SUPERSYMMETRY ALGEBRAS THAT INCLUDE TOPOLOGICAL CHARGES

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We show that in supersymmetric theories with solitons, the usual supersymmetry algebra is not valid; the algebra is modified to include the topological quantum numbers as central charges. Using the corrected algebra, we are able to show that in certain four dimensional gauge theories, there are no quantum corrections to the classical mass spectrum. These are theories for which Bogomolny has derived a classical bound; the argument involves showing that Bogomolny's bound is valid quantum mechanically and that it is saturated.

In this letter, we will describe a novel phenomenon which occurs in supersymmetric theories which possess multiple vacua and topological quantum numbers: the traditionally assumed supersymmetry algebra is not correct, but is modified by the appearance of the topological quantum numbers as central charges [1]. This seems to be a general phenomenon which always occurs in supersymmetric theories which have topological charges.

One striking consequence is that, in certain four dimensional theories, we can determine the exact quantum mechanical mass spectrum. In these theories, it turns out that the classical approximation to the mass spectrum (the tree approximation for "elementary particles"; the semiclassical approximation for solitons) is exact; there are no quantum corrections. For example, in a certain supersymmetric form of the Georgi–Glashow model, with O(3) broken down to U(1) by the Higgs phenomenon, we can determine the exact mass spectrum: the mass of any particle is the vacuum expectation value of the Higgs field times $\sqrt{e^2+g^2}$, e and g being the electric and magnetic charges of that particle.

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This formula was first discovered at the classical level for monopoles and dyons by Prasad and Sommerfield; Bogomolny; Coleman et al.; and Sommerfield [2]. That it is true classically for all states was pointed out by Montonen and Olive [3], who also speculated that it might be exact quantum mechanically. That this formula, in the supersymmetric theory, receives no quantum corrections in the one-loop level has been shown in an explicit calculation by D'Adda et al. [4].

Technically, in these theories the supersymmetry algebra is modified to include central charges because certain surface terms, customarily discarded in deriving the algebra, are actually nonvanishing.

Let us consider first two dimensional examples. The supersymmetric form of a scalar field theory in two dimensions [5] is

$$L = \int \mathrm{d}^2x\, [\tfrac{1}{2}(\partial_\mu\phi)^2 + \tfrac{1}{2}\bar{\Psi}\mathrm{i}\partial\!\!\!/\Psi - \tfrac{1}{2}V^2(\phi) - \tfrac{1}{2}V'(\phi)\bar{\Psi}\Psi]\,, \tag{1}$$

where Ψ is a Majorana fermion, and $V(\phi)$ an arbitrary function. The conserved supersymmetry current is

$$S^\mu = (\partial_\alpha\phi)\gamma^\alpha\gamma^\mu\Psi + \mathrm{i}V(\phi)\gamma^\mu\Psi. \tag{2}$$

Working with chiral components $\Psi^\pm$ of the fermi field, the chiral components $Q^\pm$ of the supersymmetry charge can be written

$$Q_+ = \int \mathrm{d}x |(\partial_0\phi + \partial_1\phi)\Psi_+ - V(\phi)\Psi_-|,$$
$$Q_- = \int \mathrm{d}x |(\partial_0\phi - \partial_1\phi)\Psi_- + V(\phi)\Psi_+|. \tag{3}$$

In this notation, the standard supersymmetry algebra is $Q_+^2 = P_+$, $Q_-^2 = P_-$, $Q_+Q_- + Q_-Q_+ = 0$, where $P_\pm = P_0 \pm P_1$. Of these statements, the first two are quite correct, but the third must be considered carefully.

Keeping surface terms, we find

$$Q_+Q_- + Q_-Q_+ = \int \mathrm{d}x\ 2V(\phi)\ \partial\phi/\partial x, \tag{4}$$

which can also be written

$$Q_+Q_- + Q_-Q_+ = \int \mathrm{d}x\,(\partial/\partial x)(2H(\phi)), \tag{5}$$

where $H(\phi)$ is a function such that $H'(\phi) = V(\phi)$. Thus $Q_+Q_- + Q_-Q_+$ is the integral of a total divergence, and naively would vanish. But in a soliton state, the right hand side of eq. (5) is not necessarily zero.

In fact, letting T be the operator on the right hand side of eq. (5), a matrix element of T is the difference between the expectation value of $2H(\phi)$ at $x = +\infty$, and its expectation value at $x = -\infty$.

For a typical example, we may let $V = -\lambda(\phi^2 - a^2)$, so the potential energy is $\lambda^2(\phi^2 - a^2)^2$. The theory has two ground states, $\phi = \pm a$. $H = a^2\lambda\phi - \frac{1}{3}\lambda\phi^3$, and

$$T = \int_{-\infty}^{\infty} \mathrm{d}x \frac{\partial}{\partial x}(2a^2\lambda\phi - \tfrac{2}{3}\lambda\phi^3). \tag{6}$$

T vanishes in a topologically trivial state, and has a positive value in a kink state, a negative value in an antikink state. Although apparently different from the usual topological charge $\int_{-\infty}^{\infty} \mathrm{d}x(\partial\phi/\partial x)$, T actually coincides with it, since both depend only on the topology.

Or we may take $V(\phi) = -\sin\phi$, which describes the supersymmetric sine-Gordon equation. Then $H(\phi) = \cos\phi$ and

$$T = \int_{-\infty}^{\infty} \mathrm{d}x \frac{\partial}{\partial x}(2\cos\phi). \tag{7}$$

Now T does differ essentially from the usual topological charge $T' = \int_{-\infty}^{\infty} \mathrm{d}x(\partial\phi/\partial x)$. T' counts the soliton number, but T counts the soliton number only modulo two. In fact, the vacuum energy $\sin^2\phi$ is periodic modulo π, while H has periodicity 2π. Thus T, while not annihilating a one soliton state, will annihilate a two soliton state just as it annihilates the vacuum.

Our modified algebra,

$$Q_+^2 = P_+, \quad Q_-^2 = P_-, \quad Q_+Q_- + Q_-Q_+ = T, \tag{8}$$

has an interesting consequence.

From eq. (8)

$$P_+ + P_- = T + (Q_+ - Q_-)^2,$$
$$P_+ + P_- = -T + (Q_+ + Q_-)^2. \tag{9}$$

But $(Q_+ \pm Q_-)^2 \geqslant 0$, so $P_+ + P_- \geqslant |T|$. For a single particle of mass M at rest, $P_+ = P_- = M$ and eq. (9) implies

$$M \geqslant \tfrac{1}{2}|T|. \tag{10}$$

When is eq. (10) saturated? From eq. (9), it is clearly saturated precisely for those states $|\alpha\rangle$ such that $(Q_+ + Q_-)|\alpha\rangle = 0$ or $(Q_+ - Q_-)|\alpha\rangle = 0$. This condition may seem rather exceptional, but actually it is satisfied, at least classically, for all the soliton and antisoliton states of these models. (Indeed, a soliton for lagrangian (1) is a classical field ϕ with $\partial\phi/\partial t = 0$ and $\partial\phi/\partial x = \pm V(\phi)$. From eq. (3) one sees that such a state is annihilated by $Q_- \pm Q_+$.)

In fact, explicit calculation shows that, classically, the solitons satisfy $M = \frac{1}{2}|T|$.

Whether this will be true quantum mechanically for the soliton state $|\alpha\rangle$ depends on whether $(Q_+ \pm Q_-)|\alpha\rangle = 0$ (for some choice of the sign) even in quantum mechanics. While we suspect that this is true, we have no proof. In the analogous situation in four dimensions, we will be able to offer a (not quite rigorous) proof.

It is known, however, from an explicit calculation that $M = \frac{1}{2}|T|$ remains valid for solitons and antisolitons at the one-loop order [6].

Note that eq. (10) is Bogomolny's classical bound, here shown to be true in quantum field theory.

A manifestly Lorentz invariant way to derive eq. (10) is to note that, from eq. (8), the mass squared operator $M^2 = P_+P_- = P_-P_+$ can be written

$$M^2 = \tfrac{1}{4}(T^2 + (\bar{Q}Q)^2), \tag{11}$$

where $\bar{Q}Q$ is the Hermitian operator $\mathrm{i}(Q_+Q_- - Q_-Q_+)$. Since $(\bar{Q}Q)^2$ is positive, this establishes that $M^2 \geqslant \frac{1}{4}T^2$, a formula saturated only for states $|\alpha\rangle$ such that $\bar{Q}Q|\alpha\rangle = 0$. In the rest frame $\bar{Q}Q = \mathrm{i}(Q_+ - Q_-)(Q_+ + Q_-) = -\mathrm{i}(Q_+ + Q_-)(Q_+ - Q_-)$ and so annihilates any state that is annihilated by $(Q_+ + Q_-)$ or $(Q_+ - Q_-)$.

Turning now to four dimensions, it is the electric and magnetic charges (and their generalizations) that will appear as central charges.

We do not usually think of the electric charge as a boundary term, but using Gauss's law it can be written as one: $E = \int \mathrm{d}^3x\ \partial_i F_{0i}$. Likewise, for the magnetic charge, if this expression were not identically zero, we would write $G = \frac{1}{2}\int \mathrm{d}^3x\ \epsilon_{ijk}\,\partial_i F_{jk}$.

The analogous expression in a Higgs theory is

$$E = \frac{1}{\langle A\rangle}\int \mathrm{d}^3x\ \partial_i(A^aF^a_{0i}),$$
$$G = \frac{1}{\langle A\rangle}\int \mathrm{d}^3x\ \epsilon_{ijk}\partial_i(A^aF^a_{jk}), \tag{12}$$

where A^a is the Higgs field whose direction determines what is "electromagnetism". The magnetic charge is now not identically zero; there may be magnetic monopoles [7]. It is in the form (12) that the electric and magnetic charges will appear in the supersymmetry algebra.

The usual supersymmetry algebra in four dimensions is

$$\{Q_{\alpha i}, \bar{Q}_{\beta j}\} = \delta_{ij}\gamma^\mu_{\alpha\beta}P_\mu, \tag{13}$$

where $Q_{\alpha i}$, $i = 1, \ldots, N$, are N Majorana fermions. Haag et al. [1] showed that this algebra can be modified to include central charges; the most general form that is possible is

$$\{Q_{\alpha i}, \bar{Q}_{\beta j}\} = \delta_{ij}\gamma^\mu_{\alpha\beta}P_\mu + \delta_{\alpha\beta}U_{ij} + (\gamma_5)_{\alpha\beta}V_{ij}, \tag{14}$$

where $U_{ij} = -U_{ji}$ and $V_{ij} = -V_{ji}$; U and V are the central charges.

We will consider here the simplest four dimensional model in which boundary terms enter as central charges. It is the $N = 2$ Yang–Mills theory [8], with lagrangian

$$L = \int \mathrm{d}^4x\,[-\tfrac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \tfrac{1}{2}\bar{\Psi}^a_i\,\mathrm{i}\not{D}\Psi^a_i + \tfrac{1}{2}D_\mu A^aD_\mu A^a$$
$$+ \tfrac{1}{2}D_\mu B^aD_\mu B^a + \tfrac{1}{2}g^2\,\mathrm{Tr}[A,B]\,[A,B]$$
$$+ \tfrac{1}{2}\mathrm{i}g\,\epsilon_{ij}\,\mathrm{Tr}([\bar{\Psi}^i, \Psi^j]A + [\bar{\Psi}^i, \gamma_5\Psi^j]B)], \tag{15}$$

where Ψ_i, $i = 1, 2$, are two Majorana fermions, A is a scalar field and B a pseudoscalar field, all in the adjoint representation of the gauge group.

A very important property of this model is that the vacuum energy is independent of the values of A and B in certain directions in field space. As long as A and B commute, the vacuum energy is, classically, zero. This property persists quantum mechanically [9] because of the supersymmetry, and therefore A and B may have nonzero vacuum expectation values, spontaneously breaking some of the gauge symmetries.

We will want to consider as an example the gauge group O(3). In O(3), B and A, to commute, must be proportional, and the vacuum expectation value of B may be set to zero by a chiral rotation. A nonzero vacuum expectation value of A will, as in the Georgi–Glashow model, spontaneously break O(3) down to U(1).

The supersymmetry current for this model is

$$S_{\mu i} = \mathrm{Tr}(\sigma^{\alpha\beta}F_{\alpha\beta}\gamma_\mu\Psi_i + \epsilon_{ij}D_\alpha A\gamma^\alpha\gamma_\mu\Psi_j$$
$$+ \epsilon_{ij}D_\alpha B\gamma^\alpha\gamma_5\Psi_j + g\gamma_\mu\gamma_5[A,B]\Psi_i). \tag{16}$$

If one calculates from eq. (16) the supersymmetry charges and their anticommutators, paying attention to boundary terms, one finds that the following operators appear in the supersymmetry algebra:

$$U = \int \mathrm{d}^3x\ \partial_i(A^aF^a_{0i} + B^a\,\tfrac{1}{2}\epsilon_{ijk}\,F^a_{jk}),$$
$$V = \int \mathrm{d}^3x\ \partial_i(A^a\,\tfrac{1}{2}\epsilon_{ijk}\,F^a_{jk} + B^aF^a_{0i}). \tag{17}$$

The supersymmetry algebra becomes

$$\{Q_{\alpha i}, \bar{Q}_{\beta j}\} = \delta_{ij}\gamma^\mu_{\alpha\beta}P_\mu + \epsilon_{ij}(\delta_{\alpha\beta}U + (\gamma_5)_{\alpha\beta}V). \tag{18}$$

U and V can be nonvanishing if and only if the vacuum expectation value $\langle A\rangle$ or $\langle B\rangle$ is nonzero. For F_{0i} and F_{jk} will always vanish at least as $1/|x|^2$ for large $|x|$, so nonzero integrals in eq. (17) are possible only if A or B is nonvanishing as $|x| \to \infty$.

From eq. (18) one can derive an inequality for the masses analogous to eq. (10); it will be simply a quantum version of the Bogomolny bound. Eq. (18) implies that, for each particle state, the mass M and the values of U and V are related by

$$M^2 \geqslant U^2 + V^2. \tag{19}$$

Perhaps the easiest way to derive eq. (19) is to work noncovariantly. In the rest frame, the only nonzero component of P_μ is P_0, which equals the mass M. In view of the chiral invariance of eq. (18), which enables us to rotate V into U, it is sufficient for us to consider the case $V = 0$. Then eq. (18) reads

$$\delta_{ij}\delta_{\alpha\beta}M + \epsilon_{ij}\gamma^0_{\alpha\beta}U = \{Q_{\alpha i}, Q_{\beta j}\}. \tag{20}$$

Viewed as an 8×8 matrix in (αi) space, $\{Q_{\alpha i}, Q_{\beta j}\}$ is positive definite. Therefore $\delta_{ij}\delta_{\alpha\beta}M + \epsilon_{ij}\delta^0_{\alpha\beta}U$ must be positive definite. Since $\epsilon_{ij}\gamma^0_{\alpha\beta}U$ has eigenvalues ± 1, this means $M \geqslant |U|$. In the general case, V not necessarily zero, this yields eq. (19).

To make the meaning of eq. (19) more clear, let us consider the special case of an O(3) gauge theory, the "Georgi–Glashow" model. As we have mentioned, $\langle B\rangle$ may be assumed to vanish while a nonzero $\langle A\rangle$ spontaneously breaks O(3) down to U(1). Comparing eqs. (17) and (12), we see that $U = \langle A\rangle E$, $V = \langle A\rangle G$, where E and G are the electric and magnetic charges of the unbroken U(1) gauge group. Eq. (19) becomes

$$M \geqslant \langle A\rangle \sqrt{E^2 + G^2}. \tag{21}$$

When are eqs. (19) and (21) saturated? Here we encounter the striking fact, pointed out before [3], that at the classical level, eq. (21) is saturated for all states in the theory. Photon, Higgs particles, fermions, W bosons, magnetic monopoles, dyons – they all satisfy $M = \sqrt{U^2 + V^2}$ or (for O(3)) $M = \langle A\rangle\sqrt{E^2 + G^2}$.

For example, the W boson in the O(3) theory has electric charge $E = e$, the gauge coupling constant, and magnetic charge $G = 0$. So eq. (21), if exactly saturated, says $M_W = e\langle A\rangle$. But this is the well-known Higgs formula for the W Mass.

Why are eqs. (19) and (21) saturated classically and is this likely to be true in quantum mechanics?

It is well known that the representations of the standard supersymmetry algebra are smaller for massless particles than for massive particles. An irreducible representation of eq. (13) has 2^N (helicity) states for zero mass, but 2^{2N} states for nonzero mass. For $N = 2$, an irreducible representation of eq. (13) has four states if $M = 0$ but sixteen states if $M \neq 0$.

When $\langle A\rangle$ or $\langle B\rangle$ does not vanish, some particles get masses. On the basis of eq. (13), the number of particle states would have to greatly increase, to provide massive representations of the algebra. But this does not happen; we know that when the Higgs phenomenon occurs, some particles get masses, and some scalars become the longitudinal components of vectors, but the total number of states does not change. This discrepancy with the expectation based on eq. (13) can be regarded as further evidence that eq. (13) is not correct.

Does the replacement of eq. (13) by eq. (18) solve the problem? Do irreducible representations of eq. (18) have four states, or sixteen?

The answers turns out to be: four states when eq. (19) is saturated; sixteen when it is not [‡1].

This can be seen by repeating the analysis of ref. [10] for algebras with central charges. To understand roughly why it is true, recall the derivation of eq. (19). We said $\{Q_{\alpha i}, Q_{\beta j}\}$ is a positive definite matrix in (αi) space. Eq. (19) can be saturated only if this matrix has some zero eigenvalues. Zero eigenvalues means that some linear combinations of the $Q_{\alpha i}$ are zero (that is, they annihilate the quantum states for which eq. (19) is saturated). If some of the Q's are zero there are fewer that must be represented, and the representations are likely to be smaller. Following ref. [10], one sees that they are much smaller: four states instead of sixteen.

We would not trust perturbation theory to determine the exact spectrum of particle masses. But we usually would trust perturbation theory, and semiclassical reasoning, to answer qualitative questions like the *number* of particle states and which representations of symmetry algebras appear. If we trust perturbation theory to count the "elementary particle" states, and the semiclassical reasoning of Jackiw and Rebbi [11] to count the monopole and dyon states, then we can say that the supersymmetry representations are fourfold rather than sixteen-fold. If so, eq. (19) must be saturated.

These theories are thus apparently the first examples of three space dimensional theories whose mass spectra are exactly known [‡2].

[‡1] This result has been obtained previously by P. Fayet and by B. Zumino.

[‡2] See following page.

Theories in which there are no quantum corrections to the classical mass spectrum are often completely integrable at the classical level. Perhaps these theories are in that category.

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‡2 Perhaps we have not determined the complete mass spectrum, since there may be additional bound states. We claim to have the exact masses of the known particles.

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