The expression (A.7) for R_0 shows clearly the axis $\mathbf{k}_0$ of the rotation R_0, whose angle φ_0 is immediately obtained by identifying (A.7), and

$$R_0 = \cos(\varphi_0/2) - i\hat{k}_0 \sin(\varphi_0/2). \qquad \text{(A.7')}$$

In particular, for an infinitesimal rotation around $\mathbf{n}$, with an angle of $\varphi \ll 1$,

$$\varphi_0 \simeq (\mathbf{n}\cdot\mathbf{k}_0 + \mathbf{n}\cdot\mathbf{k}/1 + \mathbf{k}\cdot\mathbf{k}_0)\varphi. \qquad \text{(A.8}$$

Choosing $\mathbf{k}_0$ as our z axis, and taking successively infinitesimal rotations around the x, y, and z axes, we obtain

$$\begin{aligned} (\varphi_0)_x &= [k_x/(1 + k_z]\varphi = [p_x/(p + p_z)]\varphi, \\ (\varphi_0)_y &= [k_y/(1 + k_z)]\varphi = [p_y/(p + p_z)]\varphi, \qquad \text{(A.9)} \\ (\varphi_0)_z &= [1 + k_z/(1 + k_z)]\varphi = \varphi. \end{aligned}$$

We can then immediately derive the expressions (V.9) for the Lie algebra in the zero-mass case.

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Principle of General Q Covariance

D. Finkelstein,*
Yeshiva University, New York, New York
J. M. Jauch,
University of Geneva and *CERN, Geneva, Switzerland*
S. Schiminovich,†
Yeshiva University, New York, New York
and
D. Speiser,‡
University of Geneva, Geneva, Switzerland
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In this paper the physical implications of quaternion quantum mechanics are further explored. In a quanternionic Hilbert space $\mathcal{H}_Q$, the lattice of subspaces has a symmetry group which is isomorphic to the group of all co-unitary transformations in $\mathcal{H}_Q$. In contrast to the complex space $\mathcal{H}_C$ (ordinary Hilbert space), this group is connected, while for $\mathcal{H}_C$ it consists of two disconnected pieces.

The subgroup of transformations in $\mathcal{H}_Q$ which associates with every quaternion q of magnitude 1, the correspondence $\psi \to q\psi q^{-1}$ for all $\psi \in \mathcal{H}_Q$ (called Q conjugations), is isomorphic to the three-dimensional rotation group. We postulate the principle of Q covariance: The physical laws are invariant under Q conjugations. The full significance of this postulate is brought to light in localizable systems where it can be generalized to the principle of general Q covariance: Physical laws are invariant under general Q conjugations. Under the latter we understand conjugation transformations which vary continuously from point to point.

The implementation of this principle forces us to construct a theory of parallel transport of quaternions. The notions of Q-covariant derivative and Q curvature are natural consequences thereof.

There is a further new structure built into the quaternionic frame through the equations of motion. These equations single out a purely imaginary quaternion $\eta(x)$ which may be a continuous function of the space-time coordinates. It corresponds to the i in the Schrödinger equation of ordinary quantum mechanics. We consider $\eta(x)$ as a fundamental field, much like the tensor $g_{\mu\nu}$ in the general theory of relativity. We give here a classical theory of this field by assuming the simplest invariant Lagrangian which can be constructed out of η and the covariant Q connection. It is shown that this theory describes three vector fields, two of them with mass and charge, and one massless and neutral. The latter is identifiable with the classical electromagnetic field.

1. INTRODUCTION

In the development from Galilean to special to general relativity, it was shown by Einstein that the concepts of Euclidean geometry have only an approximate validity and that the true laws of geometry are subject to disturbances from place to place. Still more fundamental than the laws of geometry are those of classical logic as expressed in the propositional calculus. In the development from classical to quantum physics it was shown by Bohr that the concepts of classical logic have

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† Now at the University of Buenos Aires.
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only an approximate validity due to complementarity. Indeed, as has been shown by Birkhoff and von Neumann[1] in a fundamental paper, the class calculus of classical mechanics is a Boolean algebra, while that of quantum mechanics is a nondistributive lattice.

Here we consider the next step: the true laws of logic may be subject to disturbances from place to place. A preliminary examination of such a local variation of the logical structure shows that there is no nontrivial possibility for such a generalization of ordinary quantum mechanics. It is too simple a structure.

A richer structure is obtained with quaternion quantum mechanics.

In order to illustrate the new features introduced by quaternion quantum mechanics, we consider side by side the three simplest possibilities of a propositional calculus for a quantum system. In such a calculus, the propositions are represented as subspaces (or projection operators) in an infinite-dimensional linear vector space $\mathcal{H}_F$ with coefficients from a field F. If F is continuous then there exist only three possibilities for F; the reals (R), the complex numbers (C), and the quaternions (Q). If $F = C$, we obtain ordinary quantum mechanics. It can be shown[2] that $F = R$ is in contradiction with nature unless it is supplemented by a superselection rule which makes it essentially equivalent with $F = C$. There remains finally $F = Q$ or quaternion quantum mechanics (Q quantum mechanics).

The difference in the logical structure for the three cases is best shown if we consider the groups of symmetry transformations of the lattice of subspaces. Let E be a projection operator in $\mathcal{H}_F$ and $M = E\mathcal{H}_F$ the subspace consisting of all elements of the form $E\psi$ with $\psi \in \mathcal{H}_F$. We say the projection E_1 is contained in E_2, and we write $E_1 < E_2$, if $M_1 = E_1\mathcal{H}_F$ is contained in $M_2 = E_2\mathcal{H}_F$. An alternate equivalent way of stating this is

$$E_1E_2 = E_1.$$

Consider now a permutation of the subspaces of $\mathcal{H}_F$, that is a one-to-one correspondence $\phi(E)$ of the projections, such that

$$E_1 < E_2 \quad \text{implies} \quad \phi(E_1) < \phi(E_2).$$

Such a permutation is called a *symmetry transformation* of the lattice. The transformations of the Hilbert space that induce the symmetry transformations form the symmetry group g_F of the space. These groups are known. For $F = R$, it is the group O of all orthogonal transformations in $\mathcal{H}_R$. For $F = C$, it is the direct product of the group U_C of all unitary transformations in $\mathcal{H}_C$, with a cyclic group Z_2 of order two. Finally, for $F = Q$, it is the direct product of the group U_Q of all unitary transformations in $\mathcal{H}_Q$, with the group R_3 of all rotations in three dimensions;

$$g_R = O, \qquad g_C = U_C \times Z_2, \qquad g_Q = U_Q \times R_3.$$

There is an important difference between the groups g_C and g_Q, which will be decisive for the content of this paper: g_C consists of two disconnected pieces while g_Q consists of one piece only. This difference is directly connected with the difference in the group of automorphisms for the number field F. [A one-to-one correspondence $\alpha \to \alpha'(\alpha, \alpha' \in F)$ is an automorphism of F if it is continuous and

$$(\alpha + \beta)' = \alpha' + \beta',$$
$$(\alpha\beta)' = \alpha'\beta',$$

for all $\alpha, \beta \in F$.] The following facts are well known: the only automorphism for R is the identity, for C the group of automorphisms is the cylic group of order two (the generator being in this case complex conjugation), while for Q it is the rotation group in three dimensions and it is implemented by

$$q \to q' = pqp^{-1} \quad \text{for all} \quad q, p \in Q \qquad (|p| = 1).$$

The advantage of a larger group of automorphisms of the quaternions is offset by the loss of commutativity. Quaternions do not commute. This leads to a major difficulty in Q quantum mechanics: Because of the noncommutativity of Q, a unique tensor product of $\mathcal{H}_Q$ with itself does not exist. In ordinary quantum mechanics, this tensor product is necessary in order to formulate the quantum mechanics of composite systems.

This deficiency can be turned into an advantage leading to a new basic principle, if we introduce the postulate that the laws of physics should be invariant with respect to this nonuniqueness in its mathematical description. We have called this the *principle of general Q covariance.*

We shall show that this principle leads in a natural way to a new basic field which is a generalization of the Maxwell field and contains it as a special case. This field describes in addition to the neutral photons of mass zero, a pair of charged vector mesons with finite mass and a magnetic moment. These fields have a basic character in the same

[1] G. Birkhoff and J. von Neumann, Ann. Math. **37**, 823 (1936).
[2] E. C. G. Stueckelberg, Helv. Phys. Acta **33**, 727 (1960).

sense as the $g_{\mu\nu}$-fields in general relativity. They are directly connected with the intrinsic structure of the propositional calculus in its relation to the space–time continuum. We leave open, for the time being, how the fields of stable or unstable physical particles can be fitted into this scheme. We feel that the point of view presented here could be the clue to a deeper understanding of these questions.

2. GENERAL Q COVARIANCE AND THE ELECTROMAGNETIC AXIS

In a previous publication[3,4] we have developed a quantum mechanics with noncommutative c numbers (probability amplitudes). We have shown that the only reasonable choice for these numbers is (if not the real numbers R, or the complex numbers C) the quaternion field Q. More general hypercomplex number systems may be excluded.

The most characteristic property of the quaternions is their "roundness," which expresses itself mathematically in the existence of a group of automorphisms which is identical with the rotation group R_3 in three dimensions. Switching to a number field with more automorphisms seems a natural way of obtaining multiplets with more members. This was the original motivation for the study of quaternion quantum mechanics. Therefore we postulate that the laws of the new quantum mechanics should be invariant under automorphisms of the quaternion number field. This invariance principle we call *Q covariance,* and the invariance group R_3 we refer to as the Q group.

In the attempt to apply quaternion quantum mechanics to localizable systems we are led to introduce an even stronger hypothesis which we call *general Q covariance.* We understand by it invariance of the physical laws under automorphisms which vary continuously from point to point of space–time.

In order to formalize this principle, we recall some basic notions of quaternion quantum mechanics.[3,4] The pure states are represented as normalized state vectors from a quaternion Hilbert space $\mathcal{H}_Q$. In such a space we have postulated two kinds of multiplications with scalars, one from the left and one from the right. This means if $\psi \in \mathcal{H}_Q$ and $q \in Q$, there is defined an element ψq and an element $q\psi$. However while $\psi \rightarrow q\psi$ is a *linear* operator, $\psi \rightarrow \psi p$ is only *colinear,* defined below. These two operators commute:

$$(q\psi)p = q(\psi p).$$

In general, the elements $q\psi$ and ψq are different from one another. If they are equal for all q we call ψ *real.* One can show that the real elements form a real linear space $\mathcal{H}_R$ in $\mathcal{H}_Q$ which is complete in the sense that its orthogonal complement in $\mathcal{H}_Q$ is zero: $\mathcal{H}_R^{\perp} = 0$.

For every element $\psi \in \mathcal{H}_Q$ and $q \in Q$, we can define the transformation

$$K(q)\colon \psi \rightarrow \psi^q \equiv q\psi q^{-1},$$

which we shall call *conjugation.* It leaves all the real elements in $\mathcal{H}_Q$ invariant. This transformation induces a transformation of all operators defined on $\mathcal{H}_Q$. If O is such an operator, the transformed operator O^q is defined by

$$O^q = qOq^{-1}.$$

We have previously introduced the notion of a *colinear operator.* An operator O is colinear if O is additive, and if for any $\psi \in \mathcal{H}_Q$, $\alpha \in Q$, there exists a scalar α' such that

$$O(\psi\alpha) = (O\psi)\alpha'.$$

It can then be shown there exists a $q \in Q$ determined by O such that

$$\alpha' = \alpha^q = q\alpha q^{-1}.$$

The transformation $K(q)$ is colinear in this sense, and moreover, every colinear operator O has a unique representation as a product of a linear operator O_L with a $K(q)$:

$$O = K(q)O_L.$$

An operator O is said to be real if $O^q = O$ for all q. It has then the property of leaving the set of real elements invariant.

The observables are self-adjoint operators A. They are not necessarily real. If they represent yes–no experiments (questions), they are projections. Special Q covariance requires that all the physical laws are invariant under the conjugations $\psi \rightarrow \psi^q$. At this point we might compare the situation with the much simpler one in ordinary quantum mechanics. The transformation which corresponds to $\psi \rightarrow \psi^q$ is a complex conjugation in $\mathcal{H}_C$. It assigns to every $\psi \in \mathcal{H}_C$ a vector $\psi^c = \mathcal{J}\psi$. This transformation is antilinear:

$$\mathcal{J}\lambda\psi = \lambda^*\mathcal{J}\psi,$$

$$(\mathcal{J}\psi, \mathcal{J}\varphi) = (\varphi, \psi),$$

and involutory:

$$\mathcal{J}^2 = I.$$

[3] D. Finkelstein, J. M. Jauch, and D. Speiser, "Notes on Quaternion Quantum Mechanics," CERN Repts. I, II, and III, (1959).

[4] D. Finkelstein, J. M. Jauch, S. Schiminovich, and D. Speiser, J. Math. Phys. 3, 207 (1962).

The corresponding transformation of operators is given by

$$A^c = \mathcal{J} A \mathcal{J}^{-1}.$$

An operator is *real* if $A^c = A$.

The physical laws are invariant under this transformation in the sense that all the observable consequences of the theory are expressible in real numbers. This is true even though these transformations are not canonical. The commutation rules are, for instance, not invariant.

We may paraphrase the meaning of this invariance property: There is no physical distinction between complex numbers having the same intrinsic algebraic properties. Likewise we would require for quaternion quantum mechanics: There is no physical distinction between quaternions having the same intrinsic algebraic properties. In this case, this means all those algebraic properties which are invariant under the transformation $\alpha \to \alpha^q = q\alpha q^{-1}$.

The analogy to the ordinary quantum mechanics applies only for *special* Q covariance. We go now beyond this and formulate *general* Q covariance. This notion is related to that of localizability of a system. In order to avoid too long a discussion which would obscure our main point, we shall not insist here on a precise and rigorous definition of this notion, but we shall use instead a heuristic language guided by physical interpretations and analogies.

The physical properties of a system at a given point of space–time are described by a system of operators $O(x)$ representing the observables associated with this point. The observables at different space–time points commute if the points are situated spacelike (microcausality). We can consider the algebra $\mathfrak{N}(x)$ of bounded operators generated by the $O(x)$. If we denote by $\mathfrak{N}'(x)$ the commutator algebra of $\mathfrak{N}(x)$, it follows from what has been said so far that

$$\mathfrak{N}(y) \leq \mathfrak{N}'(x) \quad \text{if} \quad x - y \quad \text{is spacelike}.$$

The simplest way to realize such commuting sets of operators associated with spacelike-situated points is to assume that we are dealing with a family of Hilbert spaces $\mathcal{H}(x)$, one for each x, and that the operators of $\mathfrak{N}(x)$ operate on $\mathcal{H}(x)$ only and are unit operators in all $\mathcal{H}(y)$ with $y \neq x$. In the manipulations of quantum field theory, one unites the family of Hilbert spaces $\mathcal{H}(x)$ to a large Hilbert space $\mathcal{H}$ which has the formal properties of a direct product of the spaces $\mathcal{H}(x)$.

If one attempts a similar thing for a quaternion field theory, one runs straight into the ambiguity of the direct product of quaternionic Hilbert spaces. This state of affairs suggests that the principle of Q covariance should be extended to a principle of *general* Q covariance which requires that *it is physically meaningless to compare quaternions at different space–time points except in their intrinsic algebraic properties.* An arbitrary choice must be made which relates quaternions at neighboring space–time points. But the physical content of the theory is independent of that choice.

The formal expression of this principle would be the following. Let $O(x)$ be the system of local observables (the shaky foundation on which this notion rests does not deter us), and consider the transformation

$$O(x) \to O^q(x) \equiv q(x)O(x)q^{-1}(x), \qquad q(x) \in Q. \quad (1)$$

Simultaneously, every $\psi(x) \in \mathcal{H}_Q(x)$ undergoes the transformation

$$\psi(x) \to \psi^q(x) \equiv q(x)\psi(x)q^{-1}(x). \quad (2)$$

The family of transformations of this kind make up the invariance group of general Q covariance; briefly the general Q group.

There are some simple consequences of such invariance requirements which appear already for special Q covariance.

Let us look for instance at the Schrödinger equation, which in Q quantum mechanics takes on the form

$$H\psi = \eta\psi, \quad (3)$$

where H, the Hamiltonian, is a nonnegative self-adjoint operator, and η is an imaginary unit quaternion operator which commutes with H;

$$\eta^2 = -1, \qquad [\eta, H] = 0.$$

If we carry out the transformations (1), (2) of the Q group, Eq. (3) transforms into

$$H^q\psi^q = \eta^q\psi^q.$$

Since in general $\eta^q \neq \eta$, we see that the Schrödinger equation is not invariant under the Q group unless we transform η as if it were a dynamical variable. The only feature that resembles this in ordinary quantum mechanics is the behavior of the Schrödinger equation under the time-reversal transformation. There the i in the Schrödinger equation changes its sign because time reversal is an antiunitary transformation.

A local field theory which satisfies the principle of general Q covariance will exhibit even more explicitly the dynamical character of η. The reason for this is that we must keep the formalism sufficiently general to permit a variation of η from point to point. Already in ordinary field theory it is possible to write the equations of motion in the form of Tomonaga. In this form, the state vector of the system is considered as a functional of a spacelike surface σ and the equation of motion which expresses a local variation of σ has the form

$$H(x)\psi[\sigma] = i\{\delta\psi[\sigma]/\delta\sigma(x)\},$$

where $H(x)$ is the Hamiltonian density. In Q quantum mechanics, the corresponding equation would be

$$H(x)\psi[\sigma] = \eta(x)\{\delta\psi[\sigma]/\delta\sigma(x)\},$$

where $\eta(x)^2 = -1$ and $[H(x), \eta(x)] = 0$.

Under general Q transformations, the operator η transforms just like any other dynamical variables according to

$$\eta(x) \to \eta(x)^q = q(x)\eta(x)q^{-1}(x), \qquad q(x) \in Q.$$

Evidently the Schrödinger equation is not invariant under general conjugations unless $q(x)$ commutes with $\eta(x)$.

What can we make of this apparent asymmetry of Q quantum mechanics under general conjugations? A clue might be obtained if we consider those conjugations which leave the $\eta(x)$ invariant. These conjugations form a certain subgroup of the group of all conjugations. They are generated by all those $q(x)$ which commute with $\eta(x)$. These transformations resemble gauge transformations of ordinary quantum field theory. In this analogy $\eta(x)$ takes the role of the isotopic spin axis.

It is well known that the requirement of the invariance of a field theory under local phase transformations leads in a natural way to a theory of the interaction of the electromagnetic field with a charged field. The above remarks indicate that the principle of general Q covariance contains the germ of a more general type of electromagnetic field theory, in which $\eta(x)$ appears as a fundamental dynamical variable.

The decision to raise the $\eta(x)$ to the status of a field variable is a step which resembles in many ways the treatment of the metric tensor $g_{\mu\nu}(x)$ in the theory of general relativity. There it is geometry which becomes a part of the dynamical structure, here it is the "logic" of propositions which is incorporated into the fundamental dynamical laws.

However there is a fundamental difference between the present attempt and general relativity which must be kept in mind: General relativity is a generalization of *classical* mechanics, Q quantum mechanics is a generalization of *quantum mechanics.*

The geometrical implications of the former are such that they lead to a *curved space* described by a Riemannian differential geometry. The logical implications of the latter are such that they lead to a logic of propositions which changes from point to point so that we may speak of a "*warped logic.*" In calling this the theory of warped logic, we have in mind that the lattices of subspaces of $\mathcal{H}_Q(x)$ make up the class calculus of quantum mechanics because of the correspondence between subspaces and properties or classes.

3. Q CONNECTION AND Q CURVATURE

We now take the next step that is necessary for the construction of physical theories with quaternionic field operators subject to the principle of general Q covariance. The problem is the formation of derivatives of quaternionic field operators with respect to space–time coordinates. If $A(x)$ is such a field operator, we can introduce the space–time derivatives $\partial_\mu A(x)$. The definition of these quantities requires the comparison of the field $A(x)$ with a field at a neighboring point according to the usual interpretation

$$A(x + dx) = A(x) + dx^\mu\, \partial_\mu A(x) + \cdots .$$

In a general Q covariant theory, the derivative $\partial_\mu A(x)$ is not a suitable formation since it is by itself not invariant under general Q transformations. Indeed we find

$$\partial_\mu(q(x)A(x)q^{-1}(x)) = q(x)\,\partial_\mu A(x)q^{-1}(x) + [\partial_\mu q(x)q^{-1}(x),\, q(x)A(x)q^{-1}(x)].$$

What we are faced with is the problem of composite systems and the tensor product all over again. We must subtract an operator at x from an operator at $x + dx$. This makes sense only as the difference of two operators on the tensor product of the Q Hilbert spaces $\mathcal{H}_Q(x)$ and $\mathcal{H}_Q(x + dx)$. To define this product uniquely we have found that we must give an isomorphism between the quaternions at x and the quaternions at $x + dx$. If general Q covariance is not to be a trivial hypothesis, this isomorphism itself must be taken as one of the dynamical variables of the theory, since it is not invariant under the general Q group.

The most general quaternionic isomorphism that goes over smoothly into the identity when the two points x and $x + dx$ merge, is given by the first-order change in dx,

$$q(x + dx) = q(x) - \tfrac{1}{2}[C_\mu(x)\, dx^\mu,\, q(x)],$$

where $C_\mu(x)$ is a covariant space–time vector with pure imaginary Q components. In the analogy to general relativity, they correspond to the affine connection $\Gamma^\lambda_{k\mu}$. We call the $C_\mu(x)$ the Q *connection*. They may be taken as the potentials of a sort of generalized electromagnetic field. However, to guarantee general Q covariance, we shall mention the C_μ as seldom as possible, working always with the covariant combination D_μ defined by

$$D_\mu \cdot A(x) \equiv \partial_\mu A(x) + \tfrac{1}{2}[C_\mu(x), A(x)]. \tag{4}$$

The most general Q connection satisfies the following identities, valid for any pair of fields $A(x)$, $B(x)$:

$$\begin{aligned} D_\mu \cdot AB &= (D_\mu \cdot A)B + A(D_\mu \cdot B), \\ D_\mu \cdot (A + B) &= D_\mu \cdot A + D_\mu \cdot B, \\ (D_\mu \cdot A)^Q &= D_\mu \cdot A^Q, \end{aligned} \tag{5}$$

where the superscript Q is the operation of quaternion conjugation. Furthermore,

$$D_\mu x^\nu = \delta^\nu_\mu.$$

A point of notation: It is useful to be able to drop the operand $A(x)$ in the definition of D_μ given by (4). If A is any operator, we shall write ΔA for the operator on operators defined by

$$\Delta A \cdot B = [A, B].$$

The properties of this Δ operation have been studied elsewhere.[5] Here we observe that Jacobi's identity takes the form

$$\Delta[A, B] = [\Delta A, \Delta B].$$

We can now rewrite (4) as $D_\mu = \partial_\mu + \frac{1}{2}\Delta C_\mu$.

The invariant Q connection provides us with a transport of the quaternion number system from one point to a neighboring point. This transport can be extended to any point along a given curve. If $q = q(0)$ is a quaternion at a point $x = 0$, the quaternion at a point x transported along a curve $\Gamma: x = x(s)$ which corresponds to q would be given by integrating the differential equation

$$Dq = 0$$

along Γ, where $D = (dx^\mu/ds)D_\mu$. In general this transport of the quaternion from one point to another will depend on the curve Γ. If the transport is independent of Γ for given end points, we say that it is *integrable*. The necessary and sufficient condition for the integrability of the transport is the vanishing of the commutator $[D_\mu, D_\nu]$, which we shall call the Q *curvature*.

[5] D. Finkelstein, Comm. Pure Appl. Math. 8, 245 (1955).

A direct evaluation of this expression shows that it has the form

$$[D_\mu, D_\nu] = \tfrac{1}{2}\Delta K_{\mu\nu}, \qquad K^Q_{\mu\nu} = -K_{\mu\nu},$$

with

$$K_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu + \tfrac{1}{2}[C_\mu, C_\nu]. \tag{6}$$

The existence of the Q curvature expresses the fact that the transport of a quaternion from one point to another is only unique up to an automorphism, belonging to the same automorphism class as the initial amplitude. The particular member of the class depends on the path of transport and the physical conditions.

This possibility does not exist in C quantum mechanics, where the automorphisms of the number form a disconnected set consisting of just two elements. Thus an amplitude cannot vary continuously within its class.

Under the general Q group, the Q connection transforms according to

$$D_\mu \to D^q_\mu, \quad \text{where} \quad D^q_\mu A^q = (D_\mu A)^q.$$

Explicit evaluation for D^q_μ shows that it is of the form

$$D^q_\mu = \partial_\mu + \tfrac{1}{2}\Delta C'_\mu,$$

with

$$C'_\mu = qC_\mu q^{-1} - 2(\partial_\mu q)q^{-1}.$$

The second term shows that C_μ does not transform like a quaternion under the general Q group. This is in complete analogy with the affine connection $\Gamma^\lambda_{\mu\nu}$ which does not transform like a tensor in spite of its appearance.

Because of this transformation law, it is possible to transform the C_μ locally to zero by a suitable transformation. It suffices indeed to choose local values for q and $\partial_\mu q$ such that they satisfy

$$0 = qC_\mu - 2\partial_\mu q.$$

This condition can be extended to a finite region surrounding a given point if the partial differential equation

$$\partial_\mu q = \tfrac{1}{2}qC_\mu$$

is integrable. The necessary and sufficient condition for this is

$$\partial_\nu(qC_\mu) - \partial_\mu(qC_\nu) = 0,$$

which can be written in the equivalent form

$$K_{\nu\mu} \equiv \partial_\nu C_\mu - \partial_\mu C_\nu + \tfrac{1}{2}[C_\nu, C_\mu] = 0.$$

Thus we see the vanishing of the Q curvature is the necessary and sufficient condition for the existence of a coordinate system such that $D_\mu = \partial_\mu$.

4. FIELD EQUATIONS FOR THE FUNDAMENTAL FIELDS

In this section we discuss the dynamical properties of the fundamental fields η and D_μ. We treat these here as classical fields, leaving the quantization to a later treatment. Just as the quantization of affine connections in general relativity, the quantization of these fields raises fundamental difficulties which have not yet been overcome. Rather than dwell on these difficulties we shall extract as much as possible from the classical part of the theory in order to gain an insight into the possible interpretation of the formalism. Thus we assume that the field $\eta(x)$ is a pure, imaginary, quaternionic scalar in each of the Hilbert spaces $\mathcal{H}(x)$.

In order to obtain a dynamical structure, that is, a set of field equations, we start with a Lagrangian which we assume in the simplest possible form. For the action density of the η field we assume $\frac{1}{2}(D^\mu \cdot \eta)(D_\mu \cdot \eta)$ and for that of the D field we assume $\frac{1}{4}K^{\mu\nu}K_{\mu\nu}$. We do not consider terms like η^2 since $\eta^2 = -1$.

Accordingly, the simplest action density for the full theory is a linear combination of the two:

$$L = -\frac{1}{4\alpha} K^{\mu\nu}K_{\mu\nu} - \frac{1}{2\beta} (D^\mu \cdot \eta)(D_\mu \cdot \eta) \qquad \alpha, \beta > 0. \tag{7}$$

We have chosen the signs in such a way that the "kinetic" terms in the energy (those quadratic in the time derivatives) make a positive-definite quadratic form.

The field equations which follow from this action density are

$$\frac{1}{\beta} D_\mu D^\mu \eta + \lambda\eta = 0, \tag{8}$$

and

$$D_\mu K^{\mu\nu} = -(\alpha/\beta)\eta D^\nu \eta. \tag{9}$$

Here $\lambda = \lambda(x)$ is the Lagrange multiplier associated with the subsidiary condition $\eta^2 = -1$. From the definition of $K_{\mu\nu}$ follows further the identity

$$D_\lambda K_{\mu\nu} + D_\mu K_{\nu\lambda} + D_\nu K_{\lambda\mu} = 0. \tag{10}$$

The steps which lead to these field equations are similar to those leading to the equations of general relativity. Here, as there, the field equations are obtained from the simplest invariants which can be formed out of the fundamental fields. But there is an important break with the analogy to general relativity. In Einstein's theory, an action for the combined system of the metric tensor and the covariant differential operator consists of just one term, the doubly-contracted curvature tensor R, and it involves just one physical constant. We need two terms and two constants α, β. The analogy broke because the metrical curvature describes the transport of one space–time direction in another space–time direction; but the Q curvature describes the transport of a quaternion in a space–time direction, and it makes no sense to contract one with the other. There is no analogue to R for us. From the action R follows by Palatini's identity that the covariant derivative of the metric tensor is zero. The corresponding equation for the η field would be $D_\mu \cdot \eta = 0$ and this equation does *not* follow from our action except in a limiting case to be discussed later.

5. ELECTROMAGNETIC FIELD AND MAXWELL'S EQUATIONS

The physical content of the Eqs. (8), (9), and (10) can best be extracted if we separate from the Q-covariant differential operator D_μ that part which does not change the η field. Any D_μ can be decomposed into

$$D_\mu = D_\mu^0 + D_\mu^1, \tag{11}$$

where the first term is a derivative having the property

and
$$D_\mu^0 \cdot \eta(x) = 0, \qquad D_\mu^1 = D_\mu - D_\mu^0. \tag{12}$$

In order to make this decomposition unique, we impose the further condition that D_μ^1, which can be shown to be a purely algebraic operator of the form

$$D_\mu^1 = \tfrac{1}{2}\Delta B_\mu, \tag{13}$$

is associated with a B_μ that anticommutes with η:

$$\{B_\mu, \eta\} = 0. \tag{14}$$

Under this restriction the decomposition (11) is unique and Q invariant. Anticipating the physical interpretation, we call D_μ^0 the "neutral" and D_μ^1 the "charged" part of the Q connection D_μ.

In order to give an explicit expression for this decomposition, we lapse into noncovariant language and write

$$D_\mu = \partial_\mu + \tfrac{1}{2}\Delta C_\mu.$$

We observe the anticommuting properties of η, $\eta_\mu \equiv \partial_\mu \eta$, and $\eta\eta_\mu$ that derive from the relation $\eta^2 = -1$:

$$0 = \partial_\mu \eta^2 = (\partial_\mu \eta)\eta + \eta\, \partial_\mu \eta.$$

Let us define C^0_μ by writing

$$D^0_\mu = \partial_\mu + \tfrac{1}{2}\Delta C^0_\mu,$$

so that

$$C^0_\mu + B_\mu = C_\mu.$$

The condition (12) together with (14) gives immediately the result

$$C^0_\mu = A_\mu \eta - \eta\eta_\mu, \tag{15}$$

where

$$A_\mu \equiv -\tfrac{1}{2}\{C_\mu, \eta\}. \tag{16}$$

Along with the decomposition of D into two parts goes a decomposition of its self-commutator and thus of the Q curvature $K_{\mu\nu}$ as well. Direct calculation shows that

$$K_{\mu\nu} = F_{\mu\nu}\eta + B_{\mu\nu}, \tag{17}$$

where $F_{\mu\nu}$ is the real skew tensor defined by

$$[D^0_\mu, D^0_\nu] = \tfrac{1}{2}F_{\mu\nu}\Delta\eta, \tag{18}$$

and given explicitly by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \tfrac{1}{2}[\eta_\mu, \eta_\nu]\eta. \tag{19}$$

The pure imaginary skew tensor $B_{\mu\nu}$ is defined by

$$B_{\mu\nu} = D^0_\mu B_\nu - D^0_\nu B_\mu + \tfrac{1}{2}[B_\mu, B_\nu]. \tag{20}$$

The tensor $F_{\mu\nu}$ identically satisfies the relations

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0. \tag{21}$$

We obtain a further equation for the $F_{\mu\nu}$ field by extracting from Eq. (9), that part which commutes with $\eta(x)$. This part is obtained by taking the anticommutator of (9) with $\eta(x)$. The result is the following:

$$\partial^\mu F_{\mu\nu} = j_\nu, \tag{22}$$

with

$$j_\nu = \tfrac{1}{2}(B_\mu \eta B^\mu{}_\nu - B_{\mu\nu}\eta B^\mu - \partial^\mu\{\eta, B_{\mu\nu}\}). \tag{23}$$

Equations (21) and (22) are Maxwell's equations for an electromagnetic field with a source field given by (23).

This result justifies identifying the $F_{\lambda\nu}$ with the electromagnetic field. The gauge transformations of the electromagnetic potentials are a subgroup of the general Q transformations consisting of all those which leave the field η invariant. Such transformations can be written

$$A(x) \to A^q(x) = q(x)A(x)q^{-1}(x),$$

where

$$q(x) = e^{\frac{1}{2}\eta(x)\varphi(x)},$$

with $\varphi(x)$ a real but otherwise arbitrary space–time function. The transformed Q connection becomes

$$C^q_\mu(x) = q(x)C_\mu(x)q^{-1}(x) - 2q_\mu(x)q^{-1}(x),$$

where

$$q_\mu(x) \equiv \partial_\mu q(x) = \tfrac{1}{2}\eta_\mu(x)\varphi(x)q(x) + \tfrac{1}{2}\eta(x)\varphi_\mu(x)q(x).$$

From this, with the help of Eq. (16), we obtain the transformed potentials $A^q_\mu(x)$ in the form

$$A^q_\mu(x) = -\tfrac{1}{2}\{C^q_\mu(x), \eta(x)\}.$$

Because $q(x)$ commutes with $\eta(x)$, this becomes

$$A^q_\mu(x) = A_\mu(x) + \tfrac{1}{2}\{(\eta_\mu(x)\varphi(x) - \eta(x)\varphi_\mu(x)), \eta(x)\},$$

and because $\eta_\mu(x)$ anticommutes with $\eta(x)$, the last term simplifies to $\varphi_\mu(x)$. There results the transformation

$$A_\mu(x) \to A^q_\mu(x) = A_\mu(x) + \partial_\mu\varphi(x),$$

which we recognize as the usual gauge transformation of electromagnetic theory. We have thus recovered the entire formalism of the classical Maxwell field, but with a special current given by the expression (23).

We now turn our attention to the "charged" part of the Q-covariant differential operator.

6. HEAVY PHOTONS

The vector field B_μ satisfies an equation of motion which is obtained from (9) by taking the quaternionic component of (9) that is "orthogonal" to (i.e., anticommutes with) the electromagnetic axis η. We therefore take the commutator of (9) with η, for this is the quaternionic transcription of the usual vector product in three dimensions; the result of the calculation is the equation

$$D^0_\mu D^{0\mu} B^\nu - 2\eta B_\mu F^{\mu\nu} + (\alpha/\beta)B^\nu + \tfrac{1}{4}[B_\mu, [B^\mu, B^\nu]] = 0. \tag{24}$$

In addition one obtains the equation

$$D^0_\mu \cdot B^\mu = 0. \tag{25}$$

These equations describe a field with a rest mass

$$M_B = (\alpha/\beta)^{\frac{1}{2}}.$$

The field is electrically charged. This is shown by the wave equation (24) which contains the electromagnetic coupling in D^0_μ in addition to an anomalous magnetic moment coupling [the second term in (24)]. The charge of the field is

$$e = \alpha^{1/2}.$$

Thus we see that the same Q connection contains the makings of both neutral massless particles and charged massive particles in a general Q-covariant way. Moreover, the mass is a necessary consequence of the dynamics of the electromagnetic axis η. It can be interpreted as due to the reaction back on the charged part B of the Q connection, resulting from the action of B on the electromagnetic axis. Both effects arise from the same term in the action density.

The analysis of the resulting field equations shows that they describe a pair of charged vector bosons with a finite rest mass together with the interacting electromagnetic field. The equations for the Maxwell field are exact and can be derived without approximations as a special case of the general field equations. But the equations for the vector bosons are nonlinear and their usual interpretation is only valid in a linear approximation.

7. THE ELECTROMAGNETIC LIMIT

Just as in general relativity where the limit of vanishing curvature is important in understanding the success of special relativity, in Q quantum mechanics the limit of vanishing Q curvature $K_{\mu\nu} = 0$ is important. We call this the case *Q flat.* In the Q-flat limit, the neutral and charged photons vanish.

There are two interesting intermediate cases between the general case and the flat limit. The first is characterized by $D_\mu \cdot \eta = 0$, the second by $\Delta K_{\mu\nu} \cdot \eta = 0$. The first of these two conditions implies the second, since from $D_\mu \cdot \eta = 0$ follows

$$0 = [D_\mu, D_\nu] \cdot \eta = \Delta K_{\mu\nu} \cdot \eta.$$

But the second condition is weaker. Indeed from the first we obtain immediately

$$0 = D_\mu \cdot \eta = D_\mu^0 \cdot \eta + \tfrac{1}{2}[B_\mu, \eta] = B_\mu \eta.$$

Since $\eta \neq 0$, it follows that $B_\mu = 0$, and consequently also $K_{\mu\nu} = F_{\mu\nu}\eta$. In this case the charged field alone vanishes. We could therefore call the case $D_\mu \cdot \eta = 0$ the *electromagnetic limit* of the theory.

[In case $\Delta K_{\mu\nu} \cdot \eta = 0$, we can only conclude

$$D_\mu^0 B_\nu - D_\nu^0 B_\mu = 0,$$

from which follows, with the help of (24) and (25),

$$(\alpha/\beta)B^\nu + \tfrac{1}{4}[B_\mu, [B^\mu, B^\nu]] - F^{\mu\nu}\eta B_\mu = 0,$$

as an algebraic relation. The physical meaning of this case is obscure.]

The electromagnetic limit characterized by $D_\mu \cdot \eta = 0$ can be interpreted more easily if we choose a Q frame in which $\eta(x)$ becomes a constant quaternion, say $\eta(x) = i_3$. Such a choice is always possible, for instance, by the explicit transformation

$$\eta \to \eta^q \equiv q\eta q^{-1},$$

with

$$q = \exp\{\tfrac{1}{2}(\cos^{-1}\eta_3)[\eta, i_3]/2(\eta_1^2 + \eta_2^2)^{\frac{1}{2}}\}$$
$$= \cos(\tfrac{1}{2}\cos^{-1}\eta_3) + \{[\eta, i_3]/2(\eta_4^2 + \eta_2^2)^{\frac{1}{2}}\}\sin(\tfrac{1}{2}\cos^{-1}\eta_3).$$

In this frame

$$D_\mu = \partial_\mu + \tfrac{1}{2}A_\mu \Delta\eta = D_\mu^0,$$
$$\eta_\mu = 0,$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Since the B field is identically zero, the quaternion components in the directions i_1 and i_2 never appear. We can therefore set $\eta \equiv i_3 = i$ and identify it with the ordinary $(-1)^{\frac{1}{2}}$. The resulting field equations are identical with the classical equations of the source-free electromagnetic field.

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