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# $N = 2$ Super Yang-Mills and Subgroups of $SL(2, Z)$

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**Abstract**

We discuss $SL(2, Z)$ subgroups appropriate for the study of $N = 2$ Super Yang-Mills with $N_f = 2n$ flavors. Hyperelliptic curves describing such theories should have coefficients that are modular forms of these subgroups. In particular, uniqueness arguments are sufficient to construct the $SU(3)$ curve, up to two numerical constants, which can be fixed by making some assumptions about strong coupling behavior. We also discuss the situation for higher groups. We also include a derivation of the closed form $\beta$-function for the $SU(2)$ and $SU(3)$ theories without matter, and the massless theories with $N_f = n$.

## 1. Introduction

In their classic papers, Seiberg and Witten found elliptic curves that describe the exact effective actions for $N = 2$ $SU(2)$ gauge theories, with and without matter [1,2]. In the case of $N_f = 0, 1, 2, 3$, the curve has coefficients that are holomorphic functions of the expectation values and bare masses of the theory. However, since this theory has a nonzero $\beta$-function, the coefficients of the curve must depend on a scale $\Lambda$ as well.

But in the case where $N_f = 4$, or when there is a hypermultiplet transforming in the adjoint of $SU(2)$, then the $\beta$-function is zero, and there no longer is dependence on the scale, instead there is dependence on a dimensionless parameter $\tau$. In the massless case, $\tau$ can be interpreted as the coupling of the theory. The coefficients of the curve turn out to be modular forms of $\tau$ under a subgroup of $SL(2, Z)$, $\Gamma(2)$.

For higher gauge groups, there is a straightforward generalization of the $SU(2)$ case for theories with nonzero $\beta$-functions [3–6], up to possible constant coefficients. However, in the case where $\beta = 0$, there are difficulties present. We expect the curve to be described by a parameter $\tau$, but one must be careful with its interpretation. For instance, a curve was presented in [6] which was derived by matching it to the $SU(2)$ curve and taking certain masses and expectation values to infinity. In taking this limit, one of the $U(1)$ subgroups decouples as its effective coupling runs to zero. What is left is the original Seiberg-Witten theory and hence the parameter $\tau$ should be identified with the coupling of the remaining $SU(2)$ subgroup. The curve is written in terms of $\Gamma(2)$ modular *functions*, reflecting the symmetries of this leftover $SU(2)$ subgroup.

On the other hand, in [7] a curve was found for the $SU(3)$ case by starting with a period matrix and finding the curve. The period matrix was assumed to be a constant $\tau$ multiplied by the Cartan matrix. The parameter $\tau$ is actually the true coupling when all bare masses are zero and the expectation value $u = \langle \mathrm{tr}\phi^2 \rangle$ satisfies $u = 0$. The curve was written in terms of genus two theta functions, and the symmetry group of this curve reflects the symmetry group of the classical $SU(3)$ coupling, not the coupling for an $SU(2)$ subgroup when a $U(1)$ decouples. As it so happens, at weak coupling the curve in [7] is equivalent to the curve in [6], once one takes into account that the coupling parameter $\tau$ runs in going from the massless case to the $SU(2)$ limit. At strong coupling, the identification of the curves becomes more problematic and basically requires redefining one or more of the gauge invariant expectation values as well as the bare masses. In any case, the curve in [7] more fully reflects the symmetries for an $SU(3)$ gauge theory. Writing the curve in

this form might also assist in finding the corresponding integrable models for $SU(n)$ with $N_f = 2n$ [8–14].

We will argue that the symmetry group of the classical coupling for $SU(n)$ is the $SL(2, Z)$ subgroup $\Gamma_1(n)$ ($\Gamma_1(2n)$) for $n$ odd (even). This suggests that the appropriate curves have coefficients that are modular forms of this subgroup. Using results from [7], we will see that the curve for $SU(3)$ is quite simple and elegant and its similarity to the $SU(2)$ curve is rather striking. The $SU(3)$ curve has coefficients that are modular forms of $\Gamma_1(3)$. The dimension of the space of such forms is sufficiently small in order to determine the curve up to constant coefficients. These coefficients depend on nonperturbative effects and can be determined by insisting on certain behavior at strong coupling.

At this point, we do not yet know how to construct the curves for higher $SU(n)$, partly because of the large number of modular forms for $\Gamma_1(n)$ or $\Gamma_1(2n)$. We will make some general observations about this case that will, hopefully, lead to a solution.

In section 2 we discuss the subgroups of $SL(2, Z)$ appropriate for the classical $SU(n)$ coupling. In section 3 we review the $SU(2)$ case with $N_f = 4$. In this section, we also include a derivation of the $N_f = 0, 1, 2$ $\beta$-functions in closed form, which can be easily derived from the massless $N_f = 4$ curve, and to the best of our knowledge, has not previously appeared in the literature. In section 4 we discuss the $SU(3)$ theory, complete with a derivation of its $\beta$-function for $N_f = 0, 3$. In section 5 we discuss some issues for $SU(n)$, $n \geq 4$.

## 2. $\Gamma_1(n)$ and $\Gamma_1(2n)$

For gauge group $SU(n)$, the classical coupling matrix is given by $T = \tau C$, where $C$ is the matrix

$$C = \begin{pmatrix} 2 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 1 & \ldots & 1 \\ \vdots & & & \ddots & \\ 1 & \ldots & 1 & 1 & 2 \end{pmatrix} \tag{2.1}$$

and $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$. We have chosen the Cartan basis to be generated by the gauge fields $A_i - A_n$. Clearly, the theory should be invariant under $\tau \to \tau + 1$, which corresponds to shifting $T$ by $C$. In fact this invariance should carry over to the true quantum coupling matrix, $T_q$, that is the theory is invariant under $T_q \to T_q + C$. $T_q$ is actually the period matrix for the hyperelliptic curve that describes the theory. The period matrix will appear in the curve

in terms of genus $n-1$ theta functions, which are invariant when any component of $T_q$ is shifted by an even integer. Hence $T_q$ is invariant under any shift that is equal to $C$ mod 2.

$T_q$ is also invariant under any $Sp(2n-2, Z)$ transformation that is conjugate to $C$ mod 2. The inverse of $C$ is

$$C^{-1} = \frac{1}{n} \begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ \vdots & & & \ddots & \\ -1 & \cdots & -1 & -1 & n-1 \end{pmatrix} \tag{2.2}$$

and hence the theory is invariant under

$$\begin{aligned} T_q &\to T_q + nC^{-1} & n \text{ odd} \\ T_q &\to T_q + 2nC^{-1} & n \text{ even.} \end{aligned} \tag{2.3}$$

But the theory is also invariant under the conjugate transformation

$$\begin{aligned} T_q &\to T_q(nC^{-1}T_q + I)^{-1} & n \text{ odd} \\ T_q &\to T_q(2nC^{-1}T_q + I)^{-1} & n \text{ even,} \end{aligned} \tag{2.4}$$

where $I$ is the identity matrix. Let us for the moment assume that $T_q$ is $\tau C$. Then under the transformation in (2.4), $T_q$ transforms to

$$\begin{aligned} T_q &\to \frac{\tau}{n\tau + 1} C & n \text{ odd} \\ T_q &\to \frac{\tau}{2n\tau + 1} C & n \text{ even.} \end{aligned} \tag{2.5}$$

Hence we see that for the classical form of the coupling matrix, the theory is invariant under the transformations $\tau \to \tau + 1$ and $\tau \to \tau/(n\tau + 1)$ (or $\tau \to \tau/(2n\tau + 1)$). These two transformations generate a subgroup of $SL(2, Z)$, $\Gamma_1(n)$ (or $\Gamma_1(2n)$.) The group elements of $\Gamma_1(n)$ are

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \qquad \text{mod } n. \tag{2.6}$$

Unlike the subgroups $\Gamma(n)$, $\Gamma_1(n)$ is not a normal subgroup of $SL(2, Z)$, however a lot is known about the modular forms under these groups. (For a nice discussion, see chapter 3 of [15].)

Before going further, we need to stress one point. If $n \geq 4$, then the true coupling matrix cannot be proportional to $C$. A way to see this is to note that $C$ is invariant under an $Sp(2n-2, Z)$ subgroup which is isomorphic to $S_n$, the permutation group on $n$

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elements. But this would imply that the period matrix, and hence the Riemann surface is invariant under such a group. But if the genus is three or greater, then a hyperelliptic surface does not have such a symmetry. The surfaces with an $S_n$ symmetry are constructed from $n$ sheets, and the permutation acts by exchanging sheets, hence the surface cannot be hyperelliptic.

For the $SU(3)$ case, $T_q$ has the classical form so long as the expectation values have a $Z_3$ symmetry. This occurs if $\langle s_k \rangle = 0$, $k \neq 3$ and if $t_k = 0$, $n \neq 3, 6$. $s_k$ is the order $k$ symmetric homogeneous polynomial of the $\phi_i$, the uncharged component fields of the adjoint scalar, and $t_k$ are the order $k$ homogeneous symmetric polynomials of the six bare masses.

## 3. Review of $SU(2)$ with $N_f = 4$

The $SU(2)$ coupling is of course a scalar, so the quantum coupling has the same form as the classical coupling. The symmetry group is $\Gamma_1(4)$, which is the same as $\Gamma(2)$ under the rescaling $\tau \to 2\tau$.

Let us define the quantities $f_\pm(\tau) = \theta_2^4(2\tau) \pm \theta_1^4(2\tau)$, where $\theta_{1,2}$ are standard genus one theta functions. $f_+$ and $f_-$ are weight two modular forms of $\Gamma_1(4)$. In other words, under the transformation $\tau \to (a\tau + b)/(c\tau + d)$, with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ an element of $\Gamma_1(4)$, $f_\pm \to (c\tau + d)^2 f_\pm$. In fact, $f_+$ and $f_-$ generate all of the even weight forms. Also, under the transformation $\tau \to -1/(4\tau)$, which is not actually in $\Gamma_1(4)$, the weight two forms transform as

$$f_\pm(\tau) \to \pm(4\tau)^2 i^{-2} f_\pm(\tau). \tag{3.1}$$

Let us suppose that $\tau$ is the coupling when all four bare masses are zero. There exists an $SL(2,C)$ transformation that maps the massless cubic curve in [2] to the quartic curve

$$
\begin{aligned}
y^2 &= (f_-(\tau)x^2 - u)^2 + (f_+^2(\tau) - f_-^2(\tau))x^4 \\
&= P(x)^2 + (f_+^2(\tau) - f_-^2(\tau))x^4.
\end{aligned}
\tag{3.2}
$$

Notice that under this parameterization, $y$ and $u$ have weight two under $\Gamma_1(4)$, while $x$ has weight zero. The period matrix for this genus one surface is $2\tau$. Notice further that the curve is invariant under $\tau \to -1/(4\tau)$, since the minus sign that $f_-$ picks up can be absorbed into $x$.

If we now turn on mass terms, we still want to preserve the $\Gamma_1(4)$ invariance, which means that any new terms that appear should be $\Gamma_1(4)$ forms with the proper weight. Assuming that the $m_i$ have weight zero, and insisting on the correct weak coupling behavior leads to the curve

$$y^2 = (f_- x^2 + a(\tau)x \sum_i m_i + b(\tau) \sum_{i<j} m_i m_j - u)^2 + (f_+^2 - f_-^2) \prod_i (x + m_i). \qquad (3.3)$$

Written this way, the curve has singularities near $u = m_i^2$ at weak coupling. The functions $a(\tau)$ and $b(\tau)$ must be modular forms of weight two that fall off to zero at weak coupling, otherwise the singularities will be at the wrong values. Since all even forms are generated by $f_+$ and $f_-$, this means that both $a$ and $b$ are proportional to $f_+ - f_-$. This form is a one instanton term, hence the terms it multiplies can be at most linear in each of the masses. The constant for $a$ is determined by looking at the Seiberg-Witten differential [6], $\lambda = 2x(ydP - Pdx)/(P^2 - x^2)$. This has poles at $x = -m_i$ with residue equal to $m_i$. There should also be a pole at infinity whose residue cancels the other residues. This requirement leads to

$$a(\tau) = -\frac{1}{2}(f_+(\tau) - f_-(\tau)). \qquad (3.4)$$

Determining $b(\tau)$ is harder. As it so happens, if

$$b(\tau) = -\frac{1}{4}(f_+(\tau) - f_-(\tau)), \qquad (3.5)$$

then the curve in (3.3) is invariant under the parity transformation $\tau \rightarrow \tau + 1/2$, $m_1 \rightarrow -m_1$. Unlike the cubic curve in [2], this symmetry is hardly manifest in (3.3). However, if one computes the discriminant of (3.3), one finds that it is invariant under this transformation if $b(\tau)$ satisfies (3.5)[1]

Unfortunately, the higher $SU(n)$ do not have this extra parity symmetry. However, we note an interesting property of (3.3) and the singularities at strong coupling if $b$ has the form in (3.5). This behavior will generalize. Suppose we choose $m_1 = -m_2 = m_3 = -m_4 = m$, then (3.3) reduces to

$$
\begin{aligned}
y^2 &= (f_- x^2 + \frac{1}{2}(f_+ - f_-)m^2 - u)^2 + (f_+^2 - f_-^2)(x^2 - m^2)^2 \\
&= \left[ (f_- + f)x^2 + (\frac{1}{2}(f_+ - f_-) - f)m^2 - u \right] \left[ (f_- - f)x^2 + (\frac{1}{2}(f_+ - f_-) + f)m^2 - u \right],
\end{aligned}
$$
$$(3.6)$$

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[1] The actual discriminant takes up over 100 pages of text and takes 4 hours on a 100 mip machine to compute.

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where $f^2 = f_-^2 - f_+^2$. The curve is singular when the roots inside one set of square brackets match with the roots inside the other set. This occurs when $u = (f_+ + f_-)m^2/2$. At weak coupling $f_\pm \approx 1$, hence the singularity occurs near $u = m^2$. However, for the strong coupling limit, $-1/\tau \to i\infty$, $f_+ + f_- \sim (-i\tau)^2 e^{2\pi i\tau}$, hence the singularity approaches the point $u = 0$. In other words, going to strong coupling runs the effective mass to zero. If the coefficient were different, then at strong coupling we would have found the singularity to occur at $u \sim (-i\tau)^2 m^2$.

### 3.1. $\beta$-functions

Using the curve for the $N_f = 4$ case, it is straightforward to compute the full non-perturbative $\beta$ function for the $N_f = 0$ and massless $N_f = 2$ cases. Although this is outside the main development of the paper, we are unaware of this calculation appearing previously in the literature, and in any case will be generalizable to the $SU(3)$ $\beta$-functions.

The curve for the massless $N_f = 4$ case can be expanded to

$$y^2 = f_+^2 x^4 - 2u f_- x + u^2 \tag{3.7}$$

and the argument $\tau$ of $f_+$ and $f_-$ is the actual coupling. By rescaling $x$ and $y$ and shifting $\tau$ by 1, one can reexpress the curve as

$$y^2 = x^4 - 2u' F(\tau)x + u'^2, \tag{3.8}$$

where $F(\tau) = \frac{\theta_3^4(2\tau) + \theta_1^4(2\tau)}{\theta_2^4(2\tau)}$. We have replaced $u$ by $u'$ in (3.8), in order to distinguish it from the expectation value $u$ that appears in the scale noninvariant theories. $\tau$ does not change when $u'$ is varied.

The curve in the $N_f = 0$ case is given by [3,4]

$$y^2 = x^4 - 2ux^2 + u^2 - \Lambda^4, \tag{3.9}$$

hence comparing (3.8) with (3.9), one finds that the coupling for the $N_f = 0$ case satisfies

$$F(\tau) = u(u - \Lambda^4)^{-1/2}. \tag{3.10}$$

Taking derivatives with respect to $\Lambda$ on both sides gives

$$\Lambda \frac{d\tau}{d\Lambda} F'(\tau) = \frac{2u\Lambda^4}{(u^2 - \Lambda^4)^{3/2}} = 2F(F+1)(F-1), \tag{3.11}$$

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where $F'$ is the derivative of $F$ with respect to $\tau$. Hence the $\beta$-function is

$$\beta = \Lambda \frac{d\tau}{d\Lambda} = \frac{2F(F+1)(F-1)}{F'}. \tag{3.12}$$

(3.12) can be further reduced by noting that $\theta_2^4 \partial_\tau \theta_1^4 - \theta_1^4 \partial_\tau \theta_2^4$ is a modular form of weight six. Even weight modular forms are generated by $\theta_1^4$ and $\theta_2^4$, hence this derivative should be a combination of these functions. By matching to the leading order behavior and to the transformation properties under $\tau \to -1/(4\tau)$ we find that

$$\theta_2^4(2\tau) \partial_\tau \theta_1^4(2\tau) - \theta_1^4(2\tau) \partial_\tau \theta_2^4(2\tau) = 2\pi i \theta_1^4(2\tau) \theta_2^4(2\tau) \theta_3^4(2\tau). \tag{3.13}$$

Plugging (3.13) into (3.12) leads to the extremely simple expression

$$\beta = \frac{2}{\pi i} \frac{\theta_3^4(2\tau) + \theta_1^4(2\tau)}{\theta_2^8(2\tau)}. \tag{3.14}$$

From (3.14), it is clear that $\beta$ is a weight negative two modular *function* of $\Gamma_1(4)$, and under the transformation $\tau \to -1/(4\tau)$, $\beta$ transforms as $\beta \to 1/(4\tau^2)\beta$. We have also verified that the first few terms in this expansion are consistent with the results in [16], where derivatives of the coordinates $a$ and $a_D$ are expressed in terms of elliptic functions.

The $\beta$-function has a zero when $\theta_3^4(2\tau) = -\theta_1^4(2\tau)$. In this case $\tau = (1+i)/2$, up to a $\Gamma(2)$ transformation. From (3.10), we see that this point corresponds to $u = 0$, hence it is not surprising to find a zero of the $\beta$-function since there is now only one scale in the theory. The $\beta$-function is also singular as $\theta_2(2\tau)$ approaches zero which corresponds to the limits $\tau = n$, where $n$ is any integer. These points are of course where the monopoles and dyons become massless.

The curve in the massless $N_f = 2$ case is given by

$$y^2 = x^4 - 2(u + 3\Lambda^2/8)x^2 + (u - \Lambda^4/8)^2, \tag{3.15}$$

hence we have that

$$F(\tau) = \frac{u + 3\Lambda^2/8}{u - \Lambda^4/8}. \tag{3.16}$$

Taking derivatives with respect to zero and substituting back in $F$ for $\Lambda^2/u$ results in

$$\beta = \Lambda \frac{d\tau}{d\Lambda} = \frac{(F-1)(F+3)}{2F'} = \frac{1}{2\pi i} \frac{\theta_3^4(2\tau) + \theta_2^4(2\tau)}{\theta_3^4(2\tau)\theta_2^4(2\tau)}. \tag{3.17}$$

The $\beta$-function blows up when $\theta_2^4$ or $\theta_3^4$ approach zero, corresponding to massless monopoles or dyons. There is also a zero when $2\theta_3^4(2\tau) = \theta_1^4(2\tau)$. This is the coupling if $u = 0$.

In principle, one should be able to compute the $\beta$-function for $N_f = 1, 3$ as well. The standard quartic equation in these cases has even and odd powers of $x$. There exists an $SL(2, C)$ transformation into the forms of (3.9) and (3.15), but it is highly nontrivial. For massless $N_f = 1$, the $\beta$-function can be found using the cubic form of the curves in [2]. Compare the curves

$$y^2 = (x - e_1 u')(x - e_2 u')(x - e_3 u') \tag{3.18}$$

and

$$y^2 = x^2(x - u) - \Lambda^6/64, \tag{3.19}$$

where the $e_i$ are given in [2]. One finds after shifting $x$ by a constant in (3.18),

$$\beta = \Lambda \frac{d\tau}{d\Lambda} = \frac{6(2 + F)}{F'}, \tag{3.20}$$

where

$$F = \frac{(2\theta_2^4 + \theta_1^4)(2\theta_1^4 + \theta_2^4)(\theta_2^4 - \theta_1^4)}{(\theta_1^8 + \theta_2^8 + \theta_1^4\theta_2^4)^{3/2}}. \tag{3.21}$$

## 4. $SU(3)$ with $N_f = 6$

Let $\tau$ be the true coupling when $m_i = 0$ and $u = 0$. In [7], it was shown that a genus two surface with period matrix $\tau C$ is given by the hyperelliptic curve

$$y^2 = (r(\tau)x^3 - v)^2 + s(\tau)x^6, \tag{4.1}$$

where

$$r(\tau) = \frac{(\vartheta_1 \vartheta_2 \vartheta_3)^2}{2}(\vartheta_2^2 + \vartheta_3^2)(\vartheta_1^2 + \vartheta_3^2)(\vartheta_1^2 - \vartheta_2^2)$$
$$s(\tau) = \frac{27}{4}\vartheta_1^8 \vartheta_2^8 \vartheta_3^8. \tag{4.2}$$

$\vartheta_i$ are the genus two theta functions

$$\vartheta_0 = \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}(\tau C) \qquad \vartheta_1 = \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}(\tau C)$$
$$\vartheta_2 = \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}(\tau C) \qquad \vartheta_3 = \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}(\tau C) \tag{4.3}$$

If we absorb a factor a factor of $\vartheta_1 \vartheta_2 \vartheta_3$ into $x$, then the curve can be rewritten as

$$y^2 = (r'(\tau)x^3 - v)^2 + s'(\tau)x^6, \tag{4.4}$$

where

$$r'(\tau) = \frac{(\vartheta_2^2 + \vartheta_3^2)(\vartheta_1^2 + \vartheta_3^2)(\vartheta_1^2 - \vartheta_2^2)}{2\vartheta_1 \vartheta_2 \vartheta_3}$$
$$s'(\tau) = \frac{27}{4}\vartheta_1^2 \vartheta_2^2 \vartheta_3^2. \tag{4.5}$$

We expect to be able to rewrite this curve in terms of $\Gamma_1(3)$ forms. Luckily, these forms can be classified. Consider the quantities

$$f_\pm(\tau) = \left(\frac{\eta^3(\tau)}{\eta(3\tau)}\right)^3 \pm \left(3\frac{\eta^3(3\tau)}{\eta(\tau)}\right)^3,$$
$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n). \tag{4.6}$$

Both $f_+$ and $f_-$ are modular forms of weight three for $\Gamma_1(3)$. In fact these forms generate all forms of weight $3m$, where $m$ is any positive integer. These forms also transform nicely under $\tau \to -1/(3\tau)$, with $f_\pm \to \pm(3\tau)^3 i^{-3} f_\pm$. The space of forms of weight one and two are one dimensional [17] and are generated by $f_1 = (f_+)^{1/3}$. Hence $f_1$ and $f_-$ generate all of the modular forms.

The functions $r'(\tau)$ and $s'(\tau)$ have very simple relations to these forms, namely $r' = f_-$ and $s' = f_+^2 - f_-^2$. The curve is then

$$y^2 = (f_- x^3 - v)^2 + (f_+^2 - f_-^2)x^6$$
$$= P(x)^2 + (f_+^2 - f_-^2)x^6. \tag{4.7}$$

The form of the curve in (4.7) is remarkably similar to the $SU(2)$ curve in (3.2).

We now wish to turn on the other expectation values. If we keep the quarks massless and turn on $u$, then at weak coupling $P(x)$ should approach $P(x) = x^3 - ux - v$. From (4.7), we see that $v$ has weight three under $\Gamma_1(3)$ if $x$ has weight zero, therefore, $ux$ has weight two. Thus, $ux$ must be multipied by a weight one form in $P(x)$ so that the curve is $\Gamma_1(3)$ invariant. The unique form with the correct weak coupling behavior is $f_1$, hence the generic massless curve is

$$y^2 = (f_- x^3 - f_1 ux - v)^2 + (f_+^2 - f_-^2)x^6. \tag{4.8}$$

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In terms of the genus two theta functions, $f_1(\tau) = \vartheta_0(\tau C)$, and hence this curve matches the curve given previously in [7] after a rescaling in $x$.

At this point the reader might be wondering why there is an $f_-$ in front of the $x^3$ term instead of $f_+$, since both functions have the same weak coupling behavior. It turns out that this is necessary in order to have the correct duality behavior. Suppose that $u = 0$. Then, in moving from weak coupling to strong coupling, we expect that quarks will be mapped to monopoles and vice versa. In order for this to happen, the integrals around the $a$ cycles of the hyperelliptic curve, which correspond to the electric coordinates $a^I$, should smoothly go to integrals around the $b$ cycles, which correspond to the magnetic coordinates $a_D^I$, when $\tau \to -1/(3\tau)$. Under this transformation, $f_-$ picks up an extra sign. Because of this, if we had chosen the function in front of the $x^3$ to be $f_+$, then we would have found that the $a^I$ map back to themselves under $\tau \to -1/(3\tau)$. But with the coefficient $f_-$, we find that the $a^I$ transform to the $a_D^I$ under $\tau \to -1/(3\tau)$.

The case with nonzero masses is similar to the situation found for $SU(2)$. The masses are assumed to have weight zero, hence any mass terms that appear in $P(x)$ must be multiplied by weight three forms and must fall off to zero at weak coupling. Hence these extra terms are proportional to $f_+ - f_-$. Since this is a one instanton term, they must be at most linear in each of the individual masses. Furthermore, there cannot be a term $u \sum m_i$, since this is a weight two form and hence has to multiply a weight one-form that falls to zero at weak coupling. No such form exists. Hence the massive curve should be of the form

$$
\begin{aligned}
y^2 = &\left( f_- x^3 + (f_+ - f_-)(ax^2 \sum m_i + bx \sum_{i<j} m_i m_j + c \sum_{i<j<k} m_i m_j m_k) - f_1 ux - v \right)^2 \\
&+ (f_+^2 - f_-^2) \prod_i (x + m_i),
\end{aligned} \tag{4.9}
$$

where $a$, $b$ and $c$ are to be determined. In order to have the correct residue in $\lambda$ at $x = \infty$, $a$ should be set to $a = -1/2$. To set $b$ and $c$, we use the argument used in the previous section for $SU(2)$. First consider the case $u = 0$ and $m_1 = e^{2\pi i/3} m_2 = e^{4\pi i/3} m_3 = m_4 = e^{2\pi i/3} m_5 = e^{4\pi i/3} m_6 = m$. Then the curve in (4.9) reduces to

$$
\begin{aligned}
y^2 &= (f_- x^3 + 2cm^3(f_+ - f_-) - v)^2 + (f_+^2 - f_-^2)(x^3 + m^3)^2 \\
&= \left[ (f_- + f)x^3 + 2cm^3(f_+ - f_-) + fm^3 - v \right] \\
&\quad \times \left[ (f_- - f)x^3 + 2cm^3(f_+ - f_-) - fm^3 - v \right].
\end{aligned} \tag{4.10}
$$

As before, $f^2 = f_-^2 - f_+^2$. The roots for a polynomial inside a set of square brackets matches the roots of the polynomial inside the other set of brackets if

$$v = m^3(2c(f_+ - f_-) - f_-).\tag{4.11}$$

The singularity approaches $v = 0$ for strong coupling if $c = -1/4$, otherwise the singularity would be found at large $v$ for fixed $m$. To find a suitable value for $b$, let $v = 0$, $m_1 = -m_2 = m_3 = -m_4 = m$ and $m_5 = m_6 = 0$. This curve is already singular, but an extra singularity arises if $u = m^2(f_- - 2b(f_+ - f_-))$. If $b = -1/4$, then the singularity approaches $u = 0$ in the strong coupling limit. Hence the final curve is

$$y^2 = \left(f_- x^3 - \frac{f_+ - f_-}{4}(2x^2\sum m_i + x\sum_{i<j} m_i m_j + \sum_{i<j<k} m_i m_j m_k) - f_1 ux - v\right)^2$$
$$+ (f_+^2 - f_-^2)\prod_i (x + m_i).$$

$$\tag{4.12}$$

If we take one of the masses to infinity while taking the coupling to zero, we can reduce this to a an $N_f = 5$ theory which has precisely the same form as in [5].

## 4.1. β-functions

It is also straightforward to find the $\beta$-functions for the $N_f = 0, 3$ cases, using the same procedure as in the previous section. For the $N_f = 0$ case with $u = 0$, we find

$$\beta = \Lambda\frac{d\tau}{d\Lambda} = \frac{3F(F+1)(F-1)}{F'},\tag{4.13}$$

where now, $F = f_-/f_+$. Using the fact that $f_- \partial_\tau f_+ - f_+ \partial_\tau f_-$ is a modular form of weight eight and based on its leading order behavior and transformation properties, one finds

$$f_- \partial_\tau f_+ - f_+ \partial_\tau f_- = \pi i (f_+^2 - f_-^2)f_1^2.\tag{4.14}$$

Hence (4.13) can be reexpressed as

$$\beta = \Lambda\frac{d\tau}{d\Lambda} = \frac{3}{\pi i}\frac{f_-}{f_1^5}\tag{4.15}$$

From (4.6), one finds that $\beta$ in (4.15) has a zero when $\eta^4(\tau) = 3\eta^4(3\tau)$. Using the fact that $\eta^2(-1/\tau) = -i\tau\eta^2(\tau)$, it then follows that $\beta = 0$, when $\tau = i/\sqrt{3}$. The $\beta$-function blows up when $f_1$ approaches zero, which occurs when $\eta^4(\tau) = -3\eta^4(3\tau)$. Using the

relation $\eta^4(-1/(\tau+1)) = e^{\pi i/3}(i\tau+i)^2\eta^4(\tau)$, we find that $\beta$ diverges if $\tau = 1/2+i/(2\sqrt{3})$. This singularity occurs at the cusp described in [3] and [18].

In the massless $N_f = 3$ case, by matching curves we find that the $u = 0$ $\beta$-function is

$$\beta = \Lambda\frac{d\tau}{d\Lambda} = \frac{3(F-1)(F+3)}{4F'} = \frac{3}{4\pi i}\frac{f_-(3f_+ + f_-)}{f_1^5(f_+ + f_-)}. \tag{4.16}$$

This still has a singularity when $\tau = 1/2 + i/(2\sqrt{3})$, but it also is singular if $f_+ + f_- = 0$, which occurs at $\tau = n$.

An interesting check of $\beta$ in (4.15) and (4.16) would be to compute the higher instanton corrections to the coupling. Work on this is in progress. At this time, we do not know the $\beta$-functions for other values of $N_f$.


**5. $SU(n)$, $N_f = 2n$**

For the $SU(n)$ groups with $n > 3$, two problems arise. The first problem is that the dimension of $\Gamma_1(n)$ forms of low weight is somewhat large. This makes it difficult to choose coefficients based on uniqueness arguments.

The second problem is that there is no region in the space of expectation values where the true quantum coupling is proportional to the matrix $C$ in (2.1). This basically means that the parameter $\tau$ that appears in the curve will not be the actual coupling for any choice of expectation values.

We have yet to overcome these problems, but we describe some of the issues involved. To understand the relation of $\tau$ to the coupling, let us consider the case where all bare masses are zero and all expectation values are zero except for the casimir $s_n$. One can calculate the perturbative quantum corrections to the coupling, giving

$$T_{qu} = \tau C + \frac{i}{\pi}G, \tag{5.1}$$

where the entries of the matrix $G$ are given by

$$G_{mm} = \log\left(4n^2\sin^2\frac{m\pi}{n}\right) \qquad G_{ml} = \log\left(-2ni\frac{\sin\frac{m\pi}{n}\sin\frac{l\pi}{n}}{\sin\frac{|l-m|\pi}{n}}\right). \tag{5.2}$$

The $\log 2n$ and $\log i$ terms can be absorbed into the classical coupling, however the log sine terms cannot be absorbed, otherwise the coupling won't have the proper behavior under

Weyl reflections. From (5.2) it is clear that for $n > 3$, the full coupling is not proportional to the Cartan matrix $C$.

Let us concentrate on the $SU(4)$ and $SU(5)$ cases. For $SU(4)$, we can rewrite the coupling as

$$T = \tau C + \epsilon B, \qquad B = \begin{pmatrix} 0 & -2 & 2 \\ -2 & -4 & -2 \\ 2 & -2 & 0 \end{pmatrix}, \qquad (5.3)$$

where in weak coupling, $\epsilon = \frac{i}{2\pi} \log \sin \frac{\pi}{4} - \frac{i}{2\pi} \log \sin \frac{2\pi}{4} = -\frac{i}{4\pi} \log 2$. Under the transformation $T \rightarrow T(8C^{-1}T + 1)^{-1}$, $T$ transforms to

$$T \rightarrow \widetilde{\tau} C + \widetilde{\epsilon} B \qquad (5.4)$$

where

$$\widetilde{\tau} = \frac{\tau + 8\tau^2 - 32\epsilon^2}{(1 + 8(\tau + 2\epsilon))(1 + 8(\tau - 2\epsilon))} \qquad \widetilde{\epsilon} = \frac{\epsilon}{(1 + 8(\tau + 2\epsilon))(1 + 8(\tau - 2\epsilon))}. \qquad (5.5)$$

Letting $\tau_1 = \tau + 2\epsilon$ and $\tau_2 = \tau - 2\epsilon$ leads to the transformations

$$\tau_1 \rightarrow \frac{\tau_1}{1 + 8\tau_1} \qquad \tau_2 \rightarrow \frac{\tau_2}{1 + 8\tau_2}. \qquad (5.6)$$

A similar situation exists for $SU(5)$. Here we can write the coupling matrix as

$$T = \tau C + \epsilon B, \qquad B = \begin{pmatrix} 2 & -1 & 1 & 3 \\ -1 & -2 & -3 & 1 \\ 1 & -3 & -2 & -1 \\ 3 & 1 & -1 & 2 \end{pmatrix}, \qquad (5.7)$$

where in this case, $\epsilon = \frac{I}{2\pi}(\log \sin \frac{\pi}{5} - \log \sin \frac{2\pi}{5})$ for weak coupling. We can then define $\tau_1 = \tau + \sqrt{5}\epsilon$ and $\tau_2 = \tau - \sqrt{5}\epsilon$, which transform as

$$\tau_1 \rightarrow \frac{\tau_1}{1 + 5\tau_1} \qquad \tau_2 \rightarrow \frac{\tau_2}{1 + 5\tau_2}, \qquad (5.8)$$

under $T \rightarrow T(5C^{-1}T + 1)^{-1}$. For higher groups, the same sort of procedure can be followed, but instead of one or two parameters that transform under $\Gamma_1(n)$ or $\Gamma_1(2n)$, there are $(n-1)/2$ $(n/2)$ parameters for $n$ odd (even).

The natural generalization of the massless $SU(2)$ and $SU(3)$ cases is to assume that the hyperelliptic curve is of the form

$$y^2 = (f_-(\tau)x^n - \sum_{i=2}^{n} s_n(f_+(\tau))^{1-i/n}x^{n-i})^2 + (f_+^2(\tau) - f_-^2(\tau))x^2 n, \qquad (5.9)$$

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where $f_+$ and $f_-$ are $\Gamma_1(n)$ ($\Gamma_1(2n)$ forms of weight $n$ for $n$ odd (even). But here is where the two difficulties arise that need to be overcome. First, it is not clear how $\tau$ should be chosen. For instance for $SU(4)$ (or $SU(5)$), there are two variables, $\tau_1$ and $\tau_2$, that transform under $\Gamma_1(8)$ (or $\Gamma_1(5)$). Second, the dimensions of the forms $f_-$ and $f_+$ are greater than 1. Hence, uniqueness arguments are not sufficient for determing the true equation.

*Note added:* As this paper was being typed, a preprint appeared [19] that has some overlap with the discussion in section 2.

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