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Effective dynamics of soft non-Abelian gauge fields at finite temperature

Dietrich Bödeker [1]

Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16, D-69120 Heidelberg, Germany

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Abstract

We consider time dependent correlation functions of non-Abelian gauge fields at finite temperature. An effective theory for the soft ($p \sim g^2 T$) field modes is derived by integrating out the field modes with momenta of order T and of order gT in a leading logarithmic approximation. In this effective theory the time evolution of the soft fields is determined by a local Langevin-type equation. As an application, the rate for hot electroweak baryon number violation is estimated as $\Gamma \sim g^2 \log(1/g)(g^2 T)^4$. Furthermore, possible consequences for non-perturbative lattice computations of unequal time correlation functions are discussed. © 1998 Elsevier Science B.V. All rights reserved.

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Finite temperature field theory for non-Abelian gauge fields is non-perturbative for soft [2] momenta $p \sim g^2 T$, where g is the gauge coupling and T is the temperature [1,2]. The dynamics of the soft gauge fields determines the rate for electroweak baryon number violation at very high temperatures [3]. This rate is determined by a real time correlation function of the type

$$C(t_1 - t_2) = \langle \mathscr{O}[A(t_1)] \mathscr{O}[A(t_2)] \rangle \tag{1}$$

where $\langle \cdots \rangle$ denotes the thermal average and $\mathscr{O}[A(t)]$ is a gauge invariant function of the gauge fields $A_\mu^a(t, \boldsymbol{x})$.

The aim of this letter is to derive an effective theory for the dynamics of the soft gauge fields by integrating out the ''hard'' $(k \sim T)$ and ''semi-hard'' $(k \sim gT)$ fields. For weak gauge coupling g this can be done perturbatively. This effective theory should

[1] E-mail: bodeker@thphys.uni-heidelberg.de.

[2] For spatial vectors we use the notation $k = |\boldsymbol{k}|$. 4-momentum vectors are denoted by $K^\mu = (k^0, \boldsymbol{k})$ and we use the metric $K^2 = k_0^2 - k^2$. P always refers to a soft momentum and K to a semi-hard or a hard one.

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allow for a non-perturbative computation of real time correlation functions like (1), e.g., on a lattice.

For time-independent physical quantities, like the free energy or correlation lengths, the effective theory for soft gauge fields is well established. It is a three-dimensional gauge theory, the parameters of which are determined by dimensional reduction [4,5]. The dimensionally reduced theory can serve as an input for Euclidean lattice simulations [6] or other non-perturbative methods [7]. The three-dimensional theory is much simpler to simulate on a lattice than the full four-dimensional one.

For time dependent quantities it may be even more important to find an effective theory for the soft field modes: There is no apparent way to calculate real time quantities like (1) in lattice simulations of finite temperature quantum field theory. Presently the only known tool to evaluate them is the classical field approximation [8–10] and variants thereof [11–14] which contain additional degrees of freedom representing the hard field modes.

The reason why the classical field approximation is expected to be reliable is that the field modes with soft momenta have a large occupation number and should therefore behave classically. The high momentum modes have occupation number of order unity and do not behave like classical *fields*. They are however weakly interacting and can be treated as almost free massless *particles* moving quasi-classically.

There has been a long discussion (see Refs. [15,16]) about whether these hard particles do affect the dynamics of the soft field modes, and, in particular, whether they play a role in electroweak baryon number violation. If all field modes with momenta larger than g^2T were irrelevant, there would be only one scale in the problem and on dimensional grounds the rate can be estimated as $\Gamma \sim (g^2T)^4$ [16]. Arnold et al. [17] demonstrated that the dynamics of the soft fields is damped by the hard particles and they obtained the estimate $\Gamma \sim g^2(g^2T)^4$. There are, however, contributions to the soft dynamics due to semi-hard field modes which are as important as the hard particles [11]. These contributions are the subject of this letter. Here only the main points of the calculation will be described. Further details can be found in Ref. [18].

The first step in deriving an effective theory for soft gauge fields is to integrate out the hard field modes with spatial momenta of order T. The dominant contributions from these fields are the so called hard thermal loops [19]. A consistent perturbative expansion for the semi-hard field modes requires the use of the so called Braaten-Pisarski scheme, or hard thermal loop effective theory [3]. In this scheme the hard thermal loop propagators and vertices must be treated on the same footing as their tree-level counterparts.

In the present context integrating out the hard field modes yields an effective theory for the semi-hard and the soft fields. As we will see below, the leading contributions to the effective soft dynamics arise from hard thermal loop induced interactions of the soft fields with the semi-hard ones. Since there are no hard thermal loop vertices in Abelian theories, the terms we are going to compute do not have an Abelian analogue (cf. Refs. [21,22]).

In order to integrate out the semi-hard fields we introduce a separation scale μ such that

$$g^2T \ll \mu \ll gT \tag{2}$$

and integrate out all field modes with $k > \mu$.

Let us first discuss what kind of diagrams are relevant to our problem. We have to consider diagrams in the hard thermal loop effective theory in which all internal loop momenta are semi-hard and the external momenta are soft. These have to be compared with the corresponding ‘‘tree-level’’ terms like, i.e., the transverse hard thermal loop polarization operator which is given by

$$\delta\Pi_{\rm t}(P) = \tfrac{1}{2}P_{\rm t}^{ij}(\boldsymbol{p})\ m_{\rm D}^2 \int \frac{d\Omega_{\rm v}}{4\pi} v^i v^j \frac{p^0}{v\cdot P}. \tag{3}$$

Here $P_{\rm t}^{ij}(\boldsymbol{p})$ is the transverse projector

$$P_{\rm t}^{ij}(\boldsymbol{p}) = \delta^{ij} - \frac{p^i p^j}{p^2} \tag{4}$$

and $m_{\rm D}^2$ is the leading order Debye mass squared. In a pure SU(N) gauge theory, it is given by $m_{\rm D}^2 =$

[3] For certain momenta $K = (k_0, \boldsymbol{k})$ with k of order gT, however, the hard thermal loop effective expansion breaks down, e.g., when K is on the light cone [20].

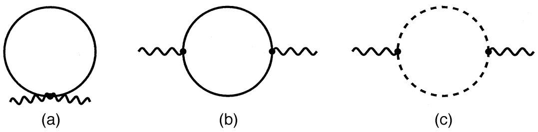

Fig. 1. Sub-leading contributions to the effective theory for the soft ($p \sim g^2T$) gauge fields. The loop momenta are semi-hard ($k \sim gT$). The small dots are ''bare'' (non hard thermal loop) vertices. The full lines denote hard thermal loop resummed propagators, the dashed lines represent ghost propagators.

$(1/3)Ng^2T^2$. In the hot electroweak theory it receives additional contributions due to the Higgs and fermion fields. Furthermore, $v^\mu \equiv (1,\boldsymbol{v})$, and the integral $\int d\Omega_v$ is over the directions of the unit vector $\boldsymbol{v}$, $|\boldsymbol{v}| = 1$.

Consider the corrections to Eq. (3) due to the semi-hard field modes. The leading order behavior of the diagrams in Fig. 1 can be easily estimated since they are completely analogous to hard thermal loop diagrams for semi-hard external momenta except that now all momenta are smaller by a factor g. The k integral behaves like $\int dk \sim gT$ which gives a suppression factor g relative to the hard thermal loops. In other words, the diagrams in Fig. 1 can be neglected relative to the ''tree-level'' term (3).

Now consider the analogous diagrams (Fig. 2) with the ''bare'' vertices replaced by hard thermal loop vertices. There are only two such diagrams since there is no hard thermal loop vertex involving ghost fields. The vertex in Fig. 2(a) contains a term proportional to $g^2 p_0/(v \cdot P)^2$ times a factor of order gT. Here the four vector v^μ is of the same type as in Eq. (3). Due to the Bose distribution function in the loop integral there is a factor T so that diagram 2(a) can be estimated as $m_{\rm D}^2 g^2 T p_0/(v \cdot P)^2$. Comparing this expression with Eq. (3) we see that both terms are of the same order of magnitude. In fact, the

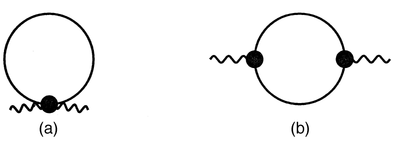

Fig. 2. Leading contributions to the effective theory for the soft ($p \sim g^2T$) gauge fields. The loop momenta are semi-hard ($k \sim gT$). The full lines denote hard thermal loop resummed propagators and the heavy dots are hard thermal loop vertices.

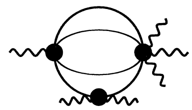

Fig. 3. Contribution to the effective theory for the soft ($p \sim g^2T$) gauge fields involving hard thermal loop n-point vertices. The notation is the same as in Fig. 2.

diagrams in Fig. 2 can be even larger by a factor of $\log(gT/\mu)$: The transverse propagator is unscreened when $|k_0| \ll k$ and in the infrared the loop integrals can be sensitive to the separation scale μ. We will see below that this logarithm indeed occurs. It is specific to the transverse semi-hard field modes. In this letter we will compute only these leading logarithmic contributions to the effective soft dynamics.

In order to obtain the effective theory for the soft field modes we have to consider diagrams with more external soft fields and with more internal semi-hard propagators like in Fig. 3. Higher hard thermal loop n-point vertices contain a large number of terms which make a diagrammatic analysis very difficult. It is more convenient to use an effective field theory description and solve the corresponding field equations of motion.

There are local [4] formulations of the hard thermal loop effective theory due to Blaizot and Iancu [23] and due to Nair [24]. These formulations make the physical content of the hard thermal loop effective theory quite transparent (see also Ref. [25]). The formulation due to Blaizot and Iancu is the non-Abelian generalization of the Vlasov equations for a QED plasma. It consists of Maxwell's equation for the semi-hard and soft gauge fields, i.e., fields with spatial momenta of order gT or less,

$$\left[D_\mu, F^{\mu\nu}(x)\right] = j^\nu(x) \tag{5}$$

where $D_\mu = \partial_\mu - igA_\mu(x)$ is the covariant derivative and $A_\mu(x) = A_\mu^a(x)T^a$ with hermitian generators normalized to $\mathrm{Tr}(T^aT^b) = (1/2)\delta^{ab}$. The current $j^\nu(x)$ is due to the hard field modes. These modes are weakly interacting and they behave as massless

[4] The hard thermal loop vertices are non-local in space and time.

particles moving at the speed of light with 3-velocity $\boldsymbol{v}$. The current can be written as

$$j^\nu(x) = m_D^2 \int \frac{d\Omega_v}{4\pi} v^\nu W(x,v). \tag{6}$$

The field $W(x,v) = W^a(x,v)T^a$ is proportional to the deviation of the distribution of hard particles from the equilibrium distribution. It is determined by the equation of motion

$$[v \cdot D, W(x,v)] = \boldsymbol{v} \cdot \boldsymbol{E}(x), \tag{7}$$

where $\boldsymbol{E}$ is the electric field strength. The conserved Hamiltonian corresponding to Eqs. (5)–(7) is [26]

$$H = \int d^3x \mathrm{Tr}\Big\{ \boldsymbol{E}(x) \cdot \boldsymbol{E}(x) + \boldsymbol{B}(x) \cdot \boldsymbol{B}(x) + m_D^2 \int \frac{d\Omega_v}{4\pi} W(x,v) W(x,v) \Big\}. \tag{8}$$

Originally, the hard thermal loop effective theory was designed for calculating Green functions with semi-hard momenta. Following [11] we assume that the leading order dynamics for both soft and semi-hard fields is correctly described by the hard thermal loop effective theory. More precisely, we assume that real time correlation functions of gauge invariant operators like (1) can be obtained as follows:

- Compute the solution $A_\mu^a(x)$ of the non-Abelian Vlasov equations (5)–(7) for given initial conditions $A_{\rm in}(\boldsymbol{x})$, $\boldsymbol{E}_{\rm in}(\boldsymbol{x})$ and $W_{\rm in}(\boldsymbol{x},v)$ at $t = 0$.
- Then $C(t_1 - t_2)$ is given by the product $\mathscr{O}[A(t_1)]\mathscr{O}[A(t_2)]$ averaged over the initial conditions with the Boltzmann weight $\exp(-\beta H[A_{\rm in}, \boldsymbol{E}_{\rm in}, W_{\rm in}])$ where Gauss' law has to be imposed as a constraint.

In the average over initial conditions ultraviolet divergences occur. These are due to the fact that a classical field theory at finite temperature is not well defined. One has to use an UV cutoff Λ which satisfies

$$gT \ll \Lambda \ll T. \tag{9}$$

After the semi-hard field modes have been integrated out, the dependence on Λ cancels against the appropriate Λ-dependent counter-terms to be included in the hard thermal loop effective theory [5].

[5] See also the discussion in Ref. [14]

To integrate out the semi-hard modes, we decompose A, $\boldsymbol{E}$ and W into

$$\begin{aligned} A &\to A + a \\ \boldsymbol{E} &\to \boldsymbol{E} + \boldsymbol{e} \\ W &\to W + w. \end{aligned} \tag{10}$$

The soft modes [6] A, $\boldsymbol{E}$ and W contain the spatial Fourier components with $p < \mu$ while the semi-hard modes a, $\boldsymbol{e}$ and w consist of those with $k > \mu$. Let us note that the separation (10) does not respect gauge invariance. Nevertheless, the results we are going to obtain, will not depend on the choice of a gauge. Furthermore, we do not specify the way the cutoff is realized precisely. The only cutoff dependence which we will find is a logarithmic one for which the precise choice should be irrelevant. Here we consider only the transverse semi-hard gauge fields.

Due to the non-linear terms in Eqs. (5) and (7) the equations of motion for the soft and the semi-hard fields are coupled. Let us see which of these couplings are relevant to our problem. The non-linear terms in Maxwell's equations (5) correspond to non-hard thermal loop vertices. For the interaction between soft and semi-hard fields these vertices can be neglected as we have already argued. The important vertices are the hard thermal loop ones. They correspond to non-linear terms in Eq. (7). Thus, substituting Eq. (10) into Eqs. (5)–(7), the equations for the soft fields read

$$[D_\mu, F^{\mu\nu}(x)] = m_D^2 \int \frac{d\Omega_v}{4\pi} v^\nu W(x,v) \tag{11}$$

and

$$[v \cdot D, W(x,v)] = \boldsymbol{v} \cdot \boldsymbol{E}(x) + \xi(x,v), \tag{12}$$

where

$$\xi^a(x,\boldsymbol{v}) = g f^{abc}\big(\boldsymbol{v} \cdot \boldsymbol{a}_t^b(x) w^c(x,v)\big)_{\rm soft}. \tag{13}$$

Here the subscript ''soft'' indicates that only spatial Fourier components with $p < \mu$ are included. The structure constants f^{abc} are defined such that $[T^a, T^b] = i f^{abc} T^c$. The transverse gauge field $\boldsymbol{a}_t(x)$

[6] For notational simplicity we do not introduce new symbols for the soft modes. From now on A, $\boldsymbol{E}$ and W will always refer to the soft fields only.

and the field $w(x,v)$ in Eq. (13) are to be understood as the solutions to the coupled equations of motion (11)–(13) and (14)–(16).

In the following we discuss how the semi-hard fields can be eliminated from Eqs. (11)–(13) leading to the final result (29). A more detailed description of the calculation can be found in Ref. [18].

The semi-hard fields are weakly interacting and can therefore be treated perturbatively. Since we are only interested in leading order results, it might appear sufficient to insert the solutions of the free equations of motion, $\boldsymbol{a}_{t0}(x)$ and $w_0(x,v)$ into $\xi(x,v)$. The corresponding term will be called $\xi_0(x,v)$. As will be seen below, a next-to-leading order term has to be included as well. For this term, which will be denoted $\xi_1(x,v)$, one can neglect those interactions in Eq. (7) which contain only semi-hard fields. In the diagrammatic language such interactions correspond to parts of hard thermal loop vertices which do not contain the enhancing factors $1/(v\cdot P)$ discussed above. The relevant non-linear terms are the ones which contain a product of a soft and a semi-hard field. Thus the equations of motion in spatial momentum space can be written as

$$\ddot{\boldsymbol{a}}_t^a(t,\boldsymbol{k}) + k^2\boldsymbol{a}_t^a(t,\boldsymbol{k}) = m_D^2\int\frac{d\Omega_v}{4\pi}\boldsymbol{v}_t w^a(t,\boldsymbol{k},v), \tag{14}$$

$$\dot{w}^a(t,\boldsymbol{k},v) + i\boldsymbol{v}\cdot\boldsymbol{k}w^a(t,\boldsymbol{k},v) = \boldsymbol{v}\cdot\boldsymbol{e}_t(t,\boldsymbol{k}) + h^a(t,\boldsymbol{k},v), \tag{15}$$

where $v_t^i = P_t^{ij}(\boldsymbol{k})v^j$ is the transverse projection of $\boldsymbol{v}$. The term $h^a(t,\boldsymbol{k},v)$ contains the interaction of the semi-hard fields with the soft ones. In coordinate space it reads

$$h^a(x,v) = gf^{abc}\big(-v\cdot A^b(x)w^c(x,v) + \boldsymbol{v}\cdot\boldsymbol{a}_t^b(x)W^c(x,v)\big). \tag{16}$$

A convenient method for solving Eqs. (14)–(16) is the Laplace transformation [7]. The Laplace transform of a function $f(t)$ is defined as

$$f(k^0) \equiv \int_0^\infty dt e^{ik^0t}f(t) \tag{17}$$

and it is an analytic function in the upper half of the complex k^0-plane. Applying (17) to Eqs. (14) and (15) one obtains

$$a_t^{ia}(K) = a_{t0}^{ia}(K) + \int\frac{d\Omega_{v_1}}{4\pi}\Delta_{12}^i(K,v_1)h^a(K,v_1) \tag{18}$$

and

$$w^a(K,v) = w_0^a(K,v) + \int\frac{d\Omega_{v_1}}{4\pi}\Delta_{22}^i(K,v,v_1)h^a(K,v_1), \tag{19}$$

where $\boldsymbol{a}_{t0}(K)$ and $w_0(K,v)$ are the solutions to the free equations of motion. They are determined by the initial values at $t=0$. The propagator functions in (18), (19) are given by

$$\Delta_{12}^i(K,v) = im_D^2 v^j G_t^{ij}(K)\frac{1}{v\cdot K}, \tag{20}$$

$$\Delta_{22}(K,v,v_1) = 4\pi\delta^{(S^2)}(\boldsymbol{v}-\boldsymbol{v}_1)\frac{i}{v\cdot K} - m_D^2 v^i v_1^j G_t^{ij}(K)\frac{ik^0}{v\cdot Kv_1\cdot K}. \tag{21}$$

Here $G_t^{ij}(K)$ is the transverse hard thermal loop resummed propagator,

$$G_t^{ij}(K) = \frac{1}{-K^2+\delta\Pi_t(K)}P_t^{ij}(\boldsymbol{k}). \tag{22}$$

Furthermore, $\delta^{(S^2)}$ is the delta function on the two dimensional unit sphere:

$$\int\frac{d\Omega_{v_1}}{4\pi}f(\boldsymbol{v}_1)\delta^{(S^2)}(\boldsymbol{v}-\boldsymbol{v}_1) = f(\boldsymbol{v}). \tag{23}$$

The solutions to the equations of motion, Eqs. (18) and (19), are still formal in the sense that $h^a(K,v)$ contains the complete solutions $\boldsymbol{a}_t(K)$ and $w(K,v)$ as well as $A(P)$ and $W(P,v)$. However, by iterating Eqs. (18) and (19) one obtains a series in

[7] Or one sided Fourier transformation, see,e.g., Ref. [29].

which each term contains only the free solutions $\boldsymbol{a}_{t0}(K)$ and $w_0(K,v)$ together with the full $A(P)$ and $W(P,v)$. Since Eqs. (14) and (15) are linear in $\boldsymbol{a}_t$ and w, the solutions are linear in $\boldsymbol{a}_{t0}$ and w_0. The function $\xi(x,v)$ is therefore bilinear in the fields $\boldsymbol{a}_{t0}$ and w_0 which always appear in the form

$$\chi = \phi(K-P)\phi'(P'-K), \tag{24}$$

where the ϕ and ϕ' are either $\boldsymbol{a}_{t0}$ or w_0. The solution to the (non-perturbative) equations of motion for the soft fields will contain products of χ's. To obtain a correlation function like (1) one has to average over the initial conditions for both soft and semi-hard fields.

In general, the thermal average of a product $\chi_1 \cdots \chi_n$ can be approximated by disconnected parts: The connected part $\langle \chi_1 \cdots \chi_n \rangle_c$ has one momentum conserving delta function for a soft momentum. A disconnected part like for instance $\langle \chi_1 \cdots \chi_m \rangle_c \langle \chi_{m+1} \cdots \chi_n \rangle_c$ contains two such delta functions. The corresponding contribution to a non-perturbative correlation function has one soft 4-momentum integration less and one additional semi-hard 4-momentum integration. We can therefore estimate [8]

$$\langle \chi_1 \cdots \chi_n \rangle_c \sim g^3 \frac{p^0}{gT} \langle \chi_1 \cdots \chi_m \rangle_c \langle \chi_{m+1} \cdots \chi_n \rangle_c . \tag{25}$$

Here p^0 is the typical frequency scale for the soft non-perturbative dynamics. This scale will be estimated from the equations of motion for the soft fields which we are going to derive (see Eq. (45)). This consideration suggests that $\xi(x,v)$ in Eq. (12) can be simplified by replacing the χ_i by their thermal averages: $\chi_i \to \langle \chi_i \rangle$. For the lowest order term ξ_0, which does not depend on the soft fields, this replacement gives zero due to the antisymmetry of the structure constants f^{abc}. Therefore we have to leave ξ_0 in Eq. (12) as it stands and the solution to the equations of motion for the soft fields will contain products of ξ_0's. The thermal averages of these products can be approximated by a product of two point correlators. We also have to include the next-to-leading order term ξ_1 which is linear in the soft fields A and W.

Neglecting the external soft 4-momentum inside loop integrals the leading logarithmic result for the two point correlator of ξ_0 in configuration space reads

$$\begin{aligned} &\langle \xi_0^a(x_1,v_1)\xi_0^b(x_2,v_2) \rangle \\ &\quad = -2N\frac{g^2T^2}{m_D^2}\log\left(\frac{gT}{\mu}\right) I(v_1,v_2) \\ &\qquad \times \delta^{ab}\delta^{(4)}(x_1-x_2), \end{aligned} \tag{26}$$

with

$$I(v,v_1) \equiv -\delta^{(S^2)}(\boldsymbol{v}-\boldsymbol{v}_1) + \frac{1}{\pi^2}\frac{(\boldsymbol{v}\cdot\boldsymbol{v}_1)^2}{\sqrt{1-(\boldsymbol{v}\cdot\boldsymbol{v}_1)^2}} . \tag{27}$$

The logarithmic μ-dependence in Eq. (26) arises because the transverse propagator (22) is unscreened for $|k^0| \ll k$. Non-logarithmic contributions have been neglected in Eq. (26). To compute them, one would have to take into account the longitudinal semi-hard gauge fields as well. Since the external soft momentum has been neglected in order to obtain Eq. (26), this expression is valid only when applied to problems with a relevant time scale much larger than $(gT)^{-1}$.

For ξ_1 we can replace the products of the free semi-hard fields $\boldsymbol{a}_{t0}$ and w_0 by their expectation values. Then the terms containing $A(x)$ cancel. With the same approximations which were used for Eq. (26) one obtains

$$\begin{aligned} \xi_1(x,v) &= Ng^2T\log\left(\frac{gT}{\mu}\right) \\ &\quad \times \int \frac{d\Omega_{v_1}}{4\pi} I(v,v_1)W(x,v_1). \end{aligned} \tag{28}$$

Note that without the cancelation of the $A(x)$-dependent terms Eq. (29) would not be gauge covariant.

[8] As we have already discussed, the integration over semi-hard momenta can give logarithms of the separation scale μ. These logarithms are not included in this estimate.

With the above approximations the equation of motion (12) reads

$$[v\cdot D,W(x,v)] = \boldsymbol{v}\cdot\boldsymbol{E}(x)+\xi_0(x,v) + Ng^2T\log\left(\frac{gT}{\mu}\right) \times\int\frac{d\Omega_{\mathrm{v}_1}}{4\pi}I(v,v_1)W(x,v_1). \tag{29}$$

Let us see whether Eq. (29) is consistent with Maxwell's equation (11) for the soft fields which requires the current on the r.h.s. of Eq. (11) to be covariantly conserved. Integrating Eq. (29) over the direction of $\boldsymbol{v}$ the third term on the r.h.s. drops out due to

$$\int\frac{d\Omega_{\mathrm{v}}}{4\pi}I(v,v_1)=0 \tag{30}$$

and one obtains

$$[D_\mu,j^\mu(x)] = m_{\mathrm{D}}^2\int\frac{d\Omega_{\mathrm{v}}}{4\pi}\xi_0(x,v). \tag{31}$$

That is, the current appears not to be conserved. However, as we have argued, only the two point function of ξ_0 should be relevant for the leading order behavior of a soft correlator. But the two point function of $\xi_0(x_1,v_1)$ with the r.h.s. of Eq. (31) vanishes due to Eq. (30). Thus we can replace

$$\int\frac{d\Omega_{\mathrm{v}}}{4\pi}\xi_0(x,v)\to 0 \tag{32}$$

so that the current is covariantly conserved within the present approximation.

Let us further note that, due to the term ξ_0, Eq. (29) appears to be non-covariant under gauge transformations. Nevertheless, a gauge invariant correlation function computed via Eqs. (26) and (29) will not depend on the choice of the gauge after the thermal average over initial conditions since the two point function (26) is invariant under gauge transformations of ξ_0.

Eq. (29) is a Boltzmann equation for the soft fluctuations of the particle distribution $W(x,v)$. The r.h.s. contains a collision term which is due to the interactions with the semi-hard fields. When $\partial_i W$ on the l.h.s. of Eq. (29) is of order $(g^2T)W$, the collision term is larger than $\partial_i W$ by a factor $\log(gT/\mu)$. The reason is that the mean free path for the hard particles interacting with fields with $k>\mu$ is of order $(g^2T\log(gT/\mu))^{-1}$ which is smaller than the soft length scale $(g^2T)^{-1}$ by the same logarithm. The collision term is accompanied by the noise term ξ_0 which is due to the thermal fluctuation of initial conditions [9] of the fields with $k>\mu$.

For a QED plasma there is no collision term at this order in the coupling constant. In this case the size of the collision term is determined by the transport cross section which corresponds to a mean free path of order order $(e^4T)^{-1}$ (cf. the discussion in Ref. [27]). For a non-Abelian plasma the relevant mean free path is determined by the total cross section which is dominated by small angle scattering: Even a scattering process which hardly changes the momentum of a hard particle can change its color charge which is what is seen by the soft gauge fields.

What can one learn from Eq. (29) about the time scale relevant to non-perturbative physics? Let us first see how many field modes can be integrated out perturbatively. The mean free path for the hard particles interacting with the modes with $k>\mu$ is of order $(g^2T\log(gT/\mu))^{-1}$. Perturbation theory for the hard particles must break down at this length scale. Thus we can decrease μ until $\mu\sim g^2T\log(gT/\mu)$. Then the logarithm in Eq. (29), in a first approximation, becomes $\log(1/g)$.

We will now simplify Eq. (29) by neglecting terms which are suppressed by inverse powers of $\log(1/g)$. We introduce moments of $W(x,v)$ and $\xi_0(x,v)$,

$$W(x)\equiv\int\frac{d\Omega_{\mathrm{v}}}{4\pi}W(x,v), \tag{33}$$

$$W^{i_1\cdots i_n}(x)\equiv\int\frac{d\Omega_{\mathrm{v}}}{4\pi}v^{i_1}\cdots v^{i_n}W(x,v), \tag{34}$$

$$\xi_0^{i_1\cdots i_n}(x)\equiv\int\frac{d\Omega_{\mathrm{v}}}{4\pi}v^{i_1}\cdots v^{i_n}\xi_0(x,v). \tag{35}$$

Multiplying Eq. (29) with the appropriate factors $v^{i_1}\cdots v^{i_n}$ and integrating over the direction of $\boldsymbol{v}$

[9] The term $\xi_0(x,v)$ should not be confused with the stochastic force discussed in Refs. [17,27,30]. The latter is due to the fluctuations of the initial conditions for the soft $W(x,v)$ and it is also present in the theory described by Eq. (29).

one obtains a set of coupled equations for the moments of $W(x,v)$. From now on we will use the temporal axial gauge $A_0=0$ which is the most convenient for our purpose. The zeroth moment of Eq. (29) is the equation for current conservation (31), (32). Taking the first moment of Eq. (29) gives

$$\partial_0 W^i(x) - \left[D^j, W^{ij}(x)\right] = \tfrac{1}{3}E^i(x) + \xi_0^i(x) - \frac{Ng^2T}{4\pi}\log(1/g)W^i(x). \tag{36}$$

The l.h.s. of Eq. (36) is logarithmically suppressed [10] relative to the term $\propto W^i(x)$ on the r.h.s. and can be neglected. In this approximation we can determine $W^i(x)$ in terms of $\boldsymbol{E}(x)$ and $\xi_0^i(x)$ without solving any differential equation. Introducing

$$\zeta^i(x) \equiv 4\pi\frac{m_{\mathrm{D}}^2}{Ng^2T}\frac{1}{\log(1/g)}\xi_0^i(x) \tag{37}$$

and

$$\gamma = \frac{4\pi}{3}\frac{m_{\mathrm{D}}^2}{Ng^2T}\frac{1}{\log(1/g)}, \tag{38}$$

the spatial components of Eq. (11) become

$$-\partial_0 E^i + \left[D_j, F^{ji}(x)\right] = \gamma E^i(x) + \zeta^i(x). \tag{39}$$

The 0 component, or Gauss' law, now reads [11]

$$\left[D_i, E^i(x)\right] = j_0(x). \tag{40}$$

The charge density $j_0(x) = m_{\mathrm{D}}^2 W(x)$ satisfies

$$\partial_0 j_0(x) = -\gamma j_0(x) - \left[D_i, \zeta^i(x)\right]. \tag{41}$$

Eqs. (39)–(41) form a closed set of equations for the gauge fields and the charge density $j_0(x)$. The stochastic force $\zeta^i(x)$ (cf. Eq. (26)) satisfies

$$\left\langle \zeta^{ia}(x_1)\zeta^{jb}(x_2)\right\rangle = 2T\gamma\delta^{ij}\delta^{ab}\delta^{(4)}(x_1-x_2). \tag{42}$$

Eqs. (39)–(42) are gauge covariant Langevin-type equations. The dynamics of the soft fields is Landau-damped by the hard field modes (see, e.g., Ref. [29]). The kinematics of the hard modes being massless particles moving on straight lines no longer appears in these equations because the semi-hard fields randomize [12] the color of the hard particles on a length scale $(\log(1/g)g^2T)^{-1}$ which is small compared to the length scale $(g^2T)^{-1}$ of non-perturbative physics. The random force $\zeta^{ia}(x)$ is due to the thermal fluctuations of the initial conditions for the semi-hard gauge field modes and for $w(x,v)$.

We will now solve the linearized form of the equation of motion (39). Even though we are interested in the non-perturbative dynamics of the soft gauge fields, this will allow us to determine the frequency scale on which the dynamics of these fields becomes non-perturbative. The solution for the transverse field reads

$$A_{\mathrm{t}}^{ia}(P) = -\frac{i}{p_0}p^2\left[\overline{G}_{\mathrm{t}}(P) - \overline{G}_{\mathrm{t}}(0,\boldsymbol{p})\right]A_{\mathrm{t,in}}^{ia}(\boldsymbol{p}) + \overline{G}_{\mathrm{t}}(P)\left[-E_{\mathrm{t,in}}^{ia}(\boldsymbol{p}) + \zeta_{\mathrm{t}}^{ia}(P)\right], \tag{43}$$

where the subscript ''in'' refers to the the initial values at $t=0$. The transverse propagator $\overline{G}_{\mathrm{t}}$ is given by

$$\overline{G}_{\mathrm{t}}(P) \equiv \frac{1}{-P^2 - i\gamma p_0}. \tag{44}$$

The typical size of the soft field is $\boldsymbol{A}_{\mathrm{t}}(x) \sim \boldsymbol{A}_{\mathrm{t,in}}(\boldsymbol{x}) \sim gT$. Thus both terms in a covariant derivative $\partial_\mu - igA_\mu$ are of the same size making the thermodynamics of the soft fields non-perturbative.

The dynamics of the soft gauge field becomes non-perturbative if it changes in time by an amount ΔA which is of the same size as A itself. Then the non-linear terms in the equations of motion are as important as the linear ones. $\Delta\boldsymbol{A}_{\mathrm{t}}$ can be estimated as $\Delta\boldsymbol{A}_{\mathrm{t}}(\boldsymbol{p}) \sim p^2\overline{G}_{\mathrm{t}}(P)\boldsymbol{A}_{\mathrm{t,in}}(\boldsymbol{p})$. From Eq. (44) one reads off that $\Delta\boldsymbol{A}_{\mathrm{t}}(\boldsymbol{p}) \sim \boldsymbol{A}_{\mathrm{t,in}}(\boldsymbol{p})$ when $p_0 \sim g^4\log(1/g)T$. Thus we find that large non-perturbative changes $\Delta\boldsymbol{A}_{\mathrm{t}}$ of the transverse (magnetic) gauge fields are associated with the frequency scale

$$p_0 \sim g^4\log(1/g)T. \tag{45}$$

Let us now use this result to estimate the rate for electroweak baryon number violation at very high

[10] Provided that $\partial_0 \lesssim g^2T$, see below.

[11] For the charge density in Eqs. (40) and (41) we use the same symbol as for the one in Eqs. (5) and (6). The latter has both soft and semi-hard Fourier components.

[12] I thank Guy Moore for this explanation.

temperatures [3,15–17], i.e., well above the critical temperature $T_c \sim 100$ GeV of the electroweak phase transition or crossover. At such high temperatures the Higgs field acquires a large thermal mass and decouples from the dynamics. Then it is sufficient to consider a pure SU(2) gauge theory. Baryon number nonconserving processes are due to topology changing transitions in the electroweak theory which are characterized by a change $\Delta N_{\rm CS} = \pm 1$ of the Chern–Simons number

$$N_{\rm CS} = \frac{g^2}{32\pi^2}\epsilon_{ijk}\int d^3x \left(F^a_{ij} A^a_k - \frac{g}{3}\epsilon^{abc} A^a_i A^b_j A^c_k \right). \tag{46}$$

Here A^a_i are now the SU(2) gauge fields and g is the weak coupling constant. The change of baryon number ΔB is related to $\Delta N_{\rm CS}$ by

$$\Delta B = n_f \Delta N_{\rm CS} \tag{47}$$

where n_f is the number of fermion families. A change $\Delta N_{\rm CS} \sim 1$ requires the formation of a magnetic field configuration with energy of order $(g^2 \Delta R)^{-1}$ where ΔR is its spatial extent. In order for this configuration not to be Boltzmann suppressed, its energy should not be larger than the temperature which requires $\Delta R \gtrsim (g^2 T)^{-1}$. On the other hand, the size of the field configuration cannot exceed the correlation length which is of order $(g^2 T)^{-1}$. Thus the length scale relevant to the problem is just the soft scale $(g^2 T)^{-1}$ at which finite temperature perturbation theory breaks down. The amplitude of this field configuration is $A \sim gT$. As we have argued above, field configuration of this size evolve with a frequency of order $g^4 \log(1/g) T$ corresponding to a time scale $\Delta t \sim (g^4 \log(1/g) T)^{-1}$. Thus the rate for a change of baryon number B per unit time and unit volume can be estimated as

$$\Gamma = \kappa g^2 \log(1/g) (g^2 T)^4, \tag{48}$$

where κ is a non-perturbative coefficient which does not depend on the gauge coupling g.

What might be the use of the effective theory for the soft gauge fields derived in this letter for non-perturbative lattice computations of real time correlators like (1)? The time scale for non-perturbative dynamics is much larger than the corresponding length scale. Therefore the time derivative on the l.h.s. of Eq. (39) can be neglected and one can write

$$\left[D_j, F^{ji}(x) \right] = -\gamma \dot{A}^i(x) + \zeta^i(x). \tag{49}$$

This equation should be easy to implement in a lattice calculation. Note that we have obtained this equation by assuming that the ultraviolet cutoff μ for the soft fields is as small as $\log(1/g) g^2 T$. Therefore the lattice cutoff should not be chosen too large. It would be interesting to see whether the results obtained via Eq. (49) depend on the cutoff. If they do, there might be a way to include counterterms such that the cutoff dependence is canceled.

It is conceivable that, unlike the hard thermal loops [11,30], the effective theory for the soft fields in a classical lattice gauge theory has a similar structure as in the quantum theory. If this turned out to be the case, one could match the coefficients in the classical counterpart of Eq. (39) with the coefficients obtained in this letter. Then a numerical computation in the classical lattice theory should give the correct leading order result for a non-perturbative correlation function. The difficulty of this approach is that sub-leading contributions to the effective theory are only logarithmically suppressed and can therefore not easily be identified in numerical data. It might be possible to improve the matching such that also non-logarithmic terms are included. This would require an extension of the calculation presented here.

In Ref. [13] an algorithm was employed which, in addition to classical gauge fields, contains massless particles which interact weakly with the gauge fields. This approach solves the problem of implementing hard thermal loops on a lattice and it appears to be equivalent to the starting point for the present calculation, i.e., the hard thermal loop effective theory described by Eqs. (5)–(8). With this algorithm it is again difficult to distinguish logarithmic from non-logarithmic contributions. Probably only a combination of the different methods discussed above can determine the rate for electroweak baryon number violation.

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