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SUPERSYMMETRY AND CENTRAL CHARGES

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Superfield representations of supersymmetry algebras with central charges are studied. An interacting field theory realization of the SU(2) invariant example is given, which is shown to be just another interpretation of P. Fayet's "hypersymmetry". For certain massless cases it is shown that the theory has a larger SU(4) invariance once the equations of motion are used, thus providing us with a possible superfield formulation of the SU(4) supersymmetric gauge theory.

1. Introduction

Several years ago, supersymmetric theories of possible physical relevance were classified [1]. While it could then be shown that other theories were not possible within the framework of massive quantum field theory, it was (and is) by no means obvious whether or not the surviving possibilities can indeed all be realized. Most of them probably cannot without introducing gauge-invariance, and thus massless fields, in order to provide a coupling mechanism (exceptions are known only for the $N = 1$ (non-extended) supersymmetry, where strictly massive field theoretical examples exist [2]). There are even indications that beyond $N = 4$ none of the schemes are realized at all [3]. Non-Abelian gauge theories with supersymmetry have been derived for $N = 1$ [4,5], for $N = 2$ [4–6] and for $N = 4$ [7]. The complete superfield structure is known for $N \leqslant 2$ [8], while for $N = 4$ only the "on-shell" superfield structure is known [9]. Fayet [6] first succeeded in coupling massive matter multiplets to the Yang-Mills fields for $N = 2$. He did, however, not give the superfield structure of his theory, nor an algebra of his "hypersymmetry". The algebra was later identified as that of an O(2) extended supersymmetry with a central charge [10].

In the present paper the superfield structure for such a multiplet is given, and the interaction with the gauge-field multiplet of SU(2) supersymmetry is constructed through minimal coupling. The $N = 2$ Yang-Mills theory with matter multiplets becomes the first known renormalizable realization of yet another class of possible supersymmetry algebras, namely those with central charges. (The coupling to supergravity was studied in ref. [10].)

Perturbation calculations for supersymmetric theories with $N > 1$ seem rather important because of their interesting short-distance behaviour [4,11]. Therefore it seems desirable to know those theories in their superspace formulation, which may eventually allow considerable simplification of Feynman diagram calculations. The present paper offers the first off-shell generalization of the particularly interesting $N = 4$ theory: we show that a certain class of $N = 2$ supersymmetric theories acquires the larger $N = 4$ supersymmetry if the equations of motion are imposed.

In sect. 2, the Lie superalgebra with internal SU(N) and central charges is introduced, and its representations on superfields are given. In sect. 3, the special case of $N = 2$ is studied together with its smallest representations on a one-particle Hilbert space. In sect. 4, the corresponding superfield is introduced and its (free) Lagrangian is derived. Sect. 5 is devoted to the gauge theory for this superfield, thus introducing interaction. The gauge-invariant Lagrangian with the desired supersymmetry is given. In sect. 6, we show that in the massless case with the matter field in the adjoint representation of the gauge group, this theory possesses $N = 4$ supersymmetry if the equations of motion are used. Concluding remarks are contained in sect. 7.

2. The algebra and its representation on superfields

It has been shown [1], that in a massive field theory the non-vanishing part of the most general supersymmetry algebra of spinor charges Q^i_α ($\alpha = 1, 2$; $i = 1, ..., N$) and their hermitian conjugates $\overline{Q}_{\dot\alpha i}$ consists of the anticommutators *

$$\begin{aligned}
\{Q^i_\alpha, \overline{Q}_{\dot\beta j}\} &= 2\delta^i_j \sigma^\mu_{\alpha\dot\beta} P_\mu \,, \\
\{Q^i_\alpha, Q^j_\beta\} &= 2\epsilon_{\alpha\beta}(a^l)^{ij} Z_l \,, \\
\{\overline{Q}_{\dot\alpha i}, \overline{Q}_{\dot\beta j}\} &= 2\epsilon_{\dot\alpha\dot\beta}(\overline{a}^l)_{ij} Z_l \,,
\end{aligned} \tag{1}$$

with $(\overline{a}^l)_{ij}$ the complex conjugate of $(a^l)^{ij}$. P_μ are the four-momenta and the Z_l are a number of hermitian central charges.

As non-trivial automorphism group of this algebra we can have $\mathrm{SL}(2, \mathrm{c}) \otimes \mathcal{G}$, where $\mathcal{G}$ is a subgroup of U(N), under which the Q^i_α transform under some N-dimensional, not necessarily irreducible, representation $\{G\}$, while the $\overline{Q}$ transform under the complex conjugate representation $\{G^*\}$.

The algebra is consistent only if every one of the matrices a^l:

(i) is skew-symmetric,

(ii) transforms $\{G\}$ into its complex conjugate: $Ga^l = a^l G^*$.

* $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} = -\epsilon^{\alpha\beta}$; $\epsilon^{12} = +1$; same for dotted indices.

The algebra can be represented on a superspace of real coordinates x^μ and z^l, and Grassmann coordinates θ_i^α and $\bar\theta^{\dot\alpha i}$ through the transformation laws

$$
\begin{aligned}
x^{\mu\prime} &= x^\mu + y^\mu + i\theta\sigma^\mu\bar\zeta - i\zeta\sigma^\mu\bar\theta\,,\\
z^{l\prime} &= z^l + w^l + i\zeta\epsilon a^l\theta - i\bar\theta\epsilon\bar a^l\bar\zeta\,,\\
\theta_i^{\alpha\prime} &= \theta_i^\alpha + \zeta_i^\alpha\,,\\
\bar\theta^{\dot\alpha i\prime} &= \bar\theta^{\dot\alpha i} + \bar\zeta^{\dot\alpha i}\,,
\end{aligned} \tag{2}
$$

where the y^μ, w^l, ζ_i^α, and $\bar\zeta^{\dot\alpha i}$ are the parameters of translations, Z-transformations, and supersymmetry transformations, respectively. Super-"covariant" derivatives are, besides ∂_μ and $\partial/\partial z^l$, the following spinorial quantities ($\not\partial \equiv \sigma^\mu\partial_\mu$):

$$
\begin{aligned}
D_\alpha^i &= \frac{\partial}{\partial\theta_i^\alpha} + i(\not\partial\bar\theta)_\alpha^i - i(\epsilon a^l\theta)_\alpha^i\frac{\partial}{\partial z^l}\,,\\
\bar D_{\dot\alpha i} &= -\frac{\partial}{\partial\bar\theta^{\dot\alpha i}} - i(\theta\not\partial)_{\dot\alpha i} + i(\bar\theta\bar a^l\epsilon)_{\dot\alpha i}\frac{\partial}{\partial z^l}\,,
\end{aligned} \tag{3}
$$

with anticommutation relations

$$
\begin{aligned}
\{D_\alpha^i, \bar D_{\dot\beta j}\} &= -2i\delta_j^i\sigma^\mu_{\alpha\dot\beta}\partial_\mu\,,\\
\{D_\alpha^i, D_\beta^j\} &= -2i\epsilon_{\alpha\beta}(a^l)^{ij}\frac{\partial}{\partial z^l}\,,\\
\{\bar D_{\dot\alpha i}, \bar D_{\dot\beta j}\} &= -2i\epsilon_{\dot\alpha\dot\beta}(\bar a^l)_{ij}\frac{\partial}{\partial z^l}\,.
\end{aligned} \tag{4}
$$

A superfield ϕ over this superspace does not in general consist of only a finite multiplet of ordinary fields over x^μ alone, since the Taylor-expansion in z^l does not break off, unless some additional condition relates the coefficients to each other. An obvious condition would be

$$
\frac{\partial}{\partial z^l}\phi = 0\,, \tag{5}
$$

which, however, would only imply that Z_l is represented trivially on ϕ.

3. The simplest non-trivial example

The simplest algebra with a central charge is for $N = 2$ and only one Z. To stay consistent with ref. [8], we use the letter g for the antisymmetric symbol in the

SU(2) space:

$$a^{ij} = g^{ij} = -g^{ji}\ ; \qquad g^{12} = 1\ , \tag{6}$$

and define

$$g_{ij} \equiv -\bar{a}_{ij} = -g^{ij}\ , \tag{7}$$

so that $g^{ij}g_{jk} = \delta^i_k$.

If the charges act on one-particle states with mass m and Z quantum number $\overset{0}{z}$, their algebra reduces to the following Clifford-algebra in the rest frame of the particle [12]:

$$\begin{aligned}
\{Q^i_\alpha, \bar{Q}_{\dot{\beta}j}\} &= 2m\delta^i_j\delta_{\alpha\dot{\beta}}\ ,\\
\{Q^i_\alpha, Q^j_\beta\} &= 2\overset{0}{z}g^{ij}\epsilon_{\alpha\beta}\ ,\\
\{\bar{Q}_{\dot{\alpha}i}, \bar{Q}_{\dot{\beta}j}\} &= -2\overset{0}{z}g_{ij}\epsilon_{\dot{\alpha}\dot{\beta}}\ ,
\end{aligned} \tag{8}$$

which is diagonalized by the following linear combinations:

$$\begin{aligned}
Q_{(1)} &= Q^1_1 - \bar{Q}_{\dot{2}2}\ , \qquad & Q_{(3)} &= Q^1_1 + \bar{Q}_{\dot{2}2}\ ,\\
Q_{(2)} &= Q^1_2 + \bar{Q}_{\dot{1}2}\ , \qquad & Q_{(4)} &= Q^1_2 - \bar{Q}_{\dot{1}2}\ ,
\end{aligned} \tag{9}$$

and their hermitian conjugates:

$$\begin{aligned}
&\{Q_{(r)}, Q_{(s)}\} = \{Q^\dagger_{(r)}, Q^\dagger_{(s)}\} = 0\ ,\\
&\{Q_{(r)}, Q^\dagger_{(s)}\} = 4(m \pm \overset{0}{z})\,\delta_{rs} \quad \text{with} \quad \begin{cases} + \text{ for } r = 1, 2\\ - \text{ for } r = 3, 4\ . \end{cases}
\end{aligned} \tag{10}$$

We see that while for $\overset{0}{z} = 0$ (no central charge) we have four raising operators in the Clifford algebra, there are only two (as in the massless case) if the relation

$$m = |\overset{0}{z}|\ , \tag{11}$$

or its operator equivalent

$$P^2 = Z^2\ , \tag{12}$$

holds *. However, while the Clifford vacuum is annihilated by Q^1_α and $\bar{Q}_{\dot{\alpha}2}$, only cer-

* This multiplet-shortening through central charges has been known to Grosser and Haag for a long time [13].

tain linear combinations of their SU(2) partners Q^2_α and $\bar{Q}_{\dot{\alpha}1}$ will do the same, i.e. the ground state is not an SU(2) scalar. To have a particle multiplet which also represents SU(2), we must introduce a second ground state with different properties, and thus double the number of states after all. However, the range in spin covered by the multiplet is still only $\Delta s = \frac{1}{2}$, as compared to $\Delta s = 1$ for $P^2 \neq Z^2$.

4. The free superfield

Let ϕ_i be a superfield with an extra SU(2)-spinor index, and fulfilling the constraints

$$D^j_\alpha \phi_i \sim \delta^j_i\,, \qquad \bar{D}_{\dot{\alpha}j}\phi_i \sim g_{ij}\,. \tag{13}$$

The tedious explicit evaluation of these constraints yields the following power expansion of ϕ_i in terms of fields over x (unwritten indices are contracted in an obvious way):

$$\begin{aligned}\phi_i &= A_i + \theta^\alpha_i \psi_\alpha + (g\bar{\theta})^{\dot{\alpha}}_i \bar{\varphi}_{\dot{\alpha}} + zF_i + i\theta_i\epsilon\theta_j g^{jk}F_k - ig_{ij}\bar{\theta}^j\epsilon\bar{\theta}^k F_k \\ &\quad + 2i\theta_i \not{\partial}\bar{\theta}^j A_j - i\theta_j\not{\partial}\bar{\theta}^j A_i - z(\theta\not{\partial}\epsilon)^{\dot{\alpha}}_i\bar{\varphi}_{\dot{\alpha}} - z(g\epsilon\not{\partial}\bar{\theta})^\alpha_i\psi_\alpha \\ &\quad + \tfrac{1}{2}z^2\,\Box A_i + \text{other terms}\,,\end{aligned} \tag{14}$$

where the "other terms" are at least trilinear in θ, $\bar{\theta}$, and z, and contain only higher derivatives of A, ψ, $\bar{\varphi}$, and F.

Since two superfields are equal if their lowest components are the same, we can see from the power expansion alone that $\Box\phi_i = (\partial^2/\partial z^2)\,\phi_i$, i.e., that eq. (12) is satisfied.

The transformation properties of the components under supersymmetry and Z transformations are given by ($\bar{\not{\partial}}^{\dot{\alpha}\beta} \equiv \bar{\sigma}^{\mu\dot{\alpha}\beta}\partial_\mu$)

$$\begin{aligned}\delta A_i &= \zeta_i\psi + g_{ij}\bar{\zeta}^j\bar{\varphi} + wF_i\,,\\ \delta\psi_\alpha &= 2i\epsilon_{\alpha\beta}\zeta^\beta_i g^{ij}F_j + 2i(\not{\partial}\bar{\zeta})^i_\alpha A_i - w(\not{\partial}\epsilon\bar{\varphi})_\alpha\,,\\ \delta\bar{\varphi}_{\dot{\alpha}} &= 2i(\zeta\not{\partial})_{\dot{\alpha}i}g^{ij}A_j - 2i\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\zeta}^{\dot{\beta}i}F_i - w(\epsilon\bar{\not{\partial}}\psi)_{\dot{\alpha}}\,,\\ \delta F_i &= -\zeta^\alpha_i(\not{\partial}\epsilon\bar{\varphi})_\alpha - g_{ij}(\epsilon\not{\partial}\bar{\zeta})^{\alpha j}\psi_\alpha + w\,\Box A_i\,.\end{aligned} \tag{15}$$

Besides ϕ_i, we also have the hermitian conjugate superfield $\bar{\phi}^i$, with

$$\bar{D}_{\dot{\alpha}j}\bar{\phi}^i \sim \delta^i_j\,, \qquad D^j_\alpha\bar{\phi}^i \sim g^{ji}\,, \tag{16}$$

whose power expansion and transformation properties are easiest obtained from (14) and (15) through hermitian conjugation.

Consider the following superfield

$$\mathcal{L}_m \equiv \tfrac{1}{24}(D\tau_A g\epsilon D - \bar{D}g\tau_A \epsilon\bar{D})\,\bar{\phi}\tau_A\phi\,, \tag{17}$$

where the $(\tau_A)_i{}^j$ are a set of Pauli matrices ($A = 1, 2, 3$) in the SU(2) space. It can be shown that *

$$\mathcal{L}_m = i\bar{\phi}\frac{\partial}{\partial z}\phi - i\frac{\partial}{\partial z}\bar{\phi}\phi - \tfrac{1}{8}(D_\alpha g\bar{\phi})\,\epsilon^{\alpha\beta}(D_\beta\phi) - \tfrac{1}{8}(\bar{D}_{\dot{\alpha}}\bar{\phi})\,\epsilon^{\dot{\alpha}\dot{\beta}}(\bar{D}_{\dot{\beta}}g\phi)\,, \tag{18}$$

and that

$$D_\alpha^i\,\mathcal{L}_m = \tfrac{1}{2}i\partial_\mu[\bar{\phi}^i\sigma^\mu\epsilon\bar{D}g\phi]_\alpha + \tfrac{1}{2}i\partial_\mu[\sigma^\mu\epsilon\bar{D}\bar{\phi}g^{ij}\phi_j]_\alpha\,, \tag{19}$$

i.e., that $D_\alpha^i\mathcal{L}_m$ is a total divergence (so is, because $\mathcal{L}_m$ is real, $\bar{D}_{\dot{\alpha}i}\mathcal{L}_m$). This implies that the lowest component of $\mathcal{L}_m$ transforms as a total divergence under supersymmetry transformations, since the θ-, $\bar{\theta}$-independent parts of D and $\bar{D}$ and of the generators of supersymmetry transformations are equal.

In the derivation of eqs. (18) and (19), no use has been made of $\bar{\phi} = \phi^+$. Since $(\partial/\partial z)\,\phi$ and $(\partial/\partial z)\,\bar{\phi}$ both fulfill the only relevant conditions (13) and (16) as well as ϕ and $\bar{\phi}$, we know that for

$$\mathcal{L}_0 \equiv \tfrac{1}{48}i(D\tau_A g\epsilon D - \bar{D}g\tau_A\epsilon\bar{D})\left(\bar{\phi}\tau_A\frac{\partial}{\partial z}\phi - \frac{\partial}{\partial z}\bar{\phi}\tau_A\phi\right), \tag{20}$$

the lowest component will also transform as a total divergence. $\mathcal{L}_0$ can be rewritten as

$$\begin{aligned}\mathcal{L}_0 = &-\tfrac{1}{2}\bar{\phi}\,\Box\,\phi - \tfrac{1}{2}\,\Box\,\bar{\phi}\phi + \frac{\partial}{\partial z}\bar{\phi}\frac{\partial}{\partial z}\phi \\ &- \tfrac{1}{16}i(Dg\bar{\phi})\,\epsilon\not{\partial}\epsilon(\bar{D}g\phi) - \tfrac{1}{16}i(\bar{D}\bar{\phi})\,\overleftrightarrow{\not{\partial}}(D\phi)\end{aligned} \tag{21}$$

and the free Lagrangian for our multiplet is now given by

$$\begin{aligned}L &= (\mathcal{L}_0 + m\mathcal{L}_m)_{\theta=\bar{\theta}=z=0} \\ &= -\tfrac{1}{2}A^{+i}\,\Box\,A_i - \tfrac{1}{2}\,\Box\,A^{+i}A_i + F^{+i}F_i + \tfrac{1}{4}i\varphi\epsilon\overleftrightarrow{\not{\partial}}\epsilon\bar{\varphi} - \tfrac{1}{4}i\bar{\psi}\overleftrightarrow{\not{\partial}}\psi \\ &\quad + imA^{+i}F_i - imF^{+i}A_i - \tfrac{1}{2}m\varphi\epsilon\psi + \tfrac{1}{2}m\bar{\psi}\epsilon\bar{\varphi}\,,\end{aligned} \tag{22}$$

* In the derivation of eqs. (18) and (19), a set of equations has been used which can be derived from (13) and (16), but have not been given here, since they can easily be read off from eqs. (33) and (34) below for the limit of gauge coupling → 0.

which describes a free multiplet, consisting of a scalar isodoublet and a Dirac-spinor isosinglet (built from the two Weyl spinors ψ and $\bar{\varphi}$), all of mass m. The doublet F_i are auxiliary fields. This is Fayet's "scalar hypermultiplet" [6]. The covariant formulation of the equations of motion derived from the Lagrangian (22) is

$$\frac{\partial}{\partial z}\phi_i = im\phi_i \,, \tag{23}$$

which, when iterated, yields the Klein-Gordon equation, using (12).

5. The gauge-covariant multiplet

We will in the end be interested in a Lagrangian theory for the superfields ϕ_i which is also invariant under super-gauge transformations [4,5] (it seems as if this is the only way to construct an interacting theory at all!):

$$\phi_i \to e^{i\Lambda}\phi_i \,, \tag{24}$$

where

$$\Lambda = \sum_l T_l \Lambda^l(x, \theta, \bar{\theta}) \,, \tag{25}$$

is a Lie algebra valued superfield of parameters. The matrices T_l generate the algebra of the gauge group and act on the (unwritten) gauge-group indices of ϕ_i. $\bar{\phi}^i$ will correspondingly transform under the complex conjugate representation.

For simplicity, we assume that

$$\frac{\partial}{\partial z}\Lambda = 0 \,, \tag{26}$$

which allows to restrict all gauge fields $\mathcal{A}$ to the known case [8] without central charge: $(\partial/\partial z)\mathcal{A} = 0$. The gauge-covariant derivatives are then $\partial/\partial z$ and

$$\mathcal{D}_\mu = \partial_\mu + i\mathcal{A}_\mu \,, \qquad \mathcal{D}^i_\alpha = D^i_\alpha + i\mathcal{A}^i_\alpha \,, \qquad \bar{\mathcal{D}}_{\dot{\alpha}i} = \bar{D}_{\dot{\alpha}i} + i\bar{\mathcal{A}}_{\dot{\alpha}i} \,. \tag{27}$$

The Yang-Mills field strengths can be derived in the usual way, and we impose the restrictions [8]

$$F^{ij}_{\alpha\beta} + F^{ij}_{\beta\alpha} = \bar{F}_{\dot{\alpha}i\dot{\beta}j} + \bar{F}_{\dot{\beta}i\dot{\alpha}j} = F^i_{\alpha\dot{\beta}j} = 0 \,, \tag{28}$$

which leave the algebra of the covariant derivatives as *

* $\sigma_{\mu\nu} = \frac{1}{2}i(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu)$, $\bar{\sigma}_{\mu\nu} = \frac{1}{2}i(\bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu)$.

$$\{\mathcal{D}^i_\alpha, \mathcal{D}^j_\beta\} = -2i\epsilon_{\alpha\beta}g^{ij}\frac{\partial}{\partial z} + i\epsilon_{\alpha\beta}g^{ij}\overline{W}\,,$$

$$\{\overline{\mathcal{D}}_{\dot\alpha i}, \overline{\mathcal{D}}_{\dot\beta j}\} = 2i\epsilon_{\dot\alpha\dot\beta}g_{ij}\frac{\partial}{\partial z} + i\epsilon_{\dot\alpha\dot\beta}g_{ij}W\,,$$

$$\{\mathcal{D}^i_\alpha, \overline{\mathcal{D}}_{\dot\beta j}\} = -2i\delta^i_j \not{\mathcal{D}}_{\alpha\dot\beta}\,,$$

$$[\mathcal{D}_\mu, \mathcal{D}^i_\alpha] = \tfrac{1}{4}(\sigma_\mu \epsilon g \overline{\mathcal{D}}\,\overline{W})^i_\alpha\,,$$

$$[\mathcal{D}_\mu, \overline{\mathcal{D}}_{\dot\alpha i}] = -\tfrac{1}{4}(g\epsilon\bar\sigma_\mu \mathcal{D} W)_{\dot\alpha i}\,,$$

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = -\tfrac{1}{32}(\mathcal{D}\epsilon\sigma_{\mu\nu}g\mathcal{D}W + \overline{\mathcal{D}}\bar\sigma_{\mu\nu}\epsilon g\overline{\mathcal{D}}\,\overline{W}) = iF_{\mu\nu}\,,$$

$$\left[\frac{\partial}{\partial z}, \text{any } \mathcal{D}\right] = 0\,, \tag{29}$$

where the W's are given in terms of the $\mathcal{A}$'s as

$$\overline{W} = \tfrac{1}{2}D\epsilon g\mathcal{A} + \tfrac{1}{2}i\mathcal{A}\epsilon g\mathcal{A}\,, \qquad W = \tfrac{1}{2}\overline{D}\epsilon g\overline{\mathcal{A}} + \tfrac{1}{2}i\overline{\mathcal{A}}\epsilon g\overline{\mathcal{A}}\,, \tag{30}$$

and fulfill the Bianchi identities [9]

$$\mathcal{D}^i_\alpha\overline{W} = \overline{\mathcal{D}}_{\dot\alpha i}W = \mathcal{D}\epsilon\tau_A g\mathcal{D}W - \overline{\mathcal{D}}\epsilon g\tau_A\overline{\mathcal{D}}\,\overline{W} = 0\,. \tag{31}$$

The gauge-covariant generalization of the conditions (13) and (16) is

$$\begin{aligned}&\mathcal{D}^i_\alpha\phi_j \sim \delta^i_j\,, &&\overline{\mathcal{D}}_{\dot\alpha i}\phi_j \sim g_{ij}\,,\\ &\mathcal{D}^i_\alpha\bar\phi^j \sim g^{ij}\,, &&\overline{\mathcal{D}}_{\dot\alpha i}\bar\phi^j \sim \delta^j_i\,.\end{aligned} \tag{32}$$

With a non-trivial amount of algebra, it is possible to derive from these the following expressions:

$$\mathcal{D}^i_\alpha\phi_j = \tfrac{1}{2}\delta^i_j(\mathcal{D}_\alpha\phi)\,, \qquad \overline{\mathcal{D}}_{\dot\alpha i}\phi_j = -\tfrac{1}{2}g_{ij}(\overline{\mathcal{D}}_{\dot\alpha}g\phi)\,,$$

$$\mathcal{D}^i_\alpha(\mathcal{D}_\beta\phi) = -4ig^{ij}\epsilon_{\alpha\beta}\frac{\partial}{\partial z}\phi_j + 2ig^{ij}\epsilon_{\alpha\beta}\overline{W}\phi_j\,,$$

$$\overline{\mathcal{D}}_{\dot\alpha i}(\overline{\mathcal{D}}_{\dot\beta}g\phi) = 4i\epsilon_{\dot\alpha\dot\beta}\frac{\partial}{\partial z}\phi_i + 2i\epsilon_{\dot\alpha\dot\beta}W\phi_i\,,$$

$$\mathcal{D}^i_\alpha(\overline{\mathcal{D}}_{\dot\beta}g\phi) = -4i\not{\mathcal{D}}_{\alpha\dot\beta}g^{ij}\phi_j\,,$$

$$\frac{\partial}{\partial z}(\mathcal{D}_\alpha\phi) = (\not{\mathcal{D}}\,\epsilon\bar{\mathcal{D}}g\phi)_\alpha - (\mathcal{D}^i_\alpha W)\,\phi_i - \tfrac{1}{2}W(\mathcal{D}_\alpha\phi)\,,$$

$$\frac{\partial}{\partial z}(\bar{\mathcal{D}}_{\dot\alpha}g\phi) = (\epsilon\bar{\not{\mathcal{D}}}\mathcal{D}\phi)_\alpha + (\bar{\mathcal{D}}_{\dot\alpha i}\bar{W})\,g^{ij}\phi_j + \tfrac{1}{2}\bar{W}(\bar{\mathcal{D}}_{\dot\alpha}g\phi)\,, \tag{33}$$

for ϕ_i, and the corresponding ones for $\bar{\phi}^i$:

$$\bar{\mathcal{D}}_{\dot\alpha i}\bar{\phi}^j = \tfrac{1}{2}\delta^j_i(\bar{\mathcal{D}}_{\dot\alpha}\bar{\phi})\,, \qquad \mathcal{D}^i_\alpha\bar{\phi}^j = -\tfrac{1}{2}g^{ij}(\mathcal{D}_\alpha g\bar{\phi})\,,$$

$$\bar{\mathcal{D}}_{\dot\alpha i}(\bar{\mathcal{D}}_{\dot\beta}\bar{\phi}) = 4i\epsilon_{\dot\alpha\dot\beta}g_{ij}\frac{\partial}{\partial z}\bar{\phi}^j + 2i\epsilon_{\dot\alpha\dot\beta}g_{ij}W\bar{\phi}^j\,,$$

$$\mathcal{D}^i_\alpha(\mathcal{D}_\beta g\bar{\phi}) = -4i\epsilon_{\alpha\beta}\frac{\partial}{\partial z}\bar{\phi}^i + 2i\epsilon_{\alpha\beta}\bar{W}\bar{\phi}^i\,,$$

$$\mathcal{D}^i_\alpha(\bar{\mathcal{D}}_{\dot\beta}\bar{\phi}) = -4i\not{\mathcal{D}}_{\alpha\dot\beta}\bar{\phi}^i\,,$$

$$\frac{\partial}{\partial z}(\bar{\mathcal{D}}_{\dot\alpha}\bar{\phi}) = (\epsilon\bar{\not{\mathcal{D}}}\mathcal{D}g\bar{\phi})_{\dot\alpha} + (\bar{\mathcal{D}}_{\dot\alpha i}\bar{W})\,\bar{\phi}^i + \tfrac{1}{2}\bar{W}(\bar{\mathcal{D}}_{\dot\alpha}\bar{\phi})\,,$$

$$\frac{\partial}{\partial z}(\mathcal{D}_\alpha g\bar{\phi}) = (\not{\mathcal{D}}\,\epsilon\bar{\mathcal{D}}\bar{\phi})_\alpha - (\mathcal{D}^i_\alpha W)\,g_{ij}\bar{\phi}^j - \tfrac{1}{2}W(\mathcal{D}_\alpha g\bar{\phi})\,. \tag{34}$$

From these again, we can derive the relations corresponding to condition (12):

$$\frac{\partial^2}{\partial z^2}\phi_i = \mathcal{D}^\mu\mathcal{D}_\mu\phi_i + \tfrac{1}{8}i(\bar{\mathcal{D}}_i\bar{W})\epsilon(\bar{\mathcal{D}}g\phi) + \tfrac{1}{8}i(g\mathcal{D}W)_i\epsilon(\mathcal{D}\phi)$$

$$+ \tfrac{1}{16}i(\mathcal{D}\epsilon\tau_A g\mathcal{D}W)(\tau_A\phi)_i + \tfrac{1}{2}\bar{W}\frac{\partial}{\partial z}\phi_i - \tfrac{1}{2}W\frac{\partial}{\partial z}\phi_i + \tfrac{1}{8}\{W,\bar{W}\}\,\phi_i\,, \tag{35}$$

$$\frac{\partial^2}{\partial z^2}\bar{\phi}^i = \mathcal{D}^\mu\mathcal{D}_\mu\bar{\phi}^i + \tfrac{1}{8}i(g\bar{\mathcal{D}}\bar{W})^i\epsilon(\bar{\mathcal{D}}\bar{\phi}) + \tfrac{1}{8}i(\mathcal{D}^iW)\epsilon(\mathcal{D}g\bar{\phi})$$

$$- \tfrac{1}{16}i(\mathcal{D}\epsilon\tau_A g\mathcal{D}W)(\bar{\phi}\tau_A)^i + \tfrac{1}{2}\bar{W}\frac{\partial}{\partial z}\bar{\phi}^i - \tfrac{1}{2}W\frac{\partial}{\partial z}\bar{\phi}^i + \tfrac{1}{8}\{W,\bar{W}\}\,\bar{\phi}^i\,. \tag{36}$$

As in the free-field case, it can be shown with the help of eqs. (33) and (34) that the gauge-invariant superfields

$$\mathcal{L}_m \equiv \tfrac{1}{24}(D\epsilon\tau_A gD - \bar{D}\epsilon g\tau_A\bar{D})(\bar{\phi}\tau_A\phi)\,,$$

$$\mathcal{L}_0 \equiv \frac{i}{48}(D\epsilon\tau_A gD - \bar{D}\epsilon g\tau_A\bar{D})\;\bar{\phi}\tau_A\frac{\overleftrightarrow{\partial}}{\partial z}\phi \tag{37}$$

can be expanded into

$$\mathcal{L}_m = i\bar{\phi}\frac{\overleftrightarrow{\partial}}{\partial z}\phi - \tfrac{1}{8}(\mathcal{D}g\bar{\phi})\epsilon(\mathcal{D}\phi) - \tfrac{1}{8}(\bar{\mathcal{D}}\bar{\phi})\epsilon(\bar{\mathcal{D}}g\phi) + \tfrac{1}{2}i\bar{\phi}(W - \bar{W})\phi\,,$$

$$\mathcal{L}_0 = -\tfrac{1}{2}\bar{\phi}\mathcal{D}^\mu\mathcal{D}_\mu\phi - \tfrac{1}{2}\mathcal{D}^\mu\mathcal{D}_\mu\bar{\phi}\phi + \frac{\partial}{\partial z}\bar{\phi}\frac{\partial}{\partial z}\phi - \tfrac{1}{16}i(\bar{\mathcal{D}}\bar{\phi})\overleftrightarrow{\bar{\not{\mathcal{D}}}}(\mathcal{D}\phi)$$

$$- \tfrac{1}{16}i(\mathcal{D}g\bar{\phi})\epsilon\overleftrightarrow{\not{\mathcal{D}}}\epsilon(\bar{\mathcal{D}}g\phi) - \tfrac{1}{8}i\bar{\phi}^i(\bar{\mathcal{D}}_i\bar{W})\epsilon(\bar{\mathcal{D}}g\phi) + \tfrac{1}{8}i(\mathcal{D}g\bar{\phi})\epsilon(\mathcal{D}^iW)\phi_i$$

$$- \tfrac{1}{8}i(\bar{\phi}g\mathcal{D}W)\epsilon\mathcal{D}\phi - \tfrac{1}{8}i(\bar{\mathcal{D}}\bar{\phi})\epsilon(\bar{\mathcal{D}}\bar{W}g\phi) - \tfrac{1}{16}i(\bar{\mathcal{D}}\bar{\phi})\epsilon\bar{W}(\bar{\mathcal{D}}g\phi)$$

$$+ \tfrac{1}{16}i(\mathcal{D}g\bar{\phi})\epsilon W(\mathcal{D}\phi) - \tfrac{1}{16}i\bar{\phi}^i(\mathcal{D}\epsilon\tau_A g\mathcal{D}W)(\tau_A\phi)_i - \tfrac{1}{8}\bar{\phi}\{W,\bar{W}\}\phi\,, \qquad (38)$$

and that their lowest components transform as four-divergences under supertransformations, and are thus candidates for terms in the matter Lagrangian, which becomes

$$L_{\text{matter}} = (\mathcal{L}_0 + m\,\mathcal{L}_m)_{\theta=\bar{\theta}=z=0}$$

$$= -\tfrac{1}{2}A^{+i}\mathcal{D}^\mu\mathcal{D}_\mu A_i - \tfrac{1}{2}\mathcal{D}^\mu\mathcal{D}_\mu A^{+i}A_i + F^{+i}F_i + \frac{i}{4}\varphi\epsilon\overleftrightarrow{\not{\mathcal{D}}}\epsilon\bar{\varphi}$$

$$- \frac{i}{4}\bar{\psi}\overleftrightarrow{\bar{\not{\mathcal{D}}}}\psi + iA^{+i}\bar{\lambda}_i\epsilon\bar{\varphi} - iA^{+i}g_{ij}\lambda^j\epsilon\psi - i\bar{\psi}\epsilon\bar{\lambda}_i g^{ij}A_j + i\varphi\epsilon\lambda^i A_i$$

$$+ i\bar{\psi}\epsilon M^+\bar{\varphi} + i\varphi\epsilon M\psi + A^{+i}C_A(\tau_A A)_i - 2A^{+i}\{M, M^+\}A_i$$

$$+ imA^{+i}F_i - imF^{+i}A_i - \tfrac{1}{2}m\varphi\epsilon\psi + \tfrac{1}{2}m\bar{\psi}\epsilon\bar{\varphi} + 2imA^{+i}(M - M^+)A_i\,, \qquad (39)$$

where we have given the following names to the lowest components of the respective superfields:

$$\phi_i = A_i + \ldots\,, \qquad \bar{\phi}^i = A^{+i} + \ldots\,,$$

$$\tfrac{1}{2}\mathcal{D}^i_\alpha\phi_i = \psi_\alpha + \ldots\,, \qquad \tfrac{1}{2}\bar{\mathcal{D}}_{\dot{\alpha}i}\bar{\phi}^i = \bar{\psi}_{\dot{\alpha}} + \ldots\,,$$

$$\tfrac{1}{2}\bar{\mathcal{D}}_{\dot{\alpha}}g\phi_i = -\bar{\varphi}_{\dot{\alpha}} + \ldots\,, \qquad \tfrac{1}{2}\mathcal{D}_\alpha g\bar{\phi} = \varphi_\alpha + \ldots\,,$$

$$\frac{\partial}{\partial z}\phi_i = F_i + \ldots\,, \qquad \frac{\partial}{\partial z}\bar{\phi}^i = F^{+i} + \ldots\,,$$

$$W = 4M + \ldots\,, \qquad \bar{W} = 4M^+ + \ldots\,,$$

$$\mathcal{D}^i_\alpha W = 4\lambda^i_\alpha + \ldots\,, \qquad \bar{\mathcal{D}}_{\dot\alpha i}\bar{W} = 4\bar\lambda_{\dot\alpha i} + \ldots\,,$$

$$\mathcal{D}\epsilon\tau_A g\mathcal{D}W = 16iC_A + \ldots\,, \qquad F_{\mu\nu} = G_{\mu\nu} + \ldots\,. \tag{40}$$

Note that the assumption $\bar{W} = W^+$ restricts the gauge freedom to real Λ. The supersymmetric equation of motion is still (23), but iteration now yields the complicated equations for an interacting theory, using eq. (35).

The Lagrangian for the Yang-Mills multiplet has already been given in ref. [8]. It is the lowest component of the superfield

$$\begin{aligned}\mathcal{L}_{\rm YM} &= -\frac{1}{3\cdot 32\cdot 32}(D\epsilon\tau_A gDD\epsilon\tau_A gD\,\mathrm{Tr}\,W^2 + \text{h.c.})\\
&= -\tfrac{1}{4}\mathrm{Tr}\,F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{16}\mathrm{Tr}(W\mathcal{D}^\mu\mathcal{D}_\mu\bar{W} + \bar{W}\mathcal{D}^\mu\mathcal{D}_\mu W)\\
&\quad -\frac{i}{32}\mathrm{Tr}\,\bar{\mathcal{D}}\bar{W}\overset{\leftrightarrow}{\bar{\not{\mathcal{D}}}}\mathcal{D}W - \frac{1}{16\cdot 32}\mathrm{Tr}(\mathcal{D}\epsilon\tau_A g\mathcal{D}W)^2 - \frac{1}{4\cdot 32}\mathrm{Tr}[\bar{W}, W]^2\\
&\quad +\frac{i}{2\cdot 32}\mathrm{Tr}\,\mathcal{D}W\epsilon g[\mathcal{D}W, \bar{W}] + \frac{i}{2\cdot 32}\mathrm{Tr}\,\bar{\mathcal{D}}\bar{W}\epsilon g[\bar{\mathcal{D}}\bar{W}, W]\,,\end{aligned} \tag{41}$$

i.e.,

$$\begin{aligned}L_{\rm YM} &= \mathcal{L}_{\rm YM}|_{\theta=\bar\theta=0}\\
&= -\tfrac{1}{4}\mathrm{Tr}\,G^{\mu\nu}G_{\mu\nu} - \mathrm{Tr}(M\mathcal{D}^\mu\mathcal{D}_\mu M^+ + M^+\mathcal{D}^\mu\mathcal{D}_\mu M)\\
&\quad -\tfrac{1}{2}i\,\mathrm{Tr}\,\bar\lambda\overset{\leftrightarrow}{\bar{\not{\mathcal{D}}}}\lambda + \tfrac{1}{2}\mathrm{Tr}\,C_A C_A - 2\,\mathrm{Tr}[M^+, M]^2\\
&\quad + i\,\mathrm{Tr}\,\lambda\epsilon g[\lambda, M^+] + i\,\mathrm{Tr}\,\bar\lambda\epsilon g[\bar\lambda, M]\,,\end{aligned} \tag{42}$$

and the supersymmetric equation of motion for the fields of the Yang-Mills multiplet can be derived from the total Lagrangian

$$L = L_{\rm YM} + L_{\rm matter} \tag{43}$$

to be

$$\tfrac{1}{16}\mathcal{D}\epsilon\tau_A g\mathcal{D}W = -i(\tau_A\phi)_i\bar\phi^i\,. \tag{44}$$

6. SU(4) invariance

Fayet [6] has shown that our Lagrangian can be obtained from an ordinary supersymmetric model with only one supersymmetry. It should contain two massive chiral multiplets ϕ and ϕ' of opposite gauge transformation properties, and a massless chiral multiplet S in the adjoint representation of the gauge group. If the gauge-invariant cubic interaction term

$$L_{\text{cub}} = h\phi' S\phi|_F + \text{h.c.}$$

is added to the otherwise minimal couplings, then one could rewrite the Lagrangian to give our L, *if* the coupling constant h is a certain multiple of the gauge coupling constant. While for arbitrary h the theory was symmetric only under one supersymmetry and a U(1) group (the so-called "R-transformation"), we now have two supersymmetries, a non-trivial SU(2), and the Z-transformation.

An obvious question now to ask is: does our L possess a higher symmetry still for some special value of the only parameter m, and some special representation of the gauge group for the matter fields? The answer is yes. If ϕ_i and $\bar\phi^i$ both transform under the adjoint representation, and if we introduce the quantities

$$\begin{aligned}
&\chi_{\alpha 1} = \lambda^1_\alpha\,, && \bar\chi^1_{\dot\alpha} = \bar\lambda_{\dot\alpha 1}\,,\\
&\chi_{\alpha 2} = \lambda^2_\alpha\,, && \bar\chi^2_{\dot\alpha} = \bar\lambda_{\dot\alpha 2}\,,\\
&\chi_{\alpha 3} = \sqrt{\tfrac{1}{2}}\psi_\alpha\,, && \bar\chi^3_{\dot\alpha} = \sqrt{\tfrac{1}{2}}\bar\psi_{\dot\alpha}\,,\\
&\chi_{\alpha 4} = \sqrt{\tfrac{1}{2}}\varphi_\alpha\,, && \bar\chi^4_{\dot\alpha} = \sqrt{\tfrac{1}{2}}\bar\varphi_{\dot\alpha}\,,\\
&M_{12} = iM\,, && M_{23} = -\sqrt{\tfrac{1}{2}}iA_1\,,\\
&M_{13} = \sqrt{\tfrac{1}{2}}iA_2\,, && M_{24} = \sqrt{\tfrac{1}{2}}iA^{+2}\,,\\
&M_{14} = \sqrt{\tfrac{1}{2}}iA^{+1}\,, && M_{34} = -iM^+\,.\\
&M_{ji} = -M_{ij}\,,
\end{aligned} \tag{46}$$

then M_{ij} has the reality properties of the $\underline{6}$ of SU(4),

$$(M_{ij})^+ = \tfrac{1}{2}\epsilon^{ijkl}M_{kl} \equiv M^{ij}\,, \tag{47}$$

and the obviously SU(4)-invariant Lagrangian

$$\begin{aligned}
L = &-\tfrac{1}{4}\mathrm{Tr}\, G^{\mu\nu}G_{\mu\nu} - \tfrac{1}{2}\mathrm{Tr}\, M_{ij}\mathcal{D}^\mu\mathcal{D}_\mu M^{ij} - \tfrac{1}{2}i\,\mathrm{Tr}\,\bar\chi^i \overleftrightarrow{\not{\mathcal{D}}}\chi_i\\
&+ \mathrm{Tr}\,\bar\chi^i\epsilon[\bar\chi^j, M_{ij}] + \mathrm{Tr}\,\chi_i\epsilon[\chi_j, M^{ij}] + \tfrac{1}{4}\mathrm{Tr}[M_{ij}, M_{kl}][M^{ij}, M^{kl}]
\end{aligned} \tag{48}$$

can be shown to be equal to our L, for $m = 0$ and after the auxiliary fields F_i and

C_A have been eliminated. The Lagrangian (48) as that of an SU(4)-extended supersymmetric gauge theory has been known for a long time [7,3].

According to the argument at the beginning of this section, the Lagrangian (48) can be interpreted as a non-extended supersymmetric model with three massless chiral multiplets in the adjoint representation, and with an additional f-coupling. This term is essential for the vanishing of the β-function up to second loop order [11], for while Ferrara and Zumino [4] have shown its vanishing in first order, it has been shown, that second order will be positive for any combination of only mass terms or d-couplings [14].

7. Conclusions

In sect. 6, we have seen how the equations of motion (in particular $(\partial/\partial z)\phi_i = 0$), established the larger $N = 4$ supersymmetry and at the same time did away with the central charge. Thus we have found a complete superfield formulation of a theory which has the spectrum and the physical properties of the interesting $N = 4$ case. It should be pointed out, however, that the theory does not possess the larger symmetry off-shell and that we have not succeeded in finding the complete auxiliary-field structure of the SU(4) case. However, once Feynman diagram techniques for superfield gauge theories have been developed, it may yet become possible to more easily obtain further information about the β-function.

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