7.A.1
7.A.2

Nuclear Physics B36 (1972) 130-140. North-Holland Publishing Company

CHIRAL MULTI-LOOPS

J. HONERKAMP
CERN - Geneva

Received 24 June 1971
(Reprepared manuscript received 24 September 1971)

Abstract: With help of functional integral methods a complete perturbation series for a non-linear chiral Lagrangian is given, in which each term, corresponding to a class of graphs with a definite number of internal loops and a definite topological structure, is chiral invariant.

1. INTRODUCTION

Recently, much work has been done regarding non-linear Lagrangians as a basis for a real field theory [1]. To do this one has to go beyond the tree-approximation and to take into account loop diagrams. If chiral invariance of a starting Lagrangian is not only an accident but a crucial assumption, one has to take care of the chiral invariance at each stage of the calculation. Despite the fact that till now nobody knows how to extract finite results from loop diagrams consistent with chiral invariance one can ask for a chiral invariant perturbation theory in the number of internal loops. If then, e.g., the one-loop diagrams can be invariantly renormalized [2] a further class of diagrams additional to the tree-diagrams is obtained which satisfy conditions deduced from currect algebra *.

The invariant expression for the one-loop diagrams is very well known. One finds [4] that naîve Feynman rules must in general be modified by terms which arise in the language of functional integral methods from a modified, chiral invariant measure.

In this paper we give a complete perturbation series for the non-linear chiral pion Lagrangian in which each term corresponding to a definite class of diagrams is chiral invariant. It turns out that this is an expansion in diagrams which have a definite number of internal loops and a definite topological structure. The crucial point is that this perturbation series is an expansion in terms which are manifestly chiral invariant. Thus, never non-covariant pieces occur for every order of the perturbation series. Because the invariant renormalization problem of the one-loop diagrams can be tackled very nicely [2], the hope is that also the graphs with a higher number of loops can be treated with the same methods to give a

* For the physical equivalence of two non-linear realizations for each class of diagrams with a definite number of loops see for example ref. [3].

more precise insight into the invariant renormalization problem of the non-linear chiral Lagrangian. In this way, more and more classes of diagrams, which also take into account unitarity and carry the contents of current algebra can be treated.

In sect. 2 we give the derivation of the perturbation series with the help of functional integral methods. This derivation makes sense if here is no internal symmetry group present. The generalization to the chiral invariant theory will be done in sect. 3.

2. NAIVE PERTURBATION THEORY

In this section we will give a brief review how to construct the perturbation series in simple cases by functional integral methods *. We regard a theory of n bosons ϕ_i. The generating functional of the Green's functions is

$$Z(J) = N^{-1} \int \prod_x \prod_{i=1}^{n} \mathrm{d}\phi_i(x)\, \mathrm{e}^{i(S(\phi) + \int J_i \phi_i \mathrm{d}^4 x)}, \tag{1}$$

where $S(\phi)$ is the action, $S(\phi) = \int \mathrm{d}^4 x L(\phi_i, \partial_\mu \phi_i)$, L the Lagrangian of the theory. The normalization factor N has to be chosen so that $Z(0) = 1$.

To get the S-matrix, one has to put in eq. (1) $J_i = \phi_i^{\mathrm{in}} K$ [7], where K is the Klein-Gordon operator; ϕ_i^{in} are fields which satisfy on-mass shell $K\phi_{\mathrm{in}} = 0$. With S_0 as the free action we have $\delta S_0/\delta\phi_i = -K\phi_i$.

The classical field is a solution of the equation of motion

$$\frac{\delta S}{\delta \phi_i} \equiv S_{\cdot i} = -K\phi_i^{\mathrm{in}}, \tag{2}$$

corresponding to the stationary point of the exponent in eq. (1).

In the following we will use an abbreviated notation in expressions where functional derivatives of $S(\phi)$ occur. An index like i or j stands there not only for itself but also for the argument x or x', and if e.g., i occurs twice in such an expression, a summation over i *and* an integration over x has to be done. Thus, e.g.,

$$S_{\cdot ij}\phi_i \phi_j = \int \mathrm{d}^4 x \mathrm{d}^4 x' \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\delta^2 S}{\delta\phi_j(x')\delta\phi_i(x)} \phi_i(x)\, \phi_j(x'). \tag{3}$$

We also introduce abbreviations like $\delta_{ik} \equiv \delta_{ik}\delta(x, x')$.

To calculate the functional integral in eq. (1) we expand $S(\phi)$ at the classical field φ. We write $\phi_i = \varphi_i + \chi_i$, then

$$S(\phi) + \int \phi_i^{\mathrm{in}} K\phi_i \mathrm{d}^4 x = S(\varphi) + \int \varphi_i K \phi_i^{\mathrm{in}} \mathrm{d}^4 x + \tfrac{1}{2} S_{\cdot ij}(\varphi)\chi_i \chi_j$$
$$+ \frac{1}{3!} S_{\cdot ijk}(\varphi)\chi_i \chi_j \chi_k + \frac{1}{4!} S_{\cdot ijkl}(\varphi)\chi_i \chi_j \chi_k \chi_l + \ldots, \tag{4}$$

* In this connection see also ref. [5], and e.g. ref. [6].

and we get

$$\int \prod_x \prod_{i=1}^{n} \mathrm{d}\phi_i(x)\, \mathrm{e}^{i(S(\phi)+\int \phi_i^{\mathrm{in}} K \phi_i \mathrm{d}^4 x)} = \mathrm{e}^{i(S(\varphi)+\int \varphi_i K \phi_i^{\mathrm{in}} \mathrm{d}^4 x)}$$

$$\times \int \prod_x \prod_i \mathrm{d}\chi_i \mathrm{e}^{\frac{1}{2} iS_{.ij}\chi_i\chi_j} \left(1+\frac{i}{3!} S_{.ijk}(\varphi)\chi_i\chi_j\chi_k + \frac{i}{4!} S_{.ijkl}\chi_i\chi_j\chi_k\chi_l\right.$$

$$\left. + \frac{i^2}{2!3!3!} S_{.ijk} S_{.lmn} \chi_i\chi_j\chi_k\chi_l\chi_m\chi_n + \dots \right. \tag{5}$$

Now we can make use of the formulae

$$\int \mathrm{d}^n x\, \mathrm{e}^{-a_{ij}x_i x_j} = \pi^{\frac{1}{2}n}\, \mathrm{Det}\,(A_{ij})^{\frac{1}{2}} , \tag{6a}$$

$$\int \mathrm{d}^n x\, \mathrm{e}^{-a_{ij}x_i x_j} x_{k_1} \dots x_{k_{2m-1}} = 0 , \qquad m \geqslant 1 , \tag{6b}$$

$$\int \mathrm{d}^n x\, \mathrm{e}^{-a_{ij}x_i x_j} x_{k_1} \dots x_{k_{2m}} = (\mathrm{Det}\, A_{ij})^{\frac{1}{2}}\, P_{n,m}(A_{k_1 k_2} A_{k_3 k_4} \dots A_{k_{2m-1}k_{2m}}$$

$$+ \text{all possible permutations of the } k_i, \text{ that are } (2m-1)!! \text{ terms}) , \tag{6c}$$

where A_{ij} is the inverse of a_{ij}: $A_{ik}a_{kj} = \delta_{ij}$ and

$$P_{n,m} = \pi^{\frac{1}{2}n}\, 2^{-m} . \tag{6d}$$

Transferring these formulae to the functional integral, we get e.g.,

$$\int \prod_x \prod_{i=1}^{n} \mathrm{d}\chi_i\, \mathrm{e}^{\frac{1}{2}iS_{.ij}\chi_i\chi_j} = \text{const.}\, (\mathrm{Det}\,(G_{ij}))^{\frac{1}{2}} , \tag{7}$$

where the Green's function G_{ij} is defined * $S_{.ij}G_{jk} = \delta_{ik}$ and the determinant includes the i, j indices as well as the coordinate variables x, x' ‡. In general it follows from eq. (5):

$$Z = N^{-1} \int \prod_x \prod_{i=1}^{n} \mathrm{d}\phi_i(x)\, \mathrm{e}^{i(S(\phi)+\int \phi_i^{\mathrm{in}} K\phi_i \mathrm{d}^4 x)}$$

$$= \mathrm{e}^{i(S(\varphi)+\int \phi_i^{\mathrm{in}} K\varphi_i \mathrm{d}^4 x)}\, (\mathrm{Det}\, G_{ij}/G_0)^{\frac{1}{2}}\, (1+\frac{i}{4!} S_{.ijkl}(\varphi)\, (G_{ij}G_{kl}$$

$$+ G_{il}G_{jk} + G_{ik}G_{jl}) + \frac{i}{2!3!3!} S_{.ijk}(\varphi) S_{.lmn}(\varphi)\, (G_{ij}(G_{kl}G_{mn}$$

$$+ G_{km}G_{ln} + G_{kn}G_{lm}) + G_{ik}(G_{jl}G_{mn} + G_{jm}G_{ln} + G_{jn}G_{lm})$$

$$+ G_{il}(G_{jk}G_{mn} + G_{jm}G_{kn} + G_{jn}G_{km}) + G_{im}(G_{jk}G_{ln} + G_{jl}G_{kn}$$

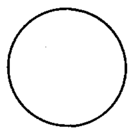
$$+ G_{jn}G_{kl}) + G_{in}(G_{jk}G_{lm} + G_{jl}G_{km} + G_{jm}G_{kl})) + \dots)$$

$$\times (1 + \frac{i}{4!} S_{.ijkl}(\varphi=0)\, (G_{ij}(\varphi=0)\, G_{lk}(\varphi=0) + \dots) + \dots)^{-1} . \tag{8}$$

* ‡ Footnotes: see next page.

The determinant of the free Green's function and the last curly bracket come in because N^{-1} is chosen, so that for $J_i = \phi_i^{\mathrm{in}} K \equiv 0$, i.e., $\varphi_i \equiv 0$ we get $Z = 1$.

The graphical representation of these terms is obvious. $S(\varphi)$ denotes the sum of all tree graphs [6], the $(\mathrm{Det}\, G/G_0)^{\frac{1}{2}}$ corresponds to the one-loop diagram (fig. 1), the other terms in the curly brackets represent the two-loop diagrams (fig. 2). All the terms of the series which are omitted correspond to graphs with more than two loops. The structure and the multiplicity of the higher loop diagrams can be immediately obtained from this expansion. Clearly, the series contains connected and disconnected diagrams.



$[\mathrm{Det}(G_{ij}/G_0)]^{1/2}$

Fig. 1.

All the expressions in eq. (8) are functionals of the classical field φ. An expansion of these functionals in φ then exhibits the structure of any diagram with any number of loops and any number of external φ lines. Furthermore, each external φ line can be displayed in its tree-graph structure, using the perturbative solution of the classical field. This solution corresponds to a connected tree-graph structure only.

We take as an example the case of neutral scalar field with cubic self-interaction [6]. We have

$$\frac{\delta S}{\delta \phi} = (-\Box + m^2)\, \phi - g\phi^2 = -K\phi^{\mathrm{in}} \,, \tag{8a}$$

$$\varphi = \phi^{\mathrm{in}} + G_0 g \phi^2 \,, \qquad G_0 = (\Box - m^2)^{-1} \,. \tag{8b}$$

The iterative solution

$$\varphi = \phi^{\mathrm{in}} + G_0 g(\phi^{\mathrm{in}})^2 + 2g^2(G_0\, \phi^{\mathrm{in}})\, G_0(\phi^{\mathrm{in}})^2 + \ldots \tag{8c}$$

* We require Feynman-boundary conditions for the Green's function, if we calculate the Det (G_{ij}) in the conventional way in momentum space performing a rotation from Minkowski space to Euclidean space. In this way non-causal loops are automatically excluded. For this, see ref. [5].

‡ That means, with $S_{.ij}\, G_0 \equiv \delta_{ij}\, \delta(x,x') + A_{ij}(x,x')$ e.g.,

$$(\mathrm{Det}\, G_{ij}/G_0)^{\frac{1}{2}} = (\mathrm{Det}\,(S_{.\,ij}\, G_0))^{-\frac{1}{2}} = \mathrm{e}^{-\frac{1}{2}\mathrm{Tr}\,\log(\delta_{ij}\,\delta(x,x') + A_{ij}(x,x'))}$$

$$= \exp\left[-\tfrac{1}{2}\left(\int \mathrm{d}^4x \sum_i A_{ii}(x,x) - \tfrac{1}{2}\int \mathrm{d}^4x\, \mathrm{d}^4x' \sum_{i,k} A_{ik}(x,x')\, A_{ki}(x',x) + \ldots\right)\right] . \tag{32}$$

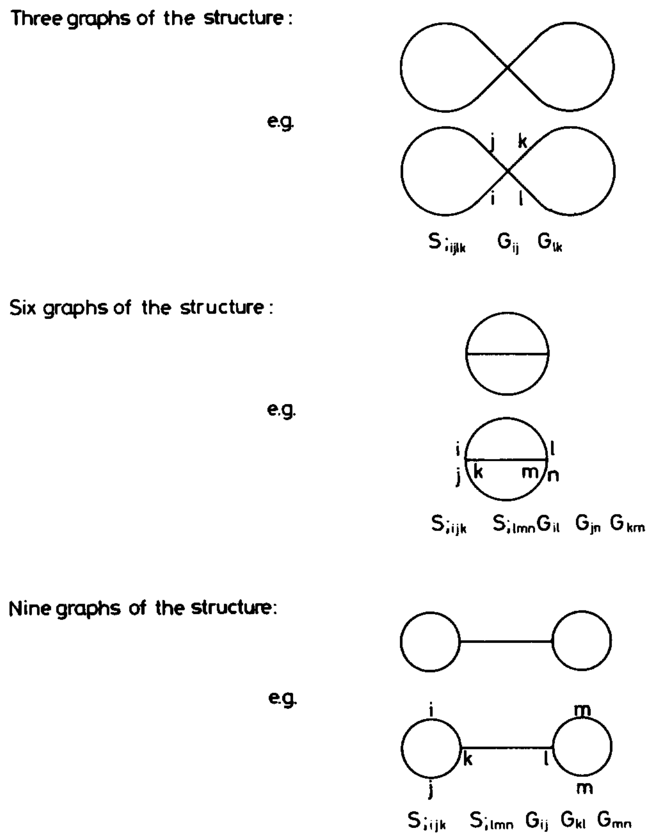


Fig. 2.

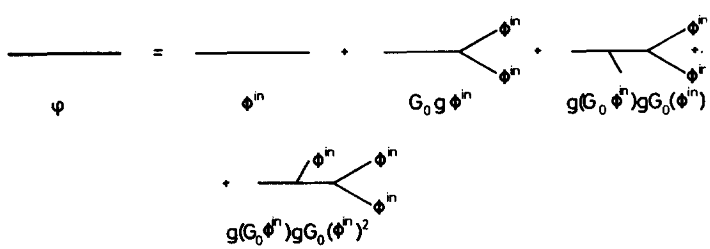


Fig. 3.

displays the decomposition of the classical field into a tree-graph structure (fig. 3).

Thus, the evaluation of the functional integral in the indicated way displays very nicely the various types of graphs and gives a concise prescription for the calculation of the higher-loop diagrams. We get an expansion in the number of loops, and by expanding the classical field into free fields, we can attach to each loop diagram any possible tree graph structure.

3. THE NON-LINEAR π LAGRANGIAN

The action for the non-linear chiral invariant π Lagrangian can be written as [10]

$$S(\pi) = \tfrac{1}{2} \int \mathrm{d}^4x\, g_{ij}(\pi)\, \partial_\mu \pi^i \partial^\mu \pi^j ; \tag{9}$$

$g_{ij}(\pi(x))$ is the metric in a curved isospace of constant curvature f_π^{-2}. We write $g = \det g_{ij}$ and we introduce three drei-bein fields $e_i^a(\pi)$ $(a = 1, 2, 3)$ via

$$g_{ij} = e_i^a\, e_{ja} . \tag{10}$$

The $e_i^a(\pi)$ obey the relations

$$e_{ia}\, e_b^i = \delta_{ab} , \qquad e_{ia} = e_i^a , \qquad e_{ia}\, b^{ja} = \delta_i^j , \tag{10a}$$

where $e_b^i = g^{ij} e_{jb}$, and they are not uniquely determined since a rotation with respect to the index a leaves g_{ij} invariant.

The classical field φ^i is a solution of the equation

$$\frac{\delta S}{\delta \pi^i} \equiv S_{,i} = -K\pi_i^{\text{in}} , \tag{11}$$

or

$$-g_{ik}(\Box\pi^k + \Gamma^k_{lj}\, \partial_\mu \pi^l \partial^\mu \pi^j) = -\Box\pi_i^{\text{in}} \tag{12}$$

where

$$\Gamma^k_{lj} = g^{kr}(\partial_l g_{rj} + \partial_j g_{rl} - \partial_r g_{lj}) , \qquad \partial_r = \frac{\partial}{\partial \pi^r} . \tag{13}$$

Following the method of sect. 2 to construct a perturbation series we would write $\pi^i = \varphi^i + \chi^i$. But now the χ^i, as well as S_{ij}, S_{ijk} are not covariant quantities. Instead of these one has to construct an expansion of $S(\pi)$ in which the covariant derivatives of $S(\pi)$ occur. This can be done with help of the geodetic interval $\Gamma(\pi, \varphi)$ [11]. Let $\zeta^i(\lambda)$ be the geodesics from φ^i to π^i, where the parameter $\lambda (0 \leqslant \lambda \leqslant s)$ measures the length of this curve and $\zeta^i(0) = \varphi^i$, $\zeta^i(s) = \pi^i$. The equation of the geodesics is

$$\frac{\mathrm{d}^2\zeta^i}{\mathrm{d}\lambda^2} + \Gamma^i_{kl} \frac{\mathrm{d}\zeta^k}{\mathrm{d}\lambda} \frac{\mathrm{d}\zeta^l}{\mathrm{d}\lambda} = 0 . \tag{14}$$

The geodetic interval is defined by

$$\Gamma(\pi, \varphi) = s \int_{0,\varphi^i}^{s,\pi^i} \mathrm{d}\lambda\, \tfrac{1}{2} g_{ij}(\zeta(\lambda)) \frac{\mathrm{d}\zeta^i}{\mathrm{d}\lambda} \frac{\mathrm{d}\zeta^j}{\mathrm{d}\lambda} ; \tag{15}$$

$\Gamma(\pi,\ \varphi)$ is a chiral bi-scalar and it is equal to one-half the square of the distance along the geodesics between φ and π.

Applying Hamilton-Jacobi theory we get*

$$s\left(\frac{d\zeta_i}{d\lambda}\right)_{\lambda=0} = \frac{\partial}{\partial\varphi^i}\Gamma(\pi,\ \varphi) \equiv \Gamma_{\cdot i}\ , \tag{16}$$

$$g_{ij}(\varphi)\Gamma^i_{\cdot}\ \Gamma^j_{\cdot} = 2\Gamma\ ; \tag{17}$$

$\Gamma_{\cdot i}$ and $\Gamma^i_{\cdot} = g^{ij}\Gamma_{\cdot j}$ are chiral bivectors, i.e.,

$${}'\Gamma_{\cdot i} = \frac{\partial\varphi^j}{\partial\varphi'^i}\Gamma_{\cdot j}\ , \tag{17a}$$

and $\Gamma^i_{\cdot}(\pi,\ \varphi)$ can be used instead of χ^i as expansion parameter. In flat iso-space we have

$$\Gamma^i_{\cdot} = \pi^i - \varphi^i = \chi^i\ . \tag{18}$$

We will also introduce

$$\Gamma^a_{\cdot} = e^a_i(\varphi)\Gamma^i_{\cdot}\ . \tag{18a}$$

Now, we can write

$$S(\pi) = S(\zeta(\lambda))\,|_{\lambda=s}\ , \tag{18b}$$

and

$$\begin{aligned} S(\zeta(\lambda)) &= S(\zeta(o)) + \lambda\left(\frac{d}{d\lambda}S(\zeta(\lambda))\right)_{\lambda=0} + \frac{\lambda^2}{2!}\left(\frac{d^2}{d\lambda^2}S(\zeta(\lambda))\right)_{\lambda=0} + \dots \\ &= S(\varphi) + \lambda\left(\frac{d\zeta^i}{d\lambda}\frac{\partial}{\partial\zeta^i}S(\zeta(\lambda))\right)_{\lambda=0} \\ &+ \frac{\lambda^2}{2!}\left[\frac{d^2\zeta^i}{d\lambda^2}\frac{\partial}{\partial\zeta^i}S(\zeta(\lambda)) + \frac{d\zeta^i}{d\lambda}\frac{d\zeta^j}{d\lambda}\frac{\partial}{\partial\zeta^i}\frac{\partial}{\partial\zeta^j}S(\zeta(\lambda))\right]_{\lambda=0} + \dots\ . \end{aligned} \tag{18c}$$

Hence, by virtue of eq. (16), we get for $\lambda = s$:

$$S(\pi) = S(\varphi) + \Gamma^i_{\cdot}\ S_{\cdot i} + \frac{1}{2!}\Gamma^i_{\cdot}\Gamma^j_{\cdot}S_{;ij} + \dots\ , \tag{19}$$

where the covariant derivative $S_{;ij}$ is defined as usual

$$S_{;ij} = \delta_j S_{\cdot i} - \Gamma^l_{ij}S_{\cdot l}\ . \tag{19a}$$

In this way a Taylor expansion like eq. (4) can be obtained in which, instead of the non-covariant quantities χ, $S_{,ijk}$ the covariant ones $\Gamma^i_{\cdot}$, $S_{;ij\dots}$ appear.

* For the concept of the geodetic interval see ref. [11].

Now, the generating functional reads

$$\int \prod_x \prod_{i=1}^{3} \mathrm{d}\pi^i(x)\sqrt{g(\pi)}\, \mathrm{e}^{i(S(\pi)+\int \pi_i^{\mathrm{in}}\Box\pi^i \mathrm{d}^4x)} \, . \tag{20}$$

In the functional integral the naîve measure

$$\prod_x \prod_i \mathrm{d}\pi^i$$

has not to be used but the chiral invariant one [4].

The source term, however, is still not invariant, and, also the equation for the classical field eq. (12) is a non-covariant equation. But we are interested only in on-mass shell amplitudes. There, the right-hand side i.e., the non-covariant part of eq. (12) vanishes and the amplitudes calculated from eq. (20) or from

$$N^{-1} \int \prod_x \prod_{i=1}^{3} \mathrm{d}\pi^i \sqrt{g}\, \mathrm{e}^{i(S(\pi)+\int \pi_i^{\mathrm{in}}\Box\pi'^i(\pi)\mathrm{d}^4x)} \, , \tag{20a}$$

are equal on-mass shell because $\pi'^i(\pi) = \pi^i$ + non-linear terms of π^i *.

Because of this irrelevance on-mass shell of non-linear terms in π as coefficients of $\pi_i^{\mathrm{in}}\Box$ we do not change the value on-mass shell of the functional integral if we write ** instead of $\pi_i^{\mathrm{in}}\Box\pi^i$

$$\pi_i^{\mathrm{in}}\Box(\varphi^i+\Gamma^i_{\cdot}(\pi,\varphi)) \, . \tag{20b}$$

Then, we insert the expansion (19) into eq. (20), expand also

$$\prod_x \sqrt{g(\pi(x))}$$

like $\mathrm{e}^{iS(\pi)}$ and substitute the integration variable π^i by the vector $\Gamma^i_{\cdot}$.

We get, using $\sqrt{g(\varphi)} = \det e_i^a(\varphi)$ and

$$\prod_x \sqrt{g(\pi(x))} = \mathrm{e}^{\delta^4(\mathrm{o})\int \log\sqrt{g}(\pi)\mathrm{d}^4x} \, ,$$

$$\prod_x \prod_i \mathrm{d}\pi^i \sqrt{g(\pi)} = \Big(\prod_x \prod_i \mathrm{d}\Gamma^i_{\cdot}\sqrt{g(\varphi)}\det\Big(\frac{\partial\pi^i}{\partial\Gamma^j_{\cdot}}\Big)\Big)\mathrm{e}^{\frac{1}{2}\delta^4(\mathrm{o})\int(\log g(\pi)-\log g(\varphi))\mathrm{d}^4x}$$

$$= \Big(\prod_x \prod_a \mathrm{d}\Gamma^a_{\cdot}\Big)\, \mathrm{e}^{\delta^4(\mathrm{o})\int[\frac{1}{2}(\log g(\pi)-\log g(\varphi))+\log\det\frac{\partial\pi^i}{\partial\Gamma^j_{\cdot}}]\mathrm{d}^4x} \, . \tag{20c}$$

* See ref. [7] and in this connection also ref. [8].

** From eq. (23) we see that $\varphi^i+\Gamma^i_{\cdot}(\pi,\varphi)$ is just an expansion of π^i, where all terms, which contribute only to the non-linear terms of $\pi^i-\varphi^i$, are omitted.

Similar to the expansion of $S(\pi)$ we get

$$\tfrac{1}{2}(\log g(\pi) - \log g(\varphi)) = \tfrac{1}{2}\Gamma^{k}_{.}(\partial_k g_{ij})g^{ij} + \tfrac{1}{4}\Gamma^{k}_{.}\Gamma^{l}_{.}((\partial_k\partial_l g_{ij})g^{ij} + (\partial_k g_{ij})(\partial_l g^{ij}) - \Gamma^{m}_{lk}(\partial_m g_{ij})g^{ij}) + \dots , \tag{21}$$

and, to get π^i as a functional of $\Gamma^i_.$ we use again similar arguments, expanding $\zeta^i(\lambda)$ at $\lambda = 0$:

$$\zeta^i(\lambda) = \zeta^i(o) + \lambda\left(\frac{d\zeta^i}{d\lambda}\right)_{\lambda=0} + \frac{\lambda^2}{2!}\left(\frac{d^2\zeta^i}{d\lambda^2}\right)_{\lambda=0} + \frac{\lambda^2}{3!}\left(\frac{d^3\zeta^i}{d\lambda^3}\right)_{\lambda=0} + \dots$$

$$= \varphi^i + \frac{\lambda}{s}\Gamma^i_. - \frac{\lambda^2}{2!s^2}\Gamma^i_{kl}\Gamma^k_.\Gamma^l_. - \frac{\lambda^3}{3!s^3}\Gamma^i_{klm}\Gamma^k_.\Gamma^l_.\Gamma^m_. + \dots , \tag{21a}$$

where the coefficients Γ^i_{kl}, Γ^i_{klm} etc., can be obtained by repeated use of the repeated differentiated equation of the geodesics [9] e.g.

$$\Gamma^i_{klm} = \tfrac{1}{3}\{(\partial_k\Gamma^i_{lm} + \partial_l\Gamma^i_{km} + \partial_m\Gamma^i_{kl}) - 2\Gamma^i_{\alpha k}\Gamma^\alpha_{lm} - 2\Gamma^i_{\alpha l}\Gamma^\alpha_{km} - 2\Gamma^i_{\alpha m}\Gamma^\alpha_{kl}\}. \tag{22}$$

Hence,

$$\pi^i = \varphi^i + \Gamma^i_. - \tfrac{1}{2}\Gamma^i_{kl}\Gamma^k_.\Gamma^l_. - \frac{1}{3!}\Gamma^i_{klm}\Gamma^k_.\Gamma^l_.\Gamma^m_. + \dots , \tag{23}$$

and, therefore

$$\frac{\partial\pi^i}{\partial\Gamma^j_.} = \delta^i_j - \Gamma^i_{jl}\Gamma^l_. - \frac{1}{2!}\Gamma^i_{jlm}\Gamma^l_.\Gamma^m_. + \dots , \tag{24}$$

$$\log\det\frac{\partial\pi^i}{\partial\Gamma^j_.} = \operatorname{tr}\log\frac{\partial\pi^i}{\partial\Gamma^j_.} = -\Gamma^i_{il}\Gamma^l_. - \tfrac{1}{2}(\Gamma^i_{ilm}\Gamma^l_.\Gamma^m_. + \Gamma^i_{rl}\Gamma^r_{im}\Gamma^l_.\Gamma^m_.) + O((\Gamma^i_.)^3) .$$

Finally,

$$\tfrac{1}{2}(\log g(\pi) - \log g(\varphi)) + \log\det\frac{\partial\pi^i}{\partial\Gamma^j_.}$$

$$= \tfrac{1}{4}\Gamma^k_.\Gamma^l_.((\partial_k\partial_l g_{ij} + (\partial_k g_{ij})(\partial_l g^{ij}) - \Gamma^m_{lk}(\partial_m g_{ij})g^{ij} - 2\Gamma^i_{ikl} - 2\Gamma^i_{rl}\Gamma^r_{ik}) + \dots = \tfrac{1}{6}\Gamma^k_.\Gamma^l_. R_{kl}(\varphi) + O((\Gamma^i_.)^3) , \tag{26}$$

where $R^i_{kl} = R^i_{k,il}$ and

$$R^i_{k,jl} = \partial_j\Gamma^i_{kl} - \partial_l\Gamma^i_{kj} - \Gamma^m_{kj}\Gamma^i_{ml} + \Gamma^m_{kl}\Gamma^i_{mj} . \tag{27}$$

Because the isospace has constant constant curvature f_π^{-2} we have [14]

$$R_{ik,jl} = f_\pi^{-2}(g_{ij}\,g_{kl} - g_{il}\,g_{kj}) \,, \tag{28}$$

$$R_{kl} = 2f_\pi^{-2}\, g_{kl} \,. \tag{29}$$

All terms under the functional integral sign are now written covariantly and by contracting each index i with $e_a^i(\varphi)$, so that e.g.,

$$S_{;ij}\Gamma^i_{\cdot}\Gamma^j_{\cdot} = S_{ab}\,\Gamma^a_{\cdot}\Gamma^b_{\cdot} \tag{30}$$

with $S_{ab} = e_a^i\, S_{;ij} e_b^j$, we obtain a perturbation series in chiral invariant terms.

The functional not reads

$$N^{-1}\int\prod_x\prod_a \mathrm{d}\Gamma^a_{\cdot}\, \exp\{i(S(\varphi)+\int \pi_i^{\mathrm{in}}\Box\varphi^i\mathrm{d}^4x) + (\tfrac{i}{2!}\,S_{ab}\Gamma^a_{\cdot}\Gamma^b_{\cdot} + \ldots)$$

$$+\,\delta^4(\mathrm{o})\,(\tfrac{1}{6}\,R_{ab}\,\Gamma^a_{\cdot}\Gamma^b_{\cdot} + \ldots)\}$$

$$= \mathrm{e}^{i(S(\varphi)+\int\pi^{\mathrm{in}}\Box\varphi^i\mathrm{d}^4x)}\,(\det G_{ab}/G_{\mathrm{o}})^{\frac{1}{2}}\,(1$$

$$+\frac{i}{4!}\,S_{abcd}(\varphi)\,(G^{ab}\,G^{cd} + G^{ac}\,G^{bd} + G^{ad}\,G^{bc})$$

$$+\frac{i}{72}\,S_{abc}\,S_{def}(G^{ab}\,G^{cd}\,G^{ef} + \ldots) + \ldots$$

$$+\,\delta^4(\mathrm{o})\,(\tfrac{1}{6}\,\frac{2}{f_\pi^2}\,G^{aa} + \ldots))\,(1 + \frac{i}{4!}\,S_{abcd}(\varphi=0)\,(G^{ab}(\varphi=0)\,G^{cd}(\varphi=0) + \ldots) + \ldots)^{-1}. \tag{31}$$

Up to the terms proportional to $\delta^4(\mathrm{o})$ we have now a similar expansion as in sect. 2. The first singular term

$$\delta^4(\mathrm{o})\,\frac{1}{3f_\pi^2}\,G_a^a$$

contributes to the two-loop diagrams. All further terms which are produced by the measure $\mathrm{d}^3\pi\sqrt{g}$ have similar structure. Because the isospace has constant curvature the covariant derivatives of the Riemann tensor are zero, hence, the further terms are of the type

$$D_{a_1\ldots a_n}\Gamma^{a_1}_{\cdot}\ldots\Gamma^{a_n}_{\cdot} \,, \tag{31a}$$

where $D_{a_1\ldots a_n}$ is an expression of only $\delta_{a_i a_j}$. Clearly, by expanding the exponent, formally higher and higher powers of $\delta^4(\mathrm{o})$ come in. The relevance of these extra terms for an invariant regularization of the two and more loop diagrams will be studied in a subsequent paper.

With eq. (31) we have an expansion in which each term is chiral invariant as well as invariant under the rotations which redefine the drei-bein fields $e_i^a(\pi)$. This expansion is a very convenient starting point for tackling the problem of an invariant renormalization of this non-linear theory.

I would to thank Professor B. Zumino for many useful and stimulating discussions, for his instruction about functional integral methods, as well as for the critical reading of the manuscript.

REFERENCES

[1] J. M. Charap, Phys. Rev. D2 (1970) 1554;
W. K. Chan and J. S. Curiale, Perturbation theory with chiral Lagrangians, Yale University preprint (1971);
R. Delbourgo, A. Salam, J. Strathdee, Phys. Rev. 187 (1969) 1999;
A. D. Hunt, K. Koller and A. Shafi, ICTP preprint 69/19, Phys. Rev., to be published.
[2] G. Ecker and J. Honerkamp, CERN TH. 1348, Nucl. Phys., to be published.
[3] S. Coleman, J. Wess and B. Zumino, Phys. Rev. 177 (1969) 2239.
[4] A. Salam and J. Strathdee, Internal report 16/70/8, Trieste;
J. Honerkamp and K. Meetz, Phys. Rev. 3D (1971) 1996;
J. S. Gerstein, R. Jackiw, B. W. Lee and S. Weinberg, Phys. Rev. 3D (1971) 2486;
J. M. Charap, Phys. Rev. 3D (1971) 1998.
[5] B. S. De Witt, Phys. Rev. 162 (1967) 1195.
[6] D. G. Boulware and L. S. Brown, Phys. Rev. 172 (1968) 1628.
[7] B. Zumino, Theory and phenomenology in particle physics, part A, p.73 ff (1968), Int. School of physics, Erice, ed. A. Zichichi, Acad. Press, New York.
[8] Y. Nambu, Phys. Letters 26B (1968) 626.
[9] L. P. Eisenhart, Riemannian geometry, Princeton University Press, Princeton (1949).
[10] K. Meetz, J. Math. Phys. 10 (1969) 589.
[11] P. Garabedian, Partial differential equations, New York, Wiley (1964);
B. S. De Witt, Dynamical theory of groups and fields, in Relativity, groups and topology, Les Houches (1963), Gordon and Breach, New York-London.