INFRARED PROBLEM IN THE THERMODYNAMICS OF THE YANG–MILLS GAS

A.D. LINDE
Lebedev Physical Institute, Moscow, USSR

Received 5 August 1980

It is shown that the infrared cutoff, which may appear in the quantum statistics of the massless Yang–Mills fields due to high temperature effects, cannot be greater than $O(g^2)T$, where g is the effective gauge coupling constant at a finite temperature. This gives rise to many difficult problems in the thermodynamics of the Yang–Mills gas.

After the discovery of asymptotic freedom in the non-abelian gauge theories [1], the thermodynamics of the Yang–Mills gas became of ultimate interest. Indeed, before this discovery it was unreasonable to expect that one could reliably predict properties of strongly interacting matter at densities $\rho \gtrsim 10^{15}$ g cm^{-3} and at temperatures $T \gtrsim 150$ MeV. At present, however, it is commonly believed, that superdense matter at temperatures much greater than 100 MeV behaves as a weakly non-ideal gas, and that one can easily obtain all the properties of this gas by means of perturbation theory improved by the use of the renormalization group equation [2]. In the high temperature limit this improvement is achieved merely by writing everywhere the effective coupling constant $g^2 = \ln^{-1}(T/\mu)$ (μ is some renormalization scale parameter) instead of the zero-temperature gauge coupling constant [2].

Any reliable information concerning the properties of superdense matter would be very important for investigation of the early stages of the universe evolution. We believe that most of the results of the works [2] are quite reliable as far as cold superdense matter is considered. However the reliability of the corresponding results at a finite temperature is much less clear. In the present paper (see also ref. [3]) we would like to point out some difficulties, encountered in the thermodynamics of superdense matter at high temperatures. The source of these difficulties is the infrared problem in the thermodynamics of the massless Yang–Mills gas [3].

At temperatures $T \gg 100$ MeV gluons are usually supposed to behave as interacting massless particles [2]. At higher temperatures $T > T_c \approx 100$ GeV $W_\mu^\pm$ and Z_μ^0 bosons also become massless [3,4]. It is known, that all difficulties, which make impossible a simple perturbation theory description of the second-order phase transitions in the theory $\lambda\varphi^4$ near the critical temperature T_c arise due to the infrared problem in the thermodynamics of interacting φ-bosons, which become massless at $T = T_c$ [5,6]. Infrared divergences, which appear in the theory of massless Yang–Mills fields, are even stronger than in the theory $\lambda\varphi^4$. It is clear therefore that if no infrared cutoff (effective mass) appears in the quantum statistics of the Yang–Mills gas, then the infrared problem similar to the one encountered in the theory of phase transitions at $T = T_c$, exists in the thermodynamics of the Yang–Mills gas at *all temperatures* at which the Yang–Mills fields are massless, and at these temperatures any perturbative description of the thermodynamics of the Yang–Mills gas is impossible [3].

To clarify this point, let us consider a contribution to the thermodynamic potential $\Omega(T)$ from the diagrams of the order g^{2N}, containing N four-gluon vertices [3,6]. After a summation over the Lorentz and isotopic indices, this contribution can be represented as

$$\Omega_N(T) \sim (2\pi T)^{N+1} g^{2N} \int d^3p_1 \dots d^3p_{N+1}$$

$$\times \sum_{n_i=-\infty}^{\infty} \prod_{k=1}^{2N} [(2\pi r_k T)^2 + \boldsymbol{q}_k^2 + m^2(T)]^{-1}, \qquad (1)$$

where $\boldsymbol{q}_k$ is a uniform linear combination of $\boldsymbol{p}_i$, r_k is the corresponding combination of integers n_i, $i = 1, 2, \dots, N+1$, $k = 1, 2, \dots, 2N$ [3], and $m(T)$ is an infrared cutoff, which may appear due to high temperature effects, see below. We remind that the summation over n_i in quantum statistics at $T \neq 0$ [eq. (1)] plays the same role as the integration over p_{0_i} in quantum field theory [7]. At $m(T) \to 0$ the leading term in the sum over n_i is the term with all $n_i = 0$ (and, consequently, $r_k = 0$). Therefore, at $m(T) \to 0$ [3]

$$\Omega_N(T) \sim (2\pi T)^{N+1} g^{2N} \int d^3p_1 \dots d^3p_{N+1}$$

$$\times \prod_{k=1}^{2N} (\boldsymbol{q}_k^2 + m^2(T))^{-1} \sim g^6 T^4 \left(\frac{g^2 T}{m(T)}\right)^{N-3}. \qquad (2)$$

From eq. (2) it is clear, that in the absence of the infrared cutoff ($m(T) = 0$) contributions to $\Omega(T)$ of the order g^8, g^{10} etc. contain power (rather than usual logarithmic) infrared divergences. A general rule obvious from eq. (2) is that the leading infrared divergences in the quantum statistics of interacting bosons at a finite temperature are those of the 3-dimensional quantum field theory [5]. At $m(T) < g^2T$ the dangerous terms (2) become finite, but due to these terms higher orders of perturbation theory for $\Omega(T)$ become greater than the lowest ones [3,6]. Therefore it is possible to compute the thermodynamic potential, energy, entropy etc. of the Yang–Mills gas by means of perturbation theory only if a sufficiently large infrared cutoff appears in the theory at a finite temperature.

It was shown by Kislinger and Morley that high temperature effects lead to the appearance of a pole of the gluon Green function $G_{\mu\nu}^{ab}$ at $k_0 \sim gT$, $\boldsymbol{k} = 0$ [8]. The value of k_0 at this pole was interpreted in ref. [8] as the value of the infrared cutoff, $m(T) \sim gT$, which would solve the above-mentioned infrared problem. However, as has been pointed out in refs. [3,6], this interpretation is incorrect: leading infrared divergences in quantum statistics of gauge fields are connected with the behaviour of the Green function $G_{\mu\nu}^{ab}$ not in the limit $k_0 \neq 0$, $\boldsymbol{k} = 0$, but in the static limit $k_0 = 0$, $\boldsymbol{k} \to 0$ [this limit just corresponds to taking $r_k = 0$ in (1)]. The Green function $G_{\mu\nu}^{ab}(k_0 = 0, \boldsymbol{k} \to 0)$ in the Coulomb gauge looks as follows [7,3]:

$$G_{00}^{ab} = \delta^{ab}(k^2 + \Pi_{00}(k))^{-1}, \quad G_{0i}^{ab} = G_{i0}^{ab} = 0,$$

$$G_{ij}^{ab} = \delta^{ab}(\delta_{ij} - k_i k_j / k^2) G(k),$$

where $k = |\boldsymbol{k}|$; a, b are isotopic indices; $i, j = 1, 2, 3$ and $\Pi_{00}(0) \sim g^2T^2$ in the order g^2. Thus in the Green function $G_{00}^{ab}(k_0 = 0, k)$ the infrared cutoff $m_0 \sim gT$ actually may appear [6,9,10], see, however, below.

This effect resembles the well-known Debye screening of electric fields in a hot plasma gas in QED [7]. However it is known, that static magnetic fields in QED plasma cannot be screened, and, consequently, no infrared cutoff appears in G_{ij} ($k_0 = 0$, $\boldsymbol{k} \to 0$) [7]. In the Yang–Mills gas the large cuttoff $\sim gT$ also does not appear in G_{ij}^{ab} ($k_0 = 0$, $\boldsymbol{k} \to 0$) [3,6,10]. However a smaller cutoff $m(T) \sim g^2T$ can appear due to the higher-order corrections to $G^{-1}(k)$ [3]. A general analysis similar to our analysis of infrared divergences of $\Omega(T)$ shows that in higher orders of perturbation theory the terms $\sim k^2(g^2T/k)^N$ appear in the expansion of $G^{-1}(k)$, so that at small k

$$G^{-1}(k) = k^2 + a_1 g^2 Tk + a_2 g^4 T^2 + a_3 g^6 T^3/k + \dots, \qquad (3)$$

where a_i are some constants ≈ 1 (in the Feynman gauge $a_1 = -9/16$ for the group SU(3) [10]). This series is divergent for $k \lesssim g^2T$. Analogously it can be shown that the series for $\Pi_{00}(k)$ is *also* divergent at $k < g^2T$. Therefore the perturbation theory can give us no information about the behaviour of the Green function $G_{\mu\nu}^{ab}(k_0 = 0, k)$ at $k < g^2T$. One can show only that the cutoff $m(T)$, which can appear in $G_{ij}^{ab}(k_0 = 0, k)$, cannot be greater than $O(g^2)T$. Indeed, at $k \gg O(g^2)T$ the value of the cutoff would be determined by the term $\sim g^4T^2$ in eq. (3), but in this case $m(T) \sim g^2T$.

In fact, eq. (3) suggests, that if the infrared cutoff $m(T)$ actually exists, it is expected to be of the order g^2T [3]. Note, that this value of $m(T)$ would be of the same order as the value of the cutoff predicted earlier by Polyakov [11].

Let us now consider three main possibilities, which may occur in the thermodynamics of the Yang–Mills gas at a finite temperature.

(i) All terms in eq. (3), which may lead to the appearance of an infrared cutoff, are cancelled, so that $G^{-1}(k) \sim k^2$ and $m(T) = 0$. In this case higher orders of perturbation theory for the thermodynamic potential, energy, entropy etc. are infrared-divergent.

(ii) $G^{-1}(k)$ becomes negative for small k. Such a situation appears e.g. if $a_1 < 0$ [10] and all $a_i = 0$ for $i > 1$ in eq. (3), or if a_2 is sufficiently large and negative. In this case the ground state of the theory may become unstable with respect to spontaneous generation of "magnetic" components of the Yang–Mills fields. Such an instability would be analogous to the instability pointed out by Savvidi et al. in their investigation of the vacuum stability problem of QCD at $T = 0$ [12]. In this case just as in the case (i), all the results concerning thermodynamic properties of the Yang–Mills gas at finite temperature obtained in ref. [2] become unreliable.

(iii) $G^{-1/2}(0) = m(T) \sim g^2 T$. According to (2), the perturbative expansion of $\Omega(T)$ contains dangerous terms $\sim g^6 T^4 (g^2 T/m)^{N-3}$. At $m(T) \sim g^2 T$ each of these terms is of the order $g^6 T^4$. Even if the sum of all these terms is finite (which seems unlikely), one cannot actually compute it, and therefore in the best case $\Omega(T)$ can be computed by means of perturbation theory only up to the order $g^6 T^4$.

It is worth noting that the effective coupling constant $g^2 = \ln^{-1}(T/\mu)$, entering all our equations, describes interaction of particles with momenta $k \sim T$ and differs from the effective coupling constant of the Yang–Mills particles at vanishing momenta. The last quantity linearly diverges at $T \neq 0$ already in the one-loop approximation [10]. A general analysis similar to our analysis of $G^{ab}_{\mu\nu}(k_0 = 0, \mathbf{k})$ shows that by means of perturbation theory one cannot reliablv obtain the vertex function $\Gamma^{abc}_{\mu\nu\lambda}$ in the static limit at spatial momenta less than $g^2 T$.

Now let us summarize our main conclusions.

(1) The infrared problem in quantum statistics of gauge fields at $T \neq 0$ is connected with properties of the Green functions $G^{ab}_{\mu\nu}(k_0, \mathbf{k})$ in the static limit $k_0 = 0, \mathbf{k} \to 0$.

(2) Perturbation theory can give us no reliable information about the behaviour of $G^{ab}_{\mu\nu}(k_0 = 0, \mathbf{k})$ at $k < g^2 T$, where $k = |\mathbf{k}|$.

(3) Infrared cutoff $m(T)$, which may appear in $G^{ab}_{ij}(k_0 = 0, \mathbf{k})$, cannot be greater than $O(g^2)T$. In the most reasonable case, when $m(T) \sim g^2 T$, the thermodynamic potential can be reliably obtained by means of perturbation theory only up to the order $g^6 T^4 \sim T^4 \ln^{-3}(T/\mu)$. In all other cases perturbation theory gives no reliable information about the thermodynamics of the Yang–Mills gas at finite temperature.

These conclusions seem rather disappointing. Quantum statistics of the massless Yang–Mills gas appears to be much more difficult than it was expected to be in ref. [2], and we can give no recipe how to overcome the above-mentioned difficulties. A pragmatically thinking reader may wonder, whether our results contain also some *positive* information about the thermodynamics of superdense matter at high temperatures. The answer is yes, and in fact this was the main motivation for the author to write the present paper.

From our investigation it follows that the infrared cutoff $m(T) \sim g^2 T$ may appear in the thermodynamics of the Yang–Mills gas [3], and that only in this case our naive expectations concerning thermodynamics of matter at high temperatures [2] will be partially true. One may regard such a cutoff as an effective mass of "magnetic" non-abelian fields. As is shown in ref. [13], the existence of this effective mass leads to confinement of magnetic monopoles at high temperatures. This effect makes it possible to remove the apparent inconsistency between cosmology and grand unified theories, connected with primordial monopoles [14]. Therefore we believe that investigation of the infrared problem in the thermodynamics of the Yang–Mills gas is very important for cosmology and for the elementary particle physics.

A more detailed analysis of the problems discussed in this paper will be contained in a separate publication.

In conclusion the author would like to express his deep gratitude to V.V. Klimov, A.M. Polyakov, E.V. Shuryak, and especially to D.A. Kirzhnits for many valuable discussions.

Note added. After this paper was written I recieved a preprint QCD and instantons at finite temperature, by Gross, Pisarski and Yaffe. Their results confirm the results obtained earlier in our paper [3], in which it has been shown that the infrared cutoff $m(T) \sim g^2 T$ may exist in the thermodynamics of the Yang–Mills gas, and that the perturbation theory treatment of the Yang–Mills gas at $T \neq 0$ is not quite reliable. They have also concluded that if the cutoff $m(T) \sim g^2T$ exists, the thermodynamic potential of the Yang–Mills gas can be obtained only up to the order g^6T^4, which is in agreement with the result obtained in the present paper. Gross et al. have also claimed that the liberation of heavy quarks at high temperatures is a quite reliable effect since the "electric" cutoff $m_0 \sim gT$ appears in $G_{00}^{ab}(k_0 = 0, \boldsymbol{k})$, and therefore the heavy quarks become noninteracting at distances greater than m_0^{-1}. However, as is argued in our paper, the series for $\Pi_{00}(k)$ is divergent at $k < g^2T$ due to the diagrams containing the Green functions G_{ij}^{ab}. Therefore the existence of the "electric" cutoff $m_0 \sim gT$ and the heavy quark liberation can be reliably proved *only* if the "magnetic" cutoff $m(T) \sim g^2T$ actually exists. We would like to stress the point mentioned also by Gross et al. that the heavy quark liberation may have almost nothing to do with the high-temperature liberation of the ordinary light quarks, since the average distance between the light quarks in the hot Yang–Mills gas is $O(T^{-1}) \ll m_0^{-1}$. Therefore the liberation of the light quarks, as we believe, occurs neither due to the Debye screening, nor due to a condensation of strings [11,15], but due to the overlap of the quark bags at high temperatures.

References

[1] D.J. Gross and F. Wilczek, Phys. Rev. Lett. 30 (1973) 1343;
H.D. Politzer, Phys. Rev. Lett. 30 (1973) 1346.

[2] J.C. Collins and M.J. Perry, Phys. Rev. Lett. 34 (1975) 1353;
M.B. Kislinger and P.D. Morley, Phys. Rev. D13 (1976) 2771;
B.A. Freedman and L.D. McLerran, Phys. Rev. D16 (1977) 1169;
V. Baluni, Phys. Rev. D17 (1978) 2092;
J. Kapusta, Nucl. Phys. B148 (1979) 461;
O.K. Kalashnikov and V.V. Klimov, Phys. Lett. 88B (1979) 328;
M.B. Kislinger and P.D. Morley, Phys. Rep. 51C (1979) 63;
E.V. Shuryak, Phys. Rep. 61C (1980) 71.

[3] A.D. Linde, Rep. Prog. Phys. 42 (1979) 389.

[4] D.A. Kirzhnits, JETP Lett. 15 (1972) 529;
D.A. Kirzhnits and A.D. Linde, Phys. Lett. 42B (1972) 471;
S. Weinberg, Phys. Rev. D9 (1974) 3357;
L. Dolan and R. Jackiw, Phys. Rev. D9 (1974) 3320;
D.A. Kirzhnits and A.D. Linde, Zh. Eksp. Teor. Fiz. 67 (1974) 1263 [JETP 40 (1975) 628].

[5] K. Wilson and J. Kogut, Phys. Rep. 12C (1974) 75.

[6] D.A. Kirzhnits and A.D. Linde, Ann. Phys. (NY) 101 (1976) 195.

[7] E.S. Fradkin, Proc. Lebedev Phys. Inst. 29 (1965) 7.

[8] M.B. Kislinger and P.D. Morley, Phys. Rev. D13 (1976) 2765.

[9] E.V. Shuryak, JETP 47 (1978) 212.

[10] O.K. Kalashnikov and V.V. Klimov, Lebedev Phys. Inst. preprint (1980).

[11] A.M. Polyakov, Phys. Lett. 72B (1978) 477.

[12] I.A. Batalin, S.G. Matinyan and G.K. Savvidy, Yad. Fiz. 26 (1977) 407;
G.K. Savvidy, Phys. Lett. 71B (1977) 133.

[13] A.D. Linde, Confinement of monopoles at high temperatures, Lebedev Phys. Inst. preprint (1980).

[14] Ya.B. Zeldovich and M.Yu. Khlopov, Phys. Lett. 79B (1978) 239;
J.P. Preskill, Phys. Rev. Lett. 43 (1979) 1365.

[15] L. Susskind, Phys. Rev. D20 (1979) 2610.