Axial-Anomaly-Induced Fermion Fractionization and Effective Gauge-Theory Actions in Odd-Dimensional Space-Times

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A new quantum field-theoretical technique is developed and used to explore the relationship between even-space-time-dimensional axial anomalies and background-field-induced fermion numbers and Euler-Heisenberg effective actions in odd-dimensional space-times.

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It is by now well known that topological objects possess surprising quantum properties such as fractional fermion number.[1-8] Examples exist in $1+1$ dimensions where kinks (solitons) provide the topologically nontrivial setting and in $3+1$ dimensions where magnetic monopoles are used as the background field. An important experimental realization of this phenomenon has already appeared in the physics of linearly conjugated polymers.[2,3]

In general, fermion fractionization is amenable to an elegant mathematical analysis: In a system where fermions interact with classical background fields the induced fermion number is the spectral asymmetry of the pertinent Dirac Hamiltonian; schematically,

$$N = -\tfrac{1}{2}\{ \sum_{E_n > 0} 1 - \sum_{E_n < 0} 1 \} \tag{1}$$

(Ref. 7), where E_n are the energy eigenvalues. Various techniques have been developed to compute (1)[4-8] and quite recently a direct topological analysis has been applied to several $(1+1)$-[6] and $(3+1)$-dimensional[7] field-theoretical models. This approach is based on the derivation of certain trace identities which relate N to an asymptotic surface integral, and the result involves only the topological properties of the background fields.

In an independent line of study it has been found that odd-space-time-dimensional gauge field theories can exhibit unexpected structures such as the appearance of topological mass terms in $2+1$ dimensions.[9,10] These extra terms, related to Chern-Simons secondary characteristic classes, were originally introduced by requiring gauge symmetry but have recently been observed to arise through radiative corrections.[11]

Here we shall explore the relationship between fermion-number fractionization and the topological properties generic to odd-space-time-dimensional gauge theories. As an illustration of the general structure we shall first analyze the $(2+1)$-dimensional QED. We then indicate how this analysis can be generalized to higher odd-dimensional space-times with non-Abelian gauge fields. Applications envisaged involve the Abrikosov-Nielsen-Olesen vortex which appears naturally in the physics of type-II superconductors, and also the string theory of hadrons and certain cosmological scenarios where stringlike structures appear. Furthermore, three-dimensional Euclidean bosonic field theories may represent the high-temperature behavior of $(3+1)$-dimensional quantum field theories.

We shall compute the fermion number induced by static classical background fields. In the course of this calculation we show that the trace-identity analysis presented in Refs. 6 and 7 is not always applicable. The operators involved are singular and in general exhibit anomalies. In the $(2+1)$-dimensional case these anomalies are governed by the two-dimensional Abelian Euclidean axial-anomaly equation[12]:

$$(i/2)\,\partial^k \operatorname{tr}\{\gamma^k\gamma^5 D(x,x)\} = M \operatorname{tr}\{\gamma^5 D(x,x)\} + (e/4\pi)\epsilon^{ij}F_{ij}\,. \tag{2}$$

(The effective dimensional reduction from $2+1$ to 2 dimensions is a consequence of time translation invariance of the background field.) Here $D(x,y)$ is the (regulated) Euclidean Dirac propagator,

$$(i\partial\!\!\!/ + eA\!\!\!/ - M)D(x,y) = \delta(x-y), \tag{3}$$

and γ^i $(i=1,2)$ and γ^5 are the two-dimensional γ

matrices. In the models studied in Refs. 6 and 7 the anomaly term is absent since there are no odd-dimensional anomalies. However, we find that here the entire fractional fermion number arises from this term. It is the two-dimensional Pontryagin density; thus the relationship between (1) and the topology of background fields. We shall use a simple covariance argument to deduce a low-momentum approximation of the induced current, and therefore the Euler-Heisenberg effective action. This effective action contains the topological mass term for (2 + 1)-dimensional gauge fields.[9,10]

The (2 +1)-dimensional QED Lagrangian is

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ + eA\!\!\!/ - m)\psi. \tag{4}$$

We assume that $A_\mu(x)$ is a static classical background field and $A_0 = 0$. The magnetic flux is

$$\Phi = (1/4\pi)\int d^2x\, \epsilon^{ij}F_{ij} = (1/2\pi)\oint dx^i A_i. \tag{5}$$

(In the case of vortices in the Abelian-Higgs model we would have $\Phi = n/q$, where n is the topological quantum number of the vortex and q is the charge of the scalar field.) We first verify (1) and then evaluate the induced fermion number density and the fermion number.

Consider the current

$$j^\mu(x) = \tfrac{1}{2}[\bar{\psi}, \gamma^\mu\psi]. \tag{6}$$

By introducing the causal propagator

$$iS(x,y) = \theta(x^0 - y^0)\langle\psi(x)\bar{\psi}(y)\rangle - \theta(y^0 - x^0)\langle\bar{\psi}(y)\psi(x)\rangle = i\theta(x^0 - y^0)S_>(x,y) - i\theta(y^0 - x^0)S_<(x,y), \tag{7}$$

we find

$$\delta\Gamma/\delta A_\mu = e\langle j^\mu(x)\rangle = -e(i/2)\operatorname{tr}\{S_>(x,x)\gamma^\mu + S_<(x,x)\gamma^\mu\}. \tag{8}$$

Here Γ is the effective action for the gauge fields obtained by eliminating the Fermi fields by functional integration, and the trace is over Dirac indices. The propagator (7) satisfies

$$(i\partial\!\!\!/ + eA\!\!\!/ - m)S(x,y) = \delta(x-y). \tag{9}$$

By defining

$$S(x,y) = \int_C (dE/2\pi)\exp[-iE(x^0 - y^0)]S_E(x,y), \tag{10}$$

where C is a causal contour in the complex E plane, and by performing a Wick rotation $E \to iE$, we find

$$\langle j^\mu(x)\rangle = -\tfrac{1}{2}\lim_{\epsilon\to 0^+}\int (dE/2\pi)e^{-iE\epsilon}\operatorname{tr}\{S_E(xx)\gamma^\mu + S_{-E}(xx)\gamma^\mu\}. \tag{11}$$

We define

$$H\psi_n = E_n\psi_n, \quad H = \alpha^k(i\nabla^k + eA^k) + \beta m, \tag{12}$$

where $\beta = \gamma^0$, $\alpha^k = \gamma^0\gamma^k$, and H is the Dirac Hamiltonian of the system (4). We get

$$\langle j^0(x)\rangle = -\tfrac{1}{2}\lim_{\epsilon\to 0^+}\sum_{E_n}\exp(-|E_n|\epsilon)\psi_n^\dagger(x)\psi_n(x)\operatorname{sgn}\{E_n\} \tag{13}$$

and

$$N = \int d^2x\langle j^0(x)\rangle = -\tfrac{1}{2}\lim_{\epsilon\to 0^+}\sum_{E_n}\exp(-|E_n|\epsilon)\operatorname{sgn}\{E_n\}. \tag{14}$$

Equation (14) is a heat-kernel regularization of (1). We shall now evaluate (13) and (14). For this we observe that

$$\operatorname{tr}\{S_E(xx)\gamma^0 + S_{-E}(xx)\gamma^0\} = 2(m/\sigma)\operatorname{tr}\langle x|\gamma^0/(i\partial\!\!\!/ + eA\!\!\!/ - \sigma)|x\rangle, \tag{15}$$

where $\sigma^2 = m^2 + E^2$. We then identify $M = \sigma$ and $\gamma^5 = \gamma^0$ in (2) and get

$$\langle j^0(x)\rangle = \frac{1}{2}\int\frac{dE}{2\pi}\left\{i\frac{m}{\sigma^2}\partial^k\operatorname{tr}[\gamma^k\gamma^5 D(x,x)] - \frac{m}{\sigma^2}\frac{e}{2\pi}\epsilon^{ij}F_{ij}\right\}. \tag{16}$$

By evaluating the integral over two-space in (14) we find that the contribution from the first term in

(16) vanishes and we have

$$N = \frac{e}{4\pi}\int \frac{dE}{2\pi}\,\frac{m}{m^2+E^2}\int d^2x\,\epsilon^{ij}F_{ij} = \tfrac{1}{2}e\Phi, \quad (17)$$

which is the relation between the fractional fermion number and the magnetic flux. It should be noted that N is directly proportional to the parameters: In odd space-time dimensions there is no chirality. Hence the extra degree of freedom necessary for more complicated relations[4,7,8] is missing.

By covariance we conclude that the part of (16) responsible for fractional fermion number is

$$\langle j^{\mu}(x)\rangle = (e/8\pi)\epsilon^{\mu\nu\sigma}F_{\nu\sigma}\,, \quad (18)$$

which is gauge invariant and conserved. If we evaluate (16) with a space-time constant F_{ij} we find that the first term vanishes. Consequently in the low-momentum limit (11) is exactly given by (18). Substituting (18) into (8) we find the functional antiderivative with respect to A_{μ}. The result is the Euler-Heisenberg effective action

$$\Gamma = \int d^3x\{-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + (e^2/16\pi)\epsilon^{\mu\nu\sigma}F_{\mu\nu}A_{\sigma}\}. \quad (19)$$

Notice that the last, radiatively induced term is the topological mass term (Chern-Simons invariant) studied in Refs. 9 and 10. We conclude that, if absent at the classical level, this term is generated through quantum corrections.[11] Consequently the theory (4) exhibits a mass gap necessary for confinement.[13,14] Furthermore, the existence of this mass gap, even as $m \to 0$, indicates a dynamical breaking of parity.[9-11] Finally, we also observe that if this effective bosonic field theory supports topological excitations they will carry a fermion number given by (17).

We can readily extend the previous results to arbitrary odd-dimensional space-times with non-Abelian background fields. As an example we consider a (4+1)-dimensional SU(N) invariant gauge theory with N_f Dirac fermions. As before we assume a static magnetic background gauge field with $A_0=0$. The four-dimensional non-Abelian Euclidean axial-anomaly equation is[15]

$$(i/2)\,\partial^k\,\mathrm{tr}\{\gamma^k\gamma^5\lambda^a D(x,x)\} = \tfrac{1}{2}\mathrm{tr}[\{\gamma^5\lambda^a, -e\not{A}+M\}D(x,x)] + (N_f g^2/16\pi^2)\,\mathrm{tr}\{\lambda^a\,{}^*F^{ij}F_{ij}\}. \quad (20)$$

In order to apply this equation we must restrict the background gauge field so that

$$[\lambda^a, A_i(x)] = 0. \quad (21)$$

We then deduce the general result by requiring both Lorentz and gauge covariance. With (21), Eq. (15) remains valid and repeating the previous steps we find

$$\langle j^a{}_0(x)\rangle = -(N_f g^2/64\pi^2)\,\mathrm{tr}\{\lambda^a\,{}^*F^{ij}F_{ij}\}. \quad (22)$$

The induced non-Abelian charge is

$$Q^a = -(N_f g^2/64\pi^2)\int d^4x\,\mathrm{tr}\{\lambda^a\,{}^*F^{ij}F_{ij}\}. \quad (23)$$

In the case of an Abelian anomaly equation we simply set $\lambda^a=1$ in (20) and (23) reduces to

$$N = -\tfrac{1}{4}N_f q, \quad (24)$$

where q is the Pontryagin index of the background field.

By implementing both Lorentz and gauge covariance in (22) we deduce the low-momentum approximation of the current,

$$\langle j^a{}_{\mu}(x)\rangle = -(N_f g^2/128\pi^2)\epsilon_{\mu\nu\rho\sigma\delta}\,\mathrm{tr}\{\lambda^a F^{\nu\rho}F^{\sigma\delta}\}. \quad (25)$$

This current then yields the Euler-Heisenberg effective action[16]

$$\Gamma = \int d^5x\{-\tfrac{1}{4}\mathrm{tr}[F^{\mu\nu}F_{\mu\nu}] - (N_f g^2/96\pi^2)\epsilon^{\alpha\beta\gamma\delta\eta}A_{\alpha}\partial_{\beta}A_{\gamma}\partial_{\delta}A_{\eta} + \tfrac{3}{2}A_{\alpha}A_{\beta}A_{\gamma}\partial_{\delta}A_{\eta} + \tfrac{3}{5}A_{\alpha}A_{\beta}A_{\gamma}A_{\delta}A_{\eta}\}. \quad (26)$$

Notice that we have arrived at a finite result in spite of the nonrenormalizability of the original field theory. The radiatively induced term also has a topological interpretation: It is the Chern-Simons invariant in five dimensions.

In Ref. 11 it has been observed that Witten's anomaly[17] is also present in (2+1)-dimensional SU(2) gauge theory. Consistency with the results of Refs. 9 and 10 requires that the number of Dirac fermions, N_f, must be even. It is straightforward to check from (26) that this is also true in 4+1 dimensions.

The analysis presented here together with the results in Ref. 11 then suggest the following relations between axial anomalies in general space-time dimensions: In $2n+2$ dimensions the Abelian anomaly term is given by the Pontryagin density which is a total divergence of a current. Take that component of this current that does not belong to a given $(2n+1)$-dimensional subspace. This gives the $(2n+1)$-dimensional Chern-Simons invariant.[9,10] The $2n$-dimensional non-Abelian anomaly term is then a functional derivative of the $(2n+1)$-dimensional Chern-Simons invariant with respect to that component of the gauge field that does not belong to the $2n$-dimensional subspace. Furthermore, the radiatively induced $(2n+1)$-dimensional Euler-Heisenberg effective action is the pertinent Chern-Simons invariant which is entirely responsible for the fermion number in that dimensionality. We have shown that this conjecture is true for $n=1,2$ and expect that our result is valid for all integer values of n.[18]

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After completing this manuscript we became aware of an investigation by B. Zumino, Wu Yang-Shi, and A. Zee (to be published) who study left-right anomalies and arrive at relations between axial anomalies in different dimensions that are very similar to the ones found here for $V-A$ anomalies.

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