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Mathematics

Deformations of instantons

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ABSTRACT A study is made of the self-dual Yang-Mills fields in Euclidean 4-space. For SU(2) gauge theory it is rigorously shown that the solutions depend on $8k - 3$ parameters, where k is the Pontrjagin index.

There has been considerable interest recently in the instanton or pseudo-particle solutions of the classical Yang–Mills equations in Euclidean 4-space (refs. 1, 2, and 3). In geometrical terms, these equations are the variational equations for the norm-square $\|F\|^2$ of the curvature F of a fiber-bundle with group G and connection A over R^4. In physics terminology, $\|F\|^2$ is the action, F the gauge field, A the gauge potential, and G the gauge group. The cases studied in most detail are for $G = SU(n)$ and, particularly, $G = SU(2)$.

The connection A is assumed to be asymptotically flat in an appropriate sense so that $F \to 0$ at ∞ and $\|F\|^2 < \infty$. Since the variational equations are conformally invariant with respect to change in the metric on R^4, the most natural geometrical restriction to impose on A at ∞ is that it extends to a connection for a bundle over the 4-sphere S^4. The topological type of such a bundle is then determined by a homotopy class of maps $S^3 \to G$, which is given by an integer k when $G = SU(n)$ (and more generally for any simple compact nonabelian Lie group): this is referred to by physicists as the Pontrjagin index (differing from the topologist's terminology by a factor of 2).

Using the duality $*$-operator on R^4 or S^4, we can decompose F into $F^+ \oplus F^-$, *where* $*F^+ = F^+$ and $*F^- = -F^-$. Clearly, $\|F\|^2 = \|F^+\|^2 + \|F^-\|^2$, while the Pontrjagin index k is given by[§]

$$k = \frac{1}{8\pi^2}\{\|F^+\|^2 - \|F^-\|^2\}.$$

Hence, $\|F^+\|^2 \geq 8\pi^2 k$, and the minimum is attained only if $F^- = 0$ or $F^+ = 0$. Solutions with $F^- = 0$ are called self-dual solutions and have been constructed for all $k \geq 0$. For $k = 0$ we have the trivial solution $F = 0$, for $k = 1$ we have the "instanton," and for $k > 1$ we have "multi-instantons." The most general explicit solutions constructed so far are those of Jackiw *et al.* (ref. 3), which depend on $5k + 4$ parameters.[¶] Our main result is that the complete set of solutions depends on $8k - 3$ parameters. This confirms some preliminary results of Jackiw and Rebbi (ref. 4) and Schwartz (ref. 5).

RESULTS

If a connection A yields a self-dual Yang–Mills field F, then so does any connection $g(A)$ where g is a bundle automorphism (or gauge transformation). The space of all solutions A modulo the action of this gauge group will, as usual in such geometric problems, be called the space of moduli. Our main result can now be formulated as a precise theorem:

[§] Physicists use a different norm and get a factor $1/(16\pi^2)$.

[¶] The formula is a little different for $k = 1,2$.

THEOREM. *The space of moduli of self-dual* $SU(2)$-*Yang–Mills fields over* S^4, *with Pontrjagin index* $k \geq 1$, *is a manifold of dimension* $8k - 3$.

The standard deformation theory approach to such problems is to consider the linearized equations modulo the infinitesimal gauge transformations. Here this leads to a three-step elliptic complex

$$0 \to \mathcal{G} \xrightarrow{D_0} \mathcal{G}\otimes\Omega^1 \xrightarrow{D_1} \mathcal{G}\otimes\Omega^2_- \to 0$$

where Ω^1 denotes 1-forms, Ω^2_- denotes anti-self-dual 2-forms on S^4, and $\mathcal{G}$ is the Lie algebra of G. The operator D_0 is the covariant derivative and D_1 is the anti-self-dual part of the covariant derivative ($D_1 D_0 = 0$ because we are using a self-dual connection). The index theorem of Atiyah–Singer (ref. 6) yields the alternating sum formula $h^0 - h^1 + h^2 = 3 - 8k$. Here h^0 is just the dimension of the null space of D_0, and this is zero unless the SU(2)-bundle is trivial (which is excluded for $k \geq 1$). h^1 is the potential number of moduli and h^2 is the dimension of the null space of D_1^*. Fortunately, in our case, a Bochner type vanishing theorem works very well and we find $h^2 = 0$. This gives $h^1 = 8k - 3$, showing that this is the dimension of the solutions of the linearized problem.

Now we appeal to the general theorem of Kuranishi (ref. 7), which, when $h^2 = 0$, guarantees that the infinitesimal variations really integrate to give genuine local variations. Moreover, the Kuranishi theorem asserts that the family of solutions thus obtained is (locally) complete and effective (non-redundant). This leads to the theorem as stated above.

Note. The theorem does not assert that, for each k, the space of moduli is connected: In principle it may have several components. For $k = 1$ it is in fact connected and is the hyperbolic 5-space. See Yang (ref. 8) for a different discussion of the case $k = 1$.

FURTHER REMARKS

The above arguments apply equally to SU(n), provided we have an irreducible connection (so that $h^0 = 0$), i.e., one that does not come trivially from SU($n - 1$). We then find $h^1 = 4nk - n^2 + 1$. Moreover, the existence of irreducible SU(n) solutions can be deduced from this formula provided $k \geq (n - 1)/2$. In the opposite direction one can deduce nonexistence for $k < n/4$.

Similar methods, i.e., index theorem plus vanishing theorem, yield a formula for the dimension d of the space of zero-eigenvalue fermions (harmonic spinors): one finds $d = k$.

The problem of explicitly constructing the $(8k - 3)$-parameter families of solutions, whose existence is asserted by our theorem, can be treated by converting it into a problem in algebraic geometry (M. F. Atiyah and R. Ward, unpublished).

The 4-sphere can be replaced by other 4-manifolds M (compact, oriented, Riemannian). If M is also a spin-manifold, then we have the two spin SU(2)-bundles, P^+, P^-. One can show that P^+ (with the Riemannian connection) is self-dual if and only if the metric on M is an Einstein metric ($R_{ij} = \lambda g_{ij}$) and that P^- is then anti-self-dual. The deformation theory above can then be applied to P^+ to obtain the number of moduli. The vanishing theorem for a self-dual field ($h^2 = 0$) still applies provided that the conformal Weyl tensor of M is self-dual and that the scalar curvature is positive.

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