Nuclear Physics B274 (1986) 183–224
North-Holland, Amsterdam

CLASSICAL AND QUANTUM DYNAMICS OF BPS MONOPOLES

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Received 7 February 1986

The dynamics of two non-relativistic BPS monopoles is described using the Atiyah-Hitchin metric on the space of collective coordinates of the monopoles, M. Classical orbits correspond to geodesics on M. Quantum states are obtained by supposing the hamiltonian is proportional to the covariant laplacian on M. We analyse in detail the asymptotic equations describing monopoles whose separation is large compared with the individual monopole radii. These give a non-trivial generalization of the Coulomb problem and are exactly soluble both in the classical and quantum regime. A variety of scattering processes and bound states are described.

1. Introduction

Recently it has become feasible to study the non-relativistic scattering of Bogomol'nyi-Prasad-Sommerfield (BPS) monopoles [1]. This is because the motion of two such monopoles is well-described by the evolution of a finite number of collective coordinates. At relative speeds much less than the speed of light, the monopoles deform adiabatically (i.e. reversibly) as they scatter, even if they come close together. Internal monopole vibrations, which would turn into radiation and carry away energy, can be neglected. The time-evolution equations for the collective coordinates are exactly known and several interesting solutions have been found.

Quantum mechanical monopole scattering should also have a relatively simple description at slow speeds, because radiation can again be neglected and we need only quantize the collective coordinates. Even here there are two physical regimes. If the impact speed is of order α times the speed of light or less, where α is the fine structure constant, then quantum effects are essential. On the other hand, at greater

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(North-Holland Physics Publishing Division)

speeds the quantum and classical scattering should be similar. We assume α is much less than 1; the value $\alpha = \frac{1}{137}$ may or may not be appropriate.

The BPS monopole is a static, spherically symmetric soliton, with smooth fields and a finite mass. It is a solution of the Bogomol'nyi equation [1]

$$B_i = D_i \Phi , \tag{1.1}$$

where B_i is the non-abelian magnetic field of an SO(3) gauge potential and $D_i\Phi$ is the covariant gradient of a triplet scalar Higgs field. This scalar field is massless. The monopole has four collective coordinates – three position coordinates and a phase angle. When these are time-dependent, the monopole acquires momentum and an electric charge – it becomes a moving dyon. Quantization of the collective coordinates leads to a quantized electric charge. We shall review this one-monopole system in sect. 2.

The Bogomol'nyi equation also has multimonopole solutions. These are much more difficult to find and describe than the single monopole solution, but a great deal is now known about them [2]. The possibility of having static configurations of separated monopoles is understood as due to the cancellation of the repulsive magnetic force by an attractive force mediated by the massless scalar field [3]. Of particular interest to us are the two-monopole solutions which have eight collective coordinates [2,4]. Roughly speaking, these parameters correspond to positions and phase angles for each monopole. Actually the set of solutions M has the form [2]

$$M = \mathbb{R}^3 \times \frac{S^1 \times \mathrm{M}}{\mathbb{Z}_2} . \tag{1.2}$$

A point in $\mathbb{R}^3 \times S^1$ specifies the centre of mass of the two-monopole system and a phase angle whose time-dependence determines the total electric charge. The four-dimensional manifold M is the interesting part of M. A point there specifies the separation of the monopoles and a relative phase angle, although things are more subtle when the monopoles coincide. The quotient by $\mathbb{Z}_2$ occurs partly because two monopoles cannot be distinguished, even classically.

Let C denote the set of all finite-energy field configurations in the two-monopole sector at a fixed time, with gauge equivalent fields identified. The fields $M \subset C$ all have the same energy, which by Bogomol'nyi's argument is the minimum possible. Because of energy conservation, time-varying fields will follow a path in C close to M if we specify as initial data a slow motion tangent to M. For example monopoles, or dyons of small electric charge, approaching at slow speed from far away would be described by such initial data. The field evolution in C can be regarded as the motion of a point in an infinite-dimensional bowl whose bottom is flat in certain directions. The principle determining the path in M which best approximates the true field evolution is the following [5]. The kinetic term of the underlying field theory

$$\int d^3x \left\{ \tfrac{1}{2}\| E_i E_i \| + \tfrac{1}{2}\| D_0\Phi\, D_0\Phi \| \right\} \tag{1.3}$$

defines a metric on C and this naturally induces a metric on M. Since the potential energy is constant on M, only this metric matters. The relevant path in M is therefore a geodesic. To study monopole scattering we need to know the metric on M and solve the geodesic equations.

It has not been possible to directly calculate the metric on M from the field theory. Nevertheless, Atiyah and Hitchin, by a remarkable indirect method, have found this metric [6]. Because it is curved there is non-trivial monopole scattering. It had been observed earlier that the Bogomol'nyi equation is related to the moment map for the group of gauge transformations acting on the set of all field configurations equipped with a natural symplectic structure, and this implies that M is hyper-Kähler [7]. Since the metric on $\mathbb{R}^3 \times S^1$ is the standard flat one, and decouples from the metric on M, the metric on M is also hyper-Kähler. But M is four-dimensional, so hyper-Kähler is equivalent to self-dual. The metric on M can therefore be thought of as a gravitational instanton.

Now, the metric on M is also rotationally invariant, because the monopoles live in ordinary flat space. It can be expressed in terms of three functions of the monopole separation, and self-duality implies that these functions obey a set of first-order ordinary differential equations. Atiyah and Hitchin found the essentially unique solution satisfying the physically acceptable boundary conditions. It involves the elliptic integral function, and gives a previously unknown type of gravitational instanton.

Because of the product structure of M and its metric, the c.m. momentum and the total electric charge of the two-monopole system are conserved and have no effect on the geodesic motion on M, that is, on the relative motion of the monopoles. Not all geodesics on M are known yet, although they could be found numerically. However, Atiyah and Hitchin discovered that M possesses some two-dimensional geodesic submanifolds, and studied the geodesics on these using simple numerical integration. Some of these geodesics correspond to pure monopole scattering through a range of angles. Others correspond to scattering processes, predicted in [5], where monopoles turn into dyons.

In the first part of this paper we shall review monopole scattering as described using the Atiyah-Hitchin metric, and then concentrate on the geodesic equations for monopoles or dyons which are never very close together. Well-separated monopoles behave like point particles, and the geodesic equations have a direct interpretation as equations of motion for these particles in $\mathbb{R}^3$ [8]. The electric charges of dyons are individually conserved in this limit, and in a subtle way the equations of motion generalize those of the classical Coulomb problem. There are bound orbits and scattering orbits, all of which are conic sections, and the differential cross section is Rutherford-like. We have also found the leading corrections to our asymptotic equations and have calculated the exponentially small electric charge exchange which occurs in monopole scattering at large impact parameter.

In the second part of the paper we analyse the quantum states of monopoles by quantizing the collective coordinates. Our two-monopole hamiltonian, which is purely kinetic, is proportional to the covariant laplacian on M constructed from the Atiyah-Hitchin metric. This is the natural generalization of the classical hamiltonian, which gives the geodesic picture of monopole scattering. Our wave function is meant to represent a field theory wave functional which depends non-trivially on the monopole collective coordinates, but which is in the ground state with respect to internal excitations of the monopole and the radiation field. Photons, massless Higgs particles and massive gauge bosons exist in the theory, but are absent in such a state. These particles are presumably emitted when monopoles scatter but the effect is small at small relative velocities.

We are not able to solve the quantum scattering problem completely. Nevertheless we find some qualitative results. For example, electric charge is integrally quantized and conserved for well-separated monopoles, but in a close collision there can be $2n$ units of charge exchanged. This can be interpreted as due to the exchange of n massive gauge bosons. We have also a complete analysis of scattering in the asymptotic limit. Here again there is a close analogy with quantum Coulomb scattering, and we find that the quantum and classical cross sections are the same.

Pure monopoles repel when they are moving, but dyons can attract. The asymptotic quantum hamiltonian has many dyon bound states, but most of these do not correspond to bound states of the complete hamiltonian. This is because the charge exchanging mechanism can reduce the charge difference between the dyons, and the energy released allows the dyons to escape. However, if the charge difference is precisely one unit then there is still an attraction and no way to reduce the charge. We show that true bound states do exist in this situation.

One problem with our discussion of quantum states is that quantization of the field theory would probably give the scalar field a small mass, so static monopoles would repel each other. This effect is ignored in our hamiltonian. A related problem is that there is a Casimir energy – the zero point energy of field fluctuations orthogonal to M – which probably is not constant over M. Both problems would be resolved by adding a potential energy term of order h to our hamiltonian, but we have no idea how it depends on the monopole separation.

It is not unreasonable to simply ignore these problems. What we want to show is that in the interaction of structured particles arising non-perturbatively in field theory, interesting physical phenomena can be attributed to the geometry of the collective coordinates alone. No doubt in more realistic situations there will be no Bogomol'nyi-type argument, and there will be a static force between two particles depending non-trivially on their separation. Nevertheless, the collective coordinate metric, which determines the kinetic energy, will probably be curved and have a crucial effect on particle scattering, especially at small separations. We believe this effect has not seriously been considered in, for example, the skyrmion model [9] of two-nucleon scattering, nor more importantly, in phenomenological investigations of the two-nucleon force. The study of monopoles suggests that it may well be misleading to use simple position vectors in $\mathbb{R}^3$ to model the collective coordinates of the nucleons.

In a supersymmetric setting, we suspect that our methods can be applied more rigorously. $N=4$ super-Yang-Mills theory has an $N=4$ multiplet of monopoles [10]. Supersymmetry implies that the Higgs particle is exactly massless, like the photon, and there are strong arguments indicating that the single monopole quantum state rigorously attains the Bogomol'nyi bound [11, 12]. Because the metric on M is hyper-Kähler there is an $N=4$ supersymmetric quantum mechanics on M [13], which very likely describes non-relativistic supersymmetric monopole scattering accurately. It has been recognized before that classical field theories with Bogomol'nyi-type solitons are naturally linked to supersymmetric quantum field theories [11]. The hyper-Kähler metric on M points to this link in a new way.

2. A brief review of the BPS monopole

The basic lagrangian is

$$L = T - V, \tag{2.1}$$

where T is the kinetic energy of the fields

$$T = \int \mathrm{d}^3x \left(\tfrac{1}{2}\|E_i E_i\| + \tfrac{1}{2}\|D_0\Phi\, D_0\Phi\|\right) \tag{2.2}$$

and V is the potential energy

$$V = \int \mathrm{d}^3x \left(\tfrac{1}{2}\|B_i B_i\| + \tfrac{1}{2}\|D_i\Phi\, D_i\Phi\|\right). \tag{2.3}$$

B_i and E_i are the non-abelian magnetic and electric field strengths for an SO(3) gauge potential A_μ, and $D_\mu\Phi$ is the covariant derivative of a triplet (Lie algebra-valued) scalar Higgs field. Thus

$$\begin{aligned} B_i &= \tfrac{1}{2}\varepsilon_{ijk}\left(\partial_j A_k - \partial_k A_j + \left[A_j, A_k\right]\right), \\ E_i &= \partial_0 A_i - D_i A_0, \\ D_\mu\Phi &= \partial_\mu\Phi + \left[A_\mu, \Phi\right]. \end{aligned} \tag{2.4}$$

We may introduce a basis for the SO(3) Lie algebra (t^a: $a = 1,2,3$) such that $[t^a, t^b] = \varepsilon^{abc}t^c$; $\|t^a t^b\| = \delta^{ab}$; and then $\Phi = \Phi^a t^a$, etc. T truly represents the gauge invariant kinetic energy of time-dependent fields only if A_0 is such that Gauss's law is satisfied,

$$D_i E_i + [\Phi, D_0\Phi] = 0. \tag{2.5}$$

There is no Higgs self-coupling in V – this is the Prasad-Sommerfield limit [1] – but we impose as a boundary condition $\|\Phi^2\| = 1$. The vacuum therefore has an unbroken electromagnetic U(1) gauge symmetry as if there were symmetry-breaking through the Higgs mechanism.

Bogomol'nyi [1] pointed out that V can be rewritten as

$$V = \int d^3x \left(\tfrac{1}{2}\|(B_i - D_i\Phi)(B_i - D_i\Phi)\| + \partial_i\|B_i\Phi\|\right). \tag{2.6}$$

The last term, which is equivalent to a surface integral at spatial infinity, is always an integer multiple of 4π if the fields are smooth. This integer is called the monopole number. Clearly, if

$$B_i = D_i\Phi, \tag{2.7}$$

V takes its minimum value in a given topological sector. Eq. (2.7) is the Bogomol'nyi equation. Static fields satisfying (2.7), with $A_0 = 0$, also satisfy the second-order field equations, which shows that V is stationary where its minimum is.

The essentially unique solution of (2.7) in the one-monopole sector is the BPS monopole [1],

$$A_i^a = \varepsilon_{iab}\frac{x^b}{x}\left(\frac{1}{x} - \frac{1}{\sinh x}\right),$$

$$\Phi^a = -\frac{x^a}{x}\left(\coth x - \frac{1}{x}\right), \tag{2.8}$$

with $x = (x^i x^i)^{1/2}$. These fields are spherically symmetric and smooth everywhere, including the origin. The short range part of A_i falls off like e^{-x}, so the core size of the monopole is 1. Well outside the core, where the long-range U(1) fields dominate, one can define the magnetic field $b_i = \|B_i\Phi\|$, whose strength is $b_i = x^i/x^3$. The BPS monopole therefore has a magnetic charge of 4π. Since the Bogomol'nyi equation is satisfied, eq. (2.6) shows that the monopole has mass 4π too. By changing the sign of Φ one obtains a monopole of charge -4π. The signs in (2.6) and (2.7) must then be reversed. (We are using the monopole to define our units of mass, length and magnetic charge. We also fix the speed of light to be 1.)

Other particles in the theory have a quantum origin. There is a photon and a scalar Higgs particle; also electrically-charged gauge bosons. In the tree approximation the Higgs particle is massless and the gauge bosons have mass $\hbar$. The charges of the gauge bosons are $\pm\hbar$. The product of the magnetic charge of the monopole and the electric charge of the gauge boson is therefore $4\pi\hbar$, the Schwinger unit (twice the Dirac unit) [14]. Note that the core size of the monopole is determined by the fields of the massive gauge bosons, and equals the Compton wavelength $(m/\hbar)$ of these particles. (Another note on units: the fine structure constant is $\alpha = Q_0^2/4\pi\hbar$, where Q_0 is a fundamental unit of electric charge. Here $Q_0 = \hbar$, so $\alpha = \hbar/4\pi$. If $\alpha \ll 1$ then $\hbar \ll 1$, and the smallness of these quantities expresses, as one would wish, the smallness of quantum effects compared with classical phenomena. $\hbar$ cannot be set to 1 because we have set the coupling constant to 1.)

A single monopole has four collective coordinates. Three of these, denoted by X, specify the position of the monopole. (It is at the origin in (2.8)). Since the mass of the monopole is 4π, its ordinary kinetic energy is $2\pi\dot{X}\cdot\dot{X}$. The fourth collective coordinate appears more subtly. Consider a time-varying field $(A_i(t), \Phi(t))$ whose time dependence is just a gauge transformation

$$\partial_0 A_i = D_i\Lambda , \qquad \partial_0\Phi = [\Phi, \Lambda] . \tag{2.9}$$

The potential energy is constant in time, so Λ is a zero mode. However Λ is not yet physically significant because normally Gauss's law implies that the kinetic energy is zero too; if one chooses $A_0 = \Lambda$, then Gauss's law is satisfied but $E_i = D_0\Phi = 0$. However, suppose $\partial_0 A_i$, $\partial_0\Phi$ have the form (2.9) and are also in background gauge, that is

$$D_i(\partial_0 A_i) + [\Phi, \partial_0\Phi] = 0 , \tag{2.10}$$

so that Gauss's law is satisfied with $A_0 = 0$. There is no non-trivial solution of (2.9) and (2.10) if Λ is also required to vanish at infinity, because $-D_iD_i - [\Phi[\Phi$ is formally a positive operator, but if this last condition is relaxed there is the essentially unique solution $\Lambda = \tau\Phi$ where τ is an arbitrary function of time. ($\partial_0 A_i = \tau D_i\Phi$, $\partial_0\Phi = 0$ and (2.10) is satisfied because Bogomol'nyi's equation implies $D_i D_i\Phi = 0$.) The kinetic energy is positive, so $\Lambda = \tau\Phi$ with $A_0 = 0$ is a physical zero mode.

We may rephrase this conclusion as follows. If we gauge transform the Prasad-Sommerfield solution by the gauge transformation

$$g = \exp(-\chi\Phi) , \tag{2.11}$$

then χ is a physical collective coordinate. If χ varies with time, *and we keep* $A_0 = 0$, then there is kinetic energy but the potential energy is constant. The nonabelian electric field is $\dot{\chi}D_i\Phi$ (so τ, above, is $\dot{\chi}$), and this equals $\dot{\chi}B_i$ by virtue of the Bogomol'nyi equation. The kinetic energy is therefore $2\pi\dot{\chi}^2$. Asymptotically we define the electric field $e_i = \| E_i\Phi\|$, which equals $\dot{\chi}b_i$. So the time dependence of the fields turns the monopole into a dyon (a particle with both magnetic and electric charge). Its electric charge is $4\pi\dot{\chi}$.

There is an alternative description of this dyon, the one used originally by Julia and Zee [15]. One simply reverses the gauge transformation (2.11). The fields are then static but A_0 is non-zero. Note that E_i feeds back into the field equations, so that to get the exact Julia-Zee solution one needs to rescale the monopole fields. If the electric charge is small we can ignore this effect, just as we ignore the Lorentz contraction of a moving monopole at non-relativistic speeds.

Summing up, the lagrangian for the monopole collective coordinates is

$$L = 2\pi\dot{X}\cdot\dot{X} + 2\pi\dot{\chi}^2 . \tag{2.12}$$

The second term is just the mass difference between a dyon and a pure (zero electric charge) monopole, but the internal motion that produced it should not be forgotten. The equations of motion derived from L imply that the momentum and electric charge are constant, but arbitrary. The electric charge is canonically conjugate to χ.

It is important to observe that when $\chi = 2\pi n$ the gauge transformation g is 1 at infinity. Fields that differ by short-range gauge transformations should be identified physically. χ is therefore an angular coordinate. This conclusion is not invalidated by the fact that the gauge transformations with $\chi = 2\pi$ and $\chi = 0$ are topologically distinct (i.e. the one cannot be continuously deformed into the other keeping $g = 1$ at infinity). The situation is rather like for points on a circle. The points with angular coordinates $\theta = 0$ and $\theta = 2\pi$ are geometrically identical but paths connecting $\theta = 0$ to $\theta = 2\pi$ are distinct from paths connecting $\theta = 0$ to $\theta = 0$. The angular nature of χ is crucial when the collective coordinate motion is quantized [16]. The quantum hamiltonian corresponding to (2.12) is

$$\frac{1}{8\pi} P\cdot P + \frac{1}{8\pi} Q^2 . \tag{2.13}$$

The operator Q, represented by $-ih\,\partial/\partial\chi$, is the electric charge operator. Q has eigenvalues hS where S is an integer. Therefore, quantization of the collective coordinate motion tells us that dyons have electric charges which are integer multiples of h [16], consistent with the Schwinger quantization condition. The mass of a dyon of unit charge exceeds by $h^2/8\pi$ the mass of a pure monopole.

3. The two-monopole parameter space and its metric

In this section we shall describe the metric discovered by Atiyah and Hitchin [6] on the two-monopole parameter space M whose form, we recall, is

$$M = \mathbb{R}^3 \times \frac{(S^1 \times \mathrm{M})}{\mathbb{Z}_2} . \tag{3.1}$$

Let us introduce coordinates X and χ on $\mathbb{R}^3 \times S^1$, with $0 \leqslant \chi \leqslant 2\pi$. X is the centre of mass of the two-monopole system. The metric on this part of M is flat, and is simply

$$4(\mathrm{d}X \cdot \mathrm{d}X + \mathrm{d}\chi^2). \tag{3.2}$$

M is a four-dimensional manifold on which SO(3) acts. Since almost all the SO(3) orbits are three-dimensional, M can be coordinatized by a radial coordinate r and Euler angles θ, ϕ and ψ, where $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$ and $0 \leqslant \psi \leqslant 2\pi$. Later we shall identify certain points with different Euler angles. Roughly speaking, r determines the separation of the monopoles, θ and ϕ are polar angles giving the direction in space of the axis joining the monopoles, and ψ is the rotation angle about this axis. ψ is significant because a two-monopole configuration is not axially symmetric in general.

The metric on M, which is independent of X and χ, is SO(3) symmetric, so we introduce the one-forms familiar in the analysis of rigid-body rotations

$$\begin{aligned}
\sigma_1 &= -\sin\psi\,\mathrm{d}\theta + \cos\psi\sin\theta\,\mathrm{d}\phi, \\
\sigma_2 &= \cos\psi\,\mathrm{d}\theta + \sin\psi\sin\theta\,\mathrm{d}\phi, \\
\sigma_3 &= \mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi,
\end{aligned} \tag{3.3}$$

with the property

$$\mathrm{d}\sigma_i = \tfrac{1}{2}\varepsilon_{ijk}\sigma_j \wedge \sigma_k. \tag{3.4}$$

The metric is then of the form

$$\mathrm{d}s^2 = f^2\,\mathrm{d}r^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2, \tag{3.5}$$

where a, b, c and f are functions of r. The metric on M is also self-dual, as explained in sect. 1. This implies

$$\frac{2bc}{f}\frac{da}{dr} = (b-c)^2 - a^2, \tag{3.6}$$

and the two related equations obtained by cyclicly permuting a, b, c.

The function f is at our disposal, since it can be altered by redefining the radial coordinate. Atiyah and Hitchin found the desired solution of (3.6), somewhat implicitly, assuming $f = abc$. By choosing instead $f = -b/r$ their solution can be made more explicit. Set

$$r = 2K(\sin\tfrac{1}{2}\beta). \tag{3.7}$$

where K is the elliptic integral

$$K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2\tau)^{-1/2} \, d\tau. \tag{3.8}$$

The range of r is $\pi \leqslant r < \infty$ as β varies between 0 and π. Now, defining

$$w_1 = bc, \qquad w_2 = ca, \qquad w_3 = ab, \tag{3.9}$$

the solution is

$$w_1 = -\sin\beta \, r \frac{dr}{d\beta} - \tfrac{1}{2}(1 + \cos\beta) r^2,$$

$$w_2 = -\sin\beta \, r \frac{dr}{d\beta},$$

$$w_3 = -\sin\beta \, r \frac{dr}{d\beta} + \tfrac{1}{2}(1 - \cos\beta) r^2. \tag{3.10}$$

In principle, by expressing β in terms of r, we can find a, b and c in terms of r. In fig. 1, graphs of the functions $a(r)$, $b(r)$ and $-c(r)$ are shown. These were

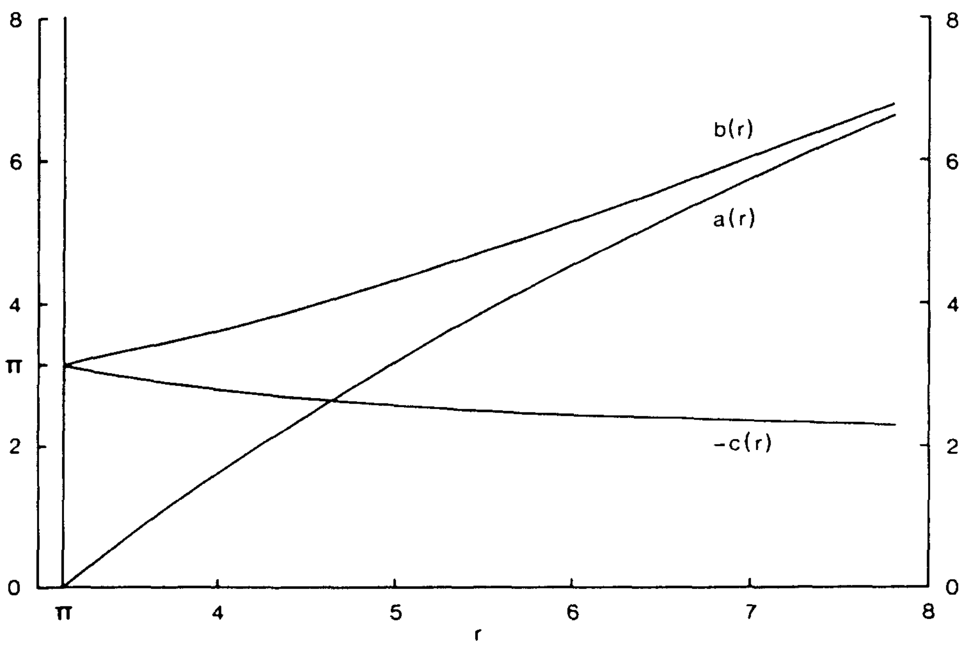

Fig. 1. The radial functions $a(r)$, $b(r)$ and $-c(r)$ of the Atiyah-Hitchin metric (cf. eq. (3.5)). $f(r)$ is $-b(r)/r$.

obtained by numerically integrating eqs. (3.6) outward from r slightly greater than π, with initial data given by eqs. (3.19) below.

To obtain the asymptotic form of the Atiyah-Hitchin metric, which is important for us, we use the asymptotic expansion of the elliptic integral when the modulus $\sin\frac{1}{2}\beta$ is near 1. Let $k' = \cos\frac{1}{2}\beta$ be the conjugate modulus, and let $\Lambda = \log(4/k')$. Then, for k' small,

$$r = 2K\big((1-k'^2)^{1/2}\big) = 2\Lambda + \tfrac{1}{2}(\Lambda - 1)k'^2 + O(k'^4) \tag{3.11}$$

or, reverting this expansion

$$k' = 4\,e^{-r/2}(1 + (2r-4)e^{-r} + \cdots). \tag{3.12}$$

Rewriting eqs. (3.10) in terms of k', and then using (3.12), one finds

$$\begin{aligned} w_1 &= -2r\left[1 + 4(r-1)e^{-r}\right] + \cdots, \\ w_2 &= -2r\left[1 - 4(r+1)e^{-r}\right] + \cdots, \\ w_3 &= r^2\left[\left(1 - \frac{2}{r}\right) - 8\left(1 - \frac{1}{r}\right)e^{-r}\right] + \cdots, \end{aligned} \tag{3.13}$$

and therefore

$$\begin{aligned} a &= r\left(1 - \frac{2}{r}\right)^{1/2} - 4r^2\left(1 - \frac{1}{2r^2}\right)e^{-r} + \cdots, \\ b &= r\left(1 - \frac{2}{r}\right)^{1/2} + 4r^2\left(1 - \frac{2}{r} - \frac{1}{2r^2}\right)e^{-r} + \cdots, \\ c &= -2\left(1 - \frac{2}{r}\right)^{-1/2} + \cdots. \end{aligned} \tag{3.14}$$

Neglected terms are of order e^{-2r} times an algebraic function of r, and in a and b there are neglected terms of order $(1/r)e^{-r}$.

By ignoring all exponentially decaying terms in a, b, c we obtain the metric at large monopole separation. The asymptotic forms

$$\begin{aligned} A = B &= r\left(1 - \frac{2}{r}\right)^{1/2}, \\ C &= -2\left(1 - \frac{2}{r}\right)^{-1/2}, \\ F &= -\frac{B}{r} = -\left(1 - \frac{2}{r}\right)^{1/2}, \end{aligned} \tag{3.15}$$

exactly satisfy eqs. (3.6), and the resulting asymptotic metric is

$$ds^2 = \left(1 - \frac{2}{r}\right)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2) + 4\left(1 - \frac{2}{r}\right)^{-1} (d\psi + \cos\theta \, d\phi)^2 . \quad (3.16)$$

This is a modified euclidean Taub-NUT metric [17], but unlike the Atiyah-Hitchin metric, it has a singularity at $r = 2$. This singularity is well inside the region where (3.15) is valid and therefore doesn't matter. (The regular Taub-NUT metric has a factor $1 + 2m/r$ with m positive, and it is smooth in the range $r \geqslant 0$. When $m = -1$ the metric is still self-dual, but singular.) Physically, by dropping the exponential terms, the metric becomes scale-free. The finite core size of the monopoles becomes unimportant and r is precisely the separation of the monopole centres. In ref. [8], it was shown that the metric (3.16) can be derived by analysing the dynamics of point-like BPS monopoles. Because $a = b$ asymptotically, $\psi \to \psi +$ constant is an isometry. The metric (3.16) therefore has an SO(2) symmetry commuting with the SO(3) rotational symmetry.

Next, let us discuss the discrete symmetries which allow the identification of points on the SO(3) orbits. These symmetries are analogous to rotations of an asymmetric rigid body by π about a principal axis. Such rotations are isometries because their effect is only to change the signs of two of the one-forms σ_1, σ_2 and σ_3. In the monopole context, two of the symmetries include a rotation by π on the S^1 factor of M. Explicitly, the symmetry operations are

$$\begin{aligned}
&I_1: \quad \theta \to \pi - \theta, \quad \phi \to \pi + \phi, \quad \psi \to -\psi, \quad \chi \to \chi, \\
&I_2: \quad \theta \to \pi - \theta, \quad \phi \to \pi + \phi, \quad \psi \to \pi - \psi, \quad \chi \to \pi + \chi, \\
&I_3: \quad \theta \to \theta, \quad \phi \to \phi, \quad \psi \to \pi + \psi, \quad \chi \to \pi + \chi,
\end{aligned} \quad (3.17)$$

which together with the identity form a viergruppe. The fields are unchanged under these operations, so the proper configuration space of the two-monopole system is obtained by identifying points related by them. The manifold M is obtained using the identification under I_1 alone. I_1 exchanges the positions of the monopoles, so classically we must regard the monopoles as identical particles. This makes sense in the field theory, since the field configuration does not provide labels for the monopoles. The symmetry I_3 is the generator of the $\mathbb{Z}_2$ which appears explicitly in (3.1). We need not discuss I_2 separately, because $I_2 = I_1 I_3$.

To investigate the metric near $r = \pi$, we use the series expansion for the elliptic integral

$$K(k) = \tfrac{1}{2}\pi\left[1 + \tfrac{1}{4}k^2 + \tfrac{9}{64}k^4 + \cdots\right] \quad (3.18)$$

and deduce that

$$a = 2(r-\pi)\left\{1 - \frac{1}{4\pi}(r-\pi) + \cdots\right\},$$

$$b = \pi\left\{1 + \frac{1}{2\pi}(r-\pi) + \frac{1}{4\pi^2}(r-\pi)^2 + \cdots\right\},$$

$$c = -\pi\left\{1 - \frac{1}{2\pi}(r-\pi) + \frac{1}{2\pi^2}(r-\pi)^2 + \cdots\right\}. \tag{3.19}$$

The manifold M is smooth at $r = \pi$, but since $a = 0$, there is a coordinate singularity there known to gravity theorists as a "Bolt" [18]. The three-dimensional orbit of SO(3), parametrized by Euler angles, collapses to a two-sphere. To understand M near the Bolt, it is helpful to introduce a new set of Euler angles $\tilde{\theta}, \tilde{\phi}, \tilde{\psi}$. These are related to the Euler angles θ, ϕ, ψ by relabelling axes so that

$$\sigma_1 = d\tilde{\psi} + \cos\tilde{\theta}\, d\tilde{\phi},$$

$$\sigma_2 = -\sin\tilde{\psi}\, d\tilde{\theta} + \cos\tilde{\psi}\sin\tilde{\theta}\, d\tilde{\phi},$$

$$\sigma_3 = \cos\tilde{\psi}\, d\tilde{\theta} + \sin\tilde{\psi}\sin\tilde{\theta}\, d\tilde{\phi}. \tag{3.20}$$

In terms of SO(3) rotation matrices,

$$R_1(\tilde{\psi})R_3(\tilde{\theta})R_1(\tilde{\phi}) = R_3(\psi)R_2(\theta)R_3(\phi), \tag{3.21}$$

where $R_m(\gamma)$ represents a rotation by γ about the mth axis.

Using the approximation $a = 2(r-\pi)$, $b = -c = \pi$, and $f = -1$, we find the metric near the Bolt is

$$ds^2 = dr^2 + 4(r-\pi)^2(d\tilde{\psi} + \cos\tilde{\theta}\, d\tilde{\phi})^2 + \pi^2(d\tilde{\theta}^2 + \sin^2\tilde{\theta}\, d\tilde{\phi}^2). \tag{3.22}$$

The last term is the metric on the Bolt itself with $\tilde{\theta}$ and $\tilde{\phi}$ spherical polar coordinates. If $\tilde{\theta}$ and $\tilde{\phi}$ are fixed, the first two terms can be written as

$$d\tilde{r}^2 + 4\tilde{r}^2 d\tilde{\psi}^2, \tag{3.23}$$

where $\tilde{r} = r - \pi$. This is the metric of $\mathbb{R}^2$ in plane polar coordinates, provided $\tilde{\psi}$ is an angle in the range $0 \leqslant \tilde{\psi} \leqslant \pi$. Because the symmetries I_1, I_2 and I_3 have the following effect in terms of the new Euler angles

$$
\begin{aligned}
&\mathrm{I}_1: && \tilde\theta \to \tilde\theta, && \tilde\phi \to \tilde\phi, && \tilde\psi \to \tilde\psi + \pi, && \tilde\chi \to \tilde\chi, \\
&\mathrm{I}_2: && \tilde\theta \to \pi - \tilde\theta, && \tilde\phi \to \tilde\phi + \pi, && \tilde\psi \to -\tilde\psi, && \tilde\chi \to \pi + \tilde\chi, \\
&\mathrm{I}_3: && \tilde\theta \to \pi - \tilde\theta, && \tilde\phi \to \tilde\phi + \pi, && \tilde\psi \to \pi - \tilde\psi, && \tilde\chi \to \pi + \tilde\chi,
\end{aligned} \tag{3.24}
$$

we see that regularity of the metric near the Bolt implies that we *must* identify the points related by the symmetry I_1. The geometry does not force the identification of points related by I_3; a smooth manifold results either way. Nevertheless, Atiyah and Hitchin's analysis shows that in the two-monopole configuration space, points related by I_3 *are* identified. This implies that on the Bolt itself, $(\tilde\theta, \tilde\phi)$ is identified with $(\pi - \tilde\theta, \tilde\phi + \pi)$ so the Bolt is the real projective plane, $\mathbb{R}\mathrm{P}_2$. The metric (3.22) is the natural metric on the tangent bundle to $\mathbb{R}\mathrm{P}_2$, with $\tilde r$ the length of the tangent vector.

Physically, the Bolt corresponds to field configurations, first constructed by Ward [2], where the monopoles are coincident. These are the only axially symmetric configurations and because the axis is unoriented, its direction is parametrized by a point on $\mathbb{R}\mathrm{P}_2$.

Finally, let us discuss the first homotopy group of M. If we concentrated on the region $r > \pi$, $\pi_1(M)$ would appear quite complicated. But M retracts onto the Bolt, where symmetry I_1 reduces to the identity. M is therefore homotopically $\mathrm{S}^1 \times \mathrm{S}^2$ with points related by I_3 identified. $\pi_1(M)$ is generated by a path which connects antipodal points on S^2 and simultaneously goes half-way round S^1. Acting with this generator twice gives a path which can be deformed to simply encircle the S^1 factor of M. Therefore $\pi_1(M) = \mathbb{Z}$. The relative two-monopole parameter space $\mathrm{M}/\mathbb{Z}_2$ is homotopically $\mathbb{R}\mathrm{P}_2$ so has first homotopy group $\mathbb{Z}_2$. This last fact is important in the quantum mechanics of two monopoles and will be discussed further below.

4. Geodesic equations for two-monopole scattering

In the slow motion limit, the lagrangian for the two-monopole system is proportional to the square of the proper velocity on the configuration space M. The constant of proportionality is π, which is half the reduced mass of a pair of monopoles. Using formulae (3.2) and (3.5) for the Atiyah-Hitchin metric on M, we obtain the lagrangian

$$
L = 4\pi(\dot X \cdot \dot X + \dot\chi^2) + \pi\left(f^2\dot r^2 + a^2 l_1^2 + b^2 l_2^2 + c^2 l_3^2\right). \tag{4.1}
$$

l_m is the component of angular velocity corresponding to the one-form σ_m, that is

$$l_1 = -\sin\psi\,\dot\theta + \cos\psi\,\sin\theta\,\dot\phi, \tag{4.2}$$

etc. The first two terms of L give the centre of mass kinetic energy, and the part of the internal energy of a pair of dyons related to the total electric charge. The remaining terms give the energy of relative motion and relative electric charge. Since L is purely kinetic, the equations of motion imply that the motion is along a geodesic at an arbitrary constant speed.

The lagrangian for the relative motion of the monopoles is similar to that for an asymmetric rigid body, the difference being that the analogues of the principal moments of inertia vary with r and hence with time, so the body changes shape. A two-monopole configuration is specified by giving the orientation in space of the principal axes, together with the radial coordinate r (see fig. 2). When r is large, the "body" is long and thin, with the monopoles near the ends.

Let us introduce

$$M_1 = a^2 l_1, \qquad M_2 = b^2 l_2, \qquad M_3 = c^2 l_3. \tag{4.3}$$

The variational equations one obtains from L are

$$\ddot X = 0, \qquad \ddot\chi = 0, \tag{4.4}$$

$$\frac{\mathrm{d}M_1}{\mathrm{d}t} = \left(\frac{1}{b^2} - \frac{1}{c^2}\right) M_2 M_3, \tag{4.5}$$

$$\frac{\mathrm{d}M_2}{\mathrm{d}t} = \left(\frac{1}{c^2} - \frac{1}{a^2}\right) M_3 M_1, \tag{4.6}$$

$$\frac{\mathrm{d}M_3}{\mathrm{d}t} = \left(\frac{1}{a^2} - \frac{1}{b^2}\right) M_1 M_2, \tag{4.7}$$

$$f\frac{\mathrm{d}}{\mathrm{d}t}\left(f\frac{\mathrm{d}r}{\mathrm{d}t}\right) = \frac{1}{a^3}\frac{\mathrm{d}a}{\mathrm{d}r}M_1^2 + \frac{1}{b^3}\frac{\mathrm{d}b}{\mathrm{d}r}M_2^2 + \frac{1}{c^3}\frac{\mathrm{d}c}{\mathrm{d}r}M_3^2. \tag{4.8}$$

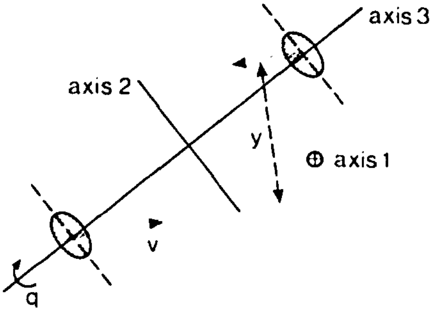

Fig. 2. Schematic diagram of a monopole collision. v is the impact speed, and y the impact parameter. The three principal axes are shown. The angular velocity about axis 3 is related to the relative electric charge q. Even at large separation there is no rotational symmetry about any axis, but there is reflection symmetry.

The total momentum, $\boldsymbol{P} = 8\pi\dot{\boldsymbol{X}}$, and the total electric charge, $Q = 8\pi\dot{\chi}$, are conserved and have no effect on the relative motion and relative electric charge of the monopoles. Another constant is the energy of the relative motion

$$E = \pi\left(f^2\dot{r}^2 + \frac{M_1^2}{a^2} + \frac{M_2^2}{b^2} + \frac{M_3^2}{c^2} \right). \tag{4.9}$$

In addition there are conserved angular momentum components J_m ($m = 1, 2, 3$) associated with the rotational symmetry of the metric. These have the form $J_m = \langle \xi_m, \Pi \rangle$ where ξ_m are rotational Killing vectors and Π is the canonical momentum one-form. (For $L = \frac{1}{2}mg_{\alpha\beta}(x)\dot{x}^\alpha\dot{x}^\beta$, $\Pi = mg_{\alpha\beta}(x)\dot{x}^\alpha\,\mathrm{d}x^\beta$.)

The Killing vectors are obtained as follows. SO(3) acts on itself by right and left multiplication, and these actions commute. In terms of Euler angles, the vector fields which generate these actions are

$$\begin{aligned}
\xi_1^{\mathrm{R}} &= -\cot\theta\cos\psi\frac{\partial}{\partial\psi} - \sin\psi\frac{\partial}{\partial\theta} + \frac{\cos\psi}{\sin\theta}\frac{\partial}{\partial\phi}, \\
\xi_2^{\mathrm{R}} &= -\cot\theta\sin\psi\frac{\partial}{\partial\psi} + \cos\psi\frac{\partial}{\partial\theta} + \frac{\sin\psi}{\sin\theta}\frac{\partial}{\partial\phi}, \\
\xi_3^{\mathrm{R}} &= \frac{\partial}{\partial\psi},
\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}
\xi_1^{\mathrm{L}} &= -\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\psi} + \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}, \\
\xi_2^{\mathrm{L}} &= -\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\psi} - \cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}, \\
\xi_3^{\mathrm{L}} &= -\frac{\partial}{\partial\phi}.
\end{aligned} \tag{4.11}$$

The Lie brackets of the right vector fields give the SO(3) Lie algebra, as do those of the left vector fields, but right and left vector fields commute

$$\left[\xi_m^{\mathrm{R}}, \xi_n^{\mathrm{R}}\right] = -\varepsilon_{mnp}\xi_p^{\mathrm{R}}, \tag{4.12}$$

$$\left[\xi_m^{\mathrm{L}}, \xi_n^{\mathrm{L}}\right] = \varepsilon_{mnp}\xi_p^{\mathrm{L}}, \tag{4.13}$$

$$\left[\xi_m^{\mathrm{R}}, \xi_n^{\mathrm{L}}\right] = 0. \tag{4.14}$$

Dual to these vector fields are sets of one-forms σ_m^{R} and σ_m^{L}, defined by $\langle \xi_m^{\mathrm{R}}, \sigma_n^{\mathrm{R}} \rangle = \delta_{mn}$, $\langle \xi_m^{\mathrm{L}}, \sigma_n^{\mathrm{L}} \rangle = \delta_{mn}$. Explicitly, σ_m^{R} are the one-forms of (3.3), and σ_m^{L} are obtained from these by exchanging ψ and ϕ and reversing the overall signs. They satisfy the Maurer-Cartan equations

$$\mathrm{d}\sigma_m^{\mathrm{R}} = \tfrac{1}{2}\epsilon_{mnp}\sigma_n^{\mathrm{R}} \wedge \sigma_p^{\mathrm{R}}, \tag{4.15}$$

$$\mathrm{d}\sigma_m^{\mathrm{L}} = -\tfrac{1}{2}\epsilon_{mnp}\sigma_n^{\mathrm{L}} \wedge \sigma_p^{\mathrm{L}}, \tag{4.16}$$

which follow from eqs. (4.12) and (4.13). Eq. (4.14) implies that the right one-forms σ_m^{R} are invariant under the left SO(3) action; their Lie derivatives with respect to ξ_m^{L} vanish. The Atiyah-Hitchin metric, constructed with the one-forms σ_m^{R}, is left-invariant too. ξ_m^{L} are therefore the Killing vectors.

The conserved quantities $J_m = \langle \xi_m^{\mathrm{L}}, \Pi \rangle$ are angular momenta relative to "space fixed axes"; on the other hand $2\pi M_m$ are angular momenta relative to "body axes" and are not conserved. One can check that $2\pi M_m$ has the form $\langle \xi_m^{\mathrm{R}}, \Pi \rangle$. The total angular momentum (squared) can be written as $J_1^2 + J_2^2 + J_3^2$ or $4\pi^2(M_1^2 + M_2^2 + M_3^2)$; these expressions are identical. The first is conserved, and eqs. (4.5)–(4.7) show that the second is conserved too.

Asymptoticaly, the lagrangian L has an additional symmetry, because $a = b$, and eq. (4.7) shows that there is an additional constant of motion, namely M_3. This is analogous to the angular momentum about the symmetry axis being constant for a symmetric rigid body. The analysis of ref. [8] shows that $2\pi M_3$ can be identified with q, where q is the relative electric charge of the monopoles. The conservation of both the total and relative charges implies that the individual charges of the monopoles are conserved. On the other hand, at small separations, the relative electric charge is neither conserved nor really well-defined.

If M_3 is constant, $4\pi^2(M_1^2 + M_2^2)$ is also constant. This last quantity equals $4\pi^2(1 - 2/r)^2 r^4(\dot\theta^2 + \sin^2\theta\,\dot\phi^2)$ asymptotically and is the square of the orbital angular momentum.

Knowing the constants of motion is useful, but we are still not able to solve the equations of motion completely. Nevertheless, Atiyah and Hitchin have shown that some interesting solutions are relatively easy to find [6]. These are analogous to rigid body rotations about a principal axis. In such a motion, two of the three quantities M_1, M_2 and M_3 vanish, and the third is constant. Eqs. (4.5)–(4.7) are clearly satisfied.

If $M_2 = M_3 = 0$ and $M_1 \neq 0$, then we have a particular case of pure monopole scattering. Conservation of energy implies

$$\dot r^2 = \frac{1}{f^2}\left(\frac{E}{\pi} - \frac{M_1^2}{a^2}\right), \tag{4.17}$$

which determines the radial motion. The constants E and M_1 are related to the relative velocity and impact parameter of the incoming monopoles. Since $a = 0$ at the Bolt, $\dot{r}$ is zero before the Bolt is reached, so the monopoles come to a point of closest approach, with $r > \pi$, and separate again. The scattering angle can be found using the coordinates $\tilde{r}, \tilde{\theta}, \tilde{\phi}, \tilde{\psi}$ adapted to the Bolt. Since $M_2 = M_3 = 0$, $\tilde{\theta}$ and $\tilde{\phi}$ are constant, and $M_1 = a^2\dot{\tilde{\psi}}$. The total change in $\tilde{\psi}$ is

$$\Delta\tilde{\psi} = M_1 \int_{-\infty}^{\infty} \frac{1}{a^2}\,\mathrm{d}t\,. \tag{4.18}$$

Knowing the radial motion, this integral can be computed, at least numerically. It converges. $\Delta\tilde{\psi}$ is the total rotation about axis 1, and is therefore the angle through which axis 3 rotates. Since axis 3 is the line joining distant monopoles (Atiyah and Hitchin call it the Higgs axis), $\Delta\tilde{\psi}$ is the scattering angle. The scattering is always repulsive, and as the impact parameter decreases to zero, the scattering angle increases monotonically to 90°.

A more interesting motion occurs if $M_1 = M_3 = 0$ and M_2 is constant. If the impact parameter is greater than π, then $M_2^2 > \pi E$, and energy conservation prevents the incoming monopoles from reaching the Bolt. The scattering is then similar to that described above. However, if the impact parameter is less than π, the Bolt is reached. Near there, the motion projects onto a motion in the $(\tilde{r}, \tilde{\psi})$ plane which passes through the origin. So, at the Bolt, $\mathrm{d}\tilde{r}/\mathrm{d}t$ changes sign and $\tilde{\psi}$ jumps by $\frac{1}{2}\pi$. (Recall $\tilde{\psi}$ has the range $0 \leqslant \tilde{\psi} \leqslant \pi$.) $\tilde{\theta}$ and $\tilde{\phi}$ vary continuously. Such a jump in $\tilde{\psi}$ exchanges M_2 and M_3 (cf. eqs. (3.20)), so in the outgoing motion M_1 and M_2 are zero and M_3 is constant. Now there are two possibilities. If the original impact parameter was less than 2, the outgoing particles are dyons with no orbital angular momenta. They come out back-to-back along a line perpendicular to the initial plane of motion of the monopoles – in fact along the original axis 2. On the other hand, if the impact parameter is between π and 2, the dyons do not have enough kinetic energy to escape to infinity. This is because $|c|$ decreases as r increases. At a finite separation the dyons stop and return to a configuration on the Bolt. There is another exchange of M_2 and M_3 and the final state consists of outgoing pure monopoles in the initial plane of motion. Clearly the scattering time can increase without bound in this type of process. The scattering angle, too, increases indefinitely. In the reverse of this type of scattering, M_3 is initially non-zero, and there is a head-on collision of dyons. This always leads to outgoing pure monopoles in the perpendicular plane.

Finally, we mention the simplest geodesic motion of all, where $M_1 = M_2 = M_3 = 0$. This is still a remarkable scattering process. Here there is a head-on collision of pure monopoles. All angles are constant until the Bolt is reached, and there $\tilde{\psi}$ jumps by $\frac{1}{2}\pi$. The outgoing particles are back-to-back pure monopoles but they have been scattered through 90° in the plane perpendicular to axis 1.

Atiyah and Hitchin have given a nice geometrical picture of all these scattering processes as geodesic motions on surfaces of revolution [6].

5. Classical monopole motion in the Taub-NUT limit

We show here that the asymptotic motion of monopoles, which corresponds to geodesic motion in euclidean Taub-NUT space, is integrable and has a remarkably close analogy with motion under a Coulomb force.

The equations for the relative motion of well-separated monopoles can be derived from eqs. (4.5)–(4.8). However it is more convenient to start with the asymptotic form of the lagrangian and deduce the dynamics from that. For $r \gg 1$ we can use the Taub-NUT forms of the metric coefficients A, B, C and F, and neglect the exponentially small corrections. The lagrangian is

$$L = \pi\left[\left(1 - \frac{2}{r}\right)\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} + 4\left(1 - \frac{2}{r}\right)^{-1}(\dot{\psi} + \cos\theta\,\dot{\phi})^2\right], \tag{5.1}$$

where $\boldsymbol{r}$ denotes the cartesian vector represented by (r, θ, ϕ) in spherical polars. ψ is now a cyclic coordinate and the relative electric charge

$$q = 8\pi\left(1 - \frac{2}{r}\right)^{-1}(\dot{\psi} + \cos\theta\,\dot{\phi}) \tag{5.2}$$

is the associated conserved quantity. In terms of $\dot{\boldsymbol{r}}$ and q, the conserved energy is

$$E = \pi\left(1 - \frac{2}{r}\right)\left(\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} + \left(\frac{q}{4\pi}\right)^2\right). \tag{5.3}$$

It is convenient to introduce the three-momentum

$$\boldsymbol{p} = 2\pi\left(1 - \frac{2}{r}\right)\dot{\boldsymbol{r}} \tag{5.4}$$

although this is only part of the momentum canonically conjugate to $\boldsymbol{r}$. Then the equation of motion is

$$\dot{\boldsymbol{p}} = 2\pi\frac{\boldsymbol{r}}{r^3}\left(\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} - \left(\frac{q}{4\pi}\right)^2\right) + q\frac{\dot{\boldsymbol{r}}\times\boldsymbol{r}}{r^3}. \tag{5.5}$$

In ref. [8], we showed that this is the expected equation for the relative motion of point dyons with magnetic and scalar charges 4π and relative electric charge q. So we have verified that the formula (5.2), with the given normalization, does represent the relative electric charge. Indirectly, the formula for the total charge, $Q = 8\pi\dot{\chi}$, is also verified. Eq. (5.5) shows that at small speeds, the dominant force between dyons is the attractive electric Coulomb force, and we shall see that there are both bound and scattering orbits for dyons. However, pure monopoles always repel, so they only have scattering orbits.

There are further constants of motion associated with eq. (5.5). One is the angular momentum vector

$$\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{p} - q\hat{\boldsymbol{r}}. \tag{5.6}$$

The first term is the orbital angular momentum, and the second is the Poincaré contribution which occurs when there are magnetic and electric charges present. Since these are orthogonal, the magnitude of the orbital angular momentum

$$l = |\boldsymbol{r} \times \boldsymbol{p}| \tag{5.7}$$

is also conserved. Finally, there is a conserved vector analogous to the Runge-Lenz vector of the Coulomb problem. Its existence is rather surprising in view of the velocity-dependent forces in (5.5) and the twisting effect of the last term. But one can easily check that

$$\boldsymbol{K} = \boldsymbol{p} \times \boldsymbol{J} + \left(4\pi E - \tfrac{1}{2}q^2\right)\hat{\boldsymbol{r}} \tag{5.8}$$

is conserved.

Using these conservation laws one can determine the orbits. Eq. (5.6) implies that

$$\boldsymbol{J} \cdot \hat{\boldsymbol{r}} = -q. \tag{5.9}$$

Therefore, as expected for dyons, the relative motion is on a cone whose vertex is at the origin, and whose axis is $\boldsymbol{J}$. The cone's opening angle is $2\cos^{-1}(|q|/|\boldsymbol{J}|)$. Also

$$\boldsymbol{K} \cdot \boldsymbol{r} = (\boldsymbol{r} \times \boldsymbol{p}) \cdot \boldsymbol{J} + \left(4\pi E - \tfrac{1}{2}q^2\right)r, \tag{5.10}$$

so, eliminating $\boldsymbol{r} \times \boldsymbol{p}$ in favour of $\boldsymbol{J}$ and using (5.9), we find

$$\left(\boldsymbol{K} + \frac{1}{q}\left(4\pi E - \tfrac{1}{2}q^2\right)\boldsymbol{J}\right) \cdot \boldsymbol{r} = J^2 - q^2. \tag{5.11}$$

The relative motion is therefore in a fixed plane. These results, combined, imply that dyon orbits are conic sections. Scattering orbits are hyperbolae and bound orbits are ellipses. (Circular, parabolic and straight-line orbits all occur as special cases.) In the c.m. frame, the individual dyons move along conics, but unlike in the Coulomb problem, the orbits lie in distinct parallel planes. This is because the plane of a non-degenerate conic section does not pass through the cone's vertex.

The case of pure monopole scattering ($q = 0$) needs to be treated separately. Here, the force is central, so both monopoles move in the same plane. Eq. (5.8) implies that

$$\boldsymbol{K} \cdot \boldsymbol{r} = J^2 + 4\pi E r, \tag{5.12}$$

which is again the equation of a conic. Since $E > 0$, $\boldsymbol{K} \cdot \boldsymbol{r}$ is always positive so the orbit must be a hyperbola.

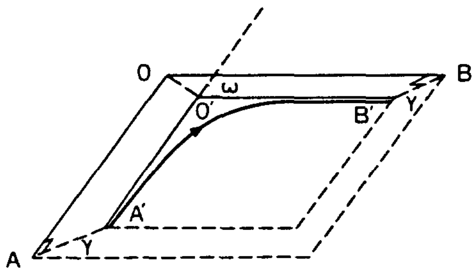

Fig. 3. A hyperbolic monopole scattering orbit in the Taub-NUT approximation. A′O′B′ is the orbit plane. ω is the scattering angle and γ the twist.

For scattering orbits, we are interested in how the scattering angle depends on the impact parameter as well as on the relative velocity and relative electric charge. Fig. 3 shows the geometry. The orbit is a hyperbola located at the intersection of a cone whose vertex is at the origin O and the plane A′O′B′. A′O′ and O′B′ are the incoming and outgoing asymptotes to the orbit. AO and OB are generators of the cone parallel to these asymptotes. The axis of the cone is not necessarily in the plane AOB. We shall refer to AO as the beam axis, because ultimately we are interested in the scattering of a beam of monopoles whose incoming asymptotes are *all* the lines parallel to AO. The impact parameter y is the perpendicular separation of the incoming asymptote from the beam axis.

If the force between monopoles were central, the orbit would lie in the impact plane AOO′A′. But unless $q = 0$, the force is not central, and the orbit is twisted. We define the twist γ to be the angle between the impact plane and the orbit plane A′O′B′.

The total energy E is $\pi(v^2 + (q/4\pi)^2)$, where v is the impact speed; the magnitude of the orbital angular momentum l is $2\pi yv$. Conservation of q, E and l imply that the final speed is v and that the outgoing asymptote O′B′ is separated from OB by the distance y. Conservation of the component of $\boldsymbol{J}$ perpendicular to the orbit plane implies that the angle between the planes BOO′B′ and A′O′B′ is γ. These last two results are also simple consequences of the geometry of conic sections.

To determine the scattering angle ω and twist γ, we use the remaining conservation laws. Conservation of the component of $\boldsymbol{J}$ in the orbit plane implies that

$$\tan\tfrac{1}{2}(\pi - \omega) = \frac{2\pi yv\sin\gamma}{q}. \tag{5.13}$$

Similarly, conservation of the component of $\boldsymbol{K}$ in the orbit plane implies that

$$\tan\tfrac{1}{2}(\pi - \omega) = -\frac{4\pi^2 yv^2\cos\gamma}{4\pi E - \frac{1}{2}q^2}. \tag{5.14}$$

We deduce from (5.13), (5.14) and the expression for E, that

$$\tan\tfrac{1}{2}\omega = \frac{1}{y}\left(1 + \left(\frac{q}{4\pi v}\right)^2\right), \tag{5.15}$$

$$\tan\tfrac{1}{2}\gamma = \frac{q}{4\pi v}. \tag{5.16}$$

Note that ω and γ only depend on the ratio of q to v. This is what the geodesic character of monopole scattering implies. When $q = 0$, one can deduce from (5.12) that $\tan\frac{1}{2}\omega = y^{-1}$, which agrees with the limit of (5.15) as $q \to 0$. When the impact parameter is large, the scattering angle is approximately

$$\frac{2}{y}\left(1 + \left(\frac{q}{4\pi v}\right)^2\right), \tag{5.17}$$

which is the formula given in ref. [8]. Rather surprisingly, the twist is independent of y.

The scattering of a uniform parallel beam of particles by a centre of force is characterized by the differential cross section, which measures the cross-sectional area element of the beam $d\sigma$ scattered into an element of solid angle $d o$. The usual formula is

$$\frac{d\sigma}{d o} = \frac{y}{\sin\omega}\left|\frac{dy}{d\omega}\right|. \tag{5.18}$$

For monopole scattering, (5.15) implies that

$$\frac{d\sigma}{d o} = \tfrac{1}{4}\left(1 + \left(\frac{q}{4\pi v}\right)^2\right)^2 \operatorname{cosec}^4\left(\tfrac{1}{2}\omega\right). \tag{5.19}$$

This is a modified Rutherford cross section. In conventional Coulomb scattering the cross section is proportional to $(q/v)^4 \operatorname{cosec}^4(\frac{1}{2}\omega)$. Eq. (5.19) has a different velocity-dependence because part of the force between monopoles is velocity-dependent.

We can use formula (5.19) in spite of the twist, because the entire incoming beam of monopoles is twisted by γ about the beam axis, and there is no effect on the number of particles scattered through an angle ω away from this axis.

Let us next consider bound orbits. The formula (5.3) for the conserved total energy shows that either $r > 2$ for all time and the energy is positive, or $r < 2$ and the energy is negative. The latter possibility is unphysical, because the Taub-NUT metric is not appropriate for such small r, and in any case the real energy is always positive. We shall therefore concentrate on positive energy bound orbits. Their energies lie in the range $0 < E < q^2/16\pi$, so only dyons can be bound.

Bound orbits are generally elliptical, but we have only investigated the circular orbits. These are conic sections perpendicular to the cone axis. On such an orbit the speed is constant, so the relative motion can be parametrized as

$$\boldsymbol{r}(t) = r(\sin\nu\cos\Omega t, -\sin\nu\sin\Omega t, \cos\nu). \tag{5.20}$$

2ν is the opening angle of the cone, r the distance from any point on the orbit to the vertex, and Ω is the angular velocity. In the c.m. frame the dyons move on parallel circles and have a constant separation r. One finds, after some algebra, that the orbit (5.20) satisfies the equation of motion (5.5), provided

$$1 - \frac{2}{r} = \sin\nu, \tag{5.21}$$

$$(r-1)^{1/2}(r-2) = \frac{q}{4\pi\Omega}. \tag{5.22}$$

On a cone of given opening angle there is therefore just one circular orbit, and r increases from 2 to ∞ as ν increases from 0 to $\frac{1}{2}\pi$. (It is likely that there is just one elliptical orbit of given eccentricity too.) Eq. (5.21) implies that the radius of the orbit is $r-2$, and the centre of the orbit is $2(r-1)^{1/2}$ from the vertex. This means that on a large orbit, the lines joining the dyon to the centre of force (the other dyon) and to the centre of the orbit have a small angular separation. This is to be expected, because the motion is slower and the electric Coulomb attraction of the dyons is the dominant force. The relative speed v is $(r-2)\Omega$, so eq. (5.22) can be rewritten as

$$\frac{q}{4\pi v} = (r-1)^{1/2}. \tag{5.23}$$

The total energy, given by (5.3), is therefore $q^2(r-2)/(16\pi(r-1))$, and the binding energy is $q^2/(16\pi(r-1))$. Finally, (5.22) implies that for large r, the period is proportional to $r^{3/2}$, which is Kepler's third law.

We have seen in this section that monopole orbits described by geodesics in euclidean Taub-NUT space have an unexpected simplicity due to the abundance of conservation laws. Schwinger and collaborators [19] have studied the scattering of "ordinary" dyons, point particles with electric and magnetic charges but no scalar charge. The Lorentz force equations are superficially simpler than (5.4) and (5.5) – the factor $(1-2/r)$ is absent from the momentum $\boldsymbol{p}$ and there are no forces quadratic in velocity. The orbits spiral around cones and are genuinely three dimensional. There are enough Poisson commuting constants of motion (E, $\boldsymbol{J}^2$ and one component of $\boldsymbol{J}$) for the equations to be integrable. Indeed, the equations are integrated in [19] and the differential cross section is evaluated. But there are not such simple analytic formulae as here, and the cross section is physically more complicated because there are "rainbow" and "glory" effects. It seems clear that the extra simplicity of BPS monopole scattering is due to the underlying self-duality of the Taub-NUT metric. This suggests that the geodesics of the self-dual Atiyah-Hitchin metric, which describe the scattering of BPS monopoles at all separations, should have an elegant mathematical description too. However, because of the lack of electric charge conservation, one is not dealing with an "integrable system", so our optimism may be unfounded.

We conclude this section by describing one correction to the results presented so far. We consider the effect, for a pure monopole scattering orbit, of the leading asymptotic term in the complete equations of motion which violates electric charge conservation. Despite the slow variation of the electric charge, the orbit itself remains approximately the same.

The relevant equation is (4.7), because asymptotically $q = 2\pi M_3$. The right-hand side of (4.7) is awkward to evaluate for a general orbit, but without loss of generality we may suppose that the orbit is a hyperbola in the plane $\phi = 0$ with its point of closest approach at $\theta = \frac{1}{2}\pi$. Any orbit can be brought to this position by a rotation. In zeroth approximation ψ is constant, because $\dot M_3 = 0$, and we find

$$M_1 M_2 \approx -8r^4\dot\theta^2 \sin 2\psi . \tag{5.24}$$

Then, using the formula (3.14) for the asymptotic metric coefficients, we deduce, to first approximation, that

$$\dot q \approx -32 r^3 e^{-r}\dot\theta^2 \sin 2\psi . \tag{5.25}$$

Orbital angular momentum conservation implies $r^2\dot\theta \approx yv$ where y and v are the impact parameter and impact speed, as before. Eliminating one factor of $\dot\theta$, we obtain

$$\dot q \approx -16\pi y v r e^{-r}\dot\theta \sin 2\psi , \tag{5.26}$$

so the change in q during the complete scattering is

$$\Delta q \approx -16\pi y v \sin 2\psi \int_{\theta_{in}}^{\theta_{out}} r e^{-r}\, d\theta . \tag{5.27}$$

The orbit does not deviate far from a straight line if $y \gg 1$, and the distance of closest approach is approximately $y + 1$. The integral in (5.27) can be estimated using a gaussian approximation near $\theta = \frac{1}{2}\pi$. The result is

$$\Delta q \approx -\frac{(8\pi)^{3/2}}{e} v y^{3/2} e^{-y} \sin 2\psi . \tag{5.28}$$

This formula has a number of interesting features. First, Δq is proportional to v, as expected. Second, charge exchange can be attributed, in quantum field theoretic terms, to exchange of charged gauge bosons. The factor e^{-r} in (5.28) is compatible with these particles having mass h. This point will become clearer when we discuss quantum scattering of monopoles below.

Finally, the factor $\sin 2\psi$ is periodic for ψ in the range $0 \leqslant \psi \leqslant \pi$ and it vanishes when $\psi = 0$ or $\frac{1}{2}\pi$. These two values correspond to initial conditions for which the monopole motion is described by a geodesic of the special type discussed at the end of sect. 4. In neither case is there any charge exchange at all if the impact parameter is larger than 2, and the estimate (5.28) is compatible with this.

6. The quantum problem

As explained in sect. 1, a two-monopole quantum state is well approximated by a complex-valued wave function Ψ defined on the configuration space M and obeying the Schrödinger equation

$$ih\frac{\partial\Psi}{\partial t} = -\frac{h^2}{4\pi}\nabla_M^2\Psi. \tag{6.1}$$

The factor 4π is twice the reduced mass and ∇_M^2 is the covariant laplacian on M

$$\nabla_M^2 = \frac{1}{4}\left(\frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial X_2^2} + \frac{\partial^2}{\partial X_3^2} + \frac{\partial^2}{\partial \chi^2}\right) + \nabla^2, \tag{6.2}$$

where ∇^2 is the covariant laplacian on the relative configuration space M with the Atiyah-Hitchin metric. We may separate off the c.m. motion and the dependence on the phase χ by setting

$$\Psi = \Phi\exp(i(N\cdot X + S\chi)), \tag{6.3}$$

where Φ is a wave function on M. $P = hN$ is the total momentum, and because the total charge Q is canonically conjugate to χ, $Q = hS$. χ is an angle whose range is $0 \leqslant \chi \leqslant 2\pi$, so S is an integer.

As in ordinary quantum mechanics we can use stationary wave functions to describe both bound states and scattering processes. In a stationary state (6.1) reduces to

$$-\frac{h^2}{4\pi}\nabla^2\Phi = E\Phi, \tag{6.4}$$

where E is the energy of the relative motion. The total energy is

$$E + \frac{|P|^2}{16\pi} + \frac{Q^2}{16\pi}. \tag{6.5}$$

In sect. 3 we saw that M is topologically $\mathbf{R}^3 \times (S^1 \times M)/\mathbf{Z}_2$, where $\mathbf{Z}_2$ acts by sending the point $(X, \chi, r, \theta, \phi, \psi)$ to $(X, \chi + \pi, r, \theta, \phi, \psi + \pi)$. Thus Φ must satisfy

$$\Phi(r, \theta, \phi, \psi + \pi) = (-1)^S \Phi(r, \theta, \phi, \psi), \tag{6.6}$$

which implies that if the total charge is odd (in units of $\hbar$), then Φ should be thought of as a section of a non-trivial line bundle over M. Since $\pi_1(M) = \mathbf{Z}_2$ there is just one such non-trivial bundle. If the total charge is even, Φ is a section of the trivial bundle over M, i.e. an ordinary function.

Physically this distinction provides us with an important selection rule. Asymptotically, for large r, $\partial/\partial\psi$ is a Killing vector and commutes with the hamiltonian. As we saw in sect. 5 (cf. eq. (5.2)), the relative electric charge q is canonically conjugate to ψ, so like the total charge, q is quantized in integer multiples of $\hbar$. Widely spaced dyons of relative charge $q = \hbar s$ have a wave function $\Phi \propto \exp(is\psi)$. Eq. (6.6) implies that S and s have the same parity modulo 2. This means that the individual electric charges of the dyons are integer multiples of $\hbar$. Relative electric charge is conserved asymptotically but not during scattering. An incoming wave with one value of q may emerge as a superposition of waves with different values of q. Nevertheless (6.6) implies that if s is originally even (odd), then s remains even (odd), i.e. the relative charge (in units of $\hbar$) is conserved modulo 2.

In the scattering of two pure monopoles, for example, the total and relative electric charges are both initially zero but during the scattering dyons may be produced. Since the total charge remains zero and the relative charge remains even, these dyons carry equal and opposite integer charges. Physically we can think of this process as the emission by one monopole of one or more massive gauge bosons carrying unit charge and their absorption by the other monopole (see fig. 4).

In addition to the scattering solutions, one can also ask about bound states, i.e. square integrable solutions of (6.4). It is easy to see, using the maximum principle, that any such solutions must have positive energy E. We may, with no loss of generality, assume that Φ is real and that $\Phi \to 0$ at infinity. Either Φ or $-\Phi$ must therefore have a positive maximum in the interior of M. Thus we may assume that there exists in M a point p and a small ball B surrounding p such that Φ is positive

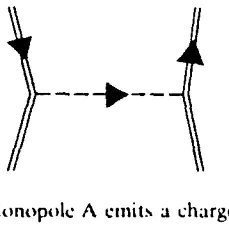

Fig. 4. Two pure monopoles scatter. Monopole A emits a charged gauge boson (dashed line), and turns into a dyon. Monopole B absorbs the gauge boson and turns into a dyon of the opposite electric charge to that of A. The arrow indicates the direction of charge flow.

in B and non-increasing on the boundary ∂B of the ball. Integrating (6.4) over B and using the divergence theorem, we find

$$\frac{-\hbar^2}{4\pi}\int_{\partial\mathrm{B}}(\partial_\mu\Phi)\,\mathrm{d}\Sigma^\mu = E\int_{\mathrm{B}}\Phi\sqrt{g}\,\mathrm{d}^4x, \tag{6.7}$$

where $\mathrm{d}\Sigma^\mu$ is the outward-pointing three-volume element of ∂B. The left-hand side of (6.7) is positive, and since the integral on the right-hand side of (6.7) is also positive we must have

$$E > 0. \tag{6.8}$$

It will be shown in sect. 7 that using the Taub-NUT metric as an approximation to the Atiyah-Hitchin metric, one obtains both positive and negative energy dyon bound states. The latter can occur because the Taub-NUT metric is not positive definite in the core region. The lightly bound positive energy states have support concentrated at large radii, whereas the tightly bound negative energy states have support concentrated near the core. It is clear from the result proved above that the negative energy bound states are an artefact of the approximation.

The classical limit of quantum scattering is most conveniently seen via the Hamilton-Jacobi equation. We write

$$\Phi = A\,\mathrm{e}^{iI/\hbar}, \tag{6.9}$$

where A is assumed slowly varying, and expand in powers of $\hbar$ to get an asymptotic expansion in what is essentially the ratio of the de Broglie wavelength of the relative motion (the scale over which I varies significantly), to a typical length scale of the metric (i.e. the size of the Bolt). If v is a typical relative velocity we therefore have an asymptotic expansion in powers of α/v where α is the "fine structure constant". For small enough v, quantum mechanical effects will certainly come into play but for $\alpha \ll v \ll 1$, we can approximate Φ with I satisfying the Hamilton-Jacobi equation

$$\frac{1}{4\pi}(\nabla I)^2 = E. \tag{6.10}$$

The orthogonal trajectories of I, defined by

$$2\pi\frac{\mathrm{d}x^\alpha}{\mathrm{d}t} = g^{\alpha\beta}\,\partial_\beta I \tag{6.11}$$

are geodesics of the metric $g_{\alpha\beta}$ with proper distance proportional to t. This situation should be contrasted with ordinary Coulomb scattering in which forces are velocity independent. There, the classical limit corresponds to low velocities. It should also be emphasised that for pure monopoles the necessary Bose symmetry under interchange of the monopoles may affect the wave function even in the classical limit. We shall discuss this later.

To proceed further we must write out the laplacian. A short calculation shows that (6.4) is equivalent to

$$\frac{1}{abcf}\frac{\partial}{\partial r}\left(\frac{abc}{f}\frac{\partial \Phi}{\partial r}\right)+\left[\frac{(\xi_1^{\mathrm{R}})^2}{a^2}+\frac{(\xi_2^{\mathrm{R}})^2}{b^2}+\frac{(\xi_3^{\mathrm{R}})^2}{c^2}\right]\Phi+\varepsilon\Phi=0, \tag{6.12}$$

where

$$\varepsilon=\frac{4\pi E}{h^2}. \tag{6.13}$$

a, b, c and f are the radial functions which appear in the Atiyah-Hitchin metric (3.5), and ξ_m^{R} are the operators defined in (4.10). Eq. (6.12) can be transformed into another form by introducing a "flat" radial coordinate ρ, defined by

$$\frac{1}{2\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}r}=\frac{f}{abc}. \tag{6.14}$$

Then (6.12) becomes

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{4\rho^4}\left[w_1^2(\xi_1^{\mathrm{R}})^2+w_2^2(\xi_2^{\mathrm{R}})^2+w_3^2(\xi_3^{\mathrm{R}})^2\right]\Phi+\varepsilon\frac{w_1w_2w_3}{4\rho^4}\Phi=0, \tag{6.15}$$

where $w_1=bc$, etc., as before. Eq. (6.15) has the advantages that the first term is like the radial part of the laplacian in flat space, and that w_1, w_2 and w_3 are more readily computable than a, b, c and f, especially asymptotically. Since $f=-b/r$, (6.14) can be rewritten as

$$\frac{1}{\rho}=2\int_r^{\infty}\frac{\mathrm{d}r'}{r'w_2(r')}, \tag{6.16}$$

and since $w_2\approx-2r(1-4(r+1)\mathrm{e}^{-r})$ asymptotically,

$$\rho\approx r(1-4\mathrm{e}^{-r}), \tag{6.17}$$

with corrections of order e^{-2r}.

It is manifest from (6.12) or (6.15) that the laplacian commutes with the action of the Killing vectors ξ_m^{L}, defined in (4.11). Eq. (6.15) may be separated by expanding

Φ in a basis of functions of the Euler angles (θ, ϕ, ψ) carrying a representation of SO(3). For our purposes the most convenient functions are built up from Wigner-type functions [20] $D^j_{sm}(\phi, \theta, \psi)$ which diagonalize the commuting operators ξ^R_3, ξ^L_3 and $(\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2$. Explicitly,

$$\xi^L_3 D^j_{sm} = -imD^j_{sm}, \tag{6.18}$$

$$\xi^R_3 D^j_{sm} = isD^j_{sm}, \tag{6.19}$$

$$\left((\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2\right) D^j_{sm} = -j(j+1) D^j_{sm}. \tag{6.20}$$

The quadratic operator should be read as having superscripts R or superscripts L, but it is the same in either case. Multiplying by $-\hbar^2$ gives the squared total angular momentum operator J^2. $i\hbar\xi^L_3$ is the third component of angular momentum J_3, and $-i\hbar\xi^R_3$ is the relative electric charge operator q. Since J^2 and J_3, but not in general q, commute with the laplacian, we may find solutions of the form (no sum over m and j)

$$\Phi^j_m = \sum_{s=-j}^{s=+j} A^j_{sm}(\rho) D^j_{sm}(\phi, \theta, \psi), \tag{6.21}$$

where the $(2j+1)$ radial functions $A^j_{sm}(\rho)$ satisfy the coupled matrix equations

$$\frac{1}{\rho^2}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^2 \frac{\mathrm{d}}{\mathrm{d}\rho} A^j_{sm}(\rho)\right) + \sum_{s'=-j}^{j} M^{s'}_s(\rho) A^j_{s'm}(\rho) + \varepsilon \frac{w_1 w_2 w_3}{4\rho^4} A^j_{sm}(\rho) = 0. \tag{6.22}$$

The radial matrix-valued functions $M^{s'}_s(\rho)$ are constructed from w_1, w_2, w_3 and the matrix elements of $(\xi^R_1)^2, (\xi^R_2)^2$ and $(\xi^R_3)^2$ in the basis $\{D^j_{sm}\}$. Since ξ^R_1, ξ^R_2 and ξ^R_3 occur quadratically in (6.15), $M^{s'}_s(\rho)$ vanishes unless s and s' have the same parity. Thus the total angular momentum j, the azimuthal quantum number m, and the parity of s are good quantum numbers. But s itself is not a good quantum number. Note that the different s' values contributing to (6.22) are finite in number.

Although the Wigner functions D^j_{sm} are sufficient to expand Φ, they do not possess the discrete symmetries of the Atiyah-Hitchin metric, that is, they do not lead to well-defined sections of either the trivial or non-trivial bundle over M. Using the conventions of Landau-Lifshitz [20], we note that

$$D^j_{sm}(\phi, \theta, \psi) = e^{is\psi} d^j_{sm}(\theta) e^{im\phi}, \tag{6.23}$$

where $d^j_{sm}(\theta)$ depends only on θ and satisfies

$$d^j_{sm}(\pi - \theta) = (-1)^{j+m} d^j_{-sm}(\theta). \tag{6.24}$$

Thus if I_1 is to act as the identity (as it must if the Bolt is to be regular), the appropriate angular functions are

$$W^j_{sm} \equiv e^{im\phi}\left[e^{is\psi}d^j_{sm}(\theta) + (-1)^j e^{-is\psi}d^j_{-sm}(\theta)\right], \tag{6.25}$$

where s may now be assumed to be nonnegative.

The functions W^j_{sm} are easily seen to be eigenfunctions of the third symmetry I_3 with eigenvalue $(-1)^s$. Therefore W^j_{sm}, multiplied by any suitable radial function, is a section of the trivial (non-trivial) bundle over M if s is even (odd), and satisfies (6.6) provided the parities of S and s are the same.

We need to impose boundary conditions at the Bolt. In its vicinity, $a \approx 2\tilde{r}$, $b \approx -c \approx \pi$, $f \approx -1$, and $\tilde{r} = r - \pi$ is approximately the proper radial distance from it. Eq. (6.12) reduces to Bessel's equation. A solution analytic at $\tilde{r} = 0$ is

$$e^{ip\tilde{\psi}}d^j_{pn}(\tilde{\theta})e^{in\tilde{\phi}}J_{p/2}\left[\left(\varepsilon - \frac{j(j+1)-p^2}{\pi^2}\right)^{1/2}\tilde{r}\right], \tag{6.26}$$

where $J_{p/2}(x)$ is the usual Bessel function and $\tilde{\psi}, \tilde{\theta}, \tilde{\phi}$ are Euler angles adapted to the Bolt. Since $0 \leqslant \tilde{\psi} \leqslant \pi$, p is an even integer. As $\tilde{r} \to 0$, $\Phi \approx \tilde{r}^{p/2}$, so if $p \neq 0$ the function (6.26) vanishes on the Bolt; otherwise it remains bounded there. The other linearly-independent solution of Bessel's equation has a logarithmic branch point and we reject it. Acceptable wave functions are those which are linear combinations of the functions (6.26) near the Bolt.

In general, eq. (6.22) is hard to solve exactly. Asymptotically, however, (6.15) reduces to

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{d\Phi}{d\rho}\right) + \frac{1}{\rho^2}\left((\xi^R_1)^2 + (\xi^R_2)^2\right)\Phi + \frac{1}{4}\left(1 - \frac{2}{\rho}\right)^2(\xi^R_3)^2\Phi$$
$$+ \frac{8}{\rho}e^{-\rho}\left((\xi^R_1)^2 - (\xi^R_2)^2\right)\Phi + \varepsilon\left(1 - \frac{2}{\rho}\right)\Phi = 0, \tag{6.27}$$

where terms of order $e^{-2\rho}$ and smaller have been neglected. The only relative charge changing term in (6.27) is the penultimate one which changes q by two units. It represents the exchange of a massive gauge boson of unit charge. The Yukawa-like factor $8e^{-\rho}/\rho$ is just what one expects for the exchange of a particle of mass h, and its strength, i.e. the factor of 8, is rather large. In the S-wave, i.e. when $j = 0$ and Φ is independent of θ, ϕ and ψ, there is no charge-changing interaction, but in all higher partial waves there is.

If we drop the charge changing term in (6.27), we are left with the wave equation in Taub-NUT space. This can be solved exactly, as we shall show in the next section. It is important to study quantum states in Taub-NUT space, because we wish to describe scattering in terms of asymptotic states with definite electric charge. Also, some bound states in Taub-NUT space approximate bound states of the exact problem. We would like to regard the charge-changing term as a small perturbation but this approach has its limitations, because it only works if the unperturbed states have support far from the core region.

7. The Schrödinger equation in Taub-NUT space

In the limit that the radial distance ρ tends to infinity and exponentially small terms are ignored, eq. (6.28) becomes the Schrödinger equation in Taub-NUT space, and ρ coincides with the standard radial coordinate r. This equation separates by setting

$$\Phi = \frac{h(r)}{r} W^{j}_{sm}(\phi, \theta, \psi). \tag{7.1}$$

It then reduces to the radial equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{j(j+1)}{r^2} - \tfrac{1}{4}s^2\left(1 - \frac{4}{r}\right) + \varepsilon\left(1 - \frac{2}{r}\right)\right]h(r) = 0. \tag{7.2}$$

W^{j}_{sm} is the symmetrized angular wave function defined in (6.25); j, m and s are the quantum numbers of total angular momentum, its azimuthal component, and the relative electric charge.

Eq. (7.2) is remarkable in that it has a centrifugal term $j(j+1)/r^2$ which depends only on j and not on s, and a Coulomb term $(s^2 - 2\varepsilon)/r$ which is quadratic in velocity (including the velocity in the relative phase direction). In more usual non-relativistic dyon-scattering problems, neither of these two properties hold. This leads to a complicated picture for the scattering involving "rainbows" and "glories" [19]. Neither of these features is present here. Another remarkable feature of eq. (7.2) is that it has singular points only at $r = 0$ (a regular singular point) and at infinity (an irregular singular point). The point $r = 2$, where the Taub-NUT metric becomes singular, since its signature changes from being $++++$ to $----$, is an *ordinary* point.

The solutions near $r = 0$ are of the form

$$h \propto r^{j+1} \qquad \text{or} \qquad h \propto r^{-j}. \tag{7.3}$$

By choosing $h \propto r^{j+1}$ as a boundary condition, we obtain a well-defined problem for which (7.2) is self-adjoint on $0 \leqslant r < \infty$. Clearly solutions which have large support for $r < 2$ cannot be expected to be good approximations to solutions of the exact matrix equations (6.22), but solutions with support in the range $r \gg 2$ should be good approximations.

Continuum solutions of (7.2) bounded at $r=0$ have the form

$$h(r) = r^{j+1} e^{ikr} F(i\lambda + j + 1, 2j+2, -2ikr), \tag{7.4}$$

where k is the effective spatial wave number at infinity defined by

$$k^2 = \varepsilon - \tfrac{1}{4}s^2, \tag{7.5}$$

and where

$$\lambda = \frac{1}{k}\left(k^2 - \tfrac{1}{4}s^2\right). \tag{7.6}$$

F is the confluent hypergeometric function

$$F(a, b, u) = 1 + \frac{a}{b}u + \frac{a(a+1)}{b(b+1)}\frac{u^2}{2!} + \cdots. \tag{7.7}$$

Square-integrable bound states correspond to values of λ such that

$$\lambda^2 = -n^2, \qquad n = 1, 2, \dots, \tag{7.8}$$

$$n > j. \tag{7.9}$$

This gives bound state energy levels

$$\varepsilon = \tfrac{1}{2}(n^2 - s^2)^{1/2}\left(\pm n - (n^2 - s^2)^{1/2}\right). \tag{7.10}$$

Since $n > j$ and, by the theory of angular momentum, $j \geqslant |s|$, we have $n^2 > s^2$. The lower sign in (7.10) gives $\varepsilon < 0$ and the upper sign $\varepsilon > 0$. If $s = 0$ there are no positive-energy bound states. If $s^2 > 0$ the tightly bound states with $\varepsilon < 0$ are probably artefacts of the Taub-NUT approximation. The lightly bound states with $\varepsilon > 0$ are more interesting. Their energy levels are shown in fig. 5. If $n \gg s$,

$$\varepsilon \approx \tfrac{1}{4}s^2 - \tfrac{1}{16}\frac{s^4}{n^2} + O\left(\frac{1}{n^4}\right), \tag{7.11}$$

so these high-lying levels are approximately ordinary Coulomb energy levels. The continuum begins at $\varepsilon = \frac{1}{4}s^2$. If $s \neq 0$ this is greater than zero and is just the electrostatic self-energy of a pair of dyons at rest.

Since the energy levels depend only on n and s and not on j and m, the bound states are degenerate with multiplicity (by (7.9)) n^2. This extra degeneracy is due to the conserved Runge-Lenz vector. We shall not explore this connection in detail in this paper but remark that there appears to be a close relation between the existence

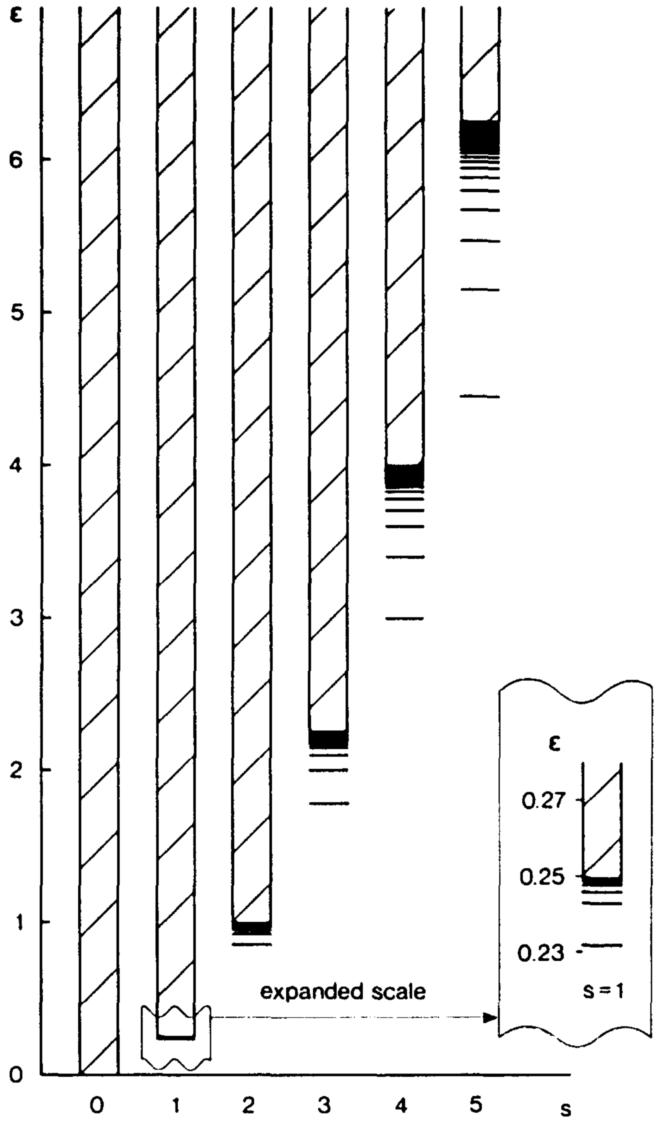

Fig. 5. The positive energy levels in the Taub-NUT metric for various values of s. The continuum begins at $\epsilon = \frac{1}{4}s^2$ and there is an accumulation of levels just below the continuum. In the exact Atiyah-Hitchin metric, s is not a good quantum number so one expects transitions between states of different s. However, s is conserved modulo 2.

of extra conserved quantities (quadratic in four-velocities) and the possibility of separating variables in two different coordinate systems. This latter fact is of special importance for us since it is possible to solve the scattering problem in this other set of coordinates – parabolic coordinates – *exactly* without recourse to a partial wave expansion. This is not possible for the more usual case of nonrelativistic dyon scattering [19].

We introduce new variables, with $z = r\cos\theta$,

$$\xi = r + z = r(1 + \cos\theta),$$

$$\eta = r - z = r(1 - \cos\theta). \tag{7.12}$$

In the coordinates (r, θ, ϕ) thought of as being in flat three-space, the surfaces $\xi =$ constant and $\eta =$ constant are paraboloids of revolution about the z-axis. We shall take this axis as the incoming beam direction of the particles. Since $\partial/\partial\psi$ and $\partial/\partial\phi$ are manifestly commuting Killing vectors, we write

$$\Phi = e^{is\psi}e^{im\phi}\Lambda(\xi, \eta), \tag{7.13}$$

and find that Λ must satisfy

$$\frac{4}{\xi + \eta}\left[\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\Lambda}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\eta\frac{\partial\Lambda}{\partial\eta}\right)\right] - \frac{1}{\xi\eta}\left[m^2 + s^2 - 2ms\frac{\xi - \eta}{\xi + \eta}\right]\Lambda$$

$$-(2\varepsilon - s^2)\frac{2\Lambda}{\xi + \eta} + \left(\varepsilon - \frac{s^2}{4}\right)\Lambda = 0. \tag{7.14}$$

One sees that (7.14) separates to give solutions of the form

$$\Phi = e^{is\psi}e^{im\phi}(r + z)^{|m+s|/2}(r - z)^{|m-s|/2}e^{-ik(r+z)/2}e^{-ik(r-z)/2}$$

$$\times F(c_1, |m + s| + 1, ik(r + z))F(c_2, |m - s| + 1, ik(r - z)), \tag{7.15}$$

where again $F(a, b, u)$ is the confluent hypergeometric function and c_1 and c_2 are separation constants satisfying

$$c_1 + c_2 = 1 + \tfrac{1}{2}|m - s| + \tfrac{1}{2}|m + s| - i\lambda. \tag{7.16}$$

We could use (7.15) to reobtain the bound states but of more interest to us are the scattering solutions. In the standard Coulomb problem, Gordon [21] and Temple [22] gave explicit scattering solutions by demanding axisymmetry, i.e. $m = 0$. In our case that is not what is required (because of the twisting of the hyperbolic orbits around the beam axis). However, one can set $m = s$ and $c_2 = 1$ (or $m = -s$ and $c_1 = 1$) in (7.15) to obtain the analogue of the Temple solutions

$$\Phi_+ = e^{is(\phi+\psi)}(r+z)^{|s|}e^{-ikz}F(|s| - i\lambda, 2|s| + 1, ik(r+z)), \tag{7.17}$$

$$\Phi_- = e^{is(\phi-\psi)}(r-z)^{|s|}e^{+ikz}F(|s| - i\lambda, 2|s| + 1, ik(r-z)). \tag{7.18}$$

Neither Φ_+ nor Φ_- is a well-defined function on the configuration space, i.e. neither satisfies

$$\Phi(r+z, r-z, \phi, \psi) = \Phi(r-z, r+z, \phi+\pi, -\psi). \tag{7.19}$$

In fact,

$$\Phi_+(r-z, r+z, \phi+\pi, -\psi) = (-1)^s\Phi_-(r+z, r-z, \phi, \psi). \tag{7.20}$$

Thus the properly symmetrized wave function is

$$\Phi = \tfrac{1}{2}(\Phi_+ + (-1)^s\Phi_-). \tag{7.21}$$

The necessity of symmetrizing the wave function as in (7.21) reduces in the pure monopole case, $s = 0$, to the effect first discussed by Mott [23] in the context of standard Coulomb scattering. The two monopoles may be thought of as indistinguishable bosons. The antipodal map $(\theta, \phi) \to (\pi - \theta, \phi + \pi)$ interchanges them and the wave function must be even under that interchange (see fig. 6). Symmetrizing the wave function gives rise to significant interference effects even in the apparently classical limit, tending to enhance scattering at right angles to the beam axis. If the monopoles were to be thought of as fermions, the scattering at right angles would be supressed.

The case of dyons is more complicated. If $s \neq 0$ the symmetrized expression (7.21) describes a dyon incident along the positive z axis and a dyon of different charge incident along the negative z axis, i.e. it describes the two processes in fig. 7, both of which conserve the relative charges. In the absence of charge exchange, these two processes are physically distinguishable, and we would expect no interference between them. This follows from the fact that the interference term is proportional

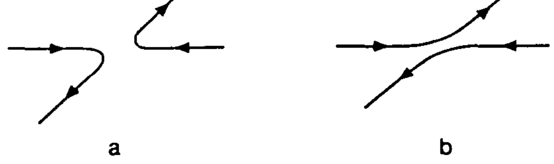

Fig. 6. The scattering of two identical monopoles. Process (a) is indistinguishable from process (b). The amplitudes must be added and one gets non-trivial interference.

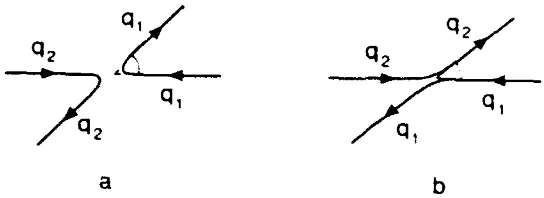

Fig. 7. The scattering of a dyon pair with one particle emerging at an angle θ relative to the beam axis. Relative charge is conserved in both processes in the Taub-NUT approximation. The amplitudes must be added but processes (a) and (b) are physically distinguishable because the charge of the outgoing particle is different. There is therefore no interference.

to $\Phi_+\Phi_-^* + \Phi_+^*\Phi_-$ which in turn is proportional to $\exp(2is\psi)$. The internal phase ψ is not measured in the scattering process so the cross section is obtained by averaging over ψ. This average vanishes.

Note that if electric charges were not conserved, the amplitudes would not simply be proportional to $\exp is\psi$ and the argument given above would break down. Indeed if we included the charge-changing terms described in sect. 6, we would expect some non-trivial interference.

To compute the cross sections we need the asymptotic forms of the solutions Φ_+ and Φ_- for large r. These depend on the asymptotic form of $F(a, b, u)$ for $|u|$ large, which is (neglecting terms of order u^{-a-2} and smaller)

$$F(a,b,u) \approx \frac{\Gamma(b)}{\Gamma(b-a)}\frac{1}{(-u)^a}\left\{1 - \frac{a(a-b+1)}{u} + \frac{\Gamma(b-a)}{\Gamma(a)}\frac{(-1)^a e^u}{u^{b-2a}}\right\}. \tag{7.22}$$

Thus, setting $u = ik(r-z)$, $a = |s| - i\lambda$ and $b = 2|s| + 1$, and dropping an overall constant factor, we obtain the asymptotic form of Φ

$$e^{is(\phi-\psi)}\left[\exp(i(kz + \lambda\log k(r-z)))\left(1 + \frac{s^2+\lambda^2}{ik(r-z)}\right)\right.$$
$$\left. + \frac{\exp(i(kr - \lambda\log k(r-z)))}{2ikr\sin^2(\frac{1}{2}\theta)} e^{i(\tau+\pi|s|)}(|s| - i\lambda)\right], \tag{7.23}$$

where

$$\tau = \arg\frac{\Gamma(1+|s|+i\lambda)}{\Gamma(1+|s|-i\lambda)}. \tag{7.24}$$

It is usual to regard the $\exp(ikz)$ term in (7.23) as a distorted plane wave incident along the z-axis and the $r^{-1}\exp(ikr)$ term as giving a scattered spherical wave with amplitude

$$f(\theta) = \frac{|s| - i\lambda}{2k} \operatorname{cosec}^2\left(\tfrac{1}{2}\theta\right). \tag{7.25}$$

θ is here the scattering angle with respect to the beam axis. For $s \neq 0$ the differential cross section is $|f(\theta)|^2$ and using (7.6) we obtain

$$\frac{d\sigma}{do} = \tfrac{1}{4}\left(1 + \frac{s^2}{4k^2}\right)^2 \operatorname{cosec}^4\left(\tfrac{1}{2}\theta\right). \tag{7.26}$$

In terms of physical variables $q = \hbar s$ and $v = \hbar k/2\pi$ where v is the impact speed and 2π is the reduced mass,

$$\frac{d\sigma}{do} = \tfrac{1}{4}\left(1 + \left(\frac{q}{4\pi v}\right)^2\right)^2 \operatorname{cosec}^4\left(\tfrac{1}{2}\theta\right). \tag{7.27}$$

This Rutherford-like cross section is exactly the same as the classical cross section which we computed in sect. 5.

In the pure monopole case, where $s = 0$ and $\lambda = k$, the effect of symmetrization gives a scattered intensity

$$\tfrac{1}{16}\left[\operatorname{cosec}^4\left(\tfrac{1}{2}\theta\right) + \sec^4\left(\tfrac{1}{2}\theta\right) + 8 \operatorname{cosec}^2\theta \cos\left(2k \log\tan\tfrac{1}{2}\theta\right)\right]. \tag{7.28}$$

Just as in ordinary Coulomb scattering of identical particles, in the classical limit ($k \to \infty$) the last term in (7.28) oscillates rapidly and would average to zero in any observation [23]. The observed cross section is therefore

$$\tfrac{1}{16}\left[\operatorname{cosec}^4\left(\tfrac{1}{2}\theta\right) + \sec^4\left(\tfrac{1}{2}\theta\right)\right] \tag{7.29}$$

rather than

$$\tfrac{1}{4}\operatorname{cosec}^4\left(\tfrac{1}{2}\theta\right), \tag{7.30}$$

which would result if one ignored the quantum mechanical interference.

8. Quantum states and the Atiyah-Hitchin metric

In the last section we described the quantum bound states and scattering states of monopoles and dyons using the Taub-NUT approximation to the Atiyah-Hitchin metric. Here we shall sketch how these results help to understand the quantum mechanics of monopoles on the true configuration space with the Atiyah-Hitchin metric.

Let us discuss bound states first. Recall the spectrum of states in the Taub-NUT approximation (fig. 5). For given relative electric charge $q = s\hbar$ the continuum starts at $\varepsilon = \frac{1}{4}s^2$ (the physical energy is $E = \hbar^2\varepsilon/4\pi$) and since the continuum represents dyons with arbitrarily large separation, their energy is independent of whether we use the exact metric or the Taub-NUT approximation. However, with the Atiyah-Hitchin metric, s is no longer a good quantum number; only $s \bmod 2$ is. In the even s sector the continuum starts at $\varepsilon = 0$, and in the odd s sector the continuum starts at $\varepsilon = \frac{1}{4}$. Since a negative energy is unattainable, there can be no true bound states in the even s sector; a pair of oppositely-charged dyons in an approximate bound state will always eventually decay into a pair of monopoles which escape to infinity. However if the dyons have a wave function supported mainly at large separation, as occurs in Taub-NUT states with large angular momentum, then they cannot rapidly turn into monopoles and escape, so they form a resonant state. It would be interesting to estimate the width of such a state, and how it is related to the evolution of the corresponding classical dyon orbit.

True bound states do exist in the odd s sector. One can verify this using the Rayleigh-Ritz inequality for the energy ε_0 of the lowest lying state, namely,

$$\varepsilon_0 \leqslant \frac{\int \Phi(-\nabla^2\Phi)\sqrt{g}\,\mathrm{d}^4x}{\int \Phi^2\sqrt{g}\,\mathrm{d}^4x}, \tag{8.1}$$

where $\sqrt{g}\,\mathrm{d}^4x$ is the volume element on M, ∇^2 is the covariant laplacian, and Φ is any wave function. By taking as a trial wave function a modified Taub-NUT state with $|s| = 1$ and large angular momentum j, we can show that ε_0 is less than $\frac{1}{4}$, i.e. below the continuum. The radial wave function needs to be strictly localized far from the Bolt where the Atiyah-Hitchin metric deviates little from the Taub-NUT metric. Such localization raises the energy a little, but not too much, as we shall verify first for a harmonic oscillator.

The harmonic oscillator with hamiltonian

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \omega^2x^2 \tag{8.2}$$

has energy levels $\varepsilon_n = (2n+1)\omega$, and the ground-state wave function is $\psi(x) = \exp(-\frac{1}{2}\omega x^2)$. The localized trial wave function

$$\begin{aligned} \psi(x) &= 1 - \tfrac{1}{2}\omega x^2, \qquad &|x| \leqslant \sqrt{2/\omega}, \\ \psi(x) &= 0, \qquad &|x| > \sqrt{2/\omega}, \end{aligned} \tag{8.3}$$

gives an energy estimate

$$\frac{\int_{-\infty}^{\infty}\psi(-\mathrm{d}^2/\mathrm{d}x^2 + \omega^2x^2)\psi\,\mathrm{d}x}{\int_{-\infty}^{\infty}\psi^2\,\mathrm{d}x} \approx 1.54\omega, \tag{8.4}$$

which lies between the lowest and first excited levels.

Let us next consider localized trial wave functions in the Taub-NUT problem. Suppose $|s| = 1$ and we use the correct angular wave function for angular momentum j with the radial wave function $h(r)/r$, as in (7.1). The exact energy levels are as in (7.10), and for large j the levels are

$$\varepsilon = \tfrac{1}{4} - \frac{1}{16}\frac{1}{(j+1+n')^2}, \qquad n' = 0, 1, \dots, \tag{8.5}$$

with corrections of order j^{-4} (cf. eq. (7.11)). On the other hand, in the Taub-NUT approximation, (8.1) reduces to

$$\frac{\int_0^\infty h\left(-\frac{d^2}{dr^2} + \frac{j(j+1)}{r^2} + \frac{1}{4}\left(1 - \frac{4}{r}\right)\right)h\,dr}{\int_0^\infty h(1-2/r)h\,dr}. \tag{8.6}$$

For a wave function with normalization

$$\int_0^\infty h^2\,dr = 1, \tag{8.7}$$

(8.6) equals

$$\frac{\frac{1}{4} + \int_0^\infty h\left(-\frac{d^2}{dr^2} + \frac{j(j+1)}{r^2} - \frac{1}{r}\right)h\,dr}{1 - \int_0^\infty (2/r)h^2\,dr}, \tag{8.8}$$

and this is approximately

$$\tfrac{1}{4} + \int_0^\infty h\left(-\frac{d^2}{dr^2} + \frac{j(j+1)}{r^2} - \frac{1}{2r}\right)h\,dr, \tag{8.9}$$

for a wave function peaked at large radius. The potential

$$\frac{j(j+1)}{r^2} - \frac{1}{2r} \tag{8.10}$$

has a minimum at $r_0 = 4j(j+1)$, and if for large j we use the harmonic approximation to the potential near r_0 we would estimate the bound state energies to be

$$\tfrac{1}{4} - \frac{1}{16j^2} + \frac{1}{8j^3}(1+n'), \qquad n' = 0, 1, \dots, \tag{8.11}$$

with an accumulated error of order j^{-4}. Indeed, (8.11) agrees with formula (8.5) to order j^{-3} provided n' is small relative to j. The exact radial wave functions are also close to harmonic oscillator wave functions. Now we argue that if j is sufficiently large and we take a localized wave function of half-width $\sqrt{32 j^3}$ in this problem, similar to the one given by (8.3), then as in the case of the harmonic oscillator, the expression (8.6) will remain below

$$\tfrac{1}{4}-\frac{1}{16j^2}+\frac{1}{4j^3}, \tag{8.12}$$

which is approximately the first excited energy level.

Finally, consider a similar trial wave function in the expression (8.1) itself. We know that in the asymptotic region, the Atiyah-Hitchin metric and corresponding laplacian differ only by exponentially small terms from their Taub-NUT approximations. For a wave function centred at $r_0 = 4j(j+1)$, the largest of these exponentially small terms contributes of order $j^2 e^{-4j^2}$ to ε_0. Our final estimate is therefore

$$\varepsilon_0 < \tfrac{1}{4}-\frac{1}{16j^2}+\frac{1}{4j^3}+Cj^2 e^{-4j^2}, \tag{8.13}$$

where C is a constant. For sufficiently large j this is less than $\frac{1}{4}$, and we conclude that in the odd s sector there are true bound states. The lowest energy eigenstate will include parts with $|s| = 3, 5$ etc. However these must be very small because the Taub-NUT bound states with $|s| = 3$ have energies well above $\frac{1}{4}$. The bound states are therefore predominantly of a pure monopole and a dyon of unit charge.

The above argument does not tell us the nature of the spectrum of bound states. Do they exist for all $j \neq 0$, and are there infinitely many for given j? $j = 0$ is not allowed in the odd s sector, but $j = 1$ bound states could occur. The radial equation for $j = 1$ is not a matrix equation but simply an ordinary differential equation for one function, and it would be interesting to know if it had bound states.

Incidentally, in the full quantum field theory it seems likely that there is just one strictly stable bound state of a monopole and a unit charge dyon. All radially excited states and those with non-minimal angular momentum will decay through the emission of radiation, as in the hydrogen atom, but the lowest state cannot decay.

The correspondence principle suggests that since there are bound quantum states, there might also be bound (and maybe closed) classical orbits on M. But we do not know if the closed geodesics in Taub-NUT space are approximations to such orbits. There is no classical conservation law to prevent dyons decaying slowly into pure monopoles and then escaping.

Let us turn now to scattering, concentrating on the quantum scattering of pure monopoles in the Atiyah-Hitchin metric. At low speeds, $v \ll \alpha$, the monopoles scatter elastically because there is not enough energy for the monopoles to turn into dyons. Suppose the incoming plane wave beam is decomposed into partial waves each with definite angular momentum. According to the radial wave equation in the Taub-NUT approximation, the closest approach of the jth partial wave is where

$$\frac{j(j+1)}{r^2}=\varepsilon\left(1-\frac{2}{r}\right), \tag{8.14}$$

i.e. $r \approx 2j\alpha/v$. The wave function oscillates outside this radius and decays inside it.

$2j\alpha/v$ is much larger than π, the Bolt radius, if v is small and $j \neq 0$. So only the S-wave ($j = 0$) penetrates to the core region. To get the correct low-energy cross section one must write the scattering amplitude as a sum of what the Taub-NUT approximation would give, which is accurate for $j > 0$, and an additional isotropic $j = 0$ piece. We have not calculated this completely. The presence of short-range forces and long-range Coulomb forces makes the calculation difficult. However, a rather rough analysis suggests that the short-range contribution to the S-wave phase shift is proportional to v^3 as $v \to 0$. One thing that is clear is that one cannot work out the differential cross section by treating the Atiyah-Hitchin metric entirely as a perturbation to the Taub-NUT metric. This approach would work for the $j > 0$ partial waves at low velocity, but not for the S-wave, and it is the S-wave that predominates in large-angle scattering.

At impact speed $v \approx \alpha$, two effects become important simultaneously. The threshold for the production of unit-charged dyons is reached at $v = 2\alpha$, and higher partial waves penetrate to the core region. The lowest partial wave where dyons can form is $j = 2$, since $s \geqslant 2$ if there is a dyon pair and $j \geqslant s$. Presumably, just above threshold, dyon production occurs predominantly in this partial wave, whose distance of closest approach is not far outside the Bolt. It follows that dyon production cannot be studied by treating the charge-changing operator in the hamiltonian as a perturbation.

One feature of dyon production is that the dyons do not emerge along the beam axis. This is a consequence of angular momentum conservation. Explicitly, j and m are conserved in the scattering, and $m = 0$ and $s = 0$ for the incoming monopole beam. The outgoing dyons have an angular wave function $W^{j}_{s0}(\phi, \theta, \psi)$ with s even and non-zero. This vanishes for $\theta = 0$ and $\theta = \pi$ and peaks at $\theta = \frac{1}{2}\pi$; for example, $W^{2}_{20} \propto \sin^2\theta \cos 2\psi$. So the dyons emerge preferentially at right angles to the monopole beam, as in a classical monopole collision.

At higher impact speed, $v \gg \alpha$, the scattering is semi-classical because many partial waves contribute. The quantized angular momenta correspond to discrete values of the impact parameter whose spacing is much less than the width of the core region. The differential elastic and inelastic cross sections should be similar to their classical counterparts, but with electric charge conservation taken into account. In sect. 5 (eq. (5.28)), we estimated the charge exchange Δq between classical monopoles approaching with impact parameter y. We found $\Delta q \leqslant 2h$ (i.e. two units) when $y \gtrsim \log(v/\alpha)$. This suggests that in the quantum theory, charge-changing processes are negligible for impact parameters greater than this, and that the inelastic total cross section for $v \gg \alpha$ is of order $\log^2(v/\alpha)$. Of course, for $v \gg \alpha$, the idea of just quantizing collective coordinates begins to fail because one expects significant radiation of photons during the scattering.

We would like to warmly thank Michael Atiyah for several discussions and for providing an early version of the forthcoming paper on monopole scattering by

Nigel Hitchin and himself. One of us (N.S.M.) is also grateful to J.J. Giambiagi for encouragement, and to Peter Rowe for correspondence on the classical limit of quantum Coulomb scattering.

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