High temperature color conductivity at next-to-leading log order

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The non-Abelian analogue of electrical conductivity at high temperature has previously been known only at leading logarithmic order — that is, neglecting effects suppressed only by an inverse logarithm of the gauge coupling. We calculate the first sub-leading correction. This has immediate application to improving, to next-to-leading log order, both effective theories of non-perturbative color dynamics, and calculations of the hot electroweak baryon number violation rate.

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I. INTRODUCTION

We will provide a next-to-leading log order (NLLO) calculation of the non-Abelian (or ''color'') conductivity in hot, weakly coupled non-Abelian plasmas. The motivation for this calculation and an overview of the strategy are presented in Ref. [1]. Here, we will simply get down to business. ''Hot'' plasma means hot enough (1) to be ultra-relativistic, (2) to ignore chemical potentials, (3) for non-Abelian gauge couplings to be small, and (4) to be in the high-temperature symmetric phase if there is a Higgs mechanism.

As discussed in Ref. [1], there is a sequence of effective theories which describe color dynamics at large distance scales and long time scales.

Theory 1: $\omega,k \ll T$,

$$(D_t+\mathbf{v}\cdot\mathbf{D})\,W-\mathbf{v}\cdot\mathbf{E}=0, \tag{1.1a}$$

$$D_\nu F^{\mu\nu}=J^\mu=m^2\,\langle v^\mu W\rangle. \tag{1.1b}$$

These ''hard-thermal-loop'' equations amount to linearized, collisionless, non-Abelian, Boltzmann-Vlasov kinetic theory. The parameter m is the Debye screening mass, which is $O(gT)$. This effective theory is valid, to leading order in the gauge coupling, for frequencies and momenta small compared to the temperature, $\omega,k \ll T$.

Theory 2: $\omega \ll k \ll m$,

$$\mathbf{v}\cdot\mathbf{D}\,W-\mathbf{v}\cdot\mathbf{E}=-\delta\hat{C}\,W+\xi, \tag{1.2a}$$

$$\langle W\rangle=0, \tag{1.2b}$$

$$\mathbf{D}\times\mathbf{B}=\mathbf{j}=m^2\,\langle\mathbf{v}W\rangle, \tag{1.2c}$$

$$\langle\!\langle \xi\xi\rangle\!\rangle=\frac{2T}{m^2}\,\delta C. \tag{1.2d}$$

This theory is a stochastic, collisional, linearized kinetic theory of hard excitations coupled to slowly varying non-Abelian gauge fields. Both the noise ξ, and the linearized collision operator $\delta\hat{C}$, arise from integrating out the effects of gauge field fluctuations below the scale of m. Here, and henceforth, $\langle\!\langle\cdots\rangle\!\rangle$ denotes an average over the (Gaussian) stochastic noise. Theory 2 is valid for spatial momenta small compared to the Debye screening mass and frequencies small compared to momenta, $\omega \ll k \ll m = O(gT)$. Equation (1.2b) implements the effects of Gauss' law in this range of ω and k and is explained in Ref. [2].

Theory 3: $\omega \ll k \ll \gamma$,

$$\sigma\mathbf{E}=\mathbf{D}\times\mathbf{B}+\boldsymbol{\zeta}, \tag{1.3a}$$

$$\langle\!\langle \boldsymbol{\zeta\zeta}\rangle\!\rangle=2\sigma T. \tag{1.3b}$$

This final theory is a stochastic Langevin equation, known as Bödeker's effective theory [3] or ''the small frequency limit of Ampere's law in a conductor'' [4]. The parameter σ is the ''color conductivity.'' Theory 3 is only valid on spatial momentum scales small compared to the hard gluon damping rate γ and frequencies small compared to momenta, $\omega \ll k \ll \gamma = O[g^2T\ln(g^{-1})]$.

Our goal is to calculate, by successive matching of these effective theories from short to large distance scales, the parameter σ of theory 3.

The spatial scale at which physics becomes non-perturbative is $k\sim g^2T$. So, by at least a logarithm, the interfaces $(m,\ \gamma)$ of the successive effective theories are associated with perturbative physics, thus making a perturbative matching calculation possible. It will be useful to keep in mind a simple result from the analysis of static properties of hot gauge theories, which is that the parameter which controls the loop expansion is g^2T/k, where k is the momentum scale of interest. So, in particular, the loop expansion for physics at the interface $k\sim\gamma$ between theory 2 and theory 3 is an expansion in inverse logarithms $[\ln(1/g)]^{-1}$.

Our notation is the same as that of Ref. [1] and is summarized in Table I. W represents the adjoint color distribution of hard particles, $\delta\hat{C}$ is a linearized collision operator, ξ and $\boldsymbol{\zeta}$ are Gaussian thermal noise, m is the leading-order Debye mass, and γ is the hard gluon damping rate. Note in particular that $\langle\cdots\rangle$ denotes averages over the direction $\mathbf{v}$ of hard particle velocities,[1] whereas $\langle\!\langle\cdots\rangle\!\rangle$ denotes averages over Gaussian noise. Also note that we use δC to represent

[1]Readers of Ref. [5] should beware that our use of the notation $\langle\cdots\rangle$ is completely different from theirs.

TABLE I. Summary of notation.

$v^\mu=(1,\mathbf{v})$, $\mathbf{v}$ a spatial unit vector
$\mathbf{A}=\mathbf{A}(\mathbf{x},t)$, the spatial non-Abelian gauge field
$W=W(\mathbf{x},\mathbf{v},t)$, the adjoint color distribution of hard excitations
$\boldsymbol{\zeta}=\boldsymbol{\zeta}(\mathbf{x},t)$ and $\xi=\xi(\mathbf{x},\mathbf{v},t)$ are Gaussian white noise
$\langle\langle\cdots\rangle\rangle$ denotes averaging over noise
$\langle\cdots\rangle\equiv\langle\cdots\rangle_{\mathbf{v}}$ denotes averaging over the direction $\mathbf{v}$
$\delta^{S_2}(\mathbf{v}-\mathbf{v}')$ is a δ function on the two-sphere normalized so that $\langle\delta^{S_2}(\mathbf{v}-\mathbf{v}')\rangle_{\mathbf{v}'}=1$
$\delta\hat{C}\,W\equiv\langle\delta C(\mathbf{v},\mathbf{v}')\,W(\mathbf{v}')\rangle_{\mathbf{v}'}$, the linearized collision operator applied to W
A caret denotes either an operator on the space of functions of $\mathbf{v}$ or a spatial unit vector
$\langle lm\|\cdots\|l'm'\rangle\equiv\int d\Omega_{\mathbf{v}}Y^*_{lm}(\mathbf{v})\cdots Y_{l'm'}(\mathbf{v})=4\pi\langle Y^*_{lm}(\mathbf{v})\cdots Y_{l'm'}(\mathbf{v})\rangle_{\mathbf{v}}$
$\langle l\|\cdots\|l'\rangle$ denotes the same for cases where the answer is proportional to $\delta_{m,m'}$
$\hat{P}_0\equiv\|00\rangle\langle 00\|$, the projection operator onto $\mathbf{v}$-independent functions
$\gamma_1\equiv\langle 1\|\delta\hat{C}\|1\rangle$, the $l=1$ eigenvalue of the linearized collision operator
$C_{\rm A}$ is the adjoint Casimir of the gauge group [N for SU(N)]
$d=3-\epsilon$ with $\epsilon\to 0$, the number of spatial dimensions
$\int_{\mathbf{q}}\equiv\int\frac{d^dq}{(2\pi)^d}$, and $\int_{q_0}\equiv\int_{-\infty}^{+\infty}\frac{dq_0}{2\pi}$
$(-+++)$ spacetime metric signature
$\lambda\equiv q^0/q$, the ratio of frequency to spatial momenta
$\approx$ denotes equality at leading-log order
$\mathbf{D}=\boldsymbol{\nabla}+g\mathbf{A}^aT^a$, the gauge covariant derivative
$T^b_{ac}=f^{abc}$, anti-Hermitian adjoint-representation generators
"ln" in an order of magnitude estimate [e.g. $O(g^2\ln)$] means $\ln(1/g)$

both a $\mathbf{v}$-space integral operator $\delta\hat{C}$ and the corresponding kernel $\delta C(\mathbf{v},\mathbf{v}')$, which is simply a function. The formulas given earlier for the noise covariances $\langle\langle\xi\xi\rangle\rangle$ and $\langle\langle\boldsymbol{\zeta}\boldsymbol{\zeta}\rangle\rangle$ should be understood as shorthand for

$$\langle\langle\xi^a(\mathbf{v},\mathbf{x},t)\,\xi^b(\mathbf{v}',\mathbf{x}',t')\rangle\rangle=\frac{2T}{m^2}\,\delta C(\mathbf{v},\mathbf{v}')\,\delta^{ab}\delta^{(3)}(\mathbf{x}-\mathbf{x}')\times\delta(t-t'),\tag{1.4}$$

$$\langle\langle\zeta^a_i(\mathbf{x},t)\,\zeta^b_j(\mathbf{x}',t')\rangle\rangle=2\sigma T\,\delta_{ij}\,\delta^{ab}\delta^{(3)}(\mathbf{x}-\mathbf{x}')\times\delta(t-t'),\tag{1.5}$$

where i,j denote vector indices and a,b are adjoint color indices.

The scale of the linearized collision operator $\delta\hat{C}$ is set by γ. At leading log order, it is given by [3,4]

$$\hat{\delta}C\,W(\mathbf{v})\equiv\langle\delta C(\mathbf{v},\mathbf{v}')\,W(\mathbf{v}')\rangle_{\mathbf{v}'},\tag{1.6a}$$

$$\delta C(\mathbf{v},\mathbf{v}')\approx\gamma\left[\delta^{S_2}(\mathbf{v}-\mathbf{v}')-\frac{4}{\pi}\frac{(\mathbf{v}\cdot\mathbf{v}')^2}{\sqrt{1-(\mathbf{v}\cdot\mathbf{v}')^2}}\right].\tag{1.6b}$$

The symbol $\approx$ denotes equalities valid only to leading-log order.

We will consider the theories 1 and 2 to be ultraviolet (UV) regulated by dimensional regularization in $d=3-\epsilon$ dimensions with gauge coupling $\mu^{\epsilon/2}g$. Theory 3 is UV finite and does not require such regularization.

In the remainder of this Introduction, we review the path integral formulation of effective theories 2 and 3, and summarize general properties of the collision operator $\delta\hat{C}$ which we will need. In Sec. II we perform the matching of theory 3 to theory 2, which is the most novel part of our calculation. The NLLO conductivity σ is calculated in terms of the collision operator $\delta\hat{C}$ of theory 2. In Sec. III, we determine the information we need about $\delta\hat{C}$ by matching theory 2 to theory 1. In the process, we explicitly calculate the hard thermal gauge boson damping rate in the presence of an infrared (IR) regulator (specifically dimensional regularization). Finally, in Sec. IV, we put everything together and discuss the result. We then also summarize the differences between this work and an earlier discussion of the NLLO color conductivity by Blaizot and Iancu [5].

A. Review of the path integral formulation

The original derivation of the Langevin equation (1.3) by Bödeker [3] and subsequent discussions [4] were performed in $A_0=0$ gauge, where the equation reads

$$-\sigma\frac{d\mathbf{A}}{dt}=\mathbf{D}\times\mathbf{B}+\boldsymbol{\zeta}.\tag{1.7}$$

However, $A_0=0$ gauge is a sick gauge for doing perturbation theory, and is consequently an inappropriate choice for our present purposes. It is therefore useful to reformulate Eq. (1.7) as a path integral, so that we can use standard

Faddeev-Popov methods to choose a more convenient gauge.[2] Equation (1.7) is a Langevin equation, and it is well known how to reformulate such equations as path integrals.[3] Specifically, Eq. (1.7) becomes

$$Z=\int [\mathcal{D}\mathbf{A}(\mathbf{x},t)]\exp\left(-\int dt\, d^3x\, L\right), \tag{1.8a}$$

with

$$L=\frac{1}{4\sigma T}\left|\sigma\frac{d\mathbf{A}}{dt}+\mathbf{D}\times\mathbf{B}\right|^2-\sigma^{-1}\,\delta^{(3)}(0)\,\mathrm{tr}\,\mathbf{D}^2, \tag{1.8b}$$

where $\mathbf{D}^2$ means $\mathbf{D}\cdot\mathbf{D}$. We will use dimensional regularization throughout our analysis, in which case one may ignore the $\mathrm{tr}\,\mathbf{D}^2$ Jacobian term because $\delta^{(d)}(0)\equiv 0$. Equation (1.8b) is still in the $A_0=0$ gauge, but we can now trivially generalize to a gauge-invariant formulation:

$$Z=\int [\mathcal{D}A_0(\mathbf{x},t)][\mathcal{D}\mathbf{A}(\mathbf{x},t)]\,\exp\left(-\int dt\, d^3x\, L\right), \tag{1.9a}$$

$$L=\frac{1}{4\sigma T}\left|-\sigma\mathbf{E}+\mathbf{D}\times\mathbf{B}\right|^2. \tag{1.9b}$$

This can be checked by using the Faddeev-Popov procedure to return to $A_0=0$ gauge. But now we can use the Faddeev-Popov procedure on Eqs. (1.9) to fix other gauges as well. Coulomb gauge, for instance, corresponds to

$$Z_{\mathrm{Coulomb}}=\int [\mathcal{D}A_0][\mathcal{D}\mathbf{A}][\mathcal{D}\bar{\eta}][\mathcal{D}\eta]\,\delta(\boldsymbol{\nabla}\cdot\mathbf{A})\times\exp\left(-\int dt\, d^3x\, L_{\mathrm{Coulomb}}\right), \tag{1.10a}$$

$$L_{\mathrm{Coulomb}}=\frac{1}{4\sigma T}\left[\left|-\sigma\mathbf{E}+\mathbf{D}\times\mathbf{B}\right|^2+\bar{\eta}\boldsymbol{\nabla}\cdot\mathbf{D}\,\eta\right], \tag{1.10b}$$

where $\bar{\eta}$ and η are anti-commuting Faddeev-Popov ghosts.

Theory 2 can also be described by a path integral. We shall find it convenient to express the theory entirely in terms of the gauge fields by eliminating W using the equations of motion. The resulting path integral formulation is discussed in detail in Ref. [2], and here we simply quote the gauge-invariant result analogous to Eqs. (1.9):[4]

$$L=\frac{1}{4T}[-\bar{\sigma}(\mathbf{D})\,\mathbf{E}+\mathbf{D}\times\mathbf{B}]^{\mathrm{T}}\bar{\sigma}(\mathbf{D})^{-1}[-\bar{\sigma}(\mathbf{D})\,\mathbf{E}+\mathbf{D}\times\mathbf{B}]+L_1[\mathbf{A}], \tag{1.11}$$

where $\bar{\sigma}(\mathbf{D})$ is now a matrix in vector-index space. It is also an operator in $\mathbf{x}$ space (and color) and is given by

$$\bar{\sigma}_{ij}(\mathbf{D})=m^2\lim_{\Lambda\to\infty}\langle v_i[\mathbf{v}\cdot\mathbf{D}+\delta\hat{C}+\Lambda\hat{P}_0]^{-1}v_j\rangle =m^2(\langle v_i\hat{G}v_j\rangle-\langle v_i\hat{G}\rangle\langle\hat{G}\rangle^{-1}\langle\hat{G}v_j\rangle), \tag{1.12}$$

where $\hat{G}$ is the W-field propagator arising from Eq. (1.2a):

$$\hat{G}\equiv[\mathbf{v}\cdot\mathbf{D}+\delta\hat{C}]^{-1}. \tag{1.13}$$

$\hat{G}$ is an operator in both $\mathbf{x}$ and $\mathbf{v}$ space (as well as color). In the first form of Eq. (1.12), $\hat{P}_0$ denotes the $\mathbf{v}$-space projection operator that projects out functions that are independent of $\mathbf{v}$. In the notation introduced below, $\hat{P}_0=|00\rangle\langle 00|$.

The term $L_1[\mathbf{A}]$ in Eq. (1.11) is complicated and is discussed in Ref. [2]. It is the analogue of the $\delta^{(3)}(0)\,\mathrm{tr}\,\mathbf{D}^2$ term in Eq. (1.8b) but is spatially non-local and does not trivially vanish in dimensional regularization. Fortunately, however, the size of $L_1[\mathbf{A}]$ is such that it will be irrelevant to our calculation of the NLLO conductivity. All we need to know about it for the present discussion is that $L_1[\mathbf{A}]$ is independent of A_0, and that it is suppressed by the loop expansion parameter compared to other terms in Eq. (1.11). (Specifically, its contribution to the $\mathbf{A}$ propagator is suppressed by one factor of the loop expansion parameter.) As we will see later, this will be enough to argue that this term does not affect our calculation of the conductivity at NLLO.

The trace of $\mathbf{E}\cdot\mathbf{D}\times\mathbf{B}$ is the space-time total derivative $-\boldsymbol{\nabla}\cdot(\mathbf{E}^a\times\mathbf{B}^a)-\partial_t(\frac{1}{2}\mathbf{B}^a\mathbf{B}^a)$. So, ignoring boundary terms, the cross term may be dropped in the action for Eq. (1.11):[5]

[2]The Langevin equation (1.3) may also be shown to be correct and unambiguous in general flow gauges of the form $A_0=R[\mathbf{A}]$, where $R[\mathbf{A}]$ depends on $\mathbf{A}(\mathbf{x},t)$ only instantaneously and so does not involve time derivatives of $\mathbf{A}$. See Ref. [6] for a discussion of flow gauges and Ref. [7] for a proof that Eq. (1.3) may be applied in any gauge of this class. There are subtleties, however, in directly interpreting the Langevin equation (1.3) in *other* gauge choices, such as Landau gauge. Using a flow gauge of the form $A_0=\lambda\boldsymbol{\nabla}\cdot\mathbf{A}$ (which is discussed further in Appendix A), it should be feasible to reproduce all the analysis of this paper directly from the Langevin equation. However, we found it most straightforward, both conceptually and computationally, to use the path integral formulations presented here.

[3]For a review, see Chaps. 4, 16, and 17, and in particular Eqs. (17.15) and (17.16), of Ref. [8]. See also Ref. [2].

[4]The path integral corresponding to this Lagrangian has time discretization ambiguities. These should be resolved by a time-symmetric prescription, that is, writing $d\mathbf{A}/dt=[\mathbf{A}(t+\epsilon)-\mathbf{A}(t)]/\epsilon$, and then interpreting $\mathbf{A}$'s without time derivatives to mean the average $[\mathbf{A}(t+\epsilon)+\mathbf{A}(t)]/2$.

[5]This reasoning depends on using a symmetric time discretization to define Eq. (1.11). See, for example, Appendix B of Ref. [9]. Time is to be regarded as running from $-\infty$ to $+\infty$, and the choice of temporal boundary condition is irrelevant for our purposes.

$$S=\int dt\,d^3x\,L=\int dt\,d^3x\left\{\frac{1}{4T}[\mathbf{E}\,\overline{\sigma}(\mathbf{D})\mathbf{E}\right.$$
$$\left.+(\mathbf{D}\times\mathbf{B})\,\overline{\sigma}(\mathbf{D})^{-1}(\mathbf{D}\times\mathbf{B})]+g^2L_1[\mathbf{A}]\right\}. \quad (1.14)$$

We should perhaps clarify our mixture of **v**-space operator notation and $\langle\cdots\rangle$ notation for averaging over **v**. Since $\delta\hat{C}$, $\hat{P}_0$, and $\hat{G}$ are operators in **v** space, they do not commute with v_i and v_j in Eq. (1.12). So, for instance, $\langle v_i\,\delta\hat{C}\,v_i\rangle\neq\langle\delta\hat{C}\rangle$ even though $v_iv_i=|\mathbf{v}|^2=1$. This particular example is made clear by rewriting $\langle v_i\,\delta\hat{C}\,v_i\rangle=\langle v_i\,\delta C(\mathbf{v},\mathbf{v}')\,v_i'\rangle_{\mathbf{v}\mathbf{v}'}$. Our notation works just like bra-ket notation in quantum mechanics if one rewrites $\langle\cdots\rangle$ as $\langle 00|\cdots|00\rangle$ with $|00\rangle$ representing the constant function $Y_{00}(\mathbf{v})=1/\sqrt{4\pi}$ in **v** space. (It will be necessary later to consider other $|lm\rangle$ spherical harmonics as well.) $|00\rangle$ is only a **v**-space entity and does not specify anything about **x** or color dependence. So, for instance, $\langle\mathbf{D}\rangle=\langle 00|\mathbf{D}|00\rangle$ is **D** and not zero.

B. General properties of $\delta\hat{C}$

It will be useful to summarize various useful properties of $\delta\hat{C}$ that follow from general considerations not restricted to the leading log approximation (1.6b). Collisions packaged in $\delta\hat{C}$ are, as far as effective theory 2 is concerned, local in space.[6] All of the **x** dependence of collision probabilities comes in the distribution functions W, and so the operator $\delta\hat{C}$ itself is independent of **x** — it is simply an operator in **v** space. Rotation invariance of the theory therefore implies **v**-space rotation invariance of the operator $\delta\hat{C}$, which in turn implies that $\delta\hat{C}$ is diagonal in the space of $|lm\rangle$'s. [That is, its eigenfunctions are the spherical harmonics $Y_{lm}(\mathbf{v})$.] It also implies that the corresponding diagonal matrix elements $\langle lm|\delta\hat{C}|lm\rangle$ depend only on l and not on m. We shall therefore write them more compactly as $\langle l|\delta\hat{C}|l\rangle$. This l decomposition of $\delta\hat{C}$ will be crucial to our later analysis, and the case $l=1$ will be of particular interest.[7]

It's useful to demonstrate this notation by returning to the leading-log approximation (1.6b). The second term of Eq. (1.6b) vanishes when applied to any **v**-parity-odd function,

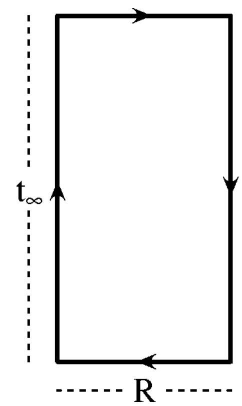

FIG. 1. A time-like Wilson loop rectangle.

and so $\langle l|\delta\hat{C}|l\rangle\approx\gamma$ for odd l. In particular, at leading-log order,

$$\langle 1|\delta\hat{C}|1\rangle\approx\gamma. \quad (1.15)$$

The relation is more complicated beyond leading-log order, and we will define a separate symbol[8]

$$\gamma_1\equiv\langle 1|\delta\hat{C}|1\rangle. \quad (1.16)$$

In Sec. II, we will find that the only pieces of $\delta\hat{C}$ that we need to calculate the NLLO conductivity are the leading-log formula (1.15), plus the NLLO result for γ_1. The eigenvalue γ_1 will appear in our analysis through the handy formulas

$$\delta\hat{C}\,\mathbf{v}\rangle=\gamma_1\mathbf{v}\rangle,\quad \langle\mathbf{v}\,\delta\hat{C}=\gamma_1\langle\mathbf{v}. \quad (1.17)$$

Here, and henceforth, we will often use a bare $\rangle$ or $\langle$ as convenient shorthand for $|00\rangle$ or $\langle 00|$, respectively. The relations (1.17) follow because $\mathbf{v}\rangle$ has $l=1$, and so is a superposition of $|1m\rangle$'s. The value of γ_1 will be calculated in Sec. III.

Rotation invariance also implies that $\delta\hat{C}$ is symmetric in **v** space, since $\delta C(\mathbf{v},\mathbf{v}')$ can only depend on $\mathbf{v}\cdot\mathbf{v}'$. Furthermore, like everything else in Eq. (1.2a), $\delta\hat{C}$ is real. As discussed in Ref. [2], this means that the conductivity operator $\overline{\sigma}^{ij}(\mathbf{D})$ of (1.12) is real and symmetric in **x**/color/vector space.

A very important property of δC is that it annihilates functions which are independent of **v**:

$$\delta\hat{C}\rangle=0 \quad (1.18)$$

or, equivalently,

$$\langle\delta\hat{C}=0\quad\text{or}\quad\langle 0|\delta\hat{C}|0\rangle=0. \quad (1.19)$$

[6]To understand the reason for this locality, it is important to realize that the δC in Eqs. (1.2) of theory 2 is a *bare* collision term. In Wilsonian language, it characterizes those collisions mediated by gauge bosons with momentum large compared to the UV cutoff scale used to define theory 2. There are also collisions mediated by the soft, dynamical gauge bosons of theory 2, and these cannot be treated locally. In our case, we will actually regularize and renormalize theory 2 by dimensional regularization and subtractions, rather than a Wilson-style momentum cutoff, but our bare δC will still be local in **x**.

[7]For a discussion of using the space of spherical harmonics $|lm\rangle$ as a basis for numerical simulations, see Ref. [10].

[8]In d dimensions, γ_1 means the eigenvalue of $\delta\hat{C}$ in the vector representation of SO(d).

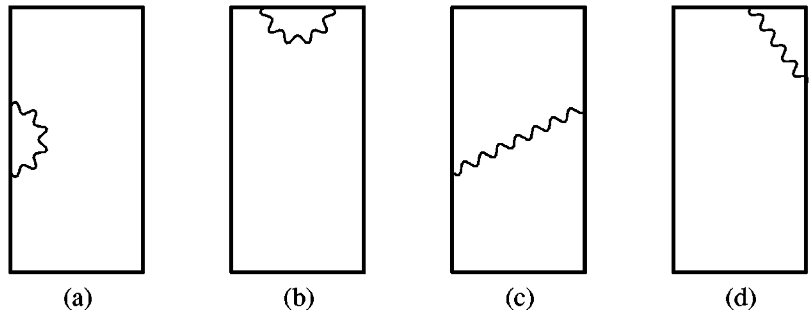

FIG. 2. Examples of first-order contribution to the expectation of a large-time Wilson rectangle.

We will see this explicitly when we discuss $\delta\hat{C}$ in Sec. III, but it is true for quite general reasons, as pointed out in Refs. [3,11]. One way to understand this is to observe that current conservation in effective theory 1 requires $0=D_\mu j^\mu = D_\mu\langle v^\mu W\rangle$. Theory 2 is a subsequent effective theory for $\omega \ll k$, meaning that time derivatives have been neglected compared to spatial derivatives. In this limit, current conservation becomes simply $\langle \mathbf{v}\cdot\mathbf{D}W\rangle=0$. Taking the $\mathbf{v}$-expectation value of Eq. (1.2a) gives $\langle \mathbf{v}\cdot\mathbf{D}W\rangle=\langle\delta\hat{C}\rangle$, and so $\langle 00|\delta\hat{C}|00\rangle=0$. Since $\delta\hat{C}$ is diagonal in $|lm\rangle$ space, it follows that $\delta\hat{C}$ has a zero mode, $\delta\hat{C}\rangle=0$. A differently packaged but related explanation of this property may be found in Sec. III C of Ref. [11].

Finally, the effect of a collision term in a Boltzmann equation is to cause the decay over time of correlations. The sign of $\delta\hat{C}$ in Eq. (1.2a) is such that decay corresponds to positive $\delta\hat{C}$, which would be more obvious if we hadn't dropped the $\partial_t W$ time derivative term in going from Eq. (1.1a) of theory 1 to Eq. (1.2a) of theory 2. Based on this, one should expect that all the eigenvalues $\langle l|\delta\hat{C}|l\rangle$ of $\delta\hat{C}$ are non-negative.[9] This can be checked explicitly [3] from the leading-log formula (1.6b). We provide simple expressions for the leading-log eigenvalues in Sec. II D. The zero eigenvalue (1.18) corresponds to the fact that charge is conserved and does not decay.

II. MATCHING THEORY 2 TO THEORY 3

To match effective theories, one must identify gauge-invariant observables which are calculable in both theories. We shall therefore spend some time discussing a gauge-invariant observable, involving Wilson loops, which can be used to determine the parameter σ of theory 3. We shall then find that, in practice, the matching problem can be conveniently simplified to the matching of the Coulomb-gauge self-energy Π_{00} of A_0 in the limit of zero frequency and small momentum.

A. Wilson loops

An example of a gauge-invariant observable that depends on the conductivity is a (real-time) Wilson loop

$$\mathcal{W}=\left\langle\left\langle \mathrm{tr}\,\mathcal{P}\exp\left(g\oint dx^\mu A_\mu\right)\right\rangle\right\rangle \tag{2.1}$$

which extends in the time direction, where $\mathcal{P}$ indicates path ordering of the exponential. Here $A_\mu \equiv A_\mu^a T^a$, and our convention is that the generators T^a are anti-Hermitian. To see the dependence on conductivity explicitly, it is convenient to focus on rectangular Wilson loops, such as depicted in Fig. 1, where one set of edges is in the time direction and the other set is purely spatial. It will also be convenient to focus on rectangles whose temporal extent t_∞ is very large compared to their spatial extent R.

1. Relation to σ in theory 3

To get a feel for these Wilson loops, let us look at their value in our final effective theory, theory 3, at first order in perturbation theory. There will be various perimeter contributions such as those of Fig. 2(a). As we shall see, these are UV divergent and should in principle be regulated.[10] But they do not depend on the separation R, and so we can ignore them if we just focus on the R dependence of the Wilson loop expectation. We will similarly ignore contributions that do not depend on t_∞, such as Fig. 2(b). R dependence is generated by propagators which connect different edges, such as in Fig. 2(c). If we pick a reasonable gauge for doing perturbation theory, then, in the large time (t_∞) limit for the Wilson loop, we can neglect diagrams such as Fig. 2(d) which attach to the far-past or far-future ends of the loop. In this case, the large-time Wilson loop, to lowest order, is determined just by Fig. 2(c).

The primary example of an *un*reasonable gauge is $A_0=0$ gauge, which is the gauge in which Bödeker's effective theory (theory 3) was originally formulated. There is no Fig. 2(c) at all in $A_0=0$ gauge, because the time-going Wilson lines only couple to A_0. But $A_0=0$ gauge is a sick gauge for perturbation theory in the first place.[11] We will instead work

[9]This is a physical statement and does not, technically, apply to the bare $\delta\hat{C}$ if one does renormalization using subtractions.

[10]As mentioned before, the dynamics of theory 3 is UV finite. But the definition of the Wilson loop operator itself requires UV regularization.

[11]That is because the free action in $A_0=0$ gauge has infinitely many zero modes associated with time-independent gauge transformations — zero modes which are not properly treated in a perturbative expansion and which manifest as spurious non-integrable singularities in propagators.

in Coulomb gauge. The calculation of this section is repeated in Appendix A in more general ''flow'' gauges, which interpolate smoothly between Coulomb gauge and $A_0=0$ gauge.

The perturbative expansion of the action (1.10b) describing Bödeker's effective theory in Coulomb gauge is

$$S_{\rm Coulomb}=\int dt\, d^3x \frac{1}{4T}\left[\sigma|\nabla A_0|^2+\frac{1}{\sigma}|(\sigma\partial_t-\nabla^2)\mathbf{A}|^2 + O(\mathbf{A}^3)+(\text{ghosts})\right]. \tag{2.2}$$

One may read off the A_0 propagator (which is instantaneous in time):

$$A_0^a \sim\!\sim\!\sim A_0^b=\frac{2T}{\sigma k^2}\,\delta^{ab}, \tag{2.3}$$

Figure 2(c) then gives a contribution to the Wilson loop of

$$d_{\rm R}^{-1}\,\delta\mathcal{W}=g^2\,\frac{{\rm tr}(T^aT^b)}{{\rm tr}(1)}\int_0^{t_\infty}dt\,dt'\,\langle\!\langle A_0^a(t,0)A_0^b(t',\mathbf{R})\rangle\!\rangle$$
$$=-g^2\,C_{\rm A}\,t_\infty\int_{\mathbf{k}}\frac{2T}{\sigma k^2}e^{i\mathbf{k}\cdot\mathbf{R}}=-\frac{2\alpha C_{\rm A}T}{\sigma R}t_\infty\,, \tag{2.4}$$

where $d_{\rm R}\equiv{\rm tr}(1)$ is the dimension of the representation associated with the Wilson loop. Because the A_0 propagator is instantaneous in time, there are no crossed graphs at higher order and this contribution exponentiates in the usual way to give

$$\ln\mathcal{W}=-\frac{2\alpha C_{\rm A}T}{\sigma R}t_\infty+(\text{higher order}) \tag{2.5}$$

in the large-time limit, up to terms independent of R or t_∞. In perturbation theory, at least, one sees that a large time-like Wilson loop provides a gauge-invariant quantity from which one may extract σ.

One can automatically remove the perimeter terms independent of either R or t_∞ by taking ratios of Wilson loops:

$$\ln\left[\frac{\mathcal{W}(R_1,t_1)\,\mathcal{W}(R_2,t_2)}{\mathcal{W}(R_1,t_2)\,\mathcal{W}(R_2,t_1)}\right]=-\frac{2\alpha C_{\rm A}T}{\sigma}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)(t_1-t_2) +(\text{higher order}). \tag{2.6}$$

This ratio is free of UV divergences in theory 3.

A warning is in order concerning the physical interpretation of the result (2.5). The result should be trustworthy whenever perturbation theory is reliable, which means whenever $R\ll 1/g^2T$. Some readers may automatically associate Wilson loops with the behavior $\exp[-V(R)\,t_\infty]$, with $V(R)$ interpreted as the potential energy (or free energy) associated with two static (i.e., infinitely massive) test charges separated by distance R. However, this interpretation only applies to Wilson loops (or Polyakov lines) in *Euclidean* time and does not apply to the case at hand of real-time loops at finite temperature. In particular, Eq. (2.5) should not be interpreted as a $1/R$ potential between static test charges for $R\ll 1/g^2T$, which would be inconsistent with Debye screening for $R\gg m^{-1}=O(1/gT)$. In fact, we know no simple physical interpretation of real-time Wilson loops at finite temperature.

Though we have only discussed a perturbative analysis of Wilson loops, this is good enough for matching theory 2 to theory 3, whose physics only differs at scales $k\gtrsim\gamma=O(g^2T\ln)$, where the physics is still perturbative (by a logarithm). Here and henceforth, in order of magnitude estimates the abbreviation ''ln'' is shorthand for $\ln(1/g)$. As discussed in our companion paper [1], and in earlier works by Braaten and Nieto [12], matching may be performed by formally computing the same quantity *perturbatively* in both theories, in the presence of some common infrared regulator. We shall consider Wilson loops with $R\gg 1/\gamma$, so that they are firmly in the region of validity of both theories, and we shall regulate the infrared behavior using dimensional regularization. For simplicity, we will focus on the formal limit $R\to\infty$ in the context of our IR-regulated perturbative calculation. (The order of limits is important: the $t_\infty\to\infty$ limit is to be taken first, so as not to invalidate the previous discussion.)

2. Relation to σ in theory 2

As far as the A_0 propagator is concerned, the perturbative expansion of the action (1.14) is much like that of theory 3 except that the color conductivity σ becomes momentum dependent. Specifically, the Coulomb gauge propagator for A_0 is now

$$A_0^a \sim\!\sim\!\sim A_0^b=\frac{2T}{k^2\bar\sigma_{\rm L}^{(0)}(k)}\,\delta^{ab}, \tag{2.7}$$

where

$$\overline{\sigma}_{\rm L}^{(0)}(k)\equiv\hat k_i\,\overline{\sigma}_{ij}^{(0)}(\mathbf{k})\,\hat k_j\,, \tag{2.8}$$

and $\overline{\sigma}_{ij}^{(0)}\equiv\overline{\sigma}_{ij}|_{\mathbf{A}=0}$ is given by Eqs. (1.12) and (1.13) with the covariant derivative $\mathbf{D}$ replaced by $i\mathbf{k}$:

$$\overline{\sigma}_{ij}^{(0)}(\mathbf{k})=m^2\lim_{\Lambda\to\infty}\langle v_i[\mathbf{v}\cdot i\mathbf{k}+\delta\hat C+\Lambda\hat P_0]^{-1}v_j\rangle$$
$$=m^2[\langle v_i\hat G_0(\mathbf{k})v_j\rangle-\langle v_i\hat G_0(\mathbf{k})\rangle \times\langle\hat G_0(\mathbf{k})\rangle^{-1}\langle\hat G_0(\mathbf{k})v_j\rangle], \tag{2.9}$$

Here,

$$\hat G_0(\mathbf{k})\equiv[\mathbf{v}\cdot\nabla+\delta\hat C]^{-1}=[\mathbf{v}\cdot i\mathbf{k}+\delta\hat C]^{-1}. \tag{2.10}$$

The first-order contribution to the Wilson loop, analogous to Eq. (2.4), is then

$$d_{\rm R}^{-1}\,\delta\mathcal{W}=-g^2C_{\rm A}t_\infty\int_{\mathbf{k}}\frac{2T}{k^2\,\overline{\sigma}_{\rm L}^{(0)}(k)}e^{i\mathbf{k}\cdot\mathbf{R}}. \tag{2.11}$$

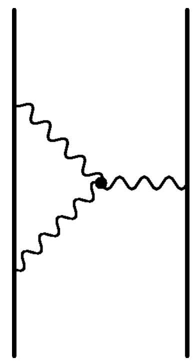

FIG. 3. A next-order correction to Fig. 2(c) that does not appear in Coulomb gauge.

One should expect that the large R behavior is dominated by the small k behavior of the integrand, and this is indeed the case, up to corrections suppressed by powers of g. (See Appendix B for an explicit argument.) The result is then

$$d_{\rm R}^{-1}\,\delta\mathcal{W}\to-\frac{2\alpha C_{\rm A}T}{\overline{\sigma}_{\rm L}^{(0)}(0)R}t_\infty \tag{2.12}$$

for $R\to\infty$. More specifically, this limit is $R\gg\gamma^{-1}$, since $\delta\hat{C}$ provides the only scale in the definition of $\overline{\sigma}^{(0)}(\mathbf{k})$ and the scale of $\delta\hat{C}$ is γ.

From Eq. (2.9), we have

$$\overline{\sigma}_{ij}^{(0)}(0)=m^2\lim_{\Lambda\to\infty}\langle v_i[\delta\hat{C}+\Lambda\hat{P}_0]^{-1}v_j\rangle. \tag{2.13}$$

As noted in Sec. I, $v_j\rangle$ has $l=1$ — that is, it is a superposition of $|1m\rangle$'s. Recalling that $\delta\hat{C}$ is diagonal in the space of $|lm\rangle$'s, as is $\hat{P}_0=|00\rangle\langle 00|$, we obtain

$$\overline{\sigma}_{ij}^{(0)}(0)=\frac{m^2}{d\gamma_1}\delta^{ij}\quad\text{and}\quad\overline{\sigma}_{\rm L}^{(0)}(0)=\frac{m^2}{d\gamma_1}, \tag{2.14}$$

in d spatial dimensions, where $\gamma_1\equiv\langle 1|\delta\hat{C}|1\rangle$ is the $l=1$ eigenvalue of $\delta\hat{C}$. [We have left the spatial dimension d arbitrary in Eq. (2.14) because the generalization away from $d=3$ will be needed later when we dimensionally regularize.] Comparison of the Wilson loop (2.12) in theory 2 and Eq. (2.4) in theory 3 then gives the leading-order result for matching the two theories:

$$\sigma\approx\frac{m^2}{d\gamma_1}\approx\frac{m^2}{d\gamma}, \tag{2.15}$$

where the last leading-log equality uses Eq. (1.15). This is precisely the leading-log result for theory 3 originally derived by Bödeker. This result could have been very quickly derived from the starting point of Eq. (1.6b) without all this discussion of Wilson loops. This approach based on Wilson loops does, however, provide a conceptually clear framework for discussing sub-leading corrections.

3. Next order in the loop expansion

As mentioned earlier, perturbation theory at a scale k is controlled by the loop expansion parameter g^2T/k. A matching calculation between theory 2 and theory 3 is a calculation of physics at the interface $k\sim\gamma$ below which theory 3 is valid, and so the loop expansion for matching calculations will be an expansion in $g^2T/\gamma\sim[\ln(1/g)]^{-1}$. To go to next-to-leading-log order in the determination of σ, we must therefore go to the next order in the loop expansion for the Wilson loops.

A nice simplification occurs in Coulomb gauge. Consider the quadratic pieces of the $\mathbf{E}\,\overline{\sigma}(\mathbf{D})\mathbf{E}$ term in the action (1.14) for theory 2:

$$\begin{aligned}\mathbf{E}\overline{\sigma}(\mathbf{D})\mathbf{E}=&-A_0\,\boldsymbol{\nabla}\cdot\overline{\sigma}(\boldsymbol{\nabla})\boldsymbol{\nabla}A_0-2A_0\,\boldsymbol{\nabla}\cdot\overline{\sigma}(\boldsymbol{\nabla})\,\partial_t\mathbf{A}\\&-\mathbf{A}\,\overline{\sigma}(\boldsymbol{\nabla})\,\partial_t^2\mathbf{A}+O(\mathbf{A}^3).\end{aligned} \tag{2.16}$$

Rotation invariance implies that $\nabla_i[\overline{\sigma}(\boldsymbol{\nabla})]_{ij}$ must be proportional to ∇_j (since $[\overline{\sigma}(\boldsymbol{\nabla})]_{ij}$ can only involve terms proportional to either δ_{ij} or $\nabla_i\nabla_j$). Therefore, the second term in Eq. (2.16) which connects A_0 and $\mathbf{A}$ actually involves $\boldsymbol{\nabla}\cdot\mathbf{A}$ and hence vanishes in Coulomb gauge. Consequently, with this gauge choice the propagator does not mix A_0 and $\mathbf{A}$.

Now notice that the entire action (1.14) is quadratic in A_0 — there are no A_0^3 or higher terms. Taken together, this means that in Coulomb gauge there is no diagram such as Fig. 3 contributing to the Wilson loop. Instead, the next-to-leading-order diagrams all have the form of self-energy corrections to the propagator in our leading-order diagram Fig. 2(c).

We shall henceforth visually distinguish A_0 and $\mathbf{A}$ propagators when drawing Feynman diagrams, representing A_0 propagators as dashed lines and $\mathbf{A}$ propagators as wavy lines. An expansion of the theory 2 action (1.14) in powers of $\mathbf{A}$ will give interaction vertices of the forms shown in Fig. 4. In Coulomb gauge, the one-loop corrections to the Wilson loop are then given by Fig. 5. (We have left out diagrams involving tadpoles, as these vanish by CP symmetry.) This diagrammatic result applies to theory 3 as well as to theory 2.

In the Coulomb gauge, the calculation of the Wilson loop at this order now reduces to the evaluation of the one-loop self-energy $\Pi_{00}(\omega,\mathbf{k})$ of A_0. In fact, for the large Wilson loops discussed earlier, all that will be important is the

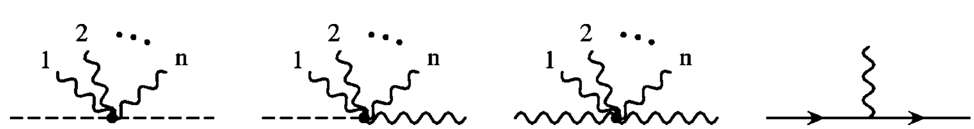

FIG. 4. The Coulomb-gauge interaction vertices of action (1.14) of theory 2. Dashed lines represent A_0, wavy lines $\mathbf{A}$, and solid lines the gauge-fixing ghost η. n is any positive integer.

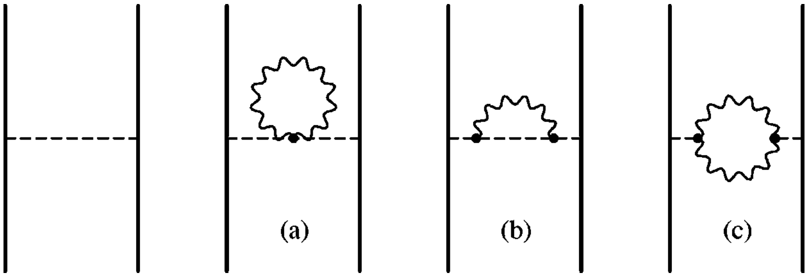

FIG. 5. The first-order and next-order contributions to the Wilson loop in the Coulomb gauge.

$\omega=0$, $\mathbf{k}\to 0$ behavior of the self-energy. Absorbing Π_{00} into the A_0 propagator (2.7) and repeating the argument that led to Eq. (2.12), we have

$$\bar{\sigma}_{\rm L}^{(0)}(k)\to\bar{\sigma}_{\rm L}^{(\rm eff)}(k)=\bar{\sigma}_{\rm L}^{(0)}(k)+\frac{2T}{k^2}\Pi_{00}(0,k) \tag{2.17}$$

and

$$d_{\rm R}^{-1}\,\delta\mathcal{W}\to-\frac{2\alpha C_{\rm A}T}{\bar{\sigma}_{\rm L}^{(\rm eff)}(0)R}t_\infty, \tag{2.18}$$

in theory 2. Corresponding expressions hold for theory 3, with $\bar{\sigma}_{\rm L}^{(0)}(k)$ replaced by σ. Matching the two theories then gives

$$\sigma=\frac{m^2}{d\gamma_1}+\lim_{k\to 0}\left\{\left[\frac{2T}{k^2}\Pi_{00}(0,k)\right]_{(\text{theory 2})}-\left[\frac{2T}{k^2}\Pi_{00}(0,k)\right]_{(\text{theory 3})}\right\}+O\left(\frac{\sigma}{\ln^2}\right). \tag{2.19}$$

Here and henceforth, all results are in Coulomb gauge unless explicitly stated otherwise.

It must be emphasized that, because this is a matching calculation, the $k\to 0$ limits of the individual Π_{00}/k^2 terms above are to be understood as taken in the presence of an infrared regulator. Suppose, for the sake of discussion, that $\Pi_{00}(k)/k^2$ were given by

$$k^{-2}\int_{\mathbf{p}}\frac{k^2}{p^2\,|\mathbf{k}+\mathbf{p}|^2}, \tag{2.20}$$

where $\mathbf{p}$ was the spatial part of some loop momentum. The integral is perfectly finite and so does not appear to require any IR regularization. Without IR regularization, the result of Eq. (2.20) goes like k^{-1} by dimensional analysis, and the $k\to 0$ limit is not well defined. Now imagine a simple momentum cutoff M on small loop momenta. The small $\mathbf{k}$ limit of Eq. (2.20) then behaves as M^{-1} rather than k^{-1}, and there is no problem with the limit. In general, in the presence of an IR regulator, we can formally expand integrands in $\mathbf{k}$, so that, for example,

$$\lim_{\mathbf{k}\to 0}k^{-2}\int_{\mathbf{p}}\frac{k^2}{p^2\,|\mathbf{k}+\mathbf{p}|^2}=\int_{\mathbf{p}}\frac{1}{p^4}, \tag{2.21}$$

in any IR regularization scheme. In dimensional regularization the result of this particular example is especially simple: Equation (2.21) is zero.

B. Calculating $\Pi_{00}(0,k\to 0)$ in theory 2

1. Perturbative expansion of the action

We will now be explicit about the perturbative expansion of the action (1.14) for theory 2. Since we are working in Coulomb gauge, there is an additional ghost piece of $\bar{\eta}\boldsymbol{\nabla}\cdot\mathbf{D}\eta$ in the action but this is irrelevant since no ghosts enter any of the diagrams in Fig. 5 that we need to calculate. In fact, we will be parsimonious in our discussion of the expansion and only explicitly keep track of those terms which are relevant to the specific diagrams of Fig. 5, namely the quadratic terms plus the $A_0\mathbf{AA}$, $A_0A_0\mathbf{A}$, and $A_0A_0\mathbf{AA}$ interactions.

Before expanding the action (1.14), it is helpful to first make some simplifications. In particular, write $\mathbf{E}=\mathbf{D}A_0-\dot{\mathbf{A}}$ and expand $\mathbf{E}\bar{\sigma}\mathbf{E}$ as

$$\int d^dx\,\mathbf{E}\bar{\sigma}\mathbf{E}=\int d^dx[-A_0\,\mathbf{D}\bar{\sigma}\mathbf{D}\,A_0+2A_0\,\mathbf{D}\bar{\sigma}\,\dot{\mathbf{A}}+\dot{\mathbf{A}}\bar{\sigma}\dot{\mathbf{A}}]. \tag{2.22}$$

From the expression (1.12) for $\bar{\sigma}$, we have

$$\mathbf{D}\bar{\sigma}=m^2(\langle\mathbf{v}\cdot\mathbf{D}\,\hat{G}\mathbf{v}\rangle-\langle\mathbf{v}\cdot\mathbf{D}\,\hat{G}\rangle\langle\hat{G}\rangle^{-1}\langle\hat{G}\mathbf{v}\rangle). \tag{2.23}$$

Now comes a trick we will use repeatedly. Since $\delta\hat{C}$ annihilates $\mathbf{v}$-independent states, as discussed in Sec. I B, we have $\langle\,\delta\hat{C}=0$ and can rewrite

$$\langle\mathbf{v}\cdot\mathbf{D}=\langle(\mathbf{v}\cdot\mathbf{D}+\delta\hat{C})=\langle\hat{G}^{-1}. \tag{2.24}$$

So

$$\langle\mathbf{v}\cdot\mathbf{D}\,\hat{G}=\langle\quad\text{and}\quad\hat{G}\,\mathbf{v}\cdot\mathbf{D}\rangle=\rangle. \tag{2.25}$$

Equation (2.23) then simplifies to

$$\mathbf{D}\bar{\sigma}=-m^2\langle\hat{G}\rangle^{-1}\langle\hat{G}\mathbf{v}\rangle \tag{2.26}$$

and

$$\mathbf{D}\bar{\sigma}\mathbf{D}=-m^2\langle\hat{G}\rangle^{-1}. \tag{2.27}$$

We may now rewrite Eq. (2.22) as

$$\int d^dx \mathbf{E}\,\overline{\sigma}\mathbf{E} = \int d^dx [m^2 A_0 \langle \hat{G}\rangle^{-1} A_0 - 2m^2 A_0 \langle \hat{G}\rangle^{-1}\langle \hat{G}\mathbf{v}\rangle\cdot\dot{\mathbf{A}} + \dot{\mathbf{A}}\overline{\sigma}\dot{\mathbf{A}}]. \tag{2.28}$$

The perturbative expansion is now generated by formally expanding the expression (1.13) for G:

$$\hat{G} \equiv [\mathbf{v}\cdot\mathbf{D} + \delta\hat{C}]^{-1} = \hat{G}_0 - g\hat{G}_0\mathbf{v}\cdot\mathbf{A}\hat{G}_0 + g^2\hat{G}_0\mathbf{v}\cdot\mathbf{A}\hat{G}_0\mathbf{v}\cdot\mathbf{A}\hat{G}_0 - \cdots, \tag{2.29}$$

with G_0 given by Eq. (2.10). (We will not explicitly show the factors of $\mu^{\epsilon/2}$ that accompany g in dimensional regularization.) Then

$$\begin{aligned}\langle \hat{G}\rangle^{-1} = {} & \langle \hat{G}_0\rangle^{-1} + g\langle \hat{G}_0\rangle^{-1}\langle \hat{G}_0\mathbf{v}\cdot\mathbf{A}\hat{G}_0\rangle\langle \hat{G}_0\rangle^{-1} \\ & + g^2\langle \hat{G}_0\rangle^{-1}\langle \hat{G}_0\mathbf{v}\cdot\mathbf{A}\hat{G}_0\rangle\langle \hat{G}_0\rangle^{-1}\langle \hat{G}_0\mathbf{v}\cdot\mathbf{A}\hat{G}_0\rangle\langle \hat{G}_0\rangle^{-1} \\ & - g^2\langle \hat{G}_0\rangle^{-1}\langle \hat{G}_0\mathbf{v}\cdot\mathbf{A}\hat{G}_0\mathbf{v}\cdot\mathbf{A}\hat{G}_0\rangle\langle \hat{G}_0\rangle^{-1} \\ & + O(g^3\mathbf{A}^3).\end{aligned} \tag{2.30}$$

Now focus for a moment on the expansion of the $A_0\langle \hat{G}\rangle^{-1}\langle \hat{G}\mathbf{v}\rangle\cdot\dot{\mathbf{A}}$ term in Eq. (2.28), and consider in particular the leading-order contribution to $\langle \hat{G}\mathbf{v}\rangle\cdot\dot{\mathbf{A}}$. By rotation invariance, $\langle \hat{G}_0 v_i\rangle$ must have a factor of ∇_i, since $\boldsymbol{\nabla}$ is the only vector quantity appearing in G_0. Combining this with

$$\langle \hat{G}_0\mathbf{v}\rangle\cdot\boldsymbol{\nabla} = \langle \hat{G}_0(\mathbf{v}\cdot\boldsymbol{\nabla} + \delta\hat{C})\rangle = 1 \tag{2.31}$$

then yields[12]

$$\langle \hat{G}_0\mathbf{v}\rangle = \langle \mathbf{v}\hat{G}_0\rangle = \frac{\boldsymbol{\nabla}}{\nabla^2}. \tag{2.32}$$

Since we are in Coulomb gauge, we thus have $\langle \hat{G}_0\mathbf{v}\rangle\cdot\dot{\mathbf{A}} = 0$. So the leading term of the expansion vanishes, and therefore

$$A_0\langle \hat{G}\rangle^{-1}\langle \hat{G}\mathbf{v}\rangle\cdot\dot{\mathbf{A}} = -gA_0\langle \hat{G}_0\rangle^{-1}\langle \hat{G}_0\mathbf{v}\cdot\mathbf{A}\hat{G}_0\mathbf{v}\rangle\cdot\dot{\mathbf{A}} + O(g^2A_0\mathbf{A}^3). \tag{2.33}$$

Putting everything together with the other terms in the action (1.14), and keeping track only of the terms that are needed for the diagrams of Fig. 5, yields $S = S_{\rm free} + S_{\rm int}$, with

$$S_{\rm free} = \int dt\, d^dx \frac{1}{4T}\{m^2 A_0\langle \hat{G}_0\rangle^{-1}A_0 + \dot{\mathbf{A}}\overline{\sigma}(\boldsymbol{\nabla})\,\dot{\mathbf{A}} + \mathbf{A}\boldsymbol{\nabla}^2[\overline{\sigma}(\boldsymbol{\nabla})]^{-1}\boldsymbol{\nabla}^2\mathbf{A} + (\text{ghosts})\}, \tag{2.34a}$$

$$S_{\rm int} = \int dt\, d^dx \frac{1}{4T}\{m^2 A_0(\langle G\rangle^{-1} - \langle \hat{G}_0\rangle^{-1})A_0 + 2gm^2A_0\langle \hat{G}_0\rangle^{-1}\langle \hat{G}_0\mathbf{v}\cdot\mathbf{A}\hat{G}_0\mathbf{v}\rangle\cdot\dot{\mathbf{A}} + (\text{not needed})\}, \tag{2.34b}$$

where $\langle G\rangle^{-1}$ is expanded as shown in Eq. (2.30). The perturbative expansion of the $L_1[\mathbf{A}]$ term of the action (1.14) falls into the ''not needed'' category since the diagrams of Fig. 5 do not contain any interaction vertices involving only $\mathbf{A}$ and not A_0, and any correction to the $\mathbf{A}$ propagator induced by $L_1[\mathbf{A}]$ will be of sub-leading order in logarithms.

2. Propagators

The perturbative $\overline{\sigma}(\boldsymbol{\nabla})$ appearing in $S_{\rm free}$ can be simplified a bit. In momentum space, we earlier called it $\overline{\sigma}^{(0)}(\mathbf{k})$, given by Eq. (2.9):

$$\overline{\sigma}^{(0)}_{ij} = m^2(\langle v_i\hat{G}_0 v_j\rangle - \langle v_i\hat{G}_0\rangle\langle \hat{G}_0\rangle^{-1}\langle \hat{G}_0 v_j\rangle). \tag{2.35}$$

Now note that the first term is transverse, because

$$\nabla_i\langle v_i\hat{G}_0 v_j\rangle = \langle(\boldsymbol{\nabla}\cdot\mathbf{v} + \delta\hat{C})\,\hat{G}_0 v_j\rangle = \langle v_j\rangle = 0 \tag{2.36}$$

(and similarly $\langle v_i\hat{G}_0 v_j\rangle\nabla_j = 0$). So we may rewrite

$$m^2\langle v_i\,\hat{G}_0(\mathbf{k})\,v_j\rangle = \overline{\sigma}^{(0)}_{\rm T}(k)\,P^{ij}_{\rm T}(\hat{\mathbf{k}}), \tag{2.37}$$

where we introduce (perturbative) transverse and longitudinal projection operators

$$P^{ij}_{\rm T}(\mathbf{k}) \equiv \delta^{ij} - \hat{k}^i\hat{k}^j, \tag{2.38}$$

$$P^{ij}_{\rm L}(\mathbf{k}) \equiv \hat{k}^i\hat{k}^j. \tag{2.39}$$

$\overline{\sigma}^{(0)}_{\rm T}$ may then be expressed as

$$\overline{\sigma}^{(0)}_{\rm T}(k) = m^2\frac{\langle v_i\,\hat{G}_0(\mathbf{k})v_i\rangle}{d-1}. \tag{2.40}$$

Similarly, the second term in Eq. (2.35) is purely longitudinal, by Eq. (2.32). So

$$\overline{\sigma}^{(0)}(\mathbf{k}) = \overline{\sigma}^{(0)}_{\rm T}(k)\,P_{\rm T}(\hat{\mathbf{k}}) + \overline{\sigma}^{(0)}_{\rm L}(k)\,P_{\rm L}(\hat{\mathbf{k}}), \tag{2.41}$$

with

$$\overline{\sigma}^{(0)}_{\rm L}(k) = \frac{m^2}{k^2}\langle \hat{G}_0(\mathbf{k})\rangle^{-1}. \tag{2.42}$$

From Eq. (2.14) for the low-momentum limit $\overline{\sigma}^{(0)}(0)$ we find

$$\overline{\sigma}^{(0)}_{\rm T}(0) = \overline{\sigma}^{(0)}_{\rm L}(0) = \frac{m^2}{d\gamma_1}. \tag{2.43}$$

[12]This formula does not easily generalize to $\langle \hat{G}\mathbf{v}\rangle$ because of the non-commutativity of the covariant derivatives contained in $\hat{G}$.

Because we are in Coulomb gauge, the longitudinal sector does not contribute to $\dot{\mathbf{A}}\overline{\sigma}(\boldsymbol{\nabla})\,\dot{\mathbf{A}}$ or $\mathbf{A}[\overline{\sigma}(\boldsymbol{\nabla})]^{-1}\mathbf{A}$. So we can replace $\overline{\sigma}$ by $\overline{\sigma}_{\mathrm{T}}^{(0)}$ in the free action (2.34a). The resulting propagator for $\mathbf{A}$ is

$$A_i^a \;\sim\!\sim\!\sim\; A_j^b = S_{ij}(\omega,\mathbf{k})\,\delta^{ab} \equiv \frac{2T\bar{\sigma}_{\mathrm{T}}^{(0)}(k)}{|i\omega\bar{\sigma}_{\mathrm{T}}^{(0)}(k)+k^2|^2}\,P_{\mathrm{T}}^{ij}(\hat{\mathbf{k}})\,\delta^{ab} = \frac{2T\bar{\sigma}_{\mathrm{T}}^{(0)}(k)}{[\omega\bar{\sigma}_{\mathrm{T}}^{(0)}(k)]^2+k^4}\,P_{\mathrm{T}}^{ij}(\hat{\mathbf{k}})\,\delta^{ab}. \tag{2.44}$$

The relations between these propagators and the retarded, equilibrium, or other types of propagators are discussed in Appendix C. The propagator for A_0 is the same as Eq. (2.7) (although we are now representing this propagator by a dashed line),

$$A_0^a\text{-----------}A_0^b = \frac{2T}{k^2\bar{\sigma}_{\mathrm{L}}^{(0)}(k)}\,\delta^{ab} = \frac{2T}{m^2}\,\langle\hat{G}_0(\mathbf{k})\rangle\,\delta^{ab}. \tag{2.45}$$

3. Transposition

It will be convenient to be able to rewrite interaction terms (2.34b) as their transposes. Under transposition in $\mathbf{x}$/color space, $\mathbf{D}^\top = -\mathbf{D}$, and so

$$\langle\hat{G}\rangle^\top = \langle[\mathbf{v}\cdot\mathbf{D}+\delta\hat{C}]^{-1}\rangle^\top = \langle[-\mathbf{v}\cdot\mathbf{D}+\delta\hat{C}]^{-1}\rangle = \langle\hat{G}\rangle, \tag{2.46}$$

where the last equality follows from taking $\mathbf{v}\to-\mathbf{v}$ in the $\mathbf{v}$ average $\langle\cdots\rangle$. The terms on the right-hand side of the perturbative expansion of Eq. (2.30) must also be symmetric, and this can be explicitly verified by recalling that the $\mathbf{A}$'s there are really color matrices $\underset{\sim}{\mathbf{A}} = \mathbf{A}^a T^a$, which are antisymmetric because $T^b_{ac} = f^{abc}$. So, for instance,

$$\langle\hat{G}_0\mathbf{v}\cdot\underset{\sim}{\mathbf{A}}\hat{G}_0\rangle^\top = -\langle\hat{G}_0^\top\mathbf{v}\cdot\underset{\sim}{\mathbf{A}}\hat{G}_0^\top\rangle = \langle\hat{G}_0\mathbf{v}\cdot\underset{\sim}{\mathbf{A}}\hat{G}_0\rangle \tag{2.47}$$

where the last equality again follows by $\mathbf{v}$ parity and where we have implicitly used the fact that $\mathbf{v}$ and $\delta\hat{C}$ are symmetric in $\mathbf{v}$ space.

Now let us check the transposition of the $A_0\langle\hat{G}_0\rangle^{-1}\langle\hat{G}_0\mathbf{v}\cdot\underset{\sim}{\mathbf{A}}\hat{G}_0\mathbf{v}\rangle\cdot\dot{\mathbf{A}}$ interaction in Eq. (2.34b). Being a little more explicit about color indices than previously, and placing an under-tilde on the $\mathbf{A}$ which is to be interpreted as a color matrix, the interaction is

$$A_0^a\,\langle\hat{G}_0^{ab}\rangle^{-1}\langle\hat{G}_0^{bc}\,\mathbf{v}\cdot\underset{\sim}{\mathbf{A}}^{cd}\,\hat{G}_0^{de}\,\mathbf{v}\rangle\cdot\dot{\mathbf{A}}^e. \tag{2.48}$$

Its transpose is

$$-\dot{\mathbf{A}}\cdot\langle\mathbf{v}\hat{G}_0^\top\mathbf{v}\cdot\underset{\sim}{\mathbf{A}}\hat{G}_0^\top\rangle\langle\hat{G}_0^\top\rangle^{-1}A_0 = -\dot{\mathbf{A}}\cdot\langle\mathbf{v}\hat{G}_0\mathbf{v}\cdot\underset{\sim}{\mathbf{A}}\hat{G}_0\rangle\langle\hat{G}_0\rangle A_0. \tag{2.49}$$

In summary, this interaction term can be written in either of two ways:

$$A_0\langle\hat{G}_0\rangle^{-1}\langle\hat{G}_0\mathbf{v}\cdot\underset{\sim}{\mathbf{A}}\hat{G}_0\mathbf{v}\rangle\cdot\dot{\mathbf{A}} = -\dot{\mathbf{A}}\cdot\langle\mathbf{v}\hat{G}_0\mathbf{v}\cdot\underset{\sim}{\mathbf{A}}\hat{G}_0\rangle\langle\hat{G}_0\rangle A_0. \tag{2.50}$$

4. Analysis of diagrams

Instead of presenting general Feynman rules for all the various vertices in the effective theory, and applying these rules to the diagrams of Fig. 5, we will simply write the expressions for the loop diagrams directly by treating S_{int} as a perturbation in the path integral, expanding the exponential $\exp(-S_{\mathrm{int}})$, and explicitly taking all possible Wick contractions of the fields. For the case at hand, this is far more convenient. The loops (a)–(c) in Fig. 5 represent the expressions

$$-\Pi_{00}^{(\mathrm{a})} = 2\left(-\frac{g^2m^2}{4T}\right)\langle\hat{G}_0\rangle^{-1}(\langle\hat{G}_0\mathbf{v}\cdot\mathbf{A}\hat{G}_0\rangle\langle\hat{G}_0\rangle^{-1}\langle\hat{G}_0\mathbf{v}\cdot\mathbf{A}\hat{G}_0\rangle - \langle\hat{G}_0\mathbf{v}\cdot\mathbf{A}\hat{G}_0\mathbf{v}\cdot\mathbf{A}\hat{G}_0\rangle)\langle\hat{G}_0\rangle^{-1}, \tag{2.51a}$$

$$-\Pi_{00}^{(\mathrm{b})} = 4\left(-\frac{gm^2}{4T}\right)^2[\langle\hat{G}_0\rangle^{-1}\langle\hat{G}_0\mathbf{v}\cdot\mathbf{A}\hat{G}_0\rangle\langle\hat{G}_0\rangle^{-1}A_0]\otimes[A_0\langle\hat{G}_0\rangle^{-1}\langle\hat{G}_0\mathbf{v}\cdot\mathbf{A}\hat{G}_0\rangle\langle\hat{G}_0\rangle^{-1}], \tag{2.51b}$$

$$-\Pi_{00}^{(\mathrm{c})}=\left(-\frac{gm^2}{2T}\right)^2\langle\hat{G}_0\rangle^{-1}[\langle\hat{G}_0\mathbf{v}\cdot\mathbf{A}\hat{G}_0\mathbf{v}\cdot\dot{\mathbf{A}}\rangle\otimes\langle-\mathbf{v}\cdot\dot{\mathbf{A}}\hat{G}_0\mathbf{v}\cdot\mathbf{A}\hat{G}_0\rangle+\langle\hat{G}_0\mathbf{v}\cdot\mathbf{A}\hat{G}_0\mathbf{v}\cdot\dot{\mathbf{A}}\rangle\otimes\langle-\mathbf{v}\cdot\dot{\mathbf{A}}\hat{G}_0\mathbf{v}\cdot\mathbf{A}\hat{G}_0\rangle]\langle\hat{G}_0\rangle^{-1}. \tag{2.51c}$$

The $\otimes$ notation above denotes where strings of color index contractions end. Specifically, $(\cdots)\otimes(\cdots)$ represents the color matrix $(\cdots)^a(\cdots)^b$, where a and b are adjoint color indices.[13]

Using the A_0 propagator (2.45), one can rewrite Eq. (2.51b) as

$$-\Pi_{00}^{(\mathrm{b})}=-2\left(-\frac{g^2m^2}{4T}\right)[\langle\hat{G}_0\rangle^{-1}\langle\hat{G}_0\mathbf{v}\cdot\mathbf{A}\hat{G}_0\rangle\langle\hat{G}_0\rangle^{-1}\langle\hat{G}_0\mathbf{v}\cdot\mathbf{A}\hat{G}_0\rangle\langle\hat{G}_0\rangle^{-1}]. \tag{2.52}$$

It neatly cancels the first term in Eq. (2.51a) to leave

$$-\Pi_{00}^{(\mathrm{a+b})}\equiv-[\Pi_{00}^{(\mathrm{a})}+\Pi_{00}^{(\mathrm{b})}]=\frac{g^2m^2}{2T}\,\langle\hat{G}_0\rangle^{-1}\langle\hat{G}_0\mathbf{v}\cdot\mathbf{A}\hat{G}_0\mathbf{v}\cdot\mathbf{A}\hat{G}_0\rangle\langle\hat{G}_0\rangle^{-1}. \tag{2.53}$$

The next step is to use the **A** propagator (2.44) and put everything into frequency and momentum space:[14]

$$[\Pi_{00}^{(\mathrm{a+b})}(0,k)]_{ab}=-\frac{g^2m^2}{2T}\int_{p_0,\mathbf{p}}\langle\hat{\mathcal{G}}_0(\mathbf{k})v_iT^c_{ad}\,\hat{G}_0(\mathbf{p})\,v_jT^c_{db}\,\hat{\mathcal{G}}_0(\mathbf{k})\rangle\,S_{ij}(-p_0,\mathbf{k}-\mathbf{p}), \tag{2.54}$$

$$\begin{aligned}[\Pi_{00}^{(\mathrm{c})}(0,k)]_{ab}=\frac{g^2m^4}{4T^2}\int_{p_0,\mathbf{p}}&\langle\hat{\mathcal{G}}_0(\mathbf{k})v_iT^c_{ad}\hat{G}_0(\mathbf{p})v_j\rangle\,p_0^2\,S_{i\bar{i}}(-p_0,\mathbf{k}-\mathbf{p})\,S_{j\bar{j}}(p_0,\mathbf{p})\\&\times[\langle v_{\bar{j}}\hat{G}_0(\mathbf{p})\,v_{\bar{i}}T^c_{db}\,\hat{\mathcal{G}}_0(\mathbf{k})\rangle-\langle v_{\bar{i}}\hat{G}_0(\mathbf{k}-\mathbf{p})\,v_{\bar{j}}T^d_{cb}\,\hat{\mathcal{G}}_0(\mathbf{k})\rangle],\end{aligned} \tag{2.55}$$

where we find it convenient to introduce

$$\hat{\mathcal{G}}_0(\mathbf{k})\equiv\frac{\hat{G}_0(\mathbf{k})}{\langle\hat{G}_0(\mathbf{k})\rangle}. \tag{2.56}$$

Remember that we are interested only in the case of zero external frequency k_0. The loop frequency integrals are easy to do using the explicit form (2.44) of S_{ij}, and give

$$\int_{p_0}S_{ij}(-p_0,\mathbf{q})=\frac{T\,P_{\mathrm{T}}^{ij}(\hat{\mathbf{q}})}{q^2}, \tag{2.57}$$

$$\int_{p_0}p_0^2\,S_{ik}(p_0,\mathbf{q}_1)\,S_{jl}(-p_0,\mathbf{q}_2)=\frac{2T^2\,P_{\mathrm{T}}^{ik}(\hat{\mathbf{q}}_1)\,P_{\mathrm{T}}^{jl}(\hat{\mathbf{q}}_2)}{q_1^2\,\overline{\sigma}_{\mathrm{T}}^{(0)}(q_2)+q_2^2\,\overline{\sigma}_{\mathrm{T}}^{(0)}(q_1)}. \tag{2.58}$$

Simplifying the color factors, we then have

$$\Pi_{00}^{(\mathrm{a+b})}(0,k)=\frac{C_{\mathrm{A}}g^2m^2}{2}\int_{\mathbf{p}}\langle\hat{\mathcal{G}}_0(\mathbf{k})\,v_i\,\hat{G}_0(\mathbf{p})\,v_j\,\hat{\mathcal{G}}_0(\mathbf{k})\rangle\frac{P_{\mathrm{T}}^{ij}(\widehat{\mathbf{k}-\mathbf{p}})}{|\mathbf{k}-\mathbf{p}|^2}, \tag{2.59}$$

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[13]So, for example, $\langle\hat{G}_0\mathbf{v}\cdot\mathbf{A}\hat{G}_0\mathbf{v}\cdot\dot{\mathbf{A}}\rangle\otimes\langle\mathbf{v}\cdot\dot{\mathbf{A}}\hat{G}_0\mathbf{v}\cdot\mathbf{A}\hat{G}_0\rangle$ represents $\langle\hat{G}_0^{ac}\mathbf{v}\cdot\mathbf{A}^{cd}\hat{G}_0^{de}\mathbf{v}\cdot\dot{\mathbf{A}}^e\rangle\langle\mathbf{v}\cdot\dot{\mathbf{A}}^f\hat{G}_0^{fg}\mathbf{v}\cdot\mathbf{A}^{gh}\hat{G}_0^{hb}\rangle$.

[14]The relative minus sign in Eq. (2.55) arises from the time derivatives in Eq. (2.51c), which give $(-ip_0)(-ip_0)$ for the first term and $(-ip_0)(+ip_0)$ for the second.

$$\Pi_{00}^{(c)}(0,k) = -\frac{C_A g^2 m^4}{2}\int_{\mathbf{p}} \langle \hat{\mathcal{G}}_0(\mathbf{k})\, v_i\, \hat{G}_0(\mathbf{p})\, v_j \rangle \frac{P_T^{i\bar{i}}(\widehat{\mathbf{k}-\mathbf{p}})\, P_T^{j\bar{j}}(\hat{\mathbf{p}})}{|\mathbf{k}-\mathbf{p}|^2\, \overline{\sigma}_T^{(0)}(p) + p^2\, \overline{\sigma}_T^{(0)}(|\mathbf{k}-\mathbf{p}|)}$$
$$\times [\langle v_{\bar{j}} \hat{G}_0(\mathbf{p})\, v_{\bar{i}} \hat{\mathcal{G}}_0(\mathbf{k})\rangle + \langle v_{\bar{i}} \hat{G}_0(\mathbf{k}-\mathbf{p})\, v_{\bar{j}} \hat{\mathcal{G}}_0(\mathbf{k})\rangle]. \tag{2.60}$$

5. The $\mathbf{k}\to 0$ *limit*

We now want to extract the $k\to 0$ limit of $\Pi_{00}(k)$ through $O(k^2)$, as this is required for the Wilson loop matching formula (2.19). This section is somewhat tedious, and some readers may wish to skip to the result (2.81).

To classify the various terms in the $\mathbf{k}$ expansion, it will help to rewrite the total self-energy given by Eqs. (2.59) and (2.60) as

$$\Pi_{00}(0,k) = \frac{C_A g^2}{2}\langle \hat{\mathcal{G}}_0(\mathbf{k})\, \hat{\mathcal{O}}(\mathbf{k}) \hat{\mathcal{G}}_0(\mathbf{k})\rangle, \tag{2.61}$$

with

$$\hat{\mathcal{O}}(\mathbf{k}) = m^2 \int_{\mathbf{p}} \left\{ v_i\, \hat{G}_0(\mathbf{p})\, v_j \frac{P_{\mathbf{k}-\mathbf{p}}^{ij}}{|\mathbf{k}-\mathbf{p}|^2} - m^2 v_i\, \hat{G}_0(\mathbf{p})\, v_j\rangle \frac{P_{\mathbf{k}-\mathbf{p}}^{i\bar{i}} P_{\mathbf{p}}^{j\bar{j}}}{|\mathbf{k}-\mathbf{p}|^2 \sigma_{\mathbf{p}} + p^2 \sigma_{\mathbf{k}-\mathbf{p}}} [\langle v_{\bar{j}} \hat{G}_0(\mathbf{p})\, v_{\bar{i}} + \langle v_{\bar{i}} \hat{G}_0(\mathbf{k}-\mathbf{p})\, v_{\bar{j}}] \right\}, \tag{2.62}$$

where in this section we use the abbreviations

$$\sigma_{\mathbf{p}} \equiv \overline{\sigma}_T^{(0)}(p), \quad P_{\mathbf{p}}^{ij} \equiv P_T^{ij}(\hat{\mathbf{p}}), \tag{2.63}$$

to keep formulas more compact. The operator $\hat{\mathcal{O}}$ is symmetric in $\mathbf{v}$ space. This is manifest for all the terms except the one involving $\hat{G}_0(\mathbf{k}-\mathbf{p})$, and it is easily seen for that term by making the change of integration variable $\mathbf{p}\to\mathbf{k}-\mathbf{p}$.

It is useful to start with the small $\mathbf{k}$ expansion of $\hat{\mathcal{G}}_0(\mathbf{k})\rangle$. This expansion is derived in Appendix D and is

$$\hat{\mathcal{G}}_0(\mathbf{k})\rangle = \frac{\hat{G}_0(\mathbf{k})\rangle}{\langle \hat{G}_0(\mathbf{k})\rangle} = \left| 1 - \frac{\mathbf{v}\cdot i\mathbf{k}}{\gamma_1} \right\rangle + O(k^2). \tag{2.64}$$

We now turn to the operator $\hat{\mathcal{O}}(\mathbf{k})$, and first consider $\hat{\mathcal{O}}(\mathbf{k})\rangle$. From the definitions (2.62) of $\hat{\mathcal{O}}(\mathbf{k})$ and (2.37) of $\sigma_{\mathbf{p}} = \overline{\sigma}_T^{(0)}(p)$, we have

$$\hat{\mathcal{O}}(\mathbf{k})\rangle = m^2 \int_{\mathbf{p}} v_i\, \hat{G}_0(\mathbf{p})\, v_j\rangle \times \left\{ \frac{P_{\mathbf{k}-\mathbf{p}}^{ij}}{|\mathbf{k}-\mathbf{p}|^2} - \frac{P_{\mathbf{k}-\mathbf{p}}^{ik} P_{\mathbf{p}}^{jk}(\sigma_{\mathbf{p}}+\sigma_{\mathbf{k}-\mathbf{p}})}{|\mathbf{k}-\mathbf{p}|^2 \sigma_{\mathbf{p}} + p^2 \sigma_{\mathbf{k}-\mathbf{p}}} \right\} \tag{2.65}$$

and, in particular,

$$\hat{\mathcal{O}}(\mathbf{0})\rangle = 0. \tag{2.66}$$

This means that $\hat{\mathcal{O}}(\mathbf{k})\rangle$ is $O(k)$, and it is the reason that we only need the expansion of $\hat{\mathcal{G}}_0(\mathbf{k})$ thru $O(k)$ and not $O(k^2)$. Specifically, using the symmetry of $\hat{\mathcal{O}}(\mathbf{k})$ and $\hat{\mathcal{G}}_0(\mathbf{k})$ under $\mathbf{v}$-space transposition, we can organize the small k expansion of Eq. (2.61) as

$$\Pi_{00}(k) = \frac{C_A g^2}{2} [\langle \hat{\mathcal{O}}(\mathbf{k})\rangle - 2\gamma_1^{-1} \langle \mathbf{v}\cdot i\mathbf{k}\, \hat{\mathcal{O}}(\mathbf{k})\rangle + \gamma_1^{-2}\langle \mathbf{v}\cdot i\mathbf{k}\, \hat{\mathcal{O}}(\mathbf{0})\, \mathbf{v}\cdot i\mathbf{k}\rangle + O(k^3)]. \tag{2.67}$$

First term. Let us begin with $\langle \hat{\mathcal{O}}(\mathbf{k})\rangle$. Starting with Eq. (2.65), one can again use the definition (2.37) of $\sigma_{\mathbf{p}} = \overline{\sigma}_T^{(0)}(p)$ to find

$$\langle \hat{\mathcal{O}}(\mathbf{k})\rangle = \int_{\mathbf{p}} P_{\mathbf{p}}^{ij} P_{\mathbf{k}-\mathbf{p}}^{ij} \sigma_{\mathbf{p}} \left[\frac{1}{|\mathbf{k}-\mathbf{p}|^2} - \frac{(\sigma_{\mathbf{p}}+\sigma_{\mathbf{k}-\mathbf{p}})}{|\mathbf{k}-\mathbf{p}|^2 \sigma_{\mathbf{p}} + p^2 \sigma_{\mathbf{k}-\mathbf{p}}} \right]$$
$$= \int_{\mathbf{p}} P_{\mathbf{p}}^{ij} P_{\mathbf{k}-\mathbf{p}}^{ij} \frac{(p^2 - |\mathbf{k}-\mathbf{p}|^2)}{|\mathbf{k}-\mathbf{p}|^2} \frac{\sigma_{\mathbf{p}} \sigma_{\mathbf{k}-\mathbf{p}}}{|\mathbf{k}-\mathbf{p}|^2 \sigma_{\mathbf{p}} + p^2 \sigma_{\mathbf{k}-\mathbf{p}}}. \tag{2.68}$$

Now symmetrize the integrand with respect to the change of integration variable $\mathbf{p}\to\mathbf{k}-\mathbf{p}$:

$$\langle \hat{\mathcal{O}}(\mathbf{k})\rangle = \int_{\mathbf{p}} P_{\mathbf{p}}^{ij} P_{\mathbf{k}-\mathbf{p}}^{ij} \frac{(p^2 - |\mathbf{k}-\mathbf{p}|^2)^2}{2p^2 |\mathbf{k}-\mathbf{p}|^2} \frac{\sigma_{\mathbf{p}} \sigma_{\mathbf{k}-\mathbf{p}}}{(|\mathbf{k}-\mathbf{p}|^2 \sigma_{\mathbf{p}} + p^2\, \sigma_{\mathbf{k}-\mathbf{p}})}$$
$$= (d-1)\int_{\mathbf{p}} \frac{(\mathbf{k}\cdot\mathbf{p})^2 \sigma_{\mathbf{p}}}{p^6} + O(k^3)$$
$$= \frac{(d-1)}{d} k^2 \int_{\mathbf{p}} \frac{\sigma_{\mathbf{p}}}{p^4} + O(k^3). \tag{2.69}$$

Second term. For the $\gamma_1^{-1}\langle \mathbf{v}\cdot i\mathbf{k}\, \hat{\mathcal{O}}(\mathbf{k})\rangle$ term of Eq. (2.67), we need the $O(k)$ piece of $\hat{\mathcal{O}}(\mathbf{k})\rangle$. Its extraction can be simplified by rewriting Eq. (2.65) as

$$\hat{\mathcal{O}}(\mathbf{k})\rangle = m^2 \int_{\mathbf{p}} v_i\, \hat{G}_0(\mathbf{p})\, v_j\rangle \left\{ \frac{1}{|\mathbf{k}-\mathbf{p}|^2}[P^{ij}_{\mathbf{k}-\mathbf{p}} - P^{ik}_{\mathbf{k}-\mathbf{p}} P^{jk}_{\mathbf{p}}] - \frac{(|\mathbf{k}-\mathbf{p}|^2 - p^2)\sigma_{\mathbf{k}-\mathbf{p}}}{|\mathbf{k}-\mathbf{p}|^2(|\mathbf{k}-\mathbf{p}|^2\, \sigma_{\mathbf{p}} + p^2\, \sigma_{\mathbf{k}-\mathbf{p}})} P^{ik}_{\mathbf{k}-\mathbf{p}} P^{jk}_{\mathbf{p}} \right\}$$
$$= m^2 \int_{\mathbf{p}} v_i\, \hat{G}_0(\mathbf{p})\, v_j\rangle \left\{ \frac{1}{p^2}[P^{ij}_{\mathbf{k}-\mathbf{p}} - P^{ik}_{\mathbf{k}-\mathbf{p}} P^{jk}_{\mathbf{p}}] + \frac{\mathbf{p}\cdot\mathbf{k}}{p^4} P^{ik}_{\mathbf{p}} P^{jk}_{\mathbf{p}} \right\} + O(k^2)$$
$$= m^2 \int_{\mathbf{p}} v_i\, \hat{G}_0(\mathbf{p})\, v_j\rangle \frac{1}{p^4}[k^i p^j - 2\hat{p}^i \hat{p}^j \mathbf{p}\cdot\mathbf{k} + \delta^{ij}\mathbf{p}\cdot\mathbf{k}] + O(k^2). \tag{2.70}$$

Therefore,

$$\langle \mathbf{v}\cdot i\mathbf{k}\, \hat{\mathcal{O}}(\mathbf{k})\rangle = m^2 \int_{\mathbf{p}} \langle v_l v_i\, \hat{G}_0(\mathbf{p})\, v_j\rangle \frac{ik^l}{p^4}[k^i p^j - 2\hat{p}^i \hat{p}^j \mathbf{p}\cdot\mathbf{k} + \delta^{ij}\mathbf{p}\cdot\mathbf{k}] + O(k^3). \tag{2.71}$$

Because of rotation invariance, $\langle \mathbf{v}\cdot i\mathbf{k}\, \hat{\mathcal{O}}(\mathbf{k})\rangle$ depends only on the magnitude k of $\mathbf{k}$, so nothing is harmed by averaging the integrand in Eq. (2.71) over the direction $\hat{\mathbf{k}}$, giving

$$\langle \mathbf{v}\cdot i\mathbf{k}\, \hat{\mathcal{O}}(\mathbf{k})\rangle = \frac{m^2 k^2}{d} \int_{\mathbf{p}} \frac{1}{p^4} [\langle \hat{G}_0(\mathbf{p})\, \mathbf{v}\cdot i\mathbf{p}\rangle - 2\langle \mathbf{v}\cdot\hat{\mathbf{p}}\, \mathbf{v}\cdot\hat{\mathbf{p}}\, \hat{G}_0(\mathbf{p})\, \mathbf{v}\cdot i\mathbf{p}\rangle + \langle \mathbf{v}\cdot i\mathbf{p}\, v_i\, \hat{G}_0(\mathbf{p})\, v_i\rangle]. \tag{2.72}$$

The factors of $\hat{G}_0(\mathbf{p})$ can be eliminated from the first and second terms by using the trick (2.24), which in the present context is

$$\hat{G}_0(\mathbf{p})\, \mathbf{v}\cdot i\mathbf{p}\rangle = \hat{G}_0(\mathbf{p})\, (\mathbf{v}\cdot i\mathbf{p} + \delta\hat{C})\rangle = \rangle. \tag{2.73}$$

Using the relations (1.17), a similar manipulation shows that

$$\langle \mathbf{v}\cdot i\mathbf{p}\, v_i\, \hat{G}_0(\mathbf{p}) = \langle v_i \mathbf{v}\cdot i\mathbf{p}\, \hat{G}_0(\mathbf{p})$$
$$= \langle v_i(\mathbf{v}\cdot i\mathbf{p} + \delta\hat{C} - \gamma_1)\, \hat{G}_0(\mathbf{p})$$
$$= \langle v_i[1 - \gamma_1\, \hat{G}_0(\mathbf{p})], \tag{2.74}$$

and so the final term of Eq. (2.72) becomes

$$\langle \mathbf{v}\cdot i\mathbf{p}\, v_i\, \hat{G}_0(\mathbf{p})\, v_i\rangle = 1 - (d-1)\frac{\gamma_1}{m^2}\sigma_{\mathbf{p}}. \tag{2.75}$$

Putting everything together,

$$\gamma_1^{-1}\langle \mathbf{v}\cdot i\mathbf{k}\, \hat{\mathcal{O}}(\mathbf{k})\rangle = \frac{(d-1)}{d} k^2 \int_{\mathbf{p}} \frac{1}{p^4}(2\sigma_0 - \sigma_{\mathbf{p}}), \tag{2.76}$$

where $\sigma_0 = m^2/d\gamma_1$ is the value of $\sigma_{\mathbf{p}}$ at $\mathbf{p}=0$, taken from Eq. (2.43).

Third term. Finally, we pursue the $\gamma_1^{-2}\langle \mathbf{v}\cdot i\mathbf{k}\, \hat{\mathcal{O}}(\mathbf{0})\, \mathbf{v}\cdot i\mathbf{k}\rangle$ term in the expansion (2.67). We are again free to average over the direction $\hat{\mathbf{k}}$, giving

$$\langle \mathbf{v}\cdot i\mathbf{k}\, \hat{\mathcal{O}}(\mathbf{0})\, \mathbf{v}\cdot i\mathbf{k}\rangle = -\frac{k^2}{d}\langle v_k\, \hat{\mathcal{O}}(\mathbf{0})\, v_k\rangle. \tag{2.77}$$

Now use the definition (2.62) of $\hat{\mathcal{O}}(\mathbf{k})$ to find

$$\langle v_k\, \hat{\mathcal{O}}(\mathbf{0})\, v_k\rangle = \int_{\mathbf{p}} \frac{m^2}{p^2} \Big\{ \langle v_k\, v_i\, \hat{G}_0(\mathbf{p})\, v_j\, v_k\rangle P^{ij}_{\mathbf{p}}$$
$$- m^2 \langle v_k\, v_i\, \hat{G}_0(\mathbf{p})\, v_j\rangle \frac{P^{i\bar{i}}_{\mathbf{p}} P^{j\bar{j}}_{\mathbf{p}}}{2\sigma_{\mathbf{p}}}$$
$$\times [\langle v_{\bar{j}}\hat{G}_0(\mathbf{p})\, v_{\bar{i}} v_k\rangle + \langle v_{\bar{i}}\hat{G}_0(-\mathbf{p})\, v_{\bar{j}} v_k\rangle] \Big\}. \tag{2.78}$$

The second term vanishes by the following symmetry argument. First, use $\mathbf{v}\to-\mathbf{v}$ and the definition (2.10) of $\hat{G}_0(\mathbf{k})$ to rewrite

$$\langle v_{\bar{j}}\hat{G}_0(\mathbf{p})\, v_{\bar{i}} v_k\rangle + \langle v_{\bar{i}}\hat{G}_0(-\mathbf{p})\, v_{\bar{j}} v_k\rangle = \langle v_{\bar{j}}\hat{G}_0(\mathbf{p})\, v_{\bar{i}} v_k\rangle - (\bar{i}\leftrightarrow\bar{j}). \tag{2.79}$$

By rotation invariance, $\langle v_{\bar{j}}\hat{G}_0(\mathbf{p})\, v_{\bar{i}} v_k\rangle$ can only depend on the direction as $\hat{\mathbf{p}}_{\bar{i}}\hat{\mathbf{p}}_{\bar{j}}\hat{\mathbf{p}}_k$, $\hat{\mathbf{p}}_{\bar{i}}\delta_{\bar{j}k}$, $\hat{\mathbf{p}}_{\bar{j}}\delta_{\bar{i}k}$, and $\hat{\mathbf{p}}_k\delta_{\bar{i}\bar{j}}$. Every one of these possibilities either has a $\hat{\mathbf{p}}_{\bar{i}}$ or a $\hat{\mathbf{p}}_{\bar{j}}$, which will annihilate against the transverse projections $P^{i\bar{i}}_{\mathbf{p}} P^{j\bar{j}}_{\mathbf{p}}$ in Eq. (2.78), or else has a $\delta^{\bar{i}\bar{j}}$, which vanishes by the antisymmetry of Eq. (2.79) in $\bar{i}\bar{j}$. In summary, Eqs. (2.77) and (2.78) become simply

$$\langle \mathbf{v}\cdot i\mathbf{k}\, \hat{\mathcal{O}}(\mathbf{0})\, \mathbf{v}\cdot i\mathbf{k}\rangle = -\frac{k^2}{d}\int_{\mathbf{p}} \frac{m^2}{p^2}\langle v_k\, v_i\, \hat{G}_0(\mathbf{p})\, v_j\, v_k\rangle P^{ij}_{\mathbf{p}}. \tag{2.80}$$

Simplifying this expression will require us to be a little more systematic about the manipulations we have been using and is the subject of a later section. We end this one by combining the results (2.69), (2.76), and (2.80) for the individual terms appearing in Eq. (2.67):

$$\Pi_{00}(k) = -\frac{C_A g^2 k^2}{2} \int_{\mathbf{p}} \left[\frac{(d-1)}{d\, p^4} (4\sigma_0 - 3\sigma_{\mathbf{p}}) + \frac{\sigma_0}{\gamma_1 p^2} \langle v_k\, v_i\, \hat{G}_0(\mathbf{p})\, v_j\, v_k \rangle P^{ij}_{\mathbf{p}} \right] + O(k^3). \tag{2.81}$$

6. Extracting UV and IR divergences

As we shall see, the integral (2.81) giving Π_{00} is both infrared and ultraviolet divergent in three spatial dimensions. We are using dimensional regularization, but it will simplify our discussion of what to do with $\langle v_k\, v_i\, \hat{G}_0(\mathbf{p})\, v_j\, v_k \rangle$ if we can instead work directly in three dimensions. Therefore, we will now isolate the pieces of the integrand responsible for the IR and UV divergences, so that we can subtract them and evaluate the remainder as a finite integral in $d-3$. Specifically, we will rewrite Eq. (2.81) as

$$\Pi_{00}(k) = -\frac{C_A g^2 k^2}{2} \left[\int_{\mathbf{p}} f_{\rm reg}(\mathbf{p}) + \int_{\mathbf{p}} f_{\rm IR}(\mathbf{p}) + \int_{\mathbf{p}} f_{\rm UV}(\mathbf{p}) \right] + O(k^3), \tag{2.82}$$

where

$$f_{\rm reg}(\mathbf{p}) = \frac{(d-1)}{d\, p^4} (4\sigma_0 - 3\sigma_{\mathbf{p}}) + \frac{\sigma_0}{\gamma_1 p^2} \langle v_k\, v_i\, \hat{G}_0(\mathbf{p})\, v_j\, v_k \rangle P^{ij}_{\mathbf{p}} - f^{(\rm IR)}(\mathbf{p}) - f^{(\rm UV)}(\mathbf{p}). \tag{2.83}$$

$f_{\rm IR}$ and $f_{\rm UV}$ will be chosen to (a) make the $f_{\rm reg}$ integral finite, and so evaluable directly in three dimensions, and (b) make the $f_{\rm IR}$ and $f_{\rm UV}$ integrals analytically tractable in dimensional regularization.

IR behavior. In Appendix D, we show that $\hat{G}_0(\mathbf{p})$ has the small $\mathbf{p}$ expansion

$$\hat{G}_0 = \frac{d}{\gamma_1 p^2} (\gamma_1 - \mathbf{v}\cdot i\mathbf{p})\, \hat{P}_0\, (\gamma_1 - \mathbf{v}\cdot i\mathbf{p}) + O(p^0). \tag{2.84}$$

We can thus expand the $\langle \mathbf{v}\mathbf{v}\hat{G}_0\mathbf{v}\mathbf{v} \rangle$ term in Eq. (2.81) as

$$\begin{aligned}\frac{\sigma_0}{\gamma_1 p^2} \langle v_k v_i\, \hat{G}_0(\mathbf{p})\, v_j v_k \rangle P^{ij}_{\mathbf{p}} &= \frac{d\sigma_0}{\gamma_1^2 p^4} \langle v_k v_i\, (\gamma_1 - \mathbf{v}\cdot i\mathbf{p}) \rangle \langle (\gamma_1 - i\mathbf{v}\cdot\mathbf{p})\, v_j v_k \rangle P^{ij}_{\mathbf{p}} + O(p^{-2}) \\ &= \frac{(d-1)}{d} \frac{\sigma_0}{p^4} + O(p^{-2}).\end{aligned} \tag{2.85}$$

As discussed earlier, the $\mathbf{p}\to 0$ limit of $\sigma_{\mathbf{p}}$ is $\sigma_0 = m^2/(d\gamma_1)$. In Appendix D, we show that the small $\mathbf{p}$ corrections to $\sigma(p)$ are $O(p^2)$. Putting everything together, we may then choose

$$f_{\rm IR}(\mathbf{p}) = 2\, \frac{(d-1)}{d}\, \frac{\sigma_0}{p^4}. \tag{2.86}$$

In dimensional regularization, the integral of $f_{\rm IR}$ vanishes:

$$\int_{\mathbf{p}} f_{\rm IR}(\mathbf{p}) = 0. \tag{2.87}$$

UV behavior. For $\mathbf{p}\to\infty$, we can treat $\delta\hat{C}$ as a perturbation to $\mathbf{v}\cdot i\mathbf{p}$, giving $\hat{G}_0 \to (\mathbf{v}\cdot i\mathbf{p})^{-1}$, except that we will need a prescription for integrating over the pole $\mathbf{v}\cdot\mathbf{p}=0$ in angular integrals. The prescription is obtained by recalling from Sec. I B that $\delta\hat{C}$ is a non-negative operator. So

$$\hat{G}_0(\mathbf{p}) \to \frac{1}{\mathbf{v}\cdot i\mathbf{p} + \epsilon} = \text{P.P.}\, \frac{1}{\mathbf{v}\cdot i\mathbf{p}} + \pi\,\delta(\mathbf{v}\cdot\mathbf{p}), \tag{2.88}$$

where ϵ is a positive infinitesimal and P.P. denotes principal part. (Higher-order corrections to this formula are discussed in Appendix E.) This limit then gives

$$\begin{aligned}\sigma_{\mathbf{p}} &= \frac{m^2}{(d-1)} \langle v_i\, \hat{G}_0(\mathbf{p})\, v_i \rangle \to \frac{m^2}{(d-1)}\, \pi \langle \delta(\mathbf{v}\cdot\mathbf{p}) \rangle \\ &= \frac{m^2}{(d-1)}\, \frac{S_{d-2}\, \pi}{S_{d-1}\, p}\end{aligned} \tag{2.89}$$

and

$$\begin{aligned}\langle v_k v_i\, \hat{G}_0(\mathbf{p})\, v_j v_k \rangle P^{ij}_{\mathbf{p}} &\to \pi \langle v_i\, \delta(\mathbf{v}\cdot\mathbf{p}) v_j \rangle P^{ij}_{\mathbf{p}} \\ &= \pi \langle \delta(\mathbf{v}\cdot\mathbf{p}) \rangle = \frac{S_{d-2}\, \pi}{S_{d-1}\, p},\end{aligned} \tag{2.90}$$

where

$$S_{d-1} \equiv \frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{2.91}$$

is the surface area of a $(d-1)$-sphere (e.g., $S_1 = 2\pi$ and $S_2 = 4\pi$). The UV piece of the integrand (2.81) is therefore

$$f_{\rm UV}(\mathbf{p}) \to \frac{S_{d-2}\, \pi\sigma_0}{S_{d-1}\, \gamma_1}\, \frac{1}{p^3}, \tag{2.92}$$

and comes only from the $\langle \mathbf{v}\mathbf{v}\hat{G}_0\mathbf{v}\mathbf{v} \rangle$ term. We do not want $f_{\rm UV}$ to mess up our IR subtraction, so we will cut it off in the infrared by choosing

$$f_{\rm UV}(\mathbf{p}) = \frac{S_{d-2}\, \pi\sigma_0}{S_{d-1}\, \gamma_1}\, \frac{1}{p\,(p^2+M^2)}, \tag{2.93}$$

where M is arbitrary. In dimensional regularization, the integral of $f_{\rm UV}$ is

$$\begin{aligned}\int_{\mathbf{p}} f_{\rm UV}(\mathbf{p}) &= \frac{\Gamma(\epsilon/2)}{2(4\pi)^{1-\epsilon/2}}\, \frac{\sigma_0}{\gamma_1} \left(\frac{\mu}{M} \right)^{\epsilon} \\ &= \frac{\sigma_0}{4\pi\gamma_1} \left[\frac{1}{\epsilon} + \ln\left(\frac{\bar{\mu}}{M} \right) + O(\epsilon) \right],\end{aligned} \tag{2.94}$$

where $\bar{\mu}$ is the modified minimal subtraction ($\overline{\rm MS}$) scale defined by

$$\bar{\mu} = \mu \sqrt{\frac{4\pi}{e^{\gamma_E}}}. \tag{2.95}$$

Putting everything together, Eq. (2.82) for $\Pi_{00}(k)$ becomes

$$\Pi_{00}(k)=-\frac{C_{\rm A}\alpha\sigma_0}{2\gamma_1}k^2\left[\frac{1}{\epsilon}+\ln\left(\frac{\bar{\mu}}{M}\right)+\frac{4\pi\gamma_1}{\sigma_0}\int_{\mathbf p}f_{\rm reg}(\mathbf{p},M)\right]+O(k^3)+O(\epsilon), \tag{2.96a}$$

where we may now set $d=3$ in

$$f_{\rm reg}(\mathbf{p},M)=\frac{1}{p^4}\left(\tfrac{4}{3}\sigma_0-2\sigma_{\mathbf p}\right)+\frac{\sigma_0}{\gamma_1 p^2}\langle v_k\, v_i\,\hat G_0(\mathbf p)\,v_j\,v_k\rangle\, P^{ij}_{\mathbf p}-\frac{\pi\sigma_0}{2\gamma_1}\frac{1}{p(p^2+M^2)}. \tag{2.96b}$$

This formula for Π_{00} does not depend on the choice of M.

7. Reducing $\langle v_{i_1}\cdots v_{i_m}\hat G_0(\mathbf p)\, v_{j_1}\cdots v_{j_m}\rangle$

In the previous section, we encountered several $\mathbf v$ averages of $\hat G_0(\mathbf p)$ flanked by various factors of $\mathbf v$. Any finite combination of $\mathbf v$'s can be rewritten as a superposition of spherical harmonics $|lm\rangle$'s, and so we can recast the problem of simplifying general expressions of the form $\langle v_{i_1}\cdots v_{i_m}\hat G_0(\mathbf p)\, v_{j_1}\cdots v_{j_m}\rangle$ to the simplification of $\langle l'm'|\hat G_0(\mathbf p)|lm\rangle$.

Choose the z axis in the direction of $\mathbf p$. Then

$$\langle l'm'|\hat G_0(\mathbf p)|lm\rangle=\langle l'm'|(ipv_z+\delta\hat C)^{-1}|lm\rangle. \tag{2.97}$$

Now recall that $\delta\hat C$ is diagonal in l and m, and note that v_z can change l but does not change the azimuthal quantum number m. So

$$\langle l'm'|\hat G_0(\mathbf p)|lm\rangle=\langle l'm|(ipv_z+\delta\hat C)^{-1}|lm\rangle\,\delta_{mm'}. \tag{2.98}$$

We can derive a recursion relation in l by writing

$$\begin{aligned}\delta_{ll'}&=\langle l'm|(ipv_z+\delta\hat C)\hat G_0|lm\rangle\\&=ip\,\langle l'm|v_z|l''m\rangle\,\langle l''m|\hat G_0|lm\rangle\\&\quad+\langle l'|\delta\hat C|l'\rangle\,\langle l'm|\hat G_0|lm\rangle,\end{aligned} \tag{2.99}$$

with an implied sum over l''. Since v_z can only change l by ± 1, this gives

$$\begin{aligned}\delta_{ll'}&=ip\langle l'm|v_z|(l'+1)m\rangle\langle(l'+1)m|\hat G_0|lm\rangle\\&\quad+ip\langle l'm|v_z|(l'-1)m\rangle\langle(l'-1)m|\hat G_0|lm\rangle\\&\quad+\langle l'|\delta\hat C|l'\rangle\langle l'm|\hat G_0|lm\rangle.\end{aligned} \tag{2.100}$$

This defines a recursion relation which allows one to rewrite matrix elements with higher l' in terms of those with lower l'. This recursion will end when l' becomes as low as it can be consistent with m—that is, at $l'=|m|$. A similar recursion can be constructed for l, and by use of these recursions, all matrix elements $\langle l'm|\hat G_0|lm\rangle$ can be rewritten in terms of $\langle |m|m|\hat G_0||m|m\rangle$. In fact, the case $m<0$ is related to the case $m>0$ by $\mathbf v$ parity:

$$\begin{aligned}\langle |m|,m|\hat G_0(\mathbf p)||m|,m\rangle&=\langle |m|,-m|\hat G_0(-\mathbf p)||m|,-m\rangle\\&=\langle |m|,-m|\hat G_0(\mathbf p)||m|,-m\rangle,\end{aligned} \tag{2.101}$$

where the final equality follows from $\mathbf v$ parity and the $[\mathbf v\cdot i\mathbf p+\delta\hat C]^{-1}$ structure of $\hat G_0(\mathbf p)$.

We now turn to the specific problem of rewriting the $\langle v_k v_i\,\hat G_0(\mathbf p)\,v_j v_k\rangle$ term of our expression (2.81) for Π_{00} in terms of the $\langle mm|\hat G_0|mm\rangle$. A product of two $\mathbf v$'s is a combination of $l=2$ and $l=0$, so we will be able to rewrite the expectation in terms of $\langle 22|\hat G_0|22\rangle$ and $\langle\hat G_0\rangle$. The advantage of this rewriting is that later analysis of how to evaluate expectations involving $\hat G_0$ will be simpler and more natural for $\langle mm|\hat G_0|mm\rangle$ than for $\langle v_k v_i\hat G_0 v_j v_k\rangle$ directly.

It is easiest to work backwards from the explicit form for $Y_{2,2}$ which gives

$$|22\rangle=-\sqrt{\frac{15}{32\pi}}(\hat v_x+i\hat v_y)^2\rangle. \tag{2.102}$$

From this, one may easily check that

$$\langle 22|\hat G_0(\mathbf p)|22\rangle=\frac{15}{8}\langle v_k v_i\,\hat G_0(\mathbf p)v_j v_l\rangle(2P^{ij}_{\mathbf p}P^{kl}_{\mathbf p}-P^{ik}_{\mathbf p}P^{jl}_{\mathbf p}), \tag{2.103}$$

for any operator $\hat G_0(\mathbf p)$, remembering that we have chosen $\hat{\mathbf p}$ to be the z direction for the purpose of defining $|lm\rangle$.[15] From Eq. (2.103), we have

$$\begin{aligned}\langle v_k v_i\hat G_0 v_j v_k\rangle P^{ij}_{\mathbf p}&=\langle v_k v_i\hat G_0 v_j v_l\rangle P^{ij}_{\mathbf p}(P^{kl}_{\mathbf p}+\hat{\mathbf p}^k\hat{\mathbf p}^l)\\&=\frac{4}{15}\langle 22|\hat G_0|22\rangle+\langle v_k v_i\hat G_0 v_j v_l\rangle\\&\quad\times(\tfrac12 P^{ik}_{\mathbf p}P^{jl}_{\mathbf p}+P^{ij}_{\mathbf p}\hat{\mathbf p}^k\hat{\mathbf p}^l).\end{aligned} \tag{2.104}$$

The second term on the right-hand side can be simplified by expanding $P^{ij}_{\mathbf p}=\delta^{ij}-\hat{\mathbf p}^i\hat{\mathbf p}^j$, and by repeated use of the relation (2.74). The result is

[15] The natural generalization to d dimensions is

$$\langle 22|\hat G_0(\mathbf p)|22\rangle\equiv\frac{d\,(d+2)}{(d^2-1)(d-2)}\langle v_k v_i\,\hat G_0(\mathbf p)v_j v_l\rangle\times[(d-1)\,P^{ij}_{\mathbf p}P^{kl}_{\mathbf p}-P^{ik}_{\mathbf p}P^{jl}_{\mathbf p}],$$

where the relative coefficient of the two $P_{\mathbf p}P_{\mathbf p}$ terms is chosen so that contraction with δ_{ik} or δ_{jl} gives zero (so as to exclude the $l=0$ combinations of $\mathbf{vv}$), and the overall normalization has been chosen so that replacing $\hat G_0(\mathbf p)$ by 1 gives $\langle 22|22\rangle=1$.

$$\langle v_k v_i \hat{G}_0 v_j v_k \rangle P^{ij}_{\mathbf{p}} = \frac{4}{15}\langle 22|\hat{G}_0|22\rangle + \frac{1}{2}\langle G_0\rangle - \frac{\gamma_1}{p^2}\left(\frac{1}{6}+\frac{2}{3}\frac{\sigma_{\mathbf{p}}}{\sigma_0}\right). \tag{2.105}$$

The analogous derivation for $\langle 11|\hat{G}_0|11\rangle$ gives[16]

$$\langle 11|\hat{G}_0|11\rangle = \frac{3}{2}\langle v_i \hat{G}_0 v_j\rangle P^{ij}_{\mathbf{p}} = \frac{3}{2}\langle v_i \hat{G}_0 v_i\rangle = \frac{3\sigma_{\mathbf{p}}}{m^2}. \tag{2.106}$$

To obtain the conductivity at NLLO, it is adequate to use leading-log approximations to the propagators in our one-loop calculation of Π_{00}. That is, we only need to evaluate the integrand $f_{\rm reg}$ in Eq. (2.96) at leading-log order. At this order, if one factors out the scale γ of $\delta\hat{C}$, functions like $\langle 22|\hat{G}_0(\mathbf{p})|22\rangle$ may be re-expressed as purely numerical functions of the single dimensionless variable p/γ. Specifically, at leading-log order,

$$\langle |m|\, m|\hat{G}_0(\mathbf{p})||m|\, m\rangle \approx \gamma^{-1}\,\Sigma_{|m|}(p/\gamma), \tag{2.107}$$

where

$$\Sigma_m(\rho) \equiv \left\langle mm\left|\frac{1}{i\rho v_z + \delta\hat{c}}\right|mm\right\rangle, \tag{2.108}$$

and $\delta\hat{c}$ is the leading-log result (1.6b) for $\delta\hat{C}/\gamma$:

$$\delta c(\mathbf{v},\mathbf{v}') = \delta^{S_2}(\mathbf{v}-\mathbf{v}') - \frac{4}{\pi}\frac{(\mathbf{v}\cdot\mathbf{v}')^2}{\sqrt{1-(\mathbf{v}\cdot\mathbf{v}')^2}}. \tag{2.109}$$

We can now combine this with Eqs. (2.96) and (2.105) to obtain

$$\Pi_{00}(k) = -\frac{C_{\rm A}\alpha\sigma_0}{2\gamma}k^2\left[\frac{1}{\epsilon}+\ln\left(\frac{\bar{\mu}}{M}\right)+I\left(\frac{M}{\gamma}\right)\right] \times[1+O(\ln^{-1})]+O(k^3)+O(\epsilon), \tag{2.110}$$

with

$$I(\nu) = \frac{2}{\pi}\int_0^\infty d\rho\left[\frac{1}{2}\left(\Sigma_0(\rho)-\frac{3}{\rho^2}\right)+\frac{8}{3\rho^2}[1-\Sigma_1(\rho)] + \frac{4}{15}\Sigma_2(\rho) - \frac{\pi\rho}{2(\rho^2+\nu^2)}\right]. \tag{2.111}$$

We have used Eq. (2.106) to rewrite $\sigma_{\mathbf{p}}$ (at the order under consideration) as $\sigma_0\,\Sigma_1(p/\gamma)$. We have ceased to distinguish between γ_1 and γ in Π_{00} since we are ignoring corrections to Π_{00} suppressed by additional powers of inverse logs [see Eq. (1.15)]. The terms in Eq. (2.111) have been arranged so that each term is individually IR safe. We will discuss how to evaluate $I(\nu)$ numerically in Sec. II D.

C. Matching to theory 3

We've now got Π_{00} in theory 2, but we still need Π_{00} in theory 3, so that we can use the matching condition (2.19) to determine the parameter σ of theory 3 at NLLO. Fortunately, the one-loop calculation in theory 3 is trivial in dimensional regularization: $\lim_{k\to 0}[k^{-2}\,\Pi_{00}(k)]=0$. The reason is simple dimensional analysis. Rescale the variables of the path integral (1.9) for theory 3 to $\bar{t}=\sigma^{-1}t$, $\bar{\mathbf{A}}=T^{-1/2}\mathbf{A}$, and $\bar{A}_0=\sigma T^{-1/2}A_0$. Here we will for once be explicit about the factors of μ^ϵ. The path action can then be rewritten as

$$S=\frac{1}{4}\int d\bar{t}\,d^dx|-\bar{\mathbf{E}}+\bar{\mathbf{D}}\times\bar{\mathbf{B}}|^2, \tag{2.112}$$

where $\bar{D}_\nu=\bar{\partial}_\nu+g\mu^{\epsilon/2}T^{1/2}\bar{A}_\nu$. In this form, the parameters of the theory appear only in the combination $g\mu^{\epsilon/2}T^{1/2}$. At one loop, the self-energy $\bar{\Pi}_{00}\equiv\sigma^{-1}T\Pi_{00}$ of $\bar{A}_0$ must be proportional to $g^2\mu^\epsilon T$, which has mass dimension $1+\epsilon$. But $\lim_{k\to 0}[\mathbf{k}^{-2}\,\bar{\Pi}_{00}(k)]$ has mass dimension zero, and there are no other dimensionful parameters in the problem that can make up the discrepancy in mass dimension. Consistency then forces $\lim_{k\to 0}[\mathbf{k}^{-2}\,\bar{\Pi}_{00}(k)]=0$ in dimensional regularization. Such simplicity is the standard virtue of dimensional regularization for matching calculations [12,13].

Our matching condition (2.19) and our theory 2 result (2.110) then give the color conductivity σ at NLLO:

$$\sigma=\frac{m^2}{d\gamma_1}\left\{1-\frac{C_{\rm A}\alpha T}{\gamma}\left[\frac{1}{\epsilon}+\ln\left(\frac{\bar{\mu}}{\gamma}\right)+I(1)\right]+O(\ln^{-2})\right\}, \tag{2.113}$$

where, for the sake of definiteness, we have fixed $M=\gamma$. As one can see, the only information we will need about $\delta\hat{C}$ at NLLO is the value of γ_1.

We now turn to methods for evaluating the functions $\Sigma_m(\rho)$ and so evaluating the numerical constant $I(1)$ from Eq. (2.111).

D. Evaluation of $\Sigma_m(\rho)\approx\gamma\langle mm|\hat{G}_0(\mathbf{p})|mm\rangle$

The dimensionless functions $\Sigma_m(\rho)$ were defined in Eq. (2.108) as $\langle mm|(iv_z\rho+\delta\hat{c})^{-1}|mm\rangle$, where $\delta\hat{c}$ is the leading-log result for $\delta\hat{C}/\gamma$. As discussed in Sec. II B 7, the operators v_z and $\delta\hat{C}$ both preserve m, $\delta\hat{C}$ preserves l as well, and v_z changes l by ± 1. For fixed m, the operator $iv_z\rho+\delta\hat{c}$ may therefore be considered as a tri-diagonal matrix in the $|lm\rangle$ basis where $l=m,m+1,m+2,\ldots$:

[16]The natural generalization to d dimensions is $\langle 11|\hat{G}_0|11\rangle \equiv[d/(d-1)]\langle v_i\,\hat{G}_0 v_j\rangle\,P^{ij}_{\mathbf{p}}=d\,\sigma_{\mathbf{p}}/m^2$.

$$
iv_z\rho+\delta\hat{c}=\begin{pmatrix} c_m & ib_m^{(m)}\rho & & & \\ ib_m^{(m)}\rho & c_{m+1} & ib_{m+1}^{(m)}\rho & & \\ & ib_{m+1}^{(m)}\rho & c_{m+2} & ib_{m+2}^{(m)}\rho & \\ & & ib_{m+2}^{(m)}\rho & c_{m+3} & \ddots \\ & & & \ddots & \ddots \end{pmatrix}, \tag{2.114}
$$

where

$$
c_l\equiv\langle l|\delta\hat{c}|l\rangle, \quad b_l^{(m)}\equiv\langle lm|v_z|(l+1)m\rangle. \tag{2.115}
$$

$\Sigma_m(\rho)$ corresponds to the upper-left element of the inverse of the matrix (2.114). Inverting tri-diagonal matrices is particularly simple, and there is a continued-fraction formula for this element:[17]

$$
\Sigma_m(\rho)=\cfrac{1}{c_m+\cfrac{(b_m^{(m)}\rho)^2}{c_{m+1}+\cfrac{(b_{m+1}^{(m)}\rho)^2}{c_{m+2}+\cfrac{(b_{m+2}^{(m)}\rho)^2}{\cdots}}}}. \tag{2.116}
$$

Note that $\Sigma_m(\rho)$ is even in ρ. All we need now are explicit formulas for the coefficients c_l and $b_l^{(m)}$. Equation (2.116) may then be used for numerical evaluation of $\Sigma_m(\rho)$.[18] The $b_l^{(m)}$ are given by

$$
b_l^{(m)}=\sqrt{\frac{(l+1)^2-m^2}{4(l+1)^2-1}}. \tag{2.117}
$$

The c_l may be evaluated from the expression (2.109) for $\delta c(\mathbf{v},\mathbf{v}')$ as [3]

$$
c_l=\langle\,\delta c(\mathbf{v},\mathbf{v}')\,P_l(\mathbf{v}\cdot\mathbf{v}')\rangle_{\mathbf{vv}'}=1-\frac{2}{\pi}\int_{-1}^{+1}dz\,\frac{z^2P_l(z)}{\sqrt{1-z^2}}, \tag{2.118}
$$

where $P_l(z)$ are Legendre polynomials. The integral vanishes if l is odd, and we obtain[19]

$$
c_{2n}=1-2\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2\left(1+\frac{1}{(2n)^2+2n-2}\right),
$$
$$
c_{2n+1}=1. \tag{2.119}
$$

The c_l are all non-negative, as was claimed in Sec. I B. Note that c_0 vanishes, as it must.

The procedure for numerical evaluation of $\Sigma_m(\rho)$ is to compute the continued-fraction formula (2.116) with some upper cut-off $l_{\max}$ on l, and then repeat the calculation, doubling $l_{\max}$ each time until the answer converges. This procedure becomes inefficient for very large ρ, however, because it then requires rather large $l_{\max}$ for good convergence. For very large ρ, it is more convenient to use asymptotic formula for $\Sigma_m(\rho)$, which are derived and presented in Appendix E.

The final result of numerical evaluation of the integral (2.111) that defines $I(\nu)$, using numerical evaluation of the functions $\Sigma_m(\rho)$ as described above, is

$$
I(1)=2.8380\ldots. \tag{2.120}
$$

III. MATCHING THEORY 1 TO THEORY 2

Our next task is to determine the operator $\delta\hat{C}$ of theory 2, appearing in Eq. (1.2a). Specifically, we want $\gamma_1=\langle 1|\delta\hat{C}|1\rangle$ to leading order in g (and all orders in logs). We will follow the general matching strategy used previously. We will temporarily introduce an infrared cutoff, then compute the total effective collision operator $\delta C_{\rm tot}$ in both theories, formally expanded to leading order in perturbation theory, and then determine what bare collision operator $\delta\hat{C}$ appearing in Eq. (1.2a) of theory 2 is required for the results to match. We will again use dimensional regularization to regulate the infrared.

A. $\delta C_{\rm tot}$ in the underlying theory (theory 1)

There are now a variety of methods for computing the effective collision operator at leading order in the underlying short-distance theory [3,15,4,16,17].[20] Previous authors have only extracted the leading log piece of their result because the leading-order result is formally log divergent in the infrared. Having regulated the infrared, we shall instead extract the entire thing.[21] So, one may now follow, in d spatial dimensions, one's favorite method of the references just cited. The method we are most intimately familiar with is our own, so our discussion will most closely parallel the presentation in Ref. [4].

At leading order in g, δC is generated by $2\leftrightarrow 2$ collisions of hard particles, mediated by semi-hard ($q_0\lesssim q\lesssim m$)

[17]This easily follows from iterating the formula for the inverse of an $(N+1)\times(N+1)$ matrix in terms of the inverse of its lower-right $N\times N$ block:

$$
\begin{pmatrix} a & b^T \\ c & d\end{pmatrix}^{-1}=\begin{pmatrix}\alpha & -\alpha\, b^Td^{-1}\\ -\alpha d^{-1}c & (1+\alpha d^{-1}cb^T)\,d^{-1}\end{pmatrix},
$$

with $\alpha\equiv(a-b^Td^{-1}c)^{-1}$. Here, a is a scalar, b and c are N-component (column) vectors, and d is an $N\times N$ matrix.

[18]From Eqs. (2.117) and (2.119), one may read off that $b_l^{(m)}\to\frac{1}{2}$ and $c_l\to 1$ as $l\to\infty$. Consequently, the tail of the continued fraction, $X\equiv\lim_{k\to\infty}c_{m+k}+(b_{m+k}^{(m)}\rho)^2/(\cdots)$, satisfies $X=1+(\rho/2)^2X^{-1}$, implying that $X=\frac{1}{2}+\frac{1}{2}\sqrt{1+\rho^2}$. The continued fraction (2.116) may be evaluated quite accurately by replacing its tail at large but finite l by this value.

[19]We evaluated the integral using Eq. 2.17.2 of Ref. [14], and verified the result numerically.

[20]Another interesting analytic approach that gives γ_1 at leading log order is that of Secs. 2 and 3 of Ref. [18]. It is not clear to us how to extend this approach beyond leading log order, and in particular how to obtain the terms of $\delta\hat{C}(\mathbf{v},\mathbf{v}')$ that are not proportional to $\delta(\mathbf{v}-\mathbf{v}')$.

[21]Reference [15] also discusses doing this but does not go so far as to extract an explicit result.

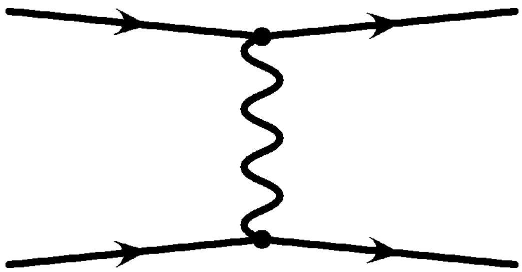

FIG. 6. The dominant scattering process: t-channel gauge boson exchange. The solid lines represent any sort of hard particles, including the non-Abelian gauge bosons themselves.

t-channel gluon exchange, such as depicted in Fig. 6. One finds[22]

$$\delta\hat{C}\, W(\mathbf{v}) = \frac{C_{\rm A} m^2 T}{2g^2\mu^\epsilon}\left\langle \int_{\mathbf{q}} |\mathcal{M}(\mathbf{v},\mathbf{v}',\mathbf{q})|^2 [W(\mathbf{v}) - W(\mathbf{v}')] \right\rangle_{\mathbf{v}'}, \tag{3.1}$$

where $\mathcal{M}$ is the amplitude for a t-channel collision between hard particles with velocities $\mathbf{v}$ and $\mathbf{v}'$, mediated by a semi-hard gauge boson with momentum $\mathbf{q}$. (This interpretation of $\mathcal{M}$ reverts to a more fundamental picture than that of theory 1, interpreting W as made up of individual hard particles. A derivation that is more directly in the framework of theory 1 may be found in Ref. [3].)

If the integrand of Eq. (3.1) is separated into two pieces, the coefficient of the first term, proportional to $W(\mathbf{v})$, is the same expression one obtains in a leading order calculation of the hard gauge boson damping rate [19],

$$\gamma = \frac{C_{\rm A} m^2 T}{2g^2\mu^\epsilon}\left\langle \int_{\mathbf{q}} |\mathcal{M}(\mathbf{v},\mathbf{v}',\mathbf{q})|^2 \right\rangle_{\mathbf{v}'}. \tag{3.2}$$

The overall coefficient in front of $\int_{\mathbf{q}}|\mathcal{M}|^2$ in Eq. (3.2) simply represents the results of group factors and the integration of the magnitude $|\mathbf{p}'|$ in the calculation of γ based on Fig. 6. The dependence on exactly what species of hard particles are present is completely isolated in the value of the Debye screening mass. The formula (3.1) for $\delta\hat{C}\, W(\mathbf{v})$ may equivalently be converted to a formula for the kernel $\delta C(\mathbf{v},\mathbf{v}')$ itself:

$$\delta C(\mathbf{v},\mathbf{v}') = \frac{C_{\rm A} m^2 T}{2g^2\mu^\epsilon}\left[\left\langle \int_{\mathbf{q}} |\mathcal{M}(\mathbf{v},\mathbf{v}',\mathbf{q})|^2 \right\rangle_{\mathbf{v}'} \delta^{S_2}(\mathbf{v}-\mathbf{v}') - \int_{\mathbf{q}} |\mathcal{M}(\mathbf{v},\mathbf{v}',\mathbf{q})|^2 \right]. \tag{3.3}$$

We have seen that, for the calculation of the NLLO conductivity, we will not need the full form of δC, but will only need the matrix element $\gamma_1 \equiv \langle 1|\delta\hat{C}|1\rangle = \langle v^i \delta\hat{C} v^i\rangle$. We will begin with a general analysis of δC, however, and specialize to γ_1 later.

If there were no screening effects to consider, the scattering amplitude $\mathcal{M}$ would be the classic Coulomb amplitude such that

$$\int_{\mathbf{q}} |\mathcal{M}|^2 = g^4 \mu^{2\epsilon} \int \frac{d^{4-\epsilon}Q}{(2\pi)^{4-\epsilon}} \left| v_\mu \frac{\delta^{\mu\nu}}{Q^2} v'_\nu \right|^2 2\pi\delta(Q\cdot v)\, 2\pi\delta(Q\cdot v') \quad \text{(no screening)}, \tag{3.4}$$

where $Q=(q^0,\mathbf{q})$ and $v=(1,\mathbf{v})$. The δ functions are simply $q^0, q \ll T$ approximations of the constraints that the final-state hard particles be on shell. To account for screening of the exchanged semi-hard gluon, however, we must replace this by

$$\int_{\mathbf{q}} |\mathcal{M}|^2 = g^4 \mu^{2\epsilon} \int \frac{d^{4-\epsilon}Q}{(2\pi)^{4-\epsilon}} \left| v_\mu \left(\frac{P_{\rm T}^{\mu\nu}(Q)}{Q^2+\Pi_{\rm T}(Q)} + \frac{P_{\rm L}^{\mu\nu}(Q)}{Q^2+\Pi_{\rm L}(Q)} \right) v'_\nu \right|^2 2\pi\delta(Q\cdot v)\, 2\pi\delta(Q\cdot v'), \tag{3.5}$$

where $P_{\rm T}$ and $P_{\rm L}$ are the transverse and longitudinal projection operators,

$$P_{\rm T}^{\mu\nu}(Q) = \begin{cases} 0, & \mu=0 \text{ or } \nu=0, \\ \delta^{ij} - \dfrac{q^i q^j}{q^2}, & \mu=i \text{ and } \nu=j, \end{cases} \tag{3.6}$$

$$P_{\rm L}^{\mu\nu}(Q) = \left(g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2} \right) - P_{\rm T}^{\mu\nu}, \tag{3.7}$$

and where $\Pi_{\rm T}$ and $\Pi_{\rm L}$ are the transverse and longitudinal pieces of the standard, leading-order, hard thermal loop self-energy [20]. This self-energy depends on Q only through the ratio $\lambda \equiv q^0/q$.

We will review the explicit formula for $\Pi(Q)$ below, but it's worth first examining some qualitative features. The longitudinal sector is screened for small Q by Debye/plasmon effects. The transverse sector, however, is unscreened in the $\lambda \to 0$ limit, reflecting the fact that charged plasmas do not screen static magnetic fields. This lack of static screening is responsible for the logarithmic infrared sensitivity that generates the usual leading-log result for the color conductivity. The logarithmic divergence appears only in the purely transverse term of the squared amplitude (3.5). It will be convenient to isolate that divergence by rewriting Eq. (3.5) as

[22]Here, as well as in Eqs. (1.1) and (1.2) defining either effective theory, m should be understood as the d-dimensional Debye mass.

$$\int_{\mathbf{q}} |\mathcal{M}|^2 = \int_{\mathbf{q}} (|\mathcal{M}|^2 - |\mathcal{M}|^2_{\rm IR}) + \int_{\mathbf{q}} |\mathcal{M}|^2_{\rm IR}, \tag{3.8}$$

where $|\mathcal{M}|^2_{\rm IR}$ is a small q^0 (small λ) limiting form of $|\mathcal{M}|^2$ that we shall discuss in a moment. Our strategy is to arrange that the integral of $|\mathcal{M}|^2_{\rm IR}$ be simple enough to evaluate in dimensional regularization, whereas the first term in Eq. (3.8) will be completely finite and evaluable directly in $d=3$ dimensions.

To continue, switch integration variables from q^0 to λ and, making use of the δ functions, rewrite Eq. (3.5) as

$$\int_{\mathbf{q}} |\mathcal{M}|^2 = 2\pi g^4 \mu^{2\epsilon} \int_{-1}^{+1} d\lambda \int_{\mathbf{q}} \frac{1}{q} \left| \frac{\mathbf{v}\cdot\mathbf{v}' - \lambda^2}{q^2(1-\lambda^2) + \Pi_{\rm T}(\lambda)} + \frac{-1+\lambda^2}{q^2(1-\lambda^2)+\Pi_{\rm L}(\lambda)} \right|^2 \delta(\lambda - \hat{\mathbf{q}}\cdot\mathbf{v})\,\delta(\lambda - \hat{\mathbf{q}}\cdot\mathbf{v}'). \tag{3.9}$$

The angular integration over $\hat{\mathbf{q}}$ is equivalent to replacing the pair of delta functions by their angular average. We will denote this average, in d dimensions, by

$$f_d(\lambda, \mathbf{v}\cdot\mathbf{v}') \equiv \langle \delta(\lambda - \hat{\mathbf{q}}\cdot\mathbf{v})\,\delta(\lambda - \hat{\mathbf{q}}\cdot\mathbf{v}')\rangle_{\hat{\mathbf{q}}}. \tag{3.10}$$

We will implement our split (3.8) by extracting the small λ behavior of Eq. (3.9),

$$\int_{\mathbf{q}} |\mathcal{M}|^2_{\rm IR} \equiv 2\pi g^4 \mu^{2\epsilon} \int_{-1}^{+1} d\lambda \int_{\mathbf{q}} \frac{1}{q} \left| \frac{\mathbf{v}\cdot\mathbf{v}'}{q^2 + \Pi_{\rm T}^{\rm IR}(\lambda)} \right|^2 \times \delta(\hat{\mathbf{q}}\cdot\mathbf{v})\,\delta(\hat{\mathbf{q}}\cdot\mathbf{v}'), \tag{3.11}$$

where $\Pi_{\rm T}^{\rm IR}$ is the limiting small λ behavior of $\Pi_{\rm T}$, to be discussed explicitly below.

1. IR piece

In terms of

$$f_d(0, \mathbf{v}\cdot\mathbf{v}') = \langle \delta(\hat{\mathbf{q}}\cdot\mathbf{v})\,\delta(\hat{\mathbf{q}}\cdot\mathbf{v}')\rangle_{\hat{\mathbf{q}}} = \frac{1-\epsilon}{2\pi\sqrt{1-(\mathbf{v}\cdot\mathbf{v}')^2}}, \tag{3.12}$$

we have

$$\int_{\mathbf{q}} |\mathcal{M}|^2_{\rm IR} = 2\pi (\mathbf{v}\cdot\mathbf{v}')^2 f_d(0, \mathbf{v}\cdot\mathbf{v}')\, g^4 \mu^{2\epsilon} \times \int_{-1}^{+1} d\lambda \int_{\mathbf{q}} \frac{1}{q|q^2 + \Pi_{\rm T}^{\rm IR}(\lambda)|^2}. \tag{3.13}$$

We now need the form of $\Pi_{\rm T}^{\rm IR}(\lambda)$. The usual $d=3$ result for the small frequency behavior of $\Pi_{\rm T}$ is $(-i\pi/4)\, m^2\lambda$. But we need the result in $3-\epsilon$ dimensions. One could derive this directly by evaluating a one-loop thermal diagram in the fundamental quantum field theory, but let us instead derive it in the usual way from effective theory 1. Working at leading order in g, formally solving the W equation (1.1a), and plugging into the Maxwell equation (1.1b) gives

$$\partial_\nu F^{\mu\nu} = m^2 \left\langle \frac{v^\mu v^i}{v\cdot\partial + \varepsilon} \right\rangle_{\mathbf{v}} E^i, \tag{3.14}$$

where the ε is simply an infinitesimal prescription specifying retarded behavior. Comparing with $\partial_\nu F^{\mu\nu} + \Pi^{\mu\nu} A_\nu = 0$, the components of the self-energy can then be read off, among which

$$\Pi^{ij}(q) = -m^2 q^0 \left\langle \frac{v^i v^j}{-q^0 + \mathbf{v}\cdot\mathbf{q} - i\varepsilon} \right\rangle_{\mathbf{v}}. \tag{3.15}$$

We want the transverse self-energy

$$\Pi_{\rm T} = \frac{1}{d-1}\,{\rm tr}\,(P_{\rm T}\Pi). \tag{3.16}$$

One can evaluate this for arbitrary frequency,[23] but here we are only interested in the small frequency limit $\Pi_{\rm T}^{\rm IR}$. Taking q^0 small, Eqs. (3.15) and (3.16) yield

$$\begin{aligned}
\Pi_{\rm T}^{\rm IR} &= -\frac{m^2 q^0}{d-1} \left\langle \frac{1-(\mathbf{v}\cdot\hat{\mathbf{q}})^2}{\mathbf{v}\cdot\mathbf{q} - i\varepsilon} \right\rangle_{\mathbf{v}} \\
&= -\frac{m^2 q^0}{d-1} \langle i\pi\,\delta(\mathbf{v}\cdot\mathbf{q})\rangle_{\mathbf{v}} \\
&= -\frac{i\pi S_{d-2}}{(d-1)\,S_{d-1}}\, m^2\lambda \\
&= -\frac{i\pi}{4}\,\kappa\, m^2\lambda,
\end{aligned} \tag{3.17}$$

where the area S_{d-1} of a $(d-1)$-sphere is given by Eq. (2.91), and where

$$\kappa \equiv \frac{2}{\pi} B\left(\frac{1}{2}, \frac{d}{2}\right) = 1 + [\ln 2 - \tfrac{1}{2}]\epsilon + O(\epsilon^2). \tag{3.18}$$

[23]The result is

$$\Pi_{\rm T} = \frac{m^2}{d-1}\left[\lambda^2 + (1-\lambda^2)\,{}_2F_1\left(\frac{1}{2}, 1; \frac{d}{2}; \lambda^{-2}\right)\right]$$

and

$$\Pi_{\rm L} = m^2(1-\lambda^2)\left[1 - {}_2F_1\left(\frac{1}{2}, 1; \frac{d}{2}; \lambda^{-2}\right)\right].$$

We can now do the momentum integral in Eq. (3.13) using

$$\int_{\mathbf{q}} \frac{1}{q\,|q^2+ic|^2} = \int_{\mathbf{q}} \frac{1}{q\,(q^4+c^2)} = \frac{S_{d-1}}{(2\pi)^d}\,\frac{\pi/4}{\cos(\epsilon\pi/4)}\,|c|^{(d-5)/2}, \tag{3.19}$$

to give

$$\begin{aligned}\int_{\mathbf{q}} |\mathcal{M}|^2_{\rm IR} &= \frac{S_{d-1}}{(2\pi)^{d-1}}\,\frac{\pi/4}{\cos(\epsilon\pi/4)}\,(\mathbf{v}\cdot\mathbf{v}')^2 f_d(0,\mathbf{v}\cdot\mathbf{v}')\, g^4\mu^{2\epsilon}\int_{-1}^{+1} d\lambda \left|\frac{\pi}{4}\kappa m^2\lambda\right|^{-1-(\epsilon/2)}\\ &= -\frac{\pi S_{d-1}}{\epsilon(2\pi)^{d-1}\cos(\epsilon\pi/4)}\,(\mathbf{v}\cdot\mathbf{v}')^2 f_d(0,\mathbf{v}\cdot\mathbf{v}')\, g^4\mu^{2\epsilon}\left(\frac{\pi}{4}\kappa m^2\right)^{-1-(\epsilon/2)}\\ &= \frac{2g^4\mu^\epsilon(\mathbf{v}\cdot\mathbf{v}')^2}{\pi^2 m^2\sqrt{1-(\mathbf{v}\cdot\mathbf{v}')^2}}\left[-\frac{1}{\epsilon}+\ln\left(\frac{m}{\mu}\right)+\tfrac{1}{2}\gamma_{\rm E}-\frac{1}{2}\right].\end{aligned} \tag{3.20}$$

2. Non-infrared piece

Now turn to the remaining piece, $\int_{\mathbf{q}}(|\mathcal{M}|^2-|\mathcal{M}|^2_{\rm IR})$, which may be evaluated directly in three dimensions. From Eqs. (3.9) and (3.13), we have

$$\begin{aligned}\int_{\mathbf{q}}(|\mathcal{M}|^2-|\mathcal{M}|^2_{\rm IR}) = 2\pi g^4\int_{-1}^{+1} d\lambda\int_{\mathbf{q}}\frac{1}{q}\Bigg\{&\left|\frac{\mathbf{v}\cdot\mathbf{v}'-\lambda^2}{q^2(1-\lambda^2)+\Pi_{\rm T}(\lambda)}+\frac{-1+\lambda^2}{q^2(1-\lambda^2)+\Pi_{\rm L}(\lambda)}\right|^2 f_3(\lambda,\mathbf{v}\cdot\mathbf{v}')\\ &-\frac{(\mathbf{v}\cdot\mathbf{v}')^2}{|q^2+\Pi_{\rm T}^{\rm IR}(\lambda)|^2} f_3(0,\mathbf{v}\cdot\mathbf{v}')\Bigg\}.\end{aligned} \tag{3.21}$$

The explicit form of f_3 is

$$f_3(\lambda,\mathbf{v}\cdot\mathbf{v}') = \frac{1}{2\pi}[1-(\mathbf{v}\cdot\mathbf{v}')^2-2\lambda^2(1-\mathbf{v}\cdot\mathbf{v}')]^{-1/2}. \tag{3.22}$$

(This is really multiplied by a step function which vanishes when the argument of the square root goes negative.) The basic 3-dimensional $\mathbf{q}$ integral required is

$$\int_{\mathbf{q}}\frac{1}{q\,(q^2+z_1)(q^2+z_2)} = \frac{\ln z_2-\ln z_1}{4\pi^2(z_2-z_1)}, \tag{3.23}$$

where the cut of the logarithm is understood to run along the negative real axis. It is convenient to rewrite $\Pi_{\rm L}$ and $\Pi_{\rm T}$ as

$$\Pi_{\rm L,T}(\lambda) \equiv m^2(1-\lambda^2)\,\rho_{\rm L,T}(\lambda). \tag{3.24}$$

We then obtain

$$\begin{aligned}\int_{\mathbf{q}}(|\mathcal{M}|^2-|\mathcal{M}|^2_{\rm IR}) = \frac{g^4}{2\pi m^2}\int_{-1}^{+1} d\lambda\Bigg\{ f_3(\lambda,\mathbf{v}\cdot\mathbf{v}')\Bigg[&\left(\frac{\mathbf{v}\cdot\mathbf{v}'-\lambda^2}{1-\lambda^2}\right)^2\frac{\arg\rho_{\rm T}(\lambda)}{\operatorname{Im}\rho_{\rm T}(\lambda)}+\frac{\arg\rho_{\rm L}(\lambda)}{\operatorname{Im}\rho_{\rm L}(\lambda)}\\ &-2\left(\frac{\mathbf{v}\cdot\mathbf{v}'-\lambda^2}{1-\lambda^2}\right)\operatorname{Re}\frac{\ln\rho_{\rm T}^*(\lambda)-\ln\rho_{\rm L}(\lambda)}{\rho_{\rm T}^*(\lambda)-\rho_{\rm L}(\lambda)}\Bigg]-\frac{2}{|\lambda|}f_3(0,\mathbf{v}\cdot\mathbf{v}')(\mathbf{v}\cdot\mathbf{v}')^2\Bigg\},\end{aligned} \tag{3.25}$$

where $\arg z\equiv\operatorname{Im}(\ln z)$ is to be understood to lie in the range $[-\pi,\pi]$.

Finally, we need explicit formulas for $\rho_{\rm T}$ and $\rho_{\rm L}$ in three dimensions. One can look up the formulas for Π [20] or easily derive them from Eq. (3.15). In the case at hand, we are interested in space-like momenta Q, and the results are

$$\rho_{\rm L}(\lambda)=1-\frac{\lambda}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right)+\frac{i\pi}{2}\lambda, \tag{3.26a}$$

$$\rho_{\rm T}(\lambda)=\frac{1}{2}\left[\frac{1}{1-\lambda^2}-\rho_{\rm L}(\lambda)\right]. \tag{3.26b}$$

By combining Eqs. (3.3), (3.8), (3.20), (3.22), (3.25), and (3.26), we now have a complete, if somewhat cumbersome and inelegant, integral formula for $\delta C(\mathbf{v},\mathbf{v}')$ at leading order in g. Because of the remaining λ integration, the functional dependence of δC on $\mathbf{v}\cdot\mathbf{v}'$ is not simple. Fortunately, we do not need the complete form of $\delta C(\mathbf{v},\mathbf{v}')$ to calculate the NLLO conductivity, and we will now specialize to the calculation of the matrix element γ_1.

3. Calculation of γ_1 and γ

Using the general formulas (3.1) or (3.3) for δC, we have

$$\gamma_1=\langle v^i\,\delta C v^i\rangle=\frac{C_{\rm A}m^2T}{2g^2\mu^{\epsilon}}\left\langle\int_{\mathbf{q}}|\mathcal{M}(\mathbf{v},\mathbf{v}',\mathbf{q})|^2\,(1-\mathbf{v}\cdot\mathbf{v}')\right\rangle_{\mathbf{v}\mathbf{v}'}. \tag{3.27}$$

As it turns out, we can easily calculate γ, defined by Eq. (3.2), at the same time as γ_1. So we will, even though we do not actually need the NLLO value of γ for our calculation of the NLLO conductivity. To this end, we define

$$\gamma^{(\eta)}=\frac{C_{\rm A}m^2T}{2g^2\mu^{\epsilon}}\left\langle\int_{\mathbf{q}}|\mathcal{M}(\mathbf{v},\mathbf{v}',\mathbf{q})|^2\,(1-\eta\,\mathbf{v}\cdot\mathbf{v}')\right\rangle_{\mathbf{v}\mathbf{v}'}, \tag{3.28}$$

where $\eta=1$ yields γ_1 and $\eta=0$ yields γ. We now apply this to the pieces (3.20) and (3.25) of $\int_{\mathbf{q}}|\mathcal{M}|^2$ using the three dimensional identities

$$\begin{aligned}\langle f_3(\lambda,\mathbf{v}\cdot\mathbf{v}')\,(1-\eta\mathbf{v}\cdot\mathbf{v}')\rangle_{\mathbf{v}\mathbf{v}'}&=\langle\delta(\lambda-\hat{\mathbf{q}}\cdot\mathbf{v})\,\delta(\lambda-\hat{\mathbf{q}}\cdot\mathbf{v}')(1-\eta\mathbf{v}\cdot\mathbf{v}')\rangle_{\hat{\mathbf{q}}\,\mathbf{v}\mathbf{v}'}\\&=\tfrac{1}{4}(1-\eta\lambda^2)\end{aligned} \tag{3.29}$$

and, similarly,

$$\langle(\mathbf{v}\cdot\mathbf{v}'-\lambda^2)\,f_3(\lambda,\mathbf{v}\cdot\mathbf{v}')\,(1-\eta\mathbf{v}\cdot\mathbf{v}')\rangle_{\mathbf{v}\mathbf{v}'}=-\tfrac{1}{8}\,\eta\,(1-\lambda^2)^2, \tag{3.30}$$

$$\begin{aligned}\langle(\mathbf{v}\cdot\mathbf{v}'-\lambda^2)^2 f_3(\lambda,\mathbf{v}\cdot\mathbf{v}')\,(1-\eta\mathbf{v}\cdot\mathbf{v}')\rangle_{\mathbf{v}\mathbf{v}'}\\=\tfrac{1}{8}(1-\lambda^2)^2\,(1-\eta\lambda^2),\end{aligned} \tag{3.31}$$

plus the $d=3-\epsilon$ identity

$$\begin{aligned}&\langle(\mathbf{v}\cdot\mathbf{v}')^2 f_d(0,\mathbf{v}\cdot\mathbf{v}')\,(1-\eta\mathbf{v}\cdot\mathbf{v}')\rangle_{\mathbf{v}\mathbf{v}'}\\&\quad=\langle(\mathbf{v}\cdot\mathbf{v}')^2\,\delta(\hat{\mathbf{q}}\cdot\mathbf{v})\,\delta(\hat{\mathbf{q}}\cdot\mathbf{v}')\rangle_{\hat{\mathbf{q}}\,\mathbf{v}\mathbf{v}'}=\frac{1}{d-1}\left(\frac{S_{d-2}}{S_{d-1}}\right)^2.\end{aligned} \tag{3.32}$$

Inserting Eq. (3.20) into Eq. (3.28) then gives

$$\begin{aligned}\gamma_{\rm IR}^{(\eta)}&=-C_{\rm A}\alpha T\,\frac{2S_{d-2}^2}{(d-1)\,\pi\,S_{d-1}\cos(\epsilon\pi/4)}\,\frac{1}{\kappa\epsilon}\left(\frac{m}{4\mu}\right)^{-\epsilon}\left(\frac{\pi}{\kappa}\right)^{\epsilon/2}\\&=C_{\rm A}\alpha T\left[-\frac{1}{\epsilon}+\ln\left(\frac{m}{\mu}\right)+\tfrac{1}{2}\gamma_{\rm E}-2\ln 2+O(\epsilon)\right].\end{aligned} \tag{3.33}$$

And inserting the non-infrared piece (3.25) into Eq. (3.28) now produces

$$\gamma^{(\eta)}-\gamma_{\rm IR}^{(\eta)}=C_{\rm A}\,\alpha T\,a^{(\eta)}, \tag{3.34}$$

where the numerical constant $a^{(\eta)}$ is given by the one dimensional integral

$$\begin{aligned}a^{(\eta)}\equiv\frac{1}{4}\int_{-1}^{+1}d\lambda\Bigg\{&(1-\eta\lambda^2)\left[\frac{1}{2}\,\frac{\arg\rho_{\rm T}(\lambda)}{{\rm Im}\,\rho_{\rm T}(\lambda)}+\frac{\arg\rho_{\rm L}(\lambda)}{{\rm Im}\,\rho_{\rm L}(\lambda)}\right]\\&+\eta\,(1-\lambda^2)\,{\rm Re}\left[\frac{\ln\rho_{\rm T}^*(\lambda)-\ln\rho_{\rm L}(\lambda)}{\rho_{\rm T}^*(\lambda)-\rho_{\rm L}(\lambda)}\right]-\frac{1}{|\lambda|}\Bigg\}.\end{aligned} \tag{3.35}$$

It is useful for numerical evaluation to split $\rho_{\rm T}$ and $\rho_{\rm L}$ into their real and imaginary parts, $\rho_{\rm T}=R_{\rm T}+iI_{\rm T}$, etc., and use

$$\frac{\arg\rho}{{\rm Im}\,\rho}=\frac{1}{|I|}\cot^{-1}\left(\frac{R}{|I|}\right) \tag{3.36}$$

and

$${\rm Re}\,\frac{\ln\rho_{\rm T}^*(\lambda)-\ln\rho_{\rm L}(\lambda)}{\rho_{\rm T}^*(\lambda)-\rho_{\rm L}(\lambda)}=\frac{1}{2}\,\frac{R_{\rm T}-R_{\rm L}}{[(R_{\rm T}-R_{\rm L})^2+(I_{\rm T}+I_{\rm L})^2]}\ln\left(\frac{R_{\rm T}^2+I_{\rm T}^2}{R_{\rm L}^2+I_{\rm L}^2}\right)+\frac{|I_{\rm T}+I_{\rm L}|}{[(R_{\rm T}-R_{\rm L})^2+(I_{\rm T}+I_{\rm L})^2]}\left[\cot^{-1}\left(\frac{R_{\rm L}}{|I_{\rm L}|}\right)-\cot^{-1}\left(\frac{R_{\rm T}}{|I_{\rm T}|}\right)\right], \tag{3.37}$$

where $\cot^{-1}x\equiv\pi/2-{\rm Tan}^{-1}x$ is defined to lie in the range $[0,\pi]$, and we have made use of the fact that the signs of $I_{\rm L}$ and $I_{\rm L}+I_{\rm T}$ are the same and are both opposite to $I_{\rm T}$. One may also note that the integrand of Eq. (3.35) is even in λ.

Putting everything together, we find

$$\gamma_1=C_{\rm A}\,\alpha T\left[-\frac{1}{\epsilon}+\ln\left(\frac{m}{\mu}\right)+\tfrac{1}{2}\ln\frac{\pi}{4}+a_1\right] \tag{3.38}$$

and

$$\gamma = C_A \alpha T\left[-\frac{1}{\epsilon} + \ln\left(\frac{m}{\bar{\mu}}\right) + \tfrac{1}{2}\ln\frac{\pi}{4} + a\right], \tag{3.39}$$

where we have now written the result in terms of the $\overline{\rm MS}$ scale $\bar{\mu} = \sqrt{4\pi}e^{-\gamma_E/2}\mu$. Numerical evaluation of Eq. (3.35) for $\eta=1$ and $\eta=0$ gives

$$a_1 = 0.323833\ldots \tag{3.40}$$

and

$$a = 0.120782\ldots\quad, \tag{3.41}$$

for $\eta=1$ and $\eta=0$, respectively. In fact, our numerical evaluation of the constant a shows that it precisely equals $-\frac{1}{2}\ln(\pi/4)$ to 12 significant digits. Surely this is an exact identity,[24] so that the dimensionally regulated hard gauge boson damping rate, to next-to-leading-log order, is simply

$$\gamma = C_A \alpha T\left[-\frac{1}{\epsilon} + \ln\left(\frac{m}{\bar{\mu}}\right)\right]. \tag{3.42}$$

B. Matching to theory 2

In theory 2, the total effective collision term is, in principal, composed of two parts. First, there is the bare collision term that appears in Eq. (1.2a), which we will call $\delta\hat{C}_{\rm bare}$ here to be explicit, and which conceptually represents collisions due to the exchange of virtual gauge bosons that were integrated out in going from theory 1 to theory 2. Second, there is a dynamical contribution to $\delta\hat{C}$, which we will call $\delta\hat{C}_{\rm dyn}$, which arises from the exchange of those gauge bosons that have *not* yet been integrated out. However, as we explain below, the same nice property of dimensional regularization which simplified the theory 3 matching calculation in Sec. II C works here as well: for the purposes of matching, $\delta\hat{C}_{\rm dyn}$ must vanish in dimensional regularization by dimensional analysis. Hence, we have simply $\delta\hat{C}_{\rm tot} = \delta\hat{C}_{\rm bare}$, and so the γ_1 we needed in order to match theory 2 to theory 3 can simply be taken directly from the theory 1 result (3.38) for $\langle 1|\delta\hat{C}_{\rm tot}|1\rangle$.

The dimensional argument can be made by rescaling to variables $\bar{t} = m^{-2}t$, $\bar{\mathbf{A}} = T^{-1/2}\mathbf{A}$, $\bar{A}_0 = m^2 T^{-1/2}A_0$, $\bar{W} = m^2 T^{-1/2}W$, and $\bar{\xi} = m^2 T^{-1/2}\xi$, so that Eqs. (1.2) for theory 2 become

$$\mathbf{v}\cdot\overline{\mathbf{D}W} - \mathbf{v}\cdot\bar{\mathbf{E}} = -\delta\hat{C}_{\rm bare}\bar{W} + \bar{\xi}, \tag{3.43a}$$

$$\langle\bar{W}\rangle = 0, \tag{3.43b}$$

$$\bar{\mathbf{D}}\times\bar{\mathbf{B}} = \langle\mathbf{v}\bar{W}\rangle, \tag{3.43c}$$

$$\langle\langle\bar{\xi}(\mathbf{x},\mathbf{v},t)\,\bar{\xi}(\mathbf{x}',\mathbf{v}',t')\rangle\rangle = 2\,\delta\hat{C}_{\rm bare}(\mathbf{v},\mathbf{v}')\,\delta(\mathbf{x}-\mathbf{x}')\,\delta(t-t'), \tag{3.43d}$$

where $\bar{D}_\nu = \bar{\partial}_\nu + g\mu^{\epsilon/2}T^{1/2}\bar{A}_\nu$. There remains a dimensional quantity, $\delta\hat{C}$, other than the effective coupling $g\mu^{\epsilon/2}T^{1/2}$. But in matching theories 1 and 2, it is important that $\delta\hat{C}_{\rm bare}$ is to be formally treated as a perturbation. This is feasible because the matching is performed at the spatial momentum scale $m=O(gT)$, whereas $\delta\hat{C}_{\rm bare}$ is of the much more infrared scale $\gamma=O(g^2T\ln)$. As discussed earlier, matching can always be thought of as taking place in a large box, serving as an infrared regulator. The box should be chosen to be large compared to the distance scale of the matching $(1/m)$, but may be small compared to more infrared scales where the physics becomes more complicated (e.g., $1/\gamma$). In the presence of such an IR cutoff, one may then treat quantities that are soft relative to the matching scale (such as $\delta\hat{C}_{\rm bare}$, in the case at hand) as perturbations. Moreover, formally treating them as perturbations works for the purposes of matching calculations even if the infrared regulator is then taken to arbitrarily large distance scales [12,13].

In summary, then, perturbation theory in the effective coupling $g\mu^{\epsilon/2}T^{1/2}$ and in $\delta\hat{C}$ can only give terms with the dimensions of $(g\mu^{\epsilon/2}T^{1/2})^m(\delta\hat{C})^n$ for integer m and n. Because of the factor of μ^ϵ, none of these can consistently match the mass dimension 1 of $\delta\hat{C}_{\rm tot}$ unless $m=0$. That means the only contribution to $\delta\hat{C}_{\rm tot}$ is the tree-level $\delta\hat{C}_{\rm bare}$.

Though the dynamical contributions to $\delta\hat{C}$ formally vanish in theory 2 in dimensional regularization, they do play an important conceptual role. We are now interpreting Eq. (3.38) as the result for the bare γ_1 in theory 2. Theory 2 requires ultraviolet regularization: the $1/\epsilon$ in Eq. (3.38) is the counter-term for a UV divergence in theory 2, and μ is the associated renormalization scale. In our matching calculation, however, the $1/\epsilon$ and the $\ln(m/\mu)$ actually arose in Eq. (3.20) from a formal *infrared* divergence of the calculation of the total γ_1 in theory 1. The discrepancy of interpretation is resolved by realizing that the theory 2 result that $\gamma_{1,\rm tot} = \gamma_{1,\rm bare}$ should really be thought of as

$$[\gamma_{1,\rm tot}]_{\rm (theory\ 2)} = \gamma_{1,\rm bare} + C_A\alpha T\left\{\left[-\frac{1}{\epsilon} + \ln\left(\frac{m}{\mu}\right)\right]_{\rm IR} + \left[\frac{1}{\epsilon} - \ln\left(\frac{m}{\mu}\right)\right]_{\rm UV}\right\}, \tag{3.44}$$

and so equating the total γ_1 in the two theories converts the the IR divergence into a UV divergence in Eq. (3.38). Equation (3.44) is an example of the generic behavior in dimensional regularization of logarithmic divergences when there

[24] We have not bothered to try proving this analytically, although we are confident one could do so. Blaizot and Iancu [5] have shown the corresponding result when using a sharp momentum IR cutoff, instead of dimensional regularization. (See also Appendix B of [21].)

is no scale to cut them off in either the IR or UV, and is typified by the simple example

$$\int \frac{d^{3-\epsilon}p}{p^3} \propto \mu^\epsilon \int_0^\infty \frac{dp}{p^{1+\epsilon}}$$

$$= \int_0^\Lambda \frac{dp}{p^{1+\epsilon}} + \int_\Lambda^\infty \frac{dp}{p^{1+\epsilon}}$$

$$= \left[-\frac{1}{\epsilon} + \ln\frac{\Lambda}{\mu} \right] + \left[\frac{1}{\epsilon} - \ln\frac{\Lambda}{\mu} \right], \qquad (3.45)$$

where in the last equality dimensional regularization was used both for the IR contribution of the first term and the UV contribution of the second.

IV. FINAL RESULTS

We now put together our NLLO result (2.113) for the color conductivity σ in terms of γ_1 and our result (3.38) for γ_1. The structure is clearest if we write an expansion for σ^{-1} (the ''color resistivity'') rather than σ directly. One finds

$$\sigma^{-1} = \frac{3C_{\rm A}\alpha T}{m^2} \left[\ln\left(\frac{m}{\gamma(\mu)} \right) + C + O(\ln^{-1}) \right], \qquad (4.1a)$$

with

$$C \equiv \tfrac{1}{2} \ln\frac{\pi}{4} + a_1 + I(1) = 3.0410\ldots, \qquad (4.1b)$$

and where the numerical constants $I(1)$ and a_1 are defined by Eqs. (2.111) and (3.35) (with $\eta=1$) and given numerically by Eqs. (2.120) and (3.40). Note that the $1/\epsilon$ divergences have canceled, as they must. Inside the logarithm of Eq. (4.1a), $\gamma(\mu)$ is to be understood as simply the leading-log formula

$$\gamma(\mu) \approx C_{\rm A}\alpha T \ln\left(\frac{m}{\mu} \right), \qquad (4.2)$$

and μ should be chosen so that it is of order γ. One may easily verify that the μ dependence in the NLLO result (4.1a) only affects that answer at order $[\ln(m/\gamma)]^{-1} \sim [\ln(1/g)]^{-1}$, which is beyond the order of this calculation.

Equation (4.1) is our final result. Although the result must be gauge independent, our derivation has been restricted to a particular choice of gauge, namely the Coulomb gauge. It would be comforting to have the calculation repeated in another gauge, perhaps the generalized flow gauges discussed in Appendix A, but this we have not attempted to do. It would also be interesting if there were any way to express the fundamental functions $\Sigma_m(\rho)$ of this problem in terms of standard mathematical functions, but we have been unable to do so.

Our result for the NLLO color conductivity may be compared against numerical simulation of the electroweak baryon number violation rate, as such simulations can in fact be used to measure the relative size of the NLLO correction to σ. We discuss this comparison in Ref. [1].

Finally, we should mention the differences between our calculation of the NLLO conductivity and that outlined in earlier work by Blaizot and Iancu [5]. In our language, Blaizot and Iancu's discussion amounts to specifying how to calculate γ_1 (or, rather, the finite difference $\gamma - \gamma_1$ which they call δ).[25] That is, it is equivalent to our discussion of matching theory 1 and theory 2. They describe this as a calculation of the conductivity, defined simply as $m^2/(3\gamma_1)$. However, as the authors of [5] clearly acknowledge, this analysis omits all the contributions that in our calculation came from integrating out physics at $k \sim \gamma$ when matching theory 2 to theory 3. Moreover, the result $m^2/(3\gamma_1)$ is not a physical (i.e., renormalization scale independent) quantity at NLLO because of the infrared divergence in the calculation of γ. Proper inclusion of the effects treated in our theory 2 to theory 3 matching is essential to obtain an answer that is independent at NLLO to the precise choice of infrared cutoff (or renormalization scale) μ.

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APPENDIX A: WILSON LOOPS IN FLOW GAUGES

A useful class of gauges for stochastic gauge theory are the flow gauges defined by the condition $\sigma A_0 = -\lambda \boldsymbol{\nabla}\cdot\mathbf{A}$. This class of gauges interpolate between $A_0=0$ gauge ($\lambda \to 0$) and Coulomb gauge ($\lambda\to\infty$) [6].[26] In this appendix, we will illustrate the use of these gauges by explicitly checking the gauge-invariance of the derivation in Sec. II A of the the first-order result for large-time rectangular Wilson loops.

The gauge-fixed path integral, analogous to Eq. (1.10), is

$$Z_{\rm flow} = \int [\mathcal{D}A_0][\mathcal{D}\mathbf{A}][\mathcal{D}\bar\eta][\mathcal{D}\eta]\,\delta(\sigma A_0 - \lambda \boldsymbol{\nabla}\cdot\mathbf{A}) \times \exp\left(-\int dt d^3x L_{\rm flow} \right), \qquad (A1a)$$

with

$$L_{\rm flow} = \frac{1}{4\sigma T}\{ |-\sigma\mathbf{E} + \mathbf{D}\times\mathbf{B}|^2 + \bar\eta(\sigma D_0 + \lambda\boldsymbol{\nabla}\cdot\mathbf{D})\eta \}. \qquad (A1b)$$

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[25] The initial preprint version of Ref. [5] did not push through the calculation to obtain a numerical value for $\gamma_1 - \gamma$, but for the final published version these authors did include the result of a numerical evaluation, and obtained a number which agrees perfectly with our value of $(\gamma_1 - \gamma)/(C_{\rm A}\alpha T) = a_1 - a = 0.20305\ldots$.

[26] The flow gauges here correspond to the $\xi\to\infty$ limit of the generalized flow gauges considered in Ref. [6].

Rewriting A_0 in terms of $\nabla\cdot\mathbf{A}$ and working in momentum space, the perturbative expansion of the action gives

$$S_{\rm flow}=\int dt\int_{\mathbf{k}}\left[\frac{1}{4\sigma T}\left|(\sigma\partial_t-\nabla^2)\mathbf{A}+(1-\lambda)\nabla\nabla\cdot\mathbf{A}\right|^2+O(\mathbf{A}^3)+(\text{ghosts})\right]. \tag{A2}$$

The propagator is

$$A_i^a \sim\!\sim\!\sim\; A_j^b=2\sigma T\delta^{ab}\left[\frac{\delta_{ij}-\hat{k}_i\hat{k}_j}{|i\sigma\omega+k^2|^2}+\frac{\hat{k}_i\hat{k}_j}{|i\sigma\omega+\lambda k^2|^2}\right]. \tag{A3}$$

One can see that the gauge $\lambda=1$ is the stochastic gauge theory analog of Feynman gauge. This Feynman-like gauge was first proposed in this context by Zwanziger [22].

In any case, let us now turn to the Wilson loop diagrams of Fig. 5(c). We need the propagator for $A_0=(\lambda/\sigma)\nabla\cdot\mathbf{A}$ which, from Eq. (A3), is

$$A_0^a \sim\!\sim\!\sim\; A_0^b=\frac{2T}{\sigma}\,\delta^{ab}\,\frac{\lambda^2k^2}{|i\sigma\omega+\lambda k^2|^2}\,. \tag{A4}$$

Then, following Eq. (2.4),

$$\begin{aligned} d_{\rm R}^{-1}\,\delta\mathcal{W}&=-\frac{g^2\mathrm{tr}(T^aT^b)}{\mathrm{tr}(1)}\int_0^{t_\infty}dt\,dt'\langle\!\langle A_0^a(t,0)A_0^b(t',\mathbf{R})\rangle\!\rangle\\ &=-g^2C_{\rm A}t_\infty\int_{\mathbf{k}}\left[\frac{2T}{\sigma}\,\frac{\lambda^2k^2}{|i\sigma\omega+\lambda k^2|^2}\right]_{\omega=0}e^{i\mathbf{k}\cdot\mathbf{R}}\\ &=-\frac{\alpha C_{\rm A}T}{\sigma R}t_\infty\,, \end{aligned} \tag{A5}$$

in the large-time limit. As required for any physical quantity, all dependence on the gauge-fixing parameter λ has disappeared.

APPENDIX B: LARGE R BEHAVIOR OF WILSON LOOPS

Consider the first-order correction to the real-time Wilson loop expectation (2.11), repeated here for convenience,

$$d_{\rm R}^{-1}\,\delta\mathcal{W}=-2g^2T\,C_{\rm A}t_\infty\int_{\mathbf{k}}\frac{e^{i\mathbf{k}\cdot\mathbf{R}}}{k^2\,\overline{\sigma}_{\rm L}^{(0)}(k)}, \tag{B1}$$

in the limit of large R. Specifically R must be large compared to the color-changing mean free path $\gamma^{-1}=O[(g^2T\ln)^{-1}]$. Physically, one should simultaneously keep R sufficiently small that physics on the scale R is still perturbative, that is, $R\ll(g^2T)^{-1}$. Formally, when performing an IR regulated matching calculation, one may simply take $R\to\infty$. But here, we will examine what happens if one is not quite so cavalier with R. For the sake of definiteness, we will consider $R\sim\delta(g^2T)^{-1}$, where $1/\ln g^{-1}\ll\delta\ll1$ and the coupling g is arbitrarily weak.

Rewrite the Fourier transform in Eq. (B1) as

$$\mathcal{F}(R)\equiv\int_{\mathbf{k}}\frac{s(k)}{k^2}\,e^{i\mathbf{k}\cdot\mathrm{R}}, \tag{B2}$$

where $s(k)\equiv[\overline{\sigma}_{\rm L}^{(0)}(k)]^{-1}$, and perform the angular integral:

$$\mathcal{F}(R)=\frac{1}{2\pi^2R}\int_0^\infty\frac{dk}{k}\,s(k)\,\sin(kR). \tag{B3}$$

The analytic continuation of $s(k)$ is an even function of k, as can be verified explicitly from the formulas (2.42) and (2.116). So Eq. (B3) can be rewritten as

$$\begin{aligned}\mathcal{F}(R)&=\lim_{\varepsilon\to0}\frac{1}{4\pi^2iR}\int_\varepsilon^\infty dk\,\frac{s(k)}{k}\,[e^{ikR}-e^{-ikR}]\\ &=\frac{1}{4\pi^2iR}\int_{-\infty}^\infty dk\,e^{ikR}\,\mathrm{P.P.}\left(\frac{s(k)}{k}\right).\end{aligned} \tag{B4}$$

Here, P.P. denotes the principal part,

$$\mathrm{P.P.}\left(\frac{1}{k}\right)=\frac{1}{2}\left(\frac{1}{k-i\varepsilon}+\frac{1}{k+i\varepsilon}\right). \tag{B5}$$

Closing the contour of Eq. (B4) in the upper half plane then picks up the $k=i\varepsilon$ pole, as well as any contributions from singularities in the function $s(k)$. The $k=i\varepsilon$ pole gives the expected result (2.12). Singularities in $s(k)=[\overline{\sigma}_{\rm L}^{(0)}(k)]^{-1}$ at some k in the upper half plane will give additional contributions suppressed by $\exp[-R\,\mathrm{Im}(k)]$. Here $\overline{\sigma}_{\rm L}^{(0)}(k)$ is analytic and non-vanishing in a neighborhood of the real axis. The only scale associated with the k dependence of $\overline{\sigma}_{\rm L}^{(0)}(k)$ is the scale γ appearing in $\delta\hat{C}$. So the only scale that can determine the imaginary parts of the singularities of s is γ. Therefore, the contribution to Eq. (B4) from singularities of s must be suppressed by $\exp[-O(\gamma R)]$. For $R\sim\delta(g^2T)^{-1}$ this is $\exp[-\delta O(\ln g^{-1})]$ which, for weak coupling, behaves as some (positive) power g and vanishes faster than any power of $1/\ln g^{-1}$.

APPENDIX C: RELATIONS BETWEEN VARIOUS CORRELATORS

Consider, for simplicity, an effective theory for a single real scalar field, governed by the classical Langevin equation $\sigma\dot{\phi}=(\nabla^2-m^2)\,\phi-dV_{\rm int}/d\phi+\zeta$ and the noise covariance $\langle\!\langle\zeta\zeta\rangle\!\rangle=2\sigma T$. Also for simplicity, take σ to be constant. The corresponding action in a path integral formulation is then[27]

$$S = \frac{1}{4\sigma T}\int dt\, d^3x \left[\sigma\dot{\phi} - (\nabla^2 - m^2)\,\phi - \frac{dV_{\rm int}}{d\phi}\right]^2, \quad \text{(C1)}$$

If $\delta V_{\rm int}/\delta\phi$ is treated as a perturbation, then the unperturbed propagator arising from this action is, in Fourier space,

$$\langle\!\langle \phi(\omega,\mathbf{k})\ \phi(\omega',\mathbf{k}')^* \rangle\!\rangle = -i\tilde{G}(\omega,\mathbf{k})(2\pi)^4\delta(\mathbf{k}-\mathbf{k}') \times \delta(\omega-\omega'). \quad \text{(C2)}$$

with

$$-i\tilde{G}(\omega,\mathbf{k}) = \frac{2\sigma T}{(\sigma\omega)^2 + (k^2+m^2)^2}. \quad \text{(C3)}$$

In the main text, the term ''propagator'' referred what we've called $-i\tilde{G}$ above, which in coordinate space is $\langle\!\langle \phi(t,\mathbf{x})\ \phi(0,0)\rangle\!\rangle$. In the underlying quantum field theory, however, it is customary to define the propagator G as i times the expectation of a product of fields, as indicated above. Note from Eq. (C3) that G is purely imaginary.

Because operators commute in a classical theory, there is no distinction between time-ordered and time-unordered propagators. So the above propagator can be regarded as the classical limit of the time-ordered propagator[28] $G(t,\mathbf{x}) = i\langle \mathcal{T}\{\phi(t,\mathbf{x})\ \phi^\dagger(0,0)\}\rangle$ or either of the Wightman functions $G_>(t,\mathbf{x}) = i\langle \phi(t,\mathbf{x})\ \phi^\dagger(0,0)\rangle$ or $G_<(t,\mathbf{x}) = i\langle \phi^\dagger(0,0)\ \phi(t,\mathbf{x})\rangle$ of the underlying quantum theory. That is,

$$G = G_> = G_< \quad \text{(classically)}. \quad \text{(C4)}$$

A classical effective theory, such as the Langevin theory (C1), is (at best) valid for low frequency, long wavelength dynamics. It is only appropriate for studying observables which are smeared over time and spatial scales large compared to $\beta\hbar$, and thus insensitive to short time (or short distance) quantum fluctuations.

In the underlying quantum theory, the spectral density $\rho(\omega,\mathbf{k})$ (defined as the Fourier transform of $\langle[\phi(t,\mathbf{x}),\phi^\dagger(0,0)]\rangle$) is related to the time-ordered propagator G by

$$\rho(\omega,\mathbf{k}) = 2\tanh(\tfrac{1}{2}\beta\hbar\omega)\ \mathrm{Im}\,\tilde{G}(\omega,\mathbf{k}). \quad \text{(C5)}$$

Although it is not essential to the discussion, we will keep track of factors of $\hbar$ in this appendix (and this appendix only). Taking the classical limit $\hbar\omega \ll T$, in which the real-time propagator G is purely imaginary, Eq. (C5) gives

$$i\rho(\omega,\mathbf{k}) = \beta\hbar\omega\tilde{G}(\omega,\mathbf{k}). \quad \text{(C6)}$$

The Fourier transform of the retarded propagator $G_{\rm R}(t,\mathbf{x}) = i\theta(t)\langle[\phi(t,\mathbf{x}),\phi^\dagger(0,0)]\rangle$ is related to ρ by the spectral representation

$$\tilde{G}_{\rm R}(\omega,\mathbf{k}) = \int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\ \frac{\rho(\omega',\mathbf{k})}{\omega'-\omega-i\varepsilon}, \quad \text{(C7)}$$

which, in the classical limit, becomes just

$$G_{\rm R}(t,\mathbf{k}) = i\,\theta(t)\ \beta\hbar\,\frac{\partial}{\partial t}G(t,\mathbf{k}). \quad \text{(C8)}$$

Similarly, the advanced propagator

$$G_A(t,\mathbf{x}) = -i\,\theta(-t)\langle[\phi(t,\mathbf{x}),\phi^\dagger(0,0)]\rangle$$

satisfies the same spectral representation (C7) except for changing $-i\varepsilon$ to $+i\varepsilon$, and in the classical limit is

$$G_{\rm A}(t,\mathbf{k}) = -i\,\theta(-t)\ \beta\hbar\,\frac{\partial}{\partial t}G(t,\mathbf{k}). \quad \text{(C9)}$$

Note that these propagators satisfy the standard relation $\tilde{G}_{\rm R} - \tilde{G}_{\rm A} = i\rho$.

In studying static equilibrium physics, one typically works in imaginary time τ rather than real time t, where Euclidean correlation functions such as $G_{\rm E}(\tau,\mathbf{x}) \equiv \langle \mathcal{T}\{\phi(-i\tau,\mathbf{x})\phi^\dagger(0,0)\}\rangle$ are periodic with period $\beta\hbar$. In the classical limit, which is the long-distance limit $\Delta x \gg \beta\hbar$, there is no substantive difference between equal-time correlations and zero-frequency correlations in imaginary time, due to the negligible extent $\beta\hbar$ of imaginary time. In this limit, the only discrepancy is a change in normalization of $\beta\hbar$ from the Fourier transform:

$$G_{\rm E}(\tau=0,\mathbf{k}) = (\beta\hbar)^{-1}\,\tilde{G}_{\rm E}(\nu=0,\mathbf{k}). \quad \text{(C10)}$$

The limit of zero time separation is the same in imaginary or real time, so one must have $G(t=0,\mathbf{k}) = iG_{\rm E}(\tau=0,\mathbf{k})$, since the real-time time-ordered propagator is related to the Euclidean propagator by appropriate analytic continuation in t. But the analytic continuation in *frequency* of $\tilde{G}_E$ from the Matsubara points $\nu_n = 2\pi i n/(\beta\hbar)$ back to real frequencies yields $\tilde{G}_{\rm R}$ or $\tilde{G}_{\rm A}$, depending on whether one continues to the real frequency axis from above or below, respectively. Hence, the zero frequency retarded, advanced, and Euclidean propagators coincide,

$$\tilde{G}_{\rm E}(\nu=0,\mathbf{k}) = \tilde{G}_{\rm R}(\omega=0,\mathbf{k}) = \tilde{G}_{\rm A}(\omega=0,\mathbf{k}).$$

Putting everything together gives the following string of equalities in the classical limit:

[27]Formally, the action should also contain the Jacobian term $-\frac{1}{2}\,\mathrm{tr}[-\nabla^2 + m^2 + V''_{\rm int}(\phi)]$. But this is proportional to $\delta^{(d)}(0)$ for local potentials $V_{\rm int}(\phi)$ and so vanishes in dimensional regularization. See, for example, Sec. 17.2 of Ref. [8].

[28]In our specific example, ϕ is real field, and so $\phi^\dagger(t,\mathbf{x})$ is no different from $\phi(t,\mathbf{x})$. In writing general quantum expressions for propagators and spectral densities and so forth, we nonetheless find it useful for the sake of reference to include daggers where they would appear for a discussion of complex fields.

$$G_{\rm E}(\tau=0,\mathbf{k})=(\beta\hbar)^{-1}\,\tilde{G}_{\rm E}(\nu=0,\mathbf{k})=(\beta\hbar)^{-1}\,\tilde{G}_{\rm R}(\omega=0,\mathbf{k})$$
$$=(\beta\hbar)^{-1}\,\tilde{G}_{\rm A}(\omega=0,\mathbf{k})=-iG(t=0,\mathbf{k})$$
$$=-i\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\tilde{G}(\omega,\mathbf{k}). \tag{C11}$$

Note that this also implies

$$G(t=0,\mathbf{k})=i(\beta\hbar)^{-1}\int_{-\infty}^{+\infty}dt\,G_{\rm R}(t,\mathbf{k})$$
$$=i(\beta\hbar)^{-1}\int_{-\infty}^{+\infty}dt\,G_{\rm A}(t,\mathbf{k}), \tag{C12}$$

which can be seen directly from Eqs. (C8) and (C9).

It is important to understand that the small frequency limit of the real-time correlator $\tilde{G}(\omega,\mathbf{k})$ is *not* directly related to the zero frequency limit of the Euclidean correlator $\tilde{G}_{\rm E}(\nu,\mathbf{k})$. As an example, for our real-time propagator (C3),

$$\tilde{G}(\omega=0,\mathbf{k})=\frac{2i\sigma T}{(k^2+m^2)^2}, \tag{C13}$$

while

$$G(t=0,\mathbf{k})=\int_{-\infty}^{+\infty}d\omega G(\omega,\mathbf{k})=\frac{iT}{k^2+m^2}. \tag{C14}$$

In this simple scalar model, the chain of relations (C11) implies that $G_{\rm E}(\tau=0,\mathbf{k})$ and $(\beta\hbar)^{-1}\,\tilde{G}_{\rm E}(\nu=0,\mathbf{k})$ likewise equal $T/(k^2+m^2)$. Some additional specific results for this model are

$$\rho(\omega,\mathbf{k})=\frac{2\sigma\hbar\omega}{(\sigma\omega)^2+(k^2+m^2)^2}, \tag{C15}$$

$$\tilde{G}_{\rm R}(\omega,\mathbf{k})=\frac{\hbar}{-i\sigma\omega+k^2+m^2}, \tag{C16}$$

$$\tilde{G}_{\rm A}(\omega,\mathbf{k})=\frac{\hbar}{i\sigma\omega+k^2+m^2}. \tag{C17}$$

Finally, if one inserts the spectral density (C15) directly into the spectral representation, in the underlying quantum theory, of the Euclidean propagator

$$\tilde{G}_{\rm E}(\nu_n,\mathbf{k})=\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\,\frac{\rho(\omega',\mathbf{k})}{\omega'+i\nu_n}, \tag{C18}$$

then one obtains

$$\tilde{G}_{\rm E}(\nu_n,\mathbf{k})=\frac{\hbar}{\sigma|\nu_n|+k^2+m^2}. \tag{C19}$$

However, this result is not sensible (for non-zero Matsubara frequencies). As noted earlier, the classical Langevin theory which led to Eq. (C15) is only valid for frequencies small compared to $(\beta\hbar)^{-1}$, whereas the result (C19) depends sensitively on the precise form of the high frequency behavior of the spectral density (C15).

APPENDIX D: SMALL p EXPANSION OF $\hat{G}_0(\mathbf{p})$

In this appendix, we discuss how to expand $\hat{G}_0(\mathbf{p}) = [\mathbf{v}\cdot i\mathbf{p}+\delta\hat{C}]^{-1}$ for small momentum $\mathbf{p}$. We cannot simply treat $i\mathbf{v}\cdot\mathbf{p}$ as a perturbation because $\delta\hat{C}$ has a zero mode (1.18) and so is not invertible. The correct way to expand is to separate $i\mathbf{v}\cdot\mathbf{p}$ into zero-mode and non-zero mode pieces,

$$\mathbf{v}\cdot i\mathbf{p}=\mathbf{v}\cdot i\mathbf{p}\,\hat{P}_0+\hat{P}_0\,\mathbf{v}\cdot i\mathbf{p}+(1-\hat{P}_0)\,\mathbf{v}\cdot i\mathbf{p}\,(1-\hat{P}_0), \tag{D1}$$

and treat only the last term as a perturbation [11]. The first two terms are rank-one operators which will lift the zero mode of $\delta\hat{C}$. This leads to the expansion

$$\hat{G}_0=[\hat{G}_{00}^{-1}+(1-\hat{P}_0)\,\mathbf{v}\cdot i\mathbf{p}\,(1-\hat{P}_0)]^{-1}$$
$$=\hat{G}_{00}-\hat{G}_{00}\,(1-\hat{P}_0)\,\mathbf{v}\cdot i\mathbf{p}\,(1-\hat{P}_0)\,\hat{G}_{00}$$
$$+\hat{G}_{00}\,(1-\hat{P}_0)\,\mathbf{v}\cdot i\mathbf{p}\,(1-\hat{P}_0)\,\hat{G}_{00}$$
$$\times(1-\hat{P}_0)\,\mathbf{v}\cdot i\mathbf{p}\,(1-\hat{P}_0)\,\hat{G}_{00}-\cdots, \tag{D2a}$$

where

$$\hat{G}_{00}\equiv[\mathbf{v}\cdot i\mathbf{p}\,\hat{P}_0+\hat{P}_0\,\mathbf{v}\cdot i\mathbf{p}+\delta\hat{C}]^{-1}$$
$$=(1-\hat{P}_0)\,\delta\hat{C}^{-1}\,(1-\hat{P}_0)$$
$$+\frac{d}{\gamma_1 p^2}\,(\gamma_1-\mathbf{v}\cdot i\mathbf{p})\,\hat{P}_0\,(\gamma_1-\mathbf{v}\cdot i\mathbf{p}). \tag{D2b}$$

To verify the last equality, note that $\delta\hat{C}\,\mathbf{v}\cdot i\mathbf{p}\hat{P}_0=\gamma_1\mathbf{v}\cdot i\mathbf{p}\hat{P}_0$ by Eq. (1.17). Observe that $\hat{G}_{00}$ is $O(p^{-2})$. That might appear problematical for the expansion (D2a), which brings along a factor of $\hat{G}_{00}$ with every factor of $\mathbf{v}\cdot i\mathbf{p}$, except that the inner factors of $\hat{G}_{00}$ always appear in the combination

$$(1-\hat{P}_0)\,\hat{G}_{00}\,(1-\hat{P}_0)=(1-\hat{P}_0)\left[\delta\hat{C}^{-1}-\frac{d}{\gamma_1}\mathbf{v}\cdot i\hat{\mathbf{p}}\hat{P}_0\,\mathbf{v}\cdot i\hat{\mathbf{p}}\right]$$
$$\times(1-\hat{P}_0), \tag{D3}$$

which is only $O(p^0)$.

The expansion (D2) yields

$$\hat{G}_0=\frac{d}{\gamma_1 p^2}\,(\gamma_1-\mathbf{v}\cdot i\mathbf{p})\,\hat{P}_0\,(\gamma_1-\mathbf{v}\cdot i\mathbf{p})+O(p^0), \tag{D4}$$

and so

$$\hat{G}_0\rangle=\frac{d}{p^2}\,(\gamma_1-\mathbf{v}\cdot i\mathbf{p})\rangle+O(p^0) \tag{D5}$$

and

$$\langle\hat{G}_0\rangle=\frac{d\gamma_1}{p^2}+O(p^0). \tag{D6}$$

Putting the last two equations together gives the expansion (2.64) for $\hat{\mathcal{G}}_0\rangle$ cited in the main text.

The expansion (D2) also gives

$$\sigma_{\mathbf{p}}=\frac{m^2}{d-1}\langle v_i \hat{G}_0 v_i\rangle=\sigma_0+O(p^2), \tag{D7}$$

which is a result used in Sec. II B 6.

APPENDIX E: LARGE p EXPANSION OF $\Sigma_m(\rho)$

Using the large **p** expansion (2.88) of $\hat{G}_0(\mathbf{p})$, one finds

$$\begin{aligned}\Sigma_m(\rho)&\sim\langle mm|\mathrm{P.P.}\frac{1}{iv_z\rho}+\pi\delta(v_z\rho)|mm\rangle\\&=\frac{\pi}{\rho}\langle mm|\;\delta(v_z)\;|mm\rangle=\frac{(2m+1)}{(2m)!}[P_m^m(0)]^2\frac{\pi}{2\rho}\\&=\frac{(2m+1)!!\,(2m-1)!!}{(2m)!}\;\frac{\pi}{2\rho},\end{aligned} \tag{E1}$$

where $(-1)!!\equiv 1$. [The P.P. terms vanishes because the $|mm\rangle$ states are invariant under $v_z\to -v_z$.]

We can make a little progress analyzing the corrections to this leading term by attempting to treat $\delta\hat{c}$ as a perturbation in $(iv_x\rho+\varepsilon)^{-1}$:

$$\begin{aligned}\Sigma_m(\rho)&=\langle mm|(iv_z\rho+\varepsilon)^{-1}|mm\rangle\\&\quad-\langle mm|(iv_z\rho+\varepsilon)^{-1}\;\delta\hat{c}\;(iv_z\rho+\varepsilon)^{-1}|mm\rangle+\cdots\;.\end{aligned} \tag{E2}$$

Call the second term of the expansion $\delta\Sigma_m$. Taking the formula (2.109) for δc, we can rewrite this term as

$$\delta\Sigma_m(\rho)=\langle mm|(iv_z\rho+\varepsilon)^{-2}|mm\rangle-4\pi\left\langle Y^*_{mm}(\mathbf{v})\left[\mathrm{P.P.}\frac{1}{iv_z\rho}+\pi\delta(v_z\rho)\right]\delta\hat{c}_2(\mathbf{v},\mathbf{v}')\left[\mathrm{P.P.}\frac{1}{iv'_z\rho}+\pi\delta(v'_z\rho)\right]Y_{mm}(\mathbf{v}')\right\rangle_{\mathbf{vv}'}, \tag{E3}$$

where

$$\delta c_2(\mathbf{v},\mathbf{v}')\equiv-\frac{4}{\pi}\,\frac{(\mathbf{v}\cdot\mathbf{v}')^2}{\sqrt{1-(\mathbf{v}\cdot\mathbf{v}')^2}}. \tag{E4}$$

By independently considering both $\mathbf{v}\to-\mathbf{v}$ and $\mathbf{v}'\to-\mathbf{v}'$ in the second term of Eq. (E3), one sees that only the δ functions will contribute for even m and only the principal part for odd m.

As we shall see, the $\mathbf{vv}'$ average in the second term of Eq. (E3) is logarithmically divergent as simultaneously $v_z\to 0$ and $\mathbf{v}'\to\pm\mathbf{v}$: a divergence which the ε prescription does not cure. In contrast, the first term of Eq. (E3) does not give rise to a logarithm. For example, for $m=0$, one gets

$$\langle 00|(iv_z\rho+\varepsilon)^{-2}|00\rangle=\frac{1}{2}\int_{-1}^{+1}\frac{dv_z}{(iv_z\rho+\epsilon)^2}=\frac{1}{p^2}. \tag{E5}$$

We will now focus on the logarithmic divergence and so ignore the first term of Eq. (E3). The divergence is cut off only by $\delta\hat{C}$ itself. That is, it is cut off when $|iv_z\rho|$ becomes of order $\delta\hat{c}=O(1)$, so that $|v_z|\sim\rho^{-1}$. The logarithmic divergence of Eq. (E3) is an artifact of our treatment of $\delta\hat{c}$ as a perturbation. However, by proceeding with Eq. (E3) and treating v_z as effectively cut off at $O(\rho^{-1})$, and so $\delta(v_z\rho)$ as being smeared to have a width of $\Delta(v_z\rho)\sim 1$, we may extract the coefficient of the logarithm.

The log divergence can be seen by setting $v_z=v'_z=0$ and considering just the integral over azimuthal angles in Eq. (E3):

$$\begin{aligned}&\int_0^{2\pi}d\phi d\phi'\,e^{-im\phi}\;\delta c_2(\mathbf{v},\mathbf{v}')\;e^{im\phi'}\Big|_{v_z=v'_z=0}\\&\quad=-8\int_0^{2\pi}d(\Delta\phi)\frac{\cos^2(\Delta\phi)}{|\sin(\Delta\phi)|}e^{im\,\Delta\phi}.\end{aligned} \tag{E6}$$

There is then a log singularity associated with $\Delta\phi\to 0$. If one repeats the above for small (rather than zero) v_z and v'_z, one finds the dominant logarithmic behavior is

$$\begin{aligned}&\int_0^{2\pi}d\phi d\phi'\,e^{-im\phi}\;\delta c_2(\mathbf{v},\mathbf{v}')\;e^{im\phi'}\\&\quad\approx-16\left[\ln\left(\frac{1}{|v_z-v'_z|}\right)+(-)^m\ln\left(\frac{1}{|v_z+v'_z|}\right)\right].\end{aligned} \tag{E7}$$

For even m, if we substitute back into Eq. (E3), treat the δ functions as smeared over $v_z\rho\sim 1$, and ignore nonlogarithmic corrections, then the logarithms of Eq. (E7) become simply $\ln(1/\rho)$'s, giving

$$\begin{aligned}\delta\Sigma_m(\rho)&\approx\frac{(2m+1)}{(2m)!}[P_m^m(0)]^2\frac{2}{\rho^2}\ln\rho\\&=\frac{(2m+1)!!\,(2m-1)!!}{(2m)!}\;\frac{2}{\rho^2}\ln\rho.\end{aligned} \tag{E8}$$

For odd m, we instead take the principal part terms of Eq. (E3), and we need to be more careful to separate the (cutoff) integral of the joint overall scale of v_z and v'_z from that of the relative scale $\beta\equiv v_z/v'_z$:

$$\begin{aligned}\delta\Sigma_m(\rho) &\approx -\frac{(2m+1)}{(2m)!}[P_m^m(0)]^2\frac{1}{\pi\rho^2}\int_{-\infty}^{+\infty}dv_z dv_z'\frac{1}{v_z}\ln\left|\frac{v_z+v_z'}{v_z-v_z'}\right|\frac{1}{v_z'}\\ &\approx -\frac{(2m+1)}{(2m)!}[P_m^m(0)]^2\frac{1}{\pi\rho^2}\int_{-\infty}^{+\infty}\frac{dv_z}{v_z}\int_{-\infty}^{+\infty}d\beta\ln\left|\frac{1+\beta}{1-\beta}\right|\\ &\approx -\frac{(2m+1)!!\,(2m-1)!!}{(2m)!}\frac{2}{\rho^2}\ln\rho. \end{aligned} \tag{E9}$$

Our final result is then

$$\Sigma_m(\rho)=\frac{(2m+1)!!\,(2m-1)!!}{(2m)!}\left[\frac{\pi}{2\rho}+(-)^m\frac{2}{\rho^2}\ln\rho+O\left(\frac{1}{\rho^2}\right)\right]. \tag{E10}$$

We do not know how to determine the non-logarithmic $O(\rho^{-2})$ piece analytically, but fits to numerical evaluation of the continued-fraction formula (2.116) at large ρ give the leading correction to Eq. (E10) to be $\ln\rho\rightarrow\ln\rho+h_m$ with $h_0\simeq-0.8$, $h_1\simeq1.16$, and $h_2\simeq-4.3$.

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[1] P. Arnold and L. Yaffe, preceding paper, Phys. Rev. D **62**, 125013 (2000).
[2] P. Arnold, Phys. Rev. D **62**, 036003 (2000).
[3] D. Bödeker, Phys. Lett. B **426**, 351 (1998); Nucl. Phys. **B559**, 502 (1999).
[4] P. Arnold, D. Son, and L. Yaffe, Phys. Rev. D **59**, 105020 (1999).
[5] J. Blaizot and E. Iancu, Nucl. Phys. **B570**, 326 (2000).
[6] H. Chan and M. Halpern, Phys. Rev. D **33**, 540 (1985).
[7] J. Zinn-Justin and D. Zwanziger, Nucl. Phys. **B295** [FS21], 297 (1988).
[8] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 2nd ed. (Oxford University Press, New York, 1993).
[9] P. Arnold, Phys. Rev. E **61**, 6091 (2000).
[10] D. Bodeker, G. Moore, and K. Rummukainen, Phys. Rev. D **61**, 056003 (2000).
[11] P. Arnold, D. Son, and L. Yaffe, Phys. Rev. D **60**, 025007 (1999).
[12] E. Braaten and A. Nieto, Phys. Rev. D **51**, 6990 (1995); **53**, 3421 (1996).
[13] See, for example, L. Brown and L. Yaffe, Phys. Rep. (to be published), physics/9911055; and H. Georgi, Annu. Rev. Nucl. Part. Sci. **43**, 209 (1993).
[14] A. Prudnikov, Yu. Brychkov, and O. Marichev, *Integrals and Series* (Gordon and Breach, New York, 1986), Vol. 2.
[15] D. Bödeker, Nucl. Phys. **B566**, 402 (2000).
[16] D. Litim and C. Manuel, Phys. Rev. Lett. **82**, 4981 (1999).
[17] M. Basagoiti, hep-ph/9903462.
[18] G. Moore, Nucl. Phys. **B568**, 367 (2000).
[19] R. Pisarski, Phys. Rev. Lett. **63**, 1129 (1989); Phys. Rev. D **47**, 5589 (1993).
[20] H. Weldon, Phys. Rev. D **26**, 1394 (1982); U. Heinz, Ann. Phys. (N.Y.) **161**, 48 (1985); **168**, 148 (1986).
[21] J. Blaizot and E. Iancu, Phys. Rev. D **56**, 7877 (1997).
[22] D. Zwanziger, Nucl. Phys. **B192**, 259 (1981).