# Quark structure and octonions*

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The octonion (Cayley) algebra is studied in a split basis by means of a formalism that brings out its quark structure. The groups $SO(8)$, $SO(7)$, and $G_2$ are represented by octonions as well as by $8 \times 8$ matrices and the principle of triality is studied in this formalism. Reduction is made through the physically important subgroups $SU(3)$ and $SU(2) \otimes SU(2)$ of $G_2$, the automorphism group of octonions.

## 1. INTRODUCTION

Octonions made their appearance in physics as a by-product of an early attempt to generalize quantum mechanics through the associativity condition for physical observables.[1,2] In their algebraic approach to quantum mechanics, Jordan, von Neumann, and Wigner focused on the properties of Hermitian density matrices. Such matrices close under the commutative "Jordan" product which can be defined as their anticommutator. Thus, in switching from the matrix algebra of density matrices, we trade associativity for commutativity. The two formulations are equivalent except in the case of octonion hermitian $3 \times 3$ density matrices which form an exceptional Jordan Algebra.[2] In the latter case the nonassociativity is intrinsic and cannot be removed by going over to a corresponding operator algebra in a finite Hilbert space. In fact, it originates in the structure of the underlying octonion algebra which is a not commutative, not associative, but, alternative division algebra.

The Jordan approach has proved to be more fruitful in mathematics than physics. It has since been quietly dropped in favor of the associative Dirac algebra of operators in Hilbert space,[3] which generalizes the algebra of finite matrices.

The story took a new turn when the charge space made its appearance two decades ago in the Gell-Mann-Nishijima[4] scheme based on isospin and strangeness. A decade later, this led to the quark structure of elementary particles, revealing the underlying $SU(3)$ symmetry.[5] Meantime another group of rank two, namely $G_2$ was tried[6] and abandoned. Now, $G_2$ is the automorphism group of the octonion algebra and it admits $SU(3)$ as a subgroup. In fact, $SU(3)$ is the automorphism group of the multiplication rules among six of the octonion units. In terms of this subgroup the generators of $G_2$ split into an $SU(3)$ octet, a triplet and an antitriplet. Furthermore $G_2$ has a $SU(2) \times SU(2)$ subgroup under which the generators decompose as (1, 0), (0, 1), and (1/2, 3/2). One of the $SU(2)$ is the isospin, while the other is a generalization of hypercharge to a rotation group. Hence the quark structure is manifest in $G_2$ and also in the other exceptional groups which are all related to octonions[7] and admit $G_2$ as a subgroup. An example is the exceptional Jordan algebra which has the exceptional group $F_4$ as its automorphism group.[8] The quark structure of this algebra was pointed out by Gamba.[9] Other possible connections of the Cayley algebra with internal symmetries were discussed by Pais[10] and others[11] while the admissibility of the elements of the exceptional Jordan algebra as observables was considered by Sherman[12] following the general algebraic framework of Segal.[13] Finally, we have shown recently[14] that the Poincaré group possesses an octonionic representation that leads to a quark structure arising from the breaking of $G_2$ with $SU(3)$ as the surviving subgroup.

An independent, and perhaps related line of research concerns the construction of Weinberg type renormalizable models[15] based on groups that do not give rise to triangular anomalies, including $G_2$, $SO(7)$, and $SO(8)$. The attempts enumerated above seem to provide sufficient motivation for a reformulation of the octonion algebra, and the groups $SO(8)$, $SO(7)$, and $G_2$ connected with it, in a manner which manifestly exhibits its quark structure and its $SU(3)$ content in charge space. Although a vast mathematical literature exists on these subjects,[7,16] it is not presented in a form directly usable by the particle physicist. The object of the present paper is to recast the mathematical theory in a quark language, in direct correspondence with Gell-Mann's treatment of $SU(3)$,[5] to develop an $8 \times 8$ matrix formalism, initiated by Seligman,[17] which allows us to treat $SO(8)$, $SO(7)$, $G_2$ in a unified way and prepare the ground for the investigation of the properties of an octonionic Hilbert space.[18]

The features which seem to be new consist of a matrix form for the octonion multiplication, the representation of $G_2$ through purely octonionic multiplication in a split basis and the reduction of $SO(8)$, $SO(7)$, and $G_2$ with respect to their physically important subgroups $SU(3)$ and $SU(2) \times SU(2)$. It is this reduction which exhibits the quark structure of the algebra.

The contents of the paper are as follows. The octonion algebra in the split basis is introduced in Sec 2 and its automorphism group $G_2$ derived in Sec. 3. The Lie algebra of $G_2$ and its imbedding in $SO(7)$ are given in Sec. 4. The following section 5 covers the $SU(3)$ and $SU(2) \times SU(2)$ subgroups of $G_2$. Section 6 is devoted to split octonions and split $G_2$ while the quark structure in split basis emerges in Sec. 7. A purely octonion representation of split $G_2$ appears in Sec. 8. The $8 \times 8$ matrix formulation of the Cayley algebra forms the object of Sec. 9. The imbedding of $G_2$ in $SO(7)$ and $SO(8)$ and its reduction with respect to its $SU(2) \times SU(2)$ subgroups are discussed respectively in Secs. 10 and 11. Finally the principle of triality is discussed within the formalism developed previously in Sec. 12. Additional details such as the structure constants of $G_2$, Zorn's vector-matrix method, theorems pertaining to triality and the realization of the Cayley algebra by means of Gell-Mann's $3 \times 3$ $\lambda$-matrices, and Dirac's $4 \times 4$ $\gamma$-matrices appear in the appendixes.

## 2. THE OCTONION ALGEBRA AND ITS SPLIT BASIS

A composition algebra is defined as an algebra $A$ with identity and with a nondegenerate quadratic form $Q$ defined over it such that $Q$ permits composition, i.e. for $x, y \in A$.

$$Q(xy) = Q(x)Q(y). \tag{2.1}$$

According to the celebrated Hurwitz theorem, there

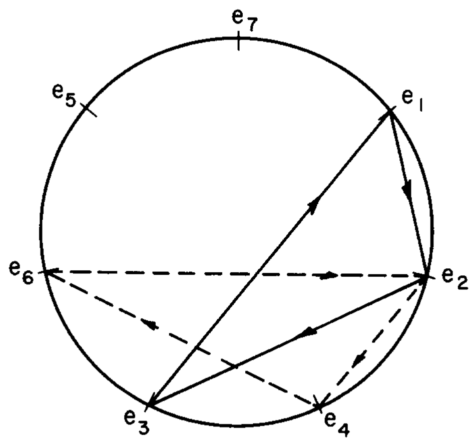


FIG. 1.

exist only four different composition algebras over the real or complex number fields. These are the real numbers **R** of dimension 1, complex numbers **C** of dimension 2, quaternions **H** of dimension 4, and octonions **O** of dimension 8. Of these algebras, the quaternions **H** are not commutative and the octonions **O** are neither commutative nor associative.[19] A composition algebra is said to be a division algebra if the quadratic form $Q$ is anisotropic i.e.,

if $Q(x) = 0$ implies that $x = 0$.

Otherwise the algebra is called *split*.

Assuming that the reader is familiar with the algebras **R, C**, and **H**, we shall review briefly the properties of octonion algebra **O**[20] (sometimes called the Cayley algebra).

A basis for the real octonion **O** will contain eight elements including the identity

$$1, e_A, \quad A = 1, \ldots, 7, \quad \text{where } e_A^2 = -1.$$

For later application to the $SU(3)$ symmetry in particle physics, we label the elements $e_A$ such that they satisfy the following multiplication table:

$$e_1e_2 = e_3, \quad e_5e_1 = e_6, \quad e_6e_2 = e_4, \quad e_4e_3 = e_5,$$
$$e_4e_7 = e_1, \quad e_6e_7 = e_3, \quad e_5e_7 = e_2$$

and

$$e_Ae_B + e_Be_A = -2\delta_{AB}$$

more concisely,

$$e_Ae_B = a_{ABC}e_C - \delta_{AB} \tag{2.2}$$

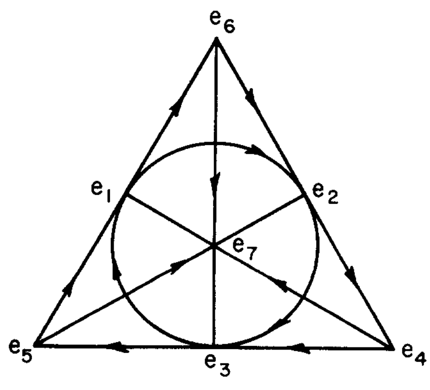


FIG. 2.

where $a_{ABC}$ is totally antisymmetric and

$$a_{ABC} = +1 \text{ for } ABC = 123, 516, 624, 435, 471, 673, 572.$$

Note here the cyclic symmetry obtained by ordering seven points clockwise on a circle with the numbering (1243657) as given in Fig. 1. Then a triangle $ABC$ is obtained from (123) by 6 successive rotations of angle $2\pi/7$. In Fig. 1 the elements corresponding to the corners of the triangle form a basis of a quaternion subalgebra (together with the identity element). Another convenient way of representing the multiplication table by singling out one of the elements is provided by the triangular diagram given in Fig. 2, where arrows show the directions along which the multiplication has a positive sign, e.g.

$$e_5e_1 = e_6, \quad e_1e_5 = -e_6, \quad e_6e_1 = -e_5, \quad e_6e_5 = e_1.$$

From the above multiplication table it is clear that the algebra **O** is not associative. Yet it satisfies a weaker condition than associativity, namely alternativity, i.e., the associator $[x, y, z]$ of the elements $x, y, z$ defined as

$$[x, y, z] = (xy)z - x(yz) \tag{2.3}$$

is an alternating function of $x, y, z$:

$$[x, y, z] = [z, x, y] = [y, z, x] = -[y, x, z) = -[x, z, y].$$

This property if trivially satisfied by associative algebras **R, C**, and **H**.

The octonion algebra **O** with the above basis considered over the real numbers **R** is a division algebra with the quadratic form $Q$ defined by

$$Q(x) = \bar{x}x = x\bar{x},$$

where $\bar{x}$ is the octonion conjugate of $x$ obtained by replacing $e_A$ in $x$ by $-e_A$.

$$x = x_0 + x_Ae_A, \quad \bar{x} = x_0 - x_Ae_A.$$

This quadratic form is also called the norm form and denoted by $N(x)$. Then

$$N(x) = x\bar{x} = \bar{x}x = x_0^2 + \sum_{A=1}^{7} x_A^2. \tag{2.4}$$

For the split octonion algebra we choose the following basis:

$$u_1 = \tfrac{1}{2}(e_1 + ie_4), \quad u_2 = \tfrac{1}{2}(e_2 + ie_5),$$
$$u_3 = \tfrac{1}{2}(e_3 + ie_6), \quad u_0 = \tfrac{1}{2}(1 + ie_7)$$
$$u_1^* = \tfrac{1}{2}(e_1 - ie_4), \quad u_2^* = \tfrac{1}{2}(e_2 - ie_5),$$
$$u_3^* = \tfrac{1}{2}(e_3 - ie_6), \quad u_0^* = \tfrac{1}{2}(1 - ie_7),$$

where $i = \sqrt{-1}$ and is assumed to commute with all $e_A$. These basis elements satisfy the multiplication table:

$$u_iu_j = \epsilon_{ijk}u_k^*, \quad i, j, k = 1, 2, 3,$$
$$u_i^*u_j^* = \epsilon_{ijk}u_k,$$
$$u_iu_j^* = -\delta_{ij}u_0, \quad u_i^*u_j = -\delta_{ij}u_0^*,$$
$$u_iu_0 = 0, \quad u_iu_0^* = u_i, \quad u_i^*u_0 = u_i^*, \quad u_i^*u_0^* = 0,$$
$$u_0u_i = u_i, \quad u_0^*u_i = 0, \quad u_0u_i^* = 0, \quad u_0^*u_i^* = u_i^*,$$
$$u_0^2 = u_0, \quad u_0^{*2} = u_0^*, \quad u_0u_0^* = u_0^*u_0 = 0.$$

Clearly, the split octonion algebra contains divisors of zero and hence is not a division algebra. In Appendix B, we give a realization of split octonion algebra in terms of Zorn's vector matrices.

## 3. $G_2$ AS THE AUTOMORPHISM GROUP OF OCTONIONS

An automorphism of an algebra $A$ is defined as an isomorphism of $A$ onto itself. Under the automorphism, multiplication table of $A$ is left invariant, i.e.,

$$x, y \in A, \qquad T \in \mathrm{Aut}A,$$

then

$$T(xy) = T(x)T(y)$$

and the automorphisms map the identity 1 into itself.

The set of all automorphisms of composition algebras form a group. For the real numbers **R** and complex numbers **C**, the groups of automorphisms are the trivial identity mapping and the cyclic group $C_2$, respectively. The automorphism group of quaternions is the $SU(2)$ group.[21] Below we shall investigate the automorphism group of the octonions, which is the exceptional Lie group $G_2$.

We use the following results of $M$. Zorn[22] as our starting point: Each automorphism of the Cayley algebra **O** is completely defined by the images of 3 "independent" basis elements.[23] Consider one such set, say $\{e_1, e_2, e_4\}$. Then there exists an automorphism $\sigma$ such that

$$\begin{aligned} &\sigma(e_1) = e_1, \\ &\sigma(e_2) = \cos\phi_1 e_2 + \sin\phi_1 e_3, \\ &\sigma(e_4) = \cos\phi_2 e_4 + \sin\phi_2 e_7, \end{aligned} \tag{3.1}$$

The images of the other basis elements are determined by the conditions:

$$\begin{aligned} &\sigma(e_2)\sigma(e_4) = \sigma(e_6), \qquad \sigma(e_1)\sigma(e_2) = \sigma(e_3), \\ &\sigma(e_1)\sigma(e_4) = \sigma(e_7), \qquad \sigma(e_4)\sigma(e_3) = \sigma(e_5). \end{aligned}$$

It can easily be checked that $\sigma(e_A)$ satisfy the same multiplication table as $e_A$ and hence $\sigma$ is an automorphism. Conversely, one has the very important result that *each automorphism of* **O** *belongs in this manner to at least one Cayley basis*.

Now let us write down all the images of all basis elements under $\sigma$ explicitly and observe some general patterns:

$$\begin{aligned} &\sigma(e_1) = e_1, \\ &\begin{pmatrix} \sigma(e_2) \\ \sigma(e_3) \end{pmatrix} = \begin{pmatrix} \cos\phi_1 & \sin\phi_1 \\ -\sin\phi_1 & \cos\phi_1 \end{pmatrix} \begin{pmatrix} e_2 \\ e_3 \end{pmatrix}, \\ &\begin{pmatrix} \sigma(e_4) \\ \sigma(e_7) \end{pmatrix} = \begin{pmatrix} \cos\phi_2 & \sin\phi_2 \\ -\sin\phi_2 & \cos\phi_2 \end{pmatrix} \begin{pmatrix} e_4 \\ e_7 \end{pmatrix}, \\ &\begin{pmatrix} \sigma(e_6) \\ \sigma(e_5) \end{pmatrix} = \begin{pmatrix} \cos(\phi_1 + \phi_2) & -\sin(\phi_1 + \phi_2) \\ \sin(\phi_1 + \phi_2) & \cos(\phi_1 + \phi_2) \end{pmatrix} \begin{pmatrix} e_6 \\ e_5 \end{pmatrix}. \end{aligned} \tag{3.2}$$

We see that under the automorphism $\sigma$ we have three invariant planes $(e_2, e_3)$, $(e_4, e_7)$, $(e_6, e_5)$ that undergo rotations through angles $\phi_1, \phi_2, \phi_3$, respectively, such that

$$\phi_1 + \phi_2 + \phi_3 = 0 \bmod 2\pi.$$

We shall call the automorphisms of the form above *canonical automorphisms*. Each canonical automorphism has a fixed point and 3 invariant planes. If we denote the fixed point by $e_k$ then the invariant planes $(e_i, e_j)$ are determined by the conditions $e_i e_j = e_k$.

For each Cayley basis, there are seven independent canonical automorphisms. The canonical automorphism given above can be written more concisely as:

$$\begin{pmatrix} \sigma(e_2) + i\sigma(e_3) \\ \sigma(e_4) + i\sigma(e_7) \\ \sigma(e_6) + i\sigma(e_5) \end{pmatrix} = e^{(\alpha^1\lambda_3 + \beta^1\lambda_8)e_1} \begin{pmatrix} e_2 + ie_3 \\ e_4 + ie_7 \\ e_6 + ie_5 \end{pmatrix}, \tag{3.3}$$

where $\lambda_3$ and $\lambda_8$ are the Gell-Mann matrices

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

and $\alpha^1$ and $\beta^1$ are related to $\phi_1$ and $\phi_2$ as:

$$\phi_1 = \alpha^1 + 3^{-1/2}\beta^1, \qquad \phi_2 = -\alpha^1 + 3^{-1/2}\beta^1.$$

This can be generalized to all the canonical automorphisms. First define seven octonionic 3-spinors $\psi(e_A)$:

$$\begin{aligned}
\psi(e_1) &= \frac{1}{2}\begin{pmatrix} e_6 + ie_5 \\ e_2 + ie_3 \\ e_4 + ie_7 \end{pmatrix} = \frac{(1 + ie_1)}{2}\begin{pmatrix} e_6 \\ e_2 \\ e_4 \end{pmatrix}, \\
\psi(e_2) &= \frac{1}{2}\begin{pmatrix} e_4 + ie_6 \\ e_3 + ie_1 \\ e_5 + ie_7 \end{pmatrix} = \frac{(1 + ie_2)}{2}\begin{pmatrix} e_4 \\ e_3 \\ e_5 \end{pmatrix}, \\
\psi(e_3) &= \frac{1}{2}\begin{pmatrix} e_5 + ie_4 \\ e_1 + ie_2 \\ e_6 + ie_7 \end{pmatrix} = \frac{(1 + ie_3)}{2}\begin{pmatrix} e_5 \\ e_1 \\ e_6 \end{pmatrix}, \\
\psi(e_4) &= \frac{1}{2}\begin{pmatrix} e_3 + ie_5 \\ e_6 + ie_2 \\ e_7 + ie_1 \end{pmatrix} = \frac{(1 + ie_4)}{2}\begin{pmatrix} e_3 \\ e_6 \\ e_7 \end{pmatrix}, \\
\psi(e_5) &= \frac{1}{2}\begin{pmatrix} e_1 + ie_6 \\ e_4 + ie_3 \\ e_7 + ie_2 \end{pmatrix} = \frac{(1 + ie_5)}{2}\begin{pmatrix} e_1 \\ e_4 \\ e_7 \end{pmatrix}, \\
\psi(e_6) &= \frac{1}{2}\begin{pmatrix} e_2 + ie_4 \\ e_5 + ie_1 \\ e_7 + ie_3 \end{pmatrix} = \frac{(1 + ie_6)}{2}\begin{pmatrix} e_2 \\ e_5 \\ e_7 \end{pmatrix}, \\
\psi(e_7) &= \frac{1}{2}\begin{pmatrix} e_1 + ie_4 \\ e_2 + ie_5 \\ e_3 + ie_6 \end{pmatrix} = \frac{(1 + ie_7)}{2}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.
\end{aligned} \tag{3.4}$$

A canonical automorphism leaving the element $e_A$ fixed is defined by its action on $\psi(e_A)$:

$$\begin{aligned} \sigma^A: \psi(e_A) \to \sigma^A\psi(e_A) = \psi'(e_A) &= e^{(\alpha^A\lambda_3 + \beta^A\lambda_8)e_A}\psi(e_A) \\ &= e^{-i(\alpha^A\lambda_3 + \beta^A\lambda_8)}\psi(e_A), \end{aligned} \tag{3.5}$$

no sum over $A$.

The rows $(e_B + ie_C)$ of $\psi(e_A)$ are determined by the invariant planes $(e_B, e_C)$ of the automorphism $\sigma^A$ and the ordering of the rows is such that the first elements along the column define imaginary units of a quaternion subalgebra in a positive sense. Hence the cyclic permutation of the rows of $\psi(e_A)$ is immaterial for the subsequent discussion. Canonical automorphisms involve two independent parameters each and hence generate a 14-parameter Lie group. That this is the complete automorphism group of octonions follows from the well-known result that the automorphism group of octonions is a 14-parameter Lie group of type $G_2$.

Consequently, every automorphism of Cayley numbers can be written as a product of canonical automorphisms and; as stated above, every automorphism can be reduced to the canonical form in a suitably chosen basis.

## 4. THE LIE ALGEBRA OF $G_2$ AND ITS IMBEDDING IN $SO(7)$

Using the result that canonical automorphisms generate the Lie group $G_2$, let us now find its Lie algebra. As parameters corresponding to the generators of $G_2$, we shall take $\alpha^A$ and $\beta^A$ rather than the angles $\phi_1^A$ and $\phi_2^A$. Now consider a canonical automorphism $\sigma_A$ with $\beta^A = 0$, then

$$\sigma_A\psi(e_A) = \psi'(e_A) = e^{\alpha^A\lambda_3 e_A}\begin{pmatrix} e_B + ie_C \\ e_D + ie_E \\ e_F + ie_G\end{pmatrix} = \begin{pmatrix} e'_B + ie'_C \\ e'_D + ie'_E \\ e'_F + ie'_G\end{pmatrix}$$

which gives

$$\begin{pmatrix} e'_B \\ e'_C\end{pmatrix} = \begin{pmatrix}\cos\alpha^A & \sin\alpha^A \\ -\sin\alpha^A & \cos\alpha^A\end{pmatrix}\begin{pmatrix} e_B \\ e_C\end{pmatrix},$$

$$\begin{pmatrix} e'_D \\ e'_E\end{pmatrix} \quad \begin{pmatrix}\cos\alpha^A & -\sin\alpha^A \\ \sin\alpha^A & \cos\alpha^A\end{pmatrix}\begin{pmatrix} e_D \\ e_E\end{pmatrix}.$$

Therefore the group action with parameter $\alpha^A$ induces rotations in the invariant planes $(e_B, e_C)$ and $(e_D, e_E)$ through angles $\alpha^A$ and $-\alpha^A$, respectively. Hence the generator of this group action is

$$(J_{BC} - J_{DE}),$$

where $J_{BC}$ and $J_{DE}$ are the anti-Hermitian rotation generators.

$$J_{BC} = -J_{CB} = -J^\dagger_{BC}.$$

Similarly, the generator corresponding to the group action with parameter $\beta^A$ is

$$(1/\sqrt{3})(J_{BC} + J_{DE} - 2J_{FG}).$$

Since the indices go from 1 to 7, the 14 generators thus constructed will form a subalgebra of $SO(7)$ if they close under commutation. As will be shown below, they indeed close under commutation and hence establish the known result that $G_2$ is a subgroup of $SO(7)$. The remaining generators of $SO(7)$ can be taken as

$$(J_{BC} + J_{DE} + J_{FG})$$

which are generated by the mappings:

$$\psi(e_A) \to e^{\gamma^A I_3 e_A}\psi(e_A) = e^{-i\gamma^A I_3}\psi(e_A),$$

where $I_3$ is the $3\times 3$ identity matrix.

For reasons that will be clear later, we shall modify the above basis for $G_2$ and $SO(7)$ and consider the following Hermitian basis:

$$\begin{aligned}
F_1 &= -i(J_{24} - J_{51}), & M_1 &= (i/\sqrt{3})(J_{24} + J_{51} - 2J_{73}), & N_1 &= i(J_{24} + J_{51} + J_{73}),\\
F_2 &= i(J_{54} - J_{12}), & M_2 &= (-i/\sqrt{3})(J_{54} + J_{12} - 2J_{67}), & N_2 &= i(J_{54} + J_{12} + J_{67}),\\
F_3 &= -i(J_{14} - J_{25}), & M_3 &= (i/\sqrt{3})(J_{14} + J_{25} - 2J_{36}), & N_3 &= i(J_{14} + J_{25} + J_{36}),\\
F_4 &= -i(J_{16} - J_{43}), & M_4 &= (i/\sqrt{3})(J_{16} + J_{43} - 2J_{72}), & N_4 &= i(J_{16} + J_{43} + J_{72}),\\
F_5 &= -i(J_{46} - J_{31}), & M_5 &= (i/\sqrt{3})(J_{46} + J_{31} - 2J_{57}), & N_5 &= i(J_{46} + J_{31} + J_{57}),\\
F_6 &= -i(J_{35} - J_{62}), & M_6 &= (i/\sqrt{3})(J_{35} + J_{62} - 2J_{71}), & N_6 &= i(J_{35} + J_{62} + J_{71}),\\
F_7 &= i(J_{65} - J_{23}), & M_7 &= (-i/\sqrt{3})(J_{65} + J_{23} - 2J_{47}), & N_7 &= i(J_{65} + J_{23} + J_{47}).
\end{aligned}$$

Note that the subscript $A$ in $M_A$ and $N_A$ does not refer to the basis element left invariant by the corresponding canonical automorphism. We have used the above numbering to be consistent with Gell-Mann's notation for $SU(3)$, which is a subgroup of $G_2$ as is shown in the next section. The generators $F_A$ and $M_A, A = 1, \ldots, 7$ close under commutation and form the Lie algebra of $G_2$. Denoting the Lie algebra of a group $G$ by $\mathcal{L}G$ we have

$$\mathcal{L}G_2 = F_A \oplus M_A,$$
$$\mathcal{L}SO(7) = F_A \oplus M_A \oplus N_A.$$

The structure constants of $\mathcal{L}G_2$ are given in Appendix A. In the following sections, we shall denote the generators of $SO(7)$ by capital Latin letters $F_A, M_A$, and $N_A$ and the corresponding $n\times n$ matrix representation of these generators by $\Lambda^{(n)}_A$, $\mu^{(n)}_A$, and $\nu^{(n)}_A$, respectively. The parameters corresponding to the generators $M_A, N_A$, and $F_A$ will be denoted by $m_A, n_A$, and $f_A$, respectively.

## 5. THE $SU(3)$ AND $SU(2)\times SU(2)$ SUBGROUPS OF $G_2$

From the above table of the generators of $G_2$ one can easily observe that there are eight generators annihilating[24] a given basis element $e_A$. The generators annihilating, say, $e_7$ are $F_A$, $A = 1, \ldots, 7$, and $F_8 = -M_3$. They close under commutation and form the Lie algebra of $SU(3)$:

$$[F_a, F_b] = 2if_{abc}F_c, \qquad a, b, c = 1, \ldots, 8,$$

where $f_{abc}$ are the usual structure constants of Gell-Mann. Hence the automorphisms of the Cayley algebra leaving a basis element $e_A$ invariant form a subgroup $SU(3)$ of $G_2$. Since $G_2$ has only real representations,[25] only the real representations of $SU(3)$ can occur in the representations of $G_2$. For example, in the seven-dimensional representations of $G_2$, the only nontrivial real representation of $SU(3)$ that can occur is the 6-dimensional representation $3\oplus\bar{3}$. This six-dimensional representation can be constructed from Gell-Mann's matrices as follows:

$$
\begin{aligned}
\Lambda_1^{(6)} &= \sigma_2 \otimes \lambda_1 = -i(\Sigma_{24} - \Sigma_{51}),\\
\Lambda_2^{(6)} &= 1_2 \otimes \lambda_2 = i(\Sigma_{54} - \Sigma_{12}),\\
\Lambda_3^{(6)} &= \sigma_2 \otimes \lambda_3 = -i(\Sigma_{14} - \Sigma_{25}),\\
\Lambda_4^{(6)} &= \sigma_2 \otimes \lambda_4 = -i(\Sigma_{16} - \Sigma_{43}),\\
\Lambda_5^{(6)} &= 1_2 \otimes \lambda_5 = -i(\Sigma_{46} - \Sigma_{31}),\\
\Lambda_6^{(6)} &= \sigma_2 \otimes \lambda_6 = -i(\Sigma_{35} - \Sigma_{62}),\\
\Lambda_7^{(6)} &= 1_2 \otimes \lambda_7 = i(\Sigma_{65} - \Sigma_{23}),\\
\Lambda_8^{(6)} &= \sigma_2 \otimes \lambda_8 = \frac{-i}{\sqrt{3}}(\Sigma_{14} + \Sigma_{25} - 2\Sigma_{36}),
\end{aligned}
\tag{5.1}
$$

where $\otimes$ denotes the direct product of matrices and $\lambda_a$ are the Gell-Mann's $\lambda$ matrices and $\sigma_2$ is the Pauli matrix

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $\Sigma_{mn}$ are the $6 \times 6$ matrix representation of the generators $J_{mn}$ of $SO(6)$. This construction shows clearly that $\Lambda_a^{(6)}$ can be imbedded into the seven-dimensional representations of $G_2$ as the representations of the generators $F_a$.

$G_2$ also has an $SU(2) \times SU(2)$ subgroup. The $SU(2) \times SU(2)$ subgroup involving the isospin subgroup of $SU(3)$ is generated by $F_1, F_2, F_3$ and $M_1, M_2, M_3$

$$
\begin{aligned}
&[F_i, F_j] = 2i\epsilon_{ijk}F_k, \qquad i,j,k = 1,2,3,\\
&[F_i, M_j] = 0, \qquad [M_i, M_j] = (2i/\sqrt{3})\epsilon_{ijk}M_k.
\end{aligned}
\tag{5.2}
$$

$SU(2) \times SU(2)$ subgroup of $G_2$ arises from the fact that octonions can be constructed from two quaternions.[26]

The $SU(3)$ subgroup can be imbedded in $G_2$ in seven different ways and for each imbedding of $SU(3)$ there are three different imbeddings of $SU(2) \times SU(2)$ involving $I$, $U$, $V$ spin subgroup of $SU(3)$.

Now consider the seven-dimensional action of $G_2'$ on the octonion units $e_A$ as the automorphism action. Then under the $SU(3)$ subgroup, six of the basis elements $e_A$ transform like the six-dimensional real representation of $SU(3)$ $(3 \oplus \bar{3})$ and the seventh element is an $SU(3)$ scalar. Under $SU(2) \times SU(2)$, four of the elements $e_A$ transform like the $(1/2, 1/2)$ representations and the remaining three transform as $(0, 1)$ representations.

## 6. SPLIT OCTONIONS AND SPLIT $G_2$

Above we have considered the automorphism group of real octonions with basis $1, e_A$. We saw that if we denote the parameters corresponding to the generators $F_A$ and $M_A$ by $f_A$ and $\sqrt{3}m_A$ and the seven-dimensional representation of these generators by $\Lambda_A^{(7)}$ and $\mu_A^{(7)}$ then the most general automorphism of real octonions are given by the transformation

$$[e] \to [e'] = \exp[-if_A\Lambda_A^{(7)} - i\sqrt{3}m_A\mu_A^{(7)}][e] = e^X[e], \tag{6.1}$$

where

$$[e] = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{pmatrix} \quad \text{and} \quad X = -i(f_A\Lambda_A^{(7)} + \sqrt{3}m_A\mu_A^{(7)}).$$

$X$ is given explicitly by

$$X = \begin{pmatrix}
0 & -(f_2+m_2) & -(f_5+m_5) & (m_3-f_3) & -(f_1+m_1) & (m_4-f_4) & 2m_6\\
(f_2+m_2) & 0 & -(f_7+m_7) & (m_1-f_1) & (f_3+m_3) & -(f_6+m_6) & 2m_4\\
(f_5+m_5) & (f_7+m_7) & 0 & -(f_4+m_4) & (m_6-f_6) & -2m_3 & 2m_1\\
(f_3-m_3) & (f_1-m_1) & (f_4+m_4) & 0 & (m_2-f_2) & (m_5-f_5) & 2m_7\\
(f_1+m_1) & -(f_3+m_3) & (f_6-m_6) & (f_2-m_2) & 0 & (m_7-f_7) & -2m_5\\
(f_4-m_4) & (f_6+m_6) & 2m_3 & (f_5-m_5) & (f_7-m_7) & 0 & 2m_2\\
-2m_6 & -2m_4 & -2m_1 & -2m_7 & 2m_5 & -2m_2 & 0
\end{pmatrix}$$

$$X^\dagger = -X$$

If we transform the real basis $[e]$ into what we shall call the split basis $[d]$ where

$$[d] = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_1^* \\ u_2^* \\ u_3^* \\ (i/\sqrt{2})e_7 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(e_1 + ie_4) \\ \frac{1}{2}(e_2 + ie_5) \\ \frac{1}{2}(e_3 + ie_6) \\ \frac{1}{2}(e_1 - ie_4) \\ \frac{1}{2}(e_2 - ie_5) \\ \frac{1}{2}(e_3 - ie_6) \\ (i/\sqrt{2})e_7 \end{bmatrix} = \begin{pmatrix} \psi(e_7) \\ \psi^*(e_7) \\ ie_7/\sqrt{2} \end{pmatrix}$$

then the automorphisms are generated by the mapping

$$[d] \to [d'] = e^{iZ}[d] \tag{6.2}$$

where $Z$ is

$$
Z = \begin{bmatrix}
(f_3-m_3) & (f_1+if_2) & (f_4+if_5) & 0 & -(m_1-im_2) & (m_4+im_5) & -\sqrt{2}(m_6+im_7) \\
(f_1-if_2) & -(f_3+m_3) & (f_6+if_7) & (m_1-im_2) & 0 & -(m_6-im_7) & -\sqrt{2}(m_4-im_5) \\
(f_4-if_5) & (f_6-if_7) & 2m_3 & -(m_4+im_5) & (m_6-im_7) & 0 & -\sqrt{2}(m_1+im_2) \\
0 & (m_1+im_2) & -(m_4-im_5) & -(f_3-m_3) & -(f_1-if_2) & -(f_4-if_5) & -\sqrt{2}(m_6-im_7) \\
-(m_1+im_2) & 0 & (m_6+im_7) & -(f_1+if_2) & (f_3+m_3) & -(f_6-if_7) & -\sqrt{2}(m_4+im_5) \\
(m_4-im_5) & -(m_6+im_7) & 0 & -(f_4+if_5) & -(f_6+if_7) & -2m_3 & -\sqrt{2}(m_1-im_2) \\
-\sqrt{2}(m_6-im_7) & -\sqrt{2}(m_4+im_5) & -\sqrt{2}(m_1-im_2) & -\sqrt{2}(m_6+im_7) & -\sqrt{2}(m_4-im_5) & -\sqrt{2}(m_1+im_2) & 0
\end{bmatrix}
$$

$$Z^\dagger = Z,$$

and $Z$ can be written in the form

$$
Z = \begin{pmatrix} U_3 & O^\dagger & \mathbf{x}^T \\ O & -U_3^* & \mathbf{x}^\dagger \\ \mathbf{x}^* & \mathbf{x} & 0 \end{pmatrix},
$$

where

$$
U_3 = \begin{pmatrix} (f_3-m_3) & (f_1+if_2) & (f_4+if_5) \\ (f_1-if_2) & -(f_3+m_3) & (f_6+if_7) \\ (f_4-if_5) & (f_6-if_7) & 2m_3 \end{pmatrix},
$$

$$
O = \begin{pmatrix} 0 & (m_1+im_2) & -(m_4-im_5) \\ -(m_1+im_2) & 0 & (m_6+im_7) \\ (m_4-im_5) & -(m_6+im_7) & 0 \end{pmatrix},
$$

$$\mathbf{x} = -\sqrt{2}[(m_6+im_7)(m_4-im_5)(m_1+im_2)]$$

and $O_{ij} = -(1/\sqrt{2})\epsilon_{ijk}x_k$

$$i,j,k = 1,2,3.$$

Note that $U_3$ and $O$ are the three-dimensional representations of the Lie algebras of $SU(3)$ and complex $SO(3)$.

If we further split the identity and consider the split octonions with basis $u_1u_2u_3u_0u_1^*u_2^*u_3^*u_0^*$, defined above, then the automorphism group $G_2$ will act on this basis by an 8-dimensional reducible representation.

$$
G_2 : [s] \to [s'] = e^{iY}[s], \quad [s] = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_0 \\ u_1^* \\ u_2^* \\ u_3^* \\ u_0^* \end{bmatrix}, \tag{6.3}
$$

where $Y$ is given explicitly as

$$
Y = \begin{bmatrix}
(f_3-m_3) & (f_1+if_2) & (f_4+if_5) & -(m_6+im_7) & 0 & -(m_1-im_2) & (m_4+im_5) & (m_6+im_7) \\
(f_1-if_2) & -(f_3+m_3) & (f_6+if_7) & -(m_4-im_5) & (m_1-im_2) & 0 & -(m_6-im_7) & (m_4-im_5) \\
(f_4-if_5) & (f_6-if_7) & 2m_3 & -(m_1+im_2) & -(m_4+im_5) & (m_6-im_7) & 0 & (m_1+im_2) \\
-(m_6-im_7) & -(m_4+im_5) & -(m_1-im_2) & 0 & -(m_6+im_7) & -(m_4-im_5) & -(m_1+im_2) & 0 \\
0 & -(m_1+im_2) & -(m_4-im_5) & -(m_6-im_7) & -(f_3-m_3) & -(f_1-if_2) & -(f_4-if_5) & (m_6-im_7) \\
-(m_1+im_2) & 0 & (m_6+im_7) & -(m_4+im_5) & -(f_1+if_2) & (f_3+m_3) & -(f_6-if_7) & (m_4+im_5) \\
(m_4-im_5) & -(m_6+im_7) & 0 & -(m_1-im_2) & -(f_4+if_5) & -(f_6+if_7) & -2m_3 & (m_1-im_2) \\
(m_6-im_7) & (m_4+im_5) & (m_1-im_2) & 0 & (m_6+im_7) & (m_4-im_5) & (m_1+im_2) & 0
\end{bmatrix}
$$

$$Y^\dagger = Y,$$

and $Y$ can be written in the form

$$
Y = \begin{bmatrix} U_3 & \mathbf{x}^T/\sqrt{2} & O^\dagger & -\mathbf{x}^T/\sqrt{2} \\ \mathbf{x}^*/\sqrt{2} & 0 & \mathbf{x}/\sqrt{2} & 0 \\ O & \mathbf{x}^\dagger/\sqrt{2} & -U_3^* & -\mathbf{x}^\dagger/\sqrt{2} \\ -\mathbf{x}^*/\sqrt{2} & 0 & -\mathbf{x}/\sqrt{2} & 0 \end{bmatrix}
$$

or alternatively as

$$
Y = \begin{pmatrix} D & -E^* \\ E & -D^* \end{pmatrix},
$$

where

$$
D = \begin{pmatrix} U_3 & \mathbf{x}^T/\sqrt{2} \\ \mathbf{x}^*/\sqrt{2} & 0 \end{pmatrix}, \quad E = \begin{pmatrix} O & \mathbf{x}^\dagger/\sqrt{2} \\ -\mathbf{x}^*/\sqrt{2} & 0 \end{pmatrix}.
$$

Note that the matrices $D$ and $E$ are not independent. $D$ involves all the parameters of $E$. Keeping this point in mind, we see that $E \in$ complex $\mathcal{L}SO(4)$, $D \in \mathcal{L}SU(4)$. Matrices $E$ close under the Lie product. Matrices $D$ need one more generator to close under Lie product to form the four-dimensional representation of the Lie algebra of $SU(4)$.

The above form of $G_2$ as the automorphism group of split octonions is called the split $G_2$. Under the $SU(3)$ subgroup of split $G_2$ leaving $u_0$ and $u_0^*$ invariant, the three split octonions $(u_1, u_2, u_3)$ transform like a unitary triplet (quarks) and the complex conjugate octonions $(u_1^*, u_2^*, u_3^*)$ transform like a unitary antitriplet (antiquarks). This property of split octonions is physically very important and plays a crucial role in obtaining a quark structure from the octonionic representations of Poincaré group.[14]

## 7. QUARK STRUCTURE IN THE SPLIT BASIS

To see another physically interesting property of split $G_2$ let us define the following basis for its Lie algebra[27]:

$$\begin{aligned}
&E_{12} = \tfrac{1}{2}(F_1 + iF_2), \quad E_{21} = \tfrac{1}{2}(F_1 - iF_2),\\
&E_{13} = \tfrac{1}{2}(F_4 + iF_5), \quad E_{31} = \tfrac{1}{2}(F_4 - iF_5),\\
&E_{23} = \tfrac{1}{2}(F_6 + iF_7), \quad E_{32} = \tfrac{1}{2}(F_6 - iF_7),\\
&F_3 = (E_{11} - E_{22}), \quad F_8 = (1/\sqrt{3})(E_{11} + E_{22} - 2E_{33}),\\
&Q_1 = \tfrac{1}{2}(M_6 + iM_7), \quad Q_1^\dagger = \tfrac{1}{2}(M_6 - iM_7),\\
&Q_2 = \tfrac{1}{2}(M_4 - iM_5), \quad Q_2^\dagger = \tfrac{1}{2}(M_4 + iM_5),\\
&Q_3 = \tfrac{1}{2}(M_1 + iM_2), \quad Q_3^\dagger = \tfrac{1}{2}(M_1 - iM_2),
\end{aligned} \tag{7.1}$$

where the expressions for $\Lambda_3$ and $\Lambda_8$ are purely formal at this point and will be explained shortly. In this basis commutation relations of split $G_2$ have the form:

$$\begin{aligned}
&[Q_i, Q_j] = -(2/\sqrt{3})\epsilon_{ijk}Q_k^\dagger,\\
&[Q_i, Q_j^\dagger] = T_{ij}, \qquad i,j = 1,2,3,\\
&[E_{ij}, Q_k] = \delta_{jk}Q_i,\\
&[T_{ii}, T_{jj}] = 0, \qquad [E_{ij}, E_{ji}] = (T_{ii} - T_{jj}),\\
&[T_{ii}, E_{ij}] = E_{ij}, \qquad [T_{jj}, E_{ij}] = -E_{ij},\\
&[E_{ij}, E_{jk}] = E_{ik}, \qquad [E_{ji}, E_{kj}] = -E_{ki}
\end{aligned} \tag{7.2}$$

where $T_{ij}$ is defined as

$$T_{ij} = E_{ij}, \qquad i \neq j,$$

and

$$\begin{aligned}
T_{11} &= \tfrac{1}{2}F_3 + (1/2\sqrt{3})F_8 = \tfrac{1}{3}(2E_{11} - E_{22} - E_{33}),\\
T_{22} &= -\tfrac{1}{2}F_3 + (1/2\sqrt{3})F_8 = \tfrac{1}{3}(-E_{11} + 2E_{22} - E_{33}),\\
T_{33} &= -(1/\sqrt{3})F_8 = \tfrac{1}{3}(-E_{11} - E_{22} + 2E_{33}).
\end{aligned} \tag{7.3}$$

The generators $T_{ij}$ form the subalgebra $SU(3)$. The generators $F_3$ and $F_8$ form a Cartan subalgebra of both $SU(3)$ and $G_2$. If we assign quantum numbers to the generators of split $G_2$, i.e., to its adjoint representation, using as the generators of third component of isospin and hypercharge the generators $I_3 = \frac{1}{2}F$ and $Y = (1/\sqrt{3})F_8$, we find that three quarks, three antiquarks, and eight mesons can be imbedded in the adjoint representation of split $G_2$, i.e., we can have the correspondence

$$\begin{aligned}
&Q_1 \leftrightarrow p \text{ quark}, \qquad Q_1^\dagger \leftrightarrow \bar{p},\\
&Q_2 \leftrightarrow n \text{ quark}, \qquad Q_2^\dagger \leftrightarrow \bar{n}\\
&Q_3 \leftrightarrow \lambda \text{ quark}, \qquad Q_3^\dagger \leftrightarrow \bar{\lambda}\\
&E_{12} \leftrightarrow \pi^+ \ (\text{or } \rho^+), \qquad E_{21} \leftrightarrow \pi^- \ (\text{or } \rho^-),\\
&E_{13} \leftrightarrow K^+ \ (K^{*+}), \qquad E_{31} \leftrightarrow K^- \ (K^{*-}),\\
&E_{23} \leftrightarrow K^0 \ (K^{*0}), \qquad E_{32} \leftrightarrow \overline{K^0} \ (\overline{K^{*0}}),\\
&\Lambda_3 \leftrightarrow \pi^0 \ (\rho^0), \qquad \Lambda_8 \leftrightarrow \eta \ (\omega_8),
\end{aligned}$$

This identification agrees with Gell-Mann's quark model in the assignment of the quantum numbers $I_3$ and $Y$ and differs from it in the assignment of baryon number. If one uses the generator $N_3$ of $SO(7)$ as the baryon number generator, one gets the result that mesons are assigned zero baryon number as they must be but that the generators $Q_i$ ($\leftrightarrow$ quarks) and the generators $Q_i^\dagger$ ($\leftrightarrow$ antiquarks) do not have well-defined baryon numbers.[28] These (pseudo-quark) generators $Q_i$ have the interesting property that they generate the (anti-pseudo-quarks) $Q_i^\dagger$ under commutation, i.e.,

$$[Q_i, Q_j] = -(2/\sqrt{3})\epsilon_{ijk}Q_k^\dagger$$

and the $SU(3)$ subalgebra (mesons) under Lie triple product, i.e.

$$[Q_i, [Q_j, Q_k]] = -(2/\sqrt{3})\epsilon_{jkl}T_{il}. \tag{7.4}$$

## 8. AN OCTONIONIC REPRESENTATION OF SPLIT $G_2$

The split octonions

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

transform as the three-dimensional irreducible representation of the $SU(3)$ subgroup of split $G_2$. But the elements

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_0 \end{pmatrix}$$

do not form a four-dimensional irreducible representation of split $G_2$. In fact the lowest nontrivial representation of $G_2$ is seven dimensional. Yet the action of $G_2$ on the basis

$$[s] = \begin{pmatrix} u \\ u^* \end{pmatrix}$$

is completely defined by its action on $u$, i.e., if

$$G_2: \begin{matrix} u \to u' \\ u^* \to (u^*)' \end{matrix},$$

then

$$(u^*)' = (u')^*.$$

The action of $G_2$ generators on $u$ can be represented by multiplication with octonion units in the following compact form:

$$E_{ij}u = u_i(u_j^* u), \qquad u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_0 \end{pmatrix}, \quad i,j = 1,2,3,$$

$$\begin{aligned}
Q_i u &= -(1/\sqrt{3})uu_i + (1/\sqrt{3})(u_i u_0)u = -(1/\sqrt{3})uu_i,\\
Q_i^\dagger u &= -(1/\sqrt{3})uu_i^* + (1/\sqrt{3})(u_i^* u_0)u = -(1/\sqrt{3})[u, u_i^*].
\end{aligned}$$

with

$$\begin{aligned}
F_3 u &= (E_{11} - E_{22})u = u_1(u_1^* u) - u_2(u_2^* u),\\
F_8 u &= (1/\sqrt{3})(E_{11} + E_{22} - 2E_{33})u\\
&= (1/\sqrt{3})\{u_1(u_1^* u) + u_2(u_2^* u) - 2u_3(u_3^* u)\},
\end{aligned}$$

which justifies the formal expressions for $F_3$ and $F_8$ given above. Thus the above form of the Lie algebra action of $G_2$ generates an octonionic representation of split $\mathcal{L}G_2$. The automorphism group of real octonions can also be shown in this form because a real octonion

$$\Phi = \Phi_0 + \Phi_A e_A$$

can be written as

$$\begin{aligned}\Phi = 2\ \mathrm{Re}\{&(\phi_0 - i\phi_7)\tfrac{1}{2}(1 + ie_7)\\ &+ (\phi_1 - i\phi_4)\tfrac{1}{2}(e_1 + ie_4)\\ &+ (\phi_2 - i\phi_5)\tfrac{1}{2}(e_2 + ie_5)\\ &+ (\phi_3 - i\phi_6)\tfrac{1}{2}(e_3 + ie_6)\},\end{aligned}$$

where Re refers to the real part with respect to the complex unit $i$. Then

$$\Phi = \phi^\dagger u + \text{c.c.} = \phi^\dagger u + \phi^T u^*,$$

where

$$\phi = \begin{pmatrix}\phi_1 + i\phi_4\\ \phi_2 + i\phi_5\\ \phi_3 + i\phi_6\\ \phi_0 + i\phi_7\end{pmatrix} \quad u = \begin{pmatrix}u_1\\ u_2\\ u_3\\ u_0\end{pmatrix}$$

Therefore, the action of Aut**O** on $e_A$ is completely defined by its action on $u$.

## 9. 8 x 8 MATRIX FORMULATION OF THE CAYLEY ALGEBRA

In Appendix D, we give two constructions of Cayley algebra in terms of $3 \times 3$ $\lambda$-matrices and $4 \times 4$ $\gamma$-matrices. In this section, we shall study the $8 \times 8$ matrix construction of octonions. Consider the column matrix

$$[s] = \begin{bmatrix}u\\ u^*\end{bmatrix}$$

of split octonions. Define the conjugate matrix $[\bar{s}]^\dagger$ as $[\bar{s}]^\dagger = [\bar{s}]^{*T}$

$$= (-u_1^*, -u_2^*, -u_3^*, u_0, -u_1, -u_2, -u_3, u_0^*),$$

where the overbar denotes octonion conjugation, * denotes complex conjugation, and $T$ is the usual transposition. Then the product $[s][\bar{s}]^\dagger$ can be written in the form

$$[s][\bar{s}]^\dagger = \tfrac{1}{2}(1 - i\boldsymbol{\Gamma}_A e_A), \quad A = 1, \ldots, 7, \tag{9.1}$$

where $\boldsymbol{\Gamma}_A$ are $8 \times 8$ matrices given by:

$$\begin{aligned}\boldsymbol{\Gamma}_1 &= -\sigma_1 \otimes \sigma_1 \otimes \sigma_2 = -\tau_1\rho_1\sigma_2,\\ \boldsymbol{\Gamma}_2 &= -\sigma_1 \otimes \sigma_2 \otimes I = -\tau_1\rho_2,\\ \boldsymbol{\Gamma}_3 &= \sigma_1 \otimes \sigma_3 \otimes \sigma_2 = \tau_1\rho_3\sigma_2,\\ \boldsymbol{\Gamma}_4 &= \sigma_2 \otimes \sigma_2 \otimes \sigma_1 = \tau_2\rho_2\sigma_1,\\ \boldsymbol{\Gamma}_5 &= -\sigma_2 \otimes \sigma_2 \otimes \sigma_3 = -\tau_2\rho_2\sigma_3,\\ \boldsymbol{\Gamma}_6 &= \sigma_2 \otimes I \otimes \sigma_2 = \tau_2\sigma_2,\\ \boldsymbol{\Gamma}_7 &= -\sigma_3 \otimes I \otimes I = -\tau_3\end{aligned} \tag{9.2}$$

($\otimes$ denotes direct product of the Pauli matrices $\sigma_1, \sigma_2, \sigma_3, I$),

where we have defined

$$\begin{aligned}I \otimes I \otimes \sigma_i &= \sigma_i,\\ I \otimes \sigma_i \otimes I &= \rho_i,\\ \sigma_i \otimes I \otimes I &= \tau_i,\end{aligned}$$

and chosen a representation in which $\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2, \boldsymbol{\Gamma}_3$ are imaginary and $\boldsymbol{\Gamma}_4, \boldsymbol{\Gamma}_5, \boldsymbol{\Gamma}_6, \boldsymbol{\Gamma}_7$ are real. These seven matrices $\boldsymbol{\Gamma}_A$ are Hermitian and satisfy the anticommutation relations

$$\{\boldsymbol{\Gamma}_A, \boldsymbol{\Gamma}_B\} = 2\delta_{AB}, \quad \boldsymbol{\Gamma}_A^\dagger = \boldsymbol{\Gamma}_A. \tag{9.3}$$

Now number the rows of the column vector $[s]$ from 1 to 8 and define a mapping $L_{u_i}$ on $[s]$ as the mapping induced by multiplication from the left by the element $u_i$.[29] Then a simple calculation gives the result that

$$\begin{aligned}L_{u_1}^{(8)} + L_{u_1^*}^{(8)} \quad L_{u_1+u_1^*}^{(8)} &= L_{e_1}^{(8)} = \bar{\Sigma}_{18} - \bar{\Sigma}_{27} + \bar{\Sigma}_{36} - \bar{\Sigma}_{45},\\ L_{u_1-u_1^*}^{(8)} &= L_{ie_4}^{(8)} = \bar{E}_{81} + \bar{E}_{18}\\ &\qquad - \bar{E}_{54} - \bar{E}_{45} - \bar{E}_{36} - \bar{E}_{63} + \bar{E}_{27} + \bar{E}_{72},\\ L_{u_2+u_2^*}^{(8)} &= L_{e_2}^{(8)} = \bar{\Sigma}_{28} - \bar{\Sigma}_{46} + \bar{\Sigma}_{53} + \bar{\Sigma}_{17},\\ L_{u_2-u_2^*}^{(8)} &= L_{ie_5}^{(8)} = -\bar{E}_{71} - \bar{E}_{17}\\ &\qquad + \bar{E}_{53} + \bar{E}_{35} - \bar{E}_{46} - \bar{E}_{64} + \bar{E}_{28} + \bar{E}_{82},\\ L_{u_3+u_3^*}^{(8)} &= L_{e_3}^{(8)} = \bar{\Sigma}_{61} - \bar{\Sigma}_{52} - \bar{\Sigma}_{47} + \bar{\Sigma}_{38},\\ L_{u_3-u_3^*}^{(8)} &= L_{ie_6}^{(8)} = \bar{E}_{61} + \bar{E}_{16}\\ &\qquad - \bar{E}_{52} - \bar{E}_{25} - \bar{E}_{47} - \bar{E}_{74} + \bar{E}_{38} + \bar{E}_{83},\\ L_{u_4-u_4^*}^{(8)} &= L_{ie_7}^{(8)} = \bar{E}_{11} + \bar{E}_{22}\\ &\qquad + \bar{E}_{33} + \bar{E}_{44} - \bar{E}_{55} - \bar{E}_{66} - \bar{E}_{77} - \bar{E}_{88},\end{aligned} \tag{9.4}$$

where $\bar{E}_{ij}$ are the $8 \times 8$ matrix units and $\bar{\Sigma}_{ab} = \bar{E}_{ab} - E_{ba}$

Comparing the matrices $\boldsymbol{\Gamma}_A$ with the mappings $L_{e_A}$ considered as matrices acting on the basis $[s]$ we have

$$\begin{aligned}L_{e_1}^{(8)} &= -i\boldsymbol{\Gamma}_1, & L_{e_4}^{(8)} &= i\boldsymbol{\Gamma}_4, &&\\ L_{e_2}^{(8)} &= -i\boldsymbol{\Gamma}_2, & L_{e_5}^{(8)} &= i\boldsymbol{\Gamma}_5, & L_{e_7}^{(8)} &= i\boldsymbol{\Gamma}_7 \quad \text{on} \begin{bmatrix}u\\ u^*\end{bmatrix}.\\ L_{e_3}^{(8)} &= -i\boldsymbol{\Gamma}_3, & L_{e_6}^{(8)} &= i\boldsymbol{\Gamma}_6, &&\end{aligned} \tag{9.5}$$

From these equalities, the anticommutation relations of $\boldsymbol{\Gamma}_A$ follow automatically, since

$$L_{e_A}L_{e_B} + L_{e_B}L_{e_A} = L_{e_Ae_B + e_Be_A} = L_{-2\delta_{AB}} \tag{9.6}$$

which in turn follows from the identity

$$O_1(O_2O_3) + O_2(O_1O_3) = (O_1O_2 + O_2O_1)O_3, \qquad O_1, O_2, O_3 \in O \tag{9.7}$$

for octonions.

Now define the matrices $\boldsymbol{\Gamma}_{AB}$ as

$$\begin{aligned}\boldsymbol{\Gamma}_{AB} &= (1/2i)[\boldsymbol{\Gamma}_A, \boldsymbol{\Gamma}_B],\\ \boldsymbol{\Gamma}_{AB} &= -\boldsymbol{\Gamma}_{BA}, \quad \boldsymbol{\Gamma}_{AB}^\dagger = \boldsymbol{\Gamma}_{AB}.\end{aligned} \tag{9.8}$$

Twenty-one matrices $\boldsymbol{\Gamma}_{AB}$ form the Lie algebra of Spin (7) and $\boldsymbol{\Gamma}_{AB} \oplus \boldsymbol{\Gamma}_A$ form the Lie algebra of $SO(8)$.

Having constructed the matrices $\Gamma_A$ and $\Gamma_{AB}$ from the spinor $[s]$, we can forget about the octonionic character of $[s]$ and consider an eight-component spinor $\Psi$. Then

$$\{V_8 = \Psi^\dagger\Psi,\ V_A = \Psi^\dagger\Gamma_A\Psi\}$$

transform like a vector under $SO(8)$. Under the subgroup Spin (7), $\Psi^\dagger\Psi$ is a scalar and $\Psi^\dagger\Gamma_A\Psi = V_A$ is a vector. To characterize $G_2$, we need one more condition in addition to the requirement that $\Psi^\dagger\Psi$ be a scalar. Now $G_2$ is the automorphism group of octonions and it leaves the identity invariant. Therefore we would expect the $G_2$ subgroup of $SO(7)$ to leave $(\Psi_4^* + \Psi_8^*)(\Psi_4 + \Psi_8)$ invariant, since $(\Psi_4 + \Psi_8)$ corresponds to the identity of the octonions in the nonoctonionic formulation considered here. Thus, under $G_2$ both $\Psi^\dagger\Psi$ and $\Psi^\dagger K\Psi$ are scalars, where $K$ is the matrix

$$K = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&1 \end{pmatrix}.$$

In fact, the assertion that $\Psi^\dagger K\Psi$ is a scalar under $G_2$ can be rigorously proved by showing that only those linear combinations of the generators $\Gamma_{AB}$ of Spin (7) that belong to $G_2$ commute with the matrix $K$. To do this it is convenient to use the following expression for $K$

$$K = \tfrac{1}{4}[\mathbf{1} - i(1/3!)a_{ABC}\Gamma_A\Gamma_B\Gamma_C], \qquad A, B, C, = 1, \dots, 7, \tag{9.9}$$

where $a_{ABC}$ is a totally antisymmetric tensor and satisfies

$$a_{ABC} = 1 \text{ for } ABC = 123, 246, 435, 516, 572, 471, 673.$$

The matrix $K$ is related to the octonion conjugation matrix $O^c$ defined by $O^c[s] = [\bar{s}]$ as

$$O^c = K - \mathbf{1} \quad \text{or} \quad K = \mathbf{1} + O^c.$$

Therefore the conditions that $\Psi^\dagger K\Psi$ be a scalar is equivalent to the condition that $\Psi^\dagger O^c\Psi$ be a scalar.

The conditions that characterize $G_2$, i.e., that $\Psi^\dagger\Psi$ and $\Psi^\dagger K\Psi$ be invariant, are equivalent to saying that $G_2$ is the common subgroup of $SO(7)$ and Spin (7),[30] i.e.,

$$G_2 = \text{Spin }(7) \cap SO(7).$$

Above we showed that the matrices $\Gamma_A$ correspond to the left multiplication by octonion units $e_A$ acting on $[s]$. This does not mean that $e_A$ can be represented by matrices $\Gamma_A$ to form a Cayley algebra under the usual matrix multiplication. To get a Cayley algebra from $\Gamma_A$, we define the product of two $\Gamma$ matrices as

$$\Gamma_A \circ \Gamma_B = \tfrac{1}{2}\{\Gamma_A, \Gamma_B\} + \tfrac{1}{2}\{[\Gamma_A, \Gamma_B], M\} + \Gamma_A M \Gamma_B - \Gamma_B M \Gamma_A, \tag{9.10}$$

where

$$M = K - \tfrac{1}{4}\mathbf{1} = -\tfrac{1}{4}i(1/3!)a_{ABC}\Gamma_A\Gamma_B\Gamma_C,$$

which gives

$$\Gamma_A \circ \Gamma_B = \delta_{AB} + i a_{ABC}\Gamma_C, \qquad A, B, C = 1, \dots, 7.$$

Hence defining the multiplication by a multiple $c$ of the identity $\mathbf{1}$ as multiplication by the scalar $c$, we get a Cayley algebra with basis

$$\begin{aligned} &e_A \equiv -i\Gamma_A, \qquad 1 \equiv \mathbf{1}, \\ &(-i\Gamma_A)\circ(-i\Gamma_B) = -\delta_{AB} + a_{ABC}(-i\Gamma_C). \end{aligned} \tag{9.11}$$

## 10. IMBEDDING IN $SO(7)$ AND $SO(8)$

In the above, we have decomposed the Lie algebra of $SO(7)$ as

$$SO(7) = F_A \oplus M_A \oplus N_A,$$

where $F_A \oplus M_A$ generate the subgroup $G_2$.

The 8-dimensional representation of $G_2$ as the automorphism group of octonions acting on the basis

$$[s] = \begin{bmatrix} u \\ u^* \end{bmatrix}$$

will induce an 8-dimensional spinor representation of $SO(7)$: In fact, after some algebra, one finds that the action of $N_A$ on $[s]$ can be represented as

$$\begin{aligned} &\nu_1^{(8)} = \tfrac{1}{2}i(L_{e_6}^{(8)} - R_{e_6}^{(8)}), && \nu_2^{(8)} = -\tfrac{1}{2}i(L_{e_3}^{(8)} - R_{e_3}^{(8)}), \\ &\nu_3^{(8)} = \tfrac{1}{2}i(L_{e_7}^{(8)} - R_{e_7}^{(8)}), && \nu_5^{(8)} = -\tfrac{1}{2}i(L_{e_2}^{(8)} - R_{e_2}^{(8)}), \\ &\nu_4^{(8)} = \tfrac{1}{2}i(L_{e_5}^{(8)} - R_{e_5}^{(8)}), && \nu_7^{(8)} = -\tfrac{1}{2}i(L_{e_1}^{(8)} - R_{e_1}^{(8)}), \\ &\nu_6^{(8)} = \tfrac{1}{2}i(L_{e_4}^{(8)} - R_{e_4}^{(8)}), \end{aligned} \tag{10.1}$$

where $L_{e_A}$ and $R_{e_A}$ are left and right multiplications by the element $e_A$, respectively. Explicit matrix form of $L_{e_A}$ was given above. For the $R_{e_A}$ we have

$$\begin{aligned} R_{e_1}^{(8)} &= \bar{\Sigma}_{27} + \bar{\Sigma}_{58} + \bar{\Sigma}_{14} + \bar{\Sigma}_{63}, \\ R_{e_2}^{(8)} &= \bar{\Sigma}_{71} + \bar{\Sigma}_{35} + \bar{\Sigma}_{24} + \bar{\Sigma}_{68}, \\ R_{e_3}^{(8)} &= \bar{\Sigma}_{78} + \bar{\Sigma}_{52} + \bar{\Sigma}_{34} + \bar{\Sigma}_{16}, \\ R_{e_4}^{(8)} &= -i\{(\bar{E}_{14} + \bar{E}_{41}) \\ &\quad + (\bar{E}_{36} + \bar{E}_{63}) - (\bar{E}_{58} + \bar{E}_{85}) - (\bar{E}_{27} + \bar{E}_{72})\}, \\ R_{e_5}^{(8)} &= -i\{(\bar{E}_{17} + \bar{E}_{71}) \\ &\quad + (\bar{E}_{24} + \bar{E}_{42}) - (\bar{E}_{35} + \bar{E}_{53}) - (\bar{E}_{68} + \bar{E}_{86})\}, \\ R_{e_6}^{(8)} &= -i\{(\bar{E}_{25} + \bar{E}_{52}) \\ &\quad + (\bar{E}_{34} + \bar{E}_{43}) - (\bar{E}_{16} + \bar{E}_{61}) - (\bar{E}_{78} + \bar{E}_{87})\}, \\ R_{e_7}^{(8)} &= -i\{(\bar{E}_{44} + \bar{E}_{55} \\ &\quad + \bar{E}_{66} + \bar{E}_{77}) - (\bar{E}_{11} + \bar{E}_{22} + \bar{E}_{33} + \bar{E}_{88})\}. \end{aligned} \tag{10.2}$$

We have shown that the $8 \times 8$ matrices $\Gamma_A$ and $\Gamma_{AB}$ form an eight-dimensional representation of the Lie algebra of $SO(8)$. Since $\Gamma_A$ correspond to the left multiplication by $e_A$ acting on $[s]$, we have the result that the eight-dimensional representation of $\mathfrak{L}SO(8)$ can also be decomposed as:

$$\mathfrak{L}SO(8) = \Lambda_A^{(8)} \oplus \mu_A^{(8)} \oplus \nu_A^{(8)} \oplus \xi_A^{(8)},$$

where $\xi_A^{(8)} = \frac{1}{2}i(L_{e_A}^{(8)} + R_{e_A}^{(8)})$ corresponding to the generator

$$Z_A = \tfrac{1}{2}i(L_{e_A} + R_{e_A}).$$

Since the group $SO(8)$ has rank four, its Cartan subalgebra will be four dimensional. One can redefine the generators of $SO(8)$ such that $F_3, M_3, N_3$, and $Z_3$ form a Cartan subalgebra.

If we take $I_3 = \frac{1}{2}F_3$, $Y_3 = -(1/\sqrt{3})M_3$, $B = -\frac{1}{3}N_3$, $\frac{1}{2}Z_3$ as the Cartan subalgebra generators, we can assign the following quantum numbers to the basis elements:

| | $I_3$ | $Y_3$ | $B$ | $\frac{1}{2}Z_3$ |
|---|---|---|---|---|
| $u_1$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 |
| $u_1^*$ | $-\frac{1}{2}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ | 0 |
| $u_2$ | $-\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 |
| $u_2^*$ | $\frac{1}{2}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ | 0 |
| $u_3$ | 0 | $-\frac{2}{3}$ | $\frac{1}{3}$ | 0 |
| $u_3^*$ | 0 | $+\frac{2}{3}$ | $-\frac{1}{3}$ | 0 |
| $u_0$ | 0 | 0 | 0 | $-1$ |
| $u_0^*$ | 0 | 0 | 0 | 1 |

Therefore, under the correspondence

$(u_1, u_2, u_3) \leftrightarrow (p, n, \lambda)$ quarks

$(u_1^*, u_2^*, u_3^*) \leftrightarrow (\bar{p}, \bar{n}, \bar{\lambda})$ antiquarks

$(u_0, u_0^*) \leftrightarrow$ (core, anticore)

we have the result that $I_3, Y_3$, and $B$ act like the generators of third component of isospin, hypercharge, and baryon number. Subscript 3 in $Y_3$ refers to the fact that within $G_2$, $Y$ is the generator of the third component of an $SU(2)$ subgroup just as $I_3$ is.

## 11. REDUCTION WITH RESPECT TO THE $SU(2) \times SU(2)$ [I-SPIN-G-SPIN] SUBGROUP OF $G_2$

The generators $I_i = F_i$, $G_i = \sqrt{3}M_i$, $i = 1, 2, 3$ form an $SU(2) \times SU(2)$ subalgebra of $\mathcal{L}G_2$:

$$[I_i, I_j] = 2i\epsilon_{ijk}I_k, \qquad [G_i, G_j] = 2i\epsilon_{ijk}G_k,$$
$$[I_i, G_j] = 0, \qquad i, j, k = 1, 2, 3. \tag{11.1}$$

$I_i$ is the isospin subalgebra of the $SU(3)$ subalgebra of $\mathcal{L}G_2$ annihilating the basis element $e_7$. Now the spinors

$$\psi = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \leftrightarrow \begin{pmatrix} p \\ n \end{pmatrix}$$

and

$$\psi^* = \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} \leftrightarrow \begin{pmatrix} \bar{p} \\ \bar{n} \end{pmatrix}$$

correspond to isospin doublets and the elements $u_3, u_3^*$, $(1/\sqrt{2})ie_7$ are isospin scalars.

Consider the infinitesimal group action generated by $G_i$

$$G: \psi \to \psi' = (1 - im^3)\psi - (m^2 + im^1)\psi^G,$$
$$\psi^{G\prime} = (m^2 - im^1)\psi + (1 + im^3)\psi^G,$$

where $\psi^G$ is the $G$ parity conjugate spinor defined by

$$\psi^G = i\tau_2\psi^* \quad \text{and} \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Therefore under the $G$-spin subgroup (generated by $G_1$) of $G_2$ the spinor $\psi$ and $\psi^G$ form a $G$-spin doublet and transform as

$$G: \begin{pmatrix} \psi \\ \psi^G \end{pmatrix} \to \begin{pmatrix} \psi' \\ \psi^{G\prime} \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}\begin{pmatrix} \psi \\ \psi^G \end{pmatrix}, \tag{11.2}$$

where

$$|a|^2 + |b|^2 = 1.$$

Similarly we find that

$$\phi = \begin{pmatrix} u_3 \\ ie_7/\sqrt{2} \\ u_3^* \end{pmatrix}$$

forms a $G$-spin triplet which transforms infinitesimally as

$$G: \phi \to \phi' = \begin{pmatrix} 1 + 2im^3 & -i\sqrt{2}(m^1 + im^2) & 0 \\ -i\sqrt{2}(m^1 - im^2) & 1 & -i\sqrt{2}(m^1 + im^2) \\ 0 & -i\sqrt{2}(m^1 - im^2) & 1 - 2im^3 \end{pmatrix}\begin{pmatrix} u_3 \\ ie_7/\sqrt{2} \\ u_3^* \end{pmatrix},$$

the global form of which is

$$G: \phi \to \phi' = \begin{pmatrix} a^2 & \sqrt{2}ab & b^2 \\ -\sqrt{2}ab^* & |a|^2 - |b|^2 & \sqrt{2}a^*b \\ b^{*2} & -\sqrt{2}a^*b^* & a^{*2} \end{pmatrix}\phi. \tag{11.3}$$

An important property of $G$-spin is that its third component is proportional to hypercharge $Y$, i.e.,

$$Y = -\tfrac{1}{3}G_3$$

and hence it should properly be called hypercharge spin. This hypercharge-spin subgroup of $G_2$ commutes with the isospin subgroup generated by $F_1, F_2, F_3$. The isospin and hypercharge spin groups together generate a four-dimensional rotation group $SU(2) \times SU(2)$ which has been considered before[31] as applied to an isotopic doublet such as the nucleon or the $p$ and $n$ quarks. The multiplets $(u_1, u_2, u_2^*, -u_1^*)$ and $(u_3, (ie_7/\sqrt{2}), u_3^*)$ form the (1/2, 1/2) and (0, 1) representations of the subgroup $SU(2)_I \otimes SU(2)_Y$, respectively. The $SU(3)$ singlet $(ie_7/\sqrt{2})$ is not an hypercharge spin singlet. It transforms like the third component of an hypercharge triplet. We shall call it the vacuan $v$. Therefore the lowest-dimensional representation of $G_2$ has the root system shown is Fig. 3. Above we defined the $G$-parity conjugate spinor $\psi^G$ of an isospin doublet $\psi$ as

$$\psi^G = Ce^{i\pi I_2}\psi,$$

where $I_2$ is the second component of isospin and $C$ is charge conjugation which in our case is taken as complex conjugation. We will generalize this $G$-parity concept to other charge space $SU(2)$ groups as follows: Write the above equation as

$$\Psi_I^{G^I} = Ce^{i\pi I_2}\psi_I; \tag{11.3a}$$

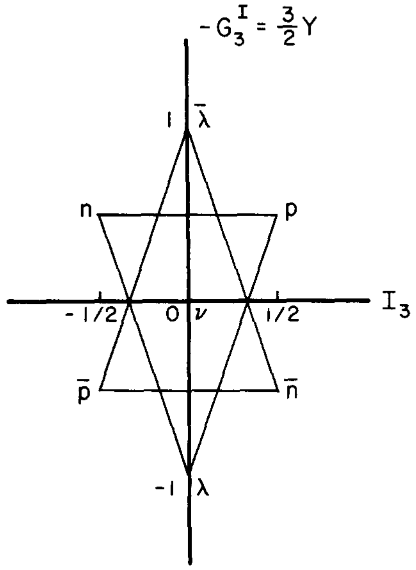


FIG. 3.

then for $U$ and $V$ spins we can define

$$\psi_U^{G^U} = Ce^{i\pi U_2}\psi_U \tag{11.3b}$$

$$\psi_V^{G^V} = Ce^{i\pi V_2}\psi_V \tag{11.3c}$$

Then under the $SU(2)_U \otimes SU(2)_{G^U}$ subgroup of $G_2$ generated by

$$\begin{aligned}
&U_1 = \tfrac{1}{2}F_6, \quad U_2 = \tfrac{1}{2}F_7,\\
&U_3 = \tfrac{1}{4}(-\sqrt{3}M_3 - F_3),\\
&G_1^U = (\sqrt{3}/2)M_6, \quad G_2^U = (\sqrt{3}/2)M_7,\\
&G_3^U = (\sqrt{3}/4)\left(\frac{1}{\sqrt{3}}F_3 - M_3\right).
\end{aligned} \tag{11.4a}$$

The spinor

$$\begin{pmatrix}\psi_U\\ \psi_U^{G^U}\end{pmatrix}$$

transforms like the (1/2, 1/2) representation, where

$$\psi_U = \begin{pmatrix}u_2\\ u_3\end{pmatrix} \leftrightarrow \begin{pmatrix}n\\ \lambda\end{pmatrix},$$

and the spinor

$$\phi_U = \begin{pmatrix}u_1\\ (i/\sqrt{2})e_7\\ u_1^*\end{pmatrix}$$

will transform as the (0, 1) representation. Same thing applies for the $SU(2)_V \otimes SU(2)_{G^V}$ subgroup generated by

$$\begin{aligned}
&V_1 = \tfrac{1}{2}F_4, \quad V_2 = \tfrac{1}{2}F_5,\\
&V_3 = \tfrac{1}{4}(F_3 - \sqrt{3}M_3),\\
&G_1^V = (\sqrt{3}/2)M_4, \quad G_2^V = (\sqrt{3}/2)M_5,
\end{aligned} \tag{11.4b}$$

$$G_3^V = (\sqrt{3}/4)(M_3 + \sqrt{3}F_3), \tag{11.4c}$$

except we have to replace $\psi_U$ by

$$\psi_V = \begin{pmatrix}u_3\\ u_1\end{pmatrix} \leftrightarrow \begin{pmatrix}\lambda\\ p\end{pmatrix}$$

and $\phi_U$ by

$$\phi_V = \begin{pmatrix}u_2\\ (i/\sqrt{2})e_7\\ u_2^*\end{pmatrix}.$$

Hence, we have the result that just as the group $SU(3)$ contains three overlapping $SU(2)$ groups corresponding to $I$, $U$, $V$ spins, the group $G_2$ contains three overlapping $SU(2) \times SU(2)_G$ groups corresponding to $I$, $U$, $V$ spins together with their generalized $G$-parity extensions. We shall call the $SU(2)_G$ groups corresponding to $I$, $U$, $V$ spins, the hypercharge spin, charge spin, and hypocharge spin groups, respectively. Under the $SU(2) \times SU(2)_G$ subgroup, the adjoint representation of $G_2$ decomposes as

$$14 = (1, 0) \oplus (0, 1) \oplus (1/2, 3/2).$$

So far we have considered the decomposition of $\mathcal{L}SO(8)$ in terms of seven anticommutating matrices $\Gamma_A$, which correspond to left multiplication by the basis elements $e_A$ of octonions acting on the basis

$$[s] = \begin{vmatrix}u\\ u^*\end{vmatrix}.$$

A more consistent approach is to consider matrices corresponding to multiplication by the split octonions: Since

$$L_{u_i} = L_{1/2(e_i + ie_{i+3})} = \tfrac{1}{2}(L_{e_i} + iL_{e_{i+3}}), \qquad i = 1, 2, 3,$$

we have

$$\begin{aligned}
&L_{u_i}^{(8)} = U_i = -\tfrac{1}{2}(\Gamma_{i+3} + \Gamma_i),\\
&L_{u_i^*}^{(8)} = \tilde{U}_i = \tfrac{1}{2}(\Gamma_{i+3} - i\Gamma_i),\\
&L_{u_0}^{(8)} = U_0 = \tfrac{1}{2}(1 - \Gamma_7),\\
&L_{u_0^*}^{(8)} = \tilde{U}_0 = \tfrac{1}{2}(1 + \Gamma_7).
\end{aligned} \tag{11.5}$$

One can also change the basis on which the octonion units act and consider the real octonion basis on which real or split octonions may act. In any case, octonion multiplication operators $L_a$ or $R_a$, $a \in \mathbf{O}$ is uniquely defined and choosing different bases on which they can act changes their matrix representations.

## 12. LIE MULTIPLICATION ALGEBRA OF OCTONIONS AND THE PRINCIPLE OF TRIALITY

A derivation $D$ of an algebra $A$ is defined as a linear transformation satisfying the property:

$$D(xy) = (Dx)y + x(Dy) \quad \text{for all } x, y \in A. \tag{12.1}$$

Derivations of an algebra $A$ form a Lie algebra under Lie product, i.e.,

$$[D_i, D_j]_- = -[D_j, D_i]$$

and for all $D_i, D_j, D_k \in \mathrm{Der}A$,

$$[D_i,[D_j,D_k]] + [D_k,[D_i,D_j]] + [D_j,[D_k,D_i]] = 0,$$

Jacobi identity.

Derivation algebra of an algebra $A$ is isomorphic to the Lie algebra of the automorphism group of $A$,[32] i.e., if $D \in \mathrm{Der}\, A$

$$D(xy) = (Dx)y + x(Dy) \Rightarrow e^D(xy) = (e^Dx)(e^Dy).$$

Therefore the derivation (Lie) algebra of octonions is isomorphic to the Lie algebra of $G_2$. Lie multiplication algebra of the octonions is defined as the Lie algebra with elements:

$$\mathcal{L}M\mathbf{O} = \mathrm{Der}\,\mathbf{O} \oplus L_{\mathbf{O}_0} \oplus R_{\mathbf{O}_0},$$

where $L_{\mathbf{O}_0}$ and $R_{\mathbf{O}_0}$ correspond to multiplication from the left and the right by traceless (or imaginary) octonion units. Since the octonions are not associative left and right multiplications do not commute. As was shown above, the Lie multiplication algebra of octonions is isomorphic to the Lie algebra of the group $SO(8)$.

$$\mathcal{L}SO(8) = F_A \oplus M_A \oplus N_A \oplus Z_A, \tag{12.2}$$

where

$$\mathrm{Der}\,\mathbf{O} \cong F_A \oplus M_A.$$

The usual real octonionic norm is invariant under the group $SO(8)$. Denoting the parameters corresponding to the generators $\Lambda_A^{(8)}, \mu_A^{(8)}, \nu_A^{(8)}, \xi_A^{(8)}$ by $f_A, m_A, n_A, z_A$, we can represent the action of $SO(8)$ on the split octonion basis $[s]$ by

$$SO(8): [s] \to e^{iL}[s], \qquad [s] = \begin{bmatrix} u \\ u^* \end{bmatrix}, \qquad u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_0 \end{pmatrix},$$

$$L = \begin{pmatrix} U & V \\ V^\dagger & -U_4^* \end{pmatrix},$$

where

$$U_4 = \begin{pmatrix} (f_3 - m_3 - n_3) & (f_1 + if_2) & (f_4 + if_5) & (\zeta_1 + i\zeta_2) \\ (f_1 - if_2) & -(f_3 + m_3 + n_3) & (f_6 + if_7) & (\zeta_3 + i\zeta_4) \\ (f_4 - if_5) & (f_6 - if_7) & (2m_3 - n_3) & (\zeta_5 + i\zeta_6) \\ (\zeta_1 - i\zeta_2) & (\zeta_3 - i\zeta_4) & (\zeta_5 - i\zeta_6) & -2z_3 \end{pmatrix}, \tag{12.3}$$

where

$$\zeta_1 = -m_6 + \tfrac{1}{2}(n_6 - z_6), \qquad \zeta_4 = m_5 - \tfrac{1}{2}(n_5 + z_5),$$
$$\zeta_2 = -m_7 - \tfrac{1}{2}(n_7 + z_7), \qquad \zeta_5 = -m_1 + \tfrac{1}{2}(n_1 - z_1),$$
$$\zeta_3 = -m_4 + \tfrac{1}{2}(n_4 - z_4), \qquad \zeta_6 = -m_2 - \tfrac{1}{2}(n_2 + z_2).$$

Matrices $U_4$ close under commutation and form the four-dimensional representation of $\mathcal{L}U(4)$.

The matrices $V$ are antisymmetric.

$V_{\mu\nu} = -V_{\nu\mu}$ and form the Lie algebra of complex $SO(4)$:

$$V_{12} = -(m_1 + n_1) + i(m_2 - n_2),$$
$$V_{13} = (m_4 + n_4) + i(m_5 + n_5),$$
$$V_{14} = [m_6 - \tfrac{1}{2}(n_6 + z_6)] + i[m_7 + \tfrac{1}{2}(n_7 - z_7)],$$
$$V_{23} = -(m_6 + n_6) + i(m_7 - n_7),$$
$$V_{24} = [m_4 - \tfrac{1}{2}(n_4 + z_4)] - i[m_5 - \tfrac{1}{2}(n_5 - z_5)],$$
$$V_{34} = [m_1 - \tfrac{1}{2}(n_1 + z_1)] + i[m_2 + \tfrac{1}{2}(n_2 - z_2)].$$

Denoting $U_4$ as $U$, we have that $U$ and $V$ can be decomposed as

$$\begin{aligned} U &= U_{G_2} \oplus U_{SO(8)/G_2} \\ L &= L_{G_2} \oplus L_{SO(8)/G_2}, \\ V &= V_{G_2} \oplus V_{SO(8)/G_2} \end{aligned} \tag{12.4}$$

where $\oplus$ refers to vector space direct sum and $V_{G_2}, U_{G_2}$ involve only the parameters $f_A$ and $m_A$ and $U_{SO(8)/G_2}$ and $V_{SO(8)/G_2}$ involve only $n_A$ and $z_A$. Below we will construct the $\mathcal{L}SO(8)$ matrices that are in local triality with each other (see Appendix C for the principle of triality.) The principle of local triality states that for a given matrix $T^L \in \mathcal{L}SO(8)$ acting on the 8-dimensional space of octonions and which is skew with respect to the natural bilinear form $(x, y)$ defined over the octonions, there exist uniquely determined matrices $T^R$ and $T^P$ belonging to the Lie algebra of $SO(8)$ (i.e., which are skew with respect to the norm form) such that

$$T^P(xy) = (T^L x)y + x(T^R y), \qquad x, y \in \mathbf{O},$$
$$T^P, T^L, T^R \in \mathcal{L}SO(8). \tag{12.5}$$

Decomposing $T^P$ and $T^R$ and $T^L$ as above,

$$T^L = T^L_{G_2} \oplus T^L_{SO(8)/G_2},$$
$$T^R = T^R_{G_2} \oplus T^R_{SO(8)/G_2},$$
$$T^P = T^P_{G_2} \oplus T^P_{SO(8)/G_2},$$

we have

$$T^P_{G_2}(xy) + T^P_{SO(8)/G_2}(xy) = (T^L_{G_2}x)y + (T^L_{SO(8)/G_2}x)y + x(T^R_{G_2}y) + x(T^R_{SO(8)/G_2}y).$$

Now since $G_2$ is the automorphism group of octonions, its Lie algebra will be the derivation algebra of octonions satisfying

$$D(xy) = (Dx)y + x(Dy), \qquad D \in \text{Lie algebra of } G_2 = \mathcal{L}G_2.$$

Hence it follows that

$$D = T^P_{G_2} = T^L_{G_2} = T^R_{G_2} \in \mathcal{L}G_2.$$

In other words under the triality mappings

$$\begin{tikzcd} L \arrow[rr] & & R \arrow[dl] \\ & P \arrow[ul] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} L \arrow[dr] & & R \arrow[ll] \\ & P \arrow[ur] & \end{tikzcd}$$

$\mathcal{L}G_2$ subalgebra of $\mathcal{L}SO(8)$ remains fixed. If we let

$$T^L_{SO(8)/G_2} = \begin{pmatrix} U_{SO(8)/G_2} & V_{SO(8)/G_2} \\ V^\dagger_{SO(8)/G_2} & -U^*_{SO(8)/G_2} \end{pmatrix}, \tag{12.6a}$$

$$T^R_{SO(8)/G_2} = \begin{pmatrix} A_{SO(8)/G_2} & B_{SO(8)/G_2} \\ B^\dagger_{SO(8)/G_2} & -A^*_{SO(8)/G_2} \end{pmatrix}, \tag{12.6b}$$

$$T^P_{SO(8)/G_2} = \begin{pmatrix} C_{SO(8)/G_2} & D_{SO(8)/G_2} \\ D^\dagger_{SO(8)/G_2} & -C^*_{SO(8)/G_2} \end{pmatrix}, \tag{12.6c}$$

where

$$U_{SO(8)/G_2}=\begin{pmatrix} -n_3 & 0 & 0 & [\frac{1}{2}(n_6-z_6)-\frac{1}{2}i(n_7+z_7)] \\ 0 & -n_3 & 0 & [\frac{1}{2}(n_4-z_4)-\frac{1}{2}i(n_5+z_5)] \\ 0 & 0 & -n_3 & [\frac{1}{2}(n_1-z_1)-\frac{1}{2}i(n_2+z_2)] \\ [\frac{1}{2}(n_6-z_6)+\frac{1}{2}i(n_7-z_7)] & [\frac{1}{2}(n_4-z_4)+\frac{1}{2}i(n_5-z_5)] & [\frac{1}{2}(n_1-z_1)+\frac{1}{2}i(n_2-z_2)] & -z_3 \end{pmatrix}, \tag{12.7a}$$

$$V_{SO(8)/G_2}=$$
$$\begin{pmatrix} 0 & -(n_1+in_2) & (n_4+in_5) & -[\frac{1}{2}(n_6+z_6)-\frac{1}{2}i(n_7-z_7)] \\ (n_1+in_2) & 0 & -(n_6+in_7) & -[\frac{1}{2}(n_4+z_4)-\frac{1}{2}i(n_5-z_5)] \\ -(n_4+in_5) & (n_6+in_7) & 0 & -[\frac{1}{2}(n_1+z_1)-\frac{1}{2}i(n_2-z_2)] \\ [\frac{1}{2}(n_6+z_6)-\frac{1}{2}i(n_7-z_7)] & [\frac{1}{2}(n_4+z_4)-\frac{1}{2}i(n_5-z_5)] & [\frac{1}{2}(n_1+z_1)-\frac{1}{2}i(n_2-z_2)] & 0 \end{pmatrix}. \tag{12.7b}$$

Then we find, after some calculation,

$$A_{SO(8)/G_2}=\begin{pmatrix} \frac{1}{2}(n_3+z_3) & 0 & 0 & -(n_6-in_7) \\ 0 & \frac{1}{2}(n_3+z_3) & 0 & -(n_4-in_5) \\ 0 & 0 & \frac{1}{2}(n_3+z_3) & -(n_1-in_2) \\ -(n_6+in_7) & -(n_4+in_5) & -(n_1+in_2) & -\frac{1}{2}(3n_3-z_3) \end{pmatrix}, \tag{12.7c}$$

$$B_{SO(8)/G_2}=$$
$$\frac{1}{2}\begin{pmatrix} 0 & [(n_1+z_1)+i(n_2-z_2)] & -[(n_4+z_4)+i(n_5-z_5)] & -[(n_6-z_6)-i(n_7+z_7)] \\ -[(n_1+z_1)+i(n_2-z_2)] & 0 & [(n_6+z_6)+i(n_7-z_7)] & -[(n_4-z_4)-i(n_5+z_5)] \\ [(n_4+z_4)+i(n_5-z_5)] & -[(n_6+z_6)+i(n_7-z_7)] & 0 & -[(n_1-z_1)-i(n_2+z_2)] \\ [(n_6-z_6)-i(n_7+z_7)] & [(n_4-z_4)-i(n_5+z_5)] & [(n_1-z_1)-i(n_2+z_2)] & 0 \end{pmatrix}, \tag{12.7d}$$

and

$$C_{SO(8)/G_2}=\begin{pmatrix} +\frac{1}{2}(n_3-z_3) & 0 & 0 & -(n_6-in_7) \\ 0 & \frac{1}{2}(n_3-z_3) & 0 & -(n_4-in_5) \\ 0 & 0 & \frac{1}{2}(n_3-z_3) & -(n_1-in_2) \\ -(n_6+in_7) & -(n_4+in_5) & -(n_1+in_2) & -\frac{1}{2}(3n_3+z_3) \end{pmatrix}, \tag{12.7e}$$

$$D_{SO(8)/G_2}=$$
$$\frac{1}{2}\begin{pmatrix} 0 & [(n_1-z_1)+i(n_2+z_2)] & -[(n_4-z_4)+i(n_5+z_5)] & -[(n_6+z_6)-i(n_7-z_7)] \\ -[(n_1-z_1)+i(n_2+z_2)] & 0 & [(n_6-z_6)+i(n_7+z_7)] & -[(n_4+z_4)-i(n_5-z_5)] \\ [(n_4-z_4)+i(n_5+z_5)] & -[(n_6-z_6)+i(n_7+z_7)] & 0 & -[(n_1+z_1)-i(n_2-z_2)] \\ [(n_6+z_6)-i(n_7-z_7)] & [(n_4+z_4)-i(n_5-z_5)] & [(n_1+z_1)-i(n_2-z_2)] & 0 \end{pmatrix}. \tag{12.7f}$$

We had shown earlier that the action of $\mathcal{L}G_2 \cong$ **DerO** on the octonion units can be represented by octonion multiplication and the action on the split octonion basis

$$[s]=\begin{pmatrix} u \\ u^* \end{pmatrix}$$

is uniquely determined by the action on $u$. Similarly, the action of $SO(8)$ on split octonions can be represented by octonion multiplication and the action on $u$ uniquely determines the action on $[s]$. Below we give the expressions for the action of $\mathcal{L}SO(8)/G_2$ matrices that are in triality with each other in terms of octonion multiplication acting on $u$:

$$u=\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_0 \end{pmatrix},$$

$$\begin{aligned} T^L_{SO(8)/G_2}u = {} & \tfrac{1}{2}n_1([u_3^*,u]+(u+uu_0^*)u_3) \\ & -\tfrac{1}{2}in_2([u_3^*,u]-(u+uu_0^*)u_3) \\ & -n_3(uu_0^*) \\ & +\tfrac{1}{2}n_4([u_2^*,u]+(u+uu_0^*)u_2) \\ & -\tfrac{1}{2}in_5([u_2^*,u]-(u+uu_0^*)u_2) \\ & +\tfrac{1}{2}n_6([u_1^*,u]+(u+uu_0^*)u_1) \\ & -\tfrac{1}{2}in_7([u_1^*,u]-(u+uu_0^*)u_1) \\ & +z_1\tfrac{1}{2}(\{u_3^*,u\}-(uu_0)u_3) \\ & -iz_2\tfrac{1}{2}(-\{u_3^*,u\}-(uu_0)u_3) \\ & -z_3(uu_0) \\ & +z_4\tfrac{1}{2}(\{u_2^*,u\}-(uu_0)u_2) \\ & -iz_5\tfrac{1}{2}(-\{u_2^*,u\}-(uu_0)u_2) \\ & +z_6\tfrac{1}{2}(\{u_1^*,u\}-(uu_0)u_1) \\ & -iz_7\tfrac{1}{2}(-\{u_1^*,u\}-(uu_0)u_1), \end{aligned} \tag{12.8}$$

$$\begin{aligned}
T^R_{SO(8)/G_2} u = {} & n_1(uu_3^* + \tfrac{1}{2}u_3^* u - (u - \tfrac{1}{2}uu_0^*)u_3) \\
& - in_2(uu_3^* + \tfrac{1}{2}u_3^* u + (u - \tfrac{1}{2}uu_0^*)u_3) \\
& + \tfrac{1}{2}n_3(uu_0^*) - \tfrac{3}{2}n_3(uu_0) \\
& + n_4(uu_2^* + \tfrac{1}{2}u_2^* u - (u - \tfrac{1}{2}uu_0^*)u_2) \\
& - in_5(uu_2^* + \tfrac{1}{2}u_2^* u + (u - \tfrac{1}{2}uu_0^*)u_2) \\
& + n_6(uu_1^* + \tfrac{1}{2}u_1^* u - (u - \tfrac{1}{2}uu_0^*)u_1) \\
& - in_7(uu_1^* + \tfrac{1}{2}u_1^* u + (u - \tfrac{1}{2}uu_0^*)u_1) \\
& + z_1\tfrac{1}{2}(-(uu_0^*)u_3 - u_3^* u) \qquad (12.9) \\
& - iz_2\tfrac{1}{2}(-(uu_0^*)u_3 + u_3^* u) \\
& + \tfrac{1}{2}z_3 u \\
& + z_4\tfrac{1}{2}(-(uu_0^*)u_2 - u_2^* u) \\
& - iz_5\tfrac{1}{2}(-(uu_0^*)u_2 + u_2^* u) \\
& + z_6\tfrac{1}{2}(-(uu_0^*)u_1 - u_1^* u) \\
& - iz_7\tfrac{1}{2}(-(uu_0^*)u_1 + u_1^* u),
\end{aligned}$$

$$\begin{aligned}
T^P_{SO(8)/G_2} u = {} & n_1(uu_3^* + \tfrac{1}{2}u_3^* u - (u - \tfrac{1}{2}uu_0^*)u_3) \\
& - in_2(uu_3^* + \tfrac{1}{2}u_3^* u + (u - \tfrac{1}{2}uu_0^*)u_3) \\
& + n_3(\tfrac{1}{2}u - 2uu_0) \\
& + n_4(uu_2^* + \tfrac{1}{2}u_2^* u - (u - \tfrac{1}{2}uu_0^*)u_2) \\
& - in_5(uu_2^* + \tfrac{1}{2}u_2^* u + (u - \tfrac{1}{2}uu_0^*)u_2) \\
& + n_6(uu_1^* + \tfrac{1}{2}u_1^* u - (u - \tfrac{1}{2}uu_0^*)u_1) \\
& - in_7(uu_1^* + \tfrac{1}{2}u_1^* u + (u - \tfrac{1}{2}uu_0^*)u_1) \\
& + z_1\tfrac{1}{2}((uu_0^*)u_3 + u_3^* u) \qquad (12.10) \\
& - iz_2\tfrac{1}{2}((uu_0^*)u_3 - u_3^* u) \\
& - \tfrac{1}{2}z_3 u \\
& + z_4\tfrac{1}{2}((uu_0^*)u_2 + u_2^* u) \\
& - iz_5\tfrac{1}{2}((uu_0^*)u_2 - u_2^* u) \\
& + z_6\tfrac{1}{2}((uu_0^*)u_1 + u_1^* u) \\
& - iz_7\tfrac{1}{2}((uu_0^*)u_1 - u_1^* u)
\end{aligned}$$

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## APPENDIX A: STRUCTURE CONSTANTS OF $G_2$

Consider the basis of $\mathcal{L}G_2$ given in Sec. 1

$$\mathcal{L}G_2 = F_A \oplus M_A, \quad A = 1, \ldots, 7.$$

As was pointed out in Sec. 2, the generators $F_A$ and $F_8 = -M_3$ form the $SU(3)$ subalgebra of $\mathcal{L}G_2$, i.e.,

$$[F_a, F_b] = 2if_{abc}F_c, \quad a, b, c = 1, 2, \ldots, 8, \tag{A1}$$

where $f_{abc}$ are the totally antisymmetric structure constants of Gell-Mann, the nonzero elements of which are given in Table A1.

Now

$$\mathcal{L}G_2 = F_a \oplus M_s, \quad a = 1, 2, \ldots, 8,$$
$$m_s = s = 1, 2, 4, 5, 6, 7,$$
$$F_a \cong \mathcal{L}SU(3).$$

TABLE A1.

| $abc$ | $f_{abc}$ |
|---|---|
| 123 | 1 |
| 147 | $\frac{1}{2}$ |
| 156 | $-\frac{1}{2}$ |
| 246 | $\frac{1}{2}$ |
| 257 | $\frac{1}{2}$ |
| 345 | $\frac{1}{2}$ |
| 367 | $-\frac{1}{2}$ |
| 458 | $\sqrt{3}/2$ |
| 678 | $\sqrt{3}/2$ |

TABLE A2.

| $a$ | $m_s m_t$ | $C_{am_sm_t}$ |
|---|---|---|
| 1 | $m_4m_7$ | 1/2 |
| 1 | $m_5m_6$ | 1/2 |
| 2 | $m_4m_6$ | − 1/2 |
| 2 | $m_5m_7$ | 1/2 |
| 3 | $m_4m_5$ | 1/2 |
| 3 | $m_6m_7$ | 1/2 |
| 4 | $m_1m_7$ | 1/2 |
| 4 | $m_2m_6$ | − 1/2 |
| 5 | $m_1m_6$ | − 1/2 |
| 5 | $m_2m_7$ | − 1/2 |
| 6 | $m_1m_5$ | − 1/2 |
| 6 | $m_2m_4$ | − 1/2 |
| 7 | $m_1m_4$ | − 1/2 |
| 7 | $m_2m_5$ | 1/2 |
| 8 | $m_4m_5$ | $-1/2\sqrt{3}$ |
| 8 | $m_6m_7$ | $1/2\sqrt{3}$ |
| 8 | $m_1m_2$ | − 1 |

TABLE A3.

| $m_sm_tm_u$ | $C_{m_sm_tm_u}$ |
|---|---|
| $m_1m_4m_7$ | $-1/\sqrt{3}$ |
| $m_1m_5m_6$ | $1/\sqrt{3}$ |
| $m_2m_4m_6$ | $-1/\sqrt{3}$ |
| $m_2m_5m_7$ | $-1/\sqrt{3}$ |

The structure constants of the form $C_{abm_s}$ vanish because $F_a$ form a subalgebra. Hence the remaining nonvanishing structure constants of $G_2$ are of the form

$$C_{am_sm_t}, \quad a = 1, \ldots, 8, \quad s, t, u = 1, 2, 4, 5, 6, 7$$

or of the form

$$C_{m_sm_tm_u},$$

$$\begin{aligned}
[F_a, F_b] &= 2if_{abc}F_c, \\
[F_a, M_s] &= 2iC_{am_sm_t}M_t, \\
[M_s, M_t] &= 2i(C_{m_sm_ta}F_a + C_{m_sm_tm_u}M_u,
\end{aligned} \tag{A2}$$

where all the structure constants are totally antisymmetric. Below we list all the nonvanishing elements of $C_{am_sm_t}$ and $C_{m_sm_tm_u}$ (Tables A2 and A3).

## APPENDIX B: ZORN'S VECTOR MATRICES

A realization of the split octonion algebra is via the Zorn's vector matrices

$$\begin{pmatrix} a & \mathbf{x} \\ \mathbf{y} & b \end{pmatrix},$$

where $a$ and $b$ are scalars and $\mathbf{x}$ and $\mathbf{y}$ are 3-vectors, with the product defined as

$$\begin{pmatrix} a & \mathbf{x} \\ \mathbf{y} & b \end{pmatrix}\begin{pmatrix} c & \mathbf{u} \\ \mathbf{v} & d \end{pmatrix} = \begin{pmatrix} ac + \mathbf{x}\cdot\mathbf{v} & a\mathbf{u} + d\mathbf{x} - \mathbf{y}\times\mathbf{v} \\ c\mathbf{y} + b\mathbf{v} + \mathbf{x}\times\mathbf{u} & \mathbf{y}\cdot\mathbf{u} + bd \end{pmatrix}. \tag{B1}$$

$\times$ denotes the usual vector product.

If the basis vectors of the three-dimensional space are $\mathbf{e}_i$, $i = 1, 2, 3$ with $\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk}\mathbf{e}_k$ and $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, then we can relate the split octonions to the vector matrices; namely

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = u_0^*, \quad \begin{pmatrix}0 & -\mathbf{e}_i\\ 0 & 0\end{pmatrix} = u_i^*,$$

$$\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = u_0, \quad \begin{pmatrix}0 & 0\\ \mathbf{e}_i & 0\end{pmatrix} = u_i.$$

Octonion conjugation defined above induces a natural involution for the vector matrices, i.e., if

$$A = \begin{pmatrix}a & -\mathbf{x}\\ \mathbf{y} & b\end{pmatrix}, \quad \overline{A} = \begin{pmatrix}b & +\mathbf{x}\\ -\mathbf{y} & a\end{pmatrix},$$

$$A = au_0^* + x_i u_i^* + bu_0 + y_i u_i,$$

$$\overline{A} = au_0 - x_i u_i^* + bu_0^* - y_i u_i,$$

$$N(A) = A\overline{A} = \overline{A}A = (ab + \mathbf{x}\cdot\mathbf{y}). \tag{B2}$$

## APPENDIX C: PRINCIPLE OF TRIALITY

The usual octonionic norm is invariant under the group $SO(8)$ or equivalently the bilinear form induced by the octonionic norm is skew with respect to the Lie algebra of $SO(8)$, i.e.,

$$(x, y) \equiv \tfrac{1}{2}(\bar{x}y + y\bar{x}) \tag{C1}$$

then for $T \in \mathfrak{L}SO(8)$

$$(Tx, y) + (x, Ty) = 0 \quad \text{for all } x, y \in \mathbf{O} \tag{C2}$$

For the elements $D$ of the derivation algebra of octonions we have

$$\begin{aligned} &D \in \operatorname{Der} \mathbf{O} \cong \mathfrak{L}G_2,\\ &D(xy) = (Dx)y + x(Dy). \end{aligned} \tag{C3}$$

Integrated form of this (local) identity gives us the automorphisms of $\mathbf{O}$, i.e.,

$$e^D(xy) = (e^Dx)(e^Dy) \tag{C4}$$

or

$$d = e^D,$$

$$d(xy) = (dx)(dy) \Rightarrow d \in G_2.$$

The principle of triality is nothing but a generalization of the identities (C3)–(4) and is unique to octonions.[8] According to the principle of local triality (PLT) it is possible to generalize identity (C3) to all the elements of the Lie multiplication algebra $\mathfrak{L}SO(8)$. Namely, given an element $T^L \in \mathfrak{L}SO(8)$ acting on the octonions there exist *unique* $T^R$ and $T^P \in \mathfrak{L}SO(8)$ such that

$$(\text{PLT}): (T^Lx)y + x(T^Ry) = T^P(xy) \quad \text{for all } x, y \in \mathbf{O}. \tag{C5}$$

Just as it is possible to integrate the derivations of octonion algebra to get its automorphisms, one can also integrate the PLT to get the principle of global triality (PGT), which is a generalization of the concept of automorphism. According to the PGT, given $t^l \in SO(8)$ acting on the octonions there exist $t^r$ and $t^p \in SO(8)$, unique up to a sign, such that[33]

$$\text{PGT}: (t^lx)(t^ry) = t^p(xy) \quad \text{for all } x, y \in \mathbf{O}. \tag{C6}$$

Since the group $SO(8)$ is the "Lie multiplication group" of octonions (i.e., that every action of $SO(8)$ on O can be represented by octonion multiplication), one can reformulate the PGT as follows[34]:

Given $d^1 \in SO(8) \quad d^2, d^3 \in SO(8)$

$$(d^1x)(d^2y) = \overline{d^3\overline{(xy)}} \qquad \text{for all } x, y \in \mathbf{O}, \tag{C7a}$$

where the overbar denotes octonion conjugation.

In this form of the PGT we have cyclic symmetry between $d^1, d^2$ and $d^3$, i.e.,

$$(d^1x)(d^2y) = \overline{d^3\overline{(xy)}}$$

implies

$$(d^2x)(d^3y) = \overline{d^1\overline{(xy)}}, \tag{C7b}$$

$$(d^3x)(d^1y) = \overline{d^2\overline{(xy)}}. \tag{C7c}$$

Since given $d^1, d^2$, and $d^3$ are determined uniquely up to a sign, the subgroup of $SO(8) \times SO(8) \times SO(8)$ consisting of elements which are in triality will form a twofold covering group of $SO(8)$, i.e., it will be isomorphic to Spin (8). The group $SO(8)$ has the subgroup $SO(7)$ and given $t \in SO(7)$ there exist $\bar{t} \in SO(8)$

$$(tx)(\bar{t}y) = \bar{t}(xy) \quad \text{for all } x, y \in \mathbf{O} \tag{C8}$$

then the elements $t$ form the covering group Spin (7) of $SO(7)$.

## APPENDIX D: REALIZATIONS OF THE CAYLEY ALGEBRA IN TERMS OF GELL-MANN λ MATRICES AND DIRAC'S γ-MATRICES

### 1. The λ-matrices

We want to define a product between the λ matrices of Gell-Mann such that they will form the nonassociative Cayley algebra. Since there are eight λ matrices and seven imaginary octonion units $e_A$, the product will be defined between seven of the λ matrices and will involve the eighth λ matrix. In view of the broken $SU(3)$, this eighth λ matrix will be taken to be $\lambda_8$. The general form of the product consistent with octonion multiplication can be parameterized as follows:

$$\begin{aligned} \lambda_A \circ \lambda_B = {}& \tfrac{1}{2}\beta \operatorname{Tr}(\lambda_A\lambda_B)1 + \tfrac{1}{2}\delta \operatorname{Tr}(\lambda_8\{\lambda_A, \lambda_B\})1\\ &- (2/\sqrt{3})(\alpha + \tfrac{1}{6}\gamma) \operatorname{Tr}(\lambda_8[\lambda_A, \lambda_B])1\\ &+ \{\alpha 1 + \sqrt{3}(\alpha + \tfrac{1}{6}\gamma)\lambda_8, [\lambda_A, \lambda_B]\}\\ &+ \gamma[\{\lambda_8, \lambda_A\}, \{\lambda_8, \lambda_B\}], \end{aligned} \tag{D1}$$

where { , } and [ , ] denote anticommutation and commutation, respectively. Then, for $A = 1, 2, 3$ we have

$$\lambda_A \circ \lambda_A = \beta + (2/\sqrt{3})\delta \equiv 1/s^2, \qquad \text{no sum over } A, \tag{D2}$$

and for $A = 4, 5, 6, 7$

$$\lambda_A \circ \lambda_A = \beta - (1/\sqrt{3})\delta \equiv 1/t^2. \tag{D3}$$

In addition, the octonion multiplication imposes the conditions:

$$\begin{aligned} &\alpha = -\tfrac{5}{4}(2\gamma/9), \quad \beta = 15(2\gamma/9)^2,\\ &\delta = 5\sqrt{3}(2\gamma/9)^2. \end{aligned} \tag{D4}$$

Hence, we get the result that the $3 \times 3$ matrices

$$e_i = is\lambda_i, \quad i = 1, 2, 3,$$
$$e_4 = it\lambda_4, \quad e_6 = -it\lambda_6,$$
$$e_5 = it\lambda_5, \quad e_7 = -it\lambda_7 \tag{D5}$$

satisfy the octonion multiplication table of the imaginary units $e_A$ under the product defined above and generate a Cayley algebra with identity being the scalar identity:

$$e_A \circ e_B = -\delta_{AB} + a_{ABC}e_C \tag{D6}$$

An interesting property of this product is that the coefficient multiplying the $\lambda$ matrices is different for different isospin multiplets.

### 2. The $\gamma$-matrices

Let us define a product between $4 \times 4$ Hermitian matrices of the form:

$$A = \begin{pmatrix} \alpha 1_2 & -i\sigma\cdot\mathbf{a} \\ i\sigma\cdot\mathbf{b} & \beta 1_2 \end{pmatrix}, \quad C = \begin{pmatrix} \gamma 1_2 & -i\sigma\cdot\mathbf{c} \\ i\sigma\cdot\mathbf{d} & \delta 1_2 \end{pmatrix}.$$

Such that they form a Cayley algebra. First, note that the matrix $A$ can be written in terms of $\gamma$ matrices as:

$$A = \tfrac{1}{2}(\alpha + \beta) + \tfrac{1}{2}(\alpha - \beta)\gamma_5 + \tfrac{1}{2}\gamma_5\boldsymbol{\gamma}\cdot(\mathbf{a} - \mathbf{b}) + \tfrac{1}{2}\boldsymbol{\gamma}\cdot(\mathbf{a} + \mathbf{b}), \tag{D7}$$

where $\gamma$ matrices are taken in the Weyl basis and the parameters $\alpha, \beta, \mathbf{a}, \mathbf{b}$ are all real.

$$\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3) = \rho_2 \otimes \boldsymbol{\sigma},$$
$$\gamma_4 = \rho_1 \otimes I, \quad \gamma_5 = \rho_3 \otimes I. \tag{D8}$$

To get a product which is not associative, we are led to defining a new operation $\sim$ over the $4 \times 4$ matrices:

$$\tilde{M} = \widetilde{\begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}} \equiv \begin{pmatrix} \mathbf{a}^\dagger & \mathbf{b} \\ \mathbf{c} & \mathbf{d}^\dagger \end{pmatrix}, \tag{D9}$$
$$= \tfrac{1}{2}(1 + \gamma_5)A^\dagger\tfrac{1}{2}(1 + \gamma_5) + \tfrac{1}{2}(1 - \gamma_5)A^\dagger\tfrac{1}{2}(1 - \gamma_5)$$
$$+ \tfrac{1}{2}(1 + \gamma_5)A\tfrac{1}{2}(1 - \gamma_5) + \tfrac{1}{2}(1 - \gamma_5)A\tfrac{1}{2}(1 + \gamma_5),$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are $2 \times 2$ matrices.

Then under the product

$$A * C = \tfrac{1}{2}(AC + \widetilde{AC}) + \tfrac{1}{2}\gamma_4(AC^\dagger - \widetilde{AC^\dagger}) \tag{D10}$$

the matrices of the form shown above form a split Cayley algebra equivalent to the Zorn's vector matrices

$$A * C = \begin{pmatrix} (\alpha\gamma + \mathbf{a}\cdot\mathbf{d}) & (-i\alpha\sigma\cdot\mathbf{c} - i\delta\sigma\cdot\mathbf{a} + i\sigma\cdot\mathbf{b}\times\mathbf{d}) \\ (i\gamma\sigma\cdot\mathbf{b} + i\beta\sigma\cdot\mathbf{d} + i\sigma\cdot(\mathbf{a}\times\mathbf{c}) & (\beta\delta + \mathbf{b}\cdot\mathbf{c}) \end{pmatrix}. \tag{D11}$$

Writing $A$ in the form

$$A = \tfrac{1}{2}(1 + \gamma_5)(\alpha + \boldsymbol{\gamma}\cdot\mathbf{a}) + \tfrac{1}{2}(1 - \gamma_5)(\beta + \boldsymbol{\gamma}\cdot\mathbf{b}),$$
$$= \tfrac{1}{2}(1 - ie_7)(\alpha + e_i a_i) + \tfrac{1}{2}(1 + ie_7)(\beta + e_i b_i),$$
$$= \alpha u_0^* + u_i^* a_i + \beta u_0 + u_i b_i, \tag{D12}$$

it is easily seen that the split octonion basis $u_i, u_0, u_i^*, u_0^*$ is realized in this case by

$$u_0^* = \tfrac{1}{2}(1 + \gamma_5), \quad u_0 = \tfrac{1}{2}(1 - \gamma_5),$$
$$u_i^* = \tfrac{1}{2}(1 + \gamma_5)\gamma_i, \quad u_i = \tfrac{1}{2}(1 - \gamma_5)\gamma_i, \; i = 1, 2, 3. \tag{D13}$$

Therefore, the role played by $ie$ in extending the quaternion algebra $(1, e_1, e_2, e_3)$ into the split octonion algebra is played in the above realization by $\gamma_5$, i.e.,

$$ie_7(1, e_1, e_2, e_3) = (ie_7, ie_4, ie_5, ie_6),$$
$$\gamma_5 * (1, \gamma_1, \gamma_2, \gamma_3) = \gamma_5(1, \gamma_1, \gamma_2, \gamma_3)$$
$$= (\gamma_5, \gamma_5\gamma_1, \gamma_5\gamma_2, \gamma_5\gamma_3).$$

$*$ multiplication by $\gamma_5$ reduces to the ordinary matrix multiplication. Conversely, the crucial role played by $\gamma_5$ in constructing projection operators into lh and rh states is reflected in the octonion algebra by the important role played by $u_0$ and $u_0^*$ as projection operators into quark and antiquark states in the octonionic representations of the Poincaré group.[14]

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[16] See, for example, A. A. Albert, *Studies in Modern Algebra* (Prentice-Hall, Princeton, NJ, 1963). N. Jacobson, "Structure and Representations of Jordan Algebras," Amer. Math. Soc. Coll. Publ. Vol. 39, and references contained therein.
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[18] See Ref. 14 and H. H. Goldstine and L. P. Horwitz, Proc. Natl. Acad. Sci. USA **48**, 1134 (1962); Math. Ann. **154**, 1 (1964); Math. Ann. **164**, 291 (1966); L. P. Horwitz and L. C. Biedenharn, Helv. Phys. Acta **38**, 385 (1965).
[19] Note the distinction between the terms nonassociative and not associative. The former is generally used to denote all the composition algebras mentioned above which satisfy the property of alternativity defined below.
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[23]By three "independent" elements we mean any three elements $e_i$, $e_j$, $e_k$ such that none of them is proportional to a product of the other two, i.e., $e_k \neq a e_i e_j$.

[24]An element left invariant by the Lie group is said to be annihilated by the Lie algebra.

[25]See, e. g., M. L. Mehta, J. Math. Phys. **7**, 1824 (1966) and M. L. Mehta and P. K. Srivastava, J. Math. Phys. **7**, 1833 (1966).

[26]For example, take $\{1, e_1, e_2, e_3\}$ as a basis generating a quaternion algebra $H$. Then each octonion can be written as $z = q_1 + q_2 e_7$, $w = r_1 + r_2 e_7 \in O$, where $q_1, q_2, r_1, r_2 \in H$, with the product defined by $zw = (q_1 r_1 - \bar{r}_2 q_2) + (r_2 q_1 + q_2 \bar{r}_1) e_7$. (The bar denotes quaternion conjugation).

[27]An equivalent form of this basis was first studied by G. Seligman as the derivation algebra of Zorn's vector matrices given in Appendix B. See Ref. 17.

[28]In fact, the generator $N_3$ extends $SU(3)$ subgroup into $U(3)$ and the group $G_2$ into $SO(7)$.

[29]Note that we put a bar over the indexed matrices when they act on the split octonions, i.e., under the numbering: $(u_1 u_2 u_3 u_0 u_1^* u_2^* u_3^* u_0^*)$ $(s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8)$ we have $\bar{E}_{ab} s_c = \delta_{bc} s_a$, $a, b, c = 1, \ldots, 8$. For the real octonions we have the numbering $(e_A, 1) \leftrightarrow (e_A, e_8)$, $A = 1, \ldots, 7$. Then $E_{ab} e_c = \delta_{bc} e_a$, $a, b, c, = 1, \ldots, 8$. We also defined $\Sigma_{ab}$ as $\Sigma_{ab} = E_{ab} - E_{ba}$, $\bar{\Sigma}_{ab} = \bar{E}_{ab} - \bar{E}_{ba}$.

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[32]For a proof and more details see N. Jacobson *Lie Algebras* (Interscience, New York, 1962.)

[33]Here we use capital letters for the elements of the Lie algebra and small letters for the group elements.

[34]See Y. Matşushima, Nagoya Math. J. **4**, 83 (1952).