Explicit Evaluation of Anomalies in Higher Dimensions

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The one-loop Kaluza-Klein anomaly is evaluated explicitly for gauge theories in six, eight, and ten dimensions. The result is well defined and unique, despite the nonrenormalizability of the theory.

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Strings have played a role in many branches of physics from polyatomic molecules to vortices in type-II superconductors. Recently the space-time analog of vortices has entered cosmology (cosmic strings) in the theory of fluctuations and galaxy formation. It is highly probable that strings may soon play a role in particle physics. There has long been the curiosity that the $D=10$ string theory may be finite and recent advances in the superstring have demonstrated the absence of tachyons, etc. Since $D=10$ dimensions is *required* for the string, a successful dimensional compactification of this theory could lead to a viable quantum gravity.

It is generally believed that all elementary-particle interactions *except* gravity are describable by non-Abelian gauge theories. Such gauge theories share with quantum electrodynamics the property of perturbative renormalizability and hence complete calculability for small couplings. This domain of applicability covers the electroweak forces for any normal energy (less than a few gigaelectronvolts) and the strong forces at high energy (more than a few gigaelectronvolts).

Historically the realization that gauge theories are renormalizable, even with the gauge group spontaneously broken, focused attention on the subject of chiral anomalies since the latter provided the only known consistency condition required for gauge theories. The anomalies had been studied much earlier and appeared then as only an arcane curiosity, but in gauge theories they become of central importance: The chiral fermions *must* be such that the anomaly cancels, otherwise the gauge theory is inconsistent and hence meaningless.

Now we wish to consider[1] a gauge theory in more than four dimensions, generally in an even number of dimensions such as 6, 8, 10, ... Such a theory is *a priori* nonrenormalizable and thus inconsistent and meaningless. The reasons for considering this inconsistent theory are two in number: First, contrary to expectations, we shall find that the one-loop anomaly in such a theory is well defined and unique, and independent of the severe ultraviolet divergences associated with nonrenormalizability. Second, this kind of nonrenormalizable gauge theory can arise as the limit of small Regge slope of a string model. We have in mind, for example, the ten-dimensional supersymmetric string.[2] Here there is the hope that the string theory is completely

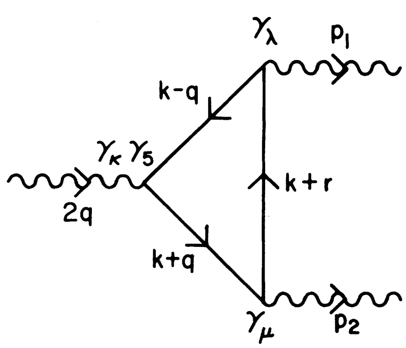

FIG. 1. Triangle diagram for $D=4$.

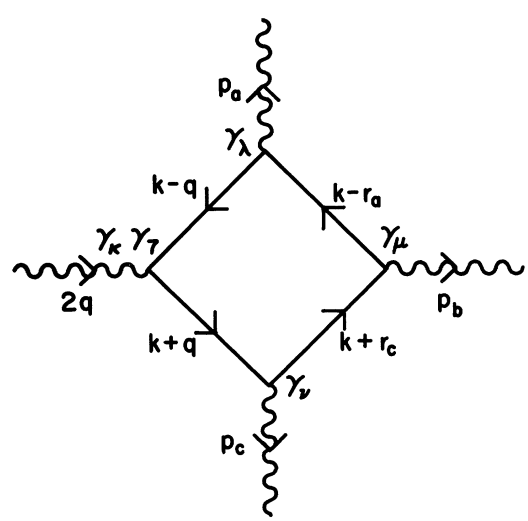

FIG. 2. Box diagram for $D=6$.

finite and hence all the ultraviolet divergences associated with the nonrenormalizability of the low-energy gauge theory are successfully cut off by the very-high-mass Regge recurrences and daughters.

In discussing the string theory, we must emphasize that although the ultraviolet divergences may be modified by the very-high-mass particles in the theory, the anomalies are *not* so modified. This is because, as has been shown elsewhere,[3] the anomaly receives contributions only from zero-mass fermions. This is why the anomalies in nonrenormalizable Kaluza-Klein gauge theory are interesting: First, they are well defined despite nonrenormalizability, and second, they are germane to finite higher-dimensional string theory.

In what follows, we shall first consider anomalies only in Abelian theories. The generalization to non-Abelian theory will be discussed later; it involves no new complication.

Let us first briefly recall the case of the triangle anomaly[4] for $D=4$. The relevant three-point function is written

$$V_{\kappa\lambda\mu}(p_1 p_2) = S\Gamma_{\kappa\lambda\mu}(p_1 p_2) + (a/8\pi^2)\epsilon_{\kappa\lambda\mu\alpha}(p_{2\alpha} - p_{1\alpha}), \tag{1}$$

where $\Gamma_{\kappa\lambda\mu}$ is the Feynman integral for Fig. 1 $[q=\frac{1}{2}(p_1+p_2),\ r=\frac{1}{2}(p_1-p_2)]$,

$$\Gamma_{\kappa\lambda\mu}(p_1 p_2) = \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{Tr}[\gamma_\lambda(\not{k}-\not{q})\gamma_\kappa\gamma_5(\not{k}+\not{q})\gamma_\mu(\not{k}+\not{r})]}{(k-q)^2(k+q)^2(k+r)^2}, \tag{2}$$

and $S\Gamma_{\kappa\lambda\mu}$ implies Bose symmetrization of $(p_{1\lambda}, p_{2\mu})$. Setting $p_{1\lambda}V_{\kappa\lambda\mu}=0$ requires $a=+1$ in Eq. (1), whereupon

$$2q_\kappa V_{\kappa\lambda\mu} = 2^3 X_3 \epsilon_{\lambda\mu\alpha\beta}\, p_{1\alpha} p_{2\beta} \tag{3}$$

with

$$X_3 = +1/2^4\pi^2. \tag{4}$$

The factor 2^3 in Eq. (3) takes account of the three $(1+\gamma_5)/2$ projection operators needed in Eq. (2).

These steps can be carried out for $D>4$. For $D=6$, we write the four-point vertex

$$V_{\kappa\lambda\mu\nu}(p_a p_b p_c) = S\Gamma_{\kappa\lambda\mu\nu}(p_a p_b p_c) + (a/2^4\pi^3)\epsilon_{\kappa\lambda\mu\nu\beta}(p_{b\alpha}p_{c\beta} + p_{c\alpha}p_{a\beta} + p_{a\alpha}p_{b\beta}). \tag{5}$$

Here the Feynman amplitude $\Gamma_{\kappa\lambda\mu\nu}(p_a p_b p_c)$ for Fig. 2 is given by

$$\Gamma_{\kappa\lambda\mu\nu}(p_a p_b p_c) = \int \frac{d^6k}{(2\pi)^6} \frac{\mathrm{Tr}[\gamma_\lambda(\not{k}-\not{q})\gamma_\kappa\gamma_7(\not{k}+\not{q})\gamma_\nu(\not{k}+\not{r}_c)\gamma_\mu(\not{k}-\not{r}_a)]}{(k-q)^2(k+q)^2(k+r_c)^2(k-r_a)^2}. \tag{6}$$

Here $q=\frac{1}{2}(p_a+p_b+p_c)$ and $r_a=(q-p_a)$, etc. To evaluate $p_{a\lambda}\Gamma_{\kappa\lambda\mu\nu}$ one uses

$$\not{p}_a = -(\not{k}-\not{q}) + (\not{k}-\not{r}_a). \tag{7}$$

The second term in Eq. (7) gives zero in conjunction with Eq. (6) since a third-rank pseudotensor depending on only two six-momenta must vanish. The first term in (6) gives

$$p_{a\lambda}\Gamma_{\kappa\lambda\mu\nu} = -\int \frac{d^6k}{(2\pi)^6} \frac{\mathrm{Tr}[\gamma_\kappa\gamma_7(\not{k}+\not{q})\gamma_\nu(\not{k}+\not{r}_c)\gamma_\mu(\not{k}-\not{r}_a)]}{D_+^{abc}(k)}, \tag{8}$$

where the denominator is in the notation

$$D_\pm^{abc} = (k\pm q)^2(k+r_c)^2(k-r_a)^2. \tag{9}$$

Shifting momentum according to $k'=k-p_a$ gives

$$D_+^{abc}(k') = D_-^{bca}(k) = (k-q)^2(k+r_a)^2(k-r_b)^2, \tag{10}$$

where D_-^{bca} is the corresponding denominator occurring in the contraction $p_{a\lambda}\Gamma_{\kappa\mu\nu\lambda}(p_b p_c p_a)$. Combining these terms and taking account of the surface term arising from the integration being linearly divergent gives

$$p_{a\lambda}(\Gamma_{\kappa\lambda\mu\nu} + \Gamma_{\kappa\mu\nu\lambda}) = -(48\pi^3)^{-1}\epsilon_{\kappa\mu\nu\alpha\beta\gamma}\, p_{a\gamma} p_{b\beta} p_{c\gamma}. \tag{11}$$

Noting that this is totally Bose symmetric and that $p_{a\,\lambda}\Gamma_{\kappa\mu\,\lambda\nu}=p_{a\,\lambda}\Gamma_{\kappa\,\nu\,\lambda\mu}=0$ gives

$$p_{a\,\lambda}S\Gamma_{\kappa\,\lambda\mu\,\nu}=-(24\pi^3)^{-1}\epsilon_{\kappa\mu\,\nu\alpha\beta\,\gamma}p_{a\,\alpha}p_{b\,\beta}p_{c\,\gamma}. \tag{12}$$

Thus, in Eq. (5), we need $a=+1$ to ensure $p_{a\,\lambda}V_{\kappa\,\lambda\mu\,\nu}=0$. To evaluate $2q_\kappa\Gamma_{\kappa\,\lambda\mu\,\nu}$ we use a similar procedure because the same denominators occur. For example,

$$2q_\kappa\Gamma_{\kappa\,\lambda\mu\,\nu}\simeq\frac{8i}{(2\pi)^6}\int d^6k\,\epsilon_{\lambda\mu\,\nu\alpha\beta\,\gamma}\left[\frac{-p_{b\,\beta}p_{c\,\gamma}}{D_+{}^{abc}(k)}+\frac{p_{a\,\beta}p_{b\,\gamma}}{D_-{}^{abc}(k)}\right] \tag{13}$$

allows us to use again $D_+{}^{abc}(k')=D_-{}^{bca}(k)$, with $k'=k-p_a$, and hence combine with another term in $2q_\kappa S\Gamma_{\kappa\,\lambda\mu\,\nu}$. The overall result is

$$2q_\kappa S\Gamma_{\kappa\,\lambda\mu\,\nu}=-(8\pi^3)^{-1}\epsilon_{\lambda\mu\,\nu\alpha\beta\,\gamma}p_{a\,\alpha}p_{b\,\beta}p_{c\,\gamma}, \tag{14}$$

and finally the square anomaly for $D=6$, normalized by

$$2q_\kappa V_{\kappa\,\lambda\mu\,\nu}(p_ap_bp_c)=2^4X_4\epsilon_{\lambda\mu\,\nu\alpha\beta\,\gamma}p_{a\,\alpha}p_{b\,\beta}p_{c\,\gamma}, \tag{15}$$

is

$$X_4=-2^{-6}\pi^{-3}. \tag{16}$$

Of relevance to the $D=10$ string[1,2] is the hexagon anomaly which we have calculated to the one-loop level. In an obvious generalization of the notation used above we have

$$V_{\kappa\,\lambda\mu\,\nu\rho\sigma}(p_ap_bp_cp_dp_e)=S\Gamma_{\kappa\,\lambda\mu\,\nu\rho\sigma}(p_a\,p_bp_cp_dp_e)+(160\pi^5)^{-1}\epsilon_{\kappa\,\lambda\mu\,\nu\rho\sigma\alpha\beta\,\gamma\delta}(p_{b\,\alpha}p_{c\,\beta}p_{d\,\gamma}p_{e\,\delta}+\ldots). \tag{17}$$

In $\Gamma_{\kappa\,\lambda\mu\,\nu\rho\sigma}(p_ap_bp_cp_dp_e)$, contraction with, say, $p_{a\,\lambda}$ gives rise to denominators of the form $[q=\frac{1}{2}(p_a+p_b+p_c+p_d+p_e)$, $r_a=(q-p_a)$, etc., $s_{ab}=(q-p_a-p_b)$, etc.$]$

$$D_\pm{}^{abcde}(k)=(k\pm q)^2(k+r_e)^2(k+s_{de})^2(k-s_{ab})^2(k-r_a)^2 \tag{18}$$

and, with $k'=k-p_a$, one finds $D_+{}^{abcde}(k')=D_-{}^{bcdea}(k)$. The other pairs of denominators obtained by permuting $(p_{b\,\mu},p_{c\,\nu},p_{d\,\rho},p_{e\,\sigma})$ all lead to additive contributions and one has finally

$$p_{a\,\lambda}S\Gamma_{\kappa\,\lambda\mu\,\nu\rho\sigma}=-(160\pi^5)^{-1}\epsilon_{\kappa\mu\,\nu\rho\sigma\alpha\beta\,\gamma\delta\epsilon}p_{b\,\alpha}p_{c\,\beta}p_{d\,\gamma}p_{e\,\delta}p_{f\,\epsilon}. \tag{19}$$

Similarly, all the $2\times5!=240$ terms in $(2q)_\kappa S\Gamma_{\kappa\,\lambda\mu\,\nu\rho\sigma}$ combine to give equal-sign surface terms when we permute the five external momenta. This sum gives for the hexagon anomaly

$$2q_\kappa V_{\kappa\,\lambda\mu\,\nu\rho\sigma}=2^6X_6\epsilon_{\lambda\mu\,\nu\rho\sigma\alpha\beta\,\gamma\delta\epsilon}p_{a\,\alpha}p_{b\,\beta}p_{c\,\gamma}p_{d\,\delta}p_{e\,\epsilon} \tag{20}$$

with

$$X_6=-2^{-10}\pi^{-5}. \tag{21}$$

The general case $D=2n$ can be shown[5] to give

$$X_{n+1}=(-1)^n2^{-2n}\pi^{-n}. \tag{22}$$

The generalization to a non-Abelian theory is straightforward since it has been shown[6] that the group-theoretic piece of the anomaly is an overall multiplicative factor so that, for example, in $D=10$

$$2q_\kappa V^{ABCDEF}_{\kappa\,\lambda\mu\,\nu\rho\sigma}(p_ap_bp_cp_dp_e)=-(16\pi^5)^{-1}S\,\mathrm{Tr}(\Lambda^A\Lambda^B\Lambda^C\Lambda^D\Lambda^E\Lambda^F)\epsilon_{\lambda\mu\,\nu\rho\sigma\alpha\beta\,\gamma\delta\epsilon}p_{a\,\alpha}p_{b\,\beta}p_{c\,\gamma}p_{d\,\delta}p_{e\,\epsilon}, \tag{23}$$

where $S\,\mathrm{Tr}$ denotes the averaged totally symmetric trace over the group generators Λ^A, $\Lambda^B,\ldots$ in the appropriate basis.

We find it remarkable that these one-loop diagrams can be calculated completely to find unique anomalies in the higher-dimensional theories despite their nonrenormalizability. Such anomalies must be absent (cancelled) in a consistent higher-dimensional string theory.

We have only summarized the calculations which are actually rather lengthy and will be published in detail elsewhere.[5] The crucial point is that the anomalies are dictated by the homotopy group of mappings of the gauge group on the $(D-1)$-sphere in Euclidean space, and this is independent of the severe ultraviolet divergences. We are throughout regarding the higher space-time dimensions as physical, and the corresponding degrees of freedom as dynamical ones which must satisfy canonical commutation relations in a quantum theory.

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[1]P. H. Frampton, Phys. Lett. 122B, 351 (1983).

[2]L. Brink, M. B. Green, and J. H. Schwarz, Nucl. Phys. B198, 474 (1982).

[3]P. H. Frampton, J. Preskill, and H. van Dam, to be published.

[4]S. Adler, Phys. Rev. 177, 2426 (1969); J. S. Bell and R. Jackiw, Nuovo Cimento 60, 47 (1969).

[5]P. H. Frampton and T. W. Kephart, University of North Carolina–Chapel Hill Report No. IFP-193-UNC, 1983 (to be published).

[6]P. H. Frampton and T. W. Kephart, following Letter [Phys. Rev. Lett. 50, 1347 (1983)].