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Complex and Quaternionic Analyticity in Chiral and Gauge Theories, I

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A comparative and systematic study is made of 2-dimensional $CP(n)$ σ-models and new 4-dimensional $HP(n)$ σ-models and their respective embedded $U(1)$ and $Sp(1)$ holonomic gauge field structures. The central theme is complex versus quaternionic analyticity. A unified formulation is achieved by way of Cartan's method of moving frames adapted to the hypercomplex geometries of the harmonic symmetric spaces $CP(n) \approx SU(n+1)/SU(n) \times U(1)$ and $HP(n) \approx Sp(n+1)/Sp(n) \times Sp(1)$ respectively. Elements of complex Kähler manifolds are applied to a detailed analysis of the $CP(n)$ σ-model and its instanton sector. Generalization to any Kählerian σ-model is manifest. On the basis of Cauchy–Riemann analyticity, Kählerian models are shown to have an infinite number of local continuity equations. In a parallel manner, new 4-dimensional conformally invariant $HP(n)$ σ-models are constructed. Focus is on the latter's hidden local gauge invariance in their holonomy group $Sp(n) \times Sp(1)$ which allows a natural embedding of the $Sp(1) \approx SU(2)$ pure Yang–Mills theory. The associated quaternionic structure is discussed in light of both quaternionic quantum mechanics and Kählerian geometry. In this chiral setting, the $SU(2)$ Yang–Mills duality equations are cast into quaternionic Cauchy–Riemann equations over $S^4 \approx HP(1)$, the conformal spacetime. In analogy to the $CP(n)$ case, their rational solutions are the most general $(8n-3)$ parameter instantons where the associated algebraic nonlinear equations of the type of Atiyah, Drinfeld, Hitchin, and Manin are now expressed in a new conformally invariant form. Geometrically, the $SU(2)$ instantons solve the Frenet–Serret equations for quaternionic holomorphic curves; they are conformal maps from $HP(1)$ into $HP(n)$ with n their second Chern index. Fueter's quaternionic analysis is presented, then applied: Fueter functions are particularly suited for the solutions of 't Hooft, of Jackiw, Nohl and Rebbi, and of Witten and Peng, as well as the self-dual finite action per unit time solution of Bogomol'nyi, Prasad and Sommerfield. Generalizing the latter, a new solution with unit Chern index and finite action per unit spacetime cell is found. It is expressed in terms of the quaternionic fourfold quasi-periodic Weierstrass Zeta function. Finally the essence of our method is revealed in terms of universal connections over Stiefel bundles; generalization to real, complex and quaternionic classifying Grassmanian σ-models with their embedded $SO(m)$, $SU(m)$ and $Sp(m)$ gauge fields is outlined in terms of gauge invariant projector valued chiral fields. Other outstanding problems are briefly discussed.

I. INTRODUCTION AND SUMMARY

In the current rebirth of field theory there has been striking progress in our understanding of the nonperturbative structure [1, 2], classical as well as quantum, of

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intrinsically nonlinear field theories. Typical illustrations are the 2-dimensional completely integrable chiral field theories [3, 4], specifically the $CP(n)$ nonlinear σ-models [5, 7], the natural extensions of the well-known $SU(2)/U(1) \approx CP(1)$ σ-model [8]. It has been often noted that chiral models are remarkable in that, in their geometric nonlinearities, they stand halfway between general relativity and gauge theories. Recent investigations have singled them out as a most promising avenue toward a unified structural understanding of the nonlinearity in gravity [9, 10] and in gauge theories [11]. It is very likely that their complete integrability will assist in proving or disproving the conjectures [12] that the latter theories are (in some sense) completely integrable as well. Thus it has been known that the exact Tominatsu–Sato solutions [13] arise from the equivalence of a $SU(1, 1)/U(1)$ σ-model to the axisymmetric gravitational problem [14]. On the other hand the uncanny resemblances between the $SU(2)/U(1) \approx CP(1)$ σ-model with the $SU(2)$ Yang–Mills theory in four dimensions have been repeatedly noted [15–17] but not systematically analyzed in full.

Cast on a lattice, the $CP(1)$ σ-model is indistinguishable from the 4-dimensional $SU(2)$ lattice gauge theory under Migdal's recursion formula [15]. In the continuum, both theories are scale invariant and asymptotically free. At the semi-classical level [17] and at zero temperature $T = 0$ [18], both possess Bloch wave-type θ vacua. Both admit exact instantion [16] and meron solutions [19]. While in two dimensions, the existence and structure of the $CP(1)$ instantons are deeply rooted in the complex analytic nature of $S^2 \simeq CP(1)$, the base and the field manifolds of the model, in the 4-dimensional $SU(2)$ Yang–Mills case, the simple connection between 't Hooft's $5n$ parameter instantons and R. Fueter's [20] quaternionic Cauchy–Riemann analyticity on S^4, the compactified 4-spacetime, had been reported previously [21]. It was then apparent that a more general form of quaternionic holomorphy is required to obtain the general $(8n - 3)$ parameter solutions of Atiyah *et al.* [22]. Besides the cited properties of the $CP(1)$ σ-model, its $CP(n)$ extension in two dimensions allows for a $1/n$ expansion [17]. In this $n \to \infty$ limit and at $T = 0$ only [18], the fundamental particles of the model are confined by a long-range topological force engendered by instantons. Another effect is the disappearance of the instanton gas which has led to Witten's conjecture [23] that similarly the 4-dimensional $SU(n)$ instanton gas may disappear in the large n limit.

The above structural parallels between 2-dimensional chiral and 4-dimensional gauge theories set in a broader framework bring to a sharper focus a string of key questions: Of the noteworthy features of 2-dimensional field theories such as the fermion–boson duality [24], the existence of an infinite number of nonlocal conservation laws [3, 4], etc., which ones do have, and under what restrictions do they have their counterparts in four dimensions? What exactly are these 4-dimensional analogs? What parallel mathematical structures will best help to lift the veil of similarities?

The present work reports on some progress we have made in a systematic effort to resolve some of these issues. Our central theme is the "magic" parallel between the complex analytic structure of the 2-dimensional $CP(n)$ σ-models and the quaternionic analytic structure of some newly constructed 4-dimensional $HP(n)$ σ-models and their embedded $Sp(1)$ pure Yang–Mills gauge substructure. In this task, essential use is made of the basic results in the geometry and topology of the n-dimensional complex and quaternionic Kählerian manifolds $CP(n)$ and $HP(n)$ respectively. Mostly we use Cartan's [25] more intuitive method of moving frames. The latter has proved both elegant and powerful in the analysis of the existence and uniqueness questions for submanifolds of a homogeneous space [26], in the study of holomorphic curves in complex projective spaces [27], in the theory of harmonic mappings [28]. It clearly is a most appropriate tool since in physics, algebraic curves and harmonic maps translate into solitons [29] or in our case, instantons. Being halfway between tensor analysis and fibre bundles, the frame method is appealing in its emphasis on direct computations. When applied to manifolds, it brings forth their basic geometrical features and their cohomology classes are explicitly given by differential forms.

Throughout this paper, we have assumed essentially no background on the part of the reader on complex and quaternionic manifolds. As much as possible, we shall take care to introduce the relevant basic concepts and derive the needed formulas. Our work can be summed up as follows:

To establish our notations and to set up the basic tools, we begin by defining the notions of the complex and quaternionic projective spaces $CP(n)$ and $HP(n)$ in a unified analytic representation. In a compact matrix formalism, Cartan's method of moving frames is used to parametrize these geometries identified as harmonic symmetric spaces of Lie groups, namely $CP(n) \approx SU(n+1)/SU(n) \times U(1)$ with $2n$ real parameters and $HP(n) \approx Sp(n+1)/Sp(n) \times Sp(1)$ with $4n$ real parameters. These spaces will be used as the chiral field manifolds of our 2-dimensional complex and 4-dimensional quaternionic σ-models.

In general [24], nonlinear σ-models invariant under an internal group G are constructed in terms of the coset representative N of the factor space G/H, where H is a maximal subgroup of G, N is taken as a function $N(x)$ of the spacetime point x (in two and four dimensions), as a set of Nambu–Goldstone fields. The Lagrangian of the model is then invariant under H with the parameters taken as arbitrary functions of x. N transforms linearly under the local group H, but nonlinearly under the global transformation G/H of G. Hence the equations of motion will have global invariance under G (more specifically under nonlinear global transformations in G/H) and *local* gauge invariance under H. This is basically the reason why a Yang–Mills theory with the gauge group $H' \subsetneq H$ can be embedded in a nonlinear σ-model with global group G. This phenomenon will be illustrated in two and four dimensions where G/H correspond to Grassmannian manifolds for which this connection with gauge fields is precise and far reaching. As a canonical example, a basis for future comparison as well as a stepping stone to 4-dimensional chiral models, we first show, in the moving frame method, the $CP(n)$ σ-model in two dimensions to be a nonlinear realization of the global symmetry $SU(n+1)$ with the isotropy subgroup $SU(n) \times U(1)$ acting linearly as a gauge group. This hidden gauge invariance corresponds geometrically to the freedom of the frames to rotate. The frame method, which brings about a "spinorization" of chiral models, has been found to constitute the key geometrical basis of the inverse scattering problem for completely integrable 2-dimensional systems [29]. It is then not surprising, though the essential reason is deep, that in its linear dress, the $CP(n)$ σ-model most closely resembles a gauge theory.

In our pedestrian approach, classical results in complex Kählerian geometry are summarized in an appendix and applied to the analysis of the instantion sector of the $CP(n)$ σ-model. In our geometrical approach, a maximal formal correspondence is thus achieved with the $5n$ parameter $SU(2)$ Yang–Mills instantons to be discussed later in light of Fueter analyticity. As long known by Eels and Sampson [30], $CP(n)$ instantons are harmonic conformal maps from $CP(1)$ into $CP(n)$, the conformal 2-spacetime into the chiral field manifold. The instanton number is the 1st Chern index, shown to be simply given, in the $CP(1)$ case, by the winding number of complex analysis through Cauchy's integral theorem as it counts the number of poles in the rational instanton field. On the other hand, when instantons are viewed as algebraic rational curves in an n-dimensional projective space, contact is made with the works of Chern on holomorphic curves. On the strength of complex analyticity we show there exists an infinite number of local continuity equations for any Kählerian σ-models.

Seeking a quaternionic counterpart of the above, we proceed to the differentiable conformally compactified 4-dimensional spacetime $E^4U(\infty) \approx S^4 \approx HP(1)$, the quaternionic projective line. After a prelude on Euclidean conformal invariance formulated in terms of real quaternions, we consider in analogy to the complex case, the Cartan structural equations induced on $HP(1)$ from the chiral field space $HP(n)$. This approach naturally leads to the construction of new conformally invariant nonlinear $HP(n)$ σ-models in four dimensions. They are the higher dimensional counterparts of the $CP(n)$ σ-models. It is shown that these quaternionic σ-models have embedded within them a pure $Sp(1)$ Yang–Mills theory. Thus deferring a more complete study of the $HP(n)$ σ-models proper to a later work [31], we focus on this embedded gauge field structure. First the quaternionic structure induced on $HP(n)$ by the holonomy group $Sp(n) \times Sp(1)$ is discussed both from the standpoint of quaternionic quantum mechanics and from that of quaternionic Kählerian geometry. We also present a new form of the quaternionic Kählerian metric in terms of the second derivatives of a scalar Kähler potential in complete analogy to the complex case. The associated generalized Liouville equation is also derived. Expressed in terms of chiral fields, the $SU(2)$ Yang–Mills self (antiself)-duality equations $F_{\mu\nu} = \pm\tilde{F}_{\mu\nu}$ are precisely given by the quaternionic counterparts of the Cauchy–Riemann equations on S^4. We show their rational solutions to be the most general $(8n - 3)$ parameter instantons. The new feature in our result is the manifest conformal invariance of the associated Atiyah *et al.* type [22] linear algebraic constrained equations (yet to be solved explicitly for $n \geqslant 3$). In our formalism Yang–Mills instantons are seen as conformal holomorphic mappings from $HP(1)$ into $HP(n)$, n being their second Chern index. As in the complex case, self (antiself)-duality is the statement of the conformal flatness of the induced quaternoinic Kählerian metric. Geometrically they correspond to quaternionic rational curves in quaternionic projective spaces. Our derivation thus provides a purely differential geometric alternative to the linear algebraic construction of $SU(2)$ instantons, of Atiyah *et al.* [22].

Noting that the quaternionic holomorphy of Atiyah *et al.* generalizes that discovered by R. Fueter [20], we apply the latter's version of quaternion analyticity to the special solutions of 't Hooft [32], Jackiw *et al.* [33], Witten [34], and Peng [35]. In particular Fueter analyticity leads to a compact, revealing, reformulation of the Bogomol'nyi, Prasad–Sommerfield [36] solution seen as a self-dual solution with finite action per unit time. Set in 't Hooft rational gauge, this quaternionic form underscores the key periodic structure of this solution and clarifies the nature of a new self dual solution which has finite action and unit Chern index per unit spacetime cell. This new "crystalline" Yang–Mills field configuration is given in terms of the quaternionic quasi-fourfold periodic Weierstrass Zeta function [37]. Thus it generalizes the BPS solution.

Finally we explain the essence of the connection between σ-models and gauge theories in light of theorems on the universality of Stiefel bundles over real, complex and quaternionic Grassmannian manifolds and the theorem of Narasimhan and Ramanan on the existence of universal connections over these bundles. This realization leads to a brief outline of the extension of our method by way of projector valued fields to general 4-dimensional Grassmannian σ-models and to a treatment of G-instantons with $G = SO(m)$, $SU(m)$ and $Sp(m)$ respectively. Other outstanding problems are also discussed.

Our paper is organized as follows: Section II gives an analytic representation of the spaces $CP(n)$ and $HP(n)$. Section III formulates the $CP(n)$ σ-model and analyzes its instanton sector by way of moving frames. Complex Kählerian geometry and topology à la Chern [38] are reviewed, then applied to the $CP(n)$ problem. A uniqueness proof of the $CP(n)$ and other Kählerian instantons is given. Section IV begins with a quaternionic formulation of conformal invariance. New 4-dimensional $HP(n)$ σ-models are then constructed. In light of quaternionic quantum mechanics and Kählerian geometry, the $(8n - 3)$ parameter $SU(2)$ Yang–Mills instanton solutions are shown to be the rational solutions to the quaternionic Cauchy–Riemann equations. Section V reviews and expands on Fueter's work. Section VI then applies Fueter's theory to special self dual solutions, old and new. Section VII concludes with a brief outline of the natural extension of our method and of other remaining problems.

Contrasting with more mathematical and less explicit papers on the subject, our presentation aims at comprehensiveness and comprehensibility. In this task, we may have been a bit overdetailed. To ease the way for the reader, four appendices are included. The first gathers some relevant elements of quaternion algebra and calculus. The second compiles some formulas on complex and quaternionic valued differential forms and their pull back properties. The third is a compendium of relevant definitions and theorems in complex manifolds and more references on quaternionic Kähler geometries. The fourth is an elaboration on Fueter's quaternionic analysis. These appendices should be taken as an integral part of the paper as they clarify and supplement the main body of an already long text.

In Part II of our work, we mainly wish to discuss the application of the present formalism to the problem of uniqueness of $SU(2)$ Yang–Mills instantons, to a more detailed study of the $HP(n)$ σ-models proper and to the question of the existence of nonlocal conservation laws.

II. Analytic Representation of $KP(n)$ Spaces

No other area of modern mathematics rests on a broader foundation as complex manifold theory [38]. Here the beauties of geometry and topology are interwoven with the subtleties of complex analysis. In physics, complex particularly Kählerian manifolds have played an ever important role in the twistor and heavenly spaces approaches to general relativity [39], in geometric quantization schemes [30], and more recently in 2-dimensional chiral theories with sophisticated topologies [7, 41]. It is for their subsequent use as chiral field spaces of Nambu–Goldstone fields that we collect in this section some necessary basic properties of two Kähler manifolds par excellence, the n-dimensional complex and quaternionic projective spaces $CP(n)$ and $HP(n)$.

While well known to algebraic geometers [37, 41, 42], these Kähler spaces may not be familiar to most physicists. Therefore our treatment will be self-contained. Since a Kählerian space, a notion yet to be defined, consists of a complex structure and a Riemannian metric, it can be viewed in a twofold way. The first is analytic and underlines the complex structure, the second is geometric and emphasizes the metric structure. For a full understanding of self-duality, both complex and Riemannian structures must be exploited jointly. For completeness and balance without distracting from our main theme, namely complex versus quaternionic analyticity, we will discuss in the text and gather in Appendix III some basic properties of Kählerian manifolds, complex and quaternionic.

The most fundamental of classical geometries, projective geometries can be completely characterized by a natural number n and the number field K with the exception of $n = 2$ in the case of CaP(2), the non-Desarguesian Cayley–Moufang octonionic plane [44]. To have a compact formulation, let K be a topological field, i.e., R the field of real numbers, C the field of complex numbers, and H the skew field of real quaternions. Here we shall only deal with $K = C$ or H. We now define the notion of an n-dimensional complex and quaternionic projective space. For any natural number n, let Q^{n+1} over K be an $(n+1)$ dimensional (for the noncommutative case right and left) linear space. Any of its points is an ordered $(n+1)$ tuple $q = (q_0, q_1, \ldots, q_n)$ where the natural homogeneous coordinates are $q_i \in K = C, H$. The reader unfamiliar with quaternions may wish to consult Appendix I.

By excluding the origin $0 = (0, 0, \ldots, 0)$ we consider the space $^*Q^{n+1} = Q^{n+1} - \{0\}$. If we fix a basis in $^*Q^{n+1}$, we can represent it as $^*Q = \{q = (q_0 \cdots q_n) \mid q_i \in K, 0 \leqslant i \leqslant n\}$, $q \neq (0, 0, \ldots, 0)$. If there is a nonzero element λ of K such that $r = q\lambda$ the elements q and r are called *equivalent*. For better visualization, let us call an "oriented line" any set $l - \{0\}$ with l the compact quotient of the space $^*Q^{n+1}$ by the above equivalence relations of these lines. More precisely if we denote by $[q]$ the equivalence class containing q and if we define the line $l([q], [r]) = \{[s] \mid s_i = q_i\lambda + h_i\mu, \forall\lambda, \mu \in K\}$ and a set of lines $\{l([q], [r] \mid [q], [r] \in KP(n)\}$, then these points and lines and the natural inclusion relation do satisfy the axioms of an n-dimensional projective geometry.

As a compact topological space, $KP(n)$ does not admit a single coordinate system defined everywhere but can be covered by at least $(n + 1)$ local open coordinate patches P_ν, $\nu = 0, 1,..., n$, the set of points $q \in KP(n)$ whose $q_\nu \neq 0$. For each such ν, the mapping

$$(q_0, q_1, ..., q_n) —— (t_\nu{}^0, t_\nu{}^1, ..., t_\nu{}^n) \tag{2.1}$$

where $t_\nu{}^i \equiv q_i q_\nu^{-1}$, is a holomorphic isomorphism of P_ν onto $^*Q^{n+1}$. If we select the set on the right in Eq. (2.1) as local inhomogeneous or affine coordinates in P_ν, we define in the process an n-dimensional complex or quaternionic structure in $KP(n)$ for $K = C$ and $K = H$. The point is that the transition of local coordinates in the overlap $P_\nu \cap P_\mu$ is given by $t_\mu{}^h = t_\nu{}^h (t_\nu{}^\mu)^{-1}$ $(0 \leqslant h \leqslant n)$ which are holomorphic functions. Being rational in the q_i the affine coordinates $t_\nu{}^i$ are thus *genuine* functions on $KP(n)$.

By assigning to a point of $^*Q^{n+1}$ the point it defines in the quotient space, $KP(n)$, we obtain a natural projection map π: $^*Q^{n+1} \to KP(n)$ defining a holomorphic line bundle. To a point $p \in KP(n)$ the coordinates of a point of π^{-1} are its homogeneous coordinates. Since only ratios of the coordinates of a point are determined, $KP(n)$ is homeomorphic to the factor space of the sphere

$$\sum_{i=0}^{n} q_i \bar{q}_i = 1 \tag{2.2}$$

by identifying the end points of each diameter. For $K = C$, Eq. (2.2) defines the sphere S^{2n+1} embedded in E^{2n+2}, the $(2n + 2)$ dimensional Euclidean space. For $K = H$ it defines the sphere S^{4n+3} in E^{4n+4}. The restriction (Eq. (2.2)) of π gives the maps π: $S^{2n+1} \to CP(n)$ and $S^{4n+3} \to HP(n)$, the Hopf fiberings of these spheres. In other words S^{2n+1}, S^{4n+3} are respectively the 1- and 3-sphere principal bundles over $CP(n)$ and $HP(n)$ with, as their fiber $S^1 \approx U(1)$, the circle group, and $S^3 \approx SU(2)$, the multiplicative group of unit quaternions [45].

As far as group actions, the group of linear K-valued transformations acting on the q_i's on the left and preserving the quadratic form Eq. (2.2) are for $K = C$ the unitary group $U(n + 1)(SU(n + 1)$ if the group action is to be effective) and for $K = H$ the symplectic group $Sp(n + 1)$. The groups $U(1)$ and $Sp(1) \approx SU(2)$ act freely on S^{2n+1} and S^{4n+3} respectively by right translation $(q_0, ..., q_n)\lambda = (q_0\lambda, ..., q_n\lambda)$, $\lambda \in U(1), Sp(1)$. The quotient of Eq. (2.2) $(S^{2n+1} \approx SU(n + 1)/SU(n)$, $S^{4n+3} \approx Sp(n + 1)/Sp(n))$by this action is just the homogeneous symmetric space of rank 1 [46], namely $SU(n + 1)/SU(n) \times U(1) \approx CP(n)$, $Sp(n + 1)/Sp(n) \times Sp(1) \approx HP(n)$. Next we proceed to a matrix parametrization of $KP(n)$.

First let us recall that a homogeneous space M of a group G has the following properties. (1) It is a topological space on which the group G acts *continuously*: if $g \in G$ and p be any point of M, then $p' = gp$ is also a point of M. (2) G acts *transitively* on M: if two points p and p' are in M, there exists a $g \in G$ such that $p' = gp$. The correspondence between the homogeneous space and the coset space of G, G/H, is the following. If H is a maximal subgroup of G such that $hp = p$, $h \in H$ is called the stabilizer or isotropy group. If we write any $g \in G$ as $g = g_c g_i$ where $g_i \in H$ and

$g_c \in G/H$. Then due to the above transitivity property any $p \in M$ can be gotten as $p = g_c g_i p = g_c y$. So the element q_c of the coset space yields a parametrization of M. The mapping $M \in G/H$ is continuous one to one and onto, the isotropy groups H and H' of two different points p and p' are conjugate as $H' = gHg^{-1}$.

From the above it follows that the analysis of a homogeneous space can be reduced to that of their coset spaces G/H, hence to the pair of groups (G, H). In turn the many properties of the latter are deduced from their Lie algebras **g** and **H**. Hence we consider the spaces $KP(n)$ as coset spaces of the Lie groups $SU(n+1)$ if $K = C$ and $Sp(n+1)$ if $K = H$. Let U denote an element of $SU(n+1)$ or $Sp(n+1)$, so that it is a $(n+1) \times (n+1)$ complex unitary or quaternionic symplectic matrix such that

$$UU^{\dagger} = U^{\dagger}U = E \tag{2.3}$$

where

$$U^{+} = \bar{U}^{T}, \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & I \end{pmatrix}. \tag{2.4}$$

The superscripts T and $-$ denote transposition and conjugation, complex or quaternionic. I_n is the $(n \times n)$ unit matrix $I = I_n$.

We now recall that a *symmetric* space is defined [46] by a triple (G, H, η) consisting of a connected Lie group G, a closed subgroup H of G and η an involutive automorphism of G such that H lies between G_η and the identity component of G, G_η being the closed subgroup of G made up of all elements left fixed by η. Explicitly we define this involution η of $U(n+1)$ and $Sp(n+1)$ by

$$\eta(A) = \eta A \eta^{-1} \tag{2.5}$$

where

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix}, \qquad \eta^2 = E \tag{2.6}$$

and $A \in U(n+1)$, $Sp(n+1)$. The subgroup of fixed elements is $SU(n) \times U(1)$ and $Sp(n) \times Sp(1)$ respectively. Hence $KP(n)$ is a symmetric space. Using η we can define a hermitian matrix

$$N = U\eta U^{\dagger} \tag{2.7}$$

with

$$N = N^{\dagger}, \qquad N^2 = E. \tag{2.8}$$

N will be a canonical chiral field in applications.

To best extract the analytic properties of the coset spaces $KP(n) = G/H$ where $G = SU(n+1)$, $Sp(n+1)$ and the isotropy group $H = SU(n) \times U(1)$, $Sp(n) \times Sp(1)$, we are led to the decomposition of any $U \in G$ into the product of an element $W \in H$ and leaving η invariant and a coset representative $V = G/H$:

$$U = VW \tag{2.9}$$

where

$$WW^\dagger = E, \qquad \eta W \eta = W \tag{2.10}$$

and

$$\eta V \eta = V^\dagger. \tag{2.11}$$

V and W can be derived from U by noting that

$$N = U\eta U^\dagger = V\eta V^\dagger, \tag{2.12}$$

$$= V^2\eta, \tag{2.13}$$

$$V = (N\eta)^{1/2} = (U\eta U^\dagger\eta)^{1/2}, \qquad W = (U\eta U^\dagger\eta)^{-1/2}\, U. \tag{2.14}$$

One can also verify that

$$\begin{aligned} V &= \frac{1}{\sqrt{2}}\left(E + \frac{1}{2}\, N\eta + \frac{1}{2}\,\eta N\right)^{-1/2} (E + N\eta), \\ &= \frac{1}{\sqrt{2}}\,(E + N\eta)\left(E + \frac{1}{2}\, N\eta + \frac{1}{2}\,\eta N\right)^{-1/2}. \end{aligned} \tag{2.15}$$

More explicitly in terms of the inhomogeneous local coordinates on $KP(n)$ $t = (t_1, \ldots, t_n)^T$ where $t_i = t_0{}^i = t^i t_0^{-1}$ in the chosen patch P_0 and t taken as a column vector, we have the representation

$$V = \gamma^{-1}\begin{pmatrix} 1 & -t^\dagger \\ t & \Lambda(t) \end{pmatrix} \tag{2.16}$$

with the definitions

$$\gamma = (1 + t^\dagger t)^{1/2} \qquad \text{and} \qquad \Lambda = \gamma(I + tt^\dagger)^{-1/2}. \tag{2.17}$$

An alternative form is available in

$$\gamma^{-2}\Lambda^2 = I - \frac{tt^\dagger}{1 + t^\dagger t} \tag{2.18}$$

from which follows

$$\gamma^{-1}\Lambda = I - \gamma^{-1}\frac{1}{\gamma - 1}\, tt^\dagger \tag{2.19}$$

which has for solution

$$\Lambda(t) = \gamma I - \frac{1}{\gamma + 1} tt^{\dagger} \tag{2.20}$$

such that $\Lambda t = t$.

If we now set

$$W = \begin{pmatrix} r & 0 \\ 0 & R \end{pmatrix}, \qquad |r| = 1, \qquad RR^{\dagger} = R^{\dagger}R = I, \tag{2.21}$$

$r \in U(1)$, $Sp(1)$ and $R \in SU(n)$, $Sp(n)$ we can write

$$U = V(t)\, W = \gamma^{-1} \begin{pmatrix} r & -t^{\dagger}R \\ tr & \Lambda R \end{pmatrix} \tag{2.22}$$

and

$$\begin{aligned} N(t) &= N^{\dagger} = V^2 \eta \\ &= (1 + t^{\dagger}t)^{-1} \begin{pmatrix} (1 - t^{\dagger}t) & 2t^{\dagger} \\ 2t & -(1 + t^{\dagger}t)\, I + 2tt^{\dagger} \end{pmatrix}. \end{aligned} \tag{2.23}$$

To link up with the familiar, we observe that Eq. (2.23) in the case of $n = 1$ and $K = C$, $q_i \equiv z_i$ reduces to the well-known stereographic projection of S^2 onto $CP(1)$, the Cauchy plane. Indeed

$$N = \begin{pmatrix} n_3 & \bar{n} \\ n & -n_3 \end{pmatrix} \tag{2.24}$$

where

$$n_3 = \frac{1 - |w|^2}{1 + |w|^2}, \qquad n = \frac{2w}{1 + |w|^2}, \qquad w = z_1 z_0^{-1}, \qquad z_0 \neq 0.$$

For $n = 1$ and $K = H$, $q_i = h_i$ it gives

$$N = \begin{pmatrix} n_5 & \bar{n} \\ n & -n_5 \end{pmatrix} \tag{2.25}$$

with

$$n_5 = \frac{1 - \bar{u}u}{1 + \bar{u}u}, \qquad n = \frac{2u}{1 + \bar{u}u}, \qquad u = h_1 h_0^{-1}, \qquad h_0 \neq 0.$$

Here the hermitian matrix N stands for the five dimensional unit vector n_i, $i = 1, 2, \ldots, 5$, which describes the mapping of the 4-sphere S^4 onto the compactified 4-Euclidean space $E^4 U\{\infty\} \approx HP(1)$, the quaternionic projective line. In general for any n, the matrix $N = V^2\eta$ represents a point of the n-dimensional projective space $KP(n)$. Its homogeneous and inhomogeneous coordinates are simply related by the formula

$$\begin{pmatrix} q_0 \\ q \end{pmatrix} = U \frac{E + \eta}{2} = \begin{pmatrix} 1 \\ t \end{pmatrix} r\gamma^{-1}. \tag{2.26}$$

Since q and $q\lambda$ ($\lambda \neq 0$) are equivalent, if we choose $\lambda = q_0^{-1} = \gamma r^{-1}$ we can represent any point of $KP(n)$ except that at infinity by the column vector $\binom{1}{t}$ simply by the vector t of the inhomogeneous coordinates t_i.

In the next two sections, we shall allow the spaces $KP(n)$ to be the space of chiral fields. Then the holomorphic coordinates t_i, $i = 1, 2,\ldots, n$, will be mappings $t_i(x)$: $E^m \to KP(n)$ from the Euclidean 2- or 4-spacetime E^2 or E^4 into the field space $KP(n)$, $K = C$ and H respectively. As will be elaborated in Section IIIb, $V(x)$ is a matrix representation of a moving frame of Cartan restricted to the space $KP(n)$. Modulo $U(n)$, $Sp(n) \times Sp(1)$ rotation, $V(x)$ is a point on $KP(n)$. It is then natural to consider the infinitesimal motion of the Cartan frame induced on spacetime. They are described by the left-invariant current

$$\Omega_\mu = -\Omega_\mu{}^\dagger = V^\dagger \partial_\mu V, \tag{2.27}$$

an antihermitian matrix over K with the spacetime derivative $\partial_\mu \equiv \partial/\partial x_\mu$, $\mu = 1, 2$ or 1, 2, 3, 4 for $K = C$ or H respectively. Explicitly it is given as

$$\Omega_\mu = -I\partial_\mu \ln \gamma + \gamma^{-2} \begin{pmatrix} t^\dagger \partial_\mu t & -\partial_\mu t^\dagger + t^\dagger \partial_\mu \Lambda \\ \Lambda \partial_\mu t & t \partial_\mu t^\dagger + \Lambda \partial_\mu \Lambda \end{pmatrix}. \tag{2.28}$$

Making use of

$$-\partial_\mu \ln \gamma = -\tfrac{1}{2}\gamma^{-2}([\partial_\mu t^\dagger]t + t^\dagger \partial_\mu t) \tag{2.29}$$

and

$$\partial_\mu(t^\dagger - t^\dagger \Lambda) = 0, \qquad \partial_\mu t^\dagger = (\partial_\mu t^\dagger \Lambda + t^\dagger \partial_\mu \Lambda) \tag{2.30}$$

we find the form

$$\Omega_\mu = \gamma^{-2} \begin{pmatrix} \tfrac{1}{2}(t^\dagger \partial_\mu t - (\partial_\mu t^\dagger)\, t) & -(\partial_\mu t^\dagger)\, \Lambda \\ \Lambda \partial_\mu t & -\tfrac{1}{2} I\, \partial_\mu(t^\dagger t) + t\, \partial_\mu t^\dagger + \Lambda\, \partial_\mu \Lambda \end{pmatrix}. \tag{2.31}$$

In view of later applications, we are led to define the fields

$$a_\mu = \frac{1}{2} \frac{t^\dagger \partial_\mu t - (\partial_\mu t^\dagger)\, t}{1 + t^\dagger t}, \tag{2.32}$$

$$B_\mu = (-\tfrac{1}{2} I\, \partial_\mu(t^\dagger t) + t\, \partial_\mu t^\dagger + \Lambda\, \partial_\mu \Lambda)(1 + t^\dagger t)^{-1}, \tag{2.33}$$

$$\phi_\mu = [(1 + t^\dagger t)(I + t t^\dagger)]^{-1/2}\, \partial_\mu t \tag{2.34}$$

such that

$$\Omega_\mu(x) = \begin{pmatrix} a_\mu & -\phi_\mu{}^\dagger \\ \phi_\mu & B_\mu \end{pmatrix}. \tag{2.35}$$

It is further useful to partition Ω_μ into its diagonal and off diagonal parts as

$$\begin{aligned} \Omega_\mu &= A_\mu + \eta F_\mu, \\ &= A_\mu - F_\mu \eta \end{aligned} \tag{2.36}$$

where

$$A_\mu = \begin{pmatrix} a_\mu & 0 \\ 0 & B_\mu \end{pmatrix} = -A_\mu{}^\dagger, \qquad [A_\mu, \eta] = 0, \tag{2.37}$$

$$F_\mu = \begin{pmatrix} 0 & \phi_\mu{}^\dagger \\ \phi_\mu & 0 \end{pmatrix} = F_\mu{}^\dagger, \qquad \{F_\mu, \eta\} = 0. \tag{2.38}$$

Noting that Ω_μ takes values in the Lie algebra $u(n+1)$, $sp(n+1)$ of $U(n+1)$, $Sp(n+1)$, the decomposition of Eq. (2.36) is most natural since it only mirrors that *canonical* decomposition of these algebra given by $g = h + m$ where $g = u(n+1)$, $sp(n+1)$

$$\begin{aligned} h &= (u(1) + u(n), sp(1) + sp(n)) \\ &= \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}, \lambda + \bar{\lambda} = 0, \lambda \in u(n), sp(1), B \in u(n), sp(n) \right\} \end{aligned} \tag{2.39}$$

and

$$m = \left\{ \begin{pmatrix} 0 & -\xi^\dagger \\ \xi & 0 \end{pmatrix}; \xi \in K^n \right\} = -\eta \begin{pmatrix} 0 & \xi^\dagger \\ \xi & 0 \end{pmatrix}. \tag{2.40}$$

Under an $U(n)$, $Sp(n) \times Sp(1)$ rotation of the moving frame, $V \to VW$ the transformations of the above fields can be readily derived as

$$\Omega'_\mu = W^\dagger \Omega_\mu W + W^\dagger \partial_\mu W, \tag{2.41}$$

$$A'_\mu = W^\dagger A_\mu W + W^\dagger \partial_\mu W, \tag{2.42}$$

$$F'_\mu = W^\dagger F_\mu W. \tag{2.43}$$

By use of form (2.21) for W we get

$$a'_\mu = \bar{r} a_\mu r + \bar{r}\, \partial_\mu r, \tag{2.44}$$

$$B'_\mu = R^\dagger B_\mu R + R^\dagger \partial_\mu R, \tag{2.45}$$

$$\phi'_\mu = R^\dagger \phi_\mu r. \tag{2.46}$$

We conclude this section with remarks pertaining to the embedding of the $SU(n+1)$ and $Sp(n+1)$ groups in the larger groups $GL(n+1, C)$ and $GL(n+1, Q)$ which are represented by general $(n+1) \times (n+1)$ complex or quaternionic matrices. Let such a matrix be denoted by M. We may decompose M with respect to the involution η defined above. Let K denote the complex or quaternionic matrix left invariant by η. We have

$$K = \eta K \eta^{-1}, \qquad [\eta, K] = 0. \tag{2.47}$$

Hence K has the form

$$K = \begin{pmatrix} l_0 & 0 \\ 0 & L \end{pmatrix} \tag{2.48}$$

where l_0 is a complex number or a quaternion and L is a complex or quaternionic $n \times n$ matrix. It follows that

$$K \in GL(1, c) \times GL(n, c) \qquad \text{or} \qquad GL(1, H) \times GL(n, H).$$

We can always write

$$M = \begin{pmatrix} m_0 & n^T \\ m & \mu \end{pmatrix} \tag{2.49}$$

where m_0 and μ are respectively (1×1) and $(n \times n)$ matrices, m an $(n \times 1)$ and n^T a $(1 \times n)$ matrix with elements from C or H.

Then we have the general Iwasawa decomposition

$$M = \begin{pmatrix} 1 & 0 \\ t & I \end{pmatrix}\begin{pmatrix} l_0 & 0 \\ 0 & L \end{pmatrix}\begin{pmatrix} 1 & s^T \\ 0 & I \end{pmatrix} = \tau(t)\, K(l_0, L)\, \tau(s)^T \tag{2.50}$$

with

$$l_0 = m_0, \qquad t = mm_0^{-1}, \qquad s^T = m_0^{-1}n^T, \qquad L = \mu - mm_0^{-1}n^T. \tag{2.51}$$

There τ, K form subgroups of $GL(n+1)$. $\tau(t)$ represents a translation by t in $E^n(C)$ or $E^n(H)$, while K is a linear deformation in the same Euclidean space. We have

$$\eta K \eta = K, \qquad \eta \tau \eta = \tau^{-1}. \tag{2.52}$$

The pair $(\tau(t), \tau(s))$ represents the coset $GL(n+1)/GL(1) \times GL(n)$ with $4n$ and $8n$ parameters respectively for the complex and quaternionic cases.

Consider now the element of $GL(n+1)$ defined by

$$\xi = \tfrac{1}{2}(\tau^2(t) + \eta\tau^{\dagger 2}\eta) = \begin{pmatrix} 1 & -t^\dagger \\ t & I \end{pmatrix} \tag{2.53}$$

which satisfies

$$\eta \xi \eta = \xi^\dagger. \tag{2.54}$$

We have

$$\xi^\dagger \xi = \xi \xi^\dagger = \eta \xi^\dagger \xi \eta = \begin{pmatrix} 1 + t^\dagger t & 0 \\ 0 & I + tt^\dagger \end{pmatrix} = \gamma^{-2}\begin{pmatrix} 1 & 0 \\ 0 & \Lambda^{-2} \end{pmatrix} \tag{2.55}$$

where we have used the definitions (2.53) and (2.17). This shows that

$$\xi^\dagger \xi \in GL(1, C) \times GL(n, C)/U(1) \tag{2.56a}$$

or

$$\xi^\dagger \xi \in GL(1, H) \times GL(n, H)/Sp(1). \tag{2.56b}$$

We also have

$$\text{Det}\,\xi = 1 + t^\dagger t, \qquad \text{Det}\,\xi^\dagger\xi = (1 + t^\dagger t)^2, \tag{2.57}$$

so that ξ and $\xi^\dagger\xi$ are non-singular. Then we obtain

$$\xi^{-1} = \xi^\dagger(\xi^\dagger\xi)^{-1} = (\xi^\dagger\xi)^{-1}\,\xi^\dagger = \gamma^{-2}\begin{pmatrix} 1 & t^\dagger \\ -t & \Lambda^{-2} \end{pmatrix} \tag{2.58}$$

and

$$V = \xi(\xi^\dagger\xi)^{-1/2} = \gamma^{-1}\begin{pmatrix} 1 & -t^\dagger \\ t & \Lambda \end{pmatrix} = (V^\dagger)^{-1} \tag{2.59}$$

where we have used $\Lambda t = t$. V is an element of $SU(n+1)$ or $Sp(n+1)$ representing the coset $SU(n+1)/SU(n) \times U(1)$ or $Sp(n+1)/Sp(n) \times Sp(1)$. In parallel to the definition of Ω_μ from V through Eq. (2.27) we can define

$$L_\mu = \xi^{-1}\,\partial_\mu \xi = \gamma^{-2}\begin{pmatrix} t^\dagger\,\partial_\mu t & -\partial_\mu t^\dagger \\ \Lambda^2\,\partial_\mu t & t\,\partial_\mu t^\dagger \end{pmatrix} = \gamma^{-2}\begin{pmatrix} t^\dagger\,\partial_\mu t & -\partial_\mu t^\dagger \\ \partial_\mu t - (\partial_\mu \Lambda^2)\,t & t\,\partial_\mu t^\dagger \end{pmatrix} \tag{2.60}$$

and

$$R_\mu = (\partial_\mu \xi)\,\xi^{-1} = \gamma^{-2}\begin{pmatrix} (\partial_\mu t^\dagger)\,t & -(\partial_\mu t^\dagger)\,\Lambda^2 \\ \partial_\mu t & (\partial_\mu t)\,t^\dagger \end{pmatrix} = \eta L_\mu{}^\dagger \eta. \tag{2.61}$$

While Ω_μ is antihermitian, L_μ has no definite hermiticity property, but, like Ω_μ obeys the integrability condition

$$\partial_\mu L_\nu - \partial_\nu L_\mu + [L_\mu\,, L_\nu] = 0. \tag{2.62}$$

We also have the general relations

$$\partial_\mu L_\nu + L_\mu L_\nu = \xi^{-1}\partial_\mu\partial_\nu\xi \tag{2.63}$$

which lead to

$$\partial_\mu L_\mu + L_\mu L_\mu = \xi^{-1}\Box\xi. \tag{2.64}$$

In the applications to CP_n and HP_n spaces the column t will turn out to be rational function of the complex variable $z \in CP_1$ or the quaternion $x \in HP_1$ which parametrizes respectively the spheres S^2 and S^4, so that the matrices ξ associated with $GL(n+1)$ will have simpler analytic properties than the matrices V associated with $SU(n+1)$ or $Sp(n+1)$. The relation between L_μ and Ω_μ follows from that between ξ and V. It reads

$$\Omega_\mu = (\xi^\dagger\xi)^{1/2}\,L_\mu(\xi^\dagger\xi)^{-1/2} + (\xi^\dagger\xi)^{1/2}\,\partial_\mu[(\xi^\dagger\xi)^{-1/2}] \tag{2.65}$$

and has the form of a $GL(1) \times GL(n)$ gauge transformation.

If we separate analytic and antianalytic parts by defining

$$X_\mu = L_\mu \frac{E+\eta}{2} = \xi^{-1}\,\partial_\mu \xi \,\frac{E+\eta}{2} \tag{2.66}$$

we find

$$X_\mu{}^\dagger X_\nu = \gamma^2 \begin{pmatrix} \phi_\mu{}^\dagger \phi_\nu & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathrm{Vec}\left(\frac{E+\eta}{2}\, L_\mu \,\frac{E+\eta}{2}\right) = \begin{pmatrix} a_\mu & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.67}$$

after using the identity

$$\Lambda^4 + tt^\dagger = \gamma^{-2}\Lambda^2. \tag{2.68}$$

Thus far no mention has been made of the more detailed differential geometry nor the topology of $KP(n)$. We purposefully defer their introduction till a later discussion. Having put down a minimal amount of the basic mathematical building blocks, we go without further delay to a reformulation of the 2-dimensional $CP(n)$ σ-model. This exercise will facilitate the subsequent quaternionic analysis of chiral and gauge models in four dimensions.

III. Complex Projective σ-Models

IIIA. *A Minimal Treatment*

The 2-dimensional $SU(2)/U(1) \approx CP(1)$ σ-model has been amply studied and its striking similarities with the 4-dimensional $SU(2)$ Yang–Mills theory have been recalled previously. As a stepping stone to the quaternionic models in four dimensions, we consider the simplest extension of this model. Since we are pursuing analogies with higher dimensional gauge systems, the $O(n)$ σ-models with $n \geqslant 4$ are of no interest as they are known *not* to admit stable instantons [17]. The work of Belavin and Polyakov [8] has brought to light the basic role of complex analyticity in determining the instanton structure in the $CP(1)$ σ-model. The $CP(1)$ instantons and anti-instantons are given by harmonic maps of $S^2 \approx CP(1)$, the conformal 2-spacetime into $CP(1)$, the chiral field space. They are labelled by the homotopy classes of $\pi_2(CP(1)) \approx Z_\infty$, the additive group of the integers. They are rational solutions to the duality equations which are simply the Cauchy–Riemann equations. If we trace their analytic structure to the fact that both the 2-spacetime and the field manifold are complex analytic manifolds, i.e., the Riemann sphere S^2, it is natural then, as an extension, to allow the Nambu–Goldstone field to take values on the complex manifold $CP(n) \approx SU(n+1)/SU(n) \times U(1)$ which is the n-dimensional generalization of $CP(1) \approx S^2$. Such $CP(n)$ σ-models have indeed been the objects of intense current investigations at both the classical and semi-classical level [3, 7]. They ,in turn, constitute but a subset of chiral models with instantons where the field space is a compact Kähler manifold [41]. Aside from their intrinsic interest, an analysis of the

$CP(n)$ σ-models suffices for our purpose to pave the way to new 4-dimensional quaternionic σ-models. Besides, our geometric formulation generalizes trivially to an Kählerian model.

Making use of the formalism laid out in Section II, we let $K = C$ and take $CP(n)$ as the space of the chiral field $V \equiv Z(x)$ seen as a mapping $Z: E^2 \to CP(n)$, from the Euclidean 2-spacetime E^2 into the field manifold $CP(n)$.

The basic geometrical objects of our construction are the induced left-invariant currents $\Omega_\mu = Z^{-1}\partial_\mu Z$ where $\Omega_\mu \, dx_\mu$ is known as a Maurer–Cartan 1-form. If we denote the complex valued holomorphic vector in Eq. (2.16) by $w = (w_1, w_2, \ldots, w_n)^T$, then

$$\Omega_\mu = \begin{pmatrix} a_\mu & -\phi_\mu{}^\dagger \\ \phi_\mu & B_\mu \end{pmatrix} \tag{3A.1}$$

where the fields a_μ, B_μ and ϕ_μ are as defined in Eqs. (2.32)–(2.34). We will also need their duals

$$\tilde{a}_\mu = i\epsilon_{\mu\nu} a_\nu\,, \qquad \tilde{B}_\mu = i\epsilon_{\mu\nu} B_\nu\,, \qquad \tilde{\phi}_\mu = i\epsilon_{\mu\nu}\phi_\nu\,, \tag{3A.2}$$

where $\epsilon_{\mu\nu}$ is the anti-symmetric Levi–Civita pseudo-tensor.

The above fields are not independent since by the very form of the current $\Omega_\mu = Z^{-1}\partial_\mu Z$, Poincaré's integrability lemma on the equality of mixed derivatives, $[\partial_\mu, \partial_\nu]Z = 0$ implies the Maurer–Cartan conditions

$$\partial_\mu \Omega_\nu - \partial_\nu \Omega_\mu + [\Omega_\mu, \Omega_\nu] = 0. \tag{3A.3}$$

They become

$$(\partial_\mu + \Omega_\mu)\, \tilde{\Omega}_\mu = 0 \tag{3A.4}$$

since a rank two skew tensor is a scalar (O-form) in two dimensions. In terms of the fields a_μ, B_μ and ϕ_μ Eq. (3A.3) is equivalent to the coupled set

$$\partial_\mu \tilde{a}_\mu = \phi_\mu{}^\dagger \tilde{\phi}_\mu\,, \tag{3A.5}$$

$$\partial_\mu \tilde{B}_\mu = \tilde{\phi}_\mu \phi_\mu{}^\dagger, \tag{3A.6}$$

$$\nabla_\mu \tilde{\phi}_\mu = 0 \tag{3A.7}$$

where $\nabla_\mu = [I\partial_\mu - (a_\mu I + B_\mu)]$ is the $(n \times n)$ matrix $U(n)$ covariant derivative. We observe that this set of equations arises solely from the local differential geometry of the space $CP(n)$ induced on the 2-dimensional spacetime. They describe, as it were, the induced motions of the Cartan frames as will be clearer subsequently. From Eqs. (2.44) and (2.45), a_μ and B_μ are recognized as the $U(1)$ and $SU(n)$ connections associated with the holonomy group $U(n)$ in $SU(n+1)/U(n)$ and ϕ_μ transforms quasi-covariantly as it interpolates between the groups $U(1)$ and $SU(n)$.

From the ϕ_μ's we can construct the following two $U(n)$ invariants,

$$A = \frac{1}{2\pi} \int d^2x \; \phi_\mu{}^\dagger \phi_\mu \,, \tag{3A.8}$$

$$\mathscr{C}_1 = \frac{1}{2\pi} \int d^2x \; \phi_\mu{}^\dagger \tilde{\phi}_\mu \,, \tag{3A.9}$$

identified as the $CP(n)$ σ-model action and its companion, the first Chern index, respectively. They can be displayed in various forms.

A canonical representation in terms of the matrix field N Eq. (2.23) is

$$A = \frac{1}{8\pi} \operatorname{Tr} \int d^2x \left(\frac{E+N}{2} \partial_\mu N \, \partial_\mu N \right) \qquad (N^2 = E). \tag{3A.10}$$

or

$$A = \frac{1}{2\pi} \operatorname{Tr} \int d^2x P(\partial_\mu P)(\partial_\mu P) = \frac{1}{4\pi} \operatorname{Tr} \int d^2x (\partial_\mu P)(\partial_\mu P) \qquad \left(P = P^2 = \frac{E+N}{2} \right)$$

$$\begin{aligned} \mathscr{C}_1 &= \frac{1}{8\pi i} \operatorname{Tr} \int d^2x \left(\frac{E+N}{2} \partial_\mu N \, \tilde{\partial}_\mu N \right) \\ &= \frac{1}{4\pi i} \operatorname{Tr} \int d^2x (\partial_\mu P \, \tilde{\partial}_\mu P). \end{aligned} \tag{3A.11}$$

These then generalize the action and index for the $CP(1)$ σ-model which are

$$A = \frac{-1}{16\pi} \operatorname{Tr} \int d^2x (\partial_\mu n \, \partial_\mu n), \tag{3A.12}$$

$$\mathscr{C}_1 = \frac{-1}{16\pi i} \operatorname{Tr} \int d^2x (n \, \partial_\mu n \, \tilde{\partial}_\mu n) \qquad (n = \mathbf{e} \cdot \mathbf{n}). \tag{3A.13}$$

More explicitly in terms of the complex vector coordinates we have

$$A = \frac{1}{2\pi} \int d^2x \left\{ \frac{\partial_\mu w^\dagger \, \partial_\mu w}{(1 + w^\dagger w)} - \frac{(\partial_\mu w^\dagger) \, w(w^\dagger \, \partial_\mu w)}{(1 + w^\dagger w)^2} \right\}, \tag{3A.14}$$

$$\mathscr{C}_1 = \frac{1}{2\pi} \int d^2x \left\{ \frac{\partial_\mu w^\dagger \, \tilde{\partial}_\mu w}{1 + w^\dagger w} - \frac{(\partial_\mu w)^\dagger \, w(w^\dagger \, \tilde{\partial}_\mu w)}{(1 + w^\dagger w)^2} \right\}, \tag{3A.15}$$

which reduce to the familiar forms of

$$A = \frac{1}{2\pi} \int d^2x \frac{\partial_\mu w \, \partial_\mu \bar{w}}{(1 + |w|^2)^2} \,, \tag{3A.16}$$

$$\mathscr{C}_1 = \frac{1}{2\pi} \int d^2x \frac{\tilde{\partial}_\mu w \, \partial_\mu \bar{w}}{(1 + |w|^2)^2} \tag{3A.17}$$

for the $CP(1)$ σ-model.

By mere inspection of Eqs. (3A.14) and (3A.15) it follows that

$$A \geqslant |\mathscr{C}_1|. \tag{3A.18}$$

Saturating this lower bound yields the corresponding "duality equations"

$$\phi_\mu^\dagger(\phi_\mu \pm \tilde{\phi}_\mu) = 0. \tag{3A.19}$$

A sufficient but not generally necessary (for $n > 1$) condition is satisfied by putting

$$\phi_\mu = \mp\tilde{\phi}_\mu\,. \tag{3A.20a}$$

or more explicitly

$$\partial_\mu w = \mp i\epsilon_{\mu\nu}\partial_\nu w \tag{3A.19b}$$

The alternatives in signs correspond to what we call *holomorphic* antiself- and self-duality, respectively.

While the set Eqs. (3A.5)–(3A.7) comes from the geometry, the variation of the action Eq. (3A.8) gives the Euler–Lagrange equations which can be cast into the compact form of

$$\nabla_\mu\phi_\mu = 0, \tag{3A.21}$$

provided we have Eqs. (3A.5)–(3A.7) with which Eq. (3A.21) forms the full set of local field equations for the $CP(n)$ σ-model. In this frame formulation, $\pm$ holomorphic duality $\phi_\mu = \pm\tilde{\phi}_\mu$ is manifestly the interchangeability between the kinematic motions of the Cartan frames and the dynamics provided by the action principle Eq. (3A.8). It is amusing to note that just such a duality, elevated to a constructive principle by Born [47], had been analyzed [48] in the context of relativistic string models and Born–Infeld theories admitting Nambu string solutions.

To make manifest the underlying complex structure of the model, it is convenient to introduce the Lorentz covariant basic units of the complex number algebra $e_\mu = (1, i)$ and their conjugates $\bar{e}_\mu = (1, -i)$. The following useful relations are of note:

$$e_\mu e_\mu = 0, \qquad e_\mu\bar{e}_\mu = 2, \qquad e_\mu\tilde{e}_\mu = 0, \qquad e_\mu\tilde{\bar{e}}_\mu = 2, \tag{3A.22}$$

$$e_\mu\bar{e}_\nu + e_\nu\bar{e}_\mu - \delta_{\mu\nu}e_\alpha\bar{e}_\alpha = 0, \qquad e_\mu\bar{e}_\nu - e_\nu\bar{e}_\mu - \epsilon_{\mu\nu}\epsilon_{\alpha\beta}e_\alpha\bar{e}_\beta = 0. \tag{3A.23}$$

A point in Euclidean 2-spacetime is then $z = e_\mu x_\mu$ and we define the Wirtinger differential operators $\partial = e_\mu\partial_\mu = 2\,\partial/\partial\bar{z}$ and $\bar{\partial} = \bar{e}_\mu\partial_\mu = 2\,\partial/\partial z$. The duality equations (3A.20) become the Cauchy–Riemann equations

$$\partial w = 0 \qquad \text{or} \qquad \bar{\partial} w = 0 \tag{3A.24}$$

of $CP(n)$ induced on E^2. As w is harmonic, $\partial\bar{\partial}w = \bar{\partial}\partial w = \nabla^2 w = 0$. The self-dual solutions are then the holomorphic, hence analytic functions $w = w(z)$, while the antiself-dual ones $w = w(\bar{z})$ are anti-holomorphic or anti-analytic. To be associated with finite action solutions they must further satisfy the boundary condition

$$N \xrightarrow[|z|\to\infty]{} \eta \text{ or } w \xrightarrow[|z|\to\infty]{} 0 \tag{3A.25}$$

which leads to an effective compactification of E^2 into $E^2U(\infty) \approx S^2 \approx CP(1)$. Hence instantons defined as finite action self (antiself) dual solutions are topological maps $w: S^2 \approx CP(1) \to CP(n)$ and are labelled by the homotopic invariants which are the integers in the homotopy group $\pi_2(CP(n)) = Z_\infty$. Condition (3A.25) implies that $w(x)$ is not only analytic and harmonic ($\partial\bar{\partial}w = \nabla^2 w = 0$) but also *rational*:

$$w(z) = (w_1(z), w_2(z),..., w_n(z))^T \tag{3A.26}$$

where each component $w_\alpha(z) = P_\alpha{}^m(z)/Q_\alpha{}^n(z)$ ($\alpha = 1,..., n$) is a rational fraction with $P_\alpha{}^m$ and $Q_\alpha{}^n$ polynomials of degree m_α and n_α in z such that $n_\alpha > m_\alpha$. The anti-analytic case is obtained by the mere replacement $z \to \bar{z}$.

For later comparison with 4-dimensional Yang–Mills instantons and their self-duality equations, it is more striking to write $w(z)$ in the manifestly rational form of

$$w_\alpha(z) = \frac{1}{2}\sum_{i=1}^{l_\alpha} \lambda_{\alpha_i}\bar{\partial}\ln|z - z_i| = \sum_{i=1}^{l_\alpha} \frac{\lambda_{\alpha_i}}{z - z_i} \tag{3A.27}$$

such that $\lambda_\alpha \in C$ and $z_i \in C$ ($i = 1, 2,...$) are complex parameters for the instanton locations and sizes, $\ln|z - z_i|$ is the Green's function for the 2-dimensional Laplacian ∇^2. If we wish to put analyticity or antianalyticity on the same footing we should replace the Cauchy–Riemann equations by quadratic homogeneous equations. Consider an element w_a of the column matrix $w(x)$. If w is analytic or antianalytic corresponding to the two cases of self-duality or antiself-duality for the column $\phi_\mu(x)$, we have either

$$\partial_\mu w_\alpha = e_\mu w_\alpha' \quad \text{or} \quad \partial_\mu w_\alpha = \bar{e}_\mu w_\alpha' \quad (w' = \partial w/\partial z). \tag{3A.28}$$

From the identities (3A.23) and their complex conjugates it follows that in either case the following homogeneous equations are satisfied:

$$\partial_\mu \bar{w}_a \partial_\nu w_b + \partial_\nu \bar{w}_a \partial_\mu w_b - \delta_{\mu\nu}\partial_\alpha \bar{w}_a \partial_\alpha w_b = 0, \tag{3A.29}$$

$$\partial_\mu \bar{w}_a \partial_\nu w_b - \partial_\nu \bar{w}_a \partial_\mu w_b + \epsilon_{\mu\nu}\epsilon_{\alpha\beta}\partial_\alpha \bar{w}_a \partial_\beta w_b = 0. \tag{3A.30}$$

Note that Eq. (3A.29) is an identity. Then we have

$$\partial_\mu \bar{w}_a \partial_\nu w_b = \tfrac{1}{2}(\delta_{\mu\nu}\delta_{\alpha\beta} - \epsilon_{\mu\nu}\epsilon_{\alpha\beta})\,\partial_\alpha \bar{w}_a \partial_\beta w_b$$

and, on contracting with $e_\mu e_\nu$ and using (3A.22) we obtain

$$(\partial \bar{w}_a)\,\partial w_b = 0 \tag{3A.31}$$

which gives either

$$\partial w_a = 0 \quad \text{or} \quad \bar{\partial} w_a = 0. \tag{3A.32}$$

We can also write the same conditions on the complex matrices

$$\xi = \begin{pmatrix} 1 & -w^\dagger \\ w & I \end{pmatrix}, \quad L = \xi^{-1}\,\partial_\mu \xi \tag{3A.33}$$

which are special cases of $\xi(t) \in GL(n+1)$ and $L_\mu(t)$, considered before. Using Eq. (3A.28) we have

$$L_\mu = \xi^{-1}\,\partial_\mu \xi = \gamma^{-2} \begin{pmatrix} w^\dagger w' & -w'^\dagger \\ \Lambda^2 w' & ww'^\dagger \end{pmatrix} E_\mu{}^{\pm} \tag{3A.34}$$

where

$$E_\mu{}^{+} = \begin{pmatrix} e_\mu & 0 \\ 0 & I\bar{e}_\mu \end{pmatrix}, \qquad E_\mu{}^{-} = \begin{pmatrix} \bar{e}_\mu & 0 \\ 0 & Ie_\mu \end{pmatrix} \tag{3A.35}$$

the sign referring to the analytic or antianalytic cases. Hence, again using identities (3A.23) we find

$$L_\mu L_\nu{}^\dagger + L_\nu L_\mu{}^\dagger - \delta_{\mu\nu} L_\alpha L_\alpha{}^\dagger = 0, \tag{3A.36}$$

$$L_\mu L_\nu{}^\dagger - L_\nu L_\mu{}^\dagger - \epsilon_{\mu\nu}\epsilon_{\alpha\beta} L_\alpha L_\beta{}^\dagger = 0. \tag{3A.37}$$

For both analytic and antianalytic w, we also have the linear equation of the second order

$$\Delta \xi = 0 \qquad (\text{or } \xi^{-1}\,\Delta\xi = 0) \tag{3A.38}$$

which is equivalent to the first order nonlinear equation

$$\partial_\mu L_\mu + L_\mu L_\mu = 0. \tag{3A.39}$$

These equations for ξ and L_μ have various generalizations to the quaternionic case.

We turn our attention finally to the first Chern index labelling the homotopy classes of $\Pi_2(CP(n))$; locally it is given through Eq. (3A.5) by

$$\mathscr{C}_1 = \frac{-i}{4\pi} \iint_{CP(1)} F_{\mu\nu}\, dx_\mu \,\Lambda\, dx_\nu = \frac{-1}{2\pi} \iint_{CP(1)} d^2x\, \partial_\mu \tilde{a}_\mu \tag{3A.40}$$

where $F_{\mu\nu}$ given locally by $(\partial_\mu a_\nu - \partial_\nu a_\mu)$ is globally defined. Hence

$$\mathscr{C}_1 = \frac{i}{2\pi} \oint_{S^1} a_\mu\, dx_\mu \tag{3a.41}$$

where a_μ is the gauge potential of the Abelian $U(1)$ group in the holonomy group $U(n)$ of $SU(n+1)/U(n)$. Of course Eq. (3A.41) is in accord with the homotopy sequence

$\pi_2(SU(n+1))/U(n)) = \pi_1((U(1)) = Z_\infty$. In the self-dual sector where $\partial w = 0$ and $\mathscr{C}_1$ is generally nonzero, the latter takes the Kähler form

$$\mathscr{C}_1 = \frac{-1}{4\pi} \iint_{S^2} d^2x \, \nabla^2 \ln(1 + w^\dagger w) \tag{3A.42}$$

or

$$\mathscr{C}_1 = \mathrm{Re}\left[\frac{-1}{4\pi i} \sum_{\alpha=1}^{n} \oint_{S^1} dz \, \frac{w_\alpha^\dagger \, \bar{\partial} w_\alpha}{1 + w^\dagger w}\right]. \tag{3A.43}$$

Antiself-duality is obtained by the substitution $z \to \bar{z}$. While it will be clear that de Rham cohomology (to be defined later) dictates that $\mathscr{C}_1$ be an integer, it is pleasing to note that in the case of $CP(1)$, Eq. (3A.43) simplifies to

$$\mathscr{C}_1 = -\frac{1}{2\pi i} \oint_c \frac{d \ln w}{dz} \, dz + \frac{1}{2\pi i} \oint_c \frac{d \ln w}{(1 + |w|^2)}. \tag{3A.44}$$

We observe that a_μ must be in a *singular* $U(1)$ gauge in order to contribute nontrivially to the index in Eq. (3A.40). So just as in the instance of monopole theory in a singular Dirac string gauge the integration path in Eq. (3A.44) consists then of infinitesimal circles around the poles of $w(x)$. This can be achieved because $\mathscr{C}_1$ is a topological invariant and by a diffeomorphism regular in S^2, the large circle S^1 can be shrunk to a contour squeezing the poles of w. Then the second term in Eq. (3A.44) gives a zero contribution since $|w|^2 \to \infty$ as z tends to the poles of w, leaving

$$\mathscr{C}_1 = \frac{-1}{2\pi i} \oint_{S^1} \frac{d \ln w}{dz} \, dz = n. \tag{3A.45}$$

By the "argument principle" of complex analysis, $\mathscr{C}_1$ just counts the number of poles in the rational n $CP(1)$ instanton field $w(x)$. Next we shall elaborate on the minimal treatment of $CP(n)$ instantons given here.

IIIB. *Complex Kählerian Geometry and Self-Duality*

So far our presentation of $CP(n)$ instantons has been rather superficial as it has bypassed much of the rich underlying geometrical and topological structure of self-duality. To fill this gap, we must come to terms with the complex Kählerian structure of $CP(n)$ and make contact with some of the most beautiful chapters in modern complex differential geometry. In this subsection and subsequently, besides quoting without proofs some basic theorems in complex, then quaternionic manifold theory, we shall make minimal use of complex and quaternionic valued differential forms. For the uninitiated, Appendix II gathers the essential properties of these forms while Appendix III compiles the basic definitions and formulas in Kählerian geometry.

Our objectives here are twofold. On the one hand, we identify the $CP(n)$ instantons as nonconstant conformal harmonic maps whose existence depends crucially on the base manifold being S^2. Thus we will be able to apply to these instantons perfectly tailored results already achieved in the theory of harmonic mappings of Riemann manifolds [50]. It seems that these results are largely unknown to physicists. On the other hand, we present the geometrical picture of instantons as holomorphic rational curves imbedded in $CP(n)$ space. In linking up mainly with the works of Chern [27] and Griffiths [26], we identify $CP(n)$ instantons as holomorphic rational curves immersed in $CP(n)$ space and display the added simplicity and depth that self-duality i.e., Cauchy–Riemann analyticity, brings to the Frenet equations for these curves. This geometrical picture is very relevant since the Frenet equations for helical curves and pseudospherical surfaces have been shown [29] to be the underlying unified equations for completely integrable 2-dimensional theories of solitons.

Because the dynamics and topology of a chiral model are induced images of the differential and topological structures of its field manifold, we begin by expanding on the treatment of Section II. The sphere S^{2n+1} in $*C^{n+1}$ is defined by the vector $Z_0 = (z_0, z_1, ..., z_n)$ with unit norm $(Z_0, Z_0) \equiv \sum_{i=0}^{n} z_i \bar{z}_i = 1$. The very use of this n-tuple to determine a point in $CP(n)$ clearly presupposes the existence of a coordinate system, here P_0. The latter is generally specified by some suitable figure called a "frame" by Cartan [26]. In the Euclidean plane, for instance, this frame is determined by two orthogonal vectors. Accordingly, in $*C^{n+1}$, we adjoint to the vector Z_0, n other Z_i ($i = 1, 2, ..., n$) such that $(Z_i, Z_j) = \delta_{ij}$. Then a *frame* is a unitary basis $\{Z_0, ..., Z_n\}$ for $*C^{n+1}$, each Z_α ($\alpha = 0, ..., n$) is defined up to a $U(n) \approx SU(n) \times U(1)$ rotation. The set of all such frames is a manifold F_{n+1} which may be identified, following Cartan, with the unitary group $U(n + 1)$ acting as a transformation group on $CP(n)$. Under this identification the projection map $F_{n+1} : U(n + 1) \to U(n + 1)/U(n)$ is defined by assigning to the frame Z_α its first vector $Z_0 = Z/|Z| e^{i\psi}$ (ψ real). This choice allows an explicit computation of the Kähler 2-form and the metric ds^2 in the homogeneous coordinates Z_0.

For a differentiable family of frame vectors, the infinitesimal motions are given by

$$dZ_\alpha = \sum_{\beta=0}^{n} \Theta_{\alpha\beta} Z_\beta, \qquad 0 \leqslant \alpha, \beta \leqslant n, \tag{3B.1}$$

where the connection forms of the group $U(n + 1)$ are $\Theta_{\alpha\beta} = (dZ_\alpha, Z_\beta)$ where $\Theta_{\alpha\beta} + \bar{\Theta}_{\beta\alpha} = 0$, resulting from $(Z_\alpha, Z_\beta) = \delta_{\alpha\beta}$. These connections obey Cartan's structure equations

$$d\Theta_{\alpha\beta} = \sum_{\gamma} \Theta_{\alpha\gamma} \wedge \Theta_{\gamma\beta}, \ 0 \leqslant \alpha, \beta, \gamma \leqslant n. \tag{3B.2}$$

Upon choosing a specific frame Z_0, the hermitian Study–Fubini metric on $CP(n)$ is given as

$$ds^2 = \sum \Theta_{0j} \bar{\Theta}_{0j}, \qquad 1 \leqslant j \leqslant n, \tag{3B.3}$$

with its globally defined Kähler 2-form

$$\hat{K} = i/2 \sum_j \Theta_{0j} \wedge \bar{\Theta}_{0j} , \tag{3B.4}$$

$$= 1/2 d\Theta_{00} . \tag{3B.5}$$

Differentiation of (Z_0, Z_0) and use of Eq. (3B.1) yield

$$\Theta_{00} = (dZ_0, Z_0) = (d' - d'') \log |Z| + i\, d\psi, \tag{3B.6}$$

$$= i |Z|^{-2} \operatorname{Im}(dZ, Z) + i\, d\psi \tag{3B.7}$$

if we write in terms of the inhomogeneous coordinates $w = (w_1, w_2, ..., w_n)$, $w_i = z_i z_0^{-1}$,

$$\Theta_{00} = i \frac{\operatorname{Im}(dw, w)}{(1 + (w, w))} + i\, d\psi. \tag{3B.8}$$

By exterior differentiation, the curvature of the connection or 1st Chern class of the holomorphic complex line bundle $^*C^{n+1} \to CP(n)$ is just

$$\Omega = \frac{1}{\pi} \hat{K} = \frac{i}{\pi} d'd'' \log |Z| . \tag{3B.9}$$

It is clearly positive definite, non-degenerate and from Eq. (3B.9) closed $d\Omega = 0$. Hence $CP(n)$ is Kählerian. Explicitly with the usual summation convention it is given in P_0 by

$$\Omega = i/2 \,(d\bar{z}_i \wedge dz_i - \bar{z}_j\, dz_j \wedge d\bar{z}_k\, z_k) \tag{3B.10}$$

while the Mannoury–Study–Fubini metric associated to Ω becomes

$$ds^2 = (d\bar{z}_i\, dz_i) - (\bar{z}_j\, dz_j)(d\bar{z}_k\, z_k). \tag{3B.11}$$

In terms of the holomorphic coordinates w_i we have

$$\Omega = \frac{i}{2} \left\{ \frac{d\bar{w}_k \wedge dw_k}{(1 + w_i \bar{w}_i)} - \frac{\bar{w}_j\, dw_j \wedge d\bar{w}_k\, w_k}{(1 + w_i \bar{w}_i)^2} \right\} \tag{3B.12}$$

and

$$ds^2 = \frac{d\bar{w}_i\, dw_i}{(1 + w_i \bar{w}_i)} - \frac{\bar{w}_i\, dw_i\, w_j\, d\bar{w}_j}{(1 + w_i \bar{w}_i)^2} , \tag{3B.13}$$

which also reads as

$$ds^2 = g_{i\bar{j}}\, dw_i\, d\bar{w}_j , \tag{3B.14}$$

$$g_{ij} = \frac{\delta_{ij}}{\rho} - \frac{w_i \bar{w}_j}{\rho} = \frac{\partial^2 \log \rho}{\partial w_i\, \partial \bar{w}_j} \tag{3B.15}$$

with $\rho = (1 + w_i\bar{w}_i)$, which is in accordance with $CP(n)$ being Kählerian (see Appendix III). Also

$$\Omega = \frac{i}{2\pi} d'd'' \log (1 + w_i\bar{w}_i). \tag{3B.16}$$

By Eq. (3B.13), at the origin of the local coordinate patch P_0, $g_{ij} = \delta_{ij}$ from which it can be verified that the curvature form of the metric is $\Theta_{\alpha\beta} = \Theta_\beta \wedge \bar{\Theta}_\alpha + \delta_{\alpha\beta}\Theta_\gamma \wedge \bar{\Theta}_\gamma$, $\Theta_\alpha \equiv \Theta_{0\alpha}$. Comparing with the $\theta_{\alpha\beta}$ given at the end of Appendix III we see that the Study–Fubini metric has constant holomorphic sectional curvature $\kappa = 4$, which is the complex generalization of the constant curvature property of $S^2 \approx CP(1)$ [51]. By contraction of indices, it follows that $CP(n)$ is an Einstein–Kähler space since the metric takes also the form

$$ds^2 = (\pm) \frac{R_{i\bar{j}}}{\pi} dw_i\, d\bar{w}_j \tag{3B.17}$$

and

$$\mathscr{C}_1(CP(n)) = \frac{1}{2\pi i} R_{i\bar{j}}\, dw_i \wedge d\bar{w}_j \tag{3B.18}$$

where the Ricci curvature is

$$R_{ij} = -\frac{\partial^2 \log (\det g_{i\bar{j}})}{\partial w_i\, \partial \bar{w}_j}, \tag{3B.19}$$

which is positive definite. Hence by Kodaira's fundamental theorem [52] Eq. (3B.13) is a Hodge metric, $CP(n)$ is a prototype algebraic variety, a Hodge manifold (also called a Kähler variety of the restricted type) with its Ω belonging to the cohomology class of an *integral* 2-cocycle over $CP(n)$, using a terminology to be defined shortly.

We now return to the $CP(n)$ σ-model. Chiral fields are maps from spacetime into the field manifold $CP(n)$, hence the geometry of chiral dynamics is that of the induced or pull back structures from $CP(n)$ to E^2. As we shall see remarkable simplifications happen when E^2 is compactified onto $CP(1)$, a complex submanifold of $CP(n)$. Let us then consider the induced complex and matrix valued 1-form $\phi(x) = \phi_\mu\, dx_\mu$, $\mu = 1, 2$ with ϕ_μ given in Eq. (2.34). It is natural to write a complex valued Hermite form

$$H = \bar{H} = \phi^\dagger\phi = dw^\dagger\, G\, dw \tag{3B.20}$$

where $G = [(I + w^\dagger w)(I + ww^\dagger)]^{-1} = (1 + w^\dagger w)^{-1} - ww^\dagger(1 + w^\dagger w)^{-2}$ is just the $CP(n)$ metric Eq. (3B.13) in matrix form. Separating out the symmetric real part and the antisymmetric imaginary part, we have

$$ds^2 = g_{\mu\nu}\, dx_\mu \otimes \delta x_\nu = \frac{dw^\dagger\, dw}{(1 + w^\dagger w)} - \frac{(dw^\dagger\, w)(w^\dagger\, dw)}{(1 + w^\dagger w)^2}, \tag{3B.21}$$

where $g_{\mu\nu}$ is the induced Riemannian metric

$$F = \frac{1}{2} F_{\mu\nu}\, dx_\mu \wedge \delta x_\nu = \frac{dw^\dagger \wedge dw}{(1 + w^\dagger w)} - \frac{(dw^\dagger w) \wedge (w^\dagger dw)}{(1 + w^\dagger w)^2} \tag{3B.22}$$

and $F_{\mu\nu}$ is the associated induced Kähler 2-form $(w^*\Omega)_{\mu\nu}$, the pull back of the fundamental 2-form on $CP(n)$ to E^2, Eq. (3B.12).

Now the action finiteness condition results in a compactification of E^2 into $CP(1)$ which is a complex analytic submanifold of the Kähler manifold $CP(n)$. It is easily seen that the fundamental two form $F = \frac{1}{2}F_{\mu\nu}\, dx_\mu \wedge \delta x_\nu$ associated with $g_{\mu\nu}$, being just the restriction of $CP(n)$ to $CP(1)$ is *closed* by the pull-back property $d(w^*\Omega) = w^*(d\Omega) = 0$ with the map $w(x)$: $CP(1) \to CP(n)$. So

$$dF = 0, \tag{3B.23}$$

which is not only the Bianchi identity that follows from the local existence of the potential a_μ, but also the kernel of a nontrivial topological structure. To see this, we must turn to the topology of $CP(n)$ which induces that of the instantons.

First we recall that a p-form ω on a manifold M is called *closed* if $d\omega = 0$ and *exact* if $\omega = d\theta$ for some $(p - 1)$ form Θ on M. While exact forms are all closed since $d^2 = 0$, the converse is globally not true [51]. The deviation from exactness reflects the topology of M and is measured by an equivalence relation on the vector space of all closed forms on M. Thus two forms on M, ω and ω' are called cohomologous $(\omega \sim \omega')$ if their difference is exact, $\omega - \omega' = d\Theta$. The set of equivalence classes is denoted by $H^p(M)$, the *pth de Rham cohomology group* of M. According to de Rham's theorem, $H^p(M)$ is the factor space of closed forms by the subspace of exact forms on M. It can be verified that cohomology classes form an additive group and even a vector space, i.e. if $\omega_1 \sim \omega_1'$ and $\omega_2 \sim \omega_2'$ then $\omega_1 + \omega_2 \sim \omega_1' + \omega_2'$, $c\omega_1 \sim c\omega_2$, c being a constant. For a compact manifold such a space is always of a finite dimension b_p, the pth Betti number. For $M = CP(n)$, it is known that $H^{2i}(CP(n), Z_\infty) \approx Z_\infty$ $(0 \leqslant i \leqslant n)$ and $H^k(CP(n), Z_\infty) \approx 0$ for odd k. Moreover by Hurewicz's theorem $H^2(CP(n), Z_\infty) \approx \pi_2(CPn)$. In fact the curvature Ω equation (3B.16) is so normalized $(\int_{CP(1)} \Omega = 1)$ that this first Chern class of the line bundle is the generator of $H^2(CP(n), Z_\infty)$ the second de Rham cohomology group of $CP(n)$ with coefficients in the ring Z_∞ of the integers. As stated previously this property identifies $CP(n)$ as a Hodge manifold, an algebraic variety. By a celebrated theorem of Chow [53], $CP(n)$ and its compact complex Kähler submanifolds are nonsingular algebraic varieties (i.e. they are given as the zeroes of homogeneous polynomials in the homogeneous variables $z_0, z_1, \ldots, z_n$). It follows that holomorphic maps between projective algebraic varieties such as instantons must be algebraic [54]. Moreover by a theorem of Goto [55] on algebraic homogeneous spaces such maps are *rational*. Thus we can sum up the $CP(n)$ instanton problem as follows: With any holomorphic instanton map $w(x)$: $CP(1) \to CP(n)$, we can associate a real number, i.e. its action

$$A = \frac{1}{2\pi} \iint_{CP(1)} d^2x g_{i\bar{j}}\, \partial_\mu w_i\, \partial_\mu \bar{w}_i, \tag{3B.24}$$

which equals (with suitable normalization) its first Chern class

$$\mathscr{C}_1 = \frac{1}{2\pi} \iint d^2x g_{i\bar{j}}\, \partial_\mu w_i\, \tilde{\partial}_\mu \bar{w}_i = n, \tag{3B.25}$$

where $g_{i\bar{j}}$ is the $CP(n)$ metric equation (3B.15). Clearly if M is any other Kähler manifold, $g_{i\bar{j}}$ will just be its metric and all the above conclusions for $CP(n)$ instantons carry over unchanged.

To see that $\mathscr{C}_1$ is the index of the map $w(z)$, we note that since $w^*d = d^*w$, it is easily checked that w^* pulls the closed 2-form $g_{i\bar{j}}\, dw_i \wedge d\bar{w}_j$ on $CP(n)$ back to a closed 2-form on $CP(1)$ w^* also pulls cohomology classes on $CP(n)$ back to cohomology classes on $CP(1)$ i.e. it defines the map w^*: $H^2(CP(n)) \approx H^2(CP(1)) = \pi_2(S^2) = Z_\infty$ by Hurewicz's theorem, a homeomorphism given by the pull back formula $\tilde{\Omega} = w^*\Omega$ such that, for $n = 1$ [56], $\mathscr{C}_1 = \int_{CP(1)} w^*\Omega = \deg(w) \int_{CP(1)} \Omega = n$, the degree of the map $w(z)$. Thus to have instantons, a 2-dimensional chiral model built on a Kähler manifold M must have $H^2(M) = \pi_2(M) \neq 0$ which is entirely a topological property of the field manifold.

In fact much is known about holomorphic maps between two compact Kähler manifolds [50]. Due to the strong holomorphic conditions imposed, nonconstant holomorphic maps are rather rare, which make instantons remarkable. A nontrivial aspect of $CP(n)$ instantons rests on its base manifold $S^2 \approx CP(1)$ being of real dimension 2 and a Riemann surface. For the dimension of the base manifold equalling 2, the equality between action and Chern index (the number of times the area of S^2 is covered by the mapping), holds if and only if the map is *conformal* [50]. A map f: $(N, g) \to (M, H)$ between two Riemannian manifolds N and M with the respective metrics g and h is said to be (weakly) conformal if and only if there is a non-negative function μ on N such that f^*h, the pull back metric on N is of the form $f^*h = \mu g$. Furthermore for $N =$ a Riemann surface such as S^2, the harmonic map is independent of the metric g since any two of them are conformally equivalent. It is proved that [57] every nonconstant harmonic map from S^2 (or from $RP(2)$, the real projective plane) to a Riemannian manifold M is conformal. From Eq. (3A.36), self-duality or analyticity implies that $g_{\mu\nu}$ is proportional to the Lagrangian density $L = (2\pi)^{-1}\, \phi^2$:

$$g_{\mu\nu} = \phi^2 \delta_{\mu\nu}\, dx_\mu \otimes \delta x_\nu, \tag{3B.26}$$

where $\phi^2 = g_{i\bar{j}} \partial_\mu w^i \partial_\mu \bar{w}_j$. Hence the induced Riemannian metric $g_{\mu\nu}$ is conformally flat. So $CP(n)$ instantons are rational *conformal* harmnoic maps. Equation (3B.20) can also be rewritten as

$$h = \pi(L\delta_{\mu\nu}\, dx_\mu \otimes \delta x_\nu + i\epsilon_{\mu\nu} \not{p}\, dx_\mu \wedge \delta x_\nu) \tag{3B.27}$$

so that the global reciprocity between the dual dynamics of L and the 1st Chern class, the divergence of the Pontryagin current $\not{p} = \partial_\mu \tilde{a}_\mu = \nabla^2 \ln(1 + w^+ w)$, occurs through the interchange of $\delta_{\mu\nu} \leftrightarrow i\epsilon_{\mu\nu}$ which is nothing else but the defining property of one *single* hermitian, complex Kählerian structure on $CP(n)$. This property can be stated as the permutability of the metric and the fundamental Kähler form through $g(X, Y) = \Omega(JX, Y)$ for all vector fields X and Y tangent to $CP(n)$ as defined in Appendix III, Eq. (AIII.7).

In four dimensions, the compactified spacetime is S^4, which like S^2, is conformally flat. Unlike S^2 and S^6, S^4 is shown *not* to admit an almost complex structure by Ehresmann [58] and Hopf [59]. However $S^4 \approx HP(1)$ does have a *quaternionic* structure. It is this remarkable parallel between the conformal flatness of the base manifolds $CP(1)$ and $HP(1)$, between the complex structure of $CP(n)$ and the quaternionic structure of $HP(n)$ which we shall exhibit in the instance of $Sp(1) \sim SU(2)$ Yang–Mills instantons.

Now we make the connection with the theory of holomorphic curves. Our treatment is just a summary of the rudiments and results of the works of Chern [27] and Griffiths [26]. A holomorphic curve is defined as a holomorphic map Z: $M \rightarrow CP(n)$ where M is a Riemann surface such as S^2. Hence $CP(n)$ instantons are just such nondegenerate holomorphic curves in $CP(n)$ space. Locally such a curve is represented standardly by a holomorphic coordinate vector $Z = (Z_0(z),..., Z_n(z))$ where $z \in S^2$. The nondegeneracy is analytically stated as a non-identically zero Wronskian of the coordinate functions

$$W(z) = \det(Z_l(z)) = Z(z) \wedge Z'(z) \wedge \cdots \wedge Z^{(n)}(z) \not\equiv 0. \tag{3B.28}$$

About a regular point z ($W(z) \neq 0$), Eq. (3B.28) allows the Frenet–Serret frames $Z_0(z) \cdots Z_n(z)$ to be defined by the condition

$$\frac{Z(z) \wedge \cdots \wedge Z^{(k)}(z)}{|Z(z) \wedge \cdots \wedge Z^{(k)}(z)|} = e^{i\psi} Z_1(z) \wedge \cdots \wedge Z_k(z). \tag{3B.29}$$

So then frames are determined up to a rotation $Z_k(z) \rightarrow e^{i\psi_k} Z_k(z)$. Due to Eq. (3B.28), the left-hand side of Eq. (3B.29) defines the kth osculating space at z. Geometrically $W_k(z) = Z(z) \wedge \cdots \wedge Z^{(k)}(z)$ defines a holomorphic curve in the Grassmannian manifold $\mathrm{Gr}_{n+k,k}(C)$ of all k-planes in $CP(n)$. It is called the kth associated curve and plays a fundamental role in the theory since it interpolates between the 1-dimensional curve and $CP(n)$. Since by Eq. (3B.29), $Z_k(z)$ is a linear combination of the $Z(z),..., Z^{(k)}(z)$ and $\partial Z^{(k)}(z) = 0$ by holomorphy or self-duality, Eq. (3B.29) simplifies to

$$dZ_k = \Theta_{k,k-1} Z_{k-1} + \Theta_{i,k} Z_k + \Theta_{k,k+1} Z_{k+1} \qquad (0 \leqslant k \leqslant n), \tag{3B.30}$$

which are the Frenet equations for a holomorphic curve. They generalize the usual Serret–Frenet equations describing the bending and twisting of a curve in R^3 (or more generally R^n) [25]. They state that the curvature, torsion, etc., of a curve in the embedding space uniquely determine that curve up to rigid motion.

As for the associated curves, their induced metric on S^2 is

$$ds_k^2 = \frac{|w_{k-1}|^2 \, |w_{k+1}|^2}{|w_k|^4} \, dz \, d\bar{z} = h_k^2 \, dz \, d\bar{z}, \tag{3B.31}$$

also written as

$$ds^2 = \Theta_{k,k+1} \bar{\Theta}_{k,k+1} \,. \tag{3B.32}$$

This metric is conformally flat and is associated with the Kähler 2-form

$$\Omega_k = \frac{\hat{K}_k}{\pi} = \frac{i}{2\pi} \Theta_{k,k+1} \wedge \bar{\Theta}_{k,k+1} = \frac{i}{2\pi} h_k^2 \, dz \wedge d\bar{z} \tag{3B.33}$$

so that $\Omega_0 = \Omega$ in Eq. (3B.29), the fundamental Kähler form on $CP(n)$.

By the same calculation which gives Eq. (3B.16), one finds that

$$\sum_{i=0}^{k} \Theta_{i,i} = (d' - d'') \log | w_k | \,, \tag{3B.34}$$

which upon exterior differentiation and use of Eq. (3B.2) yields

$$\Omega_k = \frac{i}{\pi} d'd'' \log | w_k | \,. \tag{3B.35}$$

The n forms Ω_k, $0 \leqslant k \leqslant n-1$ constitute the fundamental invariants of the holomorphic curve. From the Ricci form $\operatorname{Ric} \Omega_k = d'd'' \log h_k$ and Eq. (3B.34), one can obtain the re-cursive formula

$$\operatorname{Ric} \Omega_k = \Omega_{k-1} + \Omega_{k+1} - 2\Omega_k \tag{3B.36}$$

called Weyl's Second Main theorem [26]. It implies that given a nondegenerate holomorphic curve $Z(z)$, the osculating metrics are uniquely determined by $\Omega_0 = \Omega$, called its fundamental form. One can further prove Calabi's striking theorem, namely that a nondegenerate holomorphic curve is uniquely determined up to rigid rotation by its first fundamental form Ω_0. This is to be contrasted with the case of real curves [61] where the second fundamental form is also needed. After this brief discussion of the Frenet equations for $CP(n)$ instantons paralleling those for 2-dimensional solitons [29], we will close this section with a proof for Cauchy–Riemann analyticity that leads to the existence of an infinite number of continuity equations for our Kählerian σ-models.

IIIC. *Infinite Number of Continuity Equations.*

For compact Kähler manifolds N and M, it has been proved [50, 62] that (1) if $f: N \to M$ are $\pm$ holomorphic maps, then it is an absolute minimum of the action equation (3A.8) in its homotopy class; (2) if $W_t: N \to M$ is a smooth deformation of a holomorphic map through harmonic maps then every W_t is holomorphic. These results of course apply to our Kählerian instantons W_t: $CP(1) \to CP(n)$. Contrary to naive expectation (63), the converse of Theorem 1 above does not hold for $n > 1$; non, self-dual finite action unstable solutions to the $CP(n)$ σ-model have been exhibited recently. There we exploit further the complex analytic structure inherent in the Kählerian σ-models.

Following Perelomov [41], we start from the equations of motion which, upon use of the Kählerian conditions (Eq. (AIII.12)), can be set into the form

$$g_{\alpha\bar{\beta}}(\partial\bar{\partial}w^{\alpha}) + \frac{\partial g_{\alpha\bar{\beta}}}{\partial w^{\lambda}}\,\bar{\partial}w^{\alpha}\,\partial w^{\lambda} = 0 \tag{3C.1}$$

and its complex conjugate. We multiply Eq. (3C.1) and its conjugate by $\partial\overline{w}_{\beta}$ and ∂w_{β} respectively and sum over the index. Then upon adding the two resulting equations and using conditions (AIII.12) once again we have

$$\bar{\partial}\overline{T} = 0 \tag{3C.2}$$

with

$$\overline{T} = g_{\alpha\bar{\beta}}\,\partial w^{\alpha}\,\partial\overline{w}^{\beta}$$

or

$$\begin{aligned} T &= \bar{\partial}\overline{w}^{\alpha}\,g_{\bar{\alpha}\beta}\,\bar{\partial}w^{\beta} \\ &= \bar{e}_{\mu}(\partial_{\mu}w^{\dagger})\,G(\partial_{\nu}w)\,\bar{e}_{\nu} \\ &= (\partial w)^{\dagger}\,G(\bar{\partial}w) \end{aligned} \tag{3C.3}$$

when

$$G = \frac{1}{1 + w^{\dagger}w}\,\frac{1}{I + ww^{\dagger}}\,.$$

Equation (3C.2) is the analyticity condition on T. By Morera's theorem in complex analysis it becomes

$$\oint_{S^1} T(z)\,dz = 0. \tag{3C.4}$$

By an argument completely analogous to that in Ref. [63], the finite action condition demands that $T(z)$ must be entire hence by Liouville's Theorem T must be a *constant* everywhere. However, since

$$w \xrightarrow[|z|\to\infty]{} 0, \qquad \text{hence} \qquad T \xrightarrow[|z|\to\infty]{} 0,$$

it follows that $T \equiv 0$.

To best display the consequence of $T \equiv 0$, let us rewrite it in terms of the fields

$$\phi_{\mu} = \frac{1}{((1 + w^{\dagger}w)(I + ww^{\dagger}))^{1/2}}\,\partial_{\mu}w.$$

We consider the objects

$$\phi = e_\mu \phi_\mu = e_\mu \left(\frac{\phi_\mu + \tilde{\phi}_\mu}{2} \right),$$
$$\phi' = \bar{e}_\mu \phi_\mu = \bar{e}_\mu \left(\frac{\phi_\mu - \tilde{\phi}_\mu}{2} \right); \tag{3C.5}$$

then

$$T = (\phi')^\dagger \phi, \tag{3C.6}$$

and the Cauchy–Morera formula can be written as

$$\oint_{S^1} (\phi')^\dagger \, dz \, (\phi) = 0, \tag{3C.7}$$

which implies that either $\phi'^\dagger = 0$ (holomorphy) or $\phi = 0$ (antiself-holomorphy), which is not equivalent (except for the $CP(1)$ case) to the self-duality equation, Eq. (3A.19).

It is also instructive to show the relation with the quadratic Cauchy–Riemann relations for either analyticity or antianalyticity. From the Lagrangian we can derive the energy momentum tensor

$$T_{\mu\nu} = \frac{\partial \mathscr{L}}{\partial(\partial_\mu w_\alpha)} \partial_\nu w_\alpha + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \bar{w}_\alpha)} \partial_\nu \bar{w}_\alpha - \delta_{\mu\nu} \mathscr{L}. \tag{3C.8}$$

Denoting by G the matrix with elements $g_{\alpha\beta}$ (the metric tensor), we find

$$2\pi T_{\mu\nu} = (\partial_\mu w^\dagger) \, G(\partial_\nu w) + (\partial_\nu w^\dagger) \, G(\partial_\mu w) - \delta_{\mu\nu}(\partial_\lambda w^\dagger) \, G \partial_\lambda w. \tag{3C.9}$$

$T_{\mu\nu}$ satisfies the tracelessness condition

$$T_{\lambda\lambda} = 0 \tag{3C.10}$$

and the conservation equation

$$\partial_\mu T_{\mu\nu} = 0. \tag{3C.11}$$

In the case w is analytic, denoting $(\partial/\partial x)w$ by w' we have

$$2\pi T_{\mu\nu} = w'^\dagger (\bar{e}_\mu G e_\nu + \bar{e}_\nu G e_\mu - \delta_{\mu\nu} \bar{e}_\lambda G e_\lambda) \, w'. \tag{3C.12}$$

Now in the complex case e_ν commutes with G and w' so that $T_{\mu\nu}$ is proportional to the left-hand side of the first equation (3A.23B), namely $s_{\mu\nu} = \bar{e}_\mu e_\nu + e_\mu \bar{e}_\nu - \delta_{\mu\nu} \bar{e}_\lambda e_\lambda$, which vanishes. In the antianalytic case $T_{\mu\nu}$ is proportional to $\bar{s}_{\mu\nu}$, so that in both cases we have the equation

$$T_{\mu\nu} = 0, \qquad \bar{e}_\mu T_{\mu\nu} \bar{e}_\nu = T = 0, \tag{3C.13}$$

which is equivalent to Eq. (3A.28).

Note that the induced hermitian metric is

$$H_{\mu\nu} = \partial_\mu \bar{w}_\alpha g_{\alpha\beta} \partial_\nu w_\alpha = (\partial_\mu w^\dagger)\, G \partial_\nu w = \phi_\mu{}^\dagger \phi_\nu\,.$$

This can be decomposed into three pieces, namely a conformally flat Riemannian metric, a symmetric traceless Riemannian metric and a 2-form, so that

$$\begin{aligned} H_{\mu\nu} &= \frac{1}{2} H_{\lambda\lambda}\delta_{\mu\nu} + \frac{1}{2}(H_{\mu\nu} + H_{\nu\mu} - \delta_{\mu\nu}H_{\lambda\lambda}) + \frac{1}{2}(H_{\mu\nu} - H_{\nu\mu}) \\ &= K^2\delta_{\mu\nu} + \frac{1}{\pi} T_{\mu\nu} + \frac{1}{2} F_{\mu\nu}\,. \end{aligned} \tag{3C.14}$$

The quadratic Cauchy–Riemann relations imply the vanishing of the energy momentum tensor so that when w is analytic or antianalytic we have

$$H_{\mu\nu} = \tfrac{1}{2}\phi_\lambda{}^\dagger\phi_\lambda\delta_{\mu\nu} + \tfrac{1}{2}(\phi_\mu{}^\dagger\phi_\nu - \phi_\nu{}^\dagger\phi_\mu). \tag{3C.15}$$

Then the Riemannian part of $H_{\mu\nu}$ is conformally flat and the antisymmetrical part $F_{\mu\nu}$ associated with the closed two-form automatically satisfies

$$F_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu}\epsilon_{\alpha\beta}F_{\alpha\beta}\,. \tag{3C.16}$$

We shall see that the quadratic Cauchy–Riemann relations in the form

$$\phi_\mu{}^\dagger\phi_\nu + \phi_\nu{}^\dagger\phi_\mu - \delta_{\mu\nu}\phi_\lambda{}^\dagger\phi_\lambda = 0 \tag{3C.17}$$

and the identity

$$\phi_\mu{}^\dagger\phi_\nu - \phi_\nu{}^\dagger\phi_\mu - \epsilon_{\mu\nu}\epsilon_{\alpha\beta}\phi_\alpha{}^\dagger\phi_\beta = 0 \tag{3C.18}$$

generalize to the quaternionic case.

The above analysis clearly goes through for any Kählerian σ-model defined by the action $A = (1/2\pi) \iint d^2x(\partial_\mu w^\dagger)\, G\partial_\mu w$ where G is the metric for any simply connected compact homogeneous Kähler manifolds, which must not just be algebraic but also rational. Only the latter as pointed out by Perelomov [41] *do* admit nontrivial global holomorphic maps whose existence cannot be deduced from the local Cauchy–Riemann equations.

Model builders might be interested that for Hermitian (hence Kählerian) homogeneous spaces, the coset spaces of (semi-) simple Lie groups are (in the irreducible case) of two types: (1) G is a noncompact simple Lie group with center the identity and K a maximum compact subgroup, G/K is then a complex manifold and is analytically equivalent to Cartan's irreducible bounded symmetric domains [46] (2) G is compact and H a subgroup of maximum rank. Cartan has classified these, spaces G/H which are [46]: $\mathrm{BI}_{m,m'}$: $U(m+m')/U(m) \times U(m')$, BII_m: $SO(2m)/U(m)$ BIII_m: $Sp(m)/U(m)$, BIV_m: $SO(m+2)/SO(m) \times SO(2)$, BV: $E_6/\mathrm{Spin}(10) \times SO(2)$ and BVI: $E_7/E_6 \times SO(2)$. Of all compact symmetric spaces only this second group of spaces are rational algebraic varieties *and* have $\pi_2(G/K) = Z_\infty$. We observe that all *complex* Grassmanian manifolds, i.e. the class $\mathrm{BI}_{m,m}$, belong to the above group. Only these manifolds when taken as field manifolds of 2-dimensional field theories admit nontrivial instanton solutions.

We close this section with a derivation of an infinite number of local continuity equations for the 2-dimensional Kählerian σ-models. We formulate in Euclidean space the analog of the conservation laws in Minkowski space. Consider a general solution to the equations of motions. Unlike for instantons, such a solution does not have a zero energy-momentum tensor. Then, denoting 1 and i by e_0 and e_1 respectively we have seen that the complex quantity

$$T = \bar{e}_\mu T_{\mu\nu} \bar{e}_\nu = \phi'^\dagger \phi \tag{3C.19}$$

of Eq. (3C.13) formed from the energy-momentum tensor satisfies the analyticity condition

$$\bar{\partial} \overline{T} = 0 \tag{3C.20}$$

by virtue of the equation of motion. Further, by Morera's theorem this can be written as

$$\oint_{S^1} \phi'^\dagger \, dz \, \phi = 0, \tag{3C.21}$$

where ϕ and $\phi'^\dagger$ are separately analytic. Consider now two orthogonal families of curves

$$u = \tau, \qquad v = \xi, \tag{3C.22}$$

where τ and ξ are constant and

$$u + iv = f(z), \tag{3C.23}$$

$f(z)$ being some analytic function. Then, the line integral

$$p = p_\mu e_\mu = \int_{v=\xi} T \, dz = \int_{v=\xi} \phi'^\dagger \, dz \, \phi \tag{3C.24}$$

is a function of u only. Because of the analyticity condition, we have, however,

$$\frac{dp}{du} = 0 \tag{3C.25}$$

and the Euclidean vector p_μ is conserved with respect to the parameter u. In the instanton case $p = 0$. Now introduce the quantities

$$p^{(n)} = p_\mu^{(n)} e_\mu = \int_{v=\xi} T^n \, dz = \int_{v=\xi} (\phi'^\dagger)^n \, dz \, \phi^n \qquad (n = 1, 2, \ldots, \infty). \quad (3C.26)$$

Since T^n, ϕ^n and $(\phi'^\dagger)^n$ are all analytic, we see that the quantities $p_\mu^{(n)}$, are also conserved. Hence, by using analyticity we have generated an infinite number of conservation laws. Is there an analog of this phenomenon in four dimensions? We shall attempt to answer this question in the sequel to this paper.

IV. Quaternionic Projective σ-Models

IVA. *Conformal Invariance and Real Quaternions*

In the foregoing sections, the emergence of complex number and complex analyticity in two dimensions can be traced to the conformal invariance of the renormalizable field theory. In 4-dimensional Euclidean spacetime, we should expect the relevance of real quaternions and quaternion analyticity [21].

Let us begin by considering the physical Minkowski spacetime [64]. There the spacetime point x can be represented by a hermitian quaternion $x = x_0 + \boldsymbol{\sigma} \cdot \mathbf{x} = x^\dagger$. Its determinant $\bar{x}x$ is left invariant by a Lorentz transformation such that $x' = LxL^\dagger$ where L is a unimodular complex quaternion. The conformal transformations connected to the identity take the fractional linear form of

$$x' = \lambda L \left(\frac{1}{x - A} + \bar{C} \right)^{-1} L^+ \qquad (4A.1)$$

with $\det L = 1$ and where λ corresponds to dilatation, L to a Lorentz transformation, A to translations and C to special conformal transformations or constant accelerations. Equation (4A.1) is a nonlinear representation of $SO(4, 2)$ in the coset space $SO(4, 2)/P \otimes \Delta$ with P being the Poincaré group and Δ the dilatation subgroup. If we were to consider for simplicity the two point Wightman function $w(x_1 - x_2) = \langle 0 \mid \phi(x_1) \, \phi(x_2) \mid 0 \rangle$ where ϕ is a Lorentz covariant field and $\mid 0\rangle$ the vacuum state, then $z = x_1 - x_2$ can be extended to complex values due to causality and the spectral condition. Setting $z = x + iy$ with $x = \frac{1}{2}(z + z^\dagger)$ and $y = -i/2 \, (z - z^\dagger)$ such that $y\bar{y} > 0$, $y_0 > 0$, the z's define the tube domain of the field theory which is invariant under the conformal transformations. In the case of the $(n + 1)$ point Wightman function, one must consider a tube domain of n complex quaternionic variables. The latter is a quaternionic analog of Siegel's complex domains [65].

On the strength of analyticity in field theory, one can perform a Wick rotation on the time variable x_0 and replace it by $\tau = -ix_0 = x_4$. The corresponding Euclidean spacetime point is then represented by the real quaternion $x = j_\mu x_\mu$

$\mu = 1, 2, \ldots, 4$ where $\{j_\mu\}$ is the set of standard quaternionic units such that $j_n j_m = -\delta_{nm} + \epsilon_{nml} j_l$, $n, m, l = 1, 2, 3$. Denoting the conjugate by a crescent, we have

$$\breve{j}_4 = j_4 , \qquad \breve{j}_n = -j_n . \tag{4A.2}$$

Then

$$\begin{aligned} \breve{j}_\mu j_\nu + \breve{j}_\nu j_\mu &= 2\delta_{\mu\nu} , \\ \breve{j}_\mu j_\nu - \breve{j}_\nu j_\mu &= 2 j_{\mu\nu} , \end{aligned} \tag{4A.3}$$

where

$$j_{\mu\nu} = 1/2 \; \epsilon_{\mu\nu\alpha\beta} j_{\alpha\beta} = \tilde{j}_{\mu\nu} \tag{4A.4}$$

is self-dual, $\epsilon_{\mu\nu\alpha\beta}$ being the Levi–Civita symbol ($\epsilon_{1234} = 1$). Analogously

$$j'_{\mu\nu} = \tfrac{1}{2}(j_\mu \breve{j}_\nu - j_\nu \breve{j}_\mu) = -\tilde{j}'_{\mu\nu} \tag{4A.5}$$

is antiself-dual.

We record the following useful commutation and anticommutation relations

$$\begin{aligned} \tfrac{1}{2}[j_{\mu\nu} , j_{\alpha\beta}] &= j_{\mu\beta}\delta_{\nu\alpha} + j_{\nu\alpha}\delta_{\mu\beta} - j_{\nu\beta}\delta_{\mu\alpha} - j_{\mu\alpha}\delta_{\nu\beta} , \\ \tfrac{1}{2}\{j_{\mu\nu} , j_{\alpha\beta}\} &= -\epsilon_{\mu\nu\alpha\beta} - \delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\nu\alpha}\delta_{\mu\beta} . \end{aligned} \tag{4A.6}$$

and

$$\begin{aligned} \tfrac{1}{2}[j'_{\mu\nu} , j'_{\alpha\beta}] &= j'_{\mu\beta}\delta_{\nu\alpha} + j'_{\nu\alpha}\delta_{\mu\beta} - j'_{\nu\beta}\delta_{\mu\alpha} - j'_{\mu\alpha}\delta_{\nu\beta} , \\ \tfrac{1}{2}\{j'_{\mu\nu} , j'_{\nu\beta}\} &= \epsilon_{\mu\nu\alpha\beta} - \delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\nu\alpha}\delta_{\mu\beta} ; \end{aligned} \tag{4A.7}$$

also

$$j_{\mu\nu} j'_{\mu\nu} = 0. \tag{4A.8}$$

Other relevant properties of real quaternions are collected in Appendix I.

We recall that the conformal group in Euclidean spacetime is $SO(5, 1)$ whose covering group is $SL(2, H)$, the linear group in 2-dimensional real quaternionic space seen as a projective space. Thus, conformal invariant Euclidean field theories exhibit $SL(2, H)$ invariance and real quaternions duly make their appearance in covariant solutions to gauge theories for instance.

The group $SL(2, H)$ acts on a quaternionic 2-vector ψ as follows:

$$\psi' = L\psi, \qquad \det L = 1, \tag{4A.9}$$

$$\psi' = \begin{pmatrix} \psi'_1 \\ \psi'_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} , \tag{4A.10}$$

where α, β, γ, δ are real quaternions, e.g. $\alpha = j_\mu \alpha_\mu$ and $\det L$ is the determinant of a 4×4 matrix resulting from a matrix representation of the j_n as $j_n = -i\sigma_n$ ($n = 1, 2, 3$). L clearly depends on 15 parameters.

Let $M \in SL(2, H)$ such that

$$M = \begin{pmatrix} k & l \\ m & n \end{pmatrix}. \tag{4A.11}$$

With

$$\operatorname{Det} M = |k|^2 |n|^2 + |m|^2 |l|^2 - 2\,\mathrm{Sc}(k\bar{m}n\bar{l}) = 1 \tag{4A.12}$$

M can be decomposed as

$$\begin{pmatrix} k & l \\ m & n \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}, \tag{4A.13}$$

where $\mathrm{Vec}(\lambda) = 0$ and $|v_1| = (v_1\bar{v}_1)^{1/2} = 1$, $|v_2| = 1$. Then

$$\begin{aligned} &x = \ln^{-1}, \qquad \lambda = |n|, \qquad v_2 = n|n|^{-1}, \\ &y = m(k - \ln^{-1}m)^{-1}|n|^{-2}, \qquad v_1 = |n|(k - \ln^{-1}m) \end{aligned} \tag{4A.14}$$

provided $n \neq 0$.

Under a conformal transformation

$$M' = M = \begin{pmatrix} k' & l' \\ m' & n' \end{pmatrix}. \qquad L = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{4A.15}$$

the parameter x transforms according to the law

$$x' = l'n'^{-1} = (\alpha x + \beta)(\gamma x + \delta)^{-1}. \tag{4A.16}$$

As defined by Eq. (4A.13), x is the coset representative of the conformal group with respect to the triangular group

$$T = \begin{pmatrix} v_1\lambda^{-1} & 0 \\ y\lambda v_1 & \lambda v_2 \end{pmatrix} \tag{4A.17}$$

representing the inhomogeneous Weyl group of 4-dimensional rotations, dilatations and translations. x may thus be taken as a label of a Euclidean spacetime point. It also transforms like $\psi_1\psi_2^{-1}$, ψ being given in Eq. (4A.9), is a spinor with quaternionic components and is known as a twistor [66].

The $SO(5)$ subgroup of $SO(5, 1) \simeq SL(2, H)$ can be represented by the matrix

$$D = \frac{1}{(1 + |h|^2)^{1/2}} \begin{pmatrix} 1 & -\bar{h} \\ h & 1 \end{pmatrix} \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}, \qquad |v_1| = |v_2| = 1, \tag{4A.18}$$

which corresponds to the representation of $Sp(2, H) \equiv Sp(2)$ derived in Section II with $n = 1$, so h labels the coset space $SO(5)/SO(4) = Sp(2)/Sp(1) \times Sp(1) =$

$S^4 = HP(1)$. Taking $h = x$ corresponds to a mapping of S^4 onto the coset space $SL(2, H)/T$, T being the inhomogeneous Weyl group.

To find the metric tensor associated with a conformal transformation we can write

$$\begin{aligned} x' &= (\alpha x + \beta)(\gamma x + \delta)^{-1} \\ &= a + b(x + c)^{-1} d \end{aligned} \tag{4A.19}$$

where

$$a = \alpha\gamma^{-1}, \qquad b = \beta - \alpha\gamma^{-1}\delta, \qquad c = \gamma^{-1}\delta, \qquad d = \gamma^{-1}$$

provided that $\gamma \neq 0$.

This form yields the differential

$$dx' = -b(x + c)^{-1}\, dx(x + c)^{-1}\, d \tag{4A.20}$$

and hence the metric

$$ds^2 = d\breve{x}'\, dx' = dx'_\mu\, dx'_\mu = \phi^2\, d\breve{x}\, dx \tag{4A.21}$$

where $\phi = |\, b \,|\, |\, d \,|\, |\, x + c \,|^{-2}$. The induced, integrable metric is given by

$$h_{\mu\nu} = \phi^2\delta_{\mu\nu} \tag{4A.22}$$

so that

$$h^{\mu\nu} = \phi^{-2}\delta_{\mu\nu}, \qquad h = \det(h_{\mu\nu}) = \phi^8, \qquad h^{1/2}h^{\mu\nu} = \phi^2\delta_{\mu\nu} = h_{\mu\nu}. \tag{4A.23}$$

Consequently we can construct two conformally invariant tensors:

$$h^{-1/2}\epsilon^{\mu\nu\alpha\beta}h^{1/2}\, d^4x = \epsilon_{\mu\nu\alpha\beta}\, d^4x \tag{4A.24}$$

and

$$h^{1/2}h^{\mu\alpha}h^{\nu\beta}\, d^4x = \delta_{\mu\alpha}\delta_{\nu\beta}\, d^4x. \tag{4A.25}$$

In the event we apply the conformal transformation to a curved Euclidean space that is conformally flat with a non-integrable metric

$$g_{\mu\nu} = \chi^2\delta_{\cdot\nu} \tag{4A.26}$$

the conformal transformation changes it into

$$g'_{\mu\nu} = \chi'^2\delta_{\mu\nu} = \phi^2 g_{\mu\nu} = (\phi\chi)^2\,\delta_{\mu\nu}. \tag{4A.27}$$

So for conformally flat spaces, we still have the invariant tensors given before in Eqs. (4A.24) and (4A.25). We note that the first of these has in fact a larger invariance group under general coordinate transformations since it is just the invariant volume element in curved Euclidean spacetime.

Finally, linear combinations of the two kinds of invariant tensors given above are obtainable from

$$\{j_{\mu\nu}, j_{\alpha\beta}\}\, d^4x \text{ and } \{j'_{\mu\nu}, j'_{\alpha\beta}\}\, d^4x \tag{4A.28}$$

if Eqs. (4A.6) and (4A.7) are used. With this prelude on the connection between real quaternions and conformal structure, we are ready to proceed to quaternionic chiral models.

IIIB. *HP(n) σ-Models and Yang–Mills Embedding*

To construct our quaternionic σ-models by the method of nonlinear realization, we let the symplectic group $Sp(n+1)$ be the chiral symmetry group of the Lagrangians to be constructed, one which leads to a degeneracy of the vacuum and the appearance of Nambu–Goldstone bosons. The vacuum is left invariant by $Sp(n) \times Sp(1)$, the maximum subgroup of $Sp(n+1)$. An arbitrary transformation of $Sp(n+1)$ is decomposed in the product form $U(x) = V(x)\, W(x)$ Eq. (2.9) where $x \in E^4$, $W(x)$ is the transformation of the subgroup $Sp(n) \times Sp(1)$ and V is a transformation belonging to the left coset of $Sp(n+1)$ with respect to its holonomy group $Sp(n) \times Sp(1)$. $V(x)$ is a $(n+1) \times (n+1)$ quaternionic matrix of the Nambu–Goldstone quaternionic fields $t = u(x) = (u_1(x), u_2(x),..., u_n(x))^T$ in our projective parametrization (Eq. (2.16)) of the field space $Sp(n+1)/Sp(n) \times Sp(1) \approx HP(n)$. To distinguish the quaternionic character of the field space $HP(n)$ from that of a compactified 4-spacetime S^4 with its basis units denoted by the j_μ, we use a second set of units e_μ ($\mu = 1,..., 4$) and their conjugate $\bar{e}_\mu$ such that $[e_\mu, j_\nu] = 0$. Correspondingly we introduce the notation $Q = e_\mu q_\mu$, $\bar{Q} = \bar{e}_\mu q_\mu$.

Proceeding by analogy with the $CP(1)$ σ-model in two dimensions whose Chern index, Eq. (3A.13), is

$$\mathscr{C}_1 = -\frac{1}{8\pi i}\,\mathrm{Sc}\left\{\int d^2x (n\partial_\mu\, n\tilde{\partial}_\mu\, n)\right\} \qquad (n = \mathbf{e}\cdot\mathbf{n}) \tag{4B.1}$$

with Sc denoting the scalar part of the quaternion, we consider the following integrals which are invariant under general coordinate transformations

$$J = \tfrac{1}{2}\int d^4x \epsilon_{\mu\nu\alpha\beta} \partial_\mu\, N \partial_\nu\, N \partial_\alpha\, N \partial_\beta\, N \tag{4B.2}$$

and

$$K = \tfrac{1}{2}\int d^4x \epsilon_{\mu\nu\alpha\beta}\, N\partial_\mu\, N\partial_\nu\, N\partial_\alpha\, N\partial_\beta\, N, \tag{4B.3}$$

N being the canonical H-matrix field given by Eq. (2.23). We are also led to form the two conformally invariant integrals

$$L = \tfrac{1}{2} \int d^4x(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\nu\alpha}\delta_{\mu\beta})\, \partial_\mu N \partial_\nu N \partial_\alpha N \partial_\beta N \tag{4B.4}$$

and

$$M = \tfrac{1}{2} \int d^4x(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\nu\alpha}\delta_{\mu\beta})\, N(\partial_\mu N \partial_\nu N \partial_\alpha N \partial_\beta N). \tag{4B.5}$$

Define

$$R_{\mu\nu} = [\partial_\mu N, \partial_\nu N] \tag{4B.6}$$

and

$$R = \tfrac{1}{2} j_{\mu\nu} R_{\mu\nu}, \qquad R' = \tfrac{1}{2} j'_{\mu\nu} R_{\mu\nu}. \tag{4B.7}$$

Making use of Eqs. (4A.6) and (4A.7) we obtain

$$-\tfrac{1}{2}(R^2 + \check{R}^2) = \tfrac{1}{4}\epsilon_{\mu\nu\alpha\beta} R_{\mu\nu} R_{\alpha\beta} + \tfrac{1}{4}(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\nu\alpha}\delta_{\mu\beta})\, R_{\mu\nu} R_{\alpha\beta} \tag{4B.8}$$

and

$$-\tfrac{1}{2}(R'^2 + \check{R}'^2) = -\tfrac{1}{4}\epsilon_{\mu\nu\alpha\beta} R_{\mu\nu} R_{\alpha\beta} + \tfrac{1}{4}(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\nu\alpha}\delta_{\mu\beta})\, R_{\mu\nu} R_{\alpha\beta} \tag{4B.9}$$

so that

$$J + L = -\tfrac{1}{4} \int (R^2 + \bar{R}^2)\, d^4x, \; -J + L = -\tfrac{1}{4} \int (R'^2 + \breve{R}'^2)\, d^4x. \tag{4B.10}$$

As an action generalizing the 2-dimensional $CP(n)$ σ-models, we can take the scalar part of Tr L or Tr M. Similarly the scalar parts of Tr J and Tr K generalize the Chern–Pontryagin index Eq. (3A.11). We remark that all these integrals have conformal invariance as well as local $Sp(n) \times Sp(1)$ invariance since the latter holonomy group leaves N invariant. Moreover the index integrals are invariant under general coordinate transformations as can be readily verified from the tensor density nature of the Levi–Civita symbol. Just as $CP(n)$ is a natural choice for the field manifold of a 2-dimensional model because $\pi_i(CP(n)) \approx Z_\infty$ for $i = 2$ and $\approx \pi_i(S^{2n+1})$ otherwise with $i \geqslant 2n + 1$ for it to be nontrivial, $HP(n)$ is natural for 4-dimensional σ-models since $\pi_i(HP(n) \approx \pi_i(S^{4n+3}) \oplus \pi_{i-1}(S^3)$ [67] so that if $i < 4$, $\pi_i(HP(n)) = 0$ and when $i = 4$, $\pi_4(HP(n)) \approx \pi_3(S^3) = Z_\infty$. Having made these remarks, we defer the full investigation of the above quaternionic models till a future work. Instead we now focus exclusively on the embedded gauge field structure which will now be made explicit.

Since N depends only on the coset matrix V, we can rewrite the above invariant integral in terms of V and $\partial_\mu V$, thus generalizing the forms Eqs. (3A.14) and (3A.15) in two dimensions. We recall the antihermitian quaternionic matrix in Eq. (2.37) Ω_μ

in Eq. (2.31), $\Omega_\mu = V^\dagger \partial_\mu V = A_\mu + \eta F_\mu = A_\mu - F_\mu \eta$. Thus a_μ is a purely vectorial quaternion, B_μ is a purely vectorial $n \times n$ quaternionic matrix, and ϕ_μ is a column matrix with n quaternionic components. By construction, Ω_μ, as a consequence of the Poincaré integrability $[\partial_\mu, \partial_\nu]V = 0$, must obey the Maurer–Cartan equations

$$\partial_\mu \Omega_\nu - \partial_\nu \Omega_\mu + [\Omega_\mu, \Omega_\nu] = 0. \tag{4B.11}$$

Using Eq. (2.36) they take the form

$$\begin{aligned}\partial_\mu A_\nu - \partial_\nu A_\mu + \eta(\partial_\mu F_\nu - \partial_\nu F_\mu) + [A_\mu, A_\nu] - [F_\mu, F_\nu]\\ + \eta([F_\mu, A_\nu] + [A_\mu, F_\nu]) = 0\end{aligned} \tag{4B.12}$$

or equivalently

$$\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = [F_\mu, F_\nu] \tag{4B.13}$$

and

$$\partial_\mu F_\nu - \partial_\nu F_\mu + [F_\mu, A_\nu] + [A_\mu, F_\nu] = 0. \tag{4B.14}$$

From Eq. (4B.13), it is natural to define the quantity

$$\Phi_{\mu\nu} = [F_\mu, F_\nu]. \tag{4B.15}$$

If we further define the fields

$$f_{\mu\nu} = \mathbf{e} \cdot \mathbf{f}_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu + [a_\mu, a_\nu], \tag{4B.16}$$

$$F_{\mu\nu} = \mathbf{e} \cdot \mathbf{F}_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] \tag{4B.17}$$

we find that

$$f_{\mu\nu} = \phi_\mu{}^\dagger \phi_\nu - \phi_\nu{}^\dagger \phi_\mu, \tag{4B.18}$$

$$F_{\mu\nu} = \phi_\mu \phi_\nu{}^\dagger - \phi_\nu \phi_\mu{}^\dagger, \tag{4B.19}$$

which under a $Sp(n) \times Sp(1)$ gauge transformation via Eq. (2.46) become

$$f'_{\mu\nu} = r f_{\mu\nu} r^{-1}, \tag{4B.20}$$

$$F'_{\mu\nu} = R F_{\mu\nu} R^{-1} \tag{4B.21}$$

respectively. They transform covariantly as $Sp(1)$ and $Sp(n)$ gauge field strengths. Hence in contrast to Ω_μ which is a pure gauge potential, for a Yang–Mills theory based on $Sp(n+1)$, $f_{\mu\nu}$ and $F_{\mu\nu}$ are no longer pure gauge and therefore are nonzero. Indeed we will in this manner have succeeded in embedding solutions of the *form* of Eq. (4B.18) of a $Sp(1) \approx SU(2)$ Yang–Mills theory into the pure gauge solutions of a globally $Sp(n+1)$ chiral invariant theory, which we have associated with a quaternionic $HP(n)$ σ-model.

To see this phenomenon in some detail, we consider the induced motions

$$\partial_\mu V = V\Omega_\mu , \qquad \partial_\mu V^\dagger = -\Omega_\mu V^\dagger. \tag{4B.22}$$

Remembering expression (2.12) of N in terms of V, we can write

$$\partial_\mu N = V[\Omega_\mu , \eta]\, V^\dagger, \tag{4B.23}$$

$$N\, \partial_\mu N = V\eta[\Omega_\mu , \eta]\, V^\dagger. \tag{4B.24}$$

On the other hand Eqs. (2.37) and (2.38) give

$$[\Omega_\mu , \eta] = -2F_\mu , \qquad \eta[\Omega_\mu , \eta] = -2\eta F_\mu . \tag{4B.25}$$

Thus

$$\frac{1}{2}(E \pm N)\, \partial_\mu N\, \partial_\nu N\, \partial_\alpha N\, \partial_\beta N = 16 V \frac{E \pm \eta}{2} F_\mu F_\nu F_\alpha F_\beta V^{-1} \tag{4B.26}$$

and we find

$$\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\frac{1}{2}(E \pm N)\, \partial_\mu N\, \partial_\nu N\, \partial_\alpha N\, \partial_\beta N = 4V\frac{E \pm \eta}{2}\, \Phi_{\mu\nu}\tilde{\Phi}_{\mu\nu} V^{-1}, \tag{4B.27}$$

$$\frac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\nu\alpha}\delta_{\mu\beta})\frac{1}{2}(E \pm N)\, \partial_\mu N\, \partial_\nu N\, \partial_\alpha N\, \partial_\beta N = 4V\frac{E \pm \eta}{2}\, \Phi_{\mu\nu}\Phi_{\mu\nu} V^{-1}. \tag{4B.28}$$

Hence

$$-\frac{1}{4}\{j_{\mu\nu} , j_{\alpha\beta}\}\frac{1}{2}(E \pm N)\, \partial_\mu N\, \partial_\nu N\, \partial_\alpha N\, \partial_\beta N = 4V\frac{E \pm \eta}{2}\, \Phi_{\mu\nu}(\tilde{\Phi}_{\mu\nu} + \Phi_{\mu\nu})\, V^{-1}, \tag{4B.29}$$

$$-\frac{1}{4}\{j'_{\mu\nu} , j'_{\alpha\beta}\}\frac{1}{2}(E \pm N)\, \partial_\mu N\, \partial_\nu N\, \partial_\alpha N\, \partial_\beta N = 4V\frac{E \pm \eta}{2}\, \Phi_{\mu\nu}(-\tilde{\Phi}_{\mu\nu} + \Phi_{\mu\nu})\, V^{-1}. \tag{4B.30}$$

In terms of the $HP(n)$ σ-model field N, the embedded $Sp(1)$ Yang–Mills Lagrangian reads

$$L_{sp(1)} = \frac{1}{4} f_{\mu\nu} f_{\mu\nu} = \frac{1}{16}\frac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\nu\alpha}\delta_{\mu\beta})\, \mathrm{Tr}\left\{\frac{E+N}{2}\, \partial_\mu N\, \partial_\nu N\, \partial_\alpha N\, \partial_\beta N\right\}. \tag{4B.31}$$

While symmetry consideration might lead one to also consider $F_{\mu\nu}$ (4B.19) as a $Sp(n)$ gauge field, this is not the case. Only $f_{\mu\nu}$ has enough degrees of freedom to be a $Sp(1)$ gauge field obeying the Yang–Mills equations. Of course the same argument applied to the $U(1)$ gauge field in the $CP(n)$ case. The reason for the difference between the subgroups $U(1)$, $Sp(1)$ and $SU(n)$, $Sp(n)$ in the holonomy groups of $CP(n)$ and $HP(n)$ respectively will be made clear in Section VII. We have for

$$R = j_{\mu\nu}\, \partial_\mu N \partial_\nu N = 2Vj_{\mu\nu}[F_\mu, F_\nu]\, V^{-1} = 2Vj_{\mu\nu}\Phi_{\mu\nu}V^{-1} \tag{4B.32}$$

or putting $\Phi = (1/2)\, j_{\mu\nu}\Phi_{\mu\nu}$, $\Phi' = (1/2)\, j'_{\mu\nu}\Phi_{\mu\nu}$,

$$R = 4V\Phi V^{-1}, \qquad R' = j'_{\mu\nu}\, \partial_\mu N\, \partial_\nu N = 4V\Phi' V^{-1}. \tag{4B.33}$$

Another form is available from

$$R = (\breve{j}_\mu j_\nu - \delta_{\mu\nu})\, \partial_\mu N\, \partial_\nu N = \breve{\mathscr{D}} N \mathscr{D} N - \partial_\mu N\, \partial_\mu N, \tag{4B.34}$$

where

$$\mathscr{D} = j_\mu \partial_\mu \qquad \text{and} \qquad \breve{\mathscr{D}} = \breve{j}_\mu \partial_\mu \tag{4B.35}$$

are the Hamilton operators.

We immediately observe that for self-dual $Sp(1)$ fields, namely

$$f_{\mu\nu} = \tilde{f}_{\mu\nu}, \tag{4B.36}$$

the analog of Eq. (3A.20), we have

$$\frac{1}{2} j'_{\mu\nu} f_{\mu\nu} = \frac{E+\eta}{2}\, \Phi' = 0, \qquad \frac{E+N}{2}\, R' = 0, \qquad L - J + M - K = 0, \tag{4B.37}$$

and for antiself-dual $Sp(1)$ fields

$$f_{\mu\nu} = -\tilde{f}_{\mu\nu}, \tag{4B.38}$$

$$\frac{1}{2} j_{\mu\nu} f_{\mu\nu} = \frac{E+\eta}{2}\, \Phi = 0, \qquad \frac{E+N}{2}\, R = 0, \qquad L + J + M + K = 0, \tag{4B.39}$$

with

$$\mathrm{Tr}\,\mathrm{Sc}(\mathrm{J}) = -\mathrm{Tr}\,\mathrm{Sc}(L), \tag{4B.40}$$

or in the terms of the fields N and $P = \frac{1}{2}(E + N)$,

$$\frac{E+N}{2}\, (\mathscr{D} N \breve{\mathscr{D}} N - \partial_\mu N\, \partial_\mu N) = 0, \tag{4B.41}$$

$$(\mathscr{D} P)(E - P)\, \breve{\mathscr{D}} P - (\partial_\mu P)(E - P)\, \partial_\mu P = 0. \tag{4B.42}$$

Self-dual fields also automatically obey the Yang–Mills equations associated with the $Sp(1)$ subgroup of the holonomy group $Sp(1) \times Sp(n)$ and minimize the $Sp(1)$ Lagrangian given in Eq. (4B.31). It also follows from Eq. (4B.10) that we can solve the quaternionic σ-models based on the actions Sc Tr$(L \pm M)$, with L and M given by Eqs. (4B.4) and (4B.5) by solving the self-duality equations Eqs. (4B.36) and (4B.38). This is in perfect analogy with the method of solving the 2-dimensional $CP(n)$ σ-models by solving for the self-duality equation for ϕ_μ Eq. (3A.24).

In terms of the Nambu–Goldstone fields we recall that $\phi_\mu = [(1 + u^\dagger u)(I + uu^\dagger)]^{-1/2} \partial_\mu u$, Eq. (2.34), so the self-duality equation (4B.42) becomes

$$\mathscr{D}u^\dagger(I + uu^\dagger)^{-1} \breve{\mathscr{D}}u = \partial_\mu u^\dagger(I + uu^\dagger)^{-1} \partial_\mu u. \tag{4B.43}$$

Before solving for this quaternionic equation, it is timely to first place this local structure in the context of the global differential geometry and topology of $HP(n)$, about which much is known [60, 68–73].

IV.C. *Quaternionic Quantum Mechanics and Kähler Geometry of $HP(n)$*

A $4n$-dimensional Riemannian manifold is called [60, 69] a quaternionic manifold if its holonomy group is a subgroup of $Sp(n) \otimes Sp(1)$. So $HP(n)$ is such a manifold. In fact, it is the sole homogeneous space with holonomy group $Sp(n) \otimes Sp(1)$ $(n > 1)$ [70]. Though less familiar perhaps than complex manifolds, the differential geometry and topology of quaternionic manifolds [71, 72], of $HP(n)$ in particular, have been the objects of several studies. A survey of the literature [73] shows a few equivalent formulations of the notion of quaternionic structure on these manifolds, e.g. in terms of three almost complex structures induced by the vectorial base elements e_i $(i = 1, 2, 3)$ of the quaternion algebra. The reader is referred to Appendix III for more references.

Prior to discussing the embedding of quaternionic manifolds in Euclidean spaces, we wish to start with the general framework of quaternionic n-dimensional Hilbert space in which the group $Sp(1) \otimes Sp(n)$ arises naturally. Since the work of Von Neumann [74] it is known [75] that the postulates of quantum mechanics lead to a formulation in terms of Hilbert spaces which, for a finite number of degrees of freedom are in a one to one correspondence with projective geometries. All these finite geometries can in turn be embedded in an infinite dimensional Hilbert space. Real Hilbert spaces are the spheres $SO(n + 1)/SO(n)$, complex Hilbert spaces are the projective $CP(n)$ spaces which are the coset spaces $SU(n + 1)/SU(n) \times U(1)$, the quaternionic Hilbert spaces are the projective $HP(n) \approx Sp(n + 1)/Sp(n) \times Sp(1)$ spaces and finally the exceptional case is covered by the Cayley–Moufang projective plane $F_4/\mathrm{Spin}(9)$ [76]. All these spaces are two point homogeneous, so that a unique invariant distance (the length of the geodesic between any two poonts) can be defined between two points [77]. We shall see that the points in such projective manifolds can be associated with physical states and the distance between two points is simply related to the transition probability between the corresponding physical states. Furthermore, in the instance of $CP(n)$ and $HP(n)$, normalized kets are defined up to a phase $U(1)$ for $CP(n)$ and $Sp(1)$ for $HP(n)$. The probability amplitudes are also defined up to the same unobservable phases. We recall that the phase groups $U(1)$

and $Sp(1)$ are the fibres S^1 and S^3 of the fibre spaces S^{2n+1} and S^{4n+3} over $CP(n)$ and $HP(n)$ respectively [67].

We now review the main definitions of quaternionic quantum mechanics (Q.Q.M.) (or equivalently $HP(n)$) since Yang–Mills instantons are naturally formulated in the language of Q.Q.M. Indeed $SU(2)$ instantons may well provide the first interesting physical application of Q.Q.M. The phase group of Q.Q.M. (or the Hopf fibering of $HP(n)$) is the gauge group $SU(2)$ of the Yang–Mills field theory. The dimensionality of the Q.Q.M. is the instanton number n. The invariance group of the transition amplitude is $Sp(n)$ as opposed to $SU(n)$ on the complex case. The normalized quaternionic ket $\nu(x)$ is the quaternionic state created by the $Sp(1) \times Sp(n)$ model field applied on the vacuum. The transition amplitude $\nu^\dagger(x)\,\nu(y)$ (simply related to the distance between corresponding points) is the path ordered exponential integral $T\{\exp \int_x^y A_\mu(\xi)\, d\xi_\mu\}$ which in turn is proportional to the Green's function associated with instanton solutions $A_\mu = \nu^\dagger\, \partial_\mu \nu$. For an instanton "gas" with $n \to \infty$ the instantons will be described in an infinite quaternionic Hilbert space. Bäcklund transformations which relate solutions for n and $n+1$ instantons are associated with creation (and annihilation) operators in Q.Q.M. that change the dimensionality of the finite subspace by one. All but the last of these features will be presented here.

Let $|\,\alpha\rangle$ be an $(n+1)$-dimensional quaternionic ket. $|\,\alpha\rangle$ and $|\,\alpha\rangle q$ represent the same physical state (q multiplies $|\,\alpha\rangle$ to the right). If $|\,\alpha\rangle$ is normalized with

$$\langle\alpha\,|\,\alpha\rangle = |\,\alpha\rangle^\dagger\,|\,\alpha\rangle = |\,\bar{\alpha}\rangle^T\,|\,\alpha\rangle = 1, \tag{4C.1}$$

then $|\,\alpha\rangle$ is determined up to a quaternionic phase q with $|\,q\,| = 1$. The quaternionic transition amplitude between $|\,\alpha\rangle$ and $|\,\beta\rangle$ is given by the quaternion

$$T_{\alpha\beta} = \langle\alpha\,|\,\beta\rangle, \qquad T_{\beta\alpha} = \overline{T}_{\alpha\beta} = \langle\beta\,|\,\alpha\rangle, \tag{4C.2}$$

while the probability amplitude is the positive number

$$\Pi_{\alpha\beta} = |\langle\alpha\,|\,\beta\rangle|^2 \tag{4C.3}$$

related to the $Sp(n+1)$ invariant distance $d_{\alpha\beta}$ by

$$\cos^2 d_{\alpha\beta} = \Pi_{\alpha\beta} = |\langle\alpha\,|\,\beta\rangle|^2. \tag{4C.4}$$

If we now use non-normalized kets a, b, so that

$$|\,\alpha\rangle = a(a^\dagger a)^{-1/2}, \qquad |\,\beta\rangle = b(b^\dagger b)^{-1/2} \tag{4C.5}$$

the distance between points with homogeneous coordinates a and b is given by

$$\cos^2 d_{\alpha\beta} = \frac{(a^\dagger b)\,(b^\dagger a)}{(a^\dagger a)\,(b^\dagger b)} \tag{4C.6}$$

and the transition amplitude is

$$T_{\alpha\beta} = \frac{a^\dagger b}{(a^\dagger a)^{1/2}\,(b^\dagger b)^{1/2}}\,. \tag{4C.7}$$

The observables in Q.Q.M. are hermitian quaternionic matrices

$$M = M^\dagger = \overline{M}^T \tag{4C.8}$$

that act on kets from the left and on bras from the right. They are members of a Jordan algebra with a commutative, but non-associative, multiplication law

$$M \cdot N = N \cdot M = 1/2\,(MN + NM) \tag{4C.9}$$

and an associator

$$[LMN] = (L \cdot M) \cdot N - L \cdot (M \cdot N) \tag{4C.10}$$

with the properties

$$[LML] = 0, \qquad [LML^2] = 0. \tag{4C.11}$$

Then the quaternionic matrix elements of M between states $| \alpha\rangle$ and $| \beta\rangle$ are uniquely defined as

$$M_{\alpha\beta} = \langle \alpha M \mid \beta\rangle = \langle \alpha \mid (M \mid \beta\rangle) = (M \mid \alpha\rangle)^\dagger \mid \beta\rangle. \tag{4C.12}$$

States can also be represented by the projection operators

$$P_\alpha = \mid \alpha\rangle\,\langle \alpha \mid = \frac{aa^\dagger}{a^\dagger a} \tag{4C.13}$$

which obey the conditions

$$P_\alpha = P_\alpha{}^\dagger, \qquad \operatorname{Tr} P_\alpha = 1, \qquad P_\alpha{}^2 = P_\alpha\,. \tag{4C.14}$$

Orthogonal states $| n\rangle$ and $| m\rangle$ correspond to projection operators that satisfy the equation

$$P_n \cdot P_m = 0 \qquad (n \neq m). \tag{4C.15}$$

If n inhomogeneous coordinates $u_1, ..., u_n$ denoted by the $n \times 1$ column u are used to characterize the point u or the physical state $| u\rangle$, the ket is given by

$$\mid u\rangle = a(u)[a^\dagger(u)\, a(u)]^{-1/2}, \tag{4C.16}$$

where

$$a(u) = \begin{pmatrix} 1 \\ u \end{pmatrix}, \qquad a^\dagger a = 1 + u^\dagger u. \tag{4C.17}$$

Then the transition amplitude T_{uv} is

$$T_{uv} = \langle u \mid v \rangle = \frac{1 + u^\dagger v}{(1 + u^\dagger u)^{1/2}\,(1 + v^\dagger v)^{1/2}}, \tag{4C.18}$$

while the distance $d(u, v)$ and the transition probability are given by

$$\Pi_{uv} = \cos^2 d(u, v) = \frac{(1 + u^\dagger v)(1 + v^\dagger u)}{(1 + u^\dagger u)(1 + v^\dagger v)}. \tag{4C.19}$$

The projection operator associated with the state $\mid u \rangle$ is

$$P_u = \mid u \rangle\langle u \mid = \rho(u)^{-1} \begin{pmatrix} 1 & u^\dagger \\ u & uu^\dagger \end{pmatrix} \qquad (\rho(u) = 1 + u^\dagger u). \tag{4C.20}$$

We shall show that P_u is related to the symplectic matrix $V(u)$ which is the coset representative of $Sp(n + 1)/Sp(n) \times Sp(1)$ with

$$\eta V \eta = V^\dagger \ (\eta^2 = E), \qquad VV^\dagger = E, \tag{4C.21}$$

where η is given by Eq. (2.6) and $V(u)$ by Eq. (2.14). We find

$$N(u) = V\eta V^{-1} = V^2\eta = \begin{pmatrix} \dfrac{1 - u^\dagger u}{1 + u^\dagger u} & \dfrac{2u^\dagger}{1 + u^\dagger u} \\ \dfrac{2u}{1 + u^\dagger u} & \dfrac{1}{I + uu^\dagger} - \dfrac{uu^\dagger}{1 + u^\dagger u} \end{pmatrix} = 2P_u - E, \tag{4C.22}$$

with

$$N(u)^2 = E \qquad \text{and} \qquad N = N^\dagger. \tag{4C.23}$$

$N(u)$ is recognized as the σ-model canonical matrix occuring in the Lagrangians equations (4B.2)–(4B.5). The associated symplectic matrix V is then given by

$$V(u) = (2 \mid u \rangle\langle u \mid \eta - \eta)^{1/2} = [(2P_u - E)\eta]^{1/2}. \tag{4C.24}$$

We can also associate an antihermitian matrix $\chi(u)$ with $\mid u \rangle$. It is related to the symplectic matrix V^2 by

$$V^2(u) = \frac{2}{E - \chi(u)} - E = \frac{E + \chi}{E - \chi}, \qquad V = \left(\frac{E + \chi}{E - \chi}\right)^{1/2}, \qquad \chi = -\chi^\dagger. \tag{4C.25}$$

Explicitly

$$\chi(u) = E - \frac{2}{E + V^2} = \begin{pmatrix} 0 & -u^\dagger \\ u & 0 \end{pmatrix}. \tag{4C.26}$$

Subsequently it will be seen that when V is associated with an instanton solution, the quaternionic function $u(x)$, hence the antihermitian matrix χ, has simple analytic properties (Section VI).

The different ways of associating the physical state with inhomogeneous coordinates $(u_1, u_2, ..., u_n)$ with the $(n+1) \times (n+1)$ projection operator P (Jordan formulation of Q.Q.M.) or the quaternionic normalized $(n+1)$-dimensional ket $| u\rangle$ (Dirac–Von Neumann formulation) correspond to well-known embeddings in quaternionic projective geometry. The $u \to P_u$ embedding is the generalization of the Mannoury, Study and Fubini construction of $CP(n)$ space via an embedding in Euclidean space. In fact it generalizes to any Grassmanian manifold (Section VII). The quaternionic generalization is first due to Martinelli [78], who proceeds via the construction of the ket $| u\rangle$ which he uses for setting up the quaternionic P_u.

Note that, in the canonical parametrization we have used, $Sp(1) \times Sp(n)$ acts on u linearly and V is a rational function of u so that $Sp(n+1)$ acts on u fractional linearly. Indeed

$$V^{-1}(v)\, N(u)\, V(v) = V^\dagger(v)\, V(u)\, \eta V^\dagger(u)\, V(v) = N(u'), \tag{4C.27}$$

where

$$u' = \frac{\Lambda(v)\, u - v}{v^\dagger u + 1}, \tag{4C.28}$$

with

$$\Lambda(v) = \frac{(1 + v^\dagger v)^{1/2}}{(I + vv^\dagger)^{1/2}}. \tag{4C.29}$$

Keeping in mind the above facts about Q.Q.M., we proceed to discuss the differential geometry of $HP(n)$ in some detail. By a theorem of Kobayashi [72], every compact homogeneous Kähler manifold can be minimally embedded in a Euclidean space; the following Mannoury [79]–Martinelli [78] embeddings of $CP(n)$ and $HP(n)$ are but illustrations of this theorem. Following Tai [80], we consider in a single stroke the spaces $RP(n)$, $CP(n)$ and $HP(n)$. Let $GL(n+1, K)$ be the space of all $(n+1) \times (n+1)$ matrices over $K = R, C, H$. Let $h(n+1, K) = \{A \in GL(n+1, K), A^\dagger = A)$, the space of all $(n+1) \times (n+1)$ hermitian matrices over K. Thus if $A \in h(n+1, R)$, then A is symmetric and $U(n+1, K) = \{A \in GL(n+1, K), A^\dagger A = I\}$ then stands for the groups $O(n+1)$, $U(n+1)$ and $Sp(n+1)$ for $K = R$, C, and H respectively.

Now take $KP(n)$ as the quotient space of the unit $((n+1)d - 1)$ sphere where $d = 1, 2, 4$ for $K = R, C, H$. The latter is defined by $\{q \in K^{n+1} \mid q^\dagger q = 1\}$. We recall that $KP(n)$ is obtained by identifying q and $q\lambda$ where $\lambda \in K$ and $|\lambda| = 1$. Hence for $q \in KP(n)$, we can use the homogeneous coordinates $q = (q_0, q_1, ..., q_n)^T$ with $q^\dagger q = 1$.

Now consider the embedding map $\Pi = qq^\dagger\colon KP(n) \to h(n+1, K)$ which in the $K = H$ case is just the homogeneous coordinate version of the inclusion map $u \to P_u$ already mentioned. Π is a well-defined function from $KP(n)$ into E^m with $m = \binom{n+1}{2} d + n + 1$. Since $q^\dagger q = I = \mathrm{tr}(\Pi) = I$, the image of $KP(n)$ under Π therefore lies on the hyperplane $h_1(n+1, K) = \{Q = (q_{ij}) \in h(n+1)K) \mid \sum_0^n q_{ii} = 1\}$. If p and $q \in KP(n)$ then $\Pi(q) = \Pi(p)$, which is equivalent to $qq^\dagger = pp^\dagger$, hence $q = p\lambda$ with $\lambda \in K$ and $|\lambda| = 1$; it follows that Π is a substantial embedding of $KP(n)$ into R^N with $N = \binom{n+1}{2} d + n$. It is proved [80] that Π is not only a minimum, but also an equivariant and isometric embedding.

To be more explicit, firs consider the Mannoury embedding of $CP(n)$ coordinatized by $z = (z_0, z_1, \ldots, z_n)^T$ such that $z^\dagger z = 1$. Let $R^{(n+1)^2}$ be the Euclidean space with the coordinate system (p_r, p_{rs}, q_{rs}) with $r, s = 0, \ldots, n$ and $r < s$. Then the above $CP(n)$ embedding $CP(n) \to R^{(n+1)^2}$ is defined by the parametric equations

$$p_r = \frac{1}{2^{1/2}} | z_r |^2, \qquad p_{rs} = (z_r \bar{z}_s + z_s \bar{z}_r),$$
$$q_{rs} = i(z_r \bar{z}_s - \bar{z}_r z_s). \tag{4C.30}$$

So $CP(n)$ lies in the hyperplane of $R^{(m+)^2}$ defined by

$$p_0 + \cdots + p_n = 1. \tag{4C.31}$$

Hodge [81] shows that this embedding of $CP(n)$ into the hyperplane equation (4C.30) is the same as the map Π.

Continuing to the $HP(n)$ embedding [79]: here the homogeneous coordinates $h = (h_0, \ldots, h_n)^T$ are normalized so that $h^\dagger h = \sum_{r=1}^n \bar{h}_r h_r = 1$ defines the sphere S^{4n+3} in E^{4n+4}.

Just as the manifold of Mannoury–Study–Fubini is the real image of $CP(n)$ through the mapping (4C.3), one consider a real manifold $\mathscr{P}_{4n}$ in the real Euclidean space E^m, $m = \binom{n+1}{2} 4 + n + 1 = (2n+1)(n+1)$, which is the homeomorphic image of $HP(n)$. As in Eq. (3B.3), the manifold $\mathscr{P}_{4n}$ is defined by the parametric equations

$$p_t = \frac{1}{2^{1/2}} | h_t |^2, \qquad (p_{rs})_{(\nu)} = (h^r \bar{h}_s)_{(\nu)} \tag{4C.32}$$

with $r, s = 0, 1, \ldots, n$; $r < s$; $\nu = 1, 2, 3, 4$ labelling the four real components of the quaternion $h^r \bar{h}_s$. The coordinates $p = (p_r, (p_{rs})_\nu)$ are orthonormal coordinates in E^m. To each point $H \in HP(n)$ there corresponds in a one-to-one way a point in $p \in \mathscr{P}_{4n}$ modulo a real positive proportionality factor. In fact as $H \in HP(n)$ varies, p describes in E^m a real algebraic variety $\mathscr{P}_{4n}$ homeomorphic to $HP(n)$ [78].

Of the several immersion properties [82] of $\mathscr{P}_{4n}$ in E^m analogous to those found for the Mannoury manifold, the induced Riemannian metric is of importance to our work. It is given by

$$ds^2 = \sum_r (dp_r)^2 + \sum_{r<s,\nu} (dp_{rs\nu})^2 = \tfrac{1}{2} d\bar{y}_r{}^s \, dy_s{}^r \tag{4C.33}$$

where $y_s{}^r = (\bar{h}_r h_r)^{-1} h^r \bar{h}_s$. Hence

$$ds^2 = dh^\dagger\, dh - (h^\dagger\, dh)(dh^\dagger\, h). \tag{4C.34}$$

In terms of the inhomogeneous coordinates $u_i = h_i h_0^{-1}$, one has

$$ds^2 = \frac{du^\dagger\, du}{1 + u^\dagger u} - \frac{(du^\dagger\, u)(u^\dagger\, du)}{(1 + u^\dagger u)^2}. \tag{4C.35}$$

This canonical form of the Riemannian metric in $HP(n)$ can be alternatively [76] obtained by taking the distance function given in Eq. (4C.19) and letting $v = u + du$. Then

$$\cos^2 d(u, u + du) = 1 - ds^2 \frac{|\,1 + u^\dagger(u + du)|^2}{(1 + u^\dagger u)(1 + (u^\dagger + du^\dagger)(u + du)} \tag{4C.36}$$

or

$$ds^2 = 1 - \left(1 + \frac{du^\dagger\, du + u^\dagger\, du + du^\dagger\, u}{1 + u^\dagger u}\right)^{-1} \left(1 + \frac{u^\dagger\, du}{1 + u^\dagger u}\right)\left(1 + \frac{du^\dagger\, u}{1 + u^\dagger u}\right), \tag{4C.37}$$

which gives the Martinelli metric equation (4C.35).

If we now introduce the $(n \times n)$ hermitian matrix G (called the Mannoury–Fubini–Study metric)

$$G = G^\dagger = \frac{1}{1 + u^\dagger u}\,\frac{1}{I + uu^\dagger} = \frac{1}{1 + u^\dagger u}\left(I - \frac{uu^\dagger}{1 + u^\dagger u}\right), \tag{4C.38}$$

we can rewrite the line element as

$$ds^2 = du^\dagger\, G\, du = 1 - |\langle u \mid u + du\rangle|^2. \tag{4C.39}$$

The integrated distance between $|\, u\rangle$ and $|\, v\rangle$ is simply given by Eq. (4C.19), showing that the space $HP(n)$ is 2-point homogeneous as is known from the work of Borel [77].

Further let us consider the transition amplitude between two neighboring points with inhomogeneous coordinates u and $u + du$. It differs infinitesimally from unity. Hence we can define the quaternionic 1-form

$$\alpha = 1 - \langle u \mid u + du\rangle = 1 - \frac{1 + u^\dagger(u + du)}{((1 + u^\dagger u)(1 + (u^\dagger + du^\dagger)(u + du)))^{1/2}}, \tag{4C.40}$$

or

$$\alpha = 1 - \left(1 + \frac{u^\dagger\, du + du^\dagger\, u}{1 + u^\dagger u}\right)^{-1/2} \left(1 + \frac{u^\dagger\, du}{1 + u^\dagger u}\right) = \frac{1}{2}\,\frac{(du^\dagger)\, u - u^\dagger\, du}{(1 + u^\dagger u)}, \tag{4C.41}$$

which defines the connection (or the gauge field potential when u is a function of x). It is invariant under $Sp(n)$ but covariant under the phase group $Sp(1)$ up to an inhomogeneous term.

Since G is the hermitian metric in the quaternionic space, the components of $G\,dv$ may be regarded as the covariant components of the contravariant dv in the n-dimensional curved space. Hence the generalized scalar product between du and dv is $\mathrm{Sc}(du^{\dagger}\,G\,dv)$, giving d^2s for $du = dv$. This can be written in terms of transition probabilities as

$$\mathrm{Sc}(du^{\dagger}\,G\,dv) = \left|\left\langle u \,\middle|\, u + \frac{du - dv}{2}\right\rangle\right|^2 - \left|\left\langle u \,\middle|\, u + \frac{du + dv}{2}\right\rangle\right|^2. \tag{4C.42}$$

The related quaternionic amplitude which is $Sp(1)$ covariant is

$$\omega(u, du, dv) = du^{\dagger}G\,dv = \mathrm{Sc}(du^{\dagger}G\,dv) + \mathrm{Vec}(du^{\dagger}G\,dv). \tag{4C.43}$$

The vector part defines the generalized vector product of a contravariant du with a covariant dv and is associated with the three 2-forms of hermitian quaternionic geometry defining a surface element.

We now remark on the relation between almost quaternionic structures [78] (see Appendix III) and the existence of quaternionic "scalars" in quaternionic Hilbert space. $|\,\mu\rangle$, $|\,\nu\rangle$ be two normalized kets associated respectively with the inhomogeneous coordinate vectors $|\,m\rangle$ and $|\,n\rangle$. Their scalar product is

$$\langle m \mid n\rangle \equiv \mu^{\dagger}\nu \equiv (\mu, \nu). \tag{4C.44}$$

Since the transition probability $\Pi_{mn} = |\langle m \mid n\rangle|^2$ is invariant under $Sp(1) \times Sp(n)$, we get

$$|(\mu, \nu)l\,|^2 = |\,\bar{k}(\mu, \nu)|^2, \tag{4C.45}$$

provided that

$$|\,k\,| = |\,l\,| = 1. \tag{4C.46}$$

On the other hand

$$(\mu, \nu) = (\mu, \nu l), \qquad \bar{k}(\mu, \nu) = (\mu k, \nu). \tag{4C.47}$$

Thus

$$(\mu e_i\,, \nu) = -e_i(\mu, \nu), \qquad (\mu, \nu e_i) = (\mu, \nu)\,e_i \tag{4C.48}$$

and

$$|(\mu e_i\,, \nu)|^2 = |(\mu, \nu e_i)|^2. \tag{4C.49}$$

These are precisely the conditions [69] for the existence of almost quaternionic structures e_i ($i = 1, 2, 3$) in the $4n + 3$ real dimensional space spanned by the real parameters of the normalized ket ν.

Returning to more geometrical considerations, we note that the volume element in $HP(n)$ is given by

$$du = (\text{Det } G)^{1/2}\, d^n u, \qquad (d^n u = du_1 \cdots du_n). \tag{4C.50}$$

From the explicit form of G we find

$$\text{Det } G = (1 + u^\dagger u)^{-(n+1)} = \rho(u)^{-n-1} \tag{4C.51}$$

so that $du = \rho^{-1/2(n+1)} d^n u$. We also have

$$\text{Tr } G = \frac{n}{1 + u^\dagger u} - \frac{u^\dagger u}{(1 + u^\dagger u)^2} = \frac{n-1}{\rho} + \frac{1}{\rho^2}\,. \tag{4C.52}$$

The elements of G, given by

$$G_{\alpha\beta} = \rho^{-1}\delta_{\alpha\beta} - \rho^{-2} u_\alpha \bar{u}_\beta\,, \tag{4C.53}$$

can in turn be obtained from the single function

$$\rho = 1 + u^\dagger u = 1 + \sum_{\alpha=1}^{n} \sum_{\mu=1}^{r} u_\mu{}^\alpha u_\mu{}^\alpha. \tag{4C.54}$$

Introducing the quaternionic Kähler potential

$$f = \ln \rho = \ln(1 + u^\dagger u) = -\frac{1}{n+1} \ln \text{Det } G, \tag{4C.55}$$

and defining the differential operators

$$D_\alpha = e_\mu \frac{\partial}{\partial u_\mu{}^\alpha}\,, \qquad \bar{D}_\beta = \bar{e}_\nu \frac{\partial}{\partial u_\nu{}^\beta}\,,$$

we find that

$$G_{\alpha\beta} = \frac{1}{8}(D_\alpha \bar{D}_\beta f - D_\alpha f\, \bar{D}_\beta f) = -\frac{1}{8}\tau^{-1} D_\alpha \bar{D}_\beta \tau \qquad \left(\tau \equiv \frac{4}{1 + u^\dagger u}\right), \tag{4C.56}$$

which is the quaternionic counterpart of Eq. (3B.15) for the complex Kähler metric of $CP(n)$. The theorem (4C.56) is in fact a necessary and sufficient condition for $HP(n)$ to be a quaternionic Kähler manifold.

Noting that $\text{Tr } G = G_{\alpha\alpha} = \frac{1}{8}(D_\alpha \bar{D}_\alpha f - D_\alpha f\, \bar{D}_\alpha f) = -\frac{1}{8}\tau^{-1}\Box\tau$, and using this result, we derive the generalized Liouville equation satisfied by the Kähler potential f for $HP(n)$, i.e.

$$\tfrac{1}{8}(D_\alpha \bar{D}_\alpha f - D_\alpha f\, \bar{D}_\alpha f) = (n-1)\, e^{-f} + e^{-2f} \tag{4C.57}$$

or

$$\square_\alpha \tau = 2((1 - n)\, \tau^2 - \tfrac{1}{4}\tau^3), \tag{4C.58}$$

which is solved by $\tau = 4(1 + u^\dagger u)^{-1}$. τ is remarkable in that it is the generalization of the σ-field in the usual, say $O(4)$, σ-model and Eq. (4C.58) is the field equation for the Goldstone lagrangian with the difference that the base space is the field space u and the coupling coefficients are fixed.

Indeed Eq. (4C.57) is but the analog of the Liouville equation

$$\tfrac{1}{4}\Delta_W f = e^{-2f}, \tag{4C.59}$$

where $f = \ln(1 + w\bar{w})$, and $\lambda = 2/(1 + w\bar{w})$, $|w| < 1$ is called the metric of the 2-dimensional unit disc and where the ratio $-(\Delta \ln \lambda)/\lambda^2$, obtained from a rewriting of Eq. (4C.59), is the constant Gaussian curvature of this metric. In the same way, Eq. (4C.57) testifies to the constant holomorphic sectional curvature of $HP(n)$, a notion generalizing to hypercomplex manifolds the idea of constant curvature familiar in two dimensions [51].

A more standard treatment of quaternionic Kählerian structure is the work of Martinelli [78] from the standpoint of a G-structure [73] on a $4n$-dimensional real manifold. We shall keep to the essentials, leaving other points to the available literature listed in Appendix III.

A differential manifold M_{4n} of real dimension $4n$ is endowed with an almost quaternionic or almost hermitian quaternionic structure if the structural group of its tangent bundle is reduced from $GL(4n, R)$ to $GL(n, H) \times GL(1, H)$ hence to $Sp(n) \times Sp(1)$. Specifically an almost quaternionic structure is given if there exist on M_{4n} local linear quaternionic valued, differential forms θ^α, such that the transition between two overlapping coordinate patches is related by

$$\theta^{\beta'} = A_\alpha{}^\beta(h)\, \theta^\alpha q(h), \qquad h \in M_{4n} \tag{4C.60}$$

where $A \in GL(n, H), Sp(n)$ and $q \in GL(1, H), Sp(1)$. The almost quaternionic structure is said to be *integrable* if θ^α and $\theta^{\beta'}$ are differentials of local inhomogeneous coordinates $u^\alpha(h)$, $u^\beta(h)'$ on M_{4n}. Thus we consider the Martinelli manifold $\mathscr{P}_{4n}$ with its coordinates $u_\alpha = h_\alpha h_0^{-1}$ ($\alpha = 1,\ldots, n$) in the patch P_0. The du_α equip $\mathscr{P}_{4n}$ with a quaternionic structure in that two coordinate systems u_α and u'_α defined on the same open set lead to a $GL(n, H) \times GL(1, H)$ transformation relating their respective differentials: since $(h')^s = A_r{}^s h^r$, $A \in GL(n + 1, H)$ and $h_0 \neq 0$, $h'_0 \neq 0$, we have

$$(du')^\beta = \tilde{A}_\alpha{}^\beta\, du^\alpha\, \tilde{q}, \tag{4C.61}$$

where $\tilde{A}_\alpha{}^\beta = -u'^\beta A_\alpha{}^0 + A_\alpha{}^\beta$ and $\tilde{q} = A_\gamma{}^0 u_\gamma + A_0{}^0$. To show that the quaternionic Study–Fubini metric of Eq. (4C.35) defines in fact an almost hermitian quaternionic structure, we need to cast it in a canonical form. Let $(e_0,\ldots, e_n)$ be a canonical symplectic orthonormal basis dual to the canonical basis of $dh = (dh^0, \ldots, dh^n)^T$ of the right vector space of the H-linear differential forms $a_r\, dh^r$, $a_r \in H$. Under a $Sp(n+1)$ action we have $e_s' = e_r \Lambda_s{}^r$, which in the dual basis corresponds to

$$dh^r = \Lambda_s{}^r \theta^s \tag{4C.62}$$

such that θ^s are H-valued Maurer–Cartan differential forms and $\Lambda_s{}^r$ are functions of the point h of $\mathscr{P}_{4n}$, chosen to obey

$$\bar{\Lambda}_s{}^r \Lambda_t{}^s = \delta_t{}^r \qquad (\Lambda_s{}^r = \Lambda_t{}^s;\ r, s, t = 0, 1, \ldots, n) \tag{4C.63}$$

since the transformation equation (4C.61) is symplectic. Moreover we have

$$\Lambda_0{}^r = h^r, \qquad \Lambda_r{}^0 = \bar{h}_r \tag{4C.64}$$

for consistency with Eq. (4C.62), together with the constraint $\sum_0^n \bar{h}_r h_r = 1$. The coframes θ^s are given as $\theta^s = \bar{\Lambda}_r{}^s\, dh^r$, and $d\bar{h}_r = \bar{\theta}_s \Lambda_r{}^s$, $\bar{\theta}_0 = d\bar{h}_r\, h^r$. Then $\theta^0 = \bar{h}_r\, dh^r$ is equivalent to Eq. (4C.64). Using the latter and changing to the basis given by Eq. (4C.63), we get

$$ds^2 = \bar{\theta}_r \bar{\Lambda}_t{}^r \Lambda_s{}^t \theta^s - \theta^0 \bar{\theta}_0 = \bar{\theta}_r \theta^r - \bar{\theta}_0 \theta^0 = \bar{\theta}_\alpha \theta^\alpha \tag{4C.65}$$

which manifestly shows that the metric on $\mathscr{P}_{4n} \approx HP(n)$ is H-hermitian. It can be readily shown that the du^α go over into the coframe forms by a $GL(n, H) \times GL(1, H)$ transformation. Two forms θ^α and $\theta^{\alpha\prime}$ in the same open set P_0 are also so connected, hence they are linked by a $Sp(n) \times Sp(1)$ transformation leaving the hermitian scalar product $\sum_0^n \bar{h}_r h_r = 1$ invariant. We have the useful relations

$$dh^\alpha = u^\alpha\, dh^0 + du^\alpha\, h^0, \tag{4C.66}$$

$$du^\alpha = (\Lambda_\beta{}^\alpha - u^\alpha \Lambda_\beta{}^0)\, \theta^\beta (h^0)^{-1}, \tag{4C.67}$$

hence

$$\theta^\beta = \bar{\Lambda}_\alpha{}^\beta\, du^\alpha\, h^0, \tag{4C.68}$$

from which, by direct insertion into Eq. (4C.34) and use of Eqs. (4C.63) and (4C.64), one recovers the form (4C.35) for ds^2. In analogy to the $CP(n)$ case this metric is associated to the fundamental H-Kähler form given by

$$\Omega_{(2)} = \bar{\theta}_\alpha \wedge \theta^\alpha \tag{4C.69}$$

or

$$\Omega_{(2)} = dh^\dagger \wedge dh - (h^\dagger\, dh) \wedge (dh^\dagger\, h). \tag{4C.70}$$

since $\overline{\psi \wedge \phi} = (-1)^{pq}\, \bar{\phi} \wedge \bar{\psi}$ for any two H-valued forms of degree p and q, $\Omega_{(2)}$ is purely vectorial, $\bar{\Omega}_{(2)} + \Omega_{(2)} = 0$. In terms of u

$$\Omega_{(2)} = \frac{du^\dagger \wedge du}{(1 + u^\dagger u)} - \frac{u^\dagger\, du \wedge du^\dagger\, u}{(1 + u^\dagger u)^2}\,. \tag{4C.71}$$

From $\sum_{r=0}^{n} \bar{h}^r h_r = 1$, we get $\bar{h}^r\, dh_r + d\bar{h}_r\, h^r = 0$ or $\theta^0 + \bar{\theta}_0 = 0$, due to Eq. (4C.64). Exterior differentiation of $\Omega_{(2)}$ yields

$$d\Omega_{(2)} = -d\bar{h}_r \wedge dh^r \wedge d\bar{h}_s\, h^s - \bar{h}_r\, dh^r \wedge d\bar{h}_s \wedge dh^s, \tag{4C.72}$$

which becomes

$$d\Omega_{(2)} = \Omega_{(2)} \wedge \theta^0 - \theta^0 \wedge \Omega_{(2)}\,. \tag{4C.73}$$

Here the difference from the complex hermitian structure appears. In the latter case, θ^0 and $\Omega_{(2)}$ commute, so $\Omega_{(2)}$ is closed, $d\Omega_{(2)} = 0$, which defines a complex Kähler structure. Here the non-commutativity of the quaternions leads instead to a covariantly closed 2-form

$$d^{\mathrm{cov}}\Omega_{(2)} \equiv d\Omega_{(2)} - [\theta^0, \Omega_{(2)}] = 0, \tag{4C.74}$$

where $[\theta^0, \Omega_{(2)}] \equiv \theta^0 \wedge \Omega_{(2)} - \Omega_{(2)} \wedge \theta^0$. To have ordinary closure, we must proceed to 4-forms [69, 72, 78], namely

$$\Omega_{(4)} = \Omega_{(2)} \wedge \Omega_{(2)}\,. \tag{4C.75}$$

Then

$$\begin{aligned} d\Omega_{(4)} &= \Omega_{(4)} \wedge \theta^0 - \theta^0 \wedge \Omega_{(4)} \\ &= 0, \end{aligned} \tag{4C.76}$$

since $\bar{\Omega}_{(4)} = \Omega_{(4)}$ is pure real as $\bar{\Omega}_{(2)} = -\Omega_{(2)}$.

It follows that the existence of a closed curvature 4-form $\Omega_{(4)}$ on $HP(n)$ makes it a quaternionic Kähler manifold. Note that geometrically $d\Omega_n^{(4)}$ is a volume element of the space $HP(n)$. Equation (4C.76) may also be taken as a statement of the quaternionic Newlander–Nirenberg theorem [83] on the integrability condition for an almost quaternionic structure on $HP(n)$. As in the case of the complex line bundle, the $U(1)$ bundle over $CP(n)$, here we are dealing with a symplectic line bundle, a $SU(2)$ bundle over $HP(n)$ with the connection θ^0 in Eq. (4C.74), or α in Eq. (4C.41), corresponding to θ_{00}, Eq. (3B.8), in the complex case. Since $\Omega_{(4)}$ is closed, of degree 4 and of maximum rank, by de Rham's theorem [72], $\Omega_{(4)}^i$ is harmonic and is a generator of the ith de Rham cohomology group $H^{4i}(HP(n), R)$ whose dimension is the Betti number $B^{4i} \neq 0$ for $i = 0, 1, 2, \ldots, n$ [72]. As such $\Omega_{(4)}$ corresponds to the second universal Chern class of the holomorphic symplectic line bundle $*H^{n+1} \rightarrow$

$HP(n)$. Finally, let us mention that as $CP(n)$, $HP(n)$ has been shown [70] to have constant holomorphic sectional curvature, it is then a quaternionic Einstein–Kähler space, a generalization of the space of constant curvature $S^4 \approx HP(1)$.

When manifolds are endowed with certain structures, it is natural [84] to study their mappings such as holomorphic mappings of complex manifolds, e.g. $CP(n)$ instantons in the physicist's phraseology. In the next subsection, we will see how natural 4-dimensional spacetime $S^4 \approx HP(1)$ is as a base manifold on which to pull back the above geometric and topological properties of $HP(n)$.

IV.D. *Quaternionic Kählerian Geometry of Restricted Type and Self-Dual Instanton Solutions*

In connection with Euclidean 4-dimensional σ-models and Yang–Mills theory we return to the case in which the quaternionic inhomogeneous coordinates of $HP(n)$, u_α, are functions of the space time quaternion x.

Thus we write

$$u = u(x) \tag{4D.1}$$

and we have for two points $x + dx$ and $x + \delta x$

$$du = (\partial_\mu u)\, dx_\mu\,, \qquad dv = (\partial_\nu u)\, \delta x_\nu\,. \tag{4D.2}$$

Thus the pull-back form of the line element in quaternionic Hilbert space becomes

$$d\sigma^2 = du^\dagger(x) G\, du(x) = (\partial_\mu u^\dagger)\, G[u(x)]\, \partial_\nu u\, dx_\mu\, dx_\nu\,. \tag{4D.2a}$$

Hence we have a Riemannian metric in spacetime with the metric

$$g_{\mu\nu} = \tfrac{1}{2}(\partial_\mu u^\dagger) G\, \partial_\nu u + \tfrac{1}{2}(\partial_\nu u^\dagger) G\, \partial_\mu u. \tag{4D.3}$$

By using the $Sp(1) \times Sp(n)$ covariant column

$$\phi_\mu = G^{1/2}[u(x)]\, \partial_\nu u \tag{4D.4}$$

which arises from the gauge covariant part $[\Omega_\mu, \eta]$ of Ω_μ in the Section IV.B (see Eq. (4B.25)) we can write

$$d\sigma^2 = \phi_\mu{}^\dagger \phi_\nu\, dx_\mu\, dx_\nu = \mathrm{Sc}(\phi_\mu{}^\dagger \phi_\nu)\, dx_\mu\, dx_\nu\,, \tag{4D.5}$$

so that

$$g_{\mu\nu} = \tfrac{1}{2}(\phi_\mu{}^\dagger \phi_\nu + \phi_\nu{}^\dagger \phi_\mu). \tag{4D.6}$$

If we now consider the quaternionic combination of the generalized scalar product and the generalized vector product (surface element) between du and dv, we find

$$\begin{aligned} du^{\dagger}(x)G\,dv(x) &= (\partial_{\mu}u^{\dagger})\,G(\partial_{\nu}u)\,dx_{\mu}\,\delta x_{\nu} \\ &= g_{\mu\nu}\,dx_{\mu}\,\delta x_{\nu} + f_{\mu\nu}\,dx_{\mu}\,\delta x_{\nu}\,, \end{aligned} \tag{4D.7}$$

where

$$f_{\mu\nu} = \tfrac{1}{2}(\phi_{\mu}{}^{\dagger}\phi_{\nu} - \phi_{\nu}{}^{\dagger}\phi_{\mu}) = \mathrm{Vec}(\phi_{\mu}{}^{\dagger}\phi_{\nu}),$$

and the vectorial part of $du^{\dagger}G\,dv$ is the pull-back 2-form of $\Omega_{(2)}$ in Eq. (4C.69)

$$f = \mathrm{Vec}(du^{\dagger}G\,dv) = \tfrac{1}{2}f_{\mu\nu}\,dx_{\mu}\wedge dx_{\nu} \tag{4D.8}$$

where we have used the symbolic notation

$$dx_{\mu}\wedge dx_{\nu} = (dx_{\mu}\,\delta x_{\nu} - dx_{\nu}\,\delta x_{\mu}). \tag{4D.9}$$

Thus the quaternionic element $du^{\dagger}G\,dv$ associates naturally a Riemannian metric with a quaternionic 2-form. On the other hand the transition amplitude between points x and $x + dx$ gives the 1-form

$$\alpha = 1 - \langle u \mid u + du\rangle = -\frac{1}{2}\mathrm{Vec}\left(\frac{u^{\dagger}\,\partial_{\mu}u}{1 + u^{\dagger}u}\right)dx_{\mu} = -A_{\mu}\,dx_{\mu}\,, \tag{4D.10}$$

so that the gauge potential is also expressed in terms of u as

$$A_{\mu} = -\frac{1}{2}\,\frac{(\partial_{\mu}u^{\dagger})\,u - u^{\dagger}\,\partial_{\mu}u}{1 + u^{\dagger}u}\,. \tag{4D.11}$$

Because of the identity equation (4B.11) for Ω_{μ}, we have

$$f_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + [A_{\mu}\,, A_{\nu}], \tag{4D.12}$$

showing that $f_{\mu\nu}$ is the $SU(2)$ gauge field (the curvature field) and satisfies the Bianchi identity

$$\partial_{\mu}\tilde{f}_{\mu\nu} + [A_{\mu}\,, \tilde{f}_{\mu\nu}] = 0. \tag{4D.13}$$

Note that in terms of F_{μ} introduced in Eq. (2.38) we have

$$K_{\mu\nu} = \phi_{\mu}{}^{\dagger}\phi_{\nu} = g_{\mu\nu} + f_{\mu\nu} = \frac{E+\eta}{2}F_{\mu}F_{\nu} = \frac{E+\eta}{2}F_{\mu}F_{\nu}\frac{E+\eta}{2}\,. \tag{4D.14}$$

Here $K_{\mu\nu}$ is the hermitian metric induced by G in the 4-dimensional space x. It is called the pull-back metric. The 2-form associated with $K_{\mu\nu}$ is f given by (4D.8).

Now, because of the Bianchi identity f is covariantly closed, so that in the notation of exterior calculus we can rewrite Eq. (4D.12) as

$$df + \alpha \wedge f = 0, \tag{4D.15}$$

where α is given by (4D.10). The identity follows from Eq. (4D.12), which takes the form

$$f = d\alpha + \alpha \wedge \alpha. \tag{4D.16}$$

A quaternionic hermitian metric with an associated 2-form that is covariantly closed is quaternionic Kählerian. This proves that the induced metric $K_{\mu\nu}$ is Kählerian.

The quaternionic form (4D.7) is $Sp(n)$ invariant, but $Sp(1)$ covariant, with $g_{\mu\nu}$ being invariant and $f_{\mu\nu}$ transforming like the adjoint representation of $Sp(1)$. In order to express the Kählerian property in terms of ordinary closed forms we must introduce $Sp(1) \times Sp(n)$ invariant forms. Let us consider besides f the dual form $\tilde{f}$ defined by

$$\tilde{f} = \tfrac{1}{2}\tilde{f}_{\mu\nu}\, dx_\mu \wedge dx_\nu = \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta} f_{\alpha\beta}\, \delta' x_\mu\, \delta'' x_\nu\,. \tag{4D.17}$$

Both f and $\tilde{f}$ are purely vectorial quaternions with which we can construct the following $Sp(1) \times Sp(n)$ invariant 4-forms,

$$I_1 = \tfrac{1}{2} f_{\rho\sigma} f_{\rho\sigma} \epsilon_{\mu\nu\alpha\beta} dx_\mu \delta x_\nu \delta' x_\alpha \delta'' x_\beta = \tfrac{1}{2} f_{\mu\nu} f_{\mu\nu} d^4x, \tag{4D.18}$$

$$I_2 = \epsilon_{\kappa\lambda\rho\sigma} f_{\kappa\lambda} f_{\rho\sigma} \epsilon_{\mu\nu\alpha\beta} dx_\mu \delta x_\nu \delta' x_\alpha \delta'' x_\beta = \tfrac{1}{2} f_{\mu\nu} \tilde{f}_{\mu\nu} d^4x \tag{4D.19}$$

In compact notation I_2 can be rewritten as

$$I_2 = f \wedge f = d\kappa, \tag{4D.20}$$

where κ is the 3-form associated with the Pontryagin current κ_μ

$$\kappa = \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}\kappa_\mu\, \delta x_\nu\, \delta' x_\alpha\, \delta'' x_\beta = \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}\kappa_\mu\, dx_\nu \wedge dx_\alpha \wedge dx_\beta\,. \tag{4D.21}$$

While I_2 is a topological invariant (invariant under the diffeomorphism group of general relativity) both I_1 and I_2 are invariant under conformal transformations as shown in Section IV.A. From Eq. (4D.21) it follows that

$$dI_2 = d^2\kappa = 0, \tag{4D.22}$$

and the 4-form I_2 is closed if f is covariantly closed. I_1 is recognized as the Yang–Mills Lagrangian. In a way similar to the combination of the Riemannian metric $g_{\mu\nu}$ and the 2-form f, the Lagrangian I_1 and the 4-form I_2 can be combined by resorting to the quaternion algebra.

By using the quaternion units j_μ introduced in Section IV.A. we have seen that the use of the self-dual combination ϕ and the antiself dual combination ϕ' defined by

$$\phi = \tfrac{1}{2} j_{\mu\nu} f_{\mu\nu} = \tfrac{1}{2} \breve{j}_\mu j_\nu f_{\mu\nu}\,, \qquad \phi' = \tfrac{1}{2} j'_{\mu\nu} f_{\mu\nu} = \tfrac{1}{2} j_\mu \breve{j}_\nu f_{\mu\nu} \tag{4D.23}$$

we obtain

$$-\tfrac{1}{2}\{\phi, \phi\}\, d^4x = I_1 + I_2\,, \qquad -\tfrac{1}{2}\{\phi', \phi'\}\, d^4x = I_1 - I_2\,, \tag{4D.24}$$

where we have used the identities (4A.6) and (4A.7). The Kählerian property is now reflected by the closure of the topologically invariant part of the two conformally invariant expressions in Eq. (4D.24).

In order to make contact with the theory of Yang–Mills instantons we turn to the consideration of a restricted Kählerian geometry where the associated 2-form is not only covariantly closed but also self-dual or antiself-dual and the Riemannian metric is conformally flat.

Such restricted Kählerian geometries with metric $K_{\mu\nu}$ can be characterized both algebraically and geometrically. Let us split $g_{\mu\nu}$ into a conformally flat part and a traceless part while we split $f_{\mu\nu}$ into self-dual and antiself-dual parts

$$g_{\mu\nu} = \tfrac{1}{4} g_{\lambda\lambda}\delta_{\mu\nu} + \theta_{\mu\nu} \qquad (\theta_{\lambda\lambda} = 0), \tag{4D.25}$$

$$f_{\mu\nu} = \phi_{\mu\nu} + \phi'_{\mu\nu}\,, \tag{4D.26}$$

where

$$\theta_{\mu\nu} = g_{\mu\nu} - \tfrac{1}{4}\delta_{\mu\nu} g_{\lambda\lambda} = \tfrac{1}{2}(\phi_\mu{}^\dagger\phi_\nu + \phi_\nu{}^\dagger\phi_\mu) - \tfrac{1}{4}\phi_\lambda{}^\dagger\phi_\lambda\delta_{\mu\nu}\,, \tag{4D.27}$$

$$\phi_{\mu\nu} = \mathbf{e}\cdot\boldsymbol{\phi}_{\mu\nu} = \tfrac{1}{2}(f_{\mu\nu} + \tilde{f}_{\mu\nu}) = \tfrac{1}{2}(\phi_\mu{}^\dagger\phi_\nu - \phi_\nu{}^\dagger\phi_\mu) + \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}\phi_\alpha{}^\dagger\phi_\beta\,, \tag{4D.28}$$

$$\phi'_{\mu\nu} = \mathbf{e}\cdot\boldsymbol{\phi}'_{\mu\nu} = \tfrac{1}{2}(f_{\mu\nu} - \tilde{f}_{\mu\nu}) = \tfrac{1}{2}(\phi_\mu{}^\dagger\phi_\nu - \phi_\nu{}^\dagger\phi_\mu) - \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}\phi_\alpha{}^\dagger\phi_\beta\,. \tag{4D.29}$$

Using two commuting sets of real 4×4 matrix representations for j_m and e_i and denoting the 4×4 unit matrix by $I^{(4)}$ we construct the following 4×4 real matrices:

$$\breve{j}_\mu j_\nu K_{\mu\nu} = (\delta_{\mu\nu} + j_{\mu\nu})(g_{\mu\nu} + f_{\mu\nu}) = g_{\mu\mu} I^{(4)} + j_{\mu\nu}\phi_{\mu\nu}\,, \tag{4D.30}$$

$$j_\mu \breve{j}_\nu K_{\mu\nu} = (\delta_{\mu\nu} + j'_{\mu\nu})(g_{\mu\nu} + f_{\mu\nu}) = g_{\mu\mu} I^{(4)} + j'_{\mu\nu}\phi'_{\mu\nu}\,. \tag{4D.31}$$

Note that $\breve{j}_\mu j_\nu$ projects only the $K_{\lambda\lambda}$ and $\phi_{\mu\nu}$, while $j_\mu \breve{j}_\nu$ projects the $K_{\lambda\lambda}$ and $\phi'_{\mu\nu}$ parts of $K_{\mu\nu}$. Hence, we consider two kinds of restricted Kähler types

(a) $\theta_{\mu\nu} = 0$, $\phi'_{\mu\nu} = 0$ (self-dual type),

$$(\phi_\mu{}^\dagger\phi_\nu + \phi_\nu{}^\dagger\phi_\mu) - \tfrac{1}{2}\phi_\lambda{}^\dagger\phi_\lambda\delta_{\mu\nu} = 0, \tag{4D.32}$$

$$\phi_\mu{}^\dagger\phi_\nu - \phi_\nu{}^\dagger\phi_\mu - \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}\phi_\alpha{}^\dagger\phi_\beta = 0. \tag{4D.33}$$

(b) $\theta_{\mu\nu} = 0$, $\phi_{\mu\nu} = 0$ (antiself-dual type),

$$\phi_\mu{}^\dagger\phi_\nu - \phi_\nu{}^\dagger\phi_\mu + \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}\phi_\alpha{}^\dagger\phi_\beta = 0. \tag{4D.34}$$

These conditions generalize the quadratic Cauchy–Riemann relations in the complex case given in Section IIIC (Eqs. (3C.17), (3C.18)) and are identical with the quaternionic differential equations (4B.43).

It is possible to give an analytic and geometric interpretation of these conditions. The function $u(x)$ may be regarded as the parametric equation of a curve defined by a quaternionic parameter x. the differential

$$du = (\partial_\mu u)\, dx^\mu \tag{4D.35}$$

will define left and right quaternionic derivatives of u if it has the form

$$du = Y(x)\, dx\, w(x) \qquad \text{or} \qquad du = Y'\, d\bar{x}\, w', \tag{4D.36}$$

where Y is an $n \times n$ matrix and $w(x)$ is a quaternionic column with n components. Equation (4D.36) corresponds to the Frenet formula [61, 85] defining the tangent vector for an ordinary curve. Similarly the combination of the generalized scalar and vector products of the tangential vectors du and dv, namely $du^\dagger G\, dv$, will be a homogeneous form in the similar combination of the parametric increments dx and δx with Euclidean metric. Thus, in addition to (4D.36) we require

$$du^\dagger G\, dv = l^\dagger\, d\bar{x}\, \delta x\, r \qquad \text{or} \qquad du^\dagger G\, dv = l'^\dagger\, dx\, \delta\bar{x}\, r' \tag{4D.37}$$

where l, l' and r, r' are quaternionic columns. This generalizes the notion of the square of the intrinsic length of a curve to a quaternionic curve. From Eqs. (4D.36) and (4D.37) we find respectively

$$\partial_\mu u = G^{-1/2}\phi_\mu = Y e_\mu w, \tag{4D.38}$$

$$K_{\mu\nu} = (\partial_\mu u^\dagger) G\, \partial_\nu u = \phi_\mu{}^\dagger \phi_\nu = l^\dagger \bar{e}_\mu \bar{e}_\nu r \qquad \text{or} \qquad l'^\dagger e_\mu \bar{e}_\nu r'. \tag{4D.39}$$

The last equation gives

$$K_{\mu\nu} + K_{\nu\mu} = 2 l^\dagger r \delta_{\mu\nu} = \tfrac{1}{2} K_{\lambda\lambda} \delta_{\mu\nu}, \tag{4D.40}$$

which is equivalent to Eq. (4D.32) and

$$K_{\mu\nu} - K_{\nu\mu} = l^\dagger e_{\mu\nu} r \qquad (\text{or } l'^\dagger e'_{\mu\nu} r) \tag{4D.41}$$

gives Eq. (4D.33) or Eq. (4D.34) since $e_{\mu\nu}$ is self-dual and $e'_{\mu\nu}$ is antiself-dual. Hence the existence of a generalized quaternionic derivative for curves leads to the generalized quadratic Cauchy–Riemann relations. We now turn to Eq. (4D.38). The solution of Eq. (4D.36) or, equivalently, Eq. (4D.38) has the fractional linear form

$$u(x) = \mu + K \frac{1}{B - x} \lambda \qquad \text{or} \qquad u' = \mu' + K' \frac{1}{B' - \bar{x}} \lambda', \tag{4D.42}$$

where μ and λ are quaternionic columns and K and B are $n \times n$ quaternionic matrices. An alternative form for $u(x)$ is

$$u = u(x) = (A + xC)(M - xN)^{-1} \lambda \qquad \text{or} \qquad u' = u'(\bar{x}), \tag{4D.43}$$

where A, B, C, D are quaternionic $n \times n$ matrices. Quaternionic curves which admit a first differential form (4D.36) and a quadratic differential form (4D.37) will be called "Atiyah holomorphic curves." For such curves we find in the self-dual case

$$\partial_\mu u = K \frac{1}{B - x} e_\mu \frac{1}{B - x} \lambda, \tag{4D.44}$$

so that

$$du = Y\, dx\, w, \qquad Y = K \frac{1}{B - x}, \qquad w = K^{-1}(u(x) - \mu). \tag{4D.45}$$

Condition (4D.39) gives

$$du^\dagger G\, dv = w^\dagger\, d\bar{x}\, Y^\dagger G Y\, \delta x\, w = l^\dagger\, d\bar{x}\, \delta x\, r. \tag{4D.46}$$

This requires for arbitrary δx

$$[Y^\dagger G Y, \delta x] = 0, \tag{4D.47}$$

or

$$\operatorname{Vec}\left(Y^\dagger \frac{1}{I + uu^\dagger} Y \right) = 0. \tag{4D.48}$$

Note that in a conformal transformation dx becomes

$$dx' = -b(x + c)^{-1}\, dx (x + c)^{-1}\, d \tag{4D.49}$$

according to Eq. (4A.20). Hence we have

$$d\bar{x}'\, \delta x' = \bar{d}(\bar{x} + \bar{c})^{-1}\, d\bar{x}\, \delta x (x + c)^{-1}\, d \mid b \mid^2 \mid x + c \mid^{-2}. \tag{4D.50}$$

This shows that conformal transformations are Atiyah holomorphic. For the generalized vectorial function (4D.42), Atiyah holomorphy holds only if condition (4D.48) is satisfied. Since we have

$$I + uu^\dagger = I + (\mu + Y\lambda)(\lambda^\dagger Y^\dagger + \mu^\dagger) \tag{4D.51}$$

we obtain the condition

$$\operatorname{Vec}\{ Y^{-1} Y^{\dagger -1} + (Y^{-1}\mu + \lambda)(\lambda^\dagger + \mu^\dagger Y^{\dagger -1}) \} = 0. \tag{4D.52}$$

Since these conditions must hold for all x, they must hold separately for the constant term and the terms quadratic and linear in x. Consequently we find the following three conditions. Let

$$K^{-1}\mu = \nu, \qquad M = M^{\dagger} = K^{-1}(I + \mu\mu^{\dagger})\, K^{\dagger-1} = K^{-1}K^{\dagger-1} + \nu\nu^{\dagger}. \quad (4D.53)$$

Then we must have

$$\text{(a)} \quad \operatorname{Vec} M = \operatorname{Vec}(K^{-1}K^{\dagger-1} + \nu\nu^{\dagger}) = 0, \quad (4D.54)$$

$$\text{(b)} \quad \operatorname{Vec}(BMB^{\dagger} + \lambda\lambda^{\dagger} + B\nu\lambda^{\dagger} + \lambda\nu^{\dagger}B^{\dagger}) = 0, \quad (4D.55)$$

$$\text{(c)} \quad Q = Q^{T} \text{ where } Q = BM + \lambda\nu^{\dagger} \text{ (Q: quaternionic symmetric matrix).} \quad (4D.56)$$

As we shall see presently these conditions are conformally invariant. Special cases occur already in the literature. The case $\mu = 0$ has been considered by Christ *et al.* [22], and Atiyah *et al.* [22] have studied the case $\mu = 0$, $K = I$. In this case $M = I$ so that condition (a) is automatically satisfied. Conditions (b) and (c) take the form

$$\text{(b}'\text{)} \quad \operatorname{Vec}(BB^{\dagger} + \lambda\lambda^{\dagger}) = 0, \quad (4D.57)$$

$$\text{(c}'\text{)} \quad B = B^{T}. \quad (4D.58)$$

The general conditions are also automatically satisfied.

Now consider the case in which

$$\operatorname{Vec}\lambda = 0, \qquad \mu = \lambda, \qquad B = b = \text{diagonal matrix}, \qquad K = aI - b \qquad (a = \text{quaternion}). \quad (4D.59)$$

This gives

$$u(x) = (a - x)\, \frac{1}{b - x}\, \lambda, \quad (4D.60)$$

which is the $5n + 4$ parameter solution of Jackiw *et al.* [33].

Finally when $\mu = 0$, $K = I$, $\operatorname{Vec}\lambda = 0$ and $B = b$ is diagonal we recover the $5n$ parameter solution of t'Hooft [32].

The special Atiyah form [22] has $8n - 3$ parameters in accordance with the general index theorem. This is also true for the general solution which has more variables, more constraints and more symmetries. Unfortunately no explicit solution is known for $n \geqslant 3$.

Under a conformal transformation

$$x' = (\alpha x + \beta)(\gamma x + \delta)^{-1}, \quad (4D.61)$$

we have

$$(Ax + B)(Cx + D)^{-1}\,\lambda = (A'x' + B')(C'x' + D')^{-1}\,\lambda', \quad (4D.62)$$

so that $u(x)$ keeps its general form with new value of the parameters. Then the constraint equations (4D.53)–(4D.55) keep their forms with the transformed parameters. This is also true in the Jackiw–Rebbi case (4D.60). However, it is not true in the Atiyah special case. Under a special conformal transformation, for instance, both conditions (4D.57) and (4D.58) are violated. Consequently the Atiyah constraints can only be applied in a given conformal frame while our more general conditions generalize the conformally invariant Jackiw–Rebbi form.

Consider more closely the transformation laws of the parameters of $u(x)$ under a conformal transformation

$$x' = \mathscr{C}x = (\alpha x + \beta)(\gamma x + \delta)^{-1} = m\left(\frac{\xi^2}{x - l} + \bar{k}\right)^{-1}\bar{n}, \tag{4D.63}$$

where

$$m\bar{m} = n\bar{n} = 1, \qquad \operatorname{Vec}\xi = 0. \tag{4D.64}$$

Then

$$\begin{aligned} dx' &= M\,dx\,\bar{N}, & |\,dx'\,|^2 &= \chi^2\,|\,dx\,|^2, \\ dx'\,d\bar{y}' &= M\,dx\,d\bar{y}\,\bar{M}, & d\bar{x}'\,dy' &= N\,d\bar{x}\,dy\,\bar{N}, \end{aligned} \tag{4D.65}$$

where

$$\begin{aligned} M &= \xi m[\xi^2 + (x - l)\bar{k}]^{-1}, \\ N &= \xi n[\xi^2 + (\bar{x} - \bar{l})k]^{-1}, \\ \chi = M\bar{M} = N\bar{N} &= \xi^2[\xi^4 + |\,k\,|^2\,|\,x - l\,|^2 + 2\xi^2\,\mathrm{Sc}\{(x - l)\bar{k}\}]^{-1}. \end{aligned} \tag{4D.66}$$

The self-dual and antiself-dual 2-forms ϕ and ϕ' can be written respectively as

$$\phi = \phi_{\mu\nu}\,dx_\mu\,dy_\nu = v^\dagger e_{\mu\nu} v\,dx_\mu\,dy_\nu = \tfrac{1}{2}v^\dagger(d\bar{x}\,dy - d\bar{y}\,dx)v \tag{4D.67}$$

and

$$\phi' = \phi'_{\mu\nu}\,dx_\mu\,dy_\nu = w^\dagger e'_{\mu\nu} w\,dx_\mu\,dy_\nu = \tfrac{1}{2}w^\dagger(dx\,d\bar{y} - dy\,d\bar{x})\,w. \tag{4D.68}$$

Since $\phi \wedge \phi$ and $\phi' \wedge \phi'$ are invariant under conformal transformations we have the transformation laws

$$v' = \bar{N}^{-1}vq = \chi^{-1/2}\frac{N}{|\,N\,|}\,vq \qquad (|\,q\,| = 1), \tag{4D.69}$$

and similarly for w with M replacing N. The unit quaternion q is a gauge transformation that may depend on the conformal parameters. Thus a conformal transformation induces on v a dilatation with factor $\chi^{-1/2}$ and a gauge transformation belonging to a subgroup $Sp(1) \times Sp(1)$ of $Sp(1) \times Sp(n)$.

To find the explicit form of the transformation law in the parameter space we write

$$u(x) = \mu + K \frac{1}{B - x} \lambda = K \frac{1}{B - x} (a - x)\, \kappa, \tag{4D.70}$$

which is a form closed under conformal transformation. Under a translation we have

$$u(T_l x) = u(K, B, a, \kappa; x - l) = u(K, B + lI, a + l, \kappa; x). \tag{4D.71}$$

Under a Lorentz transformation we find

$$u(T_{m,n} x) = u(K, B, a, \kappa; m x \bar{n}) = n u(K', B', a', \kappa', x) \bar{n}, \tag{4D.72}$$

where

$$K' = \bar{n} K n, \qquad B' = \bar{m} B n, \qquad a' = \bar{m} a n, \qquad \nu' = \bar{n} \nu n. \tag{4D.73}$$

The behavior under a dilatation is given by

$$u(T_\xi x) = u(K, B, a, \kappa; \xi^{-2} x) = u(K, B\xi^2, a\xi^2, \kappa; x). \tag{4D.74}$$

Finally, under a special conformal transformation we have

$$u(T_k x) = u\left(K, B, a, \kappa; \frac{1}{1 + x\bar{k}}\, x\right) = Q^{-1} u(K', B', a', \kappa', x')\, q s, \tag{4D.75}$$

where

$$\begin{aligned} Q &= (\xi^\dagger \xi)^{-1/2}\, \xi, & K' &= QK(I - \bar{c}B)^{-1} = (\xi^\dagger \xi)^{-1/2}\, \xi K \xi^{-1}, \\ QQ^\dagger &= I, & B' &= B(I - \bar{k}B)^{-1}, \\ \xi &= I - \bar{k}B, & a' &= a(1 - \bar{k}a)^{-1}, \\ q &= (1 - \bar{k}a)/|\, 1 - \bar{k}a \,|, & \kappa' &= (1 - \bar{k}a)\, \kappa (1 - \bar{k}a)^{-1}. \\ s &= |\, 1 - \bar{k}a \,|, \end{aligned} \tag{4D.76}$$

Note that if B is a symmetric matrix (as in the Atiyah case) B' is not symmetric in general. However if B is diagonal (Jackiw–Rebbi case) so is B'. Again, if κ is purely scalar (Jackiw–Rebbi case), so is κ'. Hence the Jackiw–Rebbi conditions are conformally invariant. This is not the case for the Atiyah symmetry condition on B. Since Q and q together determine a $Sp(1) \times Sp(n)$ transformation, we see that the conformal transformation, besides changing the parameters K, B, a, κ, also induces a gauge transformation on u as well as as a dilatation transformation.

With the parameters K, B, a, κ, the self-duality equation reads

$$\mathrm{Vec}\{B(K^\dagger K)^{-1} B^\dagger + a\kappa\kappa^\dagger \bar{a}\} = 0, \tag{4D.77}$$

$$\mathrm{Vec}\{(K^\dagger K)^{-1} + \kappa\kappa^\dagger\} = 0, \tag{4D.78}$$

$$M = B(K^\dagger K)^{-1} + a\kappa\kappa^\dagger = M^T. \tag{4D.79}$$

These are automatically satisfied in the Jackiw–Rebbi case with

$$B = b = \text{diagonal}, \qquad K = I, \qquad \kappa = l, \qquad \mathrm{Vec}\, l = 0, \tag{4D.80}$$

showing that these conditions are conformally invariant.

We shall now show that, with the new set of parameters K, B, a, n obtained from the set K, B, μ, λ through the specialization

$$\mu = K\kappa, \qquad \lambda = (aI - B)\kappa, \tag{4D.81}$$

the counting of independent parameters can be carried out in a straightforward manner.

We first note that the self-duality conditions (4D.77)–(4D.79) are invariant under $d = 3n^2 + n + 7$ parameter group

$$G = L(n) \times Sp(n) \times Sp(1) \times Sp(1) \times T(1) \tag{4D.82}$$

where

$L(n)$:	L, $\mathrm{Vec}\, L = 0$	(n^2 parameters),	(4D.83)
$Sp(n)$:	W, $WW^\dagger = I$	($2n^2 + n$ parameters),	(4D.84)
$Sp(1) \times Sp(1)$:	$r, q, \mid r \mid = \mid q \mid = 1$	(6 parameters),	(4D.85)
$T(1)$:	s, $\mathrm{Vec}(s) = 0$	(1 parameter).	(4D.86)

If K, B, κ, a transform under G as

$$B \to rLBL^{-1}\bar{r}s, \tag{4D.87}$$

$$K \to WKL^{-1}\bar{r}, \tag{4D.88}$$

$$\kappa \to rL\kappa q, \tag{4D.89}$$

$$a \to as, \tag{4D.90}$$

then conditions (4D.77)–(4D.79) remain invariant. These conditions correspond to

$$c = 2 \times \frac{3n(n-1)}{2} + 2n(n-1) = 5n^2 - 5n \tag{4D.91}$$

constraints. The dimension of the initial parameters space K, B, a, κ is

$$\pi = 2 \times 4n^2 + 4 + 4n = 8n^2 + 4n + 4. \tag{4D.92}$$

The number of independent parameters is therefore

$$p = \pi - c - d = 8n - 3. \tag{4D.93}$$

Thus, p is in accordance with the index theorem for $Sp(1)$ gauge theories.

Next we proceed with an introduction to Fueter's quaternionic analyticity.

V. Fueter's Quaternionic Analysis

In systematically implementing the mathematical parallel between the 2-dimensional $CP(n)$ and the 4-dimensional $HP(n)\sigma$-models, we have been led to the quaternionic Cauchy–Riemann equations on S^4 and their rational instanton solutions. While our starting point is new and, in our view, crucial to a successful outcome, the basis motivation is of course very old, dating back to Hamilton himself [85, 86]. Indeed have we not all experienced the power and elegance of complex analysis in solving 2-dimensional problems in electrostatics, elasticity and hydrodynamics? Through the years such a realization has invited many searches for a 4-dimensional quaternionic calculus. While this topic may be unfamiliar, there is in fact a subtantial body of literature [87] on attempted constructions of theories of analytic functions in finite dimensional linear algebra over the real and complex fields, particularly in the real quaternion algebra. The most successful and far reaching of these attempts appears to be that of Fueter [88] and his school in Zurich in the 1930's [89].

Since the discovered quaternionic holomorphy of the Yang–Mills instantons in Section IV is a natural extension of Fueter holomorphy and since Fueter's quaternionic analysis is unknown to most physicists and mathematicians alike, we shall first introduce the relevant basic results [90, 91] of this theory before applying it in a crucial way to the gauge field equations and their solutions [92].

In contrast to other sterile attempts, Fueter's success is essentially due to his injection of sufficient algebraic structure [87], specifically the *ring* structure, into one part of his function theory of a real quaternionic variable. To underscore the importance of this key ingredient and to make subsequent contact with its possible physical implications, we shall in our presentation converge on Fueter's theory from the general framework of quaternionic series endowed with a ring structure. Some supplementary materials are also given in Appendixes I and IV.

Recalling the essential role played by the Euclidean conformal group $SO(5, 1) \approx SL(2, H)$ in the Yang–Mills instanton problem, we begin with the quaternionic homographic transformation

$$x' = (ax + b)(cx + d)^{-1}, \tag{5.1}$$

where $x = x_\mu e_\mu$ could be the spacetime position quaternion and a, b, c and d are real constant quaternions. Equation (5.1) can be rewritten in its canonical form

$$x' = \alpha + \beta(x - l)^{-1}\gamma, \tag{5.2}$$

where $\alpha = ac^{-1}$, $\gamma = c^{-1}$, $l = -c^{-1}d$ and $\beta = b - ac^{-1}d$. $(x - l)^{-1}$ can be cast as a power series in $\xi = xl^{-1}$ or $\eta = l^{-1}x$, i.e.,

$$(x - l)^{-1} = -l^{-1} \sum_{n=0}^{\infty} \xi^n = \sum_{n=0}^{\infty} \eta^n l^{-1}, \tag{5.3}$$

Its generalization is the series

$$B(x) = \sum_{n=0}^{\infty} \bar{c}_n (x\bar{c}_n)^n, \tag{5.4}$$

where $c_n \in H$.

Particularly, if the constant quaternions c_n transform like x under $SO(4)$, the Euclidean Lorentz group, that is

$$x'' = v_1 x v_2 , \qquad c_n'' = v_1 c_n v_2 \tag{5.5}$$

with $| v_1 | = | v_2 | = 1$, then we have

$$B'' = v_1 B v_2 , \tag{5.6}$$

and B is also a 4-vector.

We now observe that the product of two series of type (5.4) does *not* give a third function of the same kind; they do not have the ring property. However in the special instance where c_n are purely scalar quaternions, we have series like

$$w(x) = \sum_{n=0}^{\infty} \alpha_n x^n \qquad (\mathrm{Vec}\, \alpha_n = 0), \tag{5.7}$$

which do form a ring. These functions have as their "stem" functions (to use the terminology of Rinehart [87]) the functions of the real variable $\tau = x_4$. They transform the scalar axis into the scalar axis hence generalize ordinary analytic functions that leave the real axis invariant and obey the Schwartz reflection principle $\overline{f(x)} = f(\bar{x})$.

To be more explicit, let us define the quantities

$$\mu = \mathbf{e} \cdot \mathbf{n}, \qquad \mathbf{n} = \frac{\mathbf{r}}{r}, \qquad \mu^2 = -1, \tag{5.8}$$

as well as the projection operators

$$\begin{gathered} E_\pm = \tfrac{1}{2}(1 \pm i\mu), \\ E_\pm{}^2 = E_\pm , \qquad E_+ + E_- = 1, \qquad E_+ E_- = E_- E_+ = 0. \end{gathered} \tag{5.9}$$

μ is a pure vector quaternion of unit norm and an analog of i since $\mu^2 = -1$. $E_\pm$ are idempotent projection matrices. It follows that we can decompose x as

$$x = zE_- + \bar{z}E_+ \tag{5.10}$$

and

$$x^n = z^n E_- + \bar{z}^n E_+ , \tag{5.11}$$

where $z = t + ir$, $\bar{z} = t - ir$ $(r \geqslant 0)$ are the complex variable and its conjugate parametrizing the Poincare upper half-plane. From Eq. (5.11), the quaternionic Weierstrass series (5.7) takes the form

$$\begin{aligned} w(x) &= w(z)\, E_- + w(\bar{z})\, E_+ \\ &= \operatorname{Re} w(z) + \mu \operatorname{Im} w(z) \end{aligned} \tag{5.12}$$

generated by the stem function $w(z) = \operatorname{Re} w(z) + i \operatorname{Im} w(z)$ which is complex analytic in the upper half plane z. The function $w(x)$ will play an illuminating role in the Witten–Peng and BPS solutions. Besides the series $B(x)$, Fueter also considered the series

$$L(x) = \sum_{n=0}^{\infty} \xi^n a_n , \qquad R(x) = \sum_{n=0}^{\infty} b_n \eta^n, \tag{5.13}$$

where a_n and b_n are now quaternionic coefficients with $\xi = \bar{c}x$ and $\eta = x\bar{c}$, with $\xi = x$ being thus a special case. Due to the necessary distinction between left and right multiplication a more general series is

$$C(x) = \sum_{n=0}^{\infty} b_n \xi^n a_n . \tag{5.14}$$

We shall call $L(x)$ and $R(x)$ left and right holomorphic respectively; then $B(x)$ is both left and right holomorphic while $C(x)$ may be called cross-holomorphic. With the replacement $x \to \bar{x}$, the resulting series are anti-holomorphic while those in both x and $\bar{x}$ are mixed holomorphic. Of course the corresponding Laurent series with negative powers n can also be similarly defined.

To construct series with quaternionic coefficients endowed with the ring property, we have to consider the general quaternionic polynomial

$$\begin{aligned} p(x) = a_0 + a_1 x + x a_2 + a_3 x a_4 + b_1 x^2 \\ + x^2 b_2 + b_3 x^2 b_4 + b_5 x b_6 x b_7 + \cdots, \end{aligned} \tag{5.15}$$

which is just a finite sum of quaternionic monomials $m(x)$, e.g. $m(x) = \alpha_0 x \alpha_1 x \cdots \alpha_{r-1} x \alpha_r$ of degree r and with constant quaternions α_i, $i = 0,..., r$. $p(x)$ is called general quaternionic since it has no holomorphy property. Indeed we can generate antiholomorphic functions out of a holomorphic series by insertion of suitable quaternionic coefficients within the monomials, using the identity $e_\mu q e_\mu = -2\bar{q}$ and thus converting x into $\bar{x}$. Any general polynomial over quaternions is of the form $p(x)$ [91]. This is readily verified: Any polynomial has the form $p(x) = p_\mu e_\mu$ where p_μ $(\mu = 0, 1, 2, 3)$ are four real valued polynomials in x, y, z and t. Since by the reconstruction theorem $x_\mu = \mathrm{Sc}(\bar{e}_\mu x)$, upon substitution into p_μ we obtain a $p(x)$ given by a sum of quaternionic monomials, hence a quaternionic polynomial. $p(x)$ is also the quaternionic representation of general coordinate transformations in 4-dimensional Euclidean spacetime [21].

We are finally ready to turn to Fueter's quaternion analyticity [93], namely his generalization of the Cauchy–Riemann equations and of the harmonic property to functions of a quaternionic variable. The left, right, left–right and cross-holomorphic functions $L(x)$, $R(x)$, $B(x)$ and $C(x)$ are all biharmonic and obey the generic linear equation

$$\Box\Box\Phi(x) = 0, \tag{5.16}$$

where $\Box = \partial_\mu\partial_\mu = \mathscr{D}\bar{\mathscr{D}} = \bar{\mathscr{D}}\mathscr{D}$ is the 4-dimensional Laplacian.

Moreover $L(x)$ satisfies the linear third order equation

$$\Box\mathscr{D}L(x) = 0. \tag{5.17}$$

Correspondingly

$$\Box R(x)\underset{\leftarrow}{\mathscr{D}} = 0, \tag{5.18}$$

while the left–right function $B(x)$ obeys

$$\Box\mathscr{D}B(x) = \Box B(x)\underset{\leftarrow}{\mathscr{D}} = 0. \tag{5.19}$$

Fueter defines left and right analytic functions $l(x)$, $r(x)$ as those functions which satisfy the linear quaternionic generalizations of the Cauchy–Riemann equations

$$\mathscr{D}l(x) = 0 \qquad \text{and} \qquad r(x)\underset{\leftarrow}{\mathscr{D}} = 0 \tag{5.20}$$

respectively.

From the preceding discussion we can construct such left, right, left and right analytic functions from the holomorphic functions by taking

$$l(x) = \Box L(x), \qquad r(x) = \Box R(x), \qquad b(x) = \Box B(x). \tag{5.21}$$

Cross-holomorphic functions $C(x)$ will also yield analytic functions if we take

$$l(x) = \Box\bar{\mathscr{D}}C(x) \qquad \text{and} \qquad r(x) = \Box C(x)\underset{\leftarrow}{\bar{\mathscr{D}}}. \tag{5.22}$$

Thus, as examples of such functions

$$l_1(x) = \Box\bar{\mathscr{D}}\sum_n (a_n x + b_n)(c_n x + d_n)^{-1} \tag{5.23}$$

and

$$l_2(x) = \Box(x - l_n)^{-1}\gamma_n \tag{5.24}$$

are left analytic.

Functions which are left–right holomorphic or anti-holomorphic form a ring. They obey the quaternionic version of the Schwarz's reflection principle $\overline{f(x)} = f(\bar{x})$ and also satisfy the quadratic differential equation

$$g_{0i} = 0 \qquad (i = 1, 2, 3), \tag{5.25}$$

where $g_{\mu\nu}$, given by

$$ds^2 = (\partial_\mu f(x)\partial_\nu \bar{f}(x)\, dx_\mu\, dx_\nu = g_{\mu\nu}\, dx_\mu \otimes dx_\nu \tag{5.26}$$

is the metric of the quaternionic transformation

$$y = f(x). \tag{5.27}$$

Indeed, using the stem function $f(\xi)$, where $\xi = t + ir$, we find

$$\begin{aligned} ds^2 = dy\, d\bar{y} &= f'(\xi) f'(\bar{\xi})[(dt)^2 + (dr)^2] \\ &+ \left[\frac{f(\xi) - f(\bar{\xi})}{\xi - \bar{\xi}}\right]^2 [r^2\, d\Omega^2], \end{aligned} \tag{5.28}$$

where, using the polar coordinates r, θ and ϕ for $\mathbf{x}$, we can write

$$r^2\, d\Omega^2 = r^2(d\theta^2 + \sin^2\theta\, d\phi^2) = d\mathbf{x} \cdot d\mathbf{x} - (\mathbf{n} \cdot d\mathbf{x})^2, \tag{5.29}$$

with

$$\mathbf{n} = \frac{\mathbf{x}}{|\,\mathbf{x}\,|} = \frac{\mathbf{r}}{r}$$

and

$$(dr)^2 = (\mathbf{n} \cdot d\mathbf{x})^2.$$

Since there are no cross terms $dt\, dx^i$ in ds^2, this shows that $g_{0i} = 0$ for $y = f(x)$ or $y = f(\bar{x})$ because of the symmetry between ξ and $\bar{\xi}$. The relevance of such Fueter functions $\overline{f(x)} = f(\bar{x})$ is demonstrated recently in finding new multi S^4 gravitational and chiral instantons [95].

As shown by Fueter [90, 91] and also Moisil [93], whose proof we review, the above definition, Eq. (5.20), of quaternionic analyticity, leads to a generalization of the Cauchy–Morera integral theorem:

For all closed three dimensional surfaces S^3 completely enclosing a 4-dimensional domain R^4 where $l(x)$ is left analytic we have

$$\iiint_{S^3} d\Sigma\, l = 0 \tag{5.30}$$

where $d\Sigma = e_\mu\, d^4x/dx_\mu = e_\mu\, d\sigma_\mu$ is the quaternionic area element of S^3. This follows directly from Green's theorem

$$\iiint_\Sigma d\Sigma\, l - \iiiint_\Omega \mathscr{D}l\, d^4x = 0, \tag{5.31}$$

where Ω is the 4-dimensional domain whose boundary $\partial\Omega$ is Σ.

Similarly, we have for right analytic functions $r(x)$

$$\iiint_{S^3} r(x)\, d\Sigma = 0. \tag{5.32}$$

In fact a unified treatment of both left and right analytic functions is achieved through the Green formula

$$\iiint_\Sigma r\, d\Sigma\, l = \iiiint_\Omega (r\mathscr{D}l + \overline{(\bar{\mathscr{D}}\bar{r})}\, l)\, d^4x = 0. \tag{5.33}$$

We apply it to a special case by taking in Eq. (5A.33)

$$r(x) = -\tfrac{1}{4}\Box(x-\rho)^{-1} = |\, x-\rho\,|^{-2}(x-\rho)^{-1}, \tag{5.34}$$

which is both left and right analytic, and letting the point ρ be enclosed by a small sphere of radius R of boundary σ and x be a moving point on the boundary Σ and σ. If $l(x)$ is analytic in $\Omega + \Sigma$ then we have

$$\iiint_\Sigma \frac{1}{x-\rho}\,\frac{d\Sigma}{|\, x-\rho\,|^2}\, l(x) + \iiint_\sigma \frac{1}{x-\rho}\,\frac{d\Sigma}{|\, x-\rho\,|^2}\, l(x) = 0. \tag{5.35}$$

Since on σ we have

$$d\Sigma = -\frac{(x-\rho)}{|\, x-\rho\,|}\,|\, d\Sigma\,|$$

and

$$\iiint_\sigma |\, d\Sigma\,| = 2\pi^2 R^3, \tag{5.36}$$

as $R \to 0$ we obtain the Fueter–Moisil fundamental formula

$$l(\rho) = \frac{1}{2\pi^2}\iiint_{S^3} (x-\rho)^{-1}\,|\, x-\rho\,|^{-2}\, d\Sigma\, l(x), \tag{5.37}$$

which, since $(x-\rho)^{-1} = -4\,|\, x-\rho\,|^{-2}(x-\rho)^{-1}$, also reads

$$l(\rho) = \frac{-1}{8\pi^2}\iiint_{S^3} \Box(x-\rho)^{-1}\,(d\Sigma)\, l(x). \tag{5.38}$$

Similarly, for a right analytic function, we have

$$r(\rho) = \frac{-1}{8\pi^2} \iiint_{S^3} r(x)\, d\Sigma\, \square(x-\rho)^{-1}. \tag{5.39}$$

Finally we note the correspondence

$$2\pi \leftrightarrow 8\pi^2, \qquad i\,dz \leftrightarrow d\Sigma, \qquad (z-\xi)^{-1} \leftrightarrow \square(x-\rho)^{-1}, \tag{5.40}$$

with the 2-dimensional Cauchy formula of complex analysis

$$f(\xi) = \frac{1}{2\pi i} \int_{S^1} f(z)\, dz\, (z-\xi)^{-1}. \tag{5.41}$$

We expect the quaternionic formalism just developed will be of key importance in the uniqueness proof for $SU(2)$ Yang–Mills instantons [96–99]. In this work, we proceed to apply it to some special instantons solutions, old as well as new.

VI. Fueter Analyticity, Special Solutions, Old and New

In the general context of the construction of Atiyah *et al.* [22], the $5n$ parameter solutions of t'Hooft correspond to the case (4D.57)–(4D.58) where the symmetric quaternionic matrix B is diagonal and the column matrix λ is made up of pure scalar quaternions elements. We now exhibit the connection of these special solutions to Fueter analyticity. This shall be done from two points of view. First we can choose to write the matrix U in Eq. (2.9) as

$$U = VW = \xi(\xi^\dagger\xi)^{-1/2} \tag{6.1}$$

with

$$\xi = \begin{pmatrix} 1 & -\lambda^\dagger \\ u & B^\dagger - \bar{x} \end{pmatrix}, \qquad u = \frac{1}{B - xI}\lambda. \tag{6.2}$$

B is diagonal and $\mathrm{Vec}(\lambda) = 0$. This form of U generalizes the quaternionic phase $x(\bar{x}x)^{-1/2}$ which gives the one instanton solution of Polyakov *et al.* [100] seen as a harmonic mapping from $HP(1)$ onto $HP(1)$ [101]. Indeed just as

$$[x\bar{x}, e_\mu] = 0 \tag{6.3}$$

we now have

$$[\xi^\dagger\xi, e_\mu] = 0, \qquad \xi^\dagger\xi = \begin{pmatrix} \gamma^2 & 0 \\ 0 & S^{-2} \end{pmatrix}, \tag{6.4}$$

where

$$S^{-2} = \lambda\lambda^\dagger + (B - x)(B^\dagger - \bar{x})$$

and self-duality becomes equivalent to the Fueter analyticity of the column function $\Box\xi$, namely

$$\Box\mathscr{D}\xi = \Box\xi\overleftarrow{\mathscr{D}} = 0 \tag{6.5}$$

and

$$\Box\Box\xi = 0,$$

in complete analogy to the Cauchy–Riemann, harmonic structure of the $CP(n)$ instantons, Eq. (3A.24). Alternatively if we choose to consider the matrix $\chi(u)$ defined from V' in Eq. (4C.26). We also have

$$\Box\Box\chi(u) = 0, \qquad \Box\mathscr{D}\chi\frac{E+\eta}{2} = 0 \tag{6.6}$$

Yet another approach is to take the vector function $u(x) = (1/(\Delta - x))l$, Δ being diagonal and $\mathrm{Vec}(l) = 0$. From such a u we can construct the rational quaternionic function $B(x)$

$$B(x) = l^{+}\frac{1}{\Delta - x}\,l \equiv \sum_{i=0}^{n}\frac{\Gamma_i}{\Delta_i - x}, \qquad \mathrm{Vec}\,\Gamma_i = 0. \tag{6.7}$$

Then the function $F = \Box B$ is left–right analytic. Explicitly it reads

$$F = \Box\sum_{i=0}^{n}\frac{\Gamma_i}{\Delta_i - x} = \bar{\mathscr{D}}\sum_{i=0}^{n}K_i\,|\,\Delta_i - x\,|^{-2}, \qquad K_i \equiv -2\Gamma_i \tag{6.8}$$

where $|\,\Delta_i - x\,|^{-2}$ is recognized as the Green function of the 4-dimensional Laplacian $\Box$.

Now consider the combination $a = (\Box B)(\mathscr{D}B)^{-1} = +\frac{1}{2}\bar{\mathscr{D}}\ln\rho$ that obeys the equation [21]

$$\mathscr{D}a + a\bar{a} = 0. \tag{6.9}$$

This has precisely the form of the self-duality equations $F_{\mu\nu} = \pm\tilde{F}_{\mu\nu}$ when we take the Jackiw–Nohl–Rebbi–'t Hooft ansatz for the gauge potential A

$$A_\mu = \tfrac{1}{2}a_\nu e'_{\mu\nu} = \tfrac{1}{4}e_\mu\bar{a} - \tfrac{1}{4}a\bar{e}_\mu \tag{6.10}$$

so that

$$a = -\frac{2}{3}\,e_\mu A_\mu = -2\,\frac{\Sigma\Gamma_i\,|\,x - \Delta_i\,|^{-2}\,(x - \Delta_i)^{-1}}{\Sigma\Gamma_i\,|\,x - \Delta_i\,|^{-2}}\,. \tag{6.11}$$

the analyticity of $\Box B$ implies that ρ is harmonic, $\Box\rho = 0$. With an overall conformal transformation reducing the $i = 0$ term in F to x, ρ assumes the familiar 't Hooft form [32]

$$\rho = 1 + \sum_{i=1}^{n}\frac{\Gamma_i}{|\,x - \Delta_i\,|^2}\,, \tag{6.12}$$

where $\ln \rho \equiv \ln(1 + u^\dagger u)$ is just the quaternionic Kähler potential which generates the $HP(n)$ metric. In complete analogy to the form for the first Chern index of the $CP(n)$ σ-model instantons, Eq. (3A.42), we now have, for the 't Hooft *et al.* solution [32–35], the second Chern index in the form

$$\mathscr{C}_2 = \frac{-1}{8\pi^2} \iiiint_{S^4} d^4x \, \frac{1}{2} \, \Box\Box \ln \rho = \frac{1}{8\pi^2} \iiint_{S^3} (d\Sigma \, \Box) \, a \tag{6.13}$$

in analogy to $\mathscr{C}_1 = (1/2\pi) \int a_\mu \cdot dx^\mu$. Compare with Eqs. (3A.41)–(3A.45). We obtain

$$\mathscr{C}_2 = \frac{1}{8\pi^2} \iiint_{S^3} d\Sigma \, \Box \sum_{i=1}^{n} \left(\frac{1}{x - \Delta_i} \right) = n \tag{6.14}$$

after application of the Fueter–Moisil fundamental theorem, the integration being taken over *infinitesimal* spheres S^3 enclosing the poles at $x = \Delta_i$. We recall that, just as in the form (3A.41), Eq. (6.13) is only a *local* representation of the closed 4-form, Eq. (4D.19), and hence, it must have poles which are gauge singularities carrying the topological information about the index. Formula (6.14) says that the instanton number is simply given by the winding number by the argument principle of Fueter's quaternionic analysis.

Let us now recall that the Witten–Peng solutions [34, 35] describe $SU(2)$ multi-instantons with a $SO(3)$ rotational symmetry and alignment along the Euclidean time axis. As such they constitute a special case of 't Hooft solutions and are thereby described by a single function ρ of the variables $r = (x^2 + y^2 + z^2)^{1/2}$ and t. In this instance we are naturally led to pick for the holomorphic function w in $\rho = -\frac{1}{2}\mathscr{D}w$ the quaternionic Weierstrass function $w(x)$ in Eq. (5A.7) whose stem function $w(z) = u + iv$ is a complex analytic function in the Poincare half-plane. Clearly $\Box w$ is Fueter analytic hence ρ is harmonic since

$$-\mathscr{D}w = \frac{2v}{r} = \frac{w(t + ir) - w(t - ir)}{ir} \tag{6.15}$$

and

$$\Box \frac{w(t \pm ir)}{ir} = 0 \tag{6.16}$$

$w(z)$ must be the *rational* function

$$w(z) = \sum_{i=1}^{n} \frac{1}{z - z_i} = \bar{\partial} \sum_{i=1}^{n} \ln | z - z_i | \tag{6.17}$$

with $\operatorname{Im} z_i = r_i = 0$, which enforces alignment along the t axis. We obtain

$$\rho = \sum_{i=1}^{n} \frac{1}{r^2 + (t - t_i)^2}, \tag{6.18}$$

which is the Witten–Peng solution seen in the singular 't Hooft gauge.

For completeness we add the form of the self-dual and antiself-dual fields $\phi_{\mu\nu}$ and $\phi'_{\mu\nu}$ in the Jackiw–Rebbi case. If we write

$$\sigma = \sum_{k=0}^{n} \frac{l_k^2}{|\Delta_k - x|^2} = \mathscr{D}F \qquad (l_0 = 1), \tag{6.19}$$

with

$$F = \frac{1}{\Delta_0 - x} + l^T \frac{1}{\Delta - x} l \qquad (\Delta = \text{diagonal}), \tag{6.20}$$

we find after some calculation

$$A_\mu = \frac{\mathbf{e}}{2} \cdot \mathbf{A}_\mu = \frac{1}{4}(e_\mu \bar{\mathscr{D}} \ln \sigma - (\mathscr{D} \ln \sigma)\, \bar{e}_\mu) = \frac{1}{2}\operatorname{Vec}(e_\mu \bar{a}), \tag{6.21}$$

with $a = D \ln \sigma$,

$$\begin{aligned} \phi'_{\mu\nu} &= \frac{\mathbf{e}}{2} \cdot (\mathbf{F}_{\mu\nu} - \tilde{\mathbf{F}}_{\mu\nu}) = -\frac{1}{2}\operatorname{Vec}\{e'_{\mu\nu}(D\bar{a} + a\bar{a})\}, \\ &= -\frac{1}{2} e'_{\mu\nu} \frac{1}{\sigma} \Box\sigma = -\frac{1}{2}(\mathscr{D}F)^{-1} e'_{\mu\nu} \Box \mathscr{D}F, \end{aligned} \tag{6.22}$$

$$\begin{aligned} \phi_{\mu\nu} &= \frac{\mathbf{e}}{2}(\mathbf{F}_{\mu\nu} + \tilde{\mathbf{F}}_{\mu\nu}) = -\frac{1}{2} e_\alpha e_{\mu\nu} \bar{e}_\beta \sigma\, \partial_\alpha \partial_\beta \sigma^{-1} = -\frac{1}{2} \sigma \mathscr{D} e_{\mu\nu} \bar{\mathscr{D}} \sigma^{-1}, \\ &= -\frac{1}{2}(\mathscr{D}F)(\mathscr{D} e_{\mu\nu} \bar{\mathscr{D}})(\mathscr{D}F)^{-1}. \end{aligned} \tag{6.23}$$

Thus, self-duality implies

$$\phi'_{\mu\nu} = 0 \qquad \text{or} \qquad \sigma^{-1}\Box\sigma = 0, \tag{6.24a}$$

or the Fueter analyticity condition for F, namely

$$\Box \mathscr{D} F = 0. \tag{6.24b}$$

In this case the field is given by

$$F_{\mu\nu} = \frac{\mathbf{e}}{2} \cdot \mathbf{F}_{\mu\nu} = \frac{1}{2}(\phi_{\mu\nu} + \phi'_{\mu\nu}) = \frac{1}{2}\phi_{\mu\nu} = -\frac{1}{4} e_\alpha e_{\mu\nu} \bar{e}_\beta \sigma\, \partial_\alpha \partial_\beta \frac{1}{\sigma} \tag{6.25}$$

and has same structure as the expression of the 2-form associated with the HP_n Kählerian metric in terms of the Kähler kernel $\sigma = 1 + t^\dagger t$.

We next turn our attention to self-dual solutions where periodicity plays an essential role. Until now the only sample has been the monopole solution of Bogomol'nyi, Prasad and Sommerfield (BPS) [36] seen in the Euclidean setting. First we recast the BPS solution in the context of quaternion analyticity. The latter can be viewed as a self-dual solution which is only a function of the three space variables x, y, and z, as a finite action solution with unit Chern index per unit time interval, its action per unit time being the monopole mass [102, 103]. In terms of $SU(2)$ instantons it is given as an infinite superposition of Witten–Peng solutions equally spaced along the time axis. In the singular 't Hooft gauge where the fields are rational functions of x, the BPS solution can be simply expressed as

$$a = A_\mu e_\mu = (\Box F)(\mathscr{D}F)^{-1} \tag{6.26}$$

with

$$F(x) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} \frac{1}{y-n} = \frac{1}{2} \cot\left(\frac{\beta}{2} x\right), \qquad y = \frac{\beta}{2\pi} x. \tag{6.27}$$

β^{-1} is the constant parameter with the dimension of a length. $\rho = \mathscr{D}F$ is explicitly

$$\rho = \frac{2}{\beta} \sum_{-\infty}^{\infty} \frac{1}{r^2 + (t - 2\pi n/\beta)^2}\,. \tag{6.28}$$

While such a potential a is not readily recognized as the BPS solution which should be only a function of r, due to the periodicity in t in Eq. (6.28), the t dependence is an artifact of the 't Hooft gauge and can be removed by the gauge transformation $S(\theta) = e^{i\mathbf{T}\cdot\mathbf{n}\theta}$, where $\tan\theta = \mathrm{Im}(\cos\beta z)/(1 - \mathrm{Re}(\cos\beta z))$. The transformed field $A'_\mu = S^{-1}(A_\mu + i\partial_\mu)S$ reduces precisely to the standard form of the BPS solution [102].

The form of F in Eq. (6.27) is the quaternionic generalization of the meromorphic complex function

$$\cot z = \frac{1}{z} + \sum_{-\infty}^{\infty\,\prime} \left(\frac{1}{z - m\pi} + \frac{1}{m\pi}\right). \tag{6.29}$$

The most important property of such a circular function is its onefold periodicity $f(z + 2\pi n) = f(z)$ where $n \in Z_\infty$. We recall that if ω_1 and ω_2 are two real or complex numbers whose ratio is not purely real, then a function $f(z)$ satisfying $f(z + 2\omega_i) = f(z)$, $i = 1, 2$ for all values of z for which $f(z)$ exists, and is doubly periodic in z with period $2\omega_1$ and $2\omega_2$. A typical example is an elliptic function which is analytic everywhere except at its only singularities, the poles in the finite z plane. The simplest elliptic functions are the Weierstrassian functions $\mathscr{P}(z)$, ζ and σ related through

$$\mathscr{P}(z) = -\frac{d\zeta}{dz} = -\frac{d^2}{dz^2} \ln \sigma,$$

Indeed

$$\zeta(z) = \frac{1}{z} + \sum_{\omega}{}' \left(\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right), \tag{6.30}$$

where $\omega = 2m\omega_1 + 2n\omega_2$ and $\zeta(x)$ is the elliptic function counterpart of $\cot(z)$. Thanks to Fueter, the quaternionic analogs of the Weierstrassian functions are known. We can rewrite this fourfold quasi-periodic function $\zeta(x)$ in the compact form of

$$\zeta(x) \equiv \Box Z(x) = \Box \left(\frac{1}{x} + \sum_q{}' F_q(x)\right), \tag{6.31}$$

where

$$Z(x) = \frac{1}{x} + \sum_q{}' F_q(x), \tag{6.32}$$

$$F_q(x) \equiv \frac{1}{x-q} + \sum_{\alpha=0}^{3} q^{-1}(xq^{-1})^\alpha = \left(\frac{1}{q}x\right)^4 \frac{1}{x-q} \tag{6.33}$$

and $q = \sum_{\alpha=0}^{3} n_\alpha q^{(\alpha)}$ n_α are integers and the $q^{(\alpha)}$ are linearly independent fixed quaternions. If $q^\alpha = \omega_\beta^{(\alpha)} n_\beta$ then $\det(\omega_\beta^{(\alpha)}) \neq 0$.

Since $F_q(x)$ is holomorphic, $f_q(x) = \Box F_q(x)$ is Fueter analytic and satisfies both

$$\mathscr{D} f_q(x) = f_q(x) \overleftarrow{\mathscr{D}} = 0 \tag{6.34}$$

as can be readily checked. As to other relevant properties of ζ, Fueter has shown that the generalized first Legendre relation holds

$$\frac{1}{8\pi^2}\int_{\partial c} \zeta(x)\, d\Sigma = \frac{1}{8\pi^2}\int_{S^3} \left(\Box \frac{1}{x}\right) d\Sigma = 1 \tag{6.35}$$

(∂c = boundary of cell) where the integration is performed over the boundary of a period cell. It can also be shown that derivatives of $\zeta(x)$ are quadruply periodic in q_0, q_1, q_2, q_3 and like their complex counterparts, that $\zeta(x)$ is quasi-periodic

$$\zeta(x + q_h) = \zeta(x) + \eta_h, \tag{6.36}$$

η_h being certain constant quaternions. We also have the second Legendre relation

$$\sum_{h=0}^{3} \eta_h q_h = 8\pi^2, \tag{6.37}$$

which generalizes $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$ in two dimensions.

Just as for the BPS solution, the application of $\zeta(x)$ to build new self-dual field is immediate. We simply choose $\rho = DZ$, since the Fueter analyticity of $\xi = \Box Z$ implies that $\Box\rho = 0$ except at the poles of $Z(x)$. By comparing the series form of $Z(x)$ with that of $B(x)$ in Eq. (6.7), we see that $\rho = DZ(x)$ corresponds to a highly non-trivial superposition of instantons with the very specific spacetime dependent weight

$(q^{-1}x)^4$ necessary for the fourfold periodicity. As a consequence of the Legendre-type relation (6.37), the second Chern index per unit spacetime cell is unity

$$\begin{aligned} n &= \frac{1}{8\pi^2} \int_{\text{cell}} \Box\Box \ln \rho d^4x, \\ n &= \frac{1}{8\pi^2} \int_{\partial c} d\Sigma \, \Box a = 1. \end{aligned} \tag{6.38}$$

Thus our new solution, Eq. (6.31), is a natural generalization of the BPS solution. It is a finite action solution per unit spacetime volume.

As a possible use of the quaternionic 4-periodic solutions to the Yang–Mills theory it might not be inappropriate to present a speculation.

Such solutions generalize the singly periodic exponential (plane wave) solutions to the Maxwell equations. In the same way that a localized wave packet can be constructed by Fourier expanding such a packet, i.e. by superposing plane wave solutions with different periods we could superpose the functions $\zeta_q = DZ(q)$ associated with the four fundamental periods q_h and their multiples. Thus, using the superposition property of 't Hooft-type solutions, the function ζ defined by

$$\zeta = a_1 \zeta_q + a_2 \zeta_{2q} + \cdots + a_n \zeta_{nq} + \cdots \tag{6.39}$$

could, with appropriate coefficients a_n, represent a solution localized within a spacetime volume Ω, due to the destructive interference of various solutions outside Ω. For sufficiently small q_h, the contribution of the ζ-type solutions to the topological number in Ω is approximately proportional to the volume of the domain Ω since each solution corresponds to constant topoligical number density. Hence their contribution to the Lagrangian takes the form of a cosmological constant term in the gravitational action. We already know that such an additional action can be interpreted as a bag term [104], the bag being localized in Ω. Thus, the 4-periodic solution could in principle generate a bag term in the effective Lagrangian. The effective string theory would then be obtained through an elongation of the bag between static sources that are pulled apart. A more precise mathematical formulation of these intuitive considerations is under investigation.

VII.A. *Universal Connections and Grassmannian σ-models*

In this paper we have displayed the striking structural analogy between the 2-dimensional $CP(n)$ and the 4-dimensional $HP(n)$ σ-models with their respective $U(1)$ and $SU(2)$ associated gauge fields and analytic structures. What then may we ask lies at the heart of this parallelism and what extension fo our method is required in order to obtain general Yang–Mills instantons for an arbitrary Lie group? For the mathematically disinclined, we begin by stating the main point of this subsection, namely a sourceless gauge field over a compact (S^4) spacetime of the group $O(m)$, $U(m)$ or $Sp(m)$ can always be embedded in a (4-dimensional) σ-model whose chiral fields take values on an R, a C or an H-Grassmannian manifold of sufficiently high dimension. We next outline the general geometrical framework and the essential details of this embedding in a compact fibre bundle formulation.

With the overview afforded by modern global differential geometry [105], our comparative analysis provides but the simplest detailed illustration of a larger structure, the theory of universal bundles and their classification. In essence our σ-model approach to gauge fields owes its power and generality to a deep theorem by Narasimhan and Ramanan on the existence of universal connection over Stiefel bundles. For our present purpose, we state and elaborate upon such a theorem. All relevant theorems are only quoted; their proofs are available in the quoted literature [105, 106]. A relatively brief account here suffices since our discussion primarily unifies and makes more specific the general implications for gauge theories already drawn by several authors [107–110] from the Narasimhan–Ramanan (N–R) theorem. Our own realization of its central role has been an inductive process; Our chiral formulation of gauge theories in essence generalizes the observation [101, 111] that a Dirac monopole with unit Chern index and the Belavin *et al.* instanton [100] with unit second Chern index are precisely given by the natural connection defined in the Hopf fiberings $S^3 \to S^2$ and $S^7 \to S^4$ respectively [108]. These bundles are just the simplest examples of two whole sequences of complex and quaternionic Hopf fiberings

$$S^{2n+1} \xrightarrow[U(1)\approx S^1]{} CP(n) \qquad \text{and} \qquad S^{4n+3} \xrightarrow[SU(2)\approx S^3]{} HP(n)$$

seen as principal fibre bundles over the base spaces $CP(n)$ and $HP(n)$ with the group $U(1)$ and $SU(2)$ respectively as the fiber. Trautman [111] first showed that these Hopf bundles provide an alternative way of obtaining higher charge $U(1)$ Dirac monopoles and $SU(2)$ instantons. Specifically he found explicit $U(1)$ monopoles with magnetic charge n through an embedding map $S^2 \to CP(n)$, S^2 being the compact submanifold of the spatial manifold $S^2 \otimes R$. Yet he did not exploit fully this connection between $SU(2)$ gauge n instantons and the quaternionic Hopf bundles for $n > 1$. This is what is done here as the general $(8n - 3)$ parameter $SU(2)$ instantons are obtained via the Atiyah holomorphic maps $S^4 \to HP(n)$ (Section IVD). Most importantly our method is concrete in that the gauge field equations are reformulated in terms of the gauge invariant [108] projector valued field $P_u = (N + E)/2$ of an associated classifying σ-model. As we will now explain it is this universality connection between gauge theories and Grassmannian σ-models which may have profound conceptual as well as computational consequences for gauge theories. The extension of our method to accomodate any G-gauge field over a compact spacetime manifold where $G = SO(m)$, $U(m)$, $Sp(m)$ is made possible by the existence of Stiefel bundles [104], the natural generalizations of the above Hopf bundles. We first need a few definitions and facts about the classification of principal bundles [101, 106].

Consider a topological group G with it a principal fibre bundle $\xi(n, G) = E(n, G)$, p, $B(n, G)$, G). $E(n, G)$ denotes the total or bundle space, p the projection map: $E(n, G) \to B(n, G)$, the base space, G is the structural group. If $E(n, G)$ is n-connected

$(n \leqslant \infty)$ $\xi(n, G)$ is called *n-universal* relative to a space M if every bundle over M is equivalent to a bundle induced by a mapping f: $M \to B(n, G)$ called n-classifying space and if two such induced bundles are equivalent when and only when the mappings are homotopic. The induced bundle $f^*\xi(n, G)$ is constructed by glueing over any point $x \in M$ a copy of the fiber over $y = f(x)$ and is also called the pull-back of ξ by f. f is known as the characteristic mapping of $f^*\xi(n, G)$. If for a space M, there exists a universal bundle with the base space $B(n, G)$, then the classes of bundles over $B(n, G)$ are in a 1–1 correspondence with the homotopy classes of maps $M \to B(n, G)$. Hence the bundle classification problem reduces to that of a homotopy classification. It can be shown [112, 113] that a universal bundle exists whenever the base space M is compact and the structural group is a connected Lie group, these conditions are clearly satisfied by a Euclidean gauge field theory over, say, S^4.

Thus if we now consider G-bundles over S^4 and if following Ref. [114], we denote by $\mathscr{C}(G)$ the topologically nontrivial function space of connections where gauge transformations have been factored out, by $\Omega^3(G)$ the function space of maps: $S^3 \to G$ and by $\Omega^4(B(n, G))$ that of the characteristic maps $S^4 \to B(n, G)$, we have the homotopic equivalence [114]

$$\Omega^4(B(n, G)) \approx \Omega^3(G) \approx \mathscr{C}(G); \tag{7A.1}$$

and we have the exact sequence

$$\pi_4(B(n, G)) \approx \pi_3(G), \tag{7A.2}$$

which generalizes the sequence $\pi_2(CP(n)) \approx \pi_1(U(1))$ in the $CP(n)$ instanton problem.

To be concrete about universal bundles, let us now generalize the notions of Section II. Again let K be the real number field R, the complex number field C or the real quaternion field H. Let $\lambda = \dim_R K = (1, 2, 4)$ and K^r the r-dimensional linear space over K. Then for $K = R$, C, or H we let $U(n, K)$ be respectively the orthogonal group $O(n)$, the unitary group $U(n)$ or the symplectic group $Sp(n)$. The Stiefel manifold $\mathrm{St}_{m+n,m}(K) = U(m+n, K)/I_m \otimes U(n, K)$ (I_m is the unit element of $U(m, K)$) is $(\lambda(n+1)-2)$ connected. It generalizes the spheres $S^{2n+1} \approx \mathrm{St}_{n+1,1}(C)$ and $S^{4n+3} \approx \mathrm{St}_{n+1,1}(H)$. So the principal bundle $\xi(\lambda(n+1)-2,\ U(m+n, K)) = (\mathrm{St}_{m+n,m}(K), \mathrm{Gr}_{m+n,m}(K), U(m, K))$ is a $(\lambda(n+1)-2)$-universal bundle of $U(m, K)$, the structure (gauge) group with as its classifying base space $\mathrm{Gr}_{m+n,m}(K) = U(m+n, K)/U(m, k) \otimes U(n, K)$, the K-Grassmannian manifold. The latter generalizes in turn $CP(n) \approx \mathrm{Gr}_{n+1,1}(C)$ and $HP(n) \approx \mathrm{Gr}_{n+1,1}(H)$. Geometrically [114], $\mathrm{Gr} = \mathrm{Gr}_{m+n,m}(K)$ $(m \leqslant n)$ is the manifold of m-planes defined by the equations

$$\lambda_h{}^\alpha \rho^h = 0 \qquad (\alpha = 1,..., m;\ h = 1,..., n+m) \tag{7A.3}$$

of the Euclidean space K^{n+m} with local coordinates ρ^h $(h = 1, 2,..., n+m)$ and dimension $(n+m)$. For $K = R$, the *orientable* m-plane ($\approx SO(n+m)/SO(n) \otimes SO(m)$) of R^{n+m} is the universal covering Gr which covers $\mathrm{Gr}(R)$ twice. We know [114]

$\mathrm{Gr}_{n+m,m}(K)$ to be a rational algebraic variety, of hypercomplex dimension nm, hence of real dimension λnm. It is a simply connected real manifold for $K = C$ and H for $K = R$, it is $\mathrm{Gr}_{n+m,m}(R)$ which is simply connected. Also $\mathrm{Gr}_{n+m,m}(K)$ has been shown by Cartan [115] to be a *symmetric* Riemannian space. Because it is a real algebraic variety, $\mathrm{Gr}(C)$ is necessarily a Kähler manifold of the restricted type, i.e. a Hodge variety [52].

Just as the projective space $KP^n = \mathrm{Gr}_{n+1,1}(K)$ is homeomorphic and isometric with the space of hermitian and idempotent matrices P_u (e.g. Eq. (4C.20)) of rank 1 and of order $(n + 1)$ over the field K [80], it can be shown [115] that the space $\mathrm{Gr}_{m+n,m}(K)$ is homeomorphic and isometric to the space of hermitian and idempotent matrices P_u of rank m and order $(n + m)$:

$$P_u = \begin{pmatrix} (I + u^+u)^{-1} & (I + u^+u)^{-1}\, u^+ \\ u(I + u^+u)^{-1} & u(I + u^+u)^{-1}\, u^+ \end{pmatrix}, \tag{7A.4}$$

where u, the generalization of the $|\, u\rangle$ in Section II, is now an $n \times m$ rectangular matrix. P_u is a $(n + m) \times (n + m)$ square matrix and the form (7A.4) is its rational representation with respect to the matrix u and $u^\dagger$. From P_u, we can form the canonical matrix $\dot{N} = 2P_u - I_{(n+m)}$ which generalizes the $N(u)$ in Eq. (2.23). The generalized Study–Fubini metric for $\mathrm{Gr}_{n+m}(K)$ is then

$$ds^2 = \mathrm{Sc}\,\mathrm{Tr}(dP\, dP). \tag{7A.5}$$

To complete our construction, we need to further define a $(n + m) \times m$ matrix $\Gamma = [{}^I_u]$ and $\Gamma^* = [I\, u^\dagger]$. Then

$$P_u = \Gamma(\Gamma^*\Gamma)^{-1}\, \Gamma^* = \nu\nu^\dagger, \tag{7A.6}$$

where in terms of the Pontryagin coordinates u [114]

$$\nu = \Gamma \frac{1}{(I + u^+u)^{1/2}}, \tag{7A.7}$$

which is an $(n + m) \times m$ matrix over K generalizing the $\nu(x)$ in Eq. (5B.9). Since $\partial_\mu N = 2\partial_\mu P_u$, all the formalism in Section IV carries over unchanged in the 4-dimensional Grassmannian $\mathrm{Gr}_{n+m,m}(K)$ σ-models, written in terms of the projector valued field $P_{u(x)}$ where the matrix elements are K-valued function of the spacetime variable x. So, to each set (n, m), to each Stiefel fibering

$$St_{n+m,m}(K) \xrightarrow[G=U(m,K)]{} Gr_{n+m,m}(K),$$

we can associate a $(\lambda(n + 1) - 2)$ classifying Grassmannian σ-model whose Nambu–Goldstone field is given by the matrix P_u in Eq. (7A.4). One naturally expects to find an embedded $U(m, K)$ gauge field given by the $(m \times m)$ matrix potential $A_\mu = v^\dagger \partial_\mu v$ where

$$v = \begin{bmatrix} I \\ u \end{bmatrix} \frac{1}{(I + u^+u)^{1/2}} \,.$$

In practice, since any compact group may be thought of as a subgroup of $Sp(m)$ for m sufficiently large, we should be capable of getting the gauge field over S^4 for the orthogonal or unitary group by considering the $\mathrm{Gr}_{m+n,m}(H)$. The repeat of an analysis paralleling Section IV.D is left to the interested reader; the corresponding number of parameters for $Sp(m)$ instantons should be $4(m + 1)n - m(2m + 1)$ for $n \geqslant m$ and $2n^2 + 3n$ for $n < m$ [22].

Having clearly outlined the generalization of our method, and in going from the specific to the general we now make contact with the theorem of Narasimhan and Ramanan [97] following closely the general treatment of Refs. [108] and [107] respectively:

Take as a field on a spacetime manifold M (e.g. $M = S^4$) with values in the orthogonal projection of rank m in K^N. A principal $G = U(m, K)$ bundle $\mathscr{P}$ over M with connection form is associated to the field P_u by way of the ensuing construction: take $\mathscr{P}$ the space of all pairs (x, v) with $x \in M$ and v is a K-valued $N \times m$ matrix such that

$$P = vv^\dagger = P^\dagger \qquad \text{and} \qquad v^\dagger v = I_m \,. \tag{7A.8}$$

There is a natural right action of $U(m, K)$ on $(x, v) \to (x, vg)$, $g \in U(m, K)$ and a projection π of $\mathscr{P}$ on M, $\pi(x, v) = x$. $\mathscr{P}$ is then a $U(m, K)$ principal bundle over M, and the connection form $\omega = v^\dagger\, dv$ is the canonical connection form on $\mathscr{P}$, i.e. in a gauge, $A_\mu = v^\dagger(x)\, \partial_\mu v(x)$.

Conversely, the Narasimhan–Ramanan theorem states that given any two integers l and m, there is a smallest integer $N(l, m)$ such that any $U(m, k)$ bundle with a connection over any manifold M (dim $M = l$) can be obtained by the stated construction for $N \leqslant N(l, m)$ (so that the P are indeed projectors of rank m in K^N). The bounds $N(l, m) \leqslant (l + 1)(2l + 1)\, m^3$ given in Ref. [97] for $U(m, R)$ have been improved in some cases [108, 117].

In dealing with a *compact* manifold M like S^4, the use of Milnor's construction theorem [113] quoted above then leads us to consider already discussed universal Stiefel bundles with their natural connections which are also universal by the N–R theorem. All this implies [107] that *any* gauge field with group $U(m\ K)$ defined on a compact manifold may be obtained by embedding M in a Grassmannian $\mathrm{Gr}_{N,m}(K)$ of sufficiently high dimension, the latter being the classifying space of the corresponding universal K-Stiefel bundle. For the $Sp(m)$ instanton, clearly $N = n + m$ ($n =$ the second Chern index); then P_u are indeed projectors of rank m in K^{n+m} in accord with the N–R theorem.

VII.B. *Outstanding Problems*

In closing we wish to list some related problems whose solutions may well be accessible through our chiral approach.

(a) In Section IV.D, we have translated the general $SU(2)$ n-instanton problem into one of solving for particular holomorphic curves in quaternionic projective spaces, i.e. a conformal invariant system of algebraic equations (4D.77)–(4D.79) with a high degree of symmetry, Eq. (4D.82). As is the case with the $SL(2, R)$ isometry invariance in the theory of hyperspherical surfaces [118, 129], we think that the discovered symmetry, when understood geometrically, may be exploited to yield explicit expressions for the $(8n - 3)$ parameter instantons for $n \geqslant 3$.

(b) A closely connected problem is then one of deriving the Bäcklund transformations generating these general instantons [119].

(c) The 2-dimensional classical $O(m)$ σ-models not only possess an infinite number of local and non-local conservation laws in Minkowski space; they are also known [120] to have in Euclidean space an infinite number of local continuity equations. While more general 2-dimensional σ-models built on symmetric spaces such as the $CP(n)$ model have been shown [121] to have in Minkowski space an infinite number of non-local conserved charges, it is not known [122] whether they also possess an infinite number of local conservation laws. Using complex analyticity, we have been able to show the existence of an infinite number of continuity equations $\partial_{\bar{z}} T^n = 0$ or equivalently Eq. (30.26) for $n = 1, 2, \ldots, \infty$ for the Kählerian σ-models. Among the latter are the $\mathrm{Gr}_{n+m,m}(C)$ σ-model, hence the $CP(n)$ σ-models. What, then, is the corresponding infinite set of local conservation laws in the Minkowski space for these models?

(d) The next important step is clearly to seek within our chiral formalism a uniqueness proof for the $SU(2)$ Yang–Mills instantons as well as for the 4-dimensional chiral instantons of our $HP(n)$ σ-models. On the basis of the analogy with the complex chiral case, we may expect such a uniqueness proof to bring about a bonus, the existence of an infinite number of conservation laws [123]. Fueter analyticity may play a crucial role.

(e) Our primary concern being the understanding of the embedded gauge field structure, we have put aside a detailed analysis of the embedding 4-dimensional Grassmannian σ-models constructed in Section IV.B [124]. Structurally, the latter closely parallel their completely integrable 2-dimensional counterparts. While they certainly seem to be of a higher degree of nonlinearity than Yang–Mills systems [125], being associated with zero curvature bundles through their Maurer–Cartan integrability conditions (3A.3), they may be actually easier to solve than the former. Because their field manifolds are Grassmannians which are symmetric spaces it is then natural to ask whether such conformal invariant 4-dimensional σ-models are endowed with a counterpart of Pohlmeyer's R-symmetry [3]. In two dimensions, the latter symmetry is known [121] to be a necessary and sufficient condition for the existence of an infinite number of nonlocal conservation laws in Minkowski space.

(f) The existence of instantons has been shown to be intimately tied to the existence of a complex structure on the chiral field space $CP(n)$ and to that of a quaternionic structure on $HP(n)$ respectively. It is only natural to investigate the last and exceptional Cartan symmetric space of rank 1, the Cayley–Moufang octonionic plane CaP(2) $\approx F_4/\text{spin}(9)$ as a chiral field space in an 8-dimensional nonlinear σ-model. There will be an octonionic structure and its instantons. This study may not be such an unphysical undertaking in light of the current revival of Klein–Kaluza type theories [126]. Thus the 8-dimensional space associated with the octonionic structure could then be interpreted as an internal symmetry space.

(g) We can also proceed to a supersymmetric extension of our formulation; the mathematical foundations have already been laid recently [127, 128].

(h) Besides instantons, the more exotic merons are also of interest. On the strength of complex analyticity they are well understood in 2-dimensional σ-models [19]. What necessary modifications in the geometrical structure must our quaternionic approach undergo before it will be able to attack the general meron problem for the 4-dimensional quaternionic σ-models and the $SU(2)$ gauge theory?

(i) From the standpoint of computability of the present method, it seems that a gauge theory written as a nonlinear σ-model in terms of gauge invariant projector fields should lead to significant simplications in the path integral formulation and calculations [129]. This chiralization of gauge fields may well help carry over into the continuum the similarities between gauge theories and σ-models which are already most apparent on the lattice [2, 130].

(j) The generalization to the Grassmannian $\text{Gr}_{n+m,m}(K)$ σ-models and their embedded $U(m, K)$ gauge theories immediately suggests the study of their $1/n$ expansion, which then involves perfectly well defined infinite Grassmannian and Stiefel manifolds. In the $n \to \infty$ limit, we deal with a so-called universal bundle. Note that the symmetry between the labels n and m in $\text{Gr}_{n+m,m}(K)$ may be used to an advantage in such an expansion scheme.

(k) Finally, as we already outlined at the end of Section VI, suitably chosen superposition of newly discovered $\zeta(x)$-type solutions may be of relevance to the confinement problem.

We have already made some progress in the above areas and our investigations are continuing. One thing is clear: we have in this work forged the necessary tools and have successfully tested them on only a few structures of this bountiful joint theoretical laboratory of σ-models and gauge theories. Throughout our journey, we have been pleasantly surprised at the power and beauty of the quaternionic formalism. Perhaps Maxwell was too guarded when he cautioned his friend Tait, the noted champion of the quaternionic approach: "... but the virtue of the quaternion lies not so much as yet in solving hard questions as in enabling us to see the meaning of the question and its solution."

APPENDIX I: ELEMENTS OF QUATERNION ALGEBRA AND CALCULUS

The algebra H of quaternions is a 4-dimensional algebra over R, the field of real numbers. Its four base elements e_μ obey the multiplication table

$$e_i e_j = -\delta_{ij} + \epsilon_{ijk} e_k \,, \qquad e_0 e_i = e_i e_0 = e_0 \,, \qquad 1 \leqslant i < j < k \leqslant 3,$$
$$e_i^2 = -e_0 \,. \tag{AI.1}$$

e_0 is the unit element of H. In a covariant notation, any quaternion q then takes the linear form

$$q = q_\mu e_\mu \tag{AI.2}$$

with $q_\mu \in R$. Addition and multiplication are defined by the usual distributivity law and by formula (AI.1). As $(e_i e_j)\, e_k = e_i(e_j e_k)$, H is an associative but noncommutative algebra. Alternatively seen as a 2-dimensional algebra over C, the field of complex numbers, $H = C \times C$, any q takes the form

$$q = v + e_2 w \tag{AI.3}$$

with $v, w \in C$ and $v = q_0 + e_1 q_1$, $w = q_2 - e_1 q_3$ and the basic multiplication law becomes $v e_2 = -e_2 \bar{v}$.

The conjugate of the quaternion q is

$$\begin{aligned} \bar{q} &= q_\mu \bar{e}_\mu \\ &= q_0 e_0 - \mathbf{q} \cdot \mathbf{e}. \end{aligned} \tag{AI.4}$$

In practice q is often seen as a 2×2 matrix $q = q_0 - i\boldsymbol{\sigma} \cdot \mathbf{q}$, where $e_0 = I$, $e_i = -i\sigma_i$ ($i = 1, 2, 3$), σ_i being the Pauli spin matrices. Then $\bar{q} = \sigma_2 q^T \sigma_2$ with q^T the transpose of q, with σ_2 chosen imaginary and σ_3 diagonal. We have $q\bar{q} = \bar{q}q = \sum_{i=0}^{3} q_i^2$ which is a non-negative number, equalling zero only if $q = 0$. We denote as the norm of q

$$N(q) = |\, q \,|^2 = q\bar{q} = \bar{q}q = \mathrm{Det}(q). \tag{AI.5}$$

If q and q' are two quaternions, then

$$N(qq') = N(q)\, N(q'). \tag{AI.6}$$

From the existence of the norm, we have a division algebra; i.e. every q has an inverse q^{-1} such that $qq^{-1} = q^{-1}q = e_0$ and $q^{-1} = N(q)^{-1}\, \bar{q}$. In fact the only non-commutative division ring of finite range over R^4 is (up to isomorphism) that of the quaternions. H is of rank 4 over R, and we can define a topology on H homeomorphic to R^4 by the identification of the e_μ with ϵ_μ, the canonical basis of R^4. This topology of H is compatible not only with the ring structure of H but also its division ring structure. It also results in the connectedness and compact topology of the quaternionic projective space $HP(n)$ discussed in Section II. Often it is useful to split q into its "scalar" and "vector" parts defined as

$$q = \mathrm{Sc}(q) + \mathrm{Vec}(q) \tag{AI.7}$$

where

$$\mathrm{Sc}(q) = \tfrac{1}{2}(q + \bar{q}) = \tfrac{1}{2}\,\mathrm{Tr}(q) = q_0 \tag{AI.8}$$

$$\mathrm{Vec}(q) = \tfrac{1}{2}(q - \bar{q}) = \mathbf{e} \cdot \mathbf{q}. \tag{AI.9}$$

Any q obeys its characteristic equation

$$q^2 - 2q_0 q + \bar{q}q = 0. \tag{AI.10}$$

For our work with self- (antiself)-dual fields, the following relations are of crucial importance:

$$e_\mu \bar{e}_\nu + e_\nu \bar{e}_\mu = 2\,\mathrm{Sc}(e_\mu \bar{e}_\nu) = 2\delta_{\mu\nu}\,. \tag{AI.11}$$

It is natural to define

$$\begin{aligned} e_{\mu\nu} &= \tfrac{1}{2}(\bar{e}_\mu e_\nu - \bar{e}_\nu e_\mu) = \mathrm{Vec}(\bar{e}_\mu e_\nu) \\ &= \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta} e_{\alpha\beta} \equiv \tilde{e}_{\mu\nu} \qquad (\epsilon_{1230} = 1), \end{aligned} \tag{AI.12}$$

a self-dual antisymmetric tensor. Similarly

$$e'_{\mu\nu} = \tfrac{1}{2}(e_\mu \bar{e}_\nu - e_\nu \bar{e}_\mu) = \mathrm{Vec}(e_\mu \bar{e}_\nu) = -\tilde{e}'_{\mu\nu} \tag{AI.13}$$

is antiself-dual. Often algebraic operations can be simplified using the following identities:

$$e_\mu q e_\mu = \bar{e}_\mu q \bar{e}_\mu = -2\bar{q}, \tag{AI.14}$$

$$e_\mu q \bar{e}_\mu = \bar{e}_\mu q e_\mu = 4\,\mathrm{Sc}(q) = 2(q + \bar{q}), \tag{AI.15}$$

$$e_{\mu\nu} q e_{\mu\nu} = e'_{\mu\nu} q e'_{\mu\nu} = -4(q + 2\bar{q}), \tag{AI.16}$$

$$e_{\mu\nu} q e'_{\mu\nu} = e'_{\mu\nu} q e_{\mu\nu} = 0, \tag{AI.17}$$

$$e_{\mu\nu} q e_\nu = -(q e_\mu + 2\bar{e}_\mu q), \tag{AI.18}$$

$$e_{\mu\nu} q \bar{e}_\nu = (q + 2\bar{q})\, e_\mu\,, \tag{AI.19}$$

$$\tfrac{1}{2}(\bar{e}_\mu q \bar{e}_\nu + \bar{e}_\nu q \bar{e}_\mu) = \bar{e}_\mu q_\nu + \bar{e}_\nu q_\mu - \bar{q}\delta_{\mu\nu}\,, \tag{AI.20}$$

$$\tfrac{1}{2}(\bar{e}_\mu q \bar{e}_\nu - \bar{e}_\nu q \bar{e}_\mu) = -\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}(\bar{e}_\alpha q_\beta - \bar{e}_\beta q_\alpha), \tag{AI.21}$$

$$\bar{e}_\mu \bar{q} e_{\mu\nu} q e_\nu = \mathrm{Sc}(\bar{q} e_{\mu\nu} q e'_{\mu\nu}) = 0, \tag{AI.22}$$

$$e_\mu \bar{q} e'_{\mu\nu} q e_\nu = 0, \tag{AI.22$'$}$$

$$e_\mu \bar{q} e_\nu \bar{q} - e_\nu \bar{q}_\mu = -\bar{q} q e_{\mu\nu} - q e_{\mu\nu} \bar{q}. \tag{AI.23}$$

If ν is a quaternionic column and $\phi_{\mu\nu} = \tilde{\phi}_{\mu\nu} = \mathbf{e} \cdot \boldsymbol{\phi}_{\mu\nu} = \nu^\dagger e_{\mu\nu} \nu$, then we find

$$-\tfrac{1}{4}\phi_{\mu\nu}\phi_{\mu\nu} = -\tfrac{1}{4}\nu^\dagger e_{\mu\nu} \nu \nu^\dagger e_{\mu\nu} \nu = I_1 + 2I_2 \tag{AI.24}$$

where

$$I_1 = (\nu^\dagger\nu)^2 = \bar{\nu}_\alpha\nu_\alpha\bar{\nu}_\beta\nu_\beta\,, \tag{AI.25}$$

$$I_2 = \nu^\dagger(\nu\nu^\dagger)^T\,\nu = \bar{\nu}_\alpha\nu_\beta\bar{\nu}_\alpha\nu_\beta = \mathrm{Sc}(\bar{\nu}_\alpha\nu_\beta\bar{\nu}_\alpha\nu_\beta). \tag{AI.26}$$

We also note a remarkable mapping between a symmetric traceless tensor $R_{\alpha\beta}$

$$R_{\alpha\beta} = R_{\beta\alpha}\,, \qquad R_{\alpha\alpha} = 0, \qquad \mathrm{Vec}\,R_{\alpha\beta} = 0, \tag{AI.27}$$

and the self-dual Yang–Mills field $\phi_{\mu\nu}$

$$\phi_{\mu\nu} = -\phi_{\nu\mu} = \tfrac{1}{2}\mathbf{e}\cdot\boldsymbol{\phi}_{\mu\nu}\,, \qquad \phi_{\mu\nu} = \tilde{\phi}_{\mu\nu}\,, \qquad \mathrm{Sc}\,\phi_{\mu\nu} = 0. \tag{AI.28}$$

Both depend on nine real quantities. The mapping is given by

$$\phi_{\mu\nu} = e_\alpha e_{\mu\nu}\bar{e}_\beta R_{\alpha\beta}\,. \tag{AI.29}$$

Similarly for an antiself-dual Yang–Mills field

$$\phi'_{\mu\nu} = \bar{e}_\alpha e'_{\mu\nu} e_\beta S_{\alpha\beta}\,. \tag{AI.30}$$

As in the complex case, it is natural to define the Hamilton differential operators

$$\mathscr{D} = e_\mu\partial_\mu\,, \qquad \underset{\leftarrow}{\mathscr{D}} = e_\mu\overleftarrow{\partial}_\mu \qquad (\partial_\mu = \partial/\partial x_\mu) \tag{AI.31}$$

which act from the left and from the right respectively. Their conjugates are $\bar{\mathscr{D}} = \bar{e}_\mu\partial_\mu$ and $\underset{\leftarrow}{\bar{\mathscr{D}}}$. The 4-dimensional Laplacian is then $\mathscr{D}\bar{\mathscr{D}} = \bar{\mathscr{D}}\mathscr{D} = \Box = \partial_0{}^2 + \nabla^2$. In terms of the above-discussed embedding of C in H, the Fueter equation $Df = 0$, where $q \equiv x = v + e_3 w$ and $f = g(v, w) + e_3 h(v, w)$, a quaternion left analytic function, takes the form of

$$\frac{\partial g}{\partial \bar{v}} = \frac{\partial h}{\partial \bar{w}}\,, \qquad \frac{\partial g}{\partial w} = -\frac{\partial h}{\partial v}\,, \tag{AI.32}$$

analogous to the Cauchy–Riemann equations. We record the differential formula

$$\mathscr{D}(x-a)^{-1} = -e_\mu(x-a)^{-1}\,e_\mu(x-a)^{-1}, \tag{AI.33}$$

$$= 2\,|\,x-a\,|^{-2}, \qquad x, a \,\zeta\, H,$$

$$\Box(x-a)^{-1} = -4\,|\,x-a\,|^{-2}\,(x-a)^{-1}, \tag{AI.34}$$

$$\Box\,|\,x-a\,|^{-2} = 4\pi^2\delta^4(x-a), \tag{AI.35}$$

$$\Box\mathscr{D}(x-a)^{-1} = 8\pi^2\delta^4(x-a). \tag{AI.36}$$

Also using the identity

$$\ln(x - a) = \ln | x - a | + \mathbf{e} \cdot \frac{\mathbf{x} - \mathbf{a}}{| \mathbf{x} - \mathbf{a} |} \tan^{-1} \frac{| \mathbf{x} - \mathbf{a} |}{x_0 - a_0} \tag{AI.37}$$

$$= \mathrm{Re}(\ln \xi) + \mathbf{e} \cdot \frac{\mathbf{x} - \mathbf{a}}{| \mathbf{x} - \mathbf{a} |} \mathrm{Im}(\ln \xi), \tag{AI.38}$$

where ξ is the complex number

$$\xi = x_0 - a_0 + i \, | \mathbf{x} - \mathbf{a} |, \tag{AI.39}$$

we find

$$\bar{\mathscr{D}} \ln | x - a | = \bar{\mathscr{D}} \, \mathrm{Sc} \ln(x - a) = \frac{1}{x - a}, \tag{AI.40}$$

$$\mathscr{D} \frac{\lambda}{x - a} = \Box \ln | x - a |^{\lambda}, \tag{AI.41}$$

and finally if F is the meromorphic function

$$F = \sum_i \frac{\lambda_i}{x - a_i} \qquad (\mathrm{Vec}\, \lambda_i = 0) \tag{AI.42}$$

we find

$$\Box \mathscr{D} F = \Box\Box \ln \prod_i | x - a_i |^{\lambda_i} = 8\pi^2 \sum_i \lambda_i \delta^{(4)}(x - a_i). \tag{AI.43}$$

The homographic function

$$\phi(x) = (\alpha x + \beta)(\gamma z + \delta)^{-1} \tag{AI.44}$$

which represents conformal transformations satisfies some remarkable differential equations. It is biharmonic, i.e.

$$\Box\Box \phi(x) = 0. \tag{AI.45}$$

It also satisfies third order equations that can be interpreted as being associated with the vanishing of quaternionic Schwarz derivatives, namely

$$\{\phi, x\}_L = (\Box \mathscr{D} \phi)(\mathscr{D} \phi)^{-1} + |(\Box \phi)(\mathscr{D} \phi)^{-1} |^2 + \overline{\mathscr{D}\{(\Box \phi)(\mathscr{D} \phi)^{-1}\}} = 0, \tag{AI.46}$$

$$\{\phi, x\}_R = (\phi \underset{\leftarrow}{\mathscr{D}})^{-1} (\Box \phi \underset{\leftarrow}{\mathscr{D}}) + |(\phi \underset{\leftarrow}{\mathscr{D}})^{-1} \Box \phi |^2 + \overline{\{(\phi \underset{\leftarrow}{\mathscr{D}})^{-1} \Box \phi\} \underset{\leftarrow}{\mathscr{D}}} = 0. \tag{AI.47}$$

For a function F such that

$$\mathscr{D} F = F \underset{\leftarrow}{\mathscr{D}} = \sigma \qquad (\mathrm{Vec}\, \sigma = 0) \tag{AI.48}$$

we have

$$\{F, x\} = \{F, x\}_L = \{F, x\}_R = \sigma^{-1}\Box\sigma + |\mathscr{D}\ln\sigma|^2 + \Box\ln\sigma = 2\sigma^{-1}\Box\sigma \tag{AI.49}$$

and $\{F, x\}$ vanishes for a harmonic σ. Finally if we have a 2×2 quaternionic matrix $M = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$, its inverse is given as

$$M^{-1} = \begin{pmatrix}(\alpha - \beta\delta^{-1}\gamma)^{-1} & (\gamma - \delta\beta^{-1}\alpha)^{-1}\\ (\beta - \alpha\gamma^{-1}\delta)^{-1} & (\delta - \gamma\alpha^{-1}\beta)^{-1}\end{pmatrix}, \tag{AI.50}$$

which shows that if M is symmetric, M^{-1} is not symmetric in general. On the other hand if M is hermitian so is M^{-1}. If $\alpha, \beta, \gamma, \delta$ are written as 2×2 complex matrices then M can be written as a 4×4 complex matrix m. The determinant of the quaternionic matrix M, $\det M$, is defined as the determinant of the complex matrix M through

$$\operatorname{Det} M = \det m = |\alpha|^2 |\delta|^2 + |\beta|^2 |\gamma|^2 - 2\,\mathrm{Sc}(\alpha\bar{\gamma}\,\delta\bar{\beta}) \tag{AI.51}$$

so that

$$\operatorname{Det}(M_1M_2) = \operatorname{Det} M_1 \operatorname{Det} M_2\,. \tag{AI.52}$$

For additional information on quaternionic matrices the reader may wish to consult the following selected list not referred to in the main text of our paper:

S. Wach, "Essai sur la géométrie projective quaternionienne," Thesis, Mémoires Acad. Belgique, 1936.

E. Cartan, *in* "Rudolf Fueter Festschrift," Vol. VI, p. 191, Beiblatt zur Vierteljahrsschrift der Naturforschenden Gesellschaft **30** (1940).

E. Study, *Acta Math.* **42** (1920), 1.

J. L. Brenner, *Pacific J. Math.* **1** (1951), 329.

H. C. Lee, *Proc. Roy. Irish. Acad. Sect. A* **52** (1949), 253.

N. Salingaros and M. Dresden, *Phys. Rev. Lett.* **43** (1979), 1.

APPENDIX II: Exterior Calculus of R, C and H Valued Forms

Cartan's exterior calculus is an ideal tool for physics as it formalizes differential forms, the central objects in field theory, be it Maxwell's, Yang–Mills' or Einstein's theory. We begin with

(a) *Real Valued Forms*

We recall that a p-form on a manifold M ($\dim M = n$) is a field of skew covariant tensor of rank p. Sums and products of forms are defined through the corresponding operations on the tensors. Let ω θ, and ϕ be p-, q- and r-forms respectively. Then the *exterior* (or wedge) *product* is

$$\omega \wedge \theta = \tfrac{1}{2}(\omega \otimes \theta - \theta \otimes \omega); \tag{AII.1}$$

$\otimes$ denotes tensor product.

Properties of Exterior Algebra

Associativity:

$$(\omega \wedge \theta) \wedge \phi = \omega \wedge (\theta \wedge \phi). \tag{AII.2}$$

Bilinear rules:

$$\omega \wedge (\theta + \phi) = \omega \wedge \theta + \omega \wedge \phi, \tag{AII.3}$$

$$(\omega + \theta) \wedge \phi = \omega \wedge \phi + \theta \wedge \phi, \tag{AII.4}$$

$$f(\omega \wedge \theta) = f(\omega \wedge \theta) + \omega \wedge f\theta, \tag{AII.5}$$

where f is a C^∞-scalar function, a 0-form on M.
Non-commutativity:

$$\omega \wedge \theta = (-1)^{pq}\, \theta \wedge \omega. \tag{AII.6}$$

Rules of Exterior Differentiation

The exterior derivative operator d is independent of the coordinate system. If $F^p(M)$ is the space of all p forms of M, then

$$d\colon F^p(M) \to F^{p+1}(M). \tag{AII.7}$$

We have the rules:

—If f is a 0-form, df is the 1-form defined by the differential of f. $df = (\partial_\alpha f)\, dx^\alpha$.

$$d(f\omega) = f\, d\omega + df \wedge \omega. \tag{AII.8}$$

—If $\phi = \omega + \theta$,

$$d\phi = d\omega + d\theta, \tag{AII.9}$$

$$d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^p\, \omega \wedge d\tau, \tag{AII.10}$$

$$d^2 = 0, \tag{AII.11}$$

known as the Poincaré lemma. In local coordinates a p-form ω reads

$$\omega = a_{i_1 \cdots i_p}\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \tag{AII.12}$$

The coefficients $a_{i_1 \cdots i_p}$ are smooth functions of the variables x^i and skew symmetric in the indices i_j, $j = 1,..., p$. Then

$$d\omega = d(a_{i_1 \cdots i_p})\, dx^{i_1} \wedge \cdots \wedge dx^{i_p},$$

$$d\omega = \frac{\partial a_{i_1 \cdots i_p}}{\partial x_j}\, dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \tag{AII.13}$$

So if ω is an n-form, then $d\omega = 0$ (dim $M = n$).

Integral Theorem

Differential forms are the objects on which integration operates. A p-form defines a measure over a p-dimensional manifold M_p. The corresponding integral is denoted by $\int_{M_p}$. It is the inverse of the operator d in the same way partial integration generalizes Stoke's theorem. We have

$$\int_{M_p} d\omega = \int_{\partial M_p} \omega \tag{AII.14}$$

where ∂M_p is the boundary of M.

A most striking property of forms is that they "pull back" naturally under smooth mappings: Given (a) $f: M_n \to W_m$, a smooth map from M_n with coordinates $\{x_1,..., x_n\}$ into another manifold W (dim $W = m$) with coordinates $\{y_1,..., y_m\}$, (*b*) a p-form on W_m then f^* is a p-form on M_n, called an induced or pull-back form in M_n of the form ω given in W_m. If f is defined by the local functions $y^\alpha = y^\alpha(x^i)$, $\alpha = 1,..., p$ and $\omega = a_{\alpha_1 \cdots \alpha_p}(y)\, dy^{\alpha_1} \wedge \cdots \wedge dy^{\alpha_p}$ then

$$f^*\omega = \frac{1}{p!}\, a_{\alpha_1 \cdots \alpha_p}(y^\lambda_{\alpha_p}(x))\, \frac{\partial y^{\alpha_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{\alpha_p}}{\partial x^{i_p}}\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

$$= \frac{1}{p!}\, a_{\alpha_1 \cdots \alpha_p}(x^\lambda)\, dy^{\alpha_1}(x^\lambda)) \wedge \cdots \wedge dy^r(x^\lambda). \tag{AII.15}$$

The following rules are obtained.

$$f^*(\omega_1 + \omega_2) = f^*\omega_1 + f^*\omega_2, \tag{AII.16}$$

$$f^*(\omega \wedge \theta) = (f^*\omega) \wedge (f^*\theta), \tag{AII.17}$$

$$(fh)^*\, \omega = h^* f^* \omega \qquad (h = \text{another smooth map}), \tag{AII.18}$$

$$d(f^*\omega) = f^*\, d\omega. \tag{AII.19}$$

These properties apply in particular to the instance in which M_n is immersed in W_m. The form obtained from ω by the pull back is called the form induced on M_n.

(b) *Complex Valued Forms*

Complex valued forms on a complex manifold M are defined as complex linear functions of the tangent space vectors in M. So if $w_k = x_k + iy_k = e_\mu x_\mu^{(k)}$ where $e_\mu = (1, i)$ and $\bar{e} = (1, -i)$ are the complex units, then we have

$$dw_k = e_\mu\, dx_\mu^{(k)}, \qquad d\bar{w}_k = \bar{e}_\mu\, dx_\mu^{(k)}. \tag{AII.20}$$

The space of such forms in C has complex dimension $2n$, the forms dw_k, $d\bar{w}_k$ ($k = 1, 2,..., n$) are equivalently the $2n$ real forms dx_k, dy_k constituting a C^n basis. Exterior multiplication for real forms is defined in the usual way and obeys the usual rules (AII.1)–(AII.6), i.e. $dw \wedge d\bar{w} = e_\mu \bar{e}_\nu\, dx_\mu \wedge dx_\nu = -2i\, dx \wedge dy$.

To deal with complex manifolds it is useful to distinguish 1-forms of type (1, 0), $\omega = \Omega_\alpha\, dw^\alpha$, from those of type (0, 1), $\omega = \Omega_{\bar{\alpha}}\, d\bar{w}^\alpha$. Similarly we have in $T = T_{\alpha\beta}\, dw^\alpha \otimes dw^\beta$, $T_{\alpha\bar{\beta}}\, dw^\alpha \otimes d\bar{w}^\beta + T_{\bar{\alpha}\beta}\, d\bar{w}^\alpha \otimes dw^\beta$ and $T_{\bar{\alpha}\bar{\beta}}\, d\bar{w}^\alpha \otimes d\bar{w}^\beta$ covariant rank 2 tensors of types (2, 0), (1, 1) and (0, 2) respectively. Clearly this notion of type extends to covariant rank $(p + q)$ and contravariant rank $(r + s)$ tensors of types (p, q) and (r, s). From Appendix III, by definition, in a complex manifold complex coordinate transformations between two coordinate patches z and z' are analytic; hence the complex component of a complexified tensor transform as

$$T^{\alpha'\cdots\bar{\beta}'}_{\gamma'\cdots\bar{\delta}'} = \frac{\partial w^{\alpha'}}{\partial w^k} \cdots \overline{\left(\frac{\partial w^{\beta'}}{\partial w^\lambda}\right)} \cdots \left(\frac{\partial w^\mu}{\partial w^{\gamma'}}\right) \cdots \overline{\left(\frac{\partial w^\nu}{\partial w^{\delta'}}\right)}\, T^{k\cdots\bar{\lambda}}_{\mu\cdots\bar{\nu}} \tag{AII.21}$$

It follows that this type of tensor is well defined globally since it is independent of the choice of the coordinate patch. It is readily verified that the operator d obeys the same rules as (AII.7)–(AII.10). It is natural, moreover, to split it into $d' + d'' = d$ where the differentiation d' is with respect to w_k and the differentiation d'' is with respect to $\bar{w}_k$; thus if $\Omega = \Omega_{a\bar{b}\cdots l}\, dw^a \wedge dw^b \wedge \cdots \wedge dw^l$ then

$$d'\Omega = \Omega_{a\bar{b}\cdots l,m}\, dw^m \wedge dw^a \wedge dw^b \wedge \cdots \wedge dw^l, \tag{AII.22}$$

$$d''\Omega = \Omega_{a\bar{b}\cdots l,m}\, d\bar{w}^m \wedge dw^a \wedge dw^b \wedge \cdots \wedge dw^l. \tag{AII.23}$$

So if A is a form of type (p, q) then dA is of type $(p + 1, q)$ and $\bar{d}A$ is of type $(p, q + 1)$. Since $d^2 = 0$ we have

$$d'^2 = d''^2 = 0 \tag{AII.24}$$

$$d'd'' + d''d' = 0 \qquad (\Delta = d'd''). \tag{AII.25}$$

For more on C-valued exterior calculus see Weil's notes [42].

(c) *H-Valued Forms*

Let ϕ and ψ be two quaternionic valued forms of the real forms ϕ_μ, ψ_μ ($\mu = 0,..., 3$) of degrees p and q respectively. They read

$$\phi = e_\mu \phi_\mu , \qquad \bar{\phi} = \bar{e}_\mu \phi_\mu , \tag{AII.26}$$

$$\psi = e_\mu \psi_\mu , \qquad \bar{\psi} = \bar{e}_\mu \psi_\mu . \tag{AII.27}$$

The e_μ are the canonical quaternionic units. Then the exterior derivative is given as

$$d\phi = e_\mu \, d\phi_\mu \tag{AII.28}$$

and

$$d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^p \phi \wedge d\psi. \tag{AII.29}$$

Similarly, other rules are readily derived. We only record one of note, one which reflects the non-commutative character of the H-algebra:

$$\overline{\phi \wedge \psi} = (-1)^{pq} \, \bar{\psi} \wedge \bar{\phi}. \tag{AII.30}$$

It will be useful in our work.

H. FLANDERS, "Differential Forms with Applications to the Physical Sciences," Academic Press, New York, 1963.

G. DE RHAM, "Variétés différentiables," Actualités scientifiques et industrielles, No. 1222, Hermann, Paris, 1955.

A. WEIL, "Introduction à l'étude des variétés kählériennes, Actualités scientifiques et industrielles, No. 1267, Hermann, Paris, 1958.

E. MARTINELLI, *Ann. Mat. Pura Appl.* **49** (1960), 73.

APPENDIX III: COMPLEX AND QUATERNIONIC STRUCTURES

Of all the geometrical structures which can exist on differential manifolds, the Riemannian and complex structures occupy a central position in differential geometry. Since complex manifolds constitute a vast subject, this appendix should be viewed only as a supplement to Sections II and IV. For proofs of the quoted theorems and for greater details, our selected list of references should be consulted.

We first need to define the notions of complex, hermitian, almost complex, almost hermitian and finally Kählerian structures. These will generalize in a rather natural manner to the quaternionic case. Since quaternionic structure is sufficiently discussed in the body of the paper, we shall for the sake of brevity give a list of relevant references on quaternionic manifolds.

Generalizing the notion of a Riemann surface, an n-dimensional manifold is a $2n$-dimensional real manifold M with local real coordinates x_i , $i = 1, 2, 3,..., 2n$, one which can be covered by open sets P, P',... homeomorphic to R^{2n} such that if $\{w_j = x_j + ix_{n+j} , j = 1, 2,..., n\}$ and $\{w_j' = x_j' + ix_{n+j}'\}$ are two overlapping complex coordinates patches then $w_j' = w_j'(w_k)$, $j = 1,..., n$, is analytic (holomorphic) with $\det(\partial w_j'/\partial w_k) \neq 0$. If p is any point on the complexified tangent space $T_p(M)$ of M, M is said to have an almost complex structure defined by the linear action of the

Weil tensor field J of type (1, 1) such that $J_p^2 = -I_p$ on the basis $\{dw_\alpha\}$ of the tangent space of vectors

$$J\frac{\partial}{\partial w_\alpha} = i\frac{\partial}{\partial w_\alpha}, \qquad J\frac{\partial}{\partial \overline{w}_\alpha} = -i\frac{\partial}{\partial \overline{w}_\alpha}. \tag{AIII.1}$$

In components we have

$$J_p\left(\Sigma_\alpha \xi^\alpha \frac{\partial}{\partial x^\alpha}\right) = \sum_{\alpha i\beta} J_\beta{}^\alpha \xi^\beta \frac{\partial}{\partial x^\alpha} \tag{AIII.2}$$

and $J_p^2 = -I_p$ since $\sum_\beta J_\beta{}^\alpha J_\gamma{}^\beta = -\delta_\gamma{}^\alpha$ $(1 \leqslant \alpha, \beta, \gamma \leqslant n)$.

Note that a necessary condition for a manifold to admit an almost complex structure is even dimensionality and orientability. While a complex manifold has an almost complex structure, the converse is *not* true in general. The necessary and sufficient integrability condition for an almost complex structure to arise from a complex analytic one is that the Nijenhuis' tensor of torsion

$$t^\alpha_{\beta\gamma} = \sum_\rho (J^\alpha_{\beta\rho} J_\gamma{}^\rho - J^\alpha_{\gamma\rho} J_\beta{}^\rho), \tag{AIII.3}$$

with

$$J^\alpha_{\beta\gamma} = -J^\alpha_{\gamma\beta} = \frac{\partial J_\beta{}^\alpha}{\partial x^\gamma} - \frac{\partial J_\gamma{}^\alpha}{\partial x^\beta}, \tag{AIII.4}$$

vanishes identically, $t^\alpha_{\beta\gamma} \equiv 0$. Thus an almost complex structure is integrable $\leftrightarrow$ it comes from a complex structure.

Now an (almost) hermitian manifold M is a real manifold with mutually compatible (almost) complex and Riemannian structures. An (almost) hermitian structure is a complex valued function H, called hermitian metric, such that

$$\text{(i)} \quad H(\alpha X_1 + \beta X_2, X_3) = \alpha H(X_1, X_3) + \beta H(X_2, X_3), \tag{AIII.5}$$

$$\text{(ii)} \quad \overline{H(X, Y)} = H(Y, X), \tag{AIII.6}$$

$$\text{(iii)} \quad H(JX, Y) = iH(X, Y) \tag{AIII.7}$$

for all $\alpha, \beta \in R$ and the vectors X_i, $Y \in T(M)$, a real tangent space of M. Its real and imaginary parts are

$$(\operatorname{Re} H)_{ab} = g_{ab}, \qquad \operatorname{Im}(H)_{ab} = \Theta_{ab} = J_a{}^m g_{mb} = J_{ab}, \tag{AIII.8}$$

where $g_{ab} J_\gamma{}^a J^b = g_{\gamma\delta}$ are respectively the symmetric hermitian Riemannian metric of M and the antisymmetric almost hermitian structure tensor of type (0, 2). Clearly either g or Θ with J determines $H(X, Y)$.

In terms of local coordinates w^α, $\alpha = 1, 2, \ldots, n$, the positive definite metric is

$$ds^2 = \sum_{\alpha i\beta} g_{\alpha\beta}\, dw^\alpha\, d\overline{w}^\beta. \tag{AIII.9}$$

To this metric is associated its fundamental Kähler form, a real valued exterior form of rank $2n$

$$\Theta = i/2 \sum_{\alpha i \beta} g_{\alpha\bar{\beta}}\, dw^{\alpha} \wedge d\bar{w}^{\beta} = i/2 J_{ab}\, dx^{a} \wedge dx^{b}. \tag{AIII.10}$$

M is called Kählerian if Θ is closed, i.e.

$$d\Theta = 0, \tag{AIII.11}$$

which can also be stated through the conditions

$$\frac{\partial g_{\alpha\bar{\beta}}}{\partial w^{\gamma}} = \frac{\partial g_{\gamma\bar{\beta}}}{\partial w^{\alpha}} \quad \text{or} \quad \frac{\partial g_{\alpha\bar{\beta}}}{\partial \bar{w}^{\gamma}} = \frac{\partial g_{\alpha\bar{\gamma}}}{\partial \bar{w}^{\beta}}. \tag{AIII.12}$$

Kähler showed that Eq. (AIII.11) holds if and only if there exists a real valued local potential F such that

$$\Theta = \frac{i}{2} d'd''F = \frac{i}{2} \frac{\partial^2 F}{\partial w^{\alpha}\, \partial \bar{w}^{\beta}}\, dw^{\alpha} \wedge d\bar{w}^{\beta}. \tag{AIII.13}$$

Let M be a Kähler manifold dim $M = n$. Its metric, Eq. (AIII.9), defines an inner product in the tangent space of M. Following Chern [27] consider a field of unitary frames on M, where each is an ordered set of n tangent vector e_{α} with the same origin such that the inner product

$$(e_{\alpha}, e_{\beta}) = \delta_{\alpha\beta}, \qquad 1 \leqslant \alpha, \beta \leqslant n. \tag{AIII.14}$$

Its dual basis in the cotangent space dual to a unitary frame is called a unitary coframe consisting of n-complex valued linear differential forms θ_{α} of type (1, 0) such that $ds^2 = \sum_{\alpha=1}^{n} \theta_{\alpha}\bar{\theta}_{\alpha}$. It is a fundamental theorem of local hermitian geometry that in the bundle of all unitary frames there is a set of C-valued linear differential forms $\theta_{\alpha\beta}$, the connection forms, satisfying

$$d\theta_{\alpha} = \sum_{\beta} \theta_{\beta} \wedge \theta_{\beta\alpha} + \Omega_{\alpha} \tag{AIII.15}$$

where Ω_{α} are of type (2, 0) and are called the torsion. A Kähler manifold is torsionless $\Omega_{\alpha} = 0$. The θ_{α} are geometrically interpreted as defining the covariant differential $De_{\alpha} = \sum_{\beta} \theta_{\alpha\beta} \otimes e_{\beta}$ or $(e_{\beta}, De_{\alpha}) = \theta_{\alpha\beta}$. By taking the exterior derivative of Eq. (AIII.15), the curvature forms are defined by the structure equations

$$\Theta_{\alpha\beta} = d\theta_{\alpha\beta} - \sum_{\gamma} \theta_{\alpha\gamma} \wedge \theta_{\gamma\beta}, \tag{AIII.16}$$

where

$$\Theta_{\alpha\beta} = -\bar{\Theta}_{\alpha\beta} = \tfrac{1}{2} \sum_{\gamma,\delta} R_{\alpha\beta\gamma\delta}\theta_{\gamma} \wedge \bar{\theta}_{\delta}. \tag{AIII.17}$$

The antihermiticity of Θ is equivalent to $R_{\alpha\beta\gamma\delta} = \bar{R}_{\beta\alpha\delta\gamma}$. The latter constitutes the Riemann–Christoffel curvature tensor of the metric. By further exterior differentiation of Eqs. (AIII.15) and (AIII.16), the Bianchi identities yield more symmetry relations in

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\beta\alpha\delta} = R_{\alpha\delta\gamma\beta} \tag{AIII.18}$$

and

$$d\Theta_{\alpha\beta} = \sum_{\gamma} (\theta_{\alpha\gamma} \wedge \Theta_{\gamma\beta} - \Theta_{\alpha\gamma} \wedge \theta_{\gamma\beta}). \tag{AIII.19}$$

The metric of M is *Einsteinian* if

$$d\Big(\sum_{\alpha} \theta_{\alpha\alpha}\Big) = \sum_{\alpha} \Theta_{\alpha\alpha} = \frac{R}{n+1} \sum_{\alpha} \theta_{\alpha} \wedge \bar{\theta}_{\alpha}, \tag{AIII.20}$$

with $R = \bar{R} = \sum_{\alpha\beta} R_{\alpha\alpha\beta\beta}$ as the real scalar valued curvature; $R_{\alpha\beta} = \bar{R}_{\beta\alpha} = \sum_{\gamma} R_{\gamma\gamma\alpha\beta}$, the Ricci tensor. The Ricci curvature in the direction ξ is defined as

$$\mathrm{Ric}(\xi) = \sum_{\gamma,\delta} R_{\alpha\alpha\gamma\delta} \xi_{\gamma} \bar{\xi}_{\delta} \Big/ \Big(\sum_{\beta} \xi_{\beta} \bar{\xi}_{\beta}\Big). \tag{AIII.21}$$

The real 2-form of Chern is locally expressed as

$$\psi = \frac{i}{2\pi} R_{\alpha\beta} \theta_{\alpha} \wedge \bar{\theta}_{\beta}. \tag{AIII.22}$$

The curvature tensor $R_{\alpha\beta\gamma\delta}$ defines the holomorphic sectional curvature $R(x, \xi) = \kappa$ for any tangent vector ξ in M. If $\xi = \sum_{\alpha} \xi_{\alpha} e_{\alpha} \neq 0$ is a tangent vector at x then

$$R(n, \xi) = 2 \sum_{\alpha,\ldots,\delta} R_{\alpha\beta\gamma\delta} \xi_{\alpha} \xi_{\gamma} \bar{\xi}_{\beta} \bar{\xi}_{\gamma} \Big/ \Big(\sum_{\alpha} \xi_{\alpha} \bar{\xi}_{\alpha}\Big)^{2}, \tag{AIII.23}$$

which is real in consequence of Eq. (AIII.17). If $R(x, \xi) = \kappa$, a constant, for all (x, ξ), M is of constant holomorphic sectional curvature κ. Then $R_{\alpha\beta\gamma\delta} = \frac{1}{4}(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma})\kappa$ and $\Theta_{\alpha\beta} = \frac{1}{4}\kappa(\theta_{\beta} \wedge \bar{\theta}_{\alpha} + \delta_{\alpha\beta} \sum_{\gamma} \theta_{\gamma} \wedge \bar{\theta}_{\gamma})$. While a frame is chosen in obtaining the foregoing results, the latter are in fact independent of the frames.

Besides those already quoted in the body of the paper we give other selected references:

(I) *Complex Manifolds*

S. S. Chern, "Complex Manifolds," Lecture Notes, University of Chicago, 1955–1956.

K. Yano, "Differential Geometry on Complex and Almost Complex Spaces," Macmillan, New York, 1965.

A. Lichnerowicz, "Géométrie des groupes de transformations," Dunod, Paris, 1958.

P. Griffiths and J. Harris, "Principles of Algebraic Geometry," Wiley–Interscience Monograph, Wiley–Interscience, New York, 1978.

(II) *Quaternionic Manifolds*

P. LIBERMANN, *Bull. Soc. Math. France* **83** (1955), 195.
E. BONAN, *C. R. Acad. Sci. Paris* **258** (1964), 792, 1696, 1988; **259** (1964), 45; **260** (21) (1965), 5445.
D. V. ALEKSEEVSKII, *Funkcional. Anal. i Prilozen.* **2** (1968), 1; *Izv. Akad. Nauk USSSR. Ser. Mat.* **39** (1975), 2; *Math. USSR Izv.* **9**, No. 2 (1975), and references therein.
Y. Y. KUO, *Tohoku Math. J.* **22** (1970), 325.
S. ISHIHARA, *Kōdai Math. Sem. Rep.* **25** (1973), 321.

We recommend in particular two references:

(A) S. ISHIHARA, *J. Differential Geometry* **9** (1974), 483.
(B) E. BONAN, "Sur les *G*-structures de type quaternionien," thesis; *Cahiers Topologie Géom. Différentielle* **9**, No. 4 (1967), 389–468.

APPENDIX IV

Here we elaborate on one aspect of Fueter analytic functions, the generation of new holomorphic functions from Fueter series.

Consider a left-holomorphic Fueter series

$$L(x) = \Sigma x^n a_n \tag{AIV.1}$$

which satisfies

$$\Box\mathscr{D}(Lx) = 0, \qquad \mathscr{D} = e_\mu \partial_\mu\,, \qquad \Box = \mathscr{D}\bar{\mathscr{D}}. \tag{AIV.2}$$

Then, according to Fueter's definition

$$l(x) = \Box L(x) \tag{AIV.3}$$

is left analytic and satisfies

$$Dl(x) = 0. \tag{AIV.4}$$

Let us apply on $L(x)$ the differential operator

$$\Lambda(C_1\,, C_2\,,..., C_m) = C^{(1)}_{\mu_1} C^{(2)}_{\mu_2} \cdots C^{(m)}_{\mu_m} \frac{\partial}{\partial x^{\mu_1}} \frac{\partial}{\partial x^{\mu_2}} \cdots \frac{\partial}{\partial x^{\mu_m}}. \tag{AIV.5}$$

Then, because of linearity, the functions

$$L_\Lambda(x) = \Lambda L(x), \qquad L_\Lambda l(x) = \Lambda\Box L(x) \tag{AIV.6}$$

are also respectively Fueter holomorphic and analytic. As a first example, take

$$L(x) = x^5 a, \qquad m = 2. \tag{AIV.7}$$

Then

$$\begin{aligned} L_A(x) &= (r_\mu \partial_\mu)(s_\nu \partial_\nu)\, x^5 a \\ &= [\{ts\}\, x^3 + x^3\{ts\} + tx^3 s + sx^3 t \\ &\quad + txsx^2 + sxtx^2 + tx^2sx + sx^2tx \\ &\quad + xtx^2s + xsx^2t + x^2txs + x^2sxt \\ &\quad + x\{ts\}\, x^2 + x^2\{ts\}x + xtxsx + xsxtx]a \end{aligned} \tag{AIV.8}$$

is also left holomorphic. Note that this is of the general quaternionic series type but with constraints that symmetrize the various quaternionic coefficients inserted between powers of x. As a second example take

$$L(x) = \frac{1}{a - x}\,\lambda, \qquad m = 2. \tag{AIV.9}$$

Then

$$\begin{aligned} L_A(x) &= (r_\mu \partial_\mu)(s_\nu \partial_\nu)\,\frac{1}{a-x}\,\lambda \\ &= \frac{1}{a-x}\left(r\,\frac{1}{a-x}\,s + s\,\frac{1}{a-x}\,r\right)\frac{1}{a-x}\,\lambda \end{aligned} \tag{AIV.10}$$

is also left holomorphic.

Finally consider

$$W = (ax + h)(cx + d)^{-1} \tag{AIV.11}$$

with $c \neq 0$. Wc is right holomorphic, since

$$Wc = a + (b - ac^{-1}d)(x + c^{-1}d)^{-1}. \tag{AIV.12}$$

Then we quote Fueter's result, which can be verified directly: if $R(x)$ is right holomorphic, so is $R(Wc)$, that is,

$$\Box R(x)\,\underset{\leftarrow}{\mathscr{D}} = 0, \qquad \Box R(Wc)\,\underset{\leftarrow}{\mathscr{D}} = 0. \tag{AIV.13}$$

Fanned Derivatives

We now turn to the extension of the notion of derivative to Fueter analytic function. Consider the left-holomorphic Fueter polynomial

$$P(x) = a_0 + \sum_{k=1}^{n} x^k a_k = a_0 + \mathrm{Tr}(YA),$$

where

$$Y = \begin{pmatrix} x^n & & & \\ & x^{n-1} & & 0 \\ & & \ddots & \\ 0 & & & x \end{pmatrix}, \qquad A = \begin{pmatrix} a_n & & 0 \\ & \ddots & \\ 0 & & a_1 \end{pmatrix} \tag{AIV.14}$$

are $n \times n$ quaternionic diagonal matrices. We have

$$\Box DP(x) = 0 \qquad \text{and} \qquad \Box D(YA) = 0. \tag{AIV.15}$$

Let us now introduce matrix derivatives which also satisfy the same holomorphy equations. We have

$$dP = \mathrm{Tr}\{(dY)A\} = \mathrm{Tr}(l\, dx\, rA), \tag{AIV.16}$$

where we have introduced the triangular $n \times n$ matrices

$$l = \begin{pmatrix} 1 & x & x^2 & \cdots & x^{n-1} \\ 0 & 1 & x & \cdots & x^{n-2} \\ \hline 0 & 0 & 0 & & 1 \end{pmatrix}, \qquad r = \begin{pmatrix} x^{n-1} & 0 & 0 & \cdots & 0 \\ x^{n-2} & x^{n-2} & 0 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ x & x & & & 0 \\ 1 & 1 & & & 1 \end{pmatrix}. \tag{AIV.17}$$

We can call the Toeplitz matrix l the left fanned derivative and rA the right fanned derivative of P. We have

$$\Box Dl = 0, \qquad \Box D(rA) = 0. \tag{AIV.18}$$

Fanned derivatives can be expressed in terms of symmetric quaternionic matrices λ and ρ instead of triangular matrices l and r.

We can write

$$dP = \tfrac{1}{2}\,\mathrm{Tr}(\lambda\, dx\, \rho) \qquad (\lambda = \lambda^T,\ \rho = \rho^T), \tag{AIV.19}$$

$$\Box D\lambda = 0, \qquad \Box D\rho = 0, \tag{AIV.20}$$

where λ is a symmetric quaternionic Toeplitz matrix given by

$$\lambda = \begin{pmatrix} 1 & x & x^2 & x^3 & \cdots & x^{n-1} \\ x & 1 & x & x^2 & \cdots & x^{n-2} \\ x^2 & x & 1 & x & \cdots & \cdots \\ x^3 & x^2 & x & 1 & \cdots & \cdots \\ & \vdots & & & & \vdots \\ x^{n-1} & x^{n-2} & & \cdots & \cdots & 1 \end{pmatrix}, \tag{AIV.21}$$

and ρ has the form

$$\rho = \begin{pmatrix} 2a_1 & a_2 & \cdots & a_n \\ a_2 & 2xa_2 & \cdots & xa_n \\ \vdots & & 2x^{n-2}a_{n-1} & x^{n-2}a_n \\ a_n & xa_n \cdots & x^{n-2}a_n & 2x^{n-1}a_n \end{pmatrix}. \tag{AIV.22}$$

It may be interesting to note that symmetric quaternionic matrices that appear in the self-duality condition also make their appearance in the generalized matrix derivatives of a Fueter polynomial. The appearance of Toeplitz matrices is perhaps to be expected here since they have arisen in many areas of mathematics, such as complex analysis and K theory. Here we see their emergence in quaternionic analysis.

For a reference on Toeplitz matrices, see H. Widom, *in "Complex Analysis and Its Applications*, Vol. III, p. 319, IAEA, Vienna, 1976.

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