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ON SUPERSYMMETRY BREAKING BY INSTANTON EFFECTS

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Instanton effects on supersymmetry breaking in the O(3) non-linear σ-model are investigated in detail in the semiclassical approximation. The conclusion is that instantons cannot break the supersymmetry neither spontaneously nor explicitly for any O(3)-invariant operator under consideration. Examples are however given of operators that belong to an infinite-dimensional representation of O(3) and break the supersymmetric Ward identities explicitly via instanton effects, and such examples are connected to fermionic zero-mode condensates.

1. Introduction

It has been suggested in the recent past that supersymmetry in its global version might help to clarify various aspects of particle physics besides being used in its local edition for describing grand unified theories with gravity. Recently, Witten and others [1] proposed that nature might be described by a supersymmetric theory that could be broken spontaneously by dynamical effects, such as instantons. If it occurs at energies much lower than the Planck mass it could help to clarify the mass hierarchy problem. In such a theory global supersymmetry is unbroken at the tree level but broken by extremely small quantum corrections of a non-perturbative nature at a scale of order 10^{-16} times the Planck mass. The instantons that are thought to be responsible for the mentioned dynamical breaking of supersymmetry, as in the case of chiral symmetry [2], are usually not of any importance since their physical effects are negligible compared to ordinary perturbation theory at high energies, i.e. exponentially small in the coupling constant: $\exp(-1/2g)$. In fact in supersymmetric theories even a small contribution can be very significant. Therefore one is forced to utilize non-perturbative methods in order to clarify such instanton effects on the supersymmetric vacuum. Up to now no rigorous proof has been given of a spontaneous breakdown of supersymmetry caused by instantons, except in one-dimensional models [3] where such a breaking did occur. Of course one can get a clear answer about the possibility of a spontaneous breaking by calculating the Witten index [4] which in such a case should be equal to zero.

In this paper we shall investigate the rôle of instantons on supersymmetry. Recently it has been put forward by Casher [5] that $N=1$ global supersymmetry in 4-dimensional gauge theories is broken explicitly by instantons and therefore the effective low-energy theory will not be supersymmetric and will not include massless Goldstone fermions.

The arguments are relying on assumptions that can be tested exactly in 2-dimensional models [6]. There are immediate reasons to believe that instantons can force an explicit breakdown since they can generate multifermion interactions [2] with no bosonic counterpart. This generation of fermions is due to a chiral current anomaly. The difference between spontaneous and explicit breaking of supersymmetry can become clear by studying the Ward identities corresponding to supersymmetric transformations, since they should hold for a spontaneous breaking but be unfulfilled for an explicit one. In ref. [7] a study of the Ward identities gave a reconfirmation of the result of ref. [3] for supersymmetric quantum mechanics, but with no success in reaching a decisive conclusion for 4-dimensional gluodynamics although the Ward identities turned out to be unsatisfied due to the lack of some surface terms that otherwise in the quantum-mechanical case made the identities be satisfied. It is claimed in ref. [7] that if all the supersymmetric instanton modes, including bilinear terms in the grassmannian parameters are integrated over, then supersymmetry is restored.

Actually an old study [8] of $N=2$ supersymmetric Yang–Mills theory revealed that the vacuum expectation value of an 8-fermion field operator in an instanton vacuum could break the Ward identities. However taking integration over collective coordinates into account it was not sure that the effect would persist.

Motivated by the wish to clarify if instantons can break supersymmetry and due to the lack of non-perturbative methods in 4-dimensional Yang–Mills theory we shall in this paper resort to the study of instanton effects in the supersymmetric 2-dimensional σ-model. It is well-known that this model shares many properties with that of 4-dimensional Yang–Mills theory, such as asymptotic freedom. Furthermore, instanton effects have been calculated exactly in the σ-model (CP^{n-1}) [6]. Therefore we shall extend their techniques to the supersymmetric O(3) σ-model and then study a possible breaking of supersymmetry. Thus the studies are twofold or a combination of two techniques are involved here: techniques of breaking of supersymmetry pioneered by Witten [1] and the techniques of calculating instanton effects non-perturbatively in the σ-model. Therefore we shall devote sect. 2 to the study of supersymmetry breaking in the simplest model, 1-dimensional quantum mechanics, where we exhibit the advantage of superspace formulation. Next in sect. 3 we shall calculate the instanton effects exactly in the supersymmetric σ-model including a thorough treatment of the collective grassmannian coordinates. In sect. 4 we shall investigate a possible breaking by studying the supersymmetric Ward identities. We arrive at the conclusion that no O(3)-invariant quantity can break supersymmetry.

In sect. 5 we finally discuss the fermion zero-mode condensation that occurs in this theory. The condensation gives rise to chiral symmetry breaking, as well as breaking of supersymmetry by an operator that has an infinite-dimensional representation of O(3).

2. Supersymmetry in quantum mechanics

As was said in the introduction we are looking for an explicit breakdown of supersymmetry by instantons. Investigating explicit breaking one must be sure that the formalism used respects the symmetry at every stage. For SUSY the formalism one should use is the superfield formalism (SF). In this section we investigate a quantum mechanics model (QM) [3] as a laboratory for our superfield formalism of instantons.

The form of the supersymmetric lagrangian for QM is given by Cooper and Freedman [3]. It can be obtained from the SF of Salam and Strathdee [9].

In one dimension in $N=2$ theory the superfields are defined on the space (t, θ_+, θ_-), ($\theta_\pm$ is a chiral combination of the anticommuting spinor coordinates) where

$$\{\theta_+, \theta_-\} = \{\theta_-, \theta_+\} = 0\,,$$
$$[\theta, t] = 0\,. \tag{2.1}$$

The supersymmetry transformation is defined by

$$t' \to t + i(\varepsilon_+\theta_- + \varepsilon_-\theta_+)\,, \qquad \theta_\pm \to \theta'_\pm = \theta_\pm + \varepsilon_\pm\,. \tag{2.2}$$

Under eq. (2.2) the following derivatives are invariant

$$D_+ = \frac{\partial}{\partial\theta_+} - i\theta_-\frac{\partial}{\partial t}\,, \qquad D_- = \frac{\partial}{\partial\theta_-} - i\theta_+\frac{\partial}{\partial t}\,. \tag{2.3}$$

We define a real scalar multiplet

$$\Phi(t, \theta_+, \theta_-) = \Phi^*(t, \theta_+, \theta_-)\,, \tag{2.4}$$

where

$$\Phi = q(t) + i\theta_+\psi_+(t) + i\theta_-\psi_-(t) + \theta_+\theta_- F\,, \tag{2.5}$$

by the relation

$$\Phi'(t', \theta'_+, \theta'_-) = \Phi(t, \theta_+, \theta_-)\,; \tag{2.6}$$

using eqs. (2.5), (2.6) and

$$\delta_{\text{total}}\Phi = \delta_{\text{space}}\Phi + \delta_{\text{multiplet}}\Phi = 0\,, \tag{2.7}$$

we find

$$\delta q = i(\varepsilon_+\psi_+ + \varepsilon_-\psi_-)\,, \qquad \delta F = \frac{\partial}{\partial t}(\varepsilon_+\dot{\psi}_+ - \varepsilon_-\dot{\psi}_-)\,,$$

$$\delta\psi_+ = (-\dot{q} - iF)\varepsilon_-\,, \qquad \delta\psi_- = (-\dot{q} + iF)\varepsilon_+\,; \tag{2.8}$$

the most general invariant action is of the form

$$S=\int \mathrm{d}t\, \mathrm{d}\theta_+\, \mathrm{d}\theta_-\, L(D_+\Phi, D_-\Phi, \Phi)\,. \tag{2.9}$$

In what follows we will consider the lagrangian: $L=\frac{1}{2}D_+\Phi D_-\Phi+V(\Phi)$; V a function of Φ.

Using eqs. (2.3), (2.5) and (2.9) one gets, after the integration over the anticommuting coordinates,

$$S=\int \mathrm{d}t(\tfrac{1}{2}\dot{q}^2+i\psi_+\dot{\psi}_-+V'(q)F+F^2+V''\psi_+\psi_-)\,. \tag{2.10}$$

Integrating F in the functional integral

$$Z=\int \mathrm{D}q\, \mathrm{D}\psi_+\, \mathrm{D}\psi_-\, \mathrm{D}F\, \mathrm{e}^{iS} \tag{2.11}$$

we obtain

$$Z=\int \mathrm{D}q\, \mathrm{D}\psi_+\, \mathrm{D}\psi_-\, \mathrm{e}^{iS'}\,, \tag{2.12}$$

where S' is just the action used by Salomonson and Van Holten [3].

From eq. (2.9) we obtain the Euler–Lagrange equation by varying L:

$$\partial S=\int\left\{\frac{\partial L}{\partial(D_+\Phi)}\,\delta D_+\Phi+\frac{\partial L}{\partial(D_-\Phi)}\,\delta D_-\Phi+\frac{\partial L}{\partial\Phi}\,\delta\Phi\right\}\mathrm{d}\theta_-\,\mathrm{d}\theta_+\,\mathrm{d}t=0\,. \tag{2.13}$$

Integrating by parts we find

$$\tfrac{1}{2}[D_+, D_-]\Phi-V'\Phi=0\,; \tag{2.14}$$

using $\mathrm{d}/\mathrm{d}t=i\,\mathrm{d}/\mathrm{d}\tau$ after a Wick rotation we get the classical equations of motion in euclidean space of ref. [3]. The super-instantons are the solutions of the euclidean version of (2.14). Requiring that the action be finite one obtains two classical solutions

$$\Phi_1^{\mathrm{c}}(\tau)=q_-^{\mathrm{c}}(\tau)+i\theta_+\psi_+^{\mathrm{c}}+\theta_+\theta_-F^{\mathrm{c}}\,,\qquad \Phi_2^{\mathrm{c}}=q_+^{\mathrm{c}}+i\theta_-\psi_-^{\mathrm{c}}+\theta_+\theta_-F^{\mathrm{c}}\,, \tag{2.15}$$

where

$$\frac{\mathrm{d}q_\pm^{\mathrm{c}}}{\mathrm{d}\tau}=\pm V'(q^{\mathrm{c}})\,,\qquad \psi_+^{\mathrm{c}}=\lambda_+V'\,,\qquad \psi_-^{\mathrm{c}}=\lambda_-V'\,,\qquad F^{\mathrm{c}}=-V'\,, \tag{2.16}$$

λ_+, λ_- are anticommuting constants.

The solutions in eq. (2.15) are not those that have been chosen by Salomonson and Van Holten [3] (SV) and by Cooper and Freedman [3]; they choose $\psi_{\mathrm{c}}=0$. As we shall show below these two choices give the same results in any order of $\hbar$. Our choice is a supersymmetric one. A supersymmetry transformation on Φ^{c} amounts to the following change of the parameters of Φ^c:

$$\tau \to t + i\varepsilon_\pm \lambda_\mp , \qquad \lambda'_\pm = \lambda_\pm + \varepsilon_\pm . \tag{2.17}$$

The solution of SV is supersymmetric only on the points $\varepsilon_- = 0$ or $\varepsilon_+ = 0$. Cooper and Freedman choose the supersymmetry transformation to be of order $\hbar$. We prefer not to take this choice because we are using an expansion in $\hbar$ and we want it to preserve supersymmetry at any order.

Expanding the superfields around the classical solutions:

$$\Phi(\tau) = \Phi^c(\tau) + \sqrt{\hbar}\, \Phi^q , \tag{2.18}$$

where

$$\Phi^q = \bar{q} + i\theta_+ \bar{\psi}_+ + i\theta_- \bar{\psi}_- + \theta_+\theta_- \bar{F} ,$$

the action becomes

$$S = S^c + \hbar \int \mathrm{d}\tau [\tfrac{1}{2} D_+ \Phi^q D_- \Phi^q + \tfrac{1}{2} V''(\Phi^c)(\Phi^q)^2 + \tfrac{1}{6}\sqrt{\hbar}\, V'''(\Phi^c)(\Phi^q)^3]\, \mathrm{d}\theta_+\, \mathrm{d}\theta_- . \tag{2.20}$$

After partial integration it reduces to

$$S = S^c + S^q = S^c - \hbar \int \mathrm{d}\tau\, (\tfrac{1}{2}\dot{\bar{q}}^2 + \tfrac{1}{2}\bar{F}^2 + i\psi_+ \dot{\psi}_- + V''_c \bar{q}\bar{F} + V'''_c \bar{q}\psi^c_+ \bar{\psi}_-$$
$$+ V''_c \psi_+ \psi_- + \tfrac{1}{2}\sqrt{\hbar}\, (V''' V''_c + \tfrac{1}{3} V''''_c V'_c)\bar{q}^3 + \sqrt{\hbar}\, V'''_c \bar{q}\psi_+\psi_-) . \tag{2.21}$$

This is the form SV found, excluding the term

$$V''' \psi^c_\pm q \psi_\mp , \tag{2.22}$$

which obviously does not appear in their expansion.

Three comments are in order: (a) ψ_c do not contribute to S^c; (b) without the term in eq. (2.22) the quadratic part of the quantum fluctuation on the action is not supersymmetric; (c) supersymmetry does not mix Φ^c and Φ^q so both of them transform according to eq. (2.8) separately. To prove that we get the same result as SV we first have to deal with the problem of zero modes of the bilinear terms in (2.20). They arise from superspace translational invariance of the super-instantons,

$$\Phi_0 = \bar{q}^0 + i\theta_\pm \psi^0_\pm + \theta_+\theta_- F^0 ,$$

where

$$\bar{q}^0 = \alpha V'_c , \qquad \psi^0_\pm = n_\pm V'_c , \qquad \text{if} \qquad \dot{q}_c = \mp V'_c . \tag{2.23}$$

The existence of a zero mode gives rise to a non-gaussian behaviour of the functional integral. This problem is dealt with by introducing collective coordinate τ_0 replacing the bosonic zero mode. In supersymmetry the bosonic collective coordinate should be accompanied by the fermionic collective coordinate λ_0 in order to keep the integration measure supersymmetric.

The collective coordinate is defined by writing

$$\Phi(\tau, \lambda) = \Phi^{c}(\tau, \lambda) + \sqrt{\hbar}\,[\Phi^{0}(\tau) + \sum_n{}' \alpha_n \Phi_n(\tau)$$

$$\equiv \Phi^{c}(\tau - \tau_0, \lambda - \lambda_0) + \sqrt{\hbar} \sum_n{}' \alpha_n \Phi_n(\tau - \tau_0)\,, \tag{2.24}$$

where $\{\Phi^0, \Phi_n\}$ form a complete orthonormal set of eigenfunctions of the operator $(D_+D_- + V'(\Phi^c))$ and $\sum_n'$ denotes a sum over the non-zero modes Φ_n only.

In component fields eq. (2.24) reads: (as an example we take the $\dot{q}^c = -V'$ solution)

$$q = q^c(\tau) + \sqrt{\hbar}\left\{\frac{\beta_0}{N} V_c'(\tau) + \sum_n{}' \beta_n q_n(\tau)\right\}$$

$$\equiv q^c(\tau - \tau_0) + \sqrt{\hbar} \sum_n{}' \beta_n q_n(\tau - \tau_0)$$

$$\psi_+ = \lambda_+ V_c'(\tau) + \sqrt{\hbar}\left\{\frac{\lambda_{\mp}^0 V_c'}{N} + \sum_n{}' \gamma_n \psi_n(\tau)\right\}$$

$$\equiv (\lambda_+ - \lambda_+^0) V_c'(\tau - \tau_0) + \sqrt{\hbar} \sum_n{}' \gamma_n' \psi_n'(\tau - \tau_0)\,. \tag{2.25}$$

The functional integration measure for Φ is

$$\mathcal{D}\Phi = \prod_n \frac{\mathrm{d}\beta_n}{\sqrt{2\pi}} \prod_k \mathrm{d}\gamma_k^+\, \mathrm{d}\gamma_k^- \prod \frac{\mathrm{d}F}{\sqrt{2\pi}} = \frac{\mathrm{d}\beta_0}{\sqrt{2\pi}} \frac{\mathrm{d}\lambda_0^+}{\sqrt{2\pi}} \prod_{n,k}{}' \frac{\mathrm{d}\bar{\beta}_n}{\sqrt{2\pi}} \mathrm{d}\bar{\gamma}_k^+\, \mathrm{d}\bar{\gamma}_k^-\,. \tag{2.26}$$

Trading $(\beta_0, \lambda_0, \beta_n, \gamma_n)$ for $(\tau_0, \lambda_0, \bar{\beta}_n, \bar{\gamma}_n)$ we have to calculate a superjacobian factor [10] using the bosonic part of the jacobian calculated by SV

$$-\frac{N}{\sqrt{\hbar 2\pi}}\left(1 - \frac{\sqrt{\hbar}}{N^2} \sum_n \bar{\beta}_n \int \dot{q}_n'(\tau)\, V_c'\, \mathrm{d}\tau\right),$$

where the prime on q_n' indicates the shift $\tau - \tau_0$. and that the fermionic part is equal to $N/\sqrt{\hbar}$. One gets the following superjacobian

$$\frac{1}{\sqrt{2\pi}}\left(1 + \frac{\sqrt{\hbar}}{N^2} \sum_n \bar{\beta}_n \int \dot{q}_n'(\tau)\, V_c'\, \mathrm{d}\tau\right). \tag{2.28}$$

Finally the partition function is

$$Z = -\mathrm{e}^{-S^c} \int \mathrm{d}\tau_0\, \mathrm{d}\lambda_0 \left(-1 + \frac{\sqrt{\hbar}}{N^2} \sum_n \beta_n \int \dot{q}_n' V_c'\, \mathrm{d}\tau\right) \prod_n{}' \frac{\mathrm{d}\beta_n}{\sqrt{2\pi}} \mathrm{d}\gamma_n^+\, \mathrm{d}\gamma_n^-\, \mathrm{d}F\, \mathrm{e}^{-S_q}\,, \tag{2.29}$$

where S_q is given in eq. (2.21). This partition function gives the same result as the one used by SV for every quantity.

To get non-vanishing matrix elements with the partition function in eq. (2.29) the operator should include ψ_c^+ to reduce the integral over λ_0 (recall that $\int d\lambda_0 = 0$ and $\int d\lambda_0\, \lambda_0 = 1$). In the SV approach it reads: "the operator must be proportional to the fermion zero mode". The difference between ψ_c^+ and the fermionic zero mode is a factor $\sqrt{\hbar/N}$ which is just the factor between the superjacobian (2.28) and the jacobian (2.21); consequently the results in the two approaches are the same.

Although in the one-instanton calculation one gets a non-zero expectation value for the supersymmetric charge operator Q, which indicates a non-zero vacuum energy (recall that $H = Q^2$), one would like to get the same result directly in terms of the free energy in the expression for the partition function. Since the energy is proportional to e^{-S^c}, one will have to consider instanton–anti-instanton configurations.

At energies near the ground state the dilute gas is a valid approximation. Using the fermionic zero-mode argument one may think that the dilute gas gives no contribution to the partition function. However, the way to calculate it, using this approximation [11], is to put the instanton (anti-instanton) in finite boxes and to sum over all possible locations of the boxes. For the integration inside each finite interval T, one does not need to use the collective coordinates. Since the lowest fermion eigenvalue approaches zero as $T \to \infty$ at the same rate as the bosonic one, the "zero mode" contribution cancels and we get a finite result (independent of T). Since the result of integration inside any box is finite, independent of the box size, the dilute gas approximation gives a non-vanishing contribution which would presumably be proportional to t^n (n is the number of instanton–anti-instanton pairs). The contribution from $n = 1$, would be in agreement with the SV result obtained using the Schrödinger equation.

Before concluding this chapter we would like to add some remarks about the fulfillment of the supersymmetric "Ward identities" in spite of the fact that spontaneous breaking of the supersymmetry occurs in this model. Our arguments here will follow the discussion made in ref. [7]. We consider the contribution from the 1-instanton to the matrix element of the fermion operator:

$$\langle q_-^c | \hat{\psi} | q_+^c \rangle \neq 0 \,, \tag{2.30}$$

where the indices +, − refer to two neighbouring classical minima, and the corresponding matrix element of the bosonic counterpart $\hat{q}$ to $\hat{\psi}$:

$$\langle q_-^c | \hat{q} | q_+^c \rangle = 0 \,, \tag{2.31}$$

(the last matrix element being zero because it did not include any Grassmann variables). Naïvely one could fear that this discrepancy (i.e. $\langle \hat{\psi} \rangle \neq 0$ while $\langle \hat{q} \rangle = 0$) would mean an explicit breaking of supersymmetry because a supersymmetric transformation would bring $\hat{q}$ into $\hat{\psi}$. However since the classical vacuum states $|q_\pm^c\rangle$ are not invariant under the supersymmetry charge operator Q, i.e. $\boldsymbol{Q}|q_\pm^c\rangle \neq 0$ (which means spontaneous breakdown of the supersymmetry), this fact will compensate the non-zero fermionic contribution and altogether the "Ward identities" still hold. In the case of the O(3) σ-model such a compensation cannot occur since a spontaneous breaking is excluded by the Witten index being different from zero. In that model a situation like the one in eqs. (2.30), (2.31) would lead to a violation of the Ward identities.

3. Supersymmetric O(3) σ-model

In this section we shall derive the supersymmetric instanton gas exactly as was done without supersymmetry in ref. [6]. The euclidean action for a supersymmetric non-linear O(3) σ-model [12] is

$$S=\frac{1}{2g^2}\int \mathrm{d}^2x\int \mathrm{d}^2\theta\, \varepsilon^{\alpha\beta}D_\alpha \Phi^a D_\beta \Phi^a\,, \qquad D_\alpha=\frac{\partial}{\partial\bar{\theta}_\alpha}+i(\gamma^\mu\theta)_\alpha\frac{\partial}{\partial x_\mu}\,, \tag{3.1}$$

where ϕ^a, ψ^a and F^a are the components of three real superfields

$$\Phi^a(x,\theta)=\phi^a(x)+\bar{\theta}\psi^a+\tfrac{1}{2}\bar{\theta}\theta F^a\,, \qquad a=1,2,3\,, \tag{3.2}$$

and they satisfy the constraint

$$\sum_a \Phi^a\Phi^a=1\,, \tag{3.3}$$

and g is the coupling constant.

Since the constraints are non-linear it is convenient to use new variables obtained by stereographic projections that are constraint free.

$$\Phi=\frac{\Phi^1+i\Phi^2}{1+\Phi^3}\,, \qquad \bar{\Phi}=(\Phi)^*\,. \tag{3.4}$$

Then the action (3.1), in terms of component fields of Φ, $\bar{\Phi}$, is

$$\begin{aligned} S=\frac{2}{g^2}\int \mathrm{d}^2x\sqrt{g}\,\frac{1}{\rho_0^2}\Bigg[&\nabla_\mu\bar{\phi}\nabla^\mu\phi+\tfrac{1}{2}i\bar{\psi}\gamma^\mu\overleftrightarrow{\nabla}_\mu\psi \\ &-i\bar{\psi}\gamma^\mu\frac{(\bar{\phi}\nabla_\mu\phi-\phi\nabla_\mu\bar{\phi})}{\rho_0}\psi+\frac{1}{2\rho_0^2}(\bar{\psi}\bar{\psi})(\psi\psi)-\left(F-\frac{\bar{\phi}\psi\psi}{\rho_0}\right)\left(\bar{F}-\frac{\phi\bar{\psi}\bar{\psi}}{\rho_0}\right)\Bigg]\,, \end{aligned} \tag{3.5}$$

where $\rho_0=1+\phi\bar{\phi}$. Here and throughout the following we have used the notation of ref. [13]. We take the metric $g_{\mu\nu}$ to be the conformally flat metric $g_{\mu\nu}=\Omega^2\delta_{\mu\nu}$, $\Omega=(1+\chi^2/4R^2)^{-1}$. With this metric, space-time is a compact 2-sphere of radius R, and provides an infrared regularization. At the end of the calculations we will take the flat-space limit $R\to\infty$. ∇_μ in the above expression are the usual covariant derivatives on the curved space acting on spinor fields. More explicitly the action can be rewritten as

$$S=\frac{2}{g^2}\int d^2x\,\Omega^2\frac{1}{\rho_0^2}\Big[\Omega^{-2}\partial_\mu\bar\phi\partial_\mu\phi+\tfrac{1}{2}i\Omega^{-3/2}(\bar\psi\gamma^\mu\partial_\mu\Omega^{1/2}\psi$$
$$-(\partial_\mu(\Omega^{1/2}\bar\psi))\gamma^\mu\psi)-i\Omega^{-1}\bar\psi\gamma^\mu\left(\frac{\bar\phi\partial_\mu\phi-\phi\partial_\mu\bar\phi}{\rho_0}\right)\psi$$
$$+\frac{1}{2\rho_0^2}(\bar\psi\bar\psi)(\psi\psi)-\left(F-\frac{\bar\phi\psi\psi}{\rho_0}\right)\left(\bar F-\frac{\phi\bar\psi\bar\psi}{\rho_0}\right)\Big], \tag{3.6}$$

where γ^μ are the usual flat γ-matrices. In this form it is clear that, if in the flat space, a solution $(\phi_0, \psi_0, F_0, \bar\phi_0, \bar\psi_0, \bar F_0)$ is given, then there exists a corresponding solution on the 2-sphere with

$$\phi=\phi_0\,,\qquad \psi=\Omega^{-1/2}\psi_0\,,\qquad F=\Omega^{-1}F_0\,,$$
$$\bar\phi=\bar\phi_0\,,\qquad \bar\psi=\Omega^{-1/2}\bar\psi_0\,,\qquad \bar F=\Omega^{-1}\bar F_0\,. \tag{3.7}$$

The action S is invariant under the following supersymmetric transformation:

$$\delta\phi=\xi\psi\Omega^{1/2}\,,\qquad \delta\psi_\alpha=-i\Omega^{-1/2}\bar\xi^\beta\gamma_\beta^{\mu\delta}\varepsilon_{\delta\alpha}\partial_\mu\phi-\Omega^{1/2}\xi_\alpha F\,,$$
$$\delta\bar\phi=\bar\xi\bar\psi\Omega^{1/2}\,,\qquad \delta\bar\psi^\alpha=i\varepsilon^{\alpha\beta}\gamma_\beta^{\mu\delta}\xi_\delta\Omega^{-1/2}\partial_\mu\bar\phi-\Omega^{1/2}\bar\xi^\alpha\bar F\,,$$
$$\delta F=-i\Omega^{-1}\bar\xi\gamma^\mu\partial_\mu(\Omega^{1/2}\psi)\,,\qquad \delta\bar F=i\Omega^{-1}(\partial_\mu(\Omega^{1/2}\bar\psi))\gamma^\mu\xi\,, \tag{3.8}$$

where ξ_α, $\bar\xi^\alpha$ are constant grassmannian parameters, i.e. $\partial_\mu\xi_\alpha=\partial_\mu\bar\xi^\alpha=0$. In fact S is invariant under more general transformations, namely conformal supersymmetry, with ξ_α satisfying the condition $\gamma^\nu\gamma^\mu\partial_\nu\xi=0$, but we shall not be needing it in the following.

3.1. GENERALISED INSTANTON SOLUTIONS

Instanton solutions are solutions of certain first-order differential equations (analogous to the self-dual solutions in YM theories) which imply the equations of motion. Besides they carry certain topological numbers and minimise the action in a given topological sector. Thus, in the saddle-point approximation it is appropriate to expand the fields around such solutions. In the conventional treatment one usually takes only bosonic solutions, however as explained in the introduction, we shall here consider generalised instanton solutions containing fermionic parts as well.

Let us first take the flat-space case ($\Omega=1$). We introduce complex coordinates $z=x^0+ix^1$. General solutions to the first-order equations are given in ref. [10]

$$\psi_2=F=0\,,\qquad \frac{\partial}{\partial\bar z}\psi_1=\frac{\partial}{\partial\bar z}\phi=0\,. \tag{3.9}$$

For such solutions the action S is proportional to the topological charge q and gets a contribution only from the bosonic fields where

$$q=\frac{1}{\pi}\int \mathrm{d}z\,\mathrm{d}\bar{z}\frac{1}{\rho_0^2}(\partial_z\phi\partial_{\bar{z}}\bar{\phi}-\partial_z\bar{\phi}\partial_{\bar{z}}\phi)\,. \tag{3.10}$$

The bosonic instanton solution with topological charge q can be expressed as

$$\phi=c\prod_{i=1}^{q}\frac{z-a_i}{z-b_i}\,, \tag{3.11}$$

where a_i, b_i and c are complex parameters such that $c\neq 0$, $a_i\neq b_j$. The fermionic part ψ is of course any arbitrary function of z.

Using eq. (3.7) we then obtain the solution on the curved space. ϕ is the same as in the flat case, $\psi_2=F=0$ and $\psi_1=\Omega^{-1/2}\xi f(z)$ where ξ is a constant Grassmann parameter and $f(z)$ is an arbitrary function of z. The topology of the solution manifold is determined by the following norm of small fluctuations $(\Delta\phi, \Delta\psi, \Delta F)$ given by the measure on the underlying Kähler manifold [13]

$$\|\Delta\phi\|^2=\int \mathrm{d}^2x\Omega^2\frac{1}{\rho_0^2}|\Delta\phi|^2\,,\qquad \|\Delta\psi\|^2=\int \mathrm{d}^2x\Omega\frac{1}{\rho_0^2}(\Delta\bar{\psi},\Delta\psi)\,,$$

$$\|\Delta F\|^2=\int \mathrm{d}^2x\Omega^2\frac{1}{\rho_0^2}|\Delta F|^2\,, \tag{3.12}$$

where $\rho_0=1+\phi\bar{\phi}$ with ϕ being the bosonic part of the background solution. Note that the above norm arises naturally in the context of functional integration. Here one uses the background field method, which consists of expanding the fields around a given solution and solving the eigenvalue problem for small fluctuations. In any sensible definition of a functional integral, one restricts these fluctuations to be of finite L^2 norm with respect to some natural measure that in general depends on the background field. In the present case this norm coincides with (3.12). Solving the zero-mode problem for finite-norm fluctuations one gets the tangent space at each point of the solution manifold.

It is easy to verify that $\Delta\phi=\partial_c\phi_{\rm cl}$, $\partial_{ai}\phi_{\rm cl}$, $\partial_{bi}\phi_{\rm cl}$ have all finite norms and are zero modes. Finiteness of the norm for fermionic zero modes implies: $\Delta\psi_I=(1/\sqrt{\Omega})\varepsilon_I\partial_{\alpha_I}\phi_{\rm cl}$ where

$$\alpha_I=\begin{cases}a_i\,, & I=i\,, \qquad\qquad I=1,\dots,2q\\ b_i\,, & I=q+i\,.\end{cases}$$

and ε_I are Grassmann parameters. (Note that $(1/\sqrt{\Omega})\partial_c\phi_{\rm cl}$ is not normalisable.)

Thus the solution manifold that is connected to purely bosonic solutions (we therefore call it the B-sector) can be coordinated by the component fields:

$$\phi=c\prod_{i=1}^{q}\frac{z-a_i}{z-b_i}\,,\qquad \psi_1=\varepsilon_I\frac{1}{\sqrt{\Omega}}\frac{\partial\phi}{\partial\alpha_I}\,,\qquad \psi_2=F=0\,, \tag{3.13}$$

there are of course infinitely many sectors, each labelled by $\xi f(z)$, where $f(z)$ is such that $\int d^2x\,(\Omega/\rho_0^2)(f)^2=\infty$ and ξ is a non-zero Grassmann number. In the ξf sector ϕ, ψ_2 and F remain the same as above, but ψ_1 is

$$\overset{(\xi f)}{\psi_1}=\frac{\xi f(z)}{\sqrt{\Omega}}+\varepsilon_I\frac{1}{\sqrt{\Omega}}\partial_{\alpha_I}\phi\,.$$

These sectors are, however, not connected to purely bosonic solutions and they have no physical relevance so we shall restrict ourselves to the B-sector.

One important property of the solution (3.13) is that supersymmetric transformations (3.8) leave the form of the solution (3.13) unchanged, and hence formally induce a transformation on the parameters $(c, \alpha_I, \varepsilon_I)$ as

$$\begin{aligned} c&\to c\,,\\ \alpha_I&\to\alpha_I+\varepsilon_I\xi_2\,,\\ \varepsilon_I&\to\varepsilon_I+\bar{\xi}^1, \end{aligned}\tag{3.14}$$

where ξ_α are the parameters of the transformations. This fact will have important consequences when we discuss Ward identities.

3.2. FUNCTIONAL INTEGRATION

As usual we expand the fields around the classical solution

$$\begin{aligned} \phi&=\phi_{\rm cl}+\sqrt{2}\,g\phi_q\,,\\ \psi&=\psi_{\rm cl}+\sqrt{2}\,g\psi_q\,,\\ F&=F_{\rm cl}+\sqrt{2}\,gF_q\,, \end{aligned}\tag{3.15}$$

and keep the terms quadratic in quantum fluctuations. Higher-order terms would be needed only if one wishes to calculate quantities up to higher order in g, and this can be done by introducing source terms and taking functional derivatives of the generating functional with respect to the sources. In the following we shall only calculate up to lowest order in g, ignoring all the higher-order terms. After some algebra we obtain

$$\begin{aligned} S=S_{\rm cl}+\int d^2x\,\Omega^2\Big[&\bar{\phi}_q(-4\Omega^{-2}\partial_{\bar z}\rho_c^{-2}\partial_{\bar z})\phi_q+2i\Omega^{-3/2}\bar{\psi}_q\begin{bmatrix}0 & \partial_z\rho_c^{-2}\\ \rho_c^{-2}\partial_{\bar z} & 0\end{bmatrix}\Omega^{1/2}\psi_q\\ &-\frac{1}{\rho_c^2}\left(F_q-\frac{\bar{\phi}_c\psi_q\psi_c}{\rho_c}\right)\left(\bar{F}_q-\frac{\phi_c\bar{\psi}_q\bar{\psi}_c}{\rho_c}\right)+\mathscr{L}(\varepsilon_I)\Big]\,. \end{aligned}\tag{3.16}$$

Here $\rho_c=1+\phi_c\bar{\phi}_c$ and ϕ_c, ψ_c are classical fields, $\mathscr{L}(\varepsilon_I)$ contains all the remaining terms containing ε_I such that $\mathscr{L}(\varepsilon_I=0)=0$.

The partition function Z is defined as usual

$$Z=\int \mathscr{D}\phi_q\mathscr{D}\bar{\phi}_q\mathscr{D}\psi_q\mathscr{D}\bar{\psi}_q\mathscr{D}F_q\mathscr{D}\bar{F}_q\, e^{-S}, \tag{3.17}$$

where the measure $\mathscr{D}\phi_q$ etc. is realised explicitly by expanding the quantum fluctuation in terms of an orthonormal basis with respect to the inner product defined by (3.12). We shall deal with the fields ϕ_q, ψ_q in subsect. 3.3. Integration over F_q is straightforward. Define $\tilde{F}=F_q/\rho_c$, then the norm (3.12) is simply $\int|\tilde{F}|^2\, d^2x\,\Omega^2$. Thus the relevant part in the action is $(\tilde{F}-\bar{\phi}_c\psi_q\psi_c/\rho_c^2)\times(\bar{\tilde{F}}-\phi_c\bar{\psi}_q\bar{\psi}_c/\rho_c^2)$. This is a gaussian and the result of the integral is just one.

As for $\mathscr{L}(\varepsilon_I)$ we expand e^{-S} in a power series of $\mathscr{L}(\varepsilon_I)$. This series will have only a finite number of terms owing to the Grassmann character of ε_I. Then

$$Z=\int \mathscr{D}\phi\mathscr{D}\bar{\phi}\mathscr{D}\psi\mathscr{D}\bar{\psi}\left(1-\int\mathscr{L}(\varepsilon_I)+\tfrac{1}{2}\int\mathscr{L}(\varepsilon_I)^2+\cdots\right)e^{-(S_{cl}+S_0)},$$

where

$$S_0=\int d^2x\,\Omega^2\left[\bar{\phi}_q(-4\Omega^{-2}\partial_z\rho_c^{-2}\partial_{\bar{z}})\phi_q+2i\Omega^{-3/2}\bar{\psi}_q\begin{bmatrix}0 & \partial_z\rho_c^{-2}\\ \rho_c^{-2}\partial_{\bar{z}} & 0\end{bmatrix}\Omega^{1/2}\psi_q\right]. \tag{3.18}$$

Now $\mathscr{L}(\varepsilon_I)$ has the property that it contains at least one ψ_2 or $\bar{\psi}_2$ more than ψ_1, $\bar{\psi}_1$. This is because ε_I comes from the classical fermion solution which has only a ψ_1, $\bar{\psi}_1$ part, while every term in the action (3.6) has equal numbers of ψ_1 and $\bar{\psi}_2$ or ψ_2 and $\bar{\psi}_1$. Now to the lowest order in g fermionic Green functions in the instanton background would connect ψ_1 with $\bar{\psi}_2$ and ψ_2 with $\bar{\psi}_1$. This implies that the excess ψ_2 or $\bar{\psi}_2$ in $\mathscr{L}(\varepsilon_I)$ cannot be connected with any field. Therefore to the lowest order in g, these terms would not make any contribution. Thus the partition function is

$$Z=\int \mathscr{D}\phi\mathscr{D}\bar{\phi}\mathscr{D}\psi\mathscr{D}\bar{\psi}\,\exp[-(S_{cl}+S_0)]. \tag{3.19}$$

Now we introduce the collective coordinates.

3.3. COLLECTIVE COORDINATE JACOBIAN

The measure in the functional integral is explicitly obtained by expanding the field ϕ_q, ψ_q in terms of an orthonormal basis with respect to the inner product defined by eq. (3.12). However the bosonic and fermionic fields have a certain number of zero modes (respectively $4q+2$ and $4q$). Integration over these zero modes means integrating over the solution manifold or, in other words, over the parameters of the classical solution. The measure on the parameter space can be obtained by making a transformation from an orthonormal basis of zero modes to the tangent vectors $\partial/\partial\alpha_I$, $\partial/\partial c$, $\partial/\partial\varepsilon_I$ on the solution manifold, and calculating the corresponding jacobian.

Let α_μ denote the bosonic parameters of the solution with $\mu = 1, \dots, 2q+1$; $\alpha_{2q+1} = c$ and $\alpha_\mu = \alpha_I$ for $\mu \neq 2q+1$. Let $\phi_\mu(x)$ denote a complete orthonormal basis in the space of bosonic zero modes, and let $f_I(x)$ denote the same in the space of fermionic zero modes. Then fields in a neighbourhood of the classical solution can be expressed as

$$\phi = \phi_{\rm cl}(\alpha_\mu) + \sqrt{2}\, g(A_\mu \phi_\mu(\alpha) + \underset{\sim}{\phi}(\alpha))\,,$$
$$\psi = \varepsilon_I \frac{1}{\sqrt{\Omega}} \partial_{\alpha_I} \phi_{\rm cl}(\alpha) + \sqrt{2}\, g(\xi_I f_I(\alpha) + \underset{\sim}{\psi}(\alpha))\,, \tag{3.20}$$

where $\underset{\sim}{\phi}$ and $\underset{\sim}{\psi}$ are non-zero modes. The zero-mode part of the functional measure is then $\prod_\mu \mathrm{d}A_\mu\, \mathrm{d}\bar{A}_\mu \prod_I \mathrm{d}\xi_I\, \mathrm{d}\bar{\xi}_I$. Now we absorb the zero modes by shifting the parameters of the solution $\alpha \to \alpha + \delta\alpha$, $\varepsilon \to \varepsilon + \delta\varepsilon$. Then

$$\phi_{\rm cl}(\alpha + \delta\alpha) + \sqrt{2}\, g\underset{\sim}{\phi}(\alpha + \delta\alpha) = \phi_{\rm cl}(\alpha) + \sqrt{2}\, g(A_\mu \phi_\mu(\alpha) + \underset{\sim}{\phi}(\alpha))\,,$$
$$(\varepsilon_I + \delta\varepsilon_I) \frac{1}{\sqrt{\Omega}} \partial_{\alpha I} \phi_{\rm cl}|_{\alpha+\delta\alpha} + \sqrt{2}\, g\underset{\sim}{\psi}(\alpha + \delta\alpha) = \frac{\varepsilon_I}{\sqrt{\Omega}} \partial_{\alpha_I} \phi_{\rm cl} + \sqrt{2}\, g(\xi_1 f_I(\alpha) + \underset{\sim}{\psi}(\alpha))\,. \tag{3.21}$$

Making a Taylor expansion on the left-hand side around α, ε and using the orthonormality relations of ϕ_μ, f_I, we obtain

$$A_\mu = \frac{1}{\sqrt{2}\, g} [\delta\alpha_\nu N_{\mu\nu} + \mathrm{O}(g)]\,,$$
$$\xi_I = \frac{1}{\sqrt{2}\, g} [\delta\varepsilon_I N'_{IJ} + \varepsilon_J \delta\alpha_\mu M_{I\mu}^{\,J} + \mathrm{O}(g)]\,,$$

where

$$N_{\mu\nu} = \int \mathrm{d}^2x\, \Omega^2 \frac{1}{\rho_{\rm c}^2} \bar{\phi}_\mu \frac{\partial \phi_{\rm cl}}{\partial \alpha_\nu}\,,$$
$$N'_{IJ} = \int \mathrm{d}^2x\, \Omega^2 \frac{1}{\rho_{\rm c}^2} \frac{1}{\sqrt{\Omega}} \bar{f}_I \frac{\partial \phi_{\rm cl}}{\partial \alpha_J}\,,$$
$$M_{I\mu}^{\,J} = \int \mathrm{d}^2x \frac{\Omega^2}{\rho_{\rm c}^2} \bar{f}_x \frac{\partial^2 \phi_{\rm cl}}{\partial \alpha_\mu\, \partial \alpha_J}\,. \tag{3.22}$$

The jacobian J of the transformation [11] $(A_\mu, \bar{A}_\mu, \xi_I, \bar{\xi}_I) \to (\alpha_\mu, \tilde{\alpha}_\mu, \varepsilon_I, \bar{\varepsilon}_I)$ is

$$J = \begin{vmatrix} \dfrac{\partial A_\mu}{\partial \alpha_\nu} & \dfrac{\partial A_\mu}{\partial \varepsilon_J} \\ \dfrac{\partial \xi_I}{\partial \alpha_\nu} & \dfrac{\partial \xi_I}{\partial \varepsilon_J} \end{vmatrix}^2 = \left| \operatorname{Sup\,det} \begin{bmatrix} N_{\mu\nu} & 0 \\ M_{I\nu}^{J} \varepsilon_J & N'_{IJ} \end{bmatrix} \right|^2 = \left| \frac{\det N}{\det N'} \right|^2. \tag{3.23}$$

Let

$$M_{\mu\nu} = \int d^2x \frac{\Omega^2}{\rho_c^2} \frac{\partial\bar{\phi}_{cl}}{\partial\alpha_\mu} \frac{\partial\phi_{cl}}{\partial\alpha_\nu},$$

$$M'_{IJ} = \int d^2x \frac{\Omega}{\rho_c^2} \frac{\partial\phi_{cl}}{\partial\alpha_I} \frac{\partial\phi_{cl}}{\partial\alpha_J}, \tag{3.24}$$

then clearly M and M' are hermitian matrices and moreover from the orthonormality of ϕ_μ and f_I it follows that

$$M = N^+N, \qquad M' = N'^+N'.$$

Thus:

$$J = \frac{\det M}{\det M'}, \tag{3.25}$$

Det M and det M' can be calculated as in ref. [9]. We write

$$\frac{\partial\phi_{cl}}{\partial\alpha_\mu} \prod_i (z-b_i)^2 = U_{\nu\mu} z^{\nu-1}, \tag{3.26}$$

where $U_{\nu\mu}$ are functions only of parameters (α). We have, further,

$$U_{2q+1,I} = 0, \qquad \forall I \text{ and } U_{2q+1,2q+1} = 1; \tag{3.27}$$

then

$$M_{\mu\nu} = U^+_{\mu\lambda} R_{\lambda\rho} U_{\rho\nu},$$

$$M'_{IJ} = U^+_{IK} R'_{KL} U_{LJ}, \tag{3.28}$$

where

$$R_{\lambda\rho} = \int \bar{z}^\lambda z^\rho \rho_c^{-2} \Omega^2 \, d^2x,$$

$$R'_{IJ} = \int \bar{z}^I z^J \rho_c^{-2} \Omega \, d^2\chi. \tag{3.29}$$

From eq. (3.28) it follows that $\det M = \det R |\det U_{\mu\nu}|^2$ and $\det M' = \det R' |\det U_{IJ}|^2$. Moreover from eq. (3.27) we see that $\det U_{\mu\nu} = \det U_{IJ}$. Thus we obtain for J,

$$J = \frac{\det R}{\det R'}. \tag{3.30}$$

3.4. INTEGRATION OVER THE NON-ZERO MODES

For convenience we define

$$\phi = \frac{\phi_q}{\rho_c} \prod_i \frac{(z-b_i)^2}{|z-b_i|^2},$$

$$\psi = \frac{\psi_q}{\rho_c} \prod_i \frac{(z-b_i)^2}{|z-b_i|^2},$$

$$\rho = \rho_c \prod_i |z-b_i|^2 . \tag{3.31}$$

Then S_0 in terms of these new variables becomes

$$S_0 = \int d^2x\, \Omega^2 [\bar{\phi} M_\beta \phi + \bar{\psi} M_f \psi],$$

where

$$M_\beta = -4\Omega^{-2} \rho \partial_z \rho^{-2} \partial_{\bar{z}} \rho ,$$

$$M_f = 2i\Omega^{-3/2} \begin{bmatrix} 0 & \rho \partial_z \rho^{-1} \\ \rho^{-1} \partial_{\bar{z}} \rho & 0 \end{bmatrix} \Omega^{1/2}, \tag{3.32}$$

and the inner products (3.12) become simply

$$\langle \tilde{\phi} \phi \rangle = \int d^2x\, \Omega^2 \bar{\tilde{\phi}} \phi ,$$

$$\langle \tilde{\psi} \psi \rangle = \int d^2x\, \Omega^2 \bar{\tilde{\psi}} \psi . \tag{3.33}$$

Integration over the non-zero modes formally will yield $\det' M_f [\det' M_\beta]^{-1}$ where the prime indicates the determinant of non-zero eigenvalues. However these quantities are ultraviolet divergent and have to be regularized. In the following we shall use proper-time regularization as in ref. [9]. The bosonic determinant has already been calculated there. In order to calculate the fermionic determinant, we proceed as follows. Consider the operator

$$D = -4\Omega^{-3/2} \rho \partial_z \rho^{-2} \Omega^{-1} \partial_{\bar{z}} \rho \Omega^{1/2}, \tag{3.34}$$

acting on normalisable functions with the norm defined by $\int S\, d^2x \Omega^2$. Let f_A be a complete orthonormal basis of non-zero mode eigenfunctions of D

$$D f_A = \lambda_A^2 f_A , \qquad \lambda_A \neq 0 . \tag{3.35}$$

Note that eigenvalues of D are ≥ 0. Now expand

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad \text{as} \qquad \psi_1 = \xi_A f_A$$

and $\psi_2 = \eta_A \chi_A$ where ξ_A, η_A are Grassmann parameters and

$$\chi_A = \frac{2i}{\lambda_A} \Omega^{-3/2} \rho^{-1} \partial_{\bar{z}} \rho \Omega^{1/2} f_A , \qquad (\lambda_A \neq 0) . \tag{3.36}$$

It is easy to check that $\{\chi_A\}$ form an orthonormal basis on the space of non-zero modes of D. Then the fermionic part in the action S_0 becomes

$$\bar{\psi}M_f\psi=\sum_A \lambda_A(\bar{\xi}_A\eta_A+\bar{\eta}_A\xi_A)\,. \tag{3.37}$$

Therefore the integral

$$\int_{\text{non-zero modes}} \mathscr{D}\psi\,\mathscr{D}\bar{\psi}\,\mathrm{e}^{-\bar{\psi}M_f\psi}=\int\prod_A \mathrm{d}^2\xi_A\,\mathrm{d}^2\eta_A\,\mathrm{e}^{-\sum_A\lambda_A(\bar{\xi}_A\eta_A+\bar{\eta}_A\xi_A)}=\prod_A\lambda_A^2=\det{}'\,D\,. \tag{3.38}$$

Of course these are formal expressions and in order to make any sense we have to regularize. The regularized $\det' D$ is defined as [9]

$$\ln\det{}'\,D=\lim_{\varepsilon\to 0}\left[-\int_\varepsilon^\infty\frac{\mathrm{d}t}{t}(\mathrm{Sp}\,\mathrm{e}^{-tD}-p)-\alpha_0\ln\varepsilon+\alpha_1\varepsilon^{-1}\right], \tag{3.39}$$

where p is the number of zero modes of D, and a_0 and α_1 are the coefficients in the expansion

$$\mathrm{Sp}\,\mathrm{e}^{-tD}\xrightarrow[t\to 0]{}\alpha_1 t^{-1}+\alpha_0\,. \tag{3.40}$$

By a standard method of heat kernel expansion one can show that

$$\alpha_0=q+1\,,\qquad \alpha_1=R^2\,. \tag{3.41}$$

Taking a variation of $\ln\det' D$ with respect to instanton parameters we obtain

$$\begin{aligned}\delta\ln\det{}'\,D&=\lim_{\varepsilon\to 0}+\int_\varepsilon^\infty \mathrm{d}t\,\mathrm{Sp}\,(\delta D\,\mathrm{e}^{-tD})\\ &=\lim_{\varepsilon\to 0}+\int_\varepsilon^\infty \mathrm{d}t\,\mathrm{Sp}\,[\rho^{-1}\delta\rho D\,\mathrm{e}^{-tD}+D\rho^{-1}\delta\rho\,\mathrm{e}^{-tD}\\ &\quad -2\Omega^{-3/2}\rho\partial_z\rho^{-3}\delta\rho\Omega^{-1}\partial_{\bar{z}}\rho\,\mathrm{e}^{-tD}]\\ &=\lim_{\varepsilon\to 0}+2\int_\varepsilon^\infty \mathrm{d}t\,\mathrm{Sp}\,\{[D\,\mathrm{e}^{-tD}-\tilde{D}\,\mathrm{e}^{-t\tilde{D}}]\rho^{-1}\delta\rho\}\,,\end{aligned} \tag{3.42}$$

where

$$\tilde{D}=-4\Omega^{-3/2}\rho^{-1}\partial_{\bar{z}}\rho^2\Omega^{-1}\partial_z\rho^{-1}\Omega^{1/2}\,, \tag{3.43}$$

and it satisfies the relation

$$D\Omega^{-3/2}\rho\partial_z\rho^{-1}\Omega^{1/2}=\Omega^{-3/2}\rho\partial_z\rho^{-1}\Omega^{1/2}\tilde{D}\,. \tag{3.44}$$

Eq. (3.42) can be further simplified

$$\begin{aligned}\delta\ln\det{}'\,D&=+2\int_0^\infty \mathrm{d}t\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Sp}\,\{[\mathrm{e}^{-tD}-\mathrm{e}^{-t\tilde{D}}]\rho^{-1}\delta\rho\}\\ &=-2\,\mathrm{Sp}\,\{(\mathrm{e}^{-tD}-\mathrm{e}^{-t\tilde{D}})\rho^{-1}\delta\rho\}|_0^\infty\,.\end{aligned} \tag{3.45}$$

As $t \to \infty$, the only contribution comes from the zero modes; D has $4q$ zero modes whereas $\tilde{D}$ has none. As $t \to 0$, we can make a heat kernel expansion using standard methods. The result is

$$e^{-tD} \underset{t\to 0}{\longrightarrow} \frac{\sqrt{g}}{4\pi} t^{-1} + \frac{1}{4\pi} \partial_\mu \partial_\mu (\ln \rho + \tfrac{1}{2} \ln \Omega) ,$$

$$e^{-t\tilde{D}} \underset{t\to 0}{\longrightarrow} \frac{\sqrt{g}}{4\pi} t^{-1} - \frac{1}{4\pi} \partial_\mu \partial_\mu (\ln \rho - \tfrac{1}{2} \ln \Omega) , \tag{3.46}$$

the final result is

$$\delta \ln \det' D = A_1^{\mathrm{f}} + A_2^{\mathrm{f}} ,$$

$$A_1^{\mathrm{f}} = \frac{1}{2\pi} \delta \int \ln \rho \partial_\mu \partial_\mu \ln \rho \, d^2x + \frac{1}{2\pi} \oint [(\delta \ln \rho) \partial_\mu \ln \rho - \ln \rho \partial_\mu \delta \ln \rho] \, d\sigma_\mu ,$$

$$A_2^{\mathrm{f}} = 2 \int \rho^{-1} \delta\rho \pi^{\mathrm{f}}(x) \, d^2x , \tag{3.47}$$

where $\pi^{\mathrm{f}}(x) = \sum_I \bar{f}_I(x) f_I(x)$ is the projection operator on the space of fermionic zero modes.

The corresponding expression for the bosonic determinant is [9]

$$\delta \ln \det' M_\beta = A_1^{\mathrm{B}} + A_2^{\mathrm{B}} ,$$

where

$$A_1^{\mathrm{B}} = A_1^{\mathrm{f}} - \frac{1}{8\pi} \int (\delta \ln \rho) \partial_\mu \partial_\mu \ln g \, d^2x \equiv A_1^{\mathrm{f}} + B ,$$

$$A_2^{\mathrm{B}} = -2 \int \rho^{-1} \delta\rho \pi^{\mathrm{B}}(x) \, d^2x , \tag{3.48}$$

with $\pi^{\mathrm{B}}(x) = \sum_\mu \bar{\phi}_\mu(x) \phi_\mu(x)$ which is the projection operator on the space of bosonic zero modes.

As shown in ref. [9], A_2^{B} terms cancels with det R in the collective coordinate jacobian J (3.30). Here we show that in the same way A_2^{f} cancels with det R'. Indeed taking a basis $g_I = (1/\sqrt{\Omega}) z^{I-1}$, $I = 1, \ldots, 2q$ for fermionic zero modes we see that

$$\pi^{\mathrm{f}}(x) = \sum_{I,J} g_I(x) R'^{-1}_{IJ} \bar{g}_J(x) \Omega^2(x) , \tag{3.49}$$

where R'_{IJ} is as defined in (3.29). Substituting π^{f} in A_2^{f}

$$\begin{aligned} A_2^{\mathrm{f}} &= -2 \int \rho^{-1} \delta\rho \bar{g}_J(x) g_I(x) \Omega^2 \, d^2x \, R'^{-1}_{IJ} \\ &= -2 \int \rho^{-3} \delta\rho \bar{z}^{J-1} z^{I-1} \Omega \, d^2x R'^{-1}_{IJ} \\ &= (\delta R'_{JI}) R'^{-1}_{IJ} = \delta \ln \det R' . \end{aligned} \tag{3.50}$$

This implies that the $A_2^{\prime}$ term cancels with $\det R'$ in J (3.30). Now in the functional integral the bosonic determinant comes in the denominator while the fermionic one in the numerator, therefore all the contributions cancel out except that one coming from the term B in $\delta \ln \det' M_{\mathrm{B}}$.

Explicitly writing the term B,

$$
\begin{aligned}
B &= \delta \ln J + \delta \ln \det' D - \delta \ln \det' M_{\mathrm{B}} = \delta \int \ln \rho \partial_\mu \partial_\mu \ln g \, \mathrm{d}^2 x \\
&= -\delta \int \frac{1}{2\pi R^2} \frac{\mathrm{d}^2 x}{(1+x^2/4R^2)^2} \ln \{ [\prod_i |z-b_i|^2 + |c|^2 \prod_i |z-a_i|^2] / R^{2q} \} \\
&= \delta \ln (1+|c|^2)^{-2} - \delta \int \frac{1}{2\pi R^2} \frac{\mathrm{d}^2 x}{(1+x^2/4R^2)^2} \ln \left[\frac{\prod_i |z-b_i|^2 + |c|^2 \prod_i |z-a_i|^2}{R^{2q}(1+|c|^2)} \right] \\
&\equiv \delta \ln (1+|c|^2)^{-2} - \delta (Y - Y_0) \,.
\end{aligned}
\tag{3.51}
$$

Combining all these results we obtain for the partition function

$$
Z = \sum_q N_q \int \frac{\mathrm{d}^2 c}{(1+|c|^2)^2} \prod_{I=1}^{2q} \mathrm{d}^2 \varepsilon_I \, \mathrm{d}^2 \alpha_I \, \mathrm{e}^{-(Y-Y_0)} \mu^{2q} \, \mathrm{e}^{-4\pi q / g^2(\mu)} \,, \tag{3.52}
$$

where N_q is a constant independent of the parameters (α, ε) and R. $Y - Y_0$ is defined in eq. (3.51) and is the correction factor given in ref. [14]. Notice that this term solely comes from B. $g(\mu)$ in the above expression is the renormalized coupling constant

$$
\frac{1}{g^2(\mu)} = \frac{1}{g^2} - \frac{1}{4\pi} \ln \frac{\Lambda^2}{\mu^2} \,, \qquad (\mu \text{ being the subtraction point}) \,. \tag{3.53}
$$

The R-dependence of Z in eq. (3.52) can be explicitly verified by taking the variation of $\ln \det' M_{\mathrm{B}}$ and $\ln \det' D$ with respect to R and following the same procedure as above.

Finally the factor N_q can be shown to be

$$
N_q = N^{2q} / (q!)^2 \,, \tag{3.54}
$$

where N is a constant independent of q and if required can be computed by evaluating the 1-instanton contribution explicitly, but we shall not need it in the following. The form (3.54) can be obtained by considering q instantons that are widely separated and taking the limit $R \to \infty$. In this limit the q-instanton contribution should be equal to the product of a single-instanton contribution divided by the combinatorial $(q!)^2$ arising from the possible interchanges of a_i and b_i.

4. Ward identities for supersymmetry

We shall in this section investigate the role of Ward identities derived from supersymmetric transformations. These identities should be satisfied for any dynamical quantity in our O(3) σ-model if the supersymmetry is a true symmetry. Furthermore, if instanton effects should break the symmetry, we can tell an explicit breaking from a spontaneous one, since the first breaking should not respect the Ward identities while the last one does, as we saw in the quantum mechanics case in sect. 2. However we know already that the Witten index for this model is different from zero, so a spontaneous breaking is ruled out.

One usually derives (see e.g. ref. [7]) the Ward identities by considering the derivative of the Green function M_μ:

$$M_\mu = \langle 0|\mathrm{T}\{S_\mu(x), \phi(x_1), \ldots, \phi(x_n), \psi(x_{n+1}), \ldots, \psi(x_{n+m})\}|0\rangle\,, \qquad (4.1)$$

where we consider a time-ordered product of n bosonic and m fermionic fields together with the supercurrent

$$S_\mu = \partial_\nu \phi^a \gamma^\nu \gamma_\mu \psi^a\,.$$

Assuming that the relation (equation of motion)

$$\partial_\mu S_\mu = 0$$

also holds at the quantum level, one derives the Ward identities by differentiating M_μ:

$$\begin{aligned}\frac{\partial}{\partial x_\mu} M_\mu = \langle 0|\mathrm{T}\{\delta(x_0 - x_{01})\\ &\times [S_0(x), \phi(x_1)], \ldots, \phi(x_n), \psi(x_{n+1}), \ldots, \psi(x_{n+m})\}|0\rangle \\ &+ \cdots + \langle 0|\mathrm{T}\{\delta(x_0 - x_{0n}) \\ &\times [S_0(x), \psi(x_{n+m})(x_n)], \ldots, \phi, \psi(x_{n+1}), \ldots, \psi(x_{n+m-1})\}|0\rangle \\ &+ \langle 0|\mathrm{T}\{\partial_\mu S_\mu(x), \phi(x_1), \ldots, \phi(x_n), \psi(x_{n+1}), \ldots, \psi(x_{n+m})\}|0\rangle\,. \end{aligned} \qquad (4.2)$$

Integrating this equation over space-time, and using $\partial_\mu S_\mu = 0$ and the fact that supersymmetry is not spontaneously broken, i.e. $Q|0\rangle = 0$, the Ward identities in eq. (4.2) become

$$\langle 0|\delta O|0\rangle = 0\,, \qquad (4.3)$$

where δ is the supersymmetry transformation and O is an arbitrary operator consisting of boson and fermion fields. Symbolically we could therefore have written:

$$\delta\langle 0|O|0\rangle = \langle 0|\overleftarrow{Q}O|0\rangle + \langle 0|O\vec{Q}|0\rangle + \langle 0|\delta O|0\rangle = \langle 0|\delta O|0\rangle = 0\,.$$

Therefore we will have to show that $\langle 0|\delta O|0\rangle = 0$ for a general O.

Most operators O will, when transformed, make the Green function zero but there will be a chance for a four-fermion term $\bar\psi\psi\bar\psi\psi$ to give a non-zero vacuum expectation value, since in that case the instanton zero modes contribute (there are four fermionic ones per instanton) and of course a higher multiple of four-fermion interactions will do the same.

The operator O whose supersymmetric variation δO contains a four-fermion term must be of the form

(a) $$O=\bar{\psi}\psi\bar{\psi}\psi\psi f(\phi,\bar{\phi})\,,$$

or

(b) $$O=\bar{\psi}\psi\bar{\psi}f(\phi,\bar{\phi})\,.$$

First let us consider the case (a). Using the transformation laws for δ given in eq. (3.8) the four-fermion part in δO is

$$\delta O=-i\bar{\xi}^{\gamma}\gamma_{\gamma}^{\mu\beta}\varepsilon_{\beta\alpha_i}\partial_{\mu}\phi_i\psi_{j_{\alpha_j}}\psi_{k_{\alpha_k}}\varepsilon^{ijk}\bar{\psi}\psi\bar{\psi}f(\phi,\bar{\phi})\,. \tag{4.4}$$

Here i, j, k indices denote the space points z_i etc., and α_i denote the spinor indices. The corresponding indices for $\bar{\psi}$ and ϕ, $\bar{\phi}$ are not shown in the above because they are irrelevant for our purposes. Now in this semiclassical approximation, all the fields in δO are replaced by their classical values. Since $\psi_{i\alpha_i}=0$ for $\alpha_i=2$ and $\partial_{\bar{z}}\phi=0$, it follows that all the α_i should be equal to 1, and the result is:

$$\delta O=-i\bar{\xi}\varepsilon^{ijk}\partial_z\phi_i\partial_{\alpha_I}\phi_j\partial_{\alpha_J}\phi_k\bar{\psi}\psi\bar{\psi}f\varepsilon_I\varepsilon_J\,. \tag{4.5}$$

Here α_I, α_J are the instanton parameters a, b $(I,J=1,2)$. Using the fact that $\partial_z\phi=\sum_{k=1}^{2}\partial_{\alpha_k}\phi$ and that α_k must be equal to α_I or α_J, it follows that $\sigma O=0$.

Next let us consider the case (b). We will show here that for all O that belong to a finite-dimensional representation of the O(3) group, $\langle\delta O\rangle=0$. Note that all the operators that are polynomial in the original unprojected field variables ϕ^a, ψ^a belong to this class. The argument proceeds as follows. O(3) transformations of the projected field variables can be easily derived from their O(3) action on ϕ^a, ψ^a. The transformation rules are

$$\phi\to\frac{\phi\,e^{i\alpha}-\lambda}{1+\lambda\phi\,e^{i\alpha}}e^{i\beta}\,,\qquad \psi\to\frac{(1+\lambda^2)\psi\,e^{i\beta+i\alpha}}{(1+\lambda\phi\,e^{i\alpha})^2}\,, \tag{4.6}$$

where α, β and λ are parameters of the O(3) group. In the semiclassical approximation, ϕ and ψ are replaced by their classical values, and in that case the above transformation acts on the parameter space of the instanton solutions (a,b,c,ε_I), and it is easy to check that the functional measure $d\mu$ (a,b,c,ε_I) in eq. (3.52) is invariant under this transformation. This of course means that the O(3) group is not broken by instantons.

If O now belongs to a finite O(3) representation as assumed then we can decompose O into irreducible representations of O(3):

$O=\sum_{(i)}O^{(i)}$ where $O^{(i)}$ belongs to the ith irreducible representation. Since δ commutes with O(3), $\delta O=\sum_{(i)}\delta O^{(i)}$ where $\delta O^{(i)}$ also belongs to the same ith irreducible representation. O(3) invariance of the measure now implies that only the $\delta O^{(\text{singlet})}$ can give a non-zero result. Thus it suffices to consider only O(3) invariant O's.

Before showing that a general O(3)-invariant operator will satisfy eq. (4.3) (the supersymmetric Ward identities), we shall for pedagogical reasons first proceed with a simple example of an O(3)-invariant operator (only dependent on 2 space points) that will satisfy eq. (4.3). Consider the operator O in terms of unprojected field variables.

$$O = \phi^a(z_1)\psi^b_\alpha(z_1)\psi^a_\beta(z_2)\psi^b_\gamma(z_2)\,.$$

This is explicitly O(3) invariant. We can write O in terms of the projected field variables, using the projection formula (3.4):

$$O = \frac{1}{\rho_1^3\rho_2^4}\left[(1+\phi_1\bar{\phi}_2)(1+\bar{\phi}_1\phi_2)^2(\bar{\phi}_1-\bar{\phi}_2)\psi_{1\alpha}\bar{\psi}^\beta_2\psi_{2\gamma}+\text{c.c.}\right]\,,\qquad \begin{cases}\rho_1=(1+\phi_1\bar{\phi}_1)\\ \rho_2=(1+\phi_2\bar{\phi}_2)\,.\end{cases} \tag{4.7}$$

Each of the two terms in the bracket is separately O(3) invariant.

Therefore it is sufficient to take the first term, which we will denote, for notational simplicity, as $f(\phi_2, \phi_1, \bar{\phi}_2, \bar{\phi}_1)\psi_1\bar{\psi}_2\psi_2$. Now taking the supersymmetric variation and keeping the four-fermion terms we obtain:

$$\begin{aligned}\langle 0|\delta O|0\rangle &= \langle \delta f(\phi_2, \phi_1, \bar{\phi}_2, \bar{\phi}_1)\psi_1\psi_2\bar{\psi}_2\rangle = \left\langle \frac{\partial f}{\partial\bar{\phi}_1}\psi_1\bar{\psi}_1\psi_2\bar{\psi}_2\right\rangle\\ &= \text{const}\times\int\frac{d^2c}{(1+|c|^2)^2}\,d^2a\,d^2b\,\frac{\partial f}{\partial\bar{\phi}_1}\,|(\partial_a\phi_1\partial_b\phi_2-\partial_b\phi_1\partial_a\phi_2)|^2\,e^{-S_c}\,,\end{aligned} \tag{4.8}$$

where we here have used eq. (3.52) for the partition function Z_1 (of one instanton) and have integrated over the Grassmann parameters. The correction term $e^{-(Y-Y_0)}$ has been ignored for simplicity here, but in the appendix it is shown that the results are not changed by including $e^{-(Y-Y_0)}$. We can now change the integration from parameter space to the field space and noting that the factor $|(\partial_a\phi_1\partial_b\phi_2-\partial_b\phi_1\partial_a\phi_2)|^2$ is the jacobian of the mapping from (a, b) to (ϕ_1, ϕ_2). The winding number of this mapping is equal to 1. The integral in eq. (4.8) then becomes:

$$\begin{aligned}\langle 0|\delta O|0\rangle &= \text{const}\times\pi\,e^{-Sc}\int d^2\phi_1\,d^2\phi_2\frac{\partial f}{\partial\bar{\phi}_1}\\ &= \lim_{r\to\infty}\left[\text{const}\,\pi\,e^{-Sc}\int_{S_1}dO\,e^{i\theta}r\int d^2\phi_2 f(\phi_2, r\,e^{i\theta}, \bar{\phi}_2, r\,e^{-i\theta})\right]\,,\end{aligned} \tag{4.9}$$

and the final result is reached when we take the limit $r\to\infty$. It is clear from expression (4.7) that f goes to zero as $1/r^3$ in this limit and therefore the integral in eq. (4.9) is zero implying that O satisfies the Ward identities, i.e. $\langle\delta O\rangle = 0$.

We can now already see from this example that in order to have $\langle\delta O\rangle$ not equal to zero the function $f(\phi_2, \phi_1, \bar{\phi}_2, \bar{\phi}_1)$ should satisfy the condition:

$$f(\phi_2, \phi_1, \bar{\phi}_2, \bar{\phi}_1)\xrightarrow[r\to\infty]{}\frac{1}{r}\,e^{i\theta}g(\phi_2, \bar{\phi}_2)\,, \tag{4.10}$$

where $\phi_1 = r\,e^{i\theta}$ and g is an arbitrary integrable function. One such example is when

$$f = \frac{\bar{\phi}_1}{\rho_1 \rho_2^m}, \qquad m \geq 2\,. \tag{4.11}$$

In particular for $m = 2$, the integral in eq. (4.9) turns out to be

$$\langle 0|\delta O|0\rangle = \langle 0|\frac{\bar{\psi}_1\psi_1}{\rho_1^2}\frac{\bar{\psi}_2\psi_2}{\rho_2^2}|0\rangle = N^2\mu^2\,e^{-4\pi/g^2(\mu)} \equiv M^2\,, \tag{4.12}$$

where M is the renormalisation group invariant mass. We thus see that for such f, $\langle\delta O\rangle \neq 0$. However, even though the four-fermion part of δO is an O(3) singlet, O itself is not O(3) invariant. Furthermore, O is not square integrable over the O(3) group, indicating that it belongs to an infinite-dimensional representation of O(3). We shall discuss this further in sect. 5.

Now we proceed with the general case of O(3)-invariant operators. An important fact is that the integrand in the functional integral $\langle 0|\delta O|0\rangle$ can be written as a total derivative with respect to the instanton parameters. This is because a supersymmetric variation induces linear transformation on the parameter space as shown in eq. (3.14). Therefore a transformation δ on an operator $O^{\beta}_{\alpha\gamma} = O^{\beta}_{\alpha\gamma}(\alpha_I, \varepsilon_I, \bar{\alpha}_I, \bar{\varepsilon}_I)$ gives

$$\delta O^{\beta}_{\alpha\gamma} = \varepsilon_I\xi_2\partial_{\alpha_I}O^{\beta}_{\alpha\gamma} + \bar{\varepsilon}_I\bar{\xi}^2\partial_{\bar{\alpha}_I}O^{\beta}_{\alpha\gamma} + \xi_1\sum_I\frac{\partial}{\partial\varepsilon_I}O^{\beta}_{\alpha_\gamma} + \bar{\xi}^1\sum_I\frac{\partial}{\partial\bar{\varepsilon}_I}O^{\beta}_{\alpha\gamma}\,, \tag{4.13}$$

where ξ_1 and ξ_2 are the two Grassmann parameters of the supersymmetric transformation, and since we require our operator $O^{\beta}_{\alpha\gamma}$ be a four-fermion one, the only term that gives a non-vanishing expectation value in (4.13) is in our case $\bar{\varepsilon}_i\bar{\xi}\partial_{\bar{\alpha}_i}O^{\beta}_{\alpha\gamma}$. This term being a total derivative will ease the further discussion of the general 4-point operator.

Theorem. All O(3)-invariant Green functions in the supersymmetric σ-model will satisfy the supersymmetric Ward identities, i.e. $\langle\delta O\rangle = 0$ and therefore not break the supersymmetry explicitly.

Proof. We shall prove this statement for a general O(3)-invariant four-point function. Here we shall not consider the correction term $e^{-(Y-Y_0)}$ which is taken care of in the appendix. Therefore we begin with the general O(3)-invariant operator

$$O = f(\phi_1, \ldots, \phi_4)\psi_2\bar{\psi}_3\psi_4\,, \qquad \phi(x_1) = \phi_1\,, \tag{4.14}$$

where f is a fractional polynomial in an arbitrary number of fields. In fact we shall assume that f is a regular function in the field space, i.e. for any finite values of ϕ_i, $f \neq \infty$, which is an obvious assumption since f otherwise does not make sense.

Now we make use of the fact that the supersymmetric variation on O can be written as a total derivative

$$\begin{aligned}\delta O &= \bar{\varepsilon}_I \frac{\partial}{\partial \bar{\alpha}_I} O = \bar{\varepsilon}_I \frac{\partial}{\partial \bar{\alpha}_I} [f(\phi_1, \ldots, \phi_4) \varepsilon_I \partial_{\alpha_I} \phi_2 \bar{\varepsilon}_k \partial_{\bar{\alpha}_k} \bar{\phi}_3 \varepsilon_2 \partial_{\alpha_2} \phi_4] \\ &= \partial_{\bar{a}} \left[f \frac{(z_2 - z_4)(a-b)}{(z_2-b)^2 (z_4 - b)^2} c^2 \frac{(\bar{z}_3 - \bar{a})\bar{c}}{(\bar{z}_3 - \bar{b})^2} \right] + \partial_{\bar{b}} \left[f \frac{(z_2 - z_4)(a-b)}{(z_2-b)^2 (z_4 - b)^2} c^2 \frac{(\bar{z}_3 - \bar{a})\bar{c}}{(\bar{z}_3 - \bar{b})^2} \right] \\ &\equiv \partial_{\bar{a}} [g(\bar{z}_3 - \bar{a})] + \partial_{\bar{b}} [g(\bar{z}_3 - \bar{b})] ,\end{aligned} \tag{4.15}$$

and hence the complex integral over the instanton parameter space can only get contributions from singularities. There are three cases of singularities arising from the $(a, \bar{a})$ and $(b, \bar{b})$ integration of (3.19).

Case (*a*). Sheet singularities: $b = z_i$ for some i, and a arbitrary (but $a \neq z_i$).

Case (*b*). Point singularities: $a = b = z_i$ for some i, and $a = z_i$ and $b = z_j$ for some $i \neq j$.

Case (*c*). Singularities at infinity: ($a \to \infty$ and/or $b \to \infty$).

Concerning the first case we shall prove the following lemma which represents the crucial step in the main proof.

Lemma. O(3) invariance of O implies absence of sheet singularities.

Proof of lemma. First we write the O(3) transformed operator O as

$$\begin{aligned} O^{\mathrm{G}} &= f(\phi_1^{\mathrm{G}}, \ldots, \phi_4^{\mathrm{G}}) \psi_2^{\mathrm{G}} \bar{\psi}_3^{\mathrm{G}} \psi_4^{\mathrm{G}} = \frac{f(\phi_1^{\mathrm{G}}, \ldots, \phi_4^{\mathrm{G}}) \psi_2 \bar{\psi}_3 \psi_4 \, \mathrm{e}^{i(\gamma+\alpha)} (1+\lambda^2)^3}{(1+\lambda\phi_2 \mathrm{e}^{i\alpha})^2 (1+\lambda\bar{\phi}_3 \mathrm{e}^{-i\alpha})^2 (1+\lambda\phi_4 \mathrm{e}^{i\alpha})^2} \\ &\equiv f^{\mathrm{G}}(\phi_1, \ldots, \phi_4) \psi_2 \bar{\psi}_3 \psi_4 .\end{aligned} \tag{4.16}$$

where

$$\psi^{\mathrm{G}} = \frac{\psi \, \mathrm{e}^{i(\gamma+\alpha)} (1+\lambda^2)}{(1+\lambda\phi \, \mathrm{e}^{i\alpha})^2} , \qquad \phi^{\mathrm{G}} = \frac{\mathrm{e}^{i\alpha}\phi - \lambda}{1+\lambda\phi \, \mathrm{e}^{i\alpha}} \mathrm{e}^{i\gamma} ,$$

as in eq. (4.6); O(3) invariance then implies that $f^{\mathrm{G}} = f$.

Now

$$\phi_i^{\mathrm{G}} = \frac{c \, \mathrm{e}^{i\alpha}(z_i - a) - \lambda(z_i - b)}{(z_i - b) + \lambda c \, \mathrm{e}^{i\alpha}(z_i - a)} \to \frac{1}{\lambda} , \tag{4.17}$$

for $b \to z_i$ and $a \neq z_i$. In fact one can always choose α and λ so that ϕ_i^{G} is finite $\forall i$ for any given value of a and b provided that a and b are not equal to the same z_i (which was the second case of singularities to be studied later). Then from the assumption that f be regular $f(\phi_1^{\mathrm{G}}, \ldots, \phi_4^{\mathrm{G}})$ is finite.

Further we write as in eq. (4.15)

$$\delta O^{\mathrm{G}} = \partial_{\bar{a}} [g^{\mathrm{G}}(\bar{z}_3 - \bar{a})] + \partial_{\bar{b}} [g^{\mathrm{G}}(\bar{z}_3 - \bar{b})] , \tag{4.18}$$

where

$$g^G = \frac{f(\phi_1^G, \ldots, \phi_4^G)\, e^{i(\gamma+\alpha)}(1+\lambda^2)^3(z_2-z_4)(a-b)}{[(z_2-b)+\lambda c(z_2-a)\,e^{i\alpha}]^2[(\bar{z}_3-\bar{b})+\lambda\, e^{-i\alpha}\bar{c}(\bar{z}_3-\bar{a})]^2[(z_4-b)+\lambda\, e^{i\alpha}c(z_4-a)]^2}. \tag{4.19}$$

Now it is apparent that g^G has no sheet-singularity since we can choose λ and α appropriately. Take for example $b=z_2$ and $a \neq z_2$, then all the denominators are non-zero and since $f(\phi_1^G, \ldots, \phi_4^G)$ is finite, g^G will therefore have no singularity. Finally O(3) invariance requires $O=O^G$ and thus $\delta O = \delta O^G$ and in particular the singularity structure should be the same. Therefore O and δO have no sheet-singularities, *QED*.

We shall now proceed to the second case of point singularities. The case where $a=z_i$ and $b=z_j$ for some $i \neq j$ poses no problem since it is already considered in the above lemma. We shall then consider the case where $a=b=z_i$ for some i and show that the integral of δO will be zero. From the assumption that f is regular we can write f as

$$f = \sum_{\{n\}} A_{n_1 n_2 n_3 n_4} \frac{\phi_1^{n_1}\phi_2^{n_2}\phi_3^{n_3}\phi_4^{n_4}}{\prod_i (\alpha_i + |\phi_i|^2)^{m_i}}, \tag{4.20}$$

where $\alpha_i > 0$, $A_{n_1 n_2 n_3 n_4}$ are constants and $\{n\}$ are positive integers. One could have $\bar{\phi}_i$ instead of ϕ_i in the numerator or a combination of both but the argument given below is insensitive to this. For simplicity we have not included derivatives of ϕ_i in eq. (4.20), but as will be clear from the following argument, the proof depends on the absence of sheet singularities, and hence with slight modification it can be carried over to such cases.

Inserting the analytic expression for ϕ_i in f, and using the fact that sheet singularities are absent, as in the lemma above, we get the following constraints on the numbers $\{n\}$, $\{m\}$:

$$n_1 \leq 2m_1, \qquad n_2 \leq 2m_2 - 2, \qquad n_3 \leq 2m_3 - 2, \qquad n_4 \leq 2m_4 - 2. \tag{4.21}$$

Now look at the point singularity $a=b=z_i$ for some i. It is convenient in the following to work with the variable $a' = ac$. Since δO is a total derivative, with respect to the parameters a, b, the integral can be expressed as surface integrals around each singularity $b=z_i$, $a'=cz_1$. Let

$$b - z_i = \varepsilon \sin(\alpha)\, e^{i\phi}, \qquad a' - cz_i = \varepsilon \cos(\alpha)\, e^{i\theta}. \tag{4.22}$$

Then the contribution to $\langle \delta O \rangle$ from the singularity at $i=1$, to the lowest order in ε goes as

$$\begin{aligned}\langle \delta O \rangle_1 \to &\int \frac{d^2 c}{(1+|c|^2)^2} \oint_{\varepsilon\text{-sphere}} d\mu\,(\alpha, \theta, \phi)\varepsilon^3 c^{n_2+n_3+n_4}\varepsilon(e^{i\phi}\cos\alpha - c\, e^{i\phi}\sin\alpha) \\ &\times \sin^{2m_1-n_1}(\alpha)\cos^{n_1}(\alpha)\frac{[\bar{c}\, e^{i\theta}\cos\alpha + e^{i\theta}\sin\alpha]}{(\alpha_1 \sin^2\alpha + \cos^2\alpha)^{m_1}}\, e^{in_1(\theta-\phi)} \\ &\times \prod_{j\neq 1}\frac{1}{(\alpha_j + |c|^2)^{m_j}}\frac{(z_2-z_4)}{(z_2-z_1)^2(z_4-z_1)^2(\bar{z}_3-\bar{z}_1)}.\end{aligned} \tag{4.23}$$

Since $2m_1 \geq n_1$ from eq. (4.21), integration over the sphere is finite. Again using the inequalities (4.21) we see that integration over c is also finite. Therefore

$$\langle \delta O \rangle_1 \sim \varepsilon^4 \xrightarrow[\varepsilon \to 0]{} 0 \, .$$

Similarly, the contribution to $\langle \delta O \rangle$ from the singularity at $i = 2$, 3 or 4 goes as

$$\langle \delta O \rangle_i \sim \int \frac{d^2 c}{(1+|c|^2)^2} \oint_{\varepsilon\text{-sphere}} d\mu\,(\alpha, \theta, \phi) \frac{\varepsilon^4}{\varepsilon^2} \sin^{2m_i - n_i - 2}(\alpha) \prod_{j \neq i} \frac{c^{n_j}}{(\alpha_j + |c|^2)^{m_j}}$$

$$\times (\text{non-singular terms}) \, .$$

Again from (4.21) we see that $2m_i \geq n_i + 2$ (for $i \neq i$), therefore integration over the ε-sphere is finite. As before, the c integration is also finite and the result is

$$\langle \delta O \rangle_i \sim \varepsilon^2 \xrightarrow[\varepsilon \to 0]{} 0 \, .$$

Therefore we conclude that the point singularities will not cause $\langle \delta O \rangle$ to be non-zero.

The remaining case to consider is the behaviour at infinity (case (c)). Let for example $b \to \infty$. Then clearly $\phi_i \to 0$. Since f is finite, g will go like $g \to 1/b^5$ and hence there will be no problem at $b \to \infty$. Furthermore by a gauge transformation one can interchange b and a. Thus O(3) invariance of O implies that there is no problem at $a \to \infty$. Therefore in all cases $\langle \delta O \rangle = 0$, *QED.*

Here we included only the one-instanton contribution, since we only considered four-fermionic operators, but the higher number of instantons will appear when higher multiples of four-fermionic fields are considered and our arguments can easily be generalised to cover such cases.

The last point to consider, in order to complete our treatment of the Ward identities, is the extra term $e^{-(Y-Y_0)}$ to the full instanton contribution which we derived in sect. 3. The extra term arises as a finite-volume effect and dies off to 1 as the radius R of the compactified space S^2 is taken to infinity. One could suspect that such a term, when added to the calculation of our vacuum expectation value of δO, would make $\langle \delta O \rangle$ non-zero because we no longer can use the Gauss theorem. However this is not the case as briefly proved in the appendix. The proof relies on the fact that the absolute value of δO belongs to $L^1(a, b, c, d\mu)$ i.e. basically:

$$\int_{-\infty}^{\infty} |\delta O| \, d^2 a \, d^2 b \frac{d^2 c}{(1+|c|^2)^2} < \infty \, .$$

5. Fermionic condensation

In this section we shall discuss the phenomena of condensation. We saw in sect. 4, eq. (4.12), that the operator $O = (1/\rho_1^2 \rho_2^2) \bar{\psi}_1 \psi_1 \bar{\psi}_2 \psi_2$ of four-fermion fields had a non-vanishing expectation value

$$\left\langle \frac{\bar{\psi}_1' \psi_1'}{\rho_1^2} \frac{\bar{\psi}_2' \psi_2'}{\rho_2^2} \right\rangle = N^2 \mu^2 \, e^{-4\pi/g^2(\mu)} = M^2 \, ,$$

$$(M \text{ is the renormalization group invariant mass}) \, . \tag{5.1}$$

Furthermore we note that any operator containing $2q$ fermion–anti-fermion pairs located at different points will get contributions from the q-instanton sector. This contribution can be calculated exactly as in eq. (4.11) and the result is

$$\left\langle \frac{\bar{\psi}'_1\psi'_1}{\rho_1^2} \cdots \frac{\bar{\psi}'_{2q}\psi'_{2q}}{\rho_{2q}^2} \right\rangle = \frac{(N)^{2q}}{(q!)^2}\,\mu^{2q}\,\mathrm{e}^{-q(4\pi/g^2(\mu))}(q!)^2\,, \tag{5.2}$$

where the last factor $(q!)^2$ comes from the winding number of the mapping

$$(a_i, b_i, i=1,\ldots,q) \to (\phi_1,\ldots,\phi_{2q})\,.$$

An important thing to note is that these expectation values are independent of the locations of $\bar{\psi}\psi/\rho^2$ even when we include the correction term. If this phenomenon persists beyond the semiclassical approximation then it indicates a condensate. More explicitly, assuming the vacuum admits cluster decomposition, we shall insert a complete set of energy eigenstates between the two-fermion pairs:

$$\sum_n \langle 0|\frac{\bar{\psi}\psi}{\rho^2}(x_1)|n\rangle\langle n|\frac{\bar{\psi}\psi}{\rho^2}(x_2)|0\rangle = M^2\,. \tag{5.3}$$

Then taking x_1 and x_2 far apart, only the vacuum state will contribute in the above sum. Therefore $\langle\bar{\psi}\psi/\rho^2\rangle^2 = M^2$. This implies that

$$\left\langle\frac{\bar{\psi}\psi}{\rho^2}\right\rangle = \pm M\,. \tag{5.4}$$

(Instantons give contributions to the positive chirality and anti-instantons to the negative chirality.) This result is in agreement with the $1/N$ expansion [15]. Moreover, this condensation will give masses to bosons and fermions, through self-energy diagrams involving 4-fermion vertices. This condensate is known explicitly to break chiral U(1) symmetry, and the fact that eq. (5.4) admits two solutions is a reflection of the residual Z(2) chiral symmetry, and the existence of solitonic states interpolating between the two vacua.

The condensation that occurs in this study is unlike that in ref. [15] caused by the fermionic zero-modes around instantons that could not have been seen in the $1/N$ expansion. In this context it is interesting to mention that Casher in ref. [5] claimed that similar condensation of fermionic zero-modes around instantons in 4-dimensional Yang–Mills theory break explicitly not only the chiral symmetry but also the supersymmetry (both currents being in the same supermultiplet). To exhibit such explicit breaking using Ward identities one must find an operator which under supersymmetric transformations produces a term proportional to the fermionic condensate. A class of such operators was presented in sect. 4 and was of the form

$$\{O_n\} = \frac{\bar{\phi}_1\psi_1\bar{\psi}_2\psi_2}{\rho_1\rho_2^n}\,, \qquad n \geq 2\,. \tag{5.5}$$

However, these operators belong to infinite-dimensional representations and since their singularity behaviour has not been investigated further the physical relevance of the operators is not clear.

In conclusion, let us point out that the technique used in this paper, i.e. utilizing the general supersymmetric instanton solutions and integrating over supersymmetric collective coordinates, are especially convenient in discussing the supersymmetric Ward identities (e.g. in 4 dimensions) because in the semiclassical approximation supersymmetric variation can be expressed as total derivatives with respect to the instanton parameters.

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Note added

After the completion of the present work, we came to know of the three recent papers [16] by Novikov et al., who have investigated Yang–Mills theory using supersymmetric instantons. The instanton measure was found to be supersymmetry invariant. They also find the phenomenon of space independence for certain n-point functions similar to what we observed in the present work.

An extended calculation for supersymmetric CP^{N-1} models [17] is in agreement with the results of $1/N$ expansion [15].

Appendix

We shall in this appendix briefly demonstrate that the extra term $e^{-(Y-Y_0)}$ when included in the calculation of $\langle\delta O\rangle$ cannot alter the result derived in sect. 4, that $\langle\delta O\rangle = 0$ in the limit R going to infinity. First let us denote: $\delta O_G = \int \prod_I d^2\varepsilon_I \delta O$. Then the proof makes use of the fact that $|\delta O_G| \in L^1(a, b, c, d\mu)$, $d\mu = d^2a\, d^2b\, d^2c\,[1/(1+|c|^2)^2]$. This is simple to verify from the general proof in sect. 4.

Now we show that

$$|\delta O_G| \in L^1(a, b, c, d\mu)\,, \qquad \text{for} \qquad O = f(\phi_1, \ldots, \phi_4)\psi_2\bar{\psi}_3\psi_4\,,$$

implies that

$$I = \int d\mu\,(a, b, c)\delta O_G(e^{-(Y-Y_0)} - 1) = 0$$

in the limit $R \to \infty$.

The proof relies on the fact that $e^{-(Y-Y_0)}$ is bounded for any value of (a, b, c) and moreover in a region where

$$|a|, |b| < R' \ll R ,$$

$$e^{-(Y-Y_0)} - 1 \sim \left(\frac{R'}{R}\right)^{\alpha}, \qquad \alpha = 2 - \varepsilon . \tag{A.1}$$

Both these facts can be checked easily by explicit calculation of $Y - Y_0$ in the one-instanton case.

For a given set of $\{z_i\}$, choose R', R such that $R \gg R' \gg |z_i|$, $\forall_i$. Then we split the parameter space into two regions:

(a) $|a|, |b| < R'$;

(b) full parameter space, region (a).

In region (a), the integral satisfies

$$\begin{aligned} I_1 &= \int_1 \mathrm{d}\mu\ (a, b, c)\delta O_G(e^{-(Y-Y_0)} - 1) \\ &\leq \int_1 \mathrm{d}\mu\ (a, b, c)|\delta O_G| \cdot |(e^{-(Y-Y_0)} - 1)| \leq |e^{-(Y-Y_0)} - 1|_{\max_1} \\ &\quad \times \int_1 \mathrm{d}\mu\ (a, b, c)|\delta O_G| . \end{aligned} \tag{A.2}$$

The second factor is finite by the assumption, and from (A.1) the first factor is of the order $(R'/R)^{\alpha}$ which goes to zero as $R \to \infty$.

In region (b), the integral satisfies

$$\begin{aligned} I_2 &= \int_2 \mathrm{d}\mu\ (a, b, c)\delta O_G(e^{-(Y-Y_0)} - 1) \\ &\leq |e^{-(Y-Y_0)} - 1|_{\max_2} \int_2 \mathrm{d}\mu\ (a, b, c)|\delta O_G| \\ &= N\left[\int \mathrm{d}\mu\ (a, b, c)|\delta O_G| - \int_1 \mathrm{d}\mu\ (a, b, c)|\delta O_G|\right] , \end{aligned} \tag{A.3}$$

where $N = |e^{-(Y-Y_0)-1}|_{\max_2}$ = finite. The first term in the bracket is finite by the assumption and the second term in the limit $R \gg R' \to \infty$, approaches the first one. (This limit can be chosen for instance by putting $R' = (R, \max\{|z_i|\})^{1/2}$ and taking $R \to \infty$.) Therefore $I_2 \to 0$ *QED*.

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