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$N = 4$ YANG–MILLS AND $N = 8$ SUPERGRAVITY AS LIMITS OF STRING THEORIES*

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The formulation of supersymmetric string theories in ten dimensions is generalized to incorporate compactified dimensions. Expressions for the one-loop four-particle S-matrix elements of $N = 4$ Yang–Mills and $N = 8$ supergravity in four dimensions are obtained by studying the string-theory loop amplitudes in the limit that the radii of the compactified dimensions and the Regge slope parameter simultaneously approach zero. If certain patterns that emerge should persist in the higher orders of perturbation theory, then $N = 4$ Yang–Mills in four dimensions would be ultraviolet finite to all orders, whereas $N = 8$ supergravity in four dimensions would have ultraviolet divergences starting at three loops.

1. Introduction

A light-cone-gauge action for supersymmetric strings in ten-dimensional space-time was recently formulated [1]. Depending on the choice of boundary conditions, this action may be used to formulate either an interacting theory of open and closed strings with simple ten-dimensional supersymmetry ("theory I") or an interacting theory of closed strings only with extended ten-dimensional supersymmetry ("theory II"). In the absence of interactions the physical open-string states of theory I lie on linear parallel Regge trajectories of slope α' (the inverse string tension), whereas the closed-string states of either theory lie on parallel Regge trajectories of slope $\alpha'_c = \frac{1}{2}\alpha'$. The theories have no ghosts or tachyons.

Formally the "zero-slope limit" $\alpha' \to 0$ sends all massive states to infinity leaving only the massless ones. In the tree approximation to the S-matrix of theory I with external open-string massless states, such a limit is well-defined and gives the same amplitudes as supersymmetrical Yang–Mills field theory in ten dimensions [2]. Similarly the tree approximation to extended supergravity in ten dimensions is obtained from theory II. Since these theories are in ten dimensions, each is expected to be singular quantum mechanically due to ultraviolet (UV) divergences of Feyn-

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man loop graphs. This is manifested as a singular zero-slope limit of the loop corrections to the string theory [3]. In any case it would be more interesting physically if the limiting theory were a four-dimensional one. To achieve this we formulate a generalization of the supersymmetric string theories in which $10-D$ of the spatial coordinates form circles of radius R. This is described in sect. 2. In that section we also discuss certain issues relevant to the limit in which α' and $R \to 0$ so as to give a D-dimensional field theory. In this way we are able to define $N=4$ Yang–Mills theory in four dimensions as a limit of theory I and $N=8$ supergravity in four dimensions as a limit of theory II.

The limit of theory I that gives $N=4$ Yang–Mills theory is studied in sect. 3. In particular, we obtain explicit expressions for the one-loop four-particle on-shell scattering amplitudes in D dimensions. We define a continuation in the number of compactified dimensions, which serves to regularize both infrared (IR) and UV divergences. The one-loop amplitudes of the Yang–Mills theory are shown to be finite for $4<D<8$, the lower limit determined by the IR behavior and the upper by UV behavior. When D approaches 4 from above, the leading IR divergence is proportional to $(D-4)^{-2}$.

In analogous fashion, the limit of theory II that gives $N=8$ supergravity is studied in sect. 4. The expressions for one-loop four-particle on-shell amplitudes in D dimensions are derived. Once again the result is finite for $4<D<8$. In this theory the leading IR divergence is proportional to $(D-4)^{-1}$.

The expressions for the amplitudes in sects. 3 and 4 have a structure that motivates certain speculations about higher-order loops, which are summarized in the concluding section.

2. Dimensional compactification of string theories

In this section we analyze the modifications of the supersymmetric string theories required by assuming that $10-D$ of the spatial dimensions are circular with radius R. Different radii could be chosen for each compactified dimension, but in order to minimize indices we do not distinguish among them. Parts of the analysis follow that carried out for the Veneziano model in ref. [4]. The compactified space that we investigate may be viewed as a product of circles. This choice is convenient because the string wave equation,

$$\partial_\tau(\eta_{ij}\partial_\tau X^j) - \partial_\sigma(\eta_{ij}\partial_\sigma X^j) = 0\,, \tag{2.1}$$

still has $\eta_{ij}=\delta_{ij}$, so that the mode expansion in the open-string case is still given by [1]

$$X^i_{\text{open}}(\sigma,\tau) = x^i + 2\alpha' p^i\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha^i_n \cos(n\sigma)\,\mathrm{e}^{-in\tau}\,. \tag{2.2}$$

The only distinction to be made is that the momentum p^i associated with a compactified dimension is quantized:

$$p^i = M_i/R\,, \qquad M_i = \text{integer}\,. \tag{2.3}$$

If one were to use a more complicated geometry, described by a metric $\eta_{ij}(X)$ with nontrivial X dependence, then the analysis of eq. (2.1) would become much more complicated.

When $10-D$ dimensions are compactified in this way, the spectrum of states may be interpreted as particles in D dimensions, distinguished in particular by their discrete momentum quantum numbers. The masses in D dimensions are given by

$$\alpha'(\text{mass})^2 = a^2 \sum M_i^2 + N_0\,, \tag{2.4}$$

where

$$a = \sqrt{\alpha'}/R \tag{2.5}$$

is a dimensionless parameter, the M_i's are the discrete momentum quantum numbers, and N_0 is an eigenvalue of the operator [1, 2]

$$N = \sum_{n=1}^{\infty} (\alpha_{-n} \cdot \alpha_n + \tfrac{1}{2} n \bar{S}_{-n} \gamma^- S_n)\,. \tag{2.6}$$

There are no tachyons since N_0 is a non-negative integer.

The mode expansion for the closed-string case is given by

$$X^i_{\text{closed}} = x^i + 2\alpha' p^i \tau + 2N_i R\sigma + \tfrac{1}{2} i \sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} [\alpha^i_n\, \mathrm{e}^{-2in(\tau-\sigma)} + \tilde{\alpha}^i_n\, \mathrm{e}^{-2in(\tau+\sigma)}]\,. \tag{2.7}$$

Here again, the momentum p^i is quantized for compactified dimensions according to eq. (2.3). Moreover, for compactified dimensions a new term $2N_iR\sigma$ has been introduced to describe a string that winds N_i times around the ith circular dimension. (As usual, $0 \leqslant \sigma \leqslant \pi$.) Strings with positive and negative winding numbers N_i are distinguished, since the notion of an orientation is physically meaningful for closed strings [4, 5]. Writing the τ and σ derivatives of eq. (2.7) in the form

$$\partial_\sigma X^i = \sqrt{2\alpha'} \sum_{n=-\infty}^{\infty} [\tilde{\alpha}^i_n\, \mathrm{e}^{-2in(\tau+\sigma)} - \alpha^i_n\, \mathrm{e}^{-2in(\tau-\sigma)}]\,, \tag{2.8a}$$

$$\partial_\tau X^i = \sqrt{2\alpha'} \sum_{n=-\infty}^{\infty} [\tilde{\alpha}^i_n\, \mathrm{e}^{-2in(\tau+\sigma)} + \alpha^i_n\, \mathrm{e}^{-2in(\tau-\sigma)}]\,, \tag{2.8b}$$

leads to the identifications (for compactified dimensions)

$$\alpha_0^i = \frac{1}{\sqrt{2\alpha'}}\left(\alpha'\frac{M_i}{R} - N_i R\right), \tag{2.9a}$$

$$\tilde{\alpha}_0^i = \frac{1}{\sqrt{2\alpha'}}\left(\alpha'\frac{M_i}{R} + N_i R\right). \tag{2.9b}$$

The D-dimensional masses of states are given by

$$\begin{aligned}\alpha_c'(\text{mass})^2 &= \tfrac{1}{2}\sum_i [(\alpha_0^i)^2 + (\tilde{\alpha}_0^i)^2] + N_0 + \tilde{N}_0 \\ &= \tfrac{1}{2}\sum_i (a^2 M_i^2 + a^{-2} N_i^2) + N_0 + \tilde{N}_0,\end{aligned} \tag{2.10}$$

where $\alpha_c' = \frac{1}{2}\alpha'$ is the closed-string Regge slope. As before N_0 is an eigenvalue of the operator in eq. (2.6) and $\tilde{N}_0$ is an eigenvalue of the corresponding operator with tildes. In fact, for physical states it is necessary [1] that $N_0 = \tilde{N}_0$, and thus $N_0 + \tilde{N}_0$ is always an even integer.

The compactification of dimensions described above does not affect the form of tree amplitudes with massless external states. It does have some interesting consequences for the loop corrections, however. For example, the planar loop for four massless open-string states (prior to compactification) is given by kinematic factors times integrals of the form [3, 6]

$$\int_0^1 \frac{dq}{q} \int_0^1 \prod_{i=1}^{3} [d\nu_i \vartheta(\nu_{i+1} - \nu_i)] \prod_{1 \le i < j \le 4} [\psi(\rho_j/\rho_i, w)]^{2\alpha' k_i \cdot k_j}, \tag{2.11}$$

where

$$\rho_i = w^{\nu_i}, \qquad i = 1, 2, 3, \tag{2.12}$$

$$\rho_4 = w = \exp(2\pi^2/\ln q), \tag{2.13}$$

$$\begin{aligned}\psi(\rho, w) &= \frac{1-\rho}{\sqrt{\rho}} \exp\left(\frac{\ln^2 \rho}{2 \ln w}\right) \prod_{n=1}^{\infty} \left[\frac{(1 - w^n/\rho)(1 - w^n \rho)}{(1 - w^n)^2}\right] \\ &= -\frac{2\pi}{\ln q} \sin(\pi\nu) \prod_{n=1}^{\infty} \left[\frac{1 - 2q^{2n}\cos(2\pi\nu) + q^{4n}}{(1 - q^{2n})^2}\right].\end{aligned} \tag{2.14}$$

The logarithmic divergence of eq. (2.11) at $q = 0$ can be cancelled by a renormalization of α'. When some dimensions are compactified, eq. (2.11) must be modified because the momentum integrations associated with the compactified dimensions are replaced by sums. In eq. (2.11) the momentum integrations are already done, so it is necessary to back up a step to isolate their contributions. Taking the external states of the loop amplitude to be massless, which implies in particular according to eq. (2.4) that their discrete momenta are all zero, it is easily seen that for each compactified dimension eq. (2.11) contains a factor

$$\int_{-\infty}^{\infty} w^{\alpha' p^2}\,\mathrm{d}p = \left(\frac{-\pi}{\alpha' \ln w}\right)^{1/2}, \tag{2.15}$$

whereas it should contain

$$\frac{1}{R}\sum_{M=-\infty}^{\infty} w^{\alpha' M^2/R^2} = \frac{1}{R}\vartheta_3\left(0\left|\frac{a^2 \ln w}{i\pi}\right.\right). \tag{2.16}$$

where ϑ_3 is a Jacobi elliptic function. It follows that for each compactified dimension the integrand in eq. (2.11) should be modified by the correction factor

$$\begin{aligned} F_1(a;q) &= a\left(\frac{-\ln w}{\pi}\right)^{1/2}\vartheta_3\left(0\left|\frac{a^2 \ln w}{i\pi}\right.\right) \\ &= \vartheta_3\left(0\left|\frac{-i\ln q}{2\pi a^2}\right.\right) \\ &= \sum_{N=-\infty}^{\infty} q^{N^2/2a^2}. \end{aligned} \tag{2.17}$$

Since $F_1(a, 0) = 1$, the divergent part of eq. (2.11) is not altered, and so the slope renormalization works as before [3].

In the case of the nonplanar loop graphs, some of the ψ's in eq. (2.11) are replaced by ψ_T, where

$$\begin{aligned} \psi_T(\rho, w) &= \frac{1+\rho}{\sqrt{\rho}}\exp\left(\frac{\ln^2 \rho}{2\ln w}\right)\prod_{n=1}^{\infty}\left[\frac{(1+w^n/\rho)(1+w^n\rho)}{(1-w^n)^2}\right] \\ &= -\frac{\pi}{\ln q}q^{-1/4}\prod_{n=1}^{\infty}\left[\frac{1-2q^{2n-1}\cos(2\pi\nu)+q^{4n-2}}{(1-q^{2n})^2}\right]. \end{aligned} \tag{2.18}$$

As a consequence the loop integrand without finite-radius correction factors for the particular graph that gives closed-string poles in the s channel, behaves as $q^{-\alpha_c' s-1}$ times an analytic function of q in the vicinity of $q = 0$. The variable s is the invariant mass squared of the channel with vacuum quantum numbers. Therefore this graph has closed-string poles with masses given by $\alpha_c' s = 0, 2, 4, \ldots$. In the finite-radius case, a correction factor $F_1(a, q)$ must be included for each compactified dimension. Therefore using eq. (2.17) we see that in this case the masses of closed-string poles are given by

$$\alpha_c' s = \frac{1}{2a^2}\sum N_i^2 + N', \tag{2.19}$$

where N' is an even non-negative integer. Comparing with eq. (2.10), we see that closed-string states with all possible winding numbers occur. (It was erroneously concluded in ref. [4] that only even winding numbers arise.) The discrete momenta are not excited, because they are additively conserved and the external open-string states are taken to be massless.

The analysis given above may be repeated for amplitudes with external closed-string states in either theory I or theory II. As before it can be shown that compactifying dimensions does not alter the amplitudes in tree approximation, although the loop amplitudes are once again modified. When closed-string states circulate around a loop, it is necessary to allow them to have nontrivial winding numbers as well as discrete momenta. As an explicit example we recall that prior to compactification the one-loop on-shell amplitude for four massless states of theory II is given by a kinematical factor times [1]

$$\int d^2\tau\,(\mathrm{Im}\,\tau)^{-2}F(\tau)\,, \tag{2.20}$$

where

$$F(\tau)=(\mathrm{Im}\,\tau)^{-3}\int\left(\prod_{i=1}^{3}d^2\nu_i\right)\prod_{1\leq i<j\leq 4}\left(\exp\left[\frac{-\pi(\mathrm{Im}\,\nu_{ji})^2}{\mathrm{Im}\,\tau}\right]|\vartheta_1(\nu_{ji}\,|\,\tau)|\right)^{2\alpha_c' k_i\cdot k_j}. \tag{2.21}$$

In this integral $\nu_{ji}=\nu_j-\nu_i$, $\nu_4=\tau$, and the integration limits are

$$0\leq \mathrm{Im}\,\nu_i\leq \mathrm{Im}\,\tau\,,\qquad -\tfrac{1}{2}\leq \mathrm{Re}\,\nu_i\leq\tfrac{1}{2}\,. \tag{2.22}$$

The function $F(\tau)$ is automorphic, invariant under the modular group

$$\tau\to\frac{a\tau+b}{c\tau+d}\,, \tag{2.23}$$

where a, b, c, and d are integers satisfying $ad-bc=1$. Accordingly, the τ integration in eq. (2.20) is restricted to the fundamental region [7]

$$-\tfrac{1}{2}\leq \mathrm{Re}\,\tau\leq\tfrac{1}{2}\,,\qquad \mathrm{Im}\,\tau\geq 0\,,\qquad |\tau|\geq 1\,. \tag{2.24}$$

It is an essential consistency requirement that the finite-radius correction factor for closed-string loop amplitudes, $F_2(a,\tau)$, also be an automorphic function invariant under the transformation in eq. (2.23).

In order to determine $F_2(a,\tau)$ we recall that for a non-compactified coordinate the momentum integration of a closed-string loop has the form

$$\int_{-\infty}^{\infty} z^{\alpha_0^2/2}\bar{z}^{\tilde{\alpha}_0^2/2}\,dp=(\tau_2\alpha')^{-1/2}\,, \tag{2.25}$$

where we have used the identifications

$$z=e^{2\pi i\tau}\,,\quad \tau=\tau_1+i\tau_2\,, \tag{2.26}$$

$$\alpha_0=\tilde{\alpha}_0=p\sqrt{\tfrac{1}{2}\alpha'} \tag{2.27}$$

[compare eqs. (2.3) and (2.9)]. In the finite-radius case eq. (2.25) must be replaced by discrete summations using eq. (2.9). Therefore, the correction factor is given by

$$F_2(a,\tau)=a\sqrt{\tau_2}\sum_{MN} z^{(aM-N/a)^2/4}\bar{z}^{(aM+N/a)^2/4}$$
$$=a\sqrt{\tau_2}\sum_{MN}\exp\left[-2\pi iMN\tau_1-\pi\tau_2(a^2M^2+N^2/a^2)\right]. \tag{2.28}$$

As we have already said, F_2 must be invariant under the modular group, which is generated by the transformations $\tau\to\tau+1$ and $\tau\to-1/\tau$. Invariance of eq. (2.28) under $\tau\to\tau+1$ is trivial. The nontrivial identity

$$F_2(a,\tau)=F_2(a,-1/\tau) \tag{2.29}$$

is proved in appendix A. We may also note that F_2 has the symmetry

$$F_2(a,\tau)=a^2F_2(1/a,\tau). \tag{2.30}$$

In the limit $a\to 0$ (corresponding to $R\to\infty$ at fixed α')

$$F_2(a,\tau)\underset{a\to 0}{\sim} a\sqrt{\tau_2}\sum_M \exp\left(-\pi\tau_2M^2a^2\right)=a\sqrt{\tau_2}\,\vartheta_3(0\,|\,i\tau_2a^2)$$
$$=\vartheta_3\left(0\left|\frac{i}{\tau_2a^2}\right.\right)\sim 1. \tag{2.31}$$

Thus $F_2\sim 1$ with exponential accuracy in this limit and the $D=10$ result is recovered, even though the phenomenon of winding does not occur for dimensions of infinite range.

To understand R dependences properly it is important to deal with quantities of the correct dimensionality. In ten dimensions the open-string coupling constant (a Yang–Mills coupling constant) has dimensions

$$g_{10}\sim(\text{length})^3. \tag{2.32}$$

The closed-string coupling constant (a gravitational coupling) has dimensions

$$\kappa_{10}\sim(\text{length})^4. \tag{2.33}$$

For a bosonic wave function φ one has

$$\varphi_{10}\sim(\text{length})^{-4}, \tag{2.34}$$

and for a fermionic wave function ψ

$$\psi_{10}\sim(\text{length})^{-9/2}. \tag{2.35}$$

When $10-D$ dimensions are compactified with radius R, the theory is effectively D-dimensional and one should define

$$g_D=g_{10}R^{(D-10)/2}, \tag{2.36a}$$

$$\kappa_D=\kappa_{10}R^{(D-10)/2}, \tag{2.36b}$$

$$\varphi_D = \varphi_{10} R^{(10-D)/2}, \tag{2.37a}$$

$$\psi_D = \psi_{10} R^{(10-D)/2}. \tag{2.37b}$$

In theory I the coupling constants are related by

$$\kappa_{10} = (\text{const}) g_{10}^2/\alpha', \tag{2.38}$$

which on reduction to four dimensions becomes

$$\kappa_4 = (\text{const}) g_4^2 R^3/\alpha'. \tag{2.39}$$

The constant coefficient is determined precisely by the theory, but we have not bothered to sort out all the factors of 2π, etc. If κ_4 is set equal to the Planck length and $g_4 \sim 1$, then eq. (2.39) gives

$$R^3/\alpha' \sim 10^{-33}\ \text{cm}. \tag{2.40}$$

It may be reasonable to assume that the experimental absence of Regge recurrences for elementary particles (spin $\frac{5}{2}$ quarks, etc.) implies that $\sqrt{\alpha'} \leqslant 10^{-15}$ cm or so, in which case eq. (2.40) implies that $R \leqslant 10^{-21}$ cm. We see nothing in the formalism that requires R and $\sqrt{\alpha'}$ to be comparable and hence of the order of the Planck length. In any case it seems interesting to investigate the limit that gives the effective theory at energies below these scales.

When we analyze the $\alpha' \to 0$ limit for loop graphs it is necessary to let $R \to 0$ at the same time. If the $\alpha' \to 0$ limit is taken first one obtains the loop graphs of ten-dimensional supersymmetric Yang–Mills theory, which is UV divergent. On the other hand, letting $R \to 0$ first gives the string theory in a lower dimension, which is also singular. The correct way to take the limits is explained in the following sections.

3. $\mathcal{N} = 4$ Yang–Mills theory as a limit of theory I

In this section we study the behavior of the S-matrix elements for the scattering of massless open-string states in the limit that α' and R approach zero. The analysis of the tree graphs is relatively simple and has already been discussed in ref. [2]. We found that the three-particle couplings coincide with those of field theory before the limit. The four-particle tree amplitudes can be decomposed into a sum of three terms

$$\begin{aligned} T_4^{(1)} = {} & [\operatorname{tr}(\lambda_1\lambda_2\lambda_3\lambda_4) + \operatorname{tr}(\lambda_4\lambda_3\lambda_2\lambda_1)] F_{st} + [\operatorname{tr}(\lambda_1\lambda_2\lambda_4\lambda_3) + \operatorname{tr}(\lambda_3\lambda_4\lambda_2\lambda_1)] F_{su} \\ & + [\operatorname{tr}(\lambda_1\lambda_3\lambda_2\lambda_4) + \operatorname{tr}(\lambda_4\lambda_2\lambda_3\lambda_1)] F_{tu}, \end{aligned} \tag{3.1}$$

where the λ's are $U(n)$ matrices in the fundamental representation describing the $U(n)$ quantum numbers of the scattered particles. We found that in tree approximation

$$F_{st}^{(\text{string theory})}=\frac{\Gamma(1-\alpha' s)\Gamma(1-\alpha' t)}{\Gamma(1-\alpha' s-\alpha' t)}F_{st}^{(\text{field theory})}\,, \tag{3.2}$$

with corresponding relations for F_{su} and F_{tu}. It is therefore trivial at this level that the limit of the string theory in which $\alpha'\to 0$ gives the field theory. Our project here is to extend this analysis to one loop.

There are other group theory factors that first appear at one loop, namely

$$T_4^{(2)}=\operatorname{tr}(\lambda_1\lambda_2)\operatorname{tr}(\lambda_3\lambda_4)G_s+\operatorname{tr}(\lambda_2\lambda_3)\operatorname{tr}(\lambda_4\lambda_1)G_t+\operatorname{tr}(\lambda_1\lambda_3)\operatorname{tr}(\lambda_2\lambda_4)G_u\,, \tag{3.3}$$

$$\begin{aligned}T_4^{(3)}=&\operatorname{tr}(\lambda_1)[\operatorname{tr}(\lambda_2\lambda_3\lambda_4)+\operatorname{tr}(\lambda_4\lambda_3\lambda_2)]H_1+\operatorname{tr}(\lambda_2)[\operatorname{tr}(\lambda_1\lambda_3\lambda_4)+\operatorname{tr}(\lambda_4\lambda_3\lambda_1)]H_2\\&+\operatorname{tr}(\lambda_3)[\operatorname{tr}(\lambda_1\lambda_2\lambda_4)+\operatorname{tr}(\lambda_4\lambda_2\lambda_1)]H_3\\&+\operatorname{tr}(\lambda_4)[\operatorname{tr}(\lambda_1\lambda_2\lambda_3)+\operatorname{tr}(\lambda_3\lambda_2\lambda_1)]H_4\,.\end{aligned} \tag{3.4}$$

Terms involving three trace factors first occur at two loops, and so forth. In the field theory the SU(n) singlet states are decoupled, which implies for the field theory that

$$H_1=H_2=H_3=H_4=-\frac{1}{n}(F_{st}+F_{tu}+F_{su})\,, \tag{3.5a}$$

$$G_s=G_t=G_u=\frac{2}{n}(F_{st}+F_{tu}+F_{su})\,. \tag{3.5b}$$

In tree approximation the G's and H's are absent in the string theory and vanish in the field theory because F_{st} is $(st)^{-1}$ times a totally symmetrical quantity and

$$\frac{1}{st}+\frac{1}{tu}+\frac{1}{su}=0\,. \tag{3.6}$$

In the string theory the singlet states are not decoupled (even in tree approximation), and the relations (3.5) are only expected to hold in the field-theory limit. In fact, if this limit is taken in such a way as to keep Newton's constant finite, then the supergravity multiplet also survives, in which case eq. (3.5) would not apply. The analysis can be extended to processes with more external lines, but this does not appear likely to yield any surprises.

In ref. [3] it is shown that the one-loop correction to the three-particle coupling vanishes. Therefore this remains true in the $\alpha'\to 0$ limit. The first nontrivial case is the one-loop contribution to four-particle amplitudes. It is also shown in ref. [3] that the amplitude F_{st} of eq. (3.1) has the same kinematical factors describing the dependence on polarization vectors and spinors at the one-loop level as at the tree level. (The same kinematical factors also occur in the G's and H's.) The kinematical factor for the case of four vector particles is given explicitly in eq. (3.6) of ref. [3]. The factors for fermions are related by supersymmetry. The easiest way to determine them is to calculate the corresponding process in the tree approximation to ten-dimensional supersymmetrical Yang–Mills theory. Using these factors we may write

$$F_{st} = (\text{kinematical factor}) \times \left(\frac{1}{st} \frac{\Gamma(1-\alpha' s)\Gamma(1-\alpha' t)}{\Gamma(1-\alpha' s-\alpha' t)} + c_1 f^{(1)}(s,t) + \cdots \right), \quad (3.7)$$

where c_1 is a numerical constant (determined by unitarity) and $f^{(1)}$ is given by

$$f^{(1)} = \frac{g_{10}^2}{\alpha'} \int_0^1 \frac{dq}{q} [F_1(a,q)]^{10-D} \int_0^1 \prod_{i=1}^{3} [d\nu_i \vartheta(\nu_{i+1}-\nu_i)]$$

$$\times \left[\frac{\psi_{12}\psi_{34}}{\psi_{13}\psi_{24}} \right]^{-\alpha' s} \left[\frac{\psi_{23}\psi_{14}}{\psi_{13}\psi_{24}} \right]^{-\alpha' t}, \quad (3.8)$$

where we have introduced the notation

$$\psi_{ij} = \psi(\rho_j/\rho_i, w) \quad (3.9)$$

and re-expressed eq. (2.11) including the finite-radius factor $F_1(a,q)$ given in eq. (2.17). Strictly speaking eq. (3.8) should have an infinite counterterm subtracted to cancel the divergence at $q=0$. We have shown in ref. [3] that this divergence corresponds to the emission of a massless scalar closed string at zero momentum. Therefore the contribution of the counterterm disappears in the $\alpha' \to 0$ limit, provided that the limit is taken so as to give vanishing gravitational coupling κ_D, holding g_D fixed. Using eqs. (2.36) and (2.38) this requires

$$\frac{1}{\alpha'} R^{(10-D)/2} \to 0 . \quad (3.10)$$

The asymptotic analysis of eq. (3.8) is easiest to perform for fixed $a = \sqrt{\alpha'}/R$, which combined with eq. (3.10) requires $D<6$. However, this is just a matter of convenience and we will be able to describe larger values of D by continuation. When a is fixed and $D<6$ the leading $\alpha' \to 0$ asymptotic behavior of eq. (3.8) is controlled by the $q=1$ [or $w=0$, see eq. (2.13)] end of the integration region. Using the asymptotic formulas

$$F_1(a,q) \underset{w\to 0}{\sim} a \left(\frac{-\ln w}{\pi} \right)^{1/2}, \quad (3.11)$$

$$\psi(x,w) \underset{w\to 0}{\sim} (1-x)(1-w/x) \exp\left(\frac{-\ln x \ln(w/x)}{2 \ln w} \right), \quad (3.12)$$

where we have allowed for the possibility that x vanishes as w does (but not faster). In terms of the ρ variables [see eqs. (2.12) and (2.13)], eq. (3.8) reduces in this limit to

$$f^{(1)} \sim \frac{g_{10}^2}{\alpha'}(a/\sqrt{\pi})^{10-D} \int_0^1 \frac{dw}{w} \prod_{i=1}^{3} \left[\frac{d\rho_i}{\rho_i} \vartheta(\rho_{i+1}-\rho_i)\right]$$

$$\times(-\ln w)^{-D/2} K \exp\left[\frac{\alpha'}{\ln w}\left(s \ln \rho_1 \ln \frac{\rho_3}{\rho_3} + t \ln \frac{\rho_2}{\rho_1} \ln \frac{w}{\rho_3}\right)\right], \qquad (3.13)$$

where

$$K = \left[\frac{(1-\rho_2/\rho_1)(1-w\rho_1/\rho_2)(1-w/\rho_3)(1-\rho_3)}{(1-\rho_3/\rho_1)(1-w\rho_1/\rho_3)(1-w/\rho_2)(1-\rho_2)}\right]^{-\alpha' s}$$

$$\times\left[\frac{(1-\rho_3/\rho_2)(1-w\rho_2/\rho_3)(1-w/\rho_1)(1-\rho_1)}{(1-\rho_3/\rho_1)(1-w\rho_1/\rho_3)(1-w/\rho_2)(1-\rho_2)}\right]^{-\alpha' t}. \qquad (3.14)$$

Introducing variables

$$\lambda = -\ln w\,, \qquad (3.15a)$$

$$\eta_1 = \ln \rho_1/\ln w\,, \qquad (3.15b)$$

$$\eta_i = \frac{\ln(\rho_i/\rho_{i-1})}{\ln w}\,, \qquad i = 2, 3, 4\,, \qquad (3.15c)$$

eq. (3.13) becomes

$$f^{(1)} \sim \frac{g_{10}^2}{\alpha'}(a/\sqrt{\pi})^{10-D} \int_0^\infty d\lambda \int_0^1 \left(\prod_{i=1}^{4} d\eta_i\right)\delta\left(1-\sum \eta_i\right)\lambda^{3-D/2}$$

$$\times K \exp\left[-\alpha'\lambda(\eta_1\eta_3 s + \eta_2\eta_4 t)\right]. \qquad (3.16)$$

The leading asymptotic behavior as $\alpha' \to 0$ is controlled by the $\lambda \to \infty$ end of the integration region provided that $D < 8$. This is true because in this limit K may be replaced by one. With this replacement the λ integration may be performed, giving

$$f_0^{(1)} = g_D^2 \pi^{\gamma-1} \Gamma(-\gamma) \int_0^1 \left(\prod_{i=1}^{4} d\eta_i\right)\delta(1-\sum \eta_i)(\eta_1\eta_3 s + \eta_2\eta_4 t)^\gamma\,, \qquad (3.17)$$

where we have used eqs. (2.36a) and (2.5) and introduced the parameter

$$\gamma = \tfrac{1}{2}D - 4\,. \qquad (3.18)$$

It is important that the dependence on the dimensionless parameter a has cancelled since it has no place in the limiting Yang–Mills field theory. It is shown in appendix B that eq. (3.17) may be simplified to the form

$$f_0^{(1)}(s, t) = g_D^2 c(\gamma)[I_\gamma(s, t) + I_\gamma(t, s)] \qquad (3.19)$$

where

$$c(\gamma)=\tfrac{1}{4}(\pi/4)^{\gamma+1/2}[\sin(\pi\gamma)\Gamma(\gamma+\tfrac{5}{2})]^{-1}, \tag{3.20}$$

$$I_\gamma(s,t)=t^{\gamma+1}\int_0^1 \frac{(1-x)^{\gamma+1}\,\mathrm{d}x}{sx-t(1-x)}. \tag{3.21}$$

It is useful to regard these expressions as analytic functions in γ. This is a sort of supersymmetric dimensional regularization – the continuation being performed in the number of compactified dimensions. Before examining $\gamma\to-2$ (corresponding to $D=4$), we first consider the easier case $\gamma\to-1$ (corresponding to $D=6$). Expanding I_γ about $\gamma=-1$,

$$I_\gamma(s,t)\underset{\gamma\to-1}{\sim}\int_0^1\frac{\mathrm{d}x}{sx-t(1-x)}+(\gamma+1)\int_0^1\frac{\ln[t(1-x)]\,\mathrm{d}x}{sx-t(1-x)}+\cdots, \tag{3.22}$$

the first term in eq. (3.22) is odd under interchange of s and t and therefore does not contribute to $f_0^{(1)}$. We therefore have

$$f_0^{(1)}(D=6)=\frac{g_6^2}{\pi^2}\int_0^1\frac{\ln(sx)-\ln[t(1-x)]}{sx-t(1-x)}\,\mathrm{d}x, \tag{3.23}$$

which is a finite expression. This shows that the one-loop approximation to the four-particle S-matrix of $N=4$ Yang–Mills theory in $D=6$ is free from both UV and IR divergences.

Now let us examine the limit $\gamma\to-2$ appropriate to four dimensions. One must be careful because I_γ diverges at $\gamma=-2$. Extracting the singularity,

$$\begin{aligned}I_\gamma(s,t)&=\frac{t^{\gamma+1}}{(\gamma+2)s}-\frac{u}{st}I_{\gamma+1}(s,t)\\&\underset{\gamma\to-2}{\sim}\frac{1}{st}\left[\frac{1}{\gamma+2}+\ln t-u\int_0^1\frac{\mathrm{d}x}{sx-t(1-x)}\right.\\&\qquad\left.+(\gamma+2)\left(\tfrac{1}{2}\ln^2 t-u\int_0^1\frac{\ln[t(1-x)]\,\mathrm{d}x}{sx-t(1-x)}\right)+\cdots\right].\end{aligned} \tag{3.24}$$

Therefore, letting $\varepsilon=\gamma+2$ and using $g_D=g_4R^\varepsilon$,

$$\begin{aligned}f_0^{(1)}\underset{\varepsilon\to0}{\sim}g_4^2R^{2\varepsilon}\frac{c(\gamma)}{st}&\left[\frac{2}{\varepsilon}+\ln(st)+\tfrac{1}{2}\varepsilon(\ln^2 s+\ln^2 t)\right.\\&\left.+\varepsilon u\int_0^1\frac{\ln(sx)-\ln[t(1-x)]}{sx-t(1-x)}\,\mathrm{d}x\right].\end{aligned} \tag{3.25}$$

Since $c(\gamma)$ has a pole at $\varepsilon=0$, this diverges as ε^{-2}, which is characteristic of the one-loop IR singularity of Yang–Mills theory in $D=4$. In calculating the cross section, the divergence presumably cancels with soft real emissions in the usual fashion.

An interesting feature of the dimensionally-continued one-loop amplitude in the $\alpha' \to 0$ limit is that it is finite for $4 < D < 8$. Assuming the kinematical factor in eq. (3.7) remains the same in all orders (as is presumably required by supersymmetry), a suggestive dimensional argument leads to the conjecture that the n-loop amplitude is finite for $4 < D < 4 + 4/n$. The upper limit would correspond to the onset of UV divergences, whereas $D = 4$ is the onset of IR divergences. If this is correct, then the $N = 4$ Yang–Mills S matrix in four dimensions is UV finite to all orders.

To complete the one-loop analysis of the four-particle S-matrix, we must still study the $\alpha' \to 0$ limit of the nonplanar graphs that give the G and H amplitudes introduced in eqs. (3.3) and (3.4). The formulas differ from eq. (3.8) only by the substitution of ψ_{T} for certain of the ψ's and the extension of the integration region to the entire ν cube, according to the results given in ref. [6]. There is no new analysis to be performed, because $w \to 0$ is still the relevant end of the integration region and

$$\psi_{\mathrm{T}}(x, w) \underset{w \to 0}{\sim} (1+x)(1+w/x) \exp\left(\frac{-\ln x \ln (w/x)}{2 \ln w}\right). \tag{3.26}$$

As before the factors in front of the exponential do not contribute to the leading behavior as $\alpha' \to 0$, and therefore in the limit $\psi_{\mathrm{T}}(x, w)$ is equivalent to $\psi(x, w)$. Thus when $\alpha' \to 0$ the G's and H's are given by eq. (3.5), as required for the Yang–Mills field theory.

4. $N = 8$ supergravity theory as a limit of theory II

Two closed-string theories with extended supersymmetry in 10 dimensions were formulated in ref. [1]. They differ primarily in the relative handedness of the two gravitino particles, which has straightforward implications for the rest of the spectrum. These differences appear in the S-matrix elements entirely in the "kinematical" factors, and not in the analytic structure of the formulas. Indeed it appears that as soon as any dimension is compactified, reducing the spatial symmetry to SO(8) or less, the two theories become completely indistinguishable. Therefore, we refer to them in the following as a single theory. The analysis of $\alpha' \to 0$ limits for theory II is actually easier than for theory I in two respects First of all there is no internal symmetry group – the entire spectrum consists of singlets. Accordingly the n-loop contribution to an S-matrix element for an arbitrary number of particles is given by a single diagram (a remarkable simplification compared to the limiting field theory!). The second feature that makes this theory easier to analyze than theory I is the absence of divergences in one-loop graphs. At this order no renormalization of α', κ, or wave functions is required. (We expect this to be true to all orders.)

Just as in the theory I, the three-particle couplings of the massless states coincide for the string theory and the limiting field theory ($N = 8$ supergravity in 10 or fewer dimensions) in tree approximation. Also there is no contribution to the three-particle coupling (on shell) from the one-loop graphs of the string theory, and hence for the limiting field theory as well. The four-particle S-matrix elements share with theory I the feature of having common kinematical factors for trees and one-loop graphs. The complete four-particle amplitude is given by

$$(\text{kinematical factor})\times\left(\frac{1}{stu}\frac{\Gamma(1-\frac{1}{2}\alpha' s)\Gamma(1-\frac{1}{2}\alpha' t)\Gamma(1-\frac{1}{2}\alpha' u)}{\Gamma(1+\frac{1}{2}\alpha' s)\Gamma(1+\frac{1}{2}\alpha' t)\Gamma(1+\frac{1}{2}\alpha' u)}+c_1 g^{(1)}+\cdots\right), \quad (4.1)$$

where c_1 is a calculable number (again determined by unitarity). The kinematical factor may be deduced either by calculating the field theory in tree approximation or by "doubling"* those of theory I. The one-loop contribution $g^{(1)}$ is given by

$$g^{(1)}=\frac{\kappa_{10}^2}{\alpha'}\int d^2\tau\,(\mathrm{Im}\,\tau)^{-2}F(\tau)[F_2(a,\tau)]^{10-D}, \quad (4.2)$$

where we have inserted the finite-radius correction factor given in eq. (2.28) into eq. (2.20). The formula for $F(\tau)$ is given in eq. (2.21).

As in the previous section the asymptotic analysis of eq. (4.2) in the limit $\alpha'\to 0$ is most conveniently done for fixed a. The algebra is very similar to that of the previous section. One finds that the limit is singular for $D\geq 8$, while for $D<8$ the dominant contribution is associated with the $\tau_2=\mathrm{Im}\,\tau\to\infty$ end of the integration region. The result (see appendix C) is that $g^{(1)}$ asymptotically approaches

$$g_0^{(1)}=\kappa_D^2 c(\gamma)[I_\gamma(s,t)+I_\gamma(t,s)+I_\gamma(s,u)+I_\gamma(u,s)+I_\gamma(t,u)+I_\gamma(u,t)], \quad (4.3)$$

with $c(\gamma)$ and $I_\gamma(s,t)$ as given in eqs. (3.20) and (3.21) (a remarkable coincidence!). Therefore, just as before, the one-loop contribution to the $N=8$ supergravity four-particle S-matrix element is finite for $4<D<8$. Using the expression in eq. (3.24) to study the $D=4$ limit one sees using eq. (3.6) that the ε^{-2} divergence cancels. The remaining IR divergence is of the form ε^{-1}, weaker in the gravity theory than in the Yang–Mills theory, as expected.

As in the previous section, if we assume that the kinematical factor in eq. (4.1) is the same at all orders in the loop expansion, then a dimensional argument suggests that in n loops the onset of UV divergences occurs at $D=2+6/n$. If correct, this would imply that $N=8$ supergravity is UV singular at three loops in $D=4$. This is the same possibility that has been suggested by considering possible supersymmetrical counterterms [8]. This divergence is presumably a consequence of letting $\alpha'\to 0$ and $R\to 0$, and would not imply that the string theory itself is singular.

* Corresponding to a kinematical factor $\zeta_1^{\mu_1}\zeta_2^{\mu_2}\zeta_3^{\mu_3}\zeta_4^{\mu_4}L_{\mu_1\mu_2\mu_3\mu_4}$ in theory I, one has $\zeta_1^{\mu_1\nu_1}\zeta_2^{\mu_2\nu_2}\zeta_3^{\mu_3\nu_3}\zeta_4^{\mu_4\nu_4}L_{\mu_1\mu_2\mu_3\mu_4}L_{\nu_1\nu_2\nu_3\nu_4}$ in theory II. There are similar rules when fermions are involved.

Another theory that could also be investigated in the $\alpha' \to 0$, $R \to 0$ limit is the closed-string sector of theory I, which corresponds to an $N = 4$ truncation of theory II. We have looked at this problem only very superficially, but it appears that the symmetrizations required by the truncation complicate matters. As a consequence certain features, such as the vanishing of the on-shell three-particle loop, are not manifest in this case.

5. Discussion

We have shown how to incorporate compactified dimensions into each of the two supersymmetrical string theories. The limit in which the Regge slope and the radii of the compactified dimensions each approach zero was studied at the one-loop level and shown to yield explicit formulas for the corresponding S-matrix elements of the limiting field theory. For the open-string theory (theory I) we found that the limit is $N = 4$ supersymmetric Yang–Mills theory, which at one loop is UV finite for $D < 8$ and IR finite for $D > 4$. Dimensional considerations led to the speculation that at n loops the limiting field theory should be UV finite for $D < 4 + 4/n$, which if true would imply the UV finiteness of the $D = 4$ theory to all orders. For closed-string theories with extended supersymmetry in ten dimensions (theory II), we found that the limiting field theory is $N = 8$ supergravity, which at one loop is also UV finite for $D < 8$ and IR finite for $D > 4$. The infrared singularity as $D \to 4$ is milder than in the Yang–Mills case. The same sort of dimensional considerations that suggest $N = 4$ Yang–Mills theory is UV finite, lead to the speculation that at n loops the limiting $N = 8$ supergravity theory is UV finite only when $D < 2 + 6/n$, which would imply that there are UV divergences in four dimensions starting at three loops. Although further development of the formalism is required, it ought to be possible to calculate multiloop string amplitudes explicitly enough to test these speculations about the limiting field theories, as well as to establish the renormalizability or finiteness of the string theories themselves.

Theory II is more constrained than theory I inasmuch as it does not contain the freedom to choose a gauge group. Another distinction is that the reasoning that gives eq. (2.40) does not apply to theory II. Even though the possible mechanisms are not at all clear, it is conceivable that the structure of the compactified dimensions is determined dynamically. If this should be the case, then either theory would contain just a single fundamental dimensionless parameter, namely $(\alpha')^2/\kappa_{10}$. Of course, it would be interesting to investigate the possibilities for compactification on spaces that are not just products of circles, although this seems to be technically difficult.

It seems possible that both supersymmetric string theories give a well-defined quantum theory containing gravity and enough structure to have a chance of describing nature. If supergravity field theories diverge (as we suspect), then apparently no conventional field theory would have this property. In order to decide whether theory I or theory II is the more promising candidate for phenomenology, some further theoretical developments may be necessary. For example, it is clearly important to incorporate symmetry breaking – perhaps along the lines suggested in ref. [9]. It may also be necessary to develop nonperturbative techniques.

Appendix A

We wish to prove that

$$F_2(a,\tau)=a\sqrt{\tau_2}\sum_{M,N=-\infty}^{\infty}\exp\left[-2\pi iMN\tau_1-\pi\tau_2(a^2M^2+N^2/a^2)\right], \tag{A.1}$$

where $\tau=\tau_1+i\tau_2$, satisfies

$$F_2(a,\tau)=F_2(a,-1/\tau). \tag{A.2}$$

This result may be proved by standard methods in the theory of Jacobi ϑ series. One first re-expresses (A.1) in a matrix notation

$$F_2(a,\tau)=a\sqrt{\tau_2}\sum_{\{M\}}\exp\left(-\pi M^{\mathrm{T}}AM\right), \tag{A.3}$$

where

$$M=\begin{pmatrix}M_1\\M_2\end{pmatrix}, \tag{A.4}$$

$$A=\begin{bmatrix}\tau_2a^2 & i\tau_1\\ i\tau_1 & \tau_2/a^2\end{bmatrix}. \tag{A.5}$$

Then one defines the more general expression

$$F_2(a,\tau;x)=a\sqrt{\tau_2}\sum_{\{M\}}\exp\left[-\pi(M+x)^{\mathrm{T}}A(M+x)\right], \tag{A.6}$$

where

$$x=\begin{pmatrix}x_1\\x_2\end{pmatrix}. \tag{A.7}$$

This function is obviously periodic in x_1 and x_2 with unit period. Therefore it may be expanded in a double Fourier series. The calculation of the Fourier coefficients just involves gaussian integrals and gives

$$F_2(a,\tau;x)=\frac{a\sqrt{\tau_2}}{\sqrt{\det A}}\sum_{\{M\}}\exp\left(-\pi M^{\mathrm{T}}A^{-1}M+2\pi iM^{\mathrm{T}}x\right). \tag{A.8}$$

Setting $x=0$ and noting that

$$A^{-1}=A(-1/\tau)\,, \qquad \text{(interchanging } M_1 \text{ and } M_2)\,, \tag{A.9}$$

$$\det A=|\tau|^2\,, \tag{A.10}$$

$$\operatorname{Im}(-1/\tau)=\tau_2/|\tau|^2\,, \tag{A.11}$$

eq. (A.2) follows.

Appendix B

We wish to simplify the integral

$$f_0^{(1)}=g_D^2\pi^{\gamma-1}\Gamma(-\gamma)\int_0^1\left(\prod \mathrm{d}\eta_i\right)\delta(1-\sum\eta_i)(\eta_1\eta_3 s+\eta_2\eta_4 t)^{\gamma}\,. \tag{B.1}$$

Doing one trivial and one elementary integration

$$f_0^{(1)}=\frac{g_D^2\pi^{\gamma-1}\Gamma(-\gamma)}{\gamma+1}\int_0^1\mathrm{d}\eta_2\int_0^{1-\eta_2}\mathrm{d}\eta_3\,(1-\eta_2-\eta_3)^{\gamma+1}\left[\frac{(\eta_2 t)^{\gamma+1}-(\eta_3 s)^{\gamma+1}}{\eta_2 t-\eta_3 s}\right], \tag{B.2}$$

it is easy to see that the two terms are related by st interchange, so setting $x=\eta_2$ and $y=\eta_3/(1-\eta_2)$,

$$f_0^{(1)}=\frac{g_D^2\pi^{\gamma-1}\Gamma(-\gamma)}{\gamma+1}\int_0^1\mathrm{d}x\,\mathrm{d}y\frac{(1-x)^{\gamma+2}(1-y)^{\gamma+1}(xt)^{\gamma+1}}{xt-(1-x)ys}+(s\leftrightarrow t)\,. \tag{B.3}$$

Expanding the denominator in a geometric series,

$$[xt-(1-x)ys]^{-1}=\sum_{n=0}^{\infty}\frac{[(1-x)ys]^n}{(xt)^{n+1}}\,, \tag{B.4}$$

gives for $f_0^{(1)}$

$$\begin{aligned}
&\frac{g_D^2\pi^{\gamma-1}\Gamma(-\gamma)}{\gamma+1}\sum_{n=0}^{\infty}t^{\gamma}(s/t)^nB(\gamma-n+1,\gamma+n+3)B(n+1,\gamma+2)+(s\leftrightarrow t)\\
&=\frac{g_D^2\pi^{\gamma-1}\Gamma(-\gamma)\Gamma^2(\gamma+2)}{(\gamma+1)\Gamma(2\gamma+4)}t^{\gamma}\sum_{n=0}^{\infty}(s/t)^nB(n+1,\gamma-n+1)+(s\leftrightarrow t)\\
&=g_D^2\frac{(\pi/4)^{\gamma+\frac{1}{2}}t^{\gamma+1}}{4\Gamma(\gamma+5/2)\sin\pi\gamma}\int_0^1(1-x)^{\gamma+1}[sx-t(1-x)]^{-1}\,\mathrm{d}x+(s\leftrightarrow t)\\
&=g_D^2c(\gamma)[I_{\gamma}(s,t)+I_{\gamma}(t,s)]\,,
\end{aligned} \tag{B.5}$$

with c and I_{γ} as given in eqs. (3.20) and (3.21).

Appendix C

We wish to show that for $D<8$ and $\alpha'\to 0$ limit of $g^{(1)}$ given in eq. (4.2), taken at fixed $a=\sqrt{\alpha'}/T$, is $g_0^{(1)}$ as given in eq. (4.3). In the limit $\operatorname{Im}\tau=\tau_2\to\infty$, one has from eq. (2.28)

$$F_2(a,\tau)\sim a\sqrt{\tau_2}\,. \tag{C.1}$$

Also, by reasoning analogous to that of sect. 3, one may asymptotically use

$$\vartheta_1(\nu_{ij},\tau)\sim 2\sin\pi\nu_{ij}\,. \tag{C.2}$$

Thus

$$g^{(1)}\sim a^{10-D}\frac{\kappa_{10}^2}{\alpha'}\int \mathrm{d}^2\tau\,\tau_2^{-D/2}\int\left(\prod_{i=1}^{3}\mathrm{d}^2\nu_i\right)S^{-\alpha's}T^{-\alpha't}\,, \tag{C.3}$$

with

$$S=\exp\left(\frac{-\pi}{\tau_2}[(\operatorname{Im}\nu_{12})^2+(\operatorname{Im}\nu_{34})^2-(\operatorname{Im}\nu_{13})^2-(\operatorname{Im}\nu_{24})^2]\right)\frac{\sin\pi\nu_{12}\sin\pi\nu_{34}}{\sin\pi\nu_{13}\sin\pi\nu_{24}}\,, \tag{C.4}$$

$$T=\exp\left(\frac{-\pi}{\tau_2}[(\operatorname{Im}\nu_{23})^2+(\operatorname{Im}\nu_{14})^2-(\operatorname{Im}\nu_{13})^2-(\operatorname{Im}\nu_{24})^2]\right)\frac{\sin\pi\nu_{23}\sin\pi\nu_{14}}{\sin\pi\nu_{13}\sin{}_\pi\nu_{24}}\,. \tag{C.5}$$

The integration on the real parts of the ν's and τ becomes trivial asymptotically, just giving unity. Then defining variables ρ_i by

$$\operatorname{Im}\nu_i=\rho_i\tau_2\,, \tag{C.6}$$

so that $0\leqslant\rho_1,\rho_2,\rho_3\leqslant 1$ and $\rho_4=1$ [see eq. (2.22)], one is left with an integral of the form

$$\begin{aligned} g^{(1)}&\sim a^{10-D}\frac{\kappa_{10}^2}{\alpha'}\int_0^1\mathrm{d}\rho_1\,\mathrm{d}\rho_2\,\mathrm{d}\rho_3\int_1^\infty\mathrm{d}\tau_2\,(\tau_2)^{3-D/2}\,\mathrm{e}^{-A\alpha'\tau_2}\\ &\sim\kappa_D^2\Gamma(-\gamma)\int_0^1\mathrm{d}\rho_1\,\mathrm{d}\rho_2\,\mathrm{d}\rho_3\,[A]^\gamma\,, \end{aligned} \tag{C.7}$$

where $\gamma=D/2-4$. In performing the τ_2 integration we have assumed that the asymptotic result does not depend on the lower limit of integration. This is the case provided that $\gamma<0 (D<8)$. The quantity A in eq. (C.7) arises by replacing the $\sin\pi\nu$'s in eqs. (C.4) and (C.5) by $\pm(1/2i)\,\mathrm{e}^{\pm i\pi\nu}$ according to whether $\operatorname{Im}\nu$ approaches $\pm\infty$. Therefore the analytic form of A depends on the ordering of the ρ variables. One finds

$$A=s\rho_1\rho_2+t\rho_2\rho_3+u\rho_3\rho_1+A_1\,, \tag{C.8}$$

where

$$A_1 = \begin{cases} t(\rho_1-\rho_2) & \rho_1<\rho_2<\rho_3 \\ t(\rho_1-\rho_3) & \rho_1<\rho_3<\rho_2 \\ u(\rho_2-\rho_1) & \rho_2<\rho_1<\rho_3 \\ u(\rho_2-\rho_3) & \rho_2<\rho_3<\rho_1 \\ s(\rho_3-\rho_1) & \rho_3<\rho_1<\rho_2 \\ s(\rho_3-\rho_2) & \rho_3<\rho_2<\rho_1 \end{cases} \quad \text{for} \quad . \tag{C.9}$$

As a result eq. (C.7) becomes asymptotically

$$g_0^{(1)} = \kappa_D^2 \Gamma(-\gamma) \int_0^1 d\rho_3 \int_0^{\rho_3} d\rho_2 \int_0^{\rho_2} d\rho_1 \, [s\rho_1\rho_2 + t\rho_2\rho_3 + u\rho_3\rho_1 + t(\rho_1-\rho_2)]^\gamma$$

$$+ 5 \text{ terms that symmetrize } s, t, \text{ and } u\,. \tag{C.10}$$

The rest of the analysis is the same as in appendix B.

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