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Extension of the Class of Integrable Dynamical Systems Connected with Semisimple Lie Algebras

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Abstract. A new class is found of completely integrable systems connected with semisimple Lie algebras. This class generalizes most of the previously-considered integrable systems describing a one-dimensional motion of interacting particles.

During the last decade, a great number [1–6] of completely integrable dynamical systems of classical mechanics have been found using the Hamiltonian

$$H = \sum_{j=1}^{N} \frac{p_j^2}{2} + U(q_1, \ldots, q_N) \tag{1}$$

(N is an arbitrary integer).

For most of them, the equations of motion have been solved explicitly. For a small number of these systems, a connection has been established between them and the motion of poles of singular (rational or elliptic) solutions to nonlinear evolution equations: KdV, Boussinesq, and Burgers–Hopf [7, 8]. (The most complete review of the results can be found in [6].)

Here we shall establish the integrability of a new class of Hamiltonian systems (1) connected with semisimple Lie algebras. A similar connection was found for the first time by Olshanetsky and Perelomov [3]. They have shown that to each classical system of roots $\{\boldsymbol{\alpha}\}$ of semisimple Lie algebras, one may put in correspondence the potential

$$U(q_1, \ldots, q_N) = \sum_{\boldsymbol{\alpha} \in R_+} g_\alpha^2 V(\mathbf{q}\boldsymbol{\alpha}) \tag{2}$$

where $\mathbf{q} = (q_1, \ldots, q_N)$, $\boldsymbol{\alpha}$ are N-dimensional root vectors, R_+ represents a set of roots positive for a certain linear ordering in $\{\boldsymbol{\alpha}\}$ space, and g_α are constants which may be different only for roots of a different length. The latter makes potential (2) invariant under transformations of the corresponding Weyl group.

We will show that a complete integrability takes place for a class of potentials much larger than (2):

$$U(q_1, \ldots, q_N) = \sum_{\alpha \in R_+} V_\alpha(\mathbf{q}\boldsymbol{\alpha}) \tag{3}$$

where functions V_α may be different for roots of a different length. As a result of the latter, Equation (3) is also invariant under transformations of the corresponding Weyl group.

The integrability of systems (1) with potentials (2), (3) is a consequence of the existence of the Lax matrices L, M which obey the relation

$$\frac{\mathrm{d}L}{\mathrm{d}t} = [L, M] \tag{4}$$

equivalent to the equations of motion. The structure of L and M is determined by an irreducible representation of one of the algebras with the root system $\{\boldsymbol{\alpha}\}$.

The main result of Olshanetsky and Perelomov [3] consists of the following: For a root system of the type BC_N with a most general form of potential,

$$U(q_1, \ldots, q_N) = g^2 \sum_{j>n} (V(q_j - q_n) + V(q_j + q_n)) + \sum_{j=1}^{N} (g_1^2 V(q_j) + g_2^2 V(2q_j)) \tag{5}$$

the Lax matrices (4) do exist provided that

$$g_1^2 + \sqrt{2}\, g g_2 - 2g^2 = 0 \tag{6}$$

or $g_1 = 0$, g, g_2 are arbitrary, and

$$V(\xi) = \left\{ \frac{1}{\xi^2}, \frac{1}{\sinh^2(a\xi)}, \wp(a\xi) \right\}. \tag{7}$$

($\wp(a\xi)$ is the Weierstrass function.)

We note that for all the types of classical irreducible root systems having subsystems of roots of a different length (B_N, C_N, BC_N) Equation (3) can be represented by

$$U(q_1, \ldots, q_N) = \sum_{j>n}^{N} (V(q_j - q_n) + V(q_j + q_n)) + \sum_{j=1}^{N} W(q_j). \tag{8}$$

For the Lax matrices we use a generalization of the ansatz obtained in [3]:

$$L = \begin{pmatrix} l & \lambda & \psi \\ \lambda^+ & 0 & -\lambda^+ \\ \psi^+ & -\lambda & -l \end{pmatrix}, \quad M = \begin{pmatrix} m & \omega & s \\ -\omega^+ & \mu & -\omega^+ \\ -s^+ & \omega & m \end{pmatrix} \tag{9}$$

where l, ψ, m, s are $N \times N$ matrices, and λ, ω are $1 \times N$ matrices dependent on dynamic variables (p_j, q_j):

$$l_{jn} = p_j \delta_{jn} + i(1 - \delta_{jn}) g \times (q_j - q_n)$$

$$\psi_{jn} = i[\delta_{jn}(v(q_j) + i\rho(q_j)) + (1 - \delta_{jn}) g x(q_j + q_n)]$$

$$s_{jn} = i\left[\frac{\delta_{jn}}{2}(v'(q_j) + i\rho'(q_j)) + (1 - \delta_{jn}) g x'(q_j + q_n)\right] \tag{10}$$

$$m_{jn} = i\delta_{jn}\left[\tau(q_j) - \sum_{n \neq j} g(z(q_j - q_n) + z(q_j + q_n))\right] + i(1 - \delta_{jn}) g x'(q_j - q_n)$$

$$\lambda_j = i\alpha(q_j), \quad \omega_j = i\alpha'(q_j), \quad \mu = i\sum_{j=1}^{N} \varkappa(q_j)$$

Note that L is an operator of an irreducible matrix representation of the Lie algebra corresponding to the $SU(N + 1, N)$ group.

From the relation $H = \frac{1}{2}\operatorname{tr}(L^2)$ it follows that the functions $V(\xi)$ and $W(\xi)$ determining potential (8) are connected with $x(\xi)$, $v(\xi)$, $\rho(\xi)$ and $\alpha(\xi)$:

$$V(\xi) = g^2 x^2(\xi), \qquad W(\xi) = \tfrac{1}{2}(v^2(\xi) + \rho^2(\xi) + \alpha^2(\xi)). \tag{11}$$

The Lax equation (3) imposes constraints on the functions x, τ, v, ρ, z, α, and $\varkappa$ in (10). First, x and z should obey the equation

$$x'(\xi)x(\eta) - x'(\eta)x(\xi) = x(\xi + \eta)(z(\xi) - z(\eta))$$

solutions to which can been found in [2]:

$$\begin{aligned} x(\xi) &= \{\xi^{-1}, (\sinh(a\xi))^{-1}, (\operatorname{sn}(a\xi, k))^{-1}\} \\ z(\xi) &= \{-\xi^{-2}, -a(\sinh(a\xi))^{-2}, -a(\operatorname{sn}(a\xi, k))^{-2}\} \end{aligned} \tag{12}$$

where $\operatorname{sn}(a\xi, \mathrm{k})$ is the Jacobi elliptic function with modulus k. Second, for other functions from (10) we obtain two sets of functional equations, and in each case some of the functions α, $\varkappa$ and ρ vanish:

(I) $\alpha(\xi) = \varkappa(\xi) = 0$

$$\begin{aligned} &2x(\xi + \varepsilon\eta)(\tau(\xi) - \tau(\eta)) - x(\xi - \varepsilon\eta)(v'(\xi) + v'(\eta)) - \\ &\quad - 2x'(\xi - \varepsilon\eta)(v(\xi) - \varepsilon v(\eta)) = 0 \end{aligned} \tag{13}$$

$$x(\xi + \varepsilon\eta)(\rho'(\xi) - \varepsilon\rho'(\eta)) + 2x'(\xi + \varepsilon\eta)(\rho(\xi) - \rho(\eta)) = 0, \quad \varepsilon = \pm 1 \tag{14}$$

(II) $\rho(\xi) = 0$

$$\begin{aligned} &2x(\xi + \varepsilon\eta)(\tau(\xi) - \tau(\eta)) - x(\xi - \varepsilon\eta)(v'(\xi) + v'(\eta)) - \\ &\quad - 2x'(\xi - \varepsilon\eta)(v(\xi) - \varepsilon v(\eta)) \\ &= 2(\alpha(\xi)\alpha'(\eta) - \varepsilon\alpha'(\xi)\alpha(\eta))g^{-1} \end{aligned} \tag{15}$$

$$\begin{aligned} &[x(\xi - \eta) + x(\xi + \eta)]\alpha'(\eta) + [x'(\xi - \eta) - x'(\xi - \eta)]\alpha(\eta) + \\ &\quad + [g(z(\xi - \eta) + z(\xi + \eta)) + \varkappa(\eta)]\alpha(\xi) = 0 \end{aligned} \tag{16}$$

$$\alpha'(\xi)\nu(\xi) = \left[\tau(\xi) - \frac{\nu'(\xi)}{2} - \varkappa(\xi)\right]\alpha(\xi) \tag{17}$$

The method of finding analytic solutions to these equations consists of expanding the latter in series about point $\eta = 0$ with due consideration to possible pole singularities of the functions to be found and in solving the resulting ordinary differential equations. We shall present here only the final results.

(I) From (12) and (13) it follows that

$$\begin{aligned} \nu(\xi) &= [\alpha_1 + \beta_1 \operatorname{sn}^2(a\xi, k) + \gamma_1 \operatorname{sn}^4(a\xi, k)][\operatorname{sn}(a\xi, k)\operatorname{cn}(a\xi, k)\operatorname{dn}(a\xi, k)]^{-1} \\ \tau(\xi) &= a\nu(\xi)[\operatorname{sn}(2a\xi, k)]^{-1} \end{aligned} \tag{18}$$

at $x(\xi) = (\operatorname{sn}(a\xi, k))^{-1}$; α_1, β_1 and γ_1 are arbitrary constants. For other values of x from (12) which are limiting cases of $(\operatorname{sn}(a\xi, k))^{-1}$ the functions $\nu(\xi)$ and $\tau(\xi)$ can be obtained from (18) by taking the corresponding limit.

All solutions to (14) at $\varepsilon = -1$ have been found in [4]. For $\varepsilon = 1$, Equation (14) has only the following solutions from the above class

$$\begin{aligned} &\rho(\xi) = \alpha_2 + \gamma_2\xi^2, \quad x(\xi) = \xi^{-1} \\ &\rho(\xi) = \alpha_2 + \gamma_2 \cosh(2a\xi), \quad x(\xi) = (\sinh(a\xi))^{-1} \end{aligned} \tag{19}$$

where α_2 and γ_2 are arbitrary; at $x(\xi) = (\operatorname{sn}(a\xi, k))^{-1}$ nontrivial solutions of (14) are absent.

According to Equations (8), (11), (18) and (19), in the considered case potential $U(q_1, \ldots, q_N)$, (8) is determined by the following sets of functions $\{V, W\}$:

$$\begin{aligned} V(\xi) &= g^2\xi^{-2}, \qquad W(\xi) = g_1^2\xi^{-2} + g_2^2\xi^2 + g_3^2\xi^4 + g_4^2\xi^6 \\ V(\xi) &= g^2(\sinh(a\xi))^{-2}, \\ W(\xi) &= g_1^2(\sinh(a\xi))^{-2} + g_2^2(\sinh(2a\xi))^{-2} + \\ &\quad + g_3^2\cosh(2a\xi) + g_4^2\cosh(4a\xi) \end{aligned} \tag{20}$$

where all the constants $g_\gamma (\gamma = 1, \ldots, 4)$ are arbitrary:

$$\begin{aligned} V(\xi) &= g^2\wp(a\xi), \\ W(\xi) &= g_1^2\wp(a\xi) + g_2^2\wp\left(a\xi + \frac{\omega_1}{2}\right) + g_3^2\wp\left(a\xi + \frac{\omega_2}{2}\right) + \\ &\quad + g_4^2\wp\left(a\xi + \frac{\omega_1 + \omega_2}{2}\right), \end{aligned} \tag{21}$$

(ω_1 and ω_2 are periods of the Weierstrass function $\wp(a\xi)$.)

Constants g_γ in (21) should satisfy the nonlinear equation

$$\left(\sum_{\gamma=1}^{4} g_\gamma^4 - \sum_{\beta\neq\gamma}^{4} g_\beta^2 g_\gamma^2\right)^2 = 64\prod_{\gamma=1}^{4} g_\gamma^2 . \tag{22}$$

(II) In this case we determine α and $\varkappa$ from Equation (16). For $x = (\operatorname{sn}(a\xi, k))^{-1}$ we have

$$\alpha(\xi) = a_1(\operatorname{sn}(a\xi, k))^{-1}, \qquad \varkappa(\xi) = 2ga(\operatorname{sn}(a\xi, k))^{-2}$$

Substituting $\tau(\xi) = \tau_1(\xi) + \tau_2(\xi)$ with $\tau_1(\xi) = aa_1^2 g^{-1}(\operatorname{sn}(a\xi, k))^{-2}$, we may transform the system of Equations (15) to (13) and make use of the known solutions of (13) to determine $\nu(\xi)$ and $\tau_2(\xi)$. Equation (17) restricts the choice of constants in the functions $\tau_2(\xi)$, $\alpha(\xi)$, $\varkappa(\xi)$, and $\nu(\xi)$ thus obtained. The general solution to Equations (15)–(17) can be written as follows:

$$\begin{aligned}
&\alpha(\xi) = a_1(\operatorname{sn}(a\xi, k))^{-1}, \qquad \kappa(\xi) = 2ga(\operatorname{sn}(a\xi, k))^{-2}\\
&\nu(\xi) = [\alpha_1 + \gamma_1 \operatorname{sn}^4(a\xi, k)][\operatorname{sn}(a\xi, k)\operatorname{cn}(a\xi, k)\operatorname{dn}(a\xi, k)]^{-1}\\
&\tau(\xi) = aa_1^2 g^{-1}(\operatorname{sn}(a\xi, k))^{-2} + a\nu(\xi)(\operatorname{sn}(2a\xi, k))^{-1}\\
&a_1^2 = 2g^2 - 2\alpha_1 g
\end{aligned} \tag{23}$$

where α_1 and γ_1 are arbitrary constants. The functions V, W corresponding to (23), have the form (22) and the admissible points in the four-dimensional space $\{g_\gamma\}$ are in a two-dimensional hypersurface defined by the equations

$$\begin{aligned}
&F(k^4 g_2^2, (1 + k^2)^2 g_3^2, g_4^2) = 0\\
&F\left(g_1^2, g_3^2 + \frac{1 - k^2}{1 + k^2}(g_2^2 - g_4^2), 2g^2\right) = 0\\
&(F(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz)
\end{aligned} \tag{24}$$

or by equations following from the latter through the interchanges:

$$\{g_1 \rightleftarrows g_2, g_3 \rightleftarrows g_4\}, \qquad \{g_1 \leftrightarrows g_3, g_2 \rightleftarrows g_4\}, \qquad \{g_1 \rightleftarrows g_4, g_2 \rightleftarrows g_3\}.$$

These interchanges correspond to a shift of all coordinates by half-periods $\wp(a\xi)$.

Thus, we have established the existence of the Lax matrices for Hamiltonian systems with potential (8) defined by the set of functions (20)–(22) and (24).

All these systems possess the additional integrals of motion $I_n = 1/4n \operatorname{tr}(L^{2n})$, $n = 1, \ldots, N$ which are in involution. This statement equivalent, according the Liouville theorem to a complete integrability of the systems under consideration, is proved by a direct calculation of the Poisson brackets of any two eigenvalues of the matrix L [9].

All the previously known integrable systems with a Hamiltonian of the form (1) with abritrary N except for the Toda chains and oscillator system with a nonlinear interaction [10], can be obtained from Equations (8), (20)–(22) and (24) by limiting processes. In particular, the result (6), (7) of [3] corresponds to particular solutions of (22) and (24) of the form

$$\begin{aligned}
&g_1^2 = g_2^2 = g_3^2 = g_4^2,\\
&g_2^2 = g_3^2 = g_4^2, \qquad g_1^2 = g_2^2 + 2g^2 \pm 2\sqrt{2}gg_2.
\end{aligned}$$

This can easily be established by using the relation

$$\wp(2a\xi) = \frac{1}{4}\left[\wp(a\xi) + \wp\left(a\xi + \frac{\omega_1}{2}\right) + \wp\left(a\xi + \frac{\omega_2}{2}\right) + \wp\left(a\xi + \frac{\omega_1 + \omega_2}{2}\right)\right].$$

In conclusion, we note that it is interesting to find a connection of the considered systems with singular solutions of nonlinear evolution equations. The study of these problems may help us to derive explicit solutions to the equations of motion defined by Hamiltonian (8).

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