Reduction of the two-dimensional O(n) nonlinear σ-model

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(Received 24 April 1979; accepted for publication 15 June 1979)

We reduce the field equations of the two-dimensional O(n) nonlinear σ-model to relativistic O($n-2$) covariant differential equations involving $n-2$ scalar fields.

I. INTRODUCTION

The classical two-dimensional O(n) nonlinear σ-models define integrable Hamiltonian systems.[1] Taking advantage of conformal invariance, the models corresponding to $n = 3$ and $n = 4$ can be "reduced" to local relativistic scalar field theories involving O(3) and O(4) invariant combinations of the chiral field vectors and some of their derivatives. The O(3) nonlinear σ-model is reduced to the sine-Gordon theory described by the Lagrangian density[2]

$$\mathscr{L}(x^0,x^1) = \tfrac{1}{2}(\partial_\mu \alpha)(\partial^\mu \alpha) + \cos\alpha - 1. \tag{1.1}$$

The O(4) nonlinear σ-model is reduced to a local relativistic theory involving two scalar fields α and β. Its dynamics is described by the Lagrangian density

$$\mathscr{L}(x^0,x^1) = \tfrac{1}{2}(\partial_\mu \alpha)(\partial^\mu \alpha) + \tfrac{1}{2}(\partial_\mu \beta)(\partial^\mu \beta)\tan^2(\alpha/2) + \cos\alpha - 1. \tag{1.2}$$

This theory is a generalization of the sine-Gordon theory, where β is identically zero. If we combine α and β into the two-component iso-vector

$$\psi = \sin(\alpha/2)\begin{pmatrix}\cos(\beta/2)\\ \sin(\beta/2)\end{pmatrix},$$

it becomes identical with Getmanov's "New Lorentz-invariant system"[3,4]

$$\mathscr{L}(x^0,x^1) = \frac{1}{2}\frac{(\partial_\mu \psi^a)(\partial^\mu \psi^a)}{1-\psi^a\psi^a} - \frac{1}{2}\psi^a\psi^a. \tag{1.3}$$

The conservation laws and the inverse scattering equations for this "complex sine-Gordon theory" were derived in Ref. 1. Nontopological soliton, multisoliton and breather solutions were obtained in Refs. 3, 5. The transformation to action-angle variables was worked out in Ref. 5.

As can be verified by crossdifferentiation, the Bäcklund transformation mapping solutions ψ of the complex sine-Gordon equation

$$\partial_\mu \partial^\mu \psi + \frac{2(\psi^b\partial_\mu\psi^b)\partial^\mu\psi - (\partial_\mu\psi^b\partial^\mu\psi^b)\psi}{1-\psi^b\psi^b} + (1-\psi^b\psi^b)\psi = 0$$

into solutions ψ' of this same equation is

$$R\frac{(\partial_0+\partial_1)\psi'}{\sqrt{1-\psi'^b\psi'^b}} + R^{-1}\frac{(\partial_0+\partial_1)\psi}{\sqrt{1-\psi^b\psi^b}} = \gamma^{-1}\{R^{-1}\psi'\sqrt{1-\psi^b\psi^b} - R\psi\sqrt{1-\psi'^b\psi'^b}\}, \tag{1.4}$$

$$R^{-1}\frac{(\partial_0-\partial_1)\psi'}{\sqrt{1-\psi'^b\psi'^b}} - R\frac{(\partial_0-\partial_1)\psi}{\sqrt{1-\psi^b\psi^b}} = -\gamma\{R\psi'\sqrt{1-\psi^b\psi^b} + R^{-1}\psi\sqrt{1-\psi'^b\psi'^b}\},$$

with γ a real constant parameter different from zero and

$$R = \begin{pmatrix}\cos\omega & -\sin\omega\\ \sin\omega & \cos\omega\end{pmatrix},$$

$$2\omega = \arcsin\left(\frac{\epsilon^{ab}\psi^a\psi'^b}{\sqrt{1-\psi^b\psi^b}\sqrt{1-\psi'^b\psi'^b}}\right).$$

In the present note we shall search for the relativistic differential equations to which the equations of motion of the two-dimensional O(n) nonlinear σ-models can be reduced for higher values of n (c.f., Refs. 6, 7).

II. HIGHER GENERALIZATIONS OF THE SINE-GORDON EQUATION

The classical O(n) nonlinear σ-model describes the motion of a string of n-dimensional classical spins $q^a(x^0,x^1)$, $a = 1,\ldots,n$, of unit length: $q^bq^b = 1$.[8] The Lagrangian density is

$$\mathscr{L}(x^0,x^1) = \tfrac{1}{2}\{\partial_\mu q^a\partial^\mu q^a + \varkappa(q^aq^a - 1)\}, \tag{2.1}$$

where $\varkappa = \varkappa(x^0,x^1)$ is a Lagrangian multiplier. The equations of motion are

$$\partial_\mu\partial^\mu q + (\partial_\mu q^b\partial^\mu q^b)q = 0, \quad q^bq^b = 1$$
$$[\varkappa(x^0,x^1) = -(\partial_\mu q^b\partial^\mu q^b)].$$

They are invariant under general conformal transformations, space and time reflections, and under the group O(n) of internal rotations and reflections.

We break the conformal invariance by requiring

$$\partial_0 q^b\partial_0 q^b + \partial_1 q^b\partial_1 q^b = 1, \quad \partial_0 q^b\partial_1 q^b = 0. \tag{2.2}$$

It is advantageous to use light-cone coordinates

$$\xi = (x^0+x^1)/2, \quad \eta = (x^0-x^1)/2,$$

in which the equations of motion and the normalization requirements read

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 0022-2488/79/122628-05$01.00

$$q_{\xi\eta} + (q^b_\xi q^b_\eta)q = 0, \quad q^b q^b = 1;$$
$$q^b_\xi q^b_\xi = 1 = q^b_\eta q^b_\eta. \tag{2.3}$$

The subscripts ξ and η denote differentiation with respect to ξ and η.

In Ref. 1, in our quest for inverse scattering equations, we started from the Bäcklund transformation for the chiral fields q. In the course of the derivation we obtained two systems of Riccati Eqs. (VII. 11.1) and (VII.11.2) the compatibility of which requires the following relations to hold

$$\alpha_{\xi\eta} + \sin\alpha(s^{(+)}s^{(-)})_{11} = 0,$$
$$(\tan\alpha s^{(+)}_{1j})_\eta + \alpha_\xi s^{(-)}_{1j} + \tan\alpha(s^{(+)}s^{(-)})_{1j} = 0,$$
$$j = 2,3,\ldots,n-2,$$
$$s^{(+)}_\eta - s^{(-)}_\xi + [s^{(+)},s^{(-)}] = 0.$$

Here

$$\alpha = \arccos(q^b_\xi q^b_\eta),$$
$$s^{(\pm)} = -s^{(\pm)T[9]}, s^{(\pm)}_{ij} = (b^a_{i\,\substack{\xi\\\eta}} b^a_j),$$
$$i,j = 1,2,\ldots,n-2,$$

with $q, q_\xi, b_1 = (q_\eta - \cos\alpha q_\xi)/\sin\alpha, b_k; k = 2,\ldots,n-2$, an orthonormal basis in $\mathbb{R}^n$.

These relations form the starting point of the present investigation. The last equation can immediately be solved:

$$s^{(\pm)}_{ij} = \sum_{b=1}^{n-2} (f^b_{i\,\substack{\xi\\\eta}} f^b_j) \tag{2.4}$$

with $f_1 = f, f_2, \ldots, f_{n-2}$ forming an orthonormal basis in $\mathbb{R}^{n-2}$. Now the first two equations read

$$f_{\xi\eta} + \cot\alpha\alpha_\xi f_\eta + (\cos\alpha \sin\alpha)^{-1}\alpha_\eta f_\xi$$
$$+ \sum_{b=1}^{n-2} (f^b_\xi f^b_\eta) f = 0, \tag{2.5a}$$
$$\sum_{b=1}^{n-2} f^b f^b = 1,$$
$$\alpha_{\xi\eta} + \sin\alpha - \tan\alpha \sum_{b=1}^{n-2} (f^b_\xi f^b_\eta) = 0. \tag{2.5b}$$

By setting

$$\sin\alpha f = \varphi$$

they can be combined into a single equation

$$\varphi_{\xi\eta} + \frac{(\varphi\cdot\varphi_\eta)\varphi_\xi}{1 - \|\varphi\|^2} + \sqrt{1 - \|\varphi\|^2}\,\varphi = 0 \tag{2.6}$$

or

$$\partial_\mu\partial^\mu\varphi + \frac{(\varphi\cdot\partial_\mu\varphi)\partial^\mu\varphi + \epsilon^{\mu\nu}(\varphi\cdot\partial_\mu\varphi)\partial_\nu\varphi}{1 - \|\varphi\|^2}$$
$$+ \sqrt{1 - \|\varphi\|^2}\,\varphi = 0. \tag{2.6$'$}$$

Here the dot denotes the Euclidean scalar product in the space $\mathbb{R}^{n-2}$ and vertical twofold bars stand for the corresponding Euclidean norm.

This equation possesses a one-parameter family of Bäcklund transformations, the transcription of the Bäcklund transformations for the chiral field vectors q to those for φ, and infinitely many local covariant conservation laws, e.g.,

$$\left\{\frac{1}{2}\left\|\frac{\varphi_\xi}{\sqrt{1 - \|\varphi\|^2}}\right\|^2\right\}_\eta = \left\{\sqrt{1 - \|\varphi\|^2}\right\}_\xi,$$
$$\left\{\tfrac{1}{2}\|\varphi_\eta\|^2 + \frac{1}{2}\frac{(\varphi\cdot\varphi_\eta)^2}{1 - \|\varphi\|^2}\right\}_\xi = \left\{\sqrt{1 - \|\varphi\|^2}\right\}_\eta, \tag{2.7}$$
$$\left\{\frac{1}{2}\left\|\left(\frac{\varphi_\xi}{\sqrt{1 - \|\varphi\|^2}}\right)_\xi\right\|^2 - \frac{1}{8}\left\|\frac{\varphi_\xi}{\sqrt{1 - \|\varphi\|^2}}\right\|^4\right\}_\eta$$
$$= -\left\{\frac{\sqrt{1 - \|\varphi\|^2}}{2}\left\|\frac{\varphi_\xi}{\sqrt{1 - \|\varphi\|^2}}\right\|^2\right\}_\xi.$$

Had we started from

$$q, q_\eta, b_1 = \frac{q_\xi - \cos\alpha q_\eta}{\sin\alpha}, \quad b_k; k = 2,\ldots,n-2$$

as the orthonormal basis in $\mathbb{R}^n$, we would have obtained in an analogous manner the equation

$$\chi_{\xi\eta} + \frac{(\chi\cdot\chi_\xi)\chi_\eta}{1 - \|\chi\|^2} + \sqrt{1 - \|\chi\|^2}\,\chi = 0 \tag{2.8}$$

or

$$\partial_\mu\partial^\mu\chi + \frac{(\chi\cdot\partial_\mu\chi)\partial^\mu\chi + \epsilon^{\mu\nu}(\chi\cdot\partial_\nu\chi)\partial_\mu\chi}{1 - \|\chi\|^2}$$
$$+ \sqrt{1 - \|\chi\|^2}\,\chi = 0. \tag{2.8$'$}$$

Though each of the two Eqs. (2.6) and (2.8) possesses infinitely many local covariant conservation laws, none of them can be considered a direct generalization of the real and complex sine-Gordon equation. We shall arrive at such a generalization (for the case $n = 6$) by studying the orthogonal transformation $\mathscr{R}$ mapping the solutions φ of Eq. (2.6) into the solutions χ of Eq. (2.8):

$$\chi = \mathscr{R}\varphi,$$
$$\mathscr{R}_\xi = \frac{-1}{\sqrt{1 - \|\varphi\|^2}\,[1 - \sqrt{1 - \|\varphi\|^2}\,]}\mathscr{R}\varphi^a\varphi^b_\xi I^{ab},$$
$$\mathscr{R}_\eta = \frac{-1}{1 - \sqrt{1 - \|\varphi\|^2}}\mathscr{R}\varphi^a\varphi^b_\eta I^{ab}, \tag{2.9}$$

where $I^{ba} = -I^{ab}$ $(a,b = 1,\ldots,n-2)$ denote the infinitesimal generators of the group $O(n-2)$ for rotations in the (a,b) planes. For later convenience we shall work with the covering group. Let $\Gamma^a, a = 1,\ldots,n-2$, stand for the lowest-dimensional matrix representation of the basis elements of the Clifford algebra[10]

$$\Gamma^a\Gamma^b + \Gamma^b\Gamma^a = 2\delta^{ab}$$

and let the symbol [,] denote the commutator. The Lie algebra with basis $J^{ab} = \frac{1}{4}[\Gamma^a,\Gamma^b]$ is a representation of the Lie algebra of the group $O(n-2)$. The corresponding representatives U of the space-time dependent rotations $\mathscr{R}^T$ satisfy the following equations

$$U_\xi = \frac{1 + \cos\alpha}{\cos\alpha}\sum_{a,b=1}^{n-2} f^a f^b_\xi J^{ab} U$$
$$U_\eta = (1 + \cos\alpha)\sum_{a,b=1}^{n-2} f^a f^b_\eta J^{ab} U \tag{2.10}$$
$$UU^+ = U^+U = \mathbf{1}, \quad \det U = 1.$$

Consistency requires the representatives U to satisfy

$$U_{\xi\eta}+\frac{\alpha_\xi U_\eta+\alpha_\eta U_\xi}{\sin\alpha}+\tfrac{1}{2}\tan^2\frac{\alpha}{2}\,[U_\eta U_\xi^+-U_\xi U_\eta^+]U+\tfrac{1}{2}[U_\eta U_\xi^++U_\xi U_\eta^+]U=0,$$

$$U^+U=UU^+=\mathbf{1},\quad \det U=1.$$

Equation (2.5b) now reads

$$\alpha_{\xi\eta}+\sin\alpha=\frac{2\tan^2(\alpha/2)}{\sin\alpha}\{U_\eta U_\xi^++U_\xi U_\eta^+\}.$$

If we set

$$\sin\frac{\alpha}{2}\,U=V,$$

we obtain

$$V_{\xi\eta}+[1-VV^+]^{-1}V_\xi V^+V_\eta+[1-VV^+]V=0, \tag{2.11}$$

$$VV^+=V^+V=\text{multiple of the unit matrix},$$
$$\det V=\text{real}.$$

Independently of its origin, this system possesses an infinite set of local covariant conservation laws, e.g.,

$$\mathrm{Tr}\{\tfrac{1}{2}[1-VV^+]^{-1}V_\xi V_\xi^+\}_\eta = -\mathrm{Tr}\{\tfrac{1}{2}VV^+\}_\xi \quad (\xi\longleftrightarrow\eta,\ V\longleftrightarrow V^+) \tag{2.12}$$

$$\begin{aligned}&\mathrm{Tr}\{\tfrac{1}{2}[1-VV^+]^{-1}V_{\xi\xi}V_{\xi\xi}^++\tfrac{1}{2}[1-VV^+]^{-2}V_\xi V_\xi^+\\&\times[(VV^+)_{\xi\xi}-4V_\xi V_\xi^+]+\tfrac{1}{2}[1-VV^+]^{-3}\\&\times(V_\xi V_\xi^+)^2\}_\eta\\&\quad=\mathrm{Tr}\{-V_\xi V_\xi^+(1-\tfrac{1}{2}[1-VV^+]^{-1})\}_\xi,\\&\qquad(\xi\longleftrightarrow\eta,\ V\longleftrightarrow V^+).\end{aligned}$$

The system is likely to be integrable. It contains the solutions of (2.10) as special cases subject to constraints, e.g., for the case $n=5$ the constraints are

$$[[V_\xi V^+,V_\eta V^+],[V_{\xi\xi}V^+,V_\eta V^+]]=0,$$
$$[[V_\xi V^+,V_\eta V^+],[V_{\eta\eta}V^+,V_\xi V^+]]=0.$$

The constraints are simple enough to be resolved only in two cases: $n-2=2$, the case discussed in Refs. 1, 3, 5, and $n-2=4$ [O(4) factorizes!]. In the following, we shall concentrate on the latter case.

III. THE REDUCED EQUATION FOR THE O(6) NONLINEAR σ-MODEL

In this section we shall derive a recursion formula for the conserved current densities valid for all $n\geqslant 3$, and calculate explicitly the first three continuity equations for $n=6$. Moreover, a general formula for the N-soliton scattering solution is derived. For a special three-soliton configuration it is written in a form which shows the space–time dependence of the field vector most transparently.

In the case under consideration, Eq. (2.10) splits into two sets of equations, each involving an SU(2) matrix. We only need to consider one of them. Parametrizing the SU(2) matrix by a four-dimensional unit vector n, we arrive at

$$\begin{aligned}n_\xi&=-\frac{1+\cos\alpha}{2\cos\alpha}\{(f_\xi\cdot n)f-(f\cdot n)f_\xi+[n,f,f_\xi]\},\\ n_\eta&=-\frac{1+\cos\alpha}{2}\{(f_\eta\cdot n)f-(f\cdot n)f_\eta+[n,f,f_\eta]\},\end{aligned} \tag{3.1}$$

where $[A,B,C]_i=\epsilon_{ijkl}A_jB_kC_l$ denotes the vector product in $\mathbb{R}^4$. Writing $\psi=\sin(\alpha/2)\cdot n$, the compatibility condition for n and the evolution equation (2.5b) for α are cast into the single equation

$$\psi_{\xi\eta}+\frac{(\psi\cdot\psi_\xi)\psi_\eta+(\psi\cdot\psi_\eta)\psi_\xi-(\psi_\xi\cdot\psi_\eta)\psi-[\psi,\psi_\xi,\psi_\eta]}{1-\|\psi\|^2}+(1-\|\psi\|^2)\psi=0. \tag{3.2}$$

The conservation laws are found essentially by a method due to Wadati, Sanuki, Konno.[11] The Riccati equations (VII.11.1) and (VII.11.2) of Ref. 1 yield the continuity equation

$$\left(\frac{\varphi_\xi}{\sqrt{1-\|\varphi\|^2}}\cdot Z\right)_\eta+\gamma\big((\varphi\cdot Z)-2\sqrt{1-\|\varphi\|^2}\,\big)_\xi=0$$

where $Z^a=-2\sum_{j=1}^{n-2}f_j^aY_j$. The expansion of Z in powers of the parameter γ around $\gamma=0$ and $\gamma=\infty$ leads to two series of conservation laws, e.g., around $\gamma=0$

$$\begin{aligned}&(Z_1\cdot Z_1)_\eta-2(\sqrt{1-\|\varphi\|^2})_\xi=0,\\&(Z_1\cdot Z_{m+1})_\eta+(\varphi\cdot Z_m)_\xi=0,\quad m\geqslant 1,\end{aligned} \tag{3.3}$$

where

$$Z_1=\frac{\varphi_\xi}{\sqrt{1-\|\varphi\|^2}},$$

$$Z_{m+1}=(Z_m)_\xi+\sum_{k+l=m}[\tfrac{1}{2}(Z_1\cdot Z_k)Z_l-\tfrac{1}{4}(Z_k\cdot Z_l)Z_1],\quad m\geqslant 1$$

[cf Eq. (2.7) above].

For the case $n=6$ the first three conservation laws in terms of ψ are[12]

$$\left\{\frac{1}{2}\frac{\|\psi_\xi\|^2}{1-\|\psi\|^2}\right\}_\eta+\left\{\frac{1}{2}\|\psi\|^2\right\}_\xi=0,$$

$$\left\{\frac{1}{2}\frac{\|\psi_{\xi\xi}\|^2}{1-\|\psi\|^2}+\frac{1}{(1-\|\psi\|^2)^2}\|\psi_\xi\|^2(\psi\cdot\psi_{\xi\xi})-\frac{1}{2}\frac{1-2\|\psi\|^2}{(1-\|\psi\|^2)^3}\|\psi_\xi\|^4\right\}_\eta+\left\{\frac{1}{2}\frac{1-2\|\psi\|^2}{1-\|\psi\|^2}\|\psi_\xi\|^2\right\}_\xi=0,$$

$$\left\{\frac{1}{2}\frac{\|\psi_{\xi\xi\xi}\|^2}{1-\|\psi\|^2}+\frac{1}{(1-\|\psi\|^2)^2}(\psi\cdot[\psi_\xi,\psi_{\xi\xi},\psi_{\xi\xi\xi}])+\frac{1}{(1-\|\psi\|^2)^2}(\psi_{\xi\xi}\cdot\psi_{\xi\xi\xi})(\psi\cdot\psi_\xi)\right.$$

$$
\begin{aligned}
&- \frac{1}{(1-\|\psi\|^2)^2}(\psi_\xi\cdot\psi_{\xi\xi\xi})(\psi\cdot\psi_{\xi\xi}) + \frac{1}{2}\frac{1-3\|\psi\|^2}{(1-\|\psi\|^2)^3}(\psi_\xi\cdot\psi_{\xi\xi\xi})\|\psi_\xi\|^2 + \frac{3}{(1-\|\psi\|^2)^2}(\psi\cdot\psi_{\xi\xi\xi})(\psi_\xi\cdot\psi_{\xi\xi})\\
&+ \frac{4}{(1-\|\psi\|^2)^3}(\psi\cdot\psi_{\xi\xi\xi})\|\psi_\xi\|^2(\psi\cdot\psi_\xi) - \frac{1}{2}\frac{7-8\|\psi\|^2}{(1-\|\psi\|^2)^3}\|\psi_{\xi\xi}\|^2\|\psi_\xi\|^2\\
&- \frac{3-7\|\psi\|^2}{(1-\|\psi\|^2)^3}(\psi_\xi\cdot\psi_{\xi\xi})^2 - 4\frac{1-3\|\psi\|^2}{(1-\|\psi\|^2)^4}(\psi_\xi\cdot\psi_{\xi\xi})\|\psi_\xi\|^2(\psi\cdot\psi_\xi) - \frac{1}{2}\frac{15-17\|\psi\|^2}{(1-\|\psi\|^2)^4}(\psi\cdot\psi_{\xi\xi})\|\psi_\xi\|^4\\
&+ \left.\frac{1}{2}\frac{2-12\|\psi\|^2+11\|\psi\|^4}{(1-\|\psi\|^2)^5}\|\psi_\xi\|^6 - \frac{1-5\|\psi\|^2}{(1-\|\psi\|^2)^5}\|\psi_\xi\|^4(\psi\cdot\psi_\xi)^2\right\}_\eta\\
&+ \left\{\frac{1}{2}\frac{1-2\|\psi\|^2}{1-\|\psi\|^2}\|\psi_{\xi\xi}\|^2 - \frac{5}{1-\|\psi\|^2}(\psi_\xi\cdot\psi_{\xi\xi})(\psi\cdot\psi_\xi) + \frac{1}{2}\frac{1-3\|\psi\|^2}{(1-\|\psi\|^2)^2}(\psi\cdot\psi_{\xi\xi})\|\psi_\xi\|^2\right.\\
&- \left.\frac{1}{2}\frac{(2-3\|\psi\|^2)^2}{(1-\|\psi\|^2)^2}\|\psi_\xi\|^4 - \frac{5}{(1-\|\psi\|^2)^2}\|\psi_\xi\|^2(\psi\cdot\psi_\xi)^2\right\}_\xi = 0. \qquad (3.4)
\end{aligned}
$$

A second series is found by interchanging ($\xi\longleftrightarrow\eta$) and ($[A,B,C]\longleftrightarrow -[A,B,C]$).

Equations (VIII.9) and (VIII.10) of Ref. 1 allow us to apply the inverse scattering method to solve the differential Eq. (3.2). Let us write the linear operators L and B in the form

$$
L(\eta) = i\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}\partial_\xi - \frac{i}{2}\begin{pmatrix}0 & \varphi_\xi/\sqrt{1-\|\varphi\|^2}\\ \varphi_\xi^+/\sqrt{1-\|\varphi\|^2} & 0\end{pmatrix},
$$

$$
B = \frac{\gamma}{2}\begin{pmatrix}-\sqrt{1-\|\varphi\|^2}\,\mathbb{1} & \varphi\\ \varphi^+ & \sqrt{1-\|\varphi\|^2}\,\mathbb{1}\end{pmatrix}
$$

where

$$
\varphi = -i\sigma^1\varphi^1 - i\sigma^2\varphi^2 - i\sigma^3\varphi^3 + \mathbb{1}\varphi^4 .
$$

In a similar way as was worked out by Takhtadzhyan to calculate the N-soliton-scattering in the sine-Gordon theory,[13] we find for $r(\lambda)=0$ the "scattering potential"

$$
\frac{\varphi_\xi}{\sqrt{1-\|\varphi\|^2}} = -4\,\mathrm{tr}\{[1-W_-(\xi,\eta)W_+(\xi,\eta)]^{-1}\partial_\xi W_-(\xi,\eta)\}, \qquad (3.5)
$$

$$
\varphi = -\left(\frac{\varphi_\xi}{\sqrt{1-\|\varphi\|^2}}\right)_\eta .
$$

Here $W_\pm$ are GL(2,$\mathbb{C}$)-valued $N\times N$ matrices with entries

$$
W_{-jk}(\xi,\eta) = \frac{1}{\varkappa_j+\varkappa_k}\exp\left((\varkappa_j+\varkappa_k)\xi - \frac{1}{2\varkappa_j}\eta\right)m_j,
$$

$$
W_{+jk}(\xi,\eta) = \frac{-1}{\varkappa_j+\varkappa_k}\exp\left((\varkappa_j+\varkappa_k)\xi - \frac{1}{2\varkappa_j}\eta\right)(m_{j'})^+ .
$$

$\varkappa_j$ are N different arbitrary complex numbers with $\mathrm{Re}\varkappa_j>0$, $\varkappa_{j'}\equiv(\varkappa_j)^*$, and m_j are arbitrary constant GL(2,$\mathbb{C}$) matrices subject to the symmetry relation $m_{j'}=\sigma^2 m_j^*\sigma^2$, which is due to a symmetry of the scattering operators $L(\eta)$. The trace "tr" denotes the sum over the diagonal matrix-valued entries of the $N\times N$ matrix. For a more detailed derivation see Ref. 12. The pairs $(\varkappa_j,\varkappa_{j'})$ correspond to N_B breathers in the asymptotic state of the solution, the $N_S=N-2N_B$ real $\varkappa_j$ are related to solitons. φ and ψ depend on the vectors m_j in an SO(4)-covariant manner. Only the real parts of m_j survive in Eq. (3.5). Hence there are just N_S+N_B independent vectors available to build the space in which ψ develops. Thus, the simplest solution of Eq. (3.2), exhibiting, however, those features which are characteristic for the case $n=6$, is the three-soliton scattering solution.

We present this solution for the case where the polarizations of the solitons are mutually perpendicular ($m_i=-i\sigma^i M_i$, $i=1,2,3$):

$$
\psi_i(\xi,\eta) = \frac{(M_i/\varkappa_i)E_i(1-\Sigma_j c_{ij}E_j^2 + c_iE_j^2E_k^2)}{1+\Sigma_j a_jE_j^2 + \Sigma_{jk}a_{jk}E_j^2E_k^2 + a_{123}E_1^2E_2^2E_3^2}, \quad i=1,2,3,
$$

$$
\psi_4(\xi,\eta) = \frac{(M_1M_2M_3/\varkappa_1\varkappa_2\varkappa_3)E_1E_2E_3(\varkappa_1-\varkappa_2)(\varkappa_2-\varkappa_3)(\varkappa_3-\varkappa_1)/(\varkappa_1+\varkappa_2)(\varkappa_2+\varkappa_3)(\varkappa_2+\varkappa_1)}{1+\Sigma_j a_jE_j^2 + \Sigma_{jk}a_{jk}E_j^2E_k^2 + a_{123}E_1^2E_2^2E_3^2} \qquad (3.6)
$$

with the notation

$$
E_i(\xi,\eta) = \exp\left(2\varkappa_i\xi - \frac{1}{2\varkappa_i}\eta\right) = \exp\left[\left(\varkappa_i+\frac{1}{4\varkappa_i}\right)x^1 + \left(\varkappa_i-\frac{1}{4\varkappa_i}\right)x^0\right],
$$

$$c_{ij} = = \frac{M_j^2}{4\varkappa_j^2} \frac{(\varkappa_i - \varkappa_j)^2}{(\varkappa_i + \varkappa_j)^2},$$

$$c_i = \frac{M_j^2 M_k^2}{16\varkappa_j^2\varkappa_k^2} \frac{(\varkappa_i - \varkappa_j)^2(\varkappa_i - \varkappa_k)^2(\varkappa_j - \varkappa_k)^4}{(\varkappa_i + \varkappa_j)^2(\varkappa_i + \varkappa_k)^2(\varkappa_j + \varkappa_k)^4}, j \neq i \neq k;$$

$$a_i = \frac{M_i^2}{4\varkappa_i^2},$$

$$a_{ij} = \frac{M_i^2 M_j^2}{16\varkappa_i^2\varkappa_j^2} \frac{(\varkappa_i - \varkappa_j)^4}{(\varkappa_i + \varkappa_j)^4},$$

$$a_{123} = \frac{M_1^2 M_2^2 M_3^2}{64\varkappa_1^2\varkappa_2^2\varkappa_3^2} \frac{(\varkappa_1 - \varkappa_2)^4(\varkappa_2 - \varkappa_3)^4(\varkappa_3 - \varkappa_1)^4}{(\varkappa_1 + \varkappa_2)^4(\varkappa_2 + \varkappa_3)^4(\varkappa_3 + \varkappa_1)^4}.$$

We observe that—at least in this example—we can write

$$\psi = 2\,\mathrm{tr}\{[1 - W_- W_+]^{-1}\partial_\xi\partial_\eta W_-\}.$$

IV. CONCLUSIONS

The field equations of the two-dimensional nonlinear O(n) σ-model can be reduced to either one of two systems of relativistic differential equations involving $(n-2)$ scalar fields in an O$(n-2)$ covariant manner. Both systems possess a denumerably infinite set of local covariant conservation laws.

The representative of the space–time dependent rotation mediating between the two reduced field vectors itself satisfies a differential equation invariant under the restricted Poincaré group. For $n - 2 = 4$ a recursion formula for its local conservation laws is derived. A formula for explicitly calculating multisoliton scattering solutions is given.

[1] K. Pohlmeyer, Commun. Math. Phys. **46**, 207 (1976).

[2] Notation: Greek indices μ,ν run from 0 to 1, Latin ones a,b from 1 to n if not stated otherwise. The metric is $g_{00} = -g_{11} = 1$. $\epsilon^{\mu\nu}$ is the antisymmetric tensor $\epsilon^{01} = -\epsilon^{10} = 1$. The summation convention is implied.

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$$\mathscr{L}(x^0,x^1) = \frac{1}{2}\frac{(\partial_\mu\psi^a)(\partial^\mu\psi^a)}{1-\lambda^2\psi^a\psi^a} - \frac{m^2}{2}\psi^a\psi^a.$$

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