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NONLINEAR σ-MODELS AND THEIR GAUGING IN AND OUT OF SUPERSPACE

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We analyze and generalize bosonic nonlinear σ-models and their $N=1,2$ supersymmetric extensions in (4 spacetime-dimensional) $N=1$ superspace. We give a general construction of nonminimal kinetic terms for gauge fields and of $N=1,2$ gauging of isometries on Kähler and hyper-Kähler manifolds. In particular, we study the gauging of noncompact groups. We derive the complete component action and supertrace formula. For $N=2$ models, the supertrace *always* vanishes.

1. Introduction

From a theoretical point of view, supersymmetry is very appealing: It is the only symmetry that unifies bosons and fermions. In contrast, from a phenomenological point of view supersymmetry has been awkward at best: In all "realistic" models, superpartners of the observed particles are added and then a great deal of effort is expended to hide them. One proposal to avoid these difficulties has focused on low-energy effective lagrangians [1] described by supersymmetric nonlinear σ-models coupled to gauge fields [2, 3].

The work of ref. [2] started the interest in this subject. They describe the *minimal* coupling of gauge fields to arbitrary supersymmetric nonlinear σ-models. In this paper we generalize these results: We give a complete superspace treatment, include nonminimal couplings and discuss the implications of extended supersymmetry. We pay particular attention to noncompact gauge groups.

The plan of the paper is as follows: We begin by describing our notation and reviewing renormalizable supersymmetric models. Sect. 3 deals with bosonic σ-models and the gauging of their isometries. In particular, we discuss the gauging

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of noncompact symmetry groups and show how symmetry breaking guarantees the absence of negative norm states. In sect. 4 we construct general nonminimal vector kinetic terms, needed for noncompact gaugings as well as extended supergravity theories. Sect. 5 describes $N=1$ supersymmetric nonlinear σ-models in superspace and the corresponding Kähler geometry, including a careful discussion of holomorphic Killing vectors. In sect. 6 we gauge these isometries*. Sect. 7 contains some examples and sect. 8 contains a discussion of holomorphic nonminimal vector kinetic terms. In sect. 9 we introduce $N=2$ supersymmetry in the context of the kählerian vector multiplet [5]. Sect. 10 describes $N=2$ supersymmetric σ-models and hyper-Kähler geometry and sect. 11 discusses the gauging of isometries on hyper-Kähler manifolds [6]. The example of the Eguchi–Hanson instanton [7] is treated in detail. In sect. 12 we derive the component action, find the vacuum conditions and discuss symmetry breaking. Sect. 13 presents the general supertrace formula [8]. In an appendix we describe rudiments of complex geometry [9].

2. Renormalizable supersymmetric models

We review our notation for globally supersymmetric models in superspace [10]. We begin by considering renormalizable systems of n chiral superfields Φ^i, $i=1,\dots n$, $\bar{D}_{\dot\alpha}\Phi^i=0$, their complex conjugates $\bar{\Phi}_i$ and scalar gauge fields V^A (where A is a group index). Here $\bar{D}_{\dot\alpha}$ is an antichiral spinor derivative obeying the anticommutation relations

$$\{D_\alpha, \bar{D}_{\dot\alpha}\} = i\partial_{\alpha\dot\alpha}\,, \qquad \{D_\alpha, D_\beta\} = \{\bar{D}_{\dot\alpha}, \bar{D}_{\dot\beta}\} = 0 \tag{2.1}$$

(we use spinor notation for vector indices). In our conventions

$$\begin{gathered} D_\alpha D_\beta = \tfrac{1}{2} C_{\beta\alpha} C^{\gamma\delta} D_\delta D_\gamma \equiv C_{\beta\alpha} D^2\,, \qquad D_\alpha D_\beta D_\gamma = 0\,, \\ \partial_{\alpha\dot\beta}\partial^\alpha{}_{\dot\gamma} = \tfrac{1}{2} C_{\dot\beta\dot\gamma}\partial_{\alpha\dot\alpha}\partial^{\alpha\dot\alpha} \equiv C_{\dot\beta\dot\gamma}\Box\,, \qquad C_{\alpha\beta} = -i\varepsilon_{\alpha\beta}\,, \\ [D_\alpha, \bar{D}^2] = i\bar{D}_{\dot\alpha}\partial^{\dot\alpha}_{\alpha}\,, \\ D^2\bar{D}^2 + \bar{D}^2 D^2 - D^\alpha \bar{D}^2 D_\alpha = \Box\,, \qquad \bar{D}^{\dot\alpha} D^2 \bar{D}_{\dot\alpha} = D^\alpha \bar{D}^2 D_\alpha\,. \end{gathered} \tag{2.2}$$

The chiral superfields Φ^i transform under a representation of the Yang–Mills gauge group:

$$\Phi'^i = (e^{i\Lambda})^i{}_j \Phi^j\,, \qquad \bar{\Phi}'_i = \bar{\Phi}_j (e^{-i\bar\Lambda})^j{}_i\,, \tag{2.3}$$

where $\Lambda^i{}_j = \Lambda^A (T_A)^i{}_j$, Λ^A are chiral parameters and $(T_A)^i{}_j$ are representation matrices of the generators of the group G:

$$[T_A, T_B] = if_{AB}{}^C T_C. \tag{2.4}$$

* A component treatment is given in ref. [2] and a partial superspace treatment is given in [4].

As the gauge parameters are complex $\Lambda \neq \bar{\Lambda}$, the group G acts on the superfields Φ^i, $\bar{\Phi}_i$ through its *complexification* G^C with generators T_A and iT_A. If G is compact then iT_A generate noncompact transformations. We introduce the notation G_Λ ($G_{\bar{\Lambda}}$) for the Λ ($\bar{\Lambda}$) group. The gauge potential V^A transforms as an "einbein" converting Λ-transformations to $\bar{\Lambda}$-transformations [11]

$$e^{V'} = e^{i\bar{\Lambda}} e^V e^{-i\Lambda}, \qquad V = V^A T_A, \qquad \Lambda = \Lambda^A T_A. \tag{2.5}$$

Here, the T_A are adjoint representation matrices. To construct invariants we define superfields $\tilde{\bar{\Phi}}_i$ in terms of $\bar{\Phi}_i$ and the gauge potential V^A that transform under G_Λ (rather than $G_{\bar{\Lambda}}$) just as Φ^i do:

$$\begin{aligned}\tilde{\bar{\Phi}}_i &= \bar{\Phi}_j (e^V)^j{}_i, \qquad V^j{}_i = V^A (T_A)^j{}_i \\ \tilde{\bar{\Phi}}'_i &= \tilde{\bar{\Phi}}_j (e^{-i\Lambda})^j{}_i.\end{aligned} \tag{2.6}$$

A renormalizable gauge invariant action is

$$\int d^4x\, d^4\theta [\tilde{\bar{\Phi}}_i \Phi^i + \nu \operatorname{Tr} V^j{}_i] + \left\{ \int d^4x\, d^2\theta [P(\Phi^i) + \tfrac{1}{4} W^{A\alpha} W^A_\alpha] + \text{h.c.} \right\}. \tag{2.7}$$

Here $\nu \operatorname{Tr} V^j{}_i$ is the Fayet-Iliopoulos term, $P(\Phi^i)$ is the gauge invariant chiral superpotential (renormalizability requires P to be at most cubic), and $W^{A\alpha} = i\bar{D}^2(e^{-V} S^\alpha e^V)^A$ is the gauge field strength.

We can define gauge covariant derivatives $\nabla_A = (\nabla_\alpha, \nabla_{\dot{\alpha}}, \nabla_{\alpha\dot{\alpha}})$ that are covariant with respect to G_Λ

$$\nabla_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}}, \qquad \nabla_\alpha = e^{-V} D_\alpha\, e^V, \qquad \nabla_{\alpha\dot{\alpha}} = -i\{\nabla_\alpha, \nabla_{\dot{\alpha}}\}. \tag{2.8}$$

Their hermitian conjugates are covariant with respect to $G_{\bar{\Lambda}}$.

We define components of the gauge multiplet by

$$A^A_{\alpha\dot{\alpha}} = i(\nabla_{\alpha\dot{\alpha}}| - \partial_{\alpha\dot{\alpha}})^A, \qquad \lambda^A_\alpha = W^A_\alpha|, \qquad D'^A = -\tfrac{1}{2} i \{\nabla^\alpha, W_\alpha\}^A|, \tag{2.9}$$

where, for an arbitrary expression H, $H|$ is the *θ-independent* part of H. From (2.8)-(2.9) it follows that

$$f_{\alpha\beta} = \tfrac{1}{2} \nabla_{(\alpha} W_{\beta)}|, \qquad i\nabla^{\dot{\alpha}}_\alpha \bar{\lambda}_{\dot{\alpha}} = \tfrac{1}{2}[\nabla^\beta, \{\nabla_\beta, W_\alpha\}]|, \tag{2.10}$$

where $f_{\alpha\beta} = \frac{1}{2}(\partial_{(\alpha\dot{\alpha}} A_{\beta)}{}^{\dot{\alpha}} - i[A_{(\alpha\dot{\alpha}}, A_{\beta)}{}^{\dot{\alpha}}])$ is the self-dual part of the component gauge field strength (we use symmetrization notation $H_{(\alpha\beta)} = H_{\alpha\beta} + H_{\beta\alpha}$). Similarly we define the components of the chiral fields as

$$A^i = \Phi^i|, \qquad \psi^i_\alpha = \nabla_\alpha \Phi^i|, \qquad F^i = \nabla^2 \Phi^i|. \tag{2.11}$$

From (2.8)-(2.11) and the relations

$$\begin{aligned}\int d^4x\, d^4\theta\, \mathscr{L}_{\text{real}} &= \tfrac{1}{4} \int d^4x \{\nabla^\alpha, [\nabla_\alpha, \{\nabla^{\dot{\alpha}}, [\nabla_{\dot{\alpha}}, \mathscr{L}_{\text{real}}]\}]\}|, \\ \int d^4x\, d^2\theta\, \mathscr{L}_{\text{chiral}} &= \tfrac{1}{2} \int d^4x \{\nabla^\alpha, [\nabla_\alpha, \mathscr{L}_{\text{chiral}}]\}|,\end{aligned} \tag{2.12}$$

valid for any gauge *invariant* lagrangians $\mathscr{L}_{\mathrm{real}}$ and $\mathscr{L}_{\mathrm{chiral}}$, we can expand the superspace action (2.7) in components. The component field equations can be found either by varying the resulting action, or by expanding the superfield equations that follow directly from (2.7) in components:

$$0=\frac{\delta S}{\delta \Phi^i}=\bar{D}^2\tilde{\Phi}_i+P_i\,,\qquad P_i=\frac{\partial P}{\partial \Phi^i}\,,$$

$$0=\frac{\delta S}{(\mathrm{e}^{-V}\delta \mathrm{e}^{V})^A}=\tilde{\Phi}_i(T_A)^i{}_j\Phi^j-i\nabla^\alpha W^A_\alpha\,. \tag{2.13}$$

To find the field equations, we have used a covariant variation with respect to V [10].

Using the transformations (2.5) we can go to Wess–Zumino gauge where

$$V|=D_\alpha V|=\bar{D}_{\dot\alpha}V|=D^2V|=\bar{D}^2V|=0\,. \tag{2.14}$$

In this gauge (2.9) and (2.11) become

$$A_{\alpha\dot\alpha}=\bar{D}_{\dot\alpha}D_\alpha V|=\tfrac{1}{2}[\bar{D}_{\dot\alpha},D_\alpha]V|\,,$$

$$\lambda_\alpha=i\bar{D}^2D_\alpha V|\,,\qquad D'=\tfrac{1}{2}D^\alpha\bar{D}^2D_\alpha V|\,,$$

$$A^i=\Phi^i|\,,\qquad \psi^i_\alpha=D_\alpha\Phi^i|\,,\qquad F^i=D^2\Phi^i| \tag{2.15}$$

and (2.12) becomes

$$\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathscr{L}_{\mathrm{real}}=\int \mathrm{d}^4x\,D^2\bar{D}^2\mathscr{L}_{\mathrm{real}}|\,,$$

$$\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathscr{L}_{\mathrm{chiral}}=\int \mathrm{d}^4x\,D^2\mathscr{L}_{\mathrm{chiral}}|\,. \tag{2.16}$$

3. Nonlinear σ-models, isometries and gaugings

In general we have to consider nonrenormalizable interactions. These can arise as low-energy effective interactions induced by either quantum corrections or supergravitational effects. To study these theories, we use some elementary geometry.

In this section we study *bosonic* nonlinear σ-models. For familiar examples the scalar fields are constrained to take values in some symmetric space, such as the surface of the sphere or the complex projective space CP^n. More generally, one can consider n real dimensionless scalar fields $\varphi^i(x)$ with kinetic term

$$S=-\tfrac{1}{4}\mu^2\int \mathrm{d}^4x\,g_{ij}(\varphi)\partial^a\varphi^i\partial_a\varphi^j\,, \tag{3.1}$$

where $g_{ij}(\varphi)$ is some positive-definite field-dependent matrix and μ is a constant with the dimension of mass. The fields φ^i can be regarded as the coordinates of some n-dimensional manifold with riemannian metric g_{ij}.

For renormalizable models, which are defined on flat internal manifolds with $g_{ij}=\delta_{ij}$, the kinetic term is invariant under translations and rotations of the scalar fields (ISO(n), the isometry group of flat space). Usually the interaction potential restricts the symmetry to a subgroup. In general, for nonlinear σ-models with $g_{ij}\neq\delta_{ij}$, the invariances of the kinetic term (3.1) are the isometries of the manifold with metric g_{ij}.

The isometry group G has, at each point φ^i of the manifold, an isotropy subgroup H_φ consisting of the symmetries that leave the point φ^i fixed. H_φ is a subgroup of SO(n), the group of rotations in n dimensions. Local coordinates can therefore be introduced where φ^i is at the origin and H_φ acts *linearly* by matrix multiplication, just as on flat manifolds:

$$\delta\varphi^i = i\lambda^X (T_X)^i{}_j\varphi^j\,, \tag{3.2}$$

where $(T_X)^i{}_j$ are the hermitian generators of the group H in some n-dimensional representation and λ^X are *constant* (x and φ independent) parameters. These symmetries leave the point $\varphi^i=0$ fixed. The remaining symmetries in G move the point $\varphi^i=0$ and are thus nonlinearly realized. The isotropy subgroup can be different at different points, e.g., when the manifold is an ellipsoid of revolution. In particular, it may of course be of zero dimension.

Choosing a point, the isotropy subgroup H at that point can be gauged by the usual minimal coupling procedure; to gauge nonlinear realizations (which leave no point fixed) we need a geometric formulation valid in any coordinate system. The infinitesimal action of G can be written

$$\delta\varphi^i = \lambda^A k^i_A(\varphi)\,, \tag{3.3}$$

where $k^i_A(\varphi)$, $A=1,2,\dots,\dim(\mathrm{G})$, are Killing vectors on the manifold. For the linearly realized subgroup H, comparison with (3.2) gives, in special coordinates,

$$k^i_X(\varphi) = i(T_X)^i{}_j\varphi^j\,, \qquad X=1,2,\dots,\dim(H)\,. \tag{3.4}$$

The action (3.1) is invariant under (3.3) if

$$\delta g_{ij} = k^k_A g_{ij,k} + k^k_{A,j}g_{ik} + k^k_{A,i}g_{jk} = 0\,, \tag{3.5}$$

which can be rewritten as

$$k_{Ai;j} + k_{Aj;i} = 0\,, \tag{3.6}$$

where the comma denotes partial differentiation and the semi-colon covariant differentiation using the usual Christoffel connection of the scalar manifold. Eq. (3.6) is Killing's equation.

The transformation (3.3) can be rewritten as

$$\delta\varphi^i = [\lambda^A k^j_A \partial/\partial\varphi^j,\, \varphi^i] \equiv L_{\lambda\cdot k}\varphi^i\,, \tag{3.7}$$

where $L_{\lambda\cdot k}$ is the Lie derivative along $\lambda^A k_A$. The Killing vectors generate the group G:

$$[k_A, k_B]^i \equiv k^j_A k^i_{B,j} - k^j_B k^i_{A,j} = f_{AB}{}^C k^i_C\,, \tag{3.8}$$

where $f_{AB}{}^C$ are the structure constants. Exponentiation yields a finite symmetry transformation:

$$\varphi'^i = \exp\left(L_{\lambda\cdot k}\right)\varphi^i\,. \tag{3.9}$$

If G is compact, all finite group elements which are connected to the identity can be written as in (3.9), while if G is noncompact, a group element is, in general, a product of a finite number of such exponentials.

We now consider gauging of an arbitrary subgroup of the isometry group of the scalar manifold. In the remainder of this section and in the next section we use G to denote the *gauge group* rather then the entire isometry group, and H to denote the restriction of the isotropy group to G. (If a potential term is present in the action, we require it to be invariant under G, see below.) The kinetic term (3.1) can be made locally gauge invariant (λ can be promoted to $\lambda(x)$, an arbitrary function of x) by replacing the partial derivative $\partial_{\underline{a}}\varphi^i$ by the gauge covariant derivative

$$\nabla_{\underline{a}}\varphi^i = \partial_{\underline{a}}\varphi^i - A^B_{\underline{a}} k^i_B = \partial_{\underline{a}}\varphi^i - [A^B_{\underline{a}} k^j_B \partial/\partial\varphi^j, \varphi^i]\,, \tag{3.10}$$

where $A^B_{\underline{a}}$ is a gauge connection for the group G, transforming under an infinitesimal gauge transformation as

$$\delta A^B_{\underline{a}} = \partial_{\underline{a}}\lambda^B(x) + f_{CD}{}^B A^C_{\underline{a}}\lambda^D(x)\,. \tag{3.11}$$

For a linearly realized subgroup H, (3.10) reduces (in special coordinates) to the usual minimal coupling prescription $\partial_{\underline{a}}\varphi^i \to \nabla_{\underline{a}}\varphi^i = \partial_{\underline{a}}\varphi^i - iA^X_{\underline{a}}(T_X)^i{}_j\varphi^j$.

Under the finite transformation $\varphi' = g\varphi$ the covariant derivative transforms as

$$\nabla'_{\underline{a}} = g\nabla_{\underline{a}} g^{-1}\,. \tag{3.12}$$

If G is compact, one can add the minimal Yang–Mills kinetic term

$$-\tfrac{1}{4}\int d^4x\, g_{AB} F^A_{\underline{ab}} F^{B\underline{ab}}\,, \tag{3.13}$$

where

$$F_{\underline{ab}}{}^B = \partial_{\underline{a}} A^B_{\underline{b}} - \partial_{\underline{b}} A^B_{\underline{a}} + f_{CD}{}^B A^C_{\underline{a}} A^D_{\underline{b}} \tag{3.14}$$

and g_{AB} is the *Killing metric* on G

$$g_{AB} = -f_{AC}{}^D f_{BD}{}^C\,. \tag{3.15}$$

If G is noncompact, however, the Killing metric will have negative or zero eigenvalues (or both), so that the gauge invariant action (3.13) will lead to ghosts or nonpropagating vector fields (or both). However, with a *nonminimal* kinetic term [12],

$$-\tfrac{1}{4}\int d^4x\, T_{AB}(\varphi) F^A_{\underline{ab}} F^{B\underline{ab}}\,, \tag{3.16}$$

where $T_{AB}(\varphi)$ is a positive-definite matrix, we obtain a ghost-free action for noncompact, as well as compact, groups. (We can of course consider nonminimal kinetic terms for compact groups as well.)

We can also add terms of the form

$$-\tfrac{1}{8}\int d^4x\, S_{AB}(\varphi)F^A_{\underline{ab}}F^B_{\underline{cd}}\varepsilon^{\underline{abcd}} \tag{3.17}$$

to the action. These are not surface terms if $S_{AB}(\varphi)$ is not constant. Nonminimal terms of the form (3.16) and (3.17) both occur in extended supergravity for $N \geq 4$. Defining

$$Q_{AB}(\varphi) = T_{AB}(\varphi) + iS_{AB}(\varphi)\,, \tag{3.18}$$

eqs. (3.16) and (3.17) can be combined to give, using 2-component spinor notation,

$$-\tfrac{1}{4}\int d^4x(Q_{AB}F^A_{\alpha\beta}F^{B\alpha\beta} + \bar{Q}_{AB}F^A_{\dot{\alpha}\dot{\beta}}F^{B\dot{\alpha}\dot{\beta}})\,. \tag{3.19}$$

The field strength $F^A_{\alpha\beta}$ transforms in the adjoint representation of G

$$\delta F^A_{\alpha\beta} = f_{BC}{}^A F^B_{\alpha\beta}\lambda^C(x)\,, \tag{3.20}$$

so that (3.19) is gauge invariant if Q_{AB} transforms as

$$\delta Q_{AB} = Q_{AB,i}k^i_C\lambda^C = \lambda^C(f_{CA}{}^D Q_{BD} + f_{CB}{}^D Q_{AD})\,. \tag{3.21}$$

The explicit construction of the $Q_{AB}(\varphi)$ is described in the next section. There it will be shown that a ghost-free gauge invariant action can be constructed for any (sufficiently well behaved) isometry group G (which need not be compact or semisimple) acting on any M (with riemannian signature).

We now consider spontaneous symmetry breaking. The action of G divides M into gauge equivalence classes or *orbits* consisting of points that can be transformed into each other. Let H_φ denote the isotropy subgroup of the *gauge* group G at the point φ. If $\varphi_0 \equiv \langle\varphi\rangle$ is the vacuum then the gauge group G breaks to $H_0 \equiv H_{\varphi_0}$ and one scalar degree of freedom is absorbed by the gauge fields for each broken generator. For example, if the scalars lie in the (connected) coset space G/H_0 then, as any point can be obtained by acting with some $g \in G$ on φ_0, all the scalars are pure gauge and can be set to zero by a G gauge transformation. In this case the orbit of φ_0 is the entire manifold, and the space of orbits is just one point. In general, the *unphysical* scalars are the coordinates of the orbit passing through φ_0, and the physical scalars are the coordinates of the space of gauge equivalence classes (i.e. orbits).

An interaction potential $V(\varphi)$ can be added to the lagrangian without altering the above analysis, provided the potential is invariant under G

$$\delta V = \lambda^A k^i_A \frac{\partial V}{\partial \varphi^i} = 0\,. \tag{3.22}$$

By choosing the potential V appropriately, we can arrange for any given value φ_0 of φ to be a critical point of V.

We can study how the unphysical scalars disappear and the vectors become massive by examining the quadratic vector lagrangian. From (3.1, 10, 19), we have

$$\mathcal{L}^{(2)} = -\tfrac{1}{4}(Q_{AB}(\varphi_0)F^A_{\alpha\beta}F^{B\alpha\beta} + \bar{Q}_{AB}(\varphi_0)F^A_{\dot\alpha\dot\beta}F^{B\dot\alpha\dot\beta} + \mu^2 g_{ij}(\varphi_0)k^i_B(\varphi_0)k^j_C(\varphi_0)A^B_{\alpha\dot\alpha}A^{C\alpha\dot\alpha}) . \tag{3.23}$$

Neglecting the total divergence $S_{AB}(\varphi_0)F^A_{\underline{ab}}F^B_{\underline{cd}}\varepsilon^{\underline{abcd}}$

$$\mathcal{L}^{(2)} = -\tfrac{1}{4}(\delta_{BC}F'^B_{\underline{ab}}F'^{C\underline{ab}} + M_{BC}A'^B_{\alpha\dot\alpha}A'^{C\alpha\dot\alpha}) , \tag{3.24}$$

where

$$A'^B_{\underline{a}} = Z^B{}_C A^C_{\underline{a}} ,$$
$$Z^A{}_B Z^D{}_C \delta_{AD} = T_{BC} ,$$
$$F'^B_{\underline{ab}} = \partial_{\underline{a}} A'^B_{\underline{b}} - \partial_{\underline{b}} A'^B_{\underline{a}} \tag{3.25}$$

and the vector mass matrix is given by

$$M_{AB} = \mu^2[g_{ij}k^i_C k^j_D (Z^{-1})^C{}_A (Z^{-1})^D{}_B]|_{\varphi=\varphi_0} . \tag{3.26}$$

As $T(\varphi_0)_{AC}$ is a positive-definite symmetric matrix, $Z^A{}_B$ can also be chosen positive-definite and symmetric. The point $\varphi = \varphi_0$ is a fixed point of the Killing vector fields that correspond to the isotropy group H_{φ_0}. The remaining Killing vectors are non-zero at $\varphi = \varphi_0$ and each has a positive norm, $g_{ij}k^i k^j|_{\varphi=\varphi_0} \geqslant 0$ because, by assumption, g_{ij} is positive-definite (see below). Since $(Z^{-1})^A{}_B$ is positive-definite, this implies that the vector fields corresponding to the unbroken gauge group H_{φ_0} remain massless, while the remaining vectors, corresponding to the spontaneously broken symmetries, acquire masses proportional to μ.

The symmetry given by some one-parameter subgroup of G can only have an unbroken phase if the corresponding Killing vector has a fixed point on the scalar manifold. The fixed point structure of a space, however, is intimately related to its topology so that there may be topological obstructions to certain symmetry breaking schemes.

It is interesting to consider whether it is possible to use indefinite metrics g_{ij} on the scalar manifold, with the gauging arranged so that all the negative energy scalars are absorbed by vector fields in the Higgs mechanism, leaving a positive-definite metric on the space of physical scalars. However, the corresponding Killing vectors would have negative norm, $k_A{}^i k_A{}^j g_{ij} < 0$, so that from (3.26) we see that the vector fields that absorb these scalars would have tachyonic masses and give a physically unacceptable model.

4. Nonminimal vector kinetic terms

In this section, we construct the matrices $Q_{AB}(\varphi)$ that appear in (3.16)-(3.19). To avoid ghosts, $Q+\bar{Q}$ must be positive definite. The construction is a generalization of that given by Julia and Luciani for coset spaces and semisimple isometry groups; we consider general manifolds with positive definite signature and arbitrary isometry groups [12]. The basic idea of the construction is to choose a gauge and then define Q_{AB} on the resulting subspace (the space of orbits) M_0. Q_{AB} is extended to the whole manifold by acting with the gauge group G (cf. (3.21)). We first give the construction when the isotropy subgroup H of G is the same at all points and M is a product space $M=M_0\times G/H$, then generalize to the case when M is locally a product space (fiber bundle) and finally we allow the isotropy group to vary over M.

Suppose M is a direct product of some manifold M_0 with the coset space G/H, where G is the gauge group (and H the isotropy group). If χ^p and σ^i are coordinates on M_0 and on G/H respectively, then (χ,σ) are coordinates on M. To find Q_{AB} on M we choose a gauge such as $\sigma=0$ and determine $Q_{AB}(\chi,0)=Q^0_{AB}(\chi)$ on this subspace M_0. M_0 is a *cross section*: Acting on it with G sweeps out the whole of M - an arbitrary point (χ,σ) can be obtained from $(\chi,0)$ by the action of a group element $g(\sigma)$ (unique up to, e.g., right multiplication by some $h\in H$, $g(\sigma)\to g(\sigma)h$). Integrating the condition (3.21) for gauge invariance of the vector kinetic term gives Q_{AB} on M:

$$Q_{AB}(\chi,\sigma)=Q^0_{CD}(\chi)D_A^{-1C}[g(\sigma)]D_B^{-1D}[g(\sigma)]\,, \tag{4.1}$$

where $D[g]$ is the adjoint representation matrix for G. For this to be consistent, Q_{AB} should be unchanged if $g(\sigma)$ is replaced by $g(\sigma)h$ in (4.1), for any $h\in H$. This requires Q^0 to be H-invariant. Hence the problem of constructing the nonminimal vector kinetic term is reduced to finding an H-invariant Q^0 on M_0.

To illustrate the construction of Q^0 on M_0, we consider the case where the isotropy group H is $SO(n-1)$ and G is $SO(n)$, $SO(n-1,1)$ or $ISO(n-1)$. When M_0 is a point, M is an $n-1$ dimensional sphere, hyperboloid or flat euclidean space, respectively. To find an H-invariant Q^0 we decompose $F^A_{\alpha\beta}F^{B\alpha\beta}\equiv F^AF^B$ (transforming as the symmetric bi-adjoint of G) into *representations of* H and select the singlets. Explicitly, for our examples we find an adjoint and a vector of $SO(n-1)$:

$$\begin{aligned} &F^{\hat{i}\hat{j}}\to F^{ij},F^{in}\,; \qquad F^{ij}=-F^{ji}\,,\\ &\hat{i}=1,\dots,n; \qquad i=1,\dots,n-1\,. \end{aligned} \tag{4.2}$$

There are two H-singlets in F^AF^B, namely $F^{ij}F^{ij}$ and $F^{in}F^{in}$ so Q^0_{AB} takes the form:

$$\begin{aligned} &Q^0_{AB}\equiv Q_{\hat{i}\hat{j},\hat{k}\hat{l}}\,,\\ &Q_{ij,kl}=\tfrac14\alpha_1(\chi)(\delta_{ki}\delta_{jl}-\delta_{kj}\delta_{il})\,,\\ &Q_{in,jn}=\alpha_2(\chi)\delta_{ij}. \end{aligned} \tag{4.3}$$

All other terms follow by symmetry or vanish. The vector kinetic term on M_0 is then $\frac{1}{2}\alpha_1(\chi)F^{ij}F^{ij}+\alpha_2(\chi)F^{in}F^{in}+\text{c.c.}$ and will be ghost free (and have positive euclidean action) if Re $\alpha_i>0$. In general, the symmetric product of two irreducible representations will contain a singlet if and only if the representations are conjugate, in which case the singlet is unique. The adjoint representation of G, (which is always real), decomposes into an adjoint representation of H, some real representations and pairs of conjugate complex representations. For each representation of multiplicity q, $\alpha(\chi)$ is a complex $q\times q$ matrix (in the example both representations have multiplicity 1). If G is a semisimple group the singlets can be found from the Killing metric (3.15). Under the decomposition G→H the Killing metric matrix of G decomposes into blocks on the diagonal, each one positive (or negative) definite, and these blocks project out the singlets:

$$g_{AB}\rightarrow\begin{pmatrix} g_1 & 0 & 0 & \cdots \\ 0 & g_2 & 0 & \\ 0 & 0 & g_3 & \\ \vdots & & & \ddots \end{pmatrix},$$

with g_1, say, the Killing metric of H. Then the general H-invariant Q^0_{AB} is:

$$Q^0_{AB}=\begin{pmatrix} \alpha_1 g_1 & 0 & 0 & \cdots \\ 0 & \alpha_2 g_2 & 0 & \\ 0 & 0 & \alpha_3 g_3 & \\ \vdots & & & \ddots \end{pmatrix},$$

with α_i complex functions on M_0. The action is ghost free and the euclidean action S_E is positive on M_0 if

$$\varepsilon_i \,\mathrm{Re}\,(\alpha_i)>0\,, \tag{4.6}$$

where $\varepsilon_i=\pm1$ gives the sign of the eigenvalue of g_i. Positivity of S_E on M_0 implies positivity on the whole of M by gauge invariance.

When G is not semisimple but can be obtained by a Wigner–Inönü contraction from a semisimple group, then the Killing metric of G is degenerate and we use the Killing metric of the uncontracted group. In our example the Killing metric is the $\frac{1}{2}n(n-1)\times\frac{1}{2}n(n-1)$ matrix

$$g_{AB}=\begin{pmatrix} \mathbb{1}_{(n-1)(n-2)/2} & 0 \\ 0 & \xi\mathbb{1}_{n-1} \end{pmatrix}, \tag{4.7}$$

where ξ multiplies the $(n-1)\times(n-1)$ identity matrix and takes the values 1, −1 and 0 respectively so that in the group contraction the limit $\xi\rightarrow0$ is taken. We use the *same* Q^0 for all three cases and (4.3) is:

$$Q^0_{AB}(\chi)=\begin{pmatrix} \alpha_1(\chi)\mathbb{1}_{(n-1)(n-2)/2} & 0 \\ 0 & \alpha_2(\chi)\mathbb{1}_{n-1} \end{pmatrix}, \tag{4.8}$$

with Re $\alpha_i > 0$ (cf. (4.6)). Having constructed an H-invariant Q^0_{AB} on M_0, it can be extended to the whole manifold M using (4.1) or integrating (3.21). We will give this explicitly for our example with $n=3$. In this case G/H is the 2-surface in (x, y, z)-space given by

$$x^2+\xi(y^2+z^2)=1\,. \tag{4.9}$$

Choosing "projective" coordinates $\sigma^1=z/x$, $\sigma^2=-y/x$ and Killing vectors k^i_A for G:

$$\begin{aligned} &k^i_1=\varepsilon^{ij}\sigma^j\,, \qquad \varepsilon^{12}=1\,,\\ &k^i_2=\delta^i_1+\xi\sigma^1\sigma^i\,,\\ &k^i_3=\delta^i_2+\xi\sigma^2\sigma^i\,. \end{aligned} \tag{4.10}$$

The Q_{AB} obtained from Q^0_{AB} is, integrating (3.21) or applying (4.1),

$$\begin{aligned} Q_{AB}(\chi,\sigma)=\xi^{-1}\alpha_2(\chi)&\begin{pmatrix}1&0&0\\0&\xi&0\\0&0&\xi\end{pmatrix}\\ &-\frac{\alpha_2(\chi)-\xi\alpha_1(\chi)}{1+\xi[(\sigma^1)^2+(\sigma^2)^2]}\begin{pmatrix}\xi^{-1}&-\sigma^2&\sigma^1\\-\sigma^2&\xi(\sigma^2)^2&-\xi\sigma^1\sigma^2\\\sigma^1&-\xi\sigma^1\sigma^2&\xi(\sigma^1)^2\end{pmatrix}. \end{aligned} \tag{4.11}$$

Even though the coordinates σ^1, σ^2 are not defined when $x=0$, Q_{AB} is, as can be seen by writing it in terms of the coordinates (x, y, z) satisfying (4.9):

$$Q_{AB}=\xi^{-1}\alpha_2\begin{pmatrix}1&0&0\\0&\xi&0\\0&0&\xi\end{pmatrix}-(\alpha_2-\xi\alpha_1)\begin{pmatrix}\xi^{-1}x^2&xy&xz\\xy&\xi y^2&\xi yz\\xz&\xi yz&\xi z^2\end{pmatrix}. \tag{4.12}$$

When M is a product space only locally, i.e., a fiber bundle with fiber G/H and base space M_0 (see e.g. [13]), we divide M into overlapping regions M_α, each of which *is* a direct product, $M_\alpha=N_\alpha\times G/H$, with N_α some (open) subspace of M_0. We construct Q^α in each region M_α using the procedure described above and ensure that the Q's agree in the overlap regions. Explicitly, if the coordinates of M_α are $(\chi^\alpha, \sigma^\alpha)$ we first choose a local cross section S_α (e.g. $(\chi^\alpha, 0)$) and define an H-invariant $Q^{0\alpha}(\chi^\alpha)$ there. Then acting with G as in (4.1) gives a definition of Q^α on the whole of M_α.

In an overlap region $M_\alpha\cap M_\beta$, the coordinates of M_α and M_β must be related by a coordinate transformation in the base M_0

$$\chi^\beta=\chi^\beta(\chi^\alpha) \tag{4.13}$$

and by a transformation along the fiber

$$g(\sigma^\beta)=k_{\alpha\beta}(\chi^\alpha)g(\sigma^\alpha)\,, \tag{4.14}$$

where $g(\sigma) \in \mathrm{G}$ is a group element representing the point σ in G/H and $k_{\alpha\beta} \in \mathrm{G}$ is a group element known as the transition function. Then Q is well defined if $Q^{\alpha}[\chi^{\alpha}, \sigma^{\alpha}] = Q^{\beta}[\chi^{\beta}(\chi^{\alpha}), \sigma^{\beta}(\sigma^{\alpha}, \chi^{\alpha})]$ which holds provided $Q^{0\alpha}$ and $Q^{0\beta}$ are related by the transition function $k_{\alpha\beta}$ (cf. (4.1)):

$$Q^{0\alpha}_{AB}[\chi^{\alpha}] = Q^{0\beta}_{CD}[\chi^{\beta}(\chi^{\alpha})] D_A^{-1C}[k_{\alpha\beta}(\chi^{\alpha})] D_B^{-1D}[k_{\alpha\beta}(\chi^{\alpha})] . \tag{4.15}$$

Finally, because the $k_{\alpha\beta}$ are consistently defined transition functions for the fiber bundle (see [13]) Q is well defined over the whole manifold, i.e., if the Q^{α} agree in $\mathrm{M}_{\alpha} \cap \mathrm{M}_{\beta}$ and $\mathrm{M}_{\beta} \cap \mathrm{M}_{\gamma}$, they also agree in $\mathrm{M}_{\gamma} \cap \mathrm{M}_{\alpha}$.

We now consider the general case where the isotropy group H_{φ} can be different at different points. Typically this means that the orbits are of different dimensions. We assume the existence of a *slice* S_{φ} at every point $\varphi \in \mathrm{M}$. A physicist might define a slice as a subspace "orthogonal" to the orbit through φ, such that acting on S_{φ} with elements $g \in \mathrm{G}$ in the neighborhood of the identity of G one generates a neighborhood of φ in M, [14]. For a precise definition, see, e.g., [15] and for a review see [16, 17].

The manifold M can be generated from a set of slices using the group G. Thus we can find Q as in the fiber bundle case. We construct Q^0 on the slices and use G to generate Q on the whole of M.

A slice is a generalization of the local cross section of a fiber bundle; when the isotropy group H_{φ} is the same for all φ, M has the fiber bundle structure discussed above. When G is compact, a slice always exists [18], but when G is noncompact this is no longer the case. For example, if there is an orbit such that each neighborhood of φ contains points on the orbit related by a *finite* group transformation, there is clearly no slice through φ. In fact there must then be an infinite set of points on the orbit arbitrarily close to φ, separated from each other by finite group transformations, so it is clear G must be noncompact. An example of when a slice fails to exist is a torus with irrational pitch. Palais [15] has found a wide class of spaces with a noncompact group action for which slices exist.

To illustrate we consider two examples [19].

The first example is euclidean 3-space with cartesian coordinates (x^1, x^2, x^3) and the flat metric $g_{ij} = \delta_{ij}$ and G is the SO(2) group of rotations about the x^3 axis. All points not on the x^3 axis have trivial isotropy group ($\mathrm{H}_{\varphi} = 1$) and lie on orbits that are circles perpendicular to and centered on the x^3 axis. The points on the x^3 axis, however, have isotropy group SO(2) and have trivial orbits consisting of that single point. A slice is then given by the euclidean half-plane with its boundary $\{(x^1, x^2, x^3): x^2 = 0, x^1 \geqslant 0\}$. This illustrates that the slice M_0 can be a manifold with boundary if the isotropy group varies with φ.

A slightly more complicated example is given by euclidean 5-space $(x^1, x^2, x^3, x^4, x^5)$ with G the $\mathrm{SO}(2) \otimes \mathrm{SO}(2)$ group of rotations about the subspace X_1 given by $x^1 = x^2 = 0$ and X_2, given by $x^3 = x^4 = 0$. Points not lying in X_1 or X_2 have trivial isotropy group and lie on orbits that are tori $\mathrm{S}^1 \otimes \mathrm{S}^1$, points lying on either X_1 or X_2 (but not both) have isotropy group SO(2) and lie on circular orbits S^1, while points on $X_1 \cap X_2$ have isotropy group SO(2)⊗SO(2) and have trivial orbits. A slice M_0 is the euclidean quarter 3-space with its boundary given by $\{(x^1, x^2, x^3, x^4, x^5): x^2 = x^4 = 0, x^1 \geq 0, x^3 \geq 0\}$. The slice M_0 can be divided into three open manifolds, the space M_0^1 (the interior of M_0) with $x^2 = x^4 = 0$, $x^1 > 0$, $x^3 > 0$ with trivial isotropy group, the space M_0^2 with $x^1 = x^2 = x^4 = 0$ and $x^3 > 0$ and the space M_0^3 with $x^2 = x^3 = x^4 = 0$ and $x^1 > 0$ both with isotropy group SO(2), and the space M_0^4 given by the x^5 axis with isotropy group SO(2)⊗SO(2). Note that M_0^2 is in the boundary of M_0^1 and M_0^3 is in the boundary of M_0^2 (and M_0^1).

This second example has the essential features of the general case. In general the slice M_0 is a *stratified manifold* (see [14], [16], [17] and references therein cited), i.e., a space that is the union of a partially ordered set of non-intersecting open manifolds:

$$M_0 = \bigcup_\alpha M_0^\alpha \,, \tag{4.16}$$

with each manifold belonging to one or several sequences such that M_0^α is contained in the boundary of all manifolds occuring earlier in the sequence. (The classic example is the space M_0 of $n \times n$ matrices with the stratum being all matrices of rank p.) The isotropy subgroup H_α is uniform over each stratum M_0 and increases in each sequence; $H_\alpha \subset H_\beta$ if $M_0^\beta \subset \partial M_0^\alpha$. This is represented graphically in fig. 1 for the second example. As each H_α is a subgroup of G, the sequence terminates. Since the isotropy group H_α is constant over M_0^α, the action of G on M_0^α generates a fiber bundle with fiber G/H_α and base space M_0^α. Thus Q_{AB} can be constructed on each such bundle exactly as above, by first giving an H_α-invariant Q^0_{AB} on M_0^α and then using the action of G. However, we must choose the Q^0_{AB} on each M_0^α so as to give a continuous Q^0_{AB} on the whole space M_0. We start by constructing Q^0_{AB}

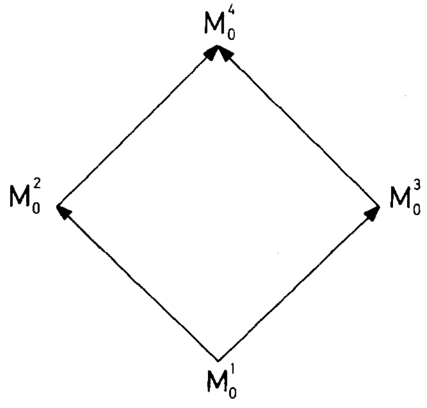

Fig. 1. Representation of a stratified manifold. Arrows indicate the sequence of increasing isotropy subgroups.

in an interior region M_0^α which is not in the boundary of any other M_0^α. When approaching the boundary $\partial\mathrm{M}_0^\alpha$ the limiting value of Q^0_{AB} must be invariant under the isotropy group of the boundary at that point. This defines Q^0_{AB} in all M_0^β's that can be reached by a sequence from M_0^α. We then repeat this construction for all other M_0^γ not in the boundary of any other. In doing this we may have the additional constraints that Q^0_{AB} is already defined on some of the boundaries.

As an example, consider the case in which M is the group manifold of some compact G with coordinates $\varphi_A(g=\exp(iT^A\varphi_A))$ transforming in the adjoint of G, where we gauge the linearly realized diagonal subgroup G of the isometry group G⊗G. In the case G = SU(2), it is easy to see from group theory that the only invariant kinetic term is

$$Q_{AB}F^AF^B+\text{c.c.}=f(|\varphi|)\delta_{AB}F^AF^B+g(|\varphi|)\varphi_A\varphi_BF^AF^B+\text{c.c.}\,, \tag{4.17}$$

where f and g are arbitrary functions of $|\varphi|=(\varphi_A\varphi_B\delta^{AB})^{1/2}$. To obtain this result by the methods of this section, we first choose the slice $\{(\varphi_1,\varphi_2,\varphi_3)\colon \varphi_1=\varphi_2=0, \varphi_3\geqslant 0\}$ which has two strata. One is given by the origin, $\varphi_A=0$, with isotropy group SU(2) so that we must have $Q^0_{AB}\propto\delta_{AB}$. The other stratum is $\{\varphi_A\colon \varphi_1=\varphi_2=0, \varphi_3>0\}$ with isotropy the U(1) generated by T^3, so that here Q^0_{AB} must be of the form

$$Q^0_{AB}(\varphi_3)=\begin{pmatrix}\alpha_1(\varphi_3) & & \\ & \alpha_1(\varphi_3) & \\ & & \alpha_2(\varphi_3)\end{pmatrix}. \tag{4.18}$$

For the correct form at the origin, we must have $\alpha_1(0)=\alpha_2(0)$, so that

$$Q^0_{AB}(\varphi_3)=\alpha_1(\varphi_3)\delta_{AB}+(\alpha_2-\alpha_1)\varphi_3\varphi_3\,. \tag{4.19}$$

Next, note that because $|\varphi|$ is a singlet under H, it is constant over each orbit, so that writing

$$f(|\varphi|)=\alpha_1(|\varphi|)\,,\qquad g(|\varphi|)=\alpha_2(|\varphi|)-\alpha_1(|\varphi|) \tag{4.20}$$

we find that under the action of an arbitrary group element, $(0,0,\varphi_3)\to\varphi_A$, and Q^0_{AB} goes to

$$Q_{AB}(\varphi_C)=f(|\varphi|)\delta_{AB}+g(|\varphi|)\varphi_A\varphi_B \tag{4.21}$$

in agreement with (4.17).

In general, when M is a group manifold G, the slice is a manifold (with boundary) of dimension $d=\text{rank}(G)$, so that we can choose the group invariants (Casimirs) c_m $(m=1,\ldots,d)$ as coordinates. Then the functions $\alpha_i(c_m)$ used in constructing Q^0_{AB} will be invariant under G. For example, if G = SU(3) and we choose the Gell-Mann basis for the algebra, a slice is given by $\{(\varphi_1,\ldots,\varphi_8), \varphi_3\geqslant 0, \varphi_8\geqslant 0, \varphi_1=\varphi_2=\varphi_4=\cdots=\varphi_7=0\}$ and can be parametrized by φ_3 and φ_8 or the invariants $\varphi^2=\varphi_A\varphi_A$ and $\varphi^3=d^{ABC}\varphi_A\varphi_B\varphi_C$. There are four strata, given by $\varphi_A=0$, with isotropy SU(3); $\varphi_3>0$ and $\varphi_8>0$ with isotropy $U(1)\otimes U(1)$ (generated by T_3 and T_8);

$\varphi_3 > 0$, $\varphi_8 = 0$ with isotropy $U(1) \otimes U(1)$ and $\varphi_3 = 0$, $\varphi_8 > 0$ with isotropy $SU(2) \otimes U(1)$. The most general form for Q^0_{AB} that is invariant under $U(1) \otimes U(1)$ is

$$Q^0_{AB} = \left(\begin{array}{ccc|cccc|c} \alpha_1 & & & & & & & \\ & \alpha_1 & & & & & & \\ & & \alpha_2 & & & & & \alpha_6 \\ \hline & & & \alpha_3 & & & & \\ & & & & \alpha_3 & & & \\ & & & & & \alpha_4 & & \\ & & & & & & \alpha_4 & \\ \hline & & \alpha_6 & & & & & \alpha_5 \end{array}\right), \tag{4.22}$$

where the α_i are functions of φ_3, φ_8. They must satisfy

$$\begin{aligned} &\alpha_1(0, \varphi_8) = \alpha_2(0, \varphi_8)\,, \\ &\alpha_3(0, \varphi_8) = \alpha_4(0, \varphi_8)\,, \\ &\alpha_6(0, \varphi_3) = 0\,, \\ &\alpha_i(0, 0) = \alpha_j(0, 0)\,, \qquad i, j = 1, \ldots, 5\,. \end{aligned} \tag{4.23}$$

To obtain Q_{AB} from Q^0_{AB}, we rewrite it in terms of functions f_i of the invariants φ^2 and φ^3 and the 'natural' symmetric tensors δ_{AB} and d^{ABC} to obtain:

$$\begin{aligned} Q_{AB}(\varphi) = {} & f_1 \delta_{AB} + f_2 \varphi_A \varphi_B + f_3 d_{AB}{}^C \varphi_C + f_4 d^{CD}{}_{(A} \varphi_{B)} \\ & + f_5 d_{ABC} d^{CDE} \varphi_D \varphi_E + f_6 d_A{}^{CD} d_B{}^{EF} \varphi_C \varphi_D \varphi_E \varphi_F\,. \end{aligned} \tag{4.24}$$

This is again what one would find using group theoretic techniques to construct singlets $Q_{AB} F^A F^B$. Terms with a different d-structure can be reduced to the terms in (4.24) using the properties of d^{ABC}: It is totally symmetric and traceless and obeys

$$d_{(AB}{}^E d_{CD)E} = \tfrac{1}{3} \delta_{(AB} \delta_{CD)}\,. \tag{4.25}$$

5. Kähler geometry

In four dimensions, a nonlinear σ-model has a supersymmetric extension if and only if the corresponding manifold is Kähler [20]. We therefore briefly review some elementary Kähler geometry [9]. Further aspects are discussed in the appendix.

We consider theories with only chiral superfields (gauge interactions are introduced in the next section). The kinetic action for a supersymmetric nonlinear σ-model is

$$\int d^4x\, d^4\theta\, K(\Phi^i, \bar{\Phi}_j) = -\tfrac{1}{2} \int d^4x\, K_i^j| \partial^{\alpha\dot{\alpha}} A^i \partial_{\alpha\dot{\alpha}} \bar{A}_j + \cdots\,, \tag{5.1}$$

$$K_i^j| \equiv \frac{\partial}{\partial A^i} \frac{\partial}{\partial \bar{A}_j} K| \tag{5.2}$$

(we use group theoretic notation: upper and lower indices are related by complex conjugation and all factors of the metric K_i^j are kept explicit). The component action in (5.1) has the form (3.1) (with additional terms involving spinors and auxiliary fields) if we identify the metric g as $\begin{pmatrix} 0 & K_i^j| \\ K_j^i| & 0 \end{pmatrix}$. Any complex manifold with a metric that can be written locally in terms of a potential K as in (5.2) is called a Kähler manifold. The line element of the Kähler manifold is

$$ds^2 = 2K_i^j| \, dA^i \, d\bar{A}_j \,. \tag{5.3}$$

The holomorphic gradients $K_{ij}|$ in general do not vanish, but simply do not appear in the metric. This form of the line element is preserved by Kähler gauge transformations

$$K \to K + \Lambda(\Phi^i) + \bar{\Lambda}(\bar{\Phi}_i) \,, \tag{5.4}$$

which also leave the action (5.1) invariant, and by chiral field redefinitions $\Phi'^i = f^i(\Phi^j)$ which, at $\theta = 0$, define holomorphic coordinate transformations $A'^i = f^i(A^j)^\star$. Arbitrary *non*holomorphic coordinate transformations in general generate terms of the form $g_{ij}\, dA^i \, dA^j$ and $\bar{g}^{ij}\, d\bar{A}_i \, d\bar{A}_j$.

We introduce the notation

$$K_{i_1 \cdots i_m}^{j_1 \cdots j_n} = \frac{\partial}{\partial \Phi^{i_1}} \cdots \frac{\partial}{\partial \Phi^{i_m}} \frac{\partial}{\partial \bar{\Phi}_{j_1}} \cdots \frac{\partial}{\partial \bar{\Phi}_{j_n}} K \,. \tag{5.5}$$

The only nonvanishing components of the connection are

$$\Gamma_{jk}{}^i = (K^{-1})^i{}_n K_{jk}^n \tag{5.6}$$

and the complex conjugates $\bar{\Gamma}^{jk}{}_i$. Here $(K^{-1})_j{}^i$ is the inverse of K_i^j and not the mixed gradient of K^{-1}. From (5.6) it follows that the covariant derivatives of covariant quantities, e.g., $P_i = \partial P / \partial A^i$, are

$$P_{i;j} = P_{i,j} - \Gamma_{ij}{}^k P_k = K_i{}^n (P_k (K^{-1})_n{}^k)_j \,,$$
$$P_i{}^{;j} = P_i{}^{,j}$$

and of contravariant quantities, e.g., $\Pi_i = (K^{-1})_i{}^j P_j$,

$$\Pi_{i;j} = \Pi_{i,j} \,,$$
$$\Pi_i{}^{;j} = \Pi_i{}^{,j} + \bar{\Gamma}_i{}^{jk} \Pi_k = (K^{-1})_i{}^k (\Pi_m K_k{}^m)^j \,. \tag{5.8}$$

The only nonvanishing components of the Riemann tensor are

$$R_i{}^k{}_j{}^p = K_{ij}^{kp} - \Gamma_{ij}{}^m \bar{\Gamma}_n{}^{kp} K_m{}^n = K_m{}^k (\Gamma_{ij}{}^m)^p \,. \tag{5.9}$$

The contracted connection and the Ricci tensor have the particularly simple forms

$$\Gamma_i \equiv K_j{}^k \Gamma_{ik}{}^j = [\ln \det (K_j{}^k)]_i \,,$$
$$R_i^j \equiv K_k{}^m R_i{}^k{}_m{}^j = [\ln \det (K_k{}^m)]_i^j \,. \tag{5.10}$$

⋆ In the remainder of this section all equations are valid *both* as *superfield equations* and as ordinary *bosonic equations*, provided that the identification $\Phi^i \leftrightarrow A^i$ is made. Chiral quantities correspond to holomorphic quantities.

Using holomorphic coordinate transformations we can transform to normal *coordinates* where at a point

$$K^j_{i_1\cdots i_m} = K_i^{j_1\cdots j_n} = 0 \qquad \forall m, n > 1\,,$$
$$K_i^j = \delta_i^j\,. \tag{5.11}$$

In normal coordinates the connection vanishes at the chosen point, and the Riemann tensor becomes

$$R_i{}^{kp}{}_j = K_{ij}^{kp}\,. \tag{5.12}$$

It is sometimes also convenient to fix the Kähler gauge invariance (5.4) by going to normal *gauge* where at a point, in addition to (5.11)*

$$K_{i_1\cdots i_m} = K^{j_1\cdots j_n} = 0 \qquad \forall m, n\,. \tag{5.13}$$

On a Kähler manifold the description of isometries has several special features. In particular, we focus on *holomorphic* isometries, i.e., symmetries that do not mix A^i with $\bar{A}_j$ (equivalently, Φ^i with $\bar{\Phi}_j$) in coordinates where (5.2, 3) holds (however, see the discussion of $N=2$ supersymmetric models in sect. 11). For a coordinate invariant characterization of holomorphic isometries see (10.17). Thus, for example, in special coordinates the action of the holomorphic *isotropy* subgroup (see (3.2)) is just as for flat manifolds:

$$\delta A^i = i\lambda^i{}_j A^j\,, \qquad \delta\bar{A}_i = -i\bar{A}_j\lambda^j{}_i\,, \qquad \lambda^i_j = \lambda^X (T_X)^i{}_j\,. \tag{5.14}$$

(The transformations (5.14) are the θ-independent components of the infinitesimal form of the transformations (2.3) for constant hermitian parameter λ.)

In general, because the variation of the metric generated by an isometry vanishes, the Kähler potential must be invariant up to a Kähler gauge transformation (5.4):

$$\delta K \equiv K_i\delta A^i + K^i\delta\bar{A}_i = \eta(A^i) + \bar{\eta}(\bar{A}_i)\,. \tag{5.15}$$

In the isotropic case there exist Kähler gauges where η vanishes

$$\delta K \equiv i\lambda^i{}_j(K_iA^j - K^j\bar{A}_i) = 0\,. \tag{5.16}$$

This is true in, e.g., normal gauge (5.13) where K, and hence K_iA^j, contain no holomorphic piece.

To describe general holomorphic isometries we introduce Killing vectors k_B with holomorphic components $k_B{}^i(A^j)$ and complex conjugate components $\bar{k}_{Bi}(\bar{A}_j)$ ($k^i_B = (k_B{}^i, \bar{k}_{Bi})$):

$$\delta A^i = \lambda^B k^i_B = [\lambda^B k^j_B \partial/\partial A^j, A^i] \equiv L_{\lambda\cdot k}A^i\,,$$
$$\delta\bar{A}_i = \lambda^B\bar{k}_{Bi} = [\lambda^B\bar{k}_{Bj}\partial/\partial\bar{A}_j, \bar{A}_i] \equiv L_{\lambda\cdot k}\bar{A}_i\,, \tag{5.17}$$

* See ref. [21]. These authors introduce the notion of normal gauge in the context of local supersymmetry as opposed to Kähler geometry.

where L_x is the Lie derivative along the vector x. (In the special coordinates where (5.17) becomes (5.14), the Killing vectors of the isotropy subgroup are $k_X{}^i = i(T_X)^i_j A^j$, $\bar{k}_{Xi} = -i\bar{A}_j(T_X)^j_i$). The holomorphic and antiholomorphic components of the Killing vectors separately generate the algebra of the isometry group:

$$k_{[A}{}^j k_{B]}{}^i{}_{,j} = f_{AB}{}^C k^i_C\,, \qquad \bar{k}_{[Aj}\bar{k}_{B]i}{}^{,j} = f_{AB}{}^C \bar{k}_{Ci}\,. \tag{5.18}$$

The condition (5.15) on the Kähler potential becomes:

$$\delta K \equiv \lambda^A(K_i k^i_A + K^i \bar{k}_{Ai}) = \lambda^A(\eta_A(\Phi^i) + \bar{\eta}_A(\bar{\Phi}_i))\,. \tag{5.19}$$

This and the analyticity of k^i_A implies that the Killing vectors fulfill Killing's equations, which for a Kähler manifold take the form

$$k^{i,j}_A = \bar{k}_{Ai,j} = 0\,, \qquad K^i{}_j k^j_{A;k} + K^j{}_k \bar{k}_{Aj}{}^{;i} = 0\,. \tag{5.20}$$

The finite form of the transformations (5.17) are obtained by exponentiation:

$$A'^i = \mathrm{e}^{L_{\lambda\cdot k}} A^i\,, \qquad \bar{A}'_i = \mathrm{e}^{L_{\lambda\cdot\bar{k}}}\bar{A}_i\,. \tag{5.21}$$

Because η_A in (5.19) is holomorphic, it is determined up to an imaginary constant. We now define a real quantity $X_A(\Phi^i, \bar{\Phi}_j) = \bar{X}_A$:

$$k^i_A K_i = iX_A + \eta_A\,, \qquad \bar{k}_{Ai}K^i = -iX_A + \bar{\eta}_A\,. \tag{5.22}$$

X_A is defined up to a real constant, reflecting the ambiguity in η_A. When we differentiate (5.22), we find, using (5.20):

$$k^i_A K_i{}^j = iX^j_A\,, \qquad \bar{k}_{Ai}K^i{}_j = -iX_{Aj}\,. \tag{5.23}$$

Thus X_A is the Killing potential introduced in [2]*.

We now discuss properties of X_A and η_A that are needed to gauge the isometries generated by k_A. Eq. (5.23) implies

$$k^i_A X_{Bi} + \bar{k}_{Bi}X^i_A = 0 \tag{5.24}$$

and hence

$$\delta X_A \equiv \lambda^B(\bar{k}_{Bi}X^i_A + k^i_B X_{Ai}) = \tfrac{1}{2}\lambda^B(\bar{k}_{[Bi}X_{A]}{}^i + k^i_{[B}X_{A]i})\,. \tag{5.25}$$

From (5.18, 19) a straightforward calculation leads to

$$k^i_{[A}\eta_{B]i} + \bar{k}_{[Ai}\bar{\eta}^i{}_{B]} = f_{AB}{}^C(\eta_C + \bar{\eta}_C)\,. \tag{5.26}$$

Analyticity then implies

$$k^i_{[A}\eta_{B]i} = f_{AB}{}^C\eta_C + ic_{AB}\,, \qquad \bar{k}_{[Ai}\bar{\eta}^i{}_{B]} = f_{AB}{}^C\bar{\eta}_C - ic_{AB}\,, \tag{5.27}$$

where $c_{AB} = -c_{BA} = c^*_{AB}$ is *constant*. The Jacobi identity imposes the constraint $c_{A[B}f_{CD]}{}^A = 0$. One solution to this constraint is $c_{AB} = f_{AB}{}^C\xi_C$ for some real constant

* No explicit procedure for computing the Killing potential was given in [2]; independently, A.N. Jourjine has found the result (5.23) [22].

ξ_C. This c_{AB} is eliminated by a shift $\eta_A + i\xi_A \to \eta_A$ (which through (5.22) implies the shift $X_A - \xi_A \to X_A$). The condition $c_{AB} = 0$, when it can be imposed, removes the ambiguity in η_A (and hence X_A) except for invariant abelian subgroups. As we shall see in the next section this ambiguity corresponds to the possibility of adding to the action a Fayet-Iliopoulos term with an arbitrary coefficient for each abelian factor [2]. When the constants c_{AB} *cannot* be removed, they are an *obstruction* to gauging (see next section). For any semisimple gauge group *including* those that are *non*compact, c_{AB} has the required form to be removable. In this case the Killing metric (3.15) is nonsingular, and because $f_{ABC} \equiv f_{AB}{}^{C} g_{DC}$ is *always* totally antisymmetric, using the Jacobi identity we find

$$\xi_A = f_{AB}{}^{D} c_{DE} g^{BE} . \tag{5.28}$$

When the metric (3.15) is degenerate the c_{AB} may be irremovable. The simplest example of this is provided by two-dimensional flat space. As Kähler potential we take

$$K = \Phi\bar{\Phi} . \tag{5.29}$$

We consider the translations generated by the Killing vectors

$$k_1 = \partial/\partial\Phi + \partial/\partial\bar{\Phi} , \qquad k_2 = i(\partial/\partial\Phi - \partial/\partial\bar{\Phi}) . \tag{5.30}$$

We compute the chiral functions η from (5.19) and find

$$\eta_1 = \Phi , \qquad \eta_2 = -i\Phi . \tag{5.31}$$

Finally, we compute the obstruction constants c from (5.27)

$$c_{12} = -2 . \tag{5.32}$$

Clearly, since the group is abelian, c cannot be removed. Only the isometry generated by *one* arbitrary linear combination of k_1 and k_2 can be gauged. This remains true even if the manifold is compactified, e.g., to the torus T^2. (In this particular example, there is a further difficulty in that X_A does not naturally satisfy the periodic boundary conditions on the torus.)

When we can set $c_{AB} = 0$, we have

$$k^i_{[A} \eta_{B]i} = f_{AB}{}^{C} \eta_C , \qquad \bar{k}_{[Ai} \bar{\eta}_{B]}{}^{i} = f_{AB}{}^{C} \bar{\eta}_C . \tag{5.33}$$

From equations (5.18, 22, 25, 33) we find the transformation of the Killing potential [2]

$$\delta X_A = \lambda^B (\bar{k}_{Bi} X^i_A + k^i_B X_{Ai}) = -\lambda^B f_{AB}{}^{C} X_C . \tag{5.34}$$

Indeed, for semisimple groups, we can use (5.34) and (5.23) to calculate X_A directly:

$$X_A = 2i f_{AB}{}^{C} k^i_D \bar{k}_{Cj} K_i{}^{j} g^{BD} . \tag{5.35}$$

6. Gauging isometries on Kähler manifolds

If an action is invariant under a group of global transformations, then it is possible to gauge arbitrary continuous subgroups of this group. In this section we discuss this procedure applied to the actions of sect. 5. We begin by treating the case where the straightforward gauging of sect. 2 can be applied. We then reformulate this in the more general geometric language of Killing vectors. Finally, we generalize the gauging to cases where the procedure of sect. 2 breaks down [2].

The gauging of a subgroup of the isotropy subgroup is straightforward: The Kähler potential can be chosen to be globally invariant (5.16), and in the special coordinates where the transformations have the flat space form (5.14) the Kähler potential is made *locally* invariant under Yang–Mills transformations by making the substitution (2.6) $\bar{\Phi}_i \to \tilde{\bar{\Phi}}_i = \bar{\Phi}_j(e^V)^j{}_i$. Since $\tilde{\bar{\Phi}}_i$ and Φ^i transform under local transformations just as $\bar{\Phi}_i$ and Φ^i transform under global transformations with the constant parameter $\lambda^i{}_j$ replaced by the chiral superfield $\Lambda^i{}_j$, and since the Kähler potential $K(\Phi^i, \tilde{\bar{\Phi}}_j)$ does not contain derivatives of the superfields, the local invariance of $K(\Phi^i, \tilde{\bar{\Phi}}_j)$ follows immediately from the global invariance of $K(\Phi^i, \bar{\Phi}_j)$. We assume the superpotential $P(\Phi^i)$ is invariant under the global transformations; this implies invariance under the local transformations. As for the bosonic case (3.17), we can generalize the kinetic term for the gauge fields in (2.7) to include an invertible symmetric chiral matrix $Q_{AB}(\Phi^i)$:

$$S_V = \tfrac{1}{4}\int d^4x\, d^2\theta\, Q_{AB} W^{A\alpha} W^B_\alpha + \text{h.c.} \tag{6.1}$$

The matrix Q_{AB} can be constructed as discussed in sect. 4, with certain special features arising because Q_{AB} is holomorphic in Φ; see sect. 8.

We have used a special coordinate system in which the Yang–Mills gauge transformations have the linear form (5.14). In general we want a geometric formulation valid in any coordinate system. Such a description of the global symmetries in terms of holomorphic Killing vectors has been given in the previous section (5.17)–(5.21).

To gauge the global isometries (promote λ to Λ and $\bar{\Lambda}$) in this formulation we must find the field $\tilde{\bar{\Phi}}_i$, which transforms with Λ rather than $\bar{\Lambda}$. Comparing (2.3) with (2.6) we see that $\tilde{\bar{\Phi}}_i$ is defined in terms of $\bar{\Phi}_i$ by making a "gauge transformation" with parameter $\bar{\Lambda}^A \to iV^A$. In arbitrary coordinates (5.21) then implies that (2.6) becomes:

$$\tilde{\bar{\Phi}}_i = e^L \bar{\Phi}_i\,, \qquad L \equiv L_{iV\cdot\bar{k}} = iL_{V\cdot\bar{k}}\,. \tag{6.2}$$

The gauging then proceeds as above by substituting $\bar{\Phi}_i \to \tilde{\bar{\Phi}}_i$ and adding the gauge kinetic term (6.1). The conditions for invariance of the superpotential and gauge kinetic term are the holomorphic versions of (3.21, 22):

$$\delta P = P_{,i} k^i_A \Lambda^A = 0\,,$$

$$\delta Q_{AB} \equiv Q_{AB,i} k^i_C \Lambda^C = (f_{CA}{}^D Q_{BD} + f_{CB}{}^D Q_{AD})\Lambda^C\,. \tag{6.3}$$

When we gauge groups that do not leave any point of the internal manifold invariant (this means that part of the group is realized nonlinearly and hence spontaneously broken), in general the Kähler potential is invariant under the Yang–Mills gauge transformation (5.17) only up to a Kähler gauge transformation $\eta=\eta_A\lambda^A$ as in (5.15) [2]. Though this suffices to guarantee the invariance of the action (3.2) under global transformations, the simple prescription given above for gauging the symmetries does not give an invariant action (e.g., $\int d^4\theta\, \lambda^A\bar{\eta}_A=0$, but the *local* variation does not vanish: $\int d^4\theta\, \Lambda^A\bar{\eta}_A\neq 0$). The solution to this problem is to introduce a new chiral superfield ζ, which transforms as

$$\delta\zeta=\eta_A(\Phi^i)\lambda^A\,,\qquad \delta\bar{\zeta}=\bar{\eta}_A(\bar{\Phi}_i)\lambda^A\,,\tag{6.4}$$

then the lagrangian

$$\mathcal{L}_\zeta\equiv K(\Phi^i,\bar{\Phi}_j)-\zeta-\bar{\zeta}\tag{6.5}$$

is invariant under global Yang–Mills transformation. Note that the action does not depend on the "auxiliary" variable ζ: $S=\int d^4x\, d^4\theta\, \mathcal{L}_\zeta=\int d^4x\, d^4\theta\, K$; our final results will be independent of ζ as well. The quantities ζ can be thought of as extra coordinates so that $(\Phi^i,\zeta,\bar{\Phi}_i,\bar{\zeta})$ are coordinates of an enlarged space in which the original Kähler manifold is embedded. The lagrangian $\mathcal{L}_\zeta$ is the Kähler potential K' of the new space. The ζ-transformations (6.4) are included in the new Killing vectors

$$\begin{aligned}k'_A&\equiv k^i_A\partial/\partial\Phi^i+\eta_A\partial/\partial\zeta\,,\\ \bar{k}'_A&\equiv\bar{k}_{Ai}\partial/\partial\bar{\Phi}_i+\bar{\eta}_A\partial/\partial\bar{\zeta}\end{aligned}\tag{6.6}$$

and the corresponding isometries leave the new Kähler potential invariant $\delta K'=0$. Since $\mathcal{L}_\zeta$ is invariant the correct gauge prescription is simply to make the substitutions $\bar{\Phi}_i\to\tilde{\bar{\Phi}}_i$ and $\bar{\zeta}\to\tilde{\bar{\zeta}}$. To do this, we compute $\tilde{\bar{\zeta}}$ in analogy with (6.2), using (6.6):

$$\begin{aligned}\tilde{\bar{\zeta}}&=e^{L'}\bar{\zeta}=\left(1+\frac{e^{L'}-1}{L'}L'\right)\bar{\zeta}=\bar{\zeta}+i\frac{e^{L'}-1}{L'}\bar{\eta}_AV^A\\ &=\bar{\zeta}+i\frac{e^{L}-1}{L}\bar{\eta}_AV^A\,,\qquad L'\equiv L_{iV^A\bar{k}'_A}\,,\qquad L\equiv L_{iV^A\bar{k}_A}\,.\end{aligned}\tag{6.7}$$

Dropping the irrelevant ζ and $\bar{\zeta}$ terms, the locally gauge invariant action becomes:

$$S=\int d^4x\, d^4\theta\left[K(\Phi^i,\tilde{\bar{\Phi}}_j)-i\frac{e^{L}-1}{L}\bar{\eta}_AV^A\right].\tag{6.8}$$

For this prescription to work, and gauging to be possible, it is necessary that (6.4) is a realization of the Yang–Mills gauge group, i.e., that (5.33) is satisfied. This is true whenever the obstruction constants c_{AB} in (5.27) can be removed. If the c_{AB} are irremovable in one group, there are *sub*groups with removable obstruction constants (see for example (5.29–33)).

We use (5.22-27) to rewrite the action (6.8) in terms of X_A. Since $L \equiv L_{iV\cdot\bar{k}}$ commutes with $\Phi^i(\mathrm{e}^L\Phi^i = \Phi^i)$ we have

$$K(\Phi^i, \tilde{\Phi}_j) = \mathrm{e}^L K(\Phi^i, \bar{\Phi}_j) = K + \frac{\mathrm{e}^L - 1}{L} K^{,i}\bar{k}_{Ai}(iV^A) . \tag{6.9}$$

Using (5.22) the action becomes (including the superpotential and the gauge kinetic term)

$$S = \int \mathrm{d}^4x\, \mathrm{d}^4\theta \left[K(\Phi^i, \bar{\Phi}_j) + \frac{\mathrm{e}^L - 1}{L} X_A V^A \right] + \left\{ \int \mathrm{d}^4x\, \mathrm{d}^2\theta [P(\Phi^i) + \tfrac{1}{4}Q_{AB}(\Phi^i) W^{A\alpha} W^B_\alpha] + \text{h.c.} \right\} . \tag{6.10}$$

The Fayet-Iliopoulos term is included through the freedom to add a real constant to X_A for every abelian factor (see discussion following (5.27)) [2]. Finally, (5.24) implies $(L_{iV\cdot\bar{k}} - L_{iV\cdot k})X_A V^A = 0$, which shows that (6.10) is hermitian*. The hermiticity can be made manifest by introducing the complex structure J defined in the appendix: $L_{iV\cdot\bar{k}} \to -\frac{1}{2}L_{V\cdot Jk}$ (where $V\cdot Jk = V^A J_\imath^{\ \jmath} k^\imath_A \partial_\jmath$, $\imath = i$ and $\bar{i}$).

By varying the gauged action we obtain the superfield equations

$$\tfrac{1}{2}\nabla^{\dot\alpha}\nabla_{\dot\alpha} K_i(\Phi^j, \tilde{\Phi}_k) + P_i + \tfrac{1}{4}Q_{ABi} W^{A\alpha} W^B_\alpha = 0 , \tag{6.11a}$$

$$X_A(\Phi^i, \tilde{\Phi}_j) - \tfrac{1}{2}i\nabla^\alpha(Q_{AB}W^B_\alpha) + \tfrac{1}{2}i\nabla^{\dot\alpha}(\tilde{Q}_{AB}\tilde{W}^B_{\dot\alpha}) = 0 , \tag{6.11b}$$

where $\tilde{W}^A T_A = \bar{W}^A \mathrm{e}^{-V} T_A \mathrm{e}^V$, etc. Eq. (6.11a) follows most easily in chiral representation (where $\nabla_{\dot\alpha} = \bar{D}_{\dot\alpha}$) from (6.8) because L and $\bar{\eta}$ are independent of Φ^i, and (6.11b) follows from

$$(\mathrm{e}^L X_A)(\mathrm{e}^{-V}\delta_V\, \mathrm{e}^V)^A = \delta_V\left(\frac{\mathrm{e}^L - 1}{L} X_A V^A\right) , \tag{6.12}$$

which is a (not at all obvious) consequence of gauge invariance.

7. Examples

The nonlinear σ-model based on the complex projective space $\mathrm{CP}^n = \mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(1)\otimes\mathrm{U}(n))$ has been much studied. CP^n is a Kähler manifold with Kähler potential

$$K = \ln(1 + A^i\bar{A}_i) \tag{7.1}$$

and isometry group $\mathrm{SU}(n+1)$, with isotropy subgroup $\mathrm{S}(\mathrm{U}(n)\otimes\mathrm{U}(1))$. A closely related Kähler manifold, with noncompact isometry group $\mathrm{SU}(n, 1)$, is the complex hyperbolic space $\mathrm{SU}(n, 1)/\mathrm{S}(\mathrm{U}(n)\otimes\mathrm{U}(1))$, which we denote CH^n. The Kähler potential for CH^n is

$$K = -\ln(1 - A^i\bar{A}_i) , \tag{7.2}$$

* Independently, S. Samuel has found the result (6.10) in its manifestly hermitian form [4].

with $A^i\bar{A}_i$ restricted to satisfy $A^i\bar{A}_i < 1$. The potentials (7.1) and (7.2) can be combined to give

$$K_\xi = \xi^{-1} \ln(1 + \xi A^i \bar{A}_i)\,, \tag{7.3}$$

with metric

$$K_{\xi i}{}^j = \frac{\delta_i^j(1+\xi A^k\bar{A}_k) - \xi A^j\bar{A}_i}{(1+\xi A^m\bar{A}_m)^2}\,, \tag{7.4}$$

with $\xi = 1$ giving CP^n and $\xi = -1$ giving CH^n. In fact, any $\xi > 0$ gives CP^n, but when $\xi \neq 1$, the coordinate system is nonstandard. Similarly, any $\xi < 0$ gives a space obtained from the standard CH^n ($\xi = -1$) by a coordinate redefinition, provided $\bar{A}_i A^i < -\xi^{-1}$. However in the limit $\xi \to 0$, (7.4) becomes the flat space metric, $K_{0i}{}^j = \delta_i{}^j$, with Kähler potential $K_0 = A^i\bar{A}_i$. Thus when $\xi = 0$ we obtain flat euclidean space $\mathrm{IU}(n)/\mathrm{S}(\mathrm{U}(n)\otimes\mathrm{U}(1))$ where $\mathrm{IU}(n)$ is the Inönü-Wigner contraction of either $\mathrm{SU}(n+1)$ or $\mathrm{SU}(n,1)$ about $\mathrm{S}(\mathrm{U}(n)\otimes\mathrm{U}(1))$ [23]. Although the isometry group of this flat space is $\mathrm{ISO}(2n)$, $\mathrm{IU}(n)$ is the maximal subgroup of this that does not mix holomorphic and antiholomorphic coordinates and so has holomorphic Killing vectors. The supersymmetric nonlinear σ-model corresponding to these geometries has action

$$\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, K_\xi(\Phi^i, \bar{\Phi}_i)\,, \tag{7.5}$$

where Φ^i is the chiral superfield with lowest component A^i. We thus obtain a one-parameter family of models, falling into three equivalence classes, in some ways analogous to the one-parameter family of $N = 4$ gauged supergravities given in [24] and the family of gauged $N = 8$ supergravities given in [25].

We now consider the gauging of the (holomorphic) isometry group G_ξ of these models. For $\xi > 0$, G_ξ is $\mathrm{SU}(n+1)$, for $\xi < 0$, G_ξ is $\mathrm{SU}(n,1)$, and for $\xi = 0$, G_ξ is $\mathrm{IU}(n)$. We determine the Killing vectors from the variation (5.15) of the Kähler potential K_ξ. Under an isometry

$$\delta K_\xi = \frac{\delta(A^i\bar{A}_i)}{1+\xi A^j\bar{A}_j} = \eta(A^i) + \bar{\eta}(\bar{A}_i)\,. \tag{7.6}$$

There are two possibilities

$$\delta(A^i\bar{A}_i) = 0 \tag{7.7}$$

and

$$\delta(A^i\bar{A}_i) = (1+\xi A^j\bar{A}_j)(\eta(A^i) + \bar{\eta}(\bar{A}_i))\,. \tag{7.8}$$

From (7.7) one finds the Killing vectors for the linearly realized isotropy subgroup (3.3, 4)

$$k_X^i = i(T_X)^i{}_j A^j\,, \tag{7.9}$$

$$k_{n^2}^i = iA^i\,, \tag{7.10}$$

where $(T_X)^i{}_j$, $X=1,\dots n^2-1$ are the hermitian traceless generators of SU(n) and $k^i_{n^2}$ is the U(1) Killing vector. The general holomorphic solution to (7.8) is

$$\delta A^i = b^i + \xi b_j A^j A^i , \qquad \eta(b) = b_j A^j , \tag{7.11}$$

where $b^i \equiv (b_i)^*$ is an arbitrary complex n-vector parameter. This leads to the $2n$ holomorphic Killing vectors of the nonlinearly realized isometries ($p, p' = 1, \dots n$)

$$k^i_p = \delta^i_p + \xi \delta^p_j A^j A^i , \tag{7.12}$$

$$k^i_{p'} = i(\delta^i_{p'} - \xi \delta^{p'}_j A^j A^i) . \tag{7.13}$$

Alternatively, we could have found (7.9, 10, 12, 13) by group theoretic methods [12, 26].

It is straightforward to construct the bosonic gauged nonlinear σ-model using these Killing vectors, as described in sect. 4.

We now consider the gauging of the supersymmetric model, as described in sect. 6. The holomorphic functions $\eta_A(A^i)$ follow from (7.6, 11),

$$\begin{aligned} \eta_X &= 0 , & \eta_{n^2} &= 0 , \\ \eta_p &= \delta^p_i A^i , & \eta_{p'} &= -i\delta^{p'}_i A^i . \end{aligned} \tag{7.14}$$

Then the constants c_{AB} defined by (5.27) all vanish except for

$$c_{pq'} = -c_{q'p} = -2\delta_{pq'} . \tag{7.15}$$

If the c_{AB} are of the form $c_{AB} = f_{AB}{}^C \beta_C$ for some real constant β_C, they can be removed by the shift $\eta_A + i\beta_A \to \eta_A$. The structure constants $f_{AB}{}^{n^2}$ are

$$f_{pq'}{}^{n^2} = -2\xi\delta_{pq'} , \qquad f_{AB}{}^{n^2} = 0 , \qquad AB \neq pq' ; \tag{7.16}$$

so we choose

$$\beta_A = \xi^{-1}\delta^{n^2}_A , \qquad (\xi \neq 0) \tag{7.17}$$

and so, when $\xi \neq 0$, all the c_{AB} can be chosen as zero by replacing η_{n^2} in (7.14) by:

$$\eta_{n^2} = i\xi^{-1} . \tag{7.18}$$

Then for $\xi \neq 0$, the supersymmetric model can be gauged as described in sect. 6, with the Killing potentials given by:

$$\begin{aligned} X_X &= \bar{A}_i (T_X)^i{}_j A^j / (1 + \xi A^k \bar{A}_k) , \\ X_p &= i\delta^i_p (A^i - \bar{A}_i)/(1 + \xi A^k \bar{A}_k) , \\ X_{p'} &= \delta^i_{p'} (A^i + \bar{A}_i)/(1 + \xi A^k \bar{A}_k) , \\ X_{n^2} &= A^i \bar{A}_i/(1 + \xi A^k \bar{A}_k) - \xi^{-1} = -1/[\xi(1 + \xi A^k \bar{A}_k)] . \end{aligned} \tag{7.19}$$

These models spontaneously break supersymmetry, the vacuum energy being proportional to ξ^{-2}. When $\xi = 0$, the $c_{pq'}$ cannot be removed and a supersymmetric gauging of the non-semisimple IU(n) is not possible. However, when $\xi = 0$, it is possible to gauge the linearly realized subgroup S(U(n)×U(1)), or the non-semisimple, non-compact subgroup ISO(n), for which all the c_{AB} vanish.

8. Nonminimal vector kinetic terms with $N=1$ supersymmetry

In sect. 4, we gave the construction of the nonminimal vector kinetic term (3.19) for the general bosonic nonlinear σ-model. A straightforward application of this to the supersymmetric σ-model gives, in general, Q_{AB} as a function of Φ *and* $\bar{\Phi}$. However, supersymmetry requires Q_{AB} to be holomorphic (6.10). The solution is to find a *subgroup* of the isometry group that leads to a holomorphic Q_{AB}.

To illustrate the problem we consider CP^1 and its noncompact forms as discussed in sect. 4. For these manifolds Q_{AB} is given in terms of two real coordinates σ^1, σ^2 by (4.12) (with no dependence on χ). Clearly, Q_{AB} (σ^1, σ^2) is *not* a holomorphic function of any complex coordinate $A(\sigma^1, \sigma^2)$. Thus there is no supersymmetric nonminimal kinetic term (6.1) for the full isometry group of CP^1 and the same is true for the isometry groups $SU(n+1)$, $SU(n,1)$ and $IU(n)$ for CP^n and its noncompact forms. (There is, of course, always the minimal kinetic term (3.13) constructed with the Killing metric, but this has ghosts for noncompact groups.) Below we construct nonminimal kinetic terms for the nonlinearly realized "real" isometry subgroups $SO(n+1)$, $SO(n,1)$ and $ISO(n)$. The general solution is analogous: We can generate nonminimal kinetic terms only for groups with a real action on the manifold. This is reminiscent of the situation in extended supergravity theories which have "complex" global symmetry groups (containing $SU(n)$) but for which only "real" orthogonal subgroups $SO(p,q)$ and their contractions can be gauged [12].

As discussed in sect. 2, the gauge group G acts on the chiral superfields through its complexification G^C:

$$\delta\Phi^i = \Lambda^A k^i_A(\Phi)\,,$$
$$\delta\bar{\Phi}_i = \bar{\Lambda}^A \bar{k}_{Ai}(\bar{\Phi})\,. \tag{8.1}$$

($\Phi^i(\bar{\Phi}_i)$ transform under $G_\Lambda(G_{\bar{\Lambda}})$.) The Kähler metric and hence the scalar kinetic term is not invariant under G^C but only under G. The Killing vectors generating the isometries of the scalar manifold are $k_A + \bar{k}_A$.

Let us now recall the essential idea in the construction of Q presented in sect. 4: One constructs an H-invariant Q on the orbit space, (actually, on a slice) and then uses the remaining (non isotropic) generators to extend Q over the whole manifold. For the construction to work, there must be a one to one correspondence between the directions perpendicular to the slice and the non isotropic Killing vectors. However, in general Killing vectors whose holomorphic parts are linearly independent over the real numbers may be linearly *dependent* over the complex numbers. We thus get a restriction on which gauge groups admit nonminimal kinetic terms.

We illustrate these considerations using the examples of sect. 7 on the manifolds CP^n, CH^n and C^n. The Killing vectors generating the maximal isometry group $SU(n+1)$, $SU(n,1)$ and $IU(n)$ respectively, are given in (7.9, 10, 12, 13). In particular, at the isotropy point $A^i=0$, the holomorphic parts of the nonisotropic

Killing vector (7.12, 13) are:

$$k_p^i = \delta_p^i\,,$$
$$k_{p'}^i = i\delta_{p'}^i\,. \tag{8.2}$$

Clearly, these are linearly independent over the real but not over the complex numbers. We must choose a subgroup including only linearly independent generators. Such a subgroup is given by SO($n+1$), SO(n, 1) and IO(n) respectively.

To explicitly obtain Q_{AB} we consider a subspace of $\mathrm{G}_\xi/\mathrm{H}$ in which the coordinates are restricted to be real, $A^i = \bar{A}_i \equiv \varphi^i$. Then the real coordinates φ^i parametrize the real projective space $\mathrm{RP}^n = \mathrm{SO}(n+1)/\mathrm{O}(n)$ if $\xi = 1$, $\mathrm{RH}^n = \mathrm{SO}(n,1)/\mathrm{O}(n)$ if $\xi = -1$ and $\mathrm{R}^n/\mathrm{Z}^2$ if $\xi = 0$ - these are locally the same as the examples considered in sect. 4. The Killing vectors of the isometry groups SO($n+1$), SO(n, 1) and ISO(n) acting on the real space are:

$$k_X^i = i(T_X)^i{}_j\varphi^j\,, \qquad X = 1, \dots, \tfrac{1}{2}n(n-1)\,,$$
$$k_p^i = \delta_p^i + \xi\delta_j^p\varphi^j\varphi^i\,, \tag{8.3}$$

where T_X are hermitian matrix representations of the generators of SO(n) in the vector representation and $i = 1, \dots, n$. The Q_{AB} on this real section was constructed in sect. 4 - for $n = 2$, it is given by (4.11). The $Q_{AB}(A^i)$ for the complex case is simply given by the analytic continuation of $Q_{AB}(\varphi^i)$. For $n = 2$

$$Q_{AB}(A) = \xi^{-1}\alpha_2\begin{pmatrix} 1 & 0 \\ 0 & \xi\mathbb{1}_{2\times 2}\end{pmatrix} - \frac{\alpha_2 - \xi\alpha_1}{1 + \xi A^i A^i}\begin{pmatrix} \xi^{-1} & -A^2 & A^1 \\ -A^2 & \xi(A^2)^2 & -\xi A^1 A^2 \\ A^1 & -\xi A^1 A^2 & \xi(A^1)^2 \end{pmatrix} \tag{8.4}$$

gives supersymmetric kinetic terms for the nonlinearly realized gauge groups SO(3), SO(2, 1) or ISO(2) for CP^2, CH^2 or C^2. For a physical theory we require $\mathrm{Re}\,\alpha_i > 0$ (4.6). For the exceptional case $n = 1$ the real isometry group is abelian and (6.3) implies that Q_{AB} must be independent of A^i.

It is now straightforward to extend this to a general Kähler manifold M. We first restrict attention to a real subspace $\hat{\mathrm{M}}$, $A^i = \bar{A}_i \equiv \varphi^i$ and construct a $Q_{AB}(\varphi^i)$ on $\hat{\mathrm{M}}$ for the isometry group $\hat{\mathrm{G}}$ of $\hat{\mathrm{M}}$ as in sect. 4, assuming a slice exists. Then $Q_{AB}(\varphi)$ satisfies (3.21) with $k_A^i(\varphi)$ the Killing vectors for $\hat{\mathrm{G}}$. Then the analytic continuations $Q_{AB}(A)$ and $k_A^i(A)$ will also satisfy (3.21). The holomorphic Killing vectors $k_A^i(A)$ give the action of $\hat{\mathrm{G}}_\Lambda$ on M. Performing the trivial generalization to superfields, $Q_{AB}(\Phi)W^{\alpha A}W_\alpha{}^B$ gives a supersymmetric kinetic term for the vector fields corresponding to the gauge group $\hat{\mathrm{G}}$ on the Kähler space M. Of course the concept of "real" subspace is coordinate dependent and in another holomorphic coordinate system we may obtain a different embedding of the subgroup $\hat{\mathrm{G}}$ in the isometry group G of M. $\hat{\mathrm{G}}$ is the maximal nonlinearly realized gauge group for which supersymmetric nonminimal kinetic terms exist, as the number of nonlinearly realized group generators equals the number of complex scalar fields A^i.

Finally we wish to ensure a ghost free theory with positive euclidean action. For $Q_{AB}(A^i)$ given by (8.4), we require Re $\alpha_i > 0$ (4.6). More generally, we consider $\alpha_i = \alpha_i(\chi)$ where $\{\chi\}$ are holomorphic coordinates on the slice M_0. If M_0 is complete, the only bounded functions $\alpha_i(\chi)$ with analytic dependence on the coordinates χ of M_0 are constant. In this case we can either choose to have a Q^0_{AB} independent of χ, or to restrict ourselves to an incomplete submanifold of M_0 in which Re $\alpha_i > 0$ - *a domain of positivity*. If M_0 is incomplete it may be possible to satisfy (4.6) with a nonconstant Q^0_{AB} on the entire manifold. In general, we can expect to find a Q^0_{AB} that has positive euclidean action if the gauge group is semisimple (in which case the determinant of Q^0_{AB} is constant over orbits; see sect. 4) and we can choose a positive definite $Q^0_{AB}(\chi)$ on the slice M_0 - and hence M_0 is incomplete or Q^0_{AB} is constant.

9. The $N=2$ kählerian vector multiplet

We start our discussion of $N=2$ supersymmetry with the kählerian vector multiplet [5]. (It has been shown that, in two dimensions, this model has an underlying hyper-Kähler structure [27]). The $N=2$ superfield construction of this multiplet is known, but as we want to use it as an example of the methods in other sections, we describe the multiplet in terms of $N=1$ superfields. It consists of vector and chiral superfields V^A and Φ^A in the adjoint representation of some group. One supersymmetry is manifest, whereas the other has to be given as explicit transformations mixing different superfields:

$$\begin{aligned} \delta\Phi &= -iW^{\alpha}D_{\alpha}\varepsilon\,, & \Phi &\equiv \Phi^A T_A\,, \\ \mathrm{e}^{-V}\delta\,\mathrm{e}^{V} &= \bar{\varepsilon}\Phi + \varepsilon\tilde{\bar{\Phi}}\,, & V &\equiv V^A T_A\,, \end{aligned} \tag{9.1}$$

where $(T_A)^B{}_C = if_{AC}{}^B$, $\tilde{\bar{\Phi}} \equiv \mathrm{e}^{-V}\bar{\Phi}\,\mathrm{e}^{V}$, $W^{A\alpha} = i\bar{D}^2(\mathrm{e}^{-V}D^{\alpha}\,\mathrm{e}^{V})^A$ and ε is a constant chiral superfield satisfying $\bar{D}\varepsilon = \partial_a\varepsilon = 0$. The algebra of these transformations closes without use of the field equations, and consequently the transformations (9.1) are *independent* of the action.

The chiral fields Φ^A are the coordinates of a manifold with the same dimension as the group. Clearly (9.1) is valid only in a special coordinate system. In these coordinates, the *gauge* transformations are linear and leave the origin fixed (the entire group is the isotropy group of the origin):

$$\begin{aligned} \delta\Phi^A &= f_{BC}{}^A\Phi^B\Lambda^C\,, \\ \mathrm{e}^{-V}\delta\,\mathrm{e}^{V} &= i(\mathrm{e}^{-V}\bar{\Lambda}\,\mathrm{e}^{V} - \Lambda)\,, \end{aligned} \tag{9.2}$$

where $\Lambda \equiv \Lambda(x, \theta)$ is a chiral superfield gauge parameter (cf. (2.3, 5).

The transformation of Φ^A in (9.2) is linear and hence the action is the straightforward covariantization (2.6) of (5.1) plus a nonminimal gauge kinetic term (6.1)

$$S = \int \mathrm{d}^4x\,\mathrm{d}^4\theta\,K(\Phi, \tilde{\bar{\Phi}}) + \left\{\tfrac{1}{4}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,Q_{AB}W^{A\alpha}W^B_{\alpha} + \text{h.c.}\right\}. \tag{9.3}$$

$N=2$ supersymmetry requires K and Q to be expressible in terms of a single holomorphic function $H(\Phi)$ [5]

$$K=\tfrac{1}{2}(\bar{\Phi}^A H_A+\Phi^A \bar{H}_A)\,, \tag{9.4}$$

$$Q_{AB}=H_{AB}\,, \tag{9.5}$$

where $H_A\equiv \partial H/\partial\Phi^A$, etc. We observe that *the metric on the scalar manifold is the coefficient of the vector kinetic term* [5]:

$$K_{A\bar{B}}=\tfrac{1}{2}(Q_{AB}+\bar{Q}_{AB})\,. \tag{9.6}$$

The function H must be a scalar under gauge transformations

$$\delta H=f_{BC}{}^{A}H_A\Phi^B\Lambda^C=0 \tag{9.7}$$

and hence is an arbitrary function of the algebraically independent invariants constructed out of Φ. As Φ is in the adjoint representation, these invariants are precisely the Casimir invariants of the Lie algebra.

10. $N=2$ supersymmetry and hyper-Kähler geometry

In four dimensions, a nonlinear σ-model has an $N=2$ supersymmetric extension if and only if the corresponding manifold is *hyper*-Kähler [20]. Here we review elements of $N=2$ supersymmetry and hyper-Kähler geometry. In contrast to the $N=1$ nonlinear σ-models and the $N=2$ vector multiplet, described in previous sections, no simple *off*-shell formulation of an arbitrary $N=2$ nonlinear σ-model is known. Correspondingly, although the metric on a Kähler manifold can be constructed algebraically (5.2)-(5.3), no such construction is known for hyper-Kähler manifolds. We therefore describe $N=2$ nonlinear σ-models in terms of $N=1$ superfields with an extra nonmanifest supersymmetry [10, 28], and hyper-Kähler manifolds as Kähler manifolds with certain additional properties.

We consider theories with only $N=1$ chiral superfields (gauge interactions are introduced in the next section). The kinetic action is still (5.1), but with an extra supersymmetry:

$$\delta\Phi^i=\bar{D}^2(\bar{\varepsilon}\bar{\Omega}^i)\,,\qquad \delta\bar{\Phi}_i=D^2(\varepsilon\Omega_i)\,,\qquad \bar{D}_{\dot{\alpha}}\varepsilon=D^2\varepsilon=\partial_{\underline{a}}\varepsilon=0\,. \tag{10.1}$$

Here ε is a constant chiral superfield parameter and $\bar{\Omega}$ is a function of Φ and $\bar{\Phi}$, defined modulo an arbitrary chiral term (see below). The algebra of these transformations closes if and only if

$$\Omega_{i,j}\bar{\Omega}^{j,k}=\Omega_{j,i}\bar{\Omega}^{k,j}=-\delta_i^k\,, \tag{10.2}$$

$$\bar{\Omega}^{j,[m|}\bar{\Omega}^{i,|k]}{}_{,j}=0\,, \tag{10.3}$$

$$\bar{D}^2\bar{\Omega}^i=\bar{\Omega}^{i,j}\bar{D}^2\bar{\Phi}_j+\tfrac{1}{2}\bar{\Omega}^{i,jk}\bar{D}\bar{\Phi}_j\bar{D}\bar{\Phi}_k=0\,. \tag{10.4}$$

The corresponding relations for Ω are found from (10.3)–(10.4) by complex conjugation. Eq. (10.4) has the form of a (super)field-equation, and must follow from varying the action. This means that *the algebra closes only on shell.*

The action (5.1) is invariant under the transformations (10.1) provided that

$$\bar{\omega}^{jm} \equiv K_i{}^j \Omega^{i,m} = -\bar{\omega}^{mj}, \tag{10.5}$$

$$K_i{}^j \bar{\Omega}^{i,mk} + K_i{}^{mk} \bar{\Omega}^{i,j} = 0, \tag{10.6a}$$

$$K_i{}^j \bar{\Omega}^{i,m}{}_k + K_{ik}{}^j \bar{\Omega}^{i,m} = 0. \tag{10.6b}$$

The field equations that follow from the action (5.1) are:

$$\bar{D}^2 K_i = K_i{}^j \bar{D}^2 \bar{\Phi}_j + \tfrac{1}{2} K_i{}^{jk} \bar{D}\bar{\Phi}_j \bar{D}\bar{\Phi}_k = 0. \tag{10.7}$$

Using (10.5) and (10.6a) we find that (10.7) is equivalent to (10.4).

We now describe the relation of (10.2–7) to hyper-Kähler geometry (see the appendix for a review of the basic notions of complex geometry). A quaternionic structure (A.13) is given by:

$$J_i^{(3)j} = \begin{pmatrix} i\delta_i^j & 0 \\ 0 & -i\delta_j^i \end{pmatrix}, \qquad J_i^{(1)j} = \begin{pmatrix} 0 & \Omega_{j,i} \\ \bar{\Omega}^{j,i} & 0 \end{pmatrix}, \qquad J_i^{(2)j} = \begin{pmatrix} 0 & i\Omega_{j,i} \\ -i\bar{\Omega}^{j,i} & 0 \end{pmatrix}. \tag{10.8}$$

In (10.8), i is a covariant and j a contravariant index. The tensors $J^{(1)}$ and $J^{(2)}$ are almost complex structures because of (10.2), and are integrable because of (10.3). The metric is hermitian with respect to these complex structures because of (10.5), and they are covariantly constant because of (10.6). Consequently, the manifold is *an hermitian manifold with a covariantly constant quaternionic structure*: this is precisely what defines a *hyper-Kähler* manifold.

On a hyper-Kähler manifold one can choose coordinates such that any one given complex structure takes the canonical form (A.6). On any Kähler manifold, when the complex structure has the canonical form, the metric can be expressed in terms of a Kähler potential, and thus there are *non*holomorphic coordinate transformations, parametrized by the 2-sphere of complex structures, between the coordinate systems where the metric has the Kähler form (5.2, 3). This gives an alternate definition of a hyper-Kähler manifold.

Hyper-Kähler manifolds have a very rich structure; in particular, we can always construct the functions Ω_i and $\bar{\Omega}^i$ (and thus the extra supersymmetries (10.1) of the corresponding nonlinear σ-model) from the Kähler potential and any one of the *non*canonical complex structures. We now explore some of the intricacies that lead to this construction. Because a noncanonical complex structure J_i^j anticommutes with the canonical one (A.13), it has the form

$$J_i^j = \begin{pmatrix} 0 & \Omega_{ji} \\ \bar{\Omega}^{ji} & 0 \end{pmatrix} \tag{10.9}$$

for some matrix Ω, ($\bar{\Omega}$ is the complex conjugate of Ω because J is real in real coordinates). It follows that J^{ij} is block diagonal

$$J^{ij} = \begin{pmatrix} \bar{\gamma}^{ij} & 0 \\ 0 & \gamma_{ij} \end{pmatrix}, \tag{10.10}$$

with

$$\bar{\gamma}^{ij} = (K^{-1})^i_k \bar{\Omega}^{kj}. \tag{10.11}$$

Because on a Kähler manifold the connection has no components with mixed holomorphic and antiholomorphic indices (see (5.6)-(5.8)), the condition that J is covariantly constant implies that $\bar{\gamma}^{jk}$ is holomorphic ($\bar{\partial}^i \bar{\gamma}^{jk} = 0$) and that γ_{jk} is antiholomorphic ($\partial_i \gamma_{jk} = 0$). We define

$$\bar{\Omega}^i \equiv \bar{\gamma}^{ij} K_j. \tag{10.12}$$

From (10.11)-(10.12) it follows that

$$\bar{\Omega}^{i,j} = \bar{\Omega}^{ij}. \tag{10.13}$$

$\bar{\Omega}^{ij}$ fulfil (10.2 − 4) because the manifold is hyper-Kähler and thus Ω^i is precisely the transformation function in (10.1) with a particular choice of the arbitrary chiral term discussed below (10.1). This choice implies

$$\begin{aligned} K_i \bar{\Omega}^i &= 0, \\ \nabla_i \bar{\Omega}^i &= 0, \end{aligned} \tag{10.14}$$

which follow from (10.12) and the antisymmetry of γ_{ij} and $\bar{\gamma}^{ij}$ implied by the hermiticity condition (A.7). We note that by the same logic used in the discussion following (10.11), $\bar{\omega}^{ij}$ introduced in (10.5) is antiholomorphic and $\omega_{ij} \equiv K_i{}^k \Omega_{kj}$ is holomorphic.

A further interesting aspect of hyper-Kähler manifolds is that the Christoffel connection can be written entirely in terms of the quaternionic structure. Comparing (10.6) and (5.6), we see that

$$\Gamma_{jk}{}^m = -\bar{\Omega}^{m,i} \Omega_{i,jk} \tag{10.15}$$

(the expression for $\bar{\Gamma}^{jk}{}_m$ follows by complex conjugation).

On a hyper-Kähler manifold, the description of isometries has several special features. In particular, holomorphic isometries can be holomorphic with respect to just one of the complex structures or with respect to all the complex structures. In the first case, $N = 2$ supersymmetric gaugings are impossible (see next section), so one chooses coordinates that put the relevant complex structure in canonical form, and then follows the discussion of sect. 6. In the second case, we say that the Killing vector that generates the isometry is *triholomorphic*. There are several equivalent ways of expressing the triholomorphic condition. The most direct statement is that

$$P_{\pm i}{}^j \nabla_j (P_{\mp m}{}^n k^m) = 0 \tag{10.16}$$

holds for all complex structures (actually, if (10.16) holds for any two linearly independent complex structures, it holds for the entire quaternionic structure). Here k is the Killing vector and $P_\pm$ are the projection operators (A.2). This is equivalent to the condition that the Lie derivative L_k of all the complex structures vanish:

$$L_k J_i{}^j \equiv k^m J_i{}^j{}_{,m} - k^j{}_{,m} J_i{}^m + k^m{}_{,i} J_m{}^j = 0\,. \tag{10.17}$$

Finally, in the coordinates used in (10.8), a Killing vector is triholomorphic if it is manifestly holomorphic and satisfies

$$\bar\Omega^{ij}\bar k_j{}^{;m} - \bar\Omega^{jm} k^i{}_{;j} = 0 \tag{10.18}$$

or equivalently

$$\bar\omega^{j[i}\bar k_j{}^{;m]} = 0 \tag{10.19}$$

(and the complex conjugate equations). The triholomorphic condition guarantees that the second supersymmetry commutes with the isometries. From (10.18) and Killing's equation (5.20), we also have:

$$k^i{}_{;i} = \bar k_i{}^{;i} = 0\,. \tag{10.20}$$

A triholomorphic Killing vector has a real Killing potential $X^{(J)}$ with respect to *each* complex structure J (5.23):

$$k^i J_{ij} = -X^{(J)}{}_{,j}\,. \tag{10.21}$$

The integrability of (10.21) is guaranteed precisely by the triholomorphic condition (10.16), (10.17). Specializing once again to the coordinates used in (10.8, 18, 19), we can define a *holomorphic* potential P and an *antiholomorphic* potential $\bar P$ with respect to $J^{(1)} \mp iJ^{(2)}$

$$k^i\omega_{ij} = -P_{,j}\,,$$
$$\bar k_i\bar\omega^{ij} = -\bar P^{,j} \tag{10.22}$$

(recall that ω is holomorphic). To compute P_A, we observe that, from (10.5, 22) and (5.23)

$$(\partial_i + i\Omega_{ji}\bar\partial^j)(\bar P_A - P_A - X_A) = 0\,, \tag{10.23}$$

which can be integrated to give

$$X_A = \bar P_A - P_A + \chi_A\,, \tag{10.24}$$

where

$$(\partial_i + i\Omega_{ji}\bar\partial^j)\chi_A = 0\,. \tag{10.25}$$

Similarly, by considering the operator $\partial_i - i\Omega_{ji}\bar\partial^j = -i\Omega_{mi}(\bar\partial^m - i\bar\Omega^{jm}\partial_j)$, we find

$$X_A = P_A - \bar P_A + \bar\chi_A\,, \tag{10.26a}$$

$$(\bar\partial^i - i\bar\Omega^{ji}\partial_j)\bar\chi_A = 0 \tag{10.26b}$$

and thus

$$2X_A = \chi_A + \bar{\chi}_A\,, \tag{10.27}$$

$$2(P_A - \bar{P}_A) = \chi_A - \bar{\chi}_A\,. \tag{10.28}$$

Eqs. (10.25) and (10.26b) state that χ (and $\bar{\chi}$) are holomorphic (and antiholomorphic) with respect to a noncanonical complex structure (cf. (A.3)). Then eq. (10.27) allows us to determine χ and $\bar{\chi}$ *up to an imaginary constant* (cf. (5.19)). In turn (10.28) allows us to determine P and $\bar{P}$ (again up to a constant; the two constants introduce an ambiguity in P by a complex constant). From (5.34) and using the analyticity, (5.27), we find the gauge transformation of χ_A and thus of P_A:

$$\delta P_A \equiv \lambda^B k^i_B P_{A,i} = -\lambda^B (f_{AB}{}^C P_C + \tilde{c}_{AB})\,. \tag{10.29}$$

Here $\tilde{c}_{AB} = -\tilde{c}_{BA}$ is a complex analog of c_{AB} in (5.27). The condition $\tilde{c}_{AB} = 0$ *when* it can be imposed, removes the ambiguity in P_A noted above, except for abelian factors. As we shall see in the next section, this residual ambiguity corresponds to the possibility of adding to the action an $N=2$ Fayet–Iliopoulos term with an arbitrary coefficient for each abelian factor. When the constants *cannot* be removed, they are an *obstruction* to $N=2$ supersymmetric gauging. As in sect. 5, whenever the group is semisimple, the obstruction is removable and (10.29) can be used to solve for P_A (cf. (5.35)):

$$P_A = f_{AB}{}^C k^i_D k^j_C \omega_{ji} g^{BD}\,. \tag{10.30}$$

The obstructions c_{AB} and $\tilde{c}_{AB}$ are independent. As an example of a case where c_{AB} but not $\tilde{c}_{AB}$ is removable, we consider 4 (real) dimensional flat space with Kähler potential

$$K = \bar{\Phi}_i \Phi^i\,, \qquad i = 1, 2\,. \tag{10.31}$$

We consider the translations generated by

$$k_A = \delta^i_A (\partial/\partial\Phi^i + \partial/\partial\bar{\Phi}_i) \qquad A = 1, 2\,. \tag{10.32}$$

We compute the chiral functions η_A from (5.19), and the real functions X_A from (5.22)

$$\begin{aligned} \eta_A &= \delta^i_A \Phi^i\,, \\ X_A &= i\delta^i_A(\Phi^i - \bar{\Phi}_i)\,. \end{aligned} \tag{10.33}$$

A simple computation shows that $c_{AB} = 0$ (cf. (5.27)). We can choose complex structures such that

$$\Omega_{ij} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \tag{10.34}$$

The operators introduced in (10.23) are

$$\partial_i + i\Omega_{ji}\bar{\partial}^j = \begin{pmatrix} \partial_1 + \bar{\partial}^2 \\ \partial_2 - \bar{\partial}^1 \end{pmatrix} . \tag{10.35}$$

A general function χ annihilated by (10.35) has the form $\chi(\Phi^1 - \bar{\Phi}_2, \Phi^2 + \bar{\Phi}_1)$. From (10.33) and (10.27) we have

$$\begin{aligned} \chi_1 &= i(\Phi^1 - \bar{\Phi}_2 - (\Phi^2 + \bar{\Phi}_1)) , \\ \chi_2 &= i(\Phi^1 - \bar{\Phi}_2 + (\Phi^2 + \bar{\Phi}_1)) . \end{aligned} \tag{10.36}$$

Comparing to (10.28) we find

$$\left. \begin{aligned} P_1 &= -2i\Phi^2 \\ P_2 &= 2i\Phi^1 \end{aligned} \right\} \leftrightarrow P_A = -2\delta^i{}_A \Omega_{ij} \Phi^j . \tag{10.37}$$

Finally, we compute $\tilde{c}$ from (10.29):

$$\tilde{c}_{12} = -k_1 P_2 = k_2 P_1 = -2i . \tag{10.38}$$

Clearly, since the group is abelian there is no shift that can remove $\tilde{c}$. Only the isometry generated by *one* arbitrary linear combination of k_1 and k_2 can be gauged in an $N=2$ supersymmetric way.

11. Gauging isometries on hyper-Kähler manifolds

In sect. 6 we discussed how to gauge isometries on a Kähler manifold in an $N=1$ supersymmetric way. In this section we discuss the gauging of isometries on hyper-Kähler manifolds, where the ungauged theory has an $N=2$ supersymmetry. The methods of section 6 still apply and give a gauging preserving $N=1$ supersymmetry. If the Killing vector of the isometry is holomorphic with respect to all three complex structures then we can perform an $N=2$ supersymmetric gauging [6]. Here we present the method along with an explicit example.

The starting point for the $N=2$ gauging is the action (5.1) invariant under the extra supersymmetry variations (10.1). To gauge the isometries, we have to introduce $N=2$ gauge multiplets (V^A, S^A), where S^A are chiral. The second supersymmetry transformations are:

$$\begin{aligned} \delta\Phi^i &= \bar{D}^2(\bar{\varepsilon}\bar{\Omega}^i(\Phi, e^L\bar{\Phi})) , \\ \delta\bar{\Phi}_i &= D^2(\varepsilon\Omega_i(\bar{\Phi}, e^{\bar{L}}\Phi)) , \\ \delta e^V &= \varepsilon\bar{S}\, e^V + e^V S\bar{\varepsilon} , \\ \delta S &= -iW^\alpha D_\alpha\varepsilon = \bar{D}^2((e^{-V}D^\alpha\, e^V)D_\alpha\varepsilon) . \end{aligned} \tag{11.1}$$

The Φ-transformations are the naive $N=1$ covariantization $\tilde{\Phi} \to \tilde{\tilde{\Phi}}$, (see (6.2)), of the transformations (10.1). The gauge transformations are:

$$
\begin{aligned}
\delta\Phi^i &= \Lambda^A k^i_A\,,\\
\delta\bar{\Phi}_i &= \bar{\Lambda}^A \bar{k}_{Ai}\,,\\
\delta \mathrm{e}^V &= i(\bar{\Lambda}\mathrm{e}^V - \mathrm{e}^V\Lambda)\,,\\
\delta S &= i[\Lambda, S]\,, \qquad \Lambda \equiv \Lambda^A T_A\,.
\end{aligned}
\tag{11.2}
$$

The action invariant under (11.1, 2) is the $N=1$ covariantized action (6.10) with additional S-dependent terms:

$$
\int \mathrm{d}^4x\,\mathrm{d}^4\theta\left[K(\Phi^i,\bar{\Phi}_j) + \frac{\mathrm{e}^L-1}{L}X_A V^A + g_{AB}S^A(\mathrm{e}^{-V}\bar{S}\,\mathrm{e}^V)^B\right] + \left\{\int \mathrm{d}^4x\,\mathrm{d}^2\theta[iS^A P_A + \tfrac{1}{4}g_{AB}W^{A\alpha}W^B_\alpha] + \text{h.c.}\right\},
\tag{11.3}
$$

where P_A are the potentials defined in (10.22). The $N=2$ invariance has been checked in Wess-Zumino gauge. Supersymmetry in an arbitrary gauge follows from the gauge invariance of the action.

It is necessary for the gauge invariance of the action that the obstructions $\tilde{c}$ in (10.29), (as well as the $N=1$ obstructions c in (5.27)), vanish. For a semisimple gauge group, the obstructions can always be removed and P_A obtained as in (10.30). For abelian factors the obstructions cannot be removed and an $N=2$ supersymmetric gauging is possible only if they vanish from the outset. In this case P_A is determined only up to a complex constant for each abelian factor. Together with the freedom of adding a real constant to X_A, (cf. discussion following (6.10)), this corresponds to the possibility of adding an $N=2$ Fayet-Iliopolous term to the action:

$$
\sum_{\substack{\text{abelian}\\ \text{factors}}}\left\{\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\nu_{3A}V^A + \left(\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,i\nu_A S^A + \text{h.c.}\right)\right\}.
\tag{11.4}
$$

As the $N=2$ vector multiplet is an *off-shell* representation, its action in (11.3) can be replaced by the action for the kählerian vector multiplet: $g_{AB}S^A(\mathrm{e}^{-V}\bar{S}\mathrm{e}^V)^B \to K_2(S, \mathrm{e}^{-V}\bar{S}\mathrm{e}^V)$, $g_{AB} \to Q_{AB}$, where K_2 and Q_{AB} are expressible in terms of a single holomorphic function $H(S)$ (see sect. 9).

To exemplify the above, we discuss gauging of isometries on the Eguchi-Hanson manifold. This is a four (real) dimensional hyper-Kähler manifold with isometry group U(2). The Kähler potential is [7]:

$$
K = \sqrt{1+\rho^4} - \ln\frac{1+\sqrt{1+\rho^4}}{\rho^2}\,,
\tag{11.5}
$$

where

$$
\rho^2 = |\eta|^2 + |\zeta|^2\,.
\tag{11.6}
$$

Here η and ζ are the complex coordinates on the manifold. The holomorphic components of the Killing vectors are (basis $\partial_\eta, \partial_\zeta$):

$$
\begin{aligned}
k_1 &= i(\eta, \zeta)\,, \\
k_2 &= i(\eta, -\zeta)\,, \\
k_3 &= i(\zeta, \eta)\,, \\
k_4 &= (-\zeta, \eta)\,.
\end{aligned}
\tag{11.7}
$$

Their Killing potentials are:

$$
\begin{aligned}
X_1 &= \sqrt{1+\rho^4}\,, \\
X_2 &= (\sqrt{1+\rho^4}/\rho^2)(|\eta|^2 - |\zeta|^2)\,, \\
X_3 &= (\sqrt{1+\rho^4}/\rho^2)(\zeta\bar{\eta} + \eta\bar{\zeta})\,, \\
X_4 &= i(\sqrt{1+\rho^4}/\rho^2)(\zeta\bar{\eta} - \eta\bar{\zeta})\,.
\end{aligned}
\tag{11.8}
$$

In this coordinate system all Killing vectors are holomorphic. The generator for the invariant U(1) subgroup is k_1 and k_2, k_3, k_4 generate SU(2). Since SU(2) is semisimple and the obstructions for the U(1) factor vanish (cf. (5.27)), we can gauge the whole U(2) in an $N=1$ supersymmetric manner and (6.10) gives the action.

The complex structures are given by (10.8), where $\bar{\Omega}^{i,j}$ is obtained from the potential $\bar{\Omega}^i$ which, using (10.6, 14), is found, in this case, to have a simple relation to the Kähler potential:

$$
\begin{aligned}
\bar{\Omega}^\eta &= K_\zeta = \left(\frac{\mathrm{d}K}{\mathrm{d}\rho^2}\right)\bar{\zeta}\,, \\
\bar{\Omega}^\zeta &= -K_\eta = -\left(\frac{\mathrm{d}K}{\mathrm{d}\rho^2}\right)\bar{\eta}\,.
\end{aligned}
\tag{11.9}
$$

Using (11.9) in (10.16) one verifies that the generators $k_2 \cdots k_4$ are triholomorphic, whereas k_1 is not. This means that k_1 is not holomorphic in a coordinate system where another complex structure is canonical and thus we cannot gauge the U(1) factor without breaking $N=2$ supersymmetry.

As SU(2) is semisimple we can remove the obstructions $\tilde{c}$ (10.29) and perform an $N=2$ supersymmetric gauging. We calculate the potentials P_A from (10.30):

$$
(P_2, P_3, P_4) = (i\eta\zeta, \tfrac{1}{2}i(\zeta^2 - \eta^2), -\tfrac{1}{2}(\zeta^2 + \eta^2))\,. \tag{11.10}
$$

The gauged action is obtained by substituting K, X_A and P_A in (11.3).

12. Components, vacuum conditions and symmetry breaking

The gauged superfield actions obtained above are suitable for deriving superfield Feynman rules; however, since they are nonrenormalizable, it is probably more appropriate to treat them as tree level effective actions. In this section we expand a general action in Wess-Zumino gauge, where the lagrangian is polynomial. We then derive the component action, which is suitable for reading off component masses, couplings, etc. We extract the vacuum field equations, and discuss aspects of gauge and supersymmetry breaking.

In Wess-Zumino gauge (2.14) $V^3=0$ and the action (6.10) becomes

$$S=\int d^4x\, d^4\theta[K(\Phi^i,\bar{\Phi}_j)+X_A V^A+\tfrac{1}{2}\bar{k}_{Ai}k^j_B K^i_j V^A V^B]$$
$$+\left\{\int d^4x\, d^2\theta[P(\Phi^i)+\tfrac{1}{4}Q_{AB}W^{A\alpha}W^B_\alpha]+\text{h.c.}\right\}, \tag{12.1}$$

where

$$W^A_\alpha=i(\bar{D}^2D_\alpha V^A+\tfrac{1}{2}[\bar{D}^2D_\alpha V,V]^A+\tfrac{1}{2}[D_\alpha V,\bar{D}^2V]^A+\tfrac{1}{2}[\bar{D}_{\dot\alpha}D_\alpha V,\bar{D}^{\dot\alpha}V]^A)\,. \tag{12.2}$$

We can now use the Wess-Zumino gauge components (2.14, 15) and the relations (2.2, 16) to derive the component action

$$S=\int d^4x\{K^j_i[-\tfrac{1}{2}\nabla^{\alpha\dot\alpha}A^i\nabla_{\alpha\dot\alpha}\bar{A}_j+i\bar{\psi}^{\dot\alpha}_j\nabla^\alpha{}_{\dot\alpha}\psi^i_\alpha+\bar{k}_{Aj}\lambda^{A\alpha}\psi^i_\alpha+k^i_A\bar{\lambda}^{A\dot\alpha}\bar{\psi}_{j\dot\alpha}+\hat{F}^i\hat{\bar{F}}_j]$$
$$+\tfrac{1}{2}[Q_{AB}(\lambda^{A\alpha}i\nabla_\alpha{}^{\dot\alpha}\bar{\lambda}^B_{\dot\alpha}-\tfrac{1}{2}f^{A\alpha\beta}f^B_{\alpha\beta}+\hat{D}^A\hat{D}^B)$$
$$-Q_{ABi}\psi^{i\alpha}\lambda^{A\beta}f^B_{\alpha\beta}+\hat{P}_{;ij}\psi^{i\alpha}\psi^j_\alpha+\text{h.c.}]$$
$$+\tfrac{1}{4}R_k{}^i{}_m{}^j\psi^{k\alpha}\psi^m_\alpha\bar{\psi}^{\dot\alpha}_i\bar{\psi}_{j\dot\alpha}-\tfrac{1}{2}(Q+\bar{Q})^{-1\,AB}[X_A-\tfrac{1}{2}i(Q_{ACi}\psi^{i\alpha}\lambda^C_\alpha-\bar{Q}^i_{AC}\bar{\psi}^{\dot\alpha}_i\bar{\lambda}^C_{\dot\alpha})]$$
$$\times[X_B-\tfrac{1}{2}i(Q_{BDj}\psi^{j\alpha}\lambda^D_\alpha-\bar{Q}^j_{BD}\bar{\psi}^{\dot\alpha}_j\bar{\lambda}^D_{\dot\alpha})]-(K^{-1})^i_j\hat{P}_i\hat{\bar{P}}^j\}\,, \tag{12.3}$$

where

$$\nabla_{\alpha\dot\alpha}\bar{\lambda}^A_{\dot\beta}\equiv\partial_{\alpha\dot\alpha}\bar{\lambda}^A_{\dot\beta}-i[A_{\alpha\dot\alpha},\bar{\lambda}_{\dot\beta}]^A=\partial_{\alpha\dot\alpha}\bar{\lambda}^A_{\dot\beta}+f_{BC}{}^A A^B_{\alpha\dot\alpha}\bar{\lambda}^C_{\dot\beta}\,,$$
$$\nabla_{\alpha\dot\alpha}A^i\equiv\partial_{\alpha\dot\alpha}A^i-k^i_B A^B_{\alpha\dot\alpha}\,,$$
$$\nabla_{\alpha\dot\alpha}\psi^i_\beta\equiv\partial_{\alpha\dot\alpha}\psi^i_\beta+\Gamma^i_{jk}(\partial_{\alpha\dot\alpha}A^j)\psi^k_\beta-k^i_{B;j}A^B_{\alpha\dot\alpha}\psi^j_\beta$$
$$=\partial_{\alpha\dot\alpha}\psi^i_\beta+\Gamma^i_{jk}(\nabla_{\alpha\dot\alpha}A^j)\psi^k_\beta-k^i_{B,j}A^B_{\alpha\dot\alpha}\psi^j_\beta\,,$$
$$\hat{F}^i\equiv F^i+(K^{-1})^i_j(\bar{P}^j+\tfrac{1}{4}\bar{Q}^j_{AB}\bar{\lambda}^{A\dot\alpha}\bar{\lambda}^B_{\dot\alpha})+\tfrac{1}{2}\Gamma^i{}_{jk}\psi^{j\alpha}\psi^k_\alpha\,,$$
$$\hat{D}^A\equiv D'^A-(Q+\bar{Q})^{-1\,AB}[\tfrac{1}{2}i(Q_{BCi}\psi^{i\alpha}\lambda^C_\alpha-\bar{Q}^i_{BC}\bar{\psi}^{\dot\alpha}_i\bar{\lambda}^C_{\dot\alpha})-X_B]\,,$$
$$\hat{P}\equiv P+\tfrac{1}{4}Q_{AB}\lambda^{A\alpha}\lambda^B_\alpha\,. \tag{12.4}$$

The connection and curvature are defined in sect. 5. This generalizes the results of [2] to include a nontrivial matrix Q_{AB}.

The component field equations can be derived either from the superfield equations (6.11) [8, 10], or directly from the component action (12.3). For discussing symmetry breaking it is sufficient to consider the vacuum field equations. The tree-level effective potential is

$$U_{\text{eff}}=\tfrac{1}{2}(Q+\bar{Q})^{-1\,AB}X_AX_B+(K^{-1})^i{}_jP_i\bar{P}^j-K^i{}_j\hat{F}^j\hat{\bar{F}}_i-\tfrac{1}{2}(Q+\bar{Q})_{AB}\hat{D}^A\hat{D}^B\,, \quad (12.5)$$

from which we obtain the vacuum field equations:

$$\begin{gathered} id^A\bar{k}_{Ai}K^i{}_j+\tfrac{1}{2}d^Ad^BQ_{ABj}+f^iP_{;ij}=0\,,\\ K^i{}_jf^j=-\bar{P}^i\,,\qquad (Q+\bar{Q})_{AB}d^A=-X_B\,, \end{gathered} \quad (12.6)$$

where $f^j\equiv\langle F^j\rangle_0$ and $d^A\equiv\langle D'^A\rangle_0$. Thus the vacuum expectation value of the effective potential is

$$\langle U_{\text{eff}}\rangle_0=\tfrac{1}{2}(Q+\bar{Q})_{AB}d^Ad^B+K^i{}_jf_i\bar{f}^j\,. \quad (12.7)$$

Supersymmetry is spontaneously broken if the supercharge does not annihilate the vacuum, i.e., the vacuum expectation value of the supersymmetric variation of some observable operator is nonvanishing. Since the supersymmetric transformation laws depend only on the gradients of physical scalars but depend directly on the values of auxiliary scalars, supersymmetry is spontaneously broken if some auxiliary scalar field develops a vacuum expectation value. From (12.7) we see that the vacuum energy density vanishes whenever supersymmetry is unbroken, while it is strictly positive for spontaneously broken supersymmetry. In particular, if a supersymmetric solution to the vacuum equations exists, it is the stable vacuum. Spontaneous breaking of global supersymmetry gives rise to a massless Goldstone fermion (goldstino). This is reflected by the transformation properties of the fields: In general, when supersymmetry is broken the fermion sector can be divided into "transverse" fields with variations that have vanishing expectation values, and the goldstino, which transforms as the supersymmetry parameter times some nonvanishing linear combination of f^i and d^A. (Explicit mass matrices can be found in [8, 10]).

Whether the spontaneous breaking of supersymmetry is linked to the breaking of gauge symmetry depends on whether the auxiliary fields that break supersymmetry are singlets under the gauge group. One can further distinguish between f^i breaking and d^A breaking. The first model with breaking used a U(1) singlet d^A term (the Fayet-Iliopoulous term) [29]. As discussed in sect. 6, this term reflects the freedom to add a constant to the Killing potential of a U(1) factor [2]. More drastic types of breaking can be obtained through nonsinglet d^A terms. These appear to require nonrenormalizable models [2] or "triggering" by a Fayet-Iliopoulous term [30]. Breaking through f^i terms was first considered in [31]. This is the mechanism most frequently used in globally supersymmetric "phenomenological" models. An interesting discussion of certain kinds of supersymmetry breaking can be found in [32].

13. Supertrace formulae

The detailed information about masses and couplings is contained in the action (6.10) or its component expansion (12.3). Some of this information is conveniently coded into supertrace mass relations [33, 1], which can be calculated efficiently using superfield methods [8, 10]. In this section we give selected details of the computation. We use the observation that the quadratic divergence Γ_∞ in the 1-loop effective action is proportional to the supertrace Str M^2 [34]

$$\begin{aligned}\Gamma_\infty &= -\tfrac{1}{2}\,\mathrm{Str}\, M^2 \cdot I_2\,,\\ I_2 &\equiv \int \frac{\mathrm{d}^4 p}{(2\pi)^4 p^2}\,,\\ \mathrm{Str}\, M^2 &\equiv \mathrm{Tr} \sum_j (-)^{2j}(2j+1) M_j^2 = \mathrm{Tr}\,\{M_0^2 - 2M_{1/2}M_{1/2}^* + 3M_1^2 + \cdots\}\,. \end{aligned} \tag{13.1}$$

We find the supertrace

$$\begin{aligned}-\tfrac{1}{2}\,\mathrm{Str}\, M^2 = \{&R^i{}_j + \mathrm{Tr}\,[Q_i(Q+\bar Q)^{-1}\bar Q^j(Q+\bar Q)^{-1}]\} f^i \bar f_j\\ &+ i\{\bar k_{Ai}{}^{;i} - \mathrm{Tr}\,[\bar Q^i(Q+\bar Q)^{-1}]\bar k_{Ai}\} d^A\,. \end{aligned} \tag{13.2}$$

Using Killing's equation and (13.10) below the second term can be rewritten in the covariant form

$$-\tfrac{1}{2} J_i{}^j \{k^i_{A;j} - [\mathrm{Tr}\ln\{\tfrac{1}{2}(Q+\bar Q)\}]_{,j} k^i_A\} d^A\,. \tag{13.3}$$

Remarkably, for the $N=2$ models in sect. 9, 10 and 11 we find that *the supertrace vanishes*. For the hyper-Kähler manifolds this follows because they are Ricci flat and from (10.20). For the kählerian vector multiplet the Q-terms cancel the curvature and the connection terms in (13.2) because the Kähler metric is $\tfrac{1}{2}(Q+\bar Q)_{AB}$ (9.6), and $\bar k_{Ai}{}^{;i} = f_{AB}{}^B$ (see (9.2)) vanishes for semisimple groups. (For the kählerian vector multiplet the chiral fields are in the adjoint representation, and thus, for the kinetic terms to be nonsingular, the gauge group must be semisimple). The supertrace also vanishes for hyper-Kähler models with isometries gauged by kählerian vector multiplets. In these models the scalar manifold is a product of the matter-hyper-Kähler and gauge-scalar Kähler manifolds. One could also try to construct models with irreducible scalar manifolds, where the nonminimal vector kinetic terms Q_{AB} could depend on all scalar fields. However, such scalar models have not been constructed and probably do not exist.

To derive (13.2), we need the superfield Feynman rules (for a detailed account, see [10]). We treat all mass terms as interactions.

The propagators are:

(i) for chiral fields: $\langle \bar\Phi_i \Phi^j\rangle = \dfrac{\delta_i^j \delta^4(\theta_1-\theta_2)}{p^2}$,

(ii) for real fields in Fermi-Feynman gauge: $\langle V^A V^B\rangle = -\dfrac{\delta^{AB}\delta^4(\theta_1-\theta_2)}{p^2}$.

The vertices have:

(i) a factor of $\bar{D}^2$, D^2 acting on every internal chiral, antichiral line;

(ii) factors of D, $\bar{D}$ coming from explicit D, $\bar{D}$ in the interactions;

(iii) a missing $\bar{D}^2$, D^2 for each interaction written as a chiral, antichiral integral;

(iv) a momentum conserving δ-function and an integral $\int \mathrm{d}^4\theta$. Finally, each loop has a factor $\mathrm{d}^4p/(2\pi)^4$ and the graphs are weighted by appropriate symmetry factors.

To evaluate a loop, we partially integrate the D, $\bar{D}$-factors off a line and then perform the θ-integral at one end of the line using the δ-function. We continue this process until only two θ-integrals remain in the loop; this gives

$$\int \frac{\mathrm{d}^4p}{(2\pi)^4} f(p)\, \mathrm{d}^4\theta_1\, \mathrm{d}^4\theta_2\, \delta^4(\theta_1-\theta_2) h(D, \bar{D}) \delta^4(\theta_1-\theta_2)\,, \tag{13.4}$$

where h is some polynomial in D, $\bar{D}$. Using (2.1, 2), h can be reduced to $D^2\bar{D}^2 g(p)$ plus terms with fewer D's. Because $\delta^4(\theta_1-\theta_2) = (\bar{\theta}_1-\bar{\theta}_2)^2(\theta_1-\theta_2)^2$, only the term with $D^2\bar{D}^2$ survives.

To evaluate the 1-loop quadratic divergence, we consider only terms proportional to $\int \mathrm{d}^4p/p^2$. Since each propagator contributes p^{-2}, we must cancel all but one propagator with a numerator factor. From (2.2), we have, e.g., $D^2\bar{D}^2D^2\bar{D}^2 = -p^2D^2\bar{D}^2$; hence we require a factor $D^2\bar{D}^2$ for each propagator. Thus, each vertex must supply the equivalent of $D^2\bar{D}^2$. We now expand the action (6.10) to quadratic order in quantum fields, keeping only those terms that contribute to the quadratic divergence*:

$$S^{(2)} = \int \mathrm{d}^4x\, \mathrm{d}^4\theta [(\delta_i^{\,j} + U_i^{\,j})\Phi^i\bar{\Phi}_j + \tfrac{1}{2}(\delta_{AB} + U_{AB}) V^A D^\alpha \bar{D}^2 D_\alpha V^B]\,, \tag{13.5}$$

where

$$U_i^{\,j} = -(\delta_i^{\,j} - K_i^{\,j} - iK_i^{\,k}\langle V^A\rangle \bar{k}_{Ak}{}^{;j})\,, \tag{13.6}$$

$$\begin{aligned} \tfrac{1}{2}U_{AB} &= -\tfrac{1}{2}(\delta_{AB} - \tfrac{1}{2}(Q+\bar{Q})_{AB} - \tfrac{1}{4}i(Q-\bar{Q})_{C(A} f_{B)D}{}^C\langle V^D\rangle) \\ &= -\tfrac{1}{2}(\delta_{AB} - \tfrac{1}{2}(Q+\bar{Q})_{AB} + \tfrac{1}{4}i(Q_{ABi}k_C^i - \bar{Q}_{AB}^i\bar{k}_{Ci})\langle V^C\rangle) \end{aligned} \tag{13.7}$$

and we have used (6.3) in (13.7). In (13.5), because $\langle V^A\rangle = \theta^2\bar{\theta}^2 d^A$, only the term linear in $\langle V^A\rangle$ contributes. (Only Φ^i, $\bar{\Phi}_j$ and V^A are quantum fields.) The terms with quantum fields $\Phi^i\Phi^j$, Φ^iV^A and their hermitian conjugates are absent from (13.5) because they cannot contribute to $D^2\bar{D}^2$. After gauge fixing, the propagators are as described above, and the interactions are $U_i^{\,j}\Phi^i\bar{\Phi}_j$ and $U_{AB}V^AD^\alpha\bar{D}^2D_\alpha V^B$. The Faddeev–Popov ghosts have been neglected because the ghost and antighost terms cancel. The interaction (13.6) enters only in Φ loops whereas (13.7) enters only in V^A loops, and there is no mixing between them.

* In [8, 10], the last term is included implicitly in the first V-term by the use of the covariant $\tilde{\bar{\Phi}}_i$ instead of $\bar{\Phi}_i$.

We begin by evaluating the contribution of a Φ loop with n vertices coming from (13.6). Each vertex has a factor $U_i^j D^2\bar{D}^2$; one of the $D^2\bar{D}^2$ is required for the final θ-integration as discussed above and the rest produces a factor $(-p^2)^{n-1}$. Including the symmetry factor and summing over n, we obtain:

$$\Gamma_\infty^\Phi = \mathrm{Tr}\sum_{n=1}^{\infty}\int \frac{\mathrm{d}^4\theta\,\mathrm{d}^4p}{(2\pi)^4p^2}\left(\frac{-1}{n}\right)\left(-U_i^j\right)^n = \int \mathrm{d}^4\theta\,\mathrm{Tr}\ln(\delta_i^j+U_i^j)I_2$$

$$= \int \mathrm{d}^4\theta\,\mathrm{Tr}\ln(K_i^j + iK_i^k\langle V^A\rangle\bar{k}_{Ak}{}^{;j})I_2$$

$$= \int \mathrm{d}^4\theta[\ln\det K_i^j + i\langle V^A\rangle\bar{k}_{Ai}{}^{;i}]I_2\,, \tag{13.8}$$

where we have used (13.6) and, as discussed above, kept only terms linear in $\langle V^A\rangle$.

We now evaluate the contribution of the V^A loops with vertices coming from (13.7). Each vertex has a factor $U_{AB}D^\alpha\bar{D}^2D_\alpha$; as above, one of the $D\bar{D}^2D$ is required for the final θ-integration (it contributes twice what $D^2\bar{D}^2$ contributes, which precisely cancels a symmetry factor of $\frac{1}{2}$ that arises because V^A is real). The rest of the $D\bar{D}^2D$ factors produces a factor $(+p^2)^{n-1}$. Finally, there is a factor $(-)^n$ from the propagators, and thus we find:

$$\Gamma_\infty^V = \mathrm{Tr}\sum_{n=1}^{\infty}\int\frac{\mathrm{d}^4\theta\,\mathrm{d}^4p}{(2\pi)^4p^2}\left(\frac{1}{n}\right)(-U_{AB})^n = -\int \mathrm{d}^4\theta\,\mathrm{Tr}\ln(\delta_{AB}+U_{AB})$$

$$= -\int \mathrm{d}^4\theta\,\mathrm{Tr}\ln[\tfrac{1}{2}(Q+\bar{Q})_{AB} - \tfrac{1}{4}i(Q_{ABi}k_C^i - \bar{Q}_{AB}{}^i\bar{k}_{Ci})\langle V^C\rangle]I_2$$

$$= -\int \mathrm{d}^4\theta\{\mathrm{Tr}\ln[\tfrac{1}{2}(Q+\bar{Q})] - \tfrac{1}{2}i[\mathrm{Tr}((Q+\bar{Q})^{-1}Q_i)k_C^i$$

$$-\mathrm{Tr}((Q+\bar{Q})^{-1}\bar{Q}^i)\bar{k}_{iC}]\langle V^C\rangle\}I_2$$

$$= -\int \mathrm{d}^4\theta\{\mathrm{Tr}\ln[\tfrac{1}{2}(Q+\bar{Q})] + i\,\mathrm{Tr}[(Q+\bar{Q})^{-1}\bar{Q}^i]\bar{k}_{Ai}\langle V^A\rangle\}I_2\,. \tag{13.9}$$

In the last step, we have used (6.3), which implies

$$Q_{ABi}k_C^i + \bar{Q}_{AB}{}^i\bar{k}_{Ci} = f_{C(A}{}^D(Q+\bar{Q})_{B)D} \tag{13.10}$$

multiplying both sides with $(Q+\bar{Q})^{-1\,AB}$ gives the desired result. Collecting the contributions from (13.8) and (13.9) gives

$$-\tfrac{1}{2}\,\mathrm{Str}\,M^2 = \int \mathrm{d}^4\theta\{\ln\det K_i^j - \mathrm{Tr}\ln[\tfrac{1}{2}(Q+\bar{Q})]$$

$$+ i[\bar{k}_{Ai}{}^{;i} - \mathrm{Tr}[(Q+\bar{Q})^{-1}\bar{Q}^i]\bar{k}_{Ai}]\langle V^A\rangle\}\,. \tag{13.11}$$

Using (5.10) we obtain (13.2).

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Appendix

COMPLEX GEOMETRY

Here we list some definitions and properties in complex geometry [35]. For a readable presentation for physicists see [9]. Indices throughout the appendix run from 1 to the real dimension of the manifold ($i \sim (i, \bar{i})$) unless explicitly indicated.

An *almost complex structure* J is a mixed second rank tensor, whose components are real in a real coordinate system, and which satisfies:

$$J_i^{\ j} J_j^{\ k} = -\delta_i^{\ k} . \tag{A.1}$$

A real $2n$-dimensional manifold with an almost complex structure is called an n-dimensional *almost complex manifold.* J defines multiplication by i on vectors.

From J one can define the projection operators

$$P_\pm \equiv \tfrac{1}{2}(1 \pm iJ) , \tag{A.2}$$

which can be used to split any vector into two projections. In particular, one can split the basis of 1-forms $\mathrm{d}x^i$ into $\omega^i_{\ \pm}$:

$$\omega^i_\pm \equiv P_{\pm j}^{\ \ i}\, \mathrm{d}x^j . \tag{A.3}$$

The almost complex structure is called a *complex structure* if and only if the equations

$$\omega^i_\pm = 0 \tag{A.4}$$

are completely integrable. In this case J is said to be *integrable.* The necessary and sufficient conditions for (A.4) are, according to Frobenius' theorem*:

$$\mathrm{d}\omega^i_\pm = \Theta^i_{\ j} \wedge \omega^j_\pm \tag{A.5}$$

(where $\Theta^i_{\ j}$ are arbitrary 1-forms) or, equivalently;

$$P_{\mp i}^{\ \ k}\, \mathrm{d}P_{\pm j}^{\ \ i}\, \mathrm{d}X^j = 0 . \tag{A.6}$$

Integrating (A.4) one finds holomorphic and antiholomorphic subspaces parametrized by z^i and $\bar{z}^{\bar{i}}$ respectively. A simple calculation shows that (A.4) is equivalent to the vanishing of the torsion of J:

$$N_{ij}^{\ \ k} = J_{[i|}^{\ \ n} \partial_n J_{|j]}^{\ \ k} + J_n^{\ k} \partial_{[j} J_{i]}^{\ \ n} = 0 \tag{A.7}$$

(N is called the *Nijenhuis tensor*).

* Actually, Frobenius' theorem can be applied in the complex case only if we assume real analyticity. A proof without this assumption is given by Newlander and Nirenberg [36].

In the complex coordinates z^i, $\bar{z}^{\bar{i}}$, the complex structure has the *canonical* form:

$$J_j^{\ j} = i\begin{pmatrix} \delta_i^j & 0 \\ 0 & -\delta_{\bar{i}}^{\bar{j}} \end{pmatrix}. \tag{A.8}$$

In fact, it can be shown that the integrability of J is equivalent to the existence of a system of complex coordinate neighborhoods in which J has the form (A.8). (Clearly J in (A.8) is invariant under holomorphic coordinate transformations and the Nijenhuis tensor vanishes in these coordinates.)

A *hermitian metric* g_{ij} is a metric that is invariant under an (almost) complex structure

$$J_i^{\ k} J_j^{\ m} g_{km} = g_{ij} \Leftrightarrow J_{ij} \equiv g_{jk} J_i^{\ k} = -J_{ji}. \tag{A.9}$$

From an arbitrary metric $\tilde{g}_{ij}$ a hermitian metric can always be constructed:

$$g_{ij} = \tilde{g}_{ij} + J_i^{\ m} J_j^{\ k} \tilde{g}_{km}. \tag{A.10}$$

An (almost) complex manifold with a hermitian metric is called an (almost) *hermitian manifold.* On such a manifold one can define the *fundamental* 2-*form*:

$$\omega \equiv J_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j. \tag{A.11}$$

In special coordinates where J has its canonical form (A.8), ω becomes, for a hermitian manifold:

$$\omega = 2i g_{i\bar{j}}\, \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{j}}. \tag{A.12}$$

An (almost) *Kähler manifold* is an (almost) hermitian manifold where the fundamental 2-form ω is closed:

$$\mathrm{d}\omega = 0 \Leftrightarrow J_{[ij,k]} = 0. \tag{A.13}$$

For a Kähler manifold (A.12, 13) imply that the metric can be expressed in terms of the Kähler potential (cf. (5.2,,3)). The conditions $N = 0$ and $\mathrm{d}\omega = 0$ are equivalent to:

$$\nabla_i J_j^{\ k} = 0. \tag{A.14}$$

Further aspects of Kähler geometry are discussed in sect. 5.

An *almost quaternionic structure* consists of three linearly independent objects $J^{(X)}$, that generate an SU(2) algebra:

$$J_i^{(X)j} J_j^{(Y)k} = -\delta^{XY} \delta_i^k + \varepsilon^{XYZ} J_i^{(Z)k}. \tag{A.15}$$

The $J^{(X)}$ may be three almost complex structures, but in general, each one of them need not be well defined over the whole manifold. This occurs if a coordinate transformation of $J^{(X)}$ induces an SU(2) rotation. An example is provided by the 4-sphere with the $J^{(X)}$ given by the self-dual unit dyads;

$$J_i^{(km)j} = g^{jn} (\delta_i^{[k} \delta_n^{m]} + \varepsilon_{in}^{\ \ km}), \tag{A.16}$$

where (kl) label the three linearly independent $J^{(X)}$. If there exists a metric that is hemitian with respect to all three $J^{(X)}$ and if they are all covariantly constant, then the manifold is *hyper-Kähler.* In this case each $J^{(X)}$ is a complex structure and coordinates can be chosen such that any one has the canonical form (A.8). All hyper-Kähler manifolds are *Ricci flat.*

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