4-Dimensional Spacetime:

For Newton, $\vec{x}$ and t were separate entities of absolute meaning to all observers. In other words, two events that were simultaneous for one observer, were simultaneous for all observers. This implies that coordinate transformations between observers were

$$\vec{x}' = \vec{x}'(\vec{x})$$
$$t' = t$$

which, if the relative velocity between the two was constant, $\vec{v}$,

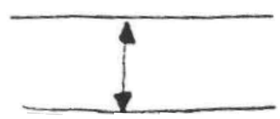
$$\vec{x}' = \vec{x} - \vec{v}\,t$$
$$t' = t$$

In particular, the speed of an object, measured to be $\vec{u}$ by one of the observers would be

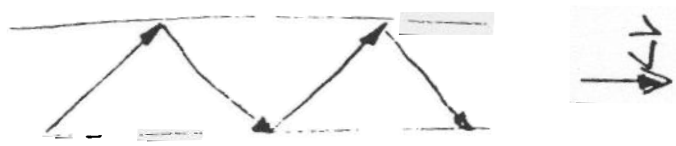
$$\frac{d\vec{x}'}{dt} = \vec{u}' = \frac{d\vec{x}}{dt} - \vec{v} \Rightarrow \vec{u}' = \vec{u} - \vec{v}$$

the surprise comes when one considers light which seems to violate this simple transformation law. Speed of light, c, in vacuum is a constant for all observers. If c is also maximum speed at which information can be transferred, then this implies that time and space coordinates must depend on the state of motion of the observer.

Imagine, for example, a clock constructed by letting a photon bounce between two parallel mirrors.



If this clock is carried by someone moving relative to the first observer



Since both observers measure the same speed for the photon but the length of the path is longer for the moving observer, the observer "at rest" concludes that the moving clock runs slower. In other words, time for the moving observer runs slower.

Einstein realized that time and space are intimately related, so that the coordinates measured by different observers is the Lorentz transformation

$$dx' = \gamma \left(dx - \frac{v}{c} c\,dt\right)$$

$$c\,dt' = \gamma \left(c\,dt - \frac{v}{c} dx\right)$$

This implies that there is a fundamental notion of distance between in 4-spacetime, that is agreed upon by all observers_

$$ds^2 = c^2 dt^2 - d\vec{x}^2 = c^2 dt'^2 - d\vec{x}'^2$$

This is referred to as the geometrical measure, or "metric" of space time, which in the general case is a 2-tensor, $\eta_{\alpha\beta}$, so that

$$ds^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta \qquad \alpha, \beta = 0, .., 3$$, summation convention assumed)

and

$$\eta_{\alpha\beta} = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}$$ in Cartesian coordinates_

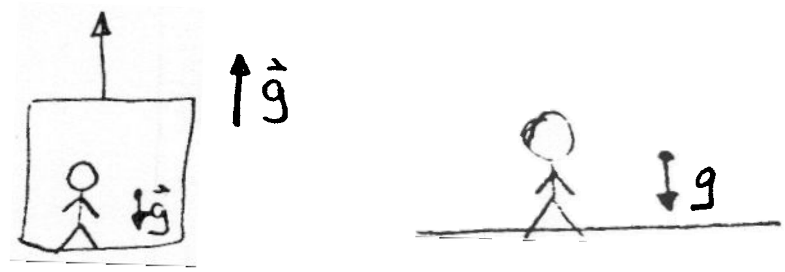
(

GENERAL RELATIVITY

Now for GR. Newton's laws of motion are rather circular, since they state that $\vec{F} = m\vec{a}$ if motion is described from an inertial frame, but an inertial frame is that where $\vec{F} = m\vec{a}$ is valid. the situation, however, is not desperate - For example, in a uniformly accelerated frame, Newton's laws could be easily modified to read

$$\vec{F} = m(\vec{a} + \vec{g})$$

so that a "free" particle will experience a force $= m\vec{g}$. Einstein genius was to realize that this is exactly the same force the mass would experience in a uniform gravitational field g.



But note that this, from symmetry considerations, means that a "free-falling" observer in a uniform grav. field is equivalent to an "elevator" at rest or moving with constant speed, i.e., an inertial frame.

This is the <u>equivalence principle</u>, which states that a freely falling observer experiences no grav. forces (locally). His spacetime geometry is Minkowski. This is true only locally; presence of gravity can be deduced from tidal forces.

—o—

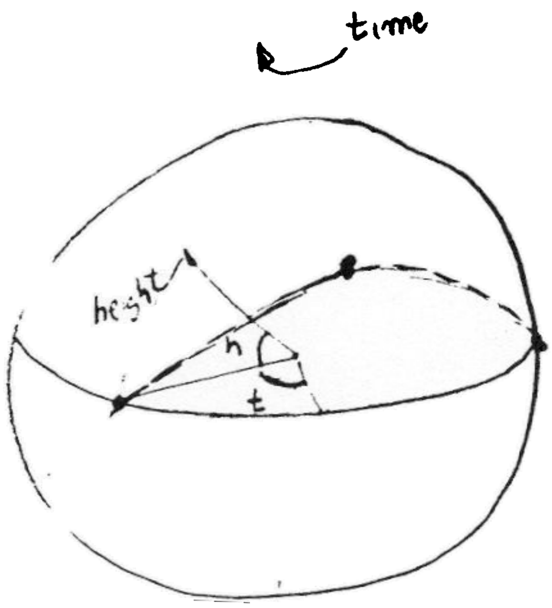
Next step is to consider the eq. of motion of a free particle in a grav. field. In a local Minkowski reference frame the particle feels no acceleration

$$\frac{d^2 \xi^\mu}{d\tau^2} = 0 \qquad \xi^\mu = (ct, x, y, z)$$

(τ is the proper time of the particle)

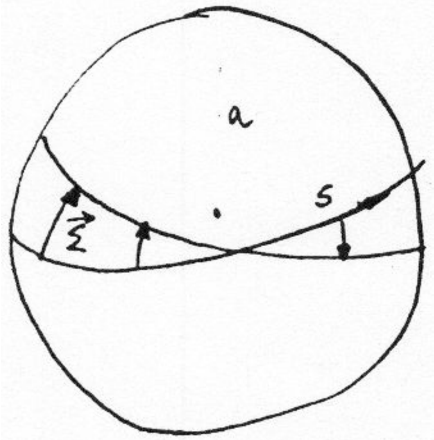
and $$ds^2 = c^2 d\tau^2 = \eta_{\alpha\beta}\, d\xi^\alpha\, d\xi^\beta$$

In the general case the local "freely falling" reference frame will change from point to point, and the particle will move along a geodesic in space time. If space time is globally non-Minkowski funny things may happen. Consider, for example, throwing a rock upwards.

Rock moves along a geodesic, but because space time is curved, rock reaches a max. height and then falls back to its original position. Note that no "forces" are acting on the rock. Its motion is the natural evolution of a free particle drifting away in a curved space time.

So in order to characterize motion we need to characterize the geometry of space time. This is usually done by looking at the behaviour of nearby geodesics. For example, in a sphere, the equation that governs the separation of 2 geodesics is.



two geodesics

$$\frac{d^2 \xi}{ds^2} + R\,\xi = 0$$

where $R = \frac{1}{a^2}$ is the "curvature" of the sphere.

Note:
- this is a second order equation
- If $R = 0$ distance between geodesics increase (decrease) linearly with s (analog of two particles moving with constant speeds).
- R fixes fully the curvature characteristics of this geometry

Actually, the orientation of the vector $\vec{\xi}$ is also important, so that in general the geodesic deviation equation is

$$\frac{d^2 \xi^\alpha}{d\tau^2} + R^\alpha{}_{\beta\gamma\delta} \frac{dx^\beta}{d\tau} \xi^\gamma \frac{dx^\delta}{d\tau} = 0$$

$R^\alpha{}_{\beta\gamma\delta}$ is the Riemann tensor. It characterizes the change in a vector "parallel-transported" around a small closed loop.

.led by non-zero second
of this curvature, and this

ontracted to the Ricci tensor

(1.61)

and different signs can arise
pted (see below). All authors,
:

(1.62)

divergence [problem 1.6]:

$= 0.$ (1.63)

by virtue of the conservation
it the two are proportional:

(1.64)

re the correct constant of
by nsidering the weak-field

First, however, we ought to
tensor. This has considerable
space: it is the fact that the
neral relativity interpretation

d? Even the simple case of a
ould agree that the surface of
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iangles of size comparable to

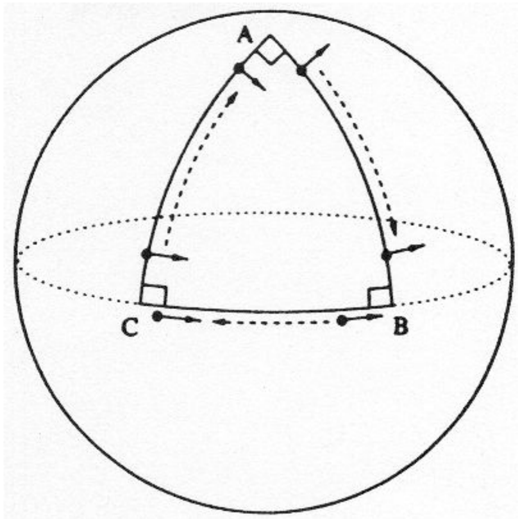


Figure 1.2. This figure illustrates the parallel transport of a vector around the closed loop ABC on the surface of a sphere. For the case of the spherical triangle with all angles equal to 90°, the vector rotates by 90° in one loop. This failure of vectors to realign under parallel transport is the fundamental signature of spatial curvature, and is used to define the affine connection and the Riemann tensor.

the radius of the sphere sample the curvature of the space, and the angular sum starts to differ from the Euclidean value.

To generalize this process, the concept of **parallel transport** is introduced. Here, imagine an observer travelling along some path, carrying with them some vector that is maintained parallel to itself as the observer moves. This is easy to imagine for small displacements, where a locally flat tangent frame can be used to apply the Euclidean concept of parallelism without difficulty. This is clearly a reversible process: a vector can be carried a large distance and back again along the same path, and will return to its original state. However, this need not be true in the case of a loop where the observer returns to the starting point along a different path, as illustrated in figure 1.2. In general, parallel transport around a loop will cause a change in a vector, and it is this that is the intrinsic signature of a curved space. We can think of the effect of parallel transport as producing a change in a vector proportional both to the vector itself (rotation), and to the distance along the loop (to first order), so that the total change in going once round a small loop can be written as

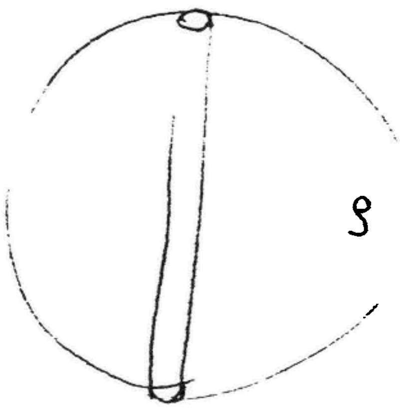
$$\delta V^\mu = -\oint \Gamma^\mu_{\alpha\beta}\, V^\alpha \, dx^\beta. \qquad (1.66)$$

Why the minus sign? The reason for this is apparent when we consider the **covariant derivative** of a vector. To differentiate involves taking the limit of $[V^\mu(x+\delta x)-V^\mu(x)]/\delta x$, but the difference of V^μ at two different points is not meaningful in the face of general coordinate transformations. A more sensible procedure is to compare the value of the vector at the new point with the result of parallel-transporting it from the old point.

So Riemann tensor seems to be the important object to compute, since it contains everything needed for the eq. of motion of particles.

Einstein noticed that some parts of the Riemann tensor must be directly related to local properties of matter.

Take, for example, a constant density sphere with a tunnel cut through it



Two free particles let go from each end will oscillate around the center with period $\left(\frac{4\pi G}{3}\rho\right)^{-1/2}$ – (True for any of the spatial coordinates)

the eq. of geodesic deviation is, then,

$$\frac{d^2\xi^j}{d\tau^2} = -\left(\frac{4\pi}{3}\rho\right)\xi^j \qquad j = x, y, z$$

and so

$$\begin{vmatrix} R^x{}_{0x0} & R^y{}_{0x0} & R^z{}_{0x0} \\ R^x{}_{0y0} & R^y{}_{0y0} & R^z{}_{0y0} \\ R^x{}_{0z0} & R^y{}_{0z0} & R^z{}_{0z0} \end{vmatrix} = \frac{4\pi\rho}{3}\begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}.$$

This illustrates the connection between geometry and matter.

Now, this eq. will cease to be valid if a grav. wave, for example, hits the sphere, but some parts of Riemann will be unaffected

In particular, the trace $R_{00} = R^x{}_{0x0} + R^y{}_{0y0} + R^z{}_{0z0} = 4\pi\rho$

In other words, a piece of Riemann, called the Einstein tensor is always generated by the local distribution of matter.

Generalizing, Einstein proposed that the Einstein tensor was identical to the stress-energy tensor of matter,

$$\boxed{G^{\alpha\beta} = 8\pi T^{\alpha\beta}}$$ (both are second-rank tensors)

— o —

The stress-energy tensor is the geometrical object that contains knowledge of energy density, momentum density, and stress as measured at each event in spacetime.

It is defined so that,

① $T(\vec{u}, \vec{u}) = \left(\text{mass-energy per unit volume as measured in frame with velocity } \vec{u}\right)$

② $T(\vec{u}, \vec{n}) = \left(\text{component, "} \vec{n} \cdot \frac{d\vec{p}}{dV} \text{", of 4 momentum density along } \vec{n} \text{ direction, as measured in Lorentz' frame of observer with 4-velocity } \vec{u}\right)$

(3) $T(\vec{u}, \dots) = \left(\right.$ density of 4-momentum $d\vec{p}/dV$", i.e. momentum per unit of 3-d volume, as measured in observer's Lorentz frame at event where T is chosen $\left.\right)$

(4) $T_{jk} = T(\vec{e}_j, \vec{e}_k) = T_{kj}$ $\vec{e}_j, \vec{e}_k$ are spacelike basis vectors of an observer's Lorentz frame

$= \left(\right.$ j-component of force acting from side $x^k - \varepsilon$ to side $x^k + \varepsilon$ across a unit surface area with perpendicular direction $\vec{e}_k$ $\left.\right)$

For a perfect fluid, (no viscosity, shear stress, anisotropic pressure, ..)

$$T_{\alpha\beta} = (\rho + p)\, u_\alpha u_\beta + p\, g_{\alpha\beta}$$

(ρ, p are rest frame density and pressure)

In the fluid-rest frame,

$$T^\alpha{}_\beta u^\beta = [(\rho+p) u^\alpha u_\beta + p\,\delta^\alpha{}_\beta]\, u^\beta = -(\rho+p)\, u^\alpha + p\, u^\alpha = -\rho\, u^\alpha$$

$(\vec{u}, \vec{u}) = -1$ (In 4-space, fluid moves at the "speed of light")

$$T_{0\beta} u^\beta = -\rho = -(\text{mass-energy density}) = -dp^0/dV$$

$$T_{j\beta} u^\beta = -(\text{momentum density}) = -dp^j/dV$$

Conservation of energy-momentum is expressed by the simple property 10

$$\nabla \cdot T = 0$$

For example for a Newtonian fluid, $(|v^j| \ll 1)$

$$T^{00} = (\rho + p) u^0 u^0 - p \simeq \rho$$

$$T^{0j} = T^{j0} = (\rho + p) u^0 u^j \simeq \rho v^j$$

$$T^{jk} = (\rho + p) u^j u^k + p \delta^{jk} \simeq \rho v^j v^k + p \delta^{jk}$$

so that the components of $\nabla \cdot T = 0$ are

$$T^{00}{}_{,0} + T^{0j}{}_{,j} = \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^j}(\rho v^j) = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

(continuity equation)

and $T^{j0}{}_{,0} + T^{jk}{}_{,k} = \frac{\partial}{\partial t}(\rho v^j) + \frac{\partial}{\partial x^k}(\rho v^j v^k) + \frac{\partial p}{\partial x^j} = 0$

or, combining with continuity eq, one finds

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho} \nabla p \qquad \text{(Euler's eq.)}$$

Einstein Field Equations

Einstein proposed that local curvature and energy-momentum were identical, so that

$$\overleftrightarrow{G} = 8\pi \overleftrightarrow{T}$$

He arrived at $\overleftrightarrow{G}$ by requiring that:

(a) G be constructed from Riemann and the metric, and nothing else

(b) G is linear in Riemann

(c) G is symmetric and of second rank (like T)

(d) G has an automatically vanishing divergence.

(e) G vanishes when spacetime is flat (but see later!)

With these conditions, there is a single tensor that satisfies these constraints. It is constructed from the Ricci tensor

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} \qquad \text{and the scalar} \qquad R = R^{\mu}{}_{\mu}$$

and reads

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}$$

$\boxed{G = 8\pi T}$ are a set of 10 second-order partial differential equations. Only six of these pose independent constraints on the ten components of the metric, the other four come from the "conservation equations

$$\nabla \cdot G = \nabla \cdot T = 0$$

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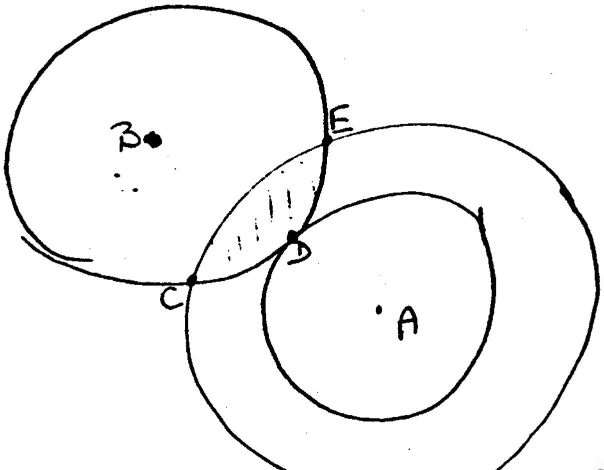
The COSMOLOGICAL PRINCIPLE

"There exist "fundamental observers" in the universe relative to which all properties of the universe, averaged on large scales, are isotropic. Provided that they set their clocks suitably, all these observers experience the same history of the Universe"

This is a refined way of saying: We are no privileged observers of the Universe.

Isotropy implies homogeneity, as this simple diagram illustrates



Isotropy about two observers A and B imply constant density in the shaded area and by extension in the whole universe

Cosmological time:

Something like a "universal time" exists in an isotropic universe. For example two 'fundamental observers' may decide to synchronize their clocks when their local densities reach a given value, they must remain synchronized afterwards. This property (a symmetry built into our Copernican assumption) means that the metric must take the form (in suitable coordinates)

$$ds^2 = c^2 dt^2 - R^2(t) \left[f^2(r)\, dr^2 + g^2(r)\, d\Omega^2 \right]$$

This form is because locally an observer sees a Minkowski space, so for him $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$

The problem is how to relate his dx, dy, dz to ours (assuming we are another fundamental observer).

Above r is a time-independent radial coordinate, and
R is a time-dependent scale factor.

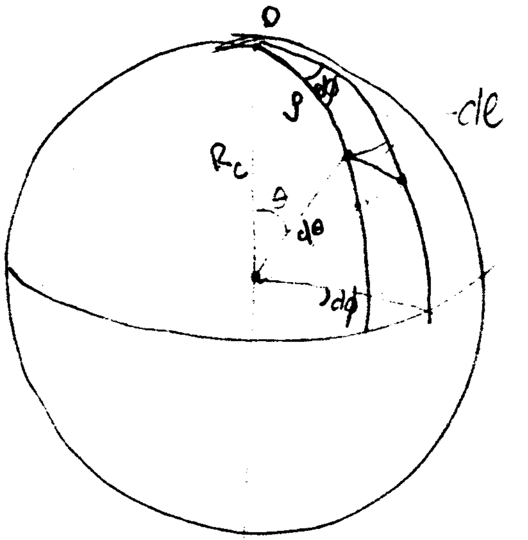
Also $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$. This is the only angular dependence allowed, otherwise ds^2 would change if a coordinate rotation is performed

Space Time Metric for Isotropic Curved Spaces

The simplest example of a 2D isotropic curved space is the surface of a sphere.

In general the distance between two neighboring points can be expressed as

$$dl^2 = R_c^2\, d\theta^2 + R_c^2 \sin^2\theta\, d\phi^2$$



in terms of the polar angles θ and ϕ and the "radius of curvature" R_c.

Using the "geodesic distance" ρ we have $\rho = \theta R_c$

and

$$dl^2 = d\rho^2 + R_c^2 \sin^2\left(\frac{\rho}{R_c}\right) d\phi^2$$

Another possible "distance measure" is

$$x = R_c \sin\left(\frac{\rho}{R_c}\right)$$

with

$$dl^2 = \frac{dx^2}{1 - K\frac{x^2}{R_c^2}} + x^2\, d\phi^2$$

($K = 0, +1,$ or -1) for Euclidean, positive and negative curvature

This can be generalized to 3D by noting that the general angular displacement perpendicular to the radial direction is

$$d\Phi^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

The spatial element is then

$$dl^2 \quad d\rho^2 + R_c^2 \sin^2\left(\frac{\rho}{R_c}\right)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]$$

or equivalently

$$dl^2 = \frac{dx^2}{K\frac{x^2}{R_c^2}} + x^2\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]$$

[x is the "angular diameter distance]

[ρ is the "geodesic distance"]

Can use this to write the Robertson Walker metric, which because it must reduce locally to Minkowski form, can be written as

$$ds^2 = c^2 dt^2 - \left[d\rho^2 \quad R_c^2\, S_K^2\left(\frac{r}{R_c}\right)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

where

$$S_K^2 \equiv \begin{cases} \sin^2 & K = 1 \\ 1 & \text{if } K = 0 \\ \sinh^2 & -1 \end{cases}$$

– Interpretation of the geodesic distance ρ

Since r is defined at a fixed cosmic time but observations are only possible along the past light cone, then ρ is not easy to characterise ... the position of a galaxy

relative to us today in relation to observations (along the light cone)
depend on how much the universe has expanded. In other words, the coordinate $\underline{\underline{\rho}}$ depends on the cosmological model.

On the other hand, our assumption of isotropy implies that the expansion is uniform. This implies that the distance to two fundamental observers change in the following manner:

$$\frac{\rho_i(t_1)}{\rho_j(t_1)} = \frac{\rho_i(t_2)}{\rho_j(t_2)} = \text{constant} = \frac{R(t_1)}{R(t_2)}$$

The function $R(t)$ is known as the "scale factor" and describes how the relative distance between two observers changes with time.

Setting $R(t_0) = 1$ we have

$$\rho(t) = R(t)\, r$$

The "distance label" r is thus attached to fundamental observer and remains constant with time.

[r is the "comoving radial distance coordinate".]

Now, proper distances perpendicular to the line of sight must also change by the same factor $R(t)$ because of isotropy.

In other words

$$\frac{\Delta \ell(t)}{\Delta \ell(t_0)} = \frac{R(t)}{R(t_0)} = R(t)$$

Using the Robertson Walker metric, this implies that

$$R(t) = \frac{R_c(t) \sin\left[S/R_c(t)\right] d\theta}{R_c(t_0) \sin\left[r/R_c(t_0)\right] d\theta}$$

or,

$$\frac{R_c(t)}{R(t)} \sin\left[\frac{R(t)\, r}{R_c(t)}\right] = R_c(t_0) \sin\left[\frac{r}{R_c(t_0)}\right]$$

which can only be true if

$$R_c(t) = R_c(t_0)\, R(t)$$

In other words, the radius of curvature is proportional to the scale factor. The curvature of space changes as the universe expands, as $k = \frac{1}{R_c^2} \propto R^{-2}$

Notice that k cannot change sign and therefore the geometry of the universe cannot change. If it starts flat, it must remain flat. If it starts closed, it must remain closed.

The radius of curvature at $t = t_0$ is usually called

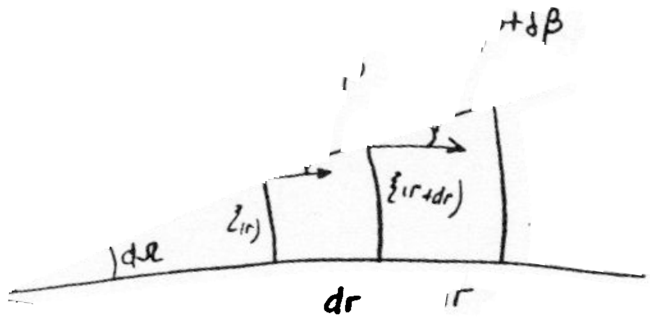
$$\mathbb{R}_0 = R_c(t_0)$$

and so $R_c = \mathbb{R}_0\, R(t)$

and the metric is then

$$ds^2 = c^2 dt^2 - R^2(t)\left[dr^2 + \mathbb{R}_0^2\, S_k^2\left(\frac{r}{\mathbb{R}_0}\right) d\Omega^2\right]$$

We are left with the task of computing $g(r)$ and $f(r)$.
One typically chooses $f=1$ [illegible] radial [illegible] as [illegible] to euclid[illegible] as possible. The form of $g(r)$ can then be found by considering the angular change for an increment dr 114



$$\xi(r) = g(r)\, d\Omega$$

From homogeneity (isotropy) one can see that the angle $\delta\beta$ (constructed by parallel transport around the square) is proportional to the area of the square. This is because it should be the same to go around the small square twice (in which case one would get $2\delta\beta$) or to go around the small square first and the second one later.

try it on a sphere!

This is also known as "spherical excess" in spherical trigonometry.

But, since $\beta = \frac{\partial \xi}{\partial r}$ and $\delta\beta \propto$ area, we have

$$\frac{\partial^2 \xi}{\partial r^2}\, dr = \pm A^2 \xi\, dr$$

($A = 0 \equiv$ Euclidean geometry)

But there are two other possibilities,

$$\xi \propto \sin Ar$$

$$\xi \propto \sinh Ar$$

(Constants of proportionality can be found imposing $\xi \to r\, d\Omega$ as $r \to 0$)

We can now write the Robertson-Walker metric,

$$ds^2 = c^2 dt^2 - R^2(t)\left[dr^2 + R_0^2 S_k^2\left(\frac{r}{R_0}\right) d\Omega^2\right]$$

where

$$S_k(r) = \begin{cases} \sin r & (k=1) \\ \sinh r & (k=-1) \\ r & (k=0) \end{cases}$$

R_0 is the "radius of curvature of the universe at the present time. R is a dimensionless scale factor

Another usual choice is to adopt $g(r) = r^2$, which gives

$$ds^2 = c^2 dt^2 - R^2(t)\left(\frac{dr^2}{1-k\left(\frac{r}{R_0}\right)^2} + r^2 d\psi\right) .$$

$k = -1, 0, +1$
$r = R_0 S_k(r)$ is used radial coordinate. Note that same "r" is used!

Yet another possibility is to scale the scale factor to the present,

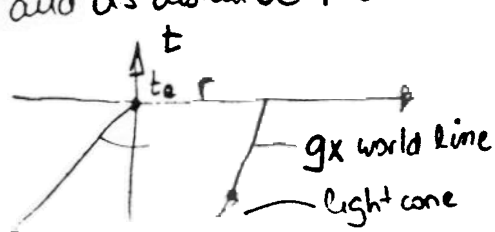
$$a(t) = \frac{R(t)}{R_0}$$

and we have *

$$ds^2 = c^2 dt^2 - a^2(t)\left(dr^2 + R_0^2 S_k^2\left(\frac{r}{R_0}\right) d\Omega^2\right)$$

$$ds^2 = c^2 dt^2 - a^2(t)\left(\frac{dr^2}{1-k\left(\frac{r}{R_0}\right)^2} + r^2 d\Omega^2\right)$$

Written in this form, r is the proper distance a galaxy would have if its world line were projected forward to the present epoch and its distance measured at that time.



I will use R or a for scale factor !!

Note

→ t is "cosmic time" i.e. the time measured by a clock carried by a fundamental observer

→ r is a "comoving radial distance coordinate", fixed to each fundamental observer (Interpretation changes according to which metric is used)

→ No assumption has been made about $a(t)$, but note that the geometry of the universe is always the same, regardless of $a(t)$; i.e. $k = -1, 0, +1$

COSMOLOGICAL REDSHIFT

Imagine a galaxy emits photons at a rate given by the period Δt_1 at time t_1. These photons, emitted at t_1, are received by an observer (at $r=0$) at time t_0 at a rate Δt_0. Since photons travel on null geodesics,

$$c\,dt = -R(t)\,dr$$

(sign is reversed because photon is travelling towards the origin)

$$\Rightarrow \int_{t_1}^{t_0} \frac{c\,dt}{R(t)} = -\int_r^0 dr = \int_{t_1+\Delta t_1}^{t_0+\Delta t_0} \frac{c\,dt}{R(t)}$$

(since r is independent of time)

or

$$\frac{c\,\Delta t_0}{R(t_0)} = \frac{c\,\Delta t_1}{R(t_1)}$$

or, $\Delta t_0 = \Delta t_1 \left(\frac{R(t_0)}{R(t_1)} \right)$ This is the phenomenon of "time dilation" in cosmology. A clock in a distant galaxy seems to run slower. Frequencies are affected in the same way. For example,

if $\Delta t_1 = \nu_1^{-1}$ and $\Delta t_0 = \nu_0^{-1}$

$$\frac{\nu_0}{\nu_1} = \frac{R(t_1)}{R(t_0)}$$

or, in terms of redshift,

$$z = \frac{\lambda_0 - \lambda_e}{\lambda_e} = \frac{\lambda_0}{\lambda_e} - 1 = \frac{\nu_1}{\nu_0} - 1$$

(Note this is not the same as $\Delta\nu/\nu$)

$$\Rightarrow \boxed{1 + z = \frac{R(t_0)}{R(t_1)}}$$

Redshifts of distant gx measure how much the size of the universe has changed, not how fast a gx is receding. This is independent of the geometry of the universe.

Is redshift really a velocity?

— o —

Hubble's law

Proper distance between two galaxies is $R(t)\, r$

These distances change as given by

$$V = \dot{R}(t)\, r = \frac{\dot{R}}{R}(R r) = \frac{\dot{R}}{R}\, d \qquad H d$$

Hubble's constant" is then $\boxed{H(t) = \frac{\dot{R}}{R}(t)}$

Angular diameter

An object of proper size d observed at a redshift z subtends an angle $d\theta$,

$$d = R(t)\, R_0\, S_k\left(\frac{r}{R_0}\right) d\theta = \frac{R_0\, S_k(r/R_0)}{1+z}\, d\theta$$

($R_0\, S_k(r/R_0)$ is sometimes called the "distance measure")

or

$$d\theta = \frac{d}{D_A} \quad \frac{d\,(1+z)}{R_0\, S_k(r/R_0)} \quad ; \quad D_A \equiv \frac{R_0 S_k(r/R_0)}{1+z} \equiv \text{angular diameter distance} -$$

Note that for $z \ll 1$; $r \ll 1$ we have the Euclidean $d = r\, d\theta$

and that for very large z $d\theta$ increases without bound (interpretation?)

Angular diameters depend on $S_k(r)$ i.e., on the particular geometry of the universe -

Apparent Intensities:

What is the flux density observed at t_0 from a source of monochromatic luminosity $L(\nu_1)$?

$$L(\nu_1) = \frac{dE}{d\nu_1\, dt_1} = \frac{dN_1\, h\nu_1}{d\nu_1\, dt_1}$$

Observed flux density is $S(\nu_0)$ $\dfrac{dE_0}{dA\, dt_0\, d\nu_0} = \dfrac{dN_0\, h\nu_0}{dA\, dt_0\, d\nu_0}$

To determine the fraction $\dfrac{dN_0}{dN_1}$ of photons received by the telescope we need to find the angle $\delta\theta$ subtended by the telescope at redshift z, which,

in units of comoving
the galaxy for all time.

(5.44)

$$\frac{c\,dt}{R(t)}. \quad (5.45)$$

(5.46)

non of *time dilation*. Dis-
time t_1 when $R(t_1) < 1$
ır frame of reference than
precisely the same as time
, relativistic muons, cre-
o have longer lifetimes in
lifetimes. The expression
lker formalism and it has
properties of supernovae
more detail in Sect. 8.4.2,
e is a remarkable narrow
y examples have ex-
of their luminosities
en more precisely defined
en luminosity and decline
maximum light that they
ws a plot of the duration
ion of redshift z, or, more
seen that the observations
of (5.46).
n for *redshift*. If $\Delta t_1 = \nu_1^{-1}$
the observed period, then

(5.47)

$- 1,$ (5.48)

(5.49)

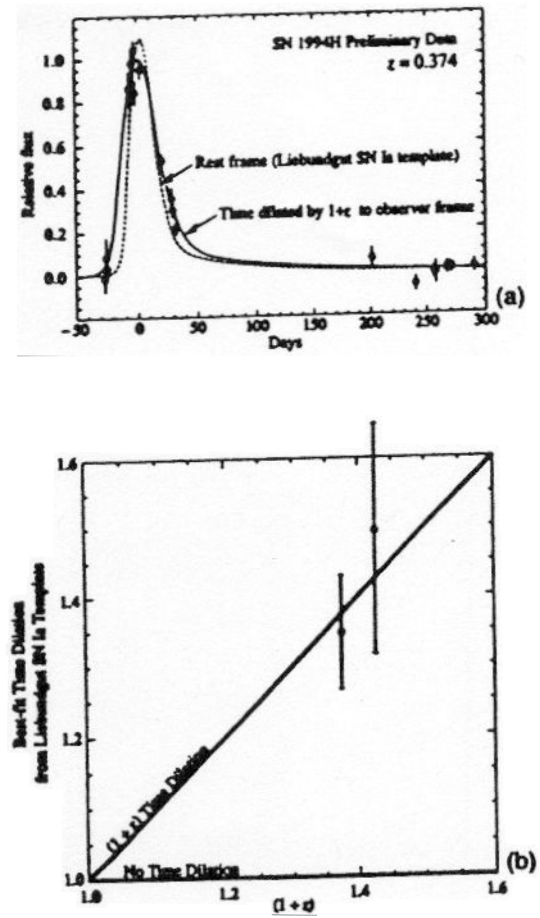


Fig. 5.6 (a) The time variation of the brightness of type 1A supernovae, showing the effect of time dilation for the supernova SN 1994H at redshift $z = 0.374$. (b) The correlation between the duration of the supernova outburst and redshift. The

This is one of the most important relations in cosmology and displays the real meaning of the redshifts of galaxies. *Redshift is a measure of the scale factor of the Universe when the radiation was emitted by the source.* When we observe a galaxy with redshift $z = 1$, the scale factor of the Universe when the light was emitted was $R(t) = 0.5$, that is, the distances between fundamental observers (or galaxies) were half their present values. Note, however, that we obtain no information about *when* the light was emitted. If we did, we could determine directly from observation the function $R(t)$. Unfortunately, from our angular diameter formula is (note that $K(t_0)=1$)

telescope diameter $\Rightarrow \delta \ell = R_0 S_k\left(\frac{r}{R_0}\right)\delta\theta$, and the telescope area is $\delta A = \frac{\pi}{4}\delta\ell^2$,

$\delta\theta$ angle measured at the source

so that the solid angle intercepted by the telescope is $\delta\Omega = \frac{\pi}{4}\delta\theta^2$,

and $dN_0 = dN_1 \frac{\delta\Omega}{4\pi} = \frac{dN_1}{4\pi}\left(\frac{\pi}{4}\right)\delta\theta^2 = \frac{dN_1}{4\pi}\left(\frac{\pi}{4}\right)\frac{\delta\ell^2}{R_0^2 S_k^2(x)}$

$x = r/R_0$

$$\Rightarrow S(\nu_0) \quad \frac{dN_1 \, h\nu_0 \, \delta\Omega}{4\pi \, dt_0 \, d\nu_0 \left(\frac{\pi}{4}\right)\delta\ell^2}$$

which we can write in terms of the source properties, using $\nu_1 = \nu_0(1+z)$ and $dt_0 = dt_1(1+z)$

$$\Rightarrow S(\nu_0) = \frac{dN_1 \, h\nu_1}{\cancel{(1+z)}} \frac{1}{4\pi}\cancel{\left(\frac{\pi}{4}\right)}\frac{\cancel{\delta\ell^2}}{R_0^2 S_k^2(x)} \; \frac{1}{dt_1(1+z)} \; \frac{1}{\cancel{\left(\frac{\pi}{4}\right)\delta\ell^2}} \; \frac{(1+z)}{d\nu_1}$$

$$\frac{dN_1 \, h\nu_1}{4\pi \, d\nu_1 \, dt_1 R_0^2 S_k^2(x)(1+z)} = \frac{L(\nu_1)}{4\pi R_0^2 S_k^2(x)(1+z)}$$

This is true for the monochromatic flux density $S(\nu_0)$. For the bolometric flux density $S_{bol} = \int S(\nu)\,d\nu$ there will be another factor of $(1+z)$ so that

$$S_{bol} \quad \frac{L_{bol}}{4\pi R_0^2 S_k^2(x)(1+z)^2} \quad \frac{L_{bol}}{4\pi D_L^2}$$

$$\boxed{D_L = R_0 S_k(r/R_0)(1+z)}$$

is the "luminosity distance", and again depends on the universe's geometry (a *standard measuring rod*) to its redshift. These relations are shown in Figure 2. Similarly, if some kind of galaxy is assumed to be uniformly distributed in space, equations (2.52) and (2.53) predict the number of such objects as a function of the redshift to which they are counted. This relation is also shown in Figure 2. Finally, because it is much easier to measure the flux of a faint object than its redshift, observational cosmologists often eliminate $a_o r$ between equations (2.19) and (2.53) to get the predicted counts of such "standard markers" as a function of the flux limit to which they are counted. As is evident from the final panel of Figure 2, this weakens the power of the test because the geometry dependence of the two relations cancels to lowest order. The application of all these tests is discussed in Section 4 below. It turns out that they are severely compromised by difficulties in defining proper standard candles or standard measuring rods.

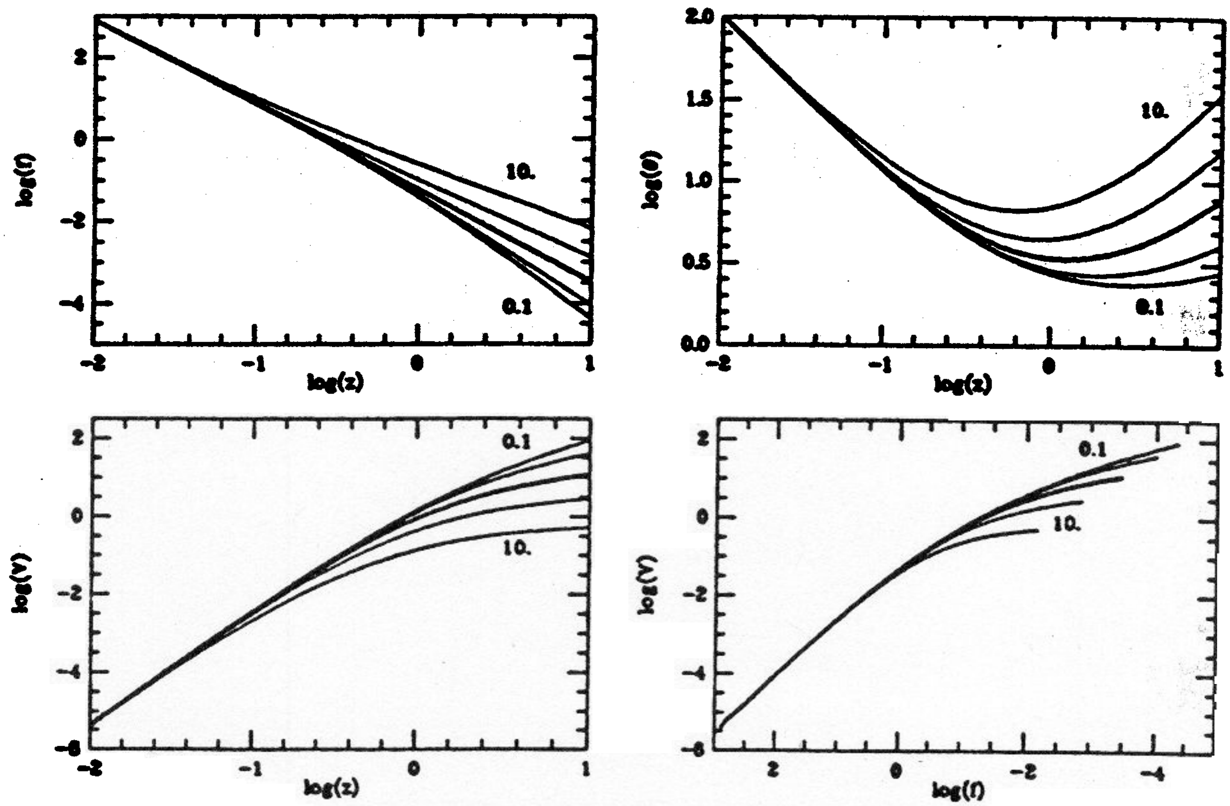
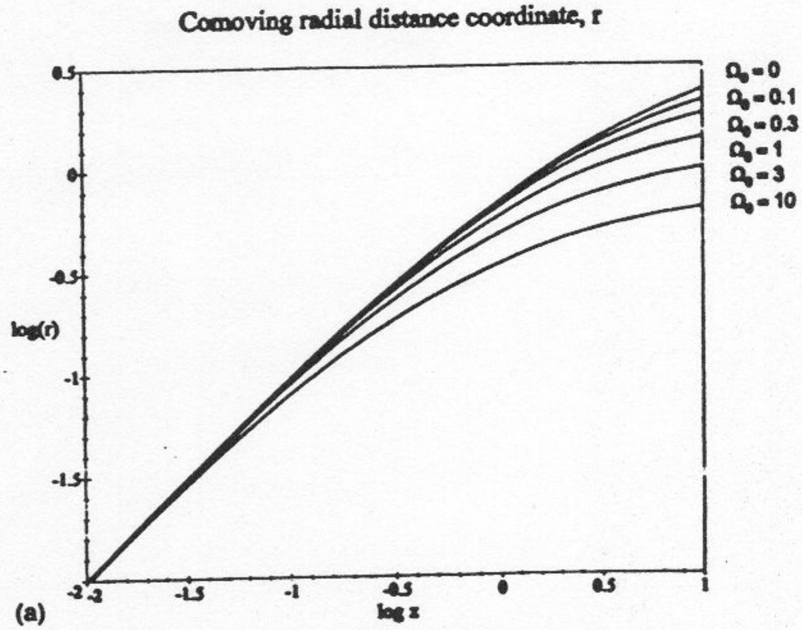


Figure 2. *Properties of standard candles and standard measuring rods in dust universes with zero cosmological constant. The quantities plotted are defined in equations (2.17), (2.19), (2.52) and (2.53) with units chosen so that $c = H_o = D = L = 1$. Models are plotted for $\Omega_o = 0.1$, 0.3, 1, 3 and 10, with the flat model shown as a heavier curve. Only points with $z < 10$ are shown in the plot of volume against limiting flux (lower right).*

2.5 Horizons

A light ray emitted by an event at (r, t_e) reaches an observer at the origin at time, t_o, given by



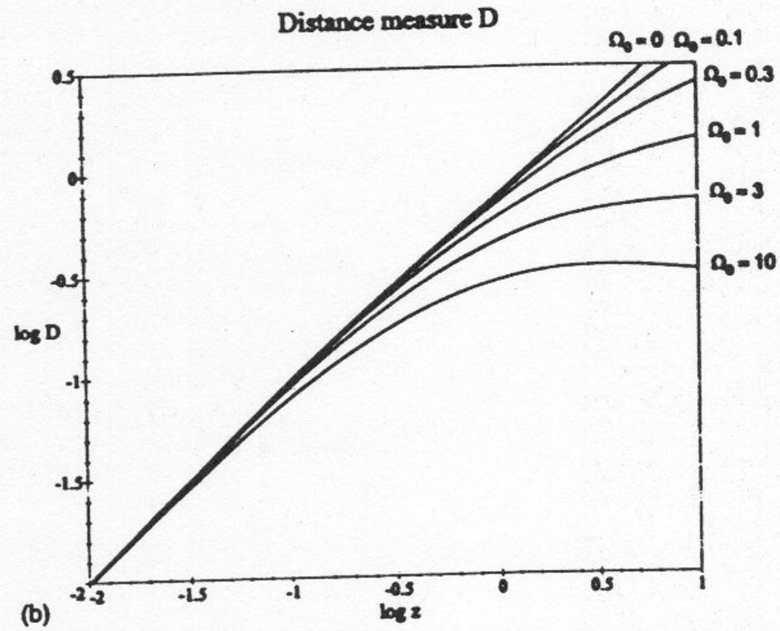


Fig. 7.3a-b. The variation with redshift of (a) the comoving radial distance coordinate r, and (b) the distance measure D for Friedman world models with $\Lambda = 0$ and $\Omega_0 = 0, 0.1, 0.3, 1, 3$ and 10. In this diagram, r and D are measured in units of c/H_0.

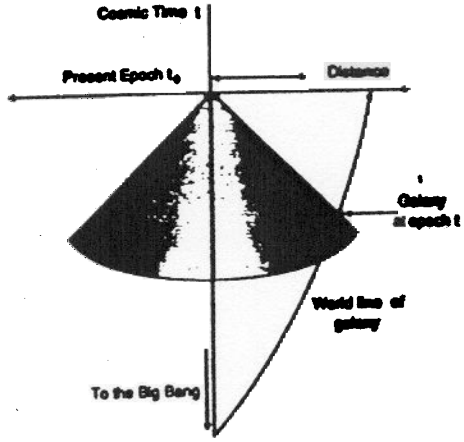


Fig. 5.5. A simple space-time diagram illustrating the definition of the comoving radial distance coordinate

distance we wish to measure. The observers are all instructed to measure the distance $d\varrho$ to the next fundamental observer at a particular cosmic time t which they read on their own clock. By adding together all the $d\varrho$s, we can find a proper distance ϱ which is measured at a single epoch and which can be used in the metric (5.27). Notice that ϱ is a *fictitious distance* in that we cannot actually measure distances in this way. We can only observe distant galaxies as they were at some epoch earlier than the present and we do not know how to project their positions relative to us forward to the present epoch without a knowledge of the kinematics of the expanding Universe. In principle, it is possible to determine the kinematics of the Universe observationally but, as we will discuss, this is not feasible at the moment. In practice, all we can do at the moment is to assume some suitable cosmological model for which the distance measure ϱ can be determined. In other words, *the distance measure ϱ depends upon the choice of cosmological model which we do not know*. We will show in Sect. 5.5 how to relate ϱ to measurable quantities.

Let us work out how the ϱ coordinates of galaxies change in a uniformly expanding Universe. The definition of a uniform expansion is that between two cosmic epochs, t_1 and t_2, the distances of any two fundamental observers, i and j, change such that

$$\frac{\varrho_i(t_1)}{\varrho_j(t_1)} = \frac{\varrho_i(t_2)}{\varrho_j(t_2)} = \text{constant}, \tag{5.28}$$

that is,

$$\frac{\varrho_i(t_1)}{\varrho_i(t_2)} = \frac{\varrho_j(t_1)}{\varrho_j(t_2)} = \ldots = \text{constant} = \frac{R(t_1)}{R(t_2)}. \tag{5.29}$$

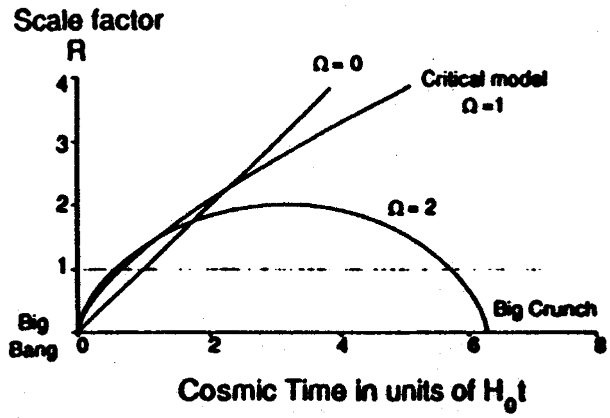


Fig. 7.2. The dynamics of the classical Friedman models parameterised by the density parameter $\Omega_0 = \varrho/\varrho_{crit}$. If $\Omega_0 > 1$, the Universe collapses to $R = 0$ as shown; if $\Omega_0 < 1$, the Universe expands to infinity and has a finite velocity of expansion as R tends to infinity. In the case $\Omega_0 = 1$, $R = (t/t_0)^{2/3}$ where $t_0 = (2/3)H_0^{-1}$. The time axis is given in terms of the dimensionless time H_0t. At the present epoch $R = 1$ and in this presentation, the three curves have the same slope of 1 at $R = 1$, corresponding to a fixed value of Hubble's constant. If t_0 is the present age of the Universe corresponding to $R = 1$, then for $\Omega_0 = 0$ $H_0t_0 = 1$, for $\Omega_0 = 1$ $H_0t_0 = 2/3$ and for $\Omega_0 = 2$ $H_0t_0 = 0.57$.

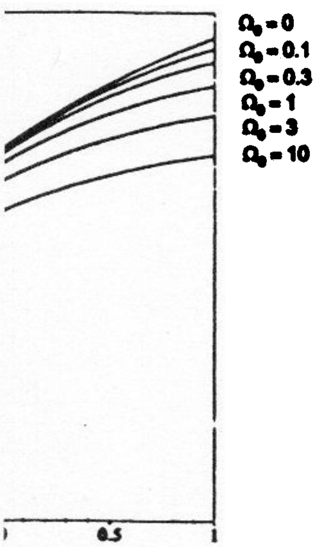
Another useful result is the function $R(t)$ for the empty world model, $\Omega_0 = 0$, $R(t) = H_0t$, $\kappa = -(H_0/c)^2$. This model is sometimes referred to as the *Milne model*. It is an interesting exercise to show why it is that, in the completely empty world model, the global geometry of the Universe is hyperbolic. The reason is that in the empty model, the galaxies partaking in the universal expansion are undecelerated and any particular galaxy always has the same velocity relative to the same fundamental observer. Therefore, the cosmic times measured in different frames of reference are related by the standard Lorentz transform $t' = \gamma(t - vr/c^2)$ where $\gamma = (1 - v^2/c^2)^{-1/2}$. The key point is that the conditions of isotropy and homogeneity apply at constant cosmic time t' in the frames of reference of all fundamental observers. The Lorentz transform shows that this cannot be achieved in flat space but it is uniquely satisfied in hyperbolic space with $\kappa = -(H_0/c)^2$. I have given a simple derivation of this result (Longair 1994).

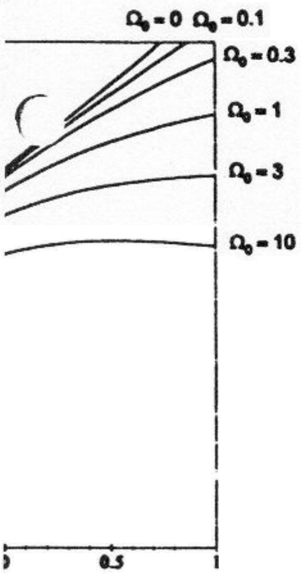
The general solutions of (7.20) are most conveniently written in parametric form. For $\Omega_0 > 1$,

$$\left.\begin{aligned} R &= a(1-\cos\theta) \qquad t = b(\theta - \sin\theta),\\ a &= \frac{\Omega_0}{2(\Omega_0 - 1)} \quad \text{and} \quad b = \frac{\Omega_0}{2H_0(\Omega_0-1)^{3/2}}. \end{aligned}\right\} \tag{7.24a}$$

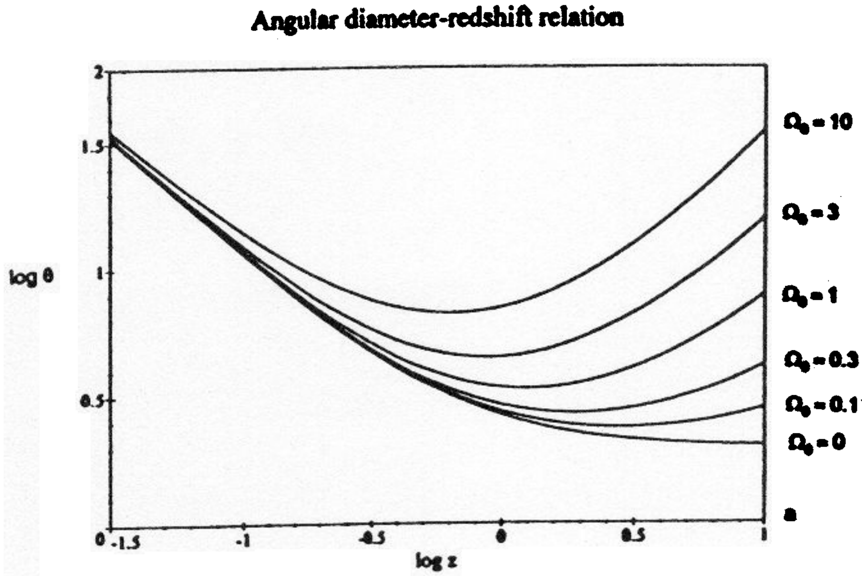
For $\Omega_0 < 1$,

ordinate, r





comoving radial distance coor-
man world models with $\Lambda = 0$
r and D are measured in units



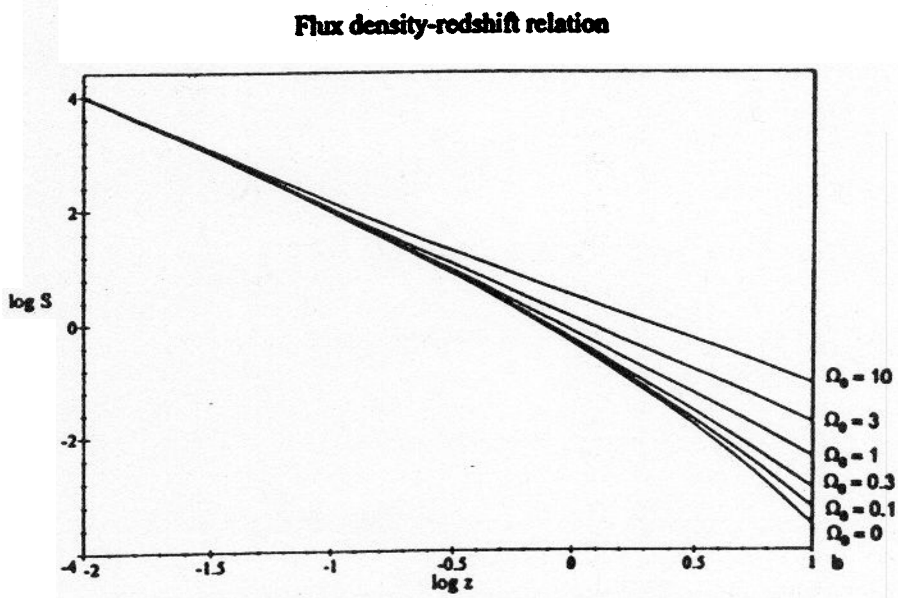


Fig. 7.4a-c. The variation of certain observables with redshift for Friedman world models with $\Lambda = 0$ and $\Omega_0 = 0, 0.1, 0.3, 1, 3$ and 10. In all three diagrams, c/H_0 has been set equal to unity. (a) The variation of the angular diameter of a rigid rod of unit proper length with redshift. (b) The variation of the flux density of a source of luminosity 1 W Hz^{-1} with a power-law spectrum $L(\nu) \propto \nu^{-1}$ with redshift. Inspection of (5.68) and (5.70) shows that this is the same as the variation of the bolometric flux densities with redshift. (c) The variation of the comoving volume within redshift z.

A common way of writing the monochromatic flux density is

$$S(\nu_0) = \frac{L(\nu_0)}{4\pi D_L^2} \left[\frac{L(\nu_1)}{L(\nu_0)} (1+z) \right]$$

monochromatic K-correction

THE FRIEDMAN WORLD MODELS

Let us recall Einstein's field equation:

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} + \left[\Lambda g_{\mu\nu} \right]$$

Recall that $R_{\mu\nu} = R^{\alpha}_{\ \mu\alpha\nu}$

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} + \frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} - \frac{\partial g_{\beta\alpha}}{\partial x^{\nu}} \right)$$

$$R^{\alpha}_{\ \beta\gamma\delta} = \frac{d \Gamma^{\alpha}_{\beta\delta}}{d x^{\gamma}} - \frac{d \Gamma^{\alpha}_{\beta\gamma}}{d x^{\delta}} \Gamma^{\alpha}_{\nu\gamma} \Gamma^{\nu}_{\delta\beta} - \Gamma^{\alpha}_{\nu\delta} \Gamma^{\nu}_{\gamma\beta}$$

This is the change ΔA_k of a vector A_k parallel-transported on a closed path around an infinitesimal area Δf^{em}

$$\Delta A_k = \frac{1}{2} R^{i}_{\ kem} A_i \Delta f^{em}$$

Substituting the Friedman-Robertson-Walker metric and assuming that the universe behaves on large scales like an ideal fluid, we obtain Friedman's equations. Because of the symmetries imposed, only two equations are non-trivia

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{1}{3}\Lambda \qquad (1)$$

and

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} - \frac{K}{R^2}\frac{c^2}{R_0^2} + \frac{1}{3}\Lambda \qquad (2)$$

Eqs' (1) and (2) have the appearance of a "force" and "energy" equation, and are indeed related by the 1st law of thermodynamics -

$$dU = -p\,dV$$

Calling $\varepsilon_{TOT} = \sum_i \varepsilon_i$ the total energy density, we have

$$\frac{d}{dR}(\varepsilon_{TOT} V) = -p\frac{dV}{dR}$$

and, using $V \propto R^3$ we find

$$\frac{d\rho}{dR} + \frac{3}{R}\left(\rho + \frac{P}{c^2}\right) = 0 \qquad (3)$$

where we have used ρ to denote the inertial mass density associated with ε_{TOT},

$$\varepsilon_{TOT} = \rho c^2$$

Differentiating ② and dividing by $\dot{R}$ we find

$$\ddot{R} = \frac{4\pi G}{3}\left[R^2 \frac{d\rho}{dR} + 2R\rho\right] + \frac{1}{3}\Lambda R$$

Substituting $\frac{d\rho}{dr}$ we have

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{1}{3}\Lambda$$ which is eq. ①

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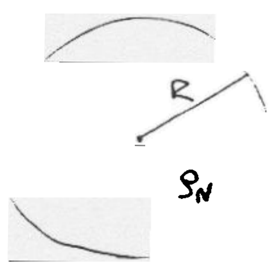
Newtonian analog.

Take a sphere of constant density ρ_N and radius R, the acceleration experienced by the outer most shell is



$$\ddot{R} = -\frac{GM}{R^2} = -\frac{G\,\frac{4}{3}\pi\rho_N R^3}{R^2} = -G\,\frac{4}{3}\pi\,\rho_N R$$

Since mass is conserved, $\rho_N = \rho_0\left(\frac{R_0}{R}\right)^3$

$$\Rightarrow \ddot{R} = -\frac{4\pi}{3} G \rho_0 R_0^3 \frac{1}{R^2}$$

which implies that $$\dot{R}^2 = \frac{8\pi G\rho_0 R_0^3}{3R} + C = \frac{8\pi G\rho_N R^2}{3} + C$$

which is identical to eq ② if we identify the constant with $-\frac{kc^2}{R_0^2}$

Note that:

- the "active gravitational mass density" is $\rho + \frac{3p}{c^2}$ not ρ, as the Newtonian analog would imply.
- Why does the Newtonian analog work? Because of our assumption of isotropy $\Rightarrow$ local physics $\equiv$ global physics, or local geometry $\equiv$ global geometry. The curvature of space is the same within one cubic metre as on the scale of the whole universe.

Dust Models with $\Lambda = 0$

("dust" $\equiv p = 0$ (no radiation))

$$\Rightarrow \quad \ddot{R} = -\frac{4\pi G \rho}{3} R$$

$$\dot{R}^2 = \frac{8\pi G \rho R^2}{3} - \frac{Kc^2}{R_0^2} \quad \Rightarrow \quad \frac{Kc^2}{R^2 R_0^2} = \frac{8\pi G}{3}\left(\rho - \frac{3H^2}{8\pi G}\right)$$

So the "curvature" K of the universe changes sign depending on whether

$$\rho \gtrless \frac{3H^2}{8\pi G} \equiv \text{critical density.}$$

It is usual to express densities in units of critical, using the parameter

$$\Omega = \frac{\rho}{\rho_{crit}}$$

Using quantities evaluated at $t = t_0$ (the present) we have

$$\ddot{R} = -\frac{\Omega_0 H_0^2}{2R^2}$$

$$\dot{R}^2 = \frac{\Omega_0 H_0^2}{R} - \frac{Kc^2}{R_0^2}$$

At $t = t_0$ this implies $R_0^2 = \frac{Kc^2}{H_0^2(\Omega_0 - 1)}$ (radius of curvature of the universe at $z=0$)

and the "curvature" $K = \frac{1}{R_0^2} = \frac{\Omega_0 - 1}{(c/H_0)^2}$

this shows the one-to-one correspondence between density and geometry, one characteristic of Friedman models with $\Lambda = 0$ –

of Friedman models - (Λ 0)

... of ... into Friedman's eq ...

Substituting ... above ...

$$\dot{R}^2 = c^2 \left[\Omega_0 \left(\frac{1}{R} - 1 \right) + 1 \right]$$

In the limit of large R, we have

$$\dot{R}^2 \to H_0^2 (1 - \Omega_0)$$

which allows us to ... the expansion in terms of present day measurements

$\boxed{\text{Case 1}}$ $\Omega_0 = 1$ $K=0$ (Einstein de Sitter model)

$R(t) = \left(\frac{t}{t_0}\right)^{2/3}$ $t_0 = \frac{2}{3H_0}$ expands for ever, expansion speed $\to 0$ as $t \to \infty$

$\boxed{\text{Case 2}}$ $\Omega_0 < 1$ $K = -1$

expands for ever, reaching a constant expansion speed $\dot{R} \approx H_0(1-\Omega_0)^{1/2}$ at $t \to \infty$. Solutions are usually given parametrically

$$R = a(\cosh\phi - 1) \; ; \quad t = b(\sinh\phi - \phi)$$

where $a = \frac{\Omega_0}{2(1-\Omega_0)}$ and $b = \frac{\Omega_0}{2H_0(1-\Omega_0)^{3/2}}$

$\boxed{\text{Case 3}}$ $\Omega_0 > 1$ $K = +1$

recollapses onto itself at $t = 2t_{max} = \frac{\pi\Omega_0}{2H_0(\Omega_0-1)^{3/2}}$

parametric solution is

$$R = a(1-\cos\theta) \qquad t \;\; b(\theta - \sin\theta)$$

$$a = \frac{\Omega_0}{2(\Omega_0-1)} \; ; \quad b = \frac{\Omega_0}{2H_0(\Omega_0-1)^{3/2}}$$

The Flatness Problem:

Using $\dot{R}^2 = H_0^2 \left[\Omega_0 \left(\frac{1}{R} - 1 \right) + 1 \right]$ and $R = \frac{1}{1+z}$

we obtain $H = \frac{\dot{R}}{R} = H_0 \left(1 + \Omega_0 z \right)^{1/2} (1+z)$

Defining at all times $\Omega \equiv \frac{\rho}{3H^2/8\pi G}$ we have

$$\Omega = \frac{\rho_0 (1+z)^3}{3H^2/8\pi G} = \Omega_0 \frac{(1+z)}{(1+\Omega_0 z)}$$

which can be rewritten as

$$\left(1 - \frac{1}{\Omega} \right) = \frac{1}{1+z} \left(1 - \frac{1}{\Omega_0} \right)$$

This implies that at large redshifts $\Omega \rightarrow 1$ regardless of its present value. In other words, at large z the universe was very close to the Einstein-de Sitter model.

This implies a fine-tuning problem. Universe today is only within a factor of ten of $\Omega_0 = 1$, so it must have been ≈ 1 to an astonishing degree of precision at early times. This is one of the "paradoxes" that an inflationary universe addresses with success.

Models with $\Lambda \neq 0$

When Einstein realized that GR implied that the universe was dynamic he sought a way in which to recover a static solution. He did this by introducing a "constant" term in his field equation.

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

This can also be viewed as a "vacuum" contribution to the stress-energy tensor

$$G_{\mu\nu} = 8\pi \left[T_{\mu\nu} + T^{(VAC)}_{\mu\nu} \right]$$

where $T^{(VAC)}_{\mu\nu} = -\left(\frac{\Lambda}{8\pi}\right) g_{\mu\nu}$.

In other words, we have given up the idea that $G = 0$ if there is no energy density present.

Friedman's equations now read:

$$\ddot{R} = -\frac{4\pi G R}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{1}{3}\Lambda R$$

To make contact with the "vacuum energy density", let's write Friedman's eq. for a non-zero ρ_{vacuum}

$$\ddot{R} = \frac{-4\pi G R}{3}\left(\rho_m + \rho_v + \frac{3P_v}{c^2}\right)$$

(assuming $P = 0$ for simplicity)

- The eq. of state of vacuum can be deduced from the 1st Law of thermodynamics and the imposition that $\frac{d\rho_v}{dR}$ = constant

(energy density of vacuum is independent of the expansion of the universe)

From $$\frac{d\rho_v}{dR} + \frac{3}{R}\left(\rho_v + \frac{P_v}{c^2}\right) = 0$$

$$\Rightarrow \quad \rho_v = -\frac{P_v}{c^2}$$

implying that $$\ddot{R} = -\frac{4\pi G R}{3}\left(\rho_m - 2\rho_v\right)$$

Since $\rho_m \propto R^{-3}$

$\rho_v = \text{const}$ $$\ddot{R} = -\frac{4\pi G}{3}\frac{\rho_0}{R^2} + \frac{8\pi G \rho_v R}{3}$$

Identifying this last term with the "cosmological constant" term in Einstein's equation we see that $\Lambda = 8\pi G \rho_v$

At the present epoch, $R = 1$,

$$\ddot{R}(t_0) = -\frac{4\pi G \rho_0}{3} + \frac{8\pi G}{3}\rho_v$$

it is common to introduce a "vacuum density parameter",

$$\Omega_\Lambda = \frac{8\pi G \rho_v}{3H_0^2} \; ; \; \Lambda = 3H_0^2\,\Omega_\Lambda$$

the dynamical equations are now

$$\ddot{R} = -\frac{\Omega_0 H_0^2}{2R^2} + \Omega_\Lambda H_0^2 R$$

$$\dot{R}^2 = \frac{\Omega_0 H_0^2}{R} - \frac{kc^2}{R_0^2} + \Omega_\Lambda H_0^2 R^2$$

Substituting values at the present epoch, we have

$$\frac{Kc^2}{\mathcal{R}_0^2} = H_0^2 \left[(\Omega_0 + \Omega_\Lambda) - 1 \right]$$

so that the universe is spatially flat if $\Omega_0 + \Omega_\Lambda = 1$

This is one of the predictions of the inflationary universe models.

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Dynamics of World Models with $\Lambda \neq 0$

$\Lambda < 0 \longrightarrow$ not very interesting dynamically. Acts like some extra mass.

$\Lambda > 0$: Interesting. Vacuum energy acts as a repulsive force opposing gravity.

Consider Friedman's energy equation,

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} - \frac{Kc^2}{R^2 \mathcal{R}_0^2}$$

Considering all forms of density,

$$\rho_m \propto R^{-3}$$

$$\rho_{rad} \propto R^{-4} \quad \left(\text{verify by using } dU = -p\,dV \text{ with } p_{rad} = \frac{1}{3}\rho_{rad} c^2\right)$$

$$\rho_{vac} = \text{const}$$

we can write it as

$$H^2 = \frac{8\pi G}{3}\left(\rho_{vac} + \rho_{m,0}\left(\frac{R_0}{R}\right)^3 + \rho_{rad}\left(\frac{R_0}{R}\right)^4\right) - \frac{kc^2}{R^2 R_0^2}$$

For large R all terms go to 0 except for the vacuum term. This implies that at late times most universes with positive cosmological constant will expand for ever $\Leftrightarrow$ $\left(\frac{\dot{R}}{R} \to \text{constant}\right)$. See exceptions below.

At small R the matter and radiation terms dominate, so universe expands like a power law

$$\left.\begin{array}{ll} R \propto t^{2/3} & \text{if matter dominates} \\ R \propto t^{1/2} & \text{if radiation dominates} \end{array}\right\} \text{decelerated expansion}$$

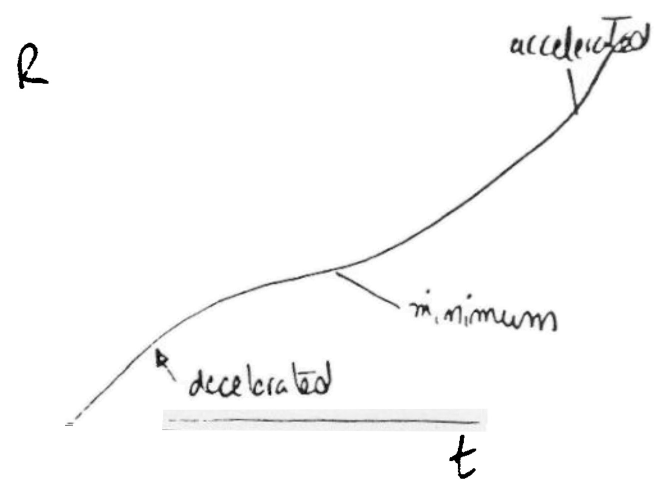
This implies that at some time the universe goes from decelerated expansion to accelerated expansion; i.e., through a minimum $\dot{R}$.

The minimum occurs at a scale factor

$$R_{min} = \left(\frac{4\pi G\rho_0}{\Lambda}\right)^{1/3} = \left(\frac{\Omega_0}{2\Omega_\Lambda}\right)^{1/3}$$

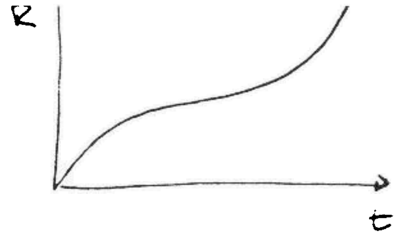
and has a value

$$\dot{R}^2_{min} = \frac{3H_0^2}{2}\left(2\Omega_\Lambda \Omega_0^2\right)^{1/3} - \frac{kc^2}{R_0^2}$$

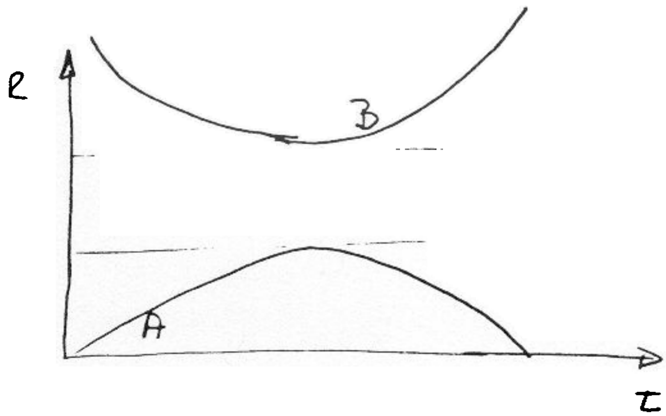


$\dot{R}^2_{min} > 0$

at late times expansion is exponential $R \propto e^{\Omega_\Lambda^{1/2} H_0 t}$



$\dot{R}^2_{min} < 0$



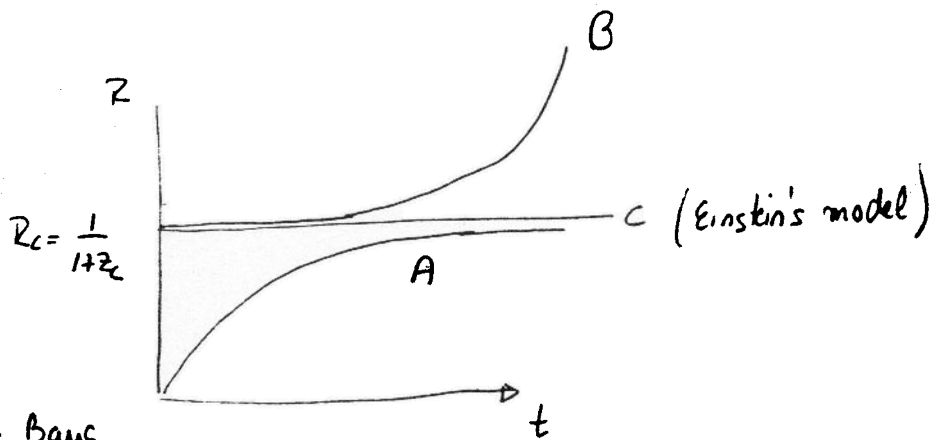
CASE A: Universe never expanded sufficiently for Λ to become important.

CASE B: Universe always dominated by Λ term. (no Big Bang)

In these "bouncing" universes, R_{min} determines the maximum redshift objects may have.

$\dot{R}^2_{min} = 0$

(Eddington-Lemaitre models)



CASE A: Universe had a Big Bang but will reach a stationary state in the infinite future

CASE B: Universe is expanding from a stationary solution in the infinite past.

CASE C: Universe has always been stationary (rather unstable: small perturbations will go into the A or B cases).

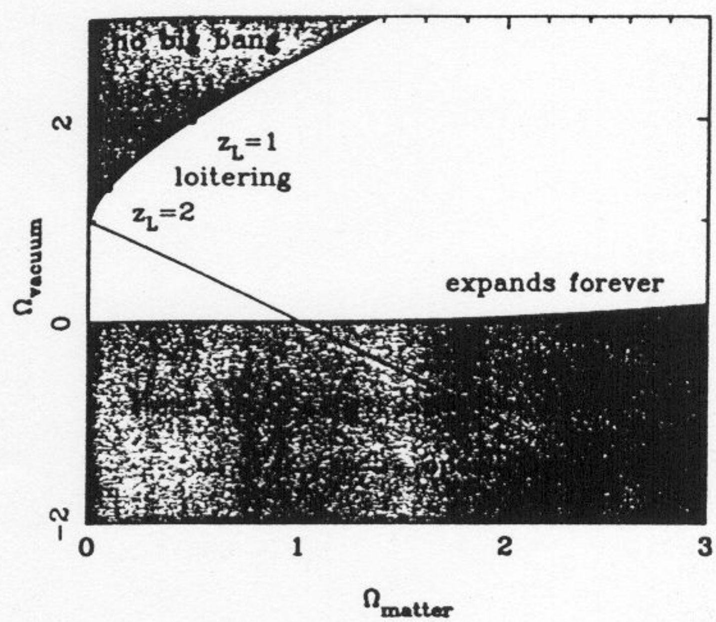


Figure 3.5. This plot shows the different possibilities for the cosmological expansion as a function of matter density and vacuum energy. Models with total $\Omega > 1$ are always spatially closed (open for $\Omega < 1$), although closed models can still expand to infinity if $\Omega_v \neq 0$. If the cosmological constant is negative, recollapse always occurs; recollapse is also possible with a positive Ω_v if $\Omega_m \gg \Omega_v$. If $\Omega_v > 1$ and Ω_m is small, there is the possibility of a 'loitering' solution with some maximum redshift and infinite age (top left); for even larger values of vacuum energy, there is no big bang singularity.

A reasonable lower limit for Ω_m of 0.1 then rules out a bounce once objects are seen at $z > 2$.

The main results of this section are summed up in figure 3.5. Since the radiation density is very small today, the main task of relativistic cosmology is to work out where on the $\Omega_{\rm matter}\Omega_{\rm vacuum}$-plane the real universe lies. The existence of high-redshift objects rules out the bounce models, so that the idea of a hot big bang cannot be evaded. As subsequent chapters will show, the data favour a position somewhere near the point (1,0), which is the worst possible situation: it means that the issues of recollapse and closure are very difficult to resolve.

FLAT UNIVERSE The most important model in cosmological research is that with $k = 0$, which implies $\Omega_{\rm total} = 1$; when dominated by matter, this is often termed the **Einstein–de Sitter model**. Paradoxically, this importance arises because it is an unstable state: as we have seen earlier, the universe will evolve away from $\Omega = 1$, given a slight perturbation. For the universe to have expanded by so many **e-foldings** (factors-of-e

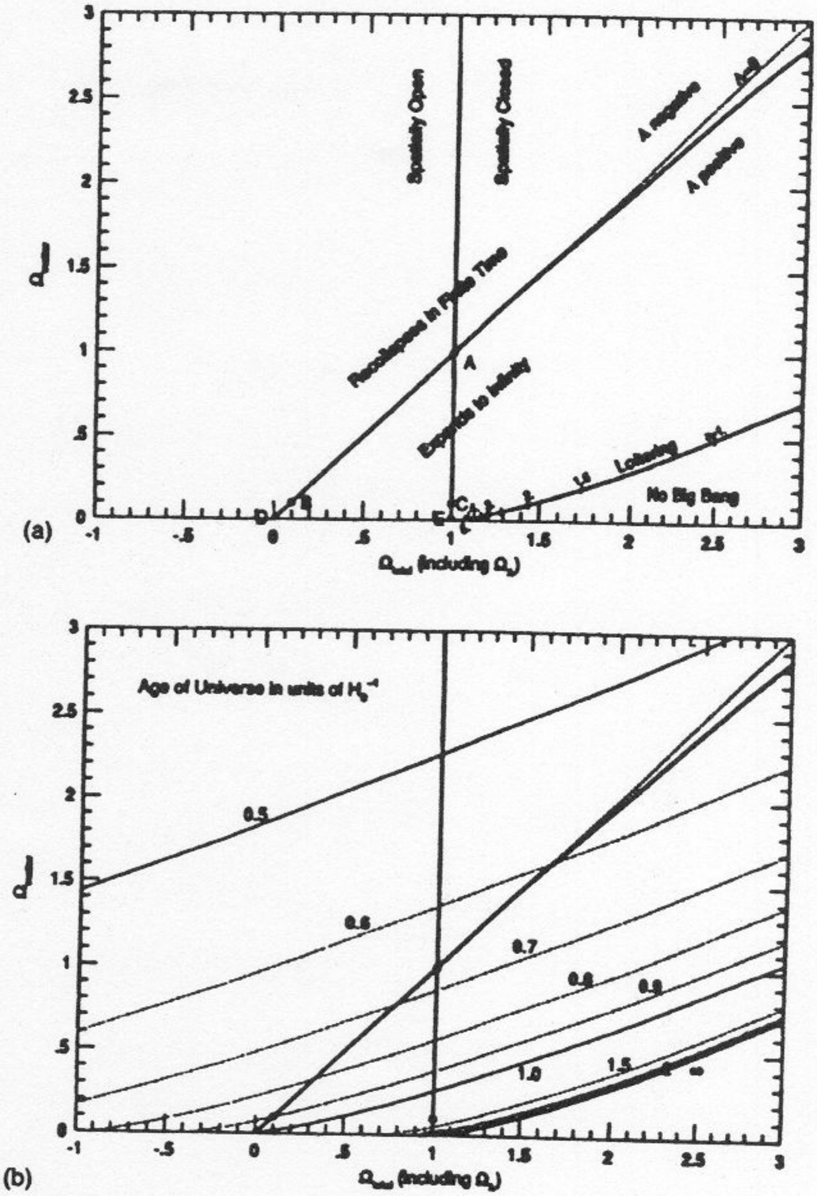


Fig. 7.7. (a) The classification of the Friedman world models with $\Omega_\Lambda \neq 0$ in a plot of Ω_0 against $\Omega_0 + \Omega_\Lambda$ (Carroll, Press and Turner 1992). The Eddington–Lemaître models lie along the line labelled 'loitering'. (b) The present age of the Universe for the range of world models displayed in (a) in units of H_0^{-1}.

$\Lambda \neq 0$

d^ls with $\Lambda \neq 0$.

mological constant are
effect is to incorporate an
down the expansion of the
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f the scale factor

$$2\Omega_\Lambda)^{1/3}, \qquad (7.69)$$

n (7.56). We then find the

$$\Omega_\Lambda \Omega_0^2)^{1/3} - \frac{c^2}{\Re^2}. \qquad (7.70)$$

namical behaviour shown in
namics become those of the

$$\Omega_\Lambda^{1/2} H_0 t \Big). \qquad (7.71)$$

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shown readily that the
n Fig. 7.5b. For the branch
large values of R that the
Universe collapsing. In the
y the Λ term – the repulsive
racted to such a scale that
s effect. In this model, there
d' under the influence of the
y of matter is zero, $\Omega_0 = 0$,

$$\cdot 1)], \qquad (7.72)$$

$$^{/2} H_0 \tau, \qquad (7.73)$$

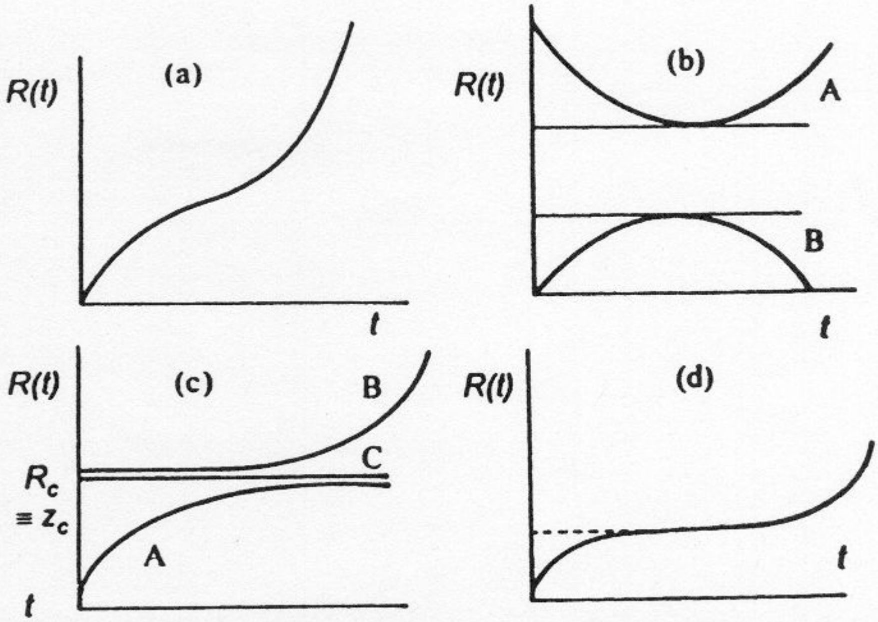


Fig. 7.5a-d. Examples of the dynamics of world models in which $\Lambda \neq 0$ (Bondi 1960). Models (a) and (d) are referred to as Lemaître models. (c) is the Eddington-Lemaître model. The Einstein static model is illustrated by the model for which $R(t)$ is constant for all time. The other models are described in the text.

where the time $\tau = t - t_{\min}$ is measured from the time at which the model 'bounced', that is, from the time at which $R = R_{\min}$. In all cases in which the models bounce, the variation of R with cosmic time is symmetrical about $R_{\min}$. Their asymptotic behaviour corresponds to exponentially collapsing and expanding de Sitter solutions

$$R = \left(\frac{\Omega_\Lambda - 1}{\Omega_\Lambda}\right)^{1/2} \exp\left(\pm \Omega_\Lambda^{1/2} H_0 \tau\right). \qquad (7.74)$$

In these 'bouncing' Universes, the smallest value of R, $R_{\min}$, corresponds to the largest redshifts which objects could have.

The most interesting cases are those for which $\dot{R}_{\min} \approx 0$. The case $\dot{R}_{\min} = 0$ is known as the *Eddington-Lemaître model* and is illustrated in Fig. 7.5c. The literal interpretation of these models is either: A, the Universe expanded from an origin at some finite time in the past and will eventually attain a stationary state in the infinite future; B, the Universe is expanding away from a stationary solution in the infinite past. The stationary state C is unstable because, if it is perturbed, the Universe moves either onto branch B, or onto the collapsing variant of branch A. In Einstein's static Universe, the stationary phase occurs at the present day. In general, from (7.69), the value of Λ corresponding to $\dot{R}_{\min} = 0$ is

A few other miscellaneous.

Matter and Radiation eras:

Since $\rho_r \propto \frac{1}{a^4}$, $\rho_m \propto \frac{1}{a^3}$ $\Rightarrow$ at early times radiation dominated the dynamics of the universe. ρ_r can be measured from the CMB, which implies

$$\Omega_r \simeq 4.2\, 10^{-5}\, h^{-2} \ll \Omega_m.$$

This implies that at present radiation is negligible, and it has only been important since

$$1 + z_{eq} \simeq 23{,}900\, \Omega h^2$$

This is not two different (for $\Omega \sim 0.1$ or so) from the "recombination" epoch $z_{rec} \simeq 1{,}000$ when the universe become neutral.

This is widely regarded as a mere coincidence.

Deceleration Parameter

One can write $R(t)$ in a Taylor series,

$$R(t) = R(t_0)\left[1 + H_0(t-t_0) - \frac{1}{2} q_0 H_0^2 (t-t_0)^2 + \dots\right]$$

where we have defined $q_0 = -\left.\frac{\ddot{R} R}{\dot{R}^2}\right|_{t=t_0}$

Since
$$\ddot{R} = -\frac{4\pi G}{3} R\left(\rho + \frac{3p}{c^2}\right)$$

and replacing $\rho = \rho_m + \rho_{rad} + \rho_v$ $\quad \left(\begin{array}{l} P_{rad} = \frac{1}{3}\rho_{rad} c^2 \\ P_v = -\rho_v c^2 \end{array} \right)$

$\Rightarrow$
$$\frac{\ddot{R}R}{\dot{R}^2} = \frac{4\pi G}{3H^2}\left(\rho_m + 2\rho_{rad} - 2\rho_v\right)$$

$\Rightarrow$
$$q_0 = -\left.\frac{\ddot{R}R}{\dot{R}^2}\right|_{t_0} = \frac{\Omega_m}{2} + \Omega_r - \Omega_v$$

Horizons:

The "particle horizon" is the distance beyond which photons emitted at $t=0$ direction of an observer (at $t=t_0$) have not yet had time to reach him.

For photons
$$c\,dt = R\,dr$$

$$\Rightarrow \Delta r_{H,t_0} = \int_0^{t_0} \frac{c\,dt}{R} = \int_0^1 \frac{c\,dR}{R\dot{R}} = \int_0 \frac{c\,dR}{R\sqrt{\rho R^2 - \frac{Kc^2}{R_0^2}}}$$

Convergence requires $\rho R^2 \xrightarrow[t\to 0]{} \infty$ $\quad \Rightarrow$ Particle horizons are finite for matter and radiation dominated universes, but not for vacuum dominated ones

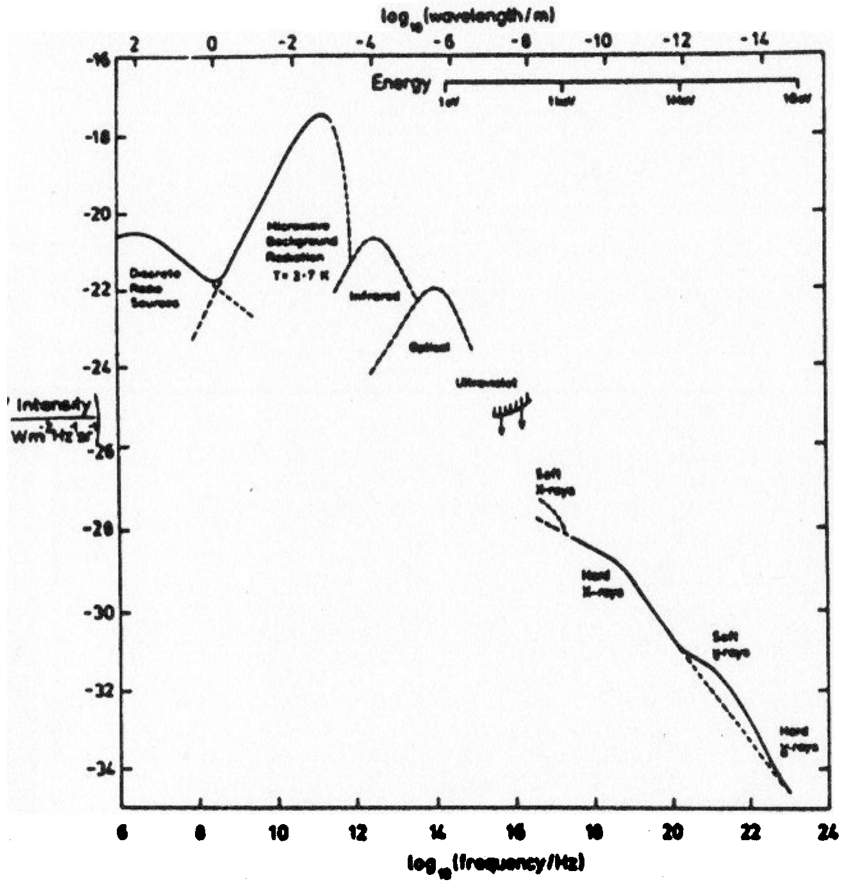


Fig. 9.1. (a) The spectrum of the extragalactic background radiation as it was known in 1969 (Longair and Sunyaev 1971). This figure still provides a good indication of the overall spectral energy distribution of the background radiation. The solid lines indicate regions of the spectrum in which extragalactic background radiation has been detected. The dashed lines were theoretical estimates of the background intensity due to discrete sources and should not be taken too seriously. (b) The spectrum of the extragalactic background radiation plotted as $I = \nu I_\nu = \lambda I_\lambda$ (Longair 1995, courtesy of Dr. Andrew Blain). This presentation shows the amount of energy $\varepsilon = 4\pi I/c$ present per unit volume throughout the Universe at the present epoch. The solid lines in the radio, millimetre, X- and γ-ray wavebands show the observed background intensities. The circles, crosses and squares correspond to upper limits to the background intensity in the far infrared to ultraviolet wavebands and are usually conservative upper limits. We will return to discuss the significance of these limits in Sect. 18.2

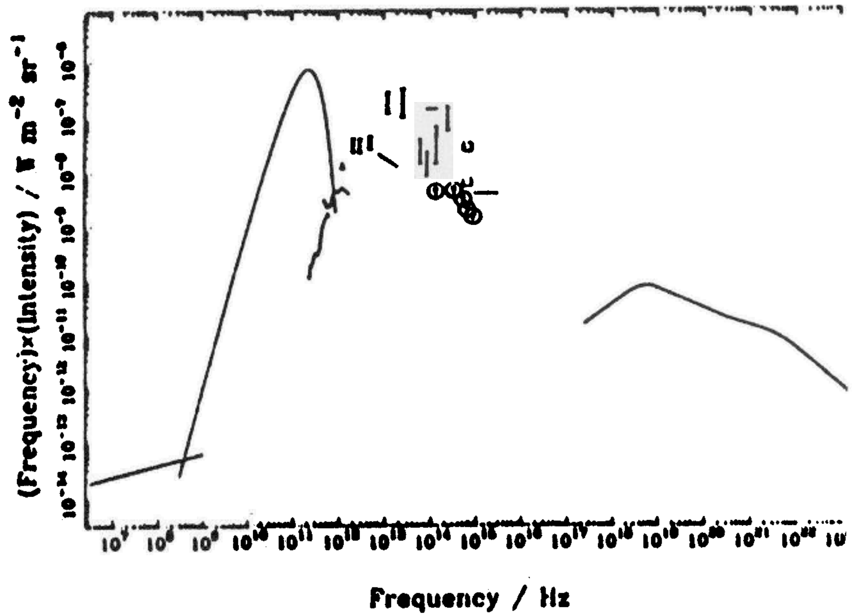


Fig. 9.1. continued

Table 9.1. The energy densities and photon number densities of the extragala[illegible] background radiation in different regions of the electromagnetic spectrum. N[illegible] that these are usually rough estimates which are only useful for making orde[illegible] magnitude calculations.

Waveband	Energy density of radiation eV m^{-3}	Number density of photons m^{-3}
Radio	$\sim 5 \times 10^{-2}$	$\sim 10^{6}$
Microwave	3×10^{5}	10^{9}
Infrared	?	?
Optical	$\sim 2 \times 10^{3}$	$\sim 10^{3}$
Ultraviolet	?	?
X-ray	75	3×10^{-3}
γ-ray	25	3×10^{-6}

various sky surveys. In Table 9.1, typical energy densities and number der[illegible] ties of the photons in each of the wavebands in which a positive detection [illegible] the extragalactic background radiation has been made are listed. It must [illegible] emphasised that these are rough estimates and, for precise work, integratic[illegible] should be taken over the appropriate regions of the spectrum. These figu[illegible] are, however, often useful for making order of magnitude estimates.

"Event Horizons" exist if

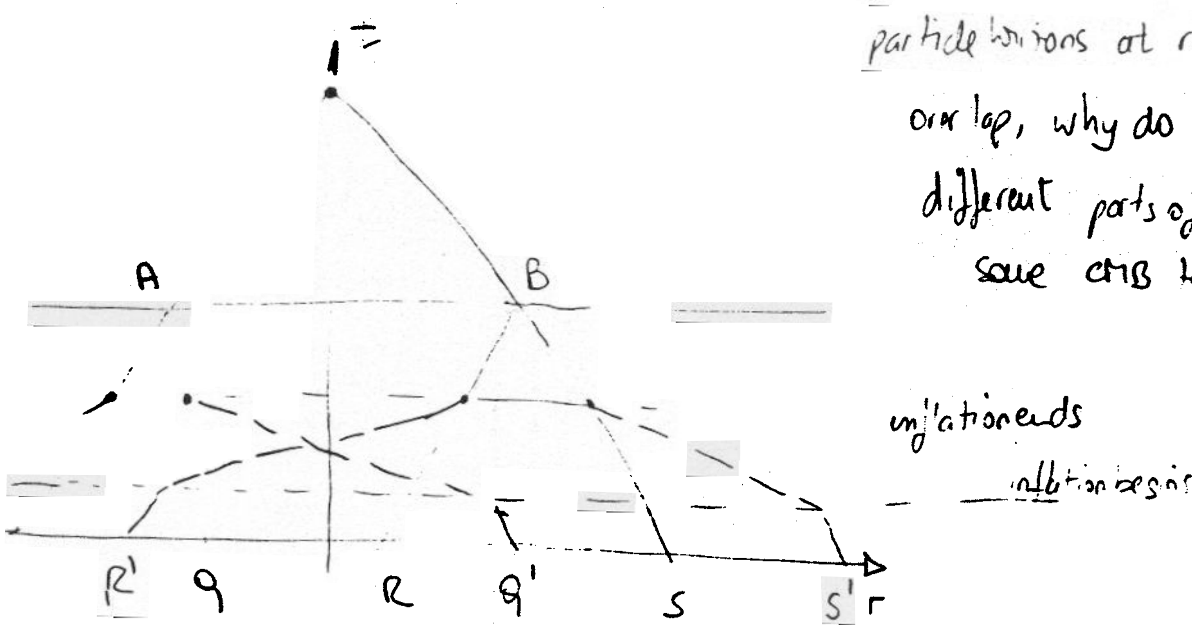
t_{coll} if universe recollapses

$$\int_{t_0}^{\infty} \frac{c\,dt}{R}$$

is finite, it implies that there are events that will never be seen by the chosen observer.

Event horizons exist in closed models and in models that are vacuum dominated at late times. They do not exist in flat models or open models with zero Λ. In these universes an event will eventually be able to influence every other fundamental observer in the universe.

Particle horizons are important in reference to one of the paradoxes posed by the Cosmic Microwave Background on Big Bang theories. In words, since particle horizons at recombination do not overlap, why do diametrically different parts of the sky have the same CMB temperature?



Inflation may provide the answer.