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The Inverse Problem in the Quantum Theory of Scattering*

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This report is a translation from the Russian of a survey article by L. D. Faddeyev, which appeared in Uspekhi Matem. Nauk., **14**, 57 (1959). Our own interest in this article lies in its relevance to the inverse scattering problem—that is, the problem of determining information about a medium from which an electromagnetic wave is reflected, given a knowledge of the reflection coefficient. Similar questions concerning scattering phenomena in other branches of physics, e.g., in quantum mechanics, can be investigated by means of the same theory. We have therefore thought it worthwhile to reproduce and distribute the translation. A good indication of the contents is given in the Introduction.

INTRODUCTION

THIS paper is devoted to a survey of the following fundamental problem arising in the quantum theory of scattering: *The solution of*

$$L\psi = -(d^2/dx^2)\psi(x, k) + q(x)\psi(x, k) = k^2\psi(x, k), \tag{0.1}$$

satisfying the condition

$$\psi(0, k) = 0, \tag{0.2}$$

behaves asymptotically like

$$\psi(x, k) \approx C(k) \sin [kx - \eta(k)], \tag{0.3}$$

provided the potential $q(x)$ decreases sufficiently fast as x tends to infinity; to what extent does the assignment of $\eta(k)$ determine $q(x)$ and how are these functions related. This problem is one of the general questions concerning the relationship between the S matrix and the energy operator in scattering theory. The operator L, defined by Eq. (0.1) and condition (0.2), is the simplest example of the energy operator occurring in scattering theory, and the function $S(k) = e^{-2i\eta(k)}$ the simplest example of the S matrix or scattering operator.

First introduced by Wheeler, the S matrix has since been frequently used in scattering theory, particularly following the publication of Heisenberg's papers.[1] In these articles, the following time-independent definition of the S matrix was given. A wavefunction, describing the steady state of a system (for simplicity, we restrict ourselves to a system of two particles), has an asymptotic representation in the space variables

$$\Psi \big|_{r\to\infty} \approx \Psi_1 + \Psi_2, \tag{0.4}$$

where r is the distance between particles. Ψ_1 and Ψ_2 are, respectively, outgoing and incoming waves, so that Ψ_1 contains the factor e^{ikr} and Ψ_2 is proportional to e^{-ikr}, k being the wavenumber characterizing the energy of the state Ψ. The quantity relating the amplitudes of these two functions is called the S matrix. In our illustration,

$$\psi(x, k) \approx [C(k)/2i][e^{ikx-i\eta(k)} - e^{-ikx+i\eta(k)}], \tag{0.5}$$

i.e., the first term corresponds to an outgoing wave, the second to an incoming wave, and their amplitudes are related by the factor

$$S(k) = e^{-2i\eta(k)}, \tag{0.6}$$

the S matrix for our example.

Heisenberg's theory of the S matrix was further developed in two papers by Møller,[2] who gave a time-dependent definition of the S matrix that is physically more justified. Since then, this time-dependent formulation of the scattering problem has received a great deal of attention (see the Appendix) and can be stated in the following way: The energy operator of a system consists of two terms

$$L = L_0 + V, \tag{0.7}$$

where L_0 corresponds to the energy of the free particles and V to their interaction energy. Long before collision, i.e., for negatively infinite time, the state of the noninteracting particles is described by a vector $\Omega_-(t)$[3] whose dependence on time is determined by the operator L_0:

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[1] W. Heisenberg, Z. Physik **120**, 513–538, 673–702 (1943).

[2] C. Møller, Kgl. Danske Videnskab Selskab, Mat.-fys. Medd. **23**, No. 1 (1945), **22**, No. 19 (1946).

[3] In conformity with established terminology, a state *vector* will be understood to be an element of Hilbert space in which all operators act.

$$\Omega_-(t) = e^{-iL_0t}\Omega_-, \tag{0.8}$$

Ω_- being a constant vector characterizing the initial state of the system. For finite time, the state $\Omega(t)$ is a solution of the Schrödinger equation

$$i\,\partial\Omega(t)/\partial t = L\Omega(t) = (L_0 + V)\Omega(t), \tag{0.9}$$

and is required to take on the initial state $\Omega_-(t)$ in the sense

$$\lim_{t\to-\infty} ||\Omega(t) - \Omega_-(t)|| = 0. \tag{0.10}$$

Over a long interval of time after collision, the motion of the particles again becomes free, so that asymptotically,

$$||\Omega(t) - \Omega_+(t)|| \to 0, \qquad (t \to \infty), \tag{0.11}$$

where $\Omega_+(t) = e^{-iL_0t}\Omega_+$. The manner in which the asymptotic state vector changes, determines the nature of the scattering process. The operator S that relates the asymptotic vectors Ω_+ and Ω_- according to the formula

$$\Omega_+ = S\Omega_-, \tag{0.12}$$

is called the *scattering operator* or S matrix.

In Sec. 3, it will be shown that this formulation holds for the example in question, the S matrix being given by the function $S(k) = e^{-2i\eta(k)}$ occurring in the time-independent definition. This fact typifies a general aspect of the stationary and nonstationary formulations of the S matrix in case both definitions are valid.

Heisenberg came to consider the S matrix as a means of overcoming the difficulties encountered in modern relativistic theory of elementary particles. He felt it was necessary to introduce a new fundamental constant having dimension length. Therefore, he analyzed the current theory and rejected as unobservable those notions which contradict the idea of a fundamental length. Only those experimentally observable quantities would be put in a future theory. In this sense, the S matrix satisfies the requirements of Heisenberg. It describes the wave function at large distances and is thus not contrary to the hypothesis of a fundamental length. Moreover, the scattering cross section, which can be measured directly is expressible in terms of the elements of the S matrix. Heisenberg also conjectured that the discrete energy levels corresponding to bound states of the particles should be determined by the analytic continuation of the S matrix into the complex energy plane.

Connected with Heisenberg's supposition that the S matrix is more fundamental than the Hamiltonian is the question of clarifying the relationship between these two characterizations of a system. In particular, in what sense should one define the Hamiltonian on the basis of the S matrix when both of these notions are used in a theory. In addition to its theoretical aspect, the inverse problem, i.e., the reconstruction of the energy operator from its S matrix, could be of great practical value in the interpretation of experimental scattering data and in the determination of various properties of the particles which are not directly measurable.

The simplest example in scattering theory is the radial equation for the scattering by a fixed, spherically symmetric center:

$$-(d^2/dx^2)\psi(x,k) = [l(l+1)/x^2 + q(x)] \times \psi(x,k) = k^2\psi(x,k), \tag{0.13}$$

which, for $l = 0$, reduces to the case already mentioned. The first attempts at solving the inverse problem were undertaken by Frødberg[4] and Hylleraas.[5] They worked out a formal procedure using a series whose convergence is highly plausible. However, Bargmann[6] constructed explicit examples in which different potentials give rise to the same $S(k)$ and to the same discrete energy levels. This showed that a potential cannot be reconstructed uniquely from prescribed energy levels and scattering function $S(k)$. Levinson[7] showed that this lack of uniqueness is related to the existence of a discrete spectrum. To wit, he proved that the reconstructed potential is unique when there is no discrete spectrum. The precise mathematical reason for this was given by Marchenko[8] who showed that the S function[9] determines the continuous portion of the so-called spectral function of equation (0.1). To find the spectral function when there is a discrete spectrum, one must not only prescribe the location of the eigenvalues, but also the values of the derivatives of the corresponding normalized eigenfunctions, for example, at $x = 0$. Marchenko[10] also showed that the spectral function uniquely determines a potential. Thus, Marchenko related the problem in question to the inverse Sturm-Liouville problem which already had been treated in the mathematical

[4] C. E. Frødberg, Phys. Rev. **72**, 519 (1947).
[5] E. A. Hylleraas, Phys. Rev. **74**, 48 (1948).
[6] V. Bargmann, Phys. Rev. **75**, 301 (1949).
[7] N. Levinson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **25**, No. 9 (1949).
[8] V. A. Marchenko, Trudy Moskov. matem. o-va **1**, 327–420 (1952).
[9] We shall call the scattering operator in our example the S function.
[10] V. A. Marchenko, Doklady Akad. Nauk S. S. S. R. 72, 457 (1950).

literature. An analogous result was obtained at approximately the same time by Borg.[11] By developing Levinson's method, Jost and Kohn[12,13] independently came to the same conclusion concerning the reason for this lack of uniqueness. They gave an explicit formula for a family of potentials which [besides $q(x)$] yield the same S function and the same discrete energy levels. An analogous formula was obtained by Holmberg.[14]

A procedure for explicitly constructing a potential without the singularity $l(l+1)/x^2$ from its spectral function was formulated by Gel'fand and Levitan.[15] They reduced the problem to a linear integral equation and gave sufficient conditions in terms of the spectral function assuring that it be the spectral function of some equation with a potential from a given class. The results of Gel'fand and Levitan on the inverse Sturm-Liouville problem were immediately applied to the inverse scattering problem by Jost and Kohn[16] and by Levinson.[17] In reference 16, a formula was given for a family of equivalent potentials each having the same S function and discrete energy levels. More precise conditions (both necessary and sufficient) on the spectral function were obtained by Krein.[18] His paper completed the general problem of reconstructing Eq. (0.1) from its spectral function. However, since the passage from the S function to the spectral function is not entirely trivial, there still remained unanswered the question of characterizing the class of possible S functions corresponding to the potentials from a given class. This problem was solved by Krein[19] and Marchenko,[20] who showed that it is convenient to formulate conditions in terms of the Fourier transform of the function $S(k) - 1$. Marchenko showed that the potential $q(x)$ possesses the same properties for x tending to zero and infinity as does the derivative of this Fourier transform. Definitive inequalities obtained by Marchenko permitted him to formulate necessary and sufficient conditions on the S function, assuring that a potential from a given class would correspond to it.

After the basic papers of Gel'fand and Levitan, and Krein, and Marchenko, a great deal of work was devoted to extending their results to an equation containing the singular term $l(l+1)/x^2$, an equation over the interval $-\infty < x < \infty$, a system of equations, and the relativistic equations. A brief survey is given in the Appendix.

It is interesting to note that the inverse problem has been studied in the U.S.S.R. almost exclusively by mathematicians and elsewhere almost exclusively by physicists, who merely use the method of Gel'fand-Levitan as interpreted by Levinson, and by Jost and Kohn. An explanation of the general features of this method which permit its application to the solution of various problems, was undertaken in a series of papers by Kay and Moses.[21–24] These authors use the general concept of transformation operator developed by Friedrichs.[25,26]

Recently, work has been devoted to applying the results of the inverse problem in the interpretation of experimental scattering data.[27–29]

Thus, the inverse scattering problem for the simplest case of the radial equation has been solved in about a decade, and a large amount of literature is devoted to it. In this survey, we shall attempt to give the results of most of these papers and also in their most general form. This will make clearer the ways in which the basic results can be carried over to other problems. In this, an essential part will be played by the general approach to transformation operators developed by Friedrichs and applied to the inverse problem by Kay and Moses.

All of the basic results on the inverse scattering problem could be obtained by the use of one of the the methods of Gel'fand-Levitan, Marchenko, or Krein. Our presentation will not stick to any particular one of these approaches, but rather, at different points, will make use of different methods. We shall attempt to establish their connection considering that each of them explains different aspects of the mathematical structure of the whole problem.

Because of the large material content, not every

[11] G. Borg, "Uniqueness theorems in the spectral theory of $y'' + (\lambda - q(x))\, y = 0$," Eleventh Congress of Scandanavian Mathematics, held at Trondheim, August 22–25, 1949, pp. 276–287.
[12] R. Jost and W. Kohn, Phys. Rev. **87**, 979 (1952).
[13] R. Jost and W. Kohn, Phys. Rev. **88**, 382 (1952).
[14] B. Holmberg, Nuovo cimento **9**, 597 (1952).
[15] I. M. Gel'fand and B. M. Levitan, Izvest. Akad Nauk S. S. S. R. Ser. matem. **15**, 309–360 (1951).
[16] R. Jost and W. Kohn, Kgl Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 9 (1953).
[17] N. Levinson, Phys. Rev. **89**, 755–757 (1953).
[18] M. G. Krein, Doklady Akad. Nauk S. S. S. R. **88**, 405 (1953).
[19] M. G. Krein, Doklady Akad. Nauk S.S.S.R. **105**, 433 (1955).
[20] V. A. Marchenko, Doklady Akad. Nauk S.S.S.R. **104**, 433 (1955).
[21] I. Kay and H. E. Moses, Nuovo cimento **2**, 917 (1955).
[22] I. Kay and H. E. Moses, Nuovo cimento **3**, 67 (1956).
[23] I. Kay and H. E. Moses, Nuovo cimento **3**, 277 (1956).
[24] I. Kay and H. E. Moses, Nuovo cimento Suppl. **5**, 230 (1957).
[25] K. O. Friedrichs, Math. Ann. **115**, 249 (1938).
[26] K. O. Friedrichs, Communs. Pure and Appl. Math. **1**, 361 (1948).
[27] R. G. Newton, Phys. Rev. **105**, 763 (1957).
[28] R. G. Newton, Phys. Rev. **107**, 1025 (1957).
[29] T. Fulton and R. G. Newton, Phys. Rev. **107**, 1102 (1957).

proof will be carried out in a completely rigorous fashion. Many of our considerations will be of a heuristic nature whenever the justification of details requires greater means than in other more standard proofs. We shall nevertheless use these heuristic proofs to avoid obscuring the conceptual side of the work with lengthy mathematical discussions. We are confident that the physicist will find our reasoning completely convincing and that the mathematician will be able to reconstruct the deficient proofs so as to make them completely rigorous. On the other hand, we have tried to state theorems in their most precise form.

Let us give a brief outline of the basic ideas and plan of the survey. The first 13 sections are devoted to the solution of the inverse-scattering problem for the operator L defined by Eq. (0.13) for $l = 0$ and the condition $\psi(0) = 0$. We consider L to be a perturbation of the operator L_0 defined by the differential expression $L_0\psi = -d^2\psi(x)/dx^2$ and the same condition $\psi(0) = 0$. According to Friedrichs, a transformation operator U is defined as the solution of the operator equation

$$LU = UL_0, \tag{0.14}$$

so that any transformation operator which has an inverse generates a similarity transformation of the the perturbed operator into the unperturbed:

$$U^{-1}LU = L_0. \tag{0.15}$$

The transformation operator U replaces the eigenfunctions of the continuous spectrum of L in all considerations. Roughly speaking, its kernel is obtained by expanding the eigenfunctions of the continuous spectrum of L in terms of the eigenfunctions of L_0.

In Secs. 4 and 5, it is shown that such transformation operators exist for our example and that the completeness theorem for the eigenfunctions of L, proved in Sec. 2, can be written in terms of a transformation operator in the form

$$UWU^* = I, \tag{0.16}$$

(for simplicity, we have restricted ourselves here to the case where L has no discrete spectrum; in the text, this restriction is not imposed). Here, W is a positive-definite self-adjoint operator commuting with L_0. W determines the "normalization" of the corresponding operator U.

A characteristic feature of our example is that among the transformation operators there exist Volterra operators of the form[30]

[30] The subscript B used throughout and which transliterates into English as V stands, of course, for Volterra.

$$U_Bf(x) = (I + K)f(x) = f(x) + \int_0^x K(x, y)f(y)\,dy. \tag{0.17}$$

The operator W corresponding to U_B is constructed using

$$W(k) = 1/[M(k)M(-k)], \tag{0.18}$$

where $M(k)$ is a certain function introduced in Sec. 1. One might call $M(\lambda^{1/2})$ the determinant of the operator $L - \lambda I$. In fact, in Sec. 2 it is shown that this function appears in the denominator of the resolvent kernel of L and determines its singularities. These consist of a branch cut corresponding to the continuous spectrum and poles at the points of the discrete spectrum.

In Sec. 3, it is shown that the time-dependent formulation of the scattering problem is valid for our example provided that L_0 is taken to be the energy operator of the free particles and the corresponding scattering operator is defined by

$$S(k) = M(-k)/M(k). \tag{0.19}$$

In Sec. 6, it is shown how to establish the relationship between $W(k)$ and $S(k)$ with the help of (0.18) and (0.19).

In Sec. 8, on the basis of the triangularity of the kernel $K(x, y)$, a linear integral equation is obtained from (0.16) connecting the kernels of the operators W and K. In Sec. 9, this equation is studied and the inverse problem is solved for the case in which L has no discrete spectrum. Supplementary facts necessary for a treatment of the general case are cited in Sec. 12.

The approach described corresponds to the Gel'fand-Levitan method. Another procedure, related to Marchenko's method, is based on the application of the operator $V_B = I + A$ introduced in Secs. 4 and 7:

$$V_Bf(x) = f(x) + \int_x^\infty A(x, y)f(y)\,dy. \tag{0.20}$$

This operator is not a transformation operator in the general sense. However, its relation to the transformation operator $\tilde{U}_B = U_BW = (U_B^*)^{-1}$ is established in Sec. 7. This relation and (0.16) are then used to show that V_B satisfies the identity

$$V_B(I - F)V_B^* = I, \tag{0.21}$$

where the operator F can be constructed directly in terms of the function $S(k)$. By means of (0.21), a linear integral equation is deduced which relates the kernel $A(x, y)$ to the function $S(k)$, thus permitting one to solve the inverse problem. This integral equation is used in Sec. 10 to investigate the connection between $S(k)$ and $q(x)$. Several aspects of Krein's method are illustrated in Sec. 11. In Sec. 13, the construction of an operator L from a known operator L_1, when the S function of L differs from that of L_1 by a rational factor, is considered. This is important in applications. In Secs. 14 and 15, the results deduced are extended to the radial equation (0.13) where $l > 0$.

In order not to interrupt the presentation, we shall not mention original papers in the text. The literature is cited in a special Appendix. A number of comments are made there and a brief review is given of work done on the inverse scattering problem that has not been included in the text.

1. THE SOLUTIONS $\varphi(x, s)$, $f(x, s)$ AND THEIR RELATIONSHIP; EXISTENCE AND INEQUALITIES. THE FUNCTION $M(s)$ AND ITS PROPERTIES

In this section, some basic properties of solutions of the equation

$$-y'' + q(x)y = s^2 y, \qquad s = \sigma + i\tau \tag{1.1}$$

are assembled which will be utilized in the subsequent presentation. In all lemmas, it is assumed without further mention that $q(x)$ is a locally summable function and satisfies the condition

$$\int_0^\infty x\,|q(x)|\,dx = C < \infty. \tag{1.2}$$

The solutions $\varphi(x, s)$ and $f(x, s)$ are determined by the conditions:

$$\varphi(x, s)\colon \varphi(0, s) = 0, \qquad \varphi'(0, s) = 1, \tag{1.3}$$

$$f(x, s)\colon \lim_{x\to\infty} e^{-isx} f(x, s) = 1. \tag{1.4}$$

Equation (1.1) and the conditions (1.3) and (1.4) are equivalent to the following integral equations:

$$\varphi(x, s) = \frac{\sin sx}{s} + \int_0^x \frac{\sin s(x-t)}{s}\, q(t)\varphi(t, s)\,dt, \tag{1.5}$$

$$f(x, s) = e^{isx} + \int_x^\infty \frac{\sin s(t-x)}{s}\, q(t) f(t, s)\,dt, \tag{1.6}$$

which can be obtained by the method of variation of parameters. With the help of these equations the following lemmas are proven:

Lemma 1.1. *For each $x \geq 0$, $\varphi(x, s)$ is an entire function of s for which the estimate*[31]

$$|\varphi(x, s)| \leq K x e^{|\tau| x}/(1 + |s|\,x) \tag{1.7}$$

holds. Moreover, $\varphi(x, s)$ is an even function of s for real s.

Lemma 1.2. *For each $x \geq 0$, $f(x, s)$ is analytic in s in the half-plane $\tau > 0$ and continuous down to the real axis. Moreover, the inequality*

$$|f(x, s)| \leq K e^{-\tau x}, \qquad \tau \geq 0 \tag{1.8}$$

holds.

Lemma 1.3. *$f(x, s)$ satisfies the following inequalities:*

$$|f(x, s) - e^{isx}| \leq K \frac{e^{-\tau x}}{|s|} \int_x^\infty |q(t)|\,dt, \qquad \tau \geq 0, \tag{1.9}$$

$$|f(x, s) - e^{isx}| \leq K e^{-\tau x} \int_x^\infty t\,|q(t)|\,dt, \qquad \tau \geq 0, \tag{1.10}$$

$$|f'(x, s) - ise^{isx}| \leq K e^{-\tau x} \int_x^\infty |q(t)|\,dt, \qquad \tau \geq 0. \tag{1.11}$$

The estimate (1.9) is suitable for $|s| \to \infty$ and may be applied when $x \neq 0$. The estimates (1.10) and (1.11) are suitable for $x \to \infty$. In addition, (1.11) implies that

$$\lim_{x\to 0} x f'(x, s) = 0. \tag{1.12}$$

In fact, as $x \to 0$,

$$x \int_x^\infty |q(t)|\,dt \leq \int_x^{x^{1/2}} t\,|q(t)|\,dt + x^{1/2} \int_{x^{1/2}}^\infty t\,|q(t)|\,dt \to 0.$$

Lemma 1.4. *For any x, the function $f(x, s)$ is continuously differentiable with respect to s down to the line $\tau = 0$ with the possible exception of the point $s = 0$. The estimate*[32]

$$|\dot{f}(x, s) - ixe^{isx}| \leq \frac{K}{|s|} e^{-\tau x}, \qquad \tau \geq 0, \tag{1.13}$$

holds uniformly in x.

Lemma 1.5. *For large $|s|$*

$$\varphi(x, s) = \sin sx/s + o(e^{|\tau| x}/|s|), \tag{1.14}$$

$$f(x, s) = e^{isx} + o(e^{-\tau x}), \qquad \tau \geq 0, \tag{1.15}$$

uniformly for all $x \geq 0$.

When s is real, it is not difficult to establish a

[31] Absolute constants depending only on C (which may be different) will be denoted by K.

[32] The dot denotes differentiation with respect to s; a bar will denote complex conjugate.

relationship between $\varphi(x, s)$ and $f(x, s)$. Without further mention, we shall write k for s whenever s is real. The solutions $f(x, k)$ and $f(x, -k) = \bar{f}(x, k)$ for $k \neq 0$ are linearly independent solutions of Eq. (1.1). In fact, their Wronskian does not vanish:

$$[f(x, k); f(x, -k)] = f'(x, k)f(x, -k) - f(x, k)f'(x, -k) = 2ik. \tag{1.16}$$

In consequence of the realness of $\varphi(x, k)$

$$\varphi(x, k) = (1/2ik)[f(x, k)\overline{M}(k) - f(x, -k)M(k)], \tag{1.17}$$

where $M(k)$ may be found by the use of the Wronskian [see (1.12)]:

$$M(k) = [\varphi(x, k); f(x, k)] = \lim_{x\to 0} [\varphi(x, k); f(x, k)] = f(0, k). \tag{1.18}$$

From this and Lemma 1.2, we conclude that $M(k)$ is the limit of the function $M(s) = f(0, s)$, analytic in the upper half-plane, and is such that $M(k) = \bar{M}(-k)$. Let us introduce the notations

$$A(k) = |M(k)|, \qquad \eta(k) = \arg M(k), \tag{1.19}$$

so that

$$A(k) = A(-k), \qquad \eta(k) = -\eta(-k). \tag{1.20}$$

By Lemma 1.3 and (1.17), we infer that for large x

$$\varphi(x, k) = [A(k)/k] \sin [kx - \eta(k)] + o(1), \tag{1.21}$$

$$\varphi'(x, k) = A(k) \cos [kx - \eta(k)] + o(1). \tag{1.22}$$

It is therefore natural to call $A(k)$ the asymptotic *amplitude* and $\eta(k)$ the asymptotic *phase*.

Let $\tau > 0$. From Eq. (1.1) for $f(x, s)$ and the equation

$$-\dot{f}''(x, s) + q(x)\dot{f}(x, s) = 2sf(x, s) + s^2\dot{f}(x, s) \tag{1.23}$$

for $\dot{f}(x, s) = df(x, s)/ds$, it is not difficult to obtain the following identities:

$$f'(0, s)\bar{f}(0, s) - f(0, s)\bar{f}'(0, s) = 4i\sigma\tau \int_0^\infty |f(t, s)|^2 \, dt, \tag{1.24}$$

$$\dot{f}'(0, s)f(0, s) - \dot{f}(0, s)f'(0, s) = 2s \int_0^\infty f^2(t, s) \, dt. \tag{1.25}$$

From the first one, we conclude that $M(s)$ can vanish only for $\sigma = 0$ or $\tau = 0$. The second possibility, however, is excluded by the fact that if $M(k) = 0$ on the real axis, (1.17) would imply that $\varphi(x, k) \equiv 0$, and this is impossible.

In the following, it will be assumed that $M(0) \neq 0$. The vanishing of $M(s)$ for $s = 0$ is equivalent to the solution of $-y'' + q(x)y = 0$, $y(0) = 0$ being bounded as $x \to \infty$ and this happens only in exceptional situations. A treatment of the case $M(0) = 0$ presents no essential difficulties but only encumbers the formulation and proof of theorems.

There still remains the possibility that $M(s) = 0$ for $\sigma = 0$ and $\tau > 0$. From the estimate

$$M(s) = 1 + o(1) \tag{1.26}$$

for large $|s|$, which follows from formula (1.15), we conclude that $M(s)$ can only have a finite number of zeros $s_n = i\kappa_n$ $(n = 1, \cdots, m)$ on the imaginary axis. When $s = s_n$, the solutions $\varphi(x, s_n)$ and $f(x, s_n)$ satisfy the same boundary condition at $x = 0$, and are therefore proportional:

$$f(x, s_n) = f'(0, s_n)\varphi(x, s_n). \tag{1.27}$$

From this and (1.25), it follows that

$$\int_0^\infty [\varphi(x, s_n)]^2 \, dx = -\frac{\dot{M}(s_n)}{2s_n f'(0, s_n)}, \tag{1.28}$$

and this implies, in particular, that $M(s)$ has only simple zeros. The above results can be formulated as follows.

Lemma 1.6. *The function $M(s)$ is analytic in the upper half-plane and has there a finite number of simple zeros $s_n = i\kappa_n$, $\kappa_n > 0$, $(n = 1, \cdots, m)$. For large $|s|$, the estimate* (1.26) *holds. The function $\dot{M}(s)$ is continuous down to the real axis with the possible exception of the point $s = 0$. Furthermore, $s\dot{M}(s)$ is continuous everywhere in the half-plane $\operatorname{Im} s \geq 0$.*

The two last assertions follow from Lemma 1.4.

2. EXPANSION THEOREM

The differential equation (1.1) together with the boundary condition, defines a self-adjoint operator in $\mathfrak{L}_2(0, \infty)$. This operator can be obtained by the extension of the symmetric operator, defined by (1.1), acting on the twice-continuously differentiable functions satisfying the boundary condition and vanishing identically outside some finite interval. We shall denote this operator by L.

Consider the kernel

$$\left.\begin{aligned} R_\lambda(x, y) &= \varphi(x, \lambda^{1/2})f(y, \lambda^{1/2})/M(\lambda^{1/2}), \quad x < y, \\ R_\lambda(x, y) &= R_\lambda(y, x), \\ &0 \leq \arg \lambda^{1/2} \leq \pi, \end{aligned}\right\} \tag{2.1}$$

which is defined for all complex λ with the exception of a finite number of points on the negative real axis corresponding to the zeros of $M(\lambda^{1/2})$. By virtue of (1.18), it is not difficult to verify that the kernel $R_\lambda(x, y)$ is a solution of the equation

$$(-d^2/dx^2 + q(x))R_\lambda(x, y) - \lambda R_\lambda(x, y) = \delta(x - y), \tag{2.2}$$

and satisfies the boundary conditions

$$R_\lambda(0, y) = R_\lambda(x, 0) = 0. \tag{2.3}$$

In consequence of (1.7) and (1.8), we have

$$|R_\lambda(x, y)| \leq K \frac{x}{1 + |\lambda^{1/2}|\, x} e^{-\tau|x-y|}, \qquad \tau = \operatorname{Im} \lambda^{1/2} > 0 \tag{2.4}$$

for complex λ and, hence, the kernel $R_\lambda(x, y)$ defines a bounded operator in $\mathfrak{L}_2(0, \infty)$, namely, the resolvent operator

$$R_\lambda = (L - \lambda I)^{-1}. \tag{2.5}$$

The singularities of R_λ in the complex λ plane consist of a cut along the positive real axis and a finite number of simple poles $\lambda_n = -\kappa_n^2$ $(n = 1, \cdots, m)$ on the negative real axis. The continuous and discrete portions of the spectrum correspond to the cut and poles, respectively. The jump in the resolvent across the cut and the residues at the poles determine the spectral function of the operator L. We come now to the following completeness theorem for the eigenfunctions of the operator L.

Theorem 2.1. *The functions* $\varphi(s, k)$ $(k \geq 0)$ *and* $\varphi_n(x) = \varphi(x, i\kappa_n)$ *form a complete orthogonal system. The completeness relationship is given by*

$$\sum_{n=1}^{m} C_n\varphi_n(x)\varphi_n(y) + \frac{2}{\pi}\int_0^\infty \varphi(x, k) \frac{1}{M(k)M(-k)} \times \varphi(y, k)k^2\, dk = \delta(x - y), \tag{2.6}$$

in which $C_n = 2i\kappa_n f'(0, i\kappa_n)/\dot{M}(i\kappa_n)$ [*see* (1.28)].

Formula (2.6) can be deduced without recourse to general operator theory. Let $f(x)$ be a twice-continuously differentiable function vanishing for large x and in the neighborhood of $x = 0$. Then

$$g(x) = -f''(x) + q(x)f(x) \tag{2.7}$$

is continuous and vanishes identically outside some finite interval not containing the origin. From (2.2) and (2.7), it follows that

$$\int_0^\infty R_\lambda(x, y)f(y)\, dy = -\frac{1}{\lambda} f(x) + \frac{1}{\lambda}\int_0^\infty R_\lambda(x, y)g(y)\, dy. \tag{2.8}$$

If we integrate both sides of (2.8) around a large circle $|\lambda| = N$, then the contribution from the second term on the right-hand side of (2.8) will approach zero as $N \to \infty$ by virtue of (1.14), (1.15), and (1.26). Thus we have

$$\lim_{N\to\infty} \frac{1}{2\pi i} \oint_{\lambda=|N|} \left[\int_0^\infty R_\lambda(x, y)f(y)\, dy\right] d\lambda = -f(x). \tag{2.9}$$

On the other hand, if we integrate the left-hand side of (2.8) along a path γ consisting of a curve encompassing the cut on the real axis and a large circle $|\lambda| = N$, we obtain

$$\begin{aligned}\frac{1}{2\pi i}\oint_\gamma \left\{\int_0^\infty R_\lambda(x, y)f(y)\, dy\right\} d\lambda &= \frac{1}{2\pi i}\oint_{|\lambda|=N} \left\{\int_0^\infty R_\lambda(x, y)f(y)\, dy\right\} d\lambda \\ &\quad + \frac{1}{2\pi i}\int_0^\infty d\sigma\left\{\int_0^\infty [R_{\sigma+i0}(x, y) - R_{\sigma-i0}(x, y)]f(y)\, dy\right\} \\ &= \sum_{n=1}^{m} \operatorname{res}\left\{\int_0^\infty R_\lambda(x, y)f(y)\, dy\right\}\Big|_{\lambda=\lambda_n}.\end{aligned}$$

Taking into consideration (2.1) and (1.17) and letting $N \to \infty$, we find, on the basis of (2.9), that

$$\begin{aligned} f(x) &= \frac{2}{\pi}\int_0^\infty k^2\, dk \times \left\{\int_0^\infty \varphi(x, k) \frac{1}{M(k)M(-k)} \varphi(y, k)f(y)\, dy\right\} \\ &\quad + \sum_{n=1}^{m}\int_0^\infty C_n\varphi_n(x)\varphi_n(y)f(y)\, dy.\end{aligned}$$

Finally, by virtue of the fact that the functions $f(x)$ are dense in $\mathfrak{L}_2(0, \infty)$, formula (2.6) is obtained.

The functions $\psi^{(+)}(x, k) = \varphi(x, k)/M(k)$ and $\psi_n(x) = C_n^{1/2}\, \varphi_n(x)$ form an orthonormal system. However, the functions $\psi^{(+)}(x, k)$ are not square integrable, and hence, are not elements of Hilbert space nor eigenfunctions in the usual sense. To attach meaning to them while still remaining in the framework of Hilbert space, we may consider them to be kernels of transformations which diagonalize the operator L. Thus, the transformation

$$T^{(+)}g = G: \quad G(k) = \int_0^\infty g(x)\psi^{(+)}(x, k)\, dx \tag{2.10}$$

carries any function $g(x)$ in $\mathfrak{L}_2(0, \infty)$ into a function $G(k)$ for which $\int_0^\infty |G(k)|^2\, k^2\, dk < \infty$. Moreover, if $Lg(x)$ belongs to $\mathfrak{L}_2(0, \infty)$, then $Lg(x)$ goes into $k^2G(k)$, and the integral $\int_0^\infty |G(k)k^2|\, k^2\, dk$ exists. In the following, the space of functions $G(k)$ with the scalar product

$$(G, G_1) = \frac{2}{\pi} \int_0^\infty \bar{G}(k) G_1(k) k^2 \, dk \qquad (2.11)$$

will be denoted by $\mathfrak{L}_k$, and the space of square integrable functions $g(x)$, which was previously called $\mathfrak{L}_2(0, \infty)$, will be denoted by $\mathfrak{L}_x$. The transformation $T^{(+)}$ acts from $\mathfrak{L}_x$ into $\mathfrak{L}_k$. The adjoint transformation $T^{(+)*}$ acts from $\mathfrak{L}_k$ into $\mathfrak{L}_x$ according to the formula

$$g = T^{(+)*}G: g(x) = \frac{2}{\pi} \int_0^\infty G(k) \bar{\psi}^{(+)}(x, k) k^2 \, dk. \quad (2.12)$$

The orthogonality of the eigenfunctions $\psi^{(+)}(x, k)$ can thus be expressed in these new terms as follows:

$$T^{(+)}T^{(+)*} = I^k. \qquad (2.13)$$

Here, I^k denotes the identity operator in L_k. The formula (2.6) can now be written in the following form:

$$T^{(+)*}T^{(+)} = I_c^x. \qquad (2.14)$$

Here, I_c^x is a projection onto the proper subspace of the operator L corresponding to its continuous spectrum. The superscript x indicates that this operator acts on $\mathfrak{L}_x$. A transformation such as $T^{(+)}$ can be associated with any solution of Eq. (1.1) satisfying a zero boundary condition. Let $\chi(x, k)$ be a solution of (1.1) such that $\chi(0, k) = 0$ and $\chi'(0, k) \neq 0$ for all k. Consider the transformation

$$Tg = G: \quad G(k) = \int_0^\infty g(x) \chi(x, k) \, dx. \qquad (2.15)$$

The transformations T and $T^{(+)}$ are related by the formula

$$T = N^k T^{(+)}, \qquad (2.16)$$

where N^k is a 'normalizing factor'; N^k is an operator in $\mathfrak{L}_k$, which multiplies by the function $N(k) = \chi'(0, k)/\psi^{(+)\prime}(0, k)$. The completeness relation and orthogonality condition can be written in terms of T as follows:

$$T^* W^k T = I_c^x, \qquad TT^*W^k = I^k, \qquad (2.17)$$

where

$$W^k = (N^{k*})^{-1}(N^k)^{-1}. \qquad (2.18)$$

The somewhat formal reasoning used at the end of this section will turn out to be useful in Sec. 5.

3. ASYMPTOTIC BEHAVIOR OF THE SOLUTION OF THE SHRÖDINGER EQUATION FOR LARGE TIME

The expansion theorem for the eigenfunctions of the operator L deduced in the preceding section enables us to use Fourier methods to solve time-dependent equations involving the operator L. We shall be particularly interested in the behavior of solutions of the Schrödinger equation

$$i \, \partial f(x, t)/\partial t = Lf(x, t), \quad f(x, t) \,|_{t=0} = f_0(x) \qquad (3.1)$$

for large $|t|$.

By expanding $f_0(x)$ in terms of the eigenfunctions of L, we can represent the solution of (3.1) in the following way:

$$f(x, t) = \frac{2}{\pi} \int_0^\infty F(k) \bar{\psi}(x, k) e^{-ik^2 t} k^2 \, dk + \sum_{n=1}^{m} F_n \bar{\psi}_n(x) e^{i\kappa_n^2 t}. \qquad (3.2)$$

Here, $\psi(x, k)$ and $\psi_n(x)$ comprise any orthonormal system of eigenfunctions of L [$\psi(x, k)$ and $\psi_n(x)$ may differ from the functions $\psi^{(+)}(x, k)$ and $\psi_n(x)$ discussed in Sec. 2 by a factor of modulus 1] and

$$F(k) = \int_0^\infty f_0(x) \psi(x, k) \, dx, \; F_n = \int_0^\infty f_0(x) \psi_n(x) \, dx. \qquad (3.3)$$

In particular, one may take as the set $\psi(x, k)$ the functions $\psi^{(+)}(x, k)$ or

$$\psi^{(-)}(x, k) = \psi^{(+)}(x, k) M(k)/M(-k) = \bar{\psi}^{(+)}(x, k).$$

The behavior of $f(x, t)$ for large $|t|$ can be analyzed on the basis of the following lemma:

Lemma 3.1. *Let $F(k)$ be an arbitrary function of $\mathfrak{L}_k$, i.e.,*

$$\int_0^\infty |F(k)|^2 \, k^2 \, dk < \infty, \qquad (3.4)$$

and set

$$f^{(\pm)}(x, t) = \int_0^\infty F(k) \bar{\psi}^{(\pm)}(x, k) e^{-ik^2 t} k^2 \, dk, \qquad (3.5)$$

$$g(x, t) = \int_0^\infty F(k) \frac{\sin kx}{k} e^{-ik^2 t} k^2 \, dk. \qquad (3.6)$$

Then

$$\lim_{t \to \pm\infty} \int_0^\infty |f^{(\pm)}(x, t) - g(x, t)|^2 \, dx = 0. \qquad (3.7)$$

It is sufficient to prove the statement for a set of functions $F(k)$ which is dense in $\mathfrak{L}_k$; the theorem then follows for any function by completion. We assume that $F(k)$ is differentiable and nonvanishing only in the interval $0 < \alpha \leq k \leq \beta < \infty$. The intervals of integration in (3.5) and (3.6) are then finite and do not contain the point $k = 0$. For definiteness, we shall suppose that $t \to \infty$. By virtue of the fact that $\psi^{(+)}(x, k)$ and $(\sin kx)/k$ are uniformly bounded for x in $0 < x < \infty$ and k in $[\alpha, \beta]$, the functions $f^{(+)}(x, t)$ and $g(x, t)$ tend to zero as $t \to \infty$ uniformly in x. Thus the integral in (3.7) vanishes as $t \to \infty$ for any finite interval of integration. We still have to show that $\int_A^\infty |f^{(+)} - g|^2\, dx$ converges for arbitrary A.

From (1.17) and the definition of $\psi^{(+)}(x, k)$,

$$\psi^{(+)}(x, k) - \frac{\sin kx}{k} = -\frac{1}{2ik} e^{-ikx}\left[\frac{M(k)}{M(-k)} - 1\right] + R(x, k), \qquad (3.8)$$

where for $x > 0$

$$|R(x, k)| \leq K \frac{1}{k^2} \int_x^\infty |q(t)|\, dt \qquad (3.9)$$

by virtue of (1.9) and (1.8). In consequence of this estimate,

$$\int_A^\infty \left| \int_\alpha^\infty F(k)R(x, k)e^{-ik^2t}k^2\, dk\right|^2 dx$$
$$\leq K \int_A^\infty dx\left(\int_x^\infty |q(t)|\, dt\right)^2 \int_\alpha^\infty |F(k)|^2\, k^2\, dk \int_\alpha^\infty \frac{dk}{k^2}$$
$$\leq K' \int_A^\infty |q(t)|\, dt \int_A^\infty t\, |q(t)|\, dt, \qquad (3.10)$$

and for sufficiently large A the integral containing $R(x, k)$ can be made as small as desired uniformly in t. We must still look at the behavior of the integral of the basic term of (3.8)

$$Q_A(t) = \int_A^\infty dx \left|\int_\alpha^\infty G(k)e^{-ikx}e^{-ik^2t}\, dk\right|^2, \qquad (3.11)$$

as $t \to \infty$. The function

$$G(k) = \frac{F(k)}{2ik}\left[\frac{M(k)}{M(-k)} - 1\right]k^2$$

is finite and continuously differentiable by Lemma 1.6. Now $Q_A(t) = \lim_{B\to\infty} Q_A^B(t)$ where

$$Q_A^B(t) = \int_A^B dx \left|\int_\alpha^\beta G(k)e^{-ikx}e^{-ik^2t}\, dk\right|^2$$
$$= \int_\alpha^\beta dk \int_\alpha^\beta dl\Big\{G(k)\overline{G}(l)e^{-i(k^2-l^2)t}$$
$$\times \frac{e^{-i(k-l)A} - e^{-i(k-l)B}}{i(k-l)}\Big\} = J_A(t) - J_B(t).$$

Here

$$J_B(t) = \int_\alpha^\beta dk \int_\alpha^\beta dl G(k)\overline{G}(l)\, \frac{e^{-i(k^2-l^2)t}e^{-i(k-l)B}}{i(k-l)}.$$

Because of the singularity in the denominator, the inner integral is understood to be a principal-valued integral, defined in consequence of the differentiability of $G(k)$.

We now transform $J_B(k)$ as follows:

$$J_B(t) = \int_\alpha^\beta dkG(k)$$
$$\times \left[\int_\alpha^\beta dl\, \frac{\overline{G}(l)e^{-i(k^2-l^2)t} - \overline{G}(k)}{i(k-l)}\, e^{-i(k-l)B} + \overline{G}(k) \int_\alpha^\beta \frac{e^{-i(k-l)B}}{i(k-l)}\, dl\right]. \qquad (3.12)$$

The first term in the integral with respect to l is continuous for $k = l$, and hence, vanishes as $B \to \infty$ by the Riemann-Lebesgue theorem. The second term in this integral tends to π as $B \to \infty$, and therefore, we find that $\lim_{B\to\infty} J_B(t)$ is independent of t and has the value

$$\pi \int_\alpha^\beta |G(k)|^2\, dk.$$

$J_A(t)$ can also be represented as the sum of two terms, one vanishing as $t \to \infty$ uniformly in A, and the other having a finite limit independent of A as $t \to \infty$ equal to $\pi \int_\alpha^\beta |G(k)|^2\, dk$. From this it follows that $Q_A(t) \to 0$ and the lemma is therefore proven.

Let us now return to the investigation of the behavior of the solution $f(x, t)$ of the Schrödinger equation. If $f_0(x)$ is orthogonal to the eigenfunctions of the discrete spectrum of L, then the sum in the second term of (3.2) does not appear. Let

$$F^{(\pm)}(k) = \int_0^\infty f_0(x)\psi^{(\pm)}(x, k)\, dx. \qquad (3.13)$$

These are the functions which occur in formula (3.2) for $f(x, t)$, $\psi^{(\pm)}(x, k)$ having been selected instead of $\psi(x, k)$. Clearly,

$$g^{(\pm)}(x, t) = \frac{2}{\pi}\int_0^\infty F^{(\pm)}(k)\, \frac{\sin kx}{k}\, e^{-ik^2t}k^2\, dk \qquad (3.14)$$

is a solution of the Schrödinger equation with an operator L_0 associated with the equation $L_0y = -y'' = k^2y$ and the boundary condition $y(0) = 0$, i.e.,

$$i\, \partial g^{(\pm)}(x, t)/\partial t = L_0 g^{(\pm)}(x, t), \qquad (3.15)$$

where

$$g^{(\pm)}(x, 0) = g_0^{(\pm)}(x) = \frac{2}{\pi}\int_0^\infty F^{(\pm)}(k)\, \frac{\sin kx}{k}\, k^2\, dk. \qquad (3.16)$$

If, in analogy with Sec. 2, we introduce a unitary transformation T_0 of $\mathfrak{L}_x$ into $\mathfrak{L}_k$ given by

$$T_0 g = G: \quad G(k) = \int_0^\infty g(x) \frac{\sin kx}{k} \, dx, \tag{3.17}$$

then $g_0^{(\pm)}(x)$ can be expressed in terms of $f_0(x)$ by the formula

$$g_0^{(\pm)}(x) = T_0^* T^{(\pm)} f_0(x). \tag{3.18}$$

From Lemma 3.1 follows Theorem 3.1.

Theorem 3.1. *If $f_0(x)$ is orthogonal to the eigenfunctions of the discrete spectrum of L, then when $t \to \pm\infty$ the solution of the Schrödinger equation* (3.1) *behaves like the solution of the Schrödinger equation* (3.15) *with initial data $g_0^{(\pm)}(x)$ given by* (3.18), *in the sense that*

$$\int_0^\infty |f(x, t) - g^{(\pm)}(x, t)|^2 \, dx \to 0 \text{ as } t \to \pm\infty. \tag{3.19}$$

4. TRANSFORMATION OPERATORS

In the following we shall need the representation of a solution of (1.1) with a potential $q(x)$ in terms of solutions of the equation with other potentials and, in particular, with $q(x) \equiv 0$, i.e., in terms of trigonometric functions.

The simplest such expression can be deduced in the following way. On the basis of (1.9) for $x \neq 0$, the function $h(x, s) = f(x, s) - e^{isx}$ is square integrable in s along any line parallel to the real axis, and in the upper half-plane

$$\int_{-\infty}^\infty |h(x, \sigma + i\tau)|^2 \, d\sigma = O(e^{-2\tau x}).$$

By a theorem of Titchmarsh,

$$A(x, y) = \frac{1}{2\pi} \int_{-\infty}^\infty [f(x, k) - e^{ikx}] e^{-iky} \, dk = 0, \qquad x > y. \tag{4.1}$$

The inversion of this Fourier transform thus yields

$$f(x, s) = e^{isx} + \int_x^\infty A(x, y) e^{isy} \, dy, \quad \tau \geq 0, \tag{4.2}$$

where $A(x, y)$ is square integrable in y for $x \neq 0$. With certain modifications, this procedure yields an expression for $\varphi(x, s)$ valid for all s:

$$\varphi(x, s) = \frac{\sin sx}{s} + \int_0^x K(x, y) \frac{\sin sy}{s} \, dy. \tag{4.3}$$

However, the derivation gives almost no information concerning the kernels $K(x, y)$ and $A(x, y)$. Equivalent equations for these kernels can be deduced by substituting expressions (4.2) and (4.3) for $f(x, s)$ and $\varphi(x, s)$ into (1.6) and (1.5), respectively, and by eliminating the trigonometric functions. This yields

$$K(x, y) = \frac{1}{2} \int_{(x-y)/2}^{(x+y)/2} q(t) \, dt + \int_{(x-y)/2}^{(x+y)/2} dt \times \int_0^{(x-y)/2} dz q(t + z) K(t + z, t - z), \qquad x \geq y, \tag{4.4}$$

$$A(x, y) = \frac{1}{2} \int_{(x+y)/2}^\infty q(t) \, dt - \int_{(x+y)/2}^\infty dt \times \int_0^{(y-x)/2} dz q(t - z) A(t - z, t + z), \qquad y \geq x. \tag{4.5}$$

By then solving these equations by the method of successive approximations, we obtain estimates for $K(x, y)$ and $A(x, y)$:

$$|K(x, y)| \leq \frac{1}{2} \int_{(x-y)/2}^x |q(t)| \, dt \times \exp \int_0^{(x+y)/2} t \, |q(t)| \, dt, \tag{4.6}$$

$$|A(x, y)| \leq \frac{1}{2} \int_{(x+y)/2}^\infty |q(t)| \, dt \times \exp \int_x^\infty t \, |q(t)| \, dt. \tag{4.7}$$

The above integral equations can also be used to show that $K(x, y)$ and $A(x, y)$ are differentiable and to derive estimates for their derivatives. For example,

$$\left| \frac{\partial}{\partial x} A(x, y) + \frac{1}{4} q\left(\frac{x + y}{2}\right) \right| \leq K \int_x^\infty |q(t)| \, dt \int_{(x+y)/2}^\infty |q(t)| \, dt. \tag{4.8}$$

A similar inequality holds for $\partial A(x, y)/\partial y$.

By virtue of (4.6) and (4.7), the theory of Volterra integral equations may be applied to integral equations having $K(x, y)$ and $A(x, y)$ as kernels. Thus, if $g(x)$ belongs to a given class of functions (which we shall not specify), then the equations

$$U_B f(x) \equiv f(x) + \int_0^x K(x, y) f(y) \, dy = g(x), \tag{4.9}$$

$$V_B f(x) \equiv f(x) + \int_x^\infty A(x, y) f(y) \, dy = g(x) \tag{4.10}$$

have solutions which may be represented in the form

$$f(x) = g(x) + \int_0^x \tilde{K}(x, y) g(y) \, dy \equiv U_B^{-1} g(x), \tag{4.11}$$

$$f(x) = g(x) + \int_x^\infty \tilde{A}(x, y)g(y)\,dy \equiv V_B^{-1}g(x). \quad (4.12)$$

In connection with this, the kernels $\tilde{K}(x, y)$ and $\tilde{A}(x, y)$ have estimates similar to (4.6) and (4.7). A more precise definition of the operators U_B and V_B as operators in Hilbert space will be given in the subsequent sections.

Relationships analagous to (4.3) exist between any two solutions of (1.1) with different potentials. Let $\varphi_1(x, k)$ and $\varphi_2(x, k)$ be solutions of (1.1) with potentials $q_1(x)$ and $q_2(x)$, respectively; two relations such as (4.3) can then be derived:

$$\varphi_1(x, k) = U_B^{(1)} \sin(kx)/k,$$
$$\varphi_2(x, k) = U_B^{(2)} \sin(kx)/k. \quad (4.13)$$

Inverting the second of these and substituting the result in the first, we obtain

$$\varphi_1(x, k) = U_B^{(1)}(U_B^{(2)})^{-1}\varphi_2(x,k) \quad (4.14)$$

or, explicitly,

$$\varphi_1(x, k) = \varphi_2(x, k) + \int_0^x K(x, y)\varphi_2(y, k)\,dy. \quad (4.15)$$

It is also possible to derive an inequality for $K(x, y)$ such as (4.6). Finally, one may also relate two different solutions $f_1(x, k)$ and $f_2(x, k)$, but such an expression will not be required in the following.

5. GENERAL THEORY OF TRANSFORMATION OPERATORS

We consider the operator L introduced in Sec. 2 to be one of the functional representations of an abstract operator, which will also be denoted by L. The operator L of Sec. 2 will now be denoted by L^x and the space $\mathfrak{L}_x$ in which it operates will be called the coordinate representation or x representation. Another representation to be considered is the so-called momentum or k representation. This space is of type $\mathfrak{L}_k$ and is defined by the condition that L_0 is an operator which multiplies an element of it by k^2:

$$L_0F(k) = k^2F(k). \quad (5.1)$$

Both representations are related to one another by the unitary transformation T_0 introduced in Sec. 3. In other words, to each element $f(x) \in \mathfrak{L}_x$, there corresponds an element $F(k)$ belonging to $\mathfrak{L}_k$:

$$F(k) = T_0f(x) = \int_0^\infty f(x)\frac{\sin kx}{k}\,dx; \quad (5.2)$$

and each operator A^x in $\mathfrak{L}_x$ is converted into an operator $A^k = T_0A^xT_0^*$. For example, in the k representation L is given by

$$L^kF(k) = k^2F(k) + \frac{2}{\pi}\int_0^\infty V(k, l)F(l)l^2\,dl, \quad (5.3)$$

where

$$V(k, l) = \int_0^\infty \frac{\sin kx}{k}\,q(x)\,\frac{\sin lx}{l}\,dx. \quad (5.4)$$

The passage back from the momentum to the coordinate representation is effected by the use of the transformation T_0^*; thus, for example, the operator which multiplies by a decreasing function $\Omega(k)$ goes into an integral operator in the x representation with a kernel

$$\Omega(x, y) = \frac{2}{\pi}\int_0^\infty \frac{\sin kx}{k}\,\Omega(k)\,\frac{\sin ky}{k}\,k^2\,dk. \quad (5.5)$$

In Sec. 2, the eigenfunction expansion theorem for L was expressed in terms of the operator T and involved various spaces. This is inconvenient. Instead of such transformations as T, we shall utilize operators acting in the same space, and which can therefore be prescribed by one of their representations. These operators are defined in the coordinate and momentum representations in the following way:

$$U^k = T_0T^* \text{ in the } k \text{ representation},$$
$$U^x = T^*T_0 \text{ in the } x \text{ representation}. \quad (5.6)$$

Such operators will be called transformation operators. In all subsequent discussions, they completely replace the eigenfunctions of the continuous spectrum. Instead of a differential equation for the eigenfunctions $\chi(x, k)$, one has for the operator U determined by them, the equation

$$LU = UL_0. \quad (5.7)$$

The completeness condition and orthogonality of the eigenfunctions in terms of U are expressed by

$$UWU^* = I_c, \qquad U^*UW = I \quad (5.8)$$

[cf. (2.17)]. The operator W in (5.8) is an operator which multiplies by the function $W(k)$ in the momentum representation. The advantage of (5.7) and (5.8) is that we do not have to write superscripts indicating the spaces in which the operators act—all operators are taken in the same representation. In many problems related to the operator L the use of different transformation operators is advantageous. Thus, the operators $U^{(\pm)}$ obtained from the transformations $T^{(\pm)}$ by means of (5.6) turn out to be useful in studying the asymptotic behavior of the solution of the Schrödinger equation

$$i\,\partial z(t)/\partial t = Lz(t), \qquad z(t)\,|_{t=0} = z_0 \quad (5.9)$$

for large $|t|$. The solution can be written in the following form:

$$z(t) = e^{-iLt}z_0, \tag{5.10}$$

and Theorem 3.1 can now be reformulated as follows.

Theorem 5.1. *If z_0 is orthogonal to the eigenfunctions of the point spectrum of L, then for the solution of the Schrödinger equation, the limiting conditions*

$$\lim_{t\to\pm\infty} e^{iL_0t}z(t) = z_\pm \tag{5.11}$$

hold where

$$z_+ = U^{(+)*}z_0, \qquad z_- = U^{(-)*}z_0. \tag{5.12}$$

If we denote the 'normalizing factor' relating $U^{(+)}$ and $U^{(-)}$ by S:

$$U^{(-)} = U^{(+)}S, \tag{5.13}$$

then (5.12) shows that z_+ and z_- are related to one another by

$$z_+ = Sz_-. \tag{5.14}$$

It is also not difficult to show from the definition of $U^{(+)}$ and (5.13) that in the momentum representation this is an operator which multiplies by the function

$$S(k) = M(-k)/M(k). \tag{5.15}$$

$S(k)$ obviously has the absolute value one. From Theorem 5.1 follows the existence of a solution of the Schrödinger equation which is asymptotic to the vector $z_-(t) = e^{-iL_0t}z_-$ as $t \to -\infty$, z_- being an arbitrary element. In addition, this solution for $t = 0$ will be orthogonal to the eigenfunctions of the point spectrum of L and consequently will behave like $e^{-iL_0t}z_+$ as $t \to \infty$, z_+ being constructed from (5.14) using the operator S. Thus, the general formulation of the scattering problem described in the introduction turns out to be valid for L, and S plays the role of the scattering operator. We have also shown that S is unitary and commutes with the energy operator.

The main objective of the survey is to establish the relationship between L and S. The operator $U_B = I + K$, defined in the coordinate representation by (4.9), will play an important role in this. By the definition of this section, this operator is a transformation operator. In fact, if we assume the transformation operator to be an integral operator, then its kernel in the x representation is obtained by expanding an appropriate solution $\chi(x, k)$ with respect to $(\sin kx)/k$:

$$\begin{aligned} U(x, y) &= \frac{2}{\pi}\int_0^\infty \bar{\chi}(x, k)\,\frac{\sin ky}{k}\,k^2\,dk \\ &= \frac{2}{\pi}\int_0^\infty \left[\chi(x, k) - \frac{\sin kx}{k}\right] \\ &\qquad \times \frac{\sin ky}{k}\,k^2\,dk + \delta(x - y). \end{aligned} \tag{5.16}$$

The kernel of the operator $U_B - I$ was derived in precisely this way in Sec. 4, where $\phi(x, k)$ is the corresponding solution. This crude argument can be put on an exact basis. In consequence of the fact that $M(k)$ is the normalizing factor for the solution $\varphi(x, k)$, with the help of which U_B is constructed, the operator W appearing in the formula for U_B, Eq. (5.8), is defined by

$$W(k) = 1/[M(k)M(-k)]. \tag{5.17}$$

A characteristic property of U_B is that it is a Volterra operator in the coordinate representation. This property is closely related to the fact that the potential $q(x)$ is diagonal in this representation (i.e., it is a multiplication operator). In fact, the triangularity of the kernel $K(x, y)$ is a consequence of the fact that $\varphi(x, s)$ is an entire function of s and this depends on the potential being diagonal. Conversely, let there exist a transformation operator U_B for some operator $L = L_0 + V$ of Volterra type in the x representation, namely,

$$U_B(x, y) = \delta(x - y) + \eta(x - y)K(x, y), \tag{5.18}$$

where

$$\eta(t) = \begin{cases} 1 & \text{for} \quad t > 0 \\ 0 & \text{for} \quad t < 0. \end{cases} \tag{5.19}$$

From (5.7), we find

$$V(I + K) = KL_0 - L_0K. \tag{5.20}$$

The operator $B = KL_0 - L_0K$ has the kernel

$$\begin{aligned} B(x, y) &= 2\,\delta(x - y)\,dK(x, x)/dx + \eta(x - y) \\ &\qquad \times [\partial^2K(x, y)/\partial x^2 - \partial^2K(x, y)/\partial y^2]. \end{aligned} \tag{5.21}$$

Multiplying (5.20) by $(I + K)^{-1}$, we find, by (5.21), that

$$\begin{aligned} V(x, y) &= \delta(x - y)2\,dK(x, x)/dx \\ &\qquad + \eta(x - y)C(x, y). \end{aligned} \tag{5.22}$$

But since V has to be self-adjoint,

$$C(x, y) = 0, \qquad V(x, y) = \delta(x - y)2\,dK(x, x)/dx, \tag{5.23}$$

i.e., L is defined by an equation of type (1.1). A precise statement of the above conclusions is contained in the following theorem:

Theorem 5.2. *Let L_0 be an operator defined in the x representation by the differential expression $L_0 y = -y''$ and the condition $y(0) = 0$. If there exists a self-adjoint operator $L = L_0 + V$ and Volterra operator $U_B = I + K$, whose kernel is differentiable and such that* (5.7) *is satisfied, then V is an operator which in the x representation, multiplies by the function*

$$q(x) = 2\, dK(x, x)/dx. \tag{5.24}$$

If the kernel of the transformation operator is not differentiable, then it is still possible to define L by the differential expression $Ly = -y'' + q(x)y$; but, in this case, $q(x)$ is a generalized function such as the derivative of an ordinary function. In particular, it is possible to obtain potentials with δ-function singularities.

Simultaneously with Theorem 5.2, we have shown that the kernel $K(x, y)$ satisfies the equation

$$q(x)K(x, y) = \partial^2 K(x, y)/\partial x^2 - \partial^2 K(x, y)/\partial y^2, \qquad x > y. \tag{5.25}$$

In fact, by substituting (5.23) into the left-hand side and (5.21) into the right-hand side of (5.20), we obtain

$$q(x)\, \delta(x - y) + \eta(x - y)q(x)K(x, y) = q(x)\, \delta(x - y) + \eta(x - y)[\partial^2 K(x, y)/\partial x^2 - \partial^2 K(x, y)/\partial y^2], \tag{5.26}$$

and this implies Eq. (5.25). Of course, whenever its derivatives do not exist, then $K(x, y)$ is a generalized solution of this equation. On the basis of this equation for $K(x, y)$, the condition $K(x, 0) = 0$ [which follows directly from the definition of $K(x, y)$], and (5.24), it is not difficult to derive (4.4), already used earlier.

6. THE FUNCTIONS $W(k)$ AND $S(k)$, THEIR PROPERTIES AND RELATIONSHIP

In this section, we study the functions $W(k)$ and $S(k)$ which were introduced in the preceding section by

$$W(k) = 1/[M(k)M(-k)] \quad (-\infty < k < \infty), \tag{6.1}$$

$$S(k) = M(-k)/M(k) \quad (-\infty < k < \infty), \tag{6.2}$$

and we shall establish how they are related. In addition to the properties of $M(k)$ assembled in Lemma 1.6, we still require one further property:

Lemma 6.1. *$M(k)$ has the representation*

$$M(k) = 1 + \int_0^\infty \Gamma(t)e^{ikt}\, dt, \tag{6.3}$$

where $|\Gamma(t)|$ is integrable.

The proof follows immediately from the definition of $M(k)$ and the expression (4.2). These imply that

$$M(k) = f(0, k) = 1 + \int_0^\infty A(0, t)e^{ikt}\, dt, \tag{6.4}$$

but $|A(0, t)|$ is integrable because of the inequality (4.7).

From (6.1) and (6.2) and by using Wiener's theorem on the Fourier transform of absolutely integrable functions, we may deduce the following lemmas:

Lemma 6.2. *$W(k)$ possesses the following properties:*

(1) *$W(k)$ is a positive even function:*

$$W(k) > 0, \qquad W(-k) = W(k), \qquad (-\infty < k < \infty); \tag{6.5}$$

(2) *$W(k) - 1$ has an absolutely integrable Fourier transform:*

$$W(k) = 1 + \int_{-\infty}^\infty H(t)e^{ikt}\, dt = 1 + 2\int_0^\infty H(t) \cos kt\, dt. \tag{6.6}$$

Lemma 6.3. *The function $S(k)$ possesses the following properties:*

(1) $|S(k)| = S(0) = S(\infty) = 1,$

$$S(-k) = S(k) = \bar{S}(k)^{-1}; \tag{6.7}$$

(2) *$S(k) - 1$ has an absolutely integrable Fourier transform:*

$$S(k) = 1 + \int_{-\infty}^\infty F(t)e^{-ikt}\, dt; \tag{6.8}$$

(3) $\arg S(k)\, |_{-\infty}^{\infty} = -4i\pi m, \qquad m \geq 0,$ (6.9)

where m is the number of discrete eigenvalues.

The last property is a consequence of the theorem for the number of zeros of an analytic function if we observe that $\arg S(k) = -2 \arg M(k)$. The properties of $W(k)$ and $S(k)$ enumerated in these lemmas are characteristic in the sense that, when they are fulfilled, there exists a unique function $M(s)$ which is analytic and bounded in the upper half-plane, and behaves asymptotically like $M(s) = 1 + o(1)$ for large $|s|$. Furthermore, $M(s)$ has a finite number of simple zeros $s_n = i\kappa_n$ $(n = 1, \cdots, m)$, and on the real axis, (6.1) and (6.2) are satisfied. In other words, there exists a function possessing the properties of the function $M(s)$ associated with some operator of type L having m discrete eigenvalues.

Let us first show that $M(s)$ is unique. Suppose there exists two functions $M_1(s)$ and $M_2(s)$, analytic and bounded in the upper half-plane $\tau > 0$, each having a finite number of zeros there at $s_n = i\kappa_n$ and such that, on the real axis,

$$S(k) = M_1(-k)/M_1(k) = M_2(-k)/M_2(k), \qquad (6.10)$$

where $S(k)$ is a given function satisfying the conditions of Lemma 6.3. Then $M_1(s)/M_2(s)$ is analytic and bounded in the upper half-plane and real on the real axis, so that it has a bounded analytic continuation into the lower half-plane. By Liouville's theorem, it follows that $M_1(s)/M_2(s) = C$. Thus, from the asymptotic behavior of M_1 and M_2 as $|s| \to \infty$, we find that $C = 1$, i.e., $M_1 = M_2$. By an analogous argument, we may prove that the function $M(s)$ satisfying (6.1) is unique [for a given $W(k)$]. It is necessary to consider ln (M_1/M_2).

We now complete the solution to our stated problem. We begin by reconstructing $M(k)$ from $S(k)$, and we consider first the case when $m = 0$ in (6.9). If we normalize the phase $\eta(k) = (i/2)$ ln $S(k)$ so that $\eta(0) = 0$, then $\eta(\infty) = 0$ and

$$\eta(k) = -\int_0^\infty \gamma(t) \sin kt\, dt, \int_0^\infty |\gamma(t)|\, dt < \infty \qquad (6.11)$$

by the Wiener-Levi theorem. The function

$$M(k) = \exp \int_0^\infty \gamma(t) e^{ikt}\, dt \qquad (6.12)$$

is the solution of our problem. Suppose now that $m \neq 0$; the function

$$\tilde{S}(k) = S(k) \prod_{n=1}^{m} \left(\frac{k - i\kappa_n}{k + i\kappa_n}\right)^2 \qquad (6.13)$$

possesses the same properties as $S(k)$ with the exception that $\tilde{m} = 0$. Therefore, the relation

$$S(k) = \tilde{M}(-k)/\tilde{M}(k) \qquad (6.14)$$

holds where $\tilde{M}(s)$ has no zeros in the upper half-plane. The solution of our problem will then be the function

$$M(k) = \tilde{M}(k) \prod_{n=1}^{m} \frac{k - i\kappa_n}{k + i\kappa_n}. \qquad (6.15)$$

Consider now the determination of $M(k)$ for a given $W(k)$. By the Wiener-Levi theorem, the function $\rho(k) = \ln W(k)$ can be represented as

$$\rho(k) = -2 \int_0^\infty \gamma(t) \cos kt\, dt. \qquad (6.16)$$

The function

$$\tilde{M}(s) = \exp \int_0^\infty \gamma(t) e^{ist}\, dt \qquad (6.17)$$

is analytic in the upper half-plane, has no zeros there, and possesses the proper asymptotic behavior for large $|s|$. The solution of our problem will be the function

$$M(k) = \tilde{M}(k) \prod_{n=1}^{m} \frac{k - i\kappa_n}{k + i\kappa_n}. \qquad (6.18)$$

In the following, the functions $M(k)$, $W(k)$, and $S(k)$, related to some operator L, will be called the M, W, and S functions of this operator. Any of these functions characterize the spectrum of the corresponding operator L, with $M(k)$ giving the most complete characterization.

The following statements are a consequence of the above discussion.

(1) If L has no discrete spectrum then its S function is uniquely determined by its W function according to (6.16), (6.17), and (6.14).

(2) If two operators L_1 and L_2 have the same W function, L_1 has no discrete spectrum, but L_2 has discrete eigenvalues at $\lambda_n = -\kappa_n^2$ $(n = 1, \cdots, m)$, then their M and S functions are related by

$$M_1(k) = M_2(k) \prod_{n=1}^{m} \frac{k + i\kappa_n}{k - i\kappa_n},$$

$$S_1(k) = S_2(k) \prod_{n=1}^{m} \left(\frac{k - i\kappa_n}{k + i\kappa_n}\right)^2. \qquad (6.19)$$

These statements will be used in Secs. 8, 11, and 12.

7. THE TRANSFORMATION OPERATOR OF $A(x, y)$

In Sec. 4, besides U_B, we introduced an operator V_B with a kernel in the x representation determined by (4.10). In contrast to U_B, the operator V_B is not a transformation operator in the sense of Sec. 5. In fact, its kernel can not be represented as the Fourier sine transform of a solution of (1.1) satisfying a zero boundary condition. The present section is devoted to a clarification of the relationship between U_B and V_B. We shall assume that there is no discrete spectrum. This allows one to use the concise operator notation which was introduced in the preceding sections.

First consider the operator $\tilde{U}_B = U_B W$ which is the transformation operator related to the solution

$$\tilde{\varphi}(x, k) = \varphi(x, k) W(k) = \frac{1}{2ik}\left[\frac{f(x, k)}{M(k)} - \frac{f(x, -k)}{M(-k)}\right]. \qquad (7.1)$$

The kernel of the operator $\tilde{K} = \tilde{U}_B - I$ in the x representation is obtained by taking the Fourier transform of $\tilde{\varphi}(x, k) - k^{-1} \sin kx$:

$$\tilde{K}(x, y) = \frac{2}{\pi}\int_0^\infty \left(\bar{\varphi}(x, k) - \frac{\sin kx}{k}\right)\frac{\sin ky}{k}\, k^2\, dk$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{f(x, k)}{M(k)} - e^{ikx}\right)e^{-iky}\, dk. \quad (7.2)$$

By virtue of the analyticity and boundedness of $f(x, s)e^{-isx}/M(s)$ in the upper half-plane, $\tilde{K}(x, y) = 0$ for $x > y$. Therefore, the operator $\tilde{U}_B$ is given in the x representation by

$$\tilde{U}_B f(x) = f(x) + \int_x^\infty \tilde{K}(x, y)f(y)\, dy. \quad (7.3)$$

In terms of $\tilde{U}_B$, the completeness and orthogonality relation

$$U_B W U_B^* = I \quad (7.4)$$

becomes

$$\tilde{U}_B \tilde{W} \tilde{U}_B^* = I, \qquad \tilde{W} = W^{-1}, \quad (7.5)$$

or

$$(\tilde{U}_B^*)^{-1} W (\tilde{U}_B)^{-1} = I. \quad (7.6)$$

We note that $(\tilde{U}_B^*)^{-1}$ is a Volterra operator such as U_B, i.e., the integration goes from 0 to x. In Sec. 9, it will be shown that this Volterra operator is uniquely determined by (7.6). Hence, by comparing (7.4) and (7.6), we may conclude that

$$(\tilde{U}_B^*)^{-1} = U_B. \quad (7.7)$$

This establishes the relationship between the two operators $\tilde{U}_B$ and U_B. On the other hand, by finding the inverse Fourier transform of

$$h(x, k) = \frac{f(x, k)}{M(k)} = e^{ikx} + \int_x^\infty \tilde{K}(x, y)e^{iky}\, dy, \quad (7.8)$$

we can easily relate $\tilde{U}_B$ and V_B. Let $\Pi(t)$ denote the function

$$\Pi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{M(k)} - 1\right)e^{-ikt}\, dk. \quad (7.9)$$

Since $1/M(s)$ is analytic in the upper half-plane (there is no discrete spectrum),

$$\Pi(t) = 0, \qquad t < 0. \quad (7.10)$$

Recalling that

$$f(x, k) = e^{ikx} + \int_x^\infty A(x, y)e^{iky}\, dy, \quad (7.11)$$

and using the convolution theorem, we find from (7.8) that

$$A(x, y) + \Pi(y - x) + \int_x^y A(x, t)\Pi(y - t)\, dt = \tilde{K}(x, y). \quad (7.12)$$

This is the desired relation. It will be convenient to write this equation in operator form. With this in mind, we associate with $\Pi(t)$ an operator Q which in the coordinate representation is given by

$$Qf(x) = (I + \Pi)f(x) \equiv f(x) + \int_x^\infty \Pi(y - x)f(y)\, dy. \quad (7.13)$$

By virtue of the boundedness of $1/M(s)$, it is not difficult to show that $Q = I + \Pi$ is a bounded operator in our Hilbert space. By means of known theorems on Volterra integral equations with difference-type kernels, it follows then that Q has an inverse

$$P = I + \Gamma = Q^{-1} \quad (7.14)$$

whose structure in the x representation is similar to Q:

$$Pf(x) = (I + \Gamma)f(x) \equiv f(x) + \int_x^\infty \Gamma(y - x)f(y)\, dy. \quad (7.15)$$

The function $\Gamma(t)$, i.e.,

$$\Gamma(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(M(k) - 1)e^{-ikt}\, dk \quad (7.16)$$

was introduced in the previous section [see (6.3)]. In terms of the operator Q, Eq. (7.12) becomes

$$\tilde{U}_B = V_B Q. \quad (7.17)$$

The completeness and orthogonality of the eigenfunctions can now be written by using the operator V_B. The substitution of (7.17) in (7.5) yields

$$V_B Q W^{-1} Q^* V_B^* = I. \quad (7.18)$$

Let us clarify the structure of the operator $QW^{-1}Q^*$. For this, the x representation is most convenient. In this representation, the operator $W^{-1} = \tilde{W}$ has the form $\tilde{W} = I + \tilde{\Omega}$ where $\tilde{\Omega}$ is an integral operator with the kernel

$$\tilde{\Omega}(x, y) = \frac{2}{\pi}\int_0^\infty \frac{\sin kx}{k}\left(\frac{1}{W(k)} - 1\right) \times \frac{\sin ky}{k}\, k^2\, dk = \tilde{H}(x - y) - \tilde{H}(x + y), \quad (7.19)$$

where

$$\tilde{H}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{W(k)} - 1\right)e^{-ikt}\, dk. \quad (7.20)$$

Denote by H_1 the operator with kernel $\tilde{H}(x - y)$ and by H_2 the operator with kernel $-\tilde{H}(x + y)$.

Then, the identity

$$[1/W(k)][1/M(k)] = M(-k) \tag{7.21}$$

can be expressed as

$$(I + \Pi)(I + H_1) = I + \Gamma^*. \tag{7.22}$$

Hence,

$$\begin{aligned} QW^{-1}Q^* &= (I + \Pi)(I + H_1 + H_2)(I + \Pi^*) \\ &= (I + \Gamma^* + (I + \Pi)H_2)(I + \Pi^*) \\ &= I + (I + \Pi)H_2(I + \Pi\ ). \end{aligned} \tag{7.23}$$

One can easily verify that the second term on the right-hand side of (7.23) is an integral operator whose kernel depends only on a sum and can be constructed using the function

$$\tilde{F}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(S(k) - \frac{1}{M^2(k)}\right)e^{ikt}\,dk. \tag{7.24}$$

For $t > 0$, $\tilde{F}(t)$ coincides with the function $F(t)$ introduced in the preceding section. We now associate with $F(t)$ the operator

$$Ff(x) = \int_0^{\infty} F(x + y)f(y)\,dy. \tag{7.25}$$

We have thus shown that the completeness and orthogonality relation for the eigenfunctions can be written in terms of V_B in the following way:

$$V_B(I - F)V_B^* = I. \tag{7.26}$$

To conclude this section, we point out the significance of the various operators in Hilbert space which have been applied. By the uniform boundedness of the functions $M(k)$, $1/M(k)$, $W(k)$, $1/W(k)$, and $S(k)$, the operators M, W, S, W^{-1}, M^{-1}, Q, P, and $I - F$ are bounded in our Hilbert space. Every other operator used, namely, U_B, $\tilde{U}_B$, and V_B, can be obtained from the unitary operator $U^{(+)}$ or $U^{(+)*}$ by multiplying by one of the above operators. Consequently, each of these operators is a bounded operator in our Hilbert space.

8. INTEGRAL EQUATIONS FOR THE KERNELS $K(x, y)$ AND $A(x, y)$

In this section, we shall show how the functions $W(k)$ and $S(k)$ and the kernels $K(x, y)$ and $A(x, y)$ are related. The completeness and orthogonality relations for the eigenfunctions expressed in terms of U_B and V_B are such expressions. However, if we write them out, for example, in the x representation, then the kernels $K(x, y)$ or $A(x, y)$ will enter nonlinearly. This is found to be unsuitable for solving the inverse problem. Nevertheless, the fact that U_B and V_B are Volterra operators in the x representation allows one to easily obtain expressions relating $W(k)$ and $S(k)$ and the kernels $K(x, y)$ and $A(x, y)$ in which the latter enter linearly.

For simplicity, we first assume that there is no discrete spectrum. Expressing the equality

$$U_BWU_B^* = I \tag{8.1}$$

in the form

$$U_BW = \tilde{U}_B, \qquad \tilde{U}_B = (U_B^*)^{-1} \tag{8.2}$$

and then writing out (8.2) in the x representation, we obtain

$$K(x, y) + \Omega(x, y) + \int_0^x K(x, t)\Omega(t, y)\,dt = \tilde{K}(x, y). \tag{8.3}$$

Here

$$\Omega(x, y) = \frac{2}{\pi}\int_0^{\infty}\frac{\sin kx}{k}\,[W(k) - 1]\,\frac{\sin ky}{k}\,k^2\,dk. \tag{8.4}$$

Since $\tilde{K}(x, y) = 0$ for $x > y$ [see (7.2)], it finally follows that

$$K(x, y) + \Omega(x, y) + \int_0^x K(x, t)\Omega(t, y)\,dt = 0, \qquad x > y. \tag{8.5}$$

In an analogous way, the relation

$$V_B(I - F)V_B^* = I \tag{8.6}$$

gives

$$A(x, y) = F(x + y) + \int_x^{\infty} A(x, t)F(t + y)\,dt, \qquad x < y, \tag{8.7}$$

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(S(k) - 1)e^{ikt}\,dk. \tag{8.8}$$

In case there is a discrete spectrum, similar equations can be obtained by starting directly from the representation (4.3) for $\varphi(x, k)$ and the completeness and orthogonality relation in the form (2.6). An equation for the transformation operator which relates the solutions $\varphi(x, k)$ of two different equations of the form (1.1) will now be derived. We begin with the following relations [cf. (2.6)]:

$$\varphi_2(x, k) = \varphi_1(x, k) + \int_0^x K(x, t)\varphi_1(t, k)\,dt, \tag{8.9}$$

$$\varphi_1(y, k) = \varphi_2(y, k) + \int_0^y \tilde{K}(t, y)\varphi_2(t, k)\,dt, \tag{8.10}$$

$$\sum_{n_i=1}^{m_i} C_{n_i}\varphi_{n_i}(x)\varphi_{n_i}(y) + \frac{2}{\pi}\int_0^\infty \varphi_i(x, k)W_i(k)\varphi_i(y, k)k^2\,dk = \delta(x - y) \qquad (i = 1, 2). \tag{8.11}$$

If both sides of (8.10) are multiplied by $\phi_2(x, k)\cdot W_2(k)k^2$ and integrated with respect to k, and then (8.11) for $i = 2$ is used, there results

$$\begin{aligned}\frac{2}{\pi}\int_0^\infty \varphi_2(x, k)W_2(k)\varphi_1(y, k)k^2\,dk &+ \sum_{n_2=1}^{m_2} C_{n_2}\varphi_1(y, i\kappa_{n_2})\varphi_{n_2}(x) \\ &= \delta(x - y) + \int_0^y \tilde{K}(t, y)\,\delta(x - t)\,dt \\ &= \delta(x - y), \qquad x > y.\end{aligned} \tag{8.12}$$

Analogous operations on (8.9) with the use of (8.12) then yield

$$\begin{aligned}\delta(x - y) &= \frac{2}{\pi}\int_0^\infty \varphi_1(x, k)W_2(k)\varphi_1(y, k)k^2\,dk \\ &+ \sum_{n_2=1}^{m_2} C_{n_2}\varphi_1(x, i\kappa_{n_2})\varphi_1(y, i\kappa_{n_2}) \\ &+ \int_0^x K(x, t)\left[\frac{2}{\pi}\int_0^\infty \varphi_1(t, k)W_2(k)\varphi_1(y, k)k^2\,dk \right. \\ &+ \left. \sum_{n_2=1}^{m_2} C_{n_2}\varphi_1(t, i\kappa_{n_2})\varphi_1(y, i\kappa_{n_2})\right] dt, \qquad x > y.\end{aligned} \tag{8.13}$$

Subtracting (8.11) for $i = 1$ from (8.13), we finally obtain, after simple computations,

$$K(x, y) + \Omega(x, y) + \int_0^x K(x, t)\Omega(t, y)\,dt = 0, \qquad x > y, \tag{8.14}$$

where

$$\begin{aligned}\Omega(x, y) &= \frac{2}{\pi}\int_0^\infty \varphi_1(x, k)[W_2(k) - W_1(k)]\varphi_1(y, k)k^2\,dk \\ &+ \sum_{n_2=1}^{m_2} C_{n_2}\varphi_1(x, i\kappa_{n_2})\varphi_1(y, i\kappa_{n_2}) \\ &- \sum_{n_1=1}^{m_1} C_{n_1}\varphi_{n_1}(x)\varphi_{n_1}(y).\end{aligned} \tag{8.15}$$

Since the generalization of (8.7) will not be required later on, its derivation is omitted. Equations (8.5) and (8.14) are called the Gel'fand-Levitan equations, and (8.7) is called the Marchenko equation.

These equations enable one to solve inverse problems, i.e., to reconstruct the operator L under different formulations. We first apply these equations in the solution of the following two problems.

(1) Given a function $W(k)$ having the properties enumerated in Lemma 6.2. To construct an operator L with no discrete spectrum for which $W(k)$ is its W function.

(2) Given an operator L_1 $[q_1(x) \not\equiv 0]$ with no discrete spectrum such that $W_1(k)$ is its W function. To construct an operator L whose W function is $W_1(k)$ and which has discrete eigenvalues at prescribed points $\lambda_n = -\kappa_n^2$ $(n = 1, \cdots, m)$.

The basic problem, i.e., the determination of L from its S function and its discrete energy levels, will be solvable if these two problems can be solved. Let there be given m distinct positive numbers κ_n $(n = 1, 2, \cdots, m)$ and a function $S(k)$ possessing the properties enumerated in Lemma 6.3. By the procedure of Sec. 6, $W(k)$ can be constructed uniquely from this data. If in conjunction with problem 1, we now construct an operator $\tilde{L}$ without a discrete spectrum for which this $W(k)$ is its W function then, by the statements at the end of Sec. 6, the S function of this operator will be

$$\tilde{S}(k) = S(k)\prod_{n=1}^{m}\left(\frac{k - i\kappa_n}{k + i\kappa_n}\right)^2. \tag{8.16}$$

If in accordance with problem 2, we now start with the operator $\tilde{L}_1$ and construct an operator L which has $W(k)$ as its W function and the m points $\lambda_n = -\kappa_n^2$ as its discrete eigenvalues, then L will have the initially-given function $S(k)$ as its S function, and the basic problem will therefore be solved.

9. EXISTENCE OF A SOLUTION OF THE GEL'FAND-LEVITAN EQUATION. SOLUTION OF THE FIRST PROBLEM

Let us proceed to solve the two problems formulated in the previous section. To this end, we shall make use of the Gel'fand-Levitan equations (8.5), (8.4) and (8.14), (8.15), and we therefore first show that they have solutions.

We go directly to the general equation (8.14). Let the operator L_1 be given. That is, its eigenfunctions $\varphi_1(x, k)$, its W function $W_1(k)$, its discrete eigenvalues $\lambda_{n_1} = -\kappa_{n_1}^2$ and corresponding normalizing factors C_{n_1} are prescribed. Also, let the function $W(k)$ having the properties enumerated in Lemma 6.2 and the arbitrary positive numbers κ_n and C_n be prescribed. Assume that the κ_n are distinct. From this data, we construct the function $\Omega(x, y)$ by means of (8.15). By virtue of our given conditions, this function will be absolutely integrable with respect to x or y in any finite interval. Equation (8.14) for $K(x, y)$ is an equation with respect to the argument y, the kernel and free term depending on x as a parameter. For fixed x, this equation is a Fredholm equation. Therefore, to show that a unique solution exists, it suffices to prove that the homogeneous equation has only a trivial solution.

Suppose that for fixed x_0, the equation

$$h_0(y) + \int_0^{x_0} h_0(t)\Omega(t, y)\, dt = 0 \tag{9.1}$$

has a solution. Denote

$$h(y) = \begin{cases} h_0(y) & \text{for } y \leq x_0, \\ 0 & \text{for } y > x_0. \end{cases} \tag{9.2}$$

Multiplying (9.1) by $h(y)$ and integrating with respect to y, we obtain

$$\int_0^\infty h^2(y)\, dy + \int_0^\infty \int_0^\infty h(t)\Omega(t, y)h(y)\, dt\, dy = 0. \tag{9.3}$$

The substitution of the expression (8.15) for $\Omega(x, y)$ in this yields

$$\begin{aligned} \int_0^\infty h^2(y)\, dy + \frac{2}{\pi}\int_0^\infty \left[\int_0^\infty h(y)\varphi_1(y, k)\, dy\right]^2 \\ \times [W_2(k) - W_1(k)]k^2\, dk \\ + \sum_{n=1}^{m} C_n\left[\int_0^\infty h(y)\varphi_1(y, i\kappa_n)\, dy\right]^2 \\ - \sum_{n_1=1}^{m_1} C_{n_1}\left[\int_0^\infty h(y)\varphi_{n_1}(y)\, dy\right]^2 = 0. \end{aligned} \tag{9.4}$$

By means of Parseval's equality for the system of functions $\varphi_1(x, k)$ this may be simplified to

$$\frac{2}{\pi}\int_0^\infty \left[\int_0^\infty h(y)\varphi_1(y, k)\, dy\right]^2 W_2(k)k^2\, dk + \sum_{n=1}^{m} C_n\left[\int_0^\infty h(y)\varphi_1(y, i\kappa_n)\, dy\right]^2 = 0. \tag{9.5}$$

Since $W_2(k)$ is positive, this implies that $h(y)$ is orthogonal to the subspace corresponding to the continuous spectrum of L. Consequently, $W_2(k)$ is a linear combination of eigenfunctions of the discrete spectrum. But this is impossible since $h(y)$ vanishes identically for $y > x_0$, and hence $h(y) \equiv 0$.

The remainder of this section and the two succeeding sections will be devoted to solving the first problem. In so doing, we must consider the equation

$$K(x, y) + \Omega(x, y) + \int_0^x K(x, t)\Omega(t, y)\, dt = 0, \qquad x > y, \tag{9.6}$$

$$\Omega(x, y) = \frac{2}{\pi}\int_0^\infty \frac{\sin kx}{k}(W(k) - 1)\frac{\sin ky}{k}k^2\, dk, \tag{9.7}$$

which was obtained from the condition

$$U_B W U_B^* = I \tag{9.8}$$

for the operator $U_B = I + K$. Conversely, we shall show that the operator $U_B = I + K$, defined in terms of the solution of (9.6), has the property (9.8), where the function $W(k)$ only has to satisfy the conditions enumerated in Lemma 6.2.

We note that the solution of (9.6) also determines a kernel $\tilde{K}(x, y)$ which is different from zero only for $x < y$:

$$\tilde{K}(x, y) = \begin{cases} K(x, y) + \Omega(x, y) + \int_0^x K(x, t)\Omega(t, y)\, dt, & x < y, \\ 0, & x > y. \end{cases} \tag{9.9}$$

With this kernel, we associate the operator $\tilde{K}$. Let $\tilde{U}_B = I + \tilde{K}$. In terms of U_B and $\tilde{U}_B$, relation (9.9) becomes

$$U_B W = \tilde{U}_B, \tag{9.10}$$

and to prove (9.8), we must now show that

$$U_B = (\tilde{U}_B^*)^{-1}. \tag{9.11}$$

Since $W(k)$ is real, W is self-adjoint. From (9.10), it follows that $\tilde{U}_B U_B^* = U_B W U_B^*$ so that $\tilde{U}_B U_B^*$ is also self-adjoint. On the other hand, since $\tilde{U}_B$ and U_B^* are Volterra operators, $\tilde{U}_B U_B^*$ is also a Volterra operator and this property together with the self-adjointness implies that $\tilde{U}_B U_B^* = I$. This is what we had to show.

The above reasoning is of a rather formal character since no estimates for $K(x, y)$ and $\tilde{K}(x, y)$ were given that would have explained the meaning of U_B and $\tilde{U}_B$ as operators in Hilbert space. However, the relation (9.8), when written out in the x representation, only involves integrals over finite intervals of integration (whose existence it is therefore not necessary to prove), and this relation may be proved starting directly from (9.6). Inasmuch as this relation is proven, it is possible to attach meaning to the operator U_B. In fact, the function $M(k)$ constructed from $W(k)$ by the procedure of Sec. 6, is a bounded function and a bounded operator can be associated with it, namely, an operator which multiplies by $M(k)$ in the momentum representation. Since $M(k) \neq 0$, the operator M^{-1} is also bounded. From (9.8), we find that $U^{(+)} = U_B M^{-1}$ is unitary and from this follows the boundedness of U_B.

With the operator thus obtained, we define an operator L by

$$L = U_B L_0 U_B^{-1}, \tag{9.12}$$

where L_0 is the operator which multiplies by k^2 in the k representation, which was introduced in Secs. 3 and 5. Now $L = U_B L_0 W U_B^*$ by virtue of (9.8), and hence, L is self-adjoint because W and L_0 commute. On the basis of Theorem 5.2, we can say that L is a differential operator in the x representation associated with an equation of the form (1.1) with a potential given by

$$q(x) = 2\, dK(x, x)/dx. \tag{9.13}$$

Equation (9.8) shows that the given function $W(k)$ is the W function for L. And so, the first problem of Sec. 8 is solved in the sense that an operator L has been shown to exist. The properties of the potential $q(x)$ obtained in the solution of this problem will be investigated in the following section.

In conclusion, we note that our considerations imply that (9.8) determines the Volterra operator U_B uniquely. This fact was previously used in Sec. 7.

10. MARCHENKO'S EQUATION. THE PROPERTIES OF THE POTENTIAL

To study the properties of $q(x)$, we find it convenient to use Marchenko's equation

$$A(x, y) = F(x + y) + \int_x^\infty A(x, t) F(t + y)\, dt, \qquad x < y, \tag{10.1}$$

in which

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [S(k) - 1] \exp(ikt)\, dk. \tag{10.2}$$

The existence of a solution of (10.1) will not be proved. Instead, we show that (10.1) is equivalent to the Gel'fand-Levitan equation (8.5) provided, of course, that the corresponding functions $W(k)$ and $S(k)$ are related by the formulas of Sec. 6. In the preceding section, it was shown that the Gel'fand-Levitan equation (8.5) and the relation $U_B W U_B^* = I$ for the operator $U_B = I + K$ are equivalent. By repeating the reasoning of Secs. 7 and 8, we find that $V_B = \tilde{U}_B Q^{-1}$ is an operator which can be constructed using the solution $A(x, y)$ of (10.1). Therefore (10.1) has a solution for any $x \geq 0$. Conversely, it is not difficult to show that if the operator V_B is formed with respect to a solution of (10.1), then $U_B = (Q^* V_B^*)^{-1}$ is the operator formed by using the solution of the Gel'fand-Levitan equation (8.5). This implies the uniqueness of a solution of (10.1).

The potential $q(x) = 2\, dK(x, x)/dx$ can be expressed very simply in terms of $A(x, y)$. In fact, from (7.12) it follows that $A(x, x) = \tilde{K}(x, x) - \Pi(0)$ and $dA(x, x)/dx = d\tilde{K}(x, x)/dx$. If (7.7) is written out in terms of $K(x, y)$ and $\tilde{K}(x, y)$, we obtain

$$\tilde{K}(y, x) + K(x, y) + \int_y^x \tilde{K}(t, x) K(t, y)\, dt = 0, \tag{10.3}$$

from which it follows that

$$dK(x, x)/dx = -d\tilde{K}(x, x)/dx.$$

Finally,

$$q(x) = 2\, dK(x, x)/dx = -2\, dA(x, x)/dx. \tag{10.4}$$

The solution of (10.1) will now be investigated. It turns out that $q(x)$ behaves in many respects like the derivative of $F(t)$ for $t > 0$. We shall derive some estimates, and we first consider how $F'(t)$ behaves in relation to $q(x)$. It is convenient to introduce the functions

$$\sigma(x) = \int_x^\infty |q(t)|\, dt, \quad \sigma_1(x) = \int_x^\infty t\, |q(t)|\, dt. \tag{10.5}$$

The estimates given in Sec. 4 for $A(x, y)$ and $\partial A(x, y)/\partial x$ can now be expressed as

$$|A(x, y)| \leq K\sigma[\tfrac{1}{2}(x + y)] \tag{10.6}$$

$$|\partial A(x, y)/\partial x + \tfrac{1}{4} q[\tfrac{1}{2}(x + y)]| \leq K\sigma[\tfrac{1}{2}(x + y)]\sigma(x). \tag{10.7}$$

In (10.1), we set $x = y$. Then

$$A(x, x) = F(2x) + \int_x^\infty A(x, t) F(t + x)\, dt \tag{10.8}$$

or

$$F(2x) = A(x, x) - 2 \int_x^\infty A(x, 2t - x) F(2t)\, dt. \tag{10.9}$$

The last equation may be solved by the method of successive approximations and the following estimate for its solution may be derived:

$$|F(2x)| \leq K\sigma(x). \tag{10.10}$$

Differentiation of (10.9) with respect to x and the use of the inequalities (10.7) and (10.10) then yield

$$|F'(2x) + \tfrac{1}{4} q(x)| \leq K\sigma^2(x). \tag{10.11}$$

Conversely, we now consider the question, how $q(x)$ relates to $F'(t)$, for which it is convenient to transform (10.1) into

$$A(x, x + y) = F(2x + y) + \int_0^\infty A(x, x + t)F(2x + t + y)\, dt. \tag{10.12}$$

Let F_x denote the operator

$$F_x g(y) = \int_0^\infty g(t)F(2x + t + y)\, dt. \tag{10.13}$$

Because (10.1) has a unique solution for any x and since the norm of F_x is small for large x, the norm of $(I - F_x)^{-1}$ is uniformly bounded in $\mathfrak{L}_1(0, \infty)$. Hence,

$$\int_0^\infty |A(x, x + y)|\, dy = \int_x^\infty |A(x, y)|\, dy \leq K \int_{2x}^\infty |F(t)|\, dt. \tag{10.14}$$

Introduce now the notation

$$\tau(x) = \int_x^\infty |F'(t)|\, dt, \qquad \tau_1(x) = \int_x^\infty t\, |F'(t)|\, dt. \tag{10.15}$$

Then,

$$|F(x)| \leq \int_x^\infty |F'(t)|\, dt = \tau(x). \tag{10.16}$$

From (10.1) and the inequality (10.14), we obtain the uniform estimate

$$|A(x, y)| \leq K\tau(x + y). \tag{10.17}$$

We next consider the differentiability of $A(x, x)$, and we set

$$B(x, y) = A(x, x + y). \tag{10.18}$$

The difference quotient

$$\Delta_x B(x, y)/h = [B(x + h, y) - B(x, y)]/h \tag{10.19}$$

satisfies an equation of the same type as does $B(x, y)$:

$$\frac{\Delta_x B(x, y)}{h} = \frac{\Delta_x F(2x + y)}{h} + \int_0^\infty B(x, t)\, \frac{\Delta_x F(2x + t + y)}{h}\, dt + \int_0^\infty \frac{\Delta_x B(x, t)}{h}\, F(2x + 2h + t + y)\, dt. \tag{10.20}$$

The free term

$$\frac{\Delta_x F(2x + y)}{h} + \int_0^\infty B(x, y)\, \frac{\Delta_x F(2x + t + y)}{h}\, dt \tag{10.21}$$

can be estimated, uniformly in h, because of the differentiability of $F(t)$, whence follows the differentiability of $B(x, y)$. Moreover, just as in the above way, we obtain an estimate for $\partial B(x, y)/\partial x$:

$$\int_0^\infty \left|\frac{\partial B}{\partial x}(x, y)\right| dy \leq K\tau(2x). \tag{10.22}$$

This, in turn, leads to the estimate

$$|dA(x, x)/dx - 2F\,(2x)| \leq K\tau^2(2x) \tag{10.23}$$

or

$$|F'(2x) + \tfrac{1}{4}q(x)| \leq K\tau^2(2x). \tag{10.24}$$

Thus the inequalities (10.11) and (10.24) explicitly show the analogous behavior of $q(x)$ and $F'(2x)$. In particular, if

$$\int_0^\infty x\, |q(x)|\, dx < \infty \tag{10.25}$$

then

$$\int_0^\infty x\, |F'(2x)|\, dx < \infty, \tag{10.26}$$

and conversely. These inequalities may also be used to establish necessary and sufficient conditions on the S function assuring that the corresponding potential decreases like x^{-n} or exponentially, or vanishes identically for $x > A$, etc. A precise formulation of these conditions will not be given here because of the lack of space.

11. KREIN'S EQUATION. ASYMPTOTIC BEHAVIOR OF $\varphi(k, x)$

The eigenfunctions $\varphi(x, k)$ of the operator L are formed according to the formula

$$\varphi(x, k) = \frac{\sin kx}{k} + \int_0^x K(x, y)\, \frac{\sin ky}{k}\, dy. \tag{11.1}$$

These functions behave asymptotically like [cf. (1.21)]

$$\varphi(x, k) \approx [A(k)/k] \times \sin\,[kx - \eta(k)], \qquad x \to \infty, \tag{11.2}$$

in which $A(k)$ and $\eta(k)$ are, respectively, the modulus and argument of the M function of L which is uniquely determined by $W(k)$ by the relation

$$W(k) = 1/[M(k)M(-k)] \tag{11.3}$$

and the condition of analyticity (see Sec. 6).

It is not easy to deduce the asymptotic expression (11.2) directly from (11.1). Another representation for $\varphi(x, k)$ will therefore be obtained from which (11.2) can be deduced without difficulty. Of course, this representation not only serves to prove (11.2), but the procedure used to obtain it is interesting *per se,* in that it reveals more completely the structure of the kernel $A(x, y)$ of the transformation operator.

The starting point is the Gel'fand-Levitan equation

$$K(x, y) + \Omega(x, y) + \int_0^x K(x, t)\Omega(t, y)\,dt = 0, \qquad y < x, \tag{11.4}$$

in which

$$\Omega(x, y) = \frac{2}{\pi}\int_0^\infty \sin kx[W(k) - 1]\sin ky\,dk. \tag{11.5}$$

We can obviously write

$$\Omega(x, y) = H(x - y) - H(x + y), \tag{11.6}$$

where the function

$$H(t) = \frac{1}{\pi}\int_0^\infty [W(k) - 1]\cos kt\,dk \tag{11.7}$$

is the one previously introduced in Sec. 6 [see (6.6)]. If a solution $K(x, y)$ is sought in the form

$$K(x, y) = \Gamma_{2x}(x - y) - \Gamma_{2x}(x + y), \tag{11.8}$$

then one obtains for $\Gamma_{2x}(t)$, the equation

$$\Gamma_{2x}(t) + H(t) + \int_0^{2x} \Gamma_{2x}(s)H(s - t)\,ds = 0, \tag{11.9}$$

which is equivalent to (11.4). This is an equation for $\Gamma_{2x}(t)$ in the argument t, x playing the role of a parameter. This fact underscores the unsymmetric dependence of $\Gamma_{2x}(t)$ on its arguments. We shall call (11.9) Krein's equation. Knowing its solution, we can easily find the kernel $K(x, y)$, and together with it, the potential $q(x)$ and the solution $\varphi(x, k)$. However, these functions can be expressed directly in terms of $\Gamma_{2x}(t)$, as follows.

If we denote by $G_a(x, t)$ the resolvent of the kernel $H(x - t)$ on the interval $(0, a)$:

$$G_a(x, t) + H(x - t) + \int_0^a G_a(x, s)H(s - t)\,ds = 0, \tag{11.10}$$

then

$$\Gamma_{2x}(t) = G_{2x}(0, t). \tag{11.11}$$

We now note some properties of $G_a(x, t)$. Differentiation of (11.10) with respect to a shows that $G_a(x, t)$ satisfies

$$\partial G_a(x, t)/\partial a = G_a(x, a)G_a(a, t). \tag{11.12}$$

Also,

$$G_a(t, s) = G_a(a - t, a - s). \tag{11.13}$$

We next express $q(x) = 2\,dK(x, x)/dx$ in terms of $\Gamma_{2x}(t) = G_{2x}(0, t)$:

$$q(x) = 2\,dG_{2x}(0, 0)/dx - 2\,dG_{2x}(0, 2x)/dx. \tag{11.14}$$

If the notation

$$A(x) = 2G_{2x}(0, 2x) = 2\Gamma_{2x}(2x) \tag{11.15}$$

is introduced, then by virtue of (11.12) and (11.13)

$$q(x) = -dA(x)/dx + A^2(x). \tag{11.16}$$

This last formula enables us to reduce our second-order differential equation containing $q(x)$ to a system of equations of the first order. Thus,

$$d^2/dx^2 - q(x) = [d/dx - A(x)][d/dx + A(x)] \tag{11.17}$$

and the equation

$$-y'' + q(x)y = k^2y \tag{11.18}$$

is equivalent to the system

$$\left.\begin{aligned} dy/dx + Ay &= kz,\\ -dz/dx + Az &= ky.\end{aligned}\right\} \tag{11.19}$$

In certain cases, this system turns out to be more convenient than the original equation (11.18). Let us now express the solution $\varphi(x, k)$ in terms of $\Gamma_{2x}(t)$:

$$\begin{aligned}\varphi(x, k) &= \frac{\sin kx}{k} + \int_0^x \Gamma_{2x}(x - y)\frac{\sin ky}{k}\,dy - \int_0^x \Gamma_{2x}(x + y)\frac{\sin ky}{k}\,dy\\ &= \frac{\sin kx}{k} + \int_0^{2x} \Gamma_{2x}(t)\frac{\sin k(x - t)}{k}\,dt\\ &= \frac{1}{k}\,\mathrm{Im}\left[e^{ikx}\left(1 + \int_0^{2x} \Gamma_{2x}(t)e^{-ikt}\,dt\right)\right].\end{aligned} \tag{11.20}$$

When $x \to \infty$, we find that

$$\varphi(x, k) \to \frac{1}{k}\,\mathrm{Im}\left[e^{ikx}\left(1 + \int_0^\infty \Gamma(t)e^{-ikt}\,dt\right)\right], \tag{11.21}$$

where $\Gamma(t) = \lim_{x\to\infty} \Gamma_{2x}(t)$ is the solution of the equation

$$\Gamma(t) + H(t) + \int_0^\infty \Gamma(s)H(t-s)\,ds = 0. \tag{11.22}$$

That is, this equation is satisfied by the function

$$\Gamma(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (M(k) - 1)e^{-ikt}\,dk, \tag{11.23}$$

introduced in Sec. 6 [see (6.3)]. This is not difficult to show if one takes the Fourier transform of the identity (11.3) re-expressed in the form

$$\begin{aligned}(W(k) - 1)(M(k) - 1) + (M(k) - 1)&\\ + (W(k) - 1) = 1/M(-k) - 1,&\end{aligned} \tag{11.24}$$

and takes into consideration that $1/M(-k)$ is analytic in the lower half-plane. Thus, from (11.21), it follows that

$$\varphi(x, k) \to (1/k)\ \mathrm{Im}\ \{M(-k)e^{-ikx}\} \tag{11.25}$$

as $x \to \infty$ and this, in turn, yields the asymptotic behavior (11.2).

12. RELATIONSHIP OF THE OPERATORS TO THE DISCRETE SPECTRA

In line with our program, we have completed the solution of the first problem and in this section we solve the second problem. We must consider the Gel'fand-Levitan equation (8.14) for the case where the given operator L_1 has no discrete spectrum, i.e. all $C_{n_1} = 0$, and where the required operator has the discrete eigenvalues $\lambda_n = -\kappa_n^2$ $(n = 1, 2, \cdots, m)$ with corresponding normalizing factors C_n and the same W function as L_1. The equation in this case is given by

$$K(x, y) + \Omega(x, y) + \int_0^x K(x, t)\Omega(t, y)\,dt = 0, \qquad x > y, \tag{12.1}$$

with

$$\Omega(x, y) = \sum_{n=1}^{m} C_n\varphi_1(x, i\kappa_n)\varphi_1(y, i\kappa_n). \tag{12.2}$$

This is an equation with a degenerate kernel and is easily solved. For this, it is advantageous to use vector notation. Thus we write $\Omega(x, y)$ in the form

$$\Omega(x, y) = (\boldsymbol{\psi}(x), \boldsymbol{\varphi}(y)), \tag{12.3}$$

where $\boldsymbol{\psi}(x)$ and $\boldsymbol{\varphi}(y)$ are vectors with the components $C_n\varphi_1(x, i\kappa_n)$ and $\varphi_1(y, i\kappa_n)$, respectively. We seek a solution of (12.1) in the form

$$K(x, y) = (\mathbf{a}(x), \boldsymbol{\varphi}(y)), \tag{12.4}$$

and the Gel'fand-Levitan equation becomes

$$(\mathbf{a}(x), \boldsymbol{\varphi}(y)) + (\boldsymbol{\psi}(x), \boldsymbol{\varphi}(y)) + \left(\int_0^x R(t)\,dt\mathbf{a}(x), \boldsymbol{\varphi}(y)\right) = 0. \tag{12.5}$$

Here $R(t)$ is the tensor product of the vectors $\boldsymbol{\varphi}(t)$ and $\boldsymbol{\psi}(t)$, i.e., the matrix with elements

$$r_{ik}(t) = \varphi_i(t)\psi_k(t) \qquad (i, k = 1, \cdots, m). \tag{12.6}$$

Consider the matrix

$$V(x) = I + \int_0^x R(t)\,dt. \tag{12.7}$$

The general result on the existence of a solution of the Gel'fand-Levitan equation enables us to say that the matrix $V(x)$ has an inverse for all x. However, from its definition, one may easily verify that $V(x)$ is positive-definite.

By virtue of (12.4) and (12.5), we have

$$K(x, y) = -[V(x)^{-1}\boldsymbol{\psi}(x), \boldsymbol{\varphi}(y)], \tag{12.8}$$

and, in particular,

$$\begin{aligned}K(x, x) &= -[V(x)^{-1}\boldsymbol{\psi}(x), \boldsymbol{\varphi}(x)] = -\mathrm{Tr}\,[V(x)^{-1}R(x)]\\ &= -\mathrm{Tr}\left[V(x)^{-1}\frac{d}{dx}V(x)\right]\\ &= -\frac{d}{dx}\ln\det V(x),\end{aligned} \tag{12.9}$$

since

$$R(x) = dV(x)/dx. \tag{12.10}$$

By an analysis of the general Gel'fand-Levitan equation (8.14) similar to that previously carried out for (8.5), it may be shown that in the x representation, the operator

$$L = U_B L_1 U_B^{-1}, \tag{12.11}$$

defined in terms of the solution $K(x, y)$ of this equation, is a differential operator with a potential $q(x) = q_1(x) + \Delta q(x)$, where

$$\Delta q(x) = 2\,dK(x, x)/dx. \tag{12.12}$$

Moreover, its eigenfunctions are determined by

$$\varphi(x, k) = \varphi_1(x, k) + \int_0^x K(x, y)\varphi_1(y, k)\,dy, \tag{12.13}$$

its W function is the function $W(k)$, and the quantities $\lambda_n = -\kappa_n^2$ $(n = 1, \cdots, m)$ are the points of its discrete spectrum. Such an analysis will not be carried out, and the above statements for the problem in question will be proved by other simpler means. In Sec. 15, it will be shown that the function $\varphi(x, k)$ determined by (12.13) is a solution of (1.1) with a potential $q(x) = q_1(x) + \Delta q(x)$, where $\Delta q(x)$ is given by (12.12). In this section, the properties of these functions will be examined.

By (12.8), (12.9), and (12.12), the following formulas for $\Delta q(x)$ and $\varphi(x, k)$ hold:

$$\Delta q(x) = -2(d^2/dx^2) \ln \|V(x)\|, \qquad \|V(x)\| = \det V(x), \tag{12.14}$$

$$\varphi(x, k) = \left\| \begin{matrix} V(x) & \psi(x) \\ \beta(x, k) & \varphi_1(x, k) \end{matrix} \right\| \Big/ \|V(x)\|. \tag{12.15}$$

The determinant in the numerator of (12.15) is that of the matrix obtained by bordering the matrix $V(x)$ with the vectors $\psi(x)$ and $\beta(x, k)$ and the function $\varphi_1(x, k)$, where

$$\beta(x, k) = \int_0^x \varphi(y)\varphi_1(y, k)\, dy. \tag{12.16}$$

The M function of L can be found from the asymptotic expansion of the solution $\varphi(x, k)$. We next investigate this asymptotic behavior and also the properties of the potential increment $\Delta q(x)$ directly from (12.14) and (12.15), confining ourselves, for simplicity, to the case $m = 1$. We make use of the fact that $\varphi_1(x, i\alpha)$, $\alpha > 0$, behaves asymptotically like [see (1.7)]

$$\varphi_1(x, i\alpha) = Ne^{\alpha x}[1 + o(1)], \qquad (x \to \infty), \tag{12.17}$$

and

$$\varphi_1(x, i\alpha) = x[1 + o(1)]x, \qquad (x \to 0). \tag{12.18}$$

Express $\varphi(x, k)$ as the sum of two terms

$$\varphi(x, k) = (1/2ik)[h(x, k) - h(x, -k)], \tag{12.19}$$

where

$$h(x, k) = M(-k) \exp(ikx) + o(1) \tag{12.20}$$

as $x \to \infty$. Equation (12.15) then yields

$$h(x, k) = h_1(x, k) - \frac{C\varphi_1(x, i\kappa)}{1 + C\int_0^x [\varphi_1(t, i\kappa)]^2\, dt} \times \int_0^x \varphi_1(x, i\kappa)h_1(x, k)\, dx$$

$$= M_1(-k)\left[e^{ikx} - \frac{CNe^{\kappa x}}{C(N^2/2\kappa)e^{2\kappa x}} \frac{N}{\kappa + ik} e^{ikx+\kappa x}\right] + o(1) = M_1(-k)\frac{k + i\kappa}{k - i\kappa} e^{ikx} + o(1), \tag{12.21}$$

so that

$$M(k) = M_1(k)(k - i\kappa)/(k + i\kappa). \tag{12.22}$$

Hence, it follows that $\lambda = -\kappa^2$ is an eigenvalue of L.

Let us now consider $\Delta q(x)$. By virtue of its definition (12.7), $V(x)$ is twice differentiable even if the potential $q_1(x)$ for the given operator is a generalized function with δ-type singularities. Therefore, $\Delta q(x)$ is always an ordinary summable function. As $x \to 0$,

$$\Delta q(x) = -2\frac{d^2}{dx^2} \ln\left\{1 + C\int_0^x t^2[1 + o(1)]\, dt\right\} = -4Cx[1 + o(1)], \tag{12.23}$$

and as $x \to \infty$,

$$\Delta q(x) = -2\frac{V''(x)V(x) - (V'(x))^2}{V^2(x)} = -(2/C)(2\kappa)^3 e^{-2\kappa x}[1 + o(1)]. \tag{12.24}$$

If $m > 1$, asymptotic forms similar to (12.23) and (12.24) still hold for $\Delta q(x)$, in which $C = \sum_{n=1}^m C_n$ in (12.23) and $C = C_r$, $\kappa = \kappa_r$ in (12.24), κ_r being the smallest of the κ_n. At all events,

$$\int_0^\infty x\,|\Delta q(x)|\, dx < \infty. \tag{12.25}$$

The solution of the second problem of Sec. 8, except for the one statement to be proved in Sec. 15, is thus complete. The solution is given by (12.14) and (12.15). In these formulas, $C_1, C_2, \cdots C_m$ may be any positive numbers, so that we have an m-parameter family of solutions. Formula (12.14) is the formula for the equivalent potentials given by Jost and Kohn.

This simultaneously completes the solution of the basic inverse problem. The results obtained in Secs. 9 to 12 are summarized in the following theorem.

Theorem 12.1. *Any function $S(k)$ with the properties*

(1) $|S(k)| = S(\infty) = S(0) = 1$,

(2) $S(-k) = \overline{S}(k) = S^{-1}(k)$,

(3) $S(k) = 1 + \int_{-\infty}^\infty F(t) \exp(-ikt)\, dt$,

where $\int_{-\infty}^\infty |F(t)|\, dt < \infty$,

(4) $\arg S(k) \,|_{-\infty}^{\infty} = -4i\pi m, \qquad m \geq 0,$

is the S function of some operator having m negative eigenvalues and a continuous spectrum along the half-ray $(0, \infty)$. In the x representation, it is a differential operator of type (1.1) *with a potential that may be a generalized function such as the derivative of a locally summable function.*

In order for the condition $\int_0^\infty x\,|q(x)|\, dx < \infty$ to hold, it is necessary and sufficient that $\int_0^\infty t\,|F'(t)|\, dt < \infty$.

If $m > 0$, the potential is not uniquely determined. An m-parameter family of potentials exists such that the associated operators have $S(k)$ as their S function and the given quantities $\lambda_n = -\kappa_n^2$ as eigenvalues.

13. OPERATORS WITH M FUNCTIONS DIFFERING BY A RATIONAL FACTOR

The general Gel'fand-Levitan equation was solved in closed form for the case treated in the preceding section. This is not the only situation in which this is possible. It turns out that one may explicitly solve the operator equation for the transformation operator relating two operators L_1 and L_2 whose M functions differ by a rational factor. We shall consider the case where the given operator L_1 with potential $q_1(x)$ has no discrete spectrum, and we shall construct the operator L_2 whose M function is given by

$$M_2(k) = M_1(k) \prod_{l=1}^{N} \frac{k + i\alpha_l}{k + i\beta_l}, \quad \alpha_l > 0, \quad \beta_l > 0. \tag{13.1}$$

The condition $\beta_l > 0$ assures that $M_2(k)$ will be regular in the upper half-plane. The condition $\alpha_l > 0$ does not disturb the generality of the discussion for since

$$\frac{k - i\alpha}{k + i\beta} = \frac{k - i\alpha}{k + i\alpha}\frac{k + i\alpha}{k + i\beta}, \tag{13.2}$$

we may first consider the transformation

$$M(k) = M_1(k)(k - i\alpha)/(k + i\beta), \tag{13.3}$$

and then perform the transformation

$$M_2(k) = M_1(k)(k - i\alpha)/(k + i\alpha), \tag{13.4}$$

as was done in the preceding section.

We shall also assume that the α_l and β_l are all distinct. The equation for the related transformation operator is given by

$$K(x, y) + \Omega(x, y) + \int_0^x K(x, t)\Omega(t, y)\, dt = 0, \qquad y < x, \tag{13.5}$$

$$\begin{aligned}\Omega(x, y) &= \frac{2}{\pi}\int_0^\infty \varphi_1(x, k)[W_2(k) - W_1(k)]\varphi_1(y, k)k^2\, dk \\ &= \frac{2}{\pi}\int_0^\infty \varphi_1(x, k)W_1(k) \\ &\quad \times \left[\prod_{l=1}^{N} \frac{k^2 + \beta_l^2}{k^2 + \alpha_l^2} - 1\right]\varphi_1(y, k)k^2\, dk. \end{aligned}\tag{13.6}$$

If the term in brackets is resolved into partial fractions:

$$\prod_{l=1}^{N} \frac{k^2 + \beta_l^2}{k^2 + \alpha_l^2} - 1 = \sum_{l=1}^{N} \frac{A_l}{k^2 + \alpha_l^2}, \tag{13.7}$$

where

$$A_j = \prod_{l=1}^{N} (\beta_l^2 - \alpha_j^2) \Big/ \prod_{j \neq l}^{N} (\alpha_l^2 - \alpha_j^2), \tag{13.8}$$

then $\Omega(x, y)$ may be expressed in the form

$$\begin{aligned}\Omega(x, y) &= \frac{2}{\pi}\int_0^\infty \varphi_1(x, k) \sum_{l=1}^{N} \frac{A_l}{k^2 + \alpha_l^2} \\ &\quad \times W_1(k)\varphi_1(y, k)k^2\, dk = \sum_{l=1}^{N} A_l g_{\alpha_l}(x, y).\end{aligned}\tag{13.9}$$

Here, $g_\alpha(x, y)$ is the resolvent kernel of the operator L_1 for $\lambda = -\alpha^2$. In Sec. 2, it was seen that

$$g_\alpha(x, y) = \varphi_1(x, i\alpha)f_1(y, i\alpha)/M_1(i\alpha),$$
$$x < y, \qquad g_\alpha(x, y) = g_\alpha(y, x), \tag{13.10}$$

is a solution of

$$[-d^2/dx^2 + \alpha^2 + q_1(x)]g_\alpha(x, y) = \delta(x - y). \tag{13.11}$$

Now, any solution of

$$-\psi''(x) - k^2\psi(x) + q_1(x)\psi(x) = 0 \tag{13.12}$$

satisfies

$$(\alpha^2 + k^2)\int_a^b g_\alpha(x, y)\psi(y)\, dy = \psi(x) + [g_\alpha(x, y); \psi(y)]\,\Big|_{y=a}^{y=b}, \tag{13.13}$$

where

$$[\varphi(x); \psi(x)] = \varphi'(x)\psi(x) - \varphi(x)\psi'(x). \tag{13.14}$$

We now seek a solution of (13.5) in the form

$$K(x, y) = \sum_{j=1}^{N} a_j(x)\varphi_j^\beta(y), \tag{13.15}$$

where

$$\varphi_j^\beta(y) = \varphi_1(y, i\beta_j) \qquad (j = 1, \cdots, N). \tag{13.16}$$

The substitution of (13.15) into (13.5) gives

$$\sum_{j=1}^{N} a_j(x)\varphi_j^\beta(y) + \sum_{l=1}^{N} A_l\varphi_l^\alpha(y)\frac{f_l^\alpha(x)}{M_l^\alpha} + \sum_{j=1}^{N}\sum_{l=1}^{N} A_l a_j(x)\int_0^x \varphi_j^\beta(t)g_l^\alpha(t, y)\, dt = 0. \tag{13.17}$$

By (13.13), the last term can be transformed as follows:

$$\begin{aligned}&\sum_{j=1}^{N}\sum_{l=1}^{N} A_l a_j(x)\int_0^x \varphi_j^\beta(t)g_l^\alpha(t, y)\, dt \\ &= \sum_{j=1}^{N}\sum_{l=1}^{N} a_j(x)\frac{A_l}{\alpha_l^2 - \beta_j^2}\varphi_j^\beta(y) \\ &\quad + \sum_{j=1}^{N}\sum_{l=1}^{N} a_j(x)[g_l^\alpha(x, y); \varphi_j^\beta(x)]\frac{A_l}{\alpha_l^2 - \beta_j^2}.\end{aligned}\tag{13.18}$$

The first term in this cancels with the first term in (13.17), and (13.17) may therefore be written as

$$\sum_{l=1}^{N} \frac{A_l \varphi_l^{\alpha}(y)}{M_l^{\alpha}} \left[f_l^{\alpha}(x) + \sum_{j=1}^{N} W_{lj}(x) a_j(x) \right] = 0, \quad (13.19)$$

where

$$W_{lj}(x) = (\alpha_l^2 - \beta_j^2)^{-1}[f_l^{\alpha}(x); \varphi_j^{\beta}(x)]. \quad (13.20)$$

It is again advantageous to use vector notation. Due to the linear independence of the φ_l^{α}, (13.19) is equivalent to

$$\mathbf{f}(x) + W(x)\mathbf{a}(x) = 0, \quad (13.21)$$

from which it follows that

$$\mathbf{a}(x) = -W^{-1}(x)\mathbf{f}(x) \quad (13.22)$$

and

$$K(x, y) = (\mathbf{a}(x), \boldsymbol{\varphi}(y)) = -(W^{-1}(x)\mathbf{f}(x), \boldsymbol{\varphi}(y)). \quad (13.23)$$

The basic property of the elements of the matrix $W(x)$ is similar to the property (12.10) of the elements of the matrix $V(x)$:

$$dW_{lj}(x)/dx = f_l^{\alpha}(x)\varphi_j^{\beta}(x). \quad (13.24)$$

This relation enables one to again obtain elegant expressions for $\Delta q(x)$ and $\varphi_2(x, k)$:

$$\Delta q(x) = -2(d^2/dx^2) \ln ||W(x)||, \quad (13.25)$$

$$\varphi_2(x, k) = \left\| \begin{matrix} W(x) & \mathbf{f}(x) \\ \boldsymbol{\beta}(x, k) & \varphi_1(x, k) \end{matrix} \right\| \Big/ ||W(x)||, \quad (13.26)$$

where

$$\beta_j(x, k) = \frac{[\varphi_j^{\beta}(x); \varphi_1(x, k)]}{\beta_j^2 + k^2} = \int_0^x \varphi_j^{\beta}(t)\varphi_1(t, k)\, dt. \quad (13.27)$$

The result (13.25) is called Bargmann's formula.

In Sec. 15, it will be shown that $\varphi_2(x, k)$ actually is a solution of (1.1) with the potential $q_2(x) = q_1(x) + \Delta q(x)$. The asymptotic form of $\varphi_2(x, k)$ can be deduced directly from the explicit form of (13.26) just as was done in the preceding section. It is given by

$$\varphi_2(x, k) = (1/(2ik)) \times [M_2(-k)e^{ikx} - M_2(k)e^{-ikx}] + o(1), \quad (13.28)$$

where

$$M_2(k) = M_1(k) \prod_{l=1}^{N} \frac{k + i\alpha_l}{k + i\beta_l}, \quad (13.29)$$

and this proves the correctness of our solution.

The behavior of $\Delta q(x)$ as $x \to 0$ or $x \to \infty$ is easily analyzed. For $x \to 0$, $\Delta q(x)$ decreases exponentially as before, however, $\Delta q(x)$ no longer tends to zero when $x \to 0$. Let us consider one factor for simplicity. From (13.24), we have

$$W(x) = W(0) + \int_0^x f_1(x, i\alpha)\varphi_1(x, i\beta)\, dx. \quad (13.30)$$

But from (13.20), it follows that

$$W(0) = M_1(i\alpha)/(\alpha^2 - \beta^2)$$

and hence

$$W(x) = \frac{M_1(i\alpha)}{\alpha^2 - \beta^2}\left(1 + (\alpha^2 - \beta^2)\frac{x^2}{2}[1 + o(1)]\right), \quad (13.31)$$

$$\Delta q(0) = -2(\alpha^2 - \beta^2). \quad (13.32)$$

The above formulas may be used to determine the potential approximately for a given $S(k)$. Any $S(k)$ may be approximated by a rational function $S_R(k)$. The corresponding approximate function $M_R(k)$ is then given by a product

$$M_R(k) = \prod_{l=1}^{N} \frac{k + i\alpha_l}{k + i\beta_l}. \quad (13.33)$$

The potential constructed from this function can be written in terms of trigonometric functions. Generally speaking, its asymptotic behavior will differ from that of the exact solution. But in some average sense, the behavior of the potential will be described by the approximate solution rather well.

As an example, take the function

$$M(k) = (k + i\alpha)/(k + i\beta). \quad (13.34)$$

The corresponding phase is given by

$$\cot \eta(k) = \frac{\alpha\beta}{\alpha - \beta}\frac{1}{k} + \frac{1}{\alpha - \beta}k, \quad (13.35)$$

and there is no discrete spectrum. For the potential $q(x)$ one obtains

$$q(x) = 2\frac{\beta^2(\beta^2 - \alpha^2)}{(\beta \cosh \beta x + \alpha \sinh \beta x)^2}. \quad (13.36)$$

The expression for $\varphi(x, k)$ is more involved and will not be given.

14. THE CASE $l > 0$

Equation (1.1), heretofore considered, is a particular case of

$$L_y^{(l)} = -y'' + [q(x) + l(l + 1)/x^2]y = s^2 y, \quad (14.1)$$

which results when variables are separated in the three-dimensional Schrödinger equation with a spherically-symmetric potential $q(x)$:

$$-\Delta u + qu = s^2 u. \tag{14.2}$$

In this section, the properties obtained in the preceding sections for the operator L will be extended to the operator $L^{(l)}$. Let

$$\varphi_0^{(l)}(x, s) = (1/s^{l+1}) j_l(sx), \tag{14.3}$$

$$f_0^{(l)}(x, s) = (i)^{l+1} h_l^{(1)}(sx), \tag{14.4}$$

where $j_l(x)$, $h_l^{(1)}(x)$ are spherical Bessel functions. The spherical function corresponding to any cylindrical function is given by

$$z_l(x) = (\tfrac{1}{2}\pi x)^{1/2} Z_{l+1/2}(x) \quad (l = 0, 1, 2, \cdots). \tag{14.5}$$

The above functions behave in the following way:

$$\left.\begin{aligned} \varphi_0^{(l)}(x, s)\,|_{x\to 0} &= \lfloor x^{l+1}/(2l+1)!!\rceil[1 + o(1)], \\ \varphi_0^{(l)}(x, s)\,|_{x\to\infty} &= (1/s^{l+1}) \sin(sx - \tfrac{1}{2}l\pi) + o(1), \end{aligned}\right\} \tag{14.6}$$

$$\begin{aligned} f_0^{(l)}(x, s)\,|_{x\to 0} &= [i^l(2l-1)!!/(sx)^l][1 + o(1)], \\ f_0^{(l)}(x, s)\,|_{x\to\infty} &= e^{isx} + o(1). \end{aligned} \tag{14.7}$$

They play roles in the case $l > 0$ analogous to $\sin(sx)/x$ and e^{isx}. The solutions $\varphi^{(l)}(x, s)$ and $f^{(l)}(x, s)$ [the generalizations of $\varphi(x, s)$ and $f(x, s)$], are determined by the conditions

$$\lim_{x\to 0} [(2l+1)!!/x^{l+1}]\varphi^{(l)}(x, s) = 1, \tag{14.8}$$

$$\lim_{x\to\infty} e^{-isx} f^{(l)}(x, s) = 1. \tag{14.9}$$

The integral equations, similar to (1.5) and (1.6) and equivalent to (14.1) with the conditions (14.8) and (14.9), are given by

$$\varphi^{(l)}(x, s) = \varphi_0^{(l)}(x, s) + \int_0^x J^{(l)}(s; x, t) q(t) \varphi^{(l)}(t, s)\, dt, \tag{14.10}$$

$$f^{(l)}(x, s) = f_0^{(l)}(x, s) - \int_x^\infty J^{(l)}(s; x, t) q(t) f^{(l)}(t, s)\, dt. \tag{14.11}$$

Here,

$$J^{(l)}(s; x, t) = (is)^l [\varphi_0^{(l)}(x, s) f_0^{(l)}(t, -s) - \varphi_0^{(l)}(t, s) f_0^{(l)}(x, -s)]. \tag{14.12}$$

By means of these equations, the results of Sec. 1 carry over with respect to the behavior of $\varphi^{(l)}(x, s)$ and $f^{(l)}(x, s)$ in the complex s plane and for large x, etc., under the supposition that the potential $q(x)$ satisfies the condition

$$\int_0^\infty x\,|q(x)|\, dx < \infty. \tag{14.13}$$

Thus, for example, $\varphi^{(l)}(x, s)$ is an entire function of s, $f^{(l)}(x, s)$ is analytic in s in the half-plane $\tau > 0$, and

$$|\varphi^{(l)}(x, s)| \le K\left(\frac{x}{1 + |s|\,x}\right)^{l+1} e^{|\tau| x}, \tag{14.14}$$

$$|f^{(l)}(x, s)| \le K\left(\frac{1 + |s|\,x}{|s|\,x}\right)^{l} e^{-\tau x}, \quad \tau \ge 0, \tag{14.15}$$

For large $|s|$, $f^{(l)}(x, s)$ behaves asymptotically like

$$f^{(l)}(x, s) = e^{isx} + o(1), \qquad \tau \ge 0. \tag{14.16}$$

For real s, $\varphi^{(l)}(x, k)$ can be expressed in terms of $f^{(l)}(x, k)$:

$$\varphi^{(l)}(x, k) = (1/2ik)(1/ik)^l [f^{(l)}(x, k) M^{(l)}(-k) - (-1)^l f^{(l)}(x, -k) M^{(l)}(k)], \tag{14.17}$$

where

$$M^{(l)}(s) = \lim_{x\to 0} \frac{(sx)^l}{i^l(2l-1)!!} f^{(l)}(x, s). \tag{14.18}$$

The function $M^{(l)}(s)$ has the same properties as $M(s)$, enumerated in Lemma 1.6. However, the formula for the normalization constant C_n becomes

$$C_n^{(l)} = \int_0^\infty [\varphi^{(l)}(x, i\kappa_n)]^2\, dx = \frac{\dot{M}(i\kappa_n)}{2i\kappa_n^{l+1}(2l+1)!!} \left[\lim_{x\to 0} \frac{f^{(l)}(x, i\kappa_n)}{x^{l+1}}\right]^{-1}. \tag{14.19}$$

As before, we assume that $M^{(l)}(0) \ne 0$. As earlier, the condition $M^{(l)}(0) = 0$ is not stable, and therefore our restriction is of no consequence. With this assumption, the solution $\varphi^{(l)}(x, 0)$ behaves asymptotically for large x like

$$\varphi^{(l)}(x, 0) = Ax^{l+1}[1 + o(1)]. \tag{14.20}$$

The results concerning the expansion theorem, the existence of transformation operators, and the asymptotic expansion of the solution of the Schrödinger equation for large time all generalize to the present case. Thus, the completeness relation for the eigenfunctions becomes

$$\sum_{n=1}^{m} C_n \varphi_n(x) \varphi_n(y) + \frac{2}{\pi} \int_0^\infty \varphi^{(l)}(x, k) \frac{1}{M^{(l)}(k) M^{(l)}(-k)} \times \varphi^{(l)}(y, k) k^{2(l+1)}\, dk = \delta(x - y). \tag{14.21}$$

Furthermore, the following representations for

$\varphi^{(l)}(x, s)$ and $f^{(l)}(x, s)$ hold:

$$\varphi^{(l)}(x, s) = \varphi_0^{(l)}(x, s) + \int_0^x K^{(l)}(x, t)\varphi_0^{(l)}(t, s)\, dt, \tag{14.22}$$

$$f^{(l)}(x, s) = f_0^{(l)}(x, s) + \int_x^\infty A^{(l)}(x, t) f_0^{(l)}(t, s)\, dt. \tag{14.23}$$

The scattering operator associated with $L^{(l)}$ is determined by the formula

$$S^{(l)}(k) = M^{(l)}(-k)/M^{(l)}(k). \tag{14.24}$$

This same function occurs in the asymptotic representation of the normalized eigenfunction $\psi^{(l)}(x, k) = (ik)^l\varphi^{(l)}(x, k)/M^{(l)}(k)$:

$$\psi^{(l)}(x, k)\,|_{x\to\infty} = \frac{1}{2ik}[S^{(l)}(k)e^{ikx} - (-1)^l e^{-ikx}] + o(1). \tag{14.25}$$

As before, these results allow one to solve the inverse problem, i.e., the reconstruction of $L^{(l)}$ from its S function. Thus the kernel $K^{(l)}(x, y)$ satisfies the equation

$$K^{(l)}(x, y) + \Omega^{(l)}(x, y) + \int_0^x K^{(l)}(x, t)\Omega^{(l)}(t, y)\, dt = 0, \quad x > y, \tag{14.26}$$

where

$$\Omega^{(l)}(x, y) = \sum_{n=1}^m C_n\varphi_0^{(l)}(x, i\kappa_n)\varphi_0^{(l)}(y, i\kappa_n) + \frac{2}{\pi}\int_0^\infty \varphi_0^{(l)}(x, k)\frac{1}{M^{(l)}(k)M^{(l)}(-k)} \times \varphi_0^{(l)}(y, k)k^{2(l+1)}\, dk. \tag{14.27}$$

By a repetition of our earlier reasoning, this equation can be shown to have a unique solution, and the completeness relation (14.21) can be derived for the functions $\varphi^{(l)}(x, k)$, determined by its solution $K^{(l)}(x, y)$ according to (14.22). Moreover, a differential equation of the form (14.11) can be derived for $\varphi^{(l)}(x, k)$. However, tracing the relationship between $S^{(l)}(k)$ and the potential $q(x)$ by means of Eq. (14.26) or the analog of the Marchenko equation turns out to be difficult. This is due to the fact that the kernel of Eq. (14.26) is expressed in terms of Bessel functions for which there is no simple addition formula such as exists for the trigonometric functions. Of course, conditions such as (6.7) and (6.9) still hold for the S function in the present case. As to the analog of (6.8), it is not clear beforehand that it is convenient to formulate one in terms of the Fourier transform of $S^{(l)}(k)$. One might have thought that the Fourier transform naturally arises in the case $l = 0$ because we are dealing with trigonometric functions. All the more surprising is the fact that the behavior of the S function for $L^{(l)}$ turns out to be no different than that of the S function for $L^{(0)}$. More precisely, we have

Theorem 14.1. *If $S(k)$ is the S function for the operator $L^{(l)}$ associated with the differential equation* (14.1) *with potential $q(x)$, then it is also the S function of an operator $L^{(m)}$ for any $m = 0, 1, 2, \cdots, l+1, \cdots$, where the corresponding potential $q^{(m)}(x)$ behaves like $q(x)$ as $x \to 0$ and $x \to \infty$.*

The proof of this theorem will be given in the following section. Also, the sense in which the behavior of the potentials $q(x)$ and $q^{(m)}(x)$ is analogous will be made more precise there.

15. TRANSFORMATION OF STURM-LIOUVILLE TYPE EQUATIONS

In the preceding sections, we mentioned repeatedly that a number of statements would be proved in Sec. 15 on the basis of a certain general method. The essence of this method will now be presented from which these statements will then follow.

Let $y_0(x)$ be some particular solution of

$$-y'' + q(x)y = \lambda y \tag{15.1}$$

for $\lambda = \lambda_0$ which does not vanish in the neighborhood of the point $x = a$. Consider the expression

$$y_1(x, \lambda) = [y(x, \lambda); y_0(x)]/[(\lambda - \lambda_0)y_0(x)], \tag{15.2}$$

where $y(x, \lambda)$ is an arbitrary solution of (15.1) and $[\varphi; \psi]$ is the Wronskian of φ and ψ:

$$[\varphi; \psi] = \varphi'(x)\psi(x) - \varphi(x)\psi'(x). \tag{15.3}$$

Lemma 15.1. *The function $y_1(x, \lambda)$ is a solution of* (15.1) *with a potential $q_1(x) = q(x) + \Delta q(x)$, where*

$$\Delta q(x) = -2\frac{d}{dx}\frac{y_0'(x)}{y_0(x)} = -2\frac{d^2}{dx^2}\ln y_0(x). \tag{15.4}$$

To prove this, we note that any two solutions of (15.1) satisfy the equality

$$\frac{d}{dx}\frac{[y(x, \lambda_1); y(x, \lambda_2)]}{\lambda_1 - \lambda_2} = y(x, \lambda_1)y(x, \lambda_2). \tag{15.5}$$

Therefore,

$$y_1'(x, \lambda) = y(x, \lambda) - \frac{[y(x, \lambda); y_0(x)]}{(\lambda - \lambda_0)y_0^2(x)}y_0'(x) = y(x, \lambda) - y_1(x, \lambda)v(x), \tag{15.6}$$

where the function $v(x) = y_0'(x)/y_0(x)$ is a solution of the Riccati equation

$$v'(x) + v^2(x) = q(x) - \lambda_0. \qquad (15.7)$$

Differentiating (15.6) once and using (15.2), (15.6), and (15.7), we find that

$$\begin{aligned} y_1''(x,\lambda) &= y'(x,\lambda) - y_1'(x,\lambda)v(x) - y_1(x,\lambda)v'(x) \\ &= y'(x,\lambda) - y(x,\lambda)y_0'(x)/y_0(x) \\ &\quad + y_1(x,\lambda)v^2(x) - y_1(x,\lambda)v'(x) \\ &= -[y(x,\lambda); y_0(x)]/y_0(x) \\ &\quad + y_1(x,\lambda)[v^2(x) + v'(x)] - 2y_1(x,\lambda)v'(x) \\ &= [\lambda_0 - \lambda + v^2(x) + v'(x) - 2v'(x)]y_1(x,\lambda), \end{aligned}$$

i.e.,

$$-y_1''(x,\lambda) + q_1(x)y_1(x,\lambda) = \lambda y_1(x,\lambda). \qquad (15.8)$$

This proves the lemma.

Formula (15.2) is meaningful only if $\lambda \neq \lambda_0$. If $\lambda = \lambda_0$, one of the solutions of the transformed equation is

$$z_{10}(x) = 1/y_0(x). \qquad (15.9)$$

As a second linearly independent solution, one may take

$$y_{10}(x) = z_{10}(x)\int^x \frac{dt}{z_{10}^2(t)} = \frac{1}{y_0(x)}\int^x y_0^2(t)\,dt. \qquad (15.10)$$

Of course, the solutions (15.9) and (15.10) may be deduced from (15.2) by a limiting process. Conversely, the function $y(x, \lambda)$ is expressible in terms of solutions of the transformed equation. For this it suffices to note that (15.9) yields

$$y_0'(x)/y_0(x) = -z_{10}'(x)/z_{10}(x) \qquad (15.11)$$

and, therefore, (15.6) can be written in the form

$$\begin{aligned} y(x,\lambda) &= y_1'(x,\lambda) - y_1(x,\lambda)z_{10}'(x)/z_{10}(x) \\ &= [y_1(x,\lambda); z_{10}(x)]/z_{10}(x). \qquad (15.12) \end{aligned}$$

This expression defines the transformation inverse to (15.2).

Thus far, we have considered the transformation (15.2) in the neighborhood of $x = a$ where the mapping solution y_0 does not vanish. We now describe the behavior of the solution of the transformed equation when this condition fails to hold. With the application to (1.1) and (14.1) in mind, we shall assume that the singular point occurs at $x = 0$ and that in the neighborhood of $x = 0$, the given potential has the singular behavior

$$q(x) = l(l+1)/x^2 + O(1/x^{2-\epsilon}), \quad \epsilon > 0. \qquad (15.13)$$

This requirement is somewhat stronger than the condition imposed on the potential in Secs. 1 and 14, namely,

$$\int_0^\infty x\,|q(x)|\,dx < \infty, \qquad (15.14)$$

but it simplifies considerably all of the calculations. In this case, (15.1) will be said to have an *l singularity* at $x = 0$. Two types of solutions exist in the neighborhood of $x = 0$, one regular:

$$y(x,\lambda) = C(\lambda)x^{l+1}[1 + O(x^\epsilon)], \qquad (15.15)$$

and the other irregular:

$$z(x,\lambda) = [D(\lambda)/x^l][1 + O(x^\epsilon)]. \qquad (15.16)$$

Evidently, all regular solutions differ only by a factor.

We now perform a transformation using a regular solution $y(x, \lambda_0)$ of (15.1). As a result, to the potential $q(x)$ is added the term

$$\begin{aligned} \Delta q(x) &= -2(d^2/dx^2)\ln x^{l+1}[1 + O(x^\epsilon)] \\ &= 2(l+1)/x^2 + O\left(\frac{1}{x^{2-\epsilon}}\right) \qquad (15.17) \end{aligned}$$

so that the equation has an $(l + 1)$ singularity at $x = 0$. In the derivation of (15.17), the asymptotic representation for $\Delta q(x)$ has been differentiated, but this can be rigorously justified.

Let us observe what happens to a regular and irregular solution under our transformation. By virtue of (15.5) and (15.15)

$$\frac{[y(x,\lambda); y(x,\lambda_0)]}{(\lambda - \lambda_0)} = \int_0^x y(t,\lambda)y(t,\lambda_0)\,dt. \qquad (15.18)$$

Therefore,

$$\begin{aligned} y_1(x,\lambda) &= C(\lambda)\int_0^x x^{2(l+1)}[1 + O(x^\epsilon)]\,dx/x^{l+1} \\ &= \frac{C(\lambda)}{2l+3}x^{l+2}[1 + O(x^\epsilon)]. \qquad (15.19) \end{aligned}$$

Furthermore,

$$\begin{aligned} &[z(x,\lambda); y(x,\lambda_0)] \\ &\qquad = (2l+1)\,D(\lambda)C(\lambda_0)[1 + O(x^\epsilon)] \qquad (15.20) \end{aligned}$$

so that

$$z_1(x,\lambda) = [(2l+1)\,D(\lambda)/x^{l+1}][1 + O(x^\epsilon)]. \qquad (15.21)$$

In the following, (15.2) will be used to transform a regular solution and (15.12) to transform an irregular one. We have thus shown that under a transformation with a regular solution, the l singularity at the origin is increased by one, and regular and irregular solutions go into regular and irregular solutions, respectively. It is not difficult to see that transforming with an irregular solution lowers the l singularity of the equation by one but again preserves the nature of the respective solutions.

Thus, to obtain an equation having the same l singularity as the given equation, we must perform two successive transformations, the first using a regular solution of the given equation and the second then using an irregular solution of the transformed equation. It turns out that the transformations constructed in Secs. 12 and 13 by means of the Gel'fand-Levitan equation could be deduced in this way. We illustrate this by transforming two operators whose spectra differ by a single eigenvalue. Consider an equation of type (1.1) with a potential satisfying condition (1.2) and assume that none of the points in the discrete spectrum of the associated operator L lie to the left of $\lambda = -\beta_0^2$. Take some $\beta > \beta_0$. Under our conditions, $\varphi_0(x) = \varphi(x, i\beta)$ vanishes only at $x = 0$. Perform the transformation (15.2) using this solution. Then the transformed equation will have an l singularity at $x = 0$ with $l = 1$. The function

$$\psi_1(x) = \frac{1}{\varphi_0(x)}\left[1 + C\int_0^x \varphi_0^2(t)\,dt\right] \tag{15.22}$$

will be an irregular solution of the transformed equation. Performing a transformation with this solution, we arrive at an equation which has no singularity at $x = 0$. Let us calculate the potential and solution $\varphi_2(x, k)$ associated with this equation. To do this, we must combine the formulas

$$\varphi_1(x, k) = \frac{[\varphi(x, k); \varphi_0(x)]}{(k^2 + \beta^2)\varphi_0(x)} = \frac{1}{\varphi_0(x)}\int_0^x \varphi(t, k)\varphi_0(t)\,dt, \tag{15.23}$$

$$\varphi_2(x, k) = [\varphi_1(x, k); \psi_1(x)]/\psi_1(x), \tag{15.24}$$

$$\Delta q(x) = -2(d^2/dx^2)\ln\varphi_0(x) - 2(d^2/dx^2)\ln\psi_1(x). \tag{15.25}$$

This leads to the following results

$$\varphi_2(x, k) = \varphi(x, k) - \left\{\varphi_0(x)\bigg/\left[1 + C\int_0^x \varphi_0^2(t)\,dt\right]\right\} \times \int_0^x \varphi(t, k)\varphi_0(t)\,dt, \tag{15.26}$$

$$\Delta q(x) = -2\frac{d^2}{dx^2}\ln\left[1 + C\int_0^x \varphi_0^2(t)\,dt\right]. \tag{15.27}$$

These formulas correspond exactly to (12.14) and (12.15). Thus by a direct verification, we have shown that the function $\varphi_2(x, k)$ determined by these formulas is a solution of (1.1) with the potential given by (15.25).

To justify the unproved statement of Sec. 13, we now perform the following operations: the transformation of the given equation using the solution $\varphi(x, i\beta)$; the transformation of the resulting equation using the solution $f_1(x, i\alpha)$ obtained from the solution $f(x, i\alpha)$ of the given equation by the first transformation. By combining the formulas

$$\varphi_1(x, k) = \frac{[\varphi(x, k); \varphi(x, i\beta)]}{(k^2 + \beta^2)\varphi(x, i\beta)} = \frac{1}{\varphi(x, i\beta)}\int_0^x \varphi(t, k)\varphi(t, i\beta)\,dt, \tag{15.28}$$

$$f_1(x, i\alpha) = \frac{[f(x, i\alpha); \varphi(x, i\beta)]}{(\beta^2 - \alpha^2)\varphi(x, i\beta)} = \frac{W(x)}{\varphi(x, i\beta)}, \tag{15.29}$$

$$\varphi_2(x, k) = \frac{[\varphi_1(x, k); f_1(x, i\alpha)]}{f_1(x, i\alpha)}, \tag{15.30}$$

$$\Delta q(x) = -2(d^2/dx^2)\ln\varphi(x, i\beta) - 2(d^2/dx^2)\ln f_1(x, i\alpha), \tag{15.31}$$

we find that

$$\varphi_2(x, k) = \varphi(x, k) - \frac{f(x, i\alpha)}{W(x)}\frac{[\varphi(x, k); \varphi(x, i\beta)]}{k^2 + \beta^2} \tag{15.32}$$

is the solution of (1.1) with the potential

$$q_2(x) = q(x) - 2(d^2/dx^2)\ln W(x) = q(x) - 2\frac{d^2}{dx^2}\ln\frac{[f(x, i\alpha); \varphi(x, i\beta)]}{\beta^2 - \alpha^2}. \tag{15.33}$$

This proves the statement of Sec. 13.

The above properties of the transformation will now be used to prove Theorem 14.1. A transformation using any regular solution changes an l singularity only at $x = 0$. The behavior of the potential increment $\Delta q(x)$ as $x \to \infty$ may be different depending on the location of the parameter λ in the complex plane. Thus, in the cases considered till now, the potential increment decreases exponentially as $x \to \infty$. However, there exists solutions which change the singularity of the equation in the same way both for $x \to 0$ and $x \to \infty$. Such a solution is the solution of (15.1) for $\lambda = 0$. If we assume that

$$q(x) = l(l + 1)/x^2 + O\left(\frac{1}{x^{2+\delta}}\right), \quad \delta > 0, \tag{15.34}$$

as $x \to \infty$, then using equations such as (14.10) and (14.11), we can show that the regular solution has the asymptotic representation

$$y(x, 0) = Cx^{l+1}[1 + O(1/x^\delta)]. \tag{15.35}$$

as $x \to \infty$ and among the irregular solutions there exists a $z(x, 0)$ such that

$$z(x, 0) = (D/x^l)[1 + O(1/x^\delta)] \qquad (15.36)$$

as $x \to \infty$. Under transformations using these solutions,

$$\Delta q(x) = -2(d^2/dx^2) \ln y(x, 0) = 2(l+1)/x^2 + O(1/x^{2+\delta}), \qquad (15.37)$$

$$\Delta q(x) = -2(d^2/dx^2) \ln z(x, 0) = -2l/x^2 + O(1/x^{2+\delta}). \qquad (15.38)$$

Thus, they have the required property of changing in an identical way the singularity of the equation for $x \to 0$ and $x \to \infty$. Let us see how the S functions change under transformations using these solutions. It is easily verified that the functions

$$f^{(l+1)}(x, k) = -[f^{(l)}(x, k); y(x, 0)]/iky(x, 0), \qquad (15.39)$$

$$f^{(l-1)}(x, k) = -[f^{(l)}(x, k); z(x, 0)]/ikz(x, 0) \qquad (15.40)$$

have the asymptotic behavior

$$f^{(l\pm 1)}(x, k) = e^{ikx} + o(1), \qquad (x \to \infty) \qquad (15.41)$$

and are thus solutions of the transformed equations analogous to $f^{(l)}(x, k)$. The function $M(k)$, determining the S function, occurs in the asymptotic formula for the solutions $f^{(l)}(x, k)$ as $x \to 0$:

$$f^{(l)}(x, k)\,|_{x\to 0} = [(2l-1)!!/(kx)^l]i^l M(k)[1 + O(x^\epsilon)]. \qquad (15.42)$$

However, by virtue of (15.20), we find that

$$f^{(l+1)}(x, k)\,|_{x\to 0} = [(2l+1)!!/(kx)^{l+1}]i^{l+1} M(k)[1 + O(x^\epsilon)], \qquad (15.43)$$

$$f^{(l-1)}(x, k)\,|_{x\to 0} = [(2l-3)!!/(kx)^{l-1}]i^{l-1} M(k)[1 + O(x^\epsilon)]. \qquad (15.44)$$

Thus, we have shown that under transformations using the solutions $y(x, 0)$ and $z(x, 0)$, M and, hence, the S function remain unchanged. This result together with formulas (15.17), (15.37), and (15.38) for the asymptotic behavior of the potential as $x \to 0$ and $x \to \infty$, proves Theorem 14.1.

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APPENDIX (COMMENTS AND NOTES ON THE LITERATURE)

1. The proofs of Lemmas 1.1–1.3 and 1.5 are found in Levinson's paper.[7] Certain of the statements are proved by Jost[33] and by Bargmann.[34] The case $M(0) = 0$ is treated in detail by Marchenko and Agranovich.[35,36]

2. The completeness theorem for the eigenfunctions of the operator L in the form (2.6) is proved by Levinson[7] for the case of no discrete spectrum. The general case is considered by Jost and Kohn.[12]

3. Many papers have been devoted by both physicists and mathematicians to the question of how the solution of the time dependent Schrödinger equation behaves for large $|t|$. A nonrigorous proof of the existence of the limits of the operator $U(0, t) = e^{iLt}e^{-iL_0t}$ as $t \to \pm\infty$, typical of physics papers, is given for example, in the survey of Gellmann and Goldberger.[37] From mathematical work, it is necessary to mention first of all the articles of Friedrichs,[26] who proved the existence of $\lim_{|t|\to\infty} U(0, t)$. He also showed that the limiting operators are unitary for a wide class of unperturbed operators L_0 on the assumption that the perturbation operator V is small. A formal presentation of his method appears in the paper of Moses.[38] Cook[39] proved the existence of $\lim_{|t|\to\infty} U(0, t)$ for the three-dimensional operator $-\Delta u + q(x)u$ assuming only that $q(x)$ is square integrable over all of space. However, he did not study the question of whether the operator $S = U(0, \infty)^*U(0, -\infty)$ is unitary. The restriction that the perturbation operator V be small is removed in the paper of Ladizhenskaya and Faddeyev[40] using the formalism of Friedrichs.

Theorem 3.1 does not follow from the results of these papers under the conditions we have imposed on the potential $q(x)$. The elementary proof cited makes use of the concrete properties of the example under consideration and does not carry over to other problems.

4. Povzner[41] and Levitan[42] first obtained and used the representation (4.3) for $\varphi(x, k)$. Formula (4.2)

[33] R. Jost, Helv. Phys. Acta **20**, 256 (1947).
[34] V. Bargmann, Revs. Modern Phys. **21**, 488 (1949).
[35] Z. S. Agranovich and V. A. Marchenko, Doklady Akad. Nauk S. S. S. R. **113**, 951 (1957).
[36] Z. S. Agranovish and V. A. Marchenko, *The Inverse Problem in the Quantum Theory of Scattering,* Izd. Kharkov Univ., 1960. (An English translation in preparation).
[37] M. Gellman and M. Goldberger, Phys. Rev. **91**, 398 (1953).
[38] H. E. Moses, Nuovo cimento **1**, 103 (1955).
[39] J. M. Cook, J. Math. and Phys. **36**, 83 (1957).
[40] O. A. Ladizhenskaya and L. D. Faddeyev, Doklady Akad. Nauk S.S. S. R. **120**, 1187 (1958).
[41] A. Ya. Povzner, Matem. Sbornik. **23**, 3 (1948).
[42] B. M. Levitan, Uspekhi Math. Nauk **4**, 3–112 (1949).

for $f(x, k)$ was deduced by Levin[43] and our first derivation repeats his argument. The method of deriving the integral equations (4.4) and (4.5) and the inequalities (4.7) and (4.8) is due to Agranovich and Marchenko.[35,36]

The theorem of Titchmarsh mentioned can be formulated in the following way: A necessary and sufficient condition for $\Phi(x)$ to be the limit of some function $\Phi(z) = \Phi(x + iy)$, which is analytic in the upper half-plane and such that

$$\int_{-\infty}^{\infty} |\Phi(x + iy)|^2 \, dx = O(e^{-2ky}),$$

is that

$$\varphi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(x) e^{-ixt} \, dx = 0, \qquad t < k.$$

5. The general concept of transformation operator, as already noted, was developed by Friedrichs.[25,26] Some of the notation and the proof of Theorem 5.2 were taken from the articles of Kay and Moses[21,23] who applied Friedrichs' method in solving inverse problems.

From the formula (5.15) for the S function, it follows that $S(k)$ cannot, in general, be continued into the complex k plane. Thus, Heisenberg's supposition that the discrete energy levels for the example in question might be determined by the analytic continuation of the S matrix is not justified. This fact was noted by Jost.[33]

The differential equation (5.25) with the condition (5.24) is the starting point for the proof of the existence of the kernel $K(x, y)$ in the paper of Gel'fand and Levitan.[15] One easily obtains the integral equation (4.4) from this equation. Chudov[44] proposed using the nonlinear equation, obtained from (5.25) by replacing $q(x)$ by $2\, dK(x, x)/dx$, to solve the inverse problem. Giving the S function for large x provides Cauchy data for this equation.

6. The method of relating $W(k)$ and $S(k)$ on the basis of the Wiener-Levi theorem is due to Krein.[19,45] The Wiener-Levi theorem can be stated in the following way. Let the function $\Phi(z)$ be analytic in a region D and then let $F(\lambda)$ be so chosen that the curve $z = F(\lambda)$ $(-\infty \leq \lambda \leq \infty)$ lies inside D. If $F(\lambda)$ is representable in the form

$$F(\lambda) = C + \int_{-\infty}^{\infty} f(t) e^{i\lambda t} \, dt,$$

where $f(t)$ is absolutely integrable, then $\Phi(F(\lambda))$ also possesses this property.

Formula (6.9) was deduced by Levinson[7] and bears his name.

7. The relationship between the kernels $K(x, y)$ and $A(x, y)$ does not appear in the literature.

8. The first derivation of (8.5) is taken from the paper of Kay and Moses.[21] The derivation of the general equation (8.14) follows the reasoning of Gel'fand and Levitan.[15]

9. The existence proof for (8.14) is taken from the paper of Jost and Kohn[16] and to a great extent follows the reasoning of Gel'fand and Levitan. The subsequent presentation with certain modifications reproduces the arguments of Kay and Moses.[21]

10. The analysis of the properties of the potential $q(x)$ is taken from the monograph of Agranovich and Marchenko.[36] The various relationships between $q(x)$ and $W(k)$ or $S(k)$ were obtained by Neuhaus,[46] Friedman, [47] Jost,[48] and Newton.[49] However the more general results follow from (10.11) and (10.24).

11. Krein's methods are given in a series of articles[50,51,19,and52] (see also his lectures presented at MGU in 1956–1957). Only certain of his results are mentioned in the survey. The system of differential equations (11.19) is the starting point of Krein's methods.

12. Formula (12.14) for the increment in the potential was obtained by Jost and Kohn.[16] The simplest formula for the solution $\varphi(x, k)$, such as (12.15), is due to Krein[19] (for the case $m = 1$).

The portion of Theorem 12.1 concerning necessary and sufficient conditions is due to Marchenko and Agranovich.[35,36]

13. In solving (13.5), we have followed the paper of Fulton and Newton[53] who refer to the work of Bargmann as the source of the method used. Expression (13.25) is called Bargmann's formula. Formulas for $\varphi(x, k)$ such as (13.26) are cited by Theiss.[54]

Another approach to the problem was developed

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[43] B. Ya. Levin, Doklady Akad. Nauk S. S. S. R. **106**, 187 (1956).
[44] L. A. Chudov, Izd. OIYaI, (1958).
[45] M. G. Krein, Integral equations on a half-line with difference type kernels, Uspekhi Math. Nauk. **13**, 3 (1958).

[46] M. G. Neuhaus, Doklady Akad. Nauk S.S.S.R. **102**, 25 (1955).
[47] B. Friedman, Michigan Math. J. **4**, 137 (1957).
[48] R. Jost, Helv. Phys. Acta **29**, 410 (1956).
[49] R. G. Newton, Phys. Rev. **101**, 1588 (1956).
[50] M. G. Krein, Doklady Akad. Nauk S.S.S.R. **94**, 987 (1953).
[51] M. G. Krein, Doklady Akad. Nauk S.S.S. R. **97**, 21 (1954).
[52] M. G. Krein, Doklady Akad. Nauk S.S.S.R. **111**, 1167 (1956).
[53] T. Fulton and R. G. Newton, Nuovo cimento **3**, 677 (1956).
[54] W. R. Theis, Z. Naturforsch. **11a**, 889 (1956).

by Krein if $M_1(k) = 1$. His results are formulated in a definitive way in reference 55, the formulas there being simpler than (13.25) and (13.26). However, these formulas are not generalized to the case $M_1(k) \neq 1$.

14. The basic properties of solutions of (14.1) for $l > 0$ are obtained by Levinson,[7] by Jost and Kohn,[16] and by Newton.[56] The papers of Stashevskaya[57] and Volk[58] are devoted to carrying over the results of Gel'fand and Levitan to equations with a singularity at $x = 0$. Theorem 14.1 is due to Marchenko (it was presented at the April, 1956 meeting of the Kharkov Mathematical Society).

15. A transformation such as (15.2) was first applied by Crum,[59] who used it to change a differential operator defined over a finite interval into an operator having one less eigenvalue than the original operator. Krein extended Crum's method and applied the results to get a complete characterization of the spectral function of an equation with the singularity $l(l + 1)/x^2$ in the potential at $x = 0$. Marchenko made use of an analogous transformation to analyze the relationship between the S function and a potential given by

$$q(x) = q_1(k) + \frac{l(l+1)}{x^2}, \quad \int_0^\infty x^{1+\epsilon} |q_1(x)| \, dx < \infty.$$

The presentation in the survey differs somewhat from the methods of the above-mentioned authors.

It is interesting to note that since the formulas (12.14), (12.15) and (13.25), (13.26) are verified by algebraical means without recourse to the properties of the general Gel'fand-Levitan equation, they still hold for complex values of the parameters κ_n, C_n, α_l, and β_l. The associated potential is, generally speaking, a complex function with a singularity of the form $m(m + 1)/(x - x_0)^2$ at any point where $||V(x)|| = 0$ or $||W(x)|| = 0$, in which m is the multiplicity of any such existing zero. This fact was noted by Krein[19] and by Theiss.[54]

We now briefly consider some of the generalizations of the problem investigated in our survey. By analyzing many of the formulas in the text, one sees that they remain valid with appropriate changes for systems of equations, i.e., for the matrix generalization of (1.1):

$$-Y'' + Q(x)Y = k^2Y.$$

Here, $Q(x)$ is a real symmetric matrix. The solutions $\varphi(x, k)$, $f(x, k)$ and the functions $W(k)$, $M(k)$, and $S(k)$ now become matrices. Therefore it is necessary to pay attention to the order of factors in generalizing formulas to the matrix case. The matrix $M(s)$ is analytic in the upper half-plane $\tau > 0$ and singular at those points corresponding to the discrete spectrum. The matrices $W(k)$ and $S(k)$ are related to it by the formulas

$$W(k) = M(k)^{-1}M^T(-k)^{-1}, \quad S(k) = M(-k)M^T(k)^{-1},$$

$M^T(k)$ being the transposed of $M(k)$. Similar systems were studied by Jost and Newton,[60] by Krein,[52] and by Agranovich and Marchenko.[35,36] A fundamental difficulty arises in carrying over the discussion of Sec. 6 to the matrix case. In consequence of the noncommutativity of the matrices, the formulas cited there no longer hold. To find how $W(k)$ and $S(k)$ are related, one has to go back and consider integral equations of the form

$$K(t) = F(t) + \int_0^\infty F(t+s)K(s) \, ds.$$

Marchenko and Agranovich derived necessary and sufficient conditions on the S matrix so that it corresponds to a matrix potential $Q(x)$ from a given class making use of analogous integral equations. The formulation of conditions directly in terms of the S matrix still remains an unsolved problem.

Newton[56] and Agranovich and Marchenko[36,61] considered a system in which the potential has the singularity $l_\alpha(l_\alpha + 1)\delta_{\alpha\beta}/x^2$; Agranovich and Marchenko reduced such a system to a regular one by transformations generalizing those introduced in Sec. 15.

The inverse problem for a system has mainly been treated for the purpose of seeing what means are needed to solve the inverse problem for the Schrödinger equation

$$-\Delta u + q(x)u = k^2u$$

in all of space when the potential decreases in all directions. However, this problem essentially differs from those treated till now. In fact, the S matrix in this case is determined by the so-called scattering amplitude $f(k; \boldsymbol{\alpha}, \boldsymbol{\beta})$ depending on the wave number $k(0 \leq k < \infty)$ and two unit vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Thus, the S matrix depends on a larger number of parameters than the potential $q(x)$ which may be regarded

[55] M. G. Krein, Doklady Akad. Nauk S.S.S.R. **113**, 970 (1957).
[56] R. G. Newton, Phys. Rev. **100**, 412 (1955).
[57] V. V. Stashevskaya, Doklady Akad. Nauk S.S.S.R. **93**, 409 (1953).
[58] V. Ya. Volk, UMN VIII, 141 (1953).
[59] M. M. Crum, Quart. J. Math. **6**, 121 (1955).
[60] R. G. Newton and R. Jost, Nuovo cimento **1**, 590 (1955).
[61] Z. S. Agranovich and V. A. Marchenko, Doklady Akad. Nauk S. S. S. R. **118**, 1055 (1958).

as a function of the distance $r(0 \leq r < \infty)$ and one unit vector. In this sense, the problem is over determined and it is necessary to look for nontrivial properties of the S matrix which would decrease the number of parameters on which it depends.

The simplest analogous problem arises in the reconstruction of a decreasing potential from the S matrix for the one-dimensional Schrödinger equation

$$-y'' + q(x)y = k^2 y \qquad (-\infty < x < \infty).$$

In this case, the S matrix is a 2×2 matrix:

$$S(k) = \begin{bmatrix} s_{11}(k) & s_{12}(k) \\ s_{21}(k) & s_{22}(k) \end{bmatrix} = \begin{bmatrix} a(k) & b(k) \\ b(k) & c(k) \end{bmatrix},$$

and due to its unitariness, is determined by giving three real functions of $k(0 \leq k < \infty)$. The potential may be regarded as being given by two real functions of $x(0 \leq x < \infty)$.

The inverse problem for this case was considered by Kay and Moses[23,24] (as an example illustrating their general approach to the inverse problem) and by the author.[62] In reference 62 it is shown that an additional condition on the S matrix follows from the analyticity of the coefficient $b(k)$ in the upper half-plane $\tau > 0$. This condition implies that the whole S matrix (and potential) is determined by one of the coefficients $a(k)$ or $c(k)$ which may be chosen as an arbitrary function. The reconstruction of the equation with an arbitrary potential from its spectral matrix function was treated by Bloch.[63]

A number of the elements of the S matrix are also analytic in the three-dimensional case. The proof of this fact is given in the papers of Khuri[64] and of the author[65] in connection with the so-called dispersion relations. However, these relations do not sufficiently reduce the number of parameters on which the S matrix depends.

It is interesting to note, in this connection, the statement of the three-dimensional inverse problem as proposed by Moses[66]: Determine the potential $q(x)$ from the back scattering amplitude $g(k, \boldsymbol{\alpha}) = f(k; \boldsymbol{\alpha}, -\boldsymbol{\alpha})$ where $\boldsymbol{\alpha}$ is a vector running over a hemisphere. This data, namely two real functions of $k(0 \leq k < \infty)$ and $\boldsymbol{\alpha}$, involve as many parameters as does the potential. It is very plausible that the process of Moses converges for sufficiently small $g(k, \boldsymbol{\alpha})$ which in other respects may be a quite arbitrary function.

A number of papers exist in which the inverse problem has been solved for the relativistic equations when the latter reduce to ordinary differential equations. The equation obtained by separating variables in the Klein-Gordon equation was studied by Corinaldesi.[67] The one-dimensional Dirac equation was considered by Kay and Moses,[68] Toll and Prats,[69] and Verdi.[70] In all of these papers, a relationship is established between the asymptotic phase and the potential both for positive and negative energies. The data, just as in the problems described above, depends on a larger number of parameters than does the potential. A correct formulation of the problem for the radial relativistic equation has still to be given.

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62 L. D. Faddeyev, Doklady Akad. Nauk S.S.S.R. **121**, 63 (1958).

63 A. Sh. Bloch, Doklady Akad. Nauk S.S.S.R. **92**, 209 (1953).

64 N. N. Khuri, Phys. Rev. **107**, 1148 (1957).

65 L. D. Faddeyev, Zhur. Eksptl 'i Teort Fiz **35**, 433 (1958).

66 H. E. Moses, Phys. Rev. **102**, 559 (1956).

67 E. Corinaldesi, Nuovo cimento **11**, 468 (1954).

68 H. E. Moses, Bull. Am. Phys. Soc. **4**, 240 (1957).

69 J. S. Toll and F. Prats, Phys. Rev. **113**, (1959).

70 M. Verdi, Nuclear Phys. **9**, 255 (1958/59).