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Stability in Gauged Extended Supergravity*

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Extended supergravity theories with gauged $SO(N)$ internal symmetry have, for $N \geqslant 4$, scalar field potentials which are unbounded below. Nevertheless, it is argued that the theories have ground states with anti-de Sitter background geometry which are stable against fluctuations which vanish sufficiently fast at spatial infinity. Stability is implied because the appropriate conserved energy functional is positive for such fluctuations. Anti-de Sitter space is not globally hyperbolic, but the boundary conditions required for positive energy are also shown to give free field theories with well-defined Cauchy problem. New information on the particle representations of $OSp(1, 4)$ supersymmetry is presented as part of the argument. Supersymmetry requires boundary conditions for spin 0 fields such that only the improved stress tensor leads to a conserved energy functional. Although the stability arguments support the view that gauged supergravity theories are acceptable quantum field theories, the problem of a large cosmological term in the Ads phase of the theories is still unsolved.

1. INTRODUCTION

The eight extended supergravity theories in the form [1–4] with gauged $SO(N)$ internal symmetry, $N = 1,..., 8$, are very attractive from the viewpoint of unification of spins with gravitation and Yang–Mills gauge invariance. However, these theories have special problems beyond those of most quantum gravity theories. Since the form of the Lagrangian and transformation rules of the maximal gauged $SO(8)$ theory have recently been found [4], it is time to address these problems, and we report in this paper on work whose goal is to find the meaning of these theories as quantum field theories.

There are three major problems of gauged extended supergravity:

(1) A large cosmological constant of the form $\Lambda \approx e^2/\kappa^2$ where e is the $SO(N)$ gauge coupling and κ^2 is Newton's constant. The expected order of magitude is of Planck size, $\Lambda \sim 10^{66}\,\mathrm{cm}^{-2} \sim 10^{38}\,\mathrm{GeV}^2$ which exceeds the experimental upper bound by a considerable factor, namely, 10^{120}.

(2) The natural background geometry of these theories is $O(3, 2)$ anti-de Sitter space, henceforth called AdS, which has problems associated with closed time-like

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geodesics (which can be eliminated by taking the covering space) and lack of a global Cauchy surface, so that the initial value problem is ill defined (although one way to avoid this difficulty is known [5]).

(3) For $N \geqslant 4$, there are fundamental scalar fields $\varphi^i(x)$. The associated potential $V(\varphi^i)$ is unbounded from below. Thus the energy spectrum would appear to be unbounded below, and one would doubt that any stable vacuum state exists.

Our work indicates that the last two problems can be solved, and that there is an at least metastable perturbative phase of these theories in n AdS background. This phase is supersymmetric under the expected supergroup $OSp(N, 4)$. Stability is obtained because the relevant energy functional [6] consists of a sum of positive kinetic terms and negative unbounded potential terms, and the former dominate for fluctuations of the fields which vanish sufficiently fast at spatial infinity. The required asymptotic falloff suggests boundary conditions which produce free field theories with well-defined Cauchy problem [5] for the various spin fields in the AdS background. Nonlinear quantum effects are not studied, but it seems that they can be at least approached in perturbation theory. This picture emerges by adapting several distinct lines of argument in the literature [5–7] to the case of gauged extended supergravity and adding some new considerations about supersymmetric field theories on an AdS manifold.

The picture developed here does indicate that the gauged extended supergravity theories are acceptable quantum field theories, but it is a disturbing picture because the AdS background with large cosmological constant is not our world, and problem (1) above is not yet solved. Here we can only speculate about possible dynamical breakdown of supersymmetry and $SO(N)$ gauge invariance or that the perturbative vacuum is unstable to the formation of a space-time foam [8] or some other vacuum field configurtion which does not have the asymptotic properties used here.

The plan of the paper is as follows. In Section 2, the relevant features of gauged supergravity theories are discussed and the background solution is obtained. In Section 3, the necessary properties of the AdS background are summarized. In Section 4, the definition of conserved $SO(3, 2)$ generators and $OSp(N, 4)$ spinor charges is reviewed. The two basic arguments for positive energy of fluctuations above the background are given in Section 5. One is a formal argument based on the supersymmetry algebra and the second is an analytic proof valid for small fluctuations. In Section 6, the free scalar field in an AdS background is discussed, and the boundary conditions required for positive energy are shown to be necessary for a unique solution of the Cauchy problem. However a conforml ambiguity in the definition of the energy functional is encountered, and to investigate this $OSp(N, 4)$ supersymmetric massless multiplets are studied in Section 7. Section 8 is devoted to a short concluding discussion. In Appendix A, we discuss the massive scalar field in an AdS background, and the massive scalar $OSp(1, N)$ supermultiplet is discussed in Appendix B. Finally in Appendix C, justification is given for the use of the improved scalar energy functional.

The stability arguments of this paper are presented in the context of gauged extended supergravity. However the small fluctuations argument is applicble to a general theory with scalar fields coupled to gravity. If the scalar potential $V(\varphi^i)$ has a critical point which is a maximum or a saddle point, the associated energy functional is still positive if this critical point leads to an AdS background and the negative eigenvalues of the "mass matrix" are not too large. A brief description of the work of this paper which stresses the implications outside of gauge supergravity will be published elsewhere [9].

2. Gauged Supergravity and the Background Solution

The full Lagrangian and transformation rules of $N \geqslant 4$ gauged supergravity theories are very complicated. The features relevant to this investigation may already be seen in the complete $N=1$ theory and in part of the $N=4$ Lagrangian which we now present. The $N=1$ theory contains the graviton and gravitino fields $V^a{}_\mu(x)$ and $\psi_\mu(x)$, and the action and transformation rules are

$$\begin{aligned}
I &= \int d^4x \left\{ -\frac{1}{4\kappa^2} \det(V^a{}_\mu) \left(R - 6\frac{e^2}{\kappa^2}\right) + \frac{1}{2}\varepsilon^{\lambda\mu\nu\rho}\bar{\psi}_\lambda \gamma^5 \gamma_\mu \tilde{D}_\nu \psi_\rho \right\}, \\
R_{\mu\nu ab} &= \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} + \omega_{\mu a}{}^c \omega_{\nu cb} - \omega_{\nu a}{}^c \omega_{\mu cb}, \\
\omega_{\nu ab} &= \frac{1}{2}(\Omega_{\nu ab} - \Omega_{\nu ba} - \Omega_{ab\nu}), \\
\Omega_{\mu\nu a} &= \partial_\mu V_{a\nu} - \partial_\nu V_{a\mu} + i\kappa^2 \bar{\psi}_\mu \gamma_a \psi_\nu, \qquad (2.1) \\
\tilde{D}_\nu \psi_\rho &= \left(\partial_\nu + \frac{1}{2}\omega_{\nu ab}\sigma^{ab} + \frac{ie}{2\kappa}\gamma_\nu\right)\psi_\rho, \\
\delta V_{a\mu} &= -i\kappa\bar{\varepsilon}(x)\gamma_a \psi_\mu, \\
\delta\psi_\mu &= \kappa^{-1}\tilde{D}_\mu \varepsilon(x) = \kappa^{-1}\left(\partial_\mu + \frac{1}{2}\omega_{\mu ab}\sigma^{ab} + \frac{ie}{2\kappa}\gamma_\mu\right)\varepsilon(x).
\end{aligned}$$

We see that the parameter e, which enters as the $SO(N)$ coupling constant for $N > 1$, determines a cosmological constant $\Lambda = 3\ e^2/\kappa^2$ and a modified [10] covariant derivative $\tilde{D}_\mu$ of spinorial quantities which are characteristic of these theories.

The full $N=4$ theory [2] contains a spin 2 field $V_{a\mu}$, four spin 3/2 fields ψ^i_μ, six spin 1 fields A^{ij}_μ, four spin 1/2 fields χ^i, and a real scalar field A and pseudoscalar field B, where $i = 1,\ldots, 4$ is an $SO(4)$ index. We are interested in the background field configuration and assume that spinor fields and vector potentials vanish in this background, so that only the spin 2 and spin 0 parts of the Lagrangian are relevant. Any other background will necessarily be less symmetric than that found under these assumptions. The spin 2–spin 0 action is

$$I = \frac{1}{2\kappa^2}\int d^4x \det(V^a{}_\mu)\left\{-\frac{1}{2}R + g^{\mu\nu}\frac{\partial_\mu \bar{z}\,\partial_\nu z}{(1-\bar{z}z)^2} + \frac{e^2}{\kappa^2}\left(3 + \frac{2\bar{z}z}{1-\bar{z}z}\right)\right\}, \qquad (2.2)$$

where $z(x)$ is related to canonically normalized spinless fields by $z(x) = \kappa(A(x) + iB(x))$. Note that the "internal space" of the scalar fields is the hyperbolic plane or homogeneous space $SU(1,1)/U(1)$ with complete metric

$$ds^2 = \frac{d\bar{z}\, dz}{(1 - \bar{z}z)^2}, \qquad |z| < 1. \tag{2.3}$$

There is a scalar field potential

$$V(\bar{z}, z) = -\frac{e^2}{\kappa^2}\left(3 + \frac{2\bar{z}z}{1 - \bar{z}z}\right) \tag{2.4}$$

which is unbounded from below as $|z| \to 1$ and which reduces the nonlinearly realized $SU(1,1)$ symmetry of the ungauged theory [11] to $U(1)$.

This situation is typical in gauged extended supergravity. The scalar field dynamics is that of a nonlinear σ-model for the manifolds [12] $SU(5,1)/U(5)$ for $N = 5$, $SO^*(12)/U(6)$ for $N = 6$, and $E(7)/SU(8)$ for $N = 8$, and in each case the gauging of $SO(N)$ introduces a scalar potential which is unbounded below and reduces the symmetry to $SO(N) \times U(1)$.

We now look for a background solution of the field equations of (2.2) which are (with $a^2 = e^2/\kappa^2$)

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + 3a^2 g_{\mu\nu} = T_{\mu\nu}, \tag{2.5}$$

$$T_{\mu\nu} = \frac{\partial_\mu \bar{z}\, \partial_\nu z + \partial_\nu \bar{z}\, \partial_\mu z}{(1 - \bar{z}z)^2} - g_{\mu\nu}\left[\frac{\partial\bar{z}\, \partial z}{(1 - \bar{z}z)^2} + 2a^2 \frac{\bar{z}z}{1 - \bar{z}z}\right], \tag{2.6}$$

$$\frac{1}{\sqrt{-g}}\, \partial_\mu(\sqrt{-g}\, g^{\mu\nu} \partial_\nu z) + \Gamma(\bar{z}, z)\, g^{\mu\nu}\, \partial_\mu \bar{z}\, \partial_\nu \bar{z} + \frac{\partial V}{\partial \bar{z}} = 0, \tag{2.7}$$

where $\Gamma(\bar{z}, z)$ is the Christoffel connection of $SU(1,1)/U(1)$ whose exact form is not relevant. We assume that the scalar field $z(x)$ is constant in the background, again with the rationale that any other background with space-time dependent $z(x)$ would be less symmetric.

The scalar field equations then require that we choose a critical point $\partial V/\partial z = 0$ of the potential which in this case is unique, namely, $z = 0$, the *maximum* of $V(\bar{z}, z)$. The stress tensor $T_{\mu\nu}$ then vanishes, and we choose the maximally symmetric solution of the Einstein equation (2.5), namely, the $SO(3,2)$ invariant anti-de Sitter manifold, which we discuss shortly. (In our conventions, $\Lambda = (3e^2/\kappa^2) > 0$ gives $SO(3,2)$ rather than $SO(4,1)$ invariance.) Thus the problem we face is to establish stability of a vacuum field configuration which is the *maximum* of a scalar potential unbounded from below.

For $N > 4$, all gauged supergravity theories have an AdS background solution with constant scalar fields invariant under $SO(N) \times U(1)$ which is analogous to the one obtained above for $N = 4$. Our analysis of stability is immediately applicable to these vacua. However, the scalar potentials are more complicated for $N > 4$, and there are other critical points [9] which require separate analysis, using the small fluctuations argument of [9] and Appendix B.

3. Anti-de Sitter Space

We summarize in this section some well-known properties [5, 13] of AdS space, which is the maximally symmetric solution of Einstein's equations with scalar curvature $R = 12a^2$. This space is the 4-dimensional hyperboloid $\eta_{AB} y^A y^B = a^{-2}$ embedded in $\mathbb{R}^5$ with Cartesian coordinates $y^A, A = 0, 1, 2, 3, 4$ and flat indefinite metric $\eta_{AB} = (+, -, -, -, +)$. This space has closed time-like curves, and to avoid the consequent causality problems one passes to the covering space CAdS, obtained by considering as distinct the sheets obtained by successive 2π rotations in the $y^0 - y^4$ plane.

One can introduce intrinsic coordinates $x^\mu = (t, x^i)$ with $\rho^2 = x^i x^i = \mathbf{x} \cdot \mathbf{x}$ by the transformation $y^A(x^\mu)$ given by

$$y^0 = \frac{1}{a} \sin t \sec \rho, \qquad y^i = \frac{1}{a\rho} \tan \rho x^i, \qquad y^4 = -\frac{1}{a} \cos t \sec \rho. \tag{3.1}$$

The radial variable ρ has the range $0 \leqslant \rho < \pi/2$, and $\rho = \pi/2$ corresponds to spatial infinity; the range of t is $-\pi \leqslant t < \pi$ for AdS and $-\infty < t < \infty$ for the covering space CAdS. One then finds a complete static metric which can be written either in terms of the quasi-Cartesian x^i using the unit vector $\hat{x} = \rho^{-1}\mathbf{x}$ and Cartesian dot products such as $\hat{x} \cdot d\mathbf{x} = \hat{x}^i \, dx^i$ or in terms of spherical coordinates ρ, θ, φ, defined by $x' = \rho \sin \theta \cos \varphi$, $x^2 = \rho \sin \theta \sin \varphi$, $x^3 = \cos \theta$. The metric is

$$ds^2 = (a \cos \rho)^{-2} \, [(dt)^2 - (d\rho)^2 - \sin^2 \rho((d\theta)^2 + \sin^2 \theta(d\varphi)^2)] \tag{3.2}$$

$$= (a \cos \rho)^{-2} \left[(dt)^2 - (\hat{x} \cdot d\mathbf{x})^2 - \left(\frac{\sin \rho}{\rho} \right)^2 (d\mathbf{x} \cdot d\mathbf{x} - (\mathbf{x} \cdot d\mathbf{x})^2) \right]. \tag{3.3}$$

For some purposes it is convenient to use the infinite range radial coordinate r given by the transformation $\operatorname{ch} ar = (\cos \rho)^{-1}$ which yields the metric

$$ds^2 = \left(\frac{\operatorname{ch} ar}{a} \right)^2 (dt)^2 - (dr)^2 - \left(\frac{\operatorname{sh} ar}{a} \right)^2 ((d\theta)^2 + \sin^2 \theta(d\varphi)^2). \tag{3.4}$$

One notes from (3.2) that the metric of constant t surfaces is a conformal factor $(\cos \rho)^{-2}$ times the standard metric of the upper hemisphere of S^3. Thus AdS is conformal [5] to half of the Einstein static universe, which has scalar curvature $R = -6a^2$.

The absence of a global Cauchy surface in AdS can be seen from the equation $dt = d\rho$ for radial null geodesics. One sees that information propagates from the origin ($\rho = 0$) to spatial infinity ($\rho = \pi/2$) in finite time $t = \pi/2$. Conversely, information leaking in from spatial infinity reaches the origin in finite time. This can be viewed as information crossing the equator of S^3 in the conformal extension to ESU, which itself is globally hyperbolic because S^3 is compact. The Cauchy problem in AdS becomes well-defined [5] if suitable boundary conditions are imposed on the equator, as we discuss below.

The isometry group of AdS space is $SO(3, 2)$ with generators

$$L_{AB} = y_A \frac{\partial}{\partial y^B} - y_B \frac{\partial}{\partial y^A} \tag{3.5}$$

in the coordinates of the embedded hyperboloid. By expressing each L_{AB} in intrinsic coordinates, one finds the 10 Killing vectors $\xi_{AB} = \xi_{AB^\mu}(\partial/\partial x^\mu)$ which satisfy the Killing condition

$$D_\mu \xi_{AB\nu} + D_\nu \xi_{AB\mu} = 0. \tag{3.6}$$

Using the coordinates $t, \mathbf{x}$ one finds the explicit Killing vectors

$$\begin{aligned}
\xi_{04} &= \frac{\partial}{\partial t}, \\
\xi_{i4} &= -\sin t \sin \rho \hat{x}^i \frac{\partial}{\partial t} \\
&\quad + \cos t \left[\frac{\rho}{\sin \rho} \left(\frac{\partial}{\partial x^i} - \hat{x}^i \hat{x} \cdot \frac{\partial}{\partial \mathbf{x}} \right) + \cos \rho \hat{x}^i \hat{x} \cdot \frac{\partial}{\partial \mathbf{x}} \right], \\
\xi_{i0} &= -\cos t \sin \rho \hat{x}^i \frac{\partial}{\partial t} \\
&\quad - \sin t \left[\frac{\rho}{\sin \rho} \left(\frac{\partial}{\partial x^i} - \hat{x}^i \hat{x} \cdot \frac{\partial}{\partial \mathbf{x}} \right) + \cos \rho \hat{x}^i \hat{x} \cdot \frac{\partial}{\partial \mathbf{x}} \right], \\
\xi_{ij} &= x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j}.
\end{aligned} \tag{3.7}$$

The $SO(3, 2)$ group generators satisfy

$$\begin{aligned}
[L_{AB}, L_{CD}] &= \eta_{BC} L_{AD} - \eta_{AC} L_{BD} - \eta_{BD} L_{AC} + \eta_{AD} L_{BC} \\
&= f_{AB;CD}^{\ EF} L_{EF},
\end{aligned} \tag{3.8}$$

and the Killing vectors determine the same Lie algebra, viz.,

$$[\xi_{AB}, \xi_{CD}] = f_{AB;CD}^{\ EF} \xi_{EF}. \tag{3.9}$$

In order to handle spinors and for some other purposes, it is useful to choose a local frame. One convenient choice is specified by the frame 1-forms $e^a = V^a{}_\mu \, dx^\mu$ with

$$\begin{aligned} ds^2 &= e^0 \cdot e^0 - e^1 \cdot e^1 - e^2 \cdot e^2 - e^3 \cdot e^3, \\ e^0 &= (a \cos \rho)^{-1} \, dt, \\ e^i &= (a \cos \rho)^{-1} \left[\hat{x}^i \hat{x} \cdot d\mathbf{x} + \frac{\sin \rho}{\rho} (dx^i - \hat{x}^i \hat{x} \cdot d\mathbf{x}) \right]. \end{aligned} \tag{3.10}$$

The connection 1-forms, defined by

$$de^a = -e^b \wedge \omega_b{}^a, \qquad \omega_{ab} = -\omega_{ba}, \tag{3.11}$$

are then given by

$$\begin{aligned} \omega^{0i} &= - \tan \rho \hat{x}^i \, dt, \\ \omega^{ij} &= \rho^{-1} (\sec \rho - 1)(\hat{x}^i \, dx^j - \hat{x}^j \, dx^i). \end{aligned} \tag{3.12}$$

Finally, we record the inverse vierbein $V_a{}^\mu$ using the vector fields $V_a = V_a{}^\mu (\partial / \partial x^\mu)$ which are given by

$$\begin{aligned} V_0 &= a \cos \rho \frac{\partial}{\partial t}, \\ V_i &= a \cos \rho \left[\hat{x}^i \hat{x} \cdot \frac{\partial}{\partial \mathbf{x}} + \frac{\rho}{\sin \rho} \left(\frac{\partial}{\partial x^i} - \hat{x}^i \hat{x} \cdot \frac{\partial}{\partial \mathbf{x}} \right) \right]. \end{aligned} \tag{3.13}$$

This is the basis dual to that given by the frame 1-forms.

Another important but less familiar concept is that of a Killing spinor [6], namely, a spinor $\varepsilon(x)$ which is covariantly constant with the modified derivative [10] suggested by the transformation rule (2.1), i.e.,

$$\tilde{D}_\mu \varepsilon(x) = \left(\partial_\mu + \frac{1}{2} \omega_{\mu ab} \sigma^{ab} + \frac{ia}{2} V_{a\mu} \gamma^a \right) \varepsilon(x) = 0. \tag{3.14}$$

This equation is integrable because the modified connection is flat in a maximally symmetric background such as AdS, i.e.,

$$\begin{aligned} [\tilde{D}_\mu, \tilde{D}_\nu] \, \varepsilon &= \tfrac{1}{2} (R_{\mu\nu ab} \sigma^{ab} - 2a^2 \sigma_{\mu\nu}) \, \varepsilon = 0, \\ R_{\mu\nu ab} &= a^2 (V_{a\mu} V_{b\nu} - V_{a\nu} V_{b\mu}). \end{aligned} \tag{3.15}$$

There are four linearly independent Killing spinors, which can be written in the form

$$\varepsilon_\alpha(x) = S_{\alpha\beta}(x) \, \xi_\beta \tag{3.16}$$

where ξ_β is any constant spinor, and the matrix $S(x)$ is

$$S(x) = (\cos\rho)^{-1/2}\left[\cos\frac{\rho}{2} + i\gamma\cdot\hat{x}\sin\frac{\rho}{2}\right]\left[\cos\frac{t}{2} - i\gamma^0\sin\frac{t}{2}\right] \tag{3.17}$$

in the frame defined above. The γ-matrices in (3.17) are flat, i.e., $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$ with $\eta_{ab} = (+, -, -, -)$. For later use we choose the γ-matrices

$$\gamma^0 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} & \sigma^i \\ -\sigma^i & \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}. \tag{3.18}$$

One should note that the spinor representation of $SO(3, 2)$ is generated by matrices l_{AB} with $l_{ab} = \sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$ and $l_{a4} = -(i/2)\gamma_a$. The matrix $S(x)$ above represents an $SO(3, 2)$ element with a simple geometric interpretation [14]. The vector fields $V_a(x)$ of (3.13) can be interpreted as tangent vector fields

$$V_a(y) = V_a{}^\mu(x)\frac{\partial y^A}{\partial x^\mu}\frac{\partial}{\partial y^A} \tag{3.19}$$

on the embedded hyperboloid via the transformation $y^A(x^\mu)$ of (3.1). Then $S(x)$ is the representative of the $SO(3, 2)$ transformation which takes the vectors $V_{\underline{a}}$ at $y(0)$ into $V_a(y(x))$. Further the connection in (3.14) can be written as $S(x)\,\partial_\mu \bar{S}(x)$. Finally there is a bilinear relation between Killing spinors and Killing vectors, which is determined by

$$\begin{aligned} -i\bar{S}(x)\,\gamma^a S(x)\,V_a &= l^{AB}\xi_{AB} = -i\gamma^a\xi_{a4} + \sigma^{ab}\xi_{ab}, \\ \bar{S}(x) &= \gamma^0 S^+(x)\,\gamma^0 = S^{-1}(x). \end{aligned} \tag{3.20}$$

4. Background Invariance and Conserved Charges

In a Minkowski background geometry the perturbative dynamics is Poincaré invariant. Analogously, the perturbative dynamics in an AdS background should be $SO(3, 2)$ invariant. For each generator L_{AB} there must be a conserved function M_{AB} of the background field configuration and of the fluctuations above the background, and these charges must generate the $SO(3, 2)$ group

$$[M_{AB}, M_{CD}] = -if_{AB;CD}^{EF} M_{EF} \tag{4.1}$$

for commutators computed in the canonical formalism. It is the time translation generator M_{AB} that is the appropriate energy functional of the system and which is shown below to be positive for fluctuations which fall off sufficiently fast.

However, more is true. The complete background field configuration in gauged supergravity, namely, $z = 0$, $\chi^i = 0$, $A_\mu^{ij} = 0$, $\psi_\mu^i = 0$, $g_{\mu\nu} = g_{\mu\nu}^{\text{AdS}}$, has a global supersymmetry. To see this, we write the local supersymmetry variations of the gauged $N=4$ theory in truncated form

$$\delta V_{a\mu} = -i\kappa \bar{\varepsilon}^i \gamma_a \psi^i_\mu, \tag{4.2}$$

$$\delta \psi^i_\mu = \frac{1}{\kappa} \tilde{D}_\mu \varepsilon^i - \frac{i}{2} \sigma^{\lambda\rho} F^{ij}_{\lambda\rho} \gamma_\mu \varepsilon^i + \cdots, \tag{4.3}$$

$$\delta A^{ij}_\mu = \frac{i}{\sqrt{2}} \varepsilon^{ijkl} \bar{\varepsilon}^k \gamma_\mu \chi^l - \bar{\varepsilon}^i \psi^j_\mu + \bar{\varepsilon}^j \psi^i_\mu + \cdots, \tag{4.4}$$

$$\delta \chi^i = -\frac{1}{\sqrt{2}} \left[i\gamma^\mu \partial_\mu (A + i\gamma_5 B) + a(A - i\gamma_5 B) \right] \varepsilon^i$$

$$- \frac{1}{2\sqrt{2}} \varepsilon^{ijkl} \sigma^{\mu\nu} \varepsilon^j F^{kl}_{\mu\nu} + \cdots, \tag{4.5}$$

$$\delta A = \frac{1}{\sqrt{2}} \bar{\varepsilon}^i \chi^i + \cdots, \tag{4.6}$$

$$\delta B = \frac{i}{\sqrt{2}} \bar{\varepsilon}^i \gamma_5 \chi^i + \cdots, \tag{4.7}$$

where omitted terms are at least of quadratic order in the fields z, χ, A_μ, ψ_μ which vanish in the background. It is easy to see that the background is invariant under supersymmetry transformation with spinors $\varepsilon^i(x)$ which satisfy $\tilde{D}_\mu \varepsilon = 0$, i.e., with Killing spinors. Thus, one expects the perturbative dynamics in this background to have a global supersymmetry. There should be a conserved spinor charge $Q_\alpha{}^i$ for each $i = 1, \ldots, 4$ and for each component of the constant spinor ξ_α of $\varepsilon_\beta(x) = S_{\beta\alpha}(x)\, \xi_\alpha$.

Clearly, the full invariance group is the supergroup $OSp(4, 4)$ with the $Q_\alpha{}^i$ as odd elements and the ten M_{AB} plus six generators I^{ij} of $SO(4)$ transformations as even elements. The basic anticommutator is

$$\{Q_\alpha{}^i, \bar{Q}^{\beta j}\} = \delta^{ij} (\gamma^a M_{a4} + i\sigma^{ab} M_{ab})_\alpha{}^\beta + i\delta_\alpha{}^\beta I_{ij}. \tag{4.8}$$

We now make these considerations more concrete and review the method of construction of conserved charges in an AdS background [6]. One divides the metric tensor (and other fields) into background values plus deviation, e.g., $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, etc. One then takes the full Einstein equation of the theory

$$R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R + 3a^2 g^{\mu\nu} = T^{\mu\nu}, \tag{4.9}$$

where $T^{\mu\nu}$ is the stress tensor due to all matter fields, and transfers all terms of second order and higher in $h_{\mu\nu}$ to the right-hand side, thus obtaining

$$R^{\mu\nu}_L - \tfrac{1}{2} \bar{g}^{\mu\nu} R_L - 3a^2 h^{\mu\nu} = t^{\mu\nu} + T^{\mu\nu} \equiv \Theta^{\mu\nu}, \tag{4.10}$$

where the indices of $h^{\mu\nu}$ are raised with the background metric and $t^{\mu\nu}$ contains terms of order h^2, h^3, etc. By expansion of the standard Bianchi identity in powers of $h_{\mu\nu}$, one sees that the left side of (4.10) satisfies the linearized Bianchi identity

$$\bar{D}_\mu(R_L^{\mu\nu} - \tfrac{1}{2}\bar{g}^{\mu\nu}R_L - 3a^2h^{\mu\nu}) = 0, \tag{4.11}$$

where $\bar{D}_\mu$ is the covariant derivative for the AdS background. Thus the right side of (4.1) must be background conserved, i.e.,

$$\bar{D}_\mu \Theta^{\mu\nu} = 0 \tag{4.12}$$

as a consequence of gravity and matter field equations. If ξ_ν is any Killing vector of the background, one sees that

$$\partial_\mu(\sqrt{-\bar{g}}\,\Theta^{\mu\nu}\xi_\nu) = 0 \tag{4.13}$$

because of (4.12) and the Killing condition (3.6). Thus the charges

$$M_{AB} = \int d^3x\,\sqrt{-\bar{g}}\,\Theta^{0\nu}\xi_{AB\nu} \tag{4.14}$$

obtained by integration over constant t hypersurfaces are conserved in t if the fields in $\Theta^{0\nu}$ fall off fast enough at spatial infinity.

Since the background fields are manifestly $O(4)$ invariant, the construction of $SO(4)$ charges I^{ij} presents no difficulty, and one may use the standard Noether method for constant gauge parameters.

The construction [6] of spinor charges $Q_\alpha{}^i$ is analogous to the one for M_{AB}. One divides the vierbein into background plus deviation and writes the full spin 3/2 field equation of the supergravity theory as

$$\varepsilon^{\lambda\mu\nu\rho}\gamma_5\bar{\gamma}_\mu\bar{\bar{D}}_\nu\psi^i_\rho = \mathscr{T}^{i\lambda} \tag{4.15}$$

where the left side contains only the background frame and connection and $\mathscr{T}^{i\lambda}$ is a "vector-spinor density" which contains the Noether current of the matter multiplets plus terms involving the deviation $V_{a\mu} - \bar{V}_{a\mu}$. The left side has identically vanishing divergence if one uses the derivative $\bar{\bar{D}}_\lambda = \partial_\lambda + \frac{1}{2}\bar{\omega}_{\lambda ab}\sigma^{ab} + (ia/2)\,\bar{V}_{a\lambda}\gamma^a$ and recalls (3.15), so that the right side satisfies the conservation law

$$\bar{\bar{D}}_\lambda\mathscr{T}^{i\lambda} = 0. \tag{4.16}$$

If we contract with a Killing spinor $\bar{\varepsilon}(x) = \bar{\xi}\bar{S}(x)$ and use (3.14), we find a conserved density

$$\partial_\lambda(\bar{\varepsilon}(x)\,\mathscr{T}^{i\lambda}(x)) = 0 \tag{4.17}$$

so that the desired conserved spinor charges are given by

$$Q_\alpha{}^i = \int d^3x\,\bar{S}_{\alpha\beta}(x)\,\mathscr{T}^{i0}_\beta(x). \tag{4.18}$$

The charges M_{AB} and $Q_\alpha{}^i$ can be expressed as surface integrals at spatial infinity [6], but this form will not be required here. All charges involve initial data for the various fields on a constant-t surface. As in any gauge theory, some of the field equations are not equations of motion but constraint equations which impose conditions on the initial data at fixed t. These constraint equations must be satisfied in order that the charges be conserved.

One expects that the structure relations of the OSp(4, 4) group, such as (4.1) and (4.8), are satisfied by the charges as Poisson bracket relations in the classical theory and as operator commutators in the quantum theory. For matter fields it is straightforward to establish this using canonical equal-time commutators, the Lie bracket relations (3.9) for Killing vectors, and the relation (3.19) between Killing spinors and Killing spinors. In the full supergravity theory, this problem requires an intricate application of the canonical formalism. This has been done for the asymptotically flat background geometry in general relativity and supergravity [15], but it seems to be an open problem for the AdS background which will depend delicately on the asymptotic behaviour as past results and our discussion below indicate. One further point is that the Killing vectors ξ_{AB} used to define M_{AB} do not leave the background frames invariant. An appropriate local Lorentz transformation to compensate this change of frames must be incorporated when the M_{AB} are expressed in terms of canonical variables. In any case, despite lack of an explicit proof, it is reasonable to assume that the OSp(4, 4) structure relations are satisfied by the Killing charges, and we shall so assume.

5. Positive Energy

There are two arguments which suggest that the Killing energy

$$\begin{aligned} E = M_{04} &= \int d^3 x \sqrt{-\bar{g}}\, \Theta^{0\nu} \xi_{04\nu} \\ &= \int d^3x \sqrt{-\bar{g}}\, \Theta^0{}_0 \end{aligned} \tag{5.1}$$

is positive despite the scalar potential which is unbounded below. The first is formal and valid to all orders in the deviations from background, and the second is a concrete argument for the theory linearized above the background, i.e., quadratic in gravitational and scalar perturbations.

The first argument [6] is an adaptation of the standard positive energy argument of Poincaré supersymmetry to the OSp(N, 4) superalgebra. The trace of the product of $\gamma^0_{\alpha\beta}$ with the fundamental anticommutator (4.8) immediately yields

$$E = M_{04} = \tfrac{1}{4} \sum_\alpha (Q_\alpha{}^i Q_\alpha{}^{i+} + Q_\alpha{}^{i+} Q_\alpha{}^i) \tag{5.2}$$

for any fixed $SO(N)$ index i. Thus, if the full quantum theory can be defined, it is reasonable to assume that the Killing energy will be a positive operator. Further in the limit [16] as $\hbar \to 0$, one would expect positive energy for any classical configuration of boson fields of the theory.

The formal argument gives little insight into the reason for positive energy, and in order to understand this we write the gravitational and scalar contribution to the energy of $N = 4$ gauged extended supergravity more explicitly as

$$E = \int d^3x \sqrt{-\bar{g}} \left\{ t^0{}_0 + \frac{\bar{g}^{00}\, \partial_0 \bar{z}\, \partial_0 z - \bar{g}^{ij}\, \partial_i \bar{z}\, \partial_j z}{(1 - \bar{z}z)^2} - 2a^2 \frac{\bar{z}z}{1 - \bar{z}z} \right\}. \tag{5.3}$$

In the full nonlinear theory there is a coupling between the terms in $t^0{}_0$ and $T^0{}_0$ through the constraint equations of the generic form

$$\nabla^2 h \sim \frac{\bar{z}z}{1 - \bar{z}z}. \tag{5.4}$$

In order to pinpoint the reason for positivity, we linearize and keep terms which are bilinear in $h_{\mu\nu}$ and z. The energy functional then decomposes into a sum of independent contributions from the metric and scalar fluctuations, viz.,

$$E = E_{\text{grav.}} + E_{\text{scalar}}, \tag{5.5}$$

where

$$E_{\text{grav.}} = \int d^3x \sqrt{-\bar{g}}\, t^0{}_0 \tag{5.6}$$

is known [6] to be positive to quadratic order in metric fluctuations and

$$\begin{aligned} E_{\text{scalar}} &= \int d^3x \sqrt{-\bar{g}}\, [\bar{g}^{00}\, \partial_0 \bar{z}\, \partial_0 z - \bar{g}^{ij}\, \partial_i \bar{z}\, \partial_j z - 2a^2 \bar{z}z] \\ &= \frac{1}{a^3} \int dt\, d^2\Omega \operatorname{ch} ar \operatorname{sh}^2 ar \left\{ \frac{a^2}{\operatorname{ch}^2 ar} \partial_0 \bar{z}\, \partial_0 z \right. \\ &\quad \left. + \frac{a^2}{\operatorname{sh}^2 ar} \left(\partial_\theta \bar{z}\, \partial_\theta z + \frac{1}{\sin^2 \theta} \partial_\varphi \bar{z}\, \partial_\varphi z \right) + \partial_r \bar{z}\, \partial_r z - 2a^2 \bar{z}z \right\}. \end{aligned} \tag{5.7}$$

The linearized approximation we are making corresponds to a scalar field with negtive mass square term. Such a theory would certainly be unstable in a flat Minkowski background since a fluctuation with an exponential falloff $z \sim e^{-\lambda r}$ gives energy $E \sim (\lambda^2 - 2a^2)$ which is negative for small λ. In AdS space the energy functional contains the volume factor $\sqrt{-\bar{g}} \sim \operatorname{ch} ar \operatorname{sh}^2 ar$. The function space of field fluctuations is defined by the requirement that the energy functional converges, and this requires exponential falloff with $\lambda > \frac{3}{2}a$. The energy of the exponential fluctuation is

$$E = (\lambda^2 - 2a^2)\frac{4\pi}{a^3}\int dr \operatorname{ch} ar \operatorname{sh}^2 ar\, e^{-2\lambda r} \tag{5.8}$$

which is positive. The energy functional is the sum of a positive kinetic term and a negative potential term. The basic reason for positivity is that allowed fluctuations are effectively confined to the region $r < a^{-1}$ and the kinetic energy cannot be made arbitrarily small.

The treatment of the exponential fluctuation $z \sim e^{-\lambda r}$ does clarify the reason for positivity, but a proof valid for general perturbations $z(t, r, \theta, \varphi)$ remains to be given. Toward this end we make the conformal transformation

$$z' = \operatorname{ch} ar\, z, \qquad \operatorname{ch} ar\, \partial_r z = \partial_r z' - a \operatorname{th} ar\, z'. \tag{5.9}$$

After substitution in (5.7) and integration by parts one finds

$$E = \frac{1}{a}\int dr\, d^2\Omega \left\{\frac{\operatorname{sh}^2 ar}{\operatorname{ch} ar}\partial_0 \bar{z}'\, \partial_0 z' + \frac{1}{\operatorname{ch} ar}\left(\partial_\theta \bar{z}'\, \partial_\theta z' + \frac{1}{\sin^2\theta}\partial_\varphi \bar{z}'\, \partial_\varphi z'\right)\right.$$
$$\left. + \frac{\operatorname{sh}^2 ar}{a^2 \operatorname{ch} ar}\left(\partial_r \bar{z}'\, \partial_r z + \frac{a^2 \bar{z}' z'}{\operatorname{ch}^2 ar}\right)\right\} + \frac{1}{a^2}\int d^2\Omega \left(\frac{\operatorname{sh}^3 ar}{\operatorname{ch}^2 ar}\bar{z}' z'\right)\Bigg|_{r=0}^{\infty}. \tag{5.10}$$

This integral converges and the surface term vanishes provided $z(t, r, \theta, \varphi) \sim e^{-\lambda r}$ as $r \to \infty$ with $\lambda > \frac{3}{2}a$, and for any such perturbation the energy is positive.

The exponential falloff required above is particular to the choice of coordinates in (5.7). The Killing energy is invariant under changes of the spatial coordinate system, and there are analogous boundary conditions at spatial infinity in any set of coordinates. The conclusion is that for any fluctuation which falls off sufficiently fast at spatial infinity that the Killing energy is finite, the energy is, in fact, positive. In order to understand more completely the role of the required boundary condition, we study in subsequent sections the free field problem in an AdS background and the requirements for supersymmetry. The picture above will turn out to be basically correct, but there are some surprises as we will see.

6. Free Scalar Field in an AdS Background

The scalar wave equation of the linearized gauged $N = 4$ supergravity theory is

$$\Box z - 2a^2 z = 0 \tag{6.1}$$

(where $\Box$ is the covariant D'Alembertian), and the first thing to notice is the "conformal value of the mass term" [5, 17], i.e., this wave equation comes from the conformal invariant action

$$I = \int d^4x \sqrt{-g}\,(g^{\mu\nu}\,\partial_\mu \bar{z}\,\partial_\nu z - \tfrac{1}{6}R\bar{z}z) \tag{6.2}$$

with $R = 12a^2$, the scalar curvature of AdS. Since the wave equation in conformal invariant and AdS is conformally flat, the Green's function [5] $G_{\text{AdS}}(x, x')$ has support only on local light cones $\sigma(x, x') = 0$ where $\sigma(x, x')$ is the geodesic distance between the points x^μ and x'^μ. This property is the best definition of a "massless" particle in AdS. The usual mass is not related to a Casimir operator of $SO(3, 2)$, yet we expect that supergravity has "massless" linearized excitations.

The basic idea of our treatment is to impose conditions on the function space of scalar fields $z(x)$, so that the Cauchy problem has a unique solution in parallel to the situation of the free field quantization in Minkowski space. We will state our assumptions in physical terms, but the procedure seems equivalent to the use of a positive self-adjoint extension of the wave operator to circumvent the lack of global hyperbolicity of the background [18]. The method is similar to that of two previous treatments [5, 7], and results coincide.

In this section it is convenient to use the coordinates t, ρ, θ, φ of (3.2) in which spatial infinity is at $\rho = \pi/2$. We use the locally conserved current density

$$j^\mu = i\sqrt{-g}\, g^{\mu\nu}(\bar{z}_1 \overleftrightarrow{\partial}_\nu z_2) \tag{6.3}$$

to define a standard scalar product

$$(z_1, z_2) = i \int d^3x \sqrt{-g}\, g^{0\nu}(\bar{z}_1 \overleftrightarrow{\partial}_\nu z_2) \tag{6.4}$$

which is formally conserved in time for solutions of (6.1). Analogously, the Killing energy

$$E = \int d^3x \sqrt{-g}\, T^0{}_0 \tag{6.5}$$

is formally conserved for solutions of (6.1). We will choose boundary conditions at $\rho = \pi/2$, such that these formally conserved quantities are actually conserved.

This will lead us to a complete set of positive frequency modes $\varphi_{\omega lm}(x) = e^{-i\omega t} f_{\omega lm}(\rho, \theta, \varphi)$ which are orthonormal in the scalar product above. Any allowed solution of the wave equation can then be expanded as

$$z(x) = \sum_{\omega lm} \{a_{\omega lm} \varphi_{\omega lm}(x) + b^*_{\omega lm} \varphi^*_{\omega lm}(x)\} \tag{6.6}$$

with coefficients obtained from the initial data $z(t_0, \rho, \theta, \varphi)$ and $\partial_0 z(t_0, \rho, \theta, \varphi)$ via

$$a_{\omega lm} = (\varphi_{\omega lm}, z), \qquad b^*_{\omega lm} = -(\varphi^*_{\omega lm}, z). \tag{6.7}$$

The Hilbert space is the set of pairs of functions $z(t_0, \rho, \theta, \varphi)$ and $\partial_0 z(t_0, \rho, \theta, \varphi)$ with square summable expansion coefficients, viz.,

$$\sum |a_{\omega lm}|^2 + \sum |b_{\omega lm}|^2 < \infty \tag{6.8}$$

and within this space the Cauchy problem has a unique solution.

A further consequence of this procedure is that the set of modes $\varphi_{\omega lm}(x)$ forms a basis of a particle representation of $SO(3, 2)$, i.e., an infinite dimensional unitary representation [7] $D(\omega_0, s)$ where ω_0 is the lowest eigenvalue of the energy (in units of a), and s is the angular momentum of the lowest energy state. For a scalar field $s = 0$, and energy is quantized in even integer steps as indicated in the weight diagram in Fig. 1.

Before determining the modes $\varphi_{\omega lm}(x)$, we must discuss an ambiguity in the definition of the energy. For the free scalar field there are two relevant stress tensors, the minimal tensor

$$T_{\mu\nu} = \partial_\mu \bar{z}\, \partial_\nu z + \partial_\nu \bar{z}\, \partial_\mu z - g_{\mu\nu}(\partial \bar{z}\, \partial z + \tfrac{1}{6} R \bar{z} z) \tag{6.9}$$

with covariant divergence (computed using (6.1))

$$D_\mu T^\mu{}_\nu = -\tfrac{1}{6}(\partial_\nu R)\, \bar{z} z \tag{6.10}$$

and the conformal stress tensor

$$\begin{aligned} \bar{T}_{\mu\nu} &= T_{\mu\nu} + \tfrac{1}{3} \Delta T_{\mu\nu}, \\ \Delta T_{\mu\nu} &= (g_{\mu\nu} \Box - D_\mu \partial_\nu + R_{\mu\nu})\, \bar{z} z, \end{aligned} \tag{6.11}$$

where $\Delta T_{\mu\nu}$ identically satisfies

$$D_\mu \Delta T^\mu{}_\nu = \tfrac{1}{2}(\partial_\nu R)\, \bar{z} z. \tag{6.12}$$

Thus $\bar{T}_{\mu\nu}$ is conserved in a general background, while both tensors are conserved in a background geometry such as AdS where $\partial_\nu R = 0$. For any Killing vector one can show that the contribution of $\Delta T_{\mu\nu}$ to the corresponding $SO(3, 2)$ generator is a surface term, namely,

$$\sqrt{-g}\, \Delta T^{0\nu} \xi_\nu = \partial_i \{ \sqrt{-g}\, [\xi^0 D^i(\bar{z} z) - (D^i \xi^0)\, \bar{z} z] \}. \tag{6.13}$$

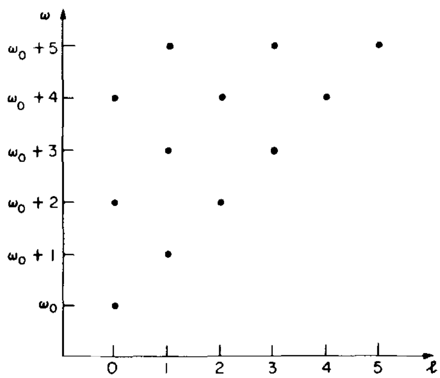

FIG. 1. Weight diagram for the $SO(3, 2)$ representation $D(\omega_0, 0)$. Each dot represents $(2l + 1)$ states with angular momentum l and frequency ω. The pattern extends upwards without bound.

The reason we must consider both stress tensors $T_{\mu\nu}$ and $\bar{T}_{\mu\nu}$ is that the boundary conditions required to make the corresponding energies finite and conserved are slightly different. A priori, one might think that the conformal stress tensor is not relevant in supergravitywhich is not conformal invariant, but it turns out that we have to reconsider this prejudice.

In the coordinates t, ρ, θ, φ the minimal Killing energy is

$$
\begin{aligned}
E(T_{\mu\nu}) &= \int d^3x\,\sqrt{-g}\,T^0{}_0 \\
&= \frac{1}{a^2}\int d\rho\, d^2\Omega \tan^2\rho \left[\partial_0\bar{z}\,\partial_0 z + \frac{1}{\sin^2\rho}\left(\partial_\theta\bar{z}\,\partial_\theta z + \frac{1}{\sin^2\theta}\partial_\varphi\bar{z}\,\partial_\varphi z\right)\right. \\
&\quad \left. + \partial_\rho\bar{z}\,\partial_\rho z - \frac{2}{\cos^2\rho}\bar{z}z\right].
\end{aligned}
\tag{6.14}
$$

After conformal scaling $z(x) = \cos\rho z'(x)$ and integration by parts similar to the steps leading to (5.10), one finds

$$
\begin{aligned}
E(T_{\mu\nu}) &= \frac{1}{a^2}\int d\rho\, d^2\Omega \sin^2\rho \left[\partial_0\bar{z}'\,\partial_0 z' + \frac{1}{\sin^2\rho}\left(\partial_\theta\bar{z}'\,\partial_\theta z' + \frac{1}{\sin^2\theta}\partial_\varphi\bar{z}'\,\partial_\varphi z'\right)\right. \\
&\quad \left. + \partial_\rho\bar{z}'\,\partial_\rho z' + \bar{z}'z'\right] - \frac{1}{a^2}\int d^2\Omega \left(\frac{\sin^3\rho}{\cos\rho}\bar{z}'z'\right)\Bigg|_{\rho=0}^{\pi/2}.
\end{aligned}
\tag{6.15}
$$

For fields $z(x)$ which vanish as $(\cos\rho)^\lambda$ as $\rho \to \pi/2$, the energy is finite and positive and the surface terms vanishes for $\lambda > 3/2$. This is simply the restatement of the exponential falloff of the previous section in the coordinates of this section. The conformal Killing energy differs only in the surface term and is given by

$$
\begin{aligned}
E(\bar{T}_{\mu\nu}) &= \frac{1}{a^2}\int d\rho\, d^2\Omega \sin^2\rho \left[\partial_0\bar{z}'\,\partial_0 z' + \frac{1}{\sin^2\rho}\left(\partial_\theta\bar{z}'\,\partial_\theta z' + \frac{1}{\sin^2\theta}\partial_\varphi\bar{z}'\,\partial_\varphi z'\right)\right. \\
&\quad \left. + \partial_\rho\bar{z}'\,\partial_\rho z' + \bar{z}'z'\right] - \frac{1}{3a^2}\int d^2\Omega(\sin^2\rho\,\partial_\rho(\bar{z}'z'))\Bigg|_{\rho=0}^{\pi/2}
\end{aligned}
\tag{6.16}
$$

and thus is finite, positive, and has vanishing surface term for $\lambda > 3/2$ and for linear falloff $z(x) \sim (\cos\rho)$, i.e., $\lambda = 1$ where the coefficient of the leading divergent contribution to the energy vanishes.

We now turn to the wave equation which separates in our coordinates. If we introduce the product ansatz

$$
\varphi_{\omega lm}(x) = e^{-i\omega t}Y_{lm}(\theta,\varphi)(\sin\rho)^l\cos\rho f_{\omega l}(\rho),
\tag{6.17}
$$

we find that the radial equation [5] is of hypergeometric type. Convergence of the integral for $(\varphi_{\omega lm}, \varphi_{\omega lm})$ at $\rho = 0$ requires that we choose the regular hypergeometric function

$$
f_{\omega l}(\rho) = F(\tfrac{1}{2}(l+1-\omega), \tfrac{1}{2}(l+1+\omega);\, l+\tfrac{3}{2};\, \sin^2\rho).
\tag{6.18}
$$

As $\rho \to \pi/2$ this function has the form

$$f_{\omega l}(\rho) \xrightarrow[\rho \to \pi/2]{} h(\omega, l) \cos\rho[1 + O(\cos^2\rho)] + \bar{h}(\omega, l)[1 + O(\cos^2\rho)] \tag{6.19}$$

with

$$\begin{aligned} h(\omega, l) &= \frac{\Gamma(l+\frac{3}{2})\,\Gamma(-\frac{1}{2})}{\Gamma(\frac{1}{2}(l+1-\omega))\,\Gamma(\frac{1}{2}(l+1+\omega))}, \\ \bar{h}(\omega, l) &= \frac{\Gamma(l+\frac{3}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}(l+2-\omega))\,\Gamma(\frac{1}{2}(l+2+\omega))}. \end{aligned} \tag{6.20}$$

We now note that the scalar product (z_1, z_2) of two solutions is conserved in time if the flux

$$\int_S dS_i \sqrt{-g}\, g^{i\nu}(\bar{z}_1 \overleftrightarrow{\partial}_\nu z_2) \tag{6.21}$$

through the closed surface at spatial infinity vanishes. If we apply this to a superposition of two modes, we find that vanishing flux requires

$$\bar{h}(\omega, l)/h(\omega, l) = q \tag{6.22}$$

where q is a fixed ratio for *all* modes. Similarly, the energy functional is conserved in time if the flux

$$\int_S dS_i \sqrt{-g}\, T^i{}_0 \tag{6.23}$$

through the boundary vanishes. For a superposition of two modes, and for the minimal Killing energy, vanishing flux requires $q = 0$ or $\bar{h}(\omega, l) = 0$, while for the conformal energy ($\bar{T}^i{}_0$ in (6.23)), we find the weaker condition: either $q = 0$ and $\bar{h}(\omega, l) = 0$ as above or $q = \infty$ and $h(\omega, l) = 0$. The vanishing of the coefficients $h(\omega, l)$ or $\bar{h}(\omega, l)$ forces quantization of the energy ω with even integer spacing for fixed l, and the hypergeometric functions become Gegenbauer or Jacobi polynomials.

Before discussing the mode functions further, we note that the procedure above can be understood [5] as the determination of boundary conditions for the conformally transformed field $z' = (\cos\rho)^{-1} z$ on the equator of S^3 which is the boundary of AdS when embedded in its conformal extension ESU. The condition (6.22) is simply a boundary condition on the logarithmic derivative of z' as in standard boundary value problems in mathematical physics. The two choices which are allowed by conservation of energy are exactly Neumann and Dirichlet boundary conditions for z'. One should also note that if the background space is AdS rather than CAdS, the periodicity condition $\omega = 2\pi n$ could have been imposed at the outset giving quantization of the frequency. On CAdS we are not entitled to impose periodicity, and quantization of the frequency is derived by requiring conservation of the scalar product and energy.

These procedures lead to two sets of mode functions which we may label by the $SO(3,2)$ representations for which they are a basis [7], namely, the representation $D(1,0)$ for the $h(\omega,l)=0$ Neumann set and the representation $D(2,0)$ for the $\bar{h}(\omega,l)=0$ Dirichlet set. The eigenfunctions are

$$D(1,0):\qquad \omega=2k+l+1,\qquad k=0,1,2,\ldots,\qquad l=0,1,2,\ldots$$
$$\Phi_{\omega lm}=i^l\left[k!(k+1)!/(\Gamma(k+\tfrac{1}{2})\,\Gamma(k+l+\tfrac{3}{2}))\right]^{1/2}$$
$$\times\, e^{i\omega t}Y_{lm}(\theta,\varphi)(\sin\rho)^l\cos\rho P_k^{(l+1/2,\,-1/2)}(\cos 2\rho) \tag{6.24}$$

and

$$D(2,0):\qquad \omega=2k+l+2,\qquad k=0,1,2,\ldots,\qquad l=0,1,2,\ldots$$
$$\Phi_{\omega lm}=i^l\left[k!(k+l+1)!/(\Gamma(k+\tfrac{3}{2})\,\Gamma(k+l+\tfrac{3}{2}))\right]^{1/2}$$
$$\times\, e^{-i\omega t}Y_{lm}(\theta,\varphi)(\sin\rho)^l\,(\cos\rho)^2\,P_k^{(l+1/2,\,1/2)}(\cos 2\rho).$$

Either set of functions can be rewritten in terms of Jacobi polynomials with argument $\cos\rho$

$$\omega=n+l+1,\qquad n=0,1,2,\ldots,\qquad l=0,1,2,\ldots$$
$$\Phi_{\omega lm}=(i/2)^l\,(\Gamma(n+l+\tfrac{3}{2}))^{-1}\,[n!(n+2l+1)!/2]^{1/2}$$
$$\times\, e^{-i\omega t}Y_{lm}(\theta,\varphi)(\sin\rho)^l\cos\rho P_n^{(l+1/2,\,l+1/2)}(\cos\rho) \tag{6.26}$$

with even $n=2k$ corresponding to the $D(1,0)$ modes (6.24) and with odd $n=2k+1$ corresponding to the $D(2,0)$ modes (6.25). In either case (6.24) or (6.25) we have a complete orthonormal set satisfying

$$(\Phi_{\omega' l' m'},\Phi_{\omega lm})=\delta_{\omega'\omega}\delta_{l'l}\delta_{m'm}, \tag{6.27}$$

but wave functions from different representations are *not* orthogonal. Note that the scalar product (z_1,z_2) contains the standard radial weight function for the "diagonal" Jacobi polynomials $P_n^{(l+1/2,l+1/2)}(\cos\rho)$, but the range $0\leqslant\cos\rho\leqslant 1$ is half the standard range. Thus, one has orthogonality only for a pair of even n or a pair of odd n modes. If we use the form (6.24) and (6.25) of the mode functions, we find that (z_1,z_2) gives both standard weight and interval of definition for the "asymmetric" Jacobi polynomials, and it is clear that each set is separately complete.

For the conformally invariant wave equation on ESU, the two sets of AdS modes would combine to give *one* complete set of modes which simply is the set of S^3 harmonics. Because AdS is conformal to the upper hemisphere of ESU, the ESU modes split into two separate orthonormal systems. The situation is analogous to that of Fourier series: The full set of modes $\cos nx$ and $\sin nx$ are required for the expansion of square integrable functions on the interval $(-\pi,\pi)$, but the modes $\cos nx$ and $\sin nx$ are separately complete systems, and not mutually orthogonal, on the interval $(0,\pi)$.

Thus, there are two complete sets of mode functions obeying distinct boundary conditions at spatial infinity and spanning two different representations of $SO(3, 2)$. Each set defines a unique solution of the Cauchy problem for the free scalar field on CAdS. There is a conformal ambiguity in the definition of the energy functional for the system, and one might think at this point that for gauged supergravity only the minimal Killing energy $E(T_{\mu\nu})$ and the modes of the representation $D(2, 0)$ of $SO(3, 2)$ are relevant. The conformal ambiguity will be resolved in the next sections where we investigate the requirement of de Sitter supersymmetry for the combined spin 0, spin 1/2, and spin 1 modes of the linearized supergravity system. It turns out that both sets of scalar modes play a role in a supersymmetric theory in an AdS background and that the conformal stress tensor can be justified. If these results are anticipated, then the significance of this section for the stability problem of gauged extended supergravity is that the boundary conditions on scalar fluctuations required for positive energy in Section 5 are exactly those for a well-defined Cauchy problem in the CAdS background.

7. Grlobal Supersymmetry in an AdS Background

We now consider the full set $z = (A + iB)/\sqrt{2}, \chi^i, A_\mu{}^{ij}, \psi_\mu{}^i$ and $h_{a\mu} = V_{a\mu} - \bar{V}_{a\mu}$ of fluctuations above the AdS background of the gauged $N = 4$ supergravity theory. If we linearize equations of motion and supersymmetry transformation rules of the full theory, we obtain a set of free fields in AdS, and we expect invariance under $OSp(4, 4)$ supersymmetry. The first requirement for $OSp(4, 4)$ invariance is that the modes for each of the fields A, B, χ^i,... form a basis for an appropriate $SO(3, 2)$ representation. Furthermore, if supersymmetry transforms spin s modes from the $SO(3, 2)$ irrep $D(\omega_0, s)$ into spin $s - 1/2$ modes from the irrep $D(\omega_0', s - 1/2)$, it must also transform $D(\omega_0', s - 1/2)$ back into $D(\omega_0, s)$ such that all modes form a system which is closed under supersymmetry. This requirement will determine which of the spin 0 representations $D(1, 0)$ and $D(2, 0)$ are compatible with $OSp(4, 4)$ invariance, i.e., supersymmetry.

For the moment we consider only $OSp(1, 4)$ and the spin 0–spin 1/2 sector with the wave equations (all derivatives are covariant over the AdS background)

$$(D^\mu D_\mu - 2a^2) A = (D^\mu D_\mu - 2a^2) B = 0, \tag{7.1}$$

$$i\gamma^\mu D_\mu \chi = 0 \tag{7.2}$$

and supersymmetry transformation rules

$$\delta A = \frac{1}{\sqrt{2}} \bar{\varepsilon}\chi, \qquad \delta B = \frac{i}{\sqrt{2}} \bar{\varepsilon}\gamma_5 \chi, \tag{7.3}$$

$$\delta\chi = -\frac{1}{\sqrt{2}} [i\gamma^\mu \partial_\mu (A + i\gamma_5 B) + a(A - i\gamma_5 B(] \varepsilon. \tag{7.4}$$

These two sets of equations are consistent only if $\varepsilon(x)$ is a Killing spinor. Only then will the transformation rules take solutions of the scalar equations (7.1) into solutions of the spinor equation (7.2) and conversely.

As was done for scalar fields in the previous section, we must now find a complete set of positive frequency solutions of the Dirac equation which form a basis for an $SO(3,2)$ representation $D(\omega_0, 1/2)$. For each angular momentum j there are modes $\chi^{(\pm)}_{\omega jm}$ with frequency ω and parity $\pm(-)^{j-1/2}$ for which we make the following ansatz using spinor spherical harmonics $\mathscr{Y}^{(\pm)}_{jm}$ with orbital angular momentum $=j-1/2$ (see, e.g., [19])

$$\chi^{(+)}_{\omega jm} = e^{-i\omega t}(\sin\rho)^{j-1/2}(\cos\rho)^{3/2}\begin{pmatrix} f_{\omega j}(\rho)\,\mathscr{Y}^{(+)}_{jm} \\ ig_{\omega j}(\rho)\,\mathscr{Y}^{(-)}_{jm} \end{pmatrix}, \tag{7.5}$$

$$\chi^{(-)}_{\omega jm} = i\gamma_5\chi^{(+)}_{\omega jm} = e^{-i\omega t}(\sin\rho)^{j-1/2}(\cos\rho)^{3/2}\begin{pmatrix} -g_{\omega j}(\rho)\,\mathscr{Y}^{(-)}_{jm} \\ if_{\omega j}(\rho)\,\mathscr{Y}^{(+)}_{jm} \end{pmatrix}. \tag{7.6}$$

The radial wave functions $f_{\omega j}(\rho)$ and $g_{\omega j}(\rho)$ satisfy coupled first-order differential equations which we have analyzed but are not given here.

The scalar product

$$(\chi_1, \chi_2) = \int d^3x\,\sqrt{-g}\,V_a{}^0\bar{\chi}_1\gamma^a\chi_2 \tag{7.7}$$

is formally conserved for solutions of the Dirac equation, and we must specify boundary conditions at $\rho=\pi/2$ so that it is actually conserved, i.e., the flux through the surface at spatial infinity vanishes. These conditions for a conserved scalar product for spinor modes are stronger than in the scalar case and ensure conservation of the spinor energy functional. Conservation of the scalar product (7.7) leads to quantization of frequencies and to radial wave functions

$$\begin{aligned} \omega &= n+j+1, \qquad n=0,1,2,\ldots, \\ f_{\omega j}(\rho) &= \left(\frac{i}{2}\right)^{j-1/2} N_{\omega j}\cos\frac{\rho}{2}\,P_n^{(j,j+1)}(\cos\rho), \\ g_{\omega j}(\rho) &= \left(\frac{i}{2}\right)^{j-1/2} N_{\omega j}\sin\frac{\rho}{2}\,P_n^{(j+1,j)}(\cos\rho), \\ N_{\omega j} &= (\Gamma(n+j+1))^{-1}\,[n!(n+2j+1)!/2]^{1/2}. \end{aligned} \tag{7.8}$$

As in the scalar case, for each angular momentum and parity the modes with even n and the modes with odd n form two separate complete orthonormal sets. Once again these two sets are not mutually orthogonal because the range of the radial variable is restricted to $0 \leqslant \rho < \pi/2$. We must now select one of these sets for each angular momentum and parity and combine them to obtain a basis for an irreducible representation of $SO(3,2)$. In this way we obtain two identical copies of the representtion $D(3/2, 1/2)$ which we call $D(3/2, 1/2)^{\pm}$ and which have the following modes as bases:

$$
\begin{aligned}
D(3/2, 1/2)^{+}: \quad & \omega = n + j + 1, \qquad n = 0, 1, 2, \ldots, \qquad j = 1/2, 3/2, 5/2, \ldots, \\
& \chi^{+}_{\omega j m} = i\chi^{(+)}_{\omega j m} \qquad \text{for even } n \\
& \quad = \chi^{(-)}_{\omega j m} \qquad \text{for odd n}
\end{aligned}
\tag{7.9}
$$

and

$$
\begin{aligned}
D(3/2, 1/2)^{-}: \quad & \omega = n + j + 1, \qquad n = 0, 1, 2, \ldots, \qquad j = 1/2, 3/2, 5/2, \ldots, \\
& \chi^{-}_{\omega j m} = i\gamma_5 \chi^{+}_{\omega j m} = i\chi^{(-)}_{\omega j m} \qquad \text{for even } n \\
& \quad = -\chi^{(+)}_{\omega j m} \qquad \text{for odd } n
\end{aligned}
\tag{7.10}
$$

which differ by the parity assignment of corresponding states: $P = \pm(-)^{\omega - 3/2}$ for $D(3/2, 1/2)^{\pm}$. Figure 2 (with $s = 1/2$) shows the weight diagram for these representations. For an alternate discussion of the spinor basis functions, see [7].

One difference between the scalar and spinor cases should be noted. In the scalar case, the boundary conditions led to modes which were bases for two distinct irreps of $SO(3, 2)$, namely, $D(1, 0)$ and $D(2, 0)$. In the spinor case, and for the vector field (and higher spins) discussed below, there are two sets of modes which carry the same $SO(3, 2)$ irrep, namely, $D(3/2, 1/2)$ for spin 1/2 and $D(2, 1)$ for spin 1.

The supersymmetry transformation (7.4) transforms scalar and pseudoscalar modes into linear combinations spinor modes, and (7.3) transforms spinor modes back into scalar and pseudoscalar modes. Indeed, instead of solving the Dirac equation explicitly, we could have used (7.4) to obtain the spinor modes. It can easily be seen from the form of $\varepsilon(x)$ in (3.17) that a supersymmetry transformation will change both frequency and angular momentum by $\pm 1/2$. Moreover the parity of a

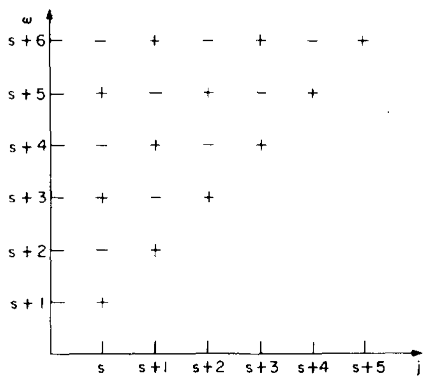

FIG. 2. Weight diagram for the "massless" $SO(3, 2)$ representation $D(s + 1, s)^{+}$. Each + and − represents states with parity $P = \pm(-)^{j-s}$ respectively. The weight diagram for $D(s + 1, s)^{-}$ contains the same states with opposite parity.

mode is preserved if ω is increased by (7.4) or if ω is decreased by (7.3), and the parity is changed otherwise. All this leads to the conclusion that spinor modes from the representation $D(3/2, 1/2)^+$ are transformed by (7.3) into scalar modes from $D(1,0)$ and pseudoscalar modes from $D(2,0)$, and that these scalar and pseudoscalar modes are transformed back into $D(3/2, 1/2)^+$ spinor modes by (7.4). Alternatively, due to the relation $\chi^- = i\gamma_5\chi^+$ in (7.10) and the invariance of (7.3)–(7.4) under the substitutions $A \to B$, $B \to -A$, $\chi \to i\gamma_5\chi$, spinor modes from $D(3/2, 1/2)^-$ are transformed into scalar modes from $D(2,0)$ and pseudoscalar modes from $D(1,0)$ and conversely. These conclusions have been verified by explicit calculations using angular momentum analysis and properties of Jacobi polynomials. Further these calculations were used to determine the reduced matrix elements of the $OSp(1,4)$ spinor charges given below.

To describe the $OSp(1,4)$ representations which result from the analysis above, it is convenient to define reducible spin 0 $SO(3,2)$ representations $D(1,0)^\pm$ which contain both scalar and pseudoscalar modes:

$$\begin{aligned} D(1,0)^+: \quad & \omega = n + l + 1, \qquad n = 0, 1, 2, \ldots, \qquad l = 0, 1, 2, \ldots, \\ & A = \Phi_{\omega l m}, \qquad B = 0 \qquad \text{for even } n, \\ & A = 0, \qquad B = i\Phi_{\omega l m} \qquad \text{for odd } n, \end{aligned} \tag{7.11}$$

$$\begin{aligned} D(1,0)^-: \quad & \omega = n + l + 1, \qquad n = 0, 1, 2, \ldots, \qquad l = 0, 1, 2, \ldots, \\ & A = 0, \qquad B = \Phi_{\omega l m} \qquad \text{for even } n, \\ & A = -i\Phi_{\omega l m}, \qquad B = 0 \qquad \text{for odd } n \end{aligned} \tag{7.12}$$

which again differ by the parity of corresponding states: $P = \pm(-)^{\omega-1}$ for $D(1,0)^\pm$. The weight diagram for these representations (Fig. 2 with $s = 0$) shows the same pattern as for the spinor representations. Supersymmetry transforms spin 0 modes from $D(1,0)^\pm$ into spin 1/2 modes from $D(3/2, 1/2)^\pm$ and vice versa. Thus, the spin 0 modes from $D(1,0)^+$ and the spin 1/2 modes from $D(3/2, 1/2)^+$ together form the basis of an irreducible representation of $OSp(1,4)$, while modes from $D(1,0)^-$ and $D(3/2, 1/2)^-$ form another basis of the same representation with opposite parity states. For a supersymmetric quantum field theory boundary conditions corresponding to one of these two bases must be chosen. Thus the chiral $U(1)$ invariance, $\delta A = \theta B$, $\delta B = -\theta A$, $\delta\chi = i\theta\gamma_5\chi$, of the linearized spin 0–spin 1/2 action and transformation rules (7.3)–(7.4) is broken by the boundary conditions required for supersymmetry.

We have found that supersymmetry has resolved the ambiguity of the previous section for the spin 0 modes in a somewhat unexpected way. Supersymmetry does not select one of the two representations $D(1,0)$ and $D(2,0)$ in favor of the other one; it rather selects both of them, one for the scalar and the other one for the pseudoscalar field. This means, in particular, that we have to use the improved stress tensor (6.11) to define a conserved positive energy functional. We would also expect that the improved supercurrent is necessary to define conserved spinor charges. Supersymmetry still leaves the ambiguity whether to choose the representations $D(1,0)^+$ and $D(3/2, 1/2)^+$ or $D(1,0)^-$ and $D(3/2, 1/2)^-$. This discrete ambiguity is related to the chiral invariance discussed above.

We now study $OSp(1,4)$ supersymmetry in the spin 1/2–spin 1 sector. Our original motivation for this was the hope that the presence of gauge potentials in this sector might resolve the remaining discrete ambiguity in the spin 0–spin 1/2 sector. The linearized equations of motion for any of the vector fields $A_\mu{}^{ij}$ and the associated Bianchi identity are just Maxwell's equations in a curved background, namely,

$$\partial_\nu(\sqrt{-g}\,F^{\mu\nu}) = 0, \tag{7.13}$$

$$\varepsilon^{\mu\nu\rho\sigma}\,\partial_\nu F_{\rho\sigma} = 0 \tag{7.14}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The supersymmetry transformation rules are

$$\delta\chi = \frac{1}{\sqrt{2}}\sigma^{\mu\nu}\varepsilon F_{\mu\nu}, \tag{7.15}$$

$$\delta A_\mu = \frac{i}{\sqrt{2}}\bar{\varepsilon}\gamma_\mu\chi \tag{7.16}$$

and again consistency between field equations and transformation rules requires $\varepsilon(x)$ to be a Killing spinor.

We need a complete set of vector modes which form a basis for an irreducible $SO(3,2)$ representation. These vector modes are unique up to a gauge transformation. We can, however, postpone questions of gauge choices by first presenting the vector modes in terms of the gauge invariant field strength. We use our local frames (3.10) to decompose $F_{\mu\nu}$ into electric and magnetic fields **E** and **B**

$$F_{\mu\nu} = E^i(V^i{}_\mu V^0{}_\nu - V^i{}_\nu V^0{}_\mu) + \varepsilon_{ijk} B^i V^j{}_\mu V^k{}_\nu. \tag{7.17}$$

The field equations are invariant under the duality transformation $\mathbf{E} \to -\mathbf{B}$, $\mathbf{B} \to \mathbf{E}$. Likewise, the transformation rule (7.15) is invariant under this duality transformation combined with $\chi \to i\gamma_5\chi$. This duality invariance extends to the transformation rule for $\delta F_{\mu\nu}$ which is the curl of (7.16).

In order to determine the vector modes explicitly we have to express **E** and **B** in terms of vector spherical harmonics (see, e.g., [20]). For each total angular momentum j there are two sets of modes with opposite parity, i.e., electric and magnetic multipoles. As for the spin 1/2 case we find that there are two complete sets of mode functions corresponding to two identical copies $D(2,1)^\pm$ of the $SO(3,2)$ representation $D(2,1)$ and again modes from one representation are not orthogonal to those of the other one. These two sets of modes are

$$\begin{array}{llll} D(2,1)^+: & \omega = n + j + 1, & n = 0, 1, 2, \ldots, & j = 1, 2, 3, \ldots, \\ \mathbf{E}^+_{\omega jm} = \mathbf{F}_{\omega jm}, & \mathbf{B}^+_{\omega jm} = \mathbf{G}_{\omega jm} & \text{for even n (electric multipole)} \\ E^+_{\omega jm} = i\mathbf{G}_{\omega jm}, & \mathbf{B}^+_{\omega jm} = -i\mathbf{F}_{\omega jm} & \text{for odd } n \text{ (magnetic multipole)} \end{array} \tag{7.18}$$

and

$$D(2,1)^-: \qquad \omega = n + j + 1, \qquad n = 0, 1, 2, \ldots, \qquad j = 1, 2, 3, \ldots,$$
$$\mathbf{E}^-_{\omega jm} = \mathbf{B}^+_{\omega jm}, \qquad \mathbf{B}^-_{\omega jm} = -\mathbf{E}^+_{\omega jm} \tag{7.19}$$

with parity $P = \pm(-)^{\omega-2}$ respectively. The electric and magnetic field component $\mathbf{F}$ and $\mathbf{G}$ of the electric multipole field are

$$\begin{aligned}\mathbf{F}_{\omega jm} = {} & e^{-i\omega t}(\cos\rho)^2 (\sin\rho)^{j-1} \left(\frac{i}{2}\right)^{j+1} N_{\omega j} \left[\frac{n+2j+2}{2}\sqrt{\frac{j}{2j+1}}\,(\sin\rho)^2\right.\\ & \times \mathbf{Y}^+_{jm} P_{n-1}^{(j+3/2,\,j+3/2)}(\cos\rho) - 2(n+1)\sqrt{\frac{j+1}{2j+1}}\,\mathbf{Y}^-_{jm} P_{n+1}^{(j-1/2,\,j-1/2)}(\cos\rho)\\ & \left. - \sqrt{j(j+1)}\,(1-\cos\rho)\,\mathbf{Y}^L_{jm} P_n^{(j+1/2,\,j+1/2)}(\cos\rho)\right],\end{aligned} \tag{7.20}$$

$$\mathbf{G}_{\omega jm} = e^{-i\omega t}(\cos\rho)^2 (\sin\rho)^j \left(\frac{i}{2}\right)^{j+1} N_{\omega j}(n+j+1)\,\mathbf{Y}^M_{jm} P_n^{(j+1/2,\,j+1/2)}(\cos\rho),$$

$$N_{\omega j} = (\Gamma(n+j+\tfrac{3}{2}))^{-1}\,[2(n+2j+1)!\,n!]^{1/2}$$

and the vector spherical harmonics ($j = 1, 2, 3, \ldots$) are given by

$$\begin{aligned}\mathbf{Y}^M_{jm} &= (j(j+1))^{-1/2}\,\mathbf{L} Y_{jm},\\ \mathbf{Y}^E_{jm} &= -i\hat{x} \times \mathbf{Y}^M_{jm} = (2j+1)^{-1/2}\,[j^{1/2}\mathbf{Y}^+_{jm} + (j+1)^{1/2}\,\mathbf{Y}^-_{jm}],\\ \mathbf{Y}^L_{jm} &= \hat{x} Y_{jm} = (2j+1)^{-1/2}\,[-(j+1)^{1/2}\,\mathbf{Y}^+_{jm} + j^{1/2}\mathbf{Y}^-_{jm}].\end{aligned} \tag{7.21}$$

The gauge potential for these modes is particularly simple in the temporal gauge, $A_0 = 0$, where

$$A^\pm_{\mu\omega jm} = \frac{i}{\omega}(\cos\rho)^{-1}\,V_{i\mu} E^{i\pm}_{\omega jm}. \tag{7.22}$$

The supersymmetry transformation rule (7.15) transforms the vector modes from the representation $D(2,1)^\pm$ into linear combinations of spinor modes from $D(3/2, 1/2)^\pm$, respectively, and the spinor modes are transformed back into vector modes by (7.16). The gauge potentials obtained from (7.16) differ, however, from those in the temporal gauge (7.22) by a gauge transformation. This gauge transformation, or equivalently δA_0, is not the same if one and the same vector mode is obtained via (7.16) from different spinor modes, it even contains combinations of frequency and angular momentum which do not correspond to any vector mode. The two possible choices of $SO(3,2)$ representations lead to δA_0's with different behaviour as $\rho \to \pi/2$: $\delta A_0 \sim \cos\rho$ if we choose $D(3/2, 1/2)^+$ and $D(2,1)^+$, and $\delta A_0 \sim$ const. if we choose $D(3/2, 1/2)^-$ and $D(2,1)^-$. We hoped that this difference in asymptotic behaviour would rule out one of the choices, because the spin 1 scalar product which is the analogue of (6.4) and (7.7), namely,

$$i\int d^3x\,\sqrt{-g}\,(F_1^{0\mu}{}^*A_{2\mu}-A_{1\mu}^*F_2^{0\mu}), \tag{7.23}$$

is gauge invariant only if the potentials satisfy appropriate boundary conditions. However, it turns out that both choices $D(3/2, 1/2)^+$ and $D(2, 1)^+$ as well as $D(3/2, 1/2)^-$ and $D(2, 1)^-$ give a gauge invariant scalar product. Thus, a discrete ambiguity still remains. It is possible that this ambiguity is resolved by consideration of interaction terms of the gauged gravity Lagrangian, but we have not investigated this.

We have not done any explicit calculations for spins 3/2 and 2, but we have already enough information to deduce the general pattern. For each spin there are two copies $D(s+1, s)^\pm$ of the $SO(3, 2)$ representation $D(s+1, s)$. (See Fig. 2.) For each of them there is a complete set of modes $\varphi^\pm_{\omega jm}$, $\omega = n+j+1$, $n=0, 1, 2,\ldots, j = s, s+1, s+2,\ldots,$ with parity $P=\pm(-)^{\omega-s-1}$, and the two sets are not mutually orthogonal. $OSp(1, 4)$ supersymmetry connects two such representations with adjacent spins s and $s-1/2$ and corresponding parity assignments. The matrix elements of the supersymmetry generators have such a simple spin dependence that one can easily deduce their general form from the known matrix elements between spin 0, spin 1/2, and spin 1. Supersymmetry transformations are generated by the Majorana spinor operator

$$Q=\begin{pmatrix}\mathscr{A}\\ \mathscr{A}^c\end{pmatrix},\qquad \mathscr{A}^c=\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\mathscr{A}^*,\qquad \mathscr{A}^{cc}=-\mathscr{A}. \tag{7.24}$$

The two component spinor operator $\mathscr{A}$ and its charge conjugate $\mathscr{A}^c$ lower and raise the energy by 1/2 unit respectively. Their reduced matrix elements between states of spin s and spin $s-1/2$ are (see, e.g., [21] for phase conventions)

$$\begin{aligned}
\langle s-\tfrac12\,\omega+\tfrac12\,j+\tfrac12\|\,\mathscr{A}^c\,\|s\omega j\rangle&=(-)^{\omega-j}\,((\omega+j+1)(j-s+1))^{1/2},\\
\langle s-\tfrac12\,\omega+\tfrac12\,j-\tfrac12\|\,\mathscr{A}^c\,\|s\omega j\rangle&=(-)^{\omega+j}\,((\omega-j)(j+s))^{1/2},\\
\langle s-\tfrac12\,\omega-\tfrac12\,j+\tfrac12\|\,\mathscr{A}\,\|s\omega j\rangle&=(-)^{\omega+j}\,((\omega-j-1)(j-s+1))^{1/2},\\
\langle s-\tfrac12\,\omega-\tfrac12\,j-\tfrac12\|\,\mathscr{A}\,\|s\omega j\rangle&=(-)^{\omega-j}\,((\omega+j)(j+s))^{1/2},
\end{aligned} \tag{7.25}$$

$$\begin{aligned}
\langle s\omega j\|\,\mathscr{A}\,\|s'\omega' j'\rangle&=(-)^{j-j'+1/2}\,\langle s'\omega' j'\|\,\mathscr{A}^c\,\|s\omega j\rangle^*,\\
\langle s\omega j\|\,\mathscr{A}^c\|\,s'\omega' j'\rangle&=(-)^{j-j'+1/2}\,\langle s'\omega' j'\|\,\mathscr{A}\,\|s\omega j\rangle^*.
\end{aligned} \tag{7.26}$$

It is easy to verify that these matrix elements generate indeed an $OSp(1, 4)$ representation which contains the two $SO(3, 2)$ representations $D(s+1, s)^\pm$ and $D(s+1/2,$

$s-1/2)^{\pm}$; $D(1,0)^{\pm}$ is reducible, all the other ones are irreducible. We also find the eigenvalues of $\bar{Q}Q$

$$\begin{aligned} \bar{Q}Q\,|s\omega jm\rangle &= 2(s-1)\,|s\omega jm\rangle, \\ \bar{Q}Q\,|s-\tfrac{1}{2}\omega jm\rangle &= -2((s-\tfrac{1}{2})+1)\,|s-\tfrac{1}{2}\omega jm\rangle. \end{aligned} \tag{7.27}$$

Together with the eigenvalue $\omega_0(\omega_0-3)+s(s+1)$ for the quadratic Casimir operator $\frac{1}{2}M_{AB}M^{AB}$ of $SO(3,2)$ in the representation $D(\omega_0,s)$, we find that the Casimir operator of $OSp(1,4)$ [22]

$$C=\tfrac{1}{2}M_{AB}M^{AB}-\tfrac{1}{2}\bar{Q}Q \tag{7.28}$$

has the eigenvalue $(s-1)(2s+1)$.

We can use these $OSp(1,4)$ representations as building blocks for $OSp(N,4)$ representations. The $SO(3,2)$ content of an $OSp(N,4)$ representation is 1 representation $D(s+1,s)^{\pm}$, N representations $D(s+1/2,s-1/2)^{\pm}$,..., 1 representation $D(s-N/2+1,s-N/2)^{\pm}$. The matrix elements of the N supersymmetry generators ${Q_\alpha}^i$ are, apart from factors ± 1 and 0 from the $SO(N)$ tensor structure, the same as for the $OSp(1,4)$ building blocks and the $OSp(N,4)$ Casimir operator

$$C=\tfrac{1}{2}M_{AB}M^{AB}-\tfrac{1}{2}\bar{Q}^iQ^i-\tfrac{1}{4}I^{ij}I^{ij} \tag{7.29}$$

has the eigenvalue $(s-1)(2s-N+2)$. We have, in this argument, assumed that $s\geqslant N/2$. The results are, however, essentially the same for $s<N/2$. Starting from one $SO(3,2)$ representation with spin s, N representations with spin $s-1/2$,..., the spins decrease until we reach, after $2s$ steps, spin zero. Then we have, in general, to continue with increasing spins until we reach one representation with spin $N/2-s$. Only in the special case where $s=N/4$ is integer (e.g., in the $N=4$ Yang–Mills multiplet or in the $N=8$ gravity multiplet) we have to stop with the appropriate number of spin 0 representations.‡

8. Conclusions

AdS space is a natural background geometry for extended supergravity theories with gauged $SO(N)$ symmetry. For $N\geqslant 4$, these theories contain scalar fields with a potential which is unbounded below. Nevertheless, the energy functional for fluctuations above the AdS background is positive if these fluctuations fall off sufficiently fast at spatial infinity, and the background is therefore stable against such fluctuations. In order to understand these asymptotic properties, we have studied free field theories for various spins in the AdS background which is the same as the linearized dynamics of the fluctuations. We have found that one must impose boundary conditions at spatial infinity which simultaneously ensure positive

‡ *Note added in proof.* After submission of this work for publication, we learned about another treatment of irreducible representations of OSp (1, 4) supersymmetry [24].

conserved energy and a well-defined Cauchy problem. Thus, a picture emerges in which perturbations of the background can be quantized consistently. This analysis was done with the expectation that the boundary conditions remain unchanged for interacting theories.

For each free field in the theory, there is an ambiguity which allows to choose one of two possible boundary conditions. However, the requirement of $OSp(N, 4)$ supersymmetry restricts the choice to one of two sets of boundary conditions within each supermultiplet. It is not clear whether this remaining ambiguity persists in the full nonlinear theory. Both sets of supersymmetric boundary conditions require the improved stress tensor to define the energy of spin 0 fluctuations.

There are several open question related to this work. The detailed role of the boundary conditions in the canonical formalism needs clarification with the goal of a rigorous derivation of the $OSp(N, 4)$ algebra in gauged supergravity. This may be related to the need for the improved energy functional which should be better understood. Study of the interacting quantum theory in perturbation theory should reinforce the view of stability taken here. Finally, there is the acute physical problem of large cosmological constant on which the present investigation has shed no light.

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APPENDIX A: "Massive" Scalars in DAdS Space

The free "massive" real scalar field $h(x)$ in an AdS background with the Sitter constant a^2 has the action

$$I_0 = \tfrac{1}{2} \int d^4x \sqrt{-\bar{g}} \{ \bar{g}^{\mu\nu} \partial_\mu h \, \partial_\nu h + \alpha a^2 h^2 \}, \tag{A.1}$$

where $\alpha \neq 2$ is regarded as "massive" (as distinct from the conformal value $\alpha = 2$). There are two resons to consider such fields in the present investigation: (1) The small fluctuation stability argument can be extended and actually applies outside of supergravity in general field theories containing scalars coupled to gravity, where "massive" fluctuation fields naturally appear. (2) An improved energy functional is still necessary to analyze stability, although we are aven further from the situation of conformal invariance. These questions were discussed in [9], but it is useful to treat them here in a way which is still brief but brings out additional details.

Let us illustrate the first point using the scalar–gravity action

$$I = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \{ -\tfrac{1}{2}R + g^{\mu\nu} \partial_\mu \varphi \, \partial_\nu \varphi - 2V(\varphi) \} \tag{A.2}$$

and suppose that $V(\varphi)$ is a scalar potential with a critical point at $\varphi = \varphi_0$ and $V(\varphi_0) < 0$. Then by analysis similar to that in Section 2, one finds that the background solution of the field equations is $\varphi(x) = \varphi_0$ and $g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x)$, and AdS metric with de Sitter constant $a^2 = -2V(\varphi_0)/3$. If the deviation of the scalar field from background is denoted by $h(x) \equiv \varphi(x) - \varphi_0$, then the linearized action of such fluctuations is I_0 of (A.1) with $\alpha a^2 = -V''(\varphi_0)$.

By the method of [9], described in Section 4, one can define a formally conserved minimal energy functional of scalar fluctuations

$$\begin{aligned} E(T_{\mu\nu}) &= \int d^3x \sqrt{-\bar{g}}\, T^0{}_0, \\ T_{\mu\nu} &= 2\partial_\mu h\, \partial_\nu h - g_{\mu\nu}(\partial^\sigma h\, \partial_\sigma h + \alpha a^2 h^2). \end{aligned} \tag{A.3}$$

However, the improved energy functional

$$\begin{aligned} E(\hat{T}_{\mu\nu}) &= \int d^3x \sqrt{-\bar{g}}\, \hat{T}^0{}_0, \\ \hat{T}_{\mu\nu} &= T_{\mu\nu} + \beta(\bar{g}_{\mu\nu}\Box - \bar{D}_\mu \partial_\nu + \bar{R}_{\mu\nu})\, h^2 \end{aligned} \tag{A4}$$

is also formally conserved for any value of the parameter β, and it will be required. For $V''(\varphi_0) > 0$ and $\alpha < 0$, i.e., the case when the critical point at $\varphi = \varphi_0$ is a minimum, it is easy to see that the integrand of (A3) is positive. Stability of the background is known in this case [6], and we can confine our discussion to $\alpha > 0$ which is the case when $\varphi = \varphi_0$ is a maximum of $V(\varphi)$.

In [9] we gave a scaling argument similar to that of Section 5 to determine the asymptotic behavior of allowed scalar fluctuations. The integrals in $E(T_{\mu\nu})$ converge for fluctuations $h(x)$ which vanish faster than $(\cos\rho)^{3/2}$ at spatial infinity. By substitution of $h(x) = (\cos\rho)^\lambda\, h'(x)$ in (A3), and choosing $\lambda = \lambda_\pm = (3 \pm \sqrt{9 - 4\alpha})/2$, one finds that $E(T_{\mu\nu})$ is positive for such fluctuations. The improved energy functional allows the same class of fluctuations for general β. However, for the special value $\beta = \lambda_-/(2\lambda_- + 1)$, which we now assume, fluctuations with asymptotic falloff $h(x) \sim (\cos\rho)^{\lambda_-}$ have positive convergent $E(\hat{T}_{\mu\nu})$ if $1/2 < \lambda_- < 3/2$. One should also note that $\alpha \leqslant 9/4$ is required in the scaling argument to maintain reality properties.

Let us now turn to the scalar wave equation

$$\Box h - \alpha a^2 h = 0 \tag{A5}$$

in the AdS background and show that there is a well-defined quantum field theory in a Hilbert space of functions satisfying the same conditions required for positive classical energy. As in Section 6 we impose boundary conditions such that the formally conserved scalar product (6.4) and energy functional are actually conserved and convergent, and we then determine a complete set of positive frequency modes $\varphi_{\omega lm}(x)$ subject to these conditions.

There are again two choices of boundary conditions which select modes corresponding to the two parts of the radial hypergeometric function as in (6.19). The mode functions are

$$\Phi^{\pm}_{\omega l m}(x) \sim e^{-i\omega t} Y_{lm}(\theta, \varphi)(\sin\rho)^l (\cos\rho)^{\lambda_\pm} P_k^{(l+1/2, \lambda_\pm - 3/2)}(\cos 2\rho),$$
$$\omega = \lambda_\pm + l + 2k. \tag{A6}$$

Note that frequencies are quantized, but the modes are not periodic in time but instead periodic up to a phase. Therefore, this treatment requires the covering space CAdS and differs from [5] where periodicity in time was imposed for massive scalar fields.

The "regular" modes $\Phi^{+}_{\omega l m}$ are required if we choose $E(T_{\mu\nu})$ as the energy functional, while $E(\hat{T}_{\mu\nu})$ allows both $\Phi^{+}_{\omega l m}$ and the "irregular" modes $\Phi^{-}_{\omega l m}$ provided $\lambda_- > 1/2$, and this last condition is also required for a conserved convergent scalar product. Thus, there is a choice of boundary conditions for $5/4 < \alpha < 9/4$, while only the regular modes are allowed for $\alpha < 5/4$. It appears impossible to satisfy physical requirements for $\alpha > 9/4$, and this regime seems to correspond to tachyons in AdS space. Thus, $\alpha = 9/4$ is the limiting ratio of negative mass square eigenvalue to de Sitter constant below which the AdS background at a maximum or saddle point of a scalar potential is stable against small fluctuations.

It is known [7] that the modes $\Phi^{\pm}_{\omega l m}(x)$ carry the representations $D(\lambda_\pm, 0)$ of $SO(3,2)$. For $\alpha < 5/4$, when the irregular modes are ruled out, the representation $D(\lambda_-, 0)$ ceases to be unitary.

APPENDIX B: Massive $OSp(1,4)$ Scalar Multiplet

This massive scalar multiplet of de Sitter supersymmetry can occur when supersymmetric grand unified theories are coupled to gravity and possibly at some critical points of the potential in $N > 5$ gauged supergravity (although no example is known). This multiplet consists of real scalar, pseudoscalar, and spinor fields, $A(x)$, $B(x)$, and $\chi(x)$, with free action (in an AdS background with de Sitter parameter a^2)

$$I = \tfrac{1}{2} \int d^4x \sqrt{-\bar{g}} \, \{ \bar{g}^{\mu\nu} \partial_\mu A \, \partial_\nu A + \bar{g}^{\mu\nu} \partial_\mu B \, \partial_\nu B + i\bar{\chi}\gamma^\mu \bar{D}_\mu \chi$$
$$+ (2a^2 + am - m^2) A^2 + (2a^2 - am - m^2) B^2 - m\bar{\chi}\chi \}. \tag{B1}$$

The fact that the three fields have three different apparent mass parameters is easily understood on the basis of auxiliary fields, and it is crucial to the result that the $SO(3,2)$ representations for the three fields are compatible with supersymmetry. One should also note that for all m, one has $\alpha_A \leqslant 9/4$ and $\alpha_B \leqslant 9/4$ so the stability condition is satisfied.

It is an exercise of joy to verify that the action (B1) is invariant under the following supersymmetry transformation rules with Killing spinor parameter $\varepsilon(x)$:

$$\delta A = \frac{1}{\sqrt{2}} \bar{\varepsilon}(x) \chi, \qquad \delta B = \frac{i}{\sqrt{2}} \bar{\varepsilon}(x) \gamma_5 \chi,$$
$$\delta \chi = -\frac{1}{\sqrt{2}} [i\gamma^\mu \partial_\mu (A + i\gamma_5 B) + a(A - i\gamma_5 B) + m(A + i\gamma_5 B)] \varepsilon(x). \tag{B2}$$

We now impose the requirement that supersymmetry transformations must connect complete sets of scalar, pseudoscalar, and spinor modes to determine the allowed $SO(3,2)$ representations $D(\omega_A, 0)$, $D(\omega_B, 0)$, and $D(\omega_\chi, 1/2)$ of the various fields. Since a Killing spinor changes frequencies by $\pm 1/2$ with correlated parity change as discussed in Section 7, it is clear that the lowest frequencies ω_A and ω_B must differ by 1 and that $\omega_\chi = (\omega_A + \omega_B)/2$.

More precise information can be obtained using the properties of scalar field modes in Appendix A. First, one finds that (with $\mu = m/a$)

$$\alpha_A = 2 + \mu - \mu^2,$$
$$\alpha_B = 2 - \mu - \mu^2 \tag{B3}$$

and

$$\lambda_{A\pm} = \tfrac{3}{2} \pm |\mu - \tfrac{1}{2}|,$$
$$\lambda_{B\pm} = \tfrac{3}{2} \pm |\mu + \tfrac{1}{2}|. \tag{B.4}$$

For $\mu > 1/2$, one must take regular spinless modes, i.e., λ_{A+} and λ_{B+} so that $\omega_A = \mu + 1$, $\omega_\chi = \mu + 3/2$, $\omega_B = \mu + 2$. As μ is decreased below $1/2$, one keeps regular modes for B but must switch to irregular modes for A. The relation $\omega_B - \omega_A = 1$ is maintained and, indeed, the values of ω_A, ω_χ, ω_B above remain valid down to $\mu = -1/2$ where the irregular modes for A become too singular. Note that the massless reducible representation $D(1,0)^+$ of Section 7 is obtained at $\mu = 0$. For negative $\mu < -1/2$, one must take regular modes for A and B and $SO(3,2)$ representations with $\omega_A = 2 - \mu$, $\omega_\chi = 3/2 - \mu$, $\omega_B = 1 - \mu$. As μ is increased above $-1/2$, these relations are maintained but one must take irregular modes for B, and this choice is valid until $\mu = 1/2$ where the irregular modes become singular. This discussion is clarified by Fig. 3.

To summarize one finds that for $|m| \geqslant a/2$ there is only one choice of scalar boundary conditions compatible with supersymmetry and regular modes are required. For $-a/2 < m < a/2$ there is a choice of boundary conditions. Either A or B must have irregular modes, and the improved energy is required.

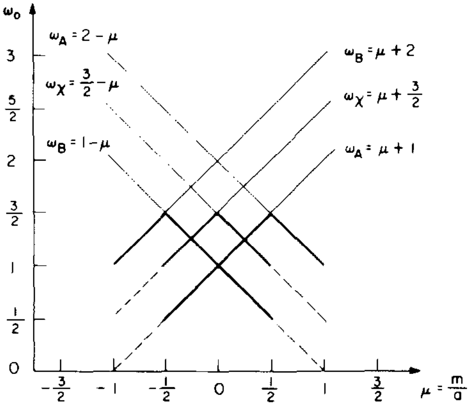

FIG. 3. Possible values for the lowest frequency ω_0 in the $SO(3, 2)$ representations for the massive scalar $OSp(N, 4)$ supermultiplet. The thin lines indicate regular modes with $\omega_0 > 3/2$ and the thick lines irregular modes with $\omega_0 < 3/2$. The dashed parts of the lines indicate where representations fail to be unitary.

APPENDIX C: THE IMPROVED ENERGY FUNCTIONAL

Although the derivation [6] of the Killing charges from the minimal form of the Einstein equations (4.9) appears natural, it leads to an energy functional which does not characterize allowed scalar fluctuations adequately and is inconsistent with $OSp(N, 4)$ supersymmetry.

In fact, it appears that the definition of the Killing energy of linearized scalar fluctuations is iherently ambiguous due to the freedom to make the change of variables

$$V_{a\mu} = A(\varphi)\, V'_{a\mu}, \qquad g_{\mu\nu} = A^2(\varphi)\, g'_{\mu\nu} \tag{C1}$$

which is a Weyl transformation of the gravitational variables with $A(\varphi)$ an arbitrary function of the scalar fields. The general scalar–graviton action (A2) can be written in terms of the variables $g'_{\mu\nu}$ nd φ as

$$I[g', \varphi] = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g'}\, \{-\tfrac{1}{2}A^2 R' - 2A^4 V(\varphi) + g'^{\mu\nu}[A^2 \partial_\mu \varphi\, \partial_\nu \varphi + 12 \partial_\mu A\, \partial_\nu A]\}. \tag{C2}$$

It is clear that if φ_0 is a stationary point of $V(\varphi)$ with de Sitter constant a^2, then the configuration $\varphi = \varphi_0$ and $g'_{\mu\nu} = \bar{g}_{\mu\nu}$ with $a'^2 = A^2(\varphi_0)\, a^2$ is the transformed background solution. This new background metric coincides with the previous if $A(\varphi_0) = 1$. The field equations for $g'_{\mu\nu}$ can be split as in Section 4 to obtain new $SO(3, 2)$ Killing charges. If one takes $A(\varphi) = 1 - \frac{1}{2}\beta(\varphi - \varphi_0)^2$, then the gravitational deviations $h_{\mu\nu}$ and $h'_{\mu\nu}$ coincide to lowest order and their contributions to the Killing energy are identical. However, the Killing energy of linearized scalar fluctuations $h = \varphi - \varphi_0$ is then the improved energy $E(\hat{T}_{\mu\nu})$ of Appendix A, which also can be derived from the action

$$I = \tfrac{1}{2} \int d^4x \sqrt{-g'} \{ g'^{\mu\nu} \partial_\mu h \, \partial_\nu h + (\alpha a^2 + 4\beta V(\varphi_0) + \tfrac{1}{2}\beta R')\, h^2 \}. \tag{C3}$$

One recalls that the special value $\beta = \lambda_-/(2\lambda_- + 1)$ with $\lambda_- = (3 - \sqrt{9 - 4\alpha})/2$ was required in the conserved energy operator for massive scalar fields. The Weyl transformation does not give any insight into the role of this special value, and this is even more puzzling for the massive $OSp(1, 4)$ supermultiplet where we would take $\Lambda = 1 - \frac{1}{2}\beta_A A^2 - \frac{1}{2}\beta_B B^2$ with $\beta_A \neq \beta_B$.

In the case of gauged $N = 4$ supergravity, one can understand the required Weyl transformation somewhat better. One can start with the action (2.2) and transform the metric with the conformal factor $\Lambda = (1 + \frac{2}{3}\bar{z}z/(1 - \bar{z}z))^{-1/4}$ which transforms the potential term into a pure cosmological constant. In linearized order this gives the stress tensor of a conformally coupled scalar field, as required in Section 6. Further one notes that this same Weyl transformation was used to derive the general coupling of the scalar multiplet to $N = 1$ supergravity using tensor calculus methods [23]. Thus, the conformal transformation may be related to the superspace structure of the theory.

Although this discussion does make it clear that there is an ambiguity in the scalar energy functional, it does not give an a priori argument that the improved energy is required. This is an open question for future work. Perhaps careful study of the asymptotic behaviour at spatial infinity within the canonical formalism would be useful.

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