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LIGHT-CONE SUPERSPACE AND THE ULTRAVIOLET FINITENESS OF THE $N = 4$ MODEL

Stanley MANDELSTAM*

Department of Physics, University of California, Berkeley, California 94720, USA

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Superspace in the light-cone frame takes a simple form. No auxiliary fields are necessary, and application to extended supersymmetries is straightforward. It is shown that the $N = 4$ model, in a certain form of the light-cone gauge, is completely free of ultraviolet divergences in any order of perturbation theory. It follows that the β-function vanishes in any gauge, to all orders of perturbation theory. Our method differs from the conventional method in that we use only half the number of θ's as there are supersymmetry operators. All fields are unconstrained and independent of the $\bar{\theta}$'s.

1. Introduction

The object of this paper is to treat superspace in light-cone coordinates. The particular advantage of the light-cone frame is that the generalization to extended supersymmetries is straightforward. We shall apply our results to examine the possible ultraviolet divergences of the $N = 4$ model. For this purpose, we shall use a form of the light-cone gauge (the "modified light-cone gauge"), which is free of some of the singularities present in the usual light-cone gauge.

As is the case in many models, the existence of a superspace for the $N = 4$ model will enable us to prove that the vertex renormalization is finite in any order of perturbation theory. Since we are dealing with a gauge theory in a physical gauge, we may combine this result with the "naive" Ward identity (i.e., not the Taylor-Slavnov identity) to show that the two-point function is perturbatively finite. The modified light-cone gauge is thus perturbatively free of all ultraviolet divergences, as originally conjectured by Gell-Mann and Schwartz (unpublished). It follows that the β-function is perturbatively finite in any gauge. Since this result had been verified up to three loops [1], it was generally believed to be true. It is worthwhile, however, to have a proof, applicable in any order, which does not rely on "miraculous" cancellations. General arguments suggesting such a result are given in refs. [2–4].

The difficulty of constructing a covariant superspace for extended supersymmetries is related to the lack of correspondence between the number of fields and the number of particles. In any supersymmetric model the particle multiplets must form

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representations of the supersymmetry algebra. In particular, the number of bosons must be equal to the number of fermions. In an explicitly supersymmetric covariant formulation, the field multiplets must also form representations of the supersymmetry algebra. It is not always clear how to form such representations. The number of boson fields is usually not equal to the number of fermion fields, and one must introduce "auxiliary fields" to restore the balance. In the Wess-Zumino model, for instance, there are two spinless particles and a Majorana fermion doublet. The spinless particles are each represented by one field, but the Majorana fermion is represented by a four-component field. One therefore introduces two spinless auxiliary fields.

The $N=4$ model contains one vector gauge multiplet, six scalar gauge multiplets, and four Majorana spinor gauge multiplets. There are thus eight bosonic and eight fermionic gauge multiplets. Going from fields to particles, we require two extra bosonic gauge multiplets for the extra polarizations of the vector field, while the number of fermionic fields is doubled. We therefore have ten bosonic and sixteen fermionic gauge multiplets of fields. It is not at present known how to introduce auxiliary fields and thereby obtain a field representation of the supersymmetry algebra. Arguments have been given to suggest that such a representation does not exist.

In the light-cone frame the number of fields is always the same as the number of particles. Hence, once we have a model with any supersymmetry algebra, we automatically have a field representation of the algebra. In fact, Schwarz and Green [5] encountered no difficulty in obtaining an explicitly supersymmetric formulation of the ten-dimensional Neveu-Schwarz-Ramond string model. Their work suggests that one could obtain a formulation of the $N=4$ model by dimensional reduction, taking the zero-slope limit, and second quantization. We shall not make explicit use of the relation between the $N=4$ model and the string model, but we could have obtained our results by the procedure just mentioned.

Besides the use of light-cone coordinates, our method of constructing superspace will differ from the usual method in a second respect. One can motivate the construction of superspace by second quantization of a supersymmetric quantum mechanical model. The fermionic dimensions correspond to the supersymmetry operators in the same way as the bosonic dimensions correspond to x and p. In the usual method, one introduces two variables θ and $\bar{\theta}$ for each conjugate pair of supersymmetry operators. One then has to work with restricted fields for non-gauge particles. It is also possible to introduce a single θ and to work with unrestricted fields, and we shall do so in this paper. The difference between the two procedures is unrelated to the use of light-cone coordinates. It is possible to apply the usual method to the $N=1$ model in light-cone coordinates, as has been discussed in a note by Siegel and Gates [6]; the method can probably be generalized to extended supersymmetries. We can also apply our method to a covariant treatment. We hope to give the details in a subsequent paper.

Once we have a superspace formulation of the $N=4$ model we shall have no difficulty in proving finiteness, provided we can count powers of p in the usual way, treating all components on the same footing. Unfortunately such power counting is not permissible in the usual form of the light-cone gauge. The numerators of the propagators contain terms proportional to $(p^+)^{-1}$, and the poles in such terms prevent us from continuing to euclidean space-time. However, the light-cone gauge condition $A^+=0$ does not define the gauge uniquely, since it is maintained by gauge transformations independent of x^-. We shall show that we can define another form of the light-cone gauge, with the $i\varepsilon$ prescription

$$(p^+)^{-1} \to (p^+ + i\varepsilon p^-)^{-1}\,.$$

Such a prescription is inconvenient for most purposes, since the invariance subgroup of the Lorentz group is smaller than in the usual light-cone gauge. On the other hand, the factors $(p^+)^{-1}$ no longer prevent continuation to euclidean space-time, and the normal power-counting rules are valid.

The $N=4$ model is thus completely free of perturbative ultraviolet divergences in this form of the light-cone gauge. Since the vanishing of the β-function is a gauge-invariant condition, we may conclude that the β-function vanishes in any gauge and in any order of perturbation theory. In most gauges the wave-function renormalization will be infinite; the divergences is a pure gauge artifact.

In the following section we shall set up our formalism and apply it to the Wess-Zumino model. We shall find that the light-cone supersymmetry algebra is simpler than the covariant algebra though, for models where an explicitly supersymmetric covariant formalism exists, the simplicity is probably offset by the difficulties associated with the lack of manifest Lorentz invariance. In sect. 3 we shall show that the method can be extended to the $N=4$ model in a straightforward manner. We shall thus have a superspace formalism from which Feynman rules with manifest $N=4$ supersymmetry can easily be constructed. The finiteness of the model will be treated in sect. 4. In the final section we shall give the superspace formalism for the $N=1$ ten-dimensional model before dimensional reduction. Unfortunately the SO(8) invariance in the eight-dimensional transverse space, while easily provable, will not be manifest; the manifest invariance group will be SO(2) × SO(6).

2. The Wess-Zumino model in light-cone coordinates

As a "guinea-pig" model to illustrate our methods we shall first study the Wess-Zumino model. The following Majorana representation for the γ's will be used:

$$\gamma^0=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\qquad \gamma^3=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},\qquad \gamma^i=\begin{pmatrix}i\sigma^i & 0\\ 0 & -i\sigma^i\end{pmatrix},\qquad \gamma^5=\begin{pmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{pmatrix},\tag{2.1}$$

with

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The indices i always take the values 1 and 2. The light-cone coordinates $x^{\pm}$ are defined in the usual way:

$$x^{\pm} = \sqrt{\tfrac{1}{2}}\,(x_0 \pm i x_3). \tag{2.2}$$

We take x^+ as the light-cone "time", x^- as a "space" coordinate.

The Wess-Zumino lagrangian without auxiliary fields is

$$\begin{aligned} \mathcal{L} = {} & (\partial_\mu A^\dagger)(\partial^\mu A) + \tfrac{1}{2} i \bar{\psi}_4 \gamma \cdot \partial \psi_4 - m^2 A^\dagger A - \tfrac{1}{2} i m \bar{\psi}_4 \psi_4 \\ & - \sqrt{\tfrac{1}{2}}\, i g \bar{\psi}_4 \{ A(1-\gamma_5) + A^\dagger (1+\gamma_5) \} \psi_4 \\ & - \sqrt{2}\, g m (A + A^\dagger) A^\dagger A - 2 g^2 (A^\dagger A)^2 , \end{aligned} \tag{2.3a}$$

where

$$A = \sqrt{\tfrac{1}{2}}\,(A_1 - i A_2), \tag{2.3b}$$

A_1 and A_2 being the scalar and pseudoscalar fields respectively. ψ_4 is a Majorana spinor.

Let us write

$$\psi_4 = \begin{pmatrix} (2)^{1/4} \psi_{\rm U} \\ (2)^{-1/4} \psi_{\rm L} \end{pmatrix}, \tag{2.4a}$$

where each of the entries have two Pauli components. The Dirac equation for $\psi_{\rm L}$ does not involve $\partial/\partial x^+$, so that $\psi_{\rm L}$ may be eliminated in favor of ψ [7]:

$$\psi_{\rm L} = -(2p^+)^{-1} \{ \sigma^i p^i - i m - \sqrt{2}\, i g [A + A^\dagger + \sigma_3 (A - A^\dagger)] \} \psi_{\rm U} . \tag{2.4b}$$

The operator p^+ is $i\, \partial/\partial x^-$, while

$$\frac{1}{p^+} f(x^-) = -\tfrac{1}{2} i \int \mathrm{d}x'^- \varepsilon(x'^- - x^-) f(x'^-). \tag{2.4c}$$

In sect. 4 we shall introduce another definition of $(p^+)^{-1}$; the difference corresponds to a change in the $i\varepsilon$ prescription. The operator $(p^+)^{-1}$ acts on all factors to the right of it which are not separated from it by brackets. It is convenient to work in terms of the helicity components of ψ_{U} and in terms of "right" and "left" derivatives:

$$\psi=(\sqrt{2})^{-1}(\psi_{\mathrm{U}1}+i\psi_{\mathrm{U}2}), \tag{2.5a}$$

$$\partial_{\mathrm{r}}=\partial_1+i\partial_2, \qquad \partial_{\ell}=\partial_1-i\partial_2. \tag{2.5b}$$

We may then express $\mathfrak{L}$ in terms of ψ_{U} as follows:

$$\begin{aligned}\mathfrak{L}=&\left(\partial_{\mu}A^{\dagger}\right)(\partial^{\mu}A)+\left(\partial_{\mu}\psi^{\dagger}\right)(2p^{+})^{-1}(\partial^{\mu}\psi)-m^2A^{\dagger}A-m^2\psi^{\dagger}(2p^{+})^{-1}\psi\\&+\sqrt{2}gi\left\{\psi\left[A,(2p^{+})^{-1}\partial_{\ell}\right]\psi\right\}-\psi^{\dagger}\left[A,(2p^{+})^{-1}\partial_{\mathrm{r}}\right]\psi^{\dagger}\\&-\sqrt{2}gm(A+A^{\dagger})AA^{\dagger}+2\sqrt{2}gm\psi^{\dagger}\left\{A^{\dagger}(2p^{+})^{-1}+(2p^{+})^{-1}A\right\}\psi\\&-2g^2(A^{\dagger}A)^2-8g^2\psi^{\dagger}A^{\dagger}(2p^{+})^{-1}\psi A.\end{aligned} \tag{2.6}$$

We note the presence of a quartic term involving two ψ's and two A's.

Under supersymmetry transformations, the four-component spinor changes as follows:

$$\delta A=\bar{\alpha}(1-\gamma_5)\psi, \tag{2.7a}$$

$$\delta\psi=\left\{i\gamma^{\mu}\partial_{\mu}\left[A+A^{\dagger}+\gamma_5(A-A^{\dagger})\right]+\sqrt{2}(F+\gamma_5G)\right\}\alpha, \tag{2.7b}$$

with $\bar{\alpha}=\alpha^{\dagger}\gamma_0$. The auxiliary fields F and G are expressed in terms of A in the usual way. Let us denote the lower and upper pair of components of α by α_{L} and α_{U} respectively. With $\alpha_{\mathrm{U}}=0$, the two-component spinors transform as follows:

$$\delta A=i\alpha_{\mathrm{L}}^{\dagger}(1-\sigma_3)\psi_{\mathrm{U}}, \tag{2.8a}$$

$$\delta\psi_{\mathrm{U}}=ip^{+}\left[A+A^{\dagger}+\sigma_3(A-A^{\dagger})\right]\alpha_{\mathrm{L}}. \tag{2.8b}$$

With $\alpha_{\mathrm{L}}=0$, the transformation is

$$\delta A=i\alpha_{\mathrm{U}}^{\dagger}(1+\sigma_3)\psi_{\mathrm{L}}, \tag{2.9a}$$

$$\delta\psi_{\mathrm{L}}=ip^{+}\left[A+A^{\dagger}-\sigma_3(A-A^{\dagger})\right]\alpha_{\mathrm{U}}. \tag{2.9b}$$

By expressing ψ_{L} in terms of ψ_{U} from eq. (2.4b), we may rewrite eq. (2.9) as a transformation involving ψ_{U}.

The transformations (2.9) are somewhat more complicated than eqs. (2.8). Since a Lorentz transformation between two different light-cone frames interchanges upper and lower components, we can obtain the transformations (2.9) by commuting a Lorentz transformation with a transformation (2.8). If the lagrangian is Lorentz invariant and invariant under the supersymmetry transformations (2.8), it will also be invariant under the transformations (2.9). Hence *we may restrict our attention to the transformations* (2.8). The subscript 1 on the α's will be dropped, and the transformations (2.9) will not be considered further.

We denote the generator of the supersymmetry transformation in eq. (2.8) by

$$\sum_{i=1}^{2} \bar{\alpha}_{Ui} Q_i . \tag{2.10}$$

The Q's then have the simple anticommutation relations

$$\{Q_i, Q_j\} = 4p^+ \delta_{ij} . \tag{2.11}$$

Expressed in terms of the helicity components of ψ_{U1} the transformations in eqs. (2.8) take the simple form

$$\delta A = i\alpha^* \psi , \tag{2.12a}$$

$$\delta\psi = 2p^+ i\alpha A , \tag{2.12b}$$

where

$$\alpha = \alpha_{L1} + i\alpha_{L2} . \tag{2.12c}$$

We now construct a superspace for the model. At this point our treatment will be somewhat different from the conventional approach. In the first-quantized theory, we may regard the Bose and Fermi particles as different states of the same particle. The fundamental operators of the model will thus be x and p, together with the supersymmetry operators Q_1 and Q_2 which connect the Bose and Fermi sectors. After second quantization, the field operators will be functions of x, together with an anticommuting variable corresponding to the Q's. Since we require one coordinate for each pair of conjugate variables in the first-quantized theory, we need only a single θ. In relating our treatment to the usual covariant treatment with four θ's $(\theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2)$, we observe that the use of light-cone coordinates halves the number of θ's, while the use of a single θ for each conjugate pair of Q's halves the number again.

Corresponding to Q_1 and Q_2, we define the two operators

$$Q_2:\ \mathcal{D}=i\left\{\frac{\partial}{\partial\theta}-2p^+\theta\right\}, \tag{2.13a}$$

$$Q_1:\ \tilde{\mathcal{D}}=\frac{\partial}{\partial\theta}+2p^+\theta. \tag{2.13b}$$

We notice that the operators do satisfy the commutation relations (2.11) and, in particular, that

$$\mathcal{D}^2=\tilde{\mathcal{D}}^2=2p^+. \tag{2.13c}$$

The superfield, which is fermionic, is defined as follows:

$$\phi=-i(2p^+)^{-1}\psi+\theta A. \tag{2.14a}$$

We could have defined a bosonic superfield, but the above choice leads to a closer correspondence between the formulas of the present model and those of the $N=4$ model. It is easily checked that the operators (2.13), when applied to eq. (2.14), do effect the transformations (2.12).

We also define the conjugate field

$$\phi^\dagger=-i(2p^+)^{-1}\psi^\dagger+\theta A^\dagger. \tag{2.14b}$$

The operator $\mathcal{D}$ is real, but $\tilde{\mathcal{D}}$ is imaginary. Hence, when applying the $\mathcal{D}$'s to $\phi^\dagger$, we must make the correspondence:

$$Q_2\phi^\dagger\sim\mathcal{D}\phi^\dagger,\qquad Q_1\phi^\dagger\sim-\tilde{\mathcal{D}}\phi^\dagger. \tag{2.15}$$

In writing down a supersymmetric action, two factors must be borne in mind. The first is that we cannot define covariant derivatives in our present formalism, since we have only a single θ at our disposal. The two operators $\mathcal{D}$ and $\tilde{\mathcal{D}}$ anticommute with one another, but they commute with themselves, of course. The second point to bear in mind is the reversal of sign that occurs when $\tilde{\mathcal{D}}$ acts on the conjugate field $\phi^\dagger$. These two possible complications cancel one another if we make the following rule: *Any term in the lagrangian can only contain factors with an even number of $\tilde{\mathcal{D}}$'s acting on a ϕ and an odd number on a $\phi^\dagger$ or vice versa.* The changes induced in the lagrangian by a supersymmetry transformation will then disappear on integration.

One might regard θ as a real variable. From the point of view of invariance under xy rotations it is preferable not to do so, since θ changes by a phase factor under such rotations. According to eq. (2.14), the superfields are analytic functions of θ, i.e., they are polynomials in θ but are independent of $\bar{\theta}$. The conjugation $\phi\to\phi^\dagger$ is defined as hermitian conjugation of the fields but not of θ. All our formulas will involve only θ, $\partial/\partial\theta$ or $\int d\theta$. It is in this sense that our superfields only involve a single θ, whereas the conventional formalism involves two variables θ and $\bar{\theta}$ (or, in general, $2n$ variables $\theta_1,\dots\theta_n$; $\theta_1,\dots\bar{\theta}_n$).*

The lagrangian is as follows:

$$\begin{aligned}\mathcal{L} = & (\partial^\mu\phi^\dagger)\tilde{\mathcal{D}}(\partial_\mu\phi) - m^2\phi^\dagger\tilde{\mathcal{D}}\phi - \tfrac{1}{3}\sqrt{2i}\,g\{\phi(\partial_\ell\phi)(2p^+)\phi - \phi^\dagger(\partial_r\phi^\dagger)(2p^+)\phi^\dagger\} \\ & -\sqrt{2}\,gm\{(\tilde{\mathcal{D}}\phi)(\tilde{\mathcal{D}}\phi)\phi^\dagger + (\tilde{\mathcal{D}}\phi^\dagger)(\tilde{\mathcal{D}}\phi^\dagger)\phi\} \\ & +2g^2(\tilde{\mathcal{D}}\phi^\dagger)(\tilde{\mathcal{D}}\phi^\dagger)\tilde{\mathcal{D}}\{(\tilde{\mathcal{D}}\phi)(\tilde{\mathcal{D}}\phi)\}. \end{aligned} \tag{2.16}$$

Expressed in terms of components, the lagrangian (2.16) does agree with eq. (2.6), except for total derivatives. It satisfies the supersymmetry rule quoted above. In light-cone coordinates, lagrangians always contain a quartic as well as a cubic term, since there are no auxiliary fields.

We could work directly with the lagrangian (2.16). It is simpler, however, to replace $\phi^\dagger$ by the field

$$\tilde{\phi} = \tilde{\mathcal{D}}\phi^\dagger = A^\dagger - i\theta\psi^\dagger. \tag{2.17}$$

Eq. (2.13) *then gives the action of the supersymmetry operators on both* ϕ *and* $\tilde{\phi}$. The field $\tilde{\phi}$ is a Bose field. In view of eq. (2.13), we can easily express $\phi^\dagger$ in terms of $\tilde{\phi}$:

$$\phi^\dagger = (2p^+)^{-1}\tilde{\mathcal{D}}\tilde{\phi}. \tag{2.18}$$

When expressing the lagrangian in terms of ϕ and $\tilde{\phi}$, we use the identities

$$\tilde{\mathcal{D}}\phi_1\tilde{\mathcal{D}}\phi_2\tilde{\mathcal{D}}\phi_3 \approx -\{(2p^+)\phi_1\partial_\theta\phi_2 \mp \partial_\theta\phi_2(2p^+)\phi_1\}\phi_3, \tag{2.19}$$

where the sign $\approx$ means equal up to total derivatives, and the $\mp$ sign is $-$ if the fields are Bose, $+$ if they are Fermi. It is convenient to use jacobian notation:

$$2i\partial_{x^-,\theta}(\phi_1\phi_2) = (2p^+\phi_1)\partial_\theta\phi_2 - (\partial_\theta\phi_1)2p^+\phi_2. \tag{2.19a}$$

Note that

$$\phi_1\partial_{x^-,\theta}(\phi_2\phi_3) \approx \phi_2\partial_{x^-,\theta}(\phi_3\phi_1) \approx \phi_3\partial_{x^-,\theta}(\phi_1\phi_2). \tag{2.19b}$$

* I am grateful to B. Zumino for suggesting this interpretation.

We then find, up to total derivatives

$$
\begin{aligned}
\mathcal{L} = & (\partial^\mu\tilde{\phi})(\partial_\mu\phi) - m^2\tilde{\phi}\phi - \tfrac{1}{3}\sqrt{2i}\,g\phi(\partial_\ell\phi)2p^+\phi \\
& + \tfrac{2}{3}\sqrt{2}\,g\tilde{\phi}\partial_{x^-,\theta}\{[(2p^+)^{-1}\partial_r\tilde{\phi}][(2p^+)^{-1}\tilde{\phi}]\} \\
& + 2\sqrt{2i}\,gm(2p^+)^{-1}\tilde{\phi}\partial_{x^-,\theta}(\phi\phi) + \sqrt{2}\,gm\tilde{\phi}\tilde{\phi}\phi \\
& + 4ig^2\tilde{\phi}\tilde{\phi}(2p^+)^{-1}\partial_{x^-\theta}(\phi\phi).
\end{aligned} \tag{2.20}
$$

This form of the lagrangian does not involve θ at all, and involves $\partial/\partial\theta$ only in the combination $\partial_{x^-,\theta}$. To examine its supersymmetry properties directly, we recombine the supersymmetry operators in the form

$$Q_a, Q_b = \tfrac{1}{2}(Q_1 \mp iQ_2),$$

which correspond to the operators $\partial/\partial\theta$, $2p^+\theta$. Thus, in x^--θ space, $\mathcal{L}$ must be invariant under the three operations $\partial/\partial x^-$, $\partial/\partial\theta$, $\theta\partial/\partial x^-$. As long as $\mathcal{L}$ does not involve θ it is invariant under Q_a, and as long as it involves $\partial/\partial\theta$ only in the combination $\partial_{x^-,\theta}$ it is invariant under Q_b. The only reason for starting with eq. (2.16) rather than eq. (2.20) is that the former lagrangian is manifestly hermitian up to total derivatives.

It is now straightforward to obtain Feynman rules, since the fields are unconstrained. As in the non-supersymmetric theory, it is convenient to write the four-point vertex as the product of two three-point vertices by multiplying and dividing by an operator, $\partial^2 + m^2$ inserted between the two $\tilde{\phi}$'s and the two ϕ's. The factors $(2p^+)^{-1}$ will then all cancel. In the massless theory, the propagator and the $\phi\phi\phi$ vertex are independent of θ and $\partial/\partial\theta$, while the $\tilde{\phi}\tilde{\phi}\tilde{\phi}$ vertex involves an operator $\partial_{x^-,\theta}$. When manipulating the operators, it is convenient to keep the θ derivatives in jacobian-like, supersymmetric combinations. For instance, if we have three lines a, b, and c, the operators combine as follows:

$$
\begin{aligned}
& \left(p_a^+\frac{\partial}{\partial\theta_b} - p_b^+\frac{\partial}{\partial\theta_a}\right)\left(p_b^+\frac{\partial}{\partial\theta_c} - p_c^+\frac{\partial}{\partial\theta_b}\right) \\
& = p_b^+\left\{p_a^+\frac{\partial}{\partial\theta_b}\frac{\partial}{\partial\theta_c} + p_b^+\frac{\partial}{\partial\theta_c}\frac{\partial}{\partial\theta_a} + p_c^+\frac{\partial}{\partial\theta_a}\frac{\partial}{\partial\theta_b}\right\}.
\end{aligned} \tag{2.21}
$$

We shall not pursue the matter further, since our main interest is the $N=4$ model.

3. The $N = 4$ model

The $N=4$ model was first proposed by Gliozzi et al. [8]. Brink et al. [9] have shown that the model possesses an internal SU(4) symmetry.

The covariant supersymmetry algebra of the $N=4$ model consists of sixteen independent elements. If we keep the transformations corresponding to eq. (2.8) and drop those corresponding to eq. (2.9), we are left with eight elements; four helicity-increasing elements $Q_{a\alpha}(1 \leqslant \alpha \leqslant 4)$ and four helicity-decreasing elements $Q_{b\alpha}$. It is convenient to define the combinations

$$Q_{1\alpha} = Q_{a\alpha} + Q_{b\alpha}, \tag{3.1a}$$

$$Q_{2\alpha} = i(Q_{a\alpha} - Q_{b\alpha}), \tag{3.1b}$$

which satisfy the anticommutation relations

$$\{Q_{i\alpha}, Q_{j\alpha}\} = 4p^{+}\delta_{ij}\delta_{\alpha\beta}. \tag{3.2}$$

We now represent the Q's in terms of four θ's as follows:

$$Q_{2\alpha}:\ \mathfrak{D}_{\alpha} = i\left(\frac{\partial}{\partial\theta^{\alpha}} - 2p^{+}\theta^{\alpha}\right), \tag{3.3a}$$

$$Q_{1\alpha}:\ \tilde{\mathfrak{D}}_{\alpha} = \frac{\partial}{\partial\theta^{\alpha}} + 2p^{+}\theta^{\alpha}. \tag{3.3b}$$

The $\mathfrak{D}$'s satisfy the commutation relations

$$\{\mathfrak{D}_{\alpha}, \mathfrak{D}_{\beta}\} = \{\tilde{\mathfrak{D}}_{\alpha}, \tilde{\mathfrak{D}}_{\beta}\} = 4p^{+}\delta_{\alpha\beta}, \tag{3.3c}$$

$$\{\mathfrak{D}_{\alpha}, \tilde{\mathfrak{D}}_{\beta}\} = 0. \tag{3.3d}$$

The supersymmetry condition will then be that, for each α, any term in the lagrangian can only contain factors with an even number of $\tilde{\mathfrak{D}}$'s acting on a ϕ and an odd number on a $\phi^{\dagger}$, or *vice versa*.

The particles in the $N=4$ multiplet consist of a vector gauge multiplet, four Majorana spinor gauge multiplets and six spinless gauge multiplets. Each gauge multiplet is in the adjoint representation of the gauge group. In the light-cone frame, there will be one field for each particle. The vector fields with helicity ± 1 are SU(4) scalars, the spinor fields with helicity $\pm\frac{1}{2}$ each form a fundamental representation of SU(4), while the scalars form a six-dimensional representation of SU(4). In writing down the superfield, we start from the highest helicity field and go downwards. Thus

$$\phi = i(2p^{+})^{-1}V + (2p^{+})^{-1}\theta^{\alpha}\psi_{\alpha} + \tfrac{1}{4}i\theta^{\alpha}\theta^{\beta}\rho^{i}_{\alpha\beta}A_{i}$$
$$+ \frac{1}{3!}\varepsilon_{\alpha\beta\gamma\delta}\theta^{\alpha}\theta^{\beta}\theta^{\gamma}\psi^{\delta\dagger} + 2ip^{+}\theta_{1}\theta_{2}\theta_{3}\theta_{4}V^{\dagger}. \tag{3.4}$$

Each field carries a gauge index which has been suppressed. The positive- and negative-helicity vector fields and the six spinless fields have been represented by the symbols V, $V^\dagger$ and A_i. The matrix $\rho^i_{\alpha\beta}$ is the appropriate Clebsch-Gordan matrix for SU(4) [or SO(6)]; an explicit form has been given by Brink et al. [9] (see sect. 5). We specify further that the ρ's satisfy the condition

$$\rho_{\alpha\beta} = -\varepsilon^{\alpha\beta\gamma\delta}\rho^*_{\gamma\delta}. \tag{3.5}$$

The supermultiplet for the $N=4$ theory, unlike those for the $N=1$ or $N=2$ theories, is *TCP* self-conjugate. The fields ϕ and $\phi^\dagger$ are therefore not independent, but are related by the equation

$$\phi^\dagger = (2p^+)^{-2}\tilde{\mathfrak{D}}\phi, \tag{3.6}$$

where

$$\tilde{\mathfrak{D}} = \tilde{\mathfrak{D}}_1\tilde{\mathfrak{D}}_2\tilde{\mathfrak{D}}_3\tilde{\mathfrak{D}}_4. \tag{3.7}$$

The relation (3.6) is easily proved from eqs. (3.4) and (3.5). Again, the dagger implies hermitian conjugation of the fields, but not of the θ's.

The lagrangian for the model is as follows:

$$\begin{aligned}\mathfrak{L} = {} & (\partial^\mu\phi)\cdot(\partial_\mu\phi) - \tfrac{1}{3}\sqrt{2}\, ig\phi\cdot\{(\partial_\ell\phi)\times(2p^+\phi)\} \\ & + \tfrac{1}{3}\sqrt{2}\, ig\phi^\dagger\cdot\{(\partial_r\phi^\dagger)\times(2p^+\phi^\dagger)\} \\ & - \tfrac{1}{64}g^2\sum_\alpha\{(\tilde{\mathfrak{D}}_\alpha\phi)\times(\tilde{\mathfrak{D}}_\alpha\phi)\}\cdot(2p^+)^{-2}\tilde{\mathfrak{D}}\{(\tilde{\mathfrak{D}}_\alpha\phi^\dagger)\times(\tilde{\mathfrak{D}}_\alpha\phi^\dagger)\}.\end{aligned} \tag{3.8}$$

All dot and cross products refer to the gauge degree of freedom. The field $\phi^\dagger$ is regarded as a function of ϕ according to eq. (3.6). All subscripts α in the last term of eq. (3.8) take the *same* value. This may at first sight seem surprising, but the number of subscripts can easily be reduced to two by integrating by parts and using eq. (3.3c). When we re-express the lagrangian in a form analogous to eq. (2.20), the SU(4) invariance will be manifest.

We notice that the lagrangian does satisfy the supersymmetry rule written above. The first term is hermitian since, by using eq. (4.6) and (4.3c) and integrating by parts, we can easily show that it is equal to $(\partial^\mu\phi^\dagger)\cdot(\partial_\mu\phi^\dagger)$. The remainder of the lagrangian is manifestly hermitian.

The lagrangian (3.8) was actually obtained partly from the component lagrangian and partly by making use of Lorentz invariance. The quadratic and cubic terms were easily obtained from the lagrangian of the component fields. The Lorentz-transformation properties of the superfield were then studied. The only non-trivial Lorentz transformation is the transformation

$$\varepsilon_{1+}j^{1+}+\varepsilon_{2+}j^{2+}\equiv\tfrac{1}{2}\varepsilon(j^{1+}-ij^{2+})+\tfrac{1}{2}\varepsilon^*(j^{1+}+ij^{2+}), \tag{3.9a}$$

where

$$\varepsilon=\varepsilon_{1+}-i\varepsilon_{2+}. \tag{3.9b}$$

In light-cone coordinates, the change of the fields under such a transformation consists of a linear and a quadratic part. The linear part is easily obtained from the transformation of the components

$$\phi\to\phi(x+\delta x)+\tfrac{1}{2}i\varepsilon(2p^+)^{-1}\partial_{\mathrm{r}}\left[\theta^\alpha,\frac{\partial}{\partial\theta^\alpha}\right]\phi, \tag{3.10a}$$

$$2p^+\to 2p^+-i\varepsilon\partial_{\mathrm{r}}+i\varepsilon^*\partial_{\ell}, \tag{3.10b}$$

$$\partial_{\mathrm{r}}\to\partial_{\mathrm{r}}+\tfrac{1}{2}i\varepsilon^*p^-,\ \partial_{\ell}\to\partial_{\ell}-\tfrac{1}{2}i\varepsilon p^-. \tag{3.10c}$$

The quadratic part of the transformation can then be found by requiring that its effect on the quadratic term in $\mathfrak{L}$ cancel the effect of (3.10) on the linear term in $\mathfrak{L}$. We thus find for the total change of ϕ

$$\begin{aligned}\phi\to{}&\phi(x+\delta x)+\tfrac{1}{2}i\varepsilon(2p^+)^{-1}\left[\theta^\alpha,\frac{\partial}{\partial\theta^\alpha}\right]\phi\\&-\tfrac{1}{2}\sqrt{2}\,g(2p^+)^{-1}\left\{\left[\theta^\alpha\frac{\partial}{\partial\theta^\alpha}-1\right]\phi\times(2p^+)\phi\right\}\\&-\tfrac{1}{2}\sqrt{2}\,g(2p^+)^{-3}\tilde{\mathfrak{D}}\left\{\left[\theta^\alpha\frac{\partial}{\partial\theta^\alpha}-1\right]\phi^\dagger\times(2p^+)\phi^\dagger\right\}.\end{aligned} \tag{3.11}$$

Finally, we found the quartic term in $\mathfrak{L}$ from the requirement that the change in that term due to the linear term in the Lorentz transformation cancel the change in the cubic term of $\mathfrak{L}$ due to the quadratic term in the Lorentz transformation. The somewhat tedious calculation was facilitated, firstly by the observation that ϕ and $\phi^\dagger$ behave like scalars under the transformations $j^{1+}+ij^{2+}$ and $j^{1+}-ij^{2+}$, respectively, and secondly by making use of the alternative form of $\mathfrak{L}$ [eq. (3.12) below]. Once the quartic term has been derived, it is not difficult to check it by resolving it into components.

As we have mentioned, the operator $\phi^\dagger$ in eq. (3.8) is to be regarded as a function of ϕ defined by eq. (3.6). By partial integration and use of eq. (2.19), we can re-express the lagrangian in a form analogous to eq. (2.20). Thus

$$\begin{aligned}\mathcal{L} = &(\partial^\mu \phi)\cdot(\partial_\mu \phi) - \tfrac{1}{3}\sqrt{2}\, ig\phi\cdot\{(\partial_{\bar{r}}\phi)\times(2p^+\phi)\}\\ &-\tfrac{1}{3}\sqrt{2}\, ig(2p^+)^{-2}\phi\cdot\prod_\alpha(2i\partial_{x^-,\theta^\alpha})\{[(2p^+)^{-2}\partial_r\phi]\times(2p^+)^{-1}\phi\}\\ &-\tfrac{1}{64}g^2\sum_\alpha(2i\partial_{x^-,\theta^\alpha})(\phi\times\phi)\cdot(2p^+)^{-2}\prod_{\beta\neq\alpha}(2i\partial_{x^-,\theta^\beta})\{(2p^+)^{-1}\phi\times(2p^+)^{-1}\phi\}\,.\end{aligned} \tag{3.12}$$

It is understood that the four anticommuting derivatives $\partial_{x^-,\theta^\alpha}$ and $\partial_{x^-,\theta^\beta}$ in the third and fourth terms of eq. (3.12) are to be written in the order 1, 2, 3, 4, or in an even permutation thereof. The derivatives with respect to θ in eq. (3.12) thus all occur combined with the symbol $\varepsilon^{\alpha\beta\gamma\delta}$ and, since this is an SU(4)-invariant combination, the lagrangian is manifestly SU(4) invariant. As in the case of the Wess-Zumino model, the θ's are not regarded as real, but all terms in $\mathcal{L}$ are analytic in the θ's, and the action is obtained by integrating over the x's and θ's, not over the $\bar{\theta}$'s.

It is now a straightforward matter to obtain Feynman rules since the fields are unconstrained.

4. Finiteness of the model

We now show that the model is finite in a certain form of the light-cone gauge in any order of perturbation theory. This implies that the β-function vanishes to all orders in any gauge.

We first prove that the above result is true if simple power counting is permissible, with all components of p counted together. We then return to the question whether such power counting is in fact valid; we shall show that it is, provided we use the "modified light-cone gauge".

The power counting is in fact simpler than usual, since the superfield is dimensionless. Factors $\partial/\partial\theta$ are considered equivalent to $p^{1/2}$ throughout this section. The number of loops is

$$n_i - n_3 - n_4 + 1\,, \tag{4.1}$$

where n_i, n_3 and n_4 are the number of internal lines, three-point vertices and four-point vertices, respectively. Each loop is, as usual, associated with a factor d^4p, but the loop also absorbs four factors $\partial/\partial\theta$. A loop thus contributes two positive powers in all. Internal lines, three-point vertices and four-point vertices contribute

-2, 2 and 2 factors, respectively. The degree of divergence is thus

$$2(n_i - n_3 - n_4 + 1) - 2n_i + 2n_3 + 2n_4 - m = 2 - m, \tag{4.2}$$

where m is the number of powers of p on external lines. For convergence, (4.2) must be strictly negative.

Now let us consider a given triangle diagram, of which we examine a particular external elementary vertex. If the vertex corresponds to the second term of eq. (3.12), we make use of the identity

$$\begin{aligned}\phi_A \cdot (\partial_\ell \phi_B \times 2p^+ \phi_C) - \phi_A \cdot (2p^+ \phi_B \times \partial_\ell \phi_C) \\ = -(\partial_\ell \phi_A) \cdot (\phi_B \times 2p^+ \phi_C) + (2p^+ \phi_A) \cdot (\phi_B \times \partial_\ell \phi_C).\end{aligned} \tag{4.3}$$

We can therefore choose a pair of lines meeting at the vertex, one of which is the external line, and apply the factors ∂_ℓ and $2p^+$ to those two lines. (The factor $\frac{1}{3}$ will then not appear, since there are three identical terms). Thus the external line has one factor p_ℓ or p^+. In other words, the vertex has one positive power of p on the external line. The third term of eq. (3.12) has an extra factor $(2p_A^+)^{-2}(2p_B^+)^{-2}(2p_C^+)^{-2}(\partial_{x^-,\theta})_{BC}$ but, by taking B or C as the external line, we see that the number of extra factors of p so introduced cannot be negative. Again, therefore, the external line has at least one factor of p. The last term of eq. (3.12) has the factors (where all four θ's are treated as equivalent)

$$(p_C^+)^{-1}(p_D^+)^{-1}(p_A^+ + p_B^+)^{-1}\left(p_A^+ \frac{\partial}{\partial\theta_B} - p_B^+ \frac{\partial}{\partial\theta_A}\right)\left(p_C^+ \frac{\partial}{\partial\theta_D} - p_D^+ \frac{\partial}{\partial\theta_C}\right)^3, \tag{4.4}$$

and, once more, there is at least one factor of p on each external line.

The entire vertex diagram thus has a power of p on each external line. According to criterion (4.2), the diagram is convergent. (We may also note that we cannot reproduce the form of the bare vertex, which has powers of p on two external lines only.) Similarly, the four- and higher-point functions have powers of p on each external line and are convergent. Thus, with the possible exception of the two-point function, which we shall examine later with the aid of the Ward identity, all Green functions are finite.

It remains to analyze the validity of the power-counting criterion, on which all our results are based. In the usual form of the light-cone gauge one cannot simply count powers of p, treating all components on the same footing [10]. The reason is that ultraviolet divergences appear from the regions

$$p^+ \text{ finite}, \qquad p^-, \boldsymbol{p}^2 \to \infty, \qquad p^+p^- - \boldsymbol{p}^2 \text{ finite}. \tag{4.5}$$

The poles in the factors $(p^+)^{-1}$ prevent us from continuing to imaginary p_0 and thus avoiding the dangerous regions (4.5). In fact, a better criterion would be to count factors of p^i and p^- only, weighting each factor p^- as equivalent to two factors of p^i.

The condition for the light-cone gauge,

$$A^+ = 0, \tag{4.6}$$

does not define the gauge uniquely, since it remains true under a gauge transformation which depends on x^i and x^+ but not on x^-. The ambiguity is reflected in the $i\varepsilon$ prescription in the factors $(p^+)^{-1}$. Usually one takes a principal-value prescription. If, however, we could use the prescription

$$(p^+)^{-1} \to (p^+ + i\varepsilon p^-)^{-1}, \tag{4.7}$$

there would be no difficulty in continuing the p^0 integration to imaginary p^0. After continuation, we could write

$$(p^+)^{-1} \to \frac{ip^4 - p^3}{(p^4)^2 + (p^3)^2}, \tag{4.7a}$$

$$(p^+p^- - \underset{\sim}{p}^2)^{-1} \to \{(p^4)^2 + (p^3)^2 + \underset{\sim}{p}^2\}^{-1}. \tag{4.7b}$$

The power-counting prescription would then be justified.

Operators in the usual light-cone gauge are defined in terms of those in some other gauge by the formula:

$$\begin{aligned} A^i_{\mathrm{LC}}(x^-) &= A^i(x^-) + \tfrac{1}{2} ig \int \mathrm{d}x^{-\prime} \varepsilon(x^- - x^{-\prime}) [A^+(x^{-\prime}), A^i(x^-)] \\ &\quad + \tfrac{1}{4}(ig)^2 \int \mathrm{d}x^{-\prime\prime} \mathrm{d}x^{-\prime} \varepsilon(x^- - x^{-\prime}) \varepsilon(x^{-\prime} - x^{-\prime\prime}) \\ &\quad \times [A^+(x^{-\prime\prime}), [A^+(x^{-\prime}), A^i(x^-)]] + \cdots . \end{aligned} \tag{4.8}$$

The variables other than x^- have been suppressed. The potentials are written as matrices in the gauge degrees of freedom. In our new light-cone gauge, we replace eq. (4.8) by the formula

$$\begin{aligned} A^i_{\mathrm{LC}}(x^-) &= A^i(x^-) + ig \int \mathrm{d}x^{-\prime} \{\varepsilon(p^-)\theta(x^- - x^{-\prime}) - \varepsilon(-p^-)\theta(x^{-\prime} - x^-)\} \\ &\quad \times [A^+(x^{-\prime}), A^i(x^-)] \\ &\quad + (ig)^2 \int \mathrm{d}x^{-\prime\prime} \mathrm{d}x^{-\prime} \{\varepsilon(p^-)\theta(x^- - x^{-\prime}) - \varepsilon(-p^-)\theta(x^{-\prime} - x^-)\} \\ &\quad \times \{\varepsilon(p^{-\prime})\theta(x^{-\prime} - x^{-\prime\prime}) - \varepsilon(-p^{-\prime})\theta(x^{-\prime\prime} - x^{-\prime})\} \\ &\quad \times A^+(x^{-\prime\prime}), [A^+(x^{-\prime}), A^i(x^-)] + \cdots . \end{aligned} \tag{4.9}$$

If we use coordinate space for all four x's, the function $\varepsilon(p^-)$ becomes replaced by

$$\tfrac{1}{2}i\int \mathrm{d}x^{+\prime}\varepsilon(x^+ - x^{+\prime})\,. \tag{4.10}$$

The definition thus involves an integration over x^+ as well as over x^-.

One can find the Feynman rules for the new light-cone gauge by obtaining the differential equations for the Green functions and solving them in a perturbation series. The differential equations are in fact identical to those of the usual light-cone gauge, since adding an x^- independent function to the function $\varepsilon(x^-)$ in eq. (4.8) does not change the equations. The boundary conditions are, however, different; the operator $(\partial/\partial x^-)^{-1}$, instead of being equal to the integral operator $\frac{1}{2}\varepsilon(x^-)$, is now equal to the integral operator $\varepsilon(p^-)\theta(x^-)-\varepsilon(-p^-)\theta(-x^-)$. In other words, the principal-value prescription for $(p^+)^{-1}$ is replaced by the prescription (4.7). This form of the light-cone gauge thus leads to the desired $i\varepsilon$ prescription, and all three- and higher-point Green functions are finite.

For most purposes other than proving finiteness, the new light-cone gauge is not convenient. The usual light-cone gauge is invariant under all Lorentz transformations which leave x^+ invariant (or which rescale this coordinate). The new light-cone gauge is invariant only with respect to transformations which leave invariant, or rescale, both x^+ and x^-. The hamiltonian in the canonical formalism can only be defined implicitly, since the definition involves the hamiltonian itself through the operator $\varepsilon(p^-)\theta(x^-)-\varepsilon(-p^-)\theta(-x^-)$. Nevertheless, our new $i\varepsilon$ prescription is just as good in principle as the usual prescription, and it does lead to Green functions which are less singular, both in the ultraviolet region and in the neighborhood of $p^+=0$.

It remains to examine the two-point function. Since we are in a physical gauge, we may use the ordinary Ward identities rather than the Taylor-Slavnov identities. Furthermore, the proper Green functions involving gluons of all four polarizations are free of singularities when any of the p^+'s become zero; such singularities cannot result from loop integrations with our present $i\varepsilon$ prescription. We can thus use the Ward identity in the form:

$$\Lambda^i(p,p,0) = -\frac{\partial}{\partial p^i}\Pi(p,p)\,, \tag{4.11}$$

where Π is any self-energy part, and Λ^i is the corresponding vertex part with an extra gluon of zero four-momentum and polarization i.

Power counting, together with Lorentz invariance under the transformation j^{ij} ($ij=1,2$) and j^{+-}, restrict the gluon self-energy part Π_{ij} to the following form:

$$\Pi_{ij} = A\delta_{ij} + Bp^2\delta_{ij} + Cp_ip_j + Dp^+p^-\delta_{ij} + \text{finite terms}\,. \tag{4.12}$$

As in our examination of the three-point function, we can show that each of the external vertices must have at least one factor p^i or p^+. Hence A and D must vanish. Eq. (4.11), together with the finiteness of Λ, shows that B and C must be finite. The self-energy parts of the other particles can be proved finite in a similar way. Thus all the Green functions of the model are finite.

5. Ten-dimensional model

One method of obtaining the $N=4$ model is by dimensional reduction of the $N=1$ model in ten space-time dimensions. The relation between the model and the supersymmetric string model is thereby made manifest. It may be of interest to treat the ten-dimensional model before dimensional reduction, and we shall do so in this section. Unfortunately we know of no way of obtaining *manifest* O(8) rotational invariance in the eight-dimensional transverse space. Our treatment of the superspace gives us naturally a manifest $SO(6)\times SO(2)$ invariance; in the four-dimensional model we were able to associate this invariance with the four-dimensional internal space $[SO(6)\equiv SU(4)]$ and the two-dimensional transverse space. It does not appear that the situation would be different if we were to obtain a formalism similar to the conventional formalism, with four θ's and four $\bar{\theta}$'s.

We begin with some remarks on the Brink et al. [9] representation of SO(6). GOS show that one can obtain two sets of four-dimensional hermitian matrices with pure imaginary elements; the matrices in each set satisfy the Pauli commutation relations, while matrices in different sets commute. Thus

$$\begin{gathered}[\alpha^i,\alpha^j]=i\varepsilon^{ijk}\alpha^k,\qquad [\beta^l,\beta^m]=i\varepsilon^{lmn}\beta^n,\qquad [\alpha^i,\beta^l]=0,\\ i,j,k=1,3,5,\qquad l,m,n=2,4,6.\end{gathered}\tag{5.1}$$

The matrices are anti-self-dual and self-dual respectively:

$$\alpha^i_{\alpha\beta}=-\varepsilon_{\alpha\beta\gamma\delta}\alpha^i_{\gamma\delta},\qquad \beta^i_{\alpha\beta}=\varepsilon_{\alpha\beta\gamma\delta}\beta^i_{\gamma\delta}.\tag{5.2}$$

One can then define O(6) σ-matrices as follows:

$$\sigma^{ij}=-\tfrac{1}{2}i[\alpha^i,\alpha^j],\qquad \sigma^{kl}=-\tfrac{1}{2}i[\beta^k,\beta^l],\qquad \sigma^{il}=\alpha^i\beta^l.\tag{5.3}$$

If ψ_α transforms as a spinor, the set of six quantities

$$\psi^{\mathrm{T}}\alpha^i_{\alpha\beta}\psi_\beta,\qquad i\psi^{\mathrm{T}}_\alpha\beta^i_{\alpha\beta}\psi_\beta$$

transforms as an O(6) vector; the matrices α^i, $i\beta^i$ have been denoted by ρ in sect. 4.

We now define a set of SO(8) matrices as follows:

$$\tau^1 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \qquad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$\tau^i = \begin{pmatrix} \alpha^{i-2} & 0 \\ 0 & -\alpha^{i-2} \end{pmatrix}, \ (i \text{ odd}), \qquad \tau^i = \begin{pmatrix} 0 & \beta^{i-2} \\ \beta^{i-2} & 0 \end{pmatrix}, \ (i \text{ even}). \tag{5.4}$$

The SO(8) σ-matrices are then

$$\sigma_8^{ij} = -i\tau^i\tau^j, \qquad (i<j). \tag{5.5}$$

All the elements are pure imaginary.

To construct a representation of the SO(8) Lie algebra within our superspace, we make use of the operators $\tilde{\mathfrak{D}}_\alpha(\alpha = 1,2,3,4)$, together with $\tilde{\mathfrak{D}}_{\alpha+4} \equiv \mathfrak{D}_\alpha$. Thus

$$J^{ij} = \frac{1}{16p^+}\sigma^{ij}_{8,\alpha\beta}\tilde{\mathfrak{D}}_\alpha\tilde{\mathfrak{D}}_\beta. \tag{5.6}$$

The J's have the required commutation relations. If i and j are both $\leqslant 2$ or > 2, the J's are

$$J^{12} = \tfrac{1}{2}\theta^\alpha\frac{\partial}{\partial\theta^\alpha}, \tag{5.7a}$$

$$J^{ij} = \tfrac{1}{2}\theta^\alpha\sigma^{ij}_{\alpha\beta}\frac{\partial}{\partial\theta^\beta}, \qquad i,j>2. \tag{5.7b}$$

The generators in the SO(2) transverse space and the SU(4) internal space are thus correctly reproduced. The additional generators are as follows:

$$J^{1i} = \frac{1}{8p^+}\alpha^i_{\alpha\beta}\left\{\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta} + (2p^+)^2\theta^\alpha\theta^\beta\right\}, \qquad i \text{ odd}, \tag{5.8a}$$

$$J^{1i} = \frac{i}{8p^+}\beta^i_{\alpha\beta}\left\{\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta} - (2p^+)^2\theta^\alpha\theta^\beta\right\}, \qquad i \text{ even}, \tag{5.8b}$$

$$J^{2i} = \frac{i}{8p^+}\alpha^i_{\alpha\beta}\left\{\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta} - (2p^+)^2\theta^\alpha\theta^\beta\right\}, \qquad i \text{ odd}, \tag{5.8c}$$

$$J^{2i} = -\frac{1}{8p^+}\beta^i_{\alpha\beta}\left\{\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta} + (2p^+)\theta^\alpha\theta^\beta\right\}, \qquad i \text{ even}. \tag{5.8d}$$

Now let us see how kinetic terms corresponding to the six extra dimensions can be added to the lagrangian. We may rewrite the cubic terms in eq. (3.8), excluding the factor $-\sqrt{2}\,ig$, as follows:

$$\tfrac{1}{6}\phi\cdot\{((\partial_1\phi)\times(2p^+\phi))\}-\tfrac{1}{6}\phi^\dagger\{((\partial_1\phi^\dagger)\times 2p^+\phi^\dagger\}$$
$$-\tfrac{1}{6}i\phi\{((\partial_2\phi)\times(2p^+\phi))\}-\tfrac{1}{6}i\phi^\dagger\{\partial_2\phi^\dagger\times 2p^+\phi^\dagger\}. \tag{5.9}$$

Thus, in the transverse two-dimensional space, the following quantities, integrated over the θ's, form components of a two-vector:

$$v^1=\tfrac{1}{6}\phi\cdot\{\phi_a\times(2p^+\phi)\}-\tfrac{1}{6}\phi^\dagger\cdot\{\phi_a\times(2p^+\phi)\}, \tag{5.10a}$$

$$v^2=-\tfrac{1}{6}i\phi\cdot\{\phi_a\times(2p^+\phi)\}-\tfrac{1}{6}i\phi^\dagger\{\phi_a^\dagger\times(2p^+\phi)\}. \tag{5.10b}$$

We have distinguished the field to be differentiated by the subscript a. We can find the other components of the vector by applying the generators (5.8) to each of the ϕ's in eq. (5.10) and adding. After integrating by parts and expressing $\phi^\dagger$ in terms of ϕ by eq. (3.6) we find, up to total derivatives

$$v^i=-\tfrac{1}{3}(2p^+)^{-1}\phi\cdot\rho^i_{\alpha\beta}(2i\partial_{x^-,\theta^\alpha})(2i\partial_{x^-,\theta^\beta})\{[(2p^+)^{-1}\phi_a]\times\phi\}. \tag{5.11}$$

[The two terms of (5.10) give identical contributions to eq. (5.11)]. It can be checked that the eight components (5.10) and (5.11) of v do transform as a vector. The lagrangian for the ten-dimensional model is thus

$$\mathcal{L}_{10}=\mathcal{L}+\mathcal{L}', \tag{5.12}$$

where $\mathcal{L}$ is given by eq. (3.12), and

$$\mathcal{L}'=\sum_{i=3}^{8}\tfrac{1}{3}\sqrt{2}\,ig(2p^+)^{-1}\phi\cdot\rho^i_{\alpha\beta}(2i\partial_{x^-,\theta^\alpha})(2i\partial_{x^-,\theta^\beta})\{[(2p^+)^{-1}\partial_i\phi]\times\phi\}. \tag{5.13}$$

In our treatment of the four-dimensional model, we associated the spatial transverse dimensions with the indices 1 and 2 of the present eight-dimensional transverse space, while the internal SO(6) dimensions were associated with the indices 3 to 8. We could also have associated the transverse dimensions with two of the indices between 3 and 8. The cubic term in the lagrangian would then be identical to eq. (5.13), with i taking two values. We would thus avoid the asymmetry between the two cubic terms of eq. (3.12). On the other hand, we would lose the manifest SO(6) invariance of the internal space. We would only be left with a manifest SO(4) invariance.

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