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PAIR PRODUCTION AT STRONG COUPLING IN WEAK EXTERNAL FIELDS*

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The rate of electron–positron pair production in a weak external electric field at large values of e is computed The method is extended to compute the rate of monopole–antimonopole pair production in a weak external magnetic field at large magnetic charge

1. Introduction

Many years ago Schwinger [1] derived an expression for the rate of scalar electron–positron pair production in an external electric field:

$$\Gamma = \frac{(eE)^2}{(2\pi)^3} \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n^2} \exp\left[-\frac{\pi m^2}{eE} n\right] . \tag{1.1}$$

This result is obtained by summing the Feynman graphs of fig. 1, representing the interaction of a single electron loop with the background field. Eq. (1.1) is only valid for weak coupling ($\alpha \ll 1$); it is corrected by graphs such as those of fig. 2. The full expression for Γ should take the form

$$\Gamma = (eE)^2 \sum_{n=0}^{\infty} e^{2n} f_n\left(\frac{eE}{m_2}\right) . \tag{1.2}$$

If we assume a weak external field ($eE \ll m^2$) as well as weak coupling, then eq. (1.1) simplifies to

$$\Gamma = \frac{(eE)^2}{(2\pi)^3} e^{-\pi m^2/eE} . \tag{1.3}$$

Two of us (I.K.A. and N.M.) [2] recently calculated the rate of magnetic monopole pair production in the Georgi–Glashow model. We found

$$\Gamma_{\mathrm{M}} = \frac{(gB)^2}{(2\pi)^3} \exp\left[\frac{-\pi M^2}{gB} + \tfrac{1}{4}g^2\right] , \tag{1.4}$$

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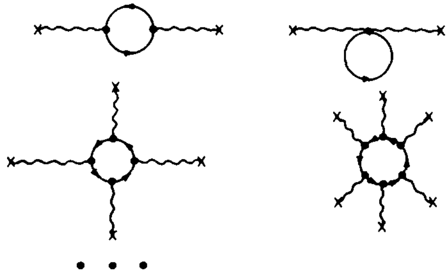

Fig 1 One loop diagrams × represents the external field

where M is the monopole mass and g is the magnetic coupling constant ($g = 4\pi/e$). This result, valid for $e^2/4\pi \ll 1$ and weak fields ($g^3 B \ll M^2$), was obtained using an instanton method that, at first sight, bears no relation to Schwinger's approach. The similarity of eqs. (1.3) and (1.4) led us to speculate that Γ_M could have been obtained by repeating Schwinger's calculation with an effective lagrangian $\mathscr{L}_{\text{eff}}$, in which the monopole is represented by a charged scalar field interacting with a dual abelian gauge field. However, there are two objections to this procedure. Firstly, one would expect $\mathscr{L}_{\text{eff}}$ to contain all sorts of higher dimension, gauge-invariant operators. Secondly, the coupling constant, g, is large in the domain of applicability of the formula, and the corrections to Schwinger's formula [eq. (1.2)] should be relevant.

This objection could be refuted if, for weak fields, but arbitrary coupling eq. (1.2) reduced to

$$\Gamma = \frac{(eE)^2}{(2\pi)^3} \exp\left[-\frac{m^2}{eE} + \tfrac{1}{4}e^2\right].$$

In sect. 2 we shall argue this to be the case. Subtleties related to renormalization are discussed in the appendix. In sect. 3, we return to the monopole problem and argue that the higher dimension operators are irrelevant for weak fields.

2. (Scalar) e^+e^- pair production

In this section we calculate the rate of (scalar) e^+e^- pair production in a weak external electric field. To begin, we rederive Schwinger's weak-coupling result by

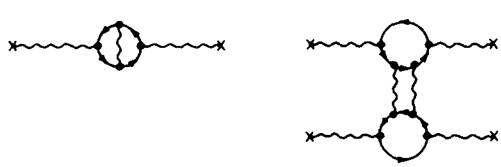

Fig 2 Higher loop diagrams

using an instanton method related to that used in ref. [1]. We then generalize to the case of arbitrary coupling.

2.1 WEAK COUPLING

Using the formula $\Gamma = 2\ \mathrm{Im}\ \varepsilon_0$, where ε_0 is the vacuum energy density, we obtain

$$V\Gamma = -2\ \mathrm{Im}\ \ln \int (\mathrm{d}A)(\mathrm{d}\phi)\ \mathrm{e}^{-S}, \tag{2.1}$$

where S is the euclidean action in the presence of the external field,

$$S = \int \mathrm{d}^4x\ [\tfrac{1}{4}F_{\mu\nu}^2 + |(\partial_\mu + ieA_\mu + ieA_\mu^{\mathrm{ex}})\phi|^2 + m^2|\phi|^2], \tag{2.2}$$

$$\begin{gathered} A_\mu^{\mathrm{ex}} = \tfrac{1}{2}iF_{\mu\nu}^{\mathrm{ex}}x_\nu, \\ F_{34}^{\mathrm{ex}} = -F_{43}^{\mathrm{ex}} = E, \qquad \text{all other components zero}. \end{gathered} \tag{2.3}$$

[$F_{\mu\nu}^{\mathrm{ex}}$ is imaginary in euclidean space and corresponds to an electric field E in the x_3 direction.] Integrating out ϕ, we obtain

$$V\Gamma = -2\ \mathrm{Im}\ \ln\,(\mathrm{d}A)\ \mathrm{e}^{-S_{\mathrm{eff}}}, \tag{2.4}$$

where

$$S_{\mathrm{eff}} = \tfrac{1}{4}\int \mathrm{d}^4x\ F_{\mu\nu}^2 + \mathrm{tr}\ \ln\,[-(\partial_\mu + ieA_\mu + ieA_\mu^{\mathrm{ex}})^2 + m^2]. \tag{2.5}$$

Perturbation theory in α is obtained by expanding $\mathrm{e}^{-S_{\mathrm{eff}}}$ in powers of eA_μ and doing gaussian integrals. In the weak coupling approximation we obtain

$$V\Gamma = 2\ \mathrm{Im}\ \mathrm{tr}\ \ln\,[-(\partial_\mu + ieA_\mu^{\mathrm{ex}})^2 + m^2]. \tag{2.6}$$

Following Schwinger we write the logarithm as a "proper time" integral:

$$V\Gamma = \mathrm{Im}\int_0^\infty \frac{\mathrm{d}T}{T}\,\mathrm{e}^{-(m^2/2)T}\ \mathrm{tr}\ \exp\,[\tfrac{1}{2}(\partial_\mu + ieA_\mu^{\mathrm{ex}})^2 T]. \tag{2.7}$$

The trace is now of the form $\mathrm{tr}\ \mathrm{e}^{-HT}$, with

$$H = \tfrac{1}{2}[P_\mu + eA_\mu^{\mathrm{ex}}(x)]^2, \tag{2.8}$$

being the hamiltonian for a particle moving in an external field in four spatial dimensions. This may be written as a quantum-mechanical path integral:

$$V\Gamma = \mathrm{Im}\int_0^\infty \frac{\mathrm{d}T}{T}\,\mathrm{e}^{-(m^2/2)T}\int [\mathrm{d}x]\ \exp\left(-\int_0^T \mathrm{d}[\tfrac{1}{2}\dot{x}^2 - ieA_\mu^{\mathrm{ex}}x_\mu]\right). \tag{2.9}$$

It will prove convenient to rescale the time variable

$$\tau \to \tau/T,$$

so that

$$V\Gamma = \mathrm{Im}\int_0^\infty \frac{\mathrm{d}T}{T}\,\mathrm{e}^{-(m^2/2)T}\int [\mathrm{d}x]\ \exp\left(-\frac{1}{2T}\int_0^1 \mathrm{d}\tau\,\dot{x}^2 + ie\oint A_\mu^{\mathrm{ex}}\,\mathrm{d}x_\mu\right). \tag{2.10}$$

[The path integral is over all periodic paths, $x_\mu(1) = x_\mu(0)$, and is normalized so that

$$\int [\mathrm{d}x] \exp\left(-\frac{1}{2T}\int_0^1 \mathrm{d}\tau\, \dot{x}^2\right) = \mathrm{tr}\exp\left(-\tfrac{1}{2}p^2 T\right) = \frac{V}{(2\pi T)^2}\,.] \tag{2.11}$$

For weak fields it is possible to do both integrals in eq. (2.10) by steepest descents. It is convenient to do the T integral first, assuming $m^2 \int_0^1 \dot{x}^2 \gg 1$, a condition which will be obeyed at the stationary point of the path integral. The T integral has a stationary point at $T_0^2 = \int \dot{x}^2/m^2$, giving

$$V\Gamma = \mathrm{Im}\int (\mathrm{d}x) \exp\left(-m\sqrt{\textstyle\int \dot{x}^2} + ie\oint A^{\mathrm{ex}}\,\mathrm{d}x\right)\frac{1}{m}\sqrt{2\pi/T_0}. \tag{2.12}$$

We now do this path integral by steepest descents. The action

$$S = m\sqrt{\textstyle\int \dot{x}^2} + ie\oint A_\mu^{\mathrm{ex}}\,\mathrm{d}x_\mu\,, \tag{2.13}$$

is stationary when

$$\frac{m\ddot{x}_\mu}{\sqrt{\int \dot{x}^2}} = -eF_{\mu\nu}\dot{x}_\nu\,. \tag{2.14}$$

Recalling the periodic boundary conditions we find the lowest action solution to be

$$x_\mu^{\mathrm{cl}} = R(0, 0, \cos 2\pi\tau, \sin 2\pi\tau)\,, \tag{2.15}$$

$$R = m/eE\,, \qquad S = \pi m^2/eE\,. \tag{2.16}$$

This is the same instanton that occurred in ref. [1]; the charged particle travels in a circle.

We now turn to calculating the one-loop factor, arising from this path integral. The second variation operator is

$$\begin{aligned} M_{\mu\nu} &\equiv \left.\frac{\delta^2 S}{\delta x_\mu(\tau)\delta x(\tau')}\right|_{x^{\mathrm{cl}}} \\ &= -\left[\frac{m\delta_{\mu\nu}}{\sqrt{\int \dot{x}^2}}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + eF_{\mu\nu}\frac{\mathrm{d}}{\mathrm{d}\tau}\right]\delta(\tau-\tau') - \frac{m\ddot{x}_\mu(\tau)\ddot{x}_\nu(\tau')}{[\int x^2]^{3/2}} \\ &= -\left[\frac{eE}{2\pi}\delta_{\mu\nu}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + eF_{\mu\nu}\frac{\mathrm{d}}{\mathrm{d}\tau}\right]\delta(\tau-\tau') - \frac{2\pi eE x_\mu^{\mathrm{cl}}(\tau)x_\nu^{\mathrm{cl}}(\tau')}{R^2}\,. \end{aligned} \tag{2.17}$$

There are four zero-modes corresponding to translation of the loop and a fifth corresponding to proper-time translation. There is also a single negative eigenvalue, $-2\pi eE$, corresponding to changing the loop radius of the classical solution, x_μ^{cl}.

The remaining positive eigenmodes are

$$\begin{gathered}(0,0,\cos 2n\pi\tau,\sin 2n\pi\tau)\,,\qquad (0,0,\sin 2n\pi\tau,-\cos 2n\pi\tau)\,,\\ \lambda=2\pi eE(n^2-n)\,,\qquad n=2,3,4,\ldots,\\ (\cos 2n\pi\tau,0,0,0)\,,\qquad (\sin 2n\pi\tau,0,0,0)\,,\qquad (0,\cos 2n\pi\tau,0,0)\,,\\ (0,\sin 2n\pi\tau,0,0)\,,\qquad \lambda=2\pi eEn^2\,,\qquad n=1,2,3,\ldots.\end{gathered} \tag{2.18}$$

To get the correct normalization of the functional integral we use eq. (2.11) (with $T=T_0$). The second variation operator is $M^0_{\mu\nu}=-(1/T_0)\,\delta_{\mu\nu}\,\mathrm{d}^2/\mathrm{d}\tau^2$, with eigenvalues $2\pi eEn^2$, $n=0,1,2,\ldots$, and multiplicity 8. Thus

$$\mathrm{Im}\int(\mathrm{d}x)\exp\left(-\tfrac{1}{2}\int\mathrm{d}\tau\,\mathrm{d}\tau'\,x_\mu(\tau)M_{\mu\nu}(\tau,\tau')x_\nu(\tau')\right)$$

$$=\frac{V}{(2\pi T_0)^2}\,\mathrm{Im}\frac{\int(\mathrm{d}x)\exp\left(-\tfrac{1}{2}\int xMx\right)}{\int(\mathrm{d}x)\exp\left(-\frac{1}{2T}\int\dot{x}^2\right)}=\frac{V}{(2\pi T_0)^2}N\frac{1}{2}\,\frac{\Pi|\lambda|^{-1/2}}{\Pi\lambda_0^{-1/2}}\,, \tag{2.19}$$

where λ, λ_0 are the non-zero eigenvalues of M and M_0, N is a factor that arises from replacing the time translation mode, by a collective coordinate,

$$N=\sqrt{\frac{\int\dot{x}^{\mathrm{cl}2}}{2\pi}}\int_0^1\mathrm{d}\tau\,\sqrt{2\pi}\,R\,, \tag{2.20}$$

and the factor of $\frac{1}{2}$ arises from analytic continuation of the functional integration contour into the complex plane [3]. The product over eigenvalues gives

$$\frac{\Pi|\lambda|^{-1/2}}{\Pi\lambda_0^{-1/2}}=\sqrt{2\pi eE}\prod_{n=2}^{\infty}\frac{n}{n-1}\prod_{n=-\infty}^{-1}\frac{n}{n-1}=\sqrt{2\pi eE}\,. \tag{2.21}$$

Putting it all together, we find

$$\Gamma=\frac{(eE)^2}{(2\pi)^3}\mathrm{e}^{-\pi m^2/eE}\,, \tag{2.22}$$

Schwinger's result.

Several comments must be made about this derivation. There are, of course, larger contributions to the vacuum energy density, arising from T near zero. These are real and do not contribute to Γ. We have presented the leading term in a saddle-point evaluation of Γ. In order to calculate higher order terms (in eE/m^2) we must use a more complicated procedure whereby we integrate over all paths, subject to constraints on the radius and the starting point, then integrate over T, and finally integrate out the constraints. If we did this here we would find that all terms in the power series in eE/m^2 vanish. This won't be the case in subsect. 2.2.

Finally, the terms in eq. (1.1) which are suppressed by additional powers of $e^{-\pi m^2/eE}$ arise in this calculation from multi-instantons, classical solutions in which the particle goes around the circle more than once.

2 2 ARBITRARY COUPLING

Eqs. (2.4) and (2.5) may be written

$$V\Gamma = -2 \operatorname{Im} \ln \langle \exp \{-\operatorname{tr} \ln [-(\partial_\mu + ieA_\mu + ieA_\mu^{\text{ex}})^2 + m^2]\}\rangle , \qquad (2.23)$$

where

$$\langle g(A)\rangle \equiv \int (\mathrm{d}A) \exp\left[-\tfrac{1}{4}\int F^2\right] g(A) \Big/ \int (\mathrm{d}A) \exp\left[-\tfrac{1}{4}\int F^2\right] . \qquad (2.24)$$

Taylor expanding the exponential corresponds to an expansion in the number of electron loops (in Feynman diagram language). Ignoring correlations between different tr ln $(-D^2 + m^2)$ factors corresponds to ignoring all photon lines connecting different electron loops, i.e. exponentiating the set of all diagrams with a single electron loop. We shall argue below that this is a good approximation for weak fields, even for sizeable couplings. In this approximation we obtain

$$V\Gamma = -2 \operatorname{Im} \langle \operatorname{tr} \ln [-(\partial_\mu + ieA_\mu + ieA_\mu^{\text{ex}})^2 + m^2]\rangle . \qquad (2.25)$$

Following the procedure of subsect. 2.1 we may turn the trace into a proper-time path integral:

$$V\Gamma = -\operatorname{Im} \int_0^\infty \frac{\mathrm{d}T}{T} \mathrm{e}^{-(m^2/2)T} \int [\mathrm{d}x] \exp\left(-\frac{1}{2T}\int_0^1 \dot{x}^2 + ie\oint A_\mu^{\text{ex}}\, \mathrm{d}x_\mu\right)$$
$$\times \left\langle \exp\left(ie\oint A_\mu \,\mathrm{d}x_\mu\right)\right\rangle . \qquad (2.26)$$

The averaging over A_μ has been reduced to a gaussian integral:

$$\left\langle \exp\left(ie\oint A_\mu\, \mathrm{d}x_\mu\right)\right\rangle = \exp\left(-\frac{e^2}{8\pi^2}\oint\oint \frac{\mathrm{d}x \cdot \mathrm{d}x'}{(x-x')^2}\right) . \qquad (2.27)$$

For smooth paths, $x(\tau)$, the exponent has only a linear ultraviolet divergence, which can be removed by dimensional regularization.

We now proceed to integrate over T and $x(\tau)$ by steepest descents as in subsect. 2.1. Doing the T integral gives

$$V\Gamma = \operatorname{Im} \int (\mathrm{d}x) \exp\left(-m\sqrt{\textstyle\int \dot{x}} + ie\oint A_\mu^{\text{ex}}\, \mathrm{d}x_\mu\right.$$
$$\left. - \frac{e^2}{8\pi^2}\oint \mathrm{d}x \cdot \mathrm{d}x' \frac{1}{(x-x')^2}\right) \frac{1}{m}\sqrt{\frac{2\pi}{T_0}}$$

The stationary point for the path integral is unchanged by the addition of this Coulomb exchange term to the action. This follows because this term does not break rotational symmetry (in the 3-4 plane); thus the stationary path must still be a circle. For a circle (after regularization)

$$\frac{1}{8\pi^2}\oint \mathrm{d}x \cdot \oint \mathrm{d}x' \frac{1}{(x-x')^2} = -\tfrac{1}{4}\,. \tag{2.28}$$

Since this is independent of the radius, the stationary radius remains $R = m/eE$. The classical action has been modified to $S = \pi m^2/eE - \frac{1}{4}e^2$. Note that for weak fields, the Coulomb interaction term makes a contribution to S that is much smaller than the first term, even for sizeable coupling. It should therefore make a small correction to the one-loop factor

$$\Gamma = \frac{(eE)^2}{(2\pi)^3}\,\mathrm{e}^{-\pi m^2/eE + e^2/4}\,. \tag{2.29}$$

This is the result obtained in ref. [1]. What about the correlations between different $\mathrm{tr}\ln[-D^2+m^2]$ factors that were dropped in going from eq. (2.24) to eq. (2.25)? Ignoring these correlations corresponds to a dilute instanton gas approximation [the density of the gas is $\sim \mathrm{e}^{-\pi m^2/eE}$]. The multi-instanton interactions are obtained by including the correlations. These should be suppressed, as usual, by extra factors of $\mathrm{e}^{-\pi m^2/eE}$.

At this point we must confess to the reader that the picture is not quite as rosy as the one we have painted. Eq. (2.29) only makes sense if the parameters e and m refer to the normalized quantities. We would expect the conventional definitions of e^2 and m^2 as the coefficient of the long-range Coulomb force and the pole in the electron propagator to be relevant here since the relevant paths involve slowly moving electrons and long-range Coulomb forces for weak fields. Our steepest descents approximations have missed renormalization effects. In fact, we believe, for weak fields, that these are the only effects they miss, and that eq. (2.29) is correct when e and m are taken to be the renormalized quantities.

To argue this point consider the one-photon exchange graph of fig. 2. In the proper-time formalism this graph is given by

$$\begin{aligned} V\Delta\Gamma = -\frac{e^2}{8\pi^2}\,\mathrm{Im}\int_0^\infty \frac{\mathrm{d}T}{2}\,\mathrm{e}^{-(m^2/2)T}\int[\mathrm{d}x]\mathrm{e}\Big(-\tfrac{1}{2}\Gamma\int_0^1 \dot{x}^2 + ie\oint A_\mu^{\mathrm{ex}}\,\mathrm{d}x_\mu \\ \times\oint \mathrm{d}x\cdot\oint \mathrm{d}x'\frac{1}{(x-x')^2}\Big)\,. \end{aligned} \tag{2.30}$$

In the steepest descents approximation this integral is dominated by circular paths

C of radius $R = m/eE$, giving

$$V\Delta\Gamma = -\frac{e^2}{8\pi^2}\frac{(eE)^2}{(2\pi)^3}\mathrm{e}^{-\pi m^2/eE}\oint_C \mathrm{d}x \cdot \oint_C \mathrm{d}x' \frac{1}{(x-x')^2} \tag{2.31}$$

$$= \tfrac{1}{4}e^2\frac{(eE)^2}{(2\pi)^3}\mathrm{e}^{-\pi m^2/eE}, \tag{2.32}$$

(using dimensional regularization). We believe that the steepest descents approximation breaks down when $|x-x'| \ll R$ because paths of the following type are important: x and x', and the path between them must be allowed to move around freely over distances $\gg 1/m$ (and not be constrained to lie on the circle). If we include the contribution of such paths we should obtain (one-loop) mass renormalization. We illustrate this point, in a somewhat simplified context, in the appendix. If two Coulomb exchange factors are taken when we must carefully integrate over path fluctuations whenever any two vertices are close to each other. Now consider the interactions of two different electron loops. It is necessary to consider small (perturbative) loops as well as large (instanton) ones. Small loops interacting with large ones should produce mass and charge renormalization.

Our belief that the only corrections to the naive steepest descents approximations are charge and mass renormalization is based on the fact that for $R \gg m^{-1}(eE \ll m^2)$ it may be possible to make a clean separation between short distance effects ($\sim m^{-1}$) which simply provide renormalization and large-distance effects ($\sim R$) that can be described semi-classically. To argue this point note that the full propagator is given (in position space) in the absence of an external field by

$$\tilde{D}(x-y) = \left\langle \frac{1}{-D^2+m^2}(x, y)\, \mathrm{e}^{-\operatorname{tr}\ln(-D^2+m^2)} \right\rangle. \tag{2.33}$$

This can again be expressed in terms of proper-time integrals, using

$$\frac{1}{-D^2+m^2}(x, y) = \tfrac{1}{2}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{-(m^2/2)T}\int [\mathrm{d}x]\exp\left(-\tfrac{1}{2}T\int \dot{x}^2 + ie\int A_\mu\, \mathrm{d}x_\mu\right), \tag{2.34}$$

where the integral is over all paths between x and y. In the no electron loops approximation, we have

$$\tilde{D}(x-y) = \tfrac{1}{2}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{-(m^2/2)T}\int [\mathrm{d}x]\exp\left(-\frac{1}{2T}\int \dot{x}^2 - \frac{e^2}{8\pi^2}\int \mathrm{d}x \cdot \int \mathrm{d}x' \frac{1}{(x-x')^2}\right). \tag{2.35}$$

In the limit $|x-y| \gg m^{-1}$ the large-distance behavior of $\tilde{D}$ is governed by the pole in momentum space at the renormalized mass:

$$\tilde{D}(x-y) \to \frac{m_R^{1/2}}{[2\pi|x-y|]^{3/2}}\mathrm{e}^{-m_R|x-y|}. \tag{2.36}$$

Now let us go back to the external field problem. If we consider short sections of path [length of $\mathrm{O}(m^{-1})$], then we may ignore the external field term $e\int_x^y A_\mu\, \mathrm{d}x_\mu \sim$

eE/m^2. But then the path integral gives the full propagator. It will achieve its asymptotic value [eq. (2.36)] before the effect of the external field term becomes significant. Thus the effect of short-range path fluctuations (and small loops) should simply be to renormalize e and m.

3. Higher dimension operators

We will now return to magnetic monopoles. It should be possible to describe their interaction with an external magnetic field by an effective lagrangian containing a charged scalar field, representing the monopole, interacting with the dual electromagnetic gauge field, $\tilde{A}_\mu$. This lagrangian could be obtained, in principle, by integrating out all other fields. It should contain not only the minimal coupling term, but also other higher dimension, gauge-invariant operators.

A naive argument can be given to explain why these operators have no effect. Consider an operator involving only $\tilde{A}_\mu$, for example $\partial_\mu F_{\mu\nu} \partial_\lambda F_{\lambda\nu}$. The coefficient of such a term in $\mathscr{L}_{\text{eff}}$ must involve some inverse powers of M, the monopole mass. Following the proper-time procedure of sect. 2 we must calculate

$$\left\langle \exp\left(\imath \oint_C A_\mu \, \mathrm{d}x_\mu\right)\right\rangle$$

using the effective lagrangian. The higher dimension operators in $\mathscr{L}_{\text{eff}}$ have a negligible effect when C is the circular path of sect. 2. This can be seen by treating these operators perturbatively. To lowest order we evaluate the operators for the field produced by the monopole loop. Since the scale of this field is R, these operators give $1/(MR)^d$ when $(4+d)$ is the dimension of the operator. Thus they are negligible for large R (weak fields). More physically, these operators reflect the short-range forces between monopoles which can be ignored for large monopole loops. There will also be higher dimension operators in $\mathscr{L}_{\text{eff}}$ involving the monopole field. However, when we integrate this field out [to obtain $\operatorname{tr}\ln(-D^2+M^2)$] we simply generate a series of local terms involving $F_{\mu\nu}$ only. Again we are being cavalier about the short-distance effects which renormalize the monopole mass and charge.

We would like to thank S. Coleman, G.P. Lepage, M. Peskin and G. Shore for helpful discussions. One of us (I.K.A.) was supported by the Harvard Society of Fellows during the time this research was carried out.

Appendix

STEEPEST DESCENTS APPROXIMATION AND MASS RENORMALIZATION

We wish to illustrate the points made in sect. 2 about the steepest descent approximation breaking down at short distance by a simplified example. Instead of QED we consider a theory of a charged scalar ("electron") interacting with a massless neutral scalar ("photon"):

$$\mathscr{L} = |\partial_\mu \psi|^2 + m^2|\psi|^2 + \tfrac{1}{2}(\partial_\mu \phi)^2 + e\psi^* \phi\psi . \tag{A.1}$$

We will consider the calculation of the electron propagator using the proper-time approach. The free propagator is given by

$$D(x-y) = \tfrac{1}{2}\int^{\infty} \mathrm{d}T\, \mathrm{e}^{-(m^2/2)T} \int [\mathrm{d}x]\, \mathrm{e}^{\;\,T\int \dot{x}^2/2} , \tag{A.2}$$

integrating over paths from x to y. For large $|x-y|m$ we may use steepest descents:

$$D(x-y) \approx \tfrac{1}{2}\int (\mathrm{d}x)\, e^{\;\,m\sqrt{\int \dot{x}^2}}\, \frac{\sqrt{2\pi T_0}}{m} , \qquad T_0 = \sqrt{\int \dot{x}^2/m} . \tag{A.3}$$

The stationary point is

$$\ddot{x} = 0 , \qquad x(\tau) = x(1-\tau) + y\tau , \tag{A.4}$$

giving

$$D(x-y) = \frac{m^{1/2}}{(2\pi|x-y|)^{3/2}}\, \mathrm{e}^{-m\,|x-y|} . \tag{A.5}$$

Now consider the one-loop correction:

$$\Delta D = -\frac{e^2}{8\pi^2}\tfrac{1}{2}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{-(m^2/2)T} \int (\mathrm{d}x)\, \mathrm{e}^{-(1/2T)\int \dot{x}^2} \int \frac{\mathrm{d}\tau\, \mathrm{d}\tau'}{[x(\tau) - x(\tau')]^2} . \tag{A.6}$$

In the steepest descents approximation this becomes

$$\frac{m^{1/2}}{(2\pi|x-y|)^{3/2}}\, \mathrm{e}^{-m|x-y|} \left(-\frac{e^2}{8\pi^2}\right) \int_0^1 \frac{\mathrm{d}\tau\, \mathrm{d}\tau'}{(y-x)^2(\tau-\tau')^2} . \tag{A.7}$$

To evaluate D more carefully let us break path integral (A.6) up into 3 parts. Denoting $x(\tau)$ by u, $x(\tau')$ by v, we have

$$\Delta D = -\frac{e^2}{8\pi^2}\int \mathrm{d}^4 u\, \mathrm{d}^4 u\, D(x-y)\, \frac{D(u-v)}{(u-v)^2}\, D(v-y) . \tag{A.8}$$

Let

$$\xi = u - v, \qquad \eta = \tfrac{1}{2}(u+v) : \tag{A.9}$$

$$\begin{aligned} \Delta D &= -\frac{e^2}{8\pi^2}\int \mathrm{d}^4\xi \frac{D(\xi)}{\xi^2} \int \mathrm{d}^4\eta\, D(x - \tfrac{1}{2}\xi - \eta) D(\eta - \tfrac{1}{2}\xi - y) \\ &= -\frac{e^2}{8\pi^2}\int \mathrm{d}^4\xi \frac{D(\xi)}{\xi^2}\, D_2(x-y-\xi) , \end{aligned} \tag{A.10}$$

where

$$D_2(x-y)=\langle x|\left(\frac{1}{-\partial^2+m^2}\right)^2|y\rangle \tag{A.11}$$

$$=-\tfrac{1}{4}\int_0^\infty \mathrm{d}T\,T\,\mathrm{e}^{\,(m^2/2)T}\int[\mathrm{d}x]\,\mathrm{e}^{-(1/2T)\int \dot{x}^2}. \tag{A.12}$$

When $|\xi|$ (and $|x-y-\xi|$) are large we may use the steepest descents approximation to D and D_2:

$$\Delta D=-\frac{e^2}{8\pi^2}\int \mathrm{d}^4\xi\,\frac{m\,\mathrm{e}^{-m(|\xi|+|x-y-\xi|)}}{(2\pi)^3\xi^{1/2}|x-y-\xi|^{1/2}}. \tag{A.13}$$

Let us choose $y=0$, $x=(0,0,0,r)$. Then, far from x and y, the integral is sharply peaked about the x_4 axis

$$\Delta D=-\frac{e^2}{8\pi^2}\int \mathrm{d}\xi_4\,\frac{m\,\mathrm{e}^{-m[\,\xi_4|+|r-\xi_4|]}}{(2\pi)^{\xi}\xi_4^{1/2}|r-\xi_4|^{1/2}}\int \mathrm{d}^3\boldsymbol{\xi}\exp\left(-\tfrac{1}{2}m\left(\frac{1}{|\xi_4|}+\frac{1}{|r-\xi_4|}\right)\boldsymbol{\xi}^2\right). \tag{A.14}$$

Eq. (A.14) is equivalent to the steepest descents approximation of eq. (A.7). It is clearly not valid for small $|\xi|$. For small ξ we may use the above approximation for $D_2(x-y-\xi)$, but not for $D(\xi)$:

$$\Delta D=+\frac{1}{2}\frac{e^2}{8\pi^2}\int \mathrm{d}^4\xi\,\frac{D(\xi)}{\xi^2}\sqrt{\frac{m}{2\pi|x-y-\xi|}}\,\mathrm{e}^{-m|x-y-\xi|} \tag{A.15}$$

$$\sim \mathrm{e}^{-m|x-y|}\frac{1}{2}\frac{e^2}{8\pi^2}\sqrt{\frac{m}{2\pi|x-y|}}\int \mathrm{d}^4\xi\,\frac{D(\xi)}{\xi^2}\,\mathrm{e}^{-m\xi_4} \tag{A.16}$$

$$=D(x-y)[-\delta m|x-y|], \tag{A.17}$$

where

$$\delta m=\frac{e^2}{8\pi^2}\int \mathrm{d}^4\xi\,\frac{D(\xi)}{\xi^2}\,\mathrm{e}^{ip\cdot\xi}\qquad (\text{at } p^2=-m^2). \tag{A.18}$$

This is the standard one-loop mass renormalization. Thus we see that when u and v are close together, the steepest descents approach breaks down and a correct evaluation gives mass renormalization.

Reference

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