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Dirac operators coupled to vector potentials

(elliptic operators/index theory/characteristic classes/anomalies/gauge fields)

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ABSTRACT **Characteristic classes for the index of the Dirac family $\not{\partial}_A$ are computed in terms of differential forms on the orbit space of vector potentials under gauge transformations. They represent obstructions to the existence of a covariant Dirac propagator. The first obstruction is related to a chiral anomaly.**

In this note we study the null spaces (zero frequency modes) of $\not{\partial}_A$, the massless Dirac operator coupled to a vector potential, as the potential A varies. We are interested in the null spaces of positive chirality as opposed to those of negative chirality. Their formal difference is a virtual bundle, Ind $\not{\partial}$; we apply the index theorem for families of operators and some infinite dimensional geometry to compute the characteristic classes of Ind $\not{\partial}$ explicitly in terms of differential forms.

The formulas obtained may be of interest in quantum chromodynamics. The path integral formulation uses gauge invariant functionals of the propagator for $\not{\partial}_A$. To define the propagator $\not{\partial}_A^{-1}$ requires some consistent identification of the null spaces of positive and negative chirality. The nonvanishing of the characteristic classes are obstructions to a consistent covariant identification of these null spaces—i.e., obstructions to the existence of a covariant propagator. The first such obstruction is related to a chiral anomaly, as discussed below. We ask whether the higher obstructions have physical significance as well.

Let M be a compact oriented Riemannian spin manifold of dimension $2n$, and P a principal bundle over M with group G. Let $\mathfrak{A}$ be the set of connections or vector potentials on P, with $\mathcal{G}$ the group of gauge transformations of P. We denote the action of $\phi \in \mathcal{G}$ on $A \in \mathfrak{A}$ by $\phi \cdot A$. Let ρ be a representation of G on $\mathbf{C}^N$ giving the associated vector bundle $E = P \times_G \mathbf{C}^N$. Each $A \in \mathfrak{A}$ gives a Dirac operator $\not{\partial}_A$: $C^\infty(S^+ \otimes E) \to C^\infty(S^- \otimes E)$ where $S^\pm$ are the spin bundles over M of positive and negative chirality, respectively. In local coordinates

$$\not{\partial}_A = \sum_{\mu=1}^{2n} \gamma_\mu(\partial_\mu + \Gamma_\mu + A_\mu)\left(\frac{1+\gamma_5}{2}\right)$$

where Γ_μ is the Riemannian connection and acts on spinorial indices, while A_μ acts on the scalar indices $1, \ldots, N$. We have the covariance $\not{\partial}_{\phi\cdot A} = \phi^{-1}\not{\partial}_A\phi$.

The analytic index of the Dirac family $\{\not{\partial}_A\}_{A\in\mathfrak{A}}$, which we denote by $\not{\partial}_{\mathfrak{A}/\mathcal{G}}$ is the formal difference $\{\ker \not{\partial}_A\}_{A\in\mathfrak{A}} - \{\ker \not{\partial}_A^*\}_{A\in\mathfrak{A}}$. Each term is not a vector bundle over $\mathfrak{A}$ because the dimensions of $\ker \not{\partial}_A$ and $\ker \not{\partial}_A^*$ can jump (the same amount) as A varies over $\mathfrak{A}$. Nevertheless, the formal difference is well defined as an element of $K(\mathfrak{A})$. Moreover, because of the covariance of $\not{\partial}_A$, $\ker \not{\partial}_{\phi\cdot A} = \phi(\ker \not{\partial}_A)$, and the formal difference is an element of $K(\mathfrak{A})$ equivariant under $\mathcal{G}$. In our case it descends to an element of $K(\mathfrak{A}/\mathcal{G})$ which we denote by Ind $\not{\partial}$.

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The analytic family indexed by $\mathfrak{A}/\mathcal{G}$ can be defined directly in terms of the Hilbert bundles $\mathcal{H}^\pm = \mathfrak{A} \times_{\mathcal{G}} L_2(S^\pm \otimes E)$ over $\mathfrak{A}/\mathcal{G}$. Covariance means $\{\not{\partial}_{\phi A}\}_{\phi\in\mathcal{G}}$ gives an elliptic operator $\not{\partial}_{\mathcal{G}\cdot A}$ mapping the fiber $\mathcal{H}^+_{\mathcal{G}\cdot A}$ to $\mathcal{H}^-_{\mathcal{G}\cdot A}$. The analytic index of this family is Ind $\not{\partial}$ above.

When $M = S^4$ and $\mathcal{G}$ is the group of gauge transformations leaving the north pole fixed, the index for the Dirac family $\not{\partial}_{\mathfrak{A}/\mathcal{G}}$ is computed topologically in ref. 1. The index theorem implies that the following two maps are homotopically equivalent. The first is given by the Dirac family

$$\{\phi\cdot A\} \xrightarrow{\not{\partial}} \not{\partial}_{\phi\cdot A}$$

mapping $\mathfrak{A}/\mathcal{G}$ into Fredholm operators. For the second, we have the composition of maps

$$\mathfrak{A}/\mathcal{G} \xrightarrow{\alpha_1} \Omega^3(G) \xrightarrow{\alpha_2} \Omega^3(U(N)) \xrightarrow{\alpha_3} \Omega^3(U(\infty)) \xrightarrow{\alpha_4} \mathcal{F}.$$

The map α_1 (which is a homotopy equivalence) is parallel transport by means of A around closed curves parameterized by the equator S^3. (Follow a fixed geodesic from the north pole to the south pole and follow a variable geodesic back.) The map α_2 is induced by the representation ρ: $G \to SU(N)$, and α_3 by the injection of $U(N) \to U(\infty)$. Finally, α_4 is a homotopy equivalence (Bott periodicity, twice).

Thus, the characteristic classes of Ind $\not{\partial}$ can be obtained by pulling back the cohomology generators in $\mathcal{F}$ via the second map. For example, if $G = U(N)$ and ρ is the identity, one obtains nonzero characteristic classes, up to degree $2N - 4$.

In general, to compute the characteristic classes of Ind $\not{\partial}$ in terms of forms, we introduce a "universal" bundle with connection. $\mathcal{G}$ acts on $P \times \mathfrak{A}$ by $(p, A) \to (\phi(p), \phi\cdot A)$. This action has no fixed points and gives a principal bundle

$$\left(P \times \mathfrak{A}, \mathcal{G}, \frac{P \times \mathfrak{A}}{\mathcal{G}} = \mathfrak{Q}\right).$$

Since the group action of G on $P \times \mathfrak{A}$ commutes with that of $\mathcal{G}$, the group G acts on $\mathfrak{Q}$. If G acts without fixed points, one obtains a principal bundle $\mathfrak{Q}$ with group G and base $\mathfrak{Q}/G = M \times \mathfrak{A}/\mathcal{G}$. That occurs when one either restricts $\mathfrak{A}$ to the space of irreducible connections or restricts $\mathcal{G}$ to be gauge transformations leaving a point of P fixed. We assume the latter. The principal G-bundle $\mathfrak{Q}$ has a natural connection w, obtained as follows. The space $P \times \mathfrak{A}$ has a Riemannian metric invariant under $G \times \mathcal{G}$. At (p, A), the metric on $T(P, p)$ is given by the metrics of G, M and the connection A; while the metric on $T(\mathfrak{A}, A)$ is the usual metric on $C^\infty(\Lambda^1 \otimes g)$. The metric on $P \times \mathfrak{A}$ descends to a metric on $\mathfrak{Q}$ invariant under G. The orthogonal complement to orbits of G gives the connection w.

$(\mathfrak{Q}, w)$ is universal in the following sense. Suppose Q is a principal G-bundle over $M \times X$, X compact and $Q|_{M\times x} \cong P$ for each $x \in X$. Suppose, moreover, that Q has a fiber connection; that is, a choice of connection on $Q_{M\times x}$ continuous for $x \in X$. Then there is a map A: $Q \to \mathfrak{Q}$ inducing the fiber connection from w. Conversely, any map β of $X \to \mathfrak{A}/\mathcal{G}$ leads to a fiber connection by pulling back $(\mathfrak{Q}, w)$ via $I \times \beta$: $M \times X \to M \times \mathfrak{A}/\mathcal{G}$.

The curvature $\mathcal{F}$ of w is easily computed. It is a horizontal 2-form with values in g, the Lie algebra of G, and has components of type (2, 0), (1, 1), and (0, 2) reflecting the product base space $M \times \mathfrak{A}/\mathcal{G}$. The formulas for $\mathcal{F}$ at (p, A) are as follows: (1) $\mathcal{F}_{t_1,t_2} = F_{t_1,t_2}(A)$ for $t_1, t_2 \in T(M, \pi(p))$; (2) $\mathcal{F}_{t,\tau} = \tau(t)$ for $t \in T(M, \pi(p))$ and $\tau \in T(\mathfrak{A}/\mathcal{G}, \{A\})$ (so that $\tau \in C^\infty(\Lambda^1 \otimes g)$ and $D_A^*\tau = 0$); and (3) $\mathcal{F}_{\tau,\sigma} = Gb_\tau^*(\sigma)$ where $G = (D_A^*D_A)^{-1}$ and b_τ: $\Lambda^0 \otimes g \to \Lambda^1 \otimes g$ is given by $f \to [\tau, f]$. In local coordinates, the (2, 0) component of the field is $F_{\mu,\nu}$. The (1, 1) component is δA_μ, while the (0, 2) component is $(D_A^*D_A)^{-1}[\delta A_\mu, \delta B_\mu](x)$ with δA and δB in the background gauge. Here, π is the projection of P onto M.

If we apply the index formula for a family (2) to $\not\partial_{\mathfrak{A}/\mathcal{G}}$ one obtains *Theorem 1*.

THEOREM 1. *$ch(Ind\ \not\partial) = \int_M \hat{\alpha}(M)\ ch(\mathcal{E})$ where $\mathcal{E} = \mathfrak{Q} \times_G C^N$ a vector bundle over $M \times \mathfrak{A}/\mathcal{G}$, $\hat{\alpha}$ is the usual characteristic class associated with the spinor index and ch is the Chern character.*

The curvature formulas above give explicit formulas for the characteristic classes of $\mathcal{E}$ in terms of differential forms (3). For example, suppose $M = S^{2n}$, $G = SU(N)$, and ρ is the identity representation. Then, $\hat{\alpha}(M) = 1$ and the Chern character of Ind $\not\partial$ is expressed in terms of the Chern classes of $\mathcal{E}$ integrated over M. These Chern classes are the invariant polynomials $k_j(\mathcal{F})$ where

$$\Sigma k_j(T)t^{N-j} = \det\left(tI_N + \frac{i}{2\pi} T\right).$$

The invariant polynomials $k_j(T)$ are also expressible in terms of $\mathrm{tr}(T^k)$, and there is some simplification for $SU(N)$ since $\mathrm{tr}(T) = 0$. So,

$$k_0 = 1,$$

$$k_1(T) = \frac{i}{2\pi}\ \mathrm{tr}(T) = 0,$$

$$k_2(T) = -\frac{1}{8\pi^2}\ \mathrm{tr}(T^2),$$

$$k_3(T) = -\frac{i}{24\pi^3}\ \mathrm{tr}(T^3),$$

$$k_4(T) = -\frac{1}{2^6\pi^4}\left(\mathrm{tr}(T)^4 - \frac{(\mathrm{tr}(T)^2)^2}{2}\right)$$

$$= -\frac{1}{2^6\pi^4}\ (\mathrm{tr}(T)^4) + k_2^2(T)/2, \text{ etc.}$$

COROLLARY 1.1. *Let $M = S^{2n}$. The Chern classes of Ind $\not\partial$ are expressible in terms of $d_{2j} = \int_{S^{2n}} k_{j+n}(\mathcal{F})_{2n,2j}$ forms of degree $2j$ on $\mathfrak{A}/\mathcal{G}$, where $k_{j+n}(\mathcal{F})_{2n,2j}$ stands for the $(2n, 2j)$ component of $k_{j+n}(\mathcal{F})$.*

For example, when $M = S^4$, the 0^{th} Chern class is

$$\int_{S^4} k_2(\mathcal{F})_{4,0} = -\frac{1}{8\pi^2}\int_{S^4} \mathrm{tr}(\mathcal{F}^2)_{4,0} = -\frac{1}{8\pi^2}\int_{S^4} \mathrm{tr}(F^2),$$

the usual Pontrjagin index. While the first Chern class c_1 of Ind $\not\partial$ equals

$$\int_{S^4} k_3(\mathcal{F})_{4,2} = -\frac{i}{24\pi^3}\int_{S^4} \mathrm{tr}(\mathcal{F}^3)_{4,2}$$

$$= -\frac{i}{24\pi^3}\int_{S^4} \varepsilon_{\alpha\beta\gamma\delta}\ \mathrm{tr}\{F_{\alpha\beta}F_{\gamma\delta}Gb_\tau^*(\sigma)$$

$$+ F_{\alpha\beta}Gb_\tau^*(\sigma)F_{\gamma\delta} + F_{\alpha\beta}\cdot(\tau_\gamma\sigma_\delta + \sigma_\delta\tau_\gamma)\}$$

as a 2 form on $\mathfrak{A}/\mathcal{G}$ evaluated on the pair of tangent vectors τ, σ.

When $\rho \neq$ Id, the above formulas hold with $\rho\cdot\mathcal{F}$ replacing $\mathcal{F}$ and $\mathrm{tr}\rho$ (the trace in the ρ-representation) replacing tr.

Suppose $G = SU(N)$, $M = S^{2n}$, and $\mathcal{G}$ is the group of gauge transformations leaving a point fixed. Then $\mathcal{G}$ is the group of the principal bundle with base $\mathfrak{A}/\mathcal{G}$ and total space $\mathfrak{A}$, which is topologically trivial. The Chern classes $d_{2j} = \int_{S^{2n}} k_{j+n}(\mathcal{F})_{2n,2j}$, which are $2j$ forms on $\mathfrak{A}/\mathcal{G}$, can be lifted to forms on $\mathfrak{A}$ that are exact on $\mathfrak{A}$: $d_{2j} = d\beta_{2j-1}$. Moreover, $\beta_{2j-1}|_{\text{Orbit}} = t_{2j-1}$ is a closed $2j - 1$ form on $\mathcal{G}$ representing a generator of $H^{2j-1}(\mathcal{G}, \mathbf{R})$ $(N \geq j + n)$, modulo products of lower order.

Although β_{2j-1} are determined only up to an exact differential, secondary characteristic classes give explicit formulas for β_{2j-1} and t_{2j-1} in terms of differential forms. Lift $k_{j+n}(\mathcal{F})$ from $M \times \mathfrak{A}/\mathcal{G}$ to $\mathfrak{Q}$, where it equals $d_Q\alpha_{2j+2n-1}$, with $\alpha_{2j+2n-1}$ the secondary characteristic class (formula 73 in ref. 3). That is, $\alpha_{2j+2n-1} = \alpha_{2j+2n-1}(w) = (j + n)$. $\int_0^1 k_{j+n}(w, \mathcal{F}_t, \ldots, \mathcal{F}_t)dt$ with $\mathcal{F}_t = t\mathcal{F} + \tfrac{1}{2}(t - t^2)[w, w]$. Lift $\alpha_{2j+2n-1}$ to $P \times \mathfrak{A}$ and denote it by $\tilde{\alpha}_{2j+2n-1}$. For simplicity, assume $P = M \times G$ (the $k = 0$ sector) so that $M \times \mathfrak{A} \subset P \times \mathfrak{A}$. Let $\beta_{2j-1} = \int_M \tilde{\alpha}_{2j+2n-1}$ a $2j - 1$ form on $\mathfrak{A}$, and let t_{2j-1} be the restriction of β_{2j-1} to an orbit $\mathcal{G}\cdot A$.

THEOREM 2. $d\beta_{2j-1} = d_{2j}$. *When $\rho = Id$, then t_{2j-1} represents a primitive element in $H^{2j-1}(\mathcal{G}, \boldsymbol{R})$ $j + n \leq N$—i.e., t_{2j-1} represents a generator modulo products of lower order.*

The nonproduct case is slightly more complicated. The G-connection w on $\mathfrak{Q}$ comes from a G-connection $\tilde{w}$ on $P \times \mathfrak{A}$. Choose a connection B on P and extend it to $P \times \mathfrak{A}$. The form $\tilde{\alpha}_{2j+2n-1}$ used above is replaced by α where

$$k_{j+n}(\mathcal{F}_{\tilde{w}}) - k_{j+n}(F_B) = d\alpha$$

and α is given by formula 70 in ref. 3—i.e.,

$$\alpha = (j + n)\int_0^1 k_{j+n}(\tilde{w} - B, \mathcal{F}_t, \ldots, \mathcal{F}_t)dt$$

and

$$\mathcal{F}_t = \mathcal{F}_{B+t(\tilde{w}-B)}.$$

It should be remarked that the characteristic classes d_{2j} are not local, for they involve the Green's operator $(D_A^*D_A)^{-1}$ in the curvature $\mathcal{F}$. However, the closed forms t_{2j-1} on $\mathcal{G}$ are local and *Theorem 2* implies they are directly expressible in terms of the Chern–Simons secondary classes. That is, suppose $f_1, \ldots, f_{2j-1}$ are elements in the Lie algebra of $\mathcal{G}$—i.e., in $C^\infty(\Lambda^0 \otimes SU(N))$; because $\mathcal{G}$ acts on P, the fs can be viewed as vertical vector fields on $P = M \times G$. They are also left invariant vector fields on $\mathcal{G}$.

Let $i(f)$ denote interior product by the vector field f and

$$i(f_1, \ldots, f_{2j-1}) = i(f_{2j-1}) \ldots i(f_1).$$

Then, at $\phi \in \mathcal{G}$,

$$t_{2j-1}(f_1, \ldots, f_{2j-1}) = \int_M i(f_1, \ldots, f_{2j-1})\alpha_{2j+2n-1}(\phi A).$$

For example, for $M = S^4$ and $j = 1$, we obtain the 1-form

$$t_1(f) = \int_{S^4} i(f)\alpha_5(\phi A) = -\frac{i}{24\pi^3}\cdot 3\int_{S^4} i(f)\int_0^1 \mathrm{tr}(\phi A(tF_{\phi A}$$

$$+ \tfrac{1}{2}(t - t^2)[\phi A, \phi A])\ (tF_{\phi A} + \tfrac{1}{2}(t - t^2)[\phi A, \phi A])dt.$$

This formula for t_1 is the formula for a nonabelian chiral anomaly (4–6). See also refs. 7–9 for a self contained account of the relationship between anomalies in all dimensions and secondary characteristic classes.

One interpretation for this anomaly involves determinants. Consider the operator $T_\phi = \partial\!\!\!/_B^* \partial\!\!\!/_{\phi A}$: $C^\infty(S^+ \otimes E) \to C^\infty(S^+ \otimes E)$, when $\partial\!\!\!/_A$ and $\partial\!\!\!/_B$ have no zero frequency modes. The operator T_ϕ is a Laplacian plus lower-order term. It has pure point spectrum $\{\lambda_j\}$, and all but a finite number of eigenvalues lie inside a wedge about the positive real axis. Hence, $\Sigma\lambda_j^{-s}$ makes sense except for a finite number of eigenvalues lying on the negative real axis.

When T has positive eigenvalues one can define log det T as

$$-\frac{d}{ds}\bigg|_{S=0} \operatorname{tr}(T^{-s}).$$

We extend this definition by letting I–P denote projection on a finite dimensional space spanned by the eigenfunctions having eigenvalues $\lambda_1, \ldots, \lambda_k$, including those eigenvalues in $[-\infty, 0]$. Let

$$\det T_\phi = e^{\log \det (PT\phi)} \cdot \prod_{j=1}^{k} \lambda_j,$$

which is well defined. Moreover, $\phi \to \det T_\phi$ is a smooth nonvanishing complex valued function on $\mathcal{G}$. Since log det T_ϕ may not be definable, det T_ϕ can give a nontrivial element in $H^1(\mathcal{G}, Z)$. A direct computation using ζ function regularization gives *Theorem 3.*

THEOREM 3. $t_1 = \frac{1}{2\pi i} d(\det T_\phi)/\det T_\phi + df$; *that is,* t_1 *and* $\frac{1}{2\pi i} d(\det T_\phi)/\det T_\phi$ *represent the same element of* $H^1(\mathfrak{A}/\mathcal{G}, \boldsymbol{R})$.

As explained above, the 1-form t_1 on $\mathcal{G}$ comes from the first Chern class d_2 of Ind $\partial\!\!\!/_{\mathfrak{A}/\mathcal{G}}$, a 2-form on $\mathfrak{A}/\mathcal{G}$ equaling $\int_{S^4} k_3(\mathcal{F})_{4,2}$. The first Chern class of Ind $\partial\!\!\!/$ is the Chern class of the determinant line bundle of Ind $\partial\!\!\!/$, and it has the following physical interpretation. Consider the fermionic path integral

$$\mathcal{I}_r(A) = \int e^{\bar\psi \partial\!\!\!/_A \psi} \bar\psi(y_1)\psi(x_1)\bar\psi(y_2)\psi(x_2) \ldots \bar\psi(y_r)\psi(x_r)\mathcal{D}\bar\psi\mathcal{D}\psi$$

which equals

$$\sum_\pi (-1)^\pi (\det \partial\!\!\!/_A^* \partial\!\!\!/_A)^{1/2} \{E_A(y_{\pi(1)}, x_1)E_A(y_{\pi(2)}, x_2) \ldots E_A(y_{\pi(r)}, x_r)\},$$

π is a permutation and E_A is the propagator for $\partial\!\!\!/_A$. Expand E_A in terms of the eigenvectors ψ_j of $\partial\!\!\!/_A^*\partial\!\!\!/_A$ and $\bar\psi_j = \partial\!\!\!/_A \psi_j/\lambda_j$ of $\partial\!\!\!/_A \partial\!\!\!/_A^*$ obtaining

$$\sum \frac{\bar\psi_j(y) \otimes \psi_j(x)}{\lambda_j}.$$

In particular

$$\int e^{\bar\psi \partial\!\!\!/_A \psi} \bar\psi(y)\psi(x) = (\Pi\lambda_j) \sum \frac{\bar\psi_j(y)\psi_j(x)}{\lambda_j},$$

all quantities depending on A. The expression makes sense when there are no zero frequency modes. Suppose $A \to B$ with $\lambda_1 \to 0$. The expression $\mathcal{I}_1(A)$ approaches $\det/\lambda_1 \cdot \bar\psi_1(y) \otimes \psi_1(x)$ with $\partial\!\!\!/_B \psi_1 = 0$. However, ψ_1 is determined only up to a phase and a consistent choice must be made.

For $r > 1$, it is easy to see that because of the exclusion principle, $\mathcal{I}_r(B)$ is indeterminate only when there are exactly r zero frequency modes for $\partial\!\!\!/_B$. Moreover, the indeterminancy depends only on a phase, the choice of a generator in $\Lambda^r(\ker \partial\!\!\!/_B)$, the 1-dimensional space of skewsymmetric r tensors of ker $\partial\!\!\!/_B$.

THEOREM 4. *A gauge covariant* $\mathcal{I}_r(A)$ *smooth in* A *exists if and only if the determinant line bundle of Ind* $\partial\!\!\!/$ *is trivial—i.e.,* $d_2 = 0$ *in* $H^2(\mathfrak{A}/\mathcal{G}, Z)$ *or* $t_1 = 0$ *in* $H^1(\mathcal{G}, Z)$.

The characteristic forms $d_{2j} \varepsilon H^{2j}(\mathfrak{A}/\mathcal{G}, Z)$ are obstructions to the existence of a covariant propagator for $\partial\!\!\!/_{\mathfrak{A}/\mathcal{G}}$. We ask the question: Do the higher obstructions have physical significance?

Using our earlier discussion of the topological index, one can show, for $M = S^{2n}$ and $G = SU(N)$, *Theorem 5.*

THEOREM 5. *If* ρ *is the identity representation, then* $d_{2j} \in H^{2j}(\mathfrak{A}/\mathcal{G}, \boldsymbol{R})$ *and* $t_{2j-1} \in H^{2j-1}(\mathcal{G}, \boldsymbol{R})$ *do not vanish for* $j \leq N - n$.

Gravitational anomalies are the subject of a recent preprint (10), especially for the Dirac operator, the Rarita–Schwinger operator, and the signature operator. These operators are dependent on the metric and are covariant under diffeomorphisms. The formulas obtained in ref. 10 by perturbative calculations at the one-loop level can also be obtained by the methods described in this paper, using the families index and secondary characteristic classes (unpublished result; O. Alvarez and B. Zumino, personal communication).

Specifically, $\mathfrak{A}$ is replaced by the space of all metrics $\mathfrak{M}$ of the manifold M. $\mathcal{G}$ is replaced by the group of diffeomorphisms of M leaving a basis at one point fixed ($\mathrm{Diff}_0(M)$). Each metric $\rho \in \mathfrak{M}$ gives a Dirac operator $\partial\!\!\!/_\rho$ (and other geometric operators) with the covariance $\partial\!\!\!/_{\phi\cdot\rho} = \phi^{-1}\partial\!\!\!/_\rho\phi$ for $\phi \in \mathrm{Diff}_0(M)$. Thus, $\mathfrak{M}/\mathrm{Diff}_0(M)$ is the parameter space for the family $\{\partial\!\!\!/_\rho\}$.

The space $P \times \mathfrak{A}$ is replaced by a sub-bundle of $B \times \mathfrak{M}$ where B is the bundle of bases of M. The sub-bundle is the set of all frames relative to each metric $\rho \in \mathfrak{M}$. The group $\mathrm{Diff}_0(M)$ acts on the sub-bundle and gives a quotient Q which is a principal $O(n)$ bundle over a base space, itself a fiber space over $\mathfrak{M}/\mathrm{Diff}_0(M)$ with fiber M. The first Chern class of the family can be promoted to a 1-form on $\mathrm{Diff}_0(M)$, which is directly expressible in terms of secondary characteristic classes. Since only Pontrjagin classes are involved, nonzero results are obtained only in dimensions $n = 4k + 2$.

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Note Added in Proof. An exposition of the first obstruction and its relation to the chiral anomaly, intended primarily for physicists, can be found in ref. 11.

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