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DUALITY TRANSFORMATIONS AND MOST GENERAL MATTER SELF-COUPLINGS IN $N = 2$ SUPERSYMMETRY

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The $N = 2$ matter off-shell representations with a finite set of field components are reformulated in the harmonic superspace approach. Their list includes all such multiplets known previously (the tensor multiplets, the relaxed hypermultiplet, the further relaxed hypermultiplet) as well as those newly introduced (higher relaxed ones, etc.). All these multiplets are described by constrained (and sometimes having a gauge freedom) analytic harmonic superfields. A manifestly $N = 2$ supersymmetric duality transformation is defined. By means of it all the self-couplings of the above superfields are shown to be dual equivalent to certain subclasses of general self-couplings of the basic analytic unconstrained $N = 2$ matter superfield having an infinite number of components, the q^+ hypermultiplet. This confirms our suggestion that the most general q^+ action yields the most general matter self-coupling in rigid $N = 2$ supersymmetry. The $N = 2$ matter actions are known to produce hyper-Kähler σ models in the physical boson sector. So, the relevant analytic superfield lagrangian density $\mathcal{L}^{(-4)}(q^+, \ldots)$ can be regarded as the general "hyper-Kähler potential", plausibly resulting in all possible hyper-Kähler metrics.

1. Introduction

A consistent description of matter supermultiplets is a necessary ingredient for future supersymmetric phenomenology. For such a description the knowledge of corresponding off-shell representations is of vital importance. Indeed, only on the latter is supersymmetry realized linearly and independently of a particular action. This in turn allows a straightforward construction of general interactions (for $N = 1$ supersymmetry see for example ref. [1, 2]).

Generally speaking, there can be several off-shell representations of the same on-shell matter multiplet which may and do lead to different self-couplings. An important problem arises: which off-shell extension provides us with the most general self-couplings? The main purpose of the present paper is to provide evidence for the fact that the complete solution for the case $N = 2$, $d = 4$ (or, equally $N = 1$, $d = 6$) is achieved within the harmonic superspace approach [3] which has opened up new avenues for the off-shell treatment of theories with extended supersymmetry.

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It is instructive to begin by recalling how the above problem is solved in $N=1$, $d=4$ supersymmetry. $N=1$ matter (two spin-0 and one spin-$\frac{1}{2}$) can be described by a chiral multiplet $\phi(x_L, \theta)$ or by a tensor multiplet $G(x, \theta, \bar{\theta})$, $G=\bar{G}$, $D^\alpha D_\alpha G = 0$ or by a complex linear multiplet $L(x, \theta, \bar{\theta})$, $D^\alpha D_\alpha L = 0$, etc. (see, e.g. the appendix in ref. [4]). Variant off-shell descriptions differ in their auxiliary fields and (sometimes) in their way of representing physical fields. For a chiral multiplet the general self-interaction is rather familiar [5]

$$S_\phi = \int d^4x\, d^4\theta\, K(\phi, \bar{\phi}) + \left(\int d^4x_L\, d^2\theta\, P(\phi) + \text{h.c.} \right), \qquad \bar{D}_{\dot{\alpha}}\phi = 0, \qquad (1.1)$$

where $K(\phi, \bar{\phi})$ and $P(\phi)$ are general functions of their arguments (the generalization to several chiral multiplets is obvious). Any self-interactions of other $N=1$ off-shell matter multiplets are known to be reduced by a duality transformation to the generic form (1.1) (see, e.g. refs. [2,6]). For instance in the general $N=1$ tensor multiplet action (with an arbitrary function f):

$$S_G = \int d^4x\, d^4\theta\, f(G), \qquad (1.2a)$$

$$D^2 G = \bar{D}^2 G = 0, \qquad (1.2b)$$

we may implement the constraint (1.2b) with the help of a Lagrange multiplier

$$S_{G,X} = \int d^4x\, d^4\theta \left[f(G) + (D^2 X + \bar{D}^2 \bar{X}) G \right]. \qquad (1.3)$$

Varying X we come back to eq. (1.2). Varying G instead, we obtain an algebraic equation $f'(G) = -D^2X - \bar{D}^2\bar{X}$ which can always be solved for G

$$G = G(D^2X + \bar{D}^2\bar{X}), \qquad (1.4)$$

provided f is nondegenerate ($f''(G) \neq 0$). Substituting it into eq. (1.3) gives

$$S_\phi = \int d^4x\, d^4\theta \left\{ f[G(\phi + \bar{\phi})] + (\phi + \bar{\phi}) G(\phi + \bar{\phi}) \right\}, \qquad (1.5)$$

where $\phi = \bar{D}^2\bar{X}$ ($\bar{\phi} = D^2X$) is a chiral superfield. So, the general self-interactions of the $N=1$ tensor multiplet (1.2) actually are equivalent to a restricted class of chiral multiplets (1.1) with $P(\phi) = 0$ and

$$K(\phi, \bar{\phi}) = f[G(\phi + \bar{\phi})] + (\phi + \bar{\phi}) G(\phi + \bar{\phi}). \qquad (1.6)$$

We observe that this particular $K(\phi, \bar{\phi})$ depends only on $\phi + \bar{\phi}$. Hence, it is invariant under

$$\phi \to \phi + ia\,, \qquad \bar{\phi} \to \bar{\phi} - ia\,, \qquad a = \bar{a} = \text{const}\,, \tag{1.7}$$

while general $K(\phi, \bar{\phi})$ in eq. (1.1) may have no such symmetry.

Similar reasoning applies to other off-shell representations of $N = 1$ matter. So, it is the chiral multiplet that provides us with the most general $N = 1$ matter self-couplings. A crucial observation is that the corresponding chiral superfield $\phi(x_{\mathrm{L}}, \theta)$ is an unconstrained nongauge function on the chiral superspace $\mathbb{C}^{4|2}$ [1,7]. As we shall argue the solution of the $N = 2$ problem is given analogously by an unconstrained nongauge superfield $q^+(\mathfrak{z}, u)$ (or $(\omega(\mathfrak{z}, u))$ defined as the general function over analytic harmonic superspace $\mathbb{A}\mathrm{R}^{4+2|4}$ [3] (the q^+ and ω-hypermultiplets represent the same entity in the first- and second-order formalisms, respectively). A key difference from the $N = 1$ case is that $q^+(\mathfrak{z}, u)$ contains an infinite number of auxiliary fields and it is unavoidable in view of a simple no-go theorem [8].

To end with the $N = 1$ case, we recall an important geometric aspect of the $N = 1$ matter description by chiral superfields. Just in terms of these superfields the Kähler geometry of $N = 1$ matter self-coupling [9] manifests itself most clearly. Indeed, the lagrangian density $K(\phi, \bar{\phi})$ in (1.1) can be viewed as Kähler potential, with $\phi, \bar{\phi}$ playing the role of complex coordinates. For any Kähler manifold, this potential completely characterizes the relevant geometry. Thus, the chiral superfield formulation visualizes the theorem stating that any $N = 1$ matter action is basically supersymmetrization of some Kähler σ model [9]. From the geometric viewpoint, to construct the most general $N = 1$ matter self-interaction, one has to take the most general Kähler σ model, change, in its Kähler potential, the complex coordinates to chiral superfields and choose the resulting expression as the lagrangian density (arbitrary nongeometric terms of the type $P(\phi)$ in (1.1) can be added).

The study of $N = 2$ matter (four spin-0 and two spin-$\frac{1}{2}$ on-shell) began with elucidating the geometry which governs its self-couplings. In a remarkable paper [10] Alvarez-Gaumé and Freedman have proved that any self-interaction of matter $N = 2$ multiplets yields a hyper-Kähler σ model in the physical boson field sector, and similarly any given hyper-Kähler σ model can be $N = 2$ supersymmetrized. However, in the $N = 2$ case the geometric considerations do not lead immediately to an explicit construction of the most general matter self-coupling. The point is that at present a complete list of hyper-Kähler metrics is lacking and no general recipe is known for construction of such metrics (in contradistinction to the $N = 1$ case, where any Kähler metric is defined by some Kähler potential $K(\phi, \bar{\phi})$ which is the primary object with no further restrictions). Knowing an adequate off-shell superfield formulation of $N = 2$ matter may fill this gap.

Until the invention of harmonic superspace all attempts at such a formulation were undertaken on the basis of $N = 2$ multiplets with a finite number of components. The first off-shell $N=2$ multiplet of this sort was found by Wess [11] and de Wit and van Holten [12]. An analysis of self-interactions of this tensor [11,12] $N=2$ multiplet by Lindström and Roček [13] allowed them to construct some known and some new hyper-Kähler metrics. However, they noticed that it is impossible to achieve the most general self-couplings of $N=2$ matter with tensor multiplets. The reason is that after a duality transformation there arises a hyper-Kähler σ model with at least one Killing vector, in complete analogy with the discussion of tensor $N=1$ multiplet above (see eq. (1.7)). At the same time, there exist hyper-Kähler metrics with no continuous isometries.

Another example of an off-shell $N=2$ matter multiplet was found by Howe, Stelle and Townsend [14] in their search for $N=2$ formulations of the $N=4$ Yang-Mills theory. Unlike the tensor $N=2$ multiplet, this relaxed hypermultiplet admits a minimal gauge coupling. Some further relaxed $N=2$ matter multiplets were discussed recently by Yamron and Siegel [4]. A common feature of corresponding actions is an inevitable presence of Killing vectors, analogous to the case of the tensor $N=2$ multiplet.

The harmonic superspace approach enables us to demonstrate that this property is not accidental. We show that all the above off-shell $N=2$ multiplets are naturally described in harmonic superspace by properly constrained (or sometimes having a gauge freedom) analytic superfields. All their self-interactions are equivalent on-shell to those of tensor multiplets. The diversity of matter multiplets with a finite number of components in $N=2$ supersymmetry is related to the existence of unconstrained harmonic multiplets with an infinite number of components. We may, e.g., easily construct further relaxations similar to those given in refs. [14,4] with increasing (but finite) arrays of auxiliary fields. The most important point is that all these actions are equivalent to some restricted class of actions of unconstrained q^+ hypermultiplet superfields, like actions of the $N=1$ tensor multiplet are equivalent to a restricted class of $N=1$ chiral superfield actions. We prove this with the help of $N=2$ duality transformations defined by us for the first time in refs. [15,16]. An equivalence to a class of q^+ actions can be proved also for other constrained $N=2$ matter actions considered recently in refs. [8,17]. At the same time, q^+ itself admits general self-couplings which cannot be implemented with any known $N=2$ multiplet having a finite number of components.

The paper is planned as follows. In sect. 2 we first succinctly recall the basics of the description of matter hypermultiplets in harmonic superspace. Then we give their general self-coupling. In q^+ language it is written down as

$$S=\frac{1}{\kappa^2}\int \mathrm{d}\zeta^{(-4)}\,\mathrm{d}u\,\mathscr{L}^{(+4)}\left(q_A^+,\bar{q}_A^+,D^{++}q_A^+,D^{++}\bar{q}_A^+,\ldots(D^{++})^n q_A^+,\ldots,u_i^{\pm}\right) \quad (1.8)$$

(for the notation, see [3] and the text). $\mathscr{L}^{(+4)}$ (1.8) is a four-fold U(1)-charged function that arbitrarily depends on superfields q_A^+ ($A=1,2,\ldots$) and any degree of their harmonic derivatives and includes explicitly harmonics u_i^-, u_i^- (also in an arbitrary way). Besides, a generalization to nonzero central charges is discussed. In sects. 3, 5 we show how to describe in harmonic superspace all other $N=2$ off-shell matter multiplets known previously. Higher relaxed multiplets are derived (sect. 3) and are shown to admit no new self-couplings (on-shell) as compared with the lower ones. Sect. 4 introduces $N=2$ duality transformations. By means of these, all the self-interactions of $N=2$ multiplets having a finite number of components (sects. 4, 5) as well as those of higher U(1)-charge analogs of the ω hypermultiplet (sect. 5) are reduced to a class of q^+ self-interactions (1.8). This class is distinguished either by having at least one isometry (for tensor and relaxed multiplets) and/or by restrictions on admissible degrees of harmonic derivatives. So, eq. (1.8) presumably describes the most general self-coupling of $N=2$ matter.

The general function $\mathscr{L}^{(+4)}$ in (1.8) is an $N=2$ analogue of the Kähler potential of $N=1$ case in (1.1) and can be called a "hyper-Kähler potential". Listing all possible $\mathscr{L}^{(+4)}$ amounts supposedly to listing all the hyper-Kähler metrics. This construction yields the metric and the corresponding complex structures explicitly only after elimination of an infinite set of auxiliary fields via their equations of motion, while in the $N=1$ case one meets the Kähler metric immediately, before removing auxiliary fields. Anyway, $\mathscr{L}^{(+4)}$ provides us with a new effective tool for studying hyper-Kähler manifolds. Perhaps, it is a more fundamental quantity than the corresponding metric itself.

2. Unconstrained $N=2$ matter multiplets

This section collects main facts concerning the basic $N=2$ matter multiplet (hypermultiplet). It is an $N=2$ analogue of the $N=1$ chiral multiplet. Like the latter, it can be described by an unconstrained superfield defined on some submanifold (on the analytic $N=2$ superspace). This hypermultiplet can be represented either in the first-order formalism (q^+) or in the second-order one (ω). We review its properties and general self couplings.

2.1. q^+ HYPERMULTIPLET

The harmonic superspace $\mathrm{HR}^{4+2|8}$ is obtained from the ordinary $N=2$ one $\mathbb{R}^{4|8}$ by adding a sphere $S^2 = SU(2)_A/U(1)$, where $SU(2)_A$ is the automorphism group of $N=2$ superalgebra. Harmonics $u_i^{\pm}$ are coordinates of this sphere ($u^{+i}u_i^- = 1$) and, consequently, they have SU(2) index i and $U(1)$ charges ± 1. There is an invariant subspace in $\mathrm{HR}^{4+2|8}$ with coordinates $(\mathfrak{z}; u) = (x_A^{\alpha\dot\alpha}, \theta^{+\alpha}, \bar\theta^{+\dot\alpha}; u_i^{\pm})$ that is called analytic superspace $\mathrm{AR}^{4+2|4}$ (see appendix A). In the first-order formalism the basic hypermultiplet is described by an unconstrained analytic superfield $q^+(\mathfrak{z}, u)$ which is complex and has U(1)-charge $+1$. To descend to ordinary fields, one has to decompose it in powers both of Grassmann and harmonic variables taking care of the strict U(1)-charge conservation in each term of the expansion:

$$q^+(\mathfrak{z}, u) = f^i(x)u_i^+ + f^{(ijk)}u_{(i}^+u_j^+u_{k)}^- + \cdots$$
$$+\theta^{+\alpha}\left(\psi_\alpha(x) + \psi_\alpha^{(ij)}(x)u_{(i}^+u_{j)}^- + \cdots\right)$$
$$+\bar\theta_{\dot\alpha}^+\left(\bar\kappa^{\dot\alpha}(x) + \bar\kappa^{\dot\alpha(ij)}u_{(i}^+u_{j)}^- + \cdots\right) + \cdots, \tag{2.1}$$

where we have kept only the physical fields (isodoublet of scalars $f^i(x)$ and two isosinglet Weyl fermions $\psi_\alpha(x), \bar\kappa^{\dot\alpha}(x)$) and the very beginning of infinite tails of auxiliary fields $f^{(ijk)}(x)$, etc. Each component field carries indices of the group $SU(2)_A$. The corresponding free action is (the bar denotes an appropriate conjugation [3])

$$S_q^{\text{free}} = \int d\mathfrak{z}^{(-4)}\,du\,\tfrac{1}{2}\bar q^+ \overleftrightarrow{D}^{++} q^+ = -\int d\mathfrak{z}^{(-4)}\,du\,\tfrac{1}{2}(q^{+a}D^{++}q_a^+), \tag{2.2}$$

where we have introduced the notation

$$q_a^+ = (q^+, \bar q^+), \qquad \bar q^{+a} \equiv q^{+a} = \varepsilon^{ab}q_b^+ . \tag{2.3}$$

Now we can easily verify that after eliminating auxiliary fields by the equation of motion

$$D^{++}q_a^+ = \left(u^{+i}\frac{\partial}{\partial u^{-i}} - 2i\theta^+\sigma^m\bar\theta^+\frac{\partial}{\partial x_A^m}\right)q_a^+ = 0, \tag{2.4}$$

there remain standard free equations for the above physical fields. The equation of motion (2.4) says that in the central basis (see appendix A) $q^+(\mathfrak{z}(z,u),u) = u_i^+q^i(z) = u_i^+q^i(x, \theta_{\alpha i}, \bar\theta_{\dot\alpha}^i)$. At the same time the manifest analyticity of q^+ in the analytic basis implies in the central basis

$$0 = D_\alpha^+ q^+(\mathfrak{z}, u) = u_i^+ D_\alpha^i u_j^+ q^j(z) \Rightarrow D_\alpha^{(i}q^{j)}(z) = 0$$

and, analogously

$$0 = \bar D_{\dot\alpha}^+ q^+(\mathfrak{z}, u) = u_i^+ \bar D_{\dot\alpha}^i u_j^+ q^j(z) \Rightarrow \bar D_{\dot\alpha}^{(i}q^{j)}(z) = 0 .$$

So we arrive at the familiar Fayet-Sohnius [18] description of hypermultiplet.

The free action (2.2) is similar to the massless Dirac action. In particular, like the latter it is invariant under a rigid SU(2) group, that is seen especially clear in the pseudoreal notation (2.3) [19].

$$\delta q_a^+ = \lambda_a^b q_b^+ , \qquad \left(\overline{\lambda_a^b}\right) = -\lambda_b^a, \qquad \lambda_a^a = 0 . \tag{2.5}$$

This SU(2) group is easily identified with the Pauli-Gürsey group because it is an off-shell extension of the known internal Pauli-Gürsey SU(2) symmetry of an on-shell component construction*.

The transformations (2.5) evidently commute with the $N=2$ supersymmetry and the SU(2) group of automorphisms acting on harmonic variables

$$\delta u_i^{\pm} = A_i^j u_j^{\pm}, \qquad \left(\overline{A_i^j}\right) = -\left(A_j^i\right), \qquad A_i^i = 0. \tag{2.6}$$

Besides, the action (2.2) is invariant under $N=2$ superconformal group SU(2,2|2) [20] and, in particular, under its subgroup $\mathrm{SU}(2)_c$ whose off-shell realization is different both from (2.5) and (2.6):

$$\delta_c^* q_a^+ = \delta_{st}^* q_a^+ - \alpha^{(ij)} u_i^- u_j^- D^{++} q_a^+, \qquad \delta^* q_a^+ = q_a^{+\prime}(\zeta, u) - q_a^+(\zeta, u),$$

$$\delta_{st}^* q_a^+ = -\alpha_i^j \left(u_j^+ \frac{\partial}{\partial u_i^+} + u_j^- \frac{\partial}{\partial u_i^-} \right) q_a^+ \tag{2.7}$$

where $\alpha^{(ij)}$ are $\mathrm{SU}(2)_c$ parameters. It is seen that $\mathrm{SU}(2)_c$ actually coincides on-shell with $\mathrm{SU}(2)_{\mathrm{A}}$.

2.2. ω HYPERMULTIPLET

In the second-order formalism the basic hypermultiplet is described by a real analytic superfield with zero U(1)-charge. Its harmonic decomposition

$$\omega(\zeta, u) = \omega(x) + \omega^{(ij)}(x) u_i^+ u_j^- + \theta^{+\alpha} u_i^- \psi_\alpha^i(x) + \bar{\theta}_{\dot\alpha}^+ u^{-i} \bar{\psi}_i^{\dot\alpha}(x) + \cdots \tag{2.8}$$

contains four physical scalars as $\underline{1} + \underline{3}$, and two Weyl fermions as $\underline{2}$ of $\mathrm{SU}(2)_{\mathrm{A}}$ (cf. SU(2) prescriptions for the q^+ case). The corresponding free action in the absence of central charges is (now an analogue of the Klein-Gordon one)

$$S_\omega^{\text{free}} = -\int d\zeta^{(-4)} du\, D^{++}\omega D^{++}\omega. \tag{2.9}$$

* Indeed, the action for free physical fields

$$S = \tfrac{1}{2}\int d^4x \left(\partial_a \bar{f}^i \partial^a f_i + i\psi \not{\partial}\bar{\psi} + i\kappa \not{\partial}\bar{\kappa} \right)$$

has a Pauli-Gürsey invariance under

$$\delta f^i = i\alpha f^i + \beta \bar{f}^i, \qquad \delta\psi = i\alpha\psi + \beta\kappa, \qquad \delta\kappa = -i\alpha\kappa - \bar{\beta}\psi,$$

$$\alpha = i\lambda_1^1, \qquad \beta = -\lambda_1^2.$$

Again, the equation of motion (now of the second order: $(D^{++})^2\omega = 0$) tells us that only fields written explicitly in eq. (2.8) are physical. An infinite tail of unwritten components is auxiliary.

Passing from q^+ to the ω hypermultiplet goes as follows [19]. Let us make a change of variables

$$q_i^+ = u_i^+\omega + u_i^- f^{++}\,; \qquad \bar{\omega} = \omega\,, \qquad \bar{f}^{++} = f^{++} \tag{2.10}$$

in (2.2). Then we obtain

$$S_q^{\text{free}} = \int d\zeta^{(-4)}\, du\, \tfrac{1}{2}\left(f^{++}f^{++} + 2f^{++}D^{--}\omega\right) \tag{2.11}$$

and this is just the first-order action for ω corresponding to (2.9). The inverse change is

$$\omega = -u_2^- q^+ + u_1^- \bar{q}^+ = u_i^- q^{+i}\,,$$

$$f^{++} = u_2^+ q^+ - u_1^+ \bar{q}^+ = u^{+i} q_i^+\,. \tag{2.12}$$

Again, the manifestly $N=2$ supersymmetric action (2.11) has two internal SU(2) symmetries (apart from the SU(2) of SU(2, 2|2)). The first is defined as*:

$$\delta u_i^{\pm} = A_i^j u_j^{\pm}\,, \qquad \omega'(\zeta, u') = \omega(\zeta, u)\,, \qquad f^{++\prime}(\zeta, u') = f^{++}(\zeta, u)\,.$$

The second one is the Pauli-Gürsey SU(2) group (2.5) which is written down in terms of ω and f^{++} as

$$\begin{aligned} \delta\omega &= -\lambda^{+-}\omega - \lambda^{--} f^{++} \\ \delta f^{++} &= \lambda^{++}\omega + \lambda^{+-} f^{++}\,, \end{aligned} \qquad \lambda^{\pm\pm} = \lambda^{(ij)} u_i^{\pm} u_j^{\pm}\,. \tag{2.13}$$

So, there are two completely equivalent ways of describing the basic $N=2$ hypermultiplet.

2.3. GENERAL SELF-COUPLINGS OF $N=2$ HYPERMULTIPLETS

Resorting to dimensionality and analyticity reasonings we can write down the most general self-interaction of any number of hypermultiplets in the form

$$S^{\text{gen}} = \frac{1}{\kappa^2}\int d\zeta^{(-4)}\, du\, \mathscr{L}^{(+4)}\left(q_A, \bar{q}_A, D^{++}q_A, D^{++}q_A^-, (D^{++})^2 q_A, \ldots, u\right). \tag{2.14}$$

* This SU(2) plays the role of automorphism group in the ω language and forms a diagonal in the direct product of (2.5) and (2.6). The decomposition (2.10) is covariant just with respect to this diagonal subgroup but not to (2.5) or (2.6) separately. The automorphism groups in the q and ω-representations coincide only modulo Pauli-Gürsey SU(2) transformations that commute with (2.6) and with supersymmetry (but not with the diagonal SU(2)-tranformations, of course).

Of course, $\mathscr{L}^{(+4)}$ must be nondegenerate to contain the free kinetic term

$$\frac{\delta\mathscr{L}^{(+4)}}{\delta\bar{q}^{+}\delta D^{++}q^{+}} - \frac{\delta\mathscr{L}^{(+4)}}{\delta q^{+}\delta D^{++}\bar{q}^{+}} \neq 0. \tag{2.15}$$

Otherwise, $\mathscr{L}^{(+4)}$ is an arbitrary U(1)-charge 4 function of harmonics, q^{+}s and their harmonic derivatives of any order (because these derivatives are dimensionless), κ is a coupling constant with dimension cm^1 for $d=4$, cm^0 for $d=2$ and cm^2 for $d=6$. In (2.14) we set q's to be dimensionless. So, spatial and spinor derivatives (dimensionalities cm^{-1} and $\mathrm{cm}^{-1/2}$, respectively) are inadmissible for dimensionality considerations, and in fact they would lead to a higher derivative action for physical fields.

One has no need to include any ω superfield into (2.14). Indeed, let some ω be present in $\mathscr{L}^{(+4)}$. We may always equivalently rewrite its kinetic term in the first order form (see eqs. (2.9)–(2.11)):

$$(D^{++}\omega)^2 \to -\tfrac{1}{2}(f^{++}f^{++} + 2f^{++}D^{++}\omega),$$

with a nonpropagating superfield f^{++} (varying the latter yields $f^{++} = -D^{++}\omega$ and we reproduce the original action). Then, f^{++} together with ω can be combined into a single q_i^{+} by eq. (2.10) that leads to a standard kinetic term for q^{+}. After this, it remains to substitute $u_i^{-}q^{+i}$ for ω everywhere in the rest of $\mathscr{L}^{(+4)}$*.

After elimination of auxiliary fields every $\mathscr{L}^{(+4)}$ results in an $N=2$ (i.e. hyper-Kähler) nonlinear sigma model [10] and hence in some hyper-Kähler metric. This suggests a new scheme of classification of hyper-Kähler manifolds according to the form of the corresponding lagrangian density in (2.14). At present, several examples have been worked out in some detail [21, 22]. For instance, the Taub-NUT manifold can be considered as the manifold coded in the action

$$S_{\mathrm{TN}} = \frac{1}{\kappa^2}\int \mathrm{d}\mathfrak{z}^{(-4)}\,\mathrm{d}u\left[\bar{q}^{+}D^{++}q^{+} + \lambda(q^{+})^2(\bar{q}^{+})^2\right], \tag{2.16}$$

where λ is a dimensionless parameter. The next example is the Eguchi-Hanson manifold that corresponds to the action

$$S_{\mathrm{EH}} = -\frac{1}{\kappa^2}\int \mathrm{d}\mathfrak{z}^{(-4)}\,\mathrm{d}u\left[(D^{++}\omega)^2 - (\xi^{++})^2\omega^{-2}\right], \tag{2.17}$$

where $\xi^{++} = \xi^{ij}u_i^{+}u_j^{+}$ stands for a dimensionless parameter. Note that in this case

* The q-ω equivalence persists, of course, at the full interaction level. In principle, we might pass from q^{+} to ω in any $\mathscr{L}^{(+4)}$. However, this procedure is more subtle than the inverse one. In fact, we do not need it at all.

the ω description turns out to be more transparent (the q description is also possible [22]). The number of examples can be enlarged [22]. The above ones, (2.16) and (2.17), possess U(2) symmetry having different origins. U(2) of (2.16) is the product of $SU(2)_A$ and U(1)-subgroup of $SU(2)_{PG}$ while U(2) of (2.17) is the product of $SU(2)_{PG}$ and U(1)-subgroup of $SU(2)_A$ *. In the general case the action (2.14) does not contain any symmetries U(1), SU(2), etc. besides $N=2$ supersymmetry. Even the SU(2) group of automorphisms can be completely violated by allowing explicit harmonics to appear in $\mathscr{L}^{(+4)}(q^+, D^{++}q^+, \ldots (D^{++})^n q^+, \ldots, u)$. This should be contrasted with the $N=2$ matter self-couplings based on superfields having a finite number of fields (the tensor multiplet, the relaxed hypermultiplet, etc.). As will be discussed in sect. 4 the lagrangian densities of tensor as well as of original and higher relaxed multiplets are reduced by duality transformations to a particular class of $\mathscr{L}^{(+4)}$ exhibiting invariance under the shifts

$$\delta q^+ = \text{const}\, u_1^+ , \qquad \delta \bar{q}^+ = \text{const}\, u_2^+ . \tag{2.18}$$

2.4. COMMENT ON CENTRAL CHARGES

Now we shall briefly discuss an extension of the above picture to nonzero central charges. As we described in the appendix to ref. [3], the central charge can be included into the harmonic superspace scheme by a standard method, i.e. by adding to $(\mathfrak{z}, u)$ an extra bosonic coordinate x_A^5.

This entails the following modification of harmonic derivative (in the analytic basis when applied to analytic superfields)

$$D^{++} \to D_{cc}^{++} + i(\theta^+\theta^+ - \bar{\theta}^+\bar{\theta}^+)\frac{\partial}{\partial x_A^5} . \tag{2.19}$$

Now analytic superfields are allowed to depend in a general way on an additional coordinate x_A^5. To preserve the number of physical fields, we follow the dimensional reduction procedure of Scherk and Schwarz [24]. Let the action (2.14) have a U(1)-symmetry commuting with $N=2$ supersymmetry and possessing a Killing vector $G^+(q^+, \ldots)$

$$\delta q^+ = \alpha G^+(q^+, \ldots) , \tag{2.20}$$

where α is the U(1)-parameter. Then x_A^5 dependence of q^+ is restricted as (m is a parameter of dimension of the mass)

$$\frac{\partial}{\partial x_A^5} q^+ = m G^+ . \tag{2.21}$$

* Looking at (2.14), it is easy to observe that it is impossible to preserve both $SU(2)_A$ and $SU(2)_{PG}$ for a single self-interacting q^+ while this becomes possible for two and more hypermultiplets.

After substituting (2.21) into the harmonic derivative (2.19) and then (2.19) into the action (2.14) we arrive at the theory invariant under $N=2$ supersymmetry with a central charge. In such a theory the potential terms (in particular, a mass term) become possible; e.g. for the free theory (2.2) such a procedure with $G^+ = iq^+$ results in a mass term: instead of (2.2) we would have

$$S^{\mathrm{cc}} = \frac{1}{2\kappa^2}\int \mathrm{d}\zeta^{(-4)}\,\mathrm{d}u\,\left[\bar{q}^+ \vec{D}^{++} q^+ + 2m(\theta^+\theta^+ - \bar{\theta}^+\bar{\theta}^+)\bar{q}^+ q^+\right]. \qquad (2.22)$$

Note that $N=2$ supersymmetry admits generally two central charges*. A corresponding extension of the above procedure can be done along similar lines with two additional coordinates x^5 and x^6 instead of x^5 and for actions (2.14) having two U(1)-symmetries commuting with each other and with supersymmetry.

3. Harmonic superspace description of off-shell matter multiplets with a finite number of fields: Tensor and relaxed $N=2$ multiplets

In this section we shall show how to describe the tensor multiplets, the relaxed hypermultiplet and further relaxed multiplets in the harmonic superspace. There all these multiplets are represented by some constrained or (and) gauge analytic superfields.

3.1. $N=2$ TENSOR MULTIPLET

This multiplet was considered in the harmonic superspace approach in [15, 16]. Here we recall some relevant facts. Its superfield strength is a real analytic superfield $L^{++}(\zeta, u) = \overline{L^{++}}(\zeta, u)$ having $U(1)$ charge $+2$ and obeying the constraint

$$D^{++}L^{++}(\zeta, u) = 0. \qquad (3.1)$$

To make a contact with the representation of tensor multiplet in ordinary $N=2$ superspace [11], we pass from the analytic basis to the central one $z = (x^m, \theta^i_\alpha, \bar{\theta}_{\dot{\alpha}i})$. There the constraint (3.1) says that

$$L^{++}(\zeta(z,u), u) = u^+_i u^+_j L^{ij}(z). \qquad (3.2)$$

On the other hand, the analyticity conditions

$$D^+_\alpha L^{++} = \bar{D}^+_{\dot{\alpha}} L^{++} = 0 \qquad (3.3)$$

in the central basis become

$$D^{(i}_\alpha L^{jk)}(z) = \bar{D}^{(i}_{\dot{\alpha}} L^{jk)}(z) = 0. \qquad (3.3')$$

So we come to the familiar picture [11].

* Harmonic superspace with both x^5 and x^6 was discussed in [23].

Let us return now to the analytic superspace and give the free action. It is bilinear in L^{++}

$$S^{\text{free}} = \frac{1}{2\kappa^2}\int d\mathfrak{z}^{(-4)}\, du\,(L^{++})^2; \qquad [\kappa] = \text{cm}^1, \qquad [L^{++}] = \text{cm}^0. \tag{3.4}$$

By dimensionality and analyticity arguments the general $N=2$ supersymmetric self-coupling is given by

$$S^{\text{int}} = \frac{1}{\kappa^2}\int d\mathfrak{z}^{(-4)}\, du\, F^{(+4)}(L^{++}, u), \tag{3.5}$$

where $F^{(+4)}\,(L^{++}, u)$ is an arbitrary dimensionless function of the field strength L^{++} and harmonics $u^{\pm}$ having U(1)-charge 4. Of course, the constraint (3.1) is implied, being the definition of tensor multiplet.

Definition (3.1) is $N=2$ superconformally covariant [20,16], while the action (3.4) is not. The theory with conformally invariant action corresponds to improved tensor multiplet and it has drawn attention because this multiplet can be used as a compensator for the $N=2$ conformal supergravity [25]. The improved tensor multiplet has been found first in terms of component fields [25], then in terms of $N=1$ and special $N=2$ superfields [13,26] and, finally, in terms of harmonic superfields [16]. In the last formulation the improved tensor multiplet action is written down as

$$S = \frac{1}{\kappa^2}\int d\mathfrak{z}^{(-4)}\, du\, \frac{(l^{++})^2}{(1+\sqrt{1+l^{++}c^{--}}\,)^2}, \qquad l^{++} \equiv L^{++} - c^{++}, \tag{3.6}$$

where

$$c_{ij} = \text{const}, \qquad c^{\pm\pm} = c^{ij}u_i^{\pm}u_j^{\pm}, \qquad c^{ij}c_{ij} = 2,$$

and (see [16] or sect. 4) eq. (3.6) is equivalent via a duality transformation to the free action for q^+ hypermultiplet. At the same time, a sum of (3.4) and (3.6) yields a nontrivial self-interaction [13,26]. This sum is known to possess an SU(2)-symmetry. The present approach allows one to simply identify this SU(2) as a subgroup of conformal supergroup SU(2,2|2) (which as a whole is broken in that case). It is realized as

$$\delta_c^* L^{++} = L^{++\prime}(\mathfrak{z}, u) - L^{++}(\mathfrak{z}, u) = -\delta_{\text{st}}^* L^{++} - \alpha^{(i,j)}u_i^- u_j^- D^{++}L^{++} = -\delta_{\text{st}}^* L^{++}, \tag{3.7}$$

where $\delta_{\text{st}}^* L^{++}$ is of the same form as in eq. (2.7) and the constraint (3.1) has been used. Thus, this SU(2) has actually the same realization on component fields as $SU(2)_A$.

It is worthwhile mentioning the impossibility of minimal Yang-Mills coupling for the $N = 2$ tensor multiplet. The reason is that this multiplet includes a gauge antisymmetric tensor field and its gauge invariance is incompatible with the Yang-Mills one.

3.2. THE RELAXED HYPERMULTIPLET

Howe, Stelle and Townsend [14] (HST) invented the relaxed matter hypermultiplet which can have a minimal Yang-Mills coupling unlike the tensor multiplet. In harmonic superspace it is described by real analytic superfields $L^{++}(\mathfrak{z}, u)$ and $V(\mathfrak{z}, u)$. The constraint now comes out as a relaxed form of (2.1)

$$(D^{++})^2 L^{++} = 0. \tag{3.8}$$

The superfield V is defined up to a gauge transformation

$$V' = V + D^{++}\lambda^{--}, \tag{3.9}$$

where λ^{--} is an analytic real gauge parameter. The free action

$$S_{\mathrm{HST}}^{\mathrm{free}} = \frac{1}{\kappa^2}\int d\mathfrak{z}^{(-4)} du \left[(L^{++})^2 + VD^{++}L^{++}\right] \tag{3.10}$$

is compatible with gauge invariance (3.9) because of the constraint (3.8).

The dimensionality, analyticity and gauge invariance reasonings lead to the following general form of self-coupling

$$S_{\mathrm{HST}} = \frac{1}{\kappa^2}\int d\mathfrak{z}^{(-4)} du \left[F^{(+4)}(L^{++}, D^{++}L^{++}, u) + VD^{++}L^{++}\right], \tag{3.11}$$

where $F^{(+4)}(L^{++}, D^{++}L^{++}, u)$ is an arbitrary U(1)-charge four function of its arguments. Note, that by varying (3.11) with respect to V one gets $D^{++}L^{++} = 0$ and thereby comes back to the general action for tensor multiplet (3.5) and the constraint (3.1). So, as to the on-shell self-interactions the relaxed hypermultiplet and tensor multiplet actions are equivalent. An advantage of the relaxed hypermultiplet is its ability to have minimal Yang-Mills coupling. To introduce it, one has to take n copies of superfields L^{++} and V, put them into a real n-dimensional representation of the Yang-Mills group and, finally, to covariantize [3] the constraint (3.8) and the gauge transformation (3.9). Note that the requirement of reality is a severe restriction on a possible choice of the representation.

Hence, one sees how simple is the description of the relaxed multiplet in harmonic superspace. To demonstrate the exact correspondence with the original description, one has again to apply to the central basis. There, eq. (3.8) says that

$$L^{++} = u_i^+ u_j^+ L^{ij}(z) + 5u^+_{(i} u_j^+ u_k^+ u^-_{l)} L^{ijkl}(z) \tag{3.12}$$

(the factor 5 is introduced here for further convenience). From analyticity of L^{++} it follows that

$$D^{(i}_{\alpha(\dot\alpha)} L^{jk)} = D_{\alpha(\dot\alpha)l} L^{ijkl},$$

$$D^{(i}_{\alpha(\dot\alpha)} L^{jklm)} = 0 \tag{3.13}$$

(the identity $u_l^+ u^+_{(i_1} u^+_{i_2} u^+_{i_3} u^-_{i_4)} = u^+_{(l} u^+_{i_1} u^+_{i_2} u^+_{i_3} u^+_{i_4)} + \frac{1}{5}\varepsilon_{l(i_4} u^+_{i_1} u^+_{i_2} u^+_{i_3)}$ was used). Finally, scalar superfield $V(z)$ of HST [14] is

$$V(z) = \int \mathrm{d}u\, V(\mathfrak{z}(z,u),u). \tag{3.14}$$

Obviously, $V(z)$ is invariant under (3.9) and solves the constraints [14]:

$$D^i_\alpha D_{\beta i} V = \overline{D}_{\dot\alpha i} \overline{D}^i_{\dot\beta} V = \left[D^i_\alpha, \overline{D}_{\dot\beta i} \right] V = 0 .$$

One can verify also that our action (3.11) in the central basis coincides with the original one (see appendix B).

3.3. FURTHER RELAXED HYPERMULTIPLETS

Now it is evident how to continue relaxing constraints. Proceeding in this way one obtains a multiplet that is described as before by real analytic superfields $L^{++}(\mathfrak{z},u)$, $V(\mathfrak{z},u)$. However, the constraint now is further relaxed (cf. (3.8))

$$(D^{++})^3 L^{++} = 0, \tag{3.15}$$

and V is defined up to a gauge transformation (cf. (3.9))

$$V' = V + (D^{++})^2 \lambda^{(-4)}, \tag{3.16}$$

where $\lambda^{(-4)}(\mathfrak{z},u)$ is a real analytic gauge parameter. The free gauge invariant action coincides with the "first-order multiplet" action of Yamron and Siegel [4] (see appendix B). In analytic superspace it has again the same form as in eq. (3.10)

$$S^{\text{free}}_{\text{YS}} = \frac{1}{\kappa^2} \int \mathrm{d}\mathfrak{z}^{(-4)} \mathrm{d}u \left[(L^{++})^2 + V D^{++} L^{++} \right]. \tag{3.17}$$

The general self-coupling is given by

$$S_{\mathrm{YS}}=\frac{1}{\kappa^2}\int d\mathfrak{z}^{(-4)}du\left[F^{(+4)}\left(L^{++},D^{++}L^{++},(D^{++})^2L^{++},u\right)+VD^{++}L^{++}\right].\tag{3.18}$$

By varying V we obtain the constraint (3.1) and so return to the $N=2$ tensor multiplet action (3.6)*. Thus, what concerns the self-interactions, all these theories (the tensor-multiplet, the relaxed hypermultiplet, the further relaxed one) are equivalent.

One could construct new multiplets by further relaxing the constraint (3.1). The first step results in a tensor multiplet, the second one in a relaxed hypermultiplet, the third one in a further relaxed hypermultiplet. Let us specify the nth step. Here we have the constraint:

$$(D^{++})^nL^{++}=0\tag{3.19}$$

and we use a scalar analytic Lagrange multiplier $V_n(\mathfrak{z},u)$ defined up to the transformation

$$V_n'=V_n+(D^{++})^{n-1}\lambda^{-2(n-1)}(\mathfrak{z},u)\tag{3.20}$$

The general action is

$$S_n=\frac{1}{\kappa^2}\int d\mathfrak{z}^{(-4)}du\left\{F^{(+4)}\left(L^{++},D^{++}L^{++},\ldots,(D^{++})^{n-1}L^{++},u\right)+V_nD^{++}L^{++}\right\}.\tag{3.21}$$

The superspin-zero superfield L^{++} carries superisospins $0,1,\ldots,n-1$ and, respectively, describes

$$(8+8)[1+3+\cdots+2n-1]=8n^2+8n^2$$

degrees of freedom. Due to the gauge freedom (3.20) the superspin zero superfield V_n really contains superisospins $1,2,\ldots n-1$ and hence furnishes us with $(8+8)(n^2-1)$ more field components. Thus, the theory arising at the nth step contains $8(2n^2-1)+8(2n^2-1)$ field degrees of freedom. We have $8+8$ for the tensor multiplet ($n=1$), $56+56$ for the relaxed hypermultiplet ($n=2$), $136+136$ for the

* One might add to (3.18) a term $\sim V^{--}(D^{++})^2L^{++}$ and equally reproduce the relaxed hypermultiplet action (3.11) and the constraint (3.8) by varying V^{--}. However, there is actually no need to consider this modification, as the modified action is reduced again to (3.18) by a redefinition of the Lagrange multiplier as $V\to V-D^{++}V^{--}$.

further relaxed hypermultiplet ($n = 3$), 248 + 248 for $n = 4$, etc. Nevertheless, for any *finite n* we obtain theories with equivalent self-couplings. Indeed, variation with respect to V_n always yields $D^{++}L^{++} = 0$ which leads to a self-interacting tensor multiplet. A radically new situation arises at $n = \infty$ when L^{++} becomes an unconstrained analytic superfield and V suffers no gauge invariance. Then $F^{(+4)}$ may contain arbitrary degrees of harmonic derivatives of L^{++} and, furthermore, one may insert into $F^{(+4)}$ an arbitrary dependence on V and its harmonic derivatives. Combining L^{++} and V into a single q^+ superfield by eq. (2.10) we actually come to the general $N = 2$ matter action (2.14). This consideration sheds more light on the key role of analytic superfields with an infinite set of components in achieving the most general $N = 2$ matter self-coupling.

We would like to note that the above relaxed $N = 2$ multiplet actions can be easily extended to nonzero central charges by the general recipes of subsect. 2.4. In particular, eqs. (3.11), (3.18), (3.21) exhibit an obvious invariance under constant shifts of gauge superfield $V_n \to V_n + k$ (in the prepotential language, this isometry is U(1) observed in [4]). One may choose this invariance to identify central charge with its generator (times a mass parameter) and thereby to produce the mass terms which coincide with those given in ref. [4].

Finally, we leave it to the reader to compare simple and transparent analytic superspace actions given here with their lengthy and somewhat ugly prototypes in conventional $N = 2$ superspace (see appendix B).

4. Duality transformations

In this section it will be shown that all the self-interactions of the tensor, the relaxed and further relaxed multiplets are classically equivalent to some restricted class of q^+ (or ω) self-interactions.

4.1. TRANSFORMING THE TENSOR MULTIPLET

$N = 2$ duality transformations are direct generalizations of the $N = 1$ ones given in the introduction (eqs. (1.2)–(1.6)). We begin with the simplest case of a tensor multiplet. The superfield L^{++} in (3.4), (3.5) is constrained by eq. (3.1). One can instead implement this constraint in the action with the help of a Lagrange multiplier ω:

$$S = \frac{1}{\kappa^2} \int \mathrm{d}\zeta^{(-4)} \mathrm{d}u \left[F^{(+4)}(L^{++}, u) + \omega D^{++} L^{++} \right]. \tag{4.1}$$

Now both L^{++} and ω are unconstrained analytic superfields and one may vary them. Varying ω, we recover the original constraint (3.1). We prefer instead to pass to q^+ hypermultiplets by means of the change of variables (2.12).

$$L^{++} = u^{+i} q_i^+ , \qquad \omega = u^{-i} q_i^+ ,$$
$$q_i^+ = -u_i^+ \omega + u_i^- L^{++} . \tag{4.2}$$

We get

$$S^{\text{dual}} = \frac{1}{\kappa^2}\int d\zeta^{(-4)} du \left[F^{(+4)}(u^+q^+, u) - \tfrac{1}{2}(u^+q^+)^2 - \tfrac{1}{2}q^{+i}D^{++}q_i^+ \right]. \quad (4.3)$$

This action certainly is not the general one (see (2.14)). Indeed, it contains no harmonic derivatives in the interactions terms and is always invariant under transformations

$$\delta q_i^+ = \text{const}\, u_i^+ . \quad (4.4)$$

This invariance implies that the corresponding hyper-Kähler metric has at least one Killing vector.

In particular, for the free tensor multiplet (3.4) the dual action is the free q^+ one. Further, for the improved tensor multiplet (3.6) we again get the free action. To see this, one needs a more sophisticated change of field variables [16]*:

$$\tilde{q}_i^+ = (c_{ij} + i\varepsilon_{ij})(u^{+j} - iu^{-j}g^{++})e^{i\omega/2} + (c_{ij} - i\varepsilon_{ij})(u^{+j} + iu^{-j}g^{+-})e^{-i\omega/2},$$

$$g^{++} \equiv l^{++}/(1 + \sqrt{1 + l^{++}c^{--}}).$$

4.2. TRANSFORMING THE RELAXED HYPERMULTIPLETS

Once again, using as a Lagrange multiplier some real analytic superfield V^{--} we introduce the constraint for the relaxed hypermultiplet (3.8) into the action (3.11):

$$S = \frac{1}{\kappa^2}\int d\zeta^{(-4)} du \left\{ F^{(+4)}(L^{++}, D^{++}L^{++}, u) + VD^{++}L^{++} + V^{--}(D^{++})^2 L^{++} \right\}. \quad (4.5)$$

Here V^{--} transforms under the gauge group (3.9) as

$$V^{--\prime} = V^{--} + \lambda^{--} \quad (4.6)$$

in order to maintain the gauge invariance of the action. Now one can combine V and V^{--} into one (nongauge) analytic superfield ω

$$\omega = V - D^{++}V^{--}. \quad (4.7)$$

Making the same change of variables (4.2) as in (4.3) we obtain now a dual form of

* In the pioneer studies in terms of component fields this equivalence was proved by indirect complicated reasonings [25]. The first direct proof of it was given by Lindström and Roček in the $N = 1$ superfield language [13].

S as

$$S^{\text{dual}} = \frac{1}{\kappa^2}\int d\mathfrak{z}^{(-4)} du \left\{ F^{(+4)}\left(u^+q^+,\, u^{+i}D^{++}q_i^+,\, u\right) - \tfrac{1}{2}\left(u^+q^+\right)^2 - \tfrac{1}{2}q^{+i}D^{++}q_i^+ \right\}.$$

Again, the action is not of a general form. Moreover it is in fact guaranteed to be equivalent to the action (4.3) for the tensor multiplet (on-shell, after eliminating auxiliary fields), and it is also invariant under translation (4.4)*.

The same procedure equally applies to the Yamron-Siegel multiplet (3.15)–(3.18) and to further relaxed multiplets. All their self-interactions turn out to be dual equivalent to some particular self-interaction of q-hypermultiplets invariant under translations (4.4) and reducible on-shell to some self-interaction of q^+ which appears in the dual description of tensor multiplet.

5. Some other off-shell representations of $N = 2$ matter

All the off-shell $N = 2$ matter multiplets considered above contained only one propagating $N = 2$ multiplet on-shell. There exist, however, $N = 2$ representations yielding on-shell at once several such multiplets [8]. We consider here all representations of this kind and demonstrate that their actions are again reduced to restricted classes of generic q^+ action (2.14). Besides, we treat, in an analogous context, the representations discussed recently in [17] (higher rank tensor $N = 2$ multiplets) which can be used to propagate on-shell one hypermultiplet. Our basic tools are as before duality transformations and/or the equivalence redefinitions of type (2.10), (2.12).

5.1. THE HIGHER SUPERISOSPIN ANALOGS OF ω HYPERMULTIPLET

According to the general formula for the superisospin content of an analytic $N = 2$ superfield with the U(1)-charge q [31]

$$I = \left|\tfrac{1}{2}q - 1\right| + n,$$

the ω superfield (2.8) is merely the first in the infinite series of superfields $\omega^{(-k)}$ ($k \geqslant 0$) having $I_k = \frac{1}{2}(k + 2)$ as the lowest superisospin. It suffices to consider the case of even $k = 2l$ only (see below). Any superfield of this sort can be chosen real, $\overline{\omega^{-2l}} = \omega^{-2l}$. It has a simple free action [27]

$$S^{l+1} = \frac{1}{\kappa^2}\int d\mathfrak{z}^{(-4)} du \left\{ \omega^{(-2l)}(\mathfrak{z}, u)\left(D^{++}\right)^{2l+2}\omega^{(-2l)}(\mathfrak{z}, u)\right\}. \qquad (5.1)$$

* This isometry reveals itself already at the level of original relaxed hypermultiplet actions (3.11), (3.18), (3.21); see a remark at the end of the previous section.

The corresponding field equation

$$(D^{++})^{2l+2}\omega^{(-2l)}=0 \tag{5.2}$$

implies that $\omega^{(-2l)}$ collects $l+1$ propagating $N=2$ multiplets. (physical components are $4l+4$ scalars $\omega^{(i_1\cdots i_{2l})}(x)$, $\omega^{(i_1\cdots i_{2l+2})}(x)$ and $2l+2$ Majorana spinors $\psi_\alpha^{(i_1\cdots i_{2l+2})}(x)$). All the components corresponding to superisospins $I>l+1$ are auxiliary. They are killed by eq. (5.2), which leads also to correct equations for physical fields. One may add the interaction terms to (5.1).

The superfields $\omega^{(-2l-1)}$ cannot be real. Instead they can be combined, together with their conjugates, in pseudoreal doublets of $SU(2)_{PG}$

$$\omega_i^{(-2l-1)}=\left(\omega^{(-2l-1)},\overline{\omega^{(-2l-1)}}\right).$$

Making a change of variables like (2.10)

$$\omega_i^{(-2l-1)}=u_i^+\omega^{(-2l-2)}+u_i^-\omega^{(-2l)},\qquad \begin{array}{l}\overline{\omega^{(-2l)}}=\omega^{(-2l)}\\ \overline{\omega^{(-2l-2)}}=\omega^{(-2l-2)}\end{array},$$

we see that $\omega^{(-2l-1)}$ is equivalent to a pair of real superfields of the previous type $\omega^{(-2l)}$, $\omega^{(-2l-2)}$. So, there is no need to consider $\omega^{(-2l-1)}$.

Let us now demonstrate that the first-order form of the action (5.1) is the free action for a q hypermultiplet with a certain multi-index of the Pauli-Gürsey SU(2)-group, namely for $q^+_{(i_1\cdots i_{2l+1})}$ subjected to the pseudoreality condition

$$q^{+(i_1\cdots i_{2l+1})}=\overline{q^+_{(i_1\cdots i_{2l+1})}}=\varepsilon^{i_1j_1}\dots\varepsilon^{i_{2l+1}j_{2l+1}}q^+_{(j_1\cdots j_{2l+1})}.$$

We start with the following q action

$$S_q^{l+1}=-\frac{1}{\kappa^2}\int d\zeta^{(-4)}\,du\,q^{+(i_1\cdots i_{2l+1})}D^{++}q^+_{(i_1\cdots i_{2l+1})}. \tag{5.3}$$

It is easy to check that on-shell $q^+_{(i_1\cdots i_{2l+1})}$ carries the same $8l+8$ degrees of freedom as $\omega^{(-2l)}$ in the action (5.1). To prove the equivalence in an explicitly superfield fashion, we expand $q^{+(i_1\cdots i_{2l+1})}$ in symmetrised products of harmonics (compare to eq. (2.10)):

$$q^{+(i_1\cdots i_{2l+1})}=\sqrt{(2l+1)!}\sum_{m=0}^{2l+1}\frac{(2l+1)!}{(2l+1-m)!m!}u^{+(i_1}\dots u^{+i_m}u^{-i_{m+1}}\dots u^{-i_{2l+1})}\omega^{2(l-m+1)},$$

$$\overline{\omega^{2(l-m+1)}}=\omega^{2(l-m+1)}, \tag{5.4}$$

where numerical factors are included for convenience. Due to the completeness properties of $u_i^\pm$ one has

$$\omega^{2(l-m+1)} = (-1)^{m-1} \frac{1}{\sqrt{(2l+1)!}} u^-_{(i_1} \ldots u^-_{i_m} u^+_{i_{m+1}} \ldots u^+_{i_{2l+1})} q^{+(i_1 \ldots i_{2l-1})} .$$

Substituting the sum (5.4) into (5.3) and eliminating nonpropagating superfields by their equations of motion we finally recover (5.1) as the second-order form of (5.3). Using the equivalence between (5.3) and (5.1) and representing $\omega^{(-2l)}$ as $\sim u^-_{i_1} \ldots u^-_{i_{2l+1}} \times q^{+(i_1 \ldots i_{2l-1})}$ one may rewrite any action of $\omega^{(-2l)}$ (with an arbitrary self-interaction) in terms of q^+ hypermultiplet, confirming once more the universal role of the latter.

5.2. CONSTRAINED HARMONIC SUPERFIELDS WITH SEVERAL PROPAGATING MULTIPLETS

It has been noted in [3, 8] that a wide variety of off-shell $N=2$ representations with a finite number of auxiliary components can be produced by imposing proper constraints on harmonic superfields. These constraints cut an infinite tail of auxiliary superisospins before employing the equations of motion. One may, e.g., restrict ω as [8]:

$$(D^{++})^k \omega = 0, \qquad k \geqslant 3 \tag{5.5}$$

still preserving the standard form (2.9) of ω action. Let us introduce (5.5) into the action with the help of Lagrange multiplier $\omega^{(-2\kappa+4)}(\mathfrak{z}, u)$ (we confine our study here to the linearized level):

$$S = \frac{1}{\kappa^2} \int d\mathfrak{z}^{(-4)} du \left\{ \omega (D^{++})^2 \omega + \omega^{(-2\kappa+4)} (D^{++})^k \omega \right\} . \tag{5.6}$$

It is diagonalized by the substitution

$$\omega = \tilde{\omega} + \tfrac{1}{2}(-1)^{k+1} (D^{++})^{k-2} \omega^{(-2k+4)} , \tag{5.7}$$

$$S = \frac{1}{\kappa^2} \int d\mathfrak{z}^{-4} du \left[\tilde{\omega} (D^{++})^2 \tilde{\omega} + \tfrac{1}{4}(-1)^{k+1} \omega^{(-2k+4)} (D^{++})^{2k+2} \omega^{(-2k+4)} \right] . \tag{5.8}$$

As it follows from the analysis of the previous subsect., such an unconstrained action propagates $1 + k - 1 = k$ hypermultiplets and it is the representation content of the model with the constraint (5.5). Needless to say, the action with any self-interaction can be equivalently rewritten via proper q^+s. For $k=3$, the general action has a three-parameter isometry $\tilde{\omega} \to \tilde{\omega} - \xi^{(ij)} u_i^+ u_j^-$, $\omega^{(-2)} \to \omega^{(-2)} + \xi^{(ij)} u_i^- u_j^-$ ($\xi^{ij} = \text{const}$), reflecting the fact that in this case ω, before performing the duality transformation (5.7), is equivalent to three tensor $N=2$ multiplets [8]* (see exercise (i) at the end of this section). By the way (5.8) implies that for even k the propagating hypermultiplets in $\omega^{(-2k+4)}$ have the kinetic terms of wrong sign and thus are ghosts.

One more example is a two-fold charged analytic superfield $R^{(+2)}$ constrained by

$$(D^{++})^k R^{(+2)} = 0. \tag{5.9}$$

The corresponding action is

$$S_R = \frac{1}{\kappa^2} \int d\zeta^{(-4)} du\, F^{(+4)}\big(R^{(+2)}, D^{++}R^{(+2)}, \ldots (D^{++})^{k-1} R^{(+2)}, u\big). \tag{5.10}$$

Generally $R^{(+2)}$ constrained by (5.9) carries k propagating multiplets [8]. It immediately follows that the Lagrange multiplier for (5.9) is the superfield $\omega^{(-2k+2)}$. So, (5.10) admits a dual form in terms of unconstrained $\omega^{(-2k+2)}$. By reasonings of subsect. 5.1, it is a second-order form of the action for $q^{+(i_1 \ldots i_{2k-1})}$. This class of q^+ actions is distinguished in that q^+ enters into the interaction terms only in a fixed contraction with harmonics:

$$R^{(+2)} \sim u^+_{(i_1} u^-_{i_2} \ldots u^-_{i_{2k-1})} q^{+(i_1 \ldots i_{2k-1})}. \tag{5.11}$$

One may consider also other constrained harmonic multiplets, e.g.

$$(D^{++})^k q_i^+(\zeta, u) = 0\ [8], \qquad k \geqslant 2$$

or

$$(D^{++})^k \omega^{(-2l)}(\zeta, u) = 0, \qquad k \geqslant 2l+3, \qquad \text{etc.}$$

with actions having the same form as in the case of unconstrained superfields. Again these actions can be dual transformed to a particular form of generic q^+ action (2.14) with a proper unconstrained q^+ for each propagating multiplet. This form is singled out either by the condition that q^+s enter into it only via a fixed contraction with harmonics or (and) by restrictions on an admissible rank of harmonic derivatives of q^+.

5.3. HIGHER RANK TENSOR $N=2$ MULTIPLETS

The last example is the generalized tensor multiplets whose possible role in describing $N=2$ matter has been pointed out recently by Ketov, Osetrin,

* For $k=3$, the ω hypermultiplet with the constraint (5.5) was discussed first in our paper [3], but with an erroneous treatment of the physical field content of the relevant action. The correct component consideration was given in ref. [8].

Lokhvitsky and Tyutin [17]. These are represented in harmonic superspace by real analytic superfields $L^{(+2l)}(\zeta, u)$ constrained by

$$D^{++}L^{(+2l)}(\zeta, u)=0 \qquad (l \geqslant 2). \tag{5.12}$$

Superfields with odd U(1)-charges need no special treatment by reasonings analogous to those in the case of superfields $\omega^{(-k)}$. Constraint (5.12) is an obvious generalization of the tensor $N=2$ multiplet condition (3.1). However, in contrast to the latter it does not give rise to conserved vectors. The general action for $L^{(+2l)}$ is as follows

$$S=\frac{1}{\kappa^2}\int \mathrm{d}\zeta^{(-4)}\,\mathrm{d}u\, G^{(+4)}(L^{(+2l)}, u), \tag{5.13}$$

and it must explicitly contain harmonics already at the linearized level.

It is a simple exercise to see that the action (5.13) is a particular case of the action (5.10). To this end, let us replace (5.13) by the following action

$$\tilde{S}=\frac{1}{\kappa^2}\int \mathrm{d}\zeta^{(-4)}\,\mathrm{d}u\left\{G^{(+4)}(L^{(+2l)}, u)+\left[L^{(+2l)}-(D^{++})^{l-1}R^{(-2)}\right]\omega^{(-2l+4)}\right\}, \tag{5.14}$$

where $R^{(+2)}$ and $\omega^{(-2l+4)}$ are, for the moment, unconstrained superfields. Let us first show that (5.14) is equivalent to (5.13). Varying $R^{(+2)}$ we obtain

$$(D^{++})^{l-1}\omega^{(-2l+4)}=0. \tag{5.15}$$

Unless $l=2$, this equation has only null solution

$$\omega^{(-2l+4)}=0 \qquad (l \neq 2), \tag{5.16}$$

while for $l=2$ the solution is

$$\omega=\omega_0=\text{const} \qquad (l=2)$$

(this is immediately checked in the central basis). Then, substituting these solutions back into (5.14) we recover (5.13) (in the case $l=2$ we are led also to modify $\tilde{S}$ by adding a "counterterm" $-L^{(+4)}\omega_0$). On the other hand, varying $\omega^{(-2l+4)}$ we obtain

$$L^{(+2l)}=(D^{++})^{l-1}R^{(+2)}, \qquad (D^{++})^{l}R^{(+2)}=0, \tag{5.17}$$

and $\tilde{S}$ is rewritten as

$$\tilde{S} = \frac{1}{\kappa^2} \int \mathrm{d}\mathfrak{z}^{(-4)} \mathrm{d}u \left\{ G^{(+4)}\left((D^{++})^{l-1} R^{(+2)}, u \right) \right\}, \qquad (D^{++})^l R^{(+2)} = 0, \quad (5.18)$$

which has to be compared with eqs. (5.10), (5.9). Similarly to general $R^{(+2)}$ action (5.10), action (5.18) admits a dual form in terms of unconstrained q^+s*.

The same reasonings show that the actions of $L^{(+2l)}$ constrained by more general conditions

$$(D^{++})^k L^{(+2l)} = 0$$

also reduce to a particular form of (5.10), and hence to a class of q^+ actions (2.14) (on performing a dual transformation).

Finally, several simple exercises for the persevering reader.

(i) Show that the free ω action (2.9) with the constraint

$$(D^{++})^3 \omega = 0$$

is equivalent to the action for real triplet of tensor $N = 2$ multiplets

$$S \sim \int \mathrm{d}\mathfrak{z}^{(-4)} \mathrm{d}u \, L^{++(ij)} L^{++}_{(ij)}, \qquad D^{++} L^{++(ij)} = 0.$$

(ii) Show that the free q^+ action (2.2) with the constraint

$$(D^{++})^2 q^+ = (D^{++})^2 \bar{q}^+ = 0$$

is equivalent to an action for a complex doublet of tensor $N = 2$ multiplets

$$S \sim \int \mathrm{d}\mathfrak{z}^{(-4)} \mathrm{d}u \, \bar{L}^{++i} L^{++}_i, \qquad D^{++} L^{++}_i = D^{++} \bar{L}^{++i} = 0.$$

(iii) Show that the action for $2l + 3$ real tensor $N = 2$ multiplets,

$$S \sim \int \mathrm{d}\mathfrak{z}^{(-4)} \mathrm{d}u \, L^{++(i_1 \ldots i_{2l+2})} L^{++}_{(i_1 \ldots i_{2l+2})}, \qquad D^{++} L^{++(i_1 \ldots i_{2l+2})} = 0,$$

* With this choice of action, there survives only one propagating hypermultiplet in $R^{(+2)}$, as opposed to the general case (5.10) with l such multiplets. The reasons are gauge freedom with respect to $R^{(+2)} \to R^{(+2)} + \lambda^{(+2)}$, $(D^{++})^{l-1} \lambda^{(+2)} = 0$ as well as the appearance of additional algebraic constraints on-shell [27].

is equivalent to the action (5.1) with the constraint

$$(D^{++})^{2l+3}\omega^{(-2l)} = 0 .$$

Hint: exploit decompositions of the type (5.4).

6. Conclusions

Thus, we have shown that the variety of $N=2$ matter off-shell representations known to date is actually reduced to a single universal representation with an infinite number of auxiliary components, the q^+ hypermultiplet. A general self-interaction of the latter encompasses all possible self-couplings of other multiplets and yields new couplings which cannot be fitted within finite-component schemes. So, q^+ seems to be most adequate to represent $N=2$ matter and it is a genuine $N=2$ analogue of the $N=1$ chiral superfield. In fact, it meets almost all the requirements demanded of the "ultimate" $N=2$ scalar multiplet in ref. [4]. It has *an internal $SU(2)$-symmetry in the free action, an internal $U(1)$-symmetry that couples to complex Yang-Mills, and no global symmetry in interactions corresponding to hyper-Kähler manifolds with no Killing vectors* (there are no reasons to expect any global symmetry in action (2.14) in general case except for $N=2$ supersymmetry). The only thing which does not agree with hopes expressed in [4] is that q^+ has an infinite number of auxiliary fields. However, just this property proves to be crucial for achieving the most general $N=2$ matter self-coupling on the basis of q^+. An infinite set of auxiliary fields is unavoidable when extending off-shell a complex form of the $N=2$ scalar multiplet [8].

The universal form (2.14) for the off-shell $N=2$ matter action looks rather suggestive. It provides e.g. a simple general proof of finiteness of $N=4$ supersymmetric hyper-Kähler σ-models to which (2.14) corresponds in $d=2$ [19]. This proof is based on manifestly supersymmetric diagram techniques for the q^+ hypermultiplet, exploits the fact that κ is dimensionless in $d=2$ and goes along the standard line of proof of the nonrenormalization theorems [2]. Intriguing questions are: what is the precise mathematical meaning of the (dimensionless) "hyper-Kähler potential" $\mathscr{L}^{(+4)}(q^+, D^{++}q^+,\ldots, (D^{++})^n q^+,\ldots, u)$ and how is the latter connected with the primary principles of hyper-Kähler geometry? In fact the harmonic superspace approach suggests a new look at this geometry. The point is that $\mathscr{L}^{(+4)}$, before employing equations of motion, depends on an infinite number of auxiliary fields contained in q^+. Conventional hyper-Kähler manifolds parameterized by a finite number of fields emerge only on-shell after elimination of the above infinite tail of auxiliary components (even with the fermionic components omitted). The latter procedure amounts to solving a system of differential equations on sphere $S^2 \sim SU(2)/U(1)$.

So, the present paper answered many questions and at the same time raised new problems. The most urgent one seems to establish a direct relation between hyper-Kähler geometry and the hyper-Kähler potential $\mathscr{L}^{(+4)}$. This would hopefully allow an immediate identification of the relevant hyper-Kähler metrics by the form of $\mathscr{L}^{(+4)}$ and provide a straightforward proof that (2.14) is indeed the most general $N=2$ matter action.

Appendix A

Here we give the relation between central and analytic bases of harmonic $N=2$ superspace [3]:

Central basis

$$\{x^m, \theta_{\alpha i}, \bar{\theta}^i_{\dot{\alpha}}, u_i^{\pm}\} \equiv \{z^M, u_i^{\pm}\},$$

$$D^i_\alpha = \frac{\partial}{\partial\theta^\alpha_i} + i\partial_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}i}, \qquad \bar{D}_{\dot{\alpha}i} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}i}} - i\theta^\alpha_i\partial_{\alpha\dot{\alpha}},$$

$$D^{++} = u^{+i}\frac{\partial}{\partial u^{-i}} \equiv \partial^{++}$$

Analytic basis

$$\{x^m_A, \theta^+_\alpha, \bar{\theta}^+_{\dot{\alpha}}, u_i^{\pm}, \theta^-_\alpha, \bar{\theta}^-_{\dot{\alpha}}\} \equiv \{(\mathfrak{z}^M, u_i^{\pm}), \theta^-_\alpha, \bar{\theta}^-_{\dot{\alpha}}\},$$

$$x^m_A = x^m - 2i\theta^{(i}\sigma^m\bar{\theta}^{j)}u^+_i u^-_j, \qquad \theta^{\pm}_\alpha = \theta^i_\alpha u_i^{\pm}, \qquad \bar{\theta}^{\pm}_{\dot{\alpha}} = \bar{\theta}^i_{\dot{\alpha}} u_i^{\pm},$$

$$D^+_\alpha = u^+_i D^i_\alpha = \frac{\partial}{\partial\theta^{-\alpha}}, \qquad \bar{D}^+_{\dot{\alpha}} = u^+_i\bar{D}^i_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{-\dot{\alpha}}},$$

$$D^-_\alpha = u^-_i D^i_\alpha = -\frac{\partial}{\partial\theta^{+\alpha}} + 2i\bar{\theta}^{-\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, \qquad \bar{D}^-_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}+}} - 2i\theta^{-\alpha}\partial_{\alpha\dot{\alpha}},$$

$$D^{++} = \partial^{++} - 2i\theta^+\sigma^m\bar{\theta}^+\frac{\partial}{\partial x^m_A} + \theta^{+\alpha}\frac{\partial}{\partial\theta^{-\alpha}} + \bar{\theta}^{+\dot{\alpha}}\frac{\partial}{\partial\bar{\theta}^{-\dot{\alpha}}}.$$

Appendix B

Here we establish a relation with the familiar description of tensor, relaxed and further relaxed $N=2$ multiplets in terms of unconstrained prepotentials [28, 14, 4, 30]

Let us begin with solving the constraint (3.10). Its solution for any n can be written in central basis as*

$$L^{++} = (D^+)^2(\bar{D}^+)^2 Y^{--}(z,u), \tag{B.1}$$

where

$$Y^{--}(z,u) = \left[D^{i\alpha}D^j_\alpha\phi(z) + \bar{D}^i_{\dot\alpha}\bar{D}^{\dot\alpha j}\bar\phi(z)\right]u_i^- u_j^- \qquad (n=1), \tag{B.2}$$

$$Y^{--}(z,u) = \left[D^{i\alpha}\psi^j_\alpha(z) + \bar{D}^i_{\dot\alpha}\bar\psi^{\dot\alpha j}(z)\right]u_i^- u_j^-, \qquad \left(\overline{\psi^j_\alpha}\right) \equiv \bar\psi_{\dot\alpha j} \qquad (n=2), \tag{B.3}$$

$$\begin{aligned} Y^{--}(z,u) = {} & Y^{i_1i_2}(z)u^-_{i_1}u^-_{i_2} + Y^{(i_1i_2i_3j_1)}(z)u^-_{i_1}u^-_{i_2}u^-_{i_3}u^+_{j_1} + \cdots \\ & + Y^{(i_1\ldots i_{n-1}j_1\ldots j_{n-3})}u^-_{i_1}\ldots u^-_{i_{n-1}}u^+_{j_1}\ldots u^+_{j_{n-3}} \qquad (n \geq 3). \end{aligned} \tag{B.4}$$

In eqs. (B.2)–(B.4), $\phi(z)$, $\psi^j_\alpha(z)$, $Y^{(i_1\ldots j_{n-3})}(z)$ are conventional unconstrained u-independent $N=2$ superfields.

To rewrite all the actions of sect. 3 in ordinary $N=2$ superspace, it suffices to use these formulae, the relation between the integration measures in analytic and central $N=2$ superspaces

$$\mathrm{d}\zeta^{(-4)}\,\mathrm{d}u\,(D^+)^2(\bar{D}^+)^2 = \mathrm{d}^{12}z\,\mathrm{d}u \tag{B.5}$$

and the prepotential representation of gauge superfield

$$V(\zeta(z,u),u) = (D^+)^2(\bar{D}^+)^2 X^{(-4)}(z,u), \tag{B.6}$$

where $X^{(-4)}(z,u)$ is a general harmonic $N=2$ superfield:

$$X^{(-4)}(z,u) = X^{(i_1i_2i_3i_4)}(z)u^-_{i_1}u^-_{i_2}u^-_{i_3}u^-_{i_4} + X^{(i_1\ldots i_5j_1)}(z)u^-_{i_1}\ldots u^-_{i_5}u^+_{j_1} + \cdots. \tag{B.7}$$

One should substitute (B.1)–(B.7) into the relevant actions, take off $(D^+)^2(\bar{D}^+)^2$ from one analytic superfield to restore the full $N=2$ integration measure (B.5), and finally integrate over $\mathrm{d}u$. As a first illustration, let us do this for the free relaxed

* Our conventions are basically the same as in our previous papers

$$\varepsilon^{ij}\varepsilon_{jk} = \delta^i_k, \qquad \left(\overline{D^i_\alpha f}\right) = (-1)^{\hat{P}(f)}\bar{D}_{\dot\alpha i}\bar f, \qquad D^\pm_\alpha = u^\pm_i D^i_\alpha,$$

$$(D^+)^2 = D^{ij}u^+_i u^+_j \equiv D^{\alpha i}D^j_\alpha u^+_i u^+_j, \qquad (\bar{D}^-)^2 = \overline{(D^+)^2}, \qquad \text{etc.}$$

hypermultiplet action (3.10). Before u-integration, the first and second terms of the action are given by the following expressions

$$S^{\mathrm{I}}_{\mathrm{HST}} = \frac{1}{\kappa^2} \int d^{12}z\, du \left\{ \left(D^{i\alpha}\psi^j_\alpha + \bar{D}^i_{\dot\alpha}\bar{\psi}^{\dot\alpha j} \right) (D^+)^2 (\bar{D}^+)^2 \left(D^{\beta k}\psi^l_\beta + \bar{D}^k_{\dot\beta}\bar{\psi}^{\dot\beta l} \right) \right. \\ \times u^-_i u^-_j u^-_k u^-_l \,, \tag{B.8}$$

$$S^{\mathrm{II}}_{\mathrm{HST}} = \frac{1}{\kappa^2} \int d^{12}z \left(D^{i\alpha}\psi_{\alpha i} + \bar{D}^i_{\dot\alpha}\bar{\psi}^{\dot\alpha}_i \right) \int du\, V(\mathfrak{z}(z,u), u) \,. \tag{B.9}$$

To do the u-integrals one should make use of the rule [29]

$$\int du\, u^{+i_1} \dots u^{+i_n} u^-_{j_1} \dots u^-_{j_n} = \frac{1}{n+1} \delta^{(i_1}_{j_1} \dots \delta^{i_n)}_{j_n} \tag{B.10}$$

and take into account that in the present case only the first term in (B.7) contributes to the u-integral in eq. (B.9). As a result, we arrive at the expressions:

$$S^{I}_{\mathrm{HST}} = \frac{1}{\kappa^2} \int d^{12}z \left[\psi_{\alpha i} D^\alpha_j L^{ij}(\psi) + \text{c.c.} \right], \tag{B.11}$$

$$L^{ij}(\psi) \equiv \tfrac{1}{5} \bar{D}^{(ij} D^{kl)} D^\alpha_k \psi_{\alpha l}(z) + \text{c.c.}, \tag{B.12}$$

$$S^{\mathrm{II}}_{\mathrm{HST}} = \frac{1}{\kappa^2} \int d^{12}z\, \tfrac{1}{5} \left(D^{i\alpha}\psi_{\alpha i} + \bar{D}^i_{\dot\alpha}\bar{\psi}^{\dot\alpha}_i \right) D^{jk} \bar{D}^{lr} X_{(jklr)}(z), \tag{B.13}$$

which up to a normalization, coincide with those of [14].

In the case of "first-order multiplet" [4] all the things go even simpler as the relevant basic unconstrained $N=2$ superfield is an isovector real scalar one $Y^{(i_1 i_2)}(z)$

$$L^{++} = (D^+)^2 (\bar{D}^+)^2 Y^{(i_1 i_2)}(z) u^-_{i_1} u^-_{i_2} \qquad (n=3). \tag{B.14}$$

Then the $(L^{++})^2$ term in the free action (3.17) is represented as

$$S^{\mathrm{I}}_{\mathrm{YS}} = \frac{1}{\kappa^2} \int d^{12}z\, \tfrac{1}{5} Y^{ij}(z) D_{ij} \bar{D}_{kl} Y^{kl}(z), \tag{B.15}$$

while the term with the Lagrange multiplier, before explicit integration over du, as

$$S_{\mathrm{YS}}^{\mathrm{II}} = \frac{2}{\kappa^2} \int d^{12}z\, Y^{(ij)}(z) \int du\, u^+_{(i} u^-_{j)} V(\mathfrak{z}(z,u),u). \tag{B.16}$$

Now the u-integral in (B.16) receives nonzero contributions both from the first and second terms in the general expansion (B.7). We obtain

$$S_{\mathrm{YS}}^{\mathrm{II}} = \frac{1}{\kappa^2} \int d^{12}z \Big[Y_i^{k_1}(z) D_{(k_1k_2} \bar{D}_{k_3k_4)} X^{(k_2k_3k_4i)}(z) + Y^{(ij)}(z) D^{k_1k_2} \bar{D}^{k_3k_4} X_{(ijk_1\ldots k_4)}(z) \Big] \tag{B.17}$$

(in this expression, we have changed a normalization of X's so as to absorb inessential numerical factors). Eqs. (B.15), (B.17) have to be compared with the corresponding formulae of [4].

The self-interaction terms, the further relaxed hypermultiplet actions, the couplings to $N=2$ Yang-Mills, the central charge modifications, etc. can be reexpressed in terms of ordinary $N=2$ superspace, by proceeding in a similar manner.

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