11/91 BUHEP-91-21

QUANTUM DEFORMATIONS OF $D = 4$ POINCARÉ AND CONFORMAL ALGEBRAS

Jerzy LUKIERSKI[*)]
Institute for Theoretical Physics, University of Wrocław,
ul. Cybulskiego 36, 50205 Wrocław, Poland

Anatol NOWICKI[*)**)]
Laboratoire de Physique Theorique,
Université de Bordeaux I, rue du Solarium,
F-33175 Gradignan, France

Henri RUEGG[***) †)]
Department of Physics, Boston University,
590 Commonwealth Ave, Boston MA 02215, USA

Content:

---

*) Partially supported by KBN grant Nr 2/0124/91/01

**) On leave of absence from Institute of Physics, Pedagogical University, Plac Slowiański 6, 65029 Zielona Góra, Poland

***) Partially supported by the Swiss National Science Foundation and the US Department of Energy, Contract No. DE-AC02-89ER40509.

†) On leave of absence from Departement de Physique Theorique, Universite de Geneve, 24 quai Ernest-Ansermet, CH-1211 Geneve 4, Switzerland

1. INTRODUCTION

Quantum deformations of Lie groups and Lie algebras (see e.g. [1-6])[1)] which appeared firstly as a byproduct of the quantum inverse scattering method [7,8] found recently many applications in both theoretical physics and mathematics. At present quantum groups appear mostly as new type of internal symmetries (see e.g. D=2 conformal field theory [9], with space-time Virasoro algebra remaining unmodified), and does not imply the generalization of standard scheme for quantum theories. It is interesting, however to take more radical point of view and to explore the quantum deformations as describing the fundamental space-time symmetries. In particular it would be desired to restrict the modifications only to very short distances, in accordance with present models of fundamental interactions based on strings, membranes etc. which are assumed to be the objects with the extension of the Planck length. Motivated by such ideas recently the formalism of quantum groups and quantum algebras[1)] has been applied to the description of quantum deformations of $D = 4$ Lorentz group and $D = 4$ Lorentz algebra [10-15]. In order to obtain the quantum deformation of semisimple Lie algebras describing Minkowski or Euclidean group of motions mostly the contraction techniques have been used. In particular there were obtained:

a) quantum deformation of $D = 2$ and $D = 3$ Euclidean and Minkowski geometries, described as quantum Lie algebra or quantum Lie group [16,17].

b) quantum deformation of $D = 4$ Poincaré algebra [18,19].

c) quantum deformations of $D = 4$ conformal algebra [20,21].

We should also note here that the inhomogeneous quantum groups were studied by contracting the Gelfand - Tzetlin basis of simple quantum Lie algebra [22] as well as by introducing the inhomogeneous quantum groups as matrix quasitriangular Hopf algebras with noncommuting entries [23][2)].

The main aim of this lecture is to present the results of our papers [18,21], where the quantum deformation of $D = 4$ Poincaré algebra is obtained by contracting the Cartan-Weyl basis of a real form $U_q(O(3,2))$ of the quantum complex $D = 4$ de-Sitter algebra $U_q(Sp(4;C))$ [18] as well as by contracting the Cartan-Weyl basis of a real form $U_q(O(4,2))$ of the quantum complex $D = 4$ conformal algebra $U_q(Sl(4;C))$ [21]. We shall describe quantum $D = 4$ Poincaré algebra as Hopf bialgebras, and the contraction limit (denoted by $\{{R\to\infty \atop q\to 1}\}$) which provides finite quantum inhomogeneous algebras is defined as follows:

$$\left\{ \begin{matrix} R \to \infty \\ q \to 1 \end{matrix} \right\}_l : \qquad R \to \infty; \qquad R^l \, log \, q \to i^\epsilon \kappa^{-l} \qquad (0 < \kappa < \infty) \tag{1.1}$$

where $l = 1$ for the contraction of $U_q(O(3,2))$ [18] and $l = 2$ for the contraction of $U_q(Sl(4;C))$; $\epsilon = 0$ for q real, and $\epsilon = 1$ for $|q| = 1$ [3)]. We see that the prescription (1.1) provides a particular way of approaching the limit $q \to 1$ in such a way that appears a new mass- like parameter κ. In this lecture we shall show that the limit (1.1) provides for finite κ a nonlinear modification of the Poincaré algebra("κ - deformation") accompanied by the consistent modification of coproduct and antipode formulae.

The quantum-algebraic structures before the contraction we shall present in Sect. 2 and 3. It should be stressed that for the quantum algebras of rank ≥ 2 physically more important seems to be the Cartan-Weyl basis, which has the following two advantages over the Cartan-Chevalley basis

i) The q-deformed Cartan-Weyl basis describes the quantum deformation of Lie algebra generators (i.e. in the limit $q \to 1$ one obtains the ordinary Lie algebra relations)

ii) The nonlinear (three-linear or fourlinear for classical Lie algebras) Serre relations are replaced by bilinear relations.

In Sect.2 we shall present the Cartan-Weyl basis for $U_q(Sp(4;C))$ as well as we shall introduce its particular real form, described by inner involution of Cartan-Weyl basis. It appears that not every involution of Cartan-Chevalley basis can be extended to the full Cartan-Weyl basis. We have shown that one can introduce 16 inner involutions of Cartan-Weyl basis of $U_q(Sp(4;C))$ [19], but only two seem to be suitable as candidates providing after contraction (1.1) the q-deformation of Poincaré algebra - one with $|q| = 1$ and second with q real. Similarly, in Sect. 3 we shall present the q-deformation of the Cartan-Weyl basis of $U_q(Sl(4;C))$ and consider its two real forms $U_q^{(i)}(O(4,2))$ $(i = 1,2)$. It appears (see Sect.4) that the real form $U_q^{(1)}(O(4,2))$ contains as its quantum subalgebras (Hopf subalgebras) the quantum Lorentz algebra $U_q(O(3,1))$ as well as quantum Weyl algebra $U_q(\mathcal{P}_4 \oplus D)$, where $\mathcal{P}_4 \oplus D$ denotes Weyl algebra ($\mathcal{P}_4$ - Poincaré algebra, D-dilatation generator). Further, in Sect. 5, using the definition of quantum contractions (1.1) with $l = 1$ (for $U_q(O(3,2))$ and $l = 2$ (for $U_q(O(4,2))$ we introduce two κ-deformations of $D = 4$ Poincaré algebra [18,21]. Short discussion is presented in the last Section.

2. QUANTUM COMPLEX $D = 4$ DE SITTER ALGEBRA $U_q(Sp(4))$ AND ITS REAL FORMS

The root space of the Lie algebra $C_2 \equiv Sp(4)$ is spanned by the simple roots

$$\alpha_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \qquad \alpha_2 = (0, \sqrt{2}) \tag{2.1}$$

Their scalar product $\alpha_{ij} = <\alpha_i, \alpha_j>$ $(i = 1,2)$ leads to the following symmetrized Cartan matrix

$$\alpha = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \tag{2.2}$$

The corresponding generators of C_2 will be denoted by $e_{\pm\alpha_i} \equiv e_{\pm i}$ and h_i.

The Drinfeld-Jimbo procedure, valid for simple Lie algebras, yields the following q-deformation $U_q(Sp(4))$ of $Sp(4)$ in the Cartan-Chevalley basis ($[x] \equiv (q-q^{-1})^{-1} \cdot (q^x - q^{-x})$):

$$
\begin{gathered}
[e_i, e_{-j}] = \delta_{ij} [h_i]_q \\
\\
[h_i, e_{\pm j}] = \pm \alpha_{ij} e_{\pm j} \qquad [h_i, h_j] = 0
\end{gathered}
\tag{2.3}
$$

restricted also by the q-Serre relations

$$
\begin{aligned}
\left[e_{\pm 1} \left[e_{\pm 1} \left[e_{\pm 1}, e_{\pm 2}\right]_{q^{\mp 1}}\right]_{q^{\pm 1}}\right]_{q^{\pm 1}} &= 0 \\
\left[e_{\pm 2} \left[e_{\pm 2}, e_{\pm 1}\right]_{q^{\mp 1}}\right]_{q^{\mp 1}} &= 0
\end{aligned}
\tag{2.4}
$$

where $[e_\alpha, e_\beta]_{q^{\pm 1}} \equiv e_\alpha e_\beta - q^{\mp <\alpha,\beta>} e_\beta e_\alpha$

In order to get a quantum Lie algebra one has to specify the coproducts

$$
\begin{aligned}
\Delta(h_i) &= h_i \otimes 1 + 1 \otimes h_i \\
\Delta(e_{\pm i}) &= e_{\pm i} \otimes q^{\frac{h_i}{2}} + q^{-\frac{h_i}{2}} \otimes e_{\pm i}
\end{aligned}
\tag{2.5}
$$

and antipodes

$$
S(h_i) = -h_i \qquad S(e_{\pm i}) = -q^{\pm \frac{1}{2} d_i} e_{\pm i}
\tag{2.6}
$$

where $d_i =< \alpha_i, \alpha_i >= (1,2)$.

We now shall complete the Cartan-Chevalley basis by introducing the generators corresponding to the nonsimple roots $\alpha_3 = \alpha_1 + \alpha_2$ and $\alpha_4 = 2\alpha_1 + \alpha_2$. It is very convenient to introduce a normal order of the roots, (for example $\alpha_1, \alpha_4, \alpha_3, \alpha_2$) and define nonsimple generators belonging to a normal order $(\alpha, \alpha+\beta, \beta)$ by [25-28]

$$
\begin{aligned}
e_{\alpha+\beta} &= e_\alpha e_\beta - q^{-<\alpha,\beta>} e_\beta e_\alpha \\
e_{-\alpha-\beta} &= e_{-\beta} e_{-\alpha} - q^{<\alpha,\beta>} e_{-\alpha} e_{-\beta}
\end{aligned}
\tag{2.7}
$$

in such a way that

$$
[e_{\alpha+\beta}, e_{-\alpha-\beta}] = [h_{\alpha+\beta}]_q \equiv [h_\alpha + h_\beta]_q
\tag{2.8}
$$

For the normal order which we have chosen, we get (defining now $[A,B]_q \equiv AB - qBA$)

$$
\begin{aligned}
e_3 = [e_1, e_2]_q \qquad & e_{-3} = [e_{-2}, e_{-1}]_{q^{-1}} \\
e_4 = [e_1, e_3] \qquad & e_4 = [e_{-3}, e_{-1}]
\end{aligned}
\tag{2.9}
$$

With these definitions, the q-Serre relations (2.4) take the linear form reminiscent of the usual Lie-algebra relations

$$
\begin{aligned}
[e_1, e_4]_{q^{-1}} \doteq 0, \qquad [e_{-3}, e_{-2}]_q = 0, \qquad [e_3, e_2]_{q^{-1}} = 0 \\
[e_4, e_3]_{q^{-1}} = 0, \qquad [e_{-4}, e_{-1}]_q = 0, \qquad [e_{-3}, e_{-4}]_q = 0
\end{aligned}
\tag{2.10}
$$

(2.3), (2.9) and (2.10) define the Cartan-Weyl basis.

The complete set of commutation relations for $U_q(Sp(4))$ were given in [18]. The coproduct and antipode are calculated using the fact that Δ is an automorphism of the algebra and S an antiautomorphism [ref 1-3].

Our aim is to define the "physical" basis for $U_q(Sp(4))$ consisting of the 10 generators $M_{AB} = -M_{BA}$ $(A, B = 0,1,2,3,4)$ in terms of the Cartan-Weyl basis. These relations will depend on the choice of real forms, which we will now discuss.

For a Hopf algebra $\mathring{A}$ over C with comultiplication Δ and antipode S one can distinguish four types of involutive homomorphisms in $\mathring{A}$:

i) The $+$ involution, which is an antiautomorphism in the algebra sector, and an automorphism in the coalgebra sector, i.e. ($\alpha_i \in \mathring{A}$):

$$(\alpha_1 \cdot \alpha_2)^+ = \alpha_2^+ \alpha_1^+ \qquad (\Delta(\alpha))^+ = \Delta(\alpha^+) \tag{2.11}$$

ii) the $*$ involution, which is an automorphism in the algebra sector, and an antiautomorphism in the coalgebra sector, i.e.

$$(\alpha_1 \cdot \alpha_2)^* = \alpha_1^* \alpha_2^* \qquad (\Delta(\alpha))^* = \Delta'(\alpha^*) \tag{2.12}$$

where $\Delta' = P\Delta = R\Delta R^{-1}$ ($P =$ permutation operator in $(\mathring{A} \otimes \mathring{A})$).

These two involutions are required to satisfy ($\times \equiv +$ or $*$)

$$S((S(a^\times))^\times) = a \Leftrightarrow S \circ \times = \times \circ S^{-1} \tag{2.13}$$

iii) The $\oplus$ involution, which is an antiautomorphism in both algebra and coalgebra sectors, i.e.

$$(\alpha_1 \cdot \alpha_2)^\oplus = (a_2)^\oplus \cdot (a_1)^\oplus \qquad (\Delta(\alpha))^\oplus = \Delta'(a^\oplus) \tag{2.14}$$

iv) The $\circledast$ involution, which is an automorphism in both algebra and coalgebra sectors, i.e.

$$(\alpha_1 \cdot \alpha_2)^\circledast = (\alpha_1)^\circledast \cdot (\alpha_2)^\circledast \qquad (\Delta(\alpha))^\circledast = \Delta(\alpha^\circledast) \tag{2.15}$$

These two involutions are assumed to satisfy ($\otimes \equiv \oplus$ or $\otimes$)

$$S^{-1}((S(a^\otimes))^\otimes = \alpha \Leftrightarrow S \circ \otimes = \otimes \circ S \tag{2.16}$$

These four types of involutions are implemented by their action on the Cartan-Weyl basis of $U_q(Sp(4))$ in the following way (for details see [19])

$$\begin{array}{ll} h'_i = \rho h_i & e'_{\pm 1} = \lambda e_{m1} \\ e'_{\pm 2} = \epsilon e_{m2} & e'_{\pm 4} = \epsilon q^{\mp n} e_{m4} \\ e'_{\pm 3} = \rho\lambda\epsilon q^{\mp n} e_{m3} & \end{array} \tag{2.17}$$

The various parameters in (2.17) take the following values:

$$\begin{array}{llllllll} + & i) & |q|=1 & \rho=-1 & m=\pm1 & n=1 & (\Delta_\pm\to\Delta_\pm) & \\ * & ii) & q\in\mathbb{R} & \rho=-1 & m=\mp1 & n=-1 & (\Delta_\pm\to\Delta_\mp) & \\ \oplus & iii) & |q|=1 & \rho=1 & m=\mp1 & n=0 & (\Delta_\pm\to\Delta_\mp) & \\ \circledast & iv) & q\in\mathbb{R} & \rho=1 & m=\pm1 & n=0 & (\Delta_\pm\to\Delta_\pm) & \end{array} \tag{2.18}$$

Further we shall choose $\oplus$-involution, corresponding to $\rho=\lambda=1$ and $\epsilon=-1$. From (2.17-18) follows that

$$\begin{array}{ll} h_1^\oplus = h_1 & h_2^\oplus = h_2 \\ e_1^\oplus = e_{-1} & e_2^\oplus = -e_{-2} \\ e_3^\oplus = -e_{-3} & e_4^\oplus = -e_{-4} \end{array} \tag{2.19}$$

and the $O(3,2)$ generators $(A,B=0,1,2,3,4)$

$$\begin{array}{ll} M_{12} = h_1 & M_{23} = \frac{1}{\sqrt{2}}(e_1+e_{-1}) \\ M_{31} = \frac{1}{i\sqrt{2}}(e_1-e_{-1}) & M_{04} = h_3 = h_1+h_2 \\ M_{34} = -\frac{1}{\sqrt{2}}(e_3-e_{-3}) & M_{03} = \frac{1}{i\sqrt{2}}(e_3+e_{-3}) \\ M_{02} = \frac{1}{2}(e_4-e_{-4}+e_2-e_{-2}) & M_{01} = -\frac{1}{2i}(e_4+e_{-4}-e_2-e_{-2}) \\ M_{24} = \frac{1}{2i}(e_4+e_{-4}+e_2+e_{-2}) & M_{14} = \frac{1}{2}(e_4-e_{-4}-e_2+e_{-2}) \end{array} \tag{2.20}$$

satisfy the relation $M_{AB}^\oplus = -M_{AB}$ and describe the $D=4$ anti-de-Sitter algebra.

$$[M_{AB}, M_{CD}] = i\,(\eta_{BC}M_{AD} - \eta_{AC}M_{BD} + \eta_{AD}M_{BC} - \eta_{BD}M_{AC}) \tag{2.21}$$

where $\eta_{AB} = diag(+,-,-,-,+)$.

The formulae describing Cartan-Weyl basis of $U_q(O(3,2))$ in terms of physical generators (2.20) can be found in [18]. We recall that for the real form (2.19) we should assume that $|q|=1$.

3. QUANTUM COMPLEX $D=4$ CONFORMAL ALGEBRA $U_q(Sl(4))$ AND ITS REAL FORMS

In order to describe the q-deformation of $SU(2,2)\simeq O(4,2)$ we introduce firstly the Cartan-Chevalley basis for $U_q(SL(4;C))$ $(i,j=1,2,3;\ [x]_q=(q-q^{-1})^{-1}(q^x-q^{-x}))$: describing quantum complexified $D=4$ conformal algebra:

$$\begin{array}{l} [h_i,h_j]=0 \\ [h_i,e_{\pm j}] = \pm a_{ij}e_{\pm j} \\ [e_i,e_{-j}] = \delta_{ij}[h_i]_q \end{array} \qquad a_{ij} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \tag{3.1}$$

where h_i describe the Cartan subalgebra, and e_i, e_{-i} ($i = 1, 2, 3$) the generators corresponding to simple roots. The generators corresponding to nonsimple roots are defined as follows (for the general scheme see [25-28]):

$$\begin{aligned} e_4 &= [e_1, e_2]_q & e_{-4} &= [e_{-2}, e_{-1}]_{q^{-1}} \\ e_5 &= [e_2, e_3]_q & e_{-5} &= [e_{-3}, e_{-2}]_{q^{-1}} \\ e_6 &= [e_1, e_5]_q & e_{-6} &= [e_{-5}, e_{-1}]_{q^{-1}} \end{aligned} \tag{3.2}$$

where $[A, B]_x \equiv AB - xBA$.

The relations (3.1) are extended to the generators (3.2) in the following way:

$$\begin{aligned} [e_4, e_{-4}] &= [h_1 + h_2]_q \equiv [h_4]_q \\ [e_5, e_{-5}] &= [h_2 + h_3] \equiv [h_5]_q \\ [e_6, e_{-6}] &= [h_1 + h_2 + h_3]_q \equiv [h_6]_q \end{aligned} \tag{3.3}$$

and $h_4 = h_1 + h_2$,$h_5 = h_2 + h_3$, $h_6 = h_1 + h_2 + h_3$, as well as ($\alpha = 4, 5, 6$)

$$[h_i, e_{\pm\alpha}] = \pm a_{i\alpha} e_{\pm\alpha} \tag{3.4a}$$

where

$$a_{i\alpha} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{3.4b}$$

The q-Serre relations produce the following collection of bilinear formulae

$$\begin{array}{lll} [e_1, e_2]_q = e_4 & [e_1, e_3] = 0 & [e_1, e_4]_{q^{-1}} = 0 \\ [e_1, e_5]_q = e_6 & & [e_1, e_6]_{q^{-1}} = 0 \\ [e_2, e_3]_q = e_5 & & [e_2, e_5]_{q^{-1}} = 0 \\ [e_2, e_4]_q = 0 & [e_3, e_5]_q = 0 & [e_2, e_6] = 0 \\ [e_4, e_3]_q = e_6 & & [e_3, e_6]_q = 0 \end{array} \tag{3.5}$$

which can be supplemented by

$$\begin{gathered} [e_4, e_5] = (q - q^{-1}) e_6 e_2 \\ [e_4, e_6]_{q^{-1}} = 0 \qquad [e_5, e_6]_q = 0 \end{gathered} \tag{3.6}$$

Further we obtain

$$\begin{array}{ll} [e_1, e_{-5}] = 0 & \\ [e_2, e_{-4}] = e_{-1} q^{h_2} & [e_2, e_{-6}] = 0 \\ [e_3, e_{-5}] = e_{-2} q^{h_3} & [e_3, e_{-6}] = e_{-4} q^{h_3} \\ [e_4, e_{-1}] = -e_2 q^{h_1} & [e_4, e_{-3}] = 0 \\ [e_4, e_{-5}] = (q - q^{-1}) q^{-h_2} e_{-3} e_1 & \\ [e_5, e_1] = -e_3 q^{h_2} & [e_5, e_{-6}] = e_{-1} q^{h_2 + h_3} \\ [e_6, e_{-1}] = -e_5 q^{h_1} & [e_6, e_{-4}] = -e_3 q^{h_1 + h_2} \end{array} \tag{3.7}$$

If we add to the relations (3.5-3.7) the conjugated ones ($h_i \to h_i, e_{\pm i} \to e_{\mp i}, q \to q^{-1}$) we obtain the q-deformation of the complete Cartan-Weyl basis of $U_q(SL(4;C))$.

In order to describe $U_q(SL(4;C))$ as quantum bialgebra we introduce the formulae for coproduct ($i = 1,2,3$)

$$\begin{aligned}\Delta(e_{\pm i}) &= e_{\pm i} \otimes k_i + k_i^{-1} \otimes e_{\pm i}\\ \Delta(k_i^{\pm 1}) &= k_i^{\pm 1} \otimes k_i^{\pm 1}\end{aligned} \tag{3.8a}$$

and further one gets

$$\begin{aligned}\Delta(e_4) &= e_4 \otimes k_4 + k_4^{-1} \otimes e_4 + (q^{-1} - q)k_2^{-1} e_1 \otimes e_2 k_1\\ \Delta(e_{-4}) &= e_{-4} \otimes k_4 + k_4^{-1} \otimes e_{-4} + (q - q^{-1})k_1^{-1} e_{-2} \otimes e_{-1} k_2\\ \Delta(e_5) &= e_5 \otimes k_5 + k_5^{-1} \otimes e_5 + (q^{-1} - q)k_3^{-1} e_2 \otimes e_3 k_2\\ \Delta(e_{-5}) &= e_{-5} \otimes k_5 + k_5^{-1} \otimes e_{-5} + (q - q^{-1})k_2^{-1} e_3 \otimes e_{-2} k_{-3}\\ \Delta(e_6) &= e_6 \otimes k_6 + k_6^{-1} \otimes e_6 + (q^{-1} - 1)\cdot\\ &\cdot \left\{k_5^{-1} e_1 \otimes e_5 k_1 + k_3^{-1} e_4 \otimes e_3 k_4\right\}\\ \Delta(e_{-6}) &= e_{-6} \otimes k_6 + k_6^{-1} \otimes e_{-6} + (q - q^{-1})\cdot\\ &\cdot \left\{k_1^{-1} e_{-5} \otimes e_{-1} k_5 + k_4^{-1} e_{-3} \otimes e_{-4} k_3\right\}\end{aligned} \tag{3.8b}$$

where $k_A = q^{\frac{1}{2}h_A}$ $(A = 1\ldots 6)$. The formulae for antipodes of the Cartan-Chevalley basis

$$S(e_{\pm i}) = -q^{\pm 1} e_{\pm i} \qquad S(k_i^{\pm 1}) = k_i^{\mp 1} \tag{3.9a}$$

are extended to the generators (2.2) as follows:

$$\begin{aligned}S(e_{\pm 4}) &= q^{\pm 2}\tilde{e}_{\pm 4} \qquad & S(e_{\pm 5}) &= q^{\pm 2}\tilde{e}_{\pm 5}\\ & & S(e_{\pm 6}) &= -q^{\pm 3}\tilde{e}_{\pm 6}\end{aligned} \tag{3.9b}$$

where

$$\begin{aligned}\tilde{e}_4 &= [e_2, e_1]_q \qquad & \tilde{e}_{-4} &= [e_{-1}, e_{-2}]_{q^{-1}}\\ \tilde{e}_5 &= [e_3, e_2]_q & \tilde{e}_{-5} &= [e_{-2}, e_{-3}]_{q^{-1}}\\ \tilde{e}_6 &= [e_3, \tilde{e}_4]_q & \tilde{e}_{-6} &= [\tilde{e}_{-4}, e_{-3}]_{q^{-1}}\end{aligned} \tag{3.10}$$

We see therefore that antipodes describe outer automorphism of the Cartan-Weyl basis.

In order to describe real quantum conformal algebra $U_q(O(4,2))$ we should restrict the Cartan-Weyl basis of $U_q(SL(4;C))$ by the reality conditions. We shall consider here the following two $\oplus$-involutions [19], describing an antiautomorphism in both algebra and coalgebra sectors:

i) First $\oplus$-involution: q real

$$\begin{aligned}h_1^{\oplus} &= -h_3 \qquad & h_2^{\oplus} &= -h_2\\ e_{\pm 1}^{\oplus} &= e_{\pm 3} & e_{\pm 2}^{\oplus} &= e_{\pm 2}\\ e_{\pm 4}^{\oplus} &= e_{\pm 5} & e_{\pm 6}^{\oplus} &= e_{\pm 6}\end{aligned} \tag{3.11}$$

The q-deformation $U_q^{(1)}(O(4,2))$ corresponding to the choice (3.11) of reality conditions will be described in Sect. 3.

2) second $\oplus$-involution: $|q|=1$

$$\begin{aligned}
&h_i^{\oplus} = h_i && i = 1,2,3 && \\
&e_1^{\oplus} = e_{-1} && e_2^{\oplus} = -e_{-2} && e_3^{\oplus} = e_{-3} \\
&e_4^{\oplus} = -e_{-4} && e_5^{\oplus} = -e_{-5} && e_6^{\oplus} = -e_{-6}
\end{aligned} \tag{3.12}$$

The q-deformation $U_q^{(2)}(O(4,2))$ corresponding to the choice (3.12) will be contracted in Sect. 4 in order to obtain the quantum deformation of the Poincaré algebra with mass-like deformation parameter κ.

Unfortunately we were not able to find a genuine $\star$-operation (+-involution, which is the antiautomorphism in algebra sector and automorphism in coalgebra sector) which provides a quantum deformation of $O(4,2) \simeq SU(2,2)$.

4. QUANTUM $D = 4$ LORENTZ AND WEYL ALGEBRA.

a) Lorentz quantum algebra.

Let us introduce the generators of the Lorentz group as follows:

$$\begin{aligned}
M_+ &= M_{23} + iM_{31} = e_1 + e_{-3} \\
M_- &= M_{23} - iM_{31} = -(e_3 + e_{-1}) \\
M_3 &= \frac{i}{2}(h_1 - h_3)
\end{aligned} \tag{4.1a}$$

$$\begin{aligned}
L_+ &= M_{20} + iM_{01} = e_1 - e_{-3} \\
L_- &= M_{20} - iM_{01} = e_{-1} - e_3 \\
L_3 &= M_{03} = \frac{1}{2}(h_1 + h_3)
\end{aligned} \tag{4.1b}$$

We obtain the following commutation relations

$$\begin{aligned}
&[M_+, M_-] = [L_3 + iM_3]_q - [L_3 - iM_3]_q \\
&[M_3, M_\pm] = \pm iM_\pm
\end{aligned} \tag{4.2a}$$

$$\begin{aligned}
&[L_+, L_-] = [L_3 - iM_3]_q - [L_3 + iM_3]_q \\
&[L_3, L_\pm] = M_\pm
\end{aligned} \tag{4.2b}$$

$$
\begin{aligned}
&[M_\pm, L_\mp] = [L_3 - iM_3]_q + [L_3 + iM_3]_q \\
&[M_\pm, L_3] = -L_\pm \qquad [M_3, L_\pm] = \pm i L_\pm \\
&[M_\pm, L_\pm] = 0 \qquad [M_3, L_3] = 0
\end{aligned}
\tag{4.2c}
$$

Using the reality conditions (3.11), we obtain that $M^{\oplus}_{\mu\nu} = -M_{\mu\nu}$ and the relations (4.2a-c) describe the q-deformation of the Lorentz algebra. The coproduct formulae take the form:

$$
\begin{aligned}
\Delta(M_\pm) &= M_\pm \otimes q^{L_3} cos(\eta M_3) + q^{-L_3} cos(\eta M_3) \otimes M_\pm \\
&\pm iL_\pm \otimes q^{L_3} sin\,(\eta M_3) \pm i q^{-L_3} sin(\eta M_3) \otimes L_\pm \\
\Delta(M_3) &= M_3 \otimes 1 + 1 \otimes M_3 \\
\Delta(L_\pm) &= L_\pm \otimes q^{L_3} cos(\eta M_3) + q^{L_3} cos(\eta M_3) \otimes L_\pm \\
&\pm iM_\pm \otimes q^{L_3} sin(\eta M_3) \pm i q^{-L_3} sin(\eta M_3) \otimes M_\pm \\
\Delta(L_3) &= L_3 \otimes 1 + 1 \otimes L_3
\end{aligned}
\tag{4.3}
$$

where $q = e^{\eta}$, and

$$
\begin{aligned}
&S(M_\pm) = -\frac{1}{2}(q + q^{-1})M_\pm + \frac{1}{2}(q^{-1} - q)L_\pm \\
&S((M_3) = -M_3 \\
&S(L_\pm) = -\frac{1}{2}(q + q^{-1})L_\pm + \frac{1}{2}(q^{-1} - q)M_\pm \\
&S(L_3) = -L_3
\end{aligned}
\tag{4.4}
$$

We see, therefore, that the q-deformation of the Lorentz subalgebra describes Hopf bialgebra, i.e. it is a genuine quantum algebra [1-3].

The Lorentz quantum algebra presented in this subsection was firstly considered in [13], and further discussed in [20].

b) Weyl quantum algebra (see also [20]).

Let us introduce the four momenta as follows:

$$
\begin{aligned}
P_0 &= -i(e_2 + e_6) \qquad & P_1 &= e_5 - e_4 \\
P_3 &= i(e_2 - e_6) \qquad & P_2 &= i(e_5 + e_4)
\end{aligned}
\tag{4.5}
$$

where $P^{\oplus}_\mu = -P_\mu$ (see (3.11)). The q-deformed algebra in the four-momentum sector look as follows:

$$
\begin{aligned}
&[P_0, P_3] = 0 \qquad & &[P_3, P_2] = i\frac{1-q}{1+q}\ \{P_3, P_1\} \\
&[P_0, P_2] = i\frac{1-q}{1+q}\ \{P_0, P_1\} \qquad & &[P_3, P_1] = i\frac{q-1}{q+1}\ \{P_3, P_2\} \\
&[P_0, P_1] = i\frac{q-1}{q+1}\ \{P_0, P_2\} \qquad & &[P_1, P_2] = \frac{i}{2}(q^{-1} - q)(P_3^2 - P_0^2)
\end{aligned}
\tag{4.6}
$$

Further the deformation of the covariance relation (for $q=1$: $[M_{\mu\nu},P_\lambda]=g_{\nu\lambda}P_\mu-g_{\mu\lambda}P_\nu$) take the form

$$\begin{aligned}
&[M_3,P_0]=0 \qquad && [L_3,P_0]=P_3\\
&[M_3,P_3]=0 && [L_3,P_3]=P_0\\
&[M_3,P_2]=P_1 && [L_3,P_2]=0\\
&[M_3,P_1]=-P_2 && [L_3,P_1]=0
\end{aligned} \tag{4.7a}$$

and further ($P_\pm=P_1\pm iP_2$)

$$\begin{aligned}
&[L_+,P_0]-[M_+,P_0]=iq^{-L_3-iM_3}P_+\\
&[L_-,P_0]-[M_-,P_0]=iP_-q^{L_3-iM_3}\\
&[M_\pm,P_0]+[L_\pm,P_0]=\frac{2i}{1+q}\,P_\pm\pm\frac{1-q}{1+q}\left(\{M_\pm,P_3\}+\{L_\pm,P_3\}\right)
\end{aligned} \tag{4.7b}$$

$$\begin{aligned}
&[M_+,P_3]-[L_+,P_3]=-iq^{-L_3-iM_3}P_+\\
&[M_-,P_3]-[L_-,P_3]=-iP_-q^{L_3-iM_3}\\
&[M_\pm,P_3]+[L_\pm,P_3]=-\frac{2i}{1+q}\,P_\pm\pm\frac{1-q}{1+q}\left(\{M_\pm,P_0\}+\{L_\pm,P_0\}\right)
\end{aligned} \tag{4.7c}$$

$$\begin{aligned}
&[M_+,P_2]-[L_+,P_2]=q^{-L_3-iM_3}(P_0-P_3)\\
&[M_-,P_2]-[L_-,P_2]=(P_3-P_0)q^{L_3-iM_3}\\
&[M_\pm,P_2]+[L_\pm,P_2]=\mp\frac{2}{1+q}\,(P_0+P_3)+i\frac{q-1}{q+1}\left(\{M_\pm,P_1\}+\{L_\pm,P_1\}\right)
\end{aligned} \tag{4.7d}$$

$$\begin{aligned}
&[M_+,P_1]-[L_+,P_1]=iq^{-L_3-iM_3}(P_3-P_0)\\
&[M_-,P_1]-[L_-,P_1]=i(P_3-P_0)q^{L_3-iM_3}\\
&[M_\pm,P_1]+[L_\pm,P_1]=\frac{2i}{1+q}\,(P_0+P_3)+i\frac{1-q}{1+q}\left(\{M_\pm,P_2\}+\{L_\pm,P_2\}\right)
\end{aligned} \tag{4.7e}$$

If we introduce the dilatation generator

$$D=\frac{1}{2}(h_1+h_3+2h_2) \tag{4.8}$$

which due to (3.11) satisfies the relation $D^\oplus=-D$, one checks easily that it enters the q-deformed Weyl algebra in undeformed way

$$[D,M_{\mu\nu}]=0 \qquad\qquad [D,P_\mu]=P_\mu \tag{4.9}$$

We see that in the algebra sector the relations (3.2a-b), (3.6) and (3,7a-e) describe q-deformed Poincaré algebra which is however not closed in the coalgebra sector:

$$
\begin{aligned}
\Delta(P_+) &= P_+ \otimes q^{\frac{1}{2}(D-iM_3)} + q^{-\frac{1}{2}(D-iM_3)} \otimes P_+ + \\
&+ \frac{i}{2}\left(q-q^{-1}\right) q^{\frac{1}{2}(L_3-D)} (M_+ + L_+) \otimes (P_0 - P_3)\, q^{\frac{1}{2}(L_3-iM_3)} \\
\Delta(P_-) &= P_- \otimes q^{\frac{1}{2}(D+iM_3)} + q^{-\frac{1}{2}(D+iM_3)} \otimes P_- + \\
&+ \frac{1}{2}\left(q^{-1}-q\right) q^{-\frac{1}{2}(L_3+iM_3)} (P_0 - P_3) \otimes (M_- + L_-)\, q^{\frac{1}{2}(D-L_3)} \\
\Delta(P_0+P_3) &= (P_0+P_3) \otimes q^{\frac{1}{2}(L_3+D)} + q^{-\frac{1}{2}(L_3+D)} \otimes (P_0+P_3) + \\
&+ \frac{i}{2}(q-q^{-1}) \left\{ q^{-\frac{1}{2}(D+iM_3)} (M_+ + L_+) \otimes P_- q^{\frac{1}{2}(L_3-iM_3)} \right. \\
&\left. - q^{-\frac{1}{2}(L_3+iM_3)} P_+ \otimes (M_- + L_-) q^{\frac{1}{2}(D-iM_3)} \right\} \\
\Delta(P_0-P_3) &= (P_0-P_3) \otimes q^{\frac{1}{2}(D-L_3)} + q^{\frac{1}{2}(L_3-D)} \otimes (P_0-P_3) \\
\Delta(D) &= D \otimes 1 + 1 \otimes D
\end{aligned}
\tag{4.10}
$$

Further we have the following formulae for antipodes

$$
\begin{aligned}
S(P_+) &= -qP_+ + \frac{i}{2} q(q^2-1)(M_+ + L_+)(P_0 - P_3) \\
S(P_-) &= -qP_- + \frac{i}{2} q(1-q^2)(P_0 - P_3)(M_- - L_-) \\
S(P_0 - P_3) &= -q(P_0) - P_3) \\
S(P_0 + P_3) &= -q(P_0 + P_3) + \frac{i}{2} q(q^2-1)(M_+ + L_+)P_- \\
&+ \frac{1}{4} q^2(q^2-1)\left[P_0 - P_3, M_+ + L_+\right]_q (M_- + L_-) \\
S(D) &= -D
\end{aligned}
\tag{4.11}
$$

We see therefore from (3.10-11) that the q-deformation of 11-dimensional Weyl algebra forms a Hopf algebra. We obtain the following sequence of quantum Hopf algebras

$$
U_q(O(3,1)) \subset U_q(\mathcal{P}_4 \oplus\!\!\!\!\supset D) \subset U_q^{(1)}(O(4,2)) \tag{4.12}
$$

where $\mathcal{P}_4 \oplus\!\!\!\!\supset D$ denotes Weyl algebra ($\mathcal{P}_4$ = Poincaré algebra).

The explicite example of q-deformed conformal algebra, presented in [20] is closely related with our choice of $U_q^{(1)}(O(4,2))$ - the main difference consists in the choice of the reality conditions.

5. QUANTUM $D = 4$ κ-DEFORMED POINCARÉ ALGEBRAS

a) Contraction of $U_q(O(3,2))$

The contraction scheme leading from the real form of $U_q(Sp(4))$ to a quantum Poincaré algebra with corresponding Lorentz metric $g_{\mu\nu}$ $(\mu,\nu = 1,2,3,4)$ is obtained by contraction (1.1) with $l = 1$. Let us leave the Lorentz generators $M_{\mu\nu}$ unchanged and define the translation generators P_μ by

$$M_{0\mu} = RP_\mu \tag{5.1}$$

Using the choice (2.20) of the $O(3,2)$ generators and the Cartan-Weyl basis of $U_q(O(3,2))$ one gets the following κ-deformed Poincaré algebra:

a) Lorentz sector $(M_\pm = M_1 \pm iM_2) \equiv M_{23} \pm iM_{31};\ M_3 = M_{12}\ \ L_\pm = M_{14} \pm iM_{24};$ $L_3 = M_{34}$

$$\begin{aligned} [M_+, M_-] &= 2M_3 \\ [M_3, M_\pm] &= \pm M_\pm \end{aligned} \tag{5.2}$$

$$\begin{aligned} [L_-, L_+] &= 2M_3 \cos\frac{P_0}{\kappa} - \frac{1}{\kappa}\{P_3, L_3\} + \frac{1}{2\kappa^2}P_3^2 \\ [L_3, L_\pm] &= \mp e^{\mp iP_0/\kappa} M_\pm \pm \frac{1}{2\kappa}L_\pm P_3 + \frac{i}{2\kappa}L_3 P_\mp \end{aligned} \tag{5.3}$$

$$\begin{aligned} &[M_3, L_3] = 0 \qquad\qquad [M_3, L_\pm] = \pm L_\pm \\ &[M_+, L_3] = -L_+ - \frac{i}{2\kappa}M_3 P_- \\ &[M_-, L_3] = L_- - \frac{i}{2\kappa}P_+ M_3 \\ &[M_+, L_-] = 2L_3 - \frac{i}{2\kappa}P_+ M_+ + \frac{1}{\kappa}P_3 M_3 \\ &[M_-, L_+] = -2L_3 + \frac{1}{2i\kappa}M_- P_- - \frac{1}{\kappa}P_3 M_3 \\ &[M_\pm, L_\pm] = \frac{1}{2i\kappa}M_\pm P_\mp \end{aligned} \tag{5.4}$$

b) Translation sector $(P_\pm = P_2 \pm P_1)$

$$[P_\mu, P_\nu] = 0 \tag{5.5}$$

$$[M_i, P_0] = 0 \qquad [M_i, P_k] = i\epsilon_{ikl}P_l \tag{5.6}$$

$$
\begin{aligned}
&[L_3,P_0]=iP_3 && [L_3,P_3]=i\kappa\sin\frac{P_0}{\kappa}\\
&[L_3,P_2]=\frac{1}{2i\kappa}P_1P_3 && [L_3,P_1]=-\frac{1}{2i\kappa}P_2P_3\\
&[L_\pm,P_0]=iP_1\mp P_2 && [L_\pm,P_3]=\frac{1}{2i\kappa}P_\mp P_3\\
&[L_\pm,P_2]=\mp\kappa\sin\frac{P_0}{\kappa}-\frac{1}{2i\kappa}P_3^2 &&\\
&[L_\pm,P_1]=i\kappa\sin\frac{P_0}{\kappa}\pm\frac{1}{2\kappa}P_3^2 &&
\end{aligned}
\tag{5.7}
$$

The coproduct is

$$
\begin{aligned}
&\Delta M_i=M_i\otimes I+I\otimes M_i \qquad i=1,2,3\\
&\Delta P_0=P_0\otimes I+I\otimes P_0\\
&\Delta P_i=P_i\otimes e^{\frac{iP_0}{2\kappa}}+e^{\frac{-iP_0}{2\kappa}}\otimes P_i \qquad i=1,2,3\\
&\Delta L_3=L_3\otimes e^{\frac{iP_0}{2\kappa}}+e^{\frac{-iP_0}{2\kappa}}\otimes L_3+\frac{i}{2\kappa}e^{\frac{-iP_0}{2\kappa}}M_+\otimes P_+-\frac{i}{2\kappa}P_-\otimes M_-e^{\frac{iP_0}{2\kappa}}\\
&\Delta L_+=L_+\otimes e^{\frac{iP_0}{2\kappa}}+e^{\frac{-iP_0}{2\kappa}}\otimes L_++\frac{i}{2\kappa}P_-\otimes M_3e^{\frac{iP_0}{2\kappa}}\\
&\qquad-\frac{i}{2\kappa}e^{\frac{-iP_0}{2\kappa}}M_3\otimes P_-+\frac{1}{\kappa}e^{\frac{-iP_0}{2\kappa}}M_+\otimes P_3\\
&\Delta L_-=L_-\otimes e^{\frac{iP_0}{2\kappa}}+e^{\frac{-iP_0}{2\kappa}}\otimes L_-+\frac{i}{2\kappa}P_+\otimes M_3e^{\frac{iP_0}{2\kappa}}\\
&\qquad-\frac{i}{2\kappa}e^{\frac{-iP_0}{2\kappa}}M_3\otimes P_++\frac{1}{\kappa}P_3\otimes M_-e^{\frac{iP_0}{2\kappa}}
\end{aligned}
\tag{5.8}
$$

The antipode is

$$
\begin{aligned}
&S(P_0)=-P_\mu\\
&S(M_i)=-M_i\\
&S(L_3)=-L_3-\frac{1}{2\kappa}P_3+\frac{i}{2\kappa}(M_+P_+-P_-M_-)\\
&S(L_+)=-L_+-\frac{i}{\kappa}P_-+\frac{1}{\kappa}M_+P_3\\
&S(L_-)=-L_-+\frac{i}{\kappa}P_++\frac{1}{\kappa}P_3M_-
\end{aligned}
\tag{5.9}
$$

b) Contraction of $U_q(O(4,2))$

We shall consider the real form (3.12) of $U_q(Sl(4;C))$ as an intermediate step in the derivation of κ-deformation of $D=4$ Poincaré algebra where κ is a mass-like parameter.

We define the generators of the $D=4$ conformal algebra in the following way:

$$\begin{aligned}
M_{12} &= \frac{i}{2}(h_1+h_3) & M_{03} &= \frac{i}{2}(e_4-e_{-4}-e_5+e_{-5}) \\
M_{23} &= \frac{i}{2}(e_1+e_{-1}+e_3+e_{-3}) & M_{02} &= \frac{1}{2}(e_6+e_{-6}-e_2-e_{-2}) \\
M_{31} &= \frac{1}{2}(e_1-e_{-1}+e_3-e_{-3}) & M_{01} &= \frac{i}{2}(e_6-e_{-6}+e_2-e_{-2})
\end{aligned} \tag{5.10}$$

$$\begin{aligned}
M_{43} &= \frac{i}{2}(h_1-h_3) & M_{54} &= -\frac{i}{2}(e_4-e_{-4}+e_5-e_{-5}) \\
M_{42} &= \frac{1}{2}(e_1-e_{-1}-e_3+e_{-3}) & M_{53} &= \frac{1}{2}(e_4+e_{-4}-e_5-e_{-5}) \\
M_{41} &= \frac{i}{2}(e_1+e_{-1}-e_3-e_{-3}) & M_{52} &= \frac{i}{2}(e_2-e_{-2}-e_6+e_{-6}) \\
M_{40} &= \frac{1}{2}(e_4+e_{-4}+e_5+e_{-5}) & M_{51} &= \frac{1}{2}(e_2+e_{-2}+e_6+e_{-6})
\end{aligned} \tag{5.11}$$

$$M_{50} = \frac{i}{2}(h_1+h_3+2h_2)$$

which due to relations (3.12) satisfy the reality condition $M^{\oplus}_{AB} = -M_{AB}$. The $O(4,2)$ q-deformed commutation relations correspond to the following assignment of the signature $g_{AB} = diag(-++++-)$ $(A,B=0,1,2,3,4,5)$ and the physical basis is given by the Lorentz generators $M_{\mu\nu}$ $(\mu\nu=0,1,2,3)$ and $P_\mu = M_{4\mu}+M_{5\mu}$, $K_\mu = M_{5\mu}-M_{4\mu}$, $D = M_{45}$. The Cartan subalgebra is described by the following three commuting generators: (M_3, P_0+K_0, P_3-K_3).

In order to obtain the κ-deformation of the $D=4$ Poincaré algebra we introduce the limit (1.1) with the following rescaling of the generators

$$\tilde{M}_{\mu\nu} = M_{\mu\nu} \qquad \tilde{K}_\mu = K_\mu \qquad \tilde{P}_\mu = \frac{1}{R}P_\mu \qquad \tilde{D} = \frac{1}{R}D \tag{5.12}$$

The rescaling (5.12) corresponds to the contraction of the nonsymmetric and nonreductive coset $K = \frac{G}{H}$, where G is a conformal group $(O(4,2))$ and H is the Poincaré group $(0(3,1) \ni T_4)$. formed by Lorentz group extended by conformal accelerations. Substituting in the formulae of Sect. 3 the relations (5.10-11), or more explicitly

$$
\begin{aligned}
e_{\pm 1} &= \frac{1}{2i}\left[M_{\pm} + \frac{1}{2}\left(R\tilde{P}_1 - K_1\right) \pm \frac{i}{2}\left(R\tilde{P}_2 - K_2\right)\right] \\
e_{\pm 2} &= \frac{1}{2i}\left[\pm L_{\mp} \pm \frac{1}{2}\left(R\tilde{P}_2 + K_2\right) + \frac{i}{2}\left(R\tilde{P}_1 + K_1\right)\right] \\
e_{\pm 3} &= \frac{1}{2i}\left[M_{\pm} + \frac{1}{2}\left(K_1 - R\tilde{P}_1\right) \mp \frac{i}{2}\left(R\tilde{P}_2 - K_2\right)\right] \\
e_{\pm 4} &= \frac{1}{2i}\left[\pm\left(L_3 + R\tilde{D}\right) + \frac{i}{2}\left(R\tilde{P}_0 - K_0\right) + \frac{i}{2}\left(R\tilde{P}_3 + K_3\right)\right] \\
e_{\pm 5} &= \frac{1}{2i}\left[\pm\left(R\tilde{D} - L_3\right) + \frac{i}{2}\left(R\tilde{P}_0 - K_0\right) - \frac{i}{2}\left(R\tilde{P}_3 + K_3\right)\right] \\
e_{\pm 6} &= \frac{1}{2i}\left[\pm L_{\pm} \mp \frac{1}{2}\left(R\tilde{P}_2 + K_2\right) + \frac{i}{2}\left(R\tilde{P}_1 + K_1\right)\right]
\end{aligned}
\tag{5.13}
$$

$$
\begin{aligned}
h_1 &= -iM_3 + \frac{i}{2}\left(R\tilde{P}_3 - K_3\right) \\
h_2 &= iM_3 - \frac{i}{2}\left(R\tilde{P}_0 + K_0\right) \\
h_3 &= -iM_3 - \frac{i}{2}\left(R\tilde{P}_3 - K_3\right)
\end{aligned}
\tag{5.14}
$$

we obtain in the contaction limit (1,1) with $l = 2$ the following relations:

i) κ-deformation of Lorentz sector (M_i, L_i)

$$
\begin{aligned}
&[M_+, M_-] = 2iM_3 \qquad && [L_+, L_-] = -2iM_3 \\
&[M_3, M_+] = iM_+ && [L_3, L_+] = -iM_+ - \frac{3}{8\kappa^2}\tilde{P}_0\left(\tilde{P}_2 - i\tilde{P}_0\right) \\
&[M_3, M_-] = -iM_- && [L_3, L_-] = iM_- + \frac{3}{8\kappa^2}\tilde{P}_0\left(\tilde{P}_0 + i\tilde{P}_1\right)
\end{aligned}
\tag{5.15}
$$

$$[M_3, L_3] = 0$$
$$[M_3, L_-] = -iL_- \qquad [M_3, L_+] = iL_+$$

$$[M_+, L_3] = -iL_+ - \frac{1}{8\kappa^2} 2\tilde{P}_3 \left(\tilde{P}_1 + i\tilde{P}_2\right)$$
$$[M_+, L_+] = \frac{1}{8\kappa^2} \left(\tilde{P}_1 + i\tilde{P}_2\right)^2$$

$$[M_-, L_3] = iL_- + \frac{1}{8\kappa^2} \tilde{P}_3 \left(\tilde{P}_1 - i\tilde{P}_2\right)$$
$$[M_-, L_-] = -\frac{1}{8\kappa^2} \left(\tilde{P}_1 - i\tilde{P}_2\right)^2$$
$$[M_\mp, L_\pm] = \mp 2iL_3 \mp \frac{1}{8\kappa^2} \left(2\tilde{P}_3^2 - \tilde{P}_1^2 - \tilde{P}_2^2\right)$$

In our contraction limit [21] we obtain the following coproduct formulae:

$$\begin{aligned}
\Delta M_3 &= M_3 \otimes 1 + 1 \otimes M_3 \\
\Delta M_\pm &= M_\pm \otimes 1 + 1 \otimes M_\pm + \frac{1}{8\kappa^2} \left(\tilde{P}_\pm \otimes \tilde{P}_3 - \tilde{P}_3 \otimes \tilde{P}_\pm\right) \\
\Delta L_3 &= L_3 \otimes 1 + 1 \otimes L_3 - \frac{i}{4\kappa^2} \left(\tilde{P}_+ \otimes \tilde{P}_- + \tilde{P}_- \otimes \tilde{P}_+\right) \\
\Delta L_\pm &= L_\pm \otimes 1 + 1 \otimes L_\pm + \frac{1}{4\kappa^2} \left(\tilde{D} \otimes \tilde{P}_\pm - -\tilde{P}_\pm \otimes \tilde{D}\right) \\
&\pm \frac{i}{8\kappa^2} \left[\left(\tilde{P}_0 \pm \tilde{P}_3\right) \otimes \tilde{P}_\pm - \tilde{P}_\pm \otimes \left(\tilde{P}_0 \mp \tilde{P}_3\right)\right]
\end{aligned} \tag{5.16}$$

and the formulae for antipodes

$$\begin{aligned}
S(M_3) &= -M_3 \qquad & S(L_3) &= -L_3 - \frac{i}{2\kappa} \tilde{P}_+ \tilde{P}_- \\
S(M_\pm) &= -M_\pm \qquad & S(L_\pm) &= -L_\pm \pm \frac{i}{4\kappa} \tilde{P}_\pm \tilde{P}_3
\end{aligned} \tag{5.17}$$

We see that in order to obtain the closed bialgebra one has to add the central generator $\tilde{D}$, which extends the Lorentz κ-algebra (5.15) in the following trivial way:

$$[M_i, \tilde{D}] = [L_i, \tilde{D}] = 0 \tag{5.18}$$

and further

$$\Delta \tilde{D} = \tilde{D} \otimes 1 + 1 \otimes \tilde{D} + \frac{i}{4\kappa^2} \left(\tilde{P}_+ \otimes \tilde{P}_- - \tilde{P}_- \otimes \tilde{P}_+\right) \tag{5.19a}$$

$$S(\tilde{D}) = -\tilde{D} \tag{5.19b}$$

We see that the generators $(M_i, L_i, \tilde{D})$ form a Hopf algebra.

ii) κ-deformation of centrally extended Poincaré algebra (Poincaré $\oplus\tilde{D}$).

The limit (1.1) with $l = 2$ implies supplementing of the algebra (5.15) by the "classical" relations

$$\begin{aligned} &[\tilde{P}_\mu, \tilde{P}_\nu] = 0 \\ &[M_{\mu\nu}, \tilde{P}_\lambda] = g_{\nu\lambda}\tilde{P}_\mu - g_{\mu\lambda}\tilde{P}_\nu \end{aligned} \tag{5.20}$$

and the "classical" formulae for the coproduct and antipode

$$\Delta\tilde{P}_\mu = \tilde{P}_\mu \otimes 1 + 1 \otimes \tilde{P}_\mu \tag{5.21a}$$

$$S(\tilde{P}_\mu) = -\tilde{P}_\mu \tag{5.21b}$$

The relations (5.15 - 5.21) describe the κ-deformed Poincaré Hopf algebra centrally extended in the coalgebra sector by the Abelian generator $\tilde{D}$.

It should be added that the contraction limit (1.1) with $l = 2$ we apply only to the generators $(M_{\mu\nu}, P_\mu, D)$, what leads to closed quantum algebra. It appears that in the conformal momenta sector this contraction limit leads to divergences.

6. FINAL REMARKS

Standard Drinfeld-Jimbo deformation [3,4] of $D = 4$ conformal algebra permits to deduce two different quantum deformations of Poincaré algebra:

a) The q-deformation of Poincaré algebra, discussed in Sect. 4. In this case

i) Quantum deformation of Lorentz algebra forms a quantum subgroup, which is a Hopf algebra (see formulae (4.2 - 4.4)).

ii) In order to introduce q-deformed Poincaré algebra as Hopf algebra one has to add eleventh dilatation generator. In such a way one obtains the q-deformed Weyl algebra as a quantum group.

iii) The four momenta are nonAbelian and form the quadratic relations (4.6), describing closed subalgebra.

It could be interesting to present the q-deformation of the theory of induced representations of Poincaré group, with the noncommutative nature of fourmomenta taken into consideration.

b) The κ-deformations of Poincaré algebra, discussed in Sect. 5. The κ-deformations have the following common features:

i) The κ-deformation of Lorentz algebra ceases to be a quantum subalgebra of the κ-deformation of Poincaré algebra.

ii) The κ-deformed Poincaré algebra can be extended to Hopf algebra without adding new generators we present another κ-deformation, with Hopf bialgebra structure requiring the addition of eleventh central generator.

iii) The fourmomenta stay Abelian (see (5.20)).

It would be interesting e.g. to relate the κ-parameter with the Planck mass.

Acknowledgements

The second author (A.N.) would like to thank Pierre Minnaert for his hospitality at Bordeaux I University, and the third author (H.R.) thanks So-Young Pi, Larry Sulak and especially Abner Shimony for their hospitality at Boston University.

FOOTNOTES:

1. Following [1] we distinguish here the quantum group G_q as the q-deformation of the (algebra of functions on the) Lie group G, and the quantum algebra $U_q(\hat{g})$ as describing the q-deformation of the universal enveloping algebra $U(\hat{g})$ of the Lie algebra $\hat{g}$.

2. For completeness see also [24].

3. For $\epsilon = 0$ and $l = 1$ such limit was firstly introduced by Firenze group [16,17] for studying the contractions of rank one quantum algebras $(U_q(Sl(2;C))$ and its real forms $U_q(SU(2))$ and $U_q(SU(1,1)))$.

REFERENCES

[1] L. Faddeev, N. Reshetikhin and L. Takhtajan, Algebra i Analiz, **1** (1989), 178.

[2] L. Woronowicz, Comm. Math. Phys. **111**, (1987) 613; ibid. **122**, (1989) 125.

[3] V.G. Drinfeld, "Quantum groups", Proceedings of the International Congress of Math. Berkeley, USA (1986), p. 793.

[4] M. Jimbo, Lett. Math. Phys. **10** (1985), 63; ibid. **11** (1986), 247.

[5] Yu T. Manin, "Quantum groups and non-commutative geometry", Centre de Recherches Math., Univ. de Montreal, 1989.

[6] Proceedings of First EIMI Workshop on Quantum Groups, October-December 1990, ed. P. Kulish, Springer Verlag, 1991.

[7] L. Faddeev in: Les Houches XXXIX, eds. J.B. Zuber and R. Stora (North-Holland, Amsterdam, 1984), course 8

[8] P. Kulish and E. Sklyanin, Lecture Notes in Physics, Vol. **151** (1984)

[9] L. Alvarez-Gaumè, C. Gomez and G. Sierra, in "Physics and Mathematics of Strings" (Memorial volume for V. Knizhnik), World Scientific, Singapore (1990), p. 16.

[10] P. Podleś and S.L. Woronowicz, Comm. Math.Phys. **130**, (1990)381

[11] V. Carow-Watamura, M. Schlieker, M. Scholl and S. Watamura, Z. Phys. **C48**, (1990)159

[12] V. Carow-Watamura, M. Schlieker, M. Scholl and S. Watamura, Int. Journ. Mod. Phys. **6A**, (1991)3081

[13] C. Gomez and G. Sierra, Phys.Lett. **255B**, (1991)51

[14] S.L. Woronowicz, "New Quantum Deformation of $SL(2;C)$ - Hopf Algebra Level", Warsaw University preprint, 1990

[15] W.P. Schmidke, J. Wess and B. Zumino, Max Planck Institute preprint MPI-Ph/91-15, March 1991

[16] E. Celeghini, R. Giacchetti, E. Sorace and M. Tarlini, J. Math.Phys. **32** (1991) 1155; ibid. 1159

[17] E. Celeghini, R. Giacchetti, E. Sorace and M. Tarlini, "Contraction of quantum

groups", contribution to [6].

[18] J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy, Phys. Lett. **264B**, 331(1991)

[19] J. Lukierski, A. Nowicki and H. Ruegg, Geneve Univ. preprint UGVA/DPT-08-740, Phys.Lett. in press.

[20] V. Dobrev, "Canonical q-Deformations of Noncompact Lie (Super-) Algebras", Goettingen Univ. preprint, July 1991

[21] J. Lukierski and A. Nowicki, "Quantum Deformations of $D = 4$ Poincare and Weyl algebra from $q-$ deformed $D = 4$ conformal algebra", Wroclaw University preprint ITP UWr 787/91, October 1991

[22] A. Chakrabarti, Journ. Math.Phys., **32**, 1227(1991)

[23] M. Schlieker, W. Weich and R. Weixler, Munich University preprint LMU-TPW 1991-3, April 1991

[24] A.M. Gavriluk and A.K. Klimyk, Lett. Math. Phys. **21**, 215(1991)

[25] M. Rosso, Comm. Math. Phys. **124**, 307(1989)

[26] H. Yamane, Publ. RIMS Kyoto Univ. **25**, 503(1989)

[27] V.N. Tolstoy, Proceedings of Summer Workshop "Quantum Groups", Clausthal, Germany, July 1989; ed. Springer

[28] S.M. Koroshkin and V.N. Tolstoy, Nuclear Physics Institute, Moscov State University, preprint 91-13/217.