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Foundations of Quaternion Quantum Mechanics

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A new kind of quantum mechanics using inner products, matrix elements, and coefficients assuming values that are quaternionic (and thus noncommutative) instead of complex is developed. This is the most general kind of quantum mechanics possessing the same kind of calculus of assertions as conventional quantum mechanics. The role played by the new imaginaries is studied. The principal conceptual difficulty concerns the theory of composite systems where the ordinary tensor product fails due to noncommutativity. It is shown that the natural resolution of this difficulty introduces new degrees of freedom similar to isospin and hypercharge. The problem of the Schrödinger equation, "which i should appear?" is studied and a generalization of Stone's theorem is used to resolve this problem.

1. WHY QUATERNION QUANTUM MECHANICS?

IN conventional quantum mechanics propositions (the result of measurements) are represented as subspaces in an infinite-dimensional Hilbert space $\mathfrak{H}$. In particular the pure states are the one-dimensional subspaces (rays) of $\mathfrak{H}$. The relation between measurements of different kinds are expressed in the lattice structure of these subspaces. One of the most important features in this structure is the absence of the distributive law which characterizes the propositional calculus of classical systems. One might say, therefore, that the lattice structure of subspaces incorporates an essential ingredient of quantum systems, viz., that measurements of different kinds may interfere leading to the well-known uncertainty relations and complementary properties.

The representation of the propositional calculus of atomic systems as a lattice structure was given as long ago as 1936 by Birkhoff and von Neumann.[1] There it is shown that a propositional calculus exists that we can call *general quantum mechanics* (as distinguished from complex quantum mechanics) in as much as no number system or vector space at all is assumed in its formulation.[2] The relevance for the present work is found in

[1] G. Birkhoff and J. von Neumann, Ann. Math. **37**, 823 (1936). The first suggestion of quaternion quantum mechanics appears in a footnote of this paper. C. N. Yang has also pointed out the interest of this possibility [*Proceedings of the Seventh Rochester Conference on High-Energy Nuclear Physics 1957* (Interscience Publishers, Inc., New York, 1957), p. IX-26].

[2] We can present the propositional calculus of general quantum mechanics as follows, if we consider finite-dimensional Hilbert spaces only, thus excluding systems with continuous variables except as limiting cases. The elements $A, B, C, \cdots$ (which may indifferently be regarded as representative ensembles, propositions about a physical system, or "operational" rules for testing the truth of statements) are subject to the basic operation of *negation* $A \rightarrow \sim A$ and the basic relation of *implication* $A \subset B$. In addition, unlike the classical propositional calculus, the propositions make up (are the points of) a topological space. The axioms are: 1. *The axioms for a complemented lattice.* Implication is reflexive, transitive, and antisymmetric (reversible only for equals). Any A, B, possess both a g.l.b. $A \cap B$ (A and B) and a l.u.b. $A \cup B$ (A or B) with respect to implication. There exists an over-all g.l.b. O and l.u.b. I. Negation is an involutory[3] anti-automorphism of the lattice. 2. *Axioms of cardinality.* To each proposition A may be associated a non-negative integer $|A|$ such that if $A \subset B$ then $|A| < |B|$; if $A \subset B$ and $|A| = |B|$ then $A = B$; $|O| = 0$; and

the following remarkable result:

It is always possible to represent the pure states of a system of "general quantum mechanics" by rays in a vector space in a one-to-one manner, and for this it is necessary and sufficient to employ orthogonal vector spaces (Hilbert spaces) over the following numl er systems:

$\mathfrak{R}$, the real numbers,
$\mathfrak{C}$, the complex numbers, and
$\mathfrak{Q}$, the quaternions.[3]

Moreover it is always possible to represent these pure states by rays in a vector space over $\mathfrak{Q}$ (not every ray may be needed for this), but not so over $\mathfrak{R}$ or $\mathfrak{C}$.

This result suggests two things. First, inasmuch as the propositional calculus reflects empirical information, some of this information appears in the number system which is used for the construction of the Hilbert space. Secondly, it is not necessary to go beyond the three possibilities $\mathfrak{R}$, $\mathfrak{C}$, and $\mathfrak{Q}$ for the representation of general quantum mechanics.

We can thus formulate the following precise problem: Which of the three possibilities for the representation of general quantum mechanics is the one most suitable for the description of the actual physical world?

A preliminary analysis of this problem might start with the question: Why has conventional quantum mechanics been developed with complex numbers instead of reals? The relation between complex and real quantum mechanics has been extensively investigated by Stueckelberg and collaborators and is now completely understood. Briefly stated the situation is as follows: Complex quantum mechanics is completely equivalent to real quantum mechanics plus a superselection rule: All observables in real quantum mechanics must commute with a fixed linear operator J, which is antisymmetrical ($J^T=-J$) and satisfies $J^2=-I$. The operator J is intimately related to the symmetry of time reversal. Indeed, the time-reversal transformations are precisely those symmetry transformations which anticommute with J. This result shows that there exists a connection between supersymmetries and the choice of the number field.

While thus the difference between complex and real quantum mechanics is relatively simple, quaternion quantum mechanics has many new features which make it a much richer theory. It is perhaps surprising that such a promising possibility has not yet been more fully developed. There appear to be reasons for this. For example, the problem of how to write a Schrödinger equation is not a trivial one in $\mathfrak{Q}$ quantum mechanics because of the appearance of a square root of minus one in the ordinary Schrödinger equation; and the description of interacting systems by a direct product is made difficult by the noncommutativity of $\mathfrak{Q}$-valued wave functions.

In this paper we shall present the general features of a quaternion quantum mechanics. In a subsequent paper we shall show how these features can be utilized

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$|\sim A|=|I|-|A|$. We assume, without loss of generality, that the least of the positive values assumed by this integer is normalized to be 1; otherwise any positive integral multiple of A would also satisfy these requirements. 3. *Axiom of superposition.* If $|A|=|B|=1$ then there exists a C with $|C|=1$ such that $A\cup B=B\cup C=C\cup A$. 4. *Axiom of continuity.* $|A|$ is a continuous function of A.

Evidently Axioms 1 and 2 are valid for the propositional calculus of a classical system with a finite number of states. Axiom 3 is the very essence of quantum logic; the C whose existence it asserts is a superposition of A and B in the quantum sense. Were the classical distributive law of logic $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$ adjoined to Axiom 1, Axiom 3 would be inconsistent. Even without continuity (Axiom 4) it would follow from Axioms 1-3 that[1] the propositions correspond one-to-one to substances of some vector space over a skew field. Since *every* subspace of the vector space is utilized in this one-to-one realization, there is no room for superselection principles in what we have called general quantum mechanics; but if Axiom 3 is simply dropped we find a "supersum" (direct sum with superselection rules between addends) of systems for each of which Axiom 3 is satisfied. Thus it is not necessary to go beyond the quaternions until Axioms 1, 2, or 4 are weakened.

We have consigned this matter to inferior print and omitted much mathematical beauty; it concerns mostly how one arrives at $\mathfrak{Q}$ quantum mechanics, and in the final analysis it is more important to know where a theoretical path leads than how one fell upon it.

[3] We give here the algebra of quaternions. Every quaternion can be written in the form

$$q=q_0+q_1i_1+q_2i_2+q_3i_3,$$

where the four coefficients q_k are real. The multiplication of quaternions is associative, distributive, and obeys

$$i_k^2=-1 \quad k=1,2,3;$$
$$i_1i_2i_3=-1.$$

In the last equation, the anticyclic order of factors might have been taken. Every quaternion q possesses an inverse q^{-1}.

In Hamilton's notation a quaternion is regarded as the sum of a "scalar" (*real*) part and a "vector" (*imaginary*) part: $q=q_0+\mathbf{q}\cdot\mathbf{i}$. The quaternions that commute with all other quaternions are just the reals. The quaternions that commute with a given nonreal quaternion form a subset isomorphic to the complex numbers. There exists an operation $q\to q^Q$ on the quaternions that is *involutory* ($q^{QQ}=q$), Hermitian definite (q^Qq is real, and vanishes only when $q=0$), and *anti-automorphic* ($p^Qq^Q=(qp)^Q$), and it is called the *quaternion conjugate* ($\mathfrak{Q}$ conjugate): $i_k^Q=-i_k$. On the other hand, the *automorphisms* of the quaternions are all of the form $q\to aqa^{-1}$. (The quaternion a associated with a particular automorphism is not uniquely defined by this equation; by requiring that the *norm* of a, meaning a^Qa, be unity, the ambiguity is reduced to an extremely important matter of sign.)

It is sometimes convenient to represent quaternions by pairs of complex numbers (c^0,c^1) according to

$$q=c^0+i_2c^1,$$

where c^0, c^1 commute with i_3, and are therefore essentially complex numbers. Treating these pairs as vectors in a two-dimensional complex vector space C^2, we find that every linear transformation of $\mathfrak{Q}$ is represented by a linear transformation of C^2, that is by a 2×2 complex matrix. In particular the *left multiplication* $q\to aq$, by a fixed quaternion a, is represented by a matrix a_{ij}, the *symplectic* representation of a. The symplectic representations of left multiplication by i_1, i_2, i_3 are just the Pauli spin operators (times i). But the symplectic representation of right multiplications by quaternions are sums of linear and antilinear operators.

Computation yields that $q^Q=c^{0*}-i_2c^1$ where Q denotes the quaternion, and the star the ordinary complex, conjugate. The "scalar product" of two quaternions p,q, $(p,q)=p^Qq$, then becomes (with $p=b^0+i_2b^1$) $p^Qq=(b^{0*}c^0+b^{1*}c^1)+i_2(b^0c^1-b^1c^0)$. We separated the quaternion with respect to i_2 and identified i_3 with the complex i. But, of course, we could have used any pair of anticommuting units as well.

for the description of the multiplicities of the strongly interacting particles, for a fundamental theory of electromagnetism, and for a possible unification of the theory of electromagnetic and Fermi interactions.

2. GENERAL QUANTUM MECHANICS

By a suitable choice of development it is possible to present three kinds of quantum mechanics to be called $\mathfrak{R}$, $\mathfrak{C}$, and $\mathfrak{Q}$ all at once. Therefore let $\mathfrak{F}$ be one of these number systems (fields) in which there are defined the notions of continuity, addition, multiplication, inverse, and conjugate (*). We term the ensuing schema $\mathfrak{F}$ quantum mechanics. We need the concept of a Hilbert space over $\mathfrak{F}$, $\mathfrak{K}(\mathfrak{F})$ and may define it by taking as fundamental the algebraic operations of vector addition, multiplication by $\mathfrak{F}$ numbers, and conjugate (†). The vector addition requires no discussion. Because $\mathfrak{F}$ may be noncommutative, multiplication of a vector Ψ or $|\cdot\rangle$ by a number a must be set up carefully.[1] We write Ψa, or $|\cdot\rangle a$, and require the associative law

$$(\Psi a)b=\Psi(ab) \tag{1}$$

and the distributive law. The conjugate $\Psi\dagger$ or $\langle\cdot|$ of a vector Ψ or $|\cdot\rangle$ is in the dual Hilbert space $\mathfrak{K}\dagger(\mathfrak{F})$, which means that it is a linear function on $\mathfrak{K}(\mathfrak{F})$:

$$\Psi\dagger(\Phi a)=(\Psi\dagger\Phi)a, \quad \Psi\dagger(\Phi+\Phi')=\Psi\dagger\Phi+\Psi\dagger\Phi'.$$

The conjugate operation is to be Hermitian, $\Psi\dagger\Phi=(\Phi\dagger\Psi)^*$; antilinear, $(\Psi a)\dagger=a^*\Psi\dagger$, $(\Psi+\Phi)\dagger=\Psi\dagger+\Phi\dagger$; and definite,

$$\Psi\dagger\Psi=0 \quad \text{if and only if } \Psi=0.$$

Thus $\Psi\dagger\Phi$ has the properties of a scalar product.[4] We use the distance $|\Psi-\Phi|=[(\Psi-\Phi)\dagger(\Psi-\Phi)]^{\frac{1}{2}}$ to define the topology of $\mathfrak{K}(\mathfrak{F})$.

A linear manifold of $\mathfrak{K}(\mathfrak{F})$ is a set of vectors closed under vector addition and multiplication by $\mathfrak{F}$ numbers. A subspace is a closed linear manifold. The subspaces play a fundamental role in the interpretation. We assume that they are in one-to-one correspondence with the propositions about the physical system[2] in the way that subsets of classical phase space are in one-to-one correspondence with propositions about the classical physical system. Momentarily we are excluding the important cases of quantum mechanics with superselection principles by this assumption. The notion of quantum mechanics defined in this manner, using such "unobservable" concepts as the Hilbert space vectors and their scalar products, coincides in the finite-dimensional case with the general quantum mechanics defined in terms of "observable" non-numerical concepts such as implication and negation.[1,2]

3. THE SCHRÖDINGER EQUATION

How do we link this structure to reality? Basic physical quantities such as energy and momentum are recognized by their relation to such symmetry properties of the physical system as time- and space-translational symmetry. We therefore consider groups of transformations acting on the system.[5]

For this end define a mapping T on $\mathfrak{K}(\mathfrak{F})$ into itself that has the property

$$T(\Psi a+\Phi b)=(T\Psi)a'+(T\Phi)b' \tag{1}$$

to be *colinear*. Here a' is to be obtained from a, and b' from b, by an automorphism of $\mathfrak{F}$ associated with T, independent of Ψ and Φ, and called the automorphism belonging to T; for $\mathfrak{F}=\mathfrak{Q}$,

$$a'=qaq^{-1},$$
$$b'=qbq^{-1},$$

q independent of Φ, Ψ. The name *colinear* is suggested by the name used for the closely related concept of a *collineation* in projective geometry. Evidently colinear transformations (and indeed only such mappings of vectors) carry subspaces into subspaces. A colinear transformation for which the associated automorphism of $\mathfrak{F}$ is the identity is termined *linear*.

The elements of groups of transformations in $\mathfrak{F}QM$ are going to be *co-unitary* transformations, which are defined to be colinear transformations U on $\mathfrak{K}(\mathfrak{F})$ enjoying the additional property

$$(U\Phi)\dagger(U\Psi)=(\Phi\dagger\Psi)'=q(\Phi\dagger\Psi)q^{-1}, \tag{2}$$

where the prime indicates that the automorphism of $\mathfrak{F}$ belonging to the colinear transformation U is applied to the $\mathfrak{F}$ number $\Phi\dagger$, Ψ. If linear, a co-unitary transformation is called *unitary*.

Let us now consider the passage of time. We shall suppose it is represented by a one-parameter group of unitary transformations U_t on $\mathfrak{K}(\mathfrak{F})$. Requiring U_t to be unitary, amounts to nothing more or less than requiring the logical relations between propositions concerning the system to be independent of the time origin.[6] For

[4] Usually it is the scalar product that is taken as fundamental, but except with the Dirac notation this leads to a doubling of symbols, and to an ambiguity about which factor is the linear one, which the antilinear. We shall find the Dirac notation extremely convenient for $\mathfrak{Q}$ quantum mechanics, since it manages automatically certain rules of order that are not important in $\mathfrak{C}$ quantum mechanics. Equivalent definitions of $\mathfrak{K}(\mathfrak{Q})$ are given by von Neumann and Birkhoff, (reference 3); E. H. Moore, *General Analysis* (American Philosophical Society, Philadelphia, Pennsylvania, 1935); and O. Teichmuller, Z. Math. 174, 73 (1935).

[5] Again two approaches present themselves, the "synthetic" and the "analytic"; just as the definition of "general quantum mechanics" in footnote 2 is the "synthetic" version of the "analytic" one given in the text of this section for $\mathfrak{F}QM$. Again we relegate the "synthetic" formulation to a footnote: An automorphism U on the propositional calculus of $\mathfrak{F}QM$ is a mapping of *propositions* to *propositions*, $U:A\rightarrow A'=A^U$, that possesses an inverse and preserves the operation of negation and the relation of implication. It is then a theorem[3] that every such mapping is effected by a mapping of vectors of the kind to be called co-unitary above. Likewise any mapping that preserves implication is represented by a colinear vector transformation.

[6] Why unitary and not simply co-unitary? Since after all the essential requirements from the point of view of the propositional calculus are that implication and negation of propositions (linear dependence and orthogonality of vectors) be preserved by the passage of time, and this is a property of the co-unitary operators.

example, suppose Ψ is a vector representing the state prepared by some definite process performed on the system by an external apparatus. For the process to be a definite one, we mean in particular that the times of operation of the parts of the apparatus are specified. Then $U_t\Psi$ represents the state that would be produced by the same apparatus operating at times retarded by the amount t.

We now wish to make the passage to a Schrödinger equation. In $\mathcal{C}$ quantum mechanics Stone's theorem is the bridge from the unitary group U_t to the Schrödinger equation for $\Psi(t)=U_{-t}\Psi(0)$,

$$d\Psi/dt=-iH\Psi. \tag{3}$$

For $\mathcal{R}$ and $\mathcal{Q}$ quantum mechanics, this equation makes no sense; quite obviously so for $\mathcal{R}$ quantum mechanics where there is no symbol i at all, but equally so for $\mathcal{Q}$ quantum mechanics as well, since the various i's that appear there are not linear operators at all and their appearance on the left of a Ψ is undefined. Therefore we require the following modification of Stone's theorem:

Theorem: Every one-parameter unitary group U_t on $\mathfrak{H}(\mathfrak{F})$ is generated by an equation of the form

$$d\Psi/dt=-\eta H\Psi, \tag{4}$$

where H is Hermitian non-negative, η is anti-Hermitian unitary, and both commute strongly with U_t. H is unique, and η is unique except on the null space of H within which it acts as an arbitrary anti-Hermitian unitary operator.

Proof: See Appendix C.

Since this decomposition results in a non-negative generator H, it is an appropriate one for the time translation but not for the other one-parameter groups, where reflection may be a physically possible process. This theorem gives us a way to construct the energy from the time translation group. How shall we construct the momentum from the space translations, etc.? Let us write A for the infinitesimal anti-Hermitian generator of time translation, so that

$$d\Psi/dt=A\Psi.$$

Then the absolute value of A gives the non-negative energy operator of the theory and the phase of A gives a unitary square root of -1:

$$H=|A|\equiv(A\dagger A)^{\frac{1}{2}},$$
$$\eta=|A|A^{-1}.$$

For ordinary quantum mechanics $\mathfrak{F}=\mathcal{C}$ and $\eta=i$. The theorem shows exactly in what directions this case can be generalized. For instance we can assume that all the observables commute with the operator η. If we do this, we obtain the following possibilities:

$\mathfrak{F}=\mathcal{R}$: This case has been studied by Stueckelberg and it was shown by him that it is equivalent to $\mathfrak{F}=\mathcal{C}$, $\eta=i$.

$\mathfrak{F}=\mathcal{C}$: If $\eta=i$ we obtain ordinary quantum mechanics without superselection rules. If $\eta\neq i$, we obtain a more general case with superselection rules.

$\mathfrak{F}=\mathcal{Q}$: This is the case we are especially concerned with in this paper. If we drop the assumption that all the observables commute with η we obtain more general cases about which little is known.

Only the case $\mathfrak{F}=\mathcal{R}$, which is equivalent to $\mathfrak{F}=\mathcal{C}$ and antilinear observables, has been investigated in some detail by Stueckelberg, Guenin, and Piron (unpublished).

At some points in the paper we shall thus make the following assumption: The operator η associated with the time-translation operator U_t by the above theorem is a superselection operator: All the observables commute with η. But it should be pointed out that in that case we are not very far from complex quantum mechanics. We state without proof the following reduction theorem.

Theorem: The propositions of $\mathcal{Q}$ quantum mechanics that commute with a fixed anti-Hermitian unitary operator are isomorphic (with respect to the logical operations of intersection, span, and orthocomplement) to the propositions of $\mathcal{C}$ quantum mechanics.

Of course there is more to quantum mechanics than its propositional calculus. Even if the η superselection principle is imposed, eliminating the new kinds of complementarity peculiar to $\mathcal{Q}$ quantum mechanics, there will still remain new dynamical variables and symmetries.

It will be observed that the points at which the operator η appears in the formulation of each particular quantum theory and the points at which Planck's constant enters are the same. The operator η can best be thought of as taking the place of the combination $i/\hbar$ in ordinary quantum mechanics. Therefore the

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The reason the unitary operators are sufficient varies slightly for the three cases $\mathfrak{F}=\mathcal{R}$, $\mathcal{C}$, $\mathcal{Q}$:

For $\mathfrak{F}=\mathcal{R}$, the only automorphism of $\mathfrak{F}$ is the identity $a\rightarrow a$; all colinear operators are linear, and all co-unitary operators are unitary.

For $\mathfrak{F}=\mathcal{C}$, the automorphisms of $\mathfrak{F}$ are the identity I and the complex conjugate $C:a\rightarrow a^*$; all colinear operators are either linear, or if not, are called *antilinear*, the two classes being disconnected. Since a one-parameter group U_t is connected and $U_0=1$ is linear, all U_t are necessarily linear.

For $\mathfrak{F}=\mathcal{Q}$, the automorphisms of $\mathfrak{F}$ are the conjugations $a\rightarrow a^q=qaq^{-1}$; any colinear operator T can be expressed in terms of an associated (nonunique) linear operator L and a quaternion q according to

$$T\Psi=L\Psi q. \tag{*}$$

Now we see that the linear operators (q real) are continuously connected to the other colinear operators (q not real). Thus the continuity argument does not work here. On the other hand the colinear T and the associated linear operator L of (*) define the same correspondence of propositions to propositions (subspaces to subspaces). Therefore, for $\mathcal{Q}$ quantum mechanics every such correspondence, being representable by a colinear operator, is representable by a linear operator. By choosing the q in (*) to be of unit norm, it is readily seen, L is determined up to sign and is unitary if T is co-unitary. We thus obtain a unitary function of time U_t obeying

$$U_tU_{t'}=\pm U_{t+t'}.$$

By continuity, it is always possible to redefine U_t so that the upper sign is chosen.

introduction of operators that fail to commute with η, as when η is a quaternion, can be regarded as taking Planck's "constant" to be a dynamical variable of the theory. It will be interesting in particular to examine the consequences of quantum fluctuations in the quantum of action. The superselection principle on the other hand makes η a more classical variable by excluding interference between its different values. In principle this makes it consistent to "freeze" the value of η and therefore to suppress these new possibilities.

4. COMPOSITE SYSTEMS

The other extension of the Birkhoff-von Neumann $\mathfrak{Q}$ quantum mechanics that is needed concerns the description of composite systems. In $\mathfrak{C}$ quantum mechanics we are accustomed to multiplying the wave functions of subsystems to get wave functions of composite systems. This multiplication is called the *tensor product*. Even in the quantum theory of fields, where other techniques are used to describe a system of several particles of the same kind, this process of multiplying wave functions is needed to set up a theory for the interaction of particles of different kinds (quanta of different fields).

This multiplication procedure is acceptable in $\mathfrak{C}$ quantum mechanics but not in $\mathfrak{Q}$ quantum mechanics. Suppose $|1\rangle$ and $|2\rangle$ are vectors describing states of distinct systems 1 and 2. Introducing bases in the corresponding Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we are led to wave functions:

$$\psi(x_1)=\langle x_1|1\rangle, \quad \psi(x_2)=\langle x_2|2\rangle, \tag{1}$$

where x_1, x_2 enumerate the basis vectors for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. In order to describe the state of the composite system 1×2, we introduce a tensor product space $\mathcal{H}_1\times\mathcal{H}_2$ by giving it a symbolic basis $\{|x_1,x_2\rangle\}$. Then the composite state is supposed to be represented by the wave function

$$\psi(x_1,x_2)=\psi(x_1)\psi(x_2).$$

In order for this coordinate-dependent method to be acceptable, it must be shown to be essentially coordinate independent. The intrinsic relations (intersection, span, orthogonality) between statements (subspaces) of the product space must depend only upon the intrinsic relations between the factors of which they are made.

This coordinate independence is proven when one shows that a change of basis in either of the factor spaces (i.e., a co-unitary transformation in $\mathcal{H}_1$ or $\mathcal{H}_2$) results merely in a change of basis in the product space (i.e., a co-unitary transformation in $\mathcal{H}_1\times\mathcal{H}_2$). The important formulas for this are the following:

$$\begin{aligned}\psi'(y_1,x_2)&=\langle y_1|1\rangle\langle x_2|2\rangle=\langle y_1|x_1\rangle(\langle x_1|1\rangle\langle x_2|2\rangle)\\&=\langle y_1|x_1\rangle\psi(x_1,x_2),\\ \psi'(x_1,y_2)&=\langle x_1|1\rangle\langle y_2|2\rangle=\langle y_2|x_2\rangle(\langle x_1|1\rangle\langle x_2|2\rangle)\\&=\langle y_2|x_2\rangle\psi(x_1,x_2).\end{aligned} \tag{2}$$

The symbol $\langle x_1|y_1\rangle$ represents the matrix elements between two bases $\{|x_1\rangle\}$ and $\{|y_1\rangle\}$ of $\mathcal{H}_1$. A summation convention is used. Obviously these formulas are valid for $\mathfrak{R}$ and $\mathfrak{C}$ Hilbert spaces, where the matrix elements, being $\mathfrak{R}$ or $\mathfrak{C}$ numbers, commute with one another. Obviously the second formula is not valid in $\mathfrak{Q}$ quantum mechanics, where the matrix elements are quaternions and do not commute.

Indeed there seems to be no satisfactory definition of the tensor product of vectors in $\mathfrak{Q}$ Hilbert spaces as an operation that is unique, commutative, and invariant under the entire unitary groups of the spaces being multiplied.

As a result, the gap between $\mathfrak{Q}$ quantum mechanics and classical physics is greater than the gap between $\mathfrak{C}$ quantum mechanics and classical physics. In classical physics there are no phase relations to be considered when systems are imbedded as subsystems in a larger system, either by adding or multiplying their phase spaces. In $\mathfrak{C}$ quantum mechanics there are phase relations between states that are important when sums are formed but not when products are formed. In $\mathfrak{Q}$ quantum mechanics the phase relations are important when states are either added or multiplied.

This novel feature of $\mathfrak{Q}$ quantum mechanics is expressed in another way in terms of complementarity. In classical physics there are no complementarity relations. In $\mathfrak{R}$ and $\mathfrak{C}$ quantum mechanics complementarity relations exist between physical properties of one physical system, but not between properties of different systems; the momentum of an electron and the position of a neutron, for example. In $\mathfrak{Q}$ quantum mechanics, there exists a complementarity between some properties of any two systems. There is no reasonable way of forming a composite system such that all the observables associated with one of the systems commute with all the observables associated with the other.

What are we to make of this peculiar unitary nature of $\mathfrak{Q}$ quantum mechanics, that prevents one at the very start from speaking of absolutely independent systems? One point of view is that in nature, after all, there are no truly independent systems. In a unitary field theory of all the elementary particles, the problem of describing composite systems is to be solved without ever introducing such an artificial concept as a tensor product. This possibility exists in $\mathfrak{Q}$ quantum mechanics as well as in $\mathfrak{C}$ quantum mechanics. Yet we know that on occasion systems can be treated as if they were very nearly independent. Therefore we must approximate such independence in $\mathfrak{Q}$ quantum mechanics.

5. NEW DEGREES OF FREEDOM

In order to express composite systems in $\mathfrak{F}$ quantum mechanics (in particular in $\mathfrak{Q}QM$ it is) necessary to understand the new degrees of freedom, that fail to

commute even when they refer to distinct and independent systems. Their physical meaning we will study later. Now we express them as follows.

There exist continuous transformations acting on the $\mathfrak{F}$ numbers that leave invariant the intrinsic relations $+, \times, ^*$. These are the *automorphisms* of $\mathfrak{F}$. For $\mathfrak{F}=\mathfrak{R}$, $\mathfrak{C}$, $\mathfrak{Q}$ they are enumerated in footnote 6. Covariance of a law under these automorphisms is intended to mean that the form of the law does not depend on the choice of a realization of the number system $\mathfrak{F}$. In the case of $\mathfrak{C}QM$, it expresses equivalence of i with $-i$. In the case of $\mathfrak{Q}QM$ it expresses the equivalence of i_1, i_2, i_3 with any other like-handed set of anticommuting units. We shall make an assumption that simultaneously introduces new degrees of freedom associated with these automorphisms of the number system and guarantees a conservation law associated with these new degrees of freedom.

For every automorphism $A: a \rightarrow a^A$ of the number field $\mathfrak{F}$, there is given a co-unitary transformation $\Psi \rightarrow \Psi^A$ of the Hilbert space such that

$$(\Psi a)^A = \Psi^A a^A. \tag{1}$$

All the laws of the system are covariant under this transformation. This is the principle of covariance under automorphism of $\mathfrak{F}$, or of $\mathfrak{F}$ covariance for short. It permits the automorphisms of $\mathfrak{F}$ to act on $\mathfrak{H}_F$ and requires covariance under them. Let us develop this somewhat elliptic expression before proceeding to deduce its consequences.

There is a subset of the number system $\mathfrak{F}$ that is invariant under all the automorphisms A, namely the reals. In parallel, let us call a state vector real if it is invariant under all the automorphisms A. It is readily shown there exist complete sets of real vectors. It is now customary to define the action of A on any operator X by decreeing that the two relations

$$X^A\Psi^A = \Phi^A \quad \text{and} \quad X\Psi = \Phi$$

are to be equivalent, and we can speak of any operator X as real for which $X^A = X$. These real operators are just those that are invariant under the automorphisms of $\mathfrak{F}$.

Now it not to be inferred from the principle of $\mathfrak{F}$ covariance that all physical quantities are invariant under all automorphisms, i.e., real. We remarked in the introduction that the most interesting new features of $\mathfrak{Q}QM$ involve nonreal Hamiltonians, for example. Rather, it is the form of the laws relating them that is to be real. We suppose there is given a fundamental list of operators Ω in terms of which all observable physical statements (projections) can be expressed. By the laws of the system we mean a collection of algebraic relations $F(\Omega)=0$ among the operators Ω from which the intrinsic relations among the observable physical properties are to be deduced. Then $\mathfrak{F}$ covariance of the physical laws is the requirement that the relations

$$F(\Omega)=0 \quad \text{and} \quad F(\Omega^A)=0$$

are equivalent.

Let us examine the consequences for the three cases $\mathfrak{F}=\mathfrak{R}$, $\mathfrak{C}$, $\mathfrak{Q}$.

The principle of $\mathfrak{R}$ covariance is null; there are no automorphisms of $\mathfrak{R}$ other than the identity I. As an example of $\mathfrak{R}QM$, take the following formulation of the linear harmonic oscillator, in which position, momentum, and energy are taken as fundamental and η is the superselection operator discussed in Sec. 3:

$$\begin{aligned} px - xp &= -\eta \\ \tfrac{1}{2}(p^2+x^2) &= H \\ \eta^2 &= -1. \end{aligned} \tag{2}$$

It is obvious that this theory is real in name only as long as η is taken to be a superselection operator. That is, its observables are isomorphic to the observables of the conventional theory of the oscillator. (Were operators that failed to commute with η accepted as observables, this would correspond to accepting antilinear operators into the usual theory as observables and would be a significant change. We shall not consider this possibility.) A representation of the relations (2) is given by the usual differential operators for x and H, a 2×2 matrix for η,

$$\eta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3}$$

and a matric-differential operator for p,

$$p = -\eta(\partial/\partial x). \tag{4}$$

The principle of $\mathfrak{C}$ covariance is not null. It requires the existence of an antilinear operation $\psi \rightarrow \psi^C$ corresponding to the automorphism $a \rightarrow a^C \equiv a^*$ of the complex numbers. It further requires invariance of the dynamical relations under C. Thus, consider a set of relations defining a harmonic oscillator. Taking position, momentum, and energy as basic, they are usually

$$\begin{aligned} \tfrac{1}{2}(p^2+x^2) &= H, \\ px - xp &= -i, \\ \eta &= i. \end{aligned} \tag{5}$$

Evidently, these are not real relations; they are not satisfied by the complex conjugate quantities. They contain the symbol i explicitly. We may construct a $\mathfrak{C}$ covariant harmonic oscillator by adjoining a real anti-Hermitian η commuting with the other operators of the theory and taking (2) instead of (5) for the basic relations. Now i no longer appears explicitly; but not x is not a complete set of commuting observables by itself. There is the additional Hermitian operator

$$\epsilon = i\eta$$

that commutes with x and has the eigenvalues ± 1.

This operator is a superselection operator since η is one.

To put it differently, the relations (2) establish a correlation between the sign of i and the sense of time, while $\mathfrak{C}$ covariance implies that the sign of i has no physical correlate. If what we have called the $\mathfrak{C}QM$ theory of the oscillator coincides with the conventional theory (because of the superselection rule), what we have called the $\mathfrak{C}QM$ theory of the oscillator corresponds to one with an internal charge degree of freedom (because of the $\mathfrak{C}$ covariance rule). This new degree of freedom is not made effective in the free oscillator, the levels with $\epsilon=+1$ being degenerate with the levels with $\epsilon=-1$, but will be effective when the oscillator is coupled to other systems with forces that depend on ϵ.

Now let us examine the consequences of $\mathfrak{Q}$ covariance. The colinear transformation $\Psi\rightarrow\Psi^A$ is not as convenient to work with as a linear transformation, especially for defining observables. However if A is the automorphism $p\rightarrow qpq^{-1}$, $\Psi^A q$ depends linearly on Ψ since by (1)

$$(\Psi a)^A q=\Psi^A a^A q=\Psi^A(qaq^{-1})q=(\Psi^A q)a.$$

Let us designate this combination by $q\Psi$:

$$q\Psi\equiv\Psi^A q \quad \text{or} \quad \Psi^A=q\Psi q^{-1}\equiv\Psi^q.$$

Thus, $\mathfrak{Q}$ covariance permits us to identify the quaternions with linear operators on $\mathfrak{K}(\mathfrak{Q})$. Henceforth they may act on vectors from the left, a license that is taken for granted for complex numbers. In particular the units i_1, i_2, i_3 are now defined linear operators when they are written on the left of a state vector.[7] A quaternion acting in this capacity will be called a quaternion operator. It is easy to prove the existence of a basis for $\mathfrak{K}(\mathfrak{Q})$ in which the quaternion operator q is represented by a diagonal matrix whose diagonal elements are q. This is the same thing as proving the existence of a real basis. By using a real basis it is easy to prove that every operator X can be uniquely expressed in terms of the operators i_α in the form

$$X=\sum_0^3 X_\alpha i_\alpha,$$

where the coefficients X_α are real linear operators.

In complex quantum mechanics we are familiar with the decomposition of a general operator X into Hermitian and anti-Hermitian parts H, A:

$$X=H+A$$
$$H=\tfrac{1}{2}(X+X\dagger)$$
$$A=\tfrac{1}{2}(X-X\dagger).$$

Of course the same decomposition exists in $\mathfrak{Q}QM$. We wish to mention here that the structure we have introduced in $\mathfrak{Q}QM$ also makes possible a decomposition of the most general operator into 8 parts.

[7] Many of the properties of a system of three anticommuting, anti-Hermitian unitary operators on $\mathfrak{K}(\mathfrak{Q})$ like i_1, i_2, i_3 have been discussed by Teichmüller, reference 4.

The general X can be expanded, we have pointed out, in the form

$$X=\sum_0^3 X_\alpha i_\alpha,$$

where the coefficients X_α commute with the quaternions i_α (i.e., are real). It is natural to define the *quaternion conjugate* of an operator X by changing the sign of its imaginary terms:

$$X^q=\sum_0^3 X_\alpha i_\alpha^q.$$

Further we introduce the transpose as an intrinsic operation according to

$$X^T=X\dagger^q=X^q\dagger.$$

Then any X is the sum of symmetric and skew-symmetric operators S, K according to

$$X=S+K$$
$$S=\tfrac{1}{2}(X+X^T)=S^T$$
$$K=\tfrac{1}{2}(X-X^T)=-K^T.$$

Finally the symmetric and skew-symmetric operators can be decomposed into four parts relative to the quaternion basis i:

$$X=\sum_0^3 S_\alpha i_\alpha+\sum_0^3 K_\alpha i_\alpha.$$

If we merely seek a theory of the linear harmonic oscillator in $\mathfrak{Q}QM$, it is sufficient to replace the symbol i in the complex theory (5) by i_3, say. If we seek a $\mathfrak{Q}$ covariant theory, this is insufficient. However the relations (2) are $\mathfrak{Q}$ covariant.

To define the theory we must somehow specify the nature of the operator η.

6. TENSOR PRODUCTS

The additional structure given the quaternion Hilbert space by the principle of $\mathfrak{Q}$ covariance also makes possible a unique definition of tensor product. The group of the geometry, which is now to leave the quaternion operators fixed as well as the other Hilbert space concepts, is thereby reduced to a real unitary group. The problem of invariance is naturally easier when the group of the geometry is reduced.

We choose real bases x_1, x_2 in the two spaces $\mathfrak{K}_1$, $\mathfrak{K}_2$ to be multiplied. We take as a formal basis of $\mathfrak{K}_1\times\mathfrak{K}_2$ the symbolic products $|x_1\rangle|x_2\rangle=|x_2\rangle|x_1\rangle=|x_1, x_2\rangle$. We define the tensor products of two vectors $|1\rangle$, $|2\rangle$ in either order by their matrix elements in the $|x_1, x_2\rangle$ basis:

$$\langle x_1, x_2|1\rangle|2\rangle=\langle x_1|1\rangle\langle x_2|2\rangle$$
$$\langle x_1, x_2|2\rangle|1\rangle=\langle x_2|2\rangle\langle x_1|1\rangle.$$

Briefly, we multiply vectors by multiplying their matrix elements in real bases. Likewise we will form the tensor products of operators by multiplying their matrix elements in real bases:

$$\langle x_1, x_2|A_1\times A_2|x_1', x_2'\rangle=\langle x_1|A_1|x_1'\rangle\langle x_2|A_2|x_2'\rangle,$$
$$\langle x_1, x_2|A_2\times A_1|x_1', x_2'\rangle=\langle x_2|A_2|x_2'\rangle\langle x_1|A_1|x_1'\rangle.$$

Since the matrix elements do not commute, the tensor product does not commute in general. The projection in $\mathcal{K}_1 \times \mathcal{K}_2$ corresponding to P_1 in $\mathcal{K}_1$ is taken to be $P_1 \times 1_2$ where 1_2 is the identity in $\mathcal{K}_2$. Likewise

$$P_2 \rightarrow P_2 \times 1_1.$$

As the result of the noncommutativity we have the unusual circumstance that the projection in $\mathcal{K}_1 \times \mathcal{K}_2$ representing the logical conjunction of two propositions P_1 and P_2 may vanish when P_1 and P_2 do not. This conjunction is to be found as the intersection of the subspaces on which $P_1 \times 1_2$ and $P_2 \times 1_1$ project, and it is easy to construct examples $[\psi_1(x_1) = \exp(i_1 x_1), \psi_2(x_2) = \exp(i_2 x_2)]$ in which this intersection is zero although P_1 and P_2 are both one-dimensional projections. This anomaly does not occur for real projections.

In ordinary quantum mechanics there are three different ways in which corresponding operators on different systems combine when the systems are composed. Unitary transformations are examples of operators that compose *multiplicatively*: $U = U_1 \times U_2$. Infinitesimal generators therefore compose *additively*, as do many physical quantities. But the underlying field elements, which are also operators, are composed by *identification*; for example, the imaginary unit i has the property $i(|1\rangle|2\rangle) = (i|1\rangle)|2\rangle = |1\rangle(i|2\rangle)$. We have come as close as possible to this situation in our formulation of the tensor product in quaternion quantum mechanics. (Namely, the effect of multiplying a tensor product by a quaternion (from either side) may be computed by letting the quaternion multiply the factor in the tensor product it abuts.) This raises a certain conceptual problem of some importance. What can it mean physically to perform such an identification?

The elements of the number field represent logical relations between possible states of the physical system under consideration. (Indeed they may be represented by ordered triples of pure states.) It is evidently possible to say when two such relations are the "same" even for states of distinct physical systems without going beyond the concepts of pure logic. However this only defines the numbers up to automorphisms, evidently. For real quantum mechanics, this is sufficient to define them completely. In complex quantum mechanics an ambiguity remains: the i of one physical system may be identified either with i or $-i$ of another by this method. (It is possible to multiply the states of one system by the time-reversed states of the other and obtain a tensor product which meets all requirements of mere logic.) In quaternion quantum mechanics the corresponding ambiguity is infinitely greater. Yet it must be resolved in order to discuss composite systems. This requires us to introduce further elements of structure into the theory.

The operator η also presents a problem in this regard. The anti-Hermitian infinitesimal generator of time is $A = -\eta H$ and is composed additively for noninteracting systems because of its meaning. The Hamiltonian H should also be composed additively for noninteracting systems. These two requirements are enough to require that the operator η be composed by identification, as is multiplication by i in complex quantum mechanics. Moreover it then follows that the relation between η and the quaternion operators must be a universal one, not involving specific dynamical features of the system under consideration. *We shall pursue the hypothesis that the operator η actually corresponds to a member of the underlying number field* as in ordinary quantum mechanics. This reduces the problem to the one previously stated.

This appears to destroy the $\mathcal{Q}$ covariance of the theory by singling out one imaginary. To preserve $\mathcal{Q}$ covariance we will have to formulate a dynamical theory of the operator η so that its value is determined by an initial condition. This will be considered in a subsequent paper. In the mean time we will speak of $\mathcal{Q}$ covariance "modulo η": agreeing to transform η as a quaternion when examining a theory for covariance.

In general, we see that the quantum theories will possess less symmetry than their classical limits. *The very process of quantization singles out an axis in the imaginary space.*

7. QUANTUM OSCILLATORS IN $\mathcal{Q}$QM

Any linear field may be regarded as an aggregate of harmonic oscillators. In the previous section we sketched the $\mathcal{R}$, $\mathcal{C}$, and $\mathcal{Q}$ quantum mechanics of a one-dimensional oscillator. This is the kind of oscillator a real scalar field gives. In the decomposition of the field in terms of some appropriate complete set of real orthogonal functions, or modes, real expansion coefficients appear as amplitudes and become the dynamical variables. Each of these amplitudes can be thought of as the displacement or coordinate of an oscillator. The principle quantum number n of one oscillator then counts the number of quanta in one mode, and is called an occupation number. Such oscillators we will call *quantum oscillators*. According to the statistics we have boson or fermion oscillators.

Sometimes it is convenient to express the field as an aggregate of two- or three-dimensional oscillators instead of a one-dimensional oscillator. If a complex field is expanded in terms of real orthogonal functions the amplitudes are complex numbers. Each amplitude can be regarded as the coordinate of an oscillator in the complex plane instead of as the coordinates of two real oscillators. This is useful when there is symmetry with respect to rotations in the complex plane. Instead of two separate occupation numbers n_1, n_2 the energy levels of the complex oscillator are most aptly described by a principal quantum number n and an angular momentum m about the origin of the complex plane. The number n is the number of quanta of that type and

the angular momentum m is the total charge that they carry:

$$n=n_+ + n_-,$$
$$m=n_+ - n_-,$$

where n_+ and n_- count positive and negative quanta.

In decomposing an isovector field in terms of a real family of (isoscalar) functions, we get oscillator amplitudes which are themselves triples (isovectors) of real numbers. Then it is convenient to work with three-dimensional oscillators. They possess a principal quantum number n, an orbital angular momentum l, and a magnetic quantum number m. The number n is again the number of quanta present of that type. The angular momentum $l \geq 0$ measures the charge multiplicity of the level, which is $2l+1$. The number m according to isospin theory gives the total charge of that member of the charge multiplet:

$$n=n_+ + n_0 + n_-,$$
$$m=n_+ - n_-,$$

where n_+, n_0, n_- count positive, neutral, and negative quanta.

Boson Oscillators

In this section we will study the oscillators that will make up a field that is quaternionic instead of real or complex. We shall call the system we are describing the quaternionic oscillator ($\mathfrak{Q}$ oscillator), having already touched on $\mathfrak{R}$ and $\mathfrak{C}$ oscillators. (It is quite possible to consider an oscillator in the general field $\mathfrak{F}$, developing all three cases in parallel. We shall not.)

The quaternionic oscillators move in the four-dimensional space $\mathfrak{Q}$. The invariance their Hamiltonians possess is called $\mathfrak{Q}$ covariance in Sec. 5. If we call the coordinate of the oscillator q,

$$q=\sum_0^3 q_\alpha i_\alpha, \quad i_0=1 \tag{1}$$

this means invariance under

$$q \longrightarrow q^a = aqa^{-1}$$

or under

$$q_0 \longrightarrow q_0$$
$$q_n \longrightarrow \sum_n r_{mn} q_n \quad (m, n=1, 2, 3). \tag{2}$$

Here we are taking note that the automorphism $q \longrightarrow q^a$ leaves the real part of q invariant and subjects the imaginary part of q to a Euclidean rotation represented by a matrix r_{mn}. The range of the variable q must be invariant under (2). There are just three non-null linear subspaces of $\mathfrak{Q}$ that are invariant under these automorphisms. The real axis is invariant: the three-dimensional space of pure imaginaries is invariant; and of course the entire space $\mathfrak{Q}$ is also. There are therefore three kinds of $\mathfrak{Q}$-covariant linear oscillators in the space $\mathfrak{Q}$; a real oscillator, an imaginary one, and a four-dimensional system we will call the full $\mathfrak{Q}$ oscillator.

The real oscillator has already been treated in Sec. 5 and we will study the imaginary quaternionic oscillator now. (For a $\mathfrak{C}$ oscillator the automorphism invariance, which is there $\mathfrak{C}$ covariance, leads again to three kinds of oscillators, but the real axis and the imaginary axis are both one dimensional, and are not essentially different.)

As a guide for the development we write the classical Lagrangian in the form

$$L=(\dot{q}^*\dot{q}-\omega^2 q^* q)/2 \tag{2}$$

and take as classical coordinates the three real coefficients q_m of the expansion

$$q=\sum i_m q_m, \quad m=1, 2, 3. \tag{4}$$

This leads to a classical Hamiltonian and Poisson brackets

$$H=(p_m p_m+\omega^2 q_m q_m)/2$$
$$[q_m, q_n]_P=[p_m, p_n]_P=0 \tag{5}$$
$$[p_m, q_n]_P=\delta_{mn}$$

where $p_m=\dot{q}_m$. We replace these Poisson brackets by η times the quantum commutator. Since η is actually a particular quaternion in disguise this temporarily destroys the $\mathfrak{Q}$ covariance of the theory as a whole, as already mentioned. From the commutation relations

$$[q_m, q_n]=[p_m, p_n]=0$$
$$[p_m, q_n]=-\eta\delta_{mn}, \tag{6}$$

it follows by use of Jacobi's identity that η commutes with q_m and p_m. The operators p_m, q_m are not completely defined yet. Their commutation relations with the quaternion units perpendicular to η are undetermined. For example we can suppose that the q_m are represented by real symmetric operators. In that case the order of factors in the expansion (4) is irrelevant. The p_m are then represented by imaginary operators.

It is readily seen that the operators H and $\mathbf{Q}=\mathbf{q}\times\mathbf{p}$ are constants of the motion. The first is the energy or total occupation number of the three oscillators, the second the "angular momentum." Moreover the operators H, Q^2, Q_3 form a complete commuting set of observables, in spite of the "extra degrees of freedom" associated with quaternions ζ, $\eta\zeta$ that anticommute with η. ζ may be used to couple the oscillator to another system, but not to construct any Hermitian operator commuting with the three enumerated.

The ground state is unique (nondegenerate) and is given by the usual real wave function of the three-dimensional isotropic oscillator in the q representation. The possibility exists of multiplying this state by the quaternionic phase $\exp \mathbf{i}\cdot\boldsymbol{\theta}$, where $\boldsymbol{\theta}$ is composed of three arbitrary real numbers, without changing the energy, but this does not change the ray in Hilbert space $\mathfrak{H}(\mathfrak{Q})$.

We notice that the group of the theory associated with its $\mathfrak{Q}$ covariance contains the transformation $\eta \longrightarrow -\eta$ usually called time reversal.

Now consider the full $\mathcal{Q}$ oscillator. It is obvious that it is a composite of a real and an imaginary $\mathcal{Q}$ oscillator, so that in principle no new behavior is to be expected. Nevertheless it possesses an additional symmetry (if all the frequencies are the same) that deserves special note:

There is a definite sense in which neither the real nor imaginary $\mathcal{Q}$ oscillators are linear. To illustrate this sense, consider the Lagrangian and equation of motion of the complex oscillator, a classical mechanical system:

$$L=\tfrac{1}{2}(\dot{z}\dagger\dot{z}-\omega^2 z\dagger z)\quad \ddot{z}=-\omega^2 z.$$

The equations of motion are invariant under the substitution

$$z\longrightarrow zc$$

where c is any complex constant. It is the $\mathcal{Q}$ analog of this linear transformation of the dynamical variables (not of the quantum state vector) that we now have in mind when we say that the real and imaginary oscillator are *nonlinear*: they will submit to the transformation

$$q\longrightarrow qk,$$

not for any constant quaternion k, but only for real k. The full $\mathcal{Q}$ oscillator can be a genuinely linear system in the sense of quaternions. It is therefore interesting; but not therefore fundamental. (The basic requirement we impose on all our mechanical systems is $\mathcal{Q}$ covariance, not $\mathcal{Q}$ linearity. All actual systems are at least a little nonlinear.)

With this forward we present the $\mathcal{Q}$ linear oscillator. The classical Lagrangian is

$$L=\tfrac{1}{2}(\dot{q}\dagger\dot{q}-\omega^2 q\dagger q),\quad q=\textstyle\sum_0^3 q_\alpha i_\alpha. \tag{7}$$

The quantum Hamiltonian and commutators are then

$$\begin{aligned} H&=\tfrac{1}{2}\textstyle\sum_0^3 (p_\alpha^2+\omega^2 q_\alpha^2) \qquad (8)\\ [p_\alpha,q_\beta]&=-\eta\delta_{\alpha\beta}\\ [p_\alpha,p_\beta]&=0\\ [q_\alpha,q_\beta]&=0. \end{aligned}$$

η is again one of the unit imaginary quaternions. For simplicity we suppose here that the q_α are symmetric operators, as well as Hermitian. Let us be more explicit about the general definition of the generator designated in a special case by $\mathbf{Q}$. The automorphism $q\rightarrow aqa^{-1}$, $p\rightarrow apa^{-1}$ $(a\in\mathcal{Q})$ acting on the quaternion units induces a transformation of the real components q_α, p_α that leaves invariant their commutation relations (8). Accordingly defining q_m' by

$$aqa^{-1}=\sum_{m=0}^{3} q_m' i_m \tag{9}$$

and defining p_m' similarly, we demand $Q(a)$ such that

$$\begin{aligned} q_m'&=Q(a)q_m Q(a)^{-1},\\ p_m'&=Q(a)p_m Q(a)^{-1}, \qquad (10)\\ \eta'&=Q(a)\eta Q(a)^{-1}=\eta. \end{aligned}$$

The infinitesimal form of this relation deals with the three infinitesimal i_1, i_2, i_3 instead of the finite a, and seeks corresponding Hermitian operators Q_1, Q_2, Q_3. The infinitesimal i_m rotation $a=1+i_m\delta\theta/2$ generates variations

$$\delta_m q=aqa^{-1}=[i_m,q]\delta\theta/2\equiv\textstyle\sum \delta_m q_n i_n \tag{11}$$

and analogously for $\delta_m p$ and $\delta_m p_n$. We require Q_m to be a Hermitian operator satisfying

$$\delta_m q_n=\eta[Q_m,q_n] \tag{12}$$

and analogously for $\delta_m p_n$. We also require

$$[Q_m,\eta]=0.$$

From this follows

$$\mathbf{Q}=\mathbf{q}\times\mathbf{p}. \tag{13}$$

For the $\mathcal{Q}$ linear oscillator, infinitesimal left multiplication by a unit quaternion is also an invariant transformation. We shall call its generator $\mathbf{T}$, requiring that for infinitesimal left multiplications

$$\delta_m q\equiv i_m\delta\theta q/2\equiv\textstyle\sum_\alpha \delta_m q_\alpha i_\alpha$$

we have

$$\delta_m q_\alpha=\eta[T_m,q_\alpha]\delta\theta. \tag{14}$$

This leads to an essentially unique real $\mathbf{T}$ which like $\mathbf{Q}$ can be expressed in terms of q and p:

$$\begin{aligned} \mathbf{T}&=\textstyle\sum \mathbf{t}_{\alpha\beta}q_\alpha p_\beta,\\ \mathbf{T}&=(T_m), \qquad (15)\\ \mathbf{t}&=(t^m). \end{aligned}$$

Substituting (15) into (14) yields the following unique forms for the coefficient matrices t:

$$\begin{aligned} {t_{01}}^1&=-\tfrac{1}{2},\\ {t_{23}}^1&=-\tfrac{1}{2}. \end{aligned} \tag{15'}$$

The coefficients $\mathbf{t}$ are unchanged by cyclic permutations of (123) and are skew symmetric in the lower index pair. The coefficients whose values do not thereby follow from (15′) are zero. The three real 4×4 matrices $\mathbf{t}_{\alpha\beta}$ generate certain orthogonal transformations in the space of the q.

Infinitesimal right multiplication is also an invariant transformation of this oscillator. We call its real generator $\mathbf{Y}$, requiring that for $\delta_m q=qi_m\delta\theta/2=\sum \delta_m q_\alpha i_\alpha$

$$\begin{aligned} \delta_m q_\alpha&=[y_m,q_\alpha]\delta\theta\\ 0&=[y_m,\eta]. \end{aligned} \tag{16}$$

This leads to an essentially unique real $\mathbf{Y}$, which can be expressed in the form

$$\mathbf{Y}=\mathbf{y}_{\alpha\beta}q_\alpha p_\beta. \tag{17}$$

Substituting (17) into (16) yields the following unique forms for the coefficient matrices $\mathbf{y}$:

$$\begin{aligned} {y_{01}}^1&=-\tfrac{1}{2},\\ {y_{23}}^1&=\tfrac{1}{2}. \end{aligned} \tag{18}$$

The coefficients $\mathbf{y}$ are unchanged by cyclic permutations of (123) and are skew symmetric in the lower index pair. The coefficients whose values do not thereby follow from (18) are zero. The three real 4×4 matrices $y_{mn}{}^{p}$ generate certain orthogonal transformations in the space of the q_α. Obviously

$$\mathbf{Q}=\mathbf{T}+\mathbf{Y}. \tag{19}$$

We will call $\mathbf{T}$ the *isospin* and $\mathbf{Y}$ the *hypercharge* vectors of this quantum oscillator. However they do not have the universal significance of the charge vector $\mathbf{Q}$, which always exists. $\mathbf{T}$ and $\mathbf{Y}$ are strictly definable as conserved quantities only for $\mathfrak{Q}$ linear systems. The decomposition (19) is closely related to the well-known decomposition of the orthogonal group O_4 (of which $\mathbf{Q}$ generates a subgroup) into two groups O_3. It is easy to verify the following commutation relations from the definitions (13), (14), and (16), without computations:

$$\begin{aligned}
&[Q_m,Q_n]=\eta\epsilon_{mnp}Q_p\\
&[T_m,T_n]=\eta\epsilon_{mnp}T_p\\
&[Y_m,Y_n]=\eta\epsilon_{mnp}Y_p\\
&[T_m,Y_n]=0.
\end{aligned} \tag{20}$$

For example the last relation holds because $\mathbf{T}$ generates a left multiplication and $\mathbf{Y}$ generates a right multiplication. Left and right multiplications always commute:

$$a(qb)=(aq)b.$$

The vector operators $\mathbf{T}$ and $\mathbf{Y}$ both have the algebraic properties of an angular momentum and commute with each other. Therefore it is possible to form commutative scalars Q, T, and Y in the familiar manner.

$$\begin{aligned}
&\mathbf{Q}^2=Q(Q+1),\\
&\mathbf{T}^2=T(T+1),\\
&\mathbf{Y}^2=Y(Y+1).
\end{aligned} \tag{21}$$

It is convenient to introduce the operators

$$\begin{aligned}
&a_\alpha=(p_\alpha+\eta\omega q_\alpha)/(2\omega)^{\frac{1}{2}},\\
&a_\alpha\dagger=(p_\alpha-\eta\omega q_\alpha)/(2\omega)^{\frac{1}{2}},
\end{aligned} \tag{22}$$

obeying

$$[a_\alpha,a_\beta]=0,\quad [a_\alpha\dagger,a_\beta]=\delta_{\alpha\beta}.$$

Except for a zero-point energy we have

$$H=\sum \omega a_\alpha\dagger a_\alpha. \tag{23}$$

Because of the relation

$$Ha_\alpha-a_\alpha H=-\omega a_\alpha, \tag{24}$$

the operators a_α reduce the number of quanta, and the operators $a_\alpha\dagger$ increase the number of quanta. All the one-quantum states are of the form

$$\Psi_1=\textstyle\sum_\alpha a_\alpha\dagger\cdot c_\alpha\Psi_0, \tag{25}$$

where the c_m are $\mathfrak{Q}$ constants and Ψ_0 is the real ground state.

There exists a frame that makes the a_α, $a_\alpha\dagger$ real instead of the q_α. This reflects the arbitrariness mentioned in connection with the commutation relations (6).

Fermion Oscillators

For the fermion oscillator we again start from the classical Poisson brackets (5), but now we replace them with anticommutators instead of commutators. We may just as well write the relations defining the full (linear) $\mathfrak{Q}$ oscillator, since the real or imaginary oscillator can be extracted easily. They have been given essentially the following form by Gursey[8]:

$$\begin{aligned}
&\{p_\alpha,q_\beta\}=\delta_{\alpha\beta}=1,\ 2,\ 3\\
&\{p_\alpha,p_\beta\}=0\\
&\{q_\alpha,q_\beta\}=0\\
&\{p_\alpha,i_\beta]=0\\
&[q_\alpha,i_\beta]=0\\
&H=\tfrac{1}{2}\eta\omega\textstyle\sum_\alpha[p_\alpha,q_\alpha].
\end{aligned}$$

The $\mathfrak{Q}$ covariance of this system is clear, modulo η, and in addition it possesses $\mathbf{T}$ and $\mathbf{Y}$ invariance, which are defined as in (14,16) in terms of left and right multiplication by quaternions. Where the boson oscillator possessed an infinity of distinct energy eigenvalues the fermion oscillator has but five in the linear case (two in the real case; four in the imaginary). These correspond to $N=0,\ 1,\ 2,\ 3,\ 4$.

There is an essential difference between the fermion and boson symmetry properties. For the boson the Hamiltonian is $\mathfrak{Q}$ covariant but the anti-Hermitian time generator involves η. For the fermion the anti-Hermitian time generator is $\mathfrak{Q}$ covariant but the Hamiltonian involves η.

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[8] F. Gürsey (private communication).

APPENDIX A. $\mathfrak{Q}$ HILBERT SPACE

Most of the statements in the theory of a complex Hilbert space can be taken over for a $\mathfrak{Q}$ Hilbert space. We will develop the elementary theory in a way that makes the analogy between the two obvious. The principle results of this Appendix are the Spectral theorem and our modification of Stone's theorem.

A Hilbert space $\mathfrak{H}$ over the field of the quaternions is a set of vectors $\varphi, \psi, \cdots$ which have the following property.

1. H is a linear space:

If $\varphi, \psi \in \mathfrak{H}$ then $\varphi+\psi \in \mathfrak{H}$.
If $\varphi \in \mathfrak{H}$ then $\varphi q \in H$, $q \in \mathfrak{Q}$.

$$(\varphi+\psi)q = \varphi q + \psi q$$
$$\varphi(p+q) = \varphi p + \varphi q$$
$$(\varphi p)q = \varphi(pq).$$

2. In $\mathfrak{H}$ a duality and therefore a scalar product are defined: To *any* vector φ is associated a dual vector $\varphi\dagger$ in the dual space, such that the *scalar product* $\varphi\dagger\psi$ is Hermitian symmetric, linear in the second factor, and antilinear in the first factor:

$$\varphi\dagger\psi = (\psi\dagger\varphi)^q$$
$$\varphi\dagger(\psi_1+\psi_2) = \varphi\dagger\psi_1 + \varphi\dagger\psi_2$$
$$\varphi\dagger(\psi q) = (\varphi\dagger\psi)q.$$

We remark that

$$(\varphi q)\dagger\psi = [\psi\dagger(\varphi q)]^q = [(\psi\dagger\varphi)q]^q = q^q(\psi\dagger\varphi)^q = q^q(\varphi\dagger\psi).$$

Finally, $\psi\dagger\psi$ is definite:

$$\psi\dagger\psi \geqq 0, \quad \text{and} \quad \psi\dagger\psi = 0 \quad \text{implies} \quad \psi = 0.$$

This is consistent because $q\dagger q \geq 0$.

3. $\mathfrak{H}$ is complete.

Schwarz's Inequality

Since in the theory of Hilbert space nearly every theorem depends on Schwarz's inequality, we will derive this relation as an example exhibiting the minor departure from complex Hilbert space. In fact, a careful handling of the scalars is all one needs.

Consider

$$0 \leq (\varphi p - \psi q)\dagger(\varphi p - \psi q)$$
$$= p^q(\varphi\dagger\varphi)p - p^q(\varphi\dagger\psi)q - q^q(\psi\dagger\varphi)p + q^q(\psi\dagger\psi)q.$$

Putting $p = (\psi\dagger\psi)$, $q = (\psi\dagger\varphi)$, we have:

$$0 \leq \psi\dagger\psi\{(\psi\dagger\psi)(\varphi\dagger\varphi) - (\psi\dagger\varphi)(\varphi\dagger\psi)\}$$

or since $(\psi\dagger\psi) \geq 0$, $(\varphi\dagger\varphi)(\psi\dagger\psi) \geq (\varphi\dagger\psi)(\psi\dagger\varphi)$. In the usual way one then derives the triangle inequality, and the statements about continuity, continuous sequences, and convergence.

Operators

A linear operator L is a mapping of $\mathfrak{H}$ into itself with the following properties

$$L(\varphi+\psi) = L\varphi + L\psi; \quad \varphi, \psi, L\varphi, L\psi \in \mathfrak{H}$$
$$L(\varphi q) = (L\varphi)q.$$

(This definition is adequate for operators in finite-dimensional spaces or for bounded operators. For unbounded operators a more precise specification of the domain of definition is necessary.) According to this definition of a linear operator, multiplication of a vector by a quaternion (from the right) is not a linear operator, for if we compute the effect of the transformation defined by $\psi \rightarrow \psi' = \psi q$ upon a linear combination

$$\psi = \varphi_1 p_1 + \varphi_2 p_2,$$

we find

$$\psi \rightarrow \psi' = \psi q = \varphi_1 p_1 q + \varphi_2 p_2 q \neq \varphi_1' p_1 + \varphi_2' p_2.$$

Instead $\psi' = \varphi_1'(q^{-1}p_1 q) + \varphi_2'(q^{-1}p_2 q)$. We therefore define a *colinear* operator Λ as a correspondence of H into H such that

$$\Lambda(\varphi+\psi) = \Lambda\varphi + \Lambda\psi,$$
$$\Lambda(\psi q) = \Lambda\psi q',$$

where q' is a quaternion associated with q and Λ. This just expresses the fact that Λ carries subspaces into subspaces. It may be shown that q' is a quaternion of the same class as q, and it will be written q^Λ.

Hermitian, anti-Hermitian, unitary, and normal opertors are defined, respectively, through the equations

$$\varphi\dagger H\psi = (H\varphi)\dagger\psi, \quad \varphi\dagger A\psi = -(A\varphi)\dagger\psi$$
$$(U\varphi)\dagger(U\psi) = \varphi\dagger\psi, \quad (N\varphi)\dagger(N\psi) = (N\dagger\varphi)\dagger(N\dagger\psi)$$

for every $\varphi, \psi \in \mathfrak{H}$, or

$$H = H\dagger, \quad A = -A\dagger, \quad U^{-1} = U\dagger, \quad NN\dagger = N\dagger N.$$

We note at this point an important difference in the relation between Hermitian and anti-Hermitian operators in the complex and the quaternionic case. In the former the imaginary unit i provides a unique transition from the Hermitian to the anti-Hermitian and vice versa: $H = iA$, $A = -iH$. (Moreover, in that case,

$$U = \frac{1-iH}{1+iH}.)$$

That an analogous relation in a $\mathfrak{Q}$ space cannot hold is easily seen from the fact that in an n-dimensional space H has $2n^2-n$, but A has $2n^2+n$, independent $\mathfrak{Q}$ elements.

However,

$$U = (1-A)/(1+A)$$

still holds. (In this case the numerator and the denominator commute.)

As in a complex space, a unitary operator does not provide us with the most general norm-preserving transformation (isometry).

This is given through a colinear operator Y with

$$(Y\varphi)\dagger(Y\psi)=(\varphi\dagger\psi)^{Y}$$

$(\varphi\dagger\psi)^{Y}$ being a quaternion of the same class as $(\varphi\dagger\psi)$. Such an operator Y is termed co-unitary.

Symplectic Representation

Finally we mention (following Chevalley) a special representation of a $\mathfrak{Q}$ operator in a n-dimensional $\mathfrak{Q}$ space $\mathfrak{H}(\mathfrak{Q})$ through a $\mathfrak{C}$ operator in a $2n$-dimensional $\mathfrak{C}$ space $\mathfrak{H}(\mathfrak{C})$. This is nothing but a straightforward generalization of the decomposition of a quaternion in an ordered pair of complex numbers.

We introduce in both spaces an orthornormal system, and associate to every vector ψ with components $(\psi_1\cdots\psi_n)$ the vector ψ_s with components $(c_1\cdots c_n, d_1\cdots d_n)$, where $\psi_m=c_m+i_2d_m$, and $[c_m,i_3]=[d_m,i_3]=0$. For each operator L on $\mathfrak{H}(\mathfrak{Q})$ we define L_s on $\mathfrak{H}(\mathfrak{C})$ by $L_s\psi_s=(L\psi)_s$. One can easily show that the correspondence $\psi\leftrightarrow\psi_s$ is bi-unique, and that to every operator L in $\mathfrak{H}(\mathfrak{Q})$ corresponds one and only one L_s in $\mathfrak{H}(\mathfrak{C})$, the *symplectic representative* of L. In particular, to an isometry in $\mathfrak{H}(\mathfrak{Q})$ corresponds an isometry in the $\mathfrak{H}(\mathfrak{C})$. We designate the isometry $\psi\rightarrow\psi i_2$ by K and its symplectic representative by K_s. K_s is anti-unitary.

Let the operator J be defined so that $J\psi$ has components $(i_3\psi_1, i_3\psi_2, \cdots, i_3\psi_n)$. The condition that an operator on $\mathfrak{H}(\mathfrak{Q})$ be unitary, $U\dagger=U^{-1}$, is expressed for the corresponding operator on $\mathfrak{H}(\mathfrak{C})$ through the two equations $U_s\dagger=U_s^{-1}$ and $U_s{}^{T}J_sU_s=J_s$, where $U_s{}^{T}$ means the transposed operator and $J_s{}^{T}=-J_s$, $J_s{}^{2}=-1$. This shows that the group of unitary transformations in a $\mathfrak{Q}$ space is isomorphic to the so-called unitary-restricted symplectic group $Sp(n)$ and therefore is simple. (The group of unitary transformations in a $\mathfrak{C}$ space, as is well known, is not.)

In order than an operator L' on $\mathfrak{H}(\mathfrak{C})$ be the symplectic representative of some L on $\mathfrak{H}(\mathfrak{Q})$, $L'=L_s$, it is necessary and sufficient that L' commute with the operator K_s:

$$L'K_s=K_sL'. \tag{1}$$

APPENDIX B. THE EIGENVALUE PROBLEM

Defining an eigenvalue as a root of the secular determinant is not very convenient here, even in the finite dimensional case, because of noncommutativity. We will treat the diagonalization problems in the finite-dimensional case with the same means as the infinite dimensional, that is to say with spectral resolution techniques. The goal of this and the next chapter will be the theorem that every normal operator admits a spectral resolution.

We give here only a short survey, concentrating on those points where the statements or the methods differ essentially from what one is used to in the complex case and referring the reader for the rest to the literature. We state the spectral theorems for bounded Hermitian, anti-Hermitian, and, finally, normal operators. These theorems then will easily be extended to unbounded operators.

We begin with some general theorems.

We state without proof: The eigenvalues of a Hermitian, anti-Hermitian, or unitary operator are real, pure imaginary, or have norm 1, respectively. Whereas in a $\mathfrak{C}$ space, eigenvectors belonging to different eigenvalues are orthogonal, here we can assert only the following theorem: eigenvectors of normal operators belonging to different *eigenclasses* are orthogonal.

Proof: If $N\varphi=\varphi n$ and $N\varphi'=\varphi n'$ (n, n' quaternions) and N is normal then $N\dagger\varphi=\varphi n^*$ and $N\dagger\varphi'=\varphi' n^*$. Thus

$$\begin{aligned} n^*\varphi\dagger(\varphi' n')&=(\varphi n)\dagger(\varphi' n')=\varphi\dagger N\dagger N\varphi'=\varphi\dagger NN\dagger\varphi'\\ &=n\varphi\dagger\varphi' n'. \end{aligned}$$

Thus $n^*qn'=nqn'^*$ where $q\equiv\varphi\dagger\varphi'$. It follows that if $q\neq 0$, then n and n' (which can only vanish together then) belong to the same class. And on the other hand, from

$$L\varphi=\varphi\alpha \quad (L \text{ any linear operator})$$

follows:

$$L\varphi q=\varphi\alpha q=(\varphi q)q^{-1}\alpha q.$$

Together with φ, φr (where r is a real number) is an eigenvector belonging to the eigenvalue α; but φq belongs to the eigenvalue $q^{-1}\alpha q$. In contrast to the complex case in general $q^{-1}\alpha q\neq\alpha$, but belongs to the same class. One expects, therefore, that only the class (i.e., the norm and the real part) of an eigenvalue might have an invariant meaning. We term the class of an eigenvalue an eigenclass. Every number in an eigenclass is an eigenvalue.

If the eigenvectors belonging to the eigenclass $\{\alpha\}$ span a subspace $\mathfrak{M}$ of dimension n, we call n the degeneracy of the eigenclass. It is always possible to span $\mathfrak{M}$ with an orthogonal system of eigenvectors *belonging to the same eigenvalue*. This can be done with a construction analogous to the Schmidt orthogonalization process. The choice of the orthogonal eigenvectors can even be made in such a way that their common eigenvalue involves only one arbitrarily selected imaginary quaternion, say i_3:

$$L\varphi_r=\varphi_r\alpha_r$$

$$(\varphi_r\dagger\varphi_s)=\delta_{rs}, \quad \alpha_r=a_r+b_ri_3$$

a_r and b_r real. b_r can even be made ≥ 0.

We now are able to state the needed spectral theorems, and to prove them where new features might appear.

Hermitian Operators

Theorem: If H is a bounded Hermitian operator, [that is, $\varphi\dagger H\psi=(H\varphi)\dagger\psi$, $||H\psi||\leq M||\psi||$] then

there exists a unique spectral family E_λ such that

$$H=\int_{-\infty}^{+\infty}\lambda dE_\lambda.$$

This theorem is generalized to the quaternion case without any change in wording whatsoever. Even the proof can be literally transferred to the quaternion case. The numbers λ occurring in the spectral resolution are real and they commute with all quaternions. The essential noncommutative property of the quaternions has therefore no chance to make itself felt in this case. This is why the analogy to the case of complex spaces is complete.

Anti-Hermitian Operators

An anti-Hermitian operator $A=-A\dagger$ admits a unique spectral resolution:

$$A=\int_0^\infty J\lambda dE_\lambda$$

where E_λ is a (unique) spectral family over the interval $0\leq\lambda$, and J a unique operator with the properties:

$$J^+=-J,\quad J^2=-E,\quad [J,E_\lambda]=[J,A]=0.$$

In contrast to the complex case, the operator J, which is the appropriate generalization of i, appears and the integral is extended only from 0 to $+\infty$.

Unitary Operators

Every unitary transformation U in the quaternion space admits a unique spectral resolution

$$U=\int_0^\pi e^{J\theta}dE(\theta)$$

where J is a uniquely determined linear operator on $E(\pi)[1-E(+0)]V$ with the properties $J\dagger=-J$, $[J,E(\theta)]=0$, and $J^2=-1$ on its domain, and $E(\theta)$ is a spectral family over the interval $0\leq\theta\leq\pi$.

APPENDIX C. STONE'S THEOREM IN A $\mathfrak{Q}$ HILBERT SPACE

A one-parameter group of unitary $\mathfrak{Q}$ operators $U(t)$ can be written in the form:

$$U(t)=e^{At}\quad\text{where}\quad A\equiv\lim_{t\to0}[U(t)-E]/t=-A\dagger.$$

Furthermore (and here it is possible to go a step further than in a $\mathfrak{C}$ Hilbert space):

$$A=\eta H\quad\text{where}\quad [\eta,H]=0,\quad \eta^2=-1,\quad \eta\dagger=-\eta,$$
$$H\dagger=H,\quad H\geq0.$$

This theorem can be proved with help of the symplectic representation. To every $U(t)$ we can associate a symplectic representative $U_s(t)$ acting on $\mathfrak{K}(\mathfrak{C})$, such that all group-relations are preserved. Stone's theorem (proved for $\mathfrak{C}$ Hilbert spaces) then asserts the existence of

$$A_s=\lim_{t\to0}[U_s(t)-E]/t=-A_s\dagger,$$

such that

$$U_s(t)=e^{A_st}.$$

But from

$$[U_s(t),K_s]=0$$

follows

$$[A_s,K_s]=0.$$

Therefore by (1) of Appendix A, there is an A of which A_s is the symplectic representative, and

$$A=\lim_{t\to0}[U(t)-E]/t.$$

Now:

$$A\equiv\int_0^\infty\eta\lambda dE_\lambda=\eta\int\lambda dE_\lambda\equiv\eta H,\quad H\geq0.$$

(Of course it is desirable to have a direct proof avoiding the complex.)