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Lax Representation with Spectral Parameter on a Torus for Integrable Particle Systems

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Abstract. Complete integrability is proved for the most general class of systems of interacting particles on a straight line with the Hamiltonian including elliptic functions of coordinates, depending on seven arbitrary parameters and having the structure defined by the root systems of the classical Lie algebras. The Lax representation for them depends on the spectral parameter given on a complex torus $\mathbb{C}/\Gamma$, where Γ is the lattice of periods of the Jacobi functions dependent on the Hamiltonian parameters. The possibility of constructing explicit solutions to the equations of motion is discussed.

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1. Introduction

In [1], we have studied the conditions of integrability of multiparticle dynamical systems with the structure of the Hamiltonian defined by the systems of roots of the classical Lie algebras

$$H = \frac{1}{2}\sum_{j>1}^{N}\frac{p_j^2}{2} + \sum_{\boldsymbol{\alpha}\in R_+} V_\alpha(\mathbf{q}\boldsymbol{\alpha}), \tag{1}$$

where $\mathbf{q}$ is an N-dimensional vector composed of the quantities q_j canonically conjugate to p_j; $\boldsymbol{\alpha}\in R_+$ are positive root vectors of one of the classical systems $(A_N - D_N)$, V_α are functions whose form depends on the length of the vector $\boldsymbol{\alpha}$ but not on its direction. For the first time, systems like (1) with $V_\alpha \equiv V$ were considered by Olshanetsky and Perelomov [2]. For systems containing the roots of different length (B_N, C_N), the Hamiltonian (1) assumes the form

$$H = \frac{1}{2}\sum_{j=1}^{N}\frac{p_j^2}{2} + \sum_{j>k}^{N}[V(q_j - q_k) + V(q_j + q_k)] + \sum_{j=1}^{N} W(q_j) \tag{2}$$

and describes the motion of $2N$ particles interacting with each other via the potential $V(\xi)$ in an external field $W(\xi)$ under the initial conditions $q_{j+N} = -q_j$ and $p_{j+N} = -p_j$ (symmetric configurations). We have found the Lax representations for the equations of motion corresponding to (2) in the following cases

$$W(\xi) = \frac{g^2}{\sinh^2 a\xi}\,,$$
$$W(\xi) = g_1^2(\sinh \alpha\xi)^{-2} + g_2^2(\sinh 2a\xi)^{-2} + g_3^2 \cosh 2a\xi + g_4^2 \cosh 4a\xi\,, \tag{3}$$

where g_α, $1 \leqslant \alpha \leqslant 4$ are arbitrary, and

$$V(\xi) = g^2 \wp(\xi)\,,$$
$$W(\xi) = g_1^2 \wp(\xi) + g_2^2 \wp\left(\xi + \frac{\omega_1}{2}\right) + g_3^2 \wp\left(\xi + \frac{\omega_2}{2}\right) + g_4^2 \wp\left(\xi + \frac{\omega_1 + \omega_2}{2}\right), \tag{4}$$

with the Weierstrass function $\wp(\xi)$ having periods ω_1, ω_2 and the constants g_α obey one or two nonlinear equations defining either three- or two-dimensional hypersurfaces in a four-dimensional space $\{g_\alpha\}$. In this case, the Lax matrices found in [1] do not depend on the spectral parameter which can easily be introduced for (3).

In this Letter, we determine the L, M pair for the Hamiltonian (2), (4) that allows us to remove the above-mentioned constraints on the constants $\{g_\alpha\}$ and, what is more, depend on the spectral parameter. The Hamiltonian with the potential (4) dependent on 7 arbitrary constants is the most general one of the family with the structure (1).

To start with, we take the matrices L and M in the form resembling that used in [1]:

$$L = \begin{pmatrix} l & \lambda & \psi \\ \lambda & 0 & -\lambda \\ -\psi & -\lambda & -l \end{pmatrix}, \quad M = \begin{pmatrix} m & \omega & s \\ -\omega & \mu & -\omega \\ s & \omega & m \end{pmatrix} \tag{5}$$

(l, ψ, s, m, λ, ω, and μ are matrices of $N \times N$ dimension). The rank of L equals $2N$ as its eigenvectors corresponding to the zeroth eigenvalue compose a linear N-dimensional space. The first and last N coordinates of these vectors coincide and are arbitrary, $y_j = y_{2N+j}$, the other coordinates are calculated by the formula

$$y_{j+N} = -\{\lambda^{-1}(l + \psi)\}_{jk} y_k\,, \quad 1 \leqslant j, k \leqslant N\,.$$

The matrices l, ψ, s, and m are almost of the same structure as in [1]:

$$\begin{aligned} l_{jk} &= p_j \delta_{jk} + i(1 - \delta_{jk}) g x(q_j - q_k)\,, \\ \psi_{jk} &= i[\delta_{jk} v(q_j) + (1 - \delta_{jk}) g x(q_j + q_k)]\,, \\ m_{jk} &= i\{\delta_{jk}(\tau(q_j) - \sum_{n \neq j} (z(q_j - q_n) + z(q_j + q_n))) + (1 - \delta_{jk}) g x'(q_j - q_k)\}\,, \\ s_{jk} &= i\left[\frac{\delta_{jk}}{2} v'(q_j) + (1 - \delta_{jk}) g x'(q_j + q_k)\right]. \end{aligned} \tag{6}$$

The principal difference from the ansatz used in [1] consists of the structure of the matrices λ, ω, and μ that here are of $N \times N$ dimension. The first two of them are diagonal:

$$\lambda_{jk} = \lambda(q_j)\delta_{jk}, \qquad \omega_{jk} = \lambda'(q_j)\delta_{jk}. \tag{7}$$

Hereafter, the prime means differentiation of a function with respect to its argument. The Lax equation $dL/dt = [L, M]$, upon substitution into it of (5)–(7), splits into four equations for the derivatives of l, λ, and ψ

$$\begin{aligned} \frac{dl}{dt} &= [l, m] + \{s, \psi\} - \{\lambda, \omega\}, \\ \frac{d\psi}{dt} &= [\psi, m] + \{l, s\}, \end{aligned} \tag{8}$$

$$\frac{d\lambda}{dt} = (l + \psi)\omega + \lambda\mu - (m - s)\lambda = \lambda(m - s) - \mu\lambda + \omega(l - \psi). \tag{9}$$

As the Hamiltonian (2) is connected with L by $H = \frac{1}{4}\operatorname{Tr} L^2$, we obtain for the functions $V(\xi)$ and $W(\xi)$ the equations

$$V(\xi) = x^2(\xi), \qquad W(\xi) = \frac{v^2}{2}(\xi) + \lambda^2(\xi). \tag{10}$$

Equations (8) have the same solutions as we have found in [1]:

$$\begin{aligned} &x(\xi) = \frac{a}{\operatorname{sn} a\xi}, \qquad z(\xi) = -\frac{ga^2}{\operatorname{sn}^2 a\xi}, \qquad \tau(\xi) = \frac{a\,v(\xi)}{\operatorname{sn} 2a\xi}, \\ &v(\xi) = a(\alpha + \beta \operatorname{sn}^2 a\xi + \gamma \operatorname{sn}^4 a\xi)(\operatorname{sn} a\xi \operatorname{cn} a\xi \operatorname{dn} a\xi)^{-1}, \end{aligned} \tag{11}$$

the constants a, α, β, γ and modulus of the elliptic functions in (11) are arbitrary.

Now consider Equations (9). As diagonal elements of the matrices in (9) are equal, we can express μ_{jj} in terms of x, z, v, τ, and λ:

$$\mu_{jj} = i\left[\tau(q_j) - \sum_{n \neq j} (z(q_j - q_n) + z(q_j + q_n)) - \frac{v'(q_j)}{2} - v(q_j)\,\frac{\lambda'(q_j)}{\lambda(q_j)}\right]. \tag{12}$$

Nondiagonal elements of (9) give two equations for μ_{jk}:

$$\begin{aligned} \mu_{jk} &= [(m_{jk} - s_{jk})\lambda(q_k) - (l_{jk} + \psi_{jk})\lambda'(q_k)]\,[\lambda(q_j)]^{-1} \\ &= [(m_{jk} - s_{jk})\lambda(q_j) - (l_{jk} - \psi_{jk})\lambda'(q_j)]\,[\lambda(q_k)]^{-1}. \end{aligned} \tag{13}$$

The condition of their compatibility is a functional equation for λ.

Denoting $\lambda^2(\xi) = \chi(\xi)$, we write it as follows

$$\begin{aligned} &x(\xi - \eta)[\chi'(\xi) + \chi'(\eta)] + x(\xi - \eta)[\chi'(\eta) - \chi'(\xi)] + \\ &\quad + 2[x'(\xi - \eta) - x'(\xi + \eta)][\chi(\xi) - \chi(\eta)] = 0 \end{aligned}$$

or, upon substitution of $x(\xi \pm \eta)$ given by (11),

$$\chi'(\xi)\,[\mathrm{sn}\,a(\xi+\eta)-\mathrm{sn}\,a(\xi-\eta)]+\chi'(\eta)\,[\mathrm{sn}\,a(\xi+\eta)+\mathrm{sn}\,a(\xi-\eta)]$$
$$=2a[\chi(\xi)-\xi(\eta)]\,[\mathrm{sn}\,a(\xi+\eta)\,\mathrm{sn}\,a(\xi-\eta)]^{-1}\times \tag{14}$$
$$\times\,[\mathrm{cn}\,a(\xi-\eta)\,\mathrm{d}\eta\,a(\xi-\eta)\,\mathrm{sn}^2a(\xi+\eta)-\mathrm{cn}\,a(\xi+\eta)\,\mathrm{dn}\,a(\xi+\eta)\,\mathrm{sn}^2a(\xi-\eta)]\,.$$

Using the addition formula for the Jacobi functions and making lengthy but not complicated computations, we reduce (14) to the form

$$\chi'(\xi)\,\frac{\mathrm{cn}\,a\xi\,\mathrm{dn}\,a\xi}{\mathrm{sn}\,a\xi}+\chi'(\eta)\,\frac{\mathrm{cn}\,a\eta\,\mathrm{dn}\,a\eta}{\mathrm{sn}\,a\eta}$$
$$=\frac{2a(\chi(\xi)-\chi(\eta))}{\mathrm{sn}^2a\xi-\mathrm{sn}^2a\eta}\,(\mathrm{cn}^2a\xi\,\mathrm{dn}^2a\eta+\mathrm{cn}^2a\eta\,\mathrm{dn}^2a\xi)\,. \tag{15}$$

Now let us look for a general solution to (15). As the right-hand side of that equation cannot have a pole for all ξ when $\eta\to 0$, we have

$$\chi(\eta)\sim\delta_0+\frac{\delta a^2}{2}\,\eta^2$$

for small η, and in the limit $\eta\to 0$, we obtain from (15)

$$\chi'(\xi)\,\frac{\mathrm{cn}\,a\xi\,\mathrm{dn}\,a\xi}{\mathrm{sn}\,a\xi}+a\delta=\frac{2a(\chi(\xi)-\delta_0)}{\mathrm{sn}^2a\xi}\,(\mathrm{cn}^2a\xi+\mathrm{dn}^2a\xi)\,,$$

i.e., a linear equation of the first order with the solution

$$\chi(\xi)=\delta_0+\frac{\delta}{2}\,\frac{\mathrm{sn}^2a\xi}{\mathrm{cn}^2a\xi\,\mathrm{dn}^2a\xi}+\delta_1\,\frac{\mathrm{sn}^4a\xi}{\mathrm{cn}^2a\xi\,\mathrm{dn}^2a\xi}\,, \tag{16}$$

where δ_1 is an integration constant. Direct substitution of the solution into (15) shows that no extra constraints arise on the constants δ_0, δ, and δ_1 and (16) is just the required general solution of the functional equation. Since, according to (10), δ_0 gives only a negligible addition to the potential $W(\xi)$, that constant may be chosen so that $\chi(\xi)$ be of the form $a^2[(\alpha_1^2\,\mathrm{cn}^4a\xi+\alpha_2^2\,\mathrm{dn}^4a\xi)(\mathrm{cn}\,a\xi\,\mathrm{dn}\,a\xi)^{-2}+2\alpha_1\alpha_2]$ or

$$\lambda(\xi)=a\,\frac{\alpha_1\,\mathrm{cn}^2a\xi+\alpha_2\,\mathrm{dn}^2a\xi}{\mathrm{cn}\,a\xi\,\mathrm{dn}\,a\xi}\,. \tag{17}$$

Going back to (10), we see that now there is a sufficient number of parameters to ensure the absence of any constraints on the constants g_1, g_2, g_3, and g_4 in (4). And what is more, their number is larger by one than the number of $\{g_\alpha\}$, therefore a certain combination of α, β, γ, α_1, α_2 can be used as a spectral parameter.

Let us transform (10) with the functions ν (11) and λ (17) to the form (4), i.e., express the squares of the Jacobi functions (10) through the Weierstrass function. This can be achieved with the relations

$$\wp(\xi) = \frac{a^2}{\mathrm{sn}^2 a\xi} + c_0, \qquad \wp\left(\xi + \frac{\omega_1}{2}\right) = \frac{a^2\,\mathrm{dn}^2 a\xi}{\mathrm{cn}^2 a\xi} + c_0,$$

$$\wp\left(\xi + \frac{\omega_2}{2}\right) = -a^2\,\mathrm{dn}^2 a\xi + c_0 + a^2, \tag{18}$$

$$\wp\left(\xi + \frac{\omega_1 + \omega_2}{2}\right) = k^2 a^2\,\frac{\mathrm{cn}^2 a\xi}{\mathrm{dn}^2 a\xi} + c_0,$$

where k is the modulus of the Jacobi function,

$$c_0 = \frac{-a^2(1 + k^2)}{3},$$

and the periods ω_1 and ω_2 of the function $\wp(\xi)$ will be used as independent parameters instead of a and k.

Taking advantage of α, β, γ, α_1, and α_2 being arbitrary, we write the functions $\lambda(\xi)$ and $\nu(\xi)$ in the form

$$\lambda(\xi) = a\,\frac{\tilde{\alpha}_1 k^2\,\mathrm{cn}^2 a\xi + \tilde{\alpha}_2\,\mathrm{dn}^2 a\xi}{\mathrm{cn}\,a\xi\,\mathrm{dn}\,a\xi},$$

$$\nu(\xi) = a\sqrt{2}\,\frac{\tilde{\alpha}\,\mathrm{cn}^2 a\xi\,\mathrm{dn}^2 a\xi + \tilde{\beta}\,\mathrm{dn}^2 a\xi - \tilde{\gamma} k^2\,\mathrm{cn}^2 a\xi\,\mathrm{sn}^2 a\xi}{\mathrm{sn}\,a\xi\,\mathrm{cn}\,a\xi\,\mathrm{dn}\,a\xi} \tag{19}$$

which allows us to find a simple connection between the constants g_1, g_2, g_3, g_4 in (4) and the Lax matrix parameters. Substituting (19) into (10) and using (18), we obtain from the comparison of (4) and (10) the following system of equations quadratic in $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$, α_1 and α_2:

$$(\tilde{\alpha} + \tilde{\beta})^2 = g_1^2, \qquad (\tilde{\alpha} + \tilde{\gamma})^2 = g_3^2, \qquad \tilde{\beta}^2 + \tilde{\alpha}_2^2 = g_2^2, \qquad \tilde{\gamma}^2 + \tilde{\alpha}_1^2 = g_4^2.$$

From these equations, we may express $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\gamma}$ through $\{g_\alpha\}$, $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$:

$$\begin{aligned}
\tilde{\alpha} &= \tfrac{1}{2}(g_1 + g_3) - \tfrac{1}{2}(g_1 - g_3)^{-1}[\tilde{\alpha}_1^2 - \tilde{\alpha}_2^2 - g_2^2 - g_4^2],\\
\tilde{\beta} &= \tfrac{1}{2}(g_1 - g_3) + \tfrac{1}{2}(g_1 - g_3)^{-1}[\tilde{\alpha}_1^2 - \tilde{\alpha}_2^2 + g_2^2 - g_4^2],\\
\tilde{\gamma} &- -\tfrac{1}{2}(g_1 - g_3) + \tfrac{1}{2}(g_1 - g_3)^{-1}[\tilde{\alpha}_1^2 - \tilde{\alpha}_2^2 + g_2^2 - g_4^2]
\end{aligned} \tag{20}$$

whereas $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are related by

$$(\tilde{\alpha}_1^2 - \tilde{\alpha}_2^2)^2 + 2A\tilde{\alpha}_1^2 + 2B\tilde{\alpha}_2^2 + C = 0, \tag{21}$$

$$\begin{aligned}
&A = g_2^2 - g_4^2 + (g_1 - g_3)^2, \qquad B = -g_2^2 + g_4^2 + (g_1 - g_3)^2,\\
&C = (g_2^2 - g_4^2)^2 - 2(g_2^2 + g_4^2)(g_1 - g_3)^2 + (g_1 - g_3)^4.
\end{aligned} \tag{22}$$

Equation (21) defines an algebraic curve of the fourth order. It is easily verified that its genus equals one and, consequently, its elliptic uniformization is feasible. To determine the explicit form of the functions accomplishing this uniformization, we substitute into (21) the following expressions

$$\tilde{\alpha}_1 = \frac{\delta\,\mathrm{sn}\,h\,\mathrm{cn}\,\varphi\,\mathrm{dn}\,\varphi}{1 - \kappa^2\,\mathrm{sn}^2 h\,\mathrm{sn}^2\varphi}\,, \qquad \tilde{\alpha}_2 = \frac{\delta\,\mathrm{cn}\,h\,\mathrm{dn}\,h\,\mathrm{sn}\,\varphi}{1 - \kappa^2\,\mathrm{sn}^2 h\,\mathrm{sn}^2\varphi}\,, \tag{23}$$

where κ is the modulus of elliptic functions in (23). Upon rather lengthy computations, it can be shown that (21) is fulfilled for arbitrary values of h if δ, κ, and φ are connected with A, B, and C (22) as follows:

$$\mathrm{sn}^2\varphi = -(A + B)^{-1}\,[-B + \sqrt{B^2 - C}\,]\left[1 + \frac{AB - (C - A^2)^{1/2}(C - B^2)^{1/2}}{C}\right], \tag{24a}$$

$$\delta = (-B + \sqrt{B^2 - C})^{1/2}(\mathrm{sn}\,\varphi)^{-1}\,, \qquad \kappa = C^{1/2}\,\mathrm{sn}^2\varphi(-B + \sqrt{B^2 - C})\,. \tag{24b}$$

Formulae (23) and (24) give explicit uniformization of the curve (21) and h is a spectral parameter. The dependence of the Lax matrices on h is given by formulae (6), (7), (12–13), (19–20), (23). The range of variation of h is a complex torus that is a factor of the plane $\mathbb{C}$ with respect to the lattice of periods Γ of the function $\mathrm{sn}\,h$ with the modulus (24b). Note that for the trigonometric degeneration (3) of the potentials (4) the Lax matrices are of $(2N \times 2N)$ dimension [1], and L, M are meromorphic functions of the spectral parameter with poles at four points of the plane $\mathbb{C}$, $h = 0, \pm 1, \infty$. In the considered elliptic case, the poles of L and M as functions of h are at the points $\mathrm{sn}\,h = \pm(\kappa\,\mathrm{sn}\,\varphi)^{-1}$ and at the points of poles of $\mathrm{sn}\,h$ belonging to $\mathbb{C}/\Gamma$.

The spectral equation $\det(L(h) - wE) = 0$ (E is a unit $3N \times 3N$ matrix) defines the curve $w(h)$ covering $\mathbb{C}/\Gamma$. The number of sheets of the covering coincides with the rank of L and equals $2N$. It is certainly of interest to consider the possibility of linearization of the Hamiltonian flow (2, 4) on the Jacobi variety of the curve $w(h)$. Investigation of the trigonometric degenerations (4) shows that this sort of linearization is feasible. Specifically, integration of the equations of motion of systems with the Hamiltonian (3) and its further degenerations (rational and other [3, 8]) can be made by the methods of algebraic geometry developed in [4] and [5]. These results will be published elsewhere. Note also that it would be of interest to perform a similar consideration for recently developed 'relativistic' generalizations of integrable systems of interacting particles [6]. The interaction potential found in [6] is defined by the root system A_N and integrability property may be valid for 'relativistic' Hamiltonians constructed by the root systems of other classical Lie algebras. Integrability of that sort of system with two degrees of freedom has been proved by myself in [7] by the method of direct construction of an extra constant of motion with no use of the Lax representation.

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