Quaternionic Representations of Compact Groups

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(Received 20 July 1962)

The main purpose of this paper is to show the conditions under which a finite dimensional representation of a group, irreducible over the complex field, is reducible over quaternions. The answer is simply stated in terms of the Frobenius-Shur classification of group representations.

I. INTRODUCTION

IN this note the attempt is made to formulate a theory of representations of groups over the skew-field of the quaternions. The whole approach is based on spectral resolution techniques which apply in the same way to the fields of real numbers, complex numbers and quaternions. It is therefore a *direct way* without the detour of transcribing the quaternion operators into complex ones through the use of the Pauli matrices. This way was described in a previous paper,[1] to which this note is a sequel.

Elsewhere[2] the particular case SU_2 (2×2, unitary, determinant $= 1$) has been investigated in detail. In Sec. II, we recall some notions and theorems of the spectral theory of operators in a Q Hilbert space. In Sec. III the basic concepts of a representation over the quaternions are developed and the main tools and lemmas are generalized, so that they do not depend upon the commutativity of the underlying field. Finally, in Sec. III.6, the main reduction theorem is formulated and proved. This theorem answers the question: When does a representation, irreducible over the field of complex numbers, reduce over the quaternions? This question is central, for the complex numbers are embedded in the quaternions as a subfield, and the problem of finding all complex representations (of finite degree) is, in principle, completely solved. It will be shown that this question has a straightforward answer in terms of the well-known tripartite classification of irreducible representations given by Frobenius and Schur[3] (see also Wigner[4]). The preparatory Sec. III.4 is devoted to an exposition of their results.

A well-known theorem (cf. Pontrjagin,[5] p. 282) says that every connected, compact group is the factor group of a direct product of simple groups over a discrete normal subgroup.

It is therefore enough to know which representations of the simple groups reduce over Q. Since the classification explained in Sec. III.4 has been applied to all simple groups, we do not enter into this subject. It will also be treated in a forthcoming paper by one of the authors from a different point of view.

This paper deals only with compact groups and therefore with finite dimensional unitary representations. We hope to return to the non-compact groups elsewhere. On the other hand the extension of the results presented here to infinite dimensional unitary representations is straightforward.

Notation

C, Q: the fields of complex numbers and quaternions.

M^H: Hermitian conjugate
M^T: transposed
M^Q: quaternion conjugate (if the m_{ik} are quaternions)
M^*: complex conmugate (if the m_{ik} are complex numbers)
M^A: adjoint $= M^{-1^T}$ of M.

* Supported by the National Science Foundation.
† Supported by the Swiss National Foundation.
[1] Foundations of Quaternion Quantum Mechanics, D. Finkelstein, J. M. Jauch, S. Schiminovitch, D. Speiser. J. Math. Phys. **3**, 1962, p. 207.
[2] Notes on Quanternion Quantum Mechanics II, III, D. Finkelstein, J. M. Jauch, D. Speiser. CERN 59-9, 59-17, 1959.
[3] G. Frobenius und J. Schur: über die reellen Darstellungen der endlichen Gruppen. Sitzungsbericht des Königl. Pr. Akademie der Wissenschaften, 1906, B II, p. 186.
[4] E. P. Wigner: Group theory and its applications to the Quantum Mechanics of Atomic Spectra, New York and London, 1959, p. 285.
[5] L. Pontrjagin: *Topological Groups*, 1946.

Subscripts (e.g., q_1) denote the components of a quaternion:

$$q = q_0 + q_1 i_1 + q_2 i_2 + q_3 i_3 \; .$$

Where a matrix is decomposed in two complex matrices, we write

$$M = M_0 + M_2, \; [M_0, i_3]_- = [M_2, i_3]_+ = 0,$$

No confusion will arise between these subscripts 0, 2 and the ones mentioned before $0 \cdots 3$

II. SOME NOTIONS AND THEOREMS OF THE SPECTRAL THEORY OF OPERATORS IN A Q SPACE

For a detailed development of a theory of operators in a Q space, and for the proofs of the following statements we refer to references 1 and 2. Here, we merely state the existence of a spectrum for Hermitian and unitary operators.

1. Every Hermitian (quaternionic) matrix H can be completely diagonalized and all eigenvalues of H are real numbers.

2. Every unitary (quaternionic) matrix U, $U^{-1} = U^H$, can be completely diagonalized with all diagonal elements of the form $e^{i_3}\varphi$ where $0 \leq \varphi \leq \pi$. The set of the resulting diagonal elements is uniquely determined by U. (Note that only half of the φ circle is used.)

3. To every n-dimensional Q space can be associated a $2n$-dimensional C space. To every Q operator then is associated a C operator. This association is biunique and called the symplectic representation. Details are presented elsewhere.[1,2,6]

III. Q REPRESENTATIONS

In the same way as in the theory of C representations we give the following definition: A *Q representation* of a (topological) group G is a homomorphic mapping of G into a (topological) group of linear operators on a Q space (cf. reference 5, p. 110). In the same way as in the complex theory (by Hurwitz integration over Hermitian form), one proves that every representation of a compact group is equivalent to a representation by unitary matrices. In the following we shall speak therefore always of unitary representations $D(G)$ for which $D^A \equiv D^{-1T} = D^{HT} = D^*$.

We note that by identifying i with i_3 every C representation is already a Q representation. *However, even if a C representation is irreducible, the associated Q representation need not be so.* It is the main purpose of this paper to indicate when and how a representation, irreducible over C, reduces over Q. This section contains the definitions of some generalized concepts and the generalizations of the well known lemmas.

1. Character and Trace

In the complex case, one defines the character by means of the trace (= diagonal sum) of a matrix. This definition is motivated by the equality

$$\mathrm{Tr}\,(A) = \mathrm{Tr}\,(B^{-1}AB), \qquad A, B \text{ complex matrices}$$

which expresses the fact that the character of a representation is independent of its matric expression; and also that the character is a class function, i.e., all group elements which can be transformed into each other have the same character. In a quaternion space this is no longer true for the diagonal sum Tr. For example, take a one-dimensional (even unitary) matrix $= i_3$, and $B = i_1$. Then $(-i_1)i_3(i_1) = -i_3$, so

$$\mathrm{Tr}\, A \neq \mathrm{Tr}\, B^{-1} A\, B.$$

But we observe that $\mathrm{Re}\,(\mathrm{Tr}\, A) = \mathrm{Re}\,(\mathrm{Tr}\, B^{-1}AB)$. Therefore we shall define the character

$$X(M) = \mathrm{Re}\,(\mathrm{Tr}\, M):$$

(a) $X(M)$ belongs to the center of the field for all M,
(b) $X(A + B) = X(A) + X(B)$,
(c) $X(AB) = X(BA)$.

Sometimes it is convenient to have a definition which is uniform for representations over all three fields R, C, Q by defining

$$X(M) \equiv \mathrm{av}_U\, U^{-1}MU,$$

where av_U involves integration over the quaternionic unitary (= symplectic) group (which of course is a compact group): i.e.,

$$X(M) = \frac{\int U^{-1}MU\, d_\mu(U)}{\int d_\mu(U)}.$$

For the characters $X_1(g)$, $X_2(g)$ of two representations $D_1(g)$, $D_2(g)$, defined by

$$X_k(g) = X(D_k(g)),$$

we may construct an "inner product"

$$X_2^* X_1 = \mathrm{av}_g\, X_2^*(g) X_1(g)$$

by averaging (integrating) over the group.

[6] C. Chevalley, *Lie Groups*, (Princeton University Press, Princeton, New Jersey, 1946), p. 16.

2. The Determinant

Although, because we used the spectral resolution techniques, there is no primary need for a determinant, one would like to have such a quantity to decide whether a given transformation is regular (i.e., has an inverse) or not.

Here at least as much care is needed as for the character. Clearly the usual definitions of the determinant of a matrix are of no use here because of noncommutativity. The determinant serves to determine the conditions under which two linearly independent vectors remain independent after a given linear transformation. Take, for simplicity, the unit vectors

$$\begin{pmatrix}1\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\end{pmatrix}$$

and the transformation $\begin{pmatrix}a & b\\c & d\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ become $\begin{pmatrix}a\\c\end{pmatrix}$ and $\begin{pmatrix}b\\d\end{pmatrix}$, respectively. By definition the question of independence reduces to the question of whether or not

$$a/b = a\cdot b^{-1} \quad \text{equals} \quad c/d = c\cdot d^{-1},$$

i.e., whether or not $ab^{-1} - cd^{-1} = 0$.

In order to get rid of the reciprocals, one might want to multiply this quantity with bd from the right, but then one arrives at $ad - cd^{-1}bd$, and now there is no way to eliminate d^{-1} in the second term. So one is forced to define another concept which will replace the usual determinant. The usual determinant $\Delta(M)$, M = any matrix, is a multiplicative mapping of the full real (complex) matrix ring on the field of real (complex) numbers, i.e., if $C = A\cdot B$ then $\Delta(C) = \Delta(A)\Delta(B)$. Instead of a mapping of the full quaternionic matrix ring on the field of quaternions, we define a "Δ mapping" of this ring on the set of non-negative real numbers. This is accomplished by the following definition:

In the quaternion theory, $\Delta(M)$ shall be the product of all the eigenvalues of $M^H M$. $M^H M$ is a Hermitian non-negative matrix which, by the spectral theorem of Sec. II has n real eigenvalues ≥ 0. These are all > 0 except when M is singular, in which case one or more of them $= 0$. For nonsingular M, therefore, $\ln M^H M$ is defined, and $\Delta(M) = \exp X(\ln M^H M)$.

That this determinant is multiplicative is proved by the remark that it reduces to the ordinary determinant in the symplectic picture, which is an isomorphism of the full quaternionic matrix M.

We note that $\Delta(U) = 1$ for every unitary matrix and $\Delta(MNM^{-1}) = \Delta(N)$ if M is nonsingular.

This determinant or Δ mapping provides a criterion for the regularity or singularity of a linear transformation. The circumstance that our definition of the quaternionic determinant $\Delta(M)$ is always positive is unavoidable for it is a homomorphism into an Abelian group. Such an Abelian group is necessarily a subgroup of the factor group of Q over its commutator subgroup, the sphere of quaternions of modulus 1.[7]

Clearly the new trace and the new determinant are a generalization to quaternion matrices of the real part and the norm of quaternion numbers. In analogy to the character $X(M)$, it is possible to give a definition of $\Delta(M)$ which has the same form for all three fields R, C, Q, reproduces the usual concepts for the usual fields R, C, and gives our new determinant in the quaternion case.

3. Schur's Lemma

Let $\{U\}$, $\{U'\}$ be irreducible sets of linear operators on Q spaces V, V'. Let there exist a linear transformation T on V into V' such that the two sets $\{U'T\}$ and $\{TU\}$ (of linear transformations on V into V') contain the same elements:

$$\{U'T\} = \{TU\}.$$

Then either $T = 0$, or $\dim V = \dim V'$ and T is nonsingular.

Proof: cf. reference 5, (second edition, 1957), p. 237.

Let $S = TV$ be the image of V in V' under T. Then S is invariant under $\{U'\}$: For let Tv be an arbitrary element of S, with v in V, and U' be an arbitrary element of $\{U'\}$; then for some u in $\{U\}$

$$u'Tv = Tuv,$$

which is again in S. Since $\{U'\}$ is irreducible, either $S = 0$ or $S = V'$. If $S = 0$ then $T = 0$, while if $S = V'$ then $\dim V' \leq \dim V$. Thus $T = 0$ or $\dim V' \leq \dim V$. Now take the Hermitian adjoint of the assumed relation

$$\{U'T\} = \{TU\}.$$

This interchanges the roles of U and U', and replaces T by T^H, so we conclude analogously that either $T^H = 0$ or $\dim V \leq \dim V'$. Combining the two results, either $T = 0$ or $\dim V = \dim V'$. In the latter case, T is nonsingular. This proves Schur's lemma for Q spaces. As in the complex

[7] We have been informed that the same concept was also introduced by Moore, and called the "norm determinant," see E. H. Moore, Mem. Am. Phil. Soc. 1, 99, 141 (1935).

case, it follows from this lemma that the characters of 2 inequivalent representations are orthogonal.

4. The Corollary to Schur's Lemma

Theorem: If a Hermitian matrix H commutes with an irreducible set D of matrices, it is a (real) multiple of the unit matrix.

Proof: We use the *ersatz* determinant. Since

$$HD = DH,$$

and $$ED = DE, \qquad E = \text{the unit matrix},$$

by subtraction $(H - \lambda E)D = D(H - \lambda E)$.

Now,

$$M^+M = (H - \lambda E)^+(H - \lambda E)$$
$$= H^2 - 2\lambda H + \lambda^2 E.$$

H is nonsingular and has real eigenvalues λ_k. Therefore H^2 has eigenvalues λ_k^2.

Since H^2, λH, and $\lambda^2 E$ all commute, we see that M^+M can be made singular by putting $\lambda = \lambda_k$, for then one of the eigenvalues of M^+M becomes $\lambda_k^2 - 2\lambda_k^2 + \lambda_k^2 = 0$, and $\Delta(M) = 0$ whence $M^+M = 0$ and $M = 0$, $H = \lambda E$.

5. The Frobenius-Schur Classification

Frobenius and Schur classified all irreducible C-representations with respect to whether or not they leave invariant a bilinear form.

Assume that a representation D leaves invariant a non-degenerate bilinear form C

$$D^T CD = C$$

or

$$CD = D^A C$$

where $D^A \equiv D^{T^{-1}}$ is the adjoint representation. Taking the adjoint of this equation, multiplying by C from the left and using the very same equation again, we get:

$$C^A D^A = DC^A$$
$$CC^A D^A = CDC^A = D^A CC^A$$

and by the corollary to Schur's lemma: $CC^A = \lambda E$, λ a scalar, that is $C = \lambda C^T$ or else $C \equiv 0$.

Taking the transpose of this equation, $C^T = \lambda C$; and inserting it, we get:

$$C = \lambda C^T = \lambda^2 C,$$

or

$$\lambda = \begin{matrix} +1 & C = +C^T \\ 0 & C = 0 \\ -1 & C = -C^T \end{matrix}$$

One may therefore say (Frobenius, and Schur, loc. cit.): an irreducible representation belongs to class $+1, 0, -1$ if it leaves invariant a symmetric bilinear form, no bilinear form, or a skew symmetric bilinear (symplectic) form, respectively. For unitary representations this may be expressed also in a different way. The first equation can be written: $CDC^{-1} = D^A$ but if D is unitary $D^A = D^{-1T} = D^*$.

Therefore if D belongs to class $+1$ or to class -1, it is equivalent to the complex conjugate representation and all characters (in the usual complex sense) $X(D)$ are real: if D belongs to class 0, D and D^* are not equivalent. Moreover it can be shown that a $D \in$ class $+1$ is equivalent to a real representation.

6. The Main Reduction Theorem

We now are able to state the main reduction theorem: A representation D, irreducible over C, reduces over Q into two representations $D_1 \oplus D_2$ if and only if $D \in$ class -1. D_1 and D_2 are equivalent and irreducible over Q.

We restrict ourselves to the case where G is compact and $D(G)$ may therefore be assumed unitary, such that $D^A = D^*$. First we prove that the condition $D \in$ class -1 is necessary.

a. Assume $D \in$ class 0, suppose that D reduces over Q. This means that there is a nonsingular (quaternionic) Hermitian operator M such that

$$MD = DM, \qquad M^H = M.$$

We decompose $M = M_0 + M_2$,

$$[M_0, i_3]_- = [M_2, i_3]_+ = 0.$$

Since $i_3 D = Di_3$,

$$M^{i_3} D = DM^{i_3}, \quad \text{where} \quad M^{i_3} \equiv i_3^{-1} M i_3.$$

But $M^{i_3} = M_0 - M_2$ and therefore both M_0 and M_2 commute with D:

$$M_0 D = DM_0\,, \tag{0}$$
$$M_2 D = DM_2\,. \tag{2}$$

By the corollary to Schur's lemma (valid in the complex case) $M_0 = \lambda E$, where λ is even and real, since $M_0^H = M_0$.

Now form $K \equiv i_1 M_2$; since M_2 is Hermitian and imaginary, it, together with K, is skew symmetric.

$$[K, i_3]_- = i_1 M_2 i_3 - i_3 i_1 M_2$$
$$= i_1 M_2 i_3 + i_1 i_3 M_2 = i_1 M_2 i_3 = 0,$$

i.e., K is a complex (and again by Schur's lemma, a nonsingular) matrix.

But $D = i_1^{-1}D^*i_1$ and from (2) we get:

$$M_2D = i_1^{-1}D^*i_1M_2,$$
$$D = K^{-1}D^*K.$$

That is: $D \simeq D^*$: D is equivalent to the complex-conjugate representation, contrary to the assumption. Of course instead of i_1 we could have used any j, $[j, i_3]_+ = 0$, $j^2 = -1$.

b. A short additional remark is sufficient to dispose of the case $D \in$ class $+1$. $C \neq 0$, $C^T = +C$ since for every matrix $M^Q = M_0^* - M_2$ (we can use here the star, since $[M_0, i_3]_- = 0$).

$$M_2^T = M_2^{HQ} = M_2^Q = -M_2$$

(here use is made of $M^H = M$)

$$K^T = -K.$$

But now

$$D = C^{-1}D^*C = C^{-1}KDK^{-1}C = (K^{-1}C)^{-1}D(K^{-1}C)$$

and by Schur's lemma:

$$K^{-1}C = \lambda E$$
$$C = \lambda K,$$

which is impossible, since a symmetric matrix cannot be a multiple of a skew one, unless $\lambda = 0$ which would mean $C = 0$, contrary to the assumption.

c. We now show that the condition $D_2 \in$ class -1 is *sufficient* for the reduction. Assume

$$CD = D^*C \qquad C^T = -C.$$

But $D^* = i_1^{-1}Di_1$ and with $K = i_1C$, $KD = DK$. We note that $K^H = K$, since $K^H = -C^Hi_1 = -i_1C^T = i_1C = K$ can be fulfilled with $C^T = -C$. This completes this part of the proof. It remains to show that D reduces into two equivalent representations irreducible over Q, $D = D_1 \oplus D_2$, $D_1 \simeq D_2$. For this purpose it is sufficient to show that $X(D_1) = X(D_2)$.

According to Frobenius and Schur,[3] it is always possible to transform the representation D so that C assumes its normal form

$$C = \begin{pmatrix} & -E \\ E & \end{pmatrix} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \otimes E.$$

The reducible representation D then is brought into the reduced form $\begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix}$ by the same matrix R which diagonalizes $K = i_1C$:

$$R^{-1}\,DR = \begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix},$$

$$R^{-1}KR = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \otimes E \text{ or } KR = R \otimes \begin{pmatrix} E & \\ & -E \end{pmatrix}.$$

From the second form of the second equation one deduces:

$$R = \begin{pmatrix} a & i_1\,a^* \\ i_1\,a & a^* \end{pmatrix} \otimes E,$$

where a is a complex number in the imaginary variable i_1, $2|a|^2 = 1$, for K is complex in the imaginary variable i_1, but we can chose a to be $2^{-1/2}$. If we now write $D = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ where $\alpha, \beta, \gamma, \delta$ are complex matrices in the imaginary variable i_3 the first and the fourth quarter of $R^{-1}DR$ are

$$D_1 = 2^{-1/2} \otimes E\cdot(\alpha - i_1\gamma + \beta i_1 - i_1\,\delta i_1)\cdot 2^{-1/2} \otimes E$$

and

$$D_2 = 2^{-1/2}\otimes E\cdot(-i_1\alpha i_1 + \gamma i_1 - i_1\beta + \delta)\cdot 2^{-1/2}\otimes E.$$

Both matrices have evidently the same real part, therefore the same character, whence it follows that they are equivalent.

Finally, D_1 is irreducible over Q. D is nothing but the symplectic picture of D_1; if D_1 were to reduce over Q, then so would D over C.

That the characters are a complete orthogonal system could probably be shown by going through the work of Peter and Weyl and demonstrating that the proofs remain valid if one considers quaternionic instead of complex integral equations. It also follows, however, from the remark that in view of the definition given in Sec. III.2 that the new D_1 has the same character the old D had, and that the system of characters therefore remains the same, up to a factor 2.

After this work was completed we received a preprint from Professor F. J. Dyson, in which this problem is also treated, but with different methods.

ACKNOWLEDGMENTS

Most of the results presented here were found by the authors during their collaboration at CERN in 1958/59 in connection with their investigation of a possible quaternion quantum mechanics. One of them (D.S.) would like to thank the Institute for Advanced Study, where part of this work was done, for the generous hospitality offered to him, and Dr. E. P. Wigner, Dr. N. Iwahori,[8] and Dr. J. Wolf for stimulating discussions. All authors have benefited from discussions of this material with G. Emch and T. Kanellopoulos.

[8] Dr. Iwahori informed us that he had obtained similar results, however, with very different methods.