PHYSICAL REVIEW D VOLUME 24, NUMBER 5 1 SEPTEMBER 1981

Minimal subtraction and momentum subtraction in quantum chromodynamics at two-loop order

Eric Braaten
Department of Physics, University of Wisconsin—Madison, Madison, Wisconsin 53706

Jacques P. Leveille
The Harrison M. Randall Laboratory of Physics, University of Michigan, Ann Arbor, Michigan 48109
(Received 22 January 1981)

The momentum-subtraction coupling constant α_{MOM} yields consistently smaller one-loop corrections to many quantum-chromodynamics (QCD) processes than the minimal-subtraction couplings α_{MS} and $\alpha_{\overline{\text{MS}}}$. By shifting the renormalization scale μ of $\alpha_{\text{MS}}(\mu)$, we obtain a minimal-subtraction coupling with the same small one-loop corrections. It is shown, by studying the effective charges of QCD, that at two-loop order this coupling constant will continue to yield corrections to physical quantities that are comparable to those obtained by momentum subtraction. We also introduce a momentum-subtraction scheme which treats the triple-gluon, quark, and ghost vertices equally at one-loop order and is more convenient for higher-order calculations than the MOM scheme.

I. INTRODUCTION

In a renormalizable quantum field theory, physical quantities are guaranteed to be independent of the choice of renormalization scheme. However, since such quantities are usually calculated using a perturbation expansion in the coupling constant, a parameter which depends on the renormalization scheme, the coefficients in the expansion are also scheme-dependent. The resulting ambiguity is phenomenologically unimportant in low-energy quantum electrodynamics (QED) because of the very small coupling constant, and because the historically chosen "mass-shell" renormalization scheme is a natural prescription for low-energy processes. However, scheme dependence becomes very important in extracting quantitative predictions from quantum chromdynamics (QCD), due to the much larger coupling constant and the lack of any natural renormalization prescription.

For massless field theories such as QCD with massless quarks, the most commonly used renormalization schemes are minimal subtraction and momentum subtraction. Minimal schemes are used in conjunction with dimensional regularization where four-dimensional divergent integrals are analytically continued to $4-2\epsilon$ dimensions.[1] In the conventional minimal-subtraction scheme (MS), only the poles in ϵ are subtracted from primitively divergent Green's functions.[2] In the modified minimal-subtraction scheme ($\overline{\text{MS}}$), certain constants that arise from the analytic continuation of angular integrals are subtracted along with the poles.[3] The coupling constants in these two schemes are actually related to all orders by a simple shift in the renormalization scale: $\alpha_{\overline{\text{MS}}}(\mu) \equiv \alpha_{\text{MS}}(\mu e^{-t})$ where $t \simeq 0.977$. (This is proved in Sec. II.) Minimal-subtraction schemes have many attractive features.[4] In gauge theories, the renormalized coupling constant and associated β function are gauge-invariant. In massive theories, the β function and the anomalous dimensions are independent of any mass parameters. Minimal schemes are very convenient for higher-order calculations involving renormalization, because the subtractions have extremely simple forms. Finally, these schemes have the additional advantage in gauge theories of automatically satisfying the constraints imposed by the Ward identities.

In momentum-subtraction schemes, the radiative corrections to a particular set of propagators and vertices are subtracted at some specified point in momentum space, commonly chosen to be the symmetric point (SP) where all external legs have the same momentum squared. Momentum schemes have the advantage of making all renormalized quantities, including the coupling constant, independent of the regularization method. However, they have many drawbacks. There is no unique choice for the set of propagator and vertex corrections which are subtracted at the SP. In gauge theories, care must be taken to make this set compatible with the Ward identities. The simple properties enjoyed in minimal schemes by the coupling constant and the renormalization-group functions are lost. Finally, the determination of the renormalized coupling constant requires the calculation of vertex corrections at the SP, which is extremely difficult beyond one-loop order.

QCD in a covariant gauge with massless quarks has three fundamental propagators (for the gluon, ghost, and quark fields) and four fundamental vertices (the triple-gluon, ghost, quark, and four-gluon vertices). An example of a momentum-subtraction scheme for this theory is the MOM scheme of Celmaster and Gonsalves.[5] In this scheme the gluon, ghost, and quark propagator corrections, together with a particular component of the triple-gluon vertex correction, are subtracted at the symmetric point. The other vertex subtractions are then fixed by the Ward identities. The coupling constant $\alpha_{MOM}(\mu, \xi)$ defined by this scheme depends weakly on the gauge parameter ξ for $|\xi| \lesssim 1$. It was suggested in Ref. 5 that this coupling constant evaluated in the Landau gauge be used as an expansion parameter for physical quantities.

Interest in the renormalization-scheme dependence of the predictions of QCD was aroused by the discovery that several physical quantities have large one-loop corrections when calculated in the MS scheme, i.e., they have relative corrections $1 + C\alpha/\pi$ with coefficients $C \gg 1$. It was found empirically that the size of the coefficients is decreased if the $\overline{\text{MS}}$ coupling constant is used as an expansion parameter.[3] Celmaster and Sivers[6] have recently considered the renormalization-scheme dependence of QCD corrections to several quantities, including the cross section for e^+e^- annihilation, the decay rate for pseudoscalar quarkonium, and the moments of structure functions for deep-inelastic scattering. All these quantities have large one-loop corrections in the $\overline{\text{MS}}$ scheme as well. However, when expanded in powers of the coupling constant $\alpha_{MOM}(\mu) \equiv \alpha_{MOM}(\mu, \xi = 0)$, defined by momentum subtraction in the Landau gauge, they all have corrections with coefficients C of order 1.

In view of the success of α_{MOM} in producing acceptable one-loop corrections for these physical quantities, it is reasonable to conjecture that this momentum-subtraction coupling constant will continue to give well-behaved perturbation expansions in higher orders. However, minimal schemes have many theoretical advantages over momentum subtraction, and it would be useful to have a minimal-subtraction coupling constant which gives similarly well-behaved expansions. Such a coupling constant is easily obtained at one-loop order by shifting the renormalization scale μ of the conventional MS coupling constant, i.e., adjusting t to make $\alpha_{MS}(\mu e^{-t})$ equal to $\alpha_{MOM}(\mu)$ to order α^2. In this paper we show that this scale-shifted MS coupling constant will be as good an expansion for QCD as typical momentum-subtraction couplings, at least to two-loop order.

The determination of the appropriate momentum scale μ for a given physical quantity is an important aspect of the problem of renormalization-scheme dependence which has been traditionally left to the intuition of the phenomenologist. Momentum-subtraction coupling constants such as $\alpha_{MOM}(\mu)$ are defined to make certain radiative corrections vanish, and this enables one to make an educated guess as to the appropriate value of μ for a given process. In minimal-subtraction schemes, poles in ϵ and certain associated constants are subtracted from the radiative corrections, which gives no clue as to the proper choice for the momentum scale. It will be shown that $\alpha_{MS}(\mu)$ is in fact an appropriate coupling constant for a momentum scale of about 7μ, and that this is responsible for the large radiative corrections encountered when it is used as an expansion parameter for physical quantities with scale μ. An alternative approach to renormalization-scheme dependence which completely eliminates the ambiguity in the momentum scale has been proposed by Stevenson.[7]

In Sec. II, we examine the ways in which the conventional minimal-subtraction scheme can be modified without sacrificing its desirable properties. A change in the analytic continuation of dimensionally regulated integrals is shown to be completely equivalent to a shift in the renormalization scale. This provides theoretical motivation for considering scale-shifted MS coupling constants. In Sec. III, the effective charges of QCD are used to study the momentum-subtraction coupling constants of the theory. It is shown that α_{MOM} can be distinguished from the poor expansion parameters α_{MS} and $\alpha_{\overline{MS}}$ in that it yields small one-loop corrections to the effective charges in gauges with $|\xi| \lesssim 1$, as well as to physical quantities. In Sec. IV, we demonstrate that these effective charges can be reliably calculated by approximating the symmetric-point vertex corrections by their values with one leg at zero momentum. We also introduce a momentum-subtraction scheme $\overline{\text{MOM}}$ for QCD, which treats the triple-gluon, ghost, and quark vertices equally at one-loop order and is more convenient for higher-order calculations than the MOM scheme. In Sec. V, the zero-momentum approximation is used to study the two-loop corrections to the effective charges of QCD. The scale-shifted MS coupling constant which coincides with α_{MOM} to order α^2 is shown to yield small corrections to the effective charges in the Feynman gauge at two loops as well as at one loop. We conclude that to two-loop order this minimal-subtraction coupling constant will yield expansions for physical quantities which are similar to those obtained in typical momentum-subtraction schemes. A summary is given in Sec. VI. The complete one- and two-loop corrections to vertices with one leg at zero momentum are given in the Appendices.

II. MINIMAL-SUBTRACTION RENORMALIZATION SCHEMES

The simplest technique for regularizing the divergent integrals encountered in field-theoretic perturbation expansions is dimensional regularization,[1] in which four-dimensional integrals are analytically continued to $D=4-2\epsilon$ dimensions. The conventional analytic continuation is

$$(2\pi)^{-4}\int d^4k \to (2\pi)^{-D}\int k^{\,-1}dk\,d\Omega_D,\quad \int d\Omega_D=\frac{2\pi^{D/2}}{\Gamma(D/2)}\,, \tag{1}$$

where $d\Omega_D$ is the angular integration element in D dimensions. To define a dimensionless renormalized coupling constant $\alpha\equiv g^2/4\pi$, we must introduce a scale parameter μ. The coupling constant $\alpha_{\rm MS}(\mu)$ in the conventional minimal-subtraction scheme (MS) is defined by

$$\alpha_0\mu^{-2\epsilon}=\alpha_{\rm MS}(\mu)Z_\alpha^{\rm MS}(\alpha_{\rm MS}(\mu),\epsilon)\,, \tag{2}$$

where α_0 is the bare charge and $Z_\alpha^{\rm MS}$ the charge renormalization constant. The renormalization constants of the MS scheme are chosen to subtract only the poles in ϵ from primitively divergent Green's functions.[2] We emphasize that there is nothing in this definition to suggest that $\alpha_{\rm MS}(\mu)$ should be a good expansion parameter for physical processes at characteristic momentum scale μ. Indeed explicit calculations have shown that it is a poor expansion parameter for one-loop corrections in QCD.

In this section we examine modifications of the MS scheme which preserve the desirable features of minimal subtraction. The two independent steps of the MS renormalization scheme are (i) the regularization of divergent integrals and (ii) the definition of the renormalization constants. The regularization of the integrals as given in Eq. (1) can be modified by the insertion of a "normalization factor" $N(\epsilon)$:

$$(2\pi)^{-4}\int d^4k \to \frac{1}{N(\epsilon)}(2\pi)^{-D}\int d^Dk\,, \tag{3}$$

where $N(\epsilon)$ is analytic at $\epsilon=0$ and $N(0)=1$. The definition of the renormalization constants can be altered to subtract positive powers of $P(\epsilon)$ from primitively divergent Green's functions, where $P(\epsilon)$ is any function with a simple pole at $\epsilon=0$. Neither of these modifications spoils the attractive features of the MS scheme. They are equivalent at one-loop order. For example, the $\overline{\rm MS}$ scheme can be defined at one loop either by $N(\epsilon)=(4\pi)^\epsilon/\Gamma(1-\epsilon)$, $P(\epsilon)=1/\epsilon$, or by $N(\epsilon)=1$, $P(\epsilon)=1/\epsilon+\ln 4\pi-\gamma$. The former is the correct definition in higher orders, since it is equivalent to normalizing the solid angle to

$$(2\pi)^{-D}\int d\Omega_D=\frac{1}{8\pi^2}\frac{1}{1-\epsilon}\ .$$

It thereby eliminates the constant $\ln 4\pi-\gamma$, which arises from the ϵ expansion of angular integrals, from renormalized quantities to all orders.

One can argue for the choice $P(\epsilon)=1/\epsilon$ on the grounds of simplicity. For example, it leads to the following simple expression for the β function[4]:

$$\beta(\alpha)\equiv\mu\frac{\partial}{\partial\mu}\alpha\bigg|_{\epsilon=0}=2\alpha\frac{\partial}{\partial\alpha}z_1(\alpha)\,, \tag{4}$$

where $z_1(\alpha)$ is the coefficient of $1/\epsilon$ in the expansion of the charge renormalization constant in inverse powers of ϵ:

$$Z_\alpha(\alpha,\epsilon)=1+\frac{z_1(\alpha)}{\epsilon}+\frac{z_2(\alpha)}{\epsilon^2}+\cdots\ . \tag{5}$$

For any other choice of $P(\epsilon)$, Eq. (4) is replaced by a complicated expression involving all the functions $z_i(\alpha)$. There is, however, no *a priori* argument for any particular normalization factor $N(\epsilon)=(4\pi)^\epsilon/\Gamma(1-\epsilon)$ removes the constant $\ln 4\pi-\gamma$ from renormalized quantities, but the same is true of $N(\epsilon)=(4\pi)^\epsilon/\Gamma(2-\epsilon)$. Consequently, we shall fix $P(\epsilon)=1/\epsilon$, and consider the effect of varying $N(\epsilon)$ on the renormalization scheme.

Variations of the function $N(\epsilon)$ generate only a one-parameter change in the renormalization scheme and this change is equivalent to a shift in the scale parameter μ. This assertion is easily proved. The analytic continuation of Eq. (3) defines a minimal-subtraction scheme whose coupling constant will be denoted by $\alpha_N(\mu)$. Since each loop integration is associated with a factor of the bare coupling constant α_0, the theory remains unchanged if we scale the bare coupling and divide the integrals by the same factor $N(\epsilon)$:

$$\alpha_0\int d^Dk=\alpha_0N(\epsilon)\,\frac{1}{N(\epsilon)}\int d^Dk\ . \tag{6}$$

Hence the coupling constant $\alpha_N(\mu)$ is defined by substituting $\alpha_0N(\epsilon)$ for α_0 in Eq. (2):

$$\alpha_0\mu^{-2\epsilon}N(\epsilon)=\alpha_N(\mu)Z_\alpha^N(\alpha_N(\mu),\epsilon). \tag{7}$$

It is known that a change in the normalization factor $N(\epsilon)$ only modifies the renormalization constants by finite amounts.[8] Since $Z_\alpha^{\rm MS}$ and Z_α^N are both defined by minimal subtraction, they have no finite parts. Consequently they must be the same functions of their respective arguments: $Z_\alpha^N(\alpha,\epsilon)=Z_\alpha^{\rm MS}(\alpha,\epsilon)$. We rewrite Eq. (7) absorbing $N(\epsilon)$ into the renormalization scale:

$$\alpha_0(\mu e^{-\ln N/2\epsilon})^{-2\epsilon}=\alpha_N(\mu)Z_\alpha^{\rm MS}(\alpha_N(\alpha),\epsilon)\ . \tag{8}$$

Shifting the scale μ in Eq. (2) by $e^{-\ln N/2\epsilon}$ gives

$$\alpha_0(\mu e^{-\ln N/2\epsilon})^{-2\epsilon}=\alpha_{\rm MS}(\mu e^{-\ln N/2\epsilon})Z_\alpha^{\rm MS}(\alpha_{\rm MS}(\mu e^{-\ln N/2\epsilon}),\epsilon). \tag{9}$$

Comparing Eqs. (8) and (9), we deduce immediately

$$\alpha_N(\mu)=\alpha_{\rm MS}(\mu e^{-\ln N/2\epsilon}) . \qquad (10)$$

Since this equation relates only finite quantities, we are free to take the limit $\epsilon \to 0$. Expanding the normalization factor in the form $N(\epsilon)=1+2t\epsilon+\cdots$, we obtain

$$\alpha_N(\mu)=\alpha_{\rm MS}(\mu e^{-t}) . \qquad (11)$$

This proves our assertion that a change in the normalization factor $N(\epsilon)$ is completely equivalent to a shift in the renormalization scale μ.

As a consequence of this fact, the MS and $\overline{\rm MS}$ coupling constants are related *to all orders* by a simple scale shift:

$$\alpha_{\overline{\rm MS}}(\mu)\equiv\alpha_{\rm MS}(\mu e^{-t_{\overline{\rm MS}}}),\quad t_{\overline{\rm MS}}=(\ln 4\pi-\gamma)/2 . \qquad (12)$$

This implies that they have the same β functions, despite the fact that the β function is known to be renormalization-scheme dependent beyond two-loop order. In particular, the three-loop β function for QCD, which was recently calculated in the $\overline{\rm MS}$ scheme,[9] is the same in the MS scheme and any other scheme which differs from it only by the normalization of divergent integrals.

As pointed out earlier, there is nothing in the definition of the coupling constant $\alpha_{\rm MS}(\mu)$ to suggest that it ought to be a good expansion parameter for processes at the momentum scale μ. On the basis of the preceding analysis, we suggest that a shift in the renormalization scale be used to make $\alpha_{\rm MS}$ a better expansion parameter. For example, it yields acceptable one-loop corrections for physical quantities in QCD if the parameter t is adjusted so that $\alpha_{\rm MS}(\mu e^{-t})$ equals the momentum-subtraction coupling constant $\alpha_{\rm MOM}(\mu)$ to one-loop order:

$$\begin{aligned}\alpha_{\rm MOM}(\mu)&=\alpha_{\rm MS}(\mu e^{-t_{\rm MOM}})+O(\alpha^3),\\ t_{\rm MOM}&=t_{\overline{\rm MS}}+1+\frac{69-32n_f}{33-2n_f}\,\frac{I}{24},\end{aligned} \qquad (13)$$

where n_f is the number of flavors of quarks and $I\simeq 2.344$.[5]

To illustrate the effect of the scale shift on one-loop corrections, we use some of the QCD processes whose prescription dependence was reviewed in Ref. 6:

(A) QCD corrections to the hadronic cross section for e^+e^- annihilation[10];

(B) the ratio of hadronic to electromagnetic decay rates for pseudoscalar quarkonium[11]; and

(C1), (C2), (C3) the second, fourth, and eighth moments of the nonsinglet structure functions for deep-inelastic lepton-hadron scattering.[3]

For each of these processes we define a "physical charge" $\alpha_i^{\rm phys}(\mu)$ by absorbing all the radiative corrections into the lowest-order expression for that process. Grunberg[12] has suggested that these physical charges can be used to obtain renormalization-group-improved predictions for the corresponding quantities. The physical charge corresponding to process B is defined by

$$\frac{\Gamma(\text{hadrons})}{\Gamma(\text{photons})}\equiv\frac{2}{9Q^4}\left(\frac{\alpha_B^{\rm phys}(M)}{\alpha_{\rm QED}}\right)^2 , \qquad (14)$$

where Q is the charge and $\mu=M$ is the mass of the heavy quark. In defining the other physical charges, we use the renormalization scales $\mu=\sqrt{s}$ (the center-of-mass energy) for e^+e^- annihilation and $\mu=(Q^2)^{1/2}$ (the momentum transfer) for the structure functions. These physical charges can be expanded in powers of the scale-shifted MS coupling constant:

$$\alpha_i^{\rm phys}(\mu)=\alpha_{\rm MS}(\mu e^{-t})+A_i(t)\alpha_{\rm MS}(\mu e^{-t})^2+\cdots . \qquad (15)$$

The t dependence of the expansion coefficients $A_i(t)$ is shown in Fig. 1 assuming four flavors of massless quarks. As demonstrated in Ref. 6, the coupling constant $\alpha_{\rm MOM}$ (corresponding to $t=t_{\rm MOM}$) yields consistently smaller coefficients than expansions in $\alpha_{\rm MS}$ or $\alpha_{\overline{\rm MS}}$ (corresponding to $t=0$ and $t=t_{\overline{\rm MS}}$, respectively). By the definition of $t_{\rm MOM}$, the minimal-subtraction expansion parameter $\alpha_{\rm MS}(\mu e^{-t_{\rm MOM}})$ yields the same small one-loop corrections to these physical charges.

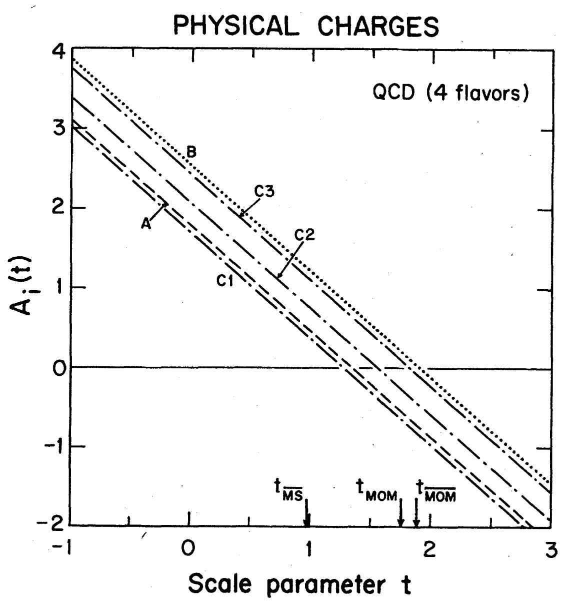


FIG. 1. Order-α^2 coefficients $A_i(t)$ for the physical charges $\alpha_i^{\rm phys}$ as a function of the scale-shift parameter t for QCD with four flavors of massless quarks.

It is possible that coupling constants defined by minimal subtraction are inherently poor expansion parameters. Shifting the renormalization scale to obtain small one-loop corrections to physical quantities could conceivably produce large higher-order corrections. We present evidence in this paper that this does not happen at two-loop order. This cannot be shown directly, since no calculation of corrections to QCD processes has been carried beyond one loop. We argue in the next section that higher-order corrections to such quantities can be studied indirectly by examining the effective charges of the theory.

III. EFFECTIVE CHARGES IN QCD

We begin our study of the effective charges of QCD by motivating the definition of the "effective charge". Consider a three-leg vertex inside a Feynman diagram. Assuming a massless theory for simplicity and ignoring any Lorentz tensor structure, the vertex contributes to the diagram a factor of the coupling constant g and a leg with momentum p_i contributes the propagator factor $1/p_i^2$. The effect of higher-order radiative corrections is to replace the vertex by the one-particle-irreducible (1PI) three-point function and the propagators by the connected two-point functions:

$$\begin{aligned} &g \to \Gamma(p_1,p_2,p_3)\,, \\ &\frac{1}{p_i^2} \to \frac{1}{p_i^2}\,\frac{1}{1+\Pi(p_i^2)}\,, \end{aligned} \tag{16}$$

where Π is the self-energy. Every propagator connects two vertices and we can associate equal parts of the propagator corrections with each vertex. All radiative corrections associated with this particular vertex can then be lumped into an "effective vertex":

$$\Gamma^{\text{eff}}(p_1,p_2,p_3) = \frac{\Gamma(p_1,p_2,p_3)}{[1+\Pi(p_1^2)]^{1/2}[1+\Pi(p_2^2)]^{1/2}[1+\Pi(p_3^2)]^{1/2}}\,. \tag{17}$$

Which momentum configurations p_1,p_2,p_3 will dominate the loop integrals of the Feynman diagram in which this vertex occurs? The answer depends on the physical process which is being calculated.

Suppose the vertex is part of a Feynman diagram for an infrared-finite physical quantity which depends on the single momentum scale μ. The internal loop momenta can be Wick-rotated so that all integrations are over Euclidean space. Since the physical quantity is infrared finite, the contributions to these integrals from configurations involving loop momenta with $p_i^2 \ll \mu^2$ must be suppressed either by small phase space or by cancellations. Any sensitivity of the physical quantity to momentum configurations with $p_i^2 \gg \mu^2$ is removed by renormalization. Therefore, the dominant contributions to the loop integrals must come from configurations with $p_1^2 \sim p_2^2 \sim p_3^2 \sim \mu^2$. This leads us to define the "effective charge" to be the value of the effective vertex at the symmetric point (SP), $p_1^2 = p_2^2 = p_3^2 = -\mu^2$:

$$g^{\text{eff}}(\mu) = \Gamma(p_1,p_2,p_3)\,|_{\text{SP}}[1+\Pi(-\mu^2)]^{-3/2}\,. \tag{18}$$

The change in sign of p_i^2 comes from having rotated back to Minkowski space. Many of the higher-order corrections are absorbed into the definition of the effective charge, and therefore $g^{\text{eff}}(\mu)$ is a logical candidate for an expansion parameter for physical quantities with momentum scale μ. This effective charge is in fact the renormalized coupling constant of the momentum-subtraction scheme in which the vertex and propagator corrections in Eq. (18) vanish at the SP.

The effective charges of a theory are natural candidates for momentum-subtraction coupling constants. We restrict our attention now to the effective charges of QCD in a covariant gauge with massless quarks. For convenience we refer to $\alpha^{\text{eff}}(\mu) \equiv g^{\text{eff}}(\mu)^2/4\pi$ as the effective charge. In QCD there are four fundamental vertices with the same coupling g. For each vertex, the radiative corrections involve several independent Lorentz tensors, and they have no unique decomposition into scalar components that can be used to define effective charges. The effective charges also have the complication of gauge dependence. Furthermore, longitudinal gluons have no propagator corrections in accordance with the Ward identities, and therefore only effective charges without longitudinal gluon legs can be used as momentum-subtraction coupling constants.

To define effective charges, we must decompose the Lorentz tensor structure of a vertex into scalar components. Each free Lorentz index at a vertex corresponds to an external gluon leg. We can therefore eliminate the tensor structure by contracting the vertex with polarization vectors for the external gluons. There is a natural choice for the basis vectors of a gluon leg at a three-point vertex since there is a preferred line, the momentum of the gluon, and a preferred plane, the plane of the momenta of the three legs. We define the "longitudinal" (L) polarization vector for a gluon of momentum p to be parallel to its momentum. The polarization vector which lies in the momentum plane but is orthogonal to p will be called "planar transverse" (P). The two remaining basis vectors can be chosen orthogonal to the momentum plane; we refer to them generically as "normal-transverse" (N) polarization vectors. These basis vectors lead to a natural decomposition of the triple-gluon and ghost vertices of QCD. The quark vertex has the extra complication of Dirac matrix structure. The lowest-order vertex, after contraction with a polarization vector ϵ for the gluon leg, is given by $(-ig)\not\epsilon$. The vertex correction can be decomposed into a component proportional to $\not\epsilon$ and a component orthogonal to $\not\epsilon$ with respect to the Dirac trace. We choose to define effective charges using only the former component. These decompositions of the triple-gluon, ghost, and quark vertices are defined in more detail in Appendix C. We do not consider the effective charges corresponding to the four-gluon vertex.

We label the effective charges for the triple-gluon vertex by the polarizations of the external gluons, i.e., $\alpha^{\text{eff}}_{NNP}(\mu,\xi)$ is the effective charge corresponding to two normal-transverse (N) and one planar-transverse (P) gluons. The only nonvanishing three-gluon effective charges in lowest order are α^{eff}_i, $i=NNP, PPP, LLP$. The lowest-order ghost vertex is nonzero for gluons with polarizations P and L and the corresponding effective charges are denoted by $\alpha^{\text{eff}}_{ggP}$ and $\alpha^{\text{eff}}_{ggL}$. Finally, the lowest-order quark vertex is nonvanishing for all three gluon polarizations, yielding three more effective charges, α^{eff}_i, $i=qqN, qqP, qqL$.

The eight effective charges defined above can be expanded in powers of the minimal-subtraction coupling constant $\alpha_{\text{MS}}(\mu e^{-t})$, where we have allowed for a shift in the scale μ by a factor e^{-t}:

$$\alpha^{\text{eff}}_i(\mu,\xi)=\alpha_{\text{MS}}(\mu e^{-t})+A_i(\xi,t)\alpha_{\text{MS}}{}^2(\mu e^{-t})+B_i(\xi,t)\alpha_{\text{MS}}{}^3(\mu e^{-t})+\cdots . \tag{19}$$

The functions $A_i(\xi,t)$ measure the order-α^2 deviations of the effective charges from the coupling constant $\alpha_{\text{MS}}(\mu e^{-t})$ and are determined by one-loop calculations of propagator and vertex corrections at the symmetric point. These calculations have been carried out by Celmaster and Gonsalves[5] for the propagators, for the complete triple-gluon and ghost vertices, and for the component of the quark vertex corresponding to $\alpha^{\text{eff}}_{qqN}$. We note in passing that the coupling constant in their MOM scheme is simply one of these effective charges: $\alpha_{\text{MOM}}(\mu) = \alpha_{NNP}(\mu,\xi=0)$.

We now examine the gauge dependence of the one-loop effective charges in the MS scheme. The same analysis was essentially carried out in Ref. 5 for the effective charges without longitudinal gluon legs. The gauge dependence of the expansion coefficients $A_i(\xi,0)$ in the MS scheme for four flavors of quarks is shown in Fig. 2 for $i=NNP$, PPP, LLP, ggP, ggL, and qqN. Since the effective charges are necessarily polynomials in the gauge parameter, the functions $A_i(\xi,0)$ diverge as powers of ξ for large $|\xi|$. However, for ξ near

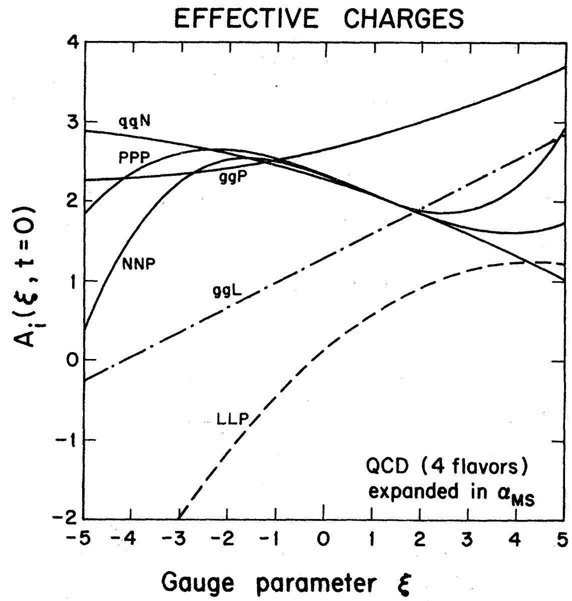


FIG. 2. Order-α^2 coefficients $A_i(\xi,t=0)$ in the MS scheme for the effective charges α_i^{eff} as a function of the gauge parameter for QCD with four flavors of massless quarks. The solid, dash-dot, and dashed lines represent charges with zero, one, and two longitudinal gluon legs, respectively.

zero, the curves NNP, PPP, ggP, and qqN are close together and are equally separated from the curve ggL and the curve LLP. The pattern seems to be that for small gauge parameters the effective charges fall into discrete bands depending upon the number of longitudinal gluon legs. This is simply due to the absence of propagator corrections for longitudinal gluons as dictated by one of the Ward identities. The bands can be made to coincide by choosing a renormalization scheme in which the gluon propagator correction is small.

The simplest change that can be made in the renormalization scheme is to shift the scale μ of the coupling constant as in Eq. (19). The expansion coefficients $A_i(\xi,t)$ vary with the scale-shift parameter t in accordance with the renormalization-group equations for the effective charges. These equations depend on the number of longitudinal gluon legs l:

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(\alpha)\frac{\partial}{\partial\alpha}-2\gamma(\alpha,\xi)\,\xi\frac{\partial}{\partial\xi}\right]\xi^l\alpha^{\text{eff}}(\mu,\xi)=0. \tag{20}$$

The β function $\beta(\alpha)$ and the gluon anomalous dimension $\gamma(\alpha,\xi)$ have expansions of the form

$$\begin{aligned}\beta(\alpha)&=-b_0\alpha^2-b_1\alpha^3-\cdots,\\ \gamma(\alpha,\xi)&=\gamma_0(\xi)\alpha+\gamma_1(\xi)\alpha^2+\cdots.\end{aligned} \tag{21}$$

Using the expansions (19) and (21) in the renormalization-group Eq. (20), we find that the coefficients $A_i(\xi, t)$ must satisfy

$$\frac{\partial}{\partial t} A_i(\xi, t) + b_0 + 2l\gamma_0(\xi) = 0\,, \tag{22}$$

which yields immediately

$$A_i(\xi, t) = A_i(\xi, 0) - [b_0 + 2l\gamma_0(\xi)]\,t\,. \tag{23}$$

Hence the $A_i(\xi, t)$ are linear in t and their slopes depend on the number of longitudinal gluon legs.

The functions $A_i(\xi = 0, t)$, which measure the deviations of the effective charges in the Landau gauge from $\alpha_{\rm MS}(\mu e^{-t})$, are plotted versus t in Fig. 3 for QCD with four flavors. We have included the charges with longitudinal gluon legs in this graph even though longitudinal gluons do not propagate in the Landau gauge. The deviations of the effective charges from $\alpha_{\rm MS}(\mu)$ (shown at $t=0$) and $\alpha_{\overline{\rm MS}}(\mu)$ (shown at $t = t_{\overline{\rm MS}}$) are all positive and some are fairly large. However, as in the corresponding graph (Fig. 1) for the physical charges, the deviations are all close to zero for $t = t_{\rm MOM}$ defined in Eq. (13). The same pattern occurs for gauges near the Landau gauge such as the Feynman gauge ($\xi = 1$), since the charges depend weakly on the gauge parameter for $|\xi| \lesssim 1$ as shown in Fig. 2. *Therefore,* $\alpha_{\rm MOM}(\mu)$ *and* $\alpha_{\rm MS}(\mu e^{-t_{\rm MOM}})$ *are characterized not only by the fact that they give consistently smaller one-loop* QCD *corrections to physical processes than* $\alpha_{\rm MS}$ *and* $\alpha_{\overline{\rm MS}}$, *but also by their yielding small one-loop corrections to all the effective charges in gauges with* $|\xi| \lesssim 1$.

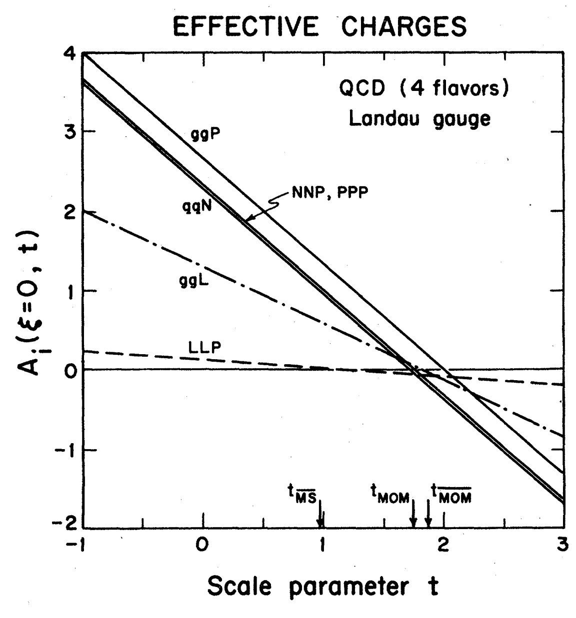


FIG. 3. Order-α^2 coefficients $A_i(\xi = 0, t)$ for the effective charges $\alpha_i^{\rm eff}$ in the Landau gauge as a function of the scale-shift parameter t for QCD with four flavors of massless quarks.

The distribution of one-loop corrections to the effective charges without longitudinal gluon legs in Fig. 3 is seen to coincide with that of the physical charges in Fig. 1. Therefore, each of these effective charges, one of which is $\alpha_{\rm MOM}$, defines a momentum-subtraction coupling constant for which the physical charges have small one-loop corrections. It is reasonable to expect such momentum-subtraction couplings to continue to give small corrections for physical quantities in higher orders. By extending our analysis of the effective charges to two-loop order, we will show that the scale-shifted minimal-subtraction coupling constant is an equally good expansion parameter to this order.

IV. APPROXIMATE EFFECTIVE CHARGES

The effective charges were defined using vertex corrections at the symmetric point (SP), which are very tedious to calculate beyond one loop. In this section it is shown that corrections to a three-point vertex can be reliably approximated by their values with one external leg at zero momentum, which reduces the calculation to the same degree of difficulty as propagator corrections. Hence we can approximate the effective charges by replacing the SP vertex corrections by their values at the zero-momentum point (ZP): $p_1^2 = p_2^2 = -\mu^2$, $p_3 = 0$. In the notation of Eq. (19), we define an "approximate effective charge" $\tilde{g}^{\rm eff}(\mu)$ by

$$\tilde{g}^{\rm eff}(\mu) = \Gamma(q, -q, 0)\,|_{q^2=-\mu^2}\,[1 + \Pi(-\mu^2)]^{-3/2}\,. \tag{24}$$

This charge is the coupling constant for a renormalization scheme in which the vertex correction is subtracted at the ZP, while the propagator corrections are subtracted at the SP as usual.

The motivation for introducing this approximation is that 1PI three-point vertices are guaranteed to be free of infrared singularities as the momentum p of one leg vanishes.[13] The coefficient of a particular Lorentz tensor in the decomposition of the vertex may diverge like $\ln p^2$ as $p \to 0$, but only if the tensor itself vanishes at the ZP. If it survives in this limit, the corresponding vertex correction is guaranteed to remain finite. Explicit calculations show further that these infrared-finite corrections are usually insensitive to p over the entire region of momentum space between the SP and the ZP. It is therefore reasonable to approximate the SP vertex corrections by the corresponding infrared-finite ZP corrections.

We illustrate this approximation using the calculations of Ball and Chiu[14] and Baker and Kim[15] for the complete one-loop corrections in the Feynman gauge to the triple-gluon and ghost vertices for

QCD without quarks. To display the effect of varying the point at which the vertex correction is evaluated, we must select an interpolating path between the SP and ZP. We choose to hold the momentum squared of two external legs fixed, $p_1^2 = p_2^2 = -\mu^2$, letting the momentum p_3 of the third leg vanish with a parameter η such that $p_3^2 = -\eta\mu^2$. We refer to this momentum configuration as the "η point"; $\eta = 1$ is just the symmetric point and $\eta = 0$ is the zero-momentum point.

The triple-gluon and ghost vertices at the η point can be decomposed into scalar components by contracting them with the longitudinal (L), planar-transverse (P), and normal-transverse (N) polarization vectors defined in Sec. III. The combinations of polarizations for which the lowest-order triple-gluon vertex does not vanish identically at the η point are $NN(P)$, $PN(N)$, $PP(P)$, $LN(N)$, $LP(P)$, $LL(P)$, and $PL(L)$, where the letter in parentheses indicates the polarization of the gluon with the asymmetric momentum p_3. The lowest-order ghost vertex vanishes for a zero-momentum outgoing ghost, and hence we consider only the cases where the asymmetric momentum is that of the gluon or incoming ghost. Furthermore, only gluons with polarizations P and L couple to the ghost in lowest order. The corresponding vertex corrections are denoted by $Pg(g)$, $Lg(g)$, $gg(P)$, and $gg(L)$. All these vertex corrections are defined in more detail in Appendix C. Six of them, $NN(P)$, $PP(P)$, $LN(N)$, $PL(L)$, $gg(P)$, and $Lg(g)$, correspond to tensors which survive in the $\eta \to 0$ limit. They are therefore guaranteed to remain finite at the ZP and can be used to define approximate effective charges.

The effective charges, with SP vertex corrections approximated by their values at the η point, can be expanded in powers of $\alpha_{\rm MS}(\mu)$, as in Eq. (20). The dependence of the expansion coefficients A_i in the Feynman gauge ($\xi = 1$) on the approximation parameter η is shown in Fig. 4. The solid curves correspond to the six effective charges whose vertex corrections are guaranteed to be finite at the ZP. The other five charges are represented by dotted lines. As $\eta \to 0$, four of the dotted lines diverge logarithmically, while all of the solid lines remain relatively flat over the entire range of η. *This shows that the effective charges can indeed be calculated reliably at one loop by approximating the SP vertex corrections by their values at the zero momentum point.*

We proceed to study the approximate effective charges of QCD obtained in the $\eta \to 0$ limit. The triple-gluon- and ghost-vertex corrections yield five distinct approximate effective charges, since the vertex corrections $NN(P)$ and $PP(P)$ coincide at $\eta = 0$. They are denoted by $\tilde{\alpha}_i^{\rm eff}$, $i = NN(P)$,

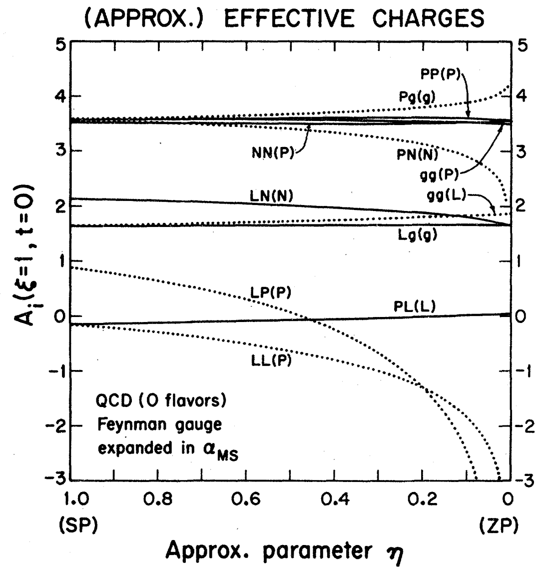


FIG. 4. Order-α^2 coefficients in the MS scheme for the effective charges in the Feynman gauge, with symmetric-point vertex corrections approximated by their values at the η point, as a function of η for QCD with no quarks. Solid (dotted) lines represent charges whose vertex corrections are (are not) guaranteed to be finite at $\eta = 0$.

$LN(N)$, $PL(L)$, $gg(P)$, and $Lg(g)$. The quark-vertex corrections determine five more: $i = Nq(q)$, $Lq(q)$, $qq(N)$, $qq(P)$, and $qq(L)$. The vertex corrections used to define these charges are given explicitly in Appendix C. The approximate effective charges $\tilde{\alpha}_i^{\rm eff}(\mu, \xi)$ can be expanded in powers of the minimal-subtraction coupling $\alpha_{\rm MS}(\mu e^{-t})$ as in Eq. (20):

$$\tilde{\alpha}_i^{\rm eff}(\mu, \xi) = \alpha_{\rm MS}(\mu e^{-t}) + \tilde{A}_i(\xi, t)\alpha_{\rm MS}(\mu e^{-t})^2 + \tilde{B}_i(\xi, t)\alpha_{\rm MS}(\mu e^{-t})^3 + \cdots . \qquad (25)$$

The gauge dependence of the expansion coefficients $\tilde{A}_i(\xi, t=0)$ in the MS scheme is shown in Fig. 5 for four flavors of quarks. Figure 5 displays the same qualitative features for $|\xi| \leq 1$ as the analogous graph for the effective charges, Fig. 2. The approximate charges fall roughly into the same three bands, according to the number of longitudinal gluons.

We next study the behavior of the expansion coefficients $\tilde{A}_i(\xi, t)$ in Eq. (23) as the scale-shift parameter t is varied. The approximate effective charges obey the same renormalization-group Eq. (20) as the effective charges, and hence the t dependence of the $\tilde{A}_i(\xi, t)$ is given by Eq. (23). In Fig. 6, these coefficients in the Feynman gauge ($\xi = 1$) are plotted versus t. This figure is similar to the corresponding graph Fig. 3 for the effective charges in the Landau gauge. The deviations of the

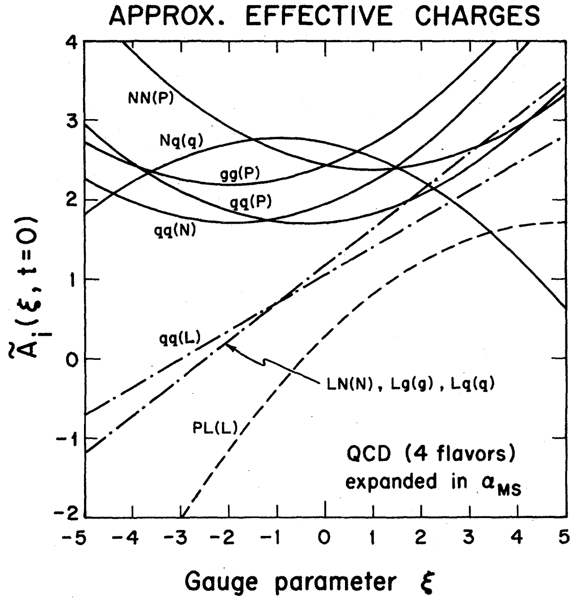


FIG. 5. Order-α^2 coefficients $\tilde{A}_i(\xi, t=0)$ in the MS scheme for the approximate effective charges $\tilde{\alpha}_i^{\text{eff}}$ as a function of the gauge parameter ξ for QCD with four flavors of massless quarks. The solid, dash-dot, and dashed lines represent charges with zero, one, and two longitudinal gluon legs, respectively.

approximate effective charges from $\alpha_{\text{MOM}}(\mu)$, which corresponds to $t = t_{\text{MOM}}$, are all close to zero. As with the effective charges, the distribution of the approximate charges without longitudinal gluon legs (shown by solid lines) coincides with that of the physical charges in Fig. 1. Therefore, the corresponding physical processes with all have small one-loop corrections in the momentum-subtraction schemes for which these charges are the coupling constants. These approximate effective charges will be used in Sec. V to study momentum-subtraction coupling constants at two-loop order, and to show that to this order the scale-shifted MS coupling $\alpha_{\text{MS}}(\mu e^{-t_{\text{MOM}}})$ is an equally good expansion parameter.

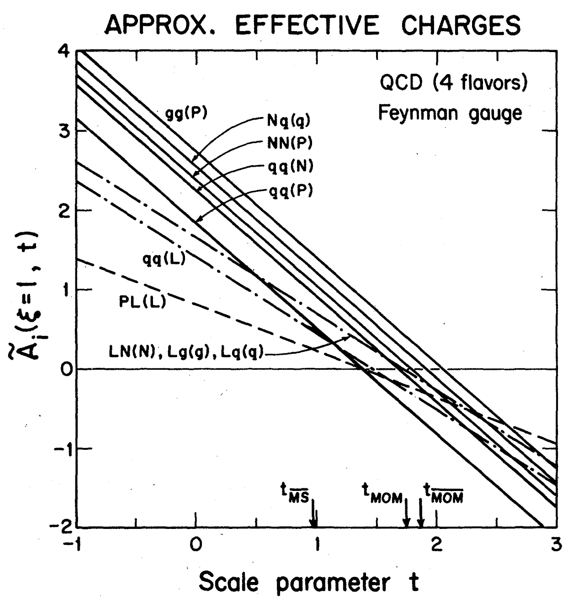


FIG. 6. Order-α^2 coefficients $\tilde{A}_i(\xi=1, t)$ for the approximate effective charges $\tilde{\alpha}_i^{\text{eff}}$ in the Feynman gauge as a function of the scale-shift parameter t for QCD with four flavors of massless quarks.

It is an interesting coincidence that the approximate effective charges $\tilde{\alpha}_i^{\text{eff}}$, $i = LN(N)$, $Lg(g)$, and $Lq(q)$, are exactly degenerate at one loop, as is seen in Figs. 5 and 6. The three corresponding vertex corrections are exactly those that appear in the Ward identities at the ZP for the triple-gluon, ghost, and quark vertices, respectively (see Appendix A). We therefore define a momentum-subtraction scheme $\overline{\text{MOM}}$ in which these three ZP vertex corrections and the three propagator corrections are all subtracted at the same momentum scale μ. The degeneracy of the effective charges at one loop guarantees that this renormalization scheme will be consistent with the Ward identities to this order. Its extension to higher orders is considered in Sec. V. An attractive feature of the $\overline{\text{MOM}}$ scheme is that it treats the triple-gluon, ghost, and quark vertices equally at one-loop order. It is also more convenient for higher-order calculations than the MOM scheme. For example, the determination of the $\overline{\text{MOM}}$ coupling constant requires the calculation of a vertex correction at the ZP instead of the SP, which reduces it to the same degree of difficulty as propagator corrections. In the ξ gauge, the coupling constant for the $\overline{\text{MOM}}$ scheme is given to one-loop order by

$$\alpha_{\overline{\text{MOM}}}(\mu, \xi) = \alpha_{\overline{\text{MS}}}(\mu)\left[1 + \left(\tfrac{169}{24} + \tfrac{9}{4}\xi + \tfrac{3}{8}\xi^2 - \tfrac{5}{9}n_f\right)\frac{\alpha_{\overline{\text{MS}}}(\mu)}{2\pi}\right], \tag{26}$$

where n_f is the number of flavors of quarks. If it is to be used as an expansion parameter for physical quantities, it must be defined in a particular gauge. We suggest that the Feynman gauge be used, since this is the simplest gauge for the higher-order calculations. We therefore define $\alpha_{\overline{\text{MOM}}}(\mu) \equiv \alpha_{\overline{\text{MOM}}}(\mu, \xi=1)$.

A minimal-subtraction coupling which equals $\alpha_{\overline{\text{MOM}}}$ to order α^2 can be obtained by shifting the renormalization scale of α_{MS}:

$$\alpha_{\overline{\text{MOM}}}(\mu) = \alpha_{\text{MS}}(\mu e^{-t_{\overline{\text{MOM}}}}) + O(\alpha^3),$$
$$t_{\overline{\text{MOM}}} = t_{\overline{\text{MS}}} + 1 - \frac{1}{3}\,\frac{12 - n_f}{33 - 2n_f}, \tag{27}$$

where $t_{\overline{\text{MS}}}$ is given in Eq. (12). The scale-shift parameter $t_{\overline{\text{MOM}}}$ is nearly equal to t_{MOM} defined in Eq.

(13) for $n_f=3$ flavors of quarks, but has a weaker dependence on the number of flavors. The flavor dependence of the one-loop corrections in the $\overline{\rm MS}$ scheme to the physical charges defined in Sec. II is shown in Fig. 7, with the corrections to the coupling constants $\alpha_{\rm MOM}$ and $\alpha_{\overline{\rm MOM}}$ superimposed. The figure indicates that $\alpha_{\overline{\rm MOM}}$ reflects the flavor dependence of the radiative corrections to physical quantities more accurately than $\alpha_{\rm MOM}$.

V. TWO-LOOP EFFECTIVE CHARGES

Using the zero-momentum-point (ZP) approximation for the vertex corrections, the analysis of the effective charges of QCD is extended to two-loop order. We have calculated all two-loop corrections to the propagators and three-point vertices at the ZP for QCD in the Feynman gauge. The calculations were checked using the relevant Ward identities. The results are tabulated in Appendix B. We use them to obtain the two-loop corrections to the ten approximate effective charges introduced in Sec. IV.

These approximate effective charges $\tilde{\alpha}_i^{\rm eff}(\mu,\xi)$ are expanded in powers of the scale-shifted MS coupling constant $\alpha_{\rm MS}(\mu e^{-t})$ in Eq. (25), We wish to study the α^2 coefficient $\tilde{B}_i(\xi,t)$ as a function of the scale-shift parameter t. Inserting the expansions Eqs. (21) and (25) into the renormalization-group equation Eq. (20), we find that $\tilde{B}_i(\xi,t)$ must satisfy

$$\frac{\partial}{\partial t}\tilde{B}_i+2\left[b_0+\gamma_0\left(l+\xi\frac{\partial}{\partial\xi}\right)\right]\tilde{A}_i+b_1+2l\gamma_1=0\,. \quad (28)$$

Using the expression for $\tilde{A}_i(\xi,t)$ given in Eq. (22), we obtain the solution

$$\tilde{B}_i(\xi,t)=\tilde{B}_i(\xi,0)-\left[b_1+2l\gamma_1+2(b_0+l\gamma_0)\tilde{A}_i(\xi,0)+2\gamma_0\xi\frac{\partial}{\partial\xi}\tilde{A}_i(\xi,0)\right]t + \left[(b_0+l\gamma_0)(b_0+2l\gamma_0)+2l\gamma_0\xi\frac{\partial}{\partial\xi}\gamma_0\right]t^2\,. \quad (29)$$

The functions $\tilde{B}_i(\xi,t)$ are all parabolas in the variable t, and the coefficient of t^2 depends on the number l of longitudinal gluon legs.

The dependence of the expansion coefficients $\tilde{B}_i(\xi=1,t)$ on the scale-shift parameter t is shown in Fig. 8 for $n_f=4$ flavors of quarks. The parabolas separate according to the number of longitudinal gluons for $t<0$. Just as in order α^2, the order-α^3 deviations of the approximate effective charges from $\alpha_{\rm MS}$ and $\alpha_{\overline{\rm MS}}$ (corresponding to $t=0$

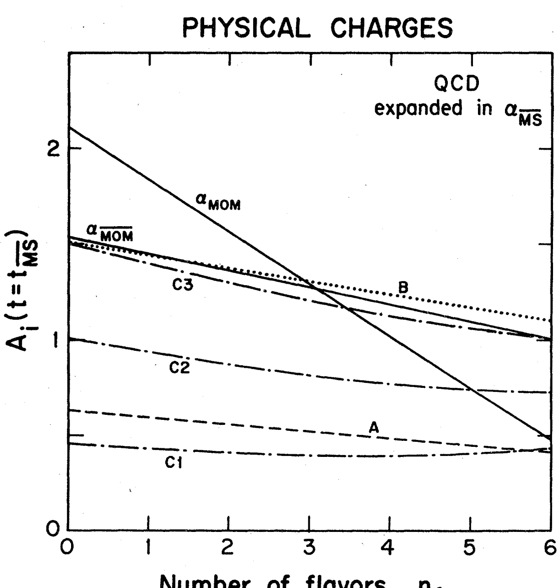


FIG. 7. Order-α^2 coefficients in the $\overline{\rm MS}$ scheme for the momentum-subtraction coupling constants $\alpha_{\rm MOM}$ and $\alpha_{\overline{\rm MOM}}$ and the physical charges $\alpha_i^{\rm phys}$ as a function of the number n_f of flavors of massless quarks.

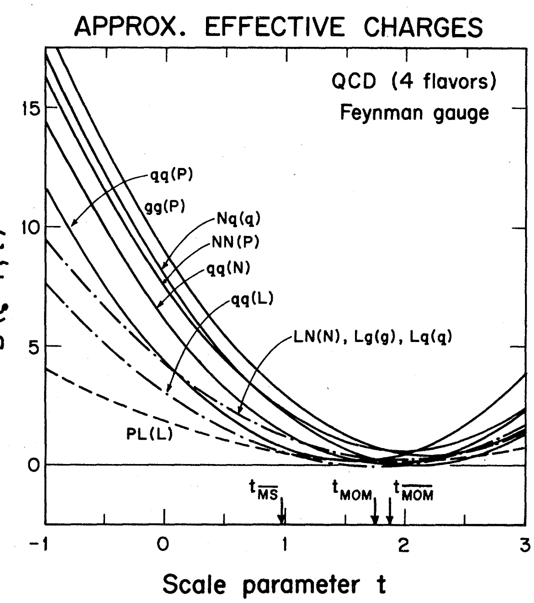


FIG. 8. Order-α^3 coefficients $\tilde{B}_i(\xi=1,t)$ for the approximate effective charges $\tilde{\alpha}_i^{\rm eff}$ in the Feynman gauge as a function of the scale-shift parameter t for QCD with four flavors of massless quarks.

and $t = t_{\overline{\rm MS}}$, respectively) are all positive and some of the deviations are large. The most striking feature of the graph, however, is that all the parabolas attain their minimum, which is close to zero, near $t = t_{\overline{\rm MOM}} \simeq t_{\rm MOM}$. Hence the minimal-subtraction coupling constant $\alpha_{\rm MS}(\mu e^{-t_{\overline{\rm MOM}}})$, for which the one-loop corrections to all the approximate effective charges in the Feynman gauge are small, yields small two-loop corrections as well. Since the approximate effective charges without longitudinal gluon legs should be fairly representative of momentum-subtraction coupling constants in general, this indicates that *to two-loop order the scale-shifted MS coupling constant* $\alpha_{\rm MS}(\mu e^{-t_{\overline{\rm MOM}}})$ *should be just as good an expansion parameter for physical quantities as typical momentum-subtraction couplings.*

In Sec. IV, it was observed that the approximate effective charges $\tilde{\alpha}_i^{\rm eff}$, $i = LN(N)$, $Lg(g)$, $Lq(q)$, are degenerate at one loop. This enabled us to define a momentum-subtraction scheme $\overline{\rm MOM}$ at one loop in which the three corresponding ZP vertex corrections and the three propagator corrections are all subtracted at the same momentum scale. The degeneracy of these three charges is broken at two loops, although the three corresponding curves cannot be distinguished in Fig. 8. Therefore, the suggested momentum-subtraction scheme is inconsistent with the Ward identities beyond one loop. The $\overline{\rm MOM}$ scheme can be extended to higher orders, however, by subtracting the three propagator corrections and only one of the three vertex corrections. We choose the ghost-vertex correction since its calculation can be greatly simplified by using one of the Ward identities (see Appendix B). The $\overline{\rm MOM}$ coupling constant in the Feynman gauge is given to two-loop order by

$$\alpha_{\overline{\rm MOM}}(\mu) = \alpha_{\overline{\rm MS}}(\mu)\left\{1 + \left(\tfrac{29}{3} - \tfrac{5}{9}n_f\right)\frac{\alpha_{\overline{\rm MS}}(\alpha)}{2\pi} + \left[\tfrac{24301}{144} - \tfrac{117}{16}\xi - \left(\tfrac{512}{27} + \tfrac{1}{3}\xi\right)n_f + \tfrac{25}{81}n_f^{\,2}\right]\left(\frac{\alpha_{\overline{\rm MS}}(\mu)}{2\pi}\right)^2\right\}, \tag{30}$$

where $\xi = \sum_{n=1}^{\infty} 1/n^3 \simeq 1.202$. We compare this coupling constant with the minimal-subtraction coupling $\alpha_{\rm MS}(\mu e^{-t_{\overline{\rm MOM}}})$ that agrees with it to order α^2:

$$\alpha_{\overline{\rm MOM}}(\mu) = \alpha_{\overline{\rm MS}}(\mu e^{-t_{\overline{\rm MOM}}})\left\{1 + \left[\tfrac{629}{16} - \tfrac{117}{16}\xi - \left(\tfrac{35}{9} + \tfrac{1}{3}\xi\right)n_f + \frac{(108 - 13n_f)12 - n_f)}{9(33 - 2n_f)}\right]\left[\frac{\alpha_{\rm MS}(\mu e^{-t_{\overline{\rm MOM}}})}{2\pi}\right]^2\right\}. \tag{31}$$

The coefficient of α^2 in the brackets is approximately $0.86 - 0.12\,n_f$ for $n_f \leq 6$. Therefore, provided that $\alpha^2 \ll 1$, this scale-shifted MS coupling constant and the momentum-subtraction coupling $\alpha_{\rm MOM}$ will yield similar radiative corrections to physical quantities through two-loop order.

VI. SUMMARY

This work was motivated by the empirical observation that the momentum-subtraction coupling constant $\alpha_{\overline{\rm MOM}}$ yields smaller one-loop corrections for many QCD processes than the MS and $\overline{\rm MS}$ coupling constants. The theoretical advantages of minimal subtraction make it desirable to find a minimal-subtraction coupling constant which is also a good expansion parameter for physical processes.

The MS and $\overline{\rm MS}$ coupling constants are related to all orders by a simple shift in the renormalization scale: $\alpha_{\overline{\rm MS}}(\mu) \equiv \alpha_{\rm MS}(\mu e^{-t})$ for $t \simeq 0.977$. By adjusting the parameter t so that $\alpha_{\rm MOM}(\mu) = \alpha_{\rm MS}(\mu e^{-t}) + O(\alpha^3)$, we obtained a minimal-subtraction coupling constant which yields the same small one-loop corrections to physical quantities in QCD as $\alpha_{\rm MOM}$. We then asked whether this scale-shifted MS coupling would continue to be a good expansion parameter for physical quantities in higher orders.

This question could not be answered directly, as calculations of corrections to QCD processes are not available beyond one loop. Instead, we decided to compare this coupling constant with the momentum-subtraction couplings of the theory, as typified by the effective charges of QCD for small values of the gauge parameter. Calculating these effective charges, however, requires the calculation of vertex corrections at the symmetric point, an arduous task beyond one loop. We avoided this difficulty by using a simple approximation to the effective charges in which we replaced the symmetric-point vertex corrections by their values with one leg at zero momentum. One-loop calculations were used to demonstrate the reliability of the approximation. The scale-shifted MS coupling constant defined above was then shown to yield small corrections to these approximate effective charges at both one-loop and two-loop order. We concluded that this minimal-subtraction coupling constant will yield expansions for physical quantities that are similar to those obtained in typical momentum-subtraction schemes, at least to two-loop order.

We also introduce a momentum-subtraction scheme $\overline{\rm MOM}$ which proves more convenient for higher-order calculations than the MOM scheme.

It has the attractive feature of treating the triple-gluon, ghost, and quark vertices equally at one-loop order.

In conclusion, we have shown that to two-loop order the conventional minimal-subtraction coupling constant $\alpha_{\overline{MS}}(\mu)$ is, for phenomenological purposes, equivalent to momentum-subtraction couplings such as α_{MOM}, provided that the renormalization scale μ is suitably shifted.

ACKNOWLEDGMENTS

We thank Bill Celmaster and Tim Jones for helpful discussions. This research was supported in part by the University of Wisconsin Research Committee with funds granted by the Wisconsin Alumni Research Foundation, and in part by the Department of Energy under Contract No. DE-AC0276ER00881-178.

APPENDIX A: ONE-LOOP CORRECTIONS AT THE ZERO-MOMENTUM POINT

We tabulate the one-loop corrections to one-particle-irreducible (1PI) three-point functions at the zero-momentum point (ZP) for a non-Abelian gauge theory in a general covariant gauge. For completeness, we also list the propagator corrections and check the relevant Ward identities.

We first establish some notation for the propagators and vertices. The gluon, ghost, and quark propagators have the respective forms

$$D^{ab}_{\mu\nu}(q)=-\frac{\delta^{ab}}{q^2}\left[\frac{1}{1+\Pi(q^2)}\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)+\xi\frac{q_\mu q_\nu}{q^2}\right],$$

$$\Delta^{ab}(q)=\frac{\delta^{ab}}{q^2}\,\frac{1}{1+B(q^2)},\qquad\text{(A1)}$$

$$S_{ij}(q)=\frac{\delta_{ij}}{q^2}\,\frac{\not q}{1+A(q^2)},$$

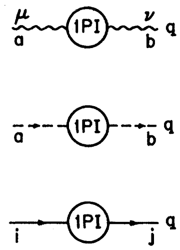
where Π, B, A are the 1PI two-point functions (self-energies) defined in Fig. 9, and ξ is the gauge parameter.

The 1PI three-point vertices are defined in Fig. 10. The triple-gluon vertex at the ZP has the form

$$T^{abc}_{\mu\nu\lambda}(-q,q,0)=gf^{abc}\Big[[1+T_1(q^2)](g_{\mu\lambda}q_\nu+g_{\nu\lambda}q_\mu-2g_{\mu\nu}q_\lambda)+T_2(q^2)q_\lambda\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)\Big].\qquad\text{(A2)}$$

The ghost vertex is

$$G^{abc}_\mu(-q,q,0)=-gf^{abc}[1+G_1(q^2)]q_\mu\,,$$
$$G^{abc}_\mu(0,q,-q)=-gf^{abc}[1+G_2(q^2)]q_\mu\qquad\text{(A3)}$$



$$\equiv -i\left[1+\Pi(q^2)\right]\delta^{ab}\left(g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right)$$

$$\equiv i\left[1+B(q^2)\right]\delta^{ab}$$

$$\equiv i\left[1+A(q^2)\right]\delta_{ij}$$

FIG. 9. Definitions of the gluon, ghost, and quark self-energies.

for a zero-momentum incoming ghost and a zero-momentum gluon, respectively. The quark vertex has the form

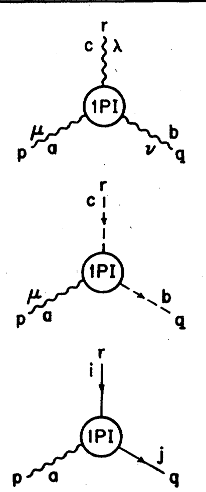
$$\Gamma^{aji}_\mu(-q,q,0)=-igT^a_{ji}\left\{[1+\Gamma_1(q^2)]\gamma_\mu+\Gamma_2(q^2)\gamma^\nu\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)\right\},$$

$$\Gamma^{aji}_\mu(0,q,-q)=-igT^a_{ji}\left\{[1+\Gamma_3(q^2)]\gamma_\mu+\Gamma_4(q^2)\gamma^\nu\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)\right\}\qquad\text{(A4)}$$

for a zero-momentum quark and a zero-momentum gluon, respectively.

The dependence of radiative corrections on the gauge group is contained in the group invariants C_A, C_F, and T defined by

$$f^{acd}f^{bcd}=C_A\delta^{ab}\,,$$
$$T^a_{ij}T^a_{jk}=C_F\delta_{ik}\,,\qquad\text{(A5)}$$
$$T^a_{ij}T^b_{ji}=T\delta^{ab}\,.$$



$$\equiv T^{abc}_{\mu\nu\lambda}(p,q,r)$$

$$\equiv G^{abc}_\mu(p,q,r)$$

$$\equiv \Gamma^{aji}_\mu(p,q,r)$$

FIG. 10. Definitions of the 1PI Green's functions for the triple-gluon, ghost, and quark vertices. All momenta are outgoing.

For an SU(N) gauge group with n_f flavors of quarks in the fundamental representation, these constants are $C_A = N$, $C_F = (N^2-1)/2N$, and $T = \frac{1}{2}n_f$.

It is sufficient to give the propagator and ZP vertex corrections at the point $q^2 = -\mu^2$, since their values for other momenta can then be determined using the renormalization-group equations for 1PI Green's functions:

$$\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial\alpha}+l\gamma+\tilde{l}\tilde{\gamma}+l_F\gamma_F-2\gamma\xi\frac{\partial}{\partial\xi}\right)\Gamma=0\,, \quad \text{(A6)}$$

where l, $\tilde{l}$, l_F are the number of gluon, ghost, and quark legs, respectively. In a minimal-subtraction renormalization scheme, the β function is gauge invariant and is given to lowest order by

$$\frac{1}{2\pi}\beta(\alpha) = -\left[\tfrac{11}{3}C_A - \tfrac{4}{3}T\right]\left(\frac{\alpha}{2\pi}\right)^2. \quad \text{(A7)}$$

The lowest-order anomalous dimensions of the gluon, ghost, and quark fields are, respectively,

$$\begin{aligned}
\gamma(\alpha,\xi) &= \left[\left(-\tfrac{13}{12}+\tfrac{1}{4}\xi\right)C_A+\tfrac{2}{3}T\right]\frac{\alpha}{2\pi}\,,\\
\tilde{\gamma}(\alpha,\xi) &= \left[\left(-\tfrac{3}{8}+\tfrac{1}{8}\xi\right)C_A\right]\frac{\alpha}{2\pi}\,, \qquad \text{(A8)}\\
\gamma_F(\alpha,\xi) &= \left(\tfrac{1}{2}\xi C_A\right)\frac{\alpha}{2\pi}\,.
\end{aligned}$$

We give the one-loop propagator and vertex corrections in the modified minimal-subtraction ($\overline{\text{MS}}$) scheme, with coupling constant $\alpha \equiv \alpha_{\overline{\text{MS}}}(\mu)$. The self-energies at momentum squared $q^2 = -\mu^2$ are

$$\begin{aligned}
\Pi(-\mu^2) &= \frac{\alpha}{2\pi}\left[\left(-\tfrac{97}{72}-\tfrac{1}{4}\xi-\tfrac{1}{8}\xi^2\right)C_A+\tfrac{10}{9}T\right],\\
B(-\mu^2) &= \frac{\alpha}{2\pi}\left(-\tfrac{1}{2}C_A\right), \qquad \text{(A9)}\\
A(-\mu^2) &= \frac{\alpha}{2\pi}\left(\tfrac{1}{2}\xi C_F\right).
\end{aligned}$$

The triple-gluon-vertex corrections at the ZP are given by

$$\begin{aligned}
T_1(-\mu^2) &= \frac{\alpha}{2\pi}\left[\left(-\tfrac{61}{72}-\tfrac{1}{8}\xi^2\right)C_A+\tfrac{10}{9}T\right],\\
T_2(-\mu^2) &= \frac{\alpha}{2\pi}\left[\left(-\tfrac{5}{3}+\xi\right)C_A+\tfrac{4}{3}T\right].
\end{aligned} \qquad \text{(A10)}$$

The ghost-vertex corrections are

$$\begin{aligned}
G_1(-\mu^2) &= \frac{\alpha}{2\pi}\left(\tfrac{1}{4}\xi C_A\right),\\
G_2(-\mu^2) &= \frac{\alpha}{2\pi}\left[\left(\tfrac{3}{8}+\tfrac{1}{8}\xi\right)C_A\right].
\end{aligned} \qquad \text{(A11)}$$

The quark-vertex corrections are given by

$$\begin{aligned}
\Gamma_1(-\mu^2) &= \frac{\alpha}{2\pi}\left[\left(\tfrac{1}{2}+\tfrac{1}{4}\xi\right)C_A+\tfrac{1}{2}\xi C_F\right],\\
\Gamma_2(-\mu^2) &= \frac{\alpha}{2\pi}\left[\left(\tfrac{9}{8}-\tfrac{1}{2}\xi-\tfrac{1}{8}\xi^2\right)C_A-C_F\right],\\
\Gamma_3(-\mu^2) &= \frac{\alpha}{2\pi}\left[\left(\tfrac{1}{8}+\tfrac{3}{8}\xi\right)C_A-\tfrac{1}{2}\xi C_F\right],\\
\Gamma_4(-\mu^2) &= \frac{\alpha}{2\pi}\left[\left(\tfrac{1}{4}-\tfrac{1}{4}\xi\right)C_A+\xi C_F\right].
\end{aligned} \qquad \text{(A12)}$$

Ward identities relate the longitudinal gluon polarization components of 1PI vertices to other Green's functions which have legs that are sources for the Becchi-Rouet-Stora[16] variations of the fields (BRS sources for short). The 1PI triple-gluon, ghost, and quark vertices are related by the Ward identities to the 1PI three-point functions having a gluon, ghost, and quark BRS source, respectively, at one leg. The Feynman rules for the three-point vertices involving BRS sources are shown in Fig. 11, and the corresponding 1PI Green's functions are defined in Fig. 12. The 1PI vertex with a gluon BRS source, a ghost, and a zero-momentum gluon has the form

$$G^{abc}_{\mu\nu}(-q,0,q) = gf^{abc}\left[g_{\mu\nu}+G_3(q^2)\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)+G_2(q^2)\frac{q_\mu q_\nu}{q^2}\right]. \quad \text{(A13)}$$

The function G_2 is the same function that appears in Eq. (A3), by a trivial Ward identity. The 1PI vertex with a ghost BRS source and two ghost legs, one with zero momentum, is

$$G^{abc}(-q,0,q) = -gf^{abc}[1+G_4(q^2)]. \quad \text{(A14)}$$

The 1PI vertex with a quark BRS source, a ghost,

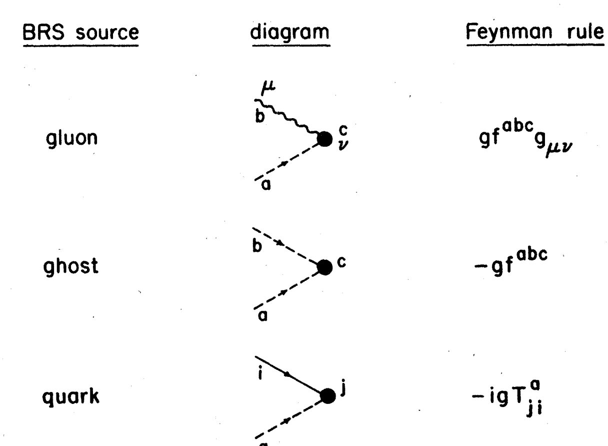


FIG. 11. Feynman rules for the three-point vertices containing gluon, ghost, and quark BRS sources.

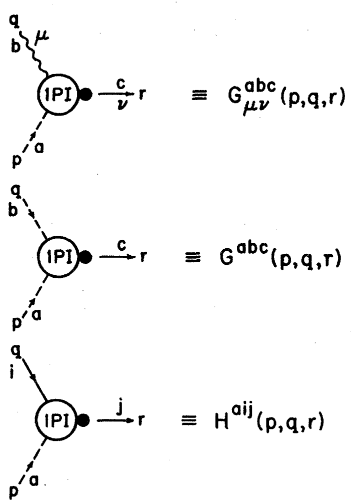


FIG. 12. Definitions of the 1PI Green's functions corresponding to the vertices in Fig. 11. All momenta are outgoing.

and a zero-momentum quark has the form

$$H^{aij}(-q,0,q)=-igT^a_{ji}[1+H(q^2)]\,. \qquad \text{(A15)}$$

The Ward identities for the gluon, ghost, and quark vertices at the ZP are, respectively,

$$\begin{aligned}
&[1+B(q^2)][1+T_1(q^2)]=[1+\Pi(q^2)][1+G_3(q^2)]\,,\\
&G_1(q^2)=G_4(q^2)\,, \qquad \text{(A16)}\\
&[1+B(q^2)][1+\Gamma_1(q^2)]=[1+A(q^2)][1+H(q^2)]\,.
\end{aligned}$$

The functions G_3, G_4, and H which appear in the Ward identities are all equal at one loop; their values at $q^2=-\mu^2$ are therefore the same as $G_1(-\mu^2)$ which is given in Eq. (A11). The equality of these functions enabled us to define a momentum-subtraction scheme $\overline{\text{MOM}}$ at one loop in which the corrections Π, B, A, T_1, G_1, and Γ_1 are all subtracted at $q^2=-\mu^2$. This is easily seen to be consistent with the Ward identities Eq. (A16). The coupling constant $\alpha_{\overline{\text{MOM}}}(\mu,\xi)$ is given to one-loop order by

$$\alpha_{\overline{\text{MOM}}}(\mu,\xi) =\alpha_{\overline{\text{MS}}}(\mu)\left\{1+[(\tfrac{169}{72}+\tfrac{3}{4}\xi+\tfrac{1}{8}\xi^2)C_A-\tfrac{10}{9}T]\frac{\alpha_{\overline{\text{MS}}}(\mu)}{2\pi}\right\}. \qquad \text{(A17)}$$

The extension of the $\overline{\text{MOM}}$ scheme to higher orders is considered in Appendix B.

APPENDIX B: TWO-LOOP CORRECTIONS AT THE ZERO-MOMENTUM POINT

We tabulate the two-loop corrections to self-energies and one-particle-irreducible (1PI) three-point functions at the zero-momentum point (ZP) for a non-Abelian gauge theory in the Feynman gauge ($\xi=1$). For completeness, we also check the relevant Ward identities.

The notation is the same as in Appendix A. Again, it is sufficient to give the corrections for momentum squared $q^2=-\mu^2$, since the renormalization-group equations can then be used to determine the corrections for other values of q^2. These renormalization-group equations require the two-loop β function

$$\begin{aligned}
\tfrac{1}{2}\pi\beta(\alpha)=&-(\tfrac{11}{3}C_A-\tfrac{4}{3}T)\left(\frac{\alpha}{2\pi}\right)^2\\
&-(\tfrac{17}{3}C_A{}^2-\tfrac{10}{3}C_AT-2C_FT)\left(\frac{\alpha}{2\pi}\right)^3 \qquad \text{(B1)}
\end{aligned}$$

and the two-loop anomalous dimensions in the Feynman gauge

$$\begin{aligned}
\gamma(\alpha,\xi=1)=&(-\tfrac{5}{6}C_A+\tfrac{2}{3}T)\frac{\alpha}{2\pi}\\
&+(-\tfrac{23}{16}C_A{}^2+\tfrac{5}{4}C_AT+C_FT)\left(\frac{\alpha}{2\pi}\right)^2,\\
\tilde{\gamma}(\alpha,\xi=1)=&(-\tfrac{1}{4}C_A)\frac{\alpha}{2\pi}+(-\tfrac{49}{96}C_A{}^2+\tfrac{5}{24}C_AT)\left(\frac{\alpha}{2\pi}\right)^2, \qquad \text{(B2)}\\
\gamma_F(\alpha,\xi=1)=&(\tfrac{1}{2}C_A)\frac{\alpha}{2\pi}\\
&+(\tfrac{17}{8}C_AC_F-\tfrac{1}{2}C_FT-\tfrac{3}{8}C_F{}^2)\left(\frac{\alpha}{2\pi}\right)^3.
\end{aligned}$$

They also require the one-loop anomalous dimensions in the general gauge which are given in Appendix A.

We give the two-loop propagator and vertex corrections in the modified minimal-subtraction ($\overline{\text{MS}}$) scheme, with coupling constant $\alpha\equiv\alpha_{\overline{\text{MS}}}(\mu)$. The self-energies in the Feynman gauge ($\xi=1$) for momentum squared $q^2=-\mu^2$ are

$$\begin{aligned}
\Pi(-\mu^2)=&\frac{\alpha}{2\pi}(-\tfrac{31}{18}C_A+\tfrac{10}{9}T)+\left(\frac{\alpha}{2\pi}\right)^2[(-\tfrac{3245}{576}+\tfrac{1}{4}\zeta)C_A{}^2+(\tfrac{451}{144}+2\zeta)C_AT+(\tfrac{55}{12}-4\zeta)C_FT]\,,\\
B(-\mu^2)=&\frac{\alpha}{2\pi}(-\tfrac{1}{2}C_A)+\left(\frac{\alpha}{2\pi}\right)^2[(-\tfrac{901}{384}+\tfrac{3}{16}\zeta)C_A{}^2+\tfrac{95}{96}C_AT]\,, \qquad \text{(B3)}\\
A(-\mu^2)=&\frac{\alpha}{2\pi}(\tfrac{1}{2}C_F)+\left(\frac{\alpha}{2\pi}\right)^2[(\tfrac{143}{32}-\tfrac{3}{2}\zeta)C_AC_F-\tfrac{7}{8}C_FT-\tfrac{5}{32}C_F{}^2]\,,
\end{aligned}$$

where $\zeta=\sum_{n=1}^{\infty}1/n^3\simeq1.202$. The triple-gluon-vertex corrections at the ZP are

$$\begin{aligned} T_1(-\mu^2) &= \frac{\alpha}{2\pi}(-\tfrac{35}{36}C_A + \tfrac{10}{9}T) + \left(\frac{\alpha}{2\pi}\right)^2[(-\tfrac{4021}{1152} - \tfrac{1}{16}\zeta)C_A{}^2 + (\tfrac{875}{288} + 2\zeta)C_AT + (\tfrac{55}{12} - 4\zeta)C_FT]\,, \\ T_2(-\mu^2) &= \frac{\alpha}{2\pi}(-\tfrac{2}{3}C_A + \tfrac{4}{3}T) + \left(\frac{\alpha}{2\pi}\right)^2[(-\tfrac{641}{144} - \tfrac{1}{4}\zeta)C_A{}^2 + \tfrac{157}{36}C_AT + 2C_FT]\,. \end{aligned} \tag{B4}$$

The ghost-vertex corrections are given by

$$\begin{aligned} G_1(-\mu^2) &= \frac{\alpha}{2\pi}(\tfrac{1}{4}C_A) + \left(\frac{\alpha}{2\pi}\right)^2[(\tfrac{25}{32} - \tfrac{3}{32}\zeta(C_A{}^2]\,, \\ G_2(-\mu^2) &= \frac{\alpha}{2\pi}(\tfrac{1}{2}C_A) + \left(\frac{\alpha}{2\pi}\right)^2[(\tfrac{457}{192} - \tfrac{3}{32}\zeta)C_A{}^2 - \tfrac{29}{48}C_AT]\,. \end{aligned} \tag{B5}$$

The quark-vertex corrections at the ZP are

$$\begin{aligned} \Gamma_1(-\mu^2) &= \frac{\alpha}{2\pi}(\tfrac{3}{4}C_A + \tfrac{1}{2}C_F) + \left(\frac{\alpha}{2\pi}\right)^2[(\tfrac{1357}{384} - \tfrac{3}{8}\zeta)C_A{}^2 - \tfrac{95}{96}C_AT + (\tfrac{155}{32} - \tfrac{3}{2}\zeta)C_AC_F - \tfrac{7}{8}C_FT - \tfrac{5}{32}C_F{}^2]\,, \\ \Gamma_2(-\mu^2) &= \frac{\alpha}{2\pi}(\tfrac{1}{2}C_A - C_F) + \left(\frac{\alpha}{2\pi}\right)^2[(\tfrac{311}{72} - \tfrac{3}{16}\zeta)C_A{}^2 + (-\tfrac{59}{36} - \zeta)C_AT - \tfrac{115}{18}C_AC_F + \tfrac{13}{9}C_FT + \tfrac{7}{4}C_F{}^2]\,, \\ \Gamma_3(-\mu^2) &= \frac{\alpha}{2\pi}(\tfrac{1}{2}C_A - \tfrac{1}{2}C_F) + \left(\frac{\alpha}{2\pi}\right)^2[(\tfrac{985}{384} - \tfrac{3}{4}\zeta)C_A{}^2 - \tfrac{71}{96}C_AT + (\tfrac{39}{32} - \tfrac{3}{2}\zeta)C_AC_F + \tfrac{1}{8}C_FT + \tfrac{19}{32}C_F{}^2]\,, \\ \Gamma_4(-\mu^2) &= \frac{\alpha}{2\pi}(C_F) + \left(\frac{\alpha}{2\pi}\right)^2[(\tfrac{71}{144} - \tfrac{3}{4}\zeta)C_A{}^2 - \tfrac{1}{36}C_AT + (\tfrac{35}{12} + \zeta)C_AC_F - C_FT - \tfrac{3}{4}C_F{}^2]\,. \end{aligned} \tag{B6}$$

The relevant Ward identities at the ZP are given in Eq. (A16) and involve 1PI three-point functions with a BRS source at one leg. The two-loop corrections to these functions in the Feynman gauge are

$$\begin{aligned} G_3(-\mu^2) &= \frac{\alpha}{2\pi}(\tfrac{1}{4}C_A) + \left(\frac{\alpha}{2\pi}\right)^2[(\tfrac{137}{192} - \tfrac{1}{8}\zeta)C_A{}^2 + \tfrac{1}{16}C_AT]\,, \quad G_4(-\mu^2) = G_1(-\mu^2)\,, \\ H(-\mu^2) &= \frac{\alpha}{2\pi}(\tfrac{1}{4}C_A) + \left(\frac{\alpha}{2\pi}\right)^2[(\tfrac{13}{16} - \tfrac{3}{16}\zeta)C_A{}^2]\,. \end{aligned} \tag{B7}$$

The Ward identities Eq. (A16) are seen to be satisfied.

The momentum-subtraction scheme $\overline{\text{MOM}}$ was defined at one loop in Appendix A by subtracting the propagator corrections Π, B, and A and the ZP vertex corrections T_1, G_1, and Γ_1 at the momentum scale $q^2 = -\mu^2$. This is consistent with the Ward identities Eq. (A16), because the functions G_3, G_4, and H are equal at one loop. However, they are no longer equal at two loops as seen in Eq. (B6), so this subtraction scheme must be modified. We chose to extend the $\overline{\text{MOM}}$ scheme to higher orders by subtracting only the functions Π, B, A, and G_1. The vertex correction G_1 is a calculationally convenient choice, because it is equal by a Ward identity to the function G_4, which is simpler to calculate. The $\overline{\text{MOM}}$ coupling constant in the Feynman gauge is then given to third order in $\alpha \equiv \alpha_{\overline{\text{MS}}}(\mu)$ by

$$\alpha_{\overline{\text{MOM}}}(\mu, \xi = 1) = \alpha\left[1 + \frac{\alpha}{2\pi}(\tfrac{29}{9}C_A - \tfrac{10}{9}T) + \left(\frac{\alpha}{2\pi}\right)^2[(\tfrac{24301}{1296} - \tfrac{13}{16}\zeta)C_A{}^2 + (-\tfrac{859}{81} - 2\zeta)C_AT + (-\tfrac{55}{12} + 4\zeta)C_FT + \tfrac{100}{81}T^2]\right]. \tag{B8}$$

APPENDIX C: SCALAR DECOMPOSITION OF THREE-POINT FUNCTIONS

We decompose the three-point vertices of QCD into scalar components corresponding to definite gluon polarizations. Since each Lorentz index corresponds to an external gluon leg, the Lorentz structure can be eliminated by contracting the vertex with a polarization vector for each external gluon. There is a natural choice of basis vectors for an external gluon leg of a three-point vertex, since there is a preferred line, the momentum of the gluon, and a preferred plane, the one determined by the momenta of the three legs. The "longitudinal" (L) polarization vector ϵ_L is defined to be parallel to the gluon momentum. The orthogonal basis vector in the momentum plane is the "planar-transverse" (P) polarization vector ϵ_P. The other two independent basis vectors can be chosen orthogonal to the momentum plane. We call them "normal-transverse" and denote them generically by ϵ_N.

In Sec. IV, we defined the η point for a vertex with external momenta p_1, p_2, p_3, to be the momentum configuration: $p_1{}^2=p_2{}^2=-\mu^2$, $p_3{}^2=-\eta\mu^2$. It interpolates between the symmetric point (SP) at $\eta=1$, and the zero-momentum point (ZP) at $\eta=0$. For the p_1 and p_3 legs, the longitudinal and planar-transverse polarization vectors, normalized to have squares ± 1, are

$$\begin{aligned} &\epsilon_L(p_1)=p_1/\mu\,,\quad \epsilon_L(p_3)=p_3/\sqrt{\eta}\mu\,,\\ &\epsilon_P(p_1)=[\eta p_1-(2-\eta)p_3]/[\eta(4-\eta)]^{1/2}\mu\,, \\ &\epsilon_P(p_3=(p_1-p_2)/(4-\eta)^{1/2}\mu\,. \end{aligned} \tag{C1}$$

The corresponding polarization vectors for the p_2 leg are obtained by using the symmetry between p_1 and p_2. The normal-transverse polarization vectors are characterized by $p_1\cdot\epsilon_N(p_i)=p_2\cdot\epsilon_N(p_i)=p_3\cdot\epsilon_N(p_i)=0$.

We decompose the triple-gluon vertex $T^{abc}_{\mu\nu\lambda}$, defined in Fig. 10, into scalar components using these polarization vectors. The scalar component $T_{LN(N)}$ corresponding to polarization vectors $\epsilon_L(p_1)$, $\epsilon_N(p_3)$, and $\epsilon_N(p_2)$ is defined, for example, by

$$\epsilon^\mu_L(p_1)\epsilon^\nu_N(p_2)\epsilon^\lambda_N(p_3)T^{abc}_{\mu\nu\lambda}(p_1,p_2,p_3)=gf^{abc}T_{LN(N)}\,. \tag{C2}$$

In lowest order, the triple-gluon vertex is

$$T^{abc}_{\mu\nu\lambda}(p_1,p_2,p_3)=gf^{abc}[(p_1-p_2)_\lambda g_{\mu\nu}+(p_2-p_3)_\mu g_{\nu\lambda}+(p_3-p_1)_\nu g_{\lambda\mu}]\,. \tag{C3}$$

The scalar components which do not vanish identically at the η point in lowest order are

$$\begin{aligned} &T_{PL(L)}=-\tfrac{1}{2}(4-\eta)^{1/2}\mu\,,\quad T_{LL(P)}=-\tfrac{1}{2}\eta(4-\eta)^{1/2}\mu\,,\quad T_{LP(P)}=-\tfrac{1}{2}\sqrt{\eta}\,(1-\eta)\mu\,,\\ &T_{PP(P)}=\tfrac{1}{2}(2+\eta)(4-\eta)^{1/2}\mu\,,\quad T_{NN(P)}=-(4-\eta)^{1/2}\mu\epsilon_N(p_1)\cdot\epsilon_N(p_2)\,,\\ &T_{LN(N)}=-(1-\eta)\mu\epsilon_N(p_2)\cdot\epsilon_N(p_3)\,,\quad T_{PN(N)}=[\eta(4-\eta)]^{1/2}\mu\epsilon_N(p_2)\cdot\epsilon_N(p_3)\,. \end{aligned} \tag{C4}$$

The components that survive at the ZP ($\eta=0$) are $T_{PL(L)}$, $T_{LN(N)}$, and $T_{PP(P)}=T_{NN(P)}$. Using the decomposition Eq. (A2) for the ZP triple-gluon vertex, they have the general expressions

$$\begin{aligned} &T_{PL(L)}=-\mu[1+T_1(-\mu^2)]\,,\\ &T_{LN(N)}=-\mu[1+T_1(-\mu^2)]\epsilon_N(p_2)\cdot\epsilon_N(p_3)\,,\\ &T_{NN(P)}=-2\mu[1+T_1(-\mu^2)-\tfrac{1}{2}T_2(-\mu^2)]\epsilon_N(p_1)\cdot\epsilon_N(p_2)\,. \end{aligned} \tag{C5}$$

The vertex corrections in brackets were used in Sec. IV to define approximate effective charges for the triple-gluon vertex.

The ghost vertex G^{abc}_μ defined in Fig. 10 is, in lowest order, proportional to the momentum of the outgoing ghost:

$$G^{abc}_\mu(p_1,p_2,p_3)=-gf^{abc}(p_2)_\mu\,. \tag{C6}$$

Since this vanishes for a zero-momentum outgoing ghost, we only consider the cases when the momentum p_3 is that of the incoming ghost or the gluon. The corresponding scalar components, denoted by $G_{Lg(g)}$ and $G_{gg(L)}$, respectively, in the case of a longitudinal gluon, are defined by

$$\begin{aligned} &\epsilon^\mu_L(p_1)G^{abc}_\mu(p_3,p_2,p_1)=-gf^{abc}G_{Lg(g)}\,,\\ &\epsilon^\mu_L(p_3)G^{abc}_\mu(p_1,p_2,p_3)=-gf^{abc}G_{gg(L)}\,. \end{aligned} \tag{C7}$$

In lowest order, the scalar components which are nonzero at the η point are

$$\begin{aligned} &G_{Lg(g)}=\tfrac{1}{2}(2-\eta)\mu,\quad G_{Pg(g)}=-\tfrac{1}{2}[\eta(4-\eta)]^{1/2}\mu\,,\\ &G_{gg(L)}=\tfrac{1}{2}\sqrt{\eta}\mu,\quad G_{gg(P)}=\tfrac{1}{2}(4-\eta)^{1/2}\mu\,. \end{aligned} \tag{C8}$$

The components $G_{Lg(g)}$ and $G_{gg(P)}$ survive at the ZP. Their general expressions, in terms of the decompositions in Eq. (A3) for the ghost vertex at the ZP, are

$$\begin{aligned} &G_{Lg(g)}(\mu)=\mu[1+G_1(-\mu^2)]\,,\\ &G_{gg(P)}(\mu)=\mu[1+G_2(-\mu^2)]\,. \end{aligned} \tag{C9}$$

The vertex corrections in brackets were used in Sec. IV to define approximate effective charges for the ghost vertex.

The quark vertex Γ^{aij}_μ defined in Fig. 10 has the additional complication of Dirac matrix structure. In lowest order, when contracted with the polarization vector $\epsilon(p_1)$, it has the form

$$\epsilon^\mu(p_1)\Gamma^{aij}_\mu(p_1,p_2,p_3)=igT^a_{ij}\not\epsilon(p_1)\,. \tag{C10}$$

In higher orders, it can still be resolved into a component proportional to $\not\epsilon$ and another orthogonal to $\not\epsilon$ with respect to the Dirac trace. We choose to use only the former component in defining effective charges. For a zero-momentum quark and a gluon of polarization L, the scalar component $\Gamma_{Lq(q)}$ is defined by

$$\tfrac{1}{4}\,\mathrm{Tr}[\not\epsilon_L(p_1)\epsilon^\mu_L(p_1)\Gamma^{aij}_\mu(p_1,p_2,p_3)]=igT^a_{ij}\Gamma_{Lq(q)}\,. \tag{C11}$$

In terms of the decompositions of the ZP quark vertices given in Eq. (A4), the general expressions for the scalar components at the ZP are

$$\begin{aligned} &\Gamma_{Lq(q)}=1+\Gamma_1(-\mu^2)\,,\\ &\Gamma_{Pq(q)}=\Gamma_{Nq(q)}=1+\Gamma_1(-\mu^2)+\Gamma_2(-\mu^2)\,,\\ &\Gamma_{qq(L)}=\Gamma_{qq(N)}=1+\Gamma_3(-\mu^2)+\Gamma_4(-\mu^2)\,,\\ &\Gamma_{qq(P)}=1+\Gamma_3(-\mu^2)\,. \end{aligned} \tag{C12}$$

These vertex corrections were used in Sec. IV to define approximate effective charges for the quark vertex.

[1]C. G. Bollini and J. J. Giambiagi, Nuovo Cimento 12B, 20 (1972); G. 't Hooft and M. Veltman, Nucl. Phys. B44, 189 (1972); J. F. Ashmore, Lett. Nuovo Cimento 4, 289 (1972); G. M. Cicuta and E. Montaldi, *ibid.* 4, 329 (1972).
[2]G. 't Hooft, Nucl. Phys. B61, 455 (1973).
[3]W. A. Bardeen, A. J. Buras, D. W. Duke, and T. Muta, Phys. D 18, 3998 (1978).
[4]See, e.g., D. J. Gross, in *Méthodes en Théorie des Champs*, edited by R. Balian and J. Zinn-Justin (North-Holland, Amsterdam, 1976).
[5]W. Celmaster and R. J. Gonsalves, Phys. Rev. D 20, 1420 (1979).
[6]W. Celmaster and D. Sivers, Phys. Rev. D 23, 227 (1981).
[7]P. M. Stevenson, Phys. Rev. D 23, 2916 (1981).
[8]See, e.g., C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980), p. 378.
[9]O. V. Tarasov, A. A. Vladimirov, and A. Yu. Zharkov, Phys. Lett. 93B, 429 (1980).
[10]W. Celmaster and R. J. Gonsalves, Phys. Rev. D 21, 3112 (1980); K. G. Chetyrkin, A. J. Kataev, and F. V. Tkachov, Phys. Lett. 85B, 277 (1979); M. Dine and J. Sapirstein, Phys. Rev. Lett. 43, 668 (1979).
[11]R. Barbieri, G. Curci, E. d'Emilio, and E. Remiddi, Nucl. Phys. B154, 535 (1979); K. Hagiwara, C. B. Kim, and T. Yoshin, *ibid.* B177, 461 (1981).
[12]G. Grunberg, Phys. Lett. 95B, 70 (1980).
[13]D. R. T. Jones, Nucl. Phys. B75, 531 (1974).
[14]J. S. Ball and T.-W. Chiu, Phys. Rev. D 22, 2550 (1980); 23, 3085(E) (1980).
[15]S. K. Kim and M. Baker, Nucl. Phys. B164, 152 (1980).
[16]C. Becchi, A. Rouet, and R. Stora, Ann. Phys. (N.Y.) 98, 287 (1976).