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Solitons with fermion number 1/2*

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We study the structure of soliton-monopole systems when Fermi fields are present. We show that the existence of a nondegenerate, isolated, zero-energy, c-number solution of the Dirac equation implies that the soliton is a degenerate doublet with Fermi number $\pm 1/2$. We find such solutions in the theory of Yang-Mills monopoles and dyons.

I. INTRODUCTION

A strategy for extracting information about solutions of a quantum field theory has been recently suggested: Operator Euler-Lagrange equations are treated as c-number field equations and are solved by methods of classical mathematical physics. Quantum mechanics is regained either by expanding the quantum theory around the classical solution in a power series of a small parameter—typically a coupling constant—or by quantizing the classical solution in a semiclassical or WKB approximation.[1]

The examples considered in the literature thus far involve Euler-Lagrange equations with *Bose* operators. The classical solution, large for weak coupling, in the quantum theory is the dominant approximation to an expectation value of the quantum field, whose operator *commutation* relations guide higher-order calculations.[2] An analogous development for *Fermi* operators with *anticommutation* relations has heretofore been lacking,[3] and is presented in this paper. In the literature there exist some solutions to Dirac-type equations, arising in various field theories with Fermi fields Ψ, where Ψ is treated as an ordinary c number.[4] We add to this list a fermionic c-number solution to the Yang-Mills theory of spontaneously broken isospin symmetry with isovector mesons and isospinor or isovector fermions—a gauge theory which is known to possess monopole and dyon solutions in the absence of fermions.[5,6] Moreover, we describe the role of the fermion solution in the quantum theory, and we find curious results.

In the models which we consider, Fermi fields couple to Bose fields with gauge-invariant bilinear interactions of strength g, a weak-coupling constant characterizing in a uniform way the boson self-couplings. In the absence of the fermions, the classical Bose solutions are $O(g^{-1})$, and are associated with soliton states. The Dirac equations, encountered in our approximation scheme for the quantum theory, are c-number equations in the external potential given by the classical boson field. The equations are linear in Ψ, the interactions with the bosons are $O(g^0)$, and the solutions for Ψ are also $O(g^0)$. Thus they are higher-order, quantum corrections to the $O(g^{-1})$ classical solution. We solve these Dirac equations, and find that they possess a normalizable, static (time-independent) solution. This is a zero-energy state, which goes into itself under fermion-number conjugation. A zero-energy mode signals degeneracy in the quantum theory, and we argue that a nondegenerate c-number zero-energy solution implies that each of the soliton states is in fact a degenerate doublet, carrying fermion number $\pm\frac{1}{2}$. It must be stressed that we do not take the viewpoint that the soliton exists independently of the fermions, which then bind to it with zero energy. Rather we say that the soliton state is doubly degenerate—a feature which becomes exposed only when the calculation is taken to a higher order. The differing interpretations lead to a different numbers of soliton states. In our interpretation there are two; in the other there would be four: the original soliton, soliton plus fermion, soliton plus antifermion and soliton plus fermion and antifermion.

We are led to our interpretation by considering the expansion of the Fermi quantum field in terms of eigenfunctions of our Dirac equation:

$$\Psi = a\psi_0 + \sum_p (b_p\psi_{p+} + d_p^\dagger \psi_{p-}^c)\,. \tag{1.1}$$

Here b_p and $d_p^\dagger$ are annihilation and creation operators associated with positive-energy solutions ψ_{p+}, and with fermion-number conjugates of negative-energy solutions ψ_{p-}^c, while ψ_0 is the fermion-number self-conjugate zero-energy solution. Since there is only one such solution, we associate only one operator with it: a. The anticommutation relations of Ψ fix those of a and $a^\dagger$:

$$\begin{aligned}&\{a, a\} = \{a^\dagger, a^\dagger\} = 0,\\ &\{a^\dagger, a\} = 1.\end{aligned} \tag{1.2}$$

But there is no requirement that a and $a^\dagger$ be particle annihilation and creation operators. The only statement one must make is that the states provide a representation of the algebra (1.2). This is achieved by allowing the lowest (soliton) state to be a degenerate doublet. Consistency of our interpretation is demonstrated in the remainder of this article.

In Sec. II we discuss simple models in one spatial dimension which accurately exemplify all the features of realistic three-dimensional theories; one of these, the SU(2) Yang-Mills theory, is treated in Sec. III. Concluding remarks comprise Sec. IV. Detailed analysis of various relevant Dirac equations is relegated to an Appendix.

II. MODELS IN ONE SPATIAL DIMENSION

A. c-number solution

We consider theories involving a scalar field Φ and a spinor field Ψ, with a Lagrangian density of the form

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{g^2} U(g\Phi) + i\bar{\Psi}\gamma^\mu \partial_\mu \Psi - G\bar{\Psi} V(g\Phi)\Psi. \tag{2.1}$$

G is a positive constant with mass dimensionality. We further suppose that, in the absence of fermions, the static field equation for Φ possesses a classically stable, finite-energy solution ϕ_c,

$$\begin{aligned} &\tfrac{1}{2}(\phi_c')^2 = \frac{1}{g^2} U(g\phi_c), \\ &\phi_c(\infty) = -\phi_c(-\infty), \end{aligned} \tag{2.2}$$

where $\phi_c(\pm\infty)$ also solve $U'(g\phi)=0$. The Yukawa interaction is assumed to have the following antisymmetric property:

$$\lim_{x\to\infty} GV(g\phi_c) = -\lim_{x\to-\infty} GV(g\phi_c) \equiv m > 0. \tag{2.3}$$

A convenient example is

$$\begin{aligned} &U(\phi) = \tfrac{1}{2}\lambda^2 (1-\phi^2)^2, \\ &V(\phi) = \phi, \\ &\phi_c(x) = \frac{1}{g}\tanh \lambda x. \end{aligned} \tag{2.4}$$

λ is a constant with mass dimensionality. We shall frequently refer to this explicit example, though our considerations are general.[7]

Observe that the Dirac equation

$$i\gamma^\mu \partial_\mu \psi - GV(g\phi_c)\psi = 0 \tag{2.5}$$

always has a unique, static, normalizable solution:

$$\begin{aligned} &i\gamma^1 \psi_0' - GV(g\phi_c)\psi_0 = 0, \\ &\psi_0(x) = N \exp\left[-G\int_0^x dx' V(g\phi_c)\right] s^+, \end{aligned} \tag{2.6}$$

N = finite normalization constant.

We have taken the representation of the Dirac matrices to be $\gamma^1 = i\sigma^3$, $\gamma^0 = \sigma^1$, while s^+ is the spinor $\binom{1}{0}$; in the following s^- will denote $\binom{0}{1}$. Since a time-dependent spinor which solves (2.5) may be written as

$$\begin{aligned} &\psi(x,t) = e^{-i\epsilon t}\psi_\epsilon(x), \\ &i\gamma^1 \psi_\epsilon' - GV(g\phi_c)\psi_\epsilon = -\epsilon\gamma^0\psi_\epsilon, \end{aligned} \tag{2.7}$$

we see that (2.6) is a nondegenerate, zero-energy eigensolution. The fermion-number conjugation matrix is σ^3, and ψ_0 is self-conjugate:

$$\psi_0^c = \sigma^3 \psi_0^* = \psi_0. \tag{2.8}$$

In general, there may be other normalizable solutions with $\epsilon \neq 0$. Also there will be a continuum spectrum with $|\epsilon| \geq (p^2+m^2)^{1/2}$ and orthonormal eigenfunctions $\psi_{p\pm}$, where the $\pm$ label signifies positive or negative energy. For simplicity we shall ignore the possibility of bound states, other than the zero-energy one, (2.6). Thus the continuum solutions satisfy

$$\int \frac{dp}{(2\pi)} [\psi_{p+}^*(x)\psi_{p+}(y) + \psi_{p-}^*(x)\psi_{p-}(y)] = \delta(x-y) - \psi_0^*(x)\psi_0(y). \tag{2.9}$$

B. Quantum interpretation

The quantum theory described by the Lagrangian (2.1) possesses in the ordinary (vacuum) sector a meson with mass $\mu = \lim_{x\to\infty}[U''(g\phi_c)]^{1/2}$ and a fermion-antifermion doublet with mass $m = \lim_{x\to\infty} GV(g\phi_c)$. The occurrence of the static solution $\phi_c(x)$ suggests that there is also a soliton sector. To expose its structure we work for definiteness with the specific theory (2.4) and postulate the existence of one-soliton states $|P\pm\rangle$, with momentum P and mass M which is $O(g^{-2})$. The additional label, $\pm$, describes a twofold degeneracy which, as we shall demonstrate below, is required by the zero-energy fermion solution: The + state is called "soliton" and the − state is called "antisoliton." We emphasize that this bifurcation has no relation to the kink-antikink dualism; rather it is a consequence of the fermions.

First we consider the quantum theory in the approximation that dominates for small g. As in previous investigations,[2] we postulate that $\langle P'\pm|\Phi|P\pm\rangle$ is $O(g^{-1})$, and that all other connected matrix elements of Φ and Ψ relevant to the one-soliton sector either vanish or are of an order in g higher than g^{-1}. In the one-soliton matrix element of the Heisenberg equation for Φ,

$$\{[E(P')-E(P)]^2-(P'-P)^2\}\langle P'\pm|\Phi|P\pm\rangle = \frac{1}{g}\langle P'\pm|U'(g\Phi)|P\pm\rangle + gG\langle P'\pm|\bar{\Psi}\Psi|P\pm\rangle\,, \tag{2.10}$$

we sum over intermediate states and keep only dominant terms for small g. The last term is $O(g)$, and hence is negligible in lowest order. Thus the classical static Bose equation is regained in a familiar fashion[2]:

$$\begin{aligned} f'' &= \frac{1}{g}U'(gf),\\ \langle P'\pm|\Phi|P\pm\rangle &= \int dx\, e^{i(P'-P)x} f(x). \end{aligned} \tag{2.11}$$

The Bose-field form factor f coincides with ϕ_c, in lowest order. The soliton or antisoliton energy $E(P)$, which in lowest order is just the mass M, can be computed from an expectation value of the Hamiltonian. The Fermi fields contribute only an order of g higher than the Bose terms. Hence the leading expression for M is the classical energy of ϕ_c:

$$\begin{aligned} M \approx E(\phi_c) &= \int dx\left[\tfrac{1}{2}(\phi_c')^2 + \frac{1}{g^2}U(g\phi_c)\right]\\ &= \int dx(\phi_c')^2\\ &= O(1/g^2). \end{aligned} \tag{2.12}$$

Contact with the fermions is made when the calculation is taken to the next order. We list the states that are relevant. They are the soliton or antisoliton plus one-meson states, $|P\pm;k\rangle$, the soliton or antisoliton plus one-fermion states, $|P\pm;p+\rangle$, and the soliton or antisoliton plus one-antifermion states, $|P\pm;p-\rangle$; P is the total momentum, k is the asymptotic meson momentum, and p is the asymptotic fermion momentum. (An additional "in" or "out" specification will be ignored.) The energy of the meson states is in first approximation $E(P)+\omega(k)$, $\omega(k)=(k^2+\mu^2)^{1/2}$; that of the fermions is $E(P)+\epsilon(p)$, $\epsilon(p)=(p^2+m^2)^{1/2}$. The following matrix elements will be needed:

$$\langle P'-|\Psi|P+\rangle = \langle P+|\Psi^\dagger|P'-\rangle^* = \int dx\, e^{i(P'-P)x}u_0(x), \tag{2.13a}$$

$$\langle P'\pm|\Psi|P\pm;p+\rangle = \langle P\pm;p+|\Psi^\dagger|P'\pm\rangle^* = \int dx\, e^{i(P'-P)x}u_p(x), \tag{2.13b}$$

$$\langle P'\pm|\Psi^\dagger|P\pm;p-\rangle = \langle P\pm;p|\Psi|P'\pm\rangle^* = \int dx\, e^{i(P'-P)x}v_p^*(x), \tag{2.13c}$$

$$\langle P'\pm|\Phi|P\pm;k\rangle = \langle P\pm;k|\Phi|P'\pm\rangle^* = \int dx\, e^{i(P'-P)x}\frac{f_k(x)}{[2\omega(k)]^{1/2}}. \tag{2.13d}$$

We shall show that these are $O(g^0)$; all other multiparticle, connected matrix elements in the soliton sector either vanish identically or are of higher order in g. Note that the Fermi field can effect a transition between the soliton and the antisoliton. The equations satisfied by the wave functions can be derived from matrix elements of the Heisenberg equations for the operators.

First we consider (2.13a):

$$\{\gamma^0[E(P)-E(P')]+\gamma^1(P-P')\}\langle P'-|\Psi|P+\rangle = Gg\langle P'-|\Phi\Psi|P+\rangle. \tag{2.14a}$$

To $O(g^0)$ the energy difference may be dropped, while the right-hand side is

$$\begin{aligned} Gg\langle P'-|\Phi\Psi|P+\rangle &= Gg\int\frac{dP''}{(2\pi)}\langle P'-|\Phi|P''-\rangle\langle P''-|\Psi|P+\rangle\\ &= G\int dx\, e^{i(P'-P)x}gf(x)u_0(x). \end{aligned} \tag{2.14b}$$

Thus we find that $u_0(x)$ satisfies the static Dirac equation:

$$i\gamma^1u_0' - Gg\phi_cu_0 = 0. \tag{2.15}$$

[The same result emerges if the operators in (2.14b) are taken in opposite order.]

In a completely analogous calculation for $\langle P'\pm|\Psi|P\pm;p+\rangle$, the interaction term may be written as

$$\begin{aligned} Gg\langle P'\pm|\Phi\Psi|P\pm;p+\rangle &= Gg\int\frac{dP''}{(2\pi)}\langle P'\pm|\Phi|P''\pm\rangle\langle P''\pm|\Psi|P\pm;p+\rangle\\ &= G\int dx\, e^{i(P'-P)x}gf(x)u_p(x) \end{aligned} \tag{2.16a}$$

or

$$Gg\langle P'\pm|\Psi\Phi|P\pm;p+\rangle$$
$$=Gg\int\frac{dP''}{(2\pi)}\frac{dp'}{(2\pi)}\langle P'\pm|\Psi|P''\pm;p+\rangle$$
$$\times\langle P''\pm;p'+|\Phi|P\pm;p+\rangle. \quad (2.16b)$$

The two are equivalent since the Φ matrix element in (2.16b) is dominated by its disconnected part:

$$\langle P\pm;p'+|\Phi|P\pm;p+\rangle=(2\pi)\delta(p'-p)\langle P''\pm|\Phi|P\pm\rangle. \quad (2.17)$$

Thus the $O(g^0)$ equation for $u_p(x)$ is derived:

$$i\gamma^1 u_p'-Gg\phi_c u_p=-\epsilon(p)\gamma^0 u_p. \quad (2.18)$$

Similarly, for v_p^*

$$i\gamma^1 v_p^{*\prime}-Gg\phi_c v_p^*=\epsilon(p)\gamma^0 v_p^*,$$
$$i\gamma^1(\sigma^3 v_p^*)'-Gg\phi_c(\sigma^3 v_p^*)=-\epsilon(p)\gamma^0(\sigma^3 v_p^*). \quad (2.19)$$

We have therefore shown that the wave functions u_0, u_p, and $\sigma^3 v_p^*$ satisfy the Dirac equations (2.6) and (2.7). Before one identifies them with the normalized solutions, ψ_0 and $\psi_{p\pm}$, the norm of the wave functions, which is arbitrary in (2.15), (2.18), and (2.19), must be determined from the field equal-time anticommutator:

$$\langle P'\pm|\{\Psi^\dagger(x),\Psi(y)\}|P\pm\rangle=\delta(x-y)(2\pi)\delta(P'-P). \quad (2.20)$$

Two terms are to be evaluated: $\langle P'\pm|\Psi^\dagger(x)\Psi(y)|P\pm\rangle$ and $\langle P'\pm|\Psi(y)\Psi^\dagger(x)|P\pm\rangle$. In the intermediate states, which saturate the product of the two operators, we retain only the no-fermion and one-fermion states. The no-fermion states contribute to the first term only for the + sign (soliton external state, antisoliton intermediate states), while to the second term the no-fermion states are present only for the − sign (antisoliton external state, soliton intermediate states). In either case, the one-soliton intermediate states contribute to the sum of the two terms

$$\int dz\, e^{i(P'-P)z}u_0^*(x+z)u_0(y+z). \quad (2.21a)$$

The one-antifermion intermediate states contribute to the first term

$$\int dz\, e^{i(P'-P)z}\int\frac{dp}{(2\pi)}v_p^*(x+z)v_p(y+z), \quad (2.21b)$$

while in the second term an analogous expression arises from the one-fermion intermediate states:

$$\int dz\, e^{i(P'-P)z}\int\frac{dp}{(2\pi)}u_p^*(x+z)u_p(y+z). \quad (2.21c)$$

Adding the three, we see from (2.9) that if u_0 is identified with the normalized, zero-energy solution ψ_0 of (2.6), u_p with the positive-energy solutions ψ_{p+} of (2.7), and v_p with the fermion-number conjugate of the negative-energy solution ψ_{p-} of (2.7), then the commutator is correctly reproduced. Thus we establish that our *Ansätze* lead to a complete, normalized set of states. (The anticommutator $\{\Psi,\Psi\}$ vanishes trivially since there are no intermediate states that can contribute.) The wave functions u_p and v_p describe the scattering of fermions off solitons. (Bound solutions of the Dirac equation, other than the one with zero energy, describe soliton-fermion bound states.)

Next we consider the meson wave function (2.13d). The equation is of the form (2.10); the interaction with the fermions can be neglected since its is of $O(g)$. Therefore $f_k(x)$ satisfies the same equation as in the absence of fermions[2]:

$$-f_k''+U''(g\phi_c)f_k=\omega^2(k)f_k \quad (2.22)$$

The zero-frequency mode ϕ_c' is associated with translations of the soliton, while the other modes describe meson-soliton bound and scattering states. Since the norm of f_k is fixed by the Bose field commutator to be unity,[2] it is established that all the one-particle wave functions are indeed of $O(g^0)$.

Finally, we calculate the fermion number of our soliton. The conserved Fermi-number current is $\bar\Psi\gamma^\mu\Psi$; the charge must be properly ordered so that it transforms correctly under fermion-number conjugation. The ordering is determined in the vacuum sector to be

$$j^0=\tfrac{1}{2}(\Psi^\dagger\Psi-\Psi\Psi^\dagger) \quad (2.23)$$

and the fermion number is given by

$$n_\pm=\tfrac{1}{2}\langle P\pm|\Psi^\dagger\Psi-\Psi\Psi^\dagger|P\pm\rangle. \quad (2.24)$$

The calculation of this matrix element is analogous to the one performed for the anticommutator, except that the two terms enter with opposite sign. Hence we find

$$n_\pm=\tfrac{1}{2}\left\{\pm\int dz\,u_0^*(z)u_0(z)+\int dz\int\frac{dp}{(2\pi)}[v_p^*(z)v_p(z)-u_p^*(z)u_p(z)]\right\}$$
$$=\pm\tfrac{1}{2}. \quad (2.25)$$

The soliton has fermion number $+\frac{1}{2}$; the antisoliton has $-\frac{1}{2}$.

These Kerman-Klein calculations can be extended to higher orders; alternatively they may be summarized by introducing collective coordinates through a canonical transformation.[8] The field transformation is

$$\Phi(x)=\hat{\Phi}(x-X)+\phi_c(x-X)\,,\qquad \Psi(x)=\hat{\Psi}(x-X)+a\psi_0(x-X)\,, \tag{2.26a}$$

with subsidiary conditions

$$\int dx\,\phi_c{}'\,\hat{\Phi}=0\,,\qquad \int dx\,\psi_0^*\hat{\Psi}=0\,. \tag{2.26b}$$

X is the collective position operator; a is an operator associated with the degenerate soliton and antisoliton states. Its anticommutation relations (1.2) are realized by

$$a\,|P+\rangle=|P-\rangle\,,\qquad a^\dagger|P-\rangle=|P+\rangle\,. \tag{2.27}$$

We do not give further details, since they are a straightforward generalization of previous research.[8]

The quantum theory possesses two discrete symmetries: fermion conjugation $\mathfrak{F}=\mathfrak{F}^{-1}=\mathfrak{F}^\dagger$,

$$\mathfrak{F}\Psi\mathfrak{F}=-i\gamma^1\Psi^\dagger\,,\qquad \mathfrak{F}\Phi\mathfrak{F}^{-1}=\Phi \tag{2.28}$$

[compare (2.8)] and discrete chirality $\mathfrak{B}=\mathfrak{B}^{-1}=\mathfrak{B}^\dagger$,

$$\mathfrak{B}\Psi\mathfrak{B}=i\gamma^5\Psi\,,\qquad \mathfrak{B}\Phi\mathfrak{B}=-\Phi\,,\qquad \gamma_5=-\gamma_5^\dagger,\;\;(\gamma_5)^2=-1\,. \tag{2.29}$$

The latter is spontaneously broken; the former is not. The soliton states transform into each other under $\mathfrak{F}$.

$$\mathfrak{F}a\mathfrak{F}=a^\dagger\,,\qquad \mathfrak{F}\,|P\pm\rangle=|P\mp\rangle\,, \tag{2.30}$$

while the fermion charge density (2.23) goes into its negative,

$$\mathfrak{F}j^0\mathfrak{F}=-j^0\,. \tag{2.31}$$

One can demonstrate quite generally that $n_\pm=\pm\frac{1}{2}$. Upon taking one-soliton matrix elements of

$$i\left[\int dx\,j^0(x),\Psi\right]=-\Psi \tag{2.32a}$$

it follows that $n_+=n_-+1$; also $n_+=-n_-$, since

$$\begin{aligned}\int dx\,j^0(x)\,|P+\rangle&=n_+|P+\rangle\\ &=\mathfrak{F}\mathfrak{F}\int dx\,j^0(x)\mathfrak{F}\mathfrak{F}\,|P+\rangle\\ &=-\mathfrak{F}\int dx\,j^0(x)\,|P-\rangle\\ &=-n_-\,|P+\rangle.\end{aligned} \tag{2.32b}$$

We see therefore that the occurence of solitons with half-integer fermion number is intimately related to the fermion-number conjugation symmetry.

C. Discussion

We have shown that c-number solutions of the Dirac equation in a soliton potential can be successfully incorporated in a quantum field theory. Furthermore, the existence of a bound zero-energy solution leads us to the surprising result that soliton states are doubly degererate and carry fermion number $\pm\frac{1}{2}$. In order to make this interpretation conclusive, we should study a two-soliton system. For example, we might calculate the bound states of solitons and antisolitons and demonstrate that there are four of them: two with fermion number 0, and one each with fermion number ± 1. We do not know at present how to carry through this analysis exactly, but we can exhibit an approximate calculation which supports our viewpoint.

Consider a widely separated kink-antikink pair,

$$\phi_2(x)=\frac{1}{g}\tanh\lambda(x-L)-\frac{1}{g}\tanh\lambda(x+L)+\frac{1}{g}\,, \tag{2.33}$$

which for sufficiently large L solves the classical Bose equation. Next consider the Dirac equation in this potential,

$$(\alpha p+Gg\phi_2\beta)\,\psi=E\psi\,,\qquad p=\frac{1}{i}\frac{\partial}{\partial x},\quad \alpha=\gamma^0\gamma^1,\quad \beta=\gamma^0\,. \tag{2.34}$$

It is easy to show that for large L there are two bound states $\psi_\pm$ with energy $E=\pm Ge^{-2GL}$. In the corresponding quantum theory it is natural to assign a zero-fermion-number state to (2.33). The positive- (negative-) energy solution of (2.34) corresponds to a state with fermion number $+1$ (-1). Finally, in the quantum theory we should also allow a fermion-antifermion state with zero fermion number. As L tends to infinity, these four states become degenerate, and it is strongly suggested that they correspond to two widely separated solitons, each with fermion number $\pm\frac{1}{2}$.

III. THREE-DIMENSIONAL MODEL

A. Preliminaries

A three-dimensional model which is known to possess static monopole (soliton) solutions with finite energy is the Yang-Mills theory of spontaneously broken isospin symmetry with a triplet of spinless mesons.[5] We add fermions to this model; they interact gauge invariantly both with the gauge field and with the mesons. We find solutions of the c-number Dirac equation in two cases: isospinor and isovector fermions. In both instances there exist isolated zero-energy solutions. In the former the solution is nondegenerate; in the latter it is doubly degenerate. We comment briefly that the same happens when the external potential is taken to be a dyon solution.[6]

The quantum expansion around the monopole, in the absence of fermions, has been given recently.[9] We do not repeat this lengthy analysis for the fermion case, beyond noting that by analogy with the one-dimensional discussion the monopole, in the presence of isospinor fermions, becomes a degenerate doublet, with fermion number $\pm\frac{1}{2}$. The isovector fermions lead to four soliton states, the properties of which we discuss at the end of this section.

B. c-number solutions

The theory with which we concern ourselves is governed by the following Lagrangian density[7]:

$$\begin{aligned}
\mathcal{L} &= -\tfrac{1}{4}F_a^{\mu\nu}F_{a\mu\nu} + \tfrac{1}{2}(D_\mu\Phi)_a(D^\mu\Phi)_a - \frac{1}{g^2}U(g\Phi) \\
&\quad + i\bar{\psi}_n\gamma^\mu(D_\mu\psi)_n - Gg\bar{\psi}_n T^a_{nm}\Psi_m\Phi_a\,, \\
F_a^{\mu\nu} &= \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g\epsilon_{abc}A_b^\mu A_c^\nu\,, \\
(D^\mu\Phi)_a &= \partial^\mu\Phi_a + g\epsilon_{abc}A_b^\mu\Phi_c\,, \\
U(\Phi) &= \frac{\lambda^2\mu^2}{2}\left(1-\frac{1}{\mu^2}\Phi_a\Phi_a\right)^2, \\
(D^\mu\Psi)_n &= \partial^\mu\Psi_n - ig\,T^a_{nm}A_a^\mu\,\Psi_m\,, \\
a &= 1,2,3\,.
\end{aligned} \tag{3.1}$$

G is a positive, dimensionless parameter; λ and μ carry dimensionality of mass. We have added a multiplet of fermions which transform under isospin rotations according to

$$\begin{aligned}
\delta^a\Psi_n &= i\,T^a_{nm}\Psi_m, \\
[T^a, T^b] &= i\epsilon^{abc}T^c\,.
\end{aligned} \tag{3.2}$$

In the absence of fermions, the Euler-Lagrange equations have the following monopole solution[5]:

$$\begin{aligned}
A_a^0 &= 0, \\
\Phi_a &= \hat{r}_a\,\phi(r)/g, \\
A_a^i &= \epsilon^{aij}\,\hat{r}_j A(r)/g.
\end{aligned} \tag{3.3}$$

Both ϕ and A vanish at $r=0$; for larger r, ϕ tends to its vacuum value μ, and A behaves as $-1/r$. The approach to asymptotic forms is exponential.

The Dirac equation in the external potential (3.3) is

$$\begin{aligned}
[\vec{\alpha}\cdot\vec{p}\,\delta_{nm} + A\,T^a_{nm}(\vec{\alpha}\times\hat{r})_a + G\phi\,T^a_{nm}\hat{r}_a\beta]\psi_m &= E\psi_n\,, \\
\vec{p} &= \frac{1}{i}\vec{\nabla}.
\end{aligned} \tag{3.4}$$

Our representation for Dirac matrices is

$$\vec{\alpha} = \begin{bmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix}, \quad \beta = -i\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}. \tag{3.5}$$

When ψ_n is decomposed into upper and lower components,

$$\psi_n = \begin{pmatrix} \chi_n^+ \\ \chi_n^- \end{pmatrix}, \tag{3.6a}$$

Eq. (3.4) becomes

$$[\vec{\sigma}\cdot\vec{p}\,\delta_{nm} + A\,T^a_{nm}(\vec{\sigma}\times\hat{r})^a \pm iG\phi\,T^a_{nm}\hat{r}_a]\chi_m^\pm = E\chi_n^\mp. \tag{3.6b}$$

This equation is analyzed in the Appendix. Here we record the zero-energy solutions. In the isospinor case $T^a = \frac{1}{2}\tau^a$; $n, m = 1, 2$. The lower component vanishes and the upper is

$$\begin{aligned}
\chi^+_{\nu n} &= N\left\{\exp\left(-\int_0^r dr'[\tfrac{1}{2}G\phi(r') - A(r')]\right)\right\} \\
&\quad \times\{s_\nu^+ s_n^- - s_\nu^- s_n^+\},
\end{aligned} \tag{3.7}$$

where ν refers to the Dirac indices and takes values 1 and 2; N is a normalization constant.

In the isovector example, $T^a_{nm} = i\epsilon_{nam}$; $n, m = 1, 2, 3$. The lower component again vanishes; the upper is

$$\chi_n^+ = N[f_1(r)\hat{r}^n\vec{\sigma}\cdot\hat{r} + f_2(r)(\sigma^n - \hat{r}^n\vec{\sigma}\cdot\hat{r})]\chi, \tag{3.8}$$

with χ an arbitrary spinor. In other words, we obtain two linearly independent solutions: $\chi = s^+$, $\chi = s^-$. The functions $f_{1,2}$ are constructed as follows. Take that solution of

$$-u'' + (F^2 + F' + 2\rho^2)u = 0 \tag{3.9a}$$

which is regular at the origin. Here ρ is the exponentially decreasing, nonasymptotic part of A,

$$\rho(r) = A(r) + \frac{1}{r}, \tag{3.9b}$$

and

$$F(r) = \frac{1}{2}\left[G\phi(r) - \frac{\rho'(r)}{\rho(r)} - \frac{1}{r}\right]. \tag{3.9c}$$

F tends to a positive constant for large r, and vanishes at $r=0$; hence u vanishes as r^2 near the origin. The functions $f_{1,2}$ are given in terms of u:

$$\begin{aligned} f_1(r) &= \frac{1}{r^2}u(r)\exp\left[-\int_0^r dr' F(r')\right], \\ f_2(r) &= \frac{1}{2r^2\rho(r)}\frac{d}{dr}(r^2 f_1(r)). \end{aligned} \tag{3.10}$$

Note that while f_2 decreases exponentially at infinity, f_1 decreases much more slowly, as r^{-2}.

The isospinor solution is nondegenerate, corresponding to zero spin. The twofold degeneracy of the isovector solution indicates that the solution has spin $\frac{1}{2}$. With the choice of Dirac matrices (3.5), fermion-number conjugation is realized by the following:

$$\text{isospinor } \psi_n^c = \begin{bmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{bmatrix} \tau^2_{nm}\psi_m^*, \tag{3.11a}$$

$$\text{isovector } \psi_n^c = \begin{bmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{bmatrix} \psi_n^* . \tag{3.11b}$$

One easily checks that the above transformation, when applied to (3.6b), changes the sign of the energy. Consequently our solutions are fermion-number self-conjugate.

Finally, we consider the classical dyon solutions.[6] The Bose fields are again as in (3.3) except that A_a^0 no longer vanishes:

$$\begin{aligned} A_a^0 &= \frac{1}{g}\hat{r}^a \mathcal{V}(r), \\ \mathcal{V}(0) &= 0, \quad \mathcal{V}(\infty) = \mathcal{V}_\infty \neq 0. \end{aligned} \tag{3.12}$$

The Dirac equation (3.4) now acquires on the right-hand side the additional term $T^a_{nm}\hat{r}^a\mathcal{V}\psi_m$. The complexity of the equations prevents us from giving an explicit construction of the solutions. However, in the Appendix we are able to show that zero-energy solutions continue to exist, for both isospinor and isovector fermions. Their forms are as in (3.7) and (3.8), except that now the lower components are nonzero. In the expanded system of equations, fermion-number conjugation remains effected by the matrices (3.11), and the zero-energy solutions are self-conjugate.

C. Quantum interpretation

We have not carried out a detailed analysis of the quantum theory associated with our solutions (3.7) and (3.8). But by combining the existing knowledge of the monopole, quantized in the absence of fermions,[9] with the analysis of the one-dimensional example, we can describe what one expects in the quantum theory.

For the isospinor case, the monopole, as well as its charged recurrences, the dyons arising from the quantization of the U(1) phase degeneracy which remains after the SU(2) symmetry has been spontaneously broken, becomes a degenerate doublet with fermion number $\pm\frac{1}{2}$. Also, the solitons are spinless, since no spin degree of freedom is found in the classical solution.

In the isovector case these are two static solutions, associated with different values of the spin degree of freedom, and correspondingly we expect to find two operators a_s ($s = \pm\frac{1}{2}$) in the expansion of the Dirac field. The basic feature that the representation of the anticommunication relation $\{a_s, a_s^\dagger\} = 1$ requires two states $|+\rangle$ and $|-\rangle$ carrying fermion number $n_\pm = \pm\frac{1}{2}$ remains true also in this case, but since we have now two independent pairs of operators the soliton states will be product vectors of the form $|i\rangle|i'\rangle$ $(i, i' = \pm)$, with fermion numbers +1 ($|+\rangle|-\rangle$), −1 ($|-\rangle|-\rangle$), and 0 ($|+\rangle|-\rangle$ and $|-\rangle|+\rangle$). Thus, we expect to have in our theory four soliton states, together with their electrically charged recurrences associated with the dyon solutions. We conjecture that the states with fermion number ± 1 have spin 0 and that those with fermion number 0 have spin $\frac{1}{2}$. We make this hypothesis because of the spin degree of freedom which is present in our solution of the Dirac equation in the isovector case. This spin degree of freedom does not lead to a further multiplicity of states, unlike the phase degree of freedom, whose quantization produces rotation-type charge bands.[10] There are no further monopoles with spin 1, $\frac{3}{2}$, etc. The reason for this difference is that the spin is not a collective phenomenon; we do not need a collective coordinate to describe it. Collective coordinates are introduced only to account for the degeneracies of the lowest $O(g^{-1})$ approximation. Any further degeneracies that are encountered in higher orders of the perturbative expansion reflect true quantum-mechanical degeneracy, and, in contrast to the translation and charge-rotation degeneracies, need not be dealt with by collective methods.

In the absence of fermions, one can expand the quantum theory around a dyon solution (3.12), and one finds that the physical content is the same as in an expansion around the neutral monopole.[9] The occurrence of zero-energy Fermi solutions for the dyon external field indicates that one is led to our interpretation of the nature of the solitons, regardless of the expansion method.

Just as in the one-dimensional example, the quantum theory possesses discrete symmetries which ensure the correctness of our interpretation. These are conventional charge conjugation $\mathfrak{C}$ (or equivalently $\mathcal{G}$ parity) and the discrete chiral transformation $\mathfrak{B}$, (2.29). Invariance under fermion-number conjugation $\mathfrak{F}$, which does not affect the boson fields, is then a consequence: $\mathfrak{F}=i\mathfrak{B}\mathcal{G}$ [see (3.11)]. Spontaneous symmetry breaking removes $\mathcal{G}$ as well as $\mathfrak{B}$. However, $\mathfrak{C}$ and $\mathfrak{F}$ remain conserved, and it is the latter that forces our interpretation.

IV. CONCLUSION

The interesting results of this investigation are threefold. First, we have shown how one can fit c-number solutions of a Dirac equation into the quantum theory of a Fermi field. This we did without introducing elements of a Grassmann algebra—anticommuting c numbers. However, we have dealt with linear Dirac equations. If the Fermi fields occur nonlinearly, when Fermi couplings are present, then the c-number solutions are large for weak coupling, just as in the Bose case, and cannot be treated by our methods.[3]

Second, as an exercise in mathematical physics, we have found solutions of the Dirac equation in the presence of a Yang-Mills monopole. There exist bound states in contrast to the Dirac monopole. This, of course, is due to the additional interaction with the scalar field.

Finally, our most provocative and puzzling results apply whenever a Dirac equation possesses a nondegenerate, fermion-number self-conjugate, zero-energy bound state. In that case the solitons are degenerate doublets with fermion number $\pm\frac{1}{2}$. We have not identified a fundamental reason for the occurrence of such solutions. The existence of states with fermion number $\pm\frac{1}{2}$ in a theory where all fundamental fields have integral fermion number is truly remarkable, yet the practical significance of this is in no way obvious.

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APPENDIX: FERMI FIELDS IN THE POTENTIAL OF YANG-MILLS MONOPOLES

The Dirac equation satisfied by the Fermi fields in the external potential provided by a monopole is [see Eqs. (3.3) and (3.4)]

$$[\vec{\alpha}\cdot\vec{p}\;\delta_{nm}+AT^a_{nm}(\vec{\alpha}\times\hat{r})_a+G\phi T^a_{nm}\hat{r}_a\beta]\psi_m=E\psi_n\,. \tag{A1}$$

In terms of upper and lower components χ_n^+ and χ_n^- of the field ψ_n, Eq. (A1) becomes

$$[\vec{\sigma}\cdot\vec{p}\;\delta_{nm}+AT^a_{nm}(\vec{\sigma}\times\hat{r})_a\pm iG\phi T^a_{nm}\hat{r}_a]\chi^\pm_m=E\chi^\mp_n\,, \tag{A2}$$

where the representation of Dirac matrices is (3.5).

The crucial observation for the study of this equation is that the operator $\vec{J}=\vec{j}+\vec{I}$, defined as the sum of ordinary angular momentum $\vec{j}=\vec{l}+\vec{s}$, and isospin $\vec{I}$, commutes with the left-hand side of Eq. (A2). This indicates that the analysis of Eq. (A2) will simplify if we introduce expansions for $\chi_n^\pm$ in eigenstates of J^2 and J_3. Parity is also a good quantum number, in the sense that the solutions of Eq. (A2) will have upper and lower components of definite, opposite parity.

A. Isospinor fermion fields

With isospinor fermion fields, $T^a_{nm}=\frac{1}{2}\tau^a_{nm}$, and the Dirac equation may be written

$$\begin{aligned}(\vec{\sigma}\cdot\vec{p})_{ij}\chi^\pm_{jn}+\tfrac{1}{2}A(\vec{\sigma}\times\hat{r})^a_{ij}\tau^a_{nm}\chi^\pm_{jm}\pm\tfrac{1}{2}iG\phi\tau^a_{nm}\hat{r}_a\chi^\pm_{im}&=(\vec{\sigma}\cdot\vec{p})_{ij}\chi^\pm_{jn}+\tfrac{1}{2}A(\vec{\sigma}\times\hat{r})^a_{ij}\chi^\pm_{jm}(\tau^{a\,\mathrm{tr}})_{mn}\\&\quad\pm\tfrac{1}{2}iG\phi\chi^\pm_{im}(\tau^{a\,\mathrm{tr}})_{mn}\hat{r}^a\\&=E\chi^\mp_{in}\,.\end{aligned} \tag{A3}$$

Upon defining 2×2 matrices $\mathfrak{M}^\pm$ by

$$\chi^\pm_{in}=\mathfrak{M}^\pm_{im}\tau^2_{mn} \tag{A4}$$

and using $\tau^2\vec{\tau}^{\mathrm{tr}}=-\vec{\tau}\tau^2$, one obtains for $\mathfrak{M}^\pm$ the matrix equation

$$\vec{\sigma}\cdot\vec{p}\,\mathfrak{M}^\pm-\tfrac{1}{2}A(\vec{\sigma}\times\hat{r})^a\mathfrak{M}^\pm\sigma^a\mp\tfrac{1}{2}iG\phi\mathfrak{M}^\pm\sigma^a\hat{r}^a=E\mathfrak{M}^\mp\,, \tag{A5}$$

where it is no longer necessary to distinguish between $\vec{\sigma}$ and $\vec{\tau}$.

We now expand $\mathfrak{M}^{\pm}$ in terms of two scalar and two vector functions:

$$\mathfrak{M}^{\pm}_{im}(\vec{r}) = g^{\pm}(\vec{r})\delta_{im} + g^{\pm}_a(\vec{r})\sigma^a_{im}. \tag{A6}$$

Equation (A5) is then equivalent to two equations for $g^{\pm}$ and $g^{\pm}_a$:

$$(\partial_a - A\hat{r}_a \pm \tfrac{1}{2}G\phi\hat{r}_a)g^{\pm} + i\epsilon_{abc}(\partial_b \mp \tfrac{1}{2}G\phi\hat{r}_b)g^{\pm}_c = iEg^{\mp}_a, \tag{A7a}$$

$$(\partial_a + A\hat{r}_a \pm \tfrac{1}{2}G\phi\hat{r}_a)g^{\pm}_a = iEg^{\mp}. \tag{A7b}$$

A partial-wave analysis of these two equations is performed quite easily by expanding $g^{\pm}$ and $g^{\pm}_a$ in terms of scalar and vector spherical harmonics:

$$g^{\pm}(\vec{r}) = \sum_{JJ_3} G^{\pm}_J(r)Y_{JJ_3}(\Omega), \tag{A8a}$$

$$g^{\pm}_a(\vec{r}) = \sum_{JJ_3}[P^{\pm}_J(r)\mathcal{P}^a_{JJ_3}(\Omega) + B^{\pm}_J(r)\mathcal{B}^a_{JJ_3}(\Omega) + C^{\pm}_J(r)\mathcal{C}^a_{JJ_3}(\Omega)]. \tag{A8b}$$

Y_{JJ_3} is the ordinary spherical harmonic, and the vector spherical harmonics are defined by

$$\mathcal{P}^a_{JJ_3} = \hat{r}_a Y_{JJ_3}, \tag{A9a}$$

$$\mathcal{B}^a_{JJ_3} = (J^2+J)^{-1/2} r\partial_a Y_{JJ_3}, \tag{A9b}$$

$$\mathcal{C}^a_{JJ_3} = (J^2+J)^{-1/2} i\epsilon^{abc}\partial_b \hat{r}_c Y_{JJ_3}. \tag{A9c}$$

Before displaying the final form of the equations, let us show how the existence of zero-energy solutions can be investigated directly from (A7). We take the scalar product of Eq. (A7a) with the operator $(\partial_a \mp \frac{1}{2}G\phi r_a)$, and find for $E=0$

$$(\partial_a \mp \tfrac{1}{2}G\phi\hat{r}_a)(\partial_a - A\hat{r}_a \pm \tfrac{1}{2}G\phi\hat{r}_a)g^{\pm} = 0. \tag{A10}$$

Upon defining

$$g^{\pm}(\vec{r}) = \exp\left[\tfrac{1}{2}\int_0^r dr' A(r')\right]\tilde{g}^{\pm}(\vec{r}) \tag{A11}$$

(A10) takes the form

$$K^{a\dagger}_{\pm}K^a_{\pm}\tilde{g}^{\pm} = 0, \tag{A12a}$$

where $K^a_{\pm}$ is the operator

$$K^a_{\pm} = p_a + \tfrac{1}{2}iA\hat{r}_a \mp \tfrac{1}{2}iG\phi\hat{r}_a. \tag{A12b}$$

But the operators $K^{a\dagger}_{\pm}K^a_{\pm}$ (no sum) are non-negative; it follows that any solution to (A12) must satisfy

$$K^a_{\pm}\tilde{g}^{\pm} = 0, \tag{A13}$$

which implies

$$\tilde{g}^{\pm}(\vec{r}) = c^{\pm}\exp\left\{\tfrac{1}{2}\int_0^r dr'[A(r') \mp G\phi(r')]\right\}. \tag{A14}$$

Since $\tilde{g}^{-}(\vec{r})$ increases exponentially for $r\to\infty$, $c^{-}=0$. Substituting (A14) into (A11), we find

$$g^{+}(\vec{r}) = c^{+}\exp\left\{\int_0^r dr'[A(r') - \tfrac{1}{2}G\phi(r')]\right\}, \tag{A15}$$

which, together with the definitions (A4) and (A6), gives the solution quoted in the text, (3.7).

Notice that (A7) and (A15) imply for $E=0$

$$\epsilon_{abc}(\partial_b \mp \tfrac{1}{2}G\phi\hat{r}_b)g^{\pm}_c = 0. \tag{A16a}$$

This means that $g^{\pm}_a$ can be expressed in terms of "potential" functions $f^{\pm}(\vec{r})$ as

$$g^{\pm}_a = (\partial_a \mp \tfrac{1}{2}G\phi\hat{r}_a)f^{\pm}, \tag{A16b}$$

which according to (A7b) satisfy

$$(\partial_a + A\hat{r}_a \pm \tfrac{1}{2}G\phi\hat{r}_a)(\partial_a \mp \tfrac{1}{2}G\phi\hat{r}_a)f^{\pm} = 0 \tag{A17}$$

or

$$K^{a\dagger}_{\mp}K^a_{\mp}\tilde{f}^{\pm} = 0, \tag{A18}$$

with

$$f^{\pm}(\vec{r}) = \exp\left[-\tfrac{1}{2}\int_0^r dr' A(r')\right]\tilde{f}^{\pm}(\vec{r}). \tag{A19}$$

The only well-behaved solution to (A18) is $f^{+}=0$,

$$f^{-}(\vec{r}) = \exp\left[-\tfrac{1}{2}G\int_0^r dr'\,\phi(r')\right], \tag{A20}$$

but Eq. (A16) then gives $g^{-}_a=0$, so that the only zero-energy solution to the Dirac equation is the one exhibited in Sec. III.

For completeness we display the form that Eqs. (A7) take after the expansion (A8):

$$\left(\frac{d}{dr} - A \pm \tfrac{1}{2}G\phi\right)G^{\pm}_J - \frac{[J(J+1)]^{1/2}}{r}C^{\pm}_J = iEP^{\mp}_J \quad (\text{all } J), \tag{A21a}$$

$$\left(\frac{d}{dr} + \frac{2}{r} + A \pm \tfrac{1}{2}G\phi\right)P^{\pm}_J - \frac{[J(J+1)]^{1/2}}{r}B^{\pm}_J = iEG^{\mp}_J \quad (\text{all } J), \tag{A21b}$$

$$\left(\frac{d}{dr} + \frac{1}{r} \mp \tfrac{1}{2}G\phi\right)B^{\pm}_J - \frac{[J(J+1)]^{1/2}}{r}P^{\pm}_J = -iEC^{\mp}_J \quad (J\geq 1), \tag{A21c}$$

$$\left(\frac{d}{dr} + \frac{1}{r} \mp \tfrac{1}{2}G\phi\right)C^{\pm}_J - \frac{[J(J+1)]^{1/2}}{r}G^{\pm}_J = -iEB^{\mp}_J \quad (J\geq 1). \tag{A21c}$$

The zero-energy solution found by us corresponds of course to

$$G_0^+ \propto \exp\left\{\int_0^r dr' \left[A(r') - \tfrac{1}{2}G\phi(r')\right]\right\};$$

all other amplitudes are zero.

Equations (A21) are very suitable for a study of fermion solutions in the dyon field. When an electric potential $T^a_{nm}\hat{r}_a \mathcal{V}\psi_m$ is present in the right-hand side of the Dirac equation (A1) [see (3.12)], the right-hand sides of (A21) become replaced by

$$iEP_J^{\mp} - \tfrac{1}{2}i\mathcal{V}G_J^{\mp}, \tag{A22a}$$

$$iEG_J^{\mp} - \tfrac{1}{2}i\mathcal{V}P_J^{\mp}, \tag{A22b}$$

$$-iEC_J^{\mp} + \tfrac{1}{2}i\mathcal{V}B_J^{\mp}, \tag{A22c}$$

$$-iEB_J^{\mp} + \tfrac{1}{2}i\mathcal{V}C_J^{\mp}. \tag{A22d}$$

For $E=0$ we expect to find a zero-energy solution that goes over into our previous solution in the limit $\mathcal{V}\to 0$. When $E=0$ but $\mathcal{V}\neq 0$, G_J^+ couples only to G_J^-, so that a zero-energy solution satisfies

$$\begin{aligned}\left(\frac{d}{dr} - A + \tfrac{1}{2}G\phi\right)G_0^+ &= -\tfrac{1}{2}i\mathcal{V}G_0^-,\\ \left(\frac{d}{dr} - A - \tfrac{1}{2}G\phi\right)G_0^- &= -\tfrac{1}{2}i\mathcal{V}G_0^+.\end{aligned} \tag{A23}$$

This system of linear differential equations has two independent solutions, one of which is well-behaved at infinity, whereas the other increases exponentially ($G_0^\pm(r) \approx c_1^\pm \exp\{\tfrac{1}{2}r[(G\mu)^2 - \mathcal{V}_\infty{}^2]^{1/2}\}$ $+c_2^\pm \exp\{-\tfrac{1}{2}r[(G\mu)^2 - \mathcal{V}_\infty{}^2]^{1/2}\}$, where μ and $\mathcal{V}_\infty$ are the constant asymptotic limits of $\phi(r)$ and $\mathcal{V}(r)$ for $r\to\infty$). However, both solutions are well-behaved at the origin. [$G_0^\pm(r)\to$ constant for $r\to 0$.] It follows that there is one normalizable zero-energy solution for $(G\mu)^2 > \mathcal{V}_\infty{}^2$, which corresponds to a range of charges Q of the dyon, around $Q=0$.

B. Isovector fermion fields

With isovector fermion fields, $T^a_{nm} = i\epsilon_{nam}$ and the Dirac equation (A2) takes the form

$$(\vec{\sigma}\cdot\vec{p}\,\delta_{nm} - A\hat{r}_n\sigma_m + A\sigma_n\hat{r}_m \mp iG\phi\epsilon_{nam}\hat{r}_a)\chi_m^\pm = iE\chi_n^\mp. \tag{A24}$$

It is convenient to expand the spinors $\chi_m^\pm$ as

$$\begin{aligned}\chi_m^\pm(\vec{r}) = {}&\hat{r}_m\Theta_{1+}^\pm(\vec{r}) + i(rp_m - \hat{r}_m\vec{r}\cdot\vec{p})\Theta_{2+}^\pm(\vec{r}) + L_m(\vec{\sigma}\cdot\hat{r})\Theta_{3+}^\pm(\vec{r})\\ &+\hat{r}_m(\vec{\sigma}\cdot\hat{r})\Theta_{1-}^\pm(\vec{r}) + i(rp_m - \hat{r}_m\vec{r}\cdot\vec{p})(\vec{\sigma}\cdot\hat{r})\Theta_{2-}^\pm(\vec{r}) + L_m\Theta_{3-}^\pm(\vec{r}),\end{aligned} \tag{A25}$$

where the (isoscalar) spinors $\Theta_{i\pm}^\pm(\vec{r})$ are further expanded in eigenfunctions of J^2 and J_3 containing only components with orbital angular momentum $l = J - \tfrac{1}{2}$.

$$\Theta_{i\pm}^\pm(\vec{r}) = \sum_{JJ_3} F_{i\pm,J}^\pm(r)\mathcal{Y}_{JJ_3}(\Omega). \tag{A26a}$$

$\mathcal{Y}_{JJ_3}$ is a spinor spherical harmonic:

$$\mathcal{Y}_{JJ_3}(\Omega) = \left(\frac{J+J_3}{2J}\right)^{1/2} Y_{J-1/2\;M-1/2}(\Omega)s^+ + \left(\frac{J-J_3}{2J}\right)^{1/2} Y_{J-1/2\;M+1/2}(\Omega)s^-. \tag{A26b}$$

The advantage of our representation is that $\vec{J}$ commutes with r_m, $i(rp_m - \hat{r}_m\vec{r}\cdot\vec{p})$, L_m, and $\vec{\sigma}\cdot\hat{r}$, so that the expansion (A26) is also an expansion of the spinors $\chi_m^\pm(\vec{r})$ into eigenfunctions of J^2 and J_3. The restriction that the spinors $\mathcal{Y}_{JJ_3}$ should have $l = J - \tfrac{1}{2}$ has been introduced to separate terms of opposite parity. The operators $\vec{\sigma}\cdot\hat{r}$ convert terms with $l = J - \tfrac{1}{2}$ into terms with $l = J + \tfrac{1}{2}$.

We insert the expansion (A25) and (A26) into (A24); it is then straigntforward but very tedious to obtain the following equations for the amplitudes $F_{i\pm,J}^\pm(r)$, $i = 1,2,3$:

$$\left[\frac{d}{dr} + (J+\tfrac{3}{2})\frac{1}{r}\right]F_{1-}^\pm - (J+\tfrac{3}{2})\rho F_{2-}^\pm - (J-\tfrac{1}{2})\rho F_{3-}^\pm = iEF_{1+}^\mp \quad (\text{all } J), \tag{A27a}$$

$$(J+\tfrac{3}{2})\left[\frac{d}{dr} + (J+\tfrac{3}{2})\frac{1}{r}\right]F_{2-}^\pm - \left[\frac{d}{dr} + \frac{1}{r} \mp (J+\tfrac{1}{2})G\phi\right]F_{3-}^\pm - \rho F_{1-}^\pm = i(J+\tfrac{1}{2})EF_{2+}^\mp \quad (J>\tfrac{1}{2}), \tag{A27b}$$

$$(J-\tfrac{1}{2})\left[\frac{d}{dr}-(J-\tfrac{1}{2})\frac{1}{r}\right]F^{\pm}_{3-}+\left[\frac{d}{dr}+\frac{1}{r}\pm(J+\tfrac{1}{2})G\phi\right]F^{\pm}_{2-}-\rho F^{\pm}_{1-}=i(J+\tfrac{1}{2})EF^{\mp}_{3+}\quad(\text{all }J),\tag{A27c}$$

$$\left[\frac{d}{dr}-(J-\tfrac{1}{2})\frac{1}{r}\right]F^{\pm}_{1+}+(J-\tfrac{1}{2})\rho F^{\pm}_{2+}+(J+\tfrac{3}{2})\rho F^{\pm}_{3+}=iEF^{\mp}_{1-}\quad(\text{all }J),\tag{A27d}$$

$$(J-\tfrac{1}{2})\left[\frac{d}{dr}-(J-\tfrac{1}{2})\frac{1}{r}\right]F^{\pm}_{2+}+\left[\frac{d}{dr}+\frac{1}{r}\pm(J+\tfrac{1}{2})G\phi\right]F^{\pm}_{3+}+\rho F^{\pm}_{1+}=i(J+\tfrac{1}{2})EF^{\mp}_{2-}\quad(\text{all }J),\tag{A27e}$$

$$(J+\tfrac{3}{2})\left[\frac{d}{dr}+(J+\tfrac{3}{2})\frac{1}{r}\right]F^{\pm}_{3+}-\left[\frac{d}{dr}+\frac{1}{r}\mp(J+\tfrac{1}{2})G\phi\right]F^{\pm}_{2+}+\rho F^{\pm}_{1+}=i(J+\tfrac{1}{2})EF^{\mp}_{3-}\quad(J>\tfrac{1}{2}).\tag{A27f}$$

We have omitted the common index J of the functions F, and have taken $\rho=1/r+A$ as in (3.9b).

If we set $E=0$ in the right-hand sides of (A27), we see that for $J>\frac{1}{2}$ they separate into sets of three equations involving only the functions $F^{s}_{is'}$ $(s,s'=+,-)$ for definite choices of s and s'. The asymptotic behaviors for $r\to\infty$ of the solutions of these equations are given by

$$c_1 r^{\alpha}+c_2 e^{G\mu r}+c_3 e^{-G\mu r},\tag{A28}$$

with $\alpha=J-\frac{1}{2}$ for the equations involving the functions F^{s}_{i+} and $\alpha=-(J+\frac{3}{2})$ for those involving the functions F^{s}_{i-}. The corresponding behaviors at the origin are given by

$$c_1' r^{\beta_{1\pm}}+c_2' r^{\beta_{2\pm}}+c_3' r^{\beta_{3\pm}},\tag{A29}$$

where

$$\beta_{1+}=(J+\tfrac{1}{2}),\quad \beta_{2+}=(J-\tfrac{3}{2}),\quad \beta_{3+}=-(J+\tfrac{3}{2}),$$
$$\beta_{1-}=(J-\tfrac{1}{2}),\quad \beta_{2-}=-(J+\tfrac{1}{2}),\quad \beta_{3-}=-(J+\tfrac{5}{2}).$$

It is apparent that in general there will be no normalizable zero-energy solution for $J>\frac{1}{2}$.

However, for $J=\frac{1}{2}$, the sets of coupled equations reduce to sets of two linear differential equations for the functions F^{s}_{1+}, F^{s}_{3+}, and F^{s}_{1-}, F^{s}_{2-}. While it is obvious that the equations for the lower components lead to exponentially increasing behaviors for $r\to\infty$, so that we must set $F^{-}_{i\pm}=0$ for a normalizable solution, the equations for the upper components take the form

$$\frac{dF^{+}_{1-}}{dr}+2\frac{1}{r}F^{+}_{1-}-2\rho F^{+}_{2-}=0,\tag{A30a}$$

$$\frac{dF^{+}_{2-}}{dr}-\rho F^{+}_{1-}+\frac{1}{r}F^{+}_{2-}+G\phi F^{+}_{2-}=0,\tag{A30b}$$

$$\frac{dF^{+}_{1+}}{dr}+2\rho F^{+}_{3+}=0,\tag{A30c}$$

$$\frac{dF^{+}_{3+}}{dr}+\rho F^{+}_{1+}+\frac{1}{r}F^{+}_{3+}+G\phi F^{+}_{3+}=0.\tag{A30d}$$

At the origin the solutions of these equations always consist of a linear combination of a regular function and a singular, non-normalizable function. For large values of r, F^{+}_{1+} and F^{+}_{3+} behave asymptotically as $c_1+c_2e^{-G\mu r}$, and we conclude that in general the solution regular at the origin will not be normalizable at $r=\infty$. However, the asymptotic behavior of the solutions of Eqs. (A30a) and (A30b) is of the form $c_1r^{-2}+c_2e^{-G\mu r}$, so that the solution regular at the origin will lead to a normalizable zero-energy solution of the Dirac equation.

We see then that also in the isovector case the Dirac equation has one normalizable zero-energy solution, or (better) two independent solutions corresponding to the two values, $J_3=\pm\frac{1}{2}$, of the spin variable in (A26). It is easy to check by setting $F^{+}_{1-}(r)=f_1(r)$ and $F^{+}_{2-}(r)=f_2(r)$, with f_1 and f_2 as given in (3.10), that (A30a) and (A30b) imply the second-order differential equation (3.9a) for the function $u(r)$.

Finally, we notice that with the inclusion of a nonvanishing electric potential term the right-hand sides of (A27b), (A27c), (A27e), and (A27f) become replaced by

$$i(J+\tfrac{1}{2})(EF^{\mp}_{2+}-\mathcal{V}F^{\mp}_{3-}),\tag{A31a}$$

$$i(J+\tfrac{1}{2})(EF^{\mp}_{3+}-\mathcal{V}F^{\mp}_{2-}),\tag{A31b}$$

$$i(J+\tfrac{1}{2})(EF^{\mp}_{2-}-\mathcal{V}F^{\mp}_{3+}),\tag{A31c}$$

$$i(J+\tfrac{1}{2})(EF^{\mp}_{3-}-\mathcal{V}F^{\mp}_{2+}).\tag{A31d}$$

In particular, the equations that determine a zero-energy solution are

$$\frac{dF^{\pm}_{1-}}{dr}+2\frac{1}{r}F^{\pm}_{1-}-2\rho F^{\pm}_{2-}=0,\tag{A32a}$$

$$\frac{dF^{\pm}_{2-}}{dr}-\rho F^{\pm}_{1-}+\frac{1}{r}F^{\pm}_{2-}+G\phi F^{\pm}_{2-}=-i\mathcal{V}F^{\mp}_{2-}.\tag{A32b}$$

This set of four linear coupled differential equations has two linearly independent solutions regular at the origin, whereas the asymptotic behavior of a general solution for $r\to\infty$ is of the form

$$c_1r^{-2}+c_2e^{r(G^2\mu^2-\mathcal{V}_\infty^2)^{1/2}}+c_3e^{-r(G^2\mu^2-\mathcal{V}_\infty^2)^{1/2}}.\tag{A33}$$

We conclude that for $\mathcal{V}_\infty^2<G^2\mu^2$ there is always one solution regular at the origin that behaves as $c_1r^{-2}+c_3e^{-r(G^2\mu^2-\mathcal{V}_\infty^2)^{1/2}}$ for $r\to\infty$. The normalizable zero-energy solution is thus present also in the field of the dyon, as we expected on physical grounds.

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