrable, because it is derived from an effective action. An interesting observation that can be made at this stage is that

$$\int \mathrm{d}^D x A_\mu^a(x) J_{W\mu}^a(x) = \int \mathrm{d}^D x A_\mu^a(x) J_\mu^a(x) \tag{4.9}$$

which follows from the antisymmetry of the functional curl. This shows that the difference between the currents $J_{W\mu}^a(x)$ and $J_\mu^a(x)$ does not lead to any ambiguity in the effective action.

It is now necessary to be more specific about $J_\mu^a(x)$. We take it to be the covariant current, i.e., the current underlying the covariant anomaly. This current is not integrable, but we have agreed to treat (4.5) as defining the effective action in terms of an arbitrary current. (4.9) asserts that this does not lead to any inconsistency.

We start by considering the functional curl of the covariant current. Now

$$\begin{aligned} &\{\delta^{cb} \partial'_\nu - f^{ceb} A_\nu^e(x')\} \frac{\delta J_\nu^b(x')}{\delta A_\mu^a(x)} \\ &= \frac{\delta}{\delta A_\mu^a(x)} \{\delta^{cb} \partial'_\nu - f^{ceb} A_\nu^e(x')\} J_\nu^b(x') \\ &\quad + f^{cab} \delta^{(D)}(x-x') J_\mu^b(x) \end{aligned} \tag{4.10}$$

where the last term comes from commuting the functional derivative operator across the covariant derivative operator. Since

$$\{\delta^{cb} \partial'_\nu - f^{ceb} A_\nu^e(x')\} \frac{\delta J_\mu^a(x)}{\delta A_\nu^b(x')} = f^{cab} \delta^{(D)}(x-x') J_\mu^b(x) \tag{4.11}$$

it follows that

$$\{\delta^{cb} \partial'_\nu - f^{ceb} A_\nu^e(x')\} \cdot \left\{ \frac{\delta J_\nu^b(x')}{\delta A_\mu^a(x)} - \frac{\delta J_\mu^a(x)}{\delta A_\nu^b(x')} \right\} = \frac{\delta G^c(x')}{\delta A_\mu^a(x)} \tag{4.12}$$

Now G^c depends on the gauge fields only through the field-strength tensor $F_{\mu\nu}^a$, hence

$$\begin{aligned} \frac{\delta G^c(x')}{\delta A_\mu^a(x)} &= \int \mathrm{d}^D x'' \frac{\delta G^c(x')}{\delta F_{\rho\nu}^b(x'')} \frac{\delta F_{\rho\nu}^b(x'')}{\delta A_\mu^a(x)} \\ &= 2\{\delta^{ab} \partial_\nu - f^{aeb} A_\nu^e(x)\} \frac{\delta G^c(x')}{\delta F_{\mu\nu}^b(x)}, \end{aligned} \tag{4.13}$$

which follows from the relation (3.9) between F and A. It must be pointed out here that in defining derivatives with respect to the variables $F_{\mu\nu}^b$, which are not all independent, we have made the convention that the derivatives be antisymmetric in μ and ν. With this convention, we obtain from (3.10)

$$\frac{\delta G^c(x')}{\delta F^b_{\mu\nu}(x)} = \frac{(-)^{[D/4]}}{(4\pi)^{D/2}\left(\frac{D}{2}\right)!}\frac{D}{2}\,\delta^{(D)}(x-x') \cdot \varepsilon_{\mu\nu\mu_1\nu_1\ldots\mu_{\frac{D}{2}-1}\nu_{\frac{D}{2}-1}} \cdot \mathrm{Str}\left(T^c T^b F_{\mu_1\nu_1}\ldots F_{\mu_{\frac{D}{2}-1}\nu_{\frac{D}{2}-1}}\right) \tag{4.14}$$

where the letter S before the symbol for trace indicates that the factors in the trace are to be symmetrized. (4.13) involves the covariant divergence of (4.14) with respect to unprimed variables. A short calculation making use of the Bianchi identity

$$\partial_{[\lambda}F_{\mu\nu]} + i[A_{[\lambda}, F_{\mu\nu]}] = 0 \tag{4.15}$$

shows that the covariant divergence can equivalently be taken with respect to primed variables if the overall sign is changed:

$$\{\delta^{ab}\partial_\nu - f^{aeb}A^e_\nu(x)\}\frac{\delta G^c(x')}{\delta F^b_{\mu\nu}(x)} = -\{\delta^{cb}\partial'_\nu - f^{ceb}A^e_\nu(x')\}\frac{\delta G^b(x')}{\delta F^a_{\mu\nu}(x)} \tag{4.16}$$

It therefore follows from (4.12) that

$$\{\delta^{cb}\partial'_\nu - f^{ceb}A^e_\nu(x')\}\left\{\frac{\delta J^b_\nu(x')}{\delta A^a_\mu(x)} - \frac{\delta J^a_\mu(x)}{\delta A^b_\nu(x')} + \frac{2\delta G^b(x')}{\delta F^a_{\mu\nu}(x)}\right\} = 0 \tag{4.17}$$

One can actually make the stronger statement that the object within braces is itself zero. To see this, note first that the functional curl involves $\delta^{(D)}(x-x')$. One may convince one-self of this by trying to calculate the functional derivative of the regularized current of Sect. 3. Thus we may write (4.17) as

$$T^{ab}_{\mu\nu}(x)\{\delta^{cb}\partial'_\nu - f^{ceb}A^e_\nu(x')\}\,\delta^{(D)}(x-x') = 0 \tag{4.18}$$

where $T^{ab}_{\mu\nu}(x)$ does not involve any differential operators. We can conclude that this tensor vanishes, i.e.,

$$\frac{\delta J^b_\nu(x')}{\delta A^a_\mu(x)} - \frac{\delta J^a_\mu(x)}{\delta A^b_\nu(x')} = -\frac{(-)^{[D/4]}D}{(4\pi)^{D/2}\left(\frac{D}{2}\right)!}\,\delta^{(D)}(x-x') \cdot \varepsilon_{\mu\nu\mu_1\nu_1\ldots\mu_{\frac{D}{2}-1}\nu_{\frac{D}{2}-1}}\,\mathrm{Str}\left(T^a T^b F_{\mu_1\nu_1}\ldots F_{\mu_{\frac{D}{2}-1}\nu_{\frac{D}{2}-1}}\right) \tag{4.19}$$

This gives

$$J^a_{W\mu}(x) = J^a_\mu(x) - \frac{(-)^{[D/4]}D}{(4\pi)^{D/2}\left(\frac{D}{2}\right)!}\,\varepsilon_{\mu\nu\mu_1\nu_1\ldots\mu_{\frac{D}{2}-1}\nu_{\frac{D}{2}-1}} \cdot \int_0^1 dg\, g\,\mathrm{Str}\left(T^a A_\nu F^g_{\mu_1\nu_1}\ldots F^g_{\mu_{\frac{D}{2}-1}\nu_{\frac{D}{2}-1}}\right) \tag{4.20}$$

The extra factor of g appears because on introducing this parameter into (4.19), the variables A on the left hand side get altered to gA.

(4.20) is the connection between the consistent and covariant currents suggested by the algebraic approach [3]. We have arrived at the same result dynamically.

The consistent anomaly is just the covariant divergence of (4.20). The first term is the covariant anomaly G^a. The second term can be simplified by applying the Bianchi identity. It is straightforward to derive the expression

$$G^a_W = \int_0^1 dg\, G^{ag} + \frac{(-)^{[D/4]}iD}{(4\pi)^{D/2}\left(\frac{D}{2}\right)!}\,\varepsilon_{\mu_1\nu_1\ldots\mu_{\frac{D}{2}}\nu_{\frac{D}{2}}} \cdot \int_0^1 dg\, g(g-1)\,\mathrm{Str}\left\{[T^a, A_{\mu_1}]A_{\nu_1}F^g_{\mu_2\nu_2}\ldots F^g_{\mu_{\frac{D}{2}}\nu_{\frac{D}{2}}}\right\} \tag{4.21}$$

This expression in general has a more complicated structure than G^a. In the special case when the gauge group is abelian, simplifications occur. The second term vanishes; moreover, G^{ag}, with its $D/2$ factors of F, involves g homogeneously to the $(D/2)^{\text{th}}$ power. Consequently

$$G_W(\text{abelian}) = \frac{2}{D+2}\,G(\text{abelian}) \tag{4.22}$$

i.e., the two anomalies are related by a Bose-symmetrization factor. Note that this does not mean that the two currents are proportional in this case: if they were, the covariant current would be integrable, so that its functional curl would vanish and the two currents would have to be *equal.*

We conclude this section with the remark that although we have calculated the consistent anomaly directly and explicitly, we have not obtained the extra terms which were obtained in [1, 6, 8–11] and subsequently removed. Our regularization procedure is such that it yields only the minimal part of the consistent anomaly.

5. Different Anomalies in V-A Theories

In this section we consider fermionic theories where both the left- and the right-handed currents are coupled to external gauge fields. Thus we take the Lagrangian density

$$\mathscr{L} = \bar\psi(\partial\!\!\!/ + \mathrm{i}\,V\!\!\!\!/ + \mathrm{i}\,A\!\!\!/\,\gamma_{D+1})\psi \tag{5.1}$$

If $V = -A$, we get back the previous form. However if we allow V and A to be independent, we have a larger invariance: the gauge group is $G \otimes G$ if the fermion fields transform under a representation of G

and generators T^a of the Lie algebra of G are understood to be contracted in the matrix notation.

Currents can as usual be defined in different ways. First we consider covariant currents, which, emphasized earlier, can be obtained directly by regularizing the propagators. The left-handed current is

$$J^a_{\mu-} = \operatorname{tr} T^a \gamma_\mu \frac{(1-\gamma_{D+1})}{2} (\not\partial + \mathrm{i}\not V + \mathrm{i}\not A \gamma_{D+1})^{-1} \tag{5.2}$$

which may be formally rewritten, because of the presence of $(1-\gamma_{D+1})/2$ as

$$J^a_{\mu-} = \operatorname{tr} T^a \gamma_\mu \frac{1-\gamma_{D+1}}{2} \left\{ \not\partial + \mathrm{i}(\not V - \not A) \frac{1-\gamma_{D+1}}{2} \right\}^{-1} \tag{5.3}$$

Similarly,

$$J^a_{\mu+} = \operatorname{tr} T^a \gamma_\mu \frac{1+\gamma_{D+1}}{2} \left\{ \not\partial + \mathrm{i}(\not V + \not A) \frac{1+\gamma_{D+1}}{2} \right\}^{-1} \tag{5.4}$$

$J^a_{\mu-}$ may be regularized in a left-invariant way and $J^a_{\mu+}$ right-invariantly; if the appropriate covariant divergences are calculated as in Sect. 3, the covariant anomalies are obtained:

$$\begin{aligned} G^a_\pm &= \partial_\mu J^a_{\mu\pm} - f^{abc}(V^b_\mu \pm A^b_\mu) J^c_{\mu\pm} \\ &= \mp \frac{(-)^{[D/4]}}{(4\pi)^{D/2}\left(\frac{D}{2}\right)!} \varepsilon_{\mu_1 \nu_1 \ldots \mu_{\frac{D}{2}} \nu_{\frac{D}{2}}} \operatorname{tr}\left(T^a F^\pm_{\mu_1\nu_1} \ldots F^\pm_{\mu_{\frac{D}{2}}\nu_{\frac{D}{2}}}\right) \end{aligned} \tag{5.5}$$

where

$$F^\pm_{\mu\nu} = -i[\partial_\mu + \mathrm{i}V_\mu \pm \mathrm{i}A_\mu, \partial_\nu + \mathrm{i}V_\nu \pm \mathrm{i}A_\nu] \tag{5.6}$$

These results can be written down by inspection of the results of Sect. 3 and vice versa. The situations in the two cases are similar, except that in Sect. 3 only the left-handed current is anomalous and the right-handed one free, whereas here both currents are anomalous.

Next we come to effective actions and consistent anomalies. It turns out that there are *two* natural ways of regularizing the effective action. These give rise to different consistent anomalies.

I. In our treatment of the covariant anomaly, we found that the left- and right-handed currents can be handled separately, and their anomalies depend on different combinations of the gauge fields. We can preserve this structure for the effective action, which then splits up into two parts depending on different combinations of the gauge fields. The analogue of (4.5) in this scheme is

$$\ln W = \mathrm{i}\int_0^1 \mathrm{d}g \int \mathrm{d}^D x (V^a_\mu + A^a_\mu) J^{ag}_{\mu+} + \mathrm{i}\int_0^1 \mathrm{d}g \int \mathrm{d}^D x (V^a_\mu - A^a_\mu) J^{ag}_{\mu-} \tag{5.7}$$

where the covariant currents are taken on the right hand side. The left and right consistent anomalies can be written down from (4.21):

$$\begin{aligned} G^a_{W\pm} &= \int_0^1 \mathrm{d}g\, G^{ag}_\pm \mp \frac{(-)^{[D/4]} iD}{(4\pi)^{D/2}\left(\frac{D}{2}\right)!} \varepsilon_{\mu_1\nu_1\ldots\mu_{\frac{D}{2}}\nu_{\frac{D}{2}}} \\ &\cdot \int_0^1 \mathrm{d}g\, g(g-1) \operatorname{Str}\left\{[T^a, V_{\mu_1} \pm A_{\mu_1}]\right. \\ &\left. \cdot (V_{\nu_1} \pm A_{\nu_1}) F^{\pm g}_{\mu_2\nu_2} \ldots F^{\pm g}_{\mu_{\frac{D}{2}}\nu_{\frac{D}{2}}}\right\} \end{aligned} \tag{5.8}$$

II. Instead of thinking along chiral lines, one may think in terms of vector and axial currents. If one calls the effective action $\ln \bar{W}$ in this scheme, and introduces an axial coupling constant g, one has

$$\begin{aligned} \frac{\partial}{\partial g} \ln \bar{W}g &= \int \mathrm{d}^D x A^a_\mu(x) \frac{\delta \ln \bar{W}g}{\delta g A^a_\mu(x)} \\ &= i\int \mathrm{d}^D x A^a_\mu(x) \{J^{ag}_{\mu+} - J^{ag}_{\mu-}\} \end{aligned} \tag{5.9}$$

It follows that

$$\ln \bar{W} = \ln \bar{W}_0 + i\int_0^1 \mathrm{d}g \int \mathrm{d}^D x A^a_\mu \{J^{ag}_{\mu+} - J^{ag}_{\mu-}\} \tag{5.10}$$

In Sect. 4 we threw away the analogue of $\bar{W}_0$ because it corresponded to a free theory. In the present context, it is a theory with pure vector coupling. This part of the effective action therefore contributes to the currents but not to the anomaly.

The currents can be found from (5.10) by the methods of Sect. 4.

$$\begin{aligned} &\frac{1}{\mathrm{i}} \frac{\delta}{\delta V^a_\mu(x)} \ln \bar{W} \\ &= \frac{1}{\mathrm{i}} \frac{\delta}{\delta V^a_\mu(x)} \ln \bar{W}_0 + \int_0^1 \mathrm{d}g \int \mathrm{d}^D x' A^b_\nu(x') \\ &\quad \cdot \left\{ \frac{\delta J^{bg}_{\nu+}(x')}{\delta V^a_\mu(x)} - \frac{\delta J^{bg}_{\nu-}(x')}{\delta V^a_\mu(x)} \right\} \\ &= \frac{1}{\mathrm{i}} \frac{\delta}{\delta V^a_\mu(x)} \ln \bar{W}_0 + \int_0^1 \mathrm{d}g \int \mathrm{d}^D x' A^b_\nu(x') \\ &\quad \cdot \left\{ \frac{\delta J^{bg}_{\nu+}(x')}{\delta g A^a_\mu(x)} + \frac{\delta J^{bg}_{\nu-}(x')}{\delta g A^a_\mu(x)} \right\} \\ &= \frac{1}{\mathrm{i}} \frac{\delta}{\delta V^a_\mu(x)} \ln \bar{W}_0 + \int_0^1 \mathrm{d}g \int \mathrm{d}^D x' A^b_\nu(x') \\ &\quad \cdot \left\{ \frac{\delta J^{ag}_{\mu+}(x)}{\delta g A^b_\nu(x')} + \frac{\delta J^{ag}_{\mu-}(x)}{\delta g A^b_\nu(x')} \right\} - 2\int_0^1 \mathrm{d}g \int \mathrm{d}^D x' A^b_\nu(x') \\ &\quad \cdot \left\{ \frac{\delta G^a_+(x)}{\delta F^{+b}_{\mu\nu}(x')} - \frac{\delta G^a_-(x)}{\delta F^{-b}_{\mu\nu}(x')} \right\}^g \end{aligned}$$

$$=\frac{1}{\mathrm{i}}\frac{\delta}{\delta V_\mu^a(x)}\ln\bar{W}_0+\{J^a_{\mu+}(x)+J^a_{\mu-}(x)\}$$
$$+\frac{(-)^{[D/4]}D}{(4\pi)^{D/2}\frac{D}{2}!}\varepsilon_{\mu\nu\mu_1\nu_1\ldots\mu_{\frac{D}{2}-1}\nu_{\frac{D}{2}-1}}\int_0^1 \mathrm{d}g\,\mathrm{Str}$$
$$\cdot\{T^aA_\nu(F^{+g}_{\mu_1\nu_1}\ldots F^{+g}_{\mu_{\frac{D}{2}-1}\nu_{\frac{D}{2}-1}}+F^{-g}_{\mu_1\nu_1}\ldots F^{-g}_{\mu_{\frac{D}{2}-1}\nu_{\frac{D}{2}-1}})\} \tag{5.11}$$

$$\frac{1}{\mathrm{i}}\frac{\delta}{\delta A_\mu^a(x)}\ln\bar{W}$$
$$=J^a_{\mu+}(x)-J^a_{\mu-}(x)+\int_0^1\mathrm{d}g\int\mathrm{d}^Dx'A^b_\nu(x')$$
$$\cdot\left\{\frac{\delta J^{bg}_{\nu+}(x)}{\delta A^a_\mu(x)}-\frac{\delta J^{ag}_{\mu+}(x)}{\delta A^b_\nu(x')}-\frac{\delta J^{bg}_{\nu-}(x')}{\delta A^a_\mu(x)}+\frac{\delta J^{ag}_{\mu-}(x)}{\delta A^b_\nu(x')}\right\}$$
$$=J^a_{\mu+}(x)-J^a_{\mu-}(x)-2\int_0^1\mathrm{d}g\,g\int\mathrm{d}^Dx'A^b_\nu(x')$$
$$\cdot\left\{\frac{\delta G^a_+(x)}{\delta F^{+b}_{\mu\nu}(x')}+\frac{\delta G^a_-(x)}{\delta F^{-b}_{\mu\nu}(x')}\right\}^g$$
$$=J^a_{\mu+}(x)-J^a_{\mu-}(x)$$
$$+\frac{(-)^{[D/4]}D}{(4\pi)^{D/2}\frac{D}{2}!}\varepsilon_{\mu\nu\mu_1\nu_1\ldots\mu_{\frac{D}{2}-1}\nu_{\frac{D}{2}-1}}\int_0^1\mathrm{d}g\,g\,\mathrm{Str}$$
$$\cdot\{T^aA_\nu(F^{+g}_{\mu_1\nu_1}\ldots F^{+g}_{\mu_{\frac{D}{2}-1}\nu_{\frac{D}{2}-1}}-F^{-g}_{\mu_1\nu_1}\ldots F^{-g}_{\mu_{\frac{D}{2}-1}\nu_{\frac{D}{2}-1}})\}. \tag{5.12}$$

As in Sect. 4 the V and A-anomalies can now be calculated, appropriate use being made of the Bianchi identity. One finds

$$G^a_{WV}\equiv\partial_\mu\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta V^a_\mu}\ln\bar{W}\right)-f^{abc}V^a_\mu\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta V^c_\mu}\ln\bar{W}\right)$$
$$-f^{abc}A^b_\mu\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta A^c_\mu}\ln\bar{W}\right)=0, \tag{5.13}$$

$$G^a_{WA}\equiv\partial_\mu\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta A^a_\mu}\ln\bar{W}\right)-f^{abc}V^b_\mu\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta A^c_\mu}\ln\bar{W}\right)$$
$$-f^{abc}A^b_\mu\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta V^c_\mu}\ln\bar{W}\right)$$
$$=\int_0^1\mathrm{d}g(G^{ag}_+-G^{ag}_-)-\frac{(-)^{[D/4]}iD}{(4\pi)^{D/2}\frac{D}{2}!}\varepsilon_{\mu\nu\mu_1\nu_1\ldots\mu_{\frac{D}{2}-1}\nu_{\frac{D}{2}-1}}$$
$$\cdot\int_0^1\mathrm{d}g(g^2-1)\,\mathrm{Str}\,\{[T^a,A_\mu]A_\nu(F^{+g}_{\mu_1\nu_1}\ldots F^{+g}_{\mu_{\frac{D}{2}-1}\nu_{\frac{D}{2}-1}}$$
$$+F^{-g}_{\mu_1\nu_1}\ldots F^{-g}_{\mu_{\frac{D}{2}-1}\nu_{\frac{D}{2}-1}})\}. \tag{5.14}$$

Thus the vector currents are conserved even in the presence of the gauge field A. Note that for $D=4$, (5.14) reduces to the well-known expression [1, 6, 8, 13]

$$\frac{1}{4\pi^2}\varepsilon_{\mu\nu\rho\sigma}\mathrm{tr}\left[T^a\left\{\frac{1}{4}F^V_{\mu\nu}F^V_{\rho\sigma}+\frac{1}{12}F^A_{\mu\nu}F^A_{\rho\sigma}\right.\right.$$
$$-\frac{2i}{3}(F^V_{\mu\nu}A_\rho A_\sigma+A_\mu F^V_{\nu\rho}A_\sigma+A_\mu A_\nu F^V_{\rho\sigma})$$
$$\left.\left.-\frac{8}{3}A_\mu A_\nu A_\rho A_\sigma\right\}\right], \tag{5.15}$$

where

$$F^V_{\mu\nu}=\tfrac{1}{2}(F^+_{\mu\nu}+F^-_{\mu\nu})$$
$$F^A_{\mu\nu}=\tfrac{1}{2}(F^+_{\mu\nu}-F^-_{\mu\nu}). \tag{5.16}$$

6. Discussion

To summarize, the covariant anomaly is obtained by regularizing the current in Fujikawa's [4] gauge invariant way. A gauge invariant effective action $(\det\mathscr{D}\mathscr{D}^+)^{\frac{1}{2}}$ is set up in this connection. A gauge-variant effective action is thereafter constructed in terms of the covariant current. It is shown that the current derived from this effective action differs from the covariant current by a local piece which can in fact be extracted from the covariant anomaly. This connection of the two currents is reminiscent of the work of Bardeen and Zumino [3]. Their approach was algebraic: they showed that the consistent current is not covariant and proceeded to add a local piece to it to make it so. We on the other hand explicitly obtain the connection from our effective action. Calculation of the covariant divergence of the consistent current finally gives the consistent anomaly. The expression obtained is the minimal one, i.e., free from the normal party terms which have been obtained by other authors and then removed by absorption in the action as counterterms. The key to the avoidance of these extra terms is the use of a good regularization. We have checked that if the effective action is constructed from a current which is *not* gauge-invariantly regularized, extra terms appear: it is only when covariance of the current is insisted upon that the essential or minimal form is directly obtained.

Finally, theories having separate left- and right-handed gauge invariances are considered. Covariant and consistent currents can be defined separately for the left and right sectors. However a new possibility is there. The effective action can be made vector-gauge-invariant, so that in the limit when the external fields coupling to the axial currents vanish, a regular vector theory emerges. All the anomalies are exhibited.

What can be learnt from all this? The expressions for the anomalies in arbitrary dimensions are not new, except in the case of the *V-A* theories with vector gauge invariance preserved. We feel that our method of calculation will provide some insight into the difference and relations between the different anomalies. Moreover, we have solved the mystery of the normal parity terms that have appeared in all fully dynamical calculations of the consistent anomaly heretofore: they can be avoided by taking a chiral-gauge-invariant regularization. The gauge-variant as well as the gauge-invariant effective actions presented by us are new. While the former is likely to be useful in perturbative calculations, the latter will at least suggest that it is a matter of convention to regard the effective action in an anomalous theory to be gauge-variant.

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