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Path Integral Quantization of Field Theories with Second-Class Constraints*,†

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Faddeev's Hamiltonian path integral method for singular Lagrangians is generalized to the case when second-class constraints appear in the theory. The general formalism is then applied to several problems: quantization of the massive Yang–Mills field theory. light-cone quantization of the self-interacting scalar field theory, and quantization of a local field theory of magnetic monopoles.

INTRODUCTION

The first systematic study of mechanical systems, including field theories, with constraints, was done by Dirac [1–3]. He showed that the algebra of Poisson brackets determines a division of constraints into two classes: the so-called first-class constraints and second-class ones. The first-class constraints are those that have zero Poisson brackets with all other constraints in the subspace of phase space in which constraints hold; constraints which are not first class are by definition second class. Dirac also showed how to redefine the Poisson brackets in such a way that all new redefined brackets (so-called Dirac brackets) of second-class constraints are zero. As we shall see, this is indeed necessary if transition to quantum theory is to be made.

The quantization of field theories with constraints based on Dirac's formalism and using the method of functional integration was performed by Faddeev [4]. Faddeev restricted his discussion to the case when only first-class constraints are present. We shall see that there are several interesting examples of field theories which contain second-class constraints, so that a generalization of Faddeev's method to these cases is warranted. The development of this generalization and its applications are the main concern of this paper.

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The paper will be organized as follows. In Section 1, we shall review Dirac's method. In Section 2, we shall give a short description of Faddeev's method. In Section 3, a generalization of Faddeev's method to the case when second-class constraints are present will be formulated. The next three sections will contain various applications of the general formalism developed in Section 3. Section 4 will be devoted to the quantization of the massive Yang–Mills field theory. In Section 5, the light-cone quantization of the self-interacting scalar field theory will be performed using our method. In Section 6, we shall quantize a local field theory of magnetic monopoles.

1. Dirac's Theory of Systems with Constraints

a. *Singular Lagrangians*

Given a mechanical system (of N degrees of freedom) with a Lagrangian L,

$$L = L(q, \dot{q}), \tag{1}$$

one defines the conjugate momenta by

$$p_n = \partial L/\partial \dot{q}_n \qquad (n = 1,..., N). \tag{2}$$

We shall dwell on the case when the expressions $\partial L/\partial \dot{q}_n$ are not independent functions of $\dot{q}_n$. Eliminating the $\dot{q}$'s one obtains a certain number of independent constraints

$$\phi_m(q, p) = 0 \qquad (m = 1, 2,..., k). \tag{3}$$

Thus, some of the p's are not independent. Solving (3) one writes

$$p_\alpha = \psi_\alpha(q, p_i) \qquad (\alpha = 1, 2,..., k), \tag{4}$$

where the p_i's ($i = 1,..., r$, $r + k = N$) are independent.

Because we are dealing with a singular Lagrangian, it is impossible to solve Eq. (2) for *all* the $\dot{q}$'s. However, if we take some of the $\dot{q}$'s to be independent (and undetermined) [we shall take those to be $\dot{q}_\alpha$ ($\alpha = 1,..., k$)], we can use Eq. (2) to solve for the remaining $\dot{q}$'s:

$$\dot{q}_i = \zeta_i(q, p_i, \dot{q}_\alpha) \qquad (i = 1, 2,..., r;\ r = N - k). \tag{5}$$

Equations (4) and (5) together have the same content as Eq. (2).

b. *Equations of Motion*

Define the Hamiltonian by

$$\bar{H} = p_n \dot{q}_n - L \equiv \tilde{W}(q, p_i, \dot{q}_\alpha). \tag{6}$$

In view of Eqs. (4) and (5) the Hamiltonian defined by (6) can be considered in general as a function of $q, p_i, \dot{q}_\alpha$. However, because of the nature of the Legendre transformation of Eq. (6) and in view of Eq. (2), it does not depend on $\dot{q}_\alpha$, which can be checked directly

$$\frac{\partial \tilde{W}}{\partial \dot{q}_\alpha} = p_\alpha + p_i \frac{\partial \zeta_i}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial \dot{q}_i} \frac{\partial \zeta_i}{\partial \dot{q}_\alpha} = 0.$$

Hamilton's principle

$$\delta \int L \, dt = 0 \tag{7}$$

can be written as

$$\delta \int (p_n \dot{q}_n - \tilde{W}(q, p_i)) \, dt = 0 \tag{8}$$

with the constraints (3)

$$\phi_m(q, p) = 0.$$

Using the Lagrange multiplier method, one is led to the equations of motion

$$\begin{aligned} \dot{p}_n &= -\frac{\partial \tilde{W}}{\partial q_n} - v_m \frac{\partial \phi_m}{\partial q_n} \\ \dot{q}_n &= \frac{\partial \tilde{W}}{\partial p_n} + v_m \frac{\partial \phi_m}{\partial q_n}, \end{aligned} \tag{9}$$

where v_m are the multipliers, which are arbitrary at this stage. These equations, together with the constraints (3) form a complete set of equations of motion.

Before proceeding with the development of the formalism, we shall introduce several important definitions. First, let us define the phase space Γ as a set whose elements are ordered $2N$-tuples $(q_1, \ldots, q_N, p_1, \ldots, p_N)$. Then, introduce the submanifold $\overline{M}$ in Γ, which by definition is the subset of Γ for which Eqs. (3) hold. Note that Eqs. (9) hold only in $\overline{M}$, which is clear from the way they were derived.

The Poisson bracket of two functions f and g of the q's and p's is defined by

$$\{f, g\} = \sum_{n=1}^{N} \left(\frac{\partial f}{\partial q_n} \frac{\partial g}{\partial p_n} - \frac{\partial f}{\partial p_n} \frac{\partial g}{\partial q_n} \right).$$

It is convenient to introduce the "total" Hamiltonian H in the following way:

$$H = \tilde{W} + v_m \phi_m. \tag{10}$$

It is then easy to see that Eqs. (9) can be transcribed into the form

$$\begin{aligned} \dot{p}_n |_{\overline{M}} &= \{p_n, H\}|_{\overline{M}} \\ \dot{q}_n |_{\overline{M}} &= \{q_n, H\}|_{\overline{M}}, \end{aligned} \tag{11}$$

where our notation is intended to emphasize that those equations hold in $\overline{M}$. For a function g of p_n and q_n we find the equation of motion

$$\dot{g}\,|_{\overline{M}} = \{g, H\}|_{\overline{M}}\,. \tag{12}$$

Hence, H as given by (10) is the generator of time translation.

c. *First-Class and Second-Class Constraints*

The constraints ϕ_m must remain zero at all times, which implies

$$\dot{\phi}_m\,|_{\overline{M}} = [\{\phi_m\,, H\,|_{\overline{M}}\} + v_n\{\phi_m\,, \phi_n\}]|_{\overline{M}}\,. \tag{13}$$

Excluding the case when those equations are contradictory either among themselves or with Eq. (3) as uninteresting, those equations (a) may be a trivial identity, (b) may be independent of the v's, or (c) may involve some of the v's.

In case they are of type (b), they represent new constraints (called secondary constraints) and may be written in the form

$$\rho_i(q, p) = 0. \tag{14}$$

Obviously, we can continue this process of generating secondary constraints until we arrive at the point when no more independent equations of type (b) are produced. After eliminating as many v's as possible from (c) type equations, we can use the remaining equations to solve for some (or all) of the v's.

Let us denote by M the subspace of phase space in which all constraints hold (i.e., both primary and secondary ones). We shall assume the irreducibility of all constraints with respect to M, i.e., a function of q's and p's vanishing in M will be expressible as a linear function of the ϕ's and the ρ's with functions of q's and p's as coefficients. We thus have in particular

$$H = H\,|_M\,(q, p) + v_l(q, p)\;\psi_l(q, p), \tag{15}$$

where we have denoted by a common symbol $\psi_l(q, p)$ all the constraints, i.e., $(\psi_l) = ((\phi_m), (\rho_i))$.

By definition, a *first-class* constraint φ^α (secondary or primary) satisfies

$$\{\varphi^\alpha, \psi_l\}|_M = 0 \tag{16}$$

for all ψ_l, and thus in view of our irreducibility hypothesis

$$\{\varphi^a, \psi_l\} = \lambda_l^{ak}\psi_k\,. \tag{17}$$

We call a constraint θ^α *second class* if it is not first class. Performing suitable linear transformations on the constraints, i.e., choosing new constraints which are linear functions of the old ones with functions of q's and p's as coefficients, let us bring as many ψ's as possible into the first class. We then claim that the following theorem holds:

THEOREM.

$$[\det \|\{\theta_a, \theta_b\}\|]|_M \neq 0, \tag{18}$$

where we have denoted the remaining constraints (all second-class) by θ_a.

To prove (18) assume the contrary, i.e.,

$$[\det \|\{\theta_a, \theta_b\}\|]|_M = 0.$$

Then there exists a set of functions λ_a, not all equal to zero, such that

$$\lambda_a[\{\theta_a, \theta_b\}]|_M = 0 \qquad \text{for all } b,$$

and thus

$$\{\lambda_a\theta_a, \theta_b\}|_M = 0,$$

so $\lambda_a\theta_a$ is first class, contrary to the assumption that we have put as many constraints as possible into the first zlass. This constitutes the proof of the theorem.

COROLLARY 1. *Equation* (18) *implies that the number of second-class constraints for a mechanical system is even, since $\{\theta_a, \theta_b\}$ is an antisymmetric matrix.*

COROLLARY 2. *All those v's in* (15) *which multiply second-class constraints (let us call them $v_b^{(\theta)}$) are determined in M.*

Indeed, we have a set of consistency conditions

$$\dot{\theta}_a|_M = \{\theta_a, H\}|_M = \{\theta_a, H|_M\}|_M + v_b^{(\theta)}|_M \{\theta_a, \theta_b\}|_M = 0 \tag{19}$$

and we can solve for $v_b^{(\theta)}|_M$ in view of (18).

d. *Dirac Brackets and Quantization*

A naive transition to quantum theory would consist in imposing the constraints as conditions on the quantum state vectors and replacing standard Poisson brackets by "$-i$" times commutators. But then if

$$\psi_1 | a\rangle = 0, \qquad \psi_2 | a\rangle = 0,$$

we find

$$[\psi_2, \psi_1] | a\rangle = 0,$$

which corresponds to a classical equation

$$\{\psi_2, \psi_1\}|_M = 0.$$

Thus, for the naive passage to quantum theory to be possible, all constraints must be first class. In case a mechanical system has second-class constraints, the remedy consists in redefining the Poisson brackets in a suitable manner:

$$\{\xi, \eta\}^* = \{\xi, \eta\} - \{\xi, \theta_a\} c_{ab}\{\theta_b, \eta\}, \tag{20}$$

where

$$c_{ab}\{\theta_b, \theta_c\} = \delta_{ac}. \tag{21}$$

$\{\xi, \eta\}^*$ is the new bracket, while the brackets on the right-hand side are standard Poisson brackets. It can be shown that the new brackets have all the standard properties of Poisson brackets.

As a consequence of the definitions (20) and (21), we find

$$\{\xi, \theta_a\}^* = \{\xi, \theta_a\} - \{\xi, \theta_b\} c_{bc}\{\theta_c, \theta_a\} = \{\xi, \theta_a\} - \{\xi, \theta_b\} \delta_{ba} = 0. \tag{22}$$

The passage to quantum theory can now be made by replacing the new brackets by "$-i$" times commutators. Then, in quantum theory, we can take $\theta_a = 0$ to hold as operator equations without any contradiction, since in view of (22) $[\theta_a, \xi] = 0$ for any operator ξ. The consistency condition (19) implies

$$\{g, H\}^*|_M = \{g, H\}|_M = \dot{g}|_M,$$

which means that the new bracket may be used to give the Hamiltonian equations of motion.

The generalization of this formalism to field theory, i.e., a mechanical system with a continuously infinite number of degrees of freedom, presents no difficulty.

2. The Feyman Path Integral for Singular Lagrangians with First-Class Constraints Only [4] (Faddeev's Method)

a. *Introduction*

As discussed in Section 1, given a certain Lagrangian, it can happen that the equations

$$p_n = \frac{\partial L(q, \dot{q})}{\partial \dot{q}_n} \tag{23}$$

cannot be solved for all of the $\dot{q}$'s. As a result (direct or indirect), the q's and the p's are constrained:

$$\varphi^a(q, p) = 0 \qquad a = 1, 2, \ldots, m. \tag{24}$$

The constraints (24) are either primary or secondary; in this section we shall limit ourselves to the case when there are no second-class ones among them. Thus,

$$\{\varphi^a, \varphi^b\} = c_c^{ab} \varphi^c. \tag{25}$$

In view of the discussion in Section 1, we must have

$$\{H, \varphi^a\} = c_b{}^a \varphi^b, \tag{26}$$

where we have changed the notation somewhat: H now stands for $H|_M$. Note that the hypersurface M in the phase space Γ is of dimension $2N - m$.

b. *Observables and Gauge Conditions*

Only those functions on M whose equations of motion contain no arbitrariness are observable quantities. The equation of motion of a quantity f is

$$\dot{f}|_M = \{f, H\}|_M + v_a|_M \{f, \varphi^a\}|_M. \tag{27}$$

This will be unique in M if

$$\{f, \varphi^a\}|_M = 0, \tag{28}$$

or equivalently,

$$\{f, \varphi^a\} = d_b{}^a \varphi^b. \tag{29}$$

The function f occurring in (27), (28), and (29) is an arbitrary continuation in Γ of a function defined on M. Since the constraints (24) are irreducible by assumption, any two such continuations will differ by a linear combination of the constraints and Eqs. (28) and (29) are independent of the continuation. Equations (28) can be viewed as a set of m first-order differential equations on M, with Eqs. (25) serving as integrability conditions. To see that, let us write Eq. (28) in terms of a noncanonical system of variables $(\varphi^a, \eta^b, q_i{}^*, p_i{}^*)$. Using (25) we obtain

$$\frac{\partial f(x)}{\partial x_l} g_a{}^l(x) = 0 \tag{30}$$

where $x = (\eta, q^*, p^*)$ and

$$g_a{}^l(x) = \{x^l, \varphi^a\}|_M. \tag{31}$$

The term containing $\partial f/\partial \varphi_a$ vanishes on account of (25). It is in this sense that Eqs. (25) serve as integrability conditions for Eqs. (28); if (25) did not hold, we would obtain a set of rather nonstandard differential equations with the unmanageable first term:

$$\left.\frac{\partial f}{\partial \varphi^b}\right|_{\varphi^c=0} \{\varphi^b, \varphi^a\}|_M + \frac{\partial f}{\partial x_l} g_a{}^l(x) = 0. \tag{32}$$

Since f satisfies a set of m first-order linear differential equations, it is completely determined by its values in the submanifold of the initial conditions for (28) (or (30)). This submanifold is of dimension $(2N - m) - m = 2(N - m)$. We can choose this submanifold to be the surface $\Gamma^*(\Gamma^* \subset M)$, defined by the equations

$$\chi_a(q, p) = 0 \qquad a = 1,..., m. \tag{33}$$

It is essential for later developments to assume that

$$\{\chi_a, \chi_b\} = 0. \tag{34}$$

In order to achieve a canonical description in Γ^*, it is necessary to require, as will be seen below (see Eq. (36))

$$\det \|\{\chi_a, \varphi^b\}\| \neq 0. \tag{35}$$

c. *Independent Canonical Variables*

If (34) holds, we can perform a canonical transformation in Γ and make a transition to new variables in which

$$\chi_a(q, p) = p_a \, .$$

In these new variables, (35) becomes

$$\det \left\| \frac{\partial \varphi^a}{\partial q_b} \right\| \neq 0, \tag{36}$$

where q^a are the coordinates conjugate to p_a. Thus Eqs. (24) can be solved for q^a. Hence, Γ^* is defined in Γ by

$$p_a = 0, \qquad q^a = \bar{q}^a(q^*, p^*),$$

where

$$\varphi^b(\bar{q}^a, 0, q^*, p^*) = 0$$

and q^* and p^* are the remaining canonical variables which act as independent variables on Γ^*.

One can show that [4]

$$\{f, g\}|_M = \frac{\partial f^*}{\partial q_i^*} \frac{\partial g^*}{\partial p_i^*} - \frac{\partial f^*}{\partial p_i^*} \frac{\partial g^*}{\partial q_i^*}, \tag{37}$$

where

$$f^* = f(\bar{q}^a(q^*, p^*), 0, q^*, p^*). \tag{38}$$

Thus q^* and p^* are canonical variables in Γ^*.

d. *Passage to Quantum Theory*

We now prove the central result of this section: For the mechanical system described hitherto, the expression for the matrix element of the S-matrix is

$$\langle \text{out} \mid S \mid \text{in} \rangle = \int \exp \left\{ i \int_{-\infty}^{+\infty} (p_i \dot{q}^i - H)\, dt \right\} \prod_t d\mu(q(t), p(t)), \tag{39}$$

where the measure of integration is given by

$$d\mu(q, p) = \left[\prod_a \delta(\chi_a)\, \delta(\varphi^a) \right] \det \|\{\chi_a, \varphi^b\}\| \prod_i dp_i\, dq^i. \tag{40}$$

The trajectories $q(t)$ coincide as $t \to \pm\infty$ with the solutions $q_{\text{in}}(t)$ and $q_{\text{out}}(t)$ of the equations describing the asymptotic motion.

The proof of the theorem goes as follows. Let us perform a canonical transformation to achieve a canonical description with the coordinates q^a, p^a, q^*, p^* as discussed above. The factor $\pi\, dp_i\, dq^i$ is invariant under a canonical transformation. In addition, we have

$$\int_{-\infty}^{+\infty} (p_i' \dot{q}^i - H')\, dt = \int_{-\infty}^{+\infty} (p_i \dot{q}^i - H)\, dt + \left(p_i \frac{\partial \Phi}{\partial p_i} - \Phi \right) \Big|_{-\infty}^{+\infty}.$$

Φ is the generating function for the canonical transformation, in the sense that

$$\delta q^i = \frac{\partial \Phi}{\partial p_i}, \qquad \delta p_i = -\frac{\partial \Phi}{\partial q^i}.$$

In field theory the interesting canonical transformations are linearized asymptotically as $t \to \pm\infty$, and then it can be shown that the change is equivalent to a unitary transformation in the operator formalism.

In the new canonical representation the measure becomes

$$\begin{aligned} &\prod_a \delta(p_a)\, \delta(\varphi^a) \det \left\| \frac{\partial \varphi^b}{\partial q^a} \right\| \prod_i dp_i\, dq^i \\ &\qquad = \prod_a \delta(p_a)\, \delta(q^a - \bar{q}^a(q^*, p^*))\, dp_a\, dq^a \prod dp^*\, dq^*. \end{aligned} \tag{41}$$

After a trivial integration over p_a and q^a the integral takes the form

$$\int \exp \left\{ i \int_{-\infty}^{+\infty} \left(\sum p^* q^* - H^* \right) dt \right\} \prod dp^*\, dq^*, \tag{42}$$

and this is indeed the functional integral representation for $\langle \text{out} \mid S \mid \text{in} \rangle$ in terms of an integration over the independent variables q^* and p^*. This constitutes the proof of our assertion.

e. *Independence of the Choice of Gauge Conditions*

The integral (39) is independent of the choice of χ_a. Indeed, for an infinitesimal change in χ_a, we easily find

$$\delta\chi_a = \{\Phi, \chi_a\} + c_{ab}\varphi^b, \tag{43}$$

where $\Phi = h_a\varphi^a$ and the h_a's are the solution of the system of equations

$$\{\chi_a, \varphi^b\} h_b = -\delta\chi_a, \tag{44}$$

which by (35) has a unique solution.

The second term is of no relevance, because of the first-class nature of the φ^b's and the factors $\delta(\varphi^a)$ in the integral, and the first term represents a canonical transformation. In this canonical transformation

$$\delta\varphi^a = \{\Phi, \varphi^a\} = \{h_b\varphi^b, \varphi^a\} = A_b{}^a\varphi^b, \tag{45}$$

so

$$\begin{gathered} \chi \to \chi + \delta\chi, \qquad \varphi \to (1 + A)\,\varphi, \qquad H \to H \\ \prod_a \delta(\varphi^a) \to (1 + \operatorname{tr} A)^{-1} \prod_a \delta(\varphi^a) \\ \det \|\{\chi_a, \varphi^b\}\| \to (1 + \operatorname{tr} A) \det \|\{\chi_a + \delta\chi_a, \varphi^b\}\|. \end{gathered} \tag{46}$$

We thus see that the integral (39) is independent of the choice of χ_a.

3. The Generalization of Faddeev's Method to the Case When Second-Class Constraints Are Present

In this section we shall generalize Faddeev's results, described in Section 2 to the case when second-class constraints are present. Thus, in our case the canonical variables do not vary throughout the phase space Γ, but satisfy the equations

$$\begin{aligned} \varphi^a(q, p) &= 0 \qquad a = 1,\ldots, m \\ \theta^l(q, p) &= 0 \qquad l = 1, 2,\ldots, 2n. \end{aligned} \tag{47}$$

We shall assume that the φ's and the θ's are independent and also irreducible in the sense that an arbitrary function h in Γ which vanishes in the subspace M in which Eqs. (47) hold is a linear combination of the constraints:

$$h = c_a(q, p)\,\varphi^a(q, p) + d_l(q, p)\,\theta^l(q, p). \tag{48}$$

The φ^a's are first class constraints, i.e.,

$$\{\varphi^a, \varphi^b\}|_M = 0 \qquad \{\varphi^a, \theta^l\}|_M = 0, \tag{49}$$

while the θ^l's are second class:

$$[\det \|\{\theta^l, \theta^k\}\|]|_M \neq 0. \tag{50}$$

Thus, in view of the irreducibility hypothesis we have

$$\begin{aligned} \{\varphi^a, \varphi^b\} &= c_c^{ab}\varphi^c + d_l^{ab}\theta^l \\ \{\varphi^a, \theta^l\} &= e_c^{al}\varphi^c + f_k^{al}\theta^k. \end{aligned} \tag{51}$$

Self-consistency requires

$$d_l^{ab}|_M = 0, \tag{52}$$

as we shall now prove; indeed, from (51) we find immediately

$$\begin{aligned} \{\theta^s, \{\varphi^a, \varphi^b\}\} &= d_l^{ab}\{\theta^s, \theta^l\} + \bar{c}_c^{sab}\varphi^c + \bar{d}_l^{sab}\theta^l \\ \{\varphi^b, \{\varphi^a, \theta^s\}\} &= \bar{e}_c^{bas}\varphi^c + \bar{f}_k^{bas}\theta^k, \end{aligned} \tag{53}$$

so that Jacobi's identity implies

$$d_l^{ab}|_M \{\theta^s, \theta^l\}|_M = 0, \tag{54}$$

and thus, in view of (50),

$$d_l^{ab}|_M = 0,$$

which was to be proved.

The equation of motion for an arbitrary quantity f is found to be, in a manner similar to that of Section 2,

$$\dot{f}|_M = \{f, H\}|_M, \tag{55}$$

where

$$H = H|_M + v_a\varphi^a + u_l\theta^l. \tag{56}$$

Self-consistency requires

$$\dot{\varphi}^a|_M = \{\varphi^a, H\}|_M = 0 \tag{57}$$

and

$$\dot{\theta}^k|_M = \{\theta^k, H\}|_M = 0. \tag{58}$$

One can easily see that (51), (57) and the irreducibility hypothesis imply

$$\{H|_M, \varphi^a\} = c_b{}^a\varphi^b + d_l{}^a\theta^l. \tag{59}$$

Equation (58), in turn, simply determines $u_l|_M$ in view of (50).

Not all quantities are observable (physical), but only those whose variation in time is not affected by the arbitrariness in the choice of the v_a's. Thus for physical quantities we must impose the requirement that

$$\dot{f}|_M = \{f, H|_M\}|_M + v_a|_M\{f, \varphi^a\}|_M + u_l|_M\{f, \theta^l\}|_M \tag{60}$$

is a uniquely determined quantity, which implies

$$\{f, \varphi^a\}|_M = 0 \tag{61}$$

or

$$\{f, \varphi^a\} = c_b{}^a\varphi^b + d_l{}^a\theta^l, \tag{62}$$

while there is no such restriction on $\{f, \theta^l\}|_M$ in view of the corollary preceding Eq. (19). (See also the comment following (59).)

Note that condition (62) is independent of the choice of continuation of a function f defined in M, that is $f|_M$, into the whole phase space Γ. Indeed, any two such continuations may differ only by a linear combination of constraints and then (51) implies that (62) holds for any such continuation.

Equations (61) can be thought of as a set of m first-order differential equations on M with (51) serving as integrability conditions. The proof of this statement is a straightforward extension of the corresponding proof in Section 2. Hence, the function f is defined uniquely by its values in the submanifold of the initial conditions of the equations which is of dimension $(2N - m - 2n) - m = 2(N - n - m)$ ($2N$ = number of canonical coordinates and momenta in Γ).

We can take this submanifold to be a surface Γ^* in M defined by

$$\chi_a(q, p) = 0 \qquad a = 1,..., m. \tag{63}$$

We shall call Eqs. (63) gauge conditions. We thus see that gauge conditions are associated with first-class constraints only. It is essential for later developments to require (see Eq. (66)):

$$\det \|\{\chi_a, \varphi^b\}\| \neq 0. \tag{64}$$

We now prove the following theorem.

THEOREM. *Let there be given a mechanical system with m first-class constraints and $2n$ second-class constraints. Let the first-class constraints be called φ_a, the* *second-class* θ_a, *and the gauge conditions associated with the first-class constraints* χ_a [5]. *Let the* χ_a*'s be chosen in such a way that* $\{\chi_a, \chi_b\} = 0$.

Then the expression for the S-matrix element is [6]

$$\langle \text{out} \mid S \mid \text{in} \rangle = \int \exp \left\{ i \int_{-\infty}^{+\infty} (p_i \dot{q}_i - H)\, dt \right\} \prod_t d\mu(q(t), p(t)), \tag{65}$$

where H is the Hamiltonian of the system and the measure of integration is defined by

$$\begin{aligned} d\mu(q, p) = \prod_a \delta(\chi_a)\, \delta(\varphi_a) \mid \det \|\{\chi_a, \varphi_b\}\| \mid \\ \times \prod_c \delta(\theta_c) \mid \det \|\{\theta_a, \theta_b\}\| \mid^{1/2} \prod_i dp_i\, dq_i . \end{aligned} \tag{66}$$

In order to prove this theorem, we need the following lemma:

LEMMA. *Let* M_ϵ *be the region of phase space in which* $\theta_a(p, q)$ *and* $\varphi_a(p, q)$ *are of order* ϵ (ϵ *is an infinitesimal quantity). Note that* $M_\epsilon \supset M$. *In the region* M_ϵ *there exists a set of function* Λ_{ab} *and* μ_{ab} *such that if we define*

$$\theta_a' = \Lambda_{ab}\theta_b + \mu_{ab}\varphi_b , \tag{67}$$

then

$$\begin{aligned} \{\theta_a', \theta_b'\} &= (\Theta')_{ab} + O(\epsilon^2) \qquad \text{in } M_\epsilon , \\ \{\theta_a', \chi_b\} &= O(\epsilon^2) \qquad \text{in } M_\epsilon , \end{aligned} \tag{68}$$

where,

$$\Theta' = \left[\begin{array}{cccc|cccc} & & & & & & & 1 \\ & & & & & & \cdot^{\cdot^{\cdot}} & \\ & & & & & 1 & & \\ & & & & 1 & & & \\ \hline & & & -1 & & & & \\ & & \cdot^{\cdot^{\cdot}} & & & & & \\ & -1 & & & & & & \\ -1 & & & & & & & \end{array} \right]. \tag{69}$$

The proof of the lemma is given in Appendix I.

Now note that due to the presence of the δ-functions $\delta(\varphi_a)$ and $\delta(\theta_c)$, the integral in (65) extends only over the region M_ϵ. Secondly, note that by (AI.8) and (AI.12)

$$\begin{aligned}\prod_a \delta(\varphi_a) \prod_c \delta(\theta_c') &= \left| \det \left\| \Lambda_{ab} + \frac{\partial \Lambda_{ac}}{\partial \theta_b} \theta_c + \frac{\partial \mu_{ac}}{\partial \theta_b} \varphi_c \right\| \right|^{-1} \prod_d \delta(\theta_d) \prod_a \delta(\varphi_a) \\ &= | \det \| \Lambda_{ab} \| |^{-1} \prod_d \delta(\theta_d) \prod_a \delta(\varphi_a) \\ &= | \det \| \Lambda^0_{ab} \| |^{-1} \prod_d \delta(\theta_d) \prod_a \delta(\varphi_a) \\ &= | \det \| \{\theta_a , \theta_b\} \| |^{1/2} \prod_d \delta(\theta_d) \prod_a \delta(\varphi_a),\end{aligned} \tag{70}$$

where we have used the relation

$$[\det \| \Lambda^0 \|]^2 \, | \det \| \{\theta_a , \theta_b\} \| | = | \det \Theta' | = 1, \tag{71}$$

which is a consequence of (AI.1) and (AI.2).

In view of Eq. (68) we can perform a canonical transformation in M_ϵ such that the new variables are $P_a = \chi_a$; $1 \leqslant a \leqslant m$, $Q_{m+a} = \theta_a'$, $P_{m+a} = \theta'_{2n-a+1}$; $1 \leqslant a \leqslant n$ (The discussion of canonical invariance following (40) applies in our case as well.) We thus find, using (65), (66), and (70), and integrating trivially over P_a, P_{m+a}, and Q_{m+a}

$$\langle \text{out} \mid S \mid \text{in} \rangle = \int \exp \left\{ i \int (\bar{P}_i \dot{\bar{Q}}_i - \bar{H}) \, dt \right\} \times \prod_t \left(\prod_a \delta(\varphi_a) \left| \det \left\| \frac{\partial \varphi_b}{\partial Q_a} \right\| \right| \right) \prod_i \mathscr{D}\bar{P}_i \, \mathscr{D}\bar{Q}_i \prod_a \mathscr{D}Q_a , \tag{72}$$

where

$$\bar{H} = H(\bar{P}_i , \bar{Q}_i , P_a = 0, Q_a , Q_{m+a} = 0, P_{m+a} = 0) \tag{73}$$

and $\bar{P}_i$'s and $\bar{Q}_i$'s are the remaining canonical variables. Finally, noting that

$$\prod_a \delta(\varphi_a) \left| \det \left\| \frac{\partial \varphi_b}{\partial Q_a} \right\| \right| = \prod_a \delta(Q_a - Q_a^*(\bar{P}_i , \bar{Q}_i)), \tag{74}$$

where $Q_a^*(\bar{P}_i , \bar{Q}_i)$ is the solution of the equation

$$\varphi_a(\bar{P}_i , \bar{Q}_i , Q_a^*, P_a = 0, Q_{m+a} = 0, P_{m+a} = 0) = 0, \tag{75}$$

we can write

$$\langle \text{out} \mid S \mid \text{in} \rangle = \int \exp \left\{ i \int (\bar{P}_i \dot{\bar{Q}}_i - \bar{H}) \, dt \right\} \prod_i \mathscr{D}\bar{P}_i \, \mathscr{D}\bar{Q}_i , \tag{76}$$

$$\bar{H} = \bar{H}(Q_a = Q_a^*(\bar{P}_i , \bar{Q}_i)). \tag{77}$$

Thus, in full analogy with the case described in Section 2, we have obtained an expression for the S-matrix elements as a functional integral over the independent canonical variables only. The weight of this integral is one, as it should be, and therefore (76) provides a justification of Eq. (65) and thus a proof of our theorem. Note the crucial role played by the determinants $|\det \|\{\chi_a, \varphi_b\}\||$ and $|\det \|\{\theta_a, \theta_b\}\||^{1/2}$.

It remains to prove that the matrix element in (65) with the measure given by (66) is independent of the choice of the gauge conditions χ_a. Again, as in Section 2, we find

$$\delta\chi_a = \{\Phi, \chi_a\} + c_{ab}\varphi_b \tag{78}$$

with

$$\Phi = h_a\varphi_a \tag{79}$$

and the h's are the solution of the system of equations

$$\{\chi_a, \varphi_b\}\, h_b = -\delta\chi_a . \tag{80}$$

The first term in (78) represents a canonical transformation. Performing such a canonical transformation results in changing φ_a and θ_a by

$$\delta\varphi_a = \{\Phi, \varphi_a\} = A_{ab}\varphi_b , \tag{81}$$

$$\delta\theta_a = \{\Phi, \theta_a\} = \{h_b\varphi_b, \theta_a\} = B_{ab}\varphi_b + D_{ab}\theta_b , \tag{82}$$

where we have used Eqs. (52). Calling

$$\delta_0\chi_a = \{\Phi, \chi_a\}, \tag{83}$$

we find

$$\{\chi_a, \varphi_b\}|_M \to \{\chi_a + \delta_0\chi_a, \varphi_b + A_{bc}\varphi_c\}|_M = \{\chi_a + \delta\chi_a, \varphi_b + A_{bc}\varphi_c\}|_M , \tag{84}$$

where we have used the first-class nature of φ_a. Thus,

$$\prod_a \delta(\varphi_a) \to (1 + \operatorname{tr} A)^{-1} \prod_a \delta(\varphi_a), \tag{85}$$

$$[\det \|\{\chi_a, \varphi_b\}\|]|_M \to (1 + \operatorname{tr} A)[\det \|\{\chi_a + \delta\chi_a, \varphi_b\}\|]|_M , \tag{86}$$

$$\prod_a \delta(\theta_a) \to (1 + \operatorname{tr} D)^{-1} \prod_a \delta(\theta_a) \tag{87}$$

and

$$[|\det \|\{\theta_a, \theta_b\}\||^{1/2}]|_M \to (1 + \operatorname{tr} D)[|\det \|\{\theta_a, \theta_b\}\||^{1/2}]|_M . \tag{88}$$

We can thus conclude that the integral (65) is independent of the choice χ_a, due to canonical invariance and Eqs. (85)–(88).

We wish to alert the reader to the fact that the extension of our results to field theory, as in Section 1 and 2, is rather straightforward.

4. Quantization of the Massive Yang–Mills Field

We shall devote this section to the study of the quantization of the massive Yang–Mills field as another example of a field theory containing second-class constraints. For an account of a different quantization scheme corresponding to the same problem, the reader is referred to the paper of Finkelstein, Kwitky, and Mouton [7].

The formalism is based on the Lagrangian density

$$\mathscr{L} = -\tfrac{1}{4}F^{\alpha}_{\mu\nu}F^{\mu\nu}_{\alpha} + \tfrac{1}{2}M^2 A_{\alpha}{}^{\mu} A_{\mu}{}^{\alpha}. \tag{89}$$

In Eq. (89), $F^{\mu\nu}_{\alpha}$ is given by the formula

$$F^{\mu\nu}_{\alpha} = \partial^{\mu} A_{\alpha}{}^{\nu} - \partial^{\nu} A_{\alpha}{}^{\mu} + g f_{\alpha\beta\gamma} A_{\beta}{}^{\mu} A_{\gamma}{}^{\nu}. \tag{90}$$

As usual, one begins by calculating the conjugate momenta:

$$\pi_{\alpha}{}^{0} = \partial\mathscr{L}/\partial\dot{A}_{0}{}^{\alpha}, \tag{91}$$

$$\pi_{\alpha}{}^{i} = \partial\mathscr{L}/\partial\dot{A}_{i}{}^{\alpha} = -F^{0i}_{\alpha}. \tag{92}$$

By inspecting (90), (91), and (92) one concludes that the only primary constraint is given by (91). To calculate the secondary constraints one needs the expression for the Hamiltonian density, which is obtained in a straightforward manner.

$$\begin{aligned}\mathscr{H} = {} & \tfrac{1}{2}\pi_{i}{}^{\alpha}\Pi_{i}^{\alpha} - \pi_{i}{}^{\alpha}\,\partial_{i}A_{\alpha}{}^{0} - g f^{\alpha\beta\gamma}\pi_{\alpha}{}^{i}A_{0}{}^{\beta}A_{i}{}^{\gamma} \\ & - \tfrac{1}{2}M^2 A_{\alpha}{}^{0}A_{\alpha}{}^{0} + \tfrac{1}{2}M^2 A_{\alpha}{}^{i}A_{\alpha}{}^{i} + \tfrac{1}{4}F^{\alpha}_{lm}F^{lm}_{\alpha} + u_{\alpha}{}^{0}\Pi^{0}_{\alpha}.\end{aligned} \tag{93}$$

The secondary constraints are then

$$\varphi^{\alpha} = \{\pi_0{}^{\alpha}, H\} = \partial_i \pi_{\alpha}{}^{i} - g f^{\alpha\beta\gamma}\pi_{\beta}{}^{i}A_{i}{}^{\gamma} + M^2 A_{\alpha}{}^{0}. \tag{94}$$

Imposing the consistency condition $\dot{\varphi}^{\alpha} = \{\varphi^{\alpha}, H\} = 0$ leads to

$$M^2 u_{\alpha}{}^{0} + \{\partial_i \pi_{\alpha}{}^{i} - g f^{\alpha\beta\gamma}\pi_{\beta}{}^{i}A_{i}{}^{\gamma}, H\} = 0, \tag{95}$$

which merely determines $u_{\alpha}{}^{0}$ and thus does not produce any new constraints.

The algebra of Poisson brackets of all the constraints is found to be

$$\{\pi_\alpha{}^0(\mathbf{x}), \pi_\beta{}^0(\mathbf{y})\} = 0, \tag{96}$$

$$\{\pi_\alpha{}^0(\mathbf{x}), \varphi^\beta(\mathbf{y})\} = -M^2\delta_{\alpha\beta}\,\delta(\mathbf{x}-\mathbf{y}), \tag{97}$$

$$\{\varphi^\alpha(\mathbf{x}), \varphi^\beta(\mathbf{y})\} = gf^{\alpha\beta\gamma}\varphi^\beta(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{y}). \tag{98}$$

The proof of (96) and (97) is trivial. To prove (98), one uses the Jacobi identity and the antisymmetry of the structure constraints of the compact semisimple Lie algebra corresponding to our Yang–Mills field theory. One can thus convince oneself that all the constraints are second class. The characteristic weight in the phase functional integral is

$$|\det \|\{\theta_a, \theta_b\}\||^{1/2} = \prod_{\mathbf{x},t,\alpha} (M^2) = \det M^2. \tag{99}$$

The general discussion of Section 3 (Eqs. (65) and (66) in particular) then leads to the expression for the S-matrix element:

$$\begin{aligned}\langle \text{out} \mid S \mid \text{in}\rangle = \int \prod_{\alpha,i} \mathscr{D}A_i{}^\alpha\, \mathscr{D}\pi_\alpha{}^i \prod_\alpha \mathscr{D}A_\alpha{}^0\, \mathscr{D}\pi_\alpha{}^0 \det M^2 \\ \times \prod_{x,\alpha} \{\delta(\pi_\alpha{}^0)\,\delta(\partial_i\pi_\alpha{}^i - gf^{\alpha\beta\gamma}\pi_\beta{}^i A_i{}^\gamma + M^2A_\alpha{}^0)\} \\ \times \exp\Big\{i\int [\pi_0{}^\alpha \dot{A}_0{}^\alpha + \pi_\alpha{}^i\dot{A}_i{}^\alpha - \tfrac{1}{2}\pi_\alpha{}^i\pi_\alpha{}^i \\ - \pi_\alpha{}^i(\partial_i A_\alpha{}^0 - gf^{\alpha\beta\gamma}A_0{}^\beta A_i{}^\gamma) + \tfrac{1}{2}M^2A_\alpha{}^\mu A_\mu{}^\alpha \\ - \tfrac{1}{4}F_{lm}^\alpha F_\alpha^{lm}]\, d^4x\Big\}.\end{aligned} \tag{100}$$

This obviously can be written as

$$\begin{aligned}\langle \text{out} \mid S \mid \text{in}\rangle = \int \prod_{\alpha,i} \mathscr{D}A_i{}^\alpha\, \mathscr{D}\pi_\alpha{}^i \prod_\alpha \mathscr{D}\lambda_\alpha\, \mathscr{D}A_\alpha{}^0 \det M^2 \\ \times \exp\Big\{i\int [\pi_\alpha{}^i(F_{0i}^\alpha - \partial_i\lambda_\alpha + gf^{\alpha\beta\gamma}\lambda^\beta A_i{}^\gamma) + M^2\lambda_\alpha A_\alpha{}^0 \\ - \tfrac{1}{2}\pi_\alpha{}^i\pi_\alpha{}^i + \tfrac{1}{2}M^2A_\alpha{}^0A_\alpha{}^0 - \tfrac{1}{2}M^2A_\alpha{}^iA_\alpha{}^i - \tfrac{1}{4}F_{lm}^\alpha F_\alpha^{lm}]\, d^4x\Big\}.\end{aligned} \tag{101}$$

After a change of variables

$$A_\alpha{}^0 \to A_\alpha{}^0 - \lambda_\alpha \tag{102}$$

and a Gaussian integration over λ_α and $\pi_\alpha{}^i$, we can establish the following result:

$$\langle \text{out} \mid S \mid \text{in} \rangle = \int \prod_\alpha \mathscr{D} A_\alpha{}^0 \prod_{\alpha, i} \mathscr{D} A_i{}^\alpha \det M$$
$$\times \exp \left\{ i \int \left[\tfrac{1}{2} F_{0i}^\alpha F_{0i}^\alpha - \tfrac{1}{4} F_{lm}^\alpha F_\alpha^{lm} + \tfrac{1}{2} M^2 A_\alpha{}^0 A_\alpha{}^0 \right.\right.$$
$$\left.\left. - \tfrac{1}{2} M^2 A_\alpha{}^i A_\alpha{}^i \right] d^4x \right\}. \tag{103}$$

The expression in the exponential is just the Lagrangian (89).

The result (103) is the basic result of this section. It shows that S-matrix elements are expressible as functional integrals solely over the basic fields of the theory. The characteristic weight of such integrals is $\det M = \prod_{\mathbf{x}, t, \alpha} (M)$ [8]. Equation (103) serves as a basis for developing perturbation theory in the path integral formalism and finding out the Feynman rules. We shall not dwell on this, since the Feynman rules have already been found by Finkelstein, Kwitky, and Mouton [7]. We merely note that since the functional measure in (103) is independent of the field variables, there will be no modification of the simple Feynman rules due to a nontrivial functional measure. We have thus rederived the basic result of Finkelstein, Kwitky, and Mouton in a more economical fashion.

The reduction to independent variables is particularly simple in this case. Integrating over $A_\alpha{}^0$ and $\pi_\alpha{}^0$, one can write (100) as

$$\langle \text{out} \mid S \mid \text{in} \rangle = \int \prod_{\alpha, i} \mathscr{D} A_i{}^\alpha \, \mathscr{D} \pi_\alpha{}^i$$
$$\times \exp \left\{ i \int \left[\pi_\alpha{}^i \dot{A}_i{}^\alpha - \frac{1}{2} \pi_\alpha{}^i \pi_\alpha{}^i - \frac{1}{2M^2} (\partial_i \pi_\alpha{}^i - g f^{\alpha\beta\gamma} \pi_\beta{}^i A_i{}^\gamma) \right.\right.$$
$$\left.\left. \times (\partial_j \pi_\alpha{}^j - g f^{\alpha\delta\epsilon} \pi_\delta{}^k A_k{}^\epsilon) - \frac{1}{2} M^2 A_\alpha{}^i A_\alpha{}^i - \frac{1}{4} F_{lm}^\alpha F_\alpha^{lm} \right] d^4x \right\}. \tag{104}$$

The weight of integration is one, as expected from our general discussion in Section 3.

It is interesting to determine the Dirac brackets for our canonical variables. One finds first

$$\| c_{ab}(\mathbf{x}, \mathbf{y}) \| = \| \{ \theta_a(\mathbf{x}), \theta_b(\mathbf{y}) \} \|^{-1}$$
$$= \left\| \begin{array}{c|c} \dfrac{1}{M^4} \Phi(\mathbf{x}) \, \delta(\mathbf{x} - \mathbf{y}) & \dfrac{1}{M^2} \delta(\mathbf{x} - \mathbf{y}) \\ \hline -\dfrac{1}{M^2} \delta(\mathbf{x} - \mathbf{y}) & 0 \end{array} \right\|. \tag{105}$$

Here,

$$\Phi_{\alpha\beta}(\mathbf{x}) = g f_{\alpha\beta\gamma}\varphi_\gamma(\mathbf{x}). \tag{106}$$

Let us introduce the notation

$$a = (\tilde{a}, \bar{a}), \tag{107}$$

so that

$$\begin{aligned} C_{\bar{a}\bar{b}}(\mathbf{x}, \mathbf{y}) &= 0, \\ C_{\substack{\tilde{a}\\(\alpha)}\substack{\tilde{b}\\(\beta)}}(\mathbf{x}, \mathbf{y}) &= \frac{g}{M^4} f_{\alpha\beta\gamma}\varphi_\gamma(\mathbf{x})\, \delta(\mathbf{x} - \mathbf{y}), \\ C_{\substack{\tilde{a}\\(\alpha)}\substack{\bar{b}\\(\beta)}}(\mathbf{x}, \mathbf{y}) &= \frac{1}{M^2}\, \delta_{\alpha\beta}\delta(\mathbf{x} - \mathbf{y}), \\ C_{\substack{\bar{a}\\(\alpha)}\substack{\tilde{b}\\(\beta)}}(\mathbf{x}, \mathbf{y}) &= -\frac{1}{M^2}\, \delta_{\alpha\beta}\delta(\mathbf{x} - \mathbf{y}). \end{aligned} \tag{108}$$

Obviously

$$\{\pi_\alpha{}^0, \text{all canonical variables}\}^* = 0, \tag{109}$$

since $\theta_\alpha = \pi_\alpha{}^0$ are second-class constraints.

Next, using Eq. (108), we obtain

$$\begin{aligned} \{\pi_\alpha{}^i(\mathbf{x}), A_\beta{}^0(\mathbf{y})\}^* &= -\int d\mathbf{z}\, d\mathbf{u}\, \{\pi_\alpha{}^i(\mathbf{x}), \theta_a(\mathbf{z})\}\, C_{ab}(\mathbf{z}, \mathbf{u})\{\theta_b(\mathbf{u}), A_\beta{}^0(\mathbf{y})\} \\ &= -\int d\mathbf{z}\, d\mathbf{u}\, \{\pi_\alpha{}^i(\mathbf{x}), \varphi_\delta(\mathbf{z})\}\, C_{\substack{\bar{a}\\(\delta)}\substack{\tilde{b}\\(\gamma)}}(\mathbf{z}, \mathbf{u})\{\pi_\gamma{}^0(\mathbf{u}), A_\beta{}^0(\mathbf{y})\} \\ &= -\frac{1}{M^2}\{\pi_\alpha{}^i(\mathbf{x}), \varphi_\beta(\mathbf{y})\}. \end{aligned} \tag{110}$$

Using (94) one finds

$$\{\pi_\alpha{}^i(\mathbf{x}), A_\beta{}^0(\mathbf{y})\}^* = -\frac{g}{M^2} f_{\alpha\beta\gamma}\pi_\gamma{}^i(\mathbf{x})\, \delta(\mathbf{x} - \mathbf{y}). \tag{111}$$

Since $C_{\bar{a}\bar{b}}(\mathbf{x}, \mathbf{y}) = 0$, one has

$$\{\pi_\alpha{}^i(\mathbf{x}), A_j{}^\beta(\mathbf{y})\}^* = -\delta_\alpha{}^\beta\delta_j{}^i\, \delta(\mathbf{x} - \mathbf{y}). \tag{112}$$

Quite similarly, we obtain

$$\begin{aligned} \{\pi_\alpha{}^i(\mathbf{x}), \pi_\beta{}^j(\mathbf{y})\}^* &= 0, \\ \{A_i{}^\alpha(\mathbf{x}), A_j{}^\beta(\mathbf{y})\}^* &= 0. \end{aligned} \tag{113}$$

Next, taking into account (108), one finds the remaining Poisson brackets

$$\{A_0^\alpha(\mathbf{x}), A_i^\beta(\mathbf{y})\}^* = -\frac{1}{M^2}\{\varphi^\alpha(\mathbf{x}), A_i^\beta(\mathbf{y})\}$$
$$= \frac{1}{M^2}\partial_i^x\delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{y}) - \frac{g}{M^2}f_{\alpha\beta\gamma}A_i^\gamma(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{y}), \quad (114)$$

$$\{A_\alpha^0(\mathbf{x}), A_\beta^0(\mathbf{y})\}^* = \frac{g}{M^4}f_{\alpha\beta\gamma}\varphi_\gamma(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{y}). \quad (115)$$

This completes the quantization of the massive Yang–Mills field theory, since in operator formalism one can obtain the basic commutators from the Dirac brackets, while in path integral formalism one uses (103) to obtain S-matrix elements.

5. Light-Cone Quantization of the Self-Interacting Scalar Theory

As we shall see in this section, quantization of field theories on the null plane leads naturally to second-class constraints. Therefore, the method we developed in Section 3 is applicable in this case. For different methods of quantization on the null plane, the reader is referred to the existing literature [9–11]. For completeness we shall also describe briefly the method of Banyai and Mezincescu [12, 13]. Both our method and the method of Banyai and Mezincescu will be illustrated in the example of the self-interacting scalar theory with a quartic coupling.

One thus starts with Lagrangian

$$\mathscr{L} = \frac{\partial\varphi}{\partial x_+}\frac{\partial\varphi}{\partial x_-} - \frac{1}{2}\left(\frac{\partial\varphi}{\partial \underline{x}}\right)^2 - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4 \quad (116)$$

and deduces the conjugate momentum

$$\pi = \frac{\partial\mathscr{L}}{\partial\left(\dfrac{\partial\varphi}{\partial x_+}\right)} = \frac{\partial\varphi}{\partial x_-}. \quad (117)$$

where we have introduced the standard null-plane variables $x_\pm = (x^0 \pm x^3)/\sqrt{2}$. Equation (117) obviously represents a constraint.

The Hamiltonian is immediately found to be

$$H = \int dx_-\, d\underline{x}\left[\frac{1}{2}\left(\frac{\partial\varphi}{\partial\underline{x}}\right)^2 + \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4 + u\left(\pi - \frac{\partial\varphi}{\partial x_-}\right)\right], \quad (118)$$

where $\underset{\sim}{x}$ are the transverse variables: $\underset{\sim}{x}^{1,2} = x^{1,2}$. As usual we require that

$$\dot{\theta} = \{\theta, H\} = 0, \tag{119}$$

where

$$\theta = \pi - \frac{\partial \varphi}{\partial x_-}. \tag{120}$$

Equation (119) results in

$$\underset{\sim}{\nabla}^2 \varphi - m^2 \varphi - \frac{\lambda}{3!} \varphi^3 - 2 \frac{\partial u}{\partial x_-} = 0, \tag{121}$$

since

$$\{\pi(\underset{\sim}{y}, y_-) - \partial_-^{y} \varphi(\underset{\sim}{y}, y_-), \pi(\underset{\sim}{x}, x_-) - \partial_-^{x} \varphi(\underset{\sim}{x}, x_-)\} = 2\partial_-^{x}\, \delta(x_- - y_-)\, \delta(\underset{\sim}{x} - \underset{\sim}{y}). \tag{122}$$

Equation (121) does not represent a new constraint but merely serves to determine u. One possible determination of u is

$$u(x) = \tfrac{1}{4} \int \epsilon(x_- - \xi)[\underset{\sim}{\nabla}^2 \varphi(\underset{\sim}{x}, \xi, x_+) - m^2 \varphi(\underset{\sim}{x}, \xi, x_+) - (\lambda/3!)\, \varphi^3(\underset{\sim}{x}, \xi, x_+)]\, d\xi. \tag{123}$$

Thus (120), taken at all $(x_-, \underset{\sim}{x})$, represents a complete set of constraints. Due to (122) these constraints are second class.

To quantize the theory in operator formalism, we have to find the Dirac brackets (this is precisely the method of Banyai and Mezincescu [13, 14]). By Eq. (20) these are given by

$$\{a(\mathbf{u}), b(\mathbf{v})\}^* = \{a(\mathbf{u}), b(\mathbf{v})\} - \int d\mathbf{x}\, d\mathbf{y}\, \{a(\mathbf{u}), \theta(\mathbf{x})\}\, c(\mathbf{x}, \mathbf{y}) \{\theta(\mathbf{y}), b(\mathbf{v})\}, \tag{124}$$

where, e.g., $\mathbf{x} = (x_-, \underset{\sim}{x})$ and

$$\int c(\mathbf{x}, \mathbf{z})\, d(\mathbf{z}, \mathbf{y})\, d\mathbf{z} = \delta(x_- - y_-)\, \delta(\underset{\sim}{x} - \underset{\sim}{y}), \tag{125}$$

$$\int d(\mathbf{x}, \mathbf{z})\, c(\mathbf{z}, \mathbf{y})\, d\mathbf{z} = \delta(x_- - y_-)\, \delta(\underset{\sim}{x} - \underset{\sim}{y}), \tag{126}$$

$$d(\mathbf{z}, \mathbf{y}) = \{\theta(\mathbf{z}), \theta(\mathbf{y})\} = 2\partial_-^{y}\, \delta(z_- - y_-)\, \delta(\underset{\sim}{z} - \underset{\sim}{y}). \tag{127}$$

Conditions (125)–(127) imply a special solution [14]

$$c(\mathbf{x}, \mathbf{y}) = -\tfrac{1}{4}\epsilon(x_- - y_-)\, \delta(\underset{\sim}{x} - \underset{\sim}{y}), \tag{128}$$

which enables us to find the basic Dirac brackets:

$$\{\varphi(\mathbf{x}), \varphi(\mathbf{y})\}^* = -\tfrac{1}{4}\epsilon(x_- - y_-)\,\delta(\underline{x} - \underline{y}), \tag{129}$$

$$\{\varphi(\mathbf{x}), \pi(\mathbf{y})\}^* = \tfrac{1}{2}\delta(x_- - y_-)\,\delta(\underline{x} - \underline{y}), \tag{130}$$

$$\{\pi(\mathbf{x}), \pi(\mathbf{y})\}^* = \tfrac{1}{2}\partial_-^x\,\delta(x_- - y_-)\,\delta(\underline{x} - \underline{y}). \tag{131}$$

The transition to quantum theory is then effected by replacing Dirac brackets by "$-i$" times the corresponding commutators, so one finds for the commutators:

$$\begin{aligned} [\varphi(x), \varphi(y)]|_{x_+=y_+} &= -\frac{i}{4}\,\epsilon(x_- - y_-)\,\delta(\underline{x} - \underline{y}), \\ [\varphi(x), \pi(y)]|_{x_+=y_+} &= \frac{1}{2}\,i\delta(x_- - y_-)\,\delta(\underline{x} - \underline{y}), \\ [\pi(x), \pi(y)]|_{x_+=y_+} &= \frac{1}{2}\,i\partial_-^x\,\delta(x_- - y_-)\,\delta(\underline{x} - \underline{y}). \end{aligned} \tag{132}$$

To quantize the theory by the method of Section 3 of this paper, we need to know the value of the determinant $|\det\{\theta(\mathbf{x}), \theta(\mathbf{y})\}|$. From (122) it is seen to be

$$|\det\{\theta(\mathbf{x}), \theta(\mathbf{y})\}| = \det(2\partial_-). \tag{133}$$

Thus we have all the ingredients necessary to write the functional integral for the S-matrix element:

$$\langle \text{out} \mid S \mid \text{in} \rangle$$

$$= \int \mathscr{D}\pi\,\mathscr{D}\varphi\,|\det(2\partial_-)|^{1/2} \prod_x \delta(\pi - \partial_-\varphi)\exp\left\{i\int(\pi\,\partial_+\varphi - \mathscr{H})\,d^4x\right\} \tag{134}$$

where

$$\mathscr{H} = \frac{1}{2}\left(\frac{\partial\varphi}{\partial\underline{x}}\right)^2 + \frac{1}{2}\,m^2\varphi^2 + \frac{\lambda}{4!}\,\varphi^4. \tag{135}$$

Equation (134) represents the main result of this section. Note the presence of the factor $|\det(2\partial_-)|^{1/2}$ in the functional measure, which cannot be obtained by naive considerations.

6. Local Lagrangian Theory of Electric and Magnetic Charges and Its Quantization

While the theory of the electromagnetic field interacting with both electric and magnetic charges has a long history [15, 16], its local Lagrangian formulation is comparatively new [17]. It is based on the introduction of *two* potentials A_μ and B_μ. The electromagnetic field tensor $F_{\mu\nu}$ is suitably expressed in terms of these.

As we shall see, this necessitates the introduction of a fixed four-vector n_μ into the theory; thus, explicit Lorentz invariance is lost and it remains to be proved that the theory is in fact n-independent.

In the case when magnetic charges are present, Maxwell's equations read

$$\partial_\mu F^{\mu\nu} = j_e^{\nu}, \tag{136}$$

$$\partial_\mu F^{d\mu\nu} = j_g^{\nu}. \tag{137}$$

In (136), $F^{d\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu}_{\kappa\lambda}F^{\kappa\lambda}$ and $\epsilon^{\mu\nu\kappa\lambda}$ is the completely antisymmetric symbol with $\epsilon^{0123} = 1$. j_e^{ν} is the electric current, while j_g^{ν} is the magnetic current. Both currents are conserved:

$$\partial_\mu j_e^{\mu} = \partial_\mu j_g^{\mu} = 0. \tag{138}$$

The general solution to (137) may be written as

$$F = \partial \wedge A - (n \cdot \partial)^{-1} (n \wedge j_g)^d. \tag{139}$$

n_μ is an arbitrary fixed four-vector and $(n \cdot \partial)^{-1}$ is an integral operator with the kernel $(n \cdot \partial)^{-1}(x - y)$ satisfying $n \cdot \partial(n \cdot \partial)^{-1}(x) = \delta^{(4)}(x)$. For arbitrary two vectors C^μ and D^ν, $(C \wedge D)^{\mu\nu} = C^\mu D^\nu - C^\nu D^\mu$. A^μ is a four-potential which depends on the choice of gauge, the choice of n and the determination of $(n \cdot \partial)^{-1}$.

The general solution to Eq. (136) is

$$F = -(\partial \wedge B)^d + (n \cdot \partial)^{-1} (n \wedge j_e). \tag{140}$$

Since any antisymmetric tensor G satisfies the identity

$$G = \frac{1}{n^2} \{[n \wedge (n \cdot G)] - [n \wedge (n \cdot G^d)]^d\}, \tag{141}$$

we can obtain from (139) and (140)

$$F = \frac{1}{n^2} (\{n \wedge [n \cdot (\partial \wedge A)]\} - \{n \wedge [n \cdot (\partial \wedge B)]\}^d), \tag{142}$$

$$F^d = \frac{1}{n^2} (\{n \wedge [n \cdot (\partial \wedge A)]\}^d + \{n \wedge [n \cdot (\partial \wedge B)]\}). \tag{143}$$

These expressions, when substituted into (136) and (137), yield

$$\begin{aligned}\frac{1}{n^2} (n \cdot \partial n \cdot \partial A^\mu &- n \cdot \partial\, \partial^\mu n \cdot A - n^\mu n \cdot \partial\, \partial \cdot A \\ &+ n^\mu\, \partial^2 n \cdot A - n \cdot \partial \epsilon^{\mu}_{\nu\kappa\lambda} n^\nu\, \partial^\kappa B^\lambda) = j_l^{\mu},\end{aligned} \tag{144}$$

$$\begin{aligned}\frac{1}{n^2} (n \cdot \partial n \cdot \partial B^\mu &- n \cdot \partial\, \partial^\mu n \cdot B - n^\mu n \cdot \partial\, \partial \cdot B \\ &+ n^\mu\, \partial^2 n \cdot B + n \cdot \partial \epsilon^{\mu}_{\nu\kappa\lambda} n^\nu\, \partial^\kappa A^\lambda) = j_g^{\mu}.\end{aligned} \tag{145}$$

These equations of motion follow from the Lagrangian density:

$$\mathscr{L} = \mathscr{L}_\gamma + \mathscr{L}_I, \tag{146}$$

where

$$\mathscr{L}_\gamma = -\frac{1}{2n^2}[n\cdot(\partial\wedge A)]\cdot[n\cdot(\partial\wedge B)^d] + \frac{1}{2n^2}[n\cdot(\partial\wedge B)]$$
$$\cdot[n\cdot(\partial\wedge A)^d] - \frac{1}{2n^2}[n\cdot(\partial\wedge A)]^2 - \frac{1}{2n^2}[n\cdot(\partial\wedge B)]^2 \tag{147}$$

and

$$\mathscr{L}_I = -j_e\cdot A - j_g\cdot B. \tag{148}$$

Using the identity

$$\operatorname{tr}(G\cdot G) = G_{\mu\nu}G^{\nu\mu} = \frac{2}{n^2}[-(n\cdot G)^2 + (n\cdot G^d)^2], \tag{149}$$

which follows from (141), we can obtain a different form for $\mathscr{L}_\gamma$:

$$\mathscr{L}_\gamma = \frac{1}{8}\operatorname{tr}[(\partial\wedge A)\cdot(\partial\wedge A)] + \frac{1}{8}\operatorname{tr}[(\partial\wedge B)\cdot(\partial\wedge B)]$$
$$-\frac{1}{4n^2}\{n\cdot[(\partial\wedge A) + (\partial\wedge B)^d]\}^2 - \frac{1}{4n^2}\{n\cdot[(\partial\wedge B) - (\partial\wedge A)^d]\}^2. \tag{150}$$

We have thus obtained a Lagrangian formulation for the electromagnetic field interacting with both electric and magnetic charges [17].

We shall now quantize this theory using the method of phase space functional integration developed in Section 3.

The quantization procedure starts, as usual, from the Lagrangian. One then goes on to derive the conjugate momenta, the Hamiltonian, and the constraints. In our case the photon Lagrangian is given by (150). After some algebra the total Lagrangian can be recast into a form more suitable for transition into the Hamiltonian formalism:

$$\mathscr{L} = \mathscr{L}_1 + \mathscr{L}_\psi \qquad \mathscr{L}_1 = \mathscr{L}_\gamma + \mathscr{L}_I \tag{151}$$

$$\mathscr{L}_1 = \frac{1}{4}(\mathbf{F}_0{}^2 - \mathbf{F}^2) + \frac{1}{4}(\mathbf{G}_0{}^2 - \mathbf{G}^2) + \frac{1}{4}(\mathbf{a}^2 + \mathbf{b}^2) - \frac{1}{2n^2}[(\mathbf{n}\cdot\mathbf{a})^2 - \mathbf{n}^2\mathbf{a}^2]$$
$$-\frac{1}{2n^2}[(\mathbf{n}\cdot\mathbf{b})^2 - \mathbf{n}^2\mathbf{b}^2] + \frac{n_0}{n^2}\epsilon_{ijk}a_ib_jn_k - j_e{}^0A_0 + \mathbf{j}_e\cdot\mathbf{A} - j_g{}^0B_0 + \mathbf{j}_g\cdot\mathbf{B}. \tag{152}$$

$\mathscr{L}_\psi$ is the free Lagrangian of the charged and monopole fields (we assume here they are spin-$\frac{1}{2}$ fields). The explanation of the notation is as follows:

$$
\begin{aligned}
n^\mu &= (n_0\,, \mathbf{n}) \\
a_i &= \bar{F}_{0i} + G_i\,, \qquad b_i = \bar{G}_{0i} - F_i \\
\bar{F}_{ij} &= \epsilon_{ijk} F_k\,, \qquad \bar{G}_{ij} = \epsilon_{ijk} G_k \\
(\mathbf{F}_0)_i &= \bar{F}_{0i} = \partial_0 A_i - \partial_i A_0\,, \\
(\mathbf{G}_0)_i &= \bar{G}_{0i} = \partial_0 B_i - \partial_i B_0\,, \\
\bar{F}_{ij} &= \partial_i A_j - \partial_j A_i \qquad \bar{G}_{ij} = \partial_i B_j - \partial_j B_i\,.
\end{aligned}
\tag{153}
$$

Equivalently:

$$
\begin{aligned}
\mathscr{L}_1 = {} & \tfrac{1}{4}(\bar{F}_{0i}\bar{F}_{0i} - F_iF_i) + \tfrac{1}{4}(\bar{G}_{0i}\bar{G}_{0i} - G_iG_i) \\
& + \tfrac{1}{4}(a_ia_i + b_ib_i) + \tfrac{1}{2}\alpha_{ij}(a_ia_j + b_ib_j) \\
& + \beta_{ij}a_ib_j - j_e{}^0 A_0 + \mathbf{j}_e \cdot \mathbf{A} - j_g{}^0 B_0 + \mathbf{j}_g \cdot \mathbf{B},
\end{aligned}
\tag{154}
$$

where

$$
\begin{aligned}
\alpha_{ij} &= \frac{\mathbf{n}^2}{n^2}\left(\delta_{ij} - \frac{n_in_j}{\mathbf{n}^2}\right) = p\left(\delta_{ij} - \frac{n_in_j}{\mathbf{n}^2}\right), \\
\beta_{ij} &= \frac{n_0}{n^2}\,\epsilon_{ijk}n_k = \frac{q}{n_0}\,\epsilon_{ijk}n_k\,.
\end{aligned}
\tag{155}
$$

(Note that $q - p = 1$.)

The derivation of conjugate momenta proceeds now as usual, with the result:

$$
\begin{aligned}
\pi^i &= \frac{\partial\mathscr{L}}{\partial\dot{A}_i} = \bar{F}_{0i} + \alpha_{ij}a_j + \beta_{ij}b_j + \tfrac{1}{2}G_i\,, \\
\sigma^i &= \frac{\partial\mathscr{L}}{\partial\dot{B}_i} = \bar{G}_{0i} + \alpha_{ij}b_j - \beta_{ij}a_j - \tfrac{1}{2}F_i\,,
\end{aligned}
\tag{156}
$$

$$
\pi^0 = \frac{\partial\mathscr{L}}{\partial\dot{A}_0} = 0, \qquad \sigma^0 = \frac{\partial\mathscr{L}}{\partial\dot{B}_0} = 0. \tag{157}
$$

Defining new quantities $\tilde{\pi}^i$ and $\tilde{\sigma}^i$ by

$$
\begin{aligned}
\tilde{\pi}^i &= \pi^i - \tfrac{1}{2}G_i - \alpha_{ij}G_j + \beta_{ij}F_j\,, \\
\tilde{\sigma}^i &= \sigma^i + \tfrac{1}{2}F_i + \alpha_{ij}F_j + \beta_{ij}G_j\,,
\end{aligned}
\tag{158}
$$

we obtain, making use of (153), (155), (156) and (158):

$$\tilde{\boldsymbol{\pi}} = q\left(\mathbf{F}_0 - \frac{\mathbf{n} \times \mathbf{G}_0}{n_0}\right) - \frac{p\mathbf{n}}{\mathbf{n}^2}(\mathbf{n} \cdot \mathbf{F}_0),$$
$$\tilde{\boldsymbol{\sigma}} = q\left(\mathbf{G}_0 + \frac{\mathbf{n} \times \mathbf{F}_0}{n_0}\right) - \frac{p\mathbf{n}}{\mathbf{n}^2}(\mathbf{n} \cdot \mathbf{G}_0). \tag{159}$$

At this stage we have to distinguish between two cases: $n_0 = 0$ and $n_0 \neq 0$. As far as the quantization procedure is concerned, the two cases are fundamentally different. For $n_0 = 0$ we obtain both first- and second-class constraints; for $n_0 \neq 0$ only first-class constraints appear. The case $n_0 \neq 0$ can thus be treated by Faddeev's method, while the case $n_0 = 0$, to which we now turn our attention, requires the generalized method described in Section 3. (The case $n_0 = 0$ was studied by Balachandran, Rupertsberger and Schecter [18].) We wish to emphasize that while those authors quantize the theory by Dirac's formalism, we perform a phase space path integral quantization as an application of Section 3 of this paper.) For $n_0 \to 0$ we have $q/n_0 = n_0/n^2 \to 0$ and also $q \to 0$, so $p = -1$. Equation (159) becomes in this case:

$$\tilde{\boldsymbol{\pi}} = \frac{\mathbf{n}}{\mathbf{n}^2}(\mathbf{n} \cdot \mathbf{F}_0), \qquad \tilde{\boldsymbol{\sigma}} = \frac{\mathbf{n}}{\mathbf{n}^2}(\mathbf{n} \cdot \mathbf{G}_0). \tag{160}$$

Therefore

$$\mathbf{n} \cdot \mathbf{F}_0 = \mathbf{n} \cdot \tilde{\boldsymbol{\pi}} = \mathbf{n} \cdot (\boldsymbol{\pi} - \tfrac{1}{2}\mathbf{G}),$$
$$\mathbf{n} \cdot \mathbf{G}_0 = \mathbf{n} \cdot \tilde{\boldsymbol{\sigma}} = \mathbf{n} \cdot (\boldsymbol{\sigma} + \tfrac{1}{2}\mathbf{F}), \tag{161}$$

and

$$\boldsymbol{\tau}^{(l)} \cdot \tilde{\boldsymbol{\pi}} = \boldsymbol{\tau}^{(l)} \cdot \tilde{\boldsymbol{\sigma}} = 0, \tag{162}$$

or, in other words

$$\boldsymbol{\tau}^{(l)} \cdot \boldsymbol{\pi}' = \boldsymbol{\tau}^{(l)} \cdot \boldsymbol{\sigma}' = 0, \tag{163}$$

where $\boldsymbol{\pi}' = \boldsymbol{\pi} + \frac{1}{2}\mathbf{G}$, $\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \frac{1}{2}\mathbf{F}$, and we have used (155) and (158). In (162) and (163), $\boldsymbol{\tau}^{(l)}$ are two vectors ($l = 1, 2$) orthogonal to $\mathbf{n}$, that is:

$$\boldsymbol{\tau}^{(l)} \cdot \mathbf{n} = 0. \tag{164}$$

We choose them in such a way that

$$\boldsymbol{\tau}^{(l)} \cdot \boldsymbol{\tau}^{(k)} = \delta_{lk}. \tag{165}$$

As usual, we have to calculate the Hamiltonian at this point. Setting $n_0 = 0$ in the Lagrangian yields:

$$\mathscr{L}_1 = -\frac{1}{2}\mathbf{a} \cdot \mathbf{G} + \frac{1}{2}\mathbf{b} \cdot \mathbf{F} + \frac{1}{2\mathbf{n}^2}[(\mathbf{n} \cdot \mathbf{a})^2 + (\mathbf{n} \cdot \mathbf{b})^2] - j_e \cdot A - j_g \cdot B. \tag{166}$$

Therefore, calling $\boldsymbol{\pi}_1' = \boldsymbol{\pi}'$, $\boldsymbol{\pi}_2' = \boldsymbol{\sigma}'$, one obtains

$$\mathscr{H}_1 |_{\overline{M}} = \tfrac{1}{2}\boldsymbol{\pi}_a' \cdot \boldsymbol{\pi}_a' - \boldsymbol{\pi}_1' \cdot \mathbf{G} + \boldsymbol{\pi}_2' \cdot \mathbf{F} + \tfrac{1}{2}\mathbf{F}^2 + \tfrac{1}{2}\mathbf{G}^2 + j_e \cdot A + j_g \cdot B + \boldsymbol{\pi} \cdot \nabla A_0 + \boldsymbol{\sigma} \cdot \nabla B_0 , \tag{167}$$

where we have used (153), (161), and (163). Therefore,

$$\mathscr{H}_1 = \mathscr{H}_1 |_{\overline{M}} + u\pi^0 + v\sigma^0 + u_b{}^l \boldsymbol{\tau}^{(l)} \cdot \boldsymbol{\pi}_b', \tag{168}$$

where u, v, and $u_b{}^l$ are, for the moment, arbitrary multipliers.

Taking Poisson brackets of primary constraints with the Hamiltonian and setting these equal to zero, one generates the following secondary first-class constraints:

$$\varphi_3 \equiv \nabla \cdot \boldsymbol{\sigma} - j_g{}^0 = 0, \qquad \varphi_4 \equiv \nabla \cdot \boldsymbol{\pi} - j_e{}^0 = 0. \tag{169}$$

Imposing the consistency conditions

$$\left\{\boldsymbol{\tau}^{(l)} \cdot \boldsymbol{\pi}_b', \int \mathscr{H}\, d\mathbf{x}\right\} = 0 \tag{170}$$

merely determines $u_b{}^l$, since

$$|\det \|\{\theta_b{}^l, \theta_c{}^j\}\| \,|^{1/2} = \det \left\{\frac{(\mathbf{n} \cdot \nabla)^2}{\mathbf{n}^2}\right\} \neq 0 \tag{171}$$

as demonstrated in Appendix II. In Eq. (171):

$$\theta_b{}^l = \boldsymbol{\tau}^{(l)} \cdot \boldsymbol{\pi}_b'. \tag{172}$$

In view of (171), the constraints $\theta_b{}^l$ are second-class.

If we choose the gauge conditions to be

$$\chi_3 = \nabla \cdot \mathbf{B}, \qquad \chi_4 = \nabla \cdot \mathbf{A}, \qquad \chi_1 = A_0 , \qquad \chi_2 = B_0 . \tag{173}$$

the Faddeev–Popov determinant is found to be

$$\Delta_f = (\det \nabla^2)^2. \tag{174}$$

On the basis of (171), (174) and the general results of Section 3, we are led to the following expression for the S-matrix element:

$$\begin{aligned}\langle \text{out} \mid S \mid \text{in} \rangle = \int & \mathscr{D}\boldsymbol{\pi}\, \mathscr{D}\boldsymbol{\sigma}\, \mathscr{D}\mathbf{A}\, \mathscr{D}\mathbf{B}\, \mathscr{D}\pi^0\, \mathscr{D}\sigma^0\, \mathscr{D}A^0\, \mathscr{D}B^0\, \Delta_f \\ & \times \det \frac{(\mathbf{n}\cdot\nabla)^2}{\mathbf{n}^2} \prod_x \{\delta(\nabla\cdot\boldsymbol{\pi} - j_l{}^0)\, \delta(\nabla\cdot\boldsymbol{\sigma} - j_g{}^0)\, \delta(\nabla\cdot\mathbf{A})\, \delta(\nabla\cdot\mathbf{B}) \\ & \times \delta(\pi^0)\, \delta(\sigma^0)\, \delta(A_0)\, \delta(B^0)\} \prod_{x,l,a} \delta(\boldsymbol{\tau}^{(l)}\cdot\boldsymbol{\pi}_a{}') \\ & \times \exp\Big\{ i \int \Big[\boldsymbol{\pi}\cdot\dot{\mathbf{A}} + \boldsymbol{\sigma}\cdot\dot{\mathbf{B}} - \frac{1}{2}(\boldsymbol{\pi}'^2 + \boldsymbol{\sigma}'^2) + \boldsymbol{\pi}'\cdot\mathbf{G} \\ & - \boldsymbol{\sigma}'\cdot\mathbf{F} - \frac{1}{2}\mathbf{F}^2 - \frac{1}{2}\mathbf{G}^2 - j_e\cdot A - j_g\cdot B \Big] d^4x \Big\}. \end{aligned} \tag{175}$$

In order to avoid cumbersome expressions, we have ignored the spinor variables except in the coupling terms $-j_e\cdot A$ and $-j_g\cdot B$.

Integrating over π^0, σ^0, A^0, B^0 and writing, e.g.,

$$\prod_x \delta(\nabla\cdot\boldsymbol{\pi} - j_l{}^0) = \int \mathscr{D}A_0 \exp\Big\{ i \int (-\boldsymbol{\pi}\cdot\nabla A_0 - A_0 j_e{}^0)\, d^4x \Big\}, \tag{176}$$

we obtain

$$\begin{aligned}\langle \text{out} \mid S \mid \text{in} \rangle = \int & \mathscr{D}\boldsymbol{\pi}\, \mathscr{D}\boldsymbol{\sigma} \prod_\mu \mathscr{D}A_\mu\, \mathscr{D}B_\mu \prod_{l,a} \mathscr{D}\lambda_a{}^l\, \Delta_f \\ & \times \det \frac{(\mathbf{n}\cdot\nabla)^2}{\mathbf{n}^2} \prod_x \{\delta(\nabla\cdot\mathbf{A})\, \delta(\nabla\cdot\mathbf{B})\} \\ & \times \exp\Big\{ i \int d^4x \Big(\boldsymbol{\pi}'\cdot\mathbf{a} + \boldsymbol{\sigma}'\cdot\mathbf{b} - \frac{1}{2}\mathbf{G}\cdot\mathbf{F}_0 + \frac{1}{2}\mathbf{F}\cdot\mathbf{G}_0 - \frac{1}{2}\boldsymbol{\pi}'^2 \\ & - \frac{1}{2}\boldsymbol{\sigma}'^2 - \frac{1}{2}\mathbf{F}^2 - \frac{1}{2}\mathbf{G}^2 - j_e\cdot A - j_g\cdot B + \lambda_a{}^l \boldsymbol{\tau}^{(l)}\cdot\boldsymbol{\pi}_a{}' \Big) \Big\}. \end{aligned} \tag{177}$$

One can now integrate first over $\boldsymbol{\pi}'$ and $\boldsymbol{\sigma}'$ and then over $\lambda_l{}^a$, and use the fact that

$$\tfrac{1}{2}\mathbf{a}^2 + \tfrac{1}{2}\mathbf{b}^2 - \tfrac{1}{2}(\boldsymbol{\tau}^{(l)}\cdot\mathbf{a})(\boldsymbol{\tau}^{(l)}\cdot\mathbf{a}) - \tfrac{1}{2}(\boldsymbol{\tau}^{(l)}\cdot\mathbf{b})(\boldsymbol{\tau}^{(l)}\cdot\mathbf{b}) = (1/(2\mathbf{n}^2))[(\mathbf{n}\cdot\mathbf{a})^2 + (\mathbf{n}\cdot\mathbf{b})^2]. \tag{178}$$

The final result of these manipulations is the following expression for the S-matrix element

$$\langle \text{out} \mid S \mid \text{in} \rangle = \int \prod_\mu \mathscr{D}A_\mu\, \mathscr{D}B_\mu (\det \nabla^2)^2 \det \frac{(\mathbf{n}\cdot\nabla)^2}{\mathbf{n}^2} \prod_x \delta(\nabla\cdot\mathbf{A})\, \delta(\nabla\cdot\mathbf{B}) \times e^{i\int \mathscr{L} d^4x}, \tag{179}$$

where $\mathscr{L}$ is the $_M$wanziger Lagrangian with the choice $n_0 = 0$. The integration over the spinor variables, although not denoted explicitly, is understood. Equivalently,

$$\langle \text{out} \mid S \mid \text{in} \rangle = \int \prod_{\mu} \mathscr{D}A_{\mu}\, \mathscr{D}B_{\mu} (\det \Box)^2 \det \left| \frac{(n \cdot \partial)^2}{n^2} \right| \prod_{x} \delta(\partial \cdot A)\, \delta(\partial \cdot B)\, e^{i\int \mathscr{L} d^4x}, \tag{180}$$

where $n^{\mu} = (0, \mathbf{n})$. We have used the standard Faddeev–Popov [19] trick to make a transition to the covariant gauge.

Conclusion

The basic result of this paper is the generalization of Faddeev's phase space (or Hamiltonian) path integral method [4] to the case when second-class constraints are present in the theory. This is the result we have obtained in Section 3. It is a general result, applicable not only to the examples worked out in this paper, but any field theory containing second-class constraints [20, 24]. Being a generalization of Faddeev's phase space method, it has all its merits, in particular: (a) it provides a bridge between the operator formalism and the Lagrangian path integral method, (b) it is a canonical method in the sense that S-matrix elements are expressed in terms of path integrals over canonical *coordinates and momenta* and that such path integrals are invariant under canonical transformations of those coordinates and momenta, (c) it supplies the functional measure for path integrals over the coordinates after integration over the canonical momenta is performed [21]. The most interesting applications of the general result obtained in Section 3 are: (a) an alternative (and simpler) derivation of the basic result of Finkelstein, Kvitky and Mouton [7] concerning the quantization of the massive Yang–Mills field; (b) light-cone quantization of the φ^4 scalar field theory via the phase space (Hamiltonian) path integral method; (c) quantization of Zwanziger's local Lagrangian formulation of magnetic monopole theory by the same method.

Appendix I

In this appendix we wish to prove the lemma we needed in Section 3 (Eq. (68)). We shall first prove that there exists a matrix Λ^0 such that

$$\Lambda^0 \Theta (\Lambda^0)^T = \Theta', \tag{AI.1}$$

where

$$\Theta_{ab} = \{\theta_a, \theta_b\} \tag{AI.2}$$

and Θ' is given by Eq. (69). The proof of (AI.1) proceeds as follows. First, since Θ is an antisymmetric matrix of even order then by a general theorem [23] there exists an orthogonal matrix R such that

$$R\Theta R^T = \tilde{\Theta}$$

and

$$\tilde{\Theta} = \begin{bmatrix} 0 & \lambda_1 & & & & & & \\ -\lambda_1 & 0 & & & & & & \\ & & 0 & \lambda_2 & & & & \\ & & -\lambda_2 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & \lambda_n \\ & & & & & & -\lambda_n & 0 \end{bmatrix} \tag{AI.3}$$

Note that in our case all the λ_i's are different from zero, since by (50)

$$\det \tilde{\Theta} = \prod_i \lambda_i{}^2 = \det \Theta \neq 0.$$

Then define the matrix R_2:

$$R_2 = \begin{bmatrix} 1 & 0 & 0 & - & - & - & - \\ 0 & 0 & 1 & 0 & - & - & - \\ 0 & 0 & 0 & 0 & 1 & - & - \\ - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & - & - & \frac{1}{\lambda_n} \\ 0 & 0 & - & - & \frac{1}{\lambda_{n-1}} & 0 & 0 \\ - & - & - & - & - & - & - \\ 0 & 0 & 0 & \frac{1}{\lambda_2} & - & - & - \\ 0 & \frac{1}{\lambda_1} & 0 & - & - & - & - \end{bmatrix}, \tag{AI.4}$$

that is

$$\begin{aligned} (R_2)_{k\ 2k-1} &= 1 && \text{for} \quad 1 \leqslant k \leqslant n \\ (R_2)_{2M-k,2k+2} &= \frac{1}{\lambda_{k+1}} && \text{for} \quad 0 \leqslant k \leqslant n-1 \\ (R_2)_{m,n} &= 0 && \text{otherwise.} \end{aligned} \tag{AI.5}$$

One can then check that

$$R_2\tilde{\Theta}R_2^T = \Theta'. \tag{AI.6}$$

This constitutes the proof of (AI.1), since we can take

$$\Lambda^0 = R_2R. \tag{AI.7}$$

To prove our lemma given by Eqs. (67), (68), and (69), let us look for solutions for Λ_{ab} and μ_{ab} in the form

$$\Lambda_{ab} = \Lambda^0_{ab} + \bar{\Lambda}_{ab}, \qquad \mu_{ab} = \mu^0_{ab} + \bar{\mu}_{ab}, \tag{AI.8}$$

where Λ^0 is the matrix introduced in (AI.1) while μ^0_{ab} is chosen in such a way that

$$\{\chi_c, \theta_b\}\, \Lambda^0_{ab} + \mu^0_{ab}\{\chi_c, \varphi_b\} = 0. \tag{AI.9}$$

Note that μ^0_{ab} can be found by solving (AI.9), since our choice of χ_a satisfies the condition

$$\det \|\{\chi_c, \varphi_b\}\| \neq 0 \qquad \text{in } M_\epsilon.$$

If we denote by Φ_{bc} a set of functions which satisfy

$$\Phi_{bc}\{\chi_c, \varphi_a\} = \delta_{ba}, \tag{AI.10}$$

we find

$$\mu^0_{ad} = -\{\chi_c, \theta_b\}\, \Lambda^0_{ab}\Phi_{dc}. \tag{AI.11}$$

We shall look for solutions of the form

$$\bar{\Lambda}_{ab} = W^e_{ab}\theta_e + U^e_{ab}\varphi_e, \tag{AI.12}$$

$$\bar{\mu}_{ab} = Z^e_{ab}\theta_e + V^e_{ab}\varphi_e. \tag{AI.13}$$

Neglecting terms of the order of ϵ^2, comparing the coefficients of θ_a and φ_a and using (51), (52), (AI.1), and (AI.9), we see that a way of satisfying (68) is having

$$\begin{aligned}\{\chi_b, \theta_e\}\, U^c_{ae} + \{\chi_b, \theta_e\}\, Z^e_{ac} + \{\chi_b, \varphi_e\}\, V^e_{ac} + \{\chi_b, \varphi_e\}\, V^c_{ae}&\\ = -\{\chi_b, \mu^0_{ac}\},&\end{aligned} \tag{AI.14}$$

$$\begin{aligned}\{\chi_b, \theta_e\}\, W^e_{ac} + \{\chi_b, \theta_e\}\, W^c_{ae} + \{\chi_b, \varphi_e\}\, U^e_{ac} + \{\chi_b, \varphi_e\}\, Z^c_{ae}&\\ = -\{\chi_b, \Lambda^0_{ac}\},&\end{aligned} \tag{AI.15}$$

$$\begin{aligned}&W^e_{ab}\Lambda^0_{cd}\{\theta_b, \theta_d\} + W^e_{cd}\Lambda^0_{ab}\{\theta_b, \theta_d\}\\ &\quad + \Lambda^0_{ab}\{\theta_b, \theta_f\}\, W^f_{ce} + \Lambda^0_{cd}W^f_{ae}\{\theta_f, \theta_d\} = \Xi_{ac,e},\end{aligned} \tag{AI.16}$$

and

$$U^{e}_{ab}\Lambda^{0}_{cd}\{\theta_b , \theta_d\} + U^{e}_{cd}\Lambda^{0}_{ab}\{\theta_b , \theta_d\}$$
$$+ Z^{f}_{ae}\{\theta_f , \theta_d\} \Lambda^{0}_{cd} - Z^{f}_{ce}\{\theta_f , \theta_b\} \Lambda^{0}_{ab} = \rho_{ac,e} , \tag{AI.17}$$

where

$$\Xi_{ac,e} = -\Lambda^{0}_{ab}\{\theta_b , \Lambda^{0}_{ce}\} - \Lambda^{0}_{cd}\{\Lambda^{0}_{ae} , \theta_d\} - \mu^{0}_{ab}\{\varphi_b , \Lambda^{0}_{ce}\}$$
$$+ \mu^{0}_{cb}\{\varphi_b , \Lambda^{0}_{ae}\} - \mu^{0}_{ab}\Lambda^{0}_{cd}f^{e}_{bd} + \mu^{0}_{cb}\Lambda^{0}_{ad}f^{e}_{bd} , \tag{AI.18}$$

and

$$\rho_{ac,e} = -\mu^{0}_{ab}\mu^{0}_{cd}c^{e}_{bd} - \{\mu^{0}_{ae} , \varphi_d\} \mu^{0}_{cd} - \{\varphi_b , \mu^{0}_{ce}\} \mu^{0}_{ab}$$
$$- \{\mu^{0}_{ae} , \theta_d\} \Lambda^{0}_{cd} - \mu^{0}_{ab}\Lambda^{0}_{cd}e^{e}_{bd} + \mu^{0}_{cb}\Lambda^{0}_{ad}e^{e}_{bd} + \{\mu^{0}_{ce} , \theta_d\} \Lambda^{0}_{ad} . \tag{AI.19}$$

In (AI.18) and (AI.19), c^{e}_{bd} , e^{e}_{bd} , f^{e}_{bd} are the functions appearing in Eq. (51); we have changed the index notation somewhat, e.g.,

$$\{\varphi_b , \theta_d\} = e^{e}_{bd}\varphi_e + f^{e}_{bd}\theta_e . \tag{AI.20}$$

Defining now

$$U^{c}_{ae} + Z^{e}_{ac} = y^{c}_{ae} , \tag{AI.21}$$

$$t^{e}_{cb} = W^{e}_{eb} + W^{b}_{ce} , \tag{AI.22}$$

(note that $t^{e}_{cb} = t^{b}_{ce}$) and

$$q^{e}_{ac} = V^{e}_{ac} + V^{c}_{ae} , \tag{AI.23}$$

we see that (AI.14) merely provides a solution for q^{e}_{ac} once the y^{c}_{ac}'s are known, while the remaining equations can be written as

$$t^{e}_{ab}X_{bc} - t^{e}_{cb}X_{ba} = \Xi_{ac,e} , \tag{AI.24}$$

$$\{\chi_b , \theta_e\} t^{e}_{ac} + \{\chi_b , \varphi_e\} y^{e}_{ac} = -\{\chi_b , \Lambda^{0}_{ac}\}, \tag{AI.25}$$

$$y^{e}_{ab}X_{bc} - y^{e}_{cb}X_{ba} = \rho_{ac,e} , \tag{AI.26}$$

$$t^{e}_{ac} = t^{c}_{ae} , \tag{AI.27}$$

where

$$X_{bc} = \{\theta_b , \theta_d\} \Lambda^{0}_{cd} . \tag{AI.28}$$

From (AI.25) we can find y^{e}_{ac} , once t^{e}_{ac} is known. Solving for y^{e}_{ac} from (AI.25), and inserting into (AI.26), we obtain (AI.24), to the lowest order in ϵ. We omit the lengthy and straight forward proof of this statement. The basic prerequisite for this proof are Eqs. (51) and (AI.9), the distributive law for Poisson brackets, i.e.,

$$\{A, BC\} = \{A, B\}C + \{A, C\}B, \tag{AI.29}$$

the Jacobi identity for Poisson brackets, i.e.,

$$\{\{A, B\}, C\} + \{\{C, A\}, B\} + \{\{B, C\}, A\} = 0, \tag{AI.30}$$

Eqs. (AI.18), (AI.19) and (AI.28), and finally the relation

$$\{\chi_f, (\Theta')_{ac}\} = \{\chi_f, \Lambda^0_{ab}\Lambda^0_{cd}\{\theta_b, \theta_d\}\} = 0, \tag{AI.31}$$

which follows from (69), (AI.1), and (AI.2).

Thus it remains to show that there exists a set of functions t^e_{ac} such that (AI.24) and (AI.27) hold. In turn, Eqs. (AI.24) and (AI.27) are equivalent to

$$h_{fac} - h_{fca} = b_{f,ca} \tag{AI.32}$$

$$h_{afc} = h_{cfa}, \tag{AI.33}$$

where

$$b_{f,ca} = X_{ef}\Xi_{ac,e} \tag{AI.34}$$

and

$$h_{fac} = X_{ef}t^e_{ab}X_{bc}. \tag{AI.35}$$

A solution for the h's is found by the following procedure:

(a) Fix h_{afc} for $a \leqslant f \leqslant c$ arbitrarily.

(b) Use (AI.32) and (AI.33) to solve for the remaining h_{afc}.

The condition for this procedure to work is easily seen to be:

$$b_{c,af} + b_{f,ca} + b_{a,fc} = 0, \tag{AI.36}$$

which does indeed hold to lowest order in ϵ, as can be seen from (AI.34), (AI.28) and (AI.18) by repeated application of (49), (51), (69), (AI.1), (AI.29), and (AI.30).

Appendix II

Introducing a suitable notation

$$\begin{aligned} &V_\nu' = A_\nu \qquad V_\nu^2 = B_\nu \\ &\epsilon_{11} = \epsilon_{22} = 0 \qquad \epsilon_{12} = -\epsilon_{21} = 1, \end{aligned} \tag{AII.1}$$

we can write a chain of identities:

$$\begin{aligned}\{\theta_b{}^l(\mathbf{x}), \theta_c{}^k(\mathbf{y})\} &= \tau_i^{(l)}\tau_j^{(k)}\{\pi_b'^i(\mathbf{x}), \pi_c'^j(\mathbf{y})\} \\ &= \tau_i^{(l)}\tau_j^{(k)}\{\pi_b{}^i(\mathbf{x}) + \tfrac{1}{2}\epsilon_{bl}\epsilon^{isr}\,\partial_s{}^{\mathbf{x}}V_r{}^l(\mathbf{x}), \pi_c{}^j(\mathbf{y}) + \tfrac{1}{2}\epsilon_{cd}\epsilon^{juv}\,\partial_u{}^{\mathbf{y}}V_v{}^d(\mathbf{y})\} \\ &= -\epsilon_{bc}\tau_i^{(l)}\tau_j^{(k)}\epsilon_{ijs}\partial_s{}^{\mathbf{x}}\delta(\mathbf{x}-\mathbf{y}) \\ &= -\epsilon_{bc}[\boldsymbol{\tau}^{(l)} \times \boldsymbol{\tau}^{(k)}] \cdot \nabla_{\mathbf{x}}\delta(\mathbf{x}-\mathbf{y}). \end{aligned} \qquad \text{(AII.2)}$$

Because of our choice (164) and (165) for the vectors $\boldsymbol{\tau}^{(l)}$, we can write

$$\{\theta_b{}^l(\mathbf{x}), \theta_c{}^k(\mathbf{y})\} = -\epsilon_{bc}\epsilon^{lk}\frac{\mathbf{n}\cdot\nabla_{\mathbf{x}}}{|\mathbf{n}|}\,\delta(\mathbf{x}-\mathbf{y}), \qquad \text{(AII.3)}$$

whereupon Eq. (171) follows directly.

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Note added in proof. Some time after this paper was submitted for publication, a similar work by H. Yabuki [25] came to my attention.

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5. In what follows, we shall not distinguish between the indices associated with the second-class constraints and those associated with the first-class constraints. The range of the respective indices will be clear in all the expressions we shall write.
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14. The general solution is

$$C(\mathbf{x}, \mathbf{y}) = -\tfrac{1}{4}\,\epsilon(x_- - y_-)\,\delta(\underset{\sim}{x} - \underset{\sim}{y}) + h(\underset{\sim}{x}, \underset{\sim}{y})$$

where $h(\underset{\sim}{x}, \underset{\sim}{y})$ is an arbitrary function of the transverse components antisymmetric in $(\underset{\sim}{x} \leftrightarrow \underset{\sim}{y})$. We choose to discuss only the simplest solution characterized by $h(\underset{\sim}{x}, \underset{\sim}{y}) = 0$.
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20. Currently, the author is working with M. Kaku on the problem of functional measure in another such theory: Quantum gravity, when quantized in the light-cone gauge and reduced to independent field variables is a theory with second-class constraints (see [22]). Part of this work will be an application of the present paper.
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