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XIV. Topology and Higher Symmetries of the Two-Dimensional Non-Linear σ-Model

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Abstract:

We formulate the O(3) non-linear σ-model and its supersymmetric extension as Abelian gauge theories. As in the Higgs model the instanton number is related to winding properties of the gauge field at infinity. In the new language, a higher symmetry becomes obvious and Pohlmeyer's conserved currents can be derived easily.

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I. The classical O(3) non-linear σ-model in euclidean space is a theory of spin fields

$$q^a(x); \qquad a = 1,2,3; \qquad x = (x^1, x^2) \in \mathbf{R}^2, \qquad q^a(x) q^a(x) = 1. \tag{1}$$

The action and the field equations are

$$S = \tfrac{1}{2}\int \mathrm{d}^2x\, \partial_\mu q^a \partial^\mu q^a; \qquad \Box q^a + (\partial_\mu q^b \partial^\mu q^b) q^a = 0. \tag{2}$$

This model is strikingly similar to the SU(2) Yang–Mills theory in four dimensions. For example, spin fields $q^a(x)$ also fall into topological classes characterized by an integer winding number [1]

$$Q = \frac{1}{8\pi}\int \mathrm{d}^2x\, \varepsilon^{\mu\nu} \boldsymbol{q}\cdot(\partial_\mu \boldsymbol{q} \times \partial_\nu \boldsymbol{q}), \tag{3}$$

that bounds the action from below:

$$S \geqslant 4\pi|Q|. \tag{4}$$

This inequality is saturated if $q^a(x)$ is selfdual (resp. anti-selfdual):

$$\partial_\mu \boldsymbol{q} = \underset{(-)}{+}\, \varepsilon_{\mu\nu}(\boldsymbol{q} \times \partial^\nu \boldsymbol{q}). \tag{5}$$

In that case q^a is called an instanton (anti-instanton) solution of the field equations (2).

The aim of this note is to derive a formulation of the σ-model in terms of an abelian gauge field A_μ and a complex vector field V_μ. The analogy with the Yang–Mills theory is then even more convincing and the higher symmetries [2] of the σ-model become simple transformations of the fields A_μ, V_μ. We shall also apply our formalism to the supersymmetric non-linear σ-model, but consider first the ordinary model.

II. For any given spin field $q^a(x)$, not necessarily a solution of the equations of motion, we can choose two smooth vector fields $e_1^a(x)$, $e_2^a(x)$ such that

$$\boldsymbol{e}_i = \varepsilon_{ij}(\boldsymbol{q} \times \boldsymbol{e}_j); \qquad \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij} \tag{6}$$

i.e. at each x the three vectors $(\boldsymbol{q}(x), \boldsymbol{e}_1(x), \boldsymbol{e}_2(x))$ form an orthonormal basis. We call it the moving frame. Of course, $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are not uniquely determined by eq. (6). Any other choice $\boldsymbol{e}_1', \boldsymbol{e}_2'$ would be related to $\boldsymbol{e}_1, \boldsymbol{e}_2$ by

$$\boldsymbol{e}_1' = \cos\alpha\, \boldsymbol{e}_1 + \sin\alpha\, \boldsymbol{e}_2, \qquad \boldsymbol{e}_2' = \cos\alpha\, \boldsymbol{e}_2 - \sin\alpha\, \boldsymbol{e}_1 \tag{7}$$

for some gauge angle $\alpha(x)$.

As x varies, the moving frame rotates:

$$\partial_\mu \boldsymbol{q} = -V_\mu^j \boldsymbol{e}_j; \qquad \partial_\mu \boldsymbol{e}_i = A_\mu \varepsilon_{ij} \boldsymbol{e}_j + V_\mu^i \boldsymbol{q}, \tag{8}$$

with

$$V_\mu^i = \boldsymbol{q}\partial_\mu \boldsymbol{e}_i; \qquad A_\mu = \boldsymbol{e}_2 \partial_\mu \boldsymbol{e}_1. \tag{9}$$

These equations define V_μ^i and A_μ in the gauge specified by $\boldsymbol{e}_1, \boldsymbol{e}_2$. In the gauge given by $\boldsymbol{e}_1', \boldsymbol{e}_2'$ (cf. eq. (7)) we would find:

$$V_\mu'^1 = \cos\alpha\, V_\mu^1 + \sin\alpha\, V_\mu^2; \qquad V_\mu'^2 = \cos\alpha\, V_\mu^2 - \sin\alpha\, V_\mu^1; \qquad A_\mu' = A_\mu + \partial_\mu \alpha. \tag{10}$$

Given the gauge field A_μ and the charged vector field $V_\mu = V_\mu^1 + \mathrm{i}V_\mu^2$, the moving frame can be reconstructed from eq. (8), provided

$$\mathrm{D}^\mu {}^*V_\mu = 0, \qquad \varepsilon^{\mu\nu}\partial_\mu A_\nu = \tfrac{1}{2}\bar{V}^\mu {}^*V_\mu. \tag{11}$$

Here, $\mathrm{D}_\mu = \partial_\mu + \mathrm{i}A_\mu$ is the covariant derivative, $\bar{V}_\mu$ the complex conjugate and ${}^*V_\mu = \mathrm{i}\varepsilon_{\mu\nu}V^\nu$ the dual of V_μ.

In terms of the new variables A_μ, V_μ the action and the equations of motion read:

$$S = \tfrac{1}{2}\int \mathrm{d}^2x\, \bar{V}_\mu V^\mu, \tag{12}$$

$$\mathrm{D}^\mu V_\mu = 0 \tag{13}$$

and the selfduality equation (5) reduces to

$$V_\mu \underset{(-)}{+} {}^*V_\mu = 0. \tag{14}$$

Thus, the non-linear σ-model is equivalent to a theory of an abelian gauge field A_μ and a charged vector field V_μ subject to the constraints (11).

III. The product $\bar{V}^\mu {}^*V_\mu$ is equal to the topological density $\varepsilon^{\mu} \boldsymbol{q}(\partial_\mu \boldsymbol{q} \times \partial_\nu \boldsymbol{q})$ so that by eq. (11)

$$Q = \frac{1}{4\pi}\int \mathrm{d}^2x\, \varepsilon^{\mu\nu}\partial_\mu A_\nu. \tag{15}$$

This formula should be compared with the Yang–Mills expression

$$Q = \frac{1}{8\pi^2}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma}\partial_\mu \partial_{\nu\rho\sigma} \tag{16}$$

where $A_{\mu\nu\rho}$ is an abelian gauge field of the fourth kind*:

$$A_{\mu\nu\rho} = \mathrm{Tr}\,\{B_{[\mu}\partial_\nu B_{\rho]} + \tfrac{2}{3}B_\mu B_\nu B_\rho\} \tag{17}$$

and $B_\mu = B_\mu^a \sigma^a/2\mathrm{i}$ is the Yang–Mills field. By its definition eq. (3), Q is the winding number of the $\mathrm{S}^2 \to \mathrm{S}^2$ mapping of x^μ into $q^a(x)$; eq. (15) states that it coincides with the winding number of the mapping of the circle at $|x| = \infty$ into the abelian gauge group manifold.

IV. The field equations (11) and (13) have an obvious symmetry, namely:

$$V'_\mu \pm {}^*V'_\mu = \mathrm{e}^{\pm \mathrm{i}\psi}(V_\mu \pm {}^*V_\mu); \qquad A'_\mu = A_\mu \tag{18}$$

with ψ = constant. This is Pohlmeyer's [2] $\mathrm{R}^{(\gamma)}$-transformation, which reveals its simple structure in the present formalism. As shown in ref. [3], this symmetry generates an infinite set of non-local conserved charges, which in turn govern the dynamics of the quantum non-linear σ-model [4].

Instanton solutions are left invariant by the transformation (18), which in this case reduces to a global gauge transformation. Conversely, any field that is invariant under (18) is necessarily

* i.e. when B_μ is gauge transformed, $A_{\mu\nu\rho}$ changes:

$$A'_{\mu\nu\rho} = A_{\mu\nu\rho} + \partial_{[\mu}\Lambda_{\nu\rho]}.$$

(anti-) selfdual, thus showing that the existence of instantons is not totally uncorrelated to the presence of a higher symmetry.

Besides the symmetry (18) there exists a Backlund transformation [2] which maps any given real solution q^a of eq. (2) onto a new complex* solution q'^a. Relative to a moving frame $(\boldsymbol{q}, \boldsymbol{e}_1, \boldsymbol{e}_2)$, $\boldsymbol{q}'$ is completely specified by a complex angle α:

$$\boldsymbol{q}' = \cos\alpha\, \boldsymbol{e}_1 + \sin\alpha\, \boldsymbol{e}_2, \tag{19}$$

with

$$\partial_\mu \alpha + A_\mu = \tfrac{1}{2}\mathrm{i}\{\mathrm{e}^{\mathrm{i}\alpha} * \bar{V}_\mu + \mathrm{e}^{-\mathrm{i}\alpha} * V_\mu\}. \tag{20}$$

If we first perform a symmetry transformation (18) on A_μ, V_μ, the angle α becomes a function of $\gamma = \mathrm{e}^{\mathrm{i}\psi}$:

$$\partial_z \alpha + A_z = \gamma(\sin\alpha\, V_z^1 - \cos\alpha\, V_z^2), \qquad \partial_{\bar z}\alpha + A_{\bar z} = -\frac{1}{\gamma}(\sin\alpha\, V_{\bar z}^1 - \cos\alpha\, V_{\bar z}^2), \tag{21}$$

where $\partial_z = \partial_1 + \mathrm{i}\partial_2$, $V_z^l = V_1^l + \mathrm{i}V_2^l$, etc. Actually, the eqs. (21) are compatible for any complex γ and we can solve them iteratively with the power series Ansatz $\alpha = \alpha_0 + \gamma\alpha_1 + \gamma^2\alpha_2 + \dots$. Inserting this expansion into the identity

$$\gamma\partial_{\bar z}(\cos\alpha\, V_z^1 + \sin\alpha\, V_z^2) + \frac{1}{\gamma}\partial_z(\cos\alpha\, V_{\bar z}^1 + \sin\alpha\, V_{\bar z}^2) = 0, \tag{22}$$

we obtain Pohlmeyer's conservation laws. For example,

$$\partial_z |V_{\bar z}| = 0. \tag{23}$$

$$\partial_{\bar z}\frac{V_{\bar z}^i V_z^i}{|V_{\bar z}|} - \partial_z \frac{(\mathrm{D}_{\bar z}V_{\bar z})^i(\mathrm{D}_{\bar z}V_{\bar z})^i}{2|V_{\bar z}|^3} = 0, \tag{24}$$

where $|V_{\bar z}| = (V_{\bar z}^i V_{\bar z}^i)^{1/2}$ and $\mathrm{D}_{\bar z}^{ij}$ is the covariant derivative $\delta^{ij}\partial_{\bar z} - \varepsilon^{ij}A_{\bar z}$.

V. The Euclidean supersymmetric σ model [5] is defined by the action

$$S = \tfrac{1}{4}\int \mathrm{d}^2x\, \mathrm{d}^2\theta\, \mathrm{D}\phi^a \gamma_5\, \mathrm{D}\phi^a, \tag{25}$$

where $\phi^a = \phi^a(x_\mu, \theta_\alpha)$ is a superfield subject to the constraint

$$\phi^a\phi^a = 1. \tag{26}$$

θ is a Grassmann spinor and D_α is the covariant derivative of supersymmetry**:

$$\mathrm{D}_\alpha = \mathrm{i}\left[\frac{\partial}{\partial\theta_\alpha} + \mathrm{i}(\gamma^\mu\theta)_\alpha\,\partial_\mu\right]. \tag{27}$$

* $\boldsymbol{q}'$ is real in Minkowski space.

** The following representation for the euclidean γ matrices is used

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma_5 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The product of two spinors χ and φ is $\chi_\alpha\varphi_\alpha$.

The second order equations of motion and the self-duality equations are:

$$(\mathrm{D}\gamma_5\mathrm{D})\boldsymbol{\phi} + (\mathrm{D}\boldsymbol{\phi}\gamma_5\mathrm{D}\boldsymbol{\phi})\boldsymbol{\phi} = 0, \tag{28}$$

$$\mathrm{D}_\alpha\phi^a \pm \varepsilon^{abc}\phi^b(\gamma_5\mathrm{D})_\alpha\phi^c = 0. \tag{29}$$

The gauge theory formulation of the σ-model can be easily generalized to the supersymmetric model by introducing a moving frame $\{\boldsymbol{\phi}(x,\theta), \boldsymbol{e}_1(x,\theta), \boldsymbol{e}_2(x,\theta)\}$ and by defining:

$$\mathrm{D}_\alpha\boldsymbol{\phi} = -\mathscr{V}_\alpha^i\boldsymbol{e}_i; \qquad \mathrm{D}_\alpha\boldsymbol{e}_i = \mathscr{V}_\alpha^i\boldsymbol{\phi} + \varepsilon^{ij}\mathscr{A}_\alpha\boldsymbol{e}_j. \tag{30}$$

Using the identity $\mathrm{D}_\alpha\mathrm{D}_\alpha = 0$ one can derive from (30) the following conditions:

$$\Delta_\alpha\mathscr{V}_\alpha = 0; \qquad \mathrm{D}_\alpha\mathscr{A}_\alpha = \tfrac{1}{2}\mathrm{i}\overline{\mathscr{V}}_\alpha\mathscr{V}_\alpha, \tag{31}$$

where $\Delta_\alpha = \mathrm{D}_\alpha + \mathrm{i}\mathscr{A}_\alpha$ is the covariant derivative and $\mathscr{V}_\alpha = \mathscr{V}_\alpha^1 + \mathrm{i}\mathscr{V}_\alpha^2$. The action, the equations of motion and the selfduality equations are respectively the following:

$$S = \tfrac{1}{4}\int \mathrm{d}^2x\,\mathrm{d}^2\theta\,\overline{\mathscr{V}}_\alpha\varepsilon_{\alpha\beta}\mathscr{V}_\beta, \tag{32}$$

$$\varepsilon_{\alpha\beta}\Delta_\alpha\mathscr{V}_\beta = 0, \tag{33}$$

$$\mathscr{V}_\alpha \underset{(-)}{+} \mathrm{i}\varepsilon_{\alpha\beta}\mathscr{V}_\beta = 0. \tag{34}$$

Everything proceeds, with rather obvious modifications, as in the conventional model, in particular the key symmetry (18) holds also here and it takes the form:

$$\mathscr{V}_\alpha \pm \mathrm{i}\varepsilon_{\alpha\beta}\mathscr{V}_\beta \to \mathrm{e}^{\pm\mathrm{i}\psi}(\mathscr{V}_\alpha \pm \mathrm{i}\varepsilon_{\alpha\beta}\mathscr{V}_\beta); \qquad \mathscr{A}_\alpha \to \mathscr{A}_\alpha \tag{35}$$

with ψ a real parameter. As before, this symmetry is related to the existence of non-local charges.

In strict analogy with the ordinary σ-model, a discrete Backlund transformation (eqs. (19), (20)) can be written down for the superfield ϕ^a, too. Combining it with the symmetry (35), we obtain the following compatible equations for the superangle α:

$$\mathrm{D}_+\alpha + \mathscr{A}_+ = \gamma(\sin\alpha\,\mathscr{V}_+^1 - \cos\alpha\,\mathscr{V}_+^2), \qquad \mathrm{D}_-\alpha + \mathscr{A}_- = -\frac{1}{\gamma}(\sin\alpha\,\mathscr{V}_-^1 - \cos\alpha\,\mathscr{V}_-^2), \tag{36}$$

($\pm$ stands for $1 \pm \mathrm{i}2$). The associated conservation law is

$$\gamma\mathrm{D}_+(\cos\alpha\,\mathscr{V}_-^1 + \sin\alpha\,\mathscr{V}_-^2) + \frac{1}{\gamma}\mathrm{D}_-(\cos\alpha\,\mathscr{V}_+^1 + \sin\alpha\,\mathscr{V}_+^2) = 0. \tag{37}$$

However, the solution of eqs. (36) by power expansion around $\gamma = 0$ leads to the impossible division by a Grassmann number, unless $\mathscr{V}_+^1\mathscr{V}_+^2 = 0$. We were therefore not able to find the analog of Pohlmeyer's local conserved currents, thus leaving the problem to the interested reader.

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