On Gauge Invariance and Vacuum Polarization

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This paper is based on the elementary remark that the extraction of gauge invariant results from a formally gauge invariant theory is ensured if one employs methods of solution that involve only gauge covariant quantities. We illustrate this statement in connection with the problem of vacuum polarization by a prescribed electromagnetic field. The vacuum current of a charged Dirac field, which can be expressed in terms of the Green's function of that field, implies an addition to the action integral of the electromagnetic field. Now these quantities can be related to the dynamical properties of a "particle" with space-time coordinates that depend upon a proper-time parameter. The proper-time equations of motion involve only electromagnetic field strengths, and provide a suitable gauge invariant basis for treating problems. Rigorous solutions of the equations of motion can be obtained for a constant field, and for a plane wave field. A renormalization of field strength and charge, applied to the modified lagrange function for constant fields, yields a finite, gauge invariant result which implies nonlinear properties for the electromagnetic field in the vacuum. The contribution of a zero spin charged field is also stated. After the same field strength renormalization, the modified physical quantities describing a plane wave in the vacuum reduce to just those of the maxwell field; there are no nonlinear phenomena for a single plane wave, of arbitrary strength and spectral composition. The results obtained for constant (that is, slowly varying fields), are then applied to treat the two-photon disintegration of a spin zero neutral meson arising from the polarization of the proton vacuum. We obtain approximate, gauge invariant expressions for the effective interaction between the meson and the electromagnetic field, in which the nuclear coupling may be scalar, pseudoscalar, or pseudovector in nature. The direct verification of equivalence between the pseudoscalar and pseudovector interactions only requires a proper statement of the limiting processes involved. For arbitrarily varying fields, perturbation methods can be applied to the equations of motion, as discussed in Appendix A, or one can employ an expansion in powers of the potential vector. The latter automatically yields gauge invariant results, provided only that the proper-time integration is reserved to the last. This indicates that the significant aspect of the proper-time method is its isolation of divergences in integrals with respect to the proper-time parameter, which is independent of the coordinate system and of the gauge. The connection between the proper-time method and the technique of "invariant regularization" is discussed. Incidentally, the probability of actual pair creation is obtained from the imaginary part of the electromagnetic field action integral. Finally, as an application of the Green's function for a constant field, we construct the mass operator of an electron in a weak, homogeneous external field, and derive the additional spin magnetic moment of $\alpha/2\pi$ magnetons by means of a perturbation calculation in which proper-mass plays the customary role of energy.

I. INTRODUCTION

QUANTUM electrodynamics is characterized by several formal invariance properties, notably relativistic and gauge invariance. Yet specific calculations by conventional methods may yield results that violate these requirements, in consequence of the divergences inherent in present field theories. Such difficulties concerning relativistic invariance have been avoided by employing formulations of the theory that are explicitly invariant under coordinate transformations, and by maintaining this generality through the course of calculations. The preservation of gauge invariance has apparently been considered to be a more formidable task. It should be evident, however, that the two problems are quite analogous, and that gauge invariance difficulties naturally disappear when methods of solution are adopted that involve only gauge invariant quantities.

We shall illustrate this assertion by applying such a gauge invariant method to treat several aspects of the problem of vacuum polarization by a prescribed electromagnetic field. The calculation of the current associated with the vacuum of a charged particle field involves the construction of the Green's function for the particle field in the prescribed electromagnetic field. This vacuum current can be exhibited as the variation of an action integral with respect to the potential vector, which action effectively adds to that of the maxwell field in describing the behavior of electromagnetic fields in the vacuum. We shall relate these problems to the solution of particle equations of motion with a proper-time parameter. The equations of motion, which involve only electromagnetic field strengths, provide the desired gauge invariant basis for our discussion.

Explicit solutions can be obtained in the two situations of constant fields, and fields propagated with the speed of light in the form of a plane wave.[1] For constant (that is, slowly varying) fields, a renormalization of field strength and charge yields a modified lagrange function differing from that of the maxwell field by terms that imply a nonlinear behavior for the electromagnetic field. The result agrees precisely with one obtained some time ago by other methods and a somewhat different viewpoint.[2] The modified physical quantities characterizing the plane wave in the vacuum revert to those of the maxwell field after the same field strength renormalization. For weak arbitrarily varying fields, perturbation methods can be applied to the equations of motion. This will be discussed in Appendix A.

The consequences thus obtained are useful in connection with a class of problems in which gauge invari-

[1] That the Dirac equation can be solved exactly, in the field of a plane wave, was recognized by D. M. Volkow, Z. Physik 94, 25 (1935).

[2] W. Heisenberg and H. Euler, Z. Physik 98, 714 (1936). V. Weisskopf, Kgl. Danske Videnskab. Selskabs. Mat.-fys. Medd. 14, No. 6 (1936).

ance difficulties have been encountered[3]—the multiple photon disintegration of a neutral meson. Without further extensive calculation, we shall obtain approximate gauge invariant expressions for the interaction of a zero-spin, neutral meson with two photons, where the intermediate nuclear interaction may be scalar, or the equivalent pseudoscalar and pseudovector couplings.

The utility of the proper-time technique to be exploited in this paper, apart from its value in obtaining rigorous solutions in a few special cases, lies in its isolation of the divergent aspects of a calculation in integrals with respect to the proper-time, a parameter that makes no reference to the coordinate system or the gauge. Indeed, we shall show that the customary perturbation procedure of expansion in powers of the potential vector does yield gauge invariant results, provided only that the proper-time integration is reserved to the last. The technique of "invariant regularization"[4] represents a partial realization of this proper-time method through the use of specially weighted integrals over the conjugate quantity, the square of the proper mass.

Finally, in Appendix B we shall employ the Green's function of an electron in a weak, homogeneous, external field to calculate the second-order electromagnetic mass, thereby providing a simple derivation of the second-order correction to the electron magnetic moment.

II. GENERAL THEORY

The field equations, commutation relations, and current vector of the Dirac field are given by[5]

$$\begin{aligned}\gamma_\mu(-i\partial_\mu-eA_\mu(x))\psi(x)+m\psi(x)=0,\\ (i\partial_\mu-eA_\mu(x))\bar{\psi}(x)\gamma_\mu+m\bar{\psi}(x)=0,\end{aligned}\tag{2.1}$$

$$\{\psi(\mathbf{x},x_0),\bar{\psi}(\mathbf{x}',x_0)\}=\gamma_0\delta(\mathbf{x}-\mathbf{x}'),\tag{2.2}$$

$$j_\mu(x)=\tfrac{1}{2}e[\bar{\psi}(x),\gamma_\mu\psi(x)],\tag{2.3}$$

where

$$\tfrac{1}{2}\{\gamma_\mu,\gamma_\nu\}=-\delta_{\mu\nu}\tag{2.4}$$

and

$$\gamma_0=-i\gamma_4,\quad \gamma_0{}^2=1.\tag{2.5}$$

The structure of the current operator,

$$j_\mu(x)=-e(\gamma_\mu)_{\beta\alpha}\tfrac{1}{2}[\psi_\alpha(x),\bar{\psi}_\beta(x)],\tag{2.6}$$

which arises from an explicit charge symmetrization, can be related to a time symmetrization by introducing chronologically ordered operators. Thus, with the notation

$$(A(x_0)B(x_0'))_+=\begin{cases}A(x_0)B(x_0'), & x_0>x_0'\\ B(x_0')A(x_0), & x_0<x_0',\end{cases}\tag{2.7}$$

and

$$\epsilon(x-x')=\begin{cases}1, & x_0>x_0'\\ -1, & x_0<x_0',\end{cases}\tag{2.8}$$

we have

$$(\psi_\alpha(x)\bar{\psi}_\beta(x'))_+\epsilon(x-x')=\begin{cases}\psi_\alpha(x)\bar{\psi}_\beta(x'), & x_0>x_0'\\ -\bar{\psi}_\beta(x')\psi_\alpha(x), & x_0<x_0'.\end{cases}\tag{2.9}$$

Therefore

$$\tfrac{1}{2}[\psi_\alpha(x),\bar{\psi}_\beta(x)]=(\psi_\alpha(x)\bar{\psi}_\beta(x'))_+\epsilon(x-x')]_{x'\to x},\tag{2.10}$$

provided one takes the average of the forms obtained by letting x' approach x from the future, and from the past. The quantity of actual interest here is the expectation value of $j_\mu(x)$ in the vacuum of the Dirac field,

$$\langle j_\mu(x)\rangle=ie\,\mathrm{tr}\gamma_\mu G(x,x')]_{x'\to x},\tag{2.11}$$

where

$$G(x,x')=i\langle(\psi(x)\bar{\psi}(x'))_+\rangle\epsilon(x-x'),\tag{2.12}$$

and tr indicates the diagonal sum with respect to the spinor indices.

The function $G(x,x')$ satisfies an inhomogeneous differential equation which is obtained by noting that

$$\begin{aligned}[\gamma(-i\partial-eA(x))+m]G(x\,x')\\ =\langle\gamma_0\{\psi(x),\bar{\psi}(x')\}\rangle\delta(x_0-x_0'),\end{aligned}\tag{2.13}$$

where the right side expresses the discontinuous change in form of $G(x,x')$ as x_0 is altered from $x_0'-0$ to $x_0'+0$. According to Eq. (2.2), therefore, we have

$$[\gamma(-i\partial-eA(x))+m]G(x,x')=\delta(x-x');\tag{2.14}$$

that is, $G(x,x')$ is a Green's function for the Dirac field. We shall not discuss which particular Green's function this is, as specified by the associated boundary conditions, since no ambiguity enters if actual pair creation in the vacuum does not occur, which we shall expressly assume.

It is useful to regard $G(x,x')$ as the matrix element of an operator G, in which states are labeled by space-time coordinates as well as by the suppressed spinor indices:

$$G(x,x')=(x|G|x').\tag{2.15}$$

The defining differential equations for the Green's function is then considered to be a matrix element of the operator equation

$$(\gamma\Pi+m)G=1,\tag{2.16}$$

where

$$\Pi_\mu=p_\mu-eA_\mu\tag{2.17}$$

is characterized by the operator properties

$$[x_\mu,\Pi_\nu]=i\delta_{\mu\nu},\quad [\Pi_\mu,\Pi_\nu]=ieF_{\mu\nu},\tag{2.18}$$

and

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\tag{2.19}$$

is the antisymmetrical field strength tensor.

With this symbolism, it is easy to show that the

[3] H. Fukuda and Y. Miyamoto, Prog. Theor. Phys. 4, 347 (1949).

[4] W. Pauli and F. Villars, Revs. Modern Phys. 21, 434 (1949).

[5] We employ units in which $\hbar=c=1$. Note that also $\bar{\psi}=\psi^\dagger\gamma_0$, since $\gamma_0\gamma_\mu$, $\mu=0,1,2,3$ form hermitian matrices.

vacuum current vector,

$$\langle j_\mu(x)\rangle = ie\,\mathrm{tr}\gamma_\mu(x|G|x), \tag{2.20}$$

is obtained from an action integral by variation of $A_\mu(x)$. This is accomplished by exhibiting

$$\delta W^{(1)} = \int (dx)\delta A_\mu(x)\langle j_\mu(x)\rangle = ie\,\mathrm{Tr}\gamma\delta AG \tag{2.21}$$

as a total differential, subject to $\delta A_\mu(x)$ vanishing at infinity. In the second version of $\delta W^{(1)}$, δA_μ denotes the operator with the matrix elements

$$(x|\delta A_\mu|x') = \delta(x-x')\delta A_\mu(x), \tag{2.22}$$

and Tr indicates the complete diagonal symmation, including spinor indices and the continuous space-time coordinates. Now

$$-e\gamma\delta A = \delta(\gamma\Pi + m), \tag{2.23}$$

and

$$G = \frac{1}{\gamma\Pi+m} = i\int_0^\infty ds\,\exp\{-i(\gamma\Pi+m)s\}, \tag{2.24}$$

so that

$$ie\,\mathrm{Tr}\gamma\delta AG = \delta\left[i\int_0^\infty dss^{-1}\,\mathrm{Tr}\exp\{-i(\gamma\Pi+m)s\}\right], \tag{2.25}$$

in virtue of the fundamental property of the trace,

$$\mathrm{Tr}AB = \mathrm{Tr}BA. \tag{2.26}$$

Thus, to within an additive constant,

$$\begin{aligned} W^{(1)} &= i\int_0^\infty dss^{-1}e^{-ims}\,\mathrm{Tr}\exp\{-i\gamma\Pi s\} \\ &= \int (dx)\mathcal{L}^{(1)}(x), \end{aligned} \tag{2.27}$$

where the lagrange function $\mathcal{L}^{(1)}(x)$ is given by

$$\mathcal{L}^{(1)}(x) = i\int_0^\infty dss^{-1}e^{-ims}\,\mathrm{tr}(x|\exp\{-i\gamma\Pi s\}|x). \tag{2.28}$$

An alternative representation, and the one we shall actually employ for calculations, is obtained by writing

$$\begin{aligned} G &= (-\gamma\Pi+m)[m^2-(\gamma\Pi)^2]^{-1} \\ &= [m^2-(\gamma\Pi)^2]^{-1}(-\gamma\Pi+m) \end{aligned} \tag{2.29}$$

or

$$\begin{aligned} G &= (-\gamma\Pi+m)i\int_0^\infty ds\,\exp[-i(m^2-(\gamma\Pi)^2)s] \\ &= i\int_0^\infty ds\,\exp[-i(m^2-(\gamma\Pi)^2)s](-\gamma\Pi+m). \end{aligned} \tag{2.30}$$

In virtue of the vanishing trace of an odd number of γ-factors, we have

$$\begin{aligned} ie\,\mathrm{Tr}\gamma\delta AG &= -\mathrm{Tr}\delta(\gamma\Pi)\gamma\Pi\int_0^\infty ds\,\exp[-i(m^2-(\gamma\Pi)^2)s] \\ &= \delta\left[\tfrac{1}{2}i\int_0^\infty dss^{-1}\exp[-i(m^2-(\gamma\Pi)^2)s]\right], \end{aligned} \tag{2.31}$$

which again involves the fundamental trace property (2.26). Thus,

$$\begin{aligned} \mathcal{L}^{(1)}(x) &= \tfrac{1}{2}i\int_0^\infty dss^{-1}\exp(-im^2s)\,\mathrm{tr}(x|U(s)|x), \\ U(s) &= \exp(-i\mathcal{H}s), \end{aligned} \tag{2.32}$$

where

$$\mathcal{H} = -(\gamma\Pi)^2 = \Pi_\mu{}^2 - \tfrac{1}{2}e\sigma_{\mu\nu}F_{\mu\nu}, \tag{2.33}$$

and

$$\sigma_{\mu\nu} = \tfrac{1}{2}i[\gamma_\mu, \gamma_\nu]. \tag{2.34}$$

We now see that the construction of $G(x, x')$ and $\mathcal{L}^{(1)}(x)$ devolves upon the evaluation of

$$(x'|U(s)|x'') = (x(s)'|x(0)''). \tag{2.35}$$

The latter notation emphasizes that $U(s)$ may be regarded as the operator describing the development of a system governed by the "hamiltonian," $\mathcal{H}$, in the "time" s, the matrix element of $U(s)$ being the transformation function from a state in which $x_\mu(s=0)$ has the value x_μ'' to a state in which $x_\mu(s)$ has the value x_μ'. Thus, we are led to an associated dynamical problem in which the space-time coordinates of a "particle" depend upon a proper time parameter, in a manner determined by the equations of motion

$$\begin{aligned} dx_\mu/ds &= -i[x_\mu, \mathcal{H}] = 2\Pi_\mu, \\ d\Pi_\mu/ds &= -i[\Pi_\mu, \mathcal{H}] = e(F_{\mu\nu}\Pi_\nu + \Pi_\nu F_{\mu\nu}) \\ &\quad + \tfrac{1}{2}e\sigma_{\lambda\nu}(\partial F_{\lambda\nu}/\partial x_\mu) = 2eF_{\mu\nu}\Pi_\nu \\ &\quad - ie(\partial F_{\mu\nu}/\partial x_\nu) + \tfrac{1}{2}e\sigma_{\lambda\nu}(\partial F_{\lambda\nu}/\partial x_\mu). \end{aligned} \tag{2.36}$$

The transformation function is characterized by the differential equations,[6]

$$i\partial_s(x(s)'|x(0)'') = (x(s)'|\mathcal{H}|x(0)''), \tag{2.37}$$

$$(-i\partial_\mu' - eA_\mu(x'))(x(s')|x(0)'') = (x(s)'|\Pi_\mu(s)|x(0)''), \tag{2.38}$$

$$(i\partial_\mu'' - eA_\mu(x''))(x(s)'|x(0)'') = (x(s)'|\Pi_\mu(0)|x(0)''), \tag{2.39}$$

and the boundary condition

$$(x(s)'|x(0)'')]_{s\to 0} = \delta(x'-x''). \tag{2.40}$$

[6] A proper time wave equation, in conjunction with the second-order Dirac operator, has been discussed by V. Fock, Physik. Z. Sowjetunion **12**, 404 (1937). See also Y. Nambu, Prog. Theor. Phys. **5**, 82 (1950).

We shall now illustrate, for the elementary situation $F_{\mu\nu}=0$, the procedure which will be employed in the following sections for constructing the transformation function.

The equations of motion read

$$d\Pi_\mu/ds=0, \quad dx_\mu/ds=2\Pi_\mu, \tag{2.41}$$

whence

$$\Pi_\mu(s)=\Pi_\mu(0), \tag{2.42}$$

and

$$(x_\mu(s)-x_\mu(0))/s=2\Pi_\mu(0). \tag{2.43}$$

Therefore

$$\begin{aligned}\mathcal{H}=\Pi^2=\tfrac{1}{4}s^{-2}(x(s)-x(0))^2&=\tfrac{1}{4}s^{-2}[x^2(s)\\&\quad-2x(s)x(0)+x^2(0)]+\tfrac{1}{4}s^{-2}[x(s),x(0)]\\&=\tfrac{1}{4}s^{-2}[x^2(s)-2x(s)x(0)+x^2(0)]-2is^{-1},\end{aligned} \tag{2.44}$$

since

$$[x_\mu(s),x_\nu(0)]=[x_\mu(0)+2s\Pi_\mu(0),x_\nu(0)]=-2is\delta_{\mu\nu}, \tag{2.45}$$

Having ordered the coordinate operators so that $x(s)$ everywhere stands to the left of $x(0)$, we can immediately evaluate the matrix element of $\mathcal{H}$ in Eq. (2.37), thus obtaining

$$\begin{aligned}&i\partial_s(x(s)'|x(0)'')\\&\quad=[\tfrac{1}{4}s^{-2}(x'-x'')^2-2is^{-1}](x(s)'|x(0)''),\end{aligned} \tag{2.46}$$

the solution of which is

$$(x(s)'|x(0)'')=C(x',x'')s^{-2}\exp[i\tfrac{1}{4}(x'-x'')^2/s]. \tag{2.47}$$

To determine the function $C(x',x'')$, we note that

$$\begin{aligned}(x(s)'|\Pi_\mu(s)|x(0)'')&=(x(s)'|\Pi_\mu(0)|x(0)'')\\&=((x_\mu'-x_\mu'')/2s)(x(s)'|x(0)''),\end{aligned} \tag{2.48}$$

which, in conjunction with Eqs. (2.38) and (2.39), implies that

$$\begin{aligned}&(-i\partial_\mu'-eA_\mu(x'))C(x',x'')\\&\quad=(i\partial_\mu''-eA_\mu(x''))C(x',x'')=0,\end{aligned} \tag{2.49}$$

or

$$\begin{aligned}C(x',x'')&=C\Phi(x',x''),\\ \Phi(x',x'')&=\exp\left[ie\int_{x''}^{x'}dx_\mu A_\mu(x)\right].\end{aligned} \tag{2.50}$$

The line integral in Eq. (2.50) is independent of the integration path, since $F_{\mu\nu}=0$. Finally, the constant C is fixed by the boundary condition (2.40). It is evident that Eq. (2.47) does have the character of a delta-function as s approaches zero, provided

$$Cs^{-2}\int(dx)\exp(i\tfrac{1}{4}x^2/s)=1; \tag{2.51}$$

that is,

$$C=-i(4\pi)^{-2}. \tag{2.52}$$

Therefore,

$$\begin{aligned}(x(s)'|x(0)'')=-i(4\pi)^{-2}&\Phi(x',x'')s^{-2}\\&\times\exp[i\tfrac{1}{4}(x'-x'')^2/s],\end{aligned} \tag{2.53}$$

and the Green's function is obtained as

$$\begin{aligned}G(x',x'')&=i\int_0^\infty ds\exp(-im^2s)\\&\qquad\times(x(s)'|(-\gamma\Pi+m)|x(0)'')\\&=(4\pi)^{-2}\Phi(x',x'')\int_0^\infty ds s^{-2}\exp(-im^2s)\\&\qquad\times\left(-\gamma\frac{(x'-x'')}{2s}+m\right)\exp\left[i\tfrac{1}{4}\frac{(x'-x'')^2}{s}\right].\end{aligned} \tag{2.54}$$

An equivalent, and more familiar procedure, is to employ the representation labeled by the eigenvalues of Π_μ. Now

$$\begin{aligned}(\Pi(s)'|\Pi(0)'')&=(\Pi(0)'|U(s)|\Pi(0)'')\\&=\delta(\Pi'-\Pi'')\exp(-i\Pi'^2s),\end{aligned} \tag{2.55}$$

while $(x(s)'|\Pi(s)')$ is determined by

$$\begin{aligned}&(-i\partial_\mu'-eA_\mu(x'))(x(s)'|\Pi(s)')\\&\quad=\Pi_\mu'(x(s)'|\Pi(s)'),\end{aligned} \tag{2.56}$$

and the normalization condition

$$\int(\Pi(s)'|x(s)')(dx')(x(s)'|\Pi(s)'')=\delta(\Pi'-\Pi''), \tag{2.57}$$

to be

$$\begin{aligned}&(x(s)'|\Pi(s)')\\&\quad=(2\pi)^{-2}\exp\left[ie\int_0^{x'}dxA\right]\exp(ix'\Pi').\end{aligned} \tag{2.58}$$

Therefore,

$$\begin{aligned}&(x(s)'|x(0)'')\\&\quad=\int(x(s)'|\Pi(s)')(d\Pi')(\Pi(s)'|\Pi(0)'')\\&\qquad\qquad\times(d\Pi'')(\Pi(0)''|x(0)'')\\&\quad=(2\pi)^{-4}\Phi(x',x'')\int(d\Pi')\\&\qquad\qquad\times\exp[i(x'-x'')\Pi'-i\Pi'^2s],\end{aligned} \tag{2.59}$$

and

$$\begin{aligned}G(x',x'')&=i(2\pi)^{-4}\Phi(x',x'')\\&\quad\times\int_0^\infty ds\int(d\Pi')\exp[i(x'-x'')\Pi']\\&\quad\times(-\gamma\Pi'+m)\exp[-i(\Pi'^2+m^2)s],\end{aligned} \tag{2.60}$$

which reduce to Eqs. (2.53) and (2.54) on performance of the Π' integration.

III. CONSTANT FIELDS

The equations of motion (2.36) here simplify to

$$dx_\mu/ds=2\Pi_\mu, \quad d\Pi_\mu/ds=2eF_{\mu\nu}\Pi_\nu, \tag{3.1}$$

or, in matrix notation,

$$dx/ds=2\Pi,\quad d\Pi/ds=2eF\Pi. \tag{3.2}$$

The symbolic solution of these equations is

$$\begin{aligned}\Pi(s)&=e^{2eFs}\Pi(0),\\ x(s)-x(0)&=[(e^{2eFs}-1)/eF]\Pi(0),\end{aligned} \tag{3.3}$$

whence

$$\begin{aligned}\Pi(0)&=eF(e^{2eFs}-1)^{-1}(x(s)-x(0))\\ &=\tfrac{1}{2}eFe^{-eFs}\sinh^{-1}(eFs)(x(s)-x(0)),\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\Pi(s)&=\tfrac{1}{2}eFe^{eFs}\sinh^{-1}(eFs)(x(s)-x(0))\\ &=(x(s)-x(0))\tfrac{1}{2}eFe^{-eFs}\sinh^{-1}(eFs).\end{aligned} \tag{3.5}$$

The latter form involves the fact that

$$\tilde{F}=-F,\quad (F_{\mu\nu}=-F_{\nu\mu}). \tag{3.6}$$

We now consider

$$\begin{aligned}\mathcal{H}+\tfrac{1}{2}e\sigma F=\Pi^2(s)&=(x(s)-x(0))K(x(s)-x(0)),\\ K&=\tfrac{1}{4}e^2F^2\sinh^{-2}(eFs).\end{aligned} \tag{3.7}$$

In rearranging the order of these operators, the following commutator is required:

$$\begin{aligned}[x(s),x(0)]&=[x(0)+(eF)^{-1}(e^{2eFs}-1)\Pi(0),x(0)]\\ &=i(eF)^{-1}(e^{2eFs}-1).\end{aligned} \tag{3.8}$$

Thus

$$\begin{aligned}\mathcal{H}+\tfrac{1}{2}e\sigma F=x(s)Kx(s)-2x(s)Kx(0)+x(0)Kx(0)\\ -\tfrac{1}{2}i\operatorname{tr}eF\coth(eFs),\end{aligned} \tag{3.9}$$

where tr again denotes a diagonal summation, and we have employed the fact that

$$\operatorname{tr}(F)=0, \tag{3.10}$$

which follows from Eq. (3.6). The resulting differential equation (2.37)

$$\begin{aligned}i\partial_s(x(s)'|x(0)'')=[-\tfrac{1}{2}e\sigma F+(x'-x'')K(x'-x'')\\ -\tfrac{1}{2}i\operatorname{tr}eF\coth(eFs)](x(s)'|x(0)''),\end{aligned} \tag{3.11}$$

has the solution

$$\begin{aligned}(x(s)'|x(0)'')=C(x',x'')e^{-L(s)}s^{-2}\\ \times\exp[\tfrac{1}{4}i(x'-x'')eF\coth(eFs)(x'-x'')]\\ \cdot\exp(i\tfrac{1}{2}e\sigma Fs),\end{aligned} \tag{3.12}$$

$$L(s)=\tfrac{1}{2}\operatorname{tr}\ln[(eFs)^{-1}\sinh(eFs)].$$

To determine $C(x',x'')$, we employ

$$\begin{aligned}(x(s)'|\Pi(s)|x(0)'')=\tfrac{1}{2}[eF\coth(eFs)+eF]\\ \times(x'-x'')(x(s)'|x(0)''),\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}(x(s)'|\Pi(0)|x(0)'')=\tfrac{1}{2}[eF\coth(eFs)-eF]\\ \times(x'-x'')(x(s)'|x(0)''),\end{aligned} \tag{3.14}$$

in conjunction with Eq. (3.12), to obtain the differential equations

$$[-i\partial_\mu'-eA_\mu(x')-\tfrac{1}{2}eF_{\mu\nu}(x'-x'')_\nu]C(x',x'')=0, \tag{3.15}$$

$$[i\partial_\mu''-eA_\mu(x'')-\tfrac{1}{2}eF_{\mu\nu}(x'-x'')_\nu]C(x',x'')=0. \tag{3.16}$$

The solution of Eq. (3.15) has the form

$$\begin{aligned}&C(x',x'')\\ &\quad=C(x'')\exp\left[ie\int_{x''}^{x'}dx(A(x)+\tfrac{1}{2}F(x-x''))\right],\end{aligned} \tag{3.17}$$

in which the integral is independent of the integration path, since $A_\mu(x)+\frac{1}{2}F_{\mu\nu}(x-x'')_\nu$ has a vanishing curl. However, by restricting the integration path to be a straight line connecting x' and x'', we may, in virtue of Eq. (3.6), simply write

$$\begin{aligned}C(x',x'')&=C\Phi(x',x''),\\ \Phi(x',x'')&=\exp\left[ie\int_{x''}^{x'}dxA(x)\right],\end{aligned} \tag{3.18}$$

and, with C a constant, attain the solution of (3.15) and (3.16). The constant C has the value

$$C=-i(4\pi)^{-2} \tag{3.19}$$

since the limiting form of $(x(s)'|x(0)'')$ as $s\rightarrow0$ is independent of the external field.

Finally, then

$$\begin{aligned}(x(s)'|x(0)'')=-i(4\pi)^{-2}\Phi(x',x'')e^{-L(s)}s^{-2}\\ \times\exp[i\tfrac{1}{4}(x'-x'')eF\coth(eFs)(x'-x'')]\\ \cdot\exp[i\tfrac{1}{2}e\sigma Fs],\end{aligned} \tag{3.20}$$

and the Green's function $G(x',x'')$ is obtained from (2.30) in the two equivalent forms,

$$\begin{aligned}G(x',x'')&=i\int_0^\infty ds\exp(-im^2s)\\ &\qquad\times[-\gamma_\mu(x(s)'|\Pi_\mu(s)|x(0)'')\\ &\qquad\qquad+m(x(s)'|x(0)'')]\\ &=i\int_0^\infty ds\exp(-im^2s)\\ &\qquad\times[-(x(s)'|\Pi_\mu(0)|x(0)'')\gamma_\mu\\ &\qquad\qquad+m(x(s)'|x(0)'')],\end{aligned} \tag{3.21}$$

which will be given explicitly on substituting Eqs. (3.13), (3.14), and (3.20).

The lagrange function $\mathcal{L}^{(1)}(x)$ is now computed as

$$\begin{aligned}\mathcal{L}^{(1)}(x)&=\tfrac{1}{2}i\int_0^\infty dss^{-1}\exp(-im^2s)\\ &\qquad\times\operatorname{tr}(x(s)'|x(0)'')]_{x',x''\rightarrow x}\\ &=(1/32\pi^2)\int_0^\infty dss^{-3}\exp(-im^2s)\\ &\qquad\times e^{-L(s)}\operatorname{tr}\exp(i\tfrac{1}{2}e\sigma Fs).\end{aligned} \tag{3.22}$$

We may exhibit this more explicitly as a real quantity by a deformation of the integration path, which is effectively the substitution $s\to -is$:

$$\mathcal{L}^{(1)}(x)=-(1/32\pi^2)\int_0^\infty ds s^{-3}\exp(-m^2 s)\times e^{-l(s)}\operatorname{tr}\exp(\tfrac{1}{2}e\sigma Fs),$$
$$l(s)=\tfrac{1}{2}\operatorname{tr}\ln[(eFs)^{-1}\sin(eFs)]. \tag{3.23}$$

Indeed, we could have initially employed the integral representation

$$[m^2-(\gamma\Pi)^2]^{-1}=\int_0^\infty ds\exp[-(m^2-(\gamma\Pi)^2)s], \tag{3.24}$$

which exists in consequence of the restriction on real pair creation. This, however, would have obscured the proper time interpretation.

To evaluate the Dirac matrix trace, we employ the following spin matrix property:

$$\tfrac{1}{2}\{\sigma_{\mu\nu},\sigma_{\lambda\kappa}\}=\delta_{\mu\lambda}\delta_{\nu\kappa}-\delta_{\mu\kappa}\delta_{\nu\lambda}+i\epsilon_{\mu\nu\lambda\kappa}\gamma_5, \tag{3.25}$$

where

$$\gamma_5=i\gamma_1\gamma_2\gamma_3\gamma_4,\quad \gamma_5{}^2=-1, \tag{3.26}$$

and $\epsilon_{\mu\nu\lambda\kappa}$ is 1, or -1, if $(\mu\nu\lambda\kappa)$ forms an even, or odd permutation of (1234), and is zero otherwise. In terms of the dual field strength tensor,

$$F_{\mu\nu}{}^*=\tfrac{1}{2}i\epsilon_{\mu\nu\lambda\kappa}F_{\lambda\kappa}, \tag{3.27}$$

we have

$$(\tfrac{1}{2}\sigma_{\mu\nu}F_{\mu\nu})^2=\tfrac{1}{2}F_{\mu\nu}{}^2+\tfrac{1}{2}\gamma_5F_{\mu\nu}F_{\mu\nu}{}^*, \tag{3.28}$$

thus introducing the fundamental scalar

$$\mathfrak{F}=\tfrac{1}{4}F_{\mu\nu}{}^2=\tfrac{1}{2}(\mathbf{H}^2-\mathbf{E}^2), \tag{3.29}$$

and pseudoscalar

$$\mathcal{G}=\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu}{}^*=\mathbf{E}\cdot\mathbf{H} \tag{3.30}$$

constructed from the field strengths. Since

$$(\tfrac{1}{2}\sigma F)^2=2(\mathfrak{F}+\gamma_5\mathcal{G}), \tag{3.31}$$

and $\gamma_5{}^2=-1$, it follows that $\frac{1}{2}\sigma F$ has the four eigenvalues

$$(\tfrac{1}{2}\sigma F)'=\pm(2(\mathfrak{F}\pm i\mathcal{G}))^{\frac{1}{2}}. \tag{3.32}$$

Therefore,

$$\operatorname{tr}\exp(\tfrac{1}{2}e\sigma Fs)=4\operatorname{Re}\cosh es(2(\mathfrak{F}+i\mathcal{G}))^{\frac{1}{2}}\equiv 4\operatorname{Re}\cosh es\mathrm{X}, \tag{3.33}$$

where Re denotes the real part of the subsequent expression. Note, incidentally, that

$$\mathrm{X}^2=(\mathbf{H}+i\mathbf{E})^2. \tag{3.34}$$

The eigenvalues of the matrix $F=(F_{\mu\nu})$ are required for the construction of $\exp(-l(s))$. They can be obtained with the aid of the easily verifiable relations,

$$F_{\mu\lambda}F_{\lambda\nu}{}^*=-\delta_{\mu\nu}\mathcal{G}, \tag{3.35}$$

and

$$F_{\mu\lambda}{}^*F_{\lambda\nu}{}^*-F_{\mu\lambda}F_{\lambda\nu}=2\delta_{\mu\nu}\mathfrak{F}. \tag{3.36}$$

From the eigenvalue equation

$$F_{\mu\nu}\psi_\nu=F'\psi_\mu, \tag{3.37}$$

and its equivalent according to Eq. (3.35),

$$F_{\mu\nu}{}^*\psi_\nu=-(1/F')\mathcal{G}\psi_\mu, \tag{3.38}$$

we obtain by iteration:

$$F_{\mu\lambda}F_{\lambda\nu}\psi_\nu=(F')^2\psi_\mu,\quad F_{\mu\lambda}{}^*F_{\lambda\nu}{}^*\psi_\nu=(1/(F')^2)\mathcal{G}^2\psi_\mu. \tag{3.39}$$

The identity (3.36) then yields the eigenvalue equation

$$(F')^4+2\mathfrak{F}(F')^2-\mathcal{G}^2=0, \tag{3.40}$$

which has the solutions $\pm F^{(1)}$, $\pm F^{(2)}$, with

$$F^{(1)}=(i/\sqrt{2})[(\mathfrak{F}+i\mathcal{G})^{\frac{1}{2}}+(\mathfrak{F}-i\mathcal{G})^{\frac{1}{2}}],$$
$$F^{(2)}=(i/\sqrt{2})[(\mathfrak{F}+i\mathcal{G})^{\frac{1}{2}}-(\mathfrak{F}-i\mathcal{G})^{\frac{1}{2}}]. \tag{3.41}$$

Expressed in terms of these eigenvalues,

$$e^{-l(s)}=(es)^2F^{(1)}F^{(2)}/\sin(eF^{(1)}s)\sin(eF^{(2)}s)=\frac{2(es)^2F^{(1)}F^{(2)}}{\cos es(F^{(1)}-F^{(2)})-\cos es(F^{(1)}+F^{(2)})}, \tag{3.42}$$

or

$$e^{-l(s)}=(es)^2\mathcal{G}/\operatorname{Im}\cosh es\mathrm{X}, \tag{3.43}$$

where Im designates the imaginary part of the following expression.

The final result for $\mathcal{L}^{(1)}$ is

$$\mathcal{L}^{(1)}=-\frac{1}{8\pi^2}\int_0^\infty ds s^{-3}\exp(-m^2s)\times\left[(es)^2\mathcal{G}\frac{\operatorname{Re}\cosh es\mathrm{X}}{\operatorname{Im}\cosh es\mathrm{X}}-1\right], \tag{3.44}$$

in which we have supplied the additive constant necessary to make $\mathcal{L}^{(1)}$ vanish in the absence of a field. The first term in the expansion of $\mathcal{L}^{(1)}$ for weak fields is

$$\mathcal{L}^{(1)}\simeq-\frac{e^2}{12\pi^2}\int_0^\infty ds s^{-1}\exp(-m^2s)\mathfrak{F}. \tag{3.45}$$

On separating this explicitly, and adding the lagrange function of the maxwell field,

$$\mathcal{L}^{(0)}=-\mathfrak{F}=\tfrac{1}{2}(\mathbf{E}^2-\mathbf{H}^2), \tag{3.46}$$

we obtain the total lagrange function

$$\mathcal{L}=-\left[1+\frac{e^2}{12\pi^2}\int_0^\infty ds s^{-1}\exp(-m^2s)\right]\mathfrak{F}-\frac{1}{8\pi^2}\int_0^\infty ds s^{-3}\exp(-m^2s)\times\left[(es)^2\mathcal{G}\frac{\operatorname{Re}\cosh es\mathrm{X}}{\operatorname{Im}\cosh es\mathrm{X}}-1-\tfrac{2}{3}(es)^2\mathfrak{F}\right]. \tag{3.47}$$

The logarithmically divergent factor that multiplies the maxwell lagrange function may be absorbed by a change of scale for all fields, and a corresponding scale change, or renormalization, of charge. If we identify the quantities thus far employed by a zero subscript, and introduce new units of field strength and charge according to

$$\begin{aligned}\mathfrak{F}+i\mathfrak{G}&=(1+Ce_0^2)(\mathfrak{F}_0+i\mathfrak{G}_0),\\ e^2&=e_0^2/(1+Ce_0^2),\end{aligned}\tag{3.48}$$

$$C=\frac{1}{12\pi^2}\int_0^\infty ds\, s^{-1}\exp(-m^2s),$$

we obtain the finite, gauge invariant result

$$\begin{aligned}\mathfrak{L}=&-\mathfrak{F}-\frac{1}{8\pi^2}\int_0^\infty dss^{-3}\exp(-m^2s)\\ &\times\left[(es)^2\mathfrak{G}\frac{\text{Re}\cosh es\mathrm{X}}{\text{Im}\cosh es\mathrm{X}}-1-\tfrac{2}{3}(es)^2\mathfrak{F}\right]\\ =&\tfrac{1}{2}(\mathbf{E}^2-\mathbf{H}^2)+\frac{2\alpha^2}{45}\frac{(\hbar/mc)^3}{mc^2}\\ &\times[(\mathbf{E}^2-\mathbf{H}^2)^2+7(\mathbf{E}\cdot\mathbf{H})^2]+\cdots.\end{aligned}\tag{3.49}$$

In the latter expansion, the conventional rationalized units have been reinstated, and $\alpha=e^2/4\pi\hbar c$.

Incidentally, the addition to the lagrange function produced by a spin zero charged field is obtained from Eq. (3.23) by omitting the Dirac trace, and multiplying by (-2). Thus,

$$\mathfrak{L}_{\text{spin }0}{}^{(1)}=\frac{1}{16\pi^2}\int_0^\infty dss^{-3}\exp(-\mu^2s)\times\left[\frac{(es)^2\mathfrak{G}}{\text{Im}\cosh es\mathrm{X}}-1\right],\tag{3.50}$$

in which μ designates the mass of the spinless particle, and an additive constant has been supplied as in Eq. (3.44). The first term in the expansion for weak fields is separated explicitly by writing

$$\begin{aligned}\mathfrak{L}_0{}^{(1)}=&-\frac{e^2}{48\pi^2}\int_0^\infty dss^{-1}\exp(-\mu^2s)\mathfrak{F}\\ &+\frac{1}{16\pi^2}\int_0^\infty dss^{-3}\exp(-\mu^2s)\\ &\times\left[\frac{(es)^2\mathfrak{G}}{\text{Im}\cosh es\mathrm{X}}-1+\tfrac{1}{3}(es)^2\mathfrak{F}\right].\end{aligned}\tag{3.51}$$

If we take into account the existence of both spin 0 and spin $\frac{1}{2}$ charged fields,

$$\mathfrak{L}=\mathfrak{L}^{(0)}+\mathfrak{L}_0{}^{(1)}+\mathfrak{L}_{\frac{1}{2}}{}^{(1)},\tag{3.52}$$

and a renormalization of the form (3.48), with

$$\begin{aligned}C=&C_0+C_{\frac{1}{2}}\\ =&\frac{1}{48\pi^2}\int_0^\infty dss^{-1}\exp(-\mu^2s)\\ &+\frac{1}{12\pi^2}\int_0^\infty dss^{-1}\exp(-m^2s),\end{aligned}\tag{3.53}$$

yields

$$\begin{aligned}\mathfrak{L}=&-\mathfrak{F}-\frac{1}{8\pi^2}\int_0^\infty dss^{-3}\exp(-m^2s)\\ &\times[(es)^2\mathfrak{G}(\text{Re}/\text{Im})-1-\tfrac{2}{3}(es)^2\mathfrak{F}]\\ &+\frac{1}{16\pi^2}\int_0^\infty dss^{-3}\exp(-\mu^2s)\\ &\times[(es)^2\mathfrak{G}(1/\text{Im})-1+\tfrac{1}{3}(es)^2\mathfrak{F}]\\ =&\tfrac{1}{2}(\mathbf{E}^2-\mathbf{H}^2)+\frac{2\alpha^2}{45}\frac{(\hbar/mc)^3}{mc^2}[(\mathbf{E}^2-\mathbf{H}^2)^2+7(\mathbf{E}\cdot\mathbf{H})^2]\\ &+\frac{\alpha^2}{90}\frac{(\hbar/\mu c)^3}{\mu c^2}\left[\frac{7}{4}(\mathbf{E}^2-\mathbf{H}^2)^2+(\mathbf{E}\cdot\mathbf{H})^2\right]+\cdots.\end{aligned}\tag{3.54}$$

The physical quantities characterizing the field are comprised in the energy momentum tensor

$$\begin{aligned}T_{\mu\nu}=&\delta_{\mu\nu}\mathfrak{L}-(\partial\mathfrak{L}/\partial F_{\mu\lambda})F_{\nu\lambda}\\ =&-(F_{\mu\lambda}F_{\nu\lambda}-\delta_{\mu\nu}\tfrac{1}{4}F_{\lambda\kappa}{}^2)(\partial\mathfrak{L}/\partial\mathfrak{F})\\ &+\delta_{\mu\nu}(\mathfrak{L}-\mathfrak{F}(\partial\mathfrak{L}/\partial\mathfrak{F})-\mathfrak{G}\partial\mathfrak{L}/\partial\mathfrak{G}).\end{aligned}\tag{3.55}$$

The maxwell tensor

$$T_{\mu\nu}{}^{(M)}=F_{\mu\nu}F_{\nu\lambda}-\delta_{\mu\nu}\tfrac{1}{4}F_{\lambda\kappa}{}^2\tag{3.56}$$

is obtained from $\mathfrak{L}=-\mathfrak{F}$, the weak field approximation of Eq. (3.49). The next terms in the expansion of $\mathfrak{L}$ yield

$$\begin{aligned}T_{\mu\nu}=&T_{\mu\nu}{}^M\left(1-\frac{16}{45}\alpha^2\frac{(\hbar/mc)^3}{mc^2}\mathfrak{F}\right)\\ &-\delta_{\mu\nu}\frac{2}{45}\alpha^2\frac{(\hbar/mc)^3}{mc^2}(4\mathfrak{F}^2+7\mathfrak{G}^2)+\cdots.\end{aligned}\tag{3.57}$$

IV. PLANE WAVE FIELDS

A plane wave, traveling with the speed of light, is characterized by the field strength tensor

$$F_{\mu\nu}=f_{\mu\nu}F(\xi),\quad \xi=n_\mu x_\mu,\tag{4.1}$$

where n_μ is a null vector,

$$n_\mu{}^2=0,\tag{4.2}$$

and $F(\xi)$ is an arbitrary function. The constant tensor $f_{\mu\nu}$, and its dual $f_{\mu\nu}{}^*$, are restricted by the conditions

$$n_\mu f_{\mu\nu}=0,\quad n_\mu f_{\mu\nu}{}^*=0,\tag{4.3}$$

from which are derived

$$f_{\mu\lambda}f_{\lambda\nu}{}^*=0,\quad f_{\mu\lambda}f_{\lambda\nu}=f_{\mu\lambda}{}^*f_{\lambda\nu}{}^*=-n_\mu n_\nu. \tag{4.4}$$

The latter statement also includes a convention concerning the scale of $f_{\mu\nu}$.

The proper time equations of motion in this external field,

$$\begin{gathered} dx_\mu/ds=2\Pi_\mu,\\ d\Pi_\mu/ds=2eF(\xi)f_{\mu\nu}\Pi_\nu+n_\mu eF'(\xi)\tfrac{1}{2}\sigma_{\lambda\nu}f_{\mu\nu}, \end{gathered} \tag{4.5}$$

admit several first integrals. Thus

$$d(n_\mu\Pi_\mu(s))/ds=0, \tag{4.6}$$

and

$$d(f_{\mu\nu}{}^*\Pi_\nu(s))/ds=0. \tag{4.7}$$

In addition,

$$d(f_{\mu\nu}\Pi_\nu(s))/ds=-n_\mu eF(\xi)d\xi/ds, \tag{4.8}$$

since

$$d\xi/ds=2n\Pi; \tag{4.9}$$

and therefore

$$d(f_{\mu\nu}\Pi_\nu+n_\mu eA(\xi))/ds=0, \tag{4.10}$$

where

$$dA(\xi)/d\xi=F(\xi). \tag{4.11}$$

In arriving at Eq. (4.10), it is necessary to recognize that $d\xi/ds$ commutes with ξ, in virtue of

$$[\xi,\,n\Pi]=[n_\mu x_\mu,\,n_\nu\Pi_\nu]=in_\mu{}^2=0. \tag{4.12}$$

Since $n\Pi$ is a constant of the motion, Eq. (4.9) can be integrated to yield

$$(\xi(s)-\xi(0))/s=2n\Pi, \tag{4.13}$$

from which we infer that

$$[\xi(s),\,\xi(0)]=2s[n\Pi,\,\xi(0)]=0. \tag{4.14}$$

The constant vector encountered on integration of Eq. (4.10),

$$f_{\mu\nu}\Pi_\nu+n_\mu eA(\xi)=C_\mu, \tag{4.15}$$

has the following evident properties:

$$\begin{gathered} n_\mu C_\mu=0,\quad f_{\mu\nu}{}^*C_\nu=0,\quad f_{\mu\nu}C_\nu=-n_\mu n\Pi,\\ C_\mu{}^2=(n\Pi)^2. \end{gathered} \tag{4.16}$$

The elimination of $f_{\mu\nu}\Pi_\nu$ from the equation of motion, with the aid of Eq. (4.15), gives

$$\begin{aligned} d\Pi_\mu/ds=(d/d\xi)[2C_\mu eA(\xi)-n_\mu e^2A^2(\xi)\\ +n_\mu eF(\xi)\tfrac{1}{2}\sigma f], \end{aligned} \tag{4.17}$$

whence

$$\begin{aligned} \Pi_\mu=\tfrac{1}{2}dx_\mu/ds=(1/2n\Pi)[2C_\mu eA(\xi)-n_\mu e^2A^2(\xi)\\ +n_\mu eF(\xi)\tfrac{1}{2}\sigma f]+D_\mu, \end{aligned} \tag{4.18}$$

where D_μ is an integration constant. Note, incidentally, that

$$f_{\mu\nu}{}^*\Pi_\nu=f_{\mu\nu}{}^*D_\nu, \tag{4.19}$$

which is independent of s, in agreement with Eq. (4.7).

On integrating Eq. (4.18) with respect to s, we find that

$$\begin{aligned} x_\mu(s)-x_\mu(0)=\frac{1}{2(n\Pi)^2}\int_{\xi(0)}^{\xi(s)}d\xi[2C_\mu eA(\xi)-n_\mu e^2A^2(\xi)\\ +n_\mu eF(\xi)\tfrac{1}{2}\sigma f]+2D_\mu s. \end{aligned} \tag{4.20}$$

With the constant D_μ determined by Eq. (4.20), Eq. (4.18) states that

$$\begin{aligned} \Pi_\mu(s)=&\frac{(x_\mu(s)-x_\mu(0))}{2s}+\frac{s}{(\xi(s)-\xi(0))}\\ &\times[2C_\mu eA(\xi(s))-n_\mu e^2A^2(\xi(s))+n_\mu eF(\xi(s))\tfrac{1}{2}\sigma f]\\ &-\frac{s}{(\xi(s)-\xi(0))^2}\int_{\xi(0)}^{\xi(s)}d\xi[2C_\mu eA(\xi)\\ &-n_\mu e^2A^2(\xi)+n_\mu eF(\xi)\tfrac{1}{2}\sigma f]. \end{aligned} \tag{4.21}$$

We can finally evaluate C_μ as

$$\begin{aligned} C_\mu&=f_{\mu\nu}\Pi_\nu+n_\mu eA\\ &=\frac{f_{\mu\nu}(x_\nu(s)-x_\nu(0))}{2s}+\frac{n_\mu}{\xi(s)-\xi(0)}\int_{\xi(0)}^{\xi(s)}d\xi eA(\xi). \end{aligned} \tag{4.22}$$

The commutation properties of these operators are involved in the construction of the transformation function. As is already indicated in the commutativity of $\xi(s)$ and $\xi(0)$, these commutation relations are greatly simplified by the special nature of the external field. Thus to evaluate $[x_\mu(0),\,x_\mu(s)]$, we employ Eq. (4.21) to express $x_\mu(0)$ in terms of $x_\mu(s)$, $\Pi_\mu(s)$, $\xi(s)$, and $\xi(0)$. Now

$$[\xi(0),\,x_\mu(s)]=[\xi(s)-2sn\Pi,\,x_\mu(s)]=2isn_\mu, \tag{4.23}$$

and, in virtue of $n_\mu C_\mu=n_\mu{}^2=0$, we have simply

$$[x_\mu(s),\,x_\mu(0)]=[-2s\Pi_\mu(s),\,x_\mu(s)]=8is. \tag{4.24}$$

No other nonvanishing commutator intervenes in bringing $\mathcal{H}$ to the form

$$\begin{aligned} \mathcal{H}=&\tfrac{1}{4}s^{-2}(x_\mu{}^2(s)-2x_\mu(s)x_\mu(0)+x_\mu{}^2(0))-2is^{-1}\\ &+\frac{1}{\xi(s)-\xi(0)}\int_{\xi(0)}^{\xi(s)}d\xi[e^2A^2(\xi)-eF(\xi)\tfrac{1}{2}\sigma f]\\ &-\frac{1}{(\xi(s)-\xi(0))^2}\left[\int_{\xi(0)}^{\xi(s)}d\xi eA(\xi)\right]^2, \end{aligned} \tag{4.25}$$

in which a constant added to $A(\xi)$ is without effect, as required by the corresponding ambiguity of Eq. (4.11).

The solution of the differential equation (2.37) is

$$\begin{aligned} (x(s)'|x(0)'')=&C(x',\,x'')s^{-2}\exp[i\tfrac{1}{4}(x'-x'')^2/s]\\ &\times\exp\left[-is/(\xi'-\xi'')\int_{\xi''}^{\xi'}d\xi[e^2A^2-eF\tfrac{1}{2}\sigma f]\right]\\ &\times\exp\left[is\left(1/(\xi'-\xi'')\int_{\xi''}^{\xi'}d\xi eA\right)^2\right], \end{aligned} \tag{4.26}$$

where

$$\xi'=n_\mu x_\mu', \quad \xi''=n_\mu x_\mu''. \tag{4.27}$$

The function $C(x', x'')$ is determined by the differential equations (2.38) and (2.39), in conjunction with Eq. (4.26). Thus,

$$\left[-i\partial_\mu'-eA_\mu(x')-f_{\mu\nu}(x'-x'')_\nu\frac{1}{\xi'-\xi''}\left(eA(\xi')-\frac{1}{\xi'-\xi''}\int_{\xi''}^{\xi'}d\xi eA(\xi)\right)\right]C(x', x'')=0. \tag{4.28}$$

The solution of this equation will be obtained in terms of a line integral which is independent of the integration path. Again choosing the path to be a straight line, we find simply

$$C(x', x'')=C\exp\left[ie\int_{x''}^{x'}dx_\mu A_\mu(x)\right]=C\Phi(x', x''), \tag{4.29}$$

in view of the antisymmetry of $f_{\mu\nu}$. It is evident that

$$C=-i(4\pi)^{-2}. \tag{4.30}$$

Only the behavior of the transformation function for $x_\mu'\simeq x_\mu''$ is of actual interest in applications to vacuum polarization phenomena. Now for $\xi'\simeq\xi''$,

$$\frac{1}{\xi'-\xi''}\int_{\xi''}^{\xi'}d\xi A^2(\xi)-\left[\frac{1}{\xi'-\xi''}\int_{\xi''}^{\xi'}d\xi A(\xi)\right]^2 \simeq\tfrac{1}{12}(\xi'-\xi'')^2F^2[\tfrac{1}{2}(\xi'+\xi'')], \tag{4.31}$$

and

$$\begin{aligned}(\xi'-\xi'')^2F^2&=(x'-x'')_\mu n_\mu F^2 n_\nu(x'-x'')_\nu\\&=-(x'-x'')_\mu F_{\mu\lambda}F_{\lambda\nu}(x'-x'')_\nu,\end{aligned} \tag{4.32}$$

according to Eqs. (4.1) and (4.4). Therefore, for $x_\mu'\simeq x_\mu''$

$$\begin{aligned}(x(s)'|x(0)'')&\simeq-i(4\pi)^{-2}\Phi(x', x'')s^{-2}\\&\times\exp[i\tfrac{1}{4}s^{-1}(x'-x'')_\mu(\delta_{\mu\nu}+\tfrac{1}{3}(es)^2F_{\mu\lambda}F_{\lambda\nu})(x'-x'')_\nu]\\&\times\exp(\tfrac{1}{2}ie\sigma_{\mu\nu}F_{\mu\nu}s),\end{aligned} \tag{4.33}$$

which is identical with the transformation function for a constant field, as simplified by the special characteristics of the field now under consideration; namely,

$$\mathfrak{F}=0, \quad \mathfrak{G}=0. \tag{4.34}$$

We can conclude, without further calculation, that the physical quantities characterizing the plane wave field, the components of the energy-momentum tensor $T_{\mu\nu}$, will be identical in form with those of a constant field that obeys Eq. (4.34). On referring to Eq. (3.57), we see that $T_{\mu\nu}$ for a plane wave is just that of the maxwell field, which may be simplified further to

$$T_{\mu\nu}=F_{\mu\lambda}F_{\nu\lambda}=n_\mu n_\nu F^2(\xi). \tag{4.35}$$

Thus, there are no nonlinear vacuum phenomena for a single plane wave, of arbitrary strength and spectral composition.

V. γ-DECAY OF NEUTRAL MESONS

In this section we shall apply the results of our proper-time method to compute the effective coupling between a zero spin neutral meson and the electromagnetic field, as produced by the polarization of the proton vacuum. This interaction manifests itself in a spontaneous decay of the neutral meson into two photons.

The lagrange function for a spinless neutral meson field, in scalar interaction with the proton-antiproton field, is given by

$$\mathfrak{L}=-\tfrac{1}{2}[(\partial_\nu\phi)^2+\mu^2\phi^2]-g\phi\tfrac{1}{2}[\bar{\psi}, \psi]. \tag{5.1}$$

To find an approximate expression for the resultant coupling between the neutral meson field and the electromagnetic field, we replace $\frac{1}{2}[\bar{\psi}, \psi]$ by its vacuum expectation value, calculated in the presence of a homogeneous electromagnetic field. The use of the latter to represent the photons emitted in the spontaneous neutral meson decay introduces a small error, which is measured by the square of the meson-proton mass ratio, $(\mu/M)^2\simeq 1/40$. On the other hand, by ignoring the effect of the meson field on the proton vacuum, we obtain only the initial approximation of a perturbation treatment. Now

$$\begin{aligned}\langle\tfrac{1}{2}[\bar{\psi}(x), \psi(x)]\rangle&=i\,\mathrm{tr}G(x, x)\\&=-M\int_0^\infty ds\exp(-iM^2s)\,\mathrm{tr}(x|U(s)|x)\\&=-\partial\mathfrak{L}^{(1)}(x)/\partial M,\end{aligned} \tag{5.2}$$

according to Eqs. (2.30) and (2.31). Thus, the effective lagrange function coupling term between the neutral meson and the electromagnetic field is given by

$$\mathfrak{L}'(x)=g\phi(x)\partial\mathfrak{L}^{(1)}(x)/\partial M, \tag{5.3}$$

which clearly also follows directly from the proton field equation of motion,

$$[\gamma(-i\partial-eA)+M+g\phi]\psi=0, \tag{5.4}$$

in the approximation which treats $\phi(x)$ as a weak, slowly varying, prescribed field. If we retain only the leading term in the expansion of $\mathfrak{L}^{(1)}$ for weak fields, Eq. (3.45), we have

$$\begin{aligned}\partial\mathfrak{L}^{(1)}/\partial M&\simeq(e^2/6\pi^2)M\int_0^\infty ds\exp(-M^2s)\mathfrak{F}\\&=(2\alpha/3\pi)(1/M)\mathfrak{F}.\end{aligned} \tag{5.5}$$

Therefore the effective coupling term is

$$\mathfrak{L}'=(\alpha/3\pi)(g/M)\phi(\mathbf{H}^2-\mathbf{E}^2), \tag{5.6}$$

which describes the decay of a stationary meson, into two parallel polarized photons, at the rate

$$1/\tau=(\alpha^2/144\pi^3)(g^2/\hbar c)(\mu/M)^2(\mu c^2/\hbar). \tag{5.7}$$

A pseudoscalar interaction between the spinless neutral meson field and the proton field is described by the term

$$g\phi(x)\tfrac{1}{2}[\bar{\psi}(x),\ \gamma_5\psi(x)] \tag{5.8}$$

in the lagrange function. For our purposes, this is replaced by

$$\begin{aligned}\mathcal{L}'(x)&=g\phi(x)\langle\tfrac{1}{2}[\bar{\psi}(x),\ \gamma_5\psi(x)]\rangle\\&=ig\phi(x)\ \mathrm{tr}\gamma_5 G(x,x)\\&=-g\phi(x)M\int_0^\infty ds\exp(-iM^2s)\\&\qquad\times\mathrm{tr}\gamma_5(x|U(s)|x).\end{aligned} \tag{5.9}$$

The transformation function (3.20), with $-is$ substituted for s, yields

$$\begin{aligned}\mathcal{L}'=-g\phi M(4\pi)^{-2}\int_0^\infty dss^{-2}\exp(-M^2s)e^{-l(s)}\\\times\mathrm{tr}\gamma_5\exp(\tfrac{1}{2}e\sigma Fs).\end{aligned} \tag{5.10}$$

Now, the eigenvalues of $\frac{1}{2}\sigma F$, as related to those of γ_5 by Eq. (3.31), give

$$\mathrm{tr}\gamma_5\exp(\tfrac{1}{2}e\sigma Fs)=-4\ \mathrm{Im}\cosh esX. \tag{5.11}$$

In view of Eq. (3.43), we obtain, without further approximation, simply

$$\begin{aligned}\mathcal{L}'&=g\phi(e^2/4\pi^2)M\int_0^\infty ds\exp(-M^2s)\mathcal{G}\\&=(\alpha/\pi)(g/M)\phi\mathbf{E}\cdot\mathbf{H}.\end{aligned} \tag{5.12}$$

This effective coupling term implies the decay of a stationary neutral meson, into two perpendicularly polarized photons, at the rate

$$1/\tau\text{-}(\alpha^2/64\pi^3)(g^2/\hbar c)(\mu/M)^2(\mu c^2/\hbar). \tag{5.13}$$

The pseudovector interaction term,

$$(g/2M)\partial_\mu\phi(x)(1/2i)[\bar{\psi}(x),\ \gamma_5\gamma_\mu\psi(x)], \tag{5.14}$$

is formally equivalent to (5.8) for the problem under discussion, in the approximation to which it is being treated. This is demonstrated by a partial integration, combined with the use of the Dirac equation (2.1). Yet it has been found difficult[3,7] to verify the equivalence in the actual results of calculation. Such discrepancies between formal and explicit calculations may be produced by insufficient attention to the limiting processes implicit in the formalism. We shall demonstrate that, with appropriate care, the proper equivalence between the pseudoscalar and pseudovector couplings is indeed exhibited.

The effective pseudovector interaction between the meson and electromagnetic field is given by

$$\begin{aligned}\mathcal{L}'(x)&=(g/2M)\partial_\mu\phi(x)\langle(1/2i)[\bar{\psi}(x),\ \gamma_5\gamma_\mu\psi(x)]\rangle\\&=(g/2M)\partial_\mu\phi(x)\ \mathrm{tr}\gamma_5\gamma_\mu G(x,x)\\&\to-(g/2M)\phi(x)\partial_\mu[\mathrm{tr}\gamma_5\gamma_\mu G(x,x)],\end{aligned} \tag{5.15}$$

where the last version represents the results of integrating by parts. We now remark that this derivative has the following meaning:

$$\begin{aligned}\partial_\mu[\mathrm{tr}\gamma_5\gamma_\mu G(x,x)]=\lim_{x',x''\to x}[(\partial_\mu'-ieA_\mu(x'))\\+(\partial_\mu''+ieA_\mu(x''))]\ \mathrm{tr}\gamma_5\gamma_\mu G(x',x''),\end{aligned} \tag{5.16}$$

in which the structure of the right side is dictated by the requirement that only gauge covariant quantities be employed. We shall verify that the straightforward evaluation of Eq. (5.16) yields the pseudoscalar coupling (5.12), without further difficulty.

According to Eq. (3.21)

$$\begin{aligned}&\mathrm{tr}\gamma_5\gamma_\mu G(x',x'')\\&=-i\ \mathrm{tr}\gamma_5\gamma_\mu\gamma_\nu\int_0^\infty ds\exp(-iM^2s)\\&\qquad\times(x(s)'|\Pi_\nu(s)|x(0)'')\\&=-i\ \mathrm{tr}\gamma_\nu\gamma_5\gamma_\mu\int_0^\infty ds\exp(-iM^2s)\\&\qquad\times(x(s)'|\Pi_\nu(0)|x(0)'').\end{aligned} \tag{5.17}$$

The result of averaging these two equivalent expressions is

$$\begin{aligned}&\mathrm{tr}\gamma_5\gamma_\mu G(x',x'')\\&=i\ \mathrm{tr}\gamma_5\int_0^\infty ds\exp(-iM^2s)\\&\qquad\times(x(s)'|\tfrac{1}{2}(\Pi_\nu(s)-\Pi_\nu(0))|x(0)'')\\&\quad-\mathrm{tr}\gamma_5\sigma_{\mu\nu}\int_0^\infty ds\exp(-iM^2s)\\&\qquad\times(x(s)'|\tfrac{1}{2}(\Pi_\nu(s)+\Pi_\nu(0))|x(0)'').\end{aligned} \tag{5.18}$$

We shall be content to evaluate Eq. (5.18) in the approximation of weak fields. On referring to Eqs. (3.4), (3.5), and (3.20), it is apparent that the leading term in this approximation is

$$\begin{aligned}&\mathrm{tr}\gamma_5\gamma_\mu G(x',x'')\\&=-(e/64\pi^2)\ \mathrm{tr}\gamma_5\sigma_{\mu\nu}\sigma_{\lambda\kappa}(x'-x'')_\nu F_{\lambda\kappa}\Phi(x',x'')\\&\quad\times\int_0^\infty dss^{-2}\exp(-iM^2s)\exp[i\tfrac{1}{4}(x'-x'')^2/s]\\&=(e/8\pi^2)F_{\mu\nu}{}^*(x'-x'')_\nu\Phi(x',x'')\\&\quad\times\int_0^\infty dss^{-2}\exp(-iM^2s)\exp[i\tfrac{1}{4}(x'-x'')^2/s],\end{aligned} \tag{5.19}$$

[7] J. Steinberger, Phys. Rev. 76, 1180 (1949).

with the aid of Eqs. (3.25) and (3.27). Since we are concerned with the behavior of this quantity only for $x'\simeq x''$, we may evaluate the proper time integral by an appropriate simplification. For $x'\simeq x''$,

$$\int_0^\infty dss^{-2}\exp(-iM^2s)\exp[i\tfrac{1}{4}(x'-x'')^2/s]$$

$$\simeq\int_0^\infty dss^{-2}\exp[i\tfrac{1}{4}(x'-x'')^2/s]$$

$$=\int_0^\infty d(s^{-1})\exp[i\tfrac{1}{4}(x'-x'')^2s^{-1}]$$

$$=4i/(x'-x'')^2. \tag{5.20}$$

Therefore,

$$\mathrm{tr}\gamma_5\gamma_\mu G(x',x'')\simeq(ie/2\pi^2)\Phi(x',x'')F_{\mu\nu}{}^*(x'-x'')_\nu(x'-x'')^{-2}. \tag{5.21}$$

To obtain the quantity of actual interest, Eq. (5.16), we observe that

$$[(\partial_\mu'-ieA_\mu(x'))+(\partial_\mu''+ieA_\mu(x''))]\Phi(x',x'')\times F_{\mu\nu}{}^*(x'-x'')_\nu(x'-x'')^{-2}=ie\Phi(x',x'')F_{\mu\nu}{}^*(x'-x'')_\nu F_{\mu\lambda}(x'-x'')_\lambda(x'-x'')^{-2}, \tag{5.22}$$

according to Eqs. (3.15) and (3.16). But, in view of Eq. (3.35),

$$F_{\mu\nu}{}^*(x'-x'')_\nu F_{\mu\lambda}(x'-x'')_\lambda=\mathcal{G}(x'-x'')^2, \tag{5.23}$$

and

$$\partial_\mu[\mathrm{tr}\gamma_5\gamma_\mu G(x,x)]=-(e^2/2\pi^2)\mathcal{G}\lim_{x'\to x''}\Phi(x',x'')=-(2\alpha/\pi)\mathcal{G}. \tag{5.24}$$

Thus, Eq. (5.15) yields

$$\mathcal{L}'=(\alpha/\pi)(g/M)\phi\mathbf{E}\cdot\mathbf{H}, \tag{5.25}$$

in complete agreement with Eq. (5.12).

VI. PERTURBATION THEORY

We shall now discuss the approximate evaluation of

$$W^{(1)}=i\tfrac{1}{2}\int_0^\infty dss^{-1}\exp(-im^2s)\,\mathrm{Tr}U(s), \tag{6.1}$$

by an expansion in powers of eA_μ and $eF_{\mu\nu}$. For this purpose, we write

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_1, \tag{6.2}$$

where

$$\mathcal{H}_0=p^2 \tag{6.3}$$

and

$$\mathcal{H}_1=-e(pA+Ap)-\tfrac{1}{2}e\sigma F+e^2A^2. \tag{6.4}$$

To obtain the expansion of $\mathrm{Tr}U(s)$ in powers of $\mathcal{H}_1$, we observe that $U(s)$ obeys the differential equation

$$i\partial_sU(s)=(\mathcal{H}_0+\mathcal{H}_1)U(s). \tag{6.5}$$

The related operator

$$V(s)=U_0{}^{-1}(s)U(s), \tag{6.6}$$

where

$$U_0(s)=\exp(-i\mathcal{H}_0s), \tag{6.7}$$

is determined by

$$i\partial_sV(s)=U_0{}^{-1}(s)\mathcal{H}_1U_0(s)V(s) \tag{6.8}$$

and

$$V(0)=1. \tag{6.9}$$

One can combine Eqs. (6.8) and (6.9) in the integral equation

$$V(s)=1-i\int_0^s ds'U_0{}^{-1}(s')\mathcal{H}_1U_0(s')V(s'), \tag{6.10}$$

and construct the solution by iteration:

$$V(s)=1-i\int_0^s ds'U_0{}^{-1}(s')\mathcal{H}_1U_0(s')+(-i)^2\int_0^s ds'U_0{}^{-1}(s')\mathcal{H}_1U_0(s')\times\int_0^{s'}ds''U_0{}^{-1}(s'')\mathcal{H}_1U_0(s'')+\cdots. \tag{6.11}$$

On introducing new variables of integration, $u_1, u_2, \cdots$, according to

$$s'=su_1,\quad s''=s'u_2,\ \cdots, \tag{6.12}$$

we obtain the expansion

$$U(s)=\exp(-i\mathcal{H}s)$$
$$=U_0(s)+(-is)\int_0^1 du_1U_0((1-u_1)s)\mathcal{H}_1U_0(u_1s)+\cdots$$
$$+(-is)^n\int_0^1 u_1{}^{n-1}du_1\cdots\int_0^1 du_n\times U_0((1-u_1)s)\mathcal{H}_1U_0(u_1(1-u_1)s)\cdots\times U_0(u_1\cdots u_{n-1}(1-u_n)s)\times\mathcal{H}_1U_0(u_1\cdots u_ns)+\cdots. \tag{6.13}$$

Instead of taking the trace of this expression directly, which would involve further simplification, we remark that

$$\mathrm{Tr}U(s)-\mathrm{Tr}U_0(s)=-is\int_0^1 d\lambda\,\mathrm{Tr}[\mathcal{H}_1\exp(-i(\mathcal{H}_0+\lambda\mathcal{H}_1)s)] \tag{6.14}$$

and insert the expansion (6.13) for $\exp[-i(\mathcal{H}_0+\lambda\mathcal{H}_1)s]$.

Thus,

$$\mathrm{Tr}U(s)=\mathrm{Tr}U_0(s)+(-is)\,\mathrm{Tr}[\mathcal{H}_1U_0(s)]$$
$$+\tfrac{1}{2}(-is)^2\int_0^1 du_1\,\mathrm{Tr}[\mathcal{H}_1U_0((1-u_1)s)\mathcal{H}_1U_0(u_1s)]+\cdots$$
$$+\frac{(-is)^{n+1}}{n+1}\int_0^1 u_1^{n-1}du_1\cdots\int_0^1 du_n$$
$$\times\mathrm{Tr}[\mathcal{H}_1U_0((1-u_1)s)\mathcal{H}_1\cdots$$
$$\times\mathcal{H}_1U_0(u_1\cdots u_ns)]+\cdots. \quad (6.15)$$

We shall retain only the first nonvanishing field dependent terms in this expansion:

$$W^{(1)}=\tfrac{1}{2}ie^2\int_0^\infty dss^{-1}\exp(-im^2s)$$
$$\times\Big\{-is\,\mathrm{Tr}[A^2\exp(-ip^2s)]$$
$$+\tfrac{1}{2}(-is)^2\int_{-1}^1\tfrac{1}{2}dv\,\mathrm{Tr}[(pA+Ap)\exp(-ip^2\tfrac{1}{2}(1-v)s)$$
$$\times(pA+Ap)\exp(-ip^2\tfrac{1}{2}(1+v)s)]$$
$$+\tfrac{1}{2}(-is)^2\int_{-1}^1\tfrac{1}{2}dv\,\mathrm{Tr}[\tfrac{1}{2}\sigma F\exp(-ip^2\tfrac{1}{2}(1-v)s)$$
$$\times\tfrac{1}{2}\sigma F\exp(-ip^2\tfrac{1}{2}(1+v)s)]\Big\}. \quad (6.16)$$

For convenience, the variable u_1 has been replaced by $\frac{1}{2}(1+v)$. The evaluation of these traces is naturally performed in a momentum representation. The matrix elements of the coordinate dependent field quantities depend only on momentum differences,

$$(p+\tfrac{1}{2}k|A_\mu|p-\tfrac{1}{2}k)=(2\pi)^{-4}\int(dx)e^{-ikx}A_\mu(x)$$
$$\equiv(2\pi)^{-2}A_\mu(k), \quad (6.17)$$

and

$$(p|A_\mu{}^2|p)=(2\pi)^{-4}\int(dx)A_\mu{}^2(x)$$
$$=(2\pi)^{-4}\int(dk)A_\mu(-k)A_\mu(k). \quad (6.18)$$

Therefore

$$W^{(1)}=\frac{2ie^2}{(2\pi)^4}\int_0^\infty dss^{-1}\exp(-im^2s)\Big\{-is\int(dk)$$
$$\times A_\mu(-k)A_\mu(k)\int(dp)\exp(-ip^2s)$$
$$+\tfrac{1}{2}(-is)^2\int_{-1}^1\tfrac{1}{2}dv\int(dk)\int(dp)2p_\mu A_\mu(-k)$$
$$\times\exp(-i(p+\tfrac{1}{2}k)^2\tfrac{1}{2}(1-v)s]$$
$$\times 2p_\nu A_\nu(k)\exp[-i(p-\tfrac{1}{2}k)^2\tfrac{1}{2}(1+v)s]$$
$$+\tfrac{1}{2}(-is)^2\int_{-1}^1\tfrac{1}{2}dv\int(dk)\int(dp)\ \tfrac{1}{4}\ \mathrm{tr}\tfrac{1}{2}\sigma F(-k)$$
$$\times\exp[-i(p+\tfrac{1}{2}k)^2\tfrac{1}{2}(1-v)s]\tfrac{1}{2}\sigma F(k)$$
$$\times\exp[-i(p-\tfrac{1}{2}k)^2\tfrac{1}{2}(1+v)s]\Big\}. \quad (6.19)$$

We thus encounter the elementary integrals

$$\int(dp\exp(-ip^2s)=-i\pi^2s^{-2}, \quad (6.20)$$

$$\int(dp)\exp[-i(p^2+(k^2/4))s+ipkvs]$$
$$=-i\pi^2s^{-2}\exp[-i(k^2/4)(1-v^2)s], \quad (6.21)$$

and

$$\int(dp)p_\mu p_\nu\exp[-i(p^2+(k^2/4))s+ipkvs]$$
$$=-\exp(-i\tfrac{1}{4}k^2s)(vs)^{-2}(\partial/\partial k_\mu)(\partial/\partial k_\nu)$$
$$\times\int(dp)\exp(-ip^2s+ipkvs)$$
$$=-i\pi^2s^{-2}(-i\tfrac{1}{2}s^{-1}\delta_{\mu\nu}+\tfrac{1}{4}v^2k_\mu k_\nu)$$
$$\times\exp[-i\tfrac{1}{4}k^2(1-v^2)s]. \quad (6.22)$$

It is convenient to replace the $\delta_{\mu\nu}$ term of the last integral by an expression which is equivalent to it in virtue of the integration with respect to v. Now

$$\int_{-1}^1\tfrac{1}{2}dv\exp[-i\tfrac{1}{4}k^2(1-v^2)s]$$
$$=1-is\tfrac{1}{2}k^2\int_{-1}^1\tfrac{1}{2}dvv^2\exp[-i\tfrac{1}{4}k^2(1-v^2)s], \quad (6.23)$$

so that, effectively

$$\int (dp)p_\mu p_\nu \exp[-i(p^2+\tfrac{1}{4}k^2)s+ipkvs]$$
$$=-\tfrac{1}{2}\pi^2 s^{-3}\delta_{\mu\nu}-i\pi^2 s^{-2}\tfrac{1}{4}v^2(k_\mu k_\nu-\delta_{\mu\nu}k^2)$$
$$\times\exp[-i\tfrac{1}{4}k^2(1-v^2)s]. \quad (6.24)$$

On inserting the values of the various integrals, and noticing that

$$(k_\mu k_\nu-\delta_{\mu\nu}k^2)A_\mu(-k)A_\nu(k)=-\tfrac{1}{2}F_{\mu\nu}(-k)F_{\mu\nu}(k), \quad (6.25)$$

we obtain immediately the gauge invariant form (with $s\rightarrow -is$)

$$W^{(1)}=-\frac{e^2}{4\pi^2}\int (dk)\tfrac{1}{2}F_{\mu\nu}(-k)F_{\mu\nu}(k)\int_0^1 dv(1-v^2)$$
$$\times\int_0^\infty dss^{-1}\exp\{-[m^2+\tfrac{1}{4}k^2(1-v^2)]s\}. \quad (6.26)$$

This has been achieved without any special device, other than that of reserving the proper-time integration to the last.

A significant separation of terms is produced by a partial integration with respect to v, according to

$$\int_0^1 dv(1-v^2)\int_0^\infty dss^{-1}\exp\{-[m^2+\tfrac{1}{4}k^2(1-v^2)]s\}$$
$$=\tfrac{2}{3}\int_0^\infty dss^{-1}\exp(-m^2s)-\tfrac{1}{2}k^2\int_0^1 dv(v^2-\tfrac{1}{3}v^4)$$
$$\times\int_0^\infty ds\exp\{-[m^2+\tfrac{1}{4}k^2(1-v^2)]s\}. \quad (6.27)$$

Adding the action integral of the maxwell field, which is expressed in momentum space by

$$W^{(0)}=-\int (dk)\tfrac{1}{4}F_{\mu\nu}(-k)F_{\mu\nu}(k), \quad (6.28)$$

we obtain the modified action integral,

$$W=-\left[1+\frac{e^2}{12\pi^2}\int_0^\infty dss^{-1}\exp(-m^2s)\right]$$
$$\times\int (dk)\tfrac{1}{4}F_{\mu\nu}(-k)F_{\mu\nu}(k)$$
$$+\frac{e^2}{4\pi^2}\int (dk)\tfrac{1}{4}F_{\mu\nu}(-k)F_{\mu\nu}(k)k^2$$
$$\times\int_0^1 dv\frac{v^2(1-\tfrac{1}{3}v^2)}{m^2+\tfrac{1}{4}k^2(1-v^2)}. \quad (6.29)$$

The field strength and charge renormalization contained in Eq. (3.48) then produces the finite gauge invariant result,[8]

$$W=-\int (dk)\tfrac{1}{4}F_{\mu\nu}(-k)F_{\mu\nu}(k)$$
$$\times\left[1-\frac{\alpha}{4\pi}\frac{k^2}{m^2}\int_0^1 dv\frac{v^2(1-\tfrac{1}{3}v^2)}{1+(k^2/4m^2)(1-v^2)}\right]. \quad (6.30)$$

The restriction which we have thus far imposed, that no actual pair creation occurs, corresponds to the requirement that $1+(k^2/4m^2)(1-v^2)$ never vanishes. This will be true if $-k^2<4m^2$, for all k_μ contained in the fourier representation of the field. Indeed, it is evident from energy and momentum considerations that to produce a pair by the absorption of a single quantum the momentum vector of the latter must be time-like and must have a magnitude exceeding $2m$. We shall now simply remark that, to extend our results to pair-producing fields, it is merely necessary to add an infinitesimal negative imaginary constant to the denominator of Eq. (6.30) and interpret the positive imaginary contribution to W thus obtained with the statement that

$$|e^{iW}|^2=e^{-2\,\mathrm{Im}W} \quad (6.31)$$

represents the probability that no actual pair creation occurs during the history of the field. The infinitesimal imaginary constant, as employed in

$$\lim_{\epsilon\rightarrow+0}\frac{1}{x-i\epsilon}=P\frac{1}{x}+\pi i\delta(x), \quad (6.32)$$

represents a familiar device for dealing with real processes. We obtain from Eq. (6.30) that

$$2\,\mathrm{Im}W=\tfrac{1}{2}\alpha\int (dk)\tfrac{1}{4}F_{\mu\nu}(-k)F_{\mu\nu}(k)\frac{k^2}{m^2}\int^1 dvv^2$$
$$\times\left(1-\frac{v^2}{3}\right)\delta\left[1+\frac{k^2}{4m^2}(1-v^2)\right]$$
$$=\alpha\int_{-k^2>4m^2}(dk)(-\tfrac{1}{4})F_{\mu\nu}(-k)F_{\mu\nu}(k)$$
$$\times\left(1-\frac{4m^2}{(-k^2)}\right)^{\frac{1}{2}}\frac{1}{3}\left(2+\frac{4m^2}{(-k^2)}\right). \quad (6.33)$$

For the weak fields that are being considered, Eq. (6.33) is just the probability that a pair is created by the field. It should be noticed, incidentally, that

$$-\tfrac{1}{4}F_{\mu\nu}(-k)F_{\mu\nu}(k)=\tfrac{1}{2}[|\mathbf{E}(k)|^2-|\mathbf{H}(k)|^2] \quad (6.34)$$

[8] The corresponding result for a spin zero charged field is obtained by omitting the spin term of Eq. (6.19), and multiplying the remainder with $(-\tfrac{1}{2})$. This effectively substitutes $\tfrac{1}{6}v^4$ for $(v^2-\tfrac{1}{3}v^4)$, in Eq. (6.30).

is actually positive for a pair-generating field. This follows, for example, from the vanishing of the magnetic field in the special coordinate system where k_μ has only a temporal component.

An alternative version of Eq. (6.33) is obtained by replacing the field with the current required to generate this field, according to the maxwell equations

$$\begin{aligned} ik_\mu F_{\mu\nu}(k) &= -J_\nu(k), \\ k_\mu F_{\nu\lambda}(k)+k_\nu F_{\lambda\mu}(k)+k_\lambda F_{\mu\nu}(k) &= 0. \end{aligned} \tag{6.35}$$

Now

$$\begin{aligned} k_\lambda{}^2 F_{\mu\nu}(-k)F_{\mu\nu}(k) &= 2k_\nu F_{\nu\mu}(-k)k_\lambda F_{\lambda\mu}(k) \\ &= 2J_\mu(-k)J_\mu(k), \end{aligned} \tag{6.36}$$

so that[9]

$$2\,\mathrm{Im}W = (\alpha/8m^2)\int_{-k^2>4m^2}(dk)J_\mu(-k)J_\mu(k) \times(1-\gamma)^{\frac{1}{2}}\gamma\tfrac{1}{3}(2+\gamma), \tag{6.37}$$

where

$$\gamma = 4m^2/(-k^2). \tag{6.38}$$

It is now appropriate to notice that the integral (3.49), representing the lagrange function for a uniform field, has singularities, unless $\mathcal{G}=0$, $\mathfrak{F}>0$, corresponding to a pure magnetic field in an appropriate coordinate system. This is the analytic expression of the fact that pairs are created by a uniform electric field. In particular, for $\mathcal{G}=0$, $-2\mathfrak{F}=\mathcal{E}^2>0$, which invariantly characterizes a pure electric field, the lagrange function proper time integral,

$$\mathfrak{L} = \tfrac{1}{2}\mathcal{E}^2-(1/8\pi^2)\int_0^\infty ds s^{-3}\exp(-m^2 s) \times[e\mathcal{E}s\cot(e\mathcal{E}s)-1+\tfrac{1}{3}(e\mathcal{E}s)^2], \tag{6.39}$$

has singularities at

$$s=s_n=n\pi/e\mathcal{E},\quad n=1,2,\cdots. \tag{6.40}$$

If the integration path is considered to lie above the real axis, which is an alternative version of the device embodied in Eq. (6.32), we obtain a positive imaginary contribution to $\mathfrak{L}$,

$$\begin{aligned} 2\,\mathrm{Im}\mathfrak{L} &= \frac{1}{4\pi}\sum_{n=1}^\infty s_n{}^{-2}\exp(-m^2 s_n) \\ &= \frac{\alpha^2}{\pi^2}\mathcal{E}^2\sum_{n=1}^\infty n^{-2}\exp\left(\frac{-n\pi m^2}{e\mathcal{E}}\right). \end{aligned} \tag{6.41}$$

This is the probability, per unit time and per unit volume, that a pair is created by the constant electric field.

We must now consider, in the framework of this special problem, the connection between the proper time method and that of "invariant regularization." The vacuum polarization addition to the action integral has the general structure

$$W^{(1)} = \int(dk)A_\mu(-k)K_{\mu\nu}(k,m^2)A_\nu(k). \tag{6.42}$$

The proper-time technique yields the coefficient $K_{\mu\nu}(k,m^2)$ in the form

$$K_{\mu\nu}(k,m^2)]_p = \int_0^\infty ds\exp(-im^2 s)K_{\mu\nu}(k,s), \tag{6.43}$$

where $K_{\mu\nu}(k,s)$ is a finite, gauge invariant quantity; infinities appear only in the final stage of integrating s to the origin. In effect, this method substitutes a lower limit, s_0, in the proper time integral and reserves the limit, $s_0\rightarrow 0$, to the end of the calculation. If, on the contrary, the proper-time technique is not explicitly introduced, $K_{\mu\nu}(k,m^2)$ will be represented by divergent integrals which lead, in general, to non-gauge invariant results. The regulator technique avoids the difficulty by introducing a suitable weighted integration with respect to the square of the proper mass, thus substituting for $K_{\mu\nu}(k,m^2)$, the quantity

$$K_{\mu\nu}(k,m^2)]_R = \int_{-\infty}^\infty d\kappa\rho(\kappa)K_{\mu\nu}(k,\kappa). \tag{6.44}$$

The "regulator" $\rho(\kappa)$ must reduce to $\delta(\kappa-m^2)$, in an appropriate limit, and will produce gauge invariant results in this problem if the following integral conditions are satisfied:

$$\int_{-\infty}^\infty d\kappa\rho(\kappa)=0,\quad \int_{-\infty}^\infty d\kappa\kappa\rho(\kappa)=0. \tag{6.45}$$

Expressed in terms of the fourier transformed quantities,

$$\begin{aligned} R(s) &= \int_{-\infty}^\infty d\kappa e^{-i\kappa s}\rho(\kappa), \\ K_{\mu\nu}(k,s) &= (1/2\pi)\int_{-\infty}^\infty d\kappa e^{i\kappa s}K_{\mu\nu}(k,\kappa), \end{aligned} \tag{6.46}$$

we have

$$K_{\mu\nu}(k,m^2)]_R = \int_{-\infty}^\infty dsR(s)K_{\mu\nu}(k,s), \tag{6.47}$$

while the conditions on $\rho(\kappa)$ appear as

$$R(0)=0,\quad R'(0)=0,\quad R(s)\rightarrow\exp(-im^2 s). \tag{6.48}$$

Now observe that the proper time method yields $K_{\mu\nu}(k,m^2)$ in the form (6.47), with

$$K_{\mu\nu}(k,s)=0,\quad s<0, \tag{6.49}$$

and

$$\begin{aligned} R(s) &= \exp(-im^2 s), & s&>s_0 \\ &= 0, & s&<s_0. \end{aligned} \tag{6.50}$$

[9] A simple example, to which this formula may be applied, is the creation of a pair in a nuclear $j=0\rightarrow 0$ transition. J. R. Oppenheimer and J. Schwinger, Phys. Rev. 56, 1066 (1939).

This $R(s)$, and all its derivatives, vanishes at the origin, thus satisfying the regulator conditions as $s_0\to 0$. It appears, then, that regularization is a procedure for inserting, into a calculation that does not employ it, enough of the structure provided by the proper time representation to ensure gauge invariant results.

APPENDIX A

It is our purpose here to use the proper time equations of motion (2.36) for the computation of the current induced in the vacuum by a weak, arbitrarily varying field:

$$F_{\mu\nu}(x)=[1/(2\pi)^2]\int(dk)e^{ikx}F_{\mu\nu}(k). \tag{A.1}$$

In the absence of a field, the equations of motions are solved by

$$\Pi_\mu(s)=\Pi_\mu(0),\quad x_\mu(s)=x_\mu(0)+2\Pi_\mu(0)s. \tag{A.2}$$

As a first approximation for weak fields, we accordingly write

$$d\Pi_\mu(s)/ds=[e/(2\pi)^2]\int(dk)F_{\mu\nu}(k)\{e^{ik(x(0)+2\Pi(0)s)},\Pi_\nu(0)\}$$
$$+e/(2\pi)^2\int(dk)ik_\mu\tfrac{1}{2}\sigma_{\lambda\nu}F_{\lambda\nu}(k)e^{ik(x(0)+2\Pi(0)s)}. \tag{A.3}$$

On integrating with respect to s, one obtains

$$\Pi_\mu(s)-\Pi_\mu(0)=[e/(2\pi)^2]\int(dk)F_{\mu\nu}(k)\times\int_0^s ds'\{e^{ik(x(0)+2\Pi(0)s')},\Pi_\nu(0)\}$$
$$+e/(2\pi)^2\int(dk)ik_\mu\tfrac{1}{2}\sigma_{\lambda\nu}F_{\lambda\nu}(k)\times\int_0^s ds'e^{ik(x(0)+2\Pi(0)s')}. \tag{A.4}$$

A second integration yields

$$\frac{x_\mu(s)-x_\mu(0)}{2s}=\Pi_\mu(0)+e/(2\pi)^2\int(dk)F_{\mu\nu}(k)\times\int_0^s ds'(1-s'/s)\{e^{ik(x(0)+2\Pi(0)s')},\Pi_\nu(0)\}$$
$$+\frac{e}{(2\pi)^2}\int(dk)ik_\mu\tfrac{1}{2}\sigma F(k)\times\int_0^s (ds'\left(1-\frac{s'}{s}\right)e^{ik(x(0)+2\Pi(0)s')}, \tag{A.5}$$

and therefore

$$\tfrac{1}{2}(\Pi_\mu(s)+\Pi_\mu(0)$$
$$=\frac{x_\mu(s)-x_\mu(0)}{2s}+\frac{e}{(2\pi)^2}\int(dk)F_{\mu\nu}(k)\times\int_0^s ds'\left(\frac{s'}{s}-\tfrac{1}{2}\right)\{e^{ik(x(0)+2\Pi(0)s')},\Pi_\nu(0)\}$$
$$+\frac{e}{(2\pi)^2}\int(dk)ik_\mu\tfrac{1}{2}\sigma F(k)\times\int_0^s ds'\left(\frac{s'}{s}-\tfrac{1}{2}\right)e^{ik(x(0)+2\Pi(0)s')}. \tag{A.6}$$

The induced current is equivalently expressed by

$$\langle j_\mu(x)\rangle=e\,\mathrm{tr}\gamma_\mu\left(x|(\gamma\Pi-m)\int_0^\infty ds\exp(-im^2s)U(s)|x\right)$$
$$=e\int_0^\infty ds\exp(-im^2s)\,\mathrm{tr}\gamma_\mu\gamma_\nu(x(s)'|\Pi_\nu(s)|x(0)'')]_{x',x''\to x}, \tag{A.7}$$

and

$$\langle j_\mu(x)\rangle=e\,\mathrm{tr}\gamma_\mu\left(x\left|\int_0^\infty ds\exp(-im^2s)U(s)(\gamma\Pi-m)\right|x\right)$$
$$=e\int_0^\infty ds\exp(-im^2s)\,\mathrm{tr}\gamma_\nu\gamma_\mu(x(s)'|\Pi_\nu(0)|x(0)'')]_{x',x''\to x}. \tag{A.8}$$

On averaging the two forms, we find that

$$\langle j_\mu(x)\rangle=-e\int_0^\infty ds\exp(-im^2s)\times\mathrm{tr}(x(s)'|\tfrac{1}{2}(\Pi_\mu(s)+\Pi_\mu(0))|x(0)'')]_{x',x''\to x}$$
$$-ie\int_0^\infty ds\exp(-im^2s)\times\mathrm{tr}\sigma_{\mu\nu}(x(s)'|\tfrac{1}{2}(\Pi_\nu(s)-\Pi_\nu(0))|x(0)'')]_{x',x''\to x}. \tag{A.9}$$

It may be noted here that no current exists in the absence of a field, since

$$\lim_{x'-x''\to\pm 0}(x(s)'|x_\mu(s)-x_\mu(0)|x(0)'')=0, \tag{A.10}$$

and, therefore, only the transformation function in the absence of a field is required for the first-order evaluation of Eq. (A.9). Now

$$\mathrm{tr}\sigma_{\mu\nu}(x(s)'|\tfrac{1}{2}(\Pi_\nu(s)-\Pi_\nu(0))|x(0)'')$$
$$=[2e/(2\pi)^2]\int(dk)(\partial F_{\mu\nu}/\partial x_\nu)(k)s\int_{-1}^1\tfrac{1}{2}dv\times(x(s)'|\exp[i(kx(s)\tfrac{1}{2}(1+v)+kx(0)\tfrac{1}{2}(1-v)]|x(0)''), \tag{A.11}$$

in which the variable s' has been replaced by v, according to

$$s'=s(1+v)/2. \tag{A.12}$$

The operators $kx(s)$ and $kx(0)$ do not commute:

$$[kx(s),kx(0)]=2s[k\Pi(0),kx(0)]=-2isk^2. \tag{A.13}$$

We may, however, employ the easily established theorem,

$$e^{A+B}=e^Ae^Be^{-\frac{1}{2}[A,B]}, \tag{A.14}$$

for operators A and B that commute with their commutator $[A,B]$. Thus,

$$\exp[i(kx(s)\tfrac{1}{2}(1+v)+kx(0)\tfrac{1}{2}(1-v))]$$
$$=\exp[ikx(s)\tfrac{1}{2}(1+v)]\exp[ikx(0)\tfrac{1}{2}(1-v)]\times\exp[-ik^2\tfrac{1}{4}(1-v^2)s], \tag{A.15}$$

and

$$\mathrm{tr}\sigma_{\mu\nu}(x(s)'|\tfrac{1}{2}(\Pi_\nu(s)-\Pi_\nu(0))|x(0)'')]_{x',x''\to x}$$
$$=[2e/(2\pi)^2]\int(dk)e^{ikx}(\partial F_{\mu\nu}/\partial x_\nu)(k)s\int_{-1}^1\tfrac{1}{2}dv\times\exp[-ik^2\tfrac{1}{4}(1-v^2)s](-i)/(4\pi)^2s^2. \tag{A.16}$$

A similar treatment applies to

$$\mathrm{tr}(x(s)'|\tfrac{1}{2}(\Pi_\nu(s)+\Pi_\nu(0))|x(0)'')]_{x',x''\to x}$$
$$=\frac{2e}{(2\pi)^2}\int(dk)F_{\mu\nu}(k)s\int_{-1}^1\tfrac{1}{2}dvv(x(s')'|\{\exp[i(kx(s)\tfrac{1}{2}(1+v)+kx(0)\tfrac{1}{2}(1-v))],(x_\nu(s)-x_\nu(0))/2s\}|x'(0))]_{x',x''\to x}. \tag{A.17}$$

With the aid of the commutation relations,

$$\begin{gathered}[e^{ikx(0)\frac{1}{2}(1-v)},x_\nu(s)]=-k_\nu(1-v)se^{ikx(0)\frac{1}{2}(1-v)},\\ [e^{ikx(s)\frac{1}{2}(1+v)},x_\nu(0)]=k_\nu(1+v)se^{ikx(s)\frac{1}{2}(1+v)},\end{gathered} \tag{A.18}$$

this reduces to

$$\mathrm{tr}(x(s)'|\tfrac{1}{2}(\Pi_\mu(s)+\Pi_\mu(0))|x(0)'')]_{x',x''\to x}$$
$$=-[2ie/(2\pi)^2]\int(dk)e^{ikx}(\partial F_{\mu\nu}/\partial x_\nu)(k)s\int_{-1}^1\tfrac{1}{2}dvv^2\times\exp[-ik^2\tfrac{1}{4}(1-v^2)s](-i)/(4\pi)^2s^2. \tag{A.19}$$

We have thus obtained

$$\langle j_\mu(x)\rangle=-(\alpha/2\pi)(2\pi)^{-2}\int(dk)e^{ikx}(\partial F_{\mu\nu}/\partial x_\nu)(k)\int_0^1 dv(1-v^2)\times\int_0^\infty dss^{-1}\exp\{-[m^2+\tfrac{1}{4}k^2(1-v^2)]s\}, \tag{A.20}$$

in which the substitution $s\to -is$ has again been introduced. This is precisely the current derived from the action integral $W^{(1)}$ of Eq. (6.26), and further discussion proceeds as in Sec. VI.

APPENDIX B

An electron in interaction with its proper radiation field, and an external field, is described by the modified Dirac equation,[10]

$$\gamma_\mu(-i\partial_\mu - eA_\mu(x))\psi(x) + \int (dx')M(x,x')\psi(x') = 0. \quad \text{(B.1)}$$

To the second order in e, the mass operator, $M(x,x')$, is given by

$$M(x,x') = m_0\delta(x-x') + ie^2\gamma_\mu G(x,x')\gamma_\mu D_+(x-x'). \quad \text{(B.2)}$$

Here $G(x,x')$ is the Green's function of the Dirac equation in the external field, and $D_+(x-x')$ is a photon Green's function, expressed by

$$D_+(x-x') = (4\pi)^{-2}\int_0^\infty dt\, t^{-2}\exp[i\tfrac{1}{4}(x-x')^2/t]. \quad \text{(B.3)}$$

We shall suppose the external field to be weak and uniform. Under these conditions, the transformation function $(x(s)|x(0)')$, involved in the construction of $G(x,x')$, may be approximated by

$$(x(s)|x(0)') \simeq -i(4\pi)^{-2}\Phi(x,x')s^{-2} \times \exp[i\tfrac{1}{4}(x-x')^2/s]\exp(i\tfrac{1}{2}e\sigma F); \quad \text{(B.4)}$$

that is, terms linear in the field strengths enter only through the Dirac spin magnetic moment. The corresponding simplification of the Green's function, obtain by averaging the two equivalent forms in Eq. (3.21), is

$$G(x,x') \simeq (4\pi)^{-2}\Phi(x,x')\int_0^\infty ds\, s^{-2}\exp(-im^2 s) \times \exp[i\tfrac{1}{4}(x-x')^2/s]\tfrac{1}{2}\left\{\frac{-\gamma(x-x')}{2s}+m,\ \exp(i\tfrac{1}{2}e\sigma F)\right\}. \quad \text{(B.5)}$$

The mass operator is thus approximately represented by

$$M(x,x') = m_0\delta(x-x') + [ie^2/(4\pi)^4]\Phi(x,x')\int_0^\infty ds\, s^{-2}\int_0^\infty dt\, t^{-2} \times \exp(-im^2 s)\exp\left[i\tfrac{1}{4}(x-x')^2\left(\frac{1}{s}+\frac{1}{t}\right)\right] \times \gamma_\lambda\tfrac{1}{2}\left\{\frac{-\gamma(x-x')}{2s}+m,\ \exp(i\tfrac{1}{2}e\sigma F)\right\}\gamma_\lambda, \quad \text{(B.6)}$$

or

$$M(x,x') = m_0\delta(x-x') + [ie^2/(4\pi)^4]\Phi(x,x') \times \int_0^\infty ds\, s^{-2}\exp(-im^2 s)\int_0^s dw\, w^{-2}\exp[i\tfrac{1}{4}(x-x')^2/w] \times [-4m - s^{-1}\gamma(x-x') + \tfrac{1}{2}i\{\gamma(x-x'),\ \tfrac{1}{2}e\sigma F\}], \quad \text{(B.7)}$$

in which we have replaced t by the variable w,

$$w^{-1} = s^{-1} + t^{-1}, \quad \text{(B.8)}$$

and employed properties of the Dirac matrices, notably

$$\gamma_\lambda\sigma_{\mu\nu}\gamma_\lambda = 0. \quad \text{(B.9)}$$

We shall also write

$$\begin{aligned}(x-x')_\mu\Phi(x,x')\exp[i\tfrac{1}{4}(x-x')^2/w] &= 2w(-i\partial_\mu - eA_\mu(x) - \tfrac{1}{2}eF_{\mu\nu}(x-x')_\nu)\Phi(x,x')\exp[i\tfrac{1}{4}(x-x')^2/w]\\ &\simeq [2w(-i\partial_\mu - eA_\mu(x)) - 2w^2 eF_{\mu\nu}(-i\partial_\nu - eA_\nu(x))] \times \Phi(x,x')\exp[i\tfrac{1}{4}(x-x')^2/w],\end{aligned} \quad \text{(B.10)}$$

which gives

$$\begin{aligned}M(x,x') = m_0\delta(x-x') + [e^2/(4\pi)^2]\int_0^\infty ds\, s^{-2}\exp(-im^2 s) &\times \int_0^s dw[2m(2-w/s) + (2w/s)(\gamma(-i\partial - eA)+m)\\ &-2mw(1-w/s)i\tfrac{1}{2}e\sigma F - iw(1+w/s)\\ &\times\{\gamma(-i\partial - eA)+m,\ \tfrac{1}{2}e\sigma F\}](x(w)|x(0)'),\end{aligned} \quad \text{(B.11)}$$

in virtue of the relation

$$[\gamma(-i\partial - eA),\ \tfrac{1}{2}\sigma F] = 2i\gamma F(-i\partial - eA). \quad \text{(B.12)}$$

We now introduce a perturbation procedure in which the mass operator assumes the role customarily played by the energy. To evaluate $\int(dx')M(x,x')\psi(x')$, we replace $\psi(x')$ by the unperturbed wave function, a solution of the Dirac equation associated with the mass m (we need not distinguish, to this approximation, between the actual mass m and the mechanical mass m_0). The x' integration can be effected immediately,

$$\int (x(w)|x(0)')(dx')\psi(x') = \int (x|U(w)|x')(dx')\psi(x') = \exp(im^2 w)\psi(x), \quad \text{(B.13)}$$

since $\psi(x)$ is an eigenfunction of $\mathcal{H}$, with the eigenvalue $-m^2$. Therefore, on discarding all terms containing the operator of the Dirac equation, which will not contribute to

$$\int (dx)(dx')\bar{\psi}(x)M(x,x')\psi(x'),$$

we obtain

$$[\gamma(-i\partial - eA) + m - \mu'\tfrac{1}{2}\sigma F]\psi = 0, \quad \text{(B.14)}$$

where

$$m = m_0 + (\alpha/2\pi)m\int_0^\infty ds\, s^{-1}\int_0^s dw\, s^{-1}(2-w/s) \times \exp[-im^2(s-w)] \quad \text{(B.15)}$$

represents the mass of a free electron, and

$$\mu' = (\alpha/2\pi)emi\int_0^\infty ds\int_0^s (dw/s)(w/s)(1-w/s) \times \exp[-im^2(s-w)] \quad \text{(B.16)}$$

describes an additional spin magnetic moment. Both integrals are conveniently evaluated by introducing

$$u = 1 - w/s, \quad \text{(B.17)}$$

and making the replacement $s \to -is$, which yields

$$\begin{aligned}m &= m_0 + (\alpha/2\pi)m\int_0^\infty ds\, s^{-1}\int_0^1 du(1+u)\exp(-m^2 us)\\ &= m_0 + (3\alpha/4\pi)m\left[\int_0^\infty ds\, s^{-1}\exp(-m^2 s) + \tfrac{5}{6}\right],\end{aligned} \quad \text{(B.18)}$$

and

$$\begin{aligned}\mu' &= (\alpha/2\pi)em\int_0^\infty ds\int_0^1 du\, u(1-u)\exp(-m^2 us)\\ &= (\alpha/2\pi)(e/m)\int_0^1 du(1-u) = (\alpha/2\pi)(e\hbar/2mc).\end{aligned} \quad \text{(B.19)}$$

We thus derive the spin magnetic moment of $\alpha/2\pi$ magnetons produced by second-order electromagnetic mass effects.

[10] The concepts employed here will be discussed at length in later publications.