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GENERAL ANALYTIC SOLUTION OF CERTAIN FUNCTIONAL EQUATIONS OF ADDITION TYPE*

M. BRUSCHI† AND F. CALOGERO†

Abstract. The general analytic solutions of the following functional equations are exhibited:

$$\alpha(x+y)/[\alpha(x)\alpha(y)]=1+\varphi(x)\varphi(y)\psi(x+y),$$

$$\beta(x+y)/[\beta(x)\beta(y)]=\gamma(x)+\gamma(y)+\chi(x+y).$$

These solutions are expressed in terms of Weierstrass elliptic functions; the special cases in which these reduce to elementary functions are also exhibited. Moreover, several remarkable formulae satisfied by Weierstrass elliptic functions are reported.

Key words. functional equations, special functions

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1. Introduction. The main purpose of this paper is to report the general analytic solution of the following two functional equations:

$$\alpha(x+y)/[\alpha(x)\alpha(y)]=1+\varphi(x)\varphi(y)\psi(x+y), \tag{1.1}$$

$$\beta(x+y)/[\beta(x)\beta(y)]=\gamma(x)+\gamma(y)+\chi(x+y). \tag{1.2}$$

Actually the second of these equations is a limiting case of the first, as we will show in § 4; we prefer nevertheless to treat them separately.

Clearly these functional equations are susceptible to many reformulations, which we obtain by redefining the a priori unknown functions, namely $\alpha(z)$, $\varphi(z)$, and $\chi(z)$ in (1.1), and $\beta(z)$, $\gamma(z)$, and $\chi(z)$ in (1.2). For instance, other avatars of (1.1) read as follows:

$$\alpha(x+y)/[\alpha(x)\alpha(y)]=1+\psi(x+y)/[\omega(x)\omega(y)], \tag{1.1a}$$

$$\alpha(x)\alpha(y)/\alpha(x+y)=1+\Phi(x)\Phi(y)\Psi(x+y), \tag{1.1b}$$

$$\alpha(x)\alpha(y)/\alpha(x+y)=1-\Phi(x)\Phi(y)/\Omega(x+y), \tag{1.1c}$$

$$\alpha(x+y)\Omega(x+y)=\Phi(x)\Phi(y)\alpha(x+y)+\alpha(x)\alpha(y)\Omega(x+y), \tag{1.1d}$$

$$\alpha(x+y)-\alpha(x)\alpha(y)=\Phi(x)\Phi(y)\psi(x+y), \tag{1.1e}$$

$$\alpha(x+y)\omega(x)\omega(y)-\psi(x+y)\alpha(x)\alpha(y)=\alpha(x)\alpha(y)\omega(x)\omega(y), \tag{1.1f}$$

$$\ln[\alpha(x+y)-\alpha(x)\alpha(y)]=f(x)+f(y)+g(x+y), \tag{1.1g}$$

$$\ln[1-\alpha(x)\alpha(y)/\alpha(x+y)]=f(x)+f(y)+h(x+y), \tag{1.1h}$$

and other avatars of (1.2) read as follows:

$$\theta(x)\theta(y)/\theta(x+y)=\gamma(x)+\gamma(y)+\chi(x+y), \tag{1.2a}$$

$$b(x+y)-b(x)-b(y)=\ln[\gamma(x)+\gamma(y)+\chi(x+y)], \tag{1.2b}$$

$$\exp\{\beta(x+y)/[\beta(x)\beta(y)]\}=G(x)G(y)H(x+y), \tag{1.2c}$$

$$\exp[\theta(x)\theta(y)/\theta(x+y)]=G(x)G(y)H(x+y). \tag{1.2d}$$

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† Istituto Nazionale di Fisica Nucleare, Sezione di Roma, Rome, Italy, and Dipartimento di Fisica, Università degli Studi di Roma "La Sapienza," Rome, Italy.

The key to these transformations reads as follows: $\omega = 1/\varphi$, $\Phi = \alpha\varphi$, $\Psi = -\psi/\alpha$, $\Omega = \alpha/\psi$, $f = \ln(\alpha\varphi)$, $g = \ln(\psi)$, $h = \ln(\psi/\alpha)$, $\theta = 1/\beta$, $b = \ln(\beta)$, $G = \exp(\gamma)$, $H = \exp(\chi)$. In the following we refer for definiteness to the canonical forms (1.1) and (1.2).

In § 2 we report the general analytic solutions of the functional equations (1.1) and (1.2). In § 3 we motivate our interest in these functional equations. In § 4 we prove our results. In § 5 we display several remarkable relations ("addition formulae" of various kinds) satisfied by the Weierstrass elliptic functions (whose definitions are collected in the Appendix, mainly to stake out our notation). Section 6 contains some concluding remarks.

2. Solutions. Before giving the solutions of the functional equations (1.1) and (1.2), let us mention some invariance properties of these equations.

It is plain that, if $\alpha(z)$, $\varphi(z)$, and $\psi(z)$ satisfy (1.1), so do

$$\begin{aligned} \tilde{\alpha}(z) &= \exp(bz)\alpha(az), \\ \tilde{\varphi}(z) &= A\exp(cz)\varphi(az), \\ \tilde{\psi}(z) &= A^{-2}\exp(-cz)\psi(az), \end{aligned} \tag{2.1}$$

with a, b, c, A arbitrary constants ($A \neq 0$), as well as

$$\begin{aligned} \tilde{\alpha}(z) &= 1/\alpha(z), \\ \tilde{\varphi}(z) &= \varphi(z)\alpha(z), \\ \tilde{\psi}(z) &= -\psi(z)/\alpha(z). \end{aligned} \tag{2.2}$$

Similarly, if $\beta(z)$, $\gamma(z)$, and $\chi(z)$ satisfy (1.2), so do

$$\begin{aligned} \tilde{\beta}(z) &= C\exp(bz)\beta(az), \\ \tilde{\gamma}(z) &= C^{-1}[\gamma(az) + Az + B], \\ \tilde{\chi}(z) &= C^{-1}[\chi(az) - Az - 2B], \end{aligned} \tag{2.3}$$

with a, b, A, B, and C arbitrary constants ($C \neq 0$).

The general analytic solution of the functional equation (1.1) reads as follows:

$$\alpha(z) = \exp(bz)\sigma(\mu)\sigma(az+\nu)/[\sigma(\nu)\sigma(az+\mu)], \tag{2.4a}$$

$$\varphi(z) = A\exp(cz)\sigma(az)/\sigma(az+\nu), \tag{2.4b}$$

$$\psi(z) = A^{-2}\exp(-cz)\sigma(\nu-\mu)\sigma(az+\mu+\nu)/[\sigma(\mu)\sigma(az+\mu)]. \tag{2.4c}$$

Here $\sigma(z) \equiv \sigma(z\,|\,\omega, \omega')$ is the Weierstrass σ-function (see the Appendix), and A, a, b, c, μ, ν, ω, and ω' are eight constants (arbitrary, except for the trivial restrictions needed to make good sense of the right-hand side (r.h.s.) of (2.4a–c)).

In § 4 we prove that functions (2.4) satisfy (1.1), and we show moreover that any analytic function $\alpha(z)$ satisfying (1.1) may depend on at most six free parameters. The fact that the expression (2.4a) of $\alpha(z)$ indeed contains six arbitrary parameters, namely a, b, μ, ν, ω, and ω', justifies our claim that formulae (2.4) provide the *general* analytic solution of the functional equation (1.1).

It may be easily verified that the solutions (2.4) are consistent with the transformations (2.1), (2.2), whose only effect is to cause a redefinition of some parameters.

For special choices of the parameters, (2.4) may be cast in simpler form. For instance the following expressions of $\alpha(z)$ in terms of Jacobi elliptic functions (see the Appendix) are all special cases of (2.4a):

(2.5a) $\alpha(z)=\mathrm{sn}\,(\mu)/\mathrm{sn}\,(z+\mu)$,

(2.5b) $\alpha(z)=\mathrm{sn}\,(\mu)\,\mathrm{cn}\,(z+\mu)/[\mathrm{cn}\,(\mu)\,\mathrm{sn}\,(z+\mu)]$,

(2.5c) $\alpha(z)=\mathrm{sn}\,(\mu)\,\mathrm{dn}\,(z+\mu)/[\mathrm{dn}\,(\mu)\,\mathrm{sn}\,(z+\mu)]$,

(2.5d) $\alpha(z)=\mathrm{cn}\,(z+\mu-\omega_3)/\mathrm{cn}\,(\mu-\omega_3)$,

(2.5e) $\alpha(z)=\mathrm{dn}\,(z+\mu-\omega_3)/\mathrm{dn}\,(\mu-\omega_3)$,

(2.5f) $\alpha(z)=\mathrm{dn}\,(\mu-\omega_2)\,\mathrm{cn}\,(z+\mu+\omega_2)/[\mathrm{cn}\,(\mu-\omega_2)\,\mathrm{dn}\,(z+\mu+\omega_2)]$.

In these formulae μ is an arbitrary constant, and we use the standard notation for the Jacobi functions and their "periods" (see the Appendix).

More special (and perhaps more interesting) cases obtain when one or both periods of the elliptic functions diverge and they reduce to elementary functions (see the Appendix). The corresponding formulae for $\alpha(z)$, $\varphi(z)$, and $\psi(z)$ read as follows:

(2.6a) $\alpha(z)=\exp\{[b+(\mu-\nu)a/3]z\}\sinh(\mu)\sinh(az+\nu)/[\sinh(\nu)\sinh(az+\mu)]$,

(2.6b) $\varphi(z)=A\exp[\nu^2/6+(c+\nu a/3)z]\sinh(az)/\sinh(az+\nu)$,

(2.6c) $\psi(z)=A^{-2}\exp[-\nu^2/3-(c+\nu a/3)z]\dfrac{\sinh(\nu-\mu)\sinh(az+\mu+\nu)}{\sinh(\mu)\sinh(az+\mu)}$,

(2.7a) $\alpha(z)=\exp(bz)(\mu/\nu)(az+\nu)/(az+\mu)$,

(2.7b) $\varphi(z)=A\exp(cz)az/(az+\nu)$,

(2.7c) $\psi(z)=A^{-2}\exp(-cz)(\nu-\mu)(az+\mu+\nu)/[\mu(az+\mu)]$.

Note that the trivial solution $\alpha(z)=1/C$, $\varphi(z)=A$, $\psi(z)=(C-1)/A^2$ with C and A arbitrary constants, obtains if, in (2.7), we set $b=c=0$, $\mu=\delta$, $\nu=C\delta$, $\delta\to 0$.

The general analytic solution of the functional equation (1.2) reads as follows:

(2.8a) $\beta(z)=C\exp(bz)\sigma(\mu)\sigma(az)/\sigma(az+\mu)$,

(2.8b) $\gamma(z)=C^{-1}[Az+B+\zeta(az)]$,

(2.8c) $\chi(z)=C^{-1}[-Az-2B+\zeta(\mu)-\zeta(az+\mu)]$.

Here $\sigma(z)\equiv\sigma(z\,|\,\omega,\omega')$ is the Weierstrass σ-function and $\zeta(z)=\zeta(z\,|\,\omega,\omega')=\sigma'(z)/\sigma(z)$ is the Weierstrass ζ-function (see the Appendix); a, b, A, B, C, μ, ω, and ω' are eight arbitrary constants ($C\neq 0$).

As mentioned in § 1 and shown in § 4, the functional equation (1.2) may be obtained by an appropriate limiting procedure from the functional equation (1.1); likewise, (2.8a–c) may be derived, by an appropriate limiting procedure, from (2.4a–c). But since the limiting procedure is not trivial, we have considered it worthwhile to exhibit separately the two functional equations (1.1) and (1.2), as well as their general solutions (2.4) and (2.8). We also report here the special cases of (2.8) analogous to the special cases of (2.4) displayed above (see (2.5)–(2.7)). These formulae read as follows:

(2.9a) $\beta(z)=\mathrm{sn}\,(z)$,

(2.9b) $\beta(z)=\mathrm{sn}\,(z)/\mathrm{cn}\,(z)$,

(2.9c) $\beta(z)=\mathrm{sn}\,(z)/\mathrm{dn}\,(z)$, (2.10a) $$\beta(z)=C\exp[(b+\mu a/3)z]\sinh(\mu)\sinh(az)/\sinh(az+\mu),$$

(2.10b) $$\gamma(z)=C^{-1}[(A-a/3)z+B+\coth(az)],$$

(2.10c) $$\chi(z)=C^{-1}[-(A-a/3)z-2B+\coth(\mu)-\coth(az+\mu)],$$

(2.11a) $$\beta(z)=C\exp(bz)\mu az/(az+\mu),$$

(2.11b) $$\gamma(z)=C^{-1}[Az+B+1/(az)],$$

(2.11c) $$\chi(z)=C^{-1}[-Az-2B+1/\mu-1/(az+\mu)].$$

Note that the trivial solution $\beta(z)=C$, with C an arbitrary constant, obtains if we set $b=0$ in (2.11) and take the limit $a\to\infty$.

3. Motivation. Some years ago the (differential) functional equation

(3.1) $$\theta(x)\theta'(y)-\theta'(x)\theta(y)=\theta(x+y)[\varepsilon(x)-\varepsilon(y)]$$

was obtained and solved in the context of the study of a certain class of integrable dynamical systems [1]. Recently, in an analogous context, an analogous (differential) functional equation was obtained and solved [2]:

(3.2) $$\alpha(x)\alpha'(y)-\alpha'(x)\alpha(y)=[\alpha(x+y)-\alpha(x)\alpha(y)][\eta(x)-\eta(y)].$$

Clearly these two functional equations may be unified by considering the following functional equation:

(3.3) $$\tilde{\alpha}(x)\tilde{\alpha}'(y)-\tilde{\alpha}'(x)\tilde{\alpha}(y)=[\tilde{\alpha}(x+y)-c\tilde{\alpha}(x)\tilde{\alpha}(y)][\tilde{\eta}(x)-\tilde{\eta}(y)].$$

Indeed (up to notational changes) this equation yields (3.1) for $c=0$ and (3.2) for $c=1$. Moreover, provided $c\neq 0$, (3.3) coincides with (3.2) after the trivial rescalings

(3.4) $$\alpha(z)=c\tilde{\alpha}(z),\qquad \eta(z)=c\tilde{\eta}(z).$$

We now show, following [2], that the functional equation (3.3) may be integrated to yield (1.1) and (1.2). Let

(3.5) $$F(x,y)=c^{-1}\ln[1-c\tilde{\alpha}(x)\tilde{\alpha}(y)/\tilde{\alpha}(x+y)].$$

It is then easily seen that (3.3) implies the first-order PDE

(3.6) $$F_x(x,y)-F_y(x,y)=\tilde{\eta}(x)-\tilde{\eta}(y),$$

whose general solution reads

(3.7) $$F(x,y)=H(x+y)+E(x)+E(y),$$

with $H(z)$ arbitrary and

(3.8) $$E(z)=\int^z dz'\,\tilde{\eta}(z').$$

Now note that (3.5) and (3.7) imply the relation

(3.9) $$1-c\tilde{\alpha}(x)\tilde{\alpha}(y)/\tilde{\alpha}(x+y)=\exp\{c[H(x+y)+E(x)+E(y)]\}.$$

For $c\neq 0$ this equation coincides with (1.1) via the positions

(3.10) $$\alpha(z)=c\tilde{\alpha}(z),\quad \varphi(z)=\exp[cE(z)]/\alpha(z),\quad \psi(z)=\alpha(z)\exp[cH(z)].$$

And the treatment remains valid also in the limit $c\to 0$, in which case (3.9) yields (1.2) (up to notational changes; see § 4 for details).

4. Proofs. The validity of the "invariance properties" (2.1)-(2.3) is verified trivially.

Our first task is to prove that (2.4) satisfies (1.1). But (2.1) implies that, to prove this, it is sufficient to verify that (1.1) is satisfied by the following functions:

$$\alpha(z)=\sigma(\mu)\sigma(z+\nu)/[\sigma(\nu)\sigma(z+\mu)], \tag{4.1a}$$

$$\varphi(z)=\sigma(z)/\sigma(z+\nu), \tag{4.1b}$$

$$\psi(z)=\sigma(\nu-\mu)\sigma(z+\mu+\nu)/[\sigma(\mu)\sigma(z+\mu)] \tag{4.1c}$$

(corresponding to (2.4) with $b=c=0$ and $a=A=1$).

This has already been proved in [2], but in a somewhat cumbersome manner. A more straightforward proof may be based on the general "addition formula" (see § 5)

$$\begin{aligned}&\sigma(u+v_1)\sigma(u-v_1)\sigma(v_2+v_3)\sigma(v_2-v_3)\\&\quad+\sigma(u+v_2)\sigma(u-v_2)\sigma(v_3+v_1)\sigma(v_3-v_1)\\&\quad+\sigma(u+v_3)\sigma(u-v_3)\sigma(v_1+v_2)\sigma(v_1-v_2)=0.\end{aligned} \tag{4.2}$$

Indeed it is easily seen that the insertion of (4.1) into (1.1) yields precisely (4.2), with

$$u=(x+y)/2,\quad v_1=(x+y)/2+\nu,\quad v_2=(x-y)/2,\quad v_3=-(x+y)/2-\mu. \tag{4.3}$$ □

We now prove that any analytic solution $\alpha(z)$ of the functional equation (1.1) may contain at most six free parameters. It is actually expedient to base this proof on the differential functional equation (3.2), which is implied by (1.1), as shown in § 3. We set $y=\delta$ in (3.2), expand around $\delta=0$, and equate to zero the coefficients of δ^n, using the ansatzen

$$\alpha(\delta)=\alpha_0+\alpha_1\delta+\tfrac{1}{2}\alpha_2\delta^2+\tfrac{1}{6}\alpha_3\delta^3+o(\delta^4), \tag{4.4a}$$

$$\eta(\delta)=\eta_{-1}\delta^{-1}+\eta_0+\eta_1\delta+o(\delta^2), \tag{4.4b}$$

whose justification is implied a posteriori by the consistency of the following results. We thus get (for $n=-1,0,1,2$)

$$\alpha_0=1, \tag{4.5a}$$

$$\eta_{-1}=1, \tag{4.5b}$$

$$\eta(z)=\eta_0+\tfrac{1}{2}[\alpha''(z)-2\alpha_1\alpha'(z)+\alpha_2\alpha(z)]/[\alpha'(z)-\alpha_1\alpha(z)], \tag{4.5c}$$

$$\begin{aligned}&2[\alpha'(z)-\alpha_1\alpha(z)]\alpha''(z)-3[\alpha''(z)-2\alpha_1\alpha'(z)]\alpha''(z)\\&\qquad+a_1[\alpha'(z)]^2+a_2\alpha'(z)\alpha(z)+a_3\alpha^2(z)=0,\end{aligned} \tag{4.5d}$$

with

$$a_1=6(2\eta_1-\alpha_2),\quad a_2=4(\alpha_3-6\alpha_1\eta_1),\quad a_3=3\alpha_2^2-4\alpha_1\alpha_3+12\alpha_1^2\eta_1. \tag{4.6}$$

Of course at each step we have used the findings from previous steps; note, incidentally, that (4.5c) provides an explicit definition of $\eta(z)$ in terms of $\alpha(z)$ (up to the parameter η_0, which remains completely arbitrary since it plays no role whatsoever; see (4.4b) and (3.2)).

This derivation implies that any analytic solution $\alpha(z)$ of the functional equation (1.1) must satisfy the constraint (4.5a) and the third-order (nonlinear) ODE (4.5d), which contains the four a priori undetermined parameters α_1, a_1, a_2, and a_3. Hence $\alpha(z)$ may depend at most on $6=4+3-1$ free parameters (the number 3 corresponds, of course, to the order of the ODE (4.5d), and -1 accounts for constraint (4.5a)). □

Analogously it can be shown that

$$\beta(z)=\sigma(\mu)\sigma(z)/\sigma(z+\mu), \tag{4.7a}$$

$$\gamma(z)=\zeta(z)\equiv\sigma'(z)/\sigma(z), \tag{4.7b}$$

$$\chi(z)=\zeta(\mu)-\zeta(z+\mu)\equiv\sigma'(\mu)/\sigma(\mu)-\sigma'(z+\mu)/\sigma(z+\mu) \tag{4.7c}$$

satisfy (1.2) (note that, via (2.3), this implies that (2.8) satisfies (1.2) as well). Indeed the insertion of these formulae in (1.2) yields the formula

$$\begin{aligned}\sigma(x+\mu)\sigma(y+\mu)\sigma(x+y)=\sigma(\mu)\sigma(x+y+\mu)[\sigma(x)\sigma'(y)+\sigma'(x)\sigma(y)]&\\ +\sigma(x)\sigma(y)[\sigma(x+y+\mu)\sigma'(\mu)&\\ -\sigma'(x+y+\mu)\sigma(\mu)],&\end{aligned} \tag{4.8}$$

whose validity is easily proved by setting in (4.2)

$$u=(x+y)/2+\mu, \quad v_1=(x+y)/2-\delta, \quad v_2=(x+y)/2, \quad v_3=(x+y)/2+\delta, \tag{4.9}$$

and then letting $\delta\to 0$.

But it is more interesting to prove that (1.2) is a limiting case of (1.1), and accordingly that the expressions (4.7) are a limiting case of (4.1). Indeed, setting

$$\alpha(z)=\delta^{-1}\beta(z), \quad \varphi(z)=1-\delta\gamma(z), \quad \psi(z)=-1+\delta\chi(z) \tag{4.10}$$

in the following equation:

$$\ln\{1-\alpha(x+y)/[\alpha(x)\alpha(y)]\}=\ln[\varphi(x)]+\ln[\varphi(y)]+\ln[-\psi(x+y)] \tag{4.11}$$

(which is clearly equivalent to (1.1)), and taking the $\delta\to 0$ limit under the assumption that in this limit the functions $\beta(z)$, $\gamma(z)$, and $\chi(z)$ remain finite, we find that (1.2) evidently obtains. On the other hand, the assumption about the finiteness of $\beta(z)$, $\gamma(z)$, and $\chi(z)$ is verified using the explicit expression (4.1) of $\alpha(z)$, $\varphi(z)$, and $\psi(z)$ with the position

$$\nu=\delta; \tag{4.12}$$

indeed, using (A.3)-(A.6) below, it is easily seen that (4.1) and (4.10) with (4.12) yield, in the $\delta\to 0$ limit, precisely (4.7). □

5. Addition formulae for Weierstrass elliptic functions. In this section we report some addition formulae for Weierstrass elliptic functions that, in spite of their remarkable neatness and generality, cannot be found in the standard compilations [3]-[5].

Foremost among these relations is the beautiful addition formula (4.2), which for completeness we report here in two equivalent forms:

$$\begin{aligned}&\sigma(u+v_1)\sigma(u-v_1)\sigma(v_2+v_3)\sigma(v_2-v_3)\\ &\quad+\sigma(u+v_2)\sigma(u-v_2)\sigma(v_3+v_1)\sigma(v_3-v_1)\\ &\quad+\sigma(u+v_3)\sigma(u-v_3)\sigma(v_1+v_2)\sigma(v_1-v_2)=0,\end{aligned} \tag{5.1a}$$

$$\begin{aligned}&\sigma(x+y)\sigma(y+z)\sigma(z+x)\sigma(2w)\\ &\quad=\sigma(x+w)\sigma(y+w)\sigma(z+w)\sigma(x+y+z-w)\\ &\quad\quad-\sigma(x-w)\sigma(y-w)\sigma(z-w)\sigma(x+y+z+w),\end{aligned} \tag{5.1b}$$

related by the change of variables

$$u+v_1=x+y+z-w, \quad u-v_1=z+w, \quad v_2+v_3=x+w, \quad v_2-v_3=-(y+w). \tag{5.2}$$

Let us emphasize that this "addition formula" features four free parameters (in addition, of course, to the two "periods" of the Weierstrass σ-functions; see the Appendix). Formula (5.1a) is not new, (see, for instance, p. 389 of [6]); a straightforward way to prove it is by equating to zero the sum of the residues of the elliptic function

$$F(z)=\prod_{k=1}^{3}[\sigma(z-z_k)/\sigma(z-p_k)], \tag{5.3}$$

with the zeros z_k and poles p_k restricted by the condition

$$\sum_{k=1}^{3}(z_k-p_k)=0, \tag{5.4}$$

which is instrumental to guaranteeing that $F(z)$, as defined by (5.3), is indeed an elliptic function (hence that its residues within a fundamental parallelogram add up to zero). It is then easy to obtain (5.1a) with

$$u+v_1=z_2-z_3,\quad u-v_1=z_1-p_1,\quad v_2+v_3=z_1-p_2,\quad v_2-v_3=z_1-p_3. \tag{5.5}$$

Since the addition formula (5.1) features four free parameters, it is easy to obtain from it, merely by reduction, myriad addition formulae with three or two arguments, including all the "classical" formulae that can be found in the standard compilations, and many others that are less advertised. For instance, setting $u=0$ in (5.1a) we get

$$\begin{aligned}\sigma^2(v_1)\sigma(v_2+v_3)\sigma(v_2-v_3)+\sigma^2(v_2)\sigma(v_3+v_1)\sigma(v_3-v_1)&\\ +\sigma^2(v_3)\sigma(v_1+v_2)\sigma(v_1-v_2)=0,&\end{aligned} \tag{5.6}$$

and setting $v_1=v_2+v_3$ in this formula yields (using (A.12))

$$\sigma(2v_1-v_2)\sigma^3(v_2)-\sigma(2v_2-v_1)\sigma^3(v_1)=\sigma(v_1-v_2)\sigma^3(v_1-v_2). \tag{5.7}$$

More generally, setting $u=\delta$ in (5.1a), expanding in δ, and equating the coefficients of δ^n obtains, in addition to (5.6) (which corresponds, of course, to $n=0$), the formula

$$\begin{aligned}&\mathcal{P}(v_1)\sigma^2(v_1)\sigma(v_2+v_3)\sigma(v_2-v_3)\\ &\quad+\mathcal{P}(v_2)\sigma^2(v_2)\sigma(v_3+v_1)\sigma(v_3-v_1)\\ &\quad+\mathcal{P}(v_3)\sigma^2(v_3)\sigma(v_1+v_2)\sigma(v_1-v_2)=0,\end{aligned} \tag{5.8}$$

which corresponds to $n=2$ ($n=1$ yields merely a trivial identity). To obtain this formula we have, of course, used the definition (A.4) of the Weierstrass $\mathcal{P}$-function. Note that, for $v_3=0$, (5.8) yields, using (A.6), (A.7), and (A.12), the standard addition formula (see (A.13))

$$\sigma(v_1+v_2)\sigma(v_1-v_2)=\sigma^2(v_1)\sigma^2(v_2)[\mathcal{P}(v_2)-\mathcal{P}(v_1)]. \tag{5.9}$$

On the other hand, taking the logarithmic derivative of (5.6) with respect to v_1 and using (A.3) obtains the formula

$$\begin{aligned}&\sigma(v_1+v_2)\sigma(v_1-v_2)\sigma^2(v_3)[2\zeta(v_1)-\zeta(v_1+v_2)-\zeta(v_1-v_2)]\\ &\quad=\sigma(v_1+v_3)\sigma(v_1-v_3)\sigma^2(v_2)[2\zeta(v_1)-\zeta(v_1+v_3)-\zeta(v_1-v_3)],\end{aligned} \tag{5.10}$$

which, setting $v_3=v_1$, yields (using (A.6) and (A.7)),

$$\zeta(v_1+v_2)+\zeta(v_1-v_2)-2\zeta(v_1)=\sigma(2v_1)\sigma^2(v_2)/[\sigma^2(v_1)\sigma(v_1+v_2)\sigma(v_1-v_2)]. \tag{5.11}$$

This formula yields, via the duplication formula (see (A.17)),

$$\sigma(2z)=-\sigma^4(z)\mathcal{P}'(z), \tag{5.12}$$

the well-known relation (A.15),

$$\zeta(v_1+v_2)+\zeta(v_1-v_2)-2\zeta(v_1)=\mathcal{P}'(v_1)/[(\mathcal{P}(v_1)-\mathcal{P}(v_2)]. \tag{5.13}$$

Note, incidentally, that the duplication formula (5.12) may itself be derived, since (5.13) may be obtained directly from (5.9) (taking the logarithmic derivative with respect to v_1), and clearly (5.13) with (5.11) yields (5.12).

On the other hand, differentiating (5.6) with respect to v_1, we obtain the relation

(5.14)

$$\begin{aligned}&2\sigma(v_1)\sigma'(v_1)\sigma(v_2+v_3)\sigma(v_2-v_3)\\ &\quad=\sigma^2(v_2)[\sigma(v_1+v_3)\sigma'(v_1-v_3)+\sigma(v_1-v_3)\sigma'(v_1+v_3)]-\sigma^2(v_3)\\ &\qquad\cdot[\sigma(v_1+v_2)\sigma'(v_1-v_2)+\sigma(v_1-v_2)\sigma'(v_1+v_2)],\end{aligned} \tag{5.14}$$

and this, via (5.9), yields the formula

$$\begin{aligned}2\zeta(v_1)[\mathcal{P}(v_2)-\mathcal{P}(v_3)]=&[\zeta(v_1+v_2)+\zeta(v_1-v_2)][\mathcal{P}(v_2)-\mathcal{P}(v_1)]\\ &-[\zeta(v_1+v_3)+\zeta(v_1-v_3)][\mathcal{P}(v_3)-\mathcal{P}(v_1)].\end{aligned} \tag{5.15}$$

Let us also report some neat relations that are more conveniently obtained from (5.1b). In the limit $w\to 0$ this yields (via (A.3) and (A.6))

$$\begin{aligned}&\zeta(x)+\zeta(y)+\zeta(z)-\zeta(x+y+z)\\ &\quad=\sigma(x+y)\sigma(y+z)\sigma(z+x)/[\sigma(x)\sigma(y)\sigma(z)\sigma(x+y+z)],\end{aligned} \tag{5.16}$$

which is essentially a more elegant version of (4.8); and in the limit $z\to 0$ this yields the standard formula (see (A.14))

$$\zeta(x+y)-\zeta(x)-\zeta(y)=\tfrac{1}{2}[\mathcal{P}'(x)-\mathcal{P}'(y)]/[\mathcal{P}(x)-\mathcal{P}(y)] \tag{5.17}$$

(which may also be easily derived from (5.13)).

On the other hand by setting $z=0$ in (5.1b) we obtain the relation

$$\begin{aligned}\sigma(x)\sigma(y)\sigma(x+y)\sigma(2w)/\sigma(w)=&\sigma(x+w)\sigma(y+w)\sigma(x+y-w)\\ &+\sigma(x-w)\sigma(y-w)\sigma(x+y+w),\end{aligned} \tag{5.18}$$

while differentiating (5.16) with respect to z yields (via (A.3)-(A.4), and again (5.16), and with the change of variables $x+y+z=u_1$, $z=u_2$, $y+z=-u_3$) the remarkably neat formula

$$\begin{aligned}\mathcal{P}(u_1)-\mathcal{P}(u_2)=&[\zeta(u_1)+\zeta(u_2)+\zeta(u_3)-\zeta(u_1+u_2+u_3)]\\ &\cdot[\zeta(u_1)-\zeta(u_1+u_3)-\zeta(u_2)+\zeta(u_2+u_3)].\end{aligned} \tag{5.19}$$

Note that the variable u_3 appears only on the right-hand side. Setting $u_3=\delta$ and expanding (5.19) around $\delta=0$, the coefficient of δ yields (5.9) while the coefficient of δ^2 yields (5.13).

6. Conclusion. It is in our opinion remarkable that the general analytic solutions of the functional equations (1.1) and (1.2), each featuring three a priori unknown functions, may be explicitly obtained.

These findings imply the possibility of obtaining solutions to more general functional equations. It is, for instance, clear from the results reported in § 2 (see, in particular, (2.4)) that the functional equation

$$\alpha(x+y)/[\alpha(x)\alpha(y)]=\prod_{n=1}^{N}[1+\varphi_n(x)\varphi_n(y)\psi_n(x+y)], \tag{6.1}$$

which features $2N+1$ a priori unknown functions and reduces to (1.1) for $N=1$, admits the solution

$$\text{(6.2a)} \qquad \alpha(z)=\exp(bz)\prod_{n=1}^{N}\{\sigma_n(\mu_n)\sigma_n(a_nz+\nu_n)/[\sigma_n(\nu_n)\sigma_n(a_nz+\mu_n)]\},$$

$$\text{(6.2b)} \qquad \varphi_n(z)=A_n\exp(c_nz)\sigma_n(a_nz)/\sigma_n(a_nz+\nu_n),$$

$$\text{(6.2c)} \qquad \varphi_n(z)=A_n^{-2}\exp(-c_nz)\sigma_n(\nu_n-\mu_n)\sigma_n(a_nz+\mu_n+\nu_n)/[\sigma_n(\mu_n)\sigma_n(a_nz+\mu_n)].$$

Here (and below) we use for the Weierstrass σ-functions the abbreviated notation $\sigma_n(z)\equiv\sigma(z\,|\,\omega_n,\omega'_n)$. Note that the solution (6.2a-c) contains $7N+1$ free parameters (namely A_n, a_n, b, c_n, μ_n, ν_n, ω_n, and ω'_n, with $A_n\neq 0$), $5N+1$ of which enter into the expression (6.2a) of $\alpha(z)$.

It is likewise plain (see (2.8)) that the functional equation

$$\text{(6.3)} \qquad \beta(x+y)/[\beta(x)\beta(y)]=\prod_{n=1}^{N}[\gamma_n(x)+\gamma_n(y)+\chi_n(x+y)],$$

which features $2N+1$ a priori unknown functions and reduces to (1.2) for $N=1$, admits the solution

$$\text{(6.4a)} \qquad \beta(z)=\exp(bz)\prod_{n=1}^{N}[C_n\sigma_n(\mu_n)\sigma_n(a_nz)/\sigma_n(a_nz+\mu_n)],$$

$$\text{(6.4b)} \qquad \gamma_n(z)=C_n^{-1}[A_nz+B_n+\zeta_n(a_nz)],$$

$$\text{(6.4c)} \qquad \chi_n(z)=C_n^{-1}[-A_nz-2B_n+\zeta_n(\mu_n)-\zeta_n(a_nz+\mu_n)].$$

Here, of course, $\zeta_n(z)\equiv\zeta(z\,|\,\omega_n,\omega'_n)$. Note also that this solution contains $7N+1$ free parameters (namely a_n, b, A_n, B_n, C_n, μ_n, ω_n, and ω'_n), $5N+1$ of which enter in the definition (6.4a) of $\beta(z)$.

A question that remains open for the moment is whether (6.2a-c), respectively, (6.4a-c), are the *general* analytic solutions of (6.1), respectively, (6.3).

Of course, many other functional equations, whose solutions can be easily found from the solutions of (1.1) and (1.2), may be manufactured combining (1.1) and (1.2) and/or their avatars (see, for instance, (1.1a-h) and (1.2a-d)).

These functional equations, together with their solutions, provide moreover a convenient tool for uncovering additional relations, satisfied by Weierstrass elliptic functions, that are generally all consequences of (5.1) but might be quite difficult to discover by direct computation. It is, for instance, plain (from the relation $\theta=1/\beta$ and from (4.7) with μ replaced by ν) that the functional equation (1.2a) admits the solution

$$\text{(6.5a)} \qquad \tilde{\theta}(z)=\sigma(z+\nu)/[\sigma(\nu)\sigma(z)],$$

$$\text{(6.5b)} \qquad \tilde{\gamma}(z)=\zeta(z),$$

$$\text{(6.5c)} \qquad \tilde{\chi}(z)=\zeta(\nu)-\zeta(z+\nu).$$

On the other hand, it is evident that the relation

$$\text{(6.6)} \qquad \alpha(z)=\beta(z)\tilde{\theta}(z)$$

holds, with $\alpha(z)$ defined by (4.1a), $\beta(z)$ defined by (4.7a), and $\tilde{\theta}(z)$ defined by (6.5a). Hence (1.1), (1.2), and (1.2a) imply the relation

$$\text{(6.7)} \qquad [\gamma(x)+\gamma(y)+\chi(x+y)]/[\tilde{\gamma}(x)+\tilde{\gamma}(y)+\tilde{\chi}(x+y)]=1+\varphi(x)\varphi(y)\psi(x+y),$$

with φ, ψ, γ, χ, $\tilde{\gamma}$, respectively, $\tilde{\chi}$, defined by (4.1b, c), (4.7b, c), and (6.5b), respectively, (6.5c), namely the neat formula

$$\begin{aligned}[\zeta(x)+\zeta(y)+\zeta(\mu)-\zeta(x+y+\mu)]/[\zeta(x)+\zeta(y)+\zeta(\nu)-\zeta(x+y+\nu)]\\ =1+\sigma(x)\sigma(y)\sigma(x+y+\mu)\sigma(\nu-\mu)/[\sigma(x+\nu)\sigma(y+\nu)\sigma(x+y+\mu)\sigma(\mu)].\end{aligned} \tag{6.8}$$

This formula, which contains four free parameters (in addition to ω and ω'), may also be obtained using (5.1a) *and* (5.16).

Let us finally mention that a natural question suggested by the main findings of this paper concerns the solvability of the simplest functional equation that encompasses (1.1) and (1.2) and involves four a priori unknown functions, namely

$$\alpha(x+y)/[\alpha(x)\alpha(y)]=\gamma(x)+\gamma(y)+\varphi(x)\varphi(y)\psi(x+y). \tag{6.9}$$

Appendix. For the sake of completeness and to standardize the notation, we report in this Appendix the relevant formulae for the Weierstrass functions $\sigma(z)$, $\zeta(z)$, $\wp(z)$.

Definitions.

$$w\equiv w_{m,n}\equiv 2m\omega+2n\omega', \tag{A.1}$$

$$\sigma(z)=\sigma(z\,|\,\omega,\omega')=z\prod{}'\{(1-z/w)\exp[z/w+\tfrac{1}{2}(z/w)^2]\}, \tag{A.2}$$

$$\zeta(z)\equiv\zeta(z\,|\,\omega,\omega')=\sigma'(z)/\sigma(z), \tag{A.3}$$

$$\wp(z)\equiv\wp(z\,|\,\omega,\omega')=-\zeta'=[\sigma'^2(z)-\sigma(z)\sigma''(z)]/\sigma^2(z). \tag{A.4}$$

Here and below, a prime appended to a function denotes differentiation, while $\prod'$ ($\sum'$) denotes the product (the sum) taken over all (positive and negative) integers m, n with the exception of $m=n=0$.

Laurent series.

$$g_2=60\sum{}' w^{-4},\qquad g_3=140\sum{}' w^{-6}, \tag{A.5}$$

$$\sigma(z)=\sum_{m=n=0}^{\infty} a_{m,n}(g_2/2)^m(2g_3)^n z^{(4m+6n+1)}/(4m+6n+1)!, \tag{A.6}$$

where

$$a_{0,0}=1,\qquad a_{m,n}=0\quad\text{if } m<0\quad\text{or}\quad n<0, \tag{A.6a}$$

$$\begin{aligned}a_{m,n}=(3m+1)a_{m+1,n-1}+\tfrac{16}{3}(n+1)a_{m-2,n+1}\\ -\tfrac{1}{3}(3m+3n-1)(4m+6n-1)a_{m-1,n},\end{aligned} \tag{A.6b}$$

$$\zeta(z)=z^{-1}-\sum_{k=2}^{\infty}c_k z^{2k-1}/(2k-1), \tag{A.7}$$

where

$$c_2=g_2/20,\qquad c_3=g_3/28, \tag{A.7a}$$

$$c_k=3/[(2k+1)(k-3)]\sum_{m=2}^{k-2}c_m c_{k-m},\qquad k\geqq 4, \tag{A.7b}$$

$$\wp(z)=z^{-2}+\sum_{k=2}^{\infty}c_k z^{2(k-1)}. \tag{A.8}$$

Period relations.

(A.9) $$\sigma(z+2m\omega+2n\omega')=(-1)^{(m+n+mn)}\sigma(z) \times \exp[(z+m\omega+n\omega')(2m\zeta(\omega)+2n\zeta(\omega'))],$$

(A.10) $$\zeta(z+2m\omega+2n\omega')=\zeta(z)+2m\zeta(\omega)+2n\zeta(\omega'),$$

(A.11) $$\mathcal{P}(z+2m\omega+2n\omega')=\mathcal{P}(z).$$

Functional relations and properties.

(A.12) $$\sigma(z)=-\sigma(-z), \quad \zeta(z)=-\zeta(-z), \quad \mathcal{P}(z)=\mathcal{P}(-z),$$

(A.13) $$\sigma(z_1+z_2)\sigma(z_1-z_2)=\sigma^2(z_1)\sigma^2(z_2)[\mathcal{P}(z_2)-\mathcal{P}(z_1)],$$

(A.14) $$\zeta(z_1+z_2)+\zeta(z_1-z_2)-2\zeta(z_1)=\mathcal{P}'(z_1)/[\mathcal{P}(z_1)-\mathcal{P}(z_2)],$$

(A.15) $$\zeta(z_1+z_2)-\zeta(z_1)-\zeta(z_2)=\tfrac{1}{2}[\mathcal{P}'(z_1)-\mathcal{P}'(z_2)]/[\mathcal{P}(z_1)-\mathcal{P}(z_2)],$$

(A.16) $$\mathcal{P}(z_1+z_2)+\mathcal{P}(z_1)+\mathcal{P}(z_2)=\tfrac{1}{4}[\mathcal{P}'(z_1)-\mathcal{P}'(z_2)]^2/[\mathcal{P}(z_1)-\mathcal{P}(z_2)]^2,$$

(A.17) $$\sigma(2z)=-\mathcal{P}'(z)\sigma^4(z),$$

(A.18) $$\zeta(2z)=2\zeta(z)+\tfrac{1}{2}\mathcal{P}''(z)/\mathcal{P}(z),$$

(A.19) $$\mathcal{P}(2z)=-2\mathcal{P}(z)+\tfrac{1}{4}[\mathcal{P}''(z)/P'(z)]^2.$$

Connection with the Jacobian elliptic functions.

(A.20a) $$\omega_1=\omega, \quad \omega_2=-(\omega+\omega'), \quad \omega_3=\omega',$$

(A.20b) $$\eta_k=\zeta(\omega_k), \qquad k=1,2,3,$$

(A.20c) $$e_k=\mathcal{P}(\omega_k), \qquad k=1,2,3,$$

(A.20d) $$\sigma_k(z)=\exp(-\eta_k z)\sigma(z+\omega_k)/\sigma(\omega_k), \qquad k=1,2,3,$$

(A.21) $$u=(e_1-e_3)^{1/2}z, \qquad m^2=(e_2-e_3)/(e_1-e_3),$$

(A.22a) $$\operatorname{sn}(u|m)=(e_1-e_3)^{1/2}\sigma(z)/\sigma_3(z),$$

(A.22b) $$\operatorname{cn}(u|m)=\sigma_1(z)/\sigma_3(z),$$

(A.22c) $$\operatorname{dn}(u|m)=\sigma_2(z)/\sigma_3(z).$$

Degenerate cases.

(A.23a) $$e_1=e_2=a, \quad e_3=-2a, \quad \omega=\infty, \quad \omega'=(12a)^{-1/2}\pi i,$$

(A.23b) $$\sigma(z)=(3a)^{-1/2}\sinh[(3a)^{1/2}z]\exp[-az^2/2],$$

(A.23c) $$\zeta(z)=-az+(3a)^{1/2}\coth[(3a)^{1/2}z],$$

(A.23d) $$\mathcal{P}(z)=a+3a\{\sinh[(3a)^{1/2}z]\}^{-2},$$

(A.24a) $$e_1=e_2=e_3=0, \qquad \omega=-i\omega'=\infty,$$

(A.24b) $$\sigma(z)=z,$$

(A.24c) $$\zeta(z)=1/z,$$

(A.24d) $$\mathcal{P}(z)=1/z^2.$$

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