

Implications of dynamical symmetry breaking

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An analysis is presented of the physical implications of theories in which the masses of the intermediate vector bosons arise from a dynamical symmetry breaking. In the absence of elementary spin-zero fields or bare fermion masses, such theories are necessarily invariant to zeroth order in the weak and electromagnetic gauge interactions under a global $U(N) \otimes U(N)$ symmetry, where N is the number of fermion types, not counting color. This symmetry is broken both intrinsically by the weak and electromagnetic interactions and spontaneously by dynamical effects of the strong interactions. An effective Lagrangian is constructed which allows the calculation of leading terms in matrix elements at low energy; this effective Lagrangian is used to analyze the relative direction of the intrinsic and spontaneous symmetry breakdown and to construct a unitarity gauge. Spontaneously broken symmetries which belong to the gauge group of the weak and electromagnetic interactions correspond to fictitious Goldstone bosons which are removed by the Higgs mechanism. Spontaneously broken symmetries of the weak and electromagnetic interactions which are not members of the gauge group correspond to true Goldstone bosons of zero mass; their presence makes it difficult to construct realistic models of this sort. Spontaneously broken elements of $U(N) \otimes U(N)$ which are not symmetries of the weak and electromagnetic interactions correspond to pseudo-Goldstone bosons, with mass comparable to that of the intermediate vector bosons and weak couplings at ordinary energies. Quark masses in these theories are typically less than 300 GeV by factors of order α . These theories require the existence of "extra-strong" gauge interactions which are not felt at energies below 300 GeV.

I. INTRODUCTION

When unified gauge theories of the weak and electromagnetic interactions were first proposed, it was assumed¹ that the spontaneous symmetry breakdown responsible for the intermediate-vector-boson masses is due to the vacuum expectation values of a set of spin-zero fields. For a variety of reasons, the attention of theorists has since been increasingly drawn to the possibility that this symmetry breaking is of a purely dynamical nature.² That is, it is supposed that there may be no elementary spin-zero fields in the Lagrangian, and that the Goldstone bosons associated with the spontaneous symmetry breakdown are bound states.

Almost all the effort that has been put into analyses of dynamical symmetry breaking has been directed to the difficult mathematical problem, of whether and how this phenomenon can occur in a variety of field-theoretic models. In this article I would like to address quite a different question: Assuming that dynamical symmetry breaking is a mathematical possibility in gauge field theories, what are the consequences for the real world?

Why should we believe that the masses of the intermediate vector bosons arise from dynamical symmetry breaking? The absence of *strongly* interacting elementary spin-zero fields is indicated by a number of requirements: asymptotic

freedom,³ electroproduction sum rules,⁴ and the naturalness of order- α parity and strangeness conservation.⁵ On the other hand, the absence of *weakly* interacting elementary spin-zero fields is much less certain. Apart from simplicity, the best reason for this assumption comes from the requirement for a natural hierarchy of gauge symmetry breaking.⁶ In order to put together the observed weak and electromagnetic interactions into a simple gauge group, it is necessary to suppose⁷ that in the spontaneous breakdown of this simple group to the nonsimple gauge group of the observed interactions, vector-boson masses are generated that are much larger than the masses expected for the W and Z ; this conclusion is even stronger if we try to include the strong interactions as well.⁸ This superstrong symmetry breakdown may well be due to the vacuum expectation values of elementary spin-zero fields. However, at ordinary energies, far below the superheavy vector-boson masses, physics is described by an effective field theory⁹ involving those fermions and vector bosons that did not get masses from the superstrong spontaneous symmetry breakdown, but no spin-zero fields. Likewise, the gauge group of this effective field theory consists of a direct product of those simple and $U(1)$ subgroups of the simple gauge group that were not broken at the superstrong level. The only way that the non-superheavy fermions and vector bosons can then

acquire masses is from a dynamical breakdown of this remaining gauge group. Furthermore, the mass scale determined by the dynamical symmetry breakdown is expected to be of the order of magnitude of the renormalization point at which the largest of the gauge couplings of the effective field theory reaches a value of order unity; this mass scale is in general enormously different from the mass scale of the superstrong symmetry breakdown.

For the purposes of this article, we will not need to commit ourselves to the general picture described above. However, our assumptions are those inspired by this picture: We assume that weak, strong, and electromagnetic interactions are described by a gauge field theory (perhaps an "effective" field theory⁹) involving fermions and gauge fields but no elementary spin-zero fields or bare fermion masses, and we suppose that the vector-boson and fermion masses arise from a dynamical breakdown of this gauge group. These assumptions are spelled out more precisely in Sec. II.

In further support of these assumptions, it should be mentioned that it is the absence of bare fermion masses that makes it natural for a spontaneous dynamical symmetry breakdown to occur. Any spontaneous symmetry breaking requires the appearance of massless Goldstone bosons,¹⁰ whether or not they are eventually eliminated by the Higgs mechanism.¹¹ For dynamical symmetry breaking, these Goldstone bosons would have to be bound states. However, we would normally expect that any bound state at zero mass would move away from zero mass if we changed the strength of the binding interactions, in which case dynamical symmetry breaking could only occur for a discrete set of coupling strengths,¹² and could not be considered "natural." In the theories considered here, with zero bare fermion mass, there is only one mass scale, defined by the renormalization point at which the gauge couplings are specified; thus a small change in the gauge coupling constant corresponds to a general change of mass scale,¹³ and cannot shift a massless bound state away from zero mass.

The consequences of our assumptions turn out to be quite striking. Before turning on the weak and electromagnetic interactions, the strong interactions are necessarily invariant, not only under the strong gauge group, but also, as shown in Sec. III, under a global $U(N) \otimes U(N)$ group, where N is the number of fermion types, not counting color or possible other strong gauge indices. This global group is broken in two different ways, described in Sec. IV. It is *spontaneously* broken down to some subgroup H by dynamical effects of the strong interactions. It is also *intrinsically*

broken to that subgroup S_w of $U(N) \otimes U(N)$ which leaves the weak and electromagnetic interactions invariant. And of course the gauge group of the weak and electromagnetic interactions is a subgroup G_w of S_w .

It is this double breakdown of $U(N) \otimes U(N)$, shown symbolically in Fig. 1, that will occupy most of our attention in this paper. Indeed, aside from the final section, the bulk of this paper can be regarded as a mathematical analysis of general theories in which there is both a strong spontaneous symmetry breaking and a weak intrinsic symmetry breaking induced by gauge interactions. The property that is specific to theories without spinless fields is that the over-all global group is $U(N) \otimes U(N)$, but most of our discussion would apply to any other global group.

Our analysis is complicated by three factors:

(1) We cannot use perturbation theory to describe the strong interactions responsible for the spontaneous symmetry breaking. This problem is evaded here by restricting ourselves to processes at relatively "low" energies, not greater than the expected masses of the intermediate vector bosons, or roughly $e \times 300$ GeV. It is shown in Sec.

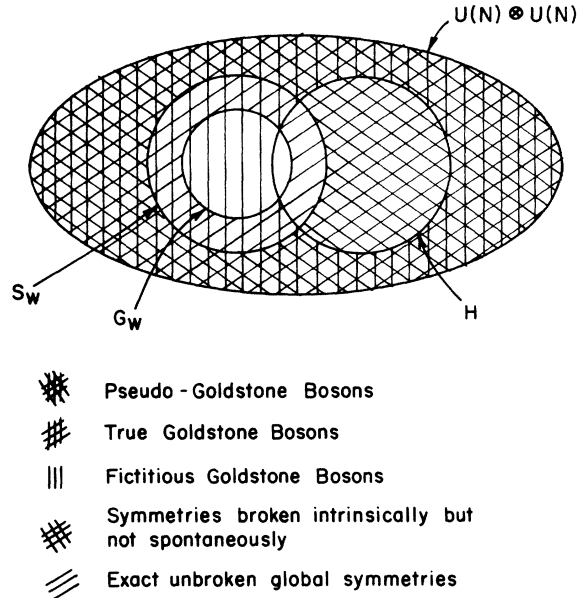


FIG. 1. Schematic representation of the various subgroups of $U(N) \otimes U(N)$. Cross hatchings indicate the various ways that global or local symmetries are broken; the unhatched lens represents the unbroken exact local symmetries, such as electromagnetic gauge invariance. As discussed in the text, H is the subgroup of $U(N) \otimes U(N)$ which is not spontaneously broken; S_w is the global symmetry group of the weak and electromagnetic interactions; and G_w is the weak and electromagnetic gauge group.

V that at such energies the terms of leading order in e in the matrix element for any process are given by calculating tree graphs, using an effective Lagrangian¹⁴ of reasonably simple structure. The effective Lagrangian involves those fermions and strong gauge vector bosons that did not acquire masses from the spontaneous dynamical symmetry breaking (which we identify as the usual quarks and gluons) plus the Goldstone bosons that accompany the spontaneous symmetry breakdown, and the gauge bosons of the weak and electromagnetic interactions.

(2) In defining a particular theory, it is not enough to specify the group structure of the non-spontaneously broken subgroup H and the gauge subgroup G_w ; we must also say how these subgroups line up with each other¹⁵ within the overall group $U(N) \otimes U(N)$. In Sec. VI it is shown that the alignment of these subgroups is determined by the condition that the Goldstone bosons must not have tadpoles; otherwise perturbation theory breaks down.

(3) There is a Goldstone boson for every independent broken symmetry¹⁰ in $U(N) \otimes U(N)$, but those Goldstone bosons that correspond to generators of the gauge group G_w are "fictitious" Goldstone bosons, which are eliminated by the Higgs mechanism.¹¹ In Sec. VII we show how in general to define a unitarity gauge¹⁶ in which these fictitious Goldstone bosons are absent. The masses of the intermediate vector bosons can be determined (to leading order in e) by inspecting the effective Lagrangian in the unitarity gauge. The other Goldstone bosons which are not eliminated by the Higgs phenomenon are studied in Sec. VIII. This class consists of "true" Goldstone bosons of zero mass, corresponding to broken symmetries in S_w but not G_w , and "pseudo"-Goldstone bosons¹⁷ with mass of order $e \times 300$ GeV, corresponding to broken symmetries of $U(N) \otimes U(N)$ which are neither in S_w nor G_w (see Fig. 1).

Different aspects of this analysis have been discussed before, but not to the best of my knowledge all together. Thus, effective Lagrangians for both broken global¹⁴ and broken gauge symmetries¹⁸ are an old story, but not for the case where the broken-symmetry group consists of a group of approximate global symmetries with an exact gauge subgroup. Also, the problem of subgroup alignment mentioned in item (2) above has been studied in the presence of strong interactions,¹⁵ but with a nongauge perturbation, and also with a gauge perturbation,¹⁹ but in the absence of strong interactions. Finally, previous attempts at a general definition of the unitarity gauge¹⁶ and the pseudo-Goldstone bosons¹⁷ dealt only with a spontaneous symmetry breakdown produced by vacuum

expectation values of elementary spin-zero fields.

In Sec. IX we take up an unrealistic example which is designed to show how this analysis can be applied to specific theories. As usual in models with spontaneously broken symmetries, we can obtain quite detailed information about the interaction of soft Goldstone bosons with quarks and vector bosons, despite the presence of strong interactions. Surprisingly, one can also solve the subgroup alignment problem explicitly. There are just two possible ways that the subgroups can line up, corresponding to the possible signs of a single unknown parameter. In one case there are two massive vector bosons, one "photon," one true Goldstone boson, two pseudo-Goldstone bosons, and a finite quark mass of second order in e ; in the other case there are three massive vector bosons, no photons, no true Goldstone bosons, two pseudo-Goldstone bosons, and an exact symmetry which keeps the quark mass zero to all orders in e . Evidently the subgroup alignment is crucial in determining the physical content of the theories with a given group structure. It is striking that for both alignments the theory contains unwelcome massless particles: in one case a true Goldstone boson, in the other a massless quark. This is a common problem in theories with dynamical symmetry breaking.

The last section offers a series of remarks about the application of the formalism developed in this article to models of the real world.

This article is not intended as an argument that the masses of the intermediate vector bosons actually do arise from dynamical symmetry breaking. Indeed, some of the difficulties of constructing realistic models based on dynamical symmetry breaking are emphasized in Sec. X. However, it would be wise at least to keep in mind that the experiments designed to find intermediate vector bosons may discover pseudo-Goldstone bosons as well.

II. GENERAL ASSUMPTIONS

The theories to be discussed in this paper are governed by the following general assumptions:

(a) The Lagrangian is locally invariant under a gauge group

$$G_S \otimes G_w. \quad (2.1)$$

Here G_S describes the strong interactions, and has gauge couplings roughly of order unity; G_w describes the weak and electromagnetic interactions, and has gauge couplings roughly of order e . [Both G_S and G_w may themselves be direct products of simple and/or $U(1)$ gauge groups.] As discussed in Sec. X, it is likely that G_S is larger than the usual color $SU(3)$ gauge group.

(b) In addition to the vector gauge fields required under (a) there is a set of fermion fields $\psi_{nm}(x)$. Here n is an N -valued row or "flavor" index labeling fermion type \mathcal{P} , \mathcal{K} , λ , \mathcal{P}' , etc., on which G_W acts, and m is a column or "color" index, on which G_S acts. (The status of the leptons is considered briefly in Sec. X.)

(c) The Lagrangian contains no fermion mass terms and no elementary spin-zero fields.

(d) The theory is renormalizable.

These assumptions are made here because they seem to be required⁶ in simple gauge theories with hierarchies of symmetry breaking, of the type described in the Introduction. However, for the purpose of this paper it will not be necessary to suppose that the strong, weak, and electromagnetic interactions arise from a superstrongly broken simple gauge group; it will only be assumed that physics at "ordinary" energies (say, up to a few thousand GeV) is governed by assumptions (a)–(d).

Under these assumptions, the Lagrangian must take the form²⁰

$$\mathcal{L} = -\bar{\psi}\gamma^\mu \mathfrak{D}_\mu \psi - \frac{1}{4} F_{\alpha\mu\nu} F_{\alpha\mu\nu} - \frac{1}{4} G_{\sigma\mu\nu} G_{\sigma\mu\nu}, \quad (2.2)$$

where $\mathfrak{D}_\mu \psi$ is the gauge-covariant derivative of the fermion field

$$(\mathfrak{D}_\mu \psi)_{nm} = \partial_\mu \psi_{nm} - i \sum_{n'\alpha} (w_\alpha)_{nn'} \psi_{n'm} W_{\alpha\mu} - i \sum_{m'\sigma} (s_\sigma)_{mm'} \psi_{nm'} S_{\sigma\mu}, \quad (2.3)$$

with $W_{\alpha\mu}$ and $S_{\sigma\mu}$ the G_W and G_S gauge fields, and w_α and s_σ the matrices representing the corresponding group generators. (The gauge coupling constants are included as factors in w_α and s_σ .) Also, $F_{\alpha\mu\nu}$ and $G_{\sigma\mu\nu}$ are the usual covariant curls of $W_{\alpha\mu}$ and $S_{\sigma\mu}$, respectively.

III. $U(N) \otimes U(N)$ SYMMETRY

The most striking consequence of the general assumptions outlined in the last section is the existence of an "accidental" approximate global symmetry of the Lagrangian. In the limit $e \rightarrow 0$ the Lagrangian is automatically invariant not only under the *local* G_S transformations on the fermion column (i.e., color) indices, but also under a group $U(N) \otimes U(N)$ of *global* transformations on the N -valued fermion row (i.e., \mathcal{P} , \mathcal{K} , λ , \mathcal{P}' , ...) index. That is, for each of the $2N^2$ independent Hermitian matrices λ_A (with Dirac matrix factor of 1 or γ_5) there is a vector or axial-vector current

$$J_A^\mu = -i \sum_{mn'n} \bar{\psi}_{n'm} \gamma^\mu (\lambda_A)_{n'n} \psi_{nm}. \quad (3.1)$$

Apart from triangle anomalies (about which more will be said later) these are all conserved:

$$\partial_\mu J_A^\mu = 0 \quad (\text{for } e = 0). \quad (3.2)$$

In what follows it will be convenient to normalize the λ_A and hence the J_A so that

$$\text{Tr}(\lambda_A \lambda_B) = 8\delta_{AB}. \quad (3.3)$$

(An unusual extra factor of 4 appears here because the trace includes a trace on Dirac indices; this is necessary because half the λ_A are proportional to the Dirac matrix γ_5 .)

It should be emphasized that the $U(N) \otimes U(N)$ symmetry arises only because of our assumptions that the Lagrangian contains no fermion mass terms $m\bar{\psi}\psi$, no scalar field couplings $\phi\bar{\psi}\psi$, and no non-renormalizable interactions, such as a Fermi interaction $\bar{\psi}\psi\bar{\psi}\psi$. Any one of these terms might in general destroy the $U(N) \otimes U(N)$ symmetry. On the other hand, once we make these assumptions, the $U(N) \otimes U(N)$ symmetry is inescapable—the fermion fields must enter the Lagrangian only in the form $\bar{\psi}\gamma^\mu \mathfrak{D}_\mu \psi$, and in the limit $e \rightarrow 0$ the covariant derivative \mathfrak{D}_μ contains only matrices which commute with all λ_A .

IV. SYMMETRY BREAKING

The $U(N) \otimes U(N)$ symmetry is in general broken by the weak and electromagnetic interactions. This intrinsic symmetry breaking can be quantitatively described by writing the generators w_α of the weak and electromagnetic gauge group as linear combinations of the $U(N) \otimes U(N)$ operators λ_A :

$$w_\alpha = \sum_A e_{\alpha A} \lambda_A. \quad (4.1)$$

In accordance with our previous assumptions, the coefficients $e_{\alpha A}$ are all of order e . Emission and absorption of virtual W bosons will produce order- e^2 perturbations which are not expected to be $U(N) \otimes U(N)$ invariant.

We shall assume that in addition to this intrinsic symmetry breaking, even in the limit $e \rightarrow 0$, there is a *spontaneous* breakdown of the symmetry group $G_S \otimes U(N) \otimes U(N)$ of the strong interactions, caused by strong forces among fermions, antifermions, and G_S gauge bosons. As usual in any spontaneous symmetry breaking, it is perfectly natural for some subgroup U to be left unbroken. For the sake of simplicity and definiteness, we shall assume that U does not mix the strong gauge group G_S with the accidental global symmetry group $U(N) \otimes U(N)$; that is, the unbroken symmetry group for $e \rightarrow 0$ is a direct product

$$U = H_S \otimes H, \quad (4.2)$$

where H_S is local and a subgroup of G_S , while H is global and a subgroup of $U(N) \otimes U(N)$. Most of the considerations below would also apply to the more general case where the unbroken subgroup

is not of the form (4.2), but our discussion would have to be made considerably more elaborate to deal with this case.

For any independent generator of G_S which is not a generator of the unbroken gauge subgroup H_S , the corresponding strongly interacting vector boson gets a mass from the Higgs phenomenon. Also, depending on the nature of the unbroken global subgroup H , some of the fermions will acquire a mass from the spontaneous breakdown of $U(N) \otimes U(N)$ to H . In the limit $e \rightarrow 0$ this theory contains no very large or very small dimensionless parameters, so we would expect all these masses to be of the same order of magnitude, say M . The only dimensional parameter in the theory for $e \rightarrow 0$ is the scale characterizing the renormalization point of the G_S gauge couplings, so we would also expect M to be determined by the condition that the largest G_S gauge coupling reaches a critical value of order unity at a renormalization point characterized by momenta of order M .

We will see below that M is likely to be quite large, of order 300 GeV. The physics of strong interactions at lower energies $E \ll M$ can therefore be described in terms of those fermions and G_S gauge bosons which do *not* pick up masses of order M from the spontaneous symmetry breakdown, and hence remain massless in the limit $e \rightarrow 0$. These may be identified as the ordinary quarks and gluons, respectively. (Of course, we do not at this point rule out the possibility that H_S is just G_S , so that *all* of the G_S gauge bosons remain massless.) As we shall see, turning on the weak and electromagnetic interactions will give the quarks masses of order $e^2 M$, while the gluons will remain massless.

In addition to quarks and gluons, this theory necessarily contains one other class of hadrons with masses which vanish for $e \rightarrow 0$, the Goldstone bosons. For every linearly independent generator of $U(N) \otimes U(N)$ which is not a generator of the unbroken subgroup H , there must appear a Goldstone boson Π_a . Since $U(N) \otimes U(N)$ is not a gauge group, there is no Higgs phenomena which can eliminate these Goldstone bosons in the limit $e \rightarrow 0$.

The coupling of the a th Goldstone boson to the A th $U(N) \otimes U(N)$ current is described by a parameter F_{aA} , defined by

$$\langle 0 | J_A^\mu(0) | \Pi_a \rangle = F_{aA} P_\mu^\dagger(2\pi)^{-3/2} (2E_\Pi)^{-1/2}. \quad (4.3)$$

These F_{aA} have the dimensions of a mass, and will play a role here like that played by the parameter F_π of current algebra. We expect that all F_{aA} are of the order of the mass M introduced earlier,

$$F_{aA} \approx M,$$

because in the limit $e \rightarrow 0$ this is the only mass in the theory.

It will be very convenient to adapt the basis for $U(N) \otimes U(N)$ to the pattern of symmetry breaking. We may define the generators of the unbroken subgroup H as linear combinations of the $2N^2$ generators λ_A of $U(N) \otimes U(N)$,

$$t_i = \sum_A C_{iA} \lambda_A, \quad (4.4)$$

with the C_{iA} chosen as orthonormal vectors so that

$$\sum_A C_{iA} C_{jA} = \delta_{ij}, \quad (4.5)$$

$$\text{Tr}(t_i t_j) = 8\delta_{ij}. \quad (4.6)$$

The unbroken-symmetry currents have no couplings to the Goldstone bosons, so that

$$\sum_A C_{iA} F_{aA} = 0. \quad (4.7)$$

Further, by a suitable unitary transformation we can always choose the Π_a states so as to diagonalize the positive Hermitian matrix

$$\sum_A F_{aA} F_{bA}.$$

If the element with $b=a$ is denoted F_a^2 , we have then

$$F_{aA} = F_a B_{aA}, \quad (4.8)$$

with

$$\sum_A B_{aA} B_{bA} = \delta_{ab}, \quad (4.9)$$

and also

$$\sum_A B_{aA} C_{iA} = 0. \quad (4.10)$$

There is one Goldstone boson for each independent broken symmetry, so the B 's and C 's form a complete orthonormal set of vectors

$$\sum_a B_{aA} B_{aB} + \sum_i C_{iA} C_{iB} = \delta_{AB}. \quad (4.11)$$

Correspondingly, we can define a set of broken symmetry generators

$$x_a \equiv \sum_A B_{aA} \lambda_A, \quad (4.12)$$

with

$$\text{Tr}(t_i x_a) = 0, \quad (4.13)$$

$$\text{Tr}(x_a x_b) = 8\delta_{ab}. \quad (4.14)$$

The generators t_i and x_a span the algebra of $U(N) \otimes U(N)$.

V. EFFECTIVE LAGRANGIAN

We now want to consider the special phenomena which arise because $U(N) \otimes U(N)$ is simultaneously broken both intrinsically by the weak and electromagnetic interactions and also spontaneously by dynamical effects of the strong interactions. This is not a mere matter of expanding in powers of e , because the theory has infrared singularities which, at momenta p much less than the characteristic mass M , introduce factors M/p which can compensate for factors e .

In order to explore this problem, let us consider the leading pole singularities produced by soft virtual quarks, Goldstone bosons, and G_W vector bosons in a general Green's function with external quark, gluon, Goldstone boson, and/or G_W vector bosons carrying momenta of order $p \ll M$.²¹ These singularities can be calculated from the sum of all tree graphs for this Green's function, constructed from an effective Lagrangian¹⁴ involving quark, G_W vector boson, gluon, and Goldstone boson fields. [For the moment, we are ignoring the effects of loops containing *hard* virtual G_W vector bosons. These produce $U(N) \otimes U(N)$ -breaking corrections of order e^2 in the effective Lagrangian, and will be considered at the end of this section.]

The effective Lagrangian here takes the general form

$$\mathcal{L} = -\frac{1}{4} F_{\alpha\mu\nu} F_{\alpha}^{\mu\nu} + \mathcal{L}_1. \quad (5.1)$$

The term \mathcal{L}_1 is subject to three conditions:

(1) In the limit $e \rightarrow 0$ the W dependence drops out, and \mathcal{L}_1 becomes invariant under $U(N) \otimes U(N)$, with fields transforming according to one of the usual nonlinear realizations of $U(N) \otimes U(N)$, in which the unbroken subgroup H is realized algebraically. It will be convenient to define the fields so that Π transforms like the so-called exponential parametrization²² of the cosets in $U(N) \otimes U(N)/H$. That is, a general $U(N) \otimes U(N)$ transformation, which is represented on the fermion fields in the original Lagrangian by a matrix g , induces on the fields in the effective Lagrangian the nonlinear transformations²²

$$\Pi_a \rightarrow \Pi'_a(\Pi, g), \quad (5.2)$$

$$q \rightarrow \exp\left(i \sum_i \mu_i(\Pi, g) t_i\right) q, \quad (5.3)$$

where q is the quark field multiplet (with components corresponding to those fermion fields that do not acquire masses of order M from the spontaneous symmetry breaking), and Π' and μ are functions defined by the relation

$$g \exp\left(i \sum_a \Pi_a x_a / F_a\right) = \exp\left(i \sum_a \Pi'_a(\Pi, g) x_a / F_a\right) \times \exp\left(i \sum_i \mu_i(\Pi, g) t_i\right). \quad (5.4)$$

The gluon fields are of course invariant under these transformations.

(2) The currents to which the products $W_{\alpha\mu} W_{\beta\nu} \cdots$ of G_W gauge fields (or their derivatives) couple in \mathcal{L}_1 take the form

$$\sum_{A,B,\dots} e_{\alpha A} e_{\beta B} \cdots T_{AB} \cdots, \quad (5.5)$$

where $T_{AB} \cdots$ is a quantity, formed out of quark, gluon, and Goldstone boson fields and their derivatives, which transforms like a $U(N) \otimes U(N)$ tensor (i.e., like $\lambda_A \lambda_B \cdots$) when Π and q undergo the transformations (5.2) and (5.3).

(3) \mathcal{L}_1 is locally as well as globally invariant under the unbroken subgroup H_S of the strong gauge group and under the weak and electromagnetic gauge group G_W .

It is shown in Appendix A that these conditions require \mathcal{L}_1 to be constructed from just the following ingredients:

(i) The quark fields q .

(ii) Their covariant derivatives (for notation, see below):

$$D_\mu q \equiv \partial_\mu q - i \sum_{ai} t_i q E_{ai}(\Pi) \partial_\mu \Pi_a F_a^{-1} - \sum_{B A \alpha i} t_i q \Lambda_{AB}(\Pi) e_{\alpha A} W_{\alpha\mu} C_{iB} + \text{gluon terms}. \quad (5.6)$$

(iii) Covariant derivatives of the Goldstone boson fields:

$$D_\mu \Pi_a \equiv F_a \left[\sum_b D_{ba}(\Pi) \partial_\mu \Pi_b F_b^{-1} + \sum_{A \alpha B} \Lambda_{AB}(\Pi) e_{\alpha A} W_{\alpha\mu} B_{aB} \right]. \quad (5.7)$$

(iv) A covariant curl of the W field:

$$\mathcal{F}_{A\mu\nu} = \sum_{B \alpha} \Lambda_{BA}(\Pi) e_{\alpha A} F_{\alpha\mu\nu}. \quad (5.8)$$

(v) The usual covariant curl of the gluon field.

Here $F_{\alpha\mu\nu}$ is the usual Yang-Mills G_W -covariant curl,²⁰ and the functions D, E , and Λ are defined by the formulas

$$S^{-1}(\Pi) \frac{\partial}{\partial \Pi_a} S(\Pi) = i F_a^{-1} \left[\sum_b D_{ab}(\Pi) x_b + \sum_i E_{ai}(\Pi) t_i \right], \quad (5.9)$$

$$S^{-1}(\Pi)\lambda_A S(\Pi) = \sum_B \Lambda_{AB}(\Pi)\lambda_B, \quad (5.10)$$

with

$$S(\Pi) \equiv \exp\left(i \sum_b \Pi_b x_b / F_b\right). \quad (5.11)$$

With our normalization convention (3.3), the Λ matrices are orthogonal,

$$\Lambda^T(\Pi) = \Lambda^{-1}(\Pi). \quad (5.12)$$

[At this point, the reader may wish to be reminded that indices A, B , etc. label all generators of $U(N) \otimes U(N)$; i, j , etc. label the unbroken generators; and a, b , etc. label the broken generators.]

In addition to these limitations on the ingredients in \mathcal{L}_1 , the conditions (1)–(3) also require that \mathcal{L}_1 must be invariant under formal global H transformations, with $D_\mu \Pi$, $D_\mu q$, and $\mathcal{F}_{A\mu\nu}$ transforming according to whatever (linear) representations of H they happen to contain.

At this point the effective Lagrangian we have derived still has an extremely complicated structure, involving unlimited numbers of q , $D_\mu q$, $D_\mu \Pi$, $G_{\sigma\mu\nu}$, and $\mathcal{F}_{A\mu\nu}$ functions. Indeed, if we were to take this Lagrangian seriously as a basis for higher-order calculations, we would have to keep all these interactions in order to provide counterterms for the infinite number of primitive divergents that would arise. However, the structure of the effective Lagrangian can be very much simplified if we use it only to determine the matrix elements to lowest order in e and p/M .

Ordinary dimensional analysis leads us to expect that an interaction v appearing in the effective Lagrangian with n_{vq} quark fields, n_{vs} gluon fields, $n_{v\Pi}$ Goldstone boson fields, n_{vw} G_W -gauge boson fields, and n_{vd} derivatives, will have a coupling constant of order

$$g_v \approx e^{n_{vw}} M^{n_{vm}}, \quad (5.13)$$

where M is the characteristic mass introduced in the last section (of order F_a) and

$$n_{vm} = 4 - \frac{3}{2}n_{vq} - n_{vs} - n_{v\Pi} - n_{vd}. \quad (5.14)$$

If a Green's function has external lines with momenta of order

$$p \approx eM, \quad (5.15)$$

then each power of M counts like a factor $1/e$. The total number of factors of e or p/M is

$$N_e = \sum_v N_v (n_{vw} - n_{vm}), \quad (5.16)$$

where N_v is the number of vertices of type v . However, a tree graph with E_q external quark lines, E_Π external Goldstone boson lines, and

E_W external G_W -gauge bosons will obey the topological relation

$$\sum_v (n_{vq} + n_{v\Pi} + n_{vw} - 2)N_v = E_q + E_\Pi + E_W - 2. \quad (5.17)$$

(Gluon terms on both sides cancel, because we exclude *internal* gluon lines.²¹) Therefore, the total number of factors of e or p/M is

$$N_e = E_q + E_\Pi + E_W - 2 + \sum_v N_v \Delta_v, \quad (5.18)$$

where

$$\Delta_v \equiv n_{vw} + \frac{1}{2}n_{vq} + n_{vd} + n_{vs} - 2. \quad (5.19)$$

Thus for any given set of external lines, the terms of lowest order in e or p/M will be given by graphs composed of vertices with the smallest possible values of Δ_v .

In fact, the smallest values of Δ_v for any allowed interaction is $\Delta_v = 0$. Keeping only terms with $\Delta_v = 0$, and normalizing fields appropriately, gives an effective Lagrangian of the general form

$$\begin{aligned} \mathcal{L}_1 = & -\bar{q}\gamma^\mu D_\mu q - \frac{1}{2} \sum_a D_\mu \Pi_a D^\mu \Pi_a - \sum_a \bar{q}\gamma^\mu \Gamma_a q D_\mu \Pi_a \\ & + \text{Fermi interactions.} \end{aligned} \quad (5.20)$$

Here Γ_a is a constant matrix, proportional to γ_5 and/or 1, and of order $1/M$, which has the same H -transformation properties as Π_a ; also, the "Fermi interactions" are $\bar{q}q\bar{q}q$ terms of order $1/M^2$ which are H invariant, but may involve any of the 16 Dirac covariants. Note that $D_\mu q$ and $D_\mu \Pi$ have just the right dependence on quark, gluon, G_W vector boson, and Goldstone boson fields, so that it is possible to construct an effective Lagrangian obeying all necessary symmetry conditions with only $\Delta_v = 0$ terms. (To the extent that a graph with loops is dominated by states with energy $E \ll M$, it can also be calculated with this effective Lagrangian.)

There is, in fact, one other term which in effect has $\Delta_v = 0$ and therefore should be added to the effective Lagrangian (5.20). The emission and reabsorption of a hard G_W vector boson produces an effective interaction of second order in e . For any such term, the Δ_v in Eq. (5.19) should be increased by two units, so that it becomes

$$\Delta_v = n_{vw} + \frac{1}{2}n_{vq} + n_{vd} + n_{vs}. \quad (5.21)$$

Thus we get a term with $\Delta_v = 0$ of second order in e if it is constructed *solely* from Goldstone boson fields, with no G_W gauge fields, quark fields, gluon fields, or derivatives. (See Fig. 2.)

This term will take the form

$$\mathcal{L}_2(\Pi) = - \sum_{\alpha AB} e_{\alpha A} e_{\alpha B} J_{AB}(\Pi), \quad (5.22)$$

where J_{AB} is the time-ordered product of two $U(N) \otimes U(N)$ currents J_A^μ and J_B^ν , integrated over momenta (including a $1/k^2$ weight factor from the W_α^μ propagator) and contracted over space-time indices. This two-point function is a $U(N) \otimes U(N)$ tensor, i.e., it transforms like $\lambda_A \lambda_B$. As shown in Appendix B, the most general form of such a tensor is

$$J_{AB}(\Pi) = - \sum_{CD} \Lambda_{AC}(\Pi) \Lambda_{BD}(\Pi) I_{CD}, \quad (5.23)$$

with I a Π -independent quantity of order M^4 which behaves as a tensor under the unbroken subgroup H . Thus, the $O(e^2)$ term with $\Delta_v = 0$ is

$$\mathcal{L}_2(\Pi) = - \sum_{\alpha ABCD} e_{\alpha A} e_{\alpha B} \Lambda_{AC}(\Pi) \Lambda_{BD}(\Pi) I_{CD}. \quad (5.24)$$

This contains both a Π mass term (with some Goldstone boson masses of order eM) and a non-derivative Π self-interaction, like that in pion-pion scattering.

Incidentally, the integral in J_{AB} is expected to receive its major contribution from momenta of order M , because in these theories there is no reason why the integral should start to converge at any lower momentum. This is why we do not include the effects of G_W vector-boson masses here; these masses will turn out to be of order eM , so their effect is a higher-order correction.²³

So far, we have seen that the leading ($\Delta_v = 0$) part of the effective Lagrangian takes the form

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} F_{\alpha\mu\nu} F_{\alpha}^{\mu\nu} + \mathcal{L}_1 + \mathcal{L}_2. \quad (5.25)$$

We could go on and describe the structure of higher terms with $\Delta_v = 1$, $\Delta_v = 2$, etc. However, we will content ourselves with describing one term of particular interest, associated with the quark masses.

The quarks are *defined* as the fermions that do not pick up a mass of order M from the spontaneous breakdown of $U(N) \otimes U(N)$ to H , so there is no quark mass term $\bar{q}q$ with $\Delta_v = -1$ in the effective Lagrangian. However, emission and absorption of a G_W vector boson can produce a term

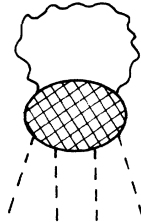


FIG. 2. Diagrams which contribute to the term \mathcal{L}_2 in the effective Lagrangian. Wavy lines are intermediate vector bosons and dashed lines are Goldstone bosons.

of order e^2 with $\Delta_v = +1$. Following the same reasoning as for \mathcal{L}_2 , this term must take the form (see Fig. 3)

$$\mathcal{L}_m = - \sum_{\alpha AB} \bar{q} N_{AB}(\Pi) q e_{\alpha A} e_{\alpha B}, \quad (5.26)$$

where $\bar{q} N q$ transforms as a tensor under $U(N) \otimes U(N)$. As shown in Appendix C, the most general form of N is

$$N_{AB}(\Pi) = \sum_{CD} \Lambda_{AC}(\Pi) \Lambda_{BD}(\Pi) Q_{CD}, \quad (5.27)$$

where Q_{CD} is a Π -independent matrix of order M which behaves as a tensor under the unbroken subgroup H . Thus the quark mass term is

$$\mathcal{L}_m = - \sum_{\alpha ABCD} \bar{q} Q_{CD} q \Lambda_{AC}(\Pi) \Lambda_{BD}(\Pi) e_{\alpha A} e_{\alpha B}. \quad (5.28)$$

The quark mass matrix is then

$$\mathcal{M} = \sum_{\alpha} Q_{AB} e_{\alpha A} e_{\alpha B}, \quad (5.29)$$

so quark masses are expected to be of order $e^2 M$. In addition, there are multilinear interactions of Goldstone bosons with quarks, including a $\Pi \bar{q} q$ coupling of order $e^2 M / F \sim e^2$.

VI. ALIGNMENT OF SUBGROUPS

There is one further step that must be taken before the effective Lagrangian can be used for actual calculations. The symmetry $U(N) \otimes U(N)$ is supposed to be spontaneously broken to some subgroup H by dynamical effects of the strong interactions. However, although we can presume that the structure of H is determined by dynamical considerations, the strong interactions alone do not determine *which* subgroup of $U(N) \otimes U(N)$ with this structure is left unbroken. Given any solution of the strong-interaction dynamics with a subgroup H left invariant, we can find another solution in which the subgroup left invariant is

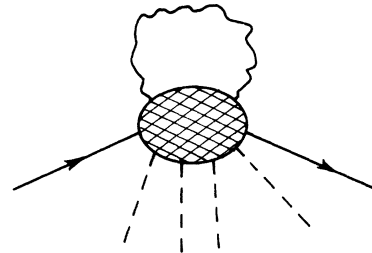


FIG. 3. Diagrams which contribute to the term \mathcal{L}_m in the effective Lagrangian. Wavy lines are intermediate vector bosons, dashed lines are Goldstone bosons, and straight lines are quarks.

$$H(g) = gHg^{-1}, \quad (6.1)$$

where g is any element of $U(N) \otimes U(N)$.

Normally we do not concern ourselves with this sort of ambiguity, for these different solutions are usually physically equivalent. However, in our case, the theory contains a perturbation, the weak and electromagnetic interactions, which also break $U(N) \otimes U(N)$ down to some fixed subgroup S_w . (Of course, S_w contains the gauge group G_w of the weak and electromagnetic interactions, but as shown in Sec. X, in general it is larger.) Thus the different solutions corresponding to the different unbroken subgroups (6.1) are physically inequivalent, and we must decide which is the correct one.

This problem was encountered in a different context some time ago by Dashen,¹⁵ who gave a general solution. It is necessary to construct a potential, given to lowest order as

$$V(g) = \langle 0, g | \mathcal{H}' | 0, g \rangle, \quad (6.2)$$

where \mathcal{H}' is the symmetry-breaking perturbation (in our case an operator of second order in the weak and electromagnetic interactions) and $|0, g\rangle$ is the vacuum corresponding to the solution that has unbroken subgroup $H(g)$:

$$H(g) |0, g\rangle = |0, g\rangle. \quad (6.3)$$

The g that defines the "correct" solution in the presence of the perturbation \mathcal{H}' is defined by the condition that $V(g)$ be a *minimum*.

For our purposes, it will be much more convenient to keep the solution of the strong- interaction symmetry breaking fixed, and instead vary the way that the weak and electromagnetic gauge group is inserted into the larger $U(N) \otimes U(N)$ global group. That is, we now fix the vacuum and the unbroken subgroup H , and instead let the gauge group be

$$G_w(g) = g^{-1} G_w g, \quad (6.4)$$

where g runs over all elements of $U(N) \otimes U(N)$. This has the advantage that we can fix the choice of the generators t_i and x_a from the beginning; the whole effect of varying the gauge group is that the "charges" $e_{\alpha A}$ are replaced with

$$e_{\alpha A}(g) = \sum_B R_{BA}(g) e_{\alpha B}, \quad (6.5)$$

where $R(g)$ is the regular representation of $U(N) \otimes U(N)$.

Using $e_{\alpha A}(g)$ in place of $e_{\alpha A}$ in Eq. (5.24), we see that the $O(e^2)$ term in the leading part of the effective Lagrangian is

$$\mathcal{L}_2(\Pi, g) = - \sum_{\alpha A B C D} e_{\alpha A}(g) e_{\alpha B}(g) \Lambda_{AC}(\Pi) \Lambda_{BD}(\Pi) I_{CD}. \quad (6.6)$$

The potential (6.2) is just given by the vacuum-fluctuation part of \mathcal{L}_2 :

$$V(g) = - \mathcal{L}_2(0, g) \quad (6.7)$$

or more explicitly

$$V(g) = - \sum_{\alpha A B} e_{\alpha A}(g) e_{\alpha B}(g) I_{AB}. \quad (6.8)$$

Another interpretation can now be put on the condition that $V(g)$ be a minimum. Using Eqs. (6.5), (B2), (B6), and (5.2) in (6.6), we have

$$\mathcal{L}_2(\Pi, g_1 g) = \mathcal{L}_2(\Pi'(\Pi, g_1), g), \quad (6.9)$$

where g_1 and g are arbitrary elements of $U(N) \otimes U(N)$, and Π' is the image of Π under g_1 . Thus the variation of $\mathcal{L}_2(\Pi, g)$ with respect to g may be determined from its variation with respect to Π . In particular, for $\Pi = 0$ and g_1 infinitesimal, Eq. (6.7) and (6.9) give

$$V\left(\left(1 + i \sum_a \frac{\epsilon_a x_a}{F_a}\right) g\right) = - \mathcal{L}_2(\epsilon, g), \quad (6.10)$$

so the condition that $V(g)$ be stationary with respect to g is equivalent to the condition that

$$\left. \frac{\partial \mathcal{L}_2(\Pi, g)}{\partial \Pi_a} \right|_{\Pi=0} = 0. \quad (6.11)$$

Graphically, this says that tadpole graphs, in which a single Goldstone boson disappears into the vacuum, necessarily vanish. The rationale for this condition is that otherwise perturbation theory in e would break down; the denominator of the propagator of a Goldstone boson at zero four-momentum is at most of order e^2 (see below) so a tadpole produced by second-order effects of the weak and electromagnetic interactions would be of zeroth order in e .²⁴

Not only must we choose the gauge group $G_w(g)$ so that $V(g)$ is stationary; we must choose it so that $V(g)$ is at least a local *minimum*. The reason is again to be found in Eq. (6.9); the condition that $V(g)$ be a minimum is equivalent to the condition that $\Pi = 0$ be a minimum of $-\mathcal{L}_2(\Pi, g)$, and this in turn ensures the positivity of the mass matrix

$$m_{ab}^2(g) = - \left. \frac{\partial^2}{\partial \Pi_a \partial \Pi_b} \mathcal{L}_2(\Pi, g) \right|_{\Pi=0}. \quad (6.12)$$

From now on we will assume that g has been chosen from the beginning so that (6.11) is satisfied and (6.12) is positive. We can therefore drop the explicit argument g everywhere.

In order to put these conditions in a more useful form, we note that

$$\Lambda(\Pi) = \exp\left(i \sum_a X_a \Pi_a / F_a\right), \quad (6.13)$$

where $(X_a)_{AB}$ is the matrix representing the generator x_a in the adjoint representation of $U(N) \otimes U(N)$. Hence (5.24) gives

$$\mathcal{L}_2(\Pi) = -\text{Tr} \left[E \exp \left(i \sum_a \frac{X_a \Pi_a}{F_a} \right) I \exp \left(-i \sum_a \frac{X_a \Pi_a}{F_a} \right) \right], \quad (6.14)$$

where E is the matrix

$$E_{AB} \equiv \sum_{\alpha} e_{\alpha A} e_{\alpha B}. \quad (6.15)$$

Condition (6.11) therefore reads

$$\text{Tr} \{ E[X_a, I] \} = 0, \quad (6.16)$$

and (6.12) gives the mass matrix as

$$m_{ab}^2 = -\frac{1}{2F_a F_b} \text{Tr} \{ E[X_a, [X_b, I]] + E[X_b, [X_a, I]] \}. \quad (6.17)$$

Since I_{AB} is H invariant, we can generalize (6.16) to

$$\text{Tr} \{ E[\Lambda_A, I] \} = 0 \quad (6.18)$$

where Λ_A is the adjoint representation of an arbitrary generator λ_A of $U(N) \otimes U(N)$. Also, Eq. (6.17) can be simplified because the two terms on the right are equal; the Jacobi identity gives their difference as

$$\begin{aligned} \text{Tr} \{ E[X_a, [X_b, I]] \} - \text{Tr} \{ E[X_b, [X_a, I]] \} \\ = \text{Tr} \{ E[[X_a, X_b], I] \}, \end{aligned}$$

and this vanishes according to (6.18). Thus, (6.17) may be written

$$m_{ab}^2 = m_{ba}^2 = -\frac{1}{F_a F_b} \text{Tr} \{ E[X_a, [X_b, I]] \}. \quad (6.19)$$

VII. UNITARITY GAUGE AND VECTOR BOSON MASSES

The effective Lagrangian derived in Sec. V is still locally invariant under the gauge group G_w of the weak and electromagnetic interactions. We are therefore free to adopt a "unitarity gauge," in which the particle content of the theory is explicitly displayed.¹⁶

Suppose for a moment that the gauge group G_w were a sufficiently large subgroup of $U(N) \otimes U(N)$, so that its generators w_α , together with the generators t_i of the unbroken subgroup H , would completely span the algebra of $U(N) \otimes U(N)$. (This includes the case usually discussed, where G_w is the whole of the original symmetry group.) Then any element of $U(N) \otimes U(N)$ could be written as a product of an element of G_w times an element of H . This would in particular be true of the element $\exp(i \sum_a x_a \Pi_a / F_a)$, and therefore we could write

$$\begin{aligned} \exp \left(i \sum_a x_a \Pi_a / F_a \right) &= \exp \left(-i \sum_\alpha \Theta_\alpha(\Pi) w_\alpha \right) \\ &\times \exp \left(i \sum_i \mu_i(\Pi) t_i \right) \end{aligned}$$

for some real parameters $\theta_\alpha(\Pi)$ and $\mu_i(\Pi)$. But by comparing this with Eq. (5.4), we see that the gauge transformation $\exp(i \sum_\alpha \theta_\alpha w_\alpha)$ would carry the Goldstone boson field Π_a into $\Pi'_a = 0$. Thus, in this case there would be a choice of gauge which eliminates all Goldstone bosons.

In the general case, we do *not* expect the generators of G_w and H to span the algebra of $U(N) \otimes U(N)$. Note that if G_w is too large, then the weak and electromagnetic interactions will not break the symmetries of H sufficiently; we would then find that any quarks which do not get masses of order M from the spontaneous breakdown of $U(N) \otimes U(N)$ to H will remain massless to all orders in e (see Sec. X).

However, as shown in Appendix D, we can always write a general element λ of the algebra of $U(N) \otimes U(N)$ in the form

$$\lambda = - \sum_\alpha \theta_\alpha w_\alpha + \sum_a \phi_a x_a + \sum_i \mu_i t_i, \quad (7.1)$$

with ϕ_a constrained by the condition that

$$\begin{aligned} 0 &= \text{Tr} \left(\theta_\alpha \sum_a \phi_a F_a^2 x_a \right) \\ &= \sum_{aA} e_{\alpha A} B_{aA} F_a^2 \phi_a. \end{aligned} \quad (7.2)$$

(The reason for adopting this particular constraint, and in particular for inserting the factor F_a^2 , will be made clear below.) Hence, since every element of $U(N) \otimes U(N)$ that is infinitesimally close to the identity may be expressed in the form

$$\exp \left(-i \sum_\alpha \theta_\alpha w_\alpha \right) \exp \left(i \sum_a \phi_a x_a \right) \exp \left(i \sum_i \mu_i t_i \right)$$

[with ϕ_a satisfying (7.2)], it follows that every element of $U(N) \otimes U(N)$ in at least some finite region around the identity may be written in this form. In particular we may write

$$\begin{aligned} \exp \left(i \sum_a x_a \Pi_a / F_a \right) &= \exp \left(-i \sum_\alpha \theta_\alpha(\Pi) w_\alpha \right) \\ &\times \exp \left(i \sum_a \phi_a(\Pi) x_a \right) \\ &\times \exp \left(i \sum_i \mu_i(\Pi) t_i \right). \end{aligned} \quad (7.3)$$

But Eq. (5.4) then tells us that the gauge transformation $\exp[+i \sum_\alpha \theta_\alpha(\Pi) w_\alpha]$ carries the Goldstone boson field Π_a into

$$\Pi'_a / F_a = \phi_a(\Pi). \quad (7.4)$$

That is, we can adopt a gauge in which, according to Eq. (7.2),

$$\sum_{aA} e_{aA} B_{aA} F_a \Pi'_a = 0 \quad (7.5)$$

or equivalently

$$\text{Tr} \left(w_\alpha \sum_a x_a F_a \Pi'_a \right) = 0. \quad (7.6)$$

This is the unitarity gauge.

The unitarity gauge as we have defined it has the crucial property of eliminating the zeroth-order mixing between the Goldstone bosons and the G_W vector bosons. From Eqs. (5.7) and (5.15), we see that the part of the effective Lagrangian that is quadratic in gauge and/or Goldstone fields is

$$-\frac{1}{2} \sum_a (D_\mu \Pi'_a)_{11n} (D^\mu \Pi'_a)_{11n}, \quad (7.7)$$

with $(D_\mu \Pi')_{11n}$ the linear part of the covariant derivative

$$(D_\mu \Pi'_a)_{11n} = \partial_\mu \Pi'_a + F_a \sum_{A\alpha} e_{aA} W'_{\alpha\mu} B_{aA}. \quad (7.8)$$

Hence Eq. (7.5) has the effect of insuring that the Π - W cross terms drop out in the quadratic part of the effective Lagrangian. It was to bring this about that we inserted the factor F_a^2 in Eq. (7.2), and it is this feature of the unitarity gauge that justifies the statement that it correctly displays the particle content of the theory.

The same result can be obtained by a simple generalization of the method of Jackiw and Johnson and Cornwall and Norton.² In a general gauge there are "black boxes," connecting a single G_W -gauge boson with a single Goldstone boson line. If we sum up the pole singularities produced by a linear chain of Π and W lines, we find that the only poles in the sum which correspond to particles of zero spin and zero mass are those in channels described by precisely the condition (7.5). It is also easy to see that the unitarity gauge as usually defined in theories with elementary scalar fields does satisfy Eq. (7.5).

Now that we have eliminated the Π - W cross terms, the mass of the G_W vector bosons may be read off from the effective Lagrangian. Equation (7.7) contains a term quadratic in W

$$-\frac{1}{2} \sum_{\alpha\beta} \mu^2_{\alpha\beta} W'_{\alpha\mu} W'^\mu_{\beta}. \quad (7.9)$$

with a vector-boson mass

$$\mu^2_{\alpha\beta} = \sum_{aAB} F_a^2 B_{aA} B_{aB} e_{aA} e_{aB} \quad (7.10)$$

or equivalently

$$\mu^2_{\alpha\beta} = \frac{1}{64} \sum_a F_a^2 \text{Tr}(x_a w_\alpha) \text{Tr}(x_a w_\beta). \quad (7.11)$$

It is immediately apparent from (7.11) that $\mu^2_{\alpha\beta}$ vanishes if either w_α or w_β is the generator of a symmetry that is not spontaneously broken. Also, the μ 's that are not zero are of order eF , and since the Fermi coupling constant G_F must be of order e^2/μ^2 , we can conclude that

$$M \approx F \approx G_F^{-1/2} \approx 300 \text{ GeV}$$

as previously indicated.

VIII. CLASSIFICATION OF THE GOLDSTONE BOSONS

The other side of the Higgs phenomenon, complementary to the appearance of vector-boson masses, is the disappearance of Goldstone bosons.¹¹ We will now consider the nature and the mass spectrum of the Goldstone bosons that are *not* eliminated by the Higgs phenomenon.

It is useful to begin by studying the eigenvectors and eigenvalues of the formal Goldstone boson mass matrix m^2_{ab} , before transformation to the unitarity gauge. First, note that there is an eigenvector of m^2_{ab} with eigenvalue zero for every independent broken gauge generator. Every generator w_α of G_W may be written in the form

$$w_\alpha = \sum_a \frac{x_a u_{a\alpha}}{F_a} + h_\alpha, \quad (8.1)$$

where h_α is a linear combination of the generators t_i of the unbroken symmetry subgroup H . But then Eq. (6.19) gives

$$\sum_b m^2_{ab} u_{b\alpha} = -\frac{1}{F_a} \text{Tr} \{ E[(W_\alpha - H_\alpha), [X_a, I]] \},$$

where W_α and H_α are the matrices representing w_α and h_α in the regular representation of $U(N) \otimes U(N)$. The W term can be rewritten in terms of $[W, E]$, which vanishes because the sum $\sum_\alpha e_{\alpha A} e_{\alpha B}$ is G_W invariant. The H term can be rewritten in terms of the double commutator $[X_a, [H_\alpha, I]]$, which vanishes because I is invariant under H , plus the double commutator $[[H_\alpha, X_a], I]$, which gives no contribution because of the "subgroup-alignment" condition (6.18). Thus, we see that $u_{a\alpha}$ is our eigenvector

$$\sum_b m^2_{ab} u_{b\alpha} = 0. \quad (8.2)$$

These will be called the *fictitious Goldstone bosons*, because as we shall see, it is just these that are eliminated by the unitarity gauge condition.

The number of independent fictitious Goldstone bosons is evidently equal to the dimensionality of G_W minus the dimensionality of that subgroup H_W of G_W which is unbroken by the spontaneous sym-

metry breakdown of $U(N) \otimes U(N)$ to H . Of course, for every generator of H_w there will be a vector boson whose mass remains zero to all orders in e , so the number of fictitious Goldstone bosons equals the number of *massive* vector bosons. (In the real world, H_w would presumably consist solely of electromagnetic gauge transformations.)

The argument that led from (8.1) to (8.2) did not actually depend on the fact that W_α is a generator of G_w , but only on the fact that it is a symmetry of the weak and electromagnetic interactions, so that W_α commutes with E . But in general there will be broken symmetries of the weak interactions that are not themselves generators of G_w . Exactly the same reasoning tells us that for each of these there will be another eigenvector of m^2_{ab} with eigenvalue zero. These will be called the *true Goldstone bosons*, because they remain massless to all orders in e but, as we shall see, they are not eliminated by the Higgs phenomenon. The number of true and fictitious Goldstone bosons is equal to the dimensionality of the complete global symmetry group S_w of the weak and electromagnetic interactions, minus the dimensionality of that subgroup of S_w which is left unbroken by the spontaneous breakdown of $U(N) \otimes U(N)$ to H . Of course, the occurrence of true Goldstone bosons would present grave difficulties for any theory that has pretensions of providing a realistic model of the actual world. These problems are further discussed in Sec. X.

Finally, there will in general be eigenvectors u_a of m^2_{ab} for which the quantity $\sum_a u_a x_a / F_a$ cannot be expressed as a linear combination of generators of S_w and generators of H . There is no reason in this case why the eigenvalue should vanish, so we expect a mass m of order

$$m^2 \approx e^2 I / F^2 \approx e^2 M^2 \quad (8.3)$$

about the same as for the vector bosons. These are called the *pseudo-Goldstone bosons*,¹⁷ because they are not the Goldstone bosons of any true symmetry of the whole theory, but only of an accidental approximate symmetry which appears exact in the limit $e \rightarrow 0$. As we shall see, the pseudo-Goldstone bosons, like the true Goldstone bosons, are not eliminated by the Higgs phenomenon. The total number of all Goldstone bosons, fictitious, true and pseudo, is simply equal to the dimensionality $2N^2$ of $U(N) \otimes U(N)$ minus the dimensionality of the unbroken subgroup H .

By use of the familiar Schmidt orthogonalization technique, we can choose an orthonormal set of eigenvectors of m^2_{ab}

$$\sum_b m^2_{ab} u_b^n = m_n^2 u_a^n, \quad (8.4)$$

$$\sum_a u_a^n u_a^m = \delta_{nm}, \quad (8.5)$$

with each u^n representing either a fictitious, true, or pseudo-Goldstone bosons. That is, we first choose an orthonormal set of u_a^n vectors corresponding to fictitious Goldstone bosons, for which $\sum u_a^n x_a / F_a$ is a linear combination of H and G_w generators; then add an orthonormal set corresponding to true Goldstone bosons, for which $\sum u_a^n x_a / F_a$ is a linear combination of H and S_w but not H and G_w generators; and finally add an orthonormal set corresponding to pseudo-Goldstone bosons, for which $\sum u_a^n x_a / F_a$ is not a linear combination of H and S_w generators. With a set of orthonormal vectors u^n constructed in this way, we can define a corresponding set of Goldstone boson fields

$$\Pi^n \equiv \sum_a u_a^n \Pi_a, \quad (8.6)$$

each of definite mass and type (fictitious, true, or pseudo). Further, since the u^n form a complete set, we can also write

$$\Pi_a \equiv \sum_n u_a^n \Pi^n. \quad (8.7)$$

In particular, we have

$$\sum_a \partial_\mu \Pi^a \partial^\mu \Pi^a = \sum_n \partial_\mu \Pi^n \partial^\mu \Pi^n,$$

so the Π^n are canonically normalized.

Now let us impose the condition (7.6) that defines the unitarity gauge. A vector u^n which corresponds to a true or a pseudo-Goldstone boson will be orthogonal to all u_a^n corresponding to the fictitious Goldstone bosons, and hence also to the vectors $u_{a\alpha}$ defined by Eq. (8.1). But it follows then that

$$\begin{aligned} \text{Tr} \left(i v_\alpha \sum_a x_a u_a^n F_a \right) &= \text{Tr} \left(\sum_b \frac{x_b u_{b\alpha}}{F_b} \sum_a x_a u_a^n F_a \right) \\ &= 8 \sum_a u_{a\alpha} u_a^n = 0. \end{aligned}$$

Hence the unitarity gauge condition (7.6) imposes no constraint on the fields Π^n representing true or pseudo-Goldstone bosons. On the other hand, a u_a^n which corresponds to a fictitious Goldstone boson allows the decomposition

$$\sum_n \frac{u_a^n x_a}{F_a} = w^n + h^n,$$

where w^n and h^n are generators of G_w and H , respectively. It follows that

$$\begin{aligned} \text{Tr} \left(w^n \sum_a x_a u_a^m F_a \right) &= \text{Tr} \left(\sum_b \frac{x_b u_b^n}{F_b} \sum_a x_a u_a^m F_a \right) \\ &= 8 \sum_a u_a^n u_a^m = 8 \delta_{nm}. \end{aligned}$$

Thus (7.5) requires that

$$0 = \text{Tr} \left(w^n \sum_a x_a \Pi'_a F_a \right) = 8 \Pi^{n'}.$$

We conclude that the whole effect of the condition of unitarity gauge is to eliminate the Π^n corresponding to fictitious Goldstone bosons, leaving the masses and fields of the true and the pseudo-Goldstone bosons unchanged.

IX. AN EXAMPLE

We shall now descend from the generality of the previous discussion to the consideration of one specific example. It probably is unnecessary to remark that this model is totally unrealistic as a theory of real particles or interactions; it is presented solely for the purposes of illustration.

Our model contains two color multiplets of fermions, called q and h . (Color indices are dropped everywhere.) In the limit $e \rightarrow 0$, the strong interactions are necessarily invariant under a group $U(2) \otimes U(2)$ of global transformations

$$\begin{pmatrix} q \\ h \end{pmatrix} \rightarrow U \begin{pmatrix} q \\ h \end{pmatrix}, \quad (9.1)$$

with U a unitary matrix (commuting with color), involving both the Dirac matrices 1 and γ_5 . The generators of this algebra are defined as

$$\begin{aligned} \tilde{\lambda}_L &\equiv \frac{1}{\sqrt{2}} (1 + \gamma_5) \tilde{\tau}, & \lambda_{0L} &\equiv \frac{1}{\sqrt{2}} (1 + \gamma_5), \\ \tilde{\lambda}_R &\equiv \frac{1}{\sqrt{2}} (1 - \gamma_5) \tilde{\tau}, & \lambda_{0R} &\equiv \frac{1}{\sqrt{2}} (1 - \gamma_5), \end{aligned} \quad (9.2)$$

with $\tilde{\tau}$ the usual 2×2 Pauli matrices.

We assume that the $U(2) \otimes U(2)$ symmetry is dynamically broken down to the largest subgroup which will allow one of the two fermion multiplets to acquire a mass:

$$H = U(1) \otimes U(1) \otimes U(1). \quad (9.3)$$

(The color gauge group G_s is assumed to remain unbroken.) We can always define the fermion fields so that it is q that remains massless; the labels “ q ” and “ h ” thus stand for “quark” and “heavy fermion.” With this definition, the generators of H are

$$-\frac{1}{F_L^2} \bar{q} \gamma^\mu (1 + \gamma_5) q (\Pi_{1L} \partial_\mu \Pi_{2L} - \Pi_{2L} \partial_\mu \Pi_{1L}) - \frac{1}{F_R^2} \bar{q} \gamma^\mu (1 - \gamma_5) q (\Pi_{1R} \partial_\mu \Pi_{2R} - \Pi_{2R} \partial_\mu \Pi_{1R}) \quad (9.9)$$

and a trilinear self-interaction of Goldstone bosons

$$-\frac{1}{\sqrt{2} F_0} \left(1 - \frac{F_0^2}{F_L^2} \right) \partial_\mu \Pi_0 (\Pi_{1L} \partial^\mu \Pi_{2L} - \Pi_{2L} \partial^\mu \Pi_{1L}) + \frac{1}{\sqrt{2} F_0} \left(1 - \frac{F_0^2}{F_R^2} \right) \partial_\mu \Pi_0 (\Pi_{1R} \partial^\mu \Pi_{2R} - \Pi_{2R} \partial^\mu \Pi_{1R}). \quad (9.10)$$

$$\begin{aligned} t_L &\equiv \frac{1}{\sqrt{2}} (\lambda_{0L} + \lambda_{3L}) = \frac{1}{2} (1 + \gamma_5) (1 + \tau_3), \\ t_R &\equiv \frac{1}{\sqrt{2}} (\lambda_{0R} + \lambda_{3R}) = \frac{1}{2} (1 - \gamma_5) (1 + \tau_3), \end{aligned} \quad (9.4)$$

$$t_0 \equiv \frac{1}{2} (\lambda_{0L} + \lambda_{0R} - \lambda_{3L} - \lambda_{3R}) = \frac{1}{\sqrt{2}} (1 - \tau_3).$$

We can complete an orthonormal basis for $U(N) \otimes U(N)$ with the five additional generators

$$\begin{aligned} x_{1L} &\equiv \lambda_{1L} = \frac{1}{\sqrt{2}} (1 + \gamma_5) \tau_1, \\ x_{2L} &\equiv \lambda_{2L} = \frac{1}{\sqrt{2}} (1 + \gamma_5) \tau_2, \\ x_{1R} &\equiv \lambda_{1R} = \frac{1}{\sqrt{2}} (1 - \gamma_5) \tau_1, \\ x_{2R} &\equiv \lambda_{2R} = \frac{1}{\sqrt{2}} (1 - \gamma_5) \tau_2, \end{aligned} \quad (9.5)$$

$$x_0 \equiv \frac{1}{2} (\lambda_{0L} - \lambda_{0R} - \lambda_{3L} + \lambda_{3R}) = \frac{1}{\sqrt{2}} \gamma_5 (1 - \tau_3).$$

With respect to the $O(2) \otimes O(2)$ group generated by t_L and t_R , these generators transform according to the representations

$$\begin{aligned} \{x_{1L}, x_{2L}\} & \quad (2, 1), \\ \{x_{1R}, x_{2R}\} & \quad (1, 2), \\ x_0 & \quad (1, 1). \end{aligned}$$

The Goldstone bosons Π_a may be chosen to belong to corresponding $O(2) \otimes O(2)$ multiplets. With this definition, we automatically have

$$\langle 0 | J_A^\mu \Pi_a \rangle \propto B_{aA} F_a, \quad (9.6)$$

where B_{aA} are the coefficients in (9.5),

$$x_a = \sum_A B_{aA} F_A \lambda_A,$$

and the F 's are equal within irreducible H multiplets,

$$F_{1L} = F_{2L} \equiv F_L, \quad F_{1R} = F_{2R} \equiv F_R. \quad (9.7)$$

We expect that the F 's are of the same order of magnitude as the heavy quark mass,

$$F_L \approx F_R \approx F_0 \approx M_h. \quad (9.8)$$

It is straightforward to calculate the covariant derivatives (5.6) and (5.7) for $e=0$ as power series in the Π_a . For instance, the effective Lagrangian contains a bilinear interaction of Goldstone bosons with quarks

The vector and axial-vector bilinear covariants that can be formed from the q field do not share the same H transformation properties as any of the Π_a , so there is no $\bar{q}qD\Pi$ term in the leading part of the effective Lagrangian. There is, however, a Fermi interaction, which after Fierz transformations may be put in the form

$$-G_{LL}[\bar{q}\gamma^\mu(1+\gamma_5)q][\bar{q}\gamma_\mu(1+\gamma_5)q] - G_{LR}[\bar{q}\gamma^\mu(1+\gamma_5)q][\bar{q}\gamma_\mu(1-\gamma_5)q] - G_{RR}[\bar{q}\gamma^\mu(1-\gamma_5)q][\bar{q}\gamma_\mu(1-\gamma_5)q]. \quad (9.11)$$

We expect the constants G_{LL} , G_{LR} , and G_{RR} to be of order $1/M_h^2$.

Now let us turn on the "weak" interactions. We will assume that the weak gauge group is

$$G_W = \text{SU}(2) \quad (9.12)$$

with both left- and right-handed fermion fields $[(1 \pm \gamma_5)q, (1 \pm \gamma_5)h]$ transforming as G_W doublets. However, we do not immediately know *which* $\text{SU}(2)$ subgroup of $\text{U}(2) \otimes \text{U}(2)$ generates the weak interactions. In general, the generators of $\text{SU}(2)$ might be any matrices of the form

$$w_\alpha = e \sum_\beta [(g_L)_{\alpha\beta} \lambda_{L\beta} + (g_R)_{\alpha\beta} \lambda_{R\beta}], \quad (9.13)$$

where e is the $\text{SU}(2)$ gauge coupling constant; α and β run over the values 1, 2, 3; λ_L and λ_R are the matrices (9.2); and g_L and g_R are *unknown* 3×3 orthogonal matrices. Nor is it arbitrary which g matrices we choose; the definition of the fermions has been fixed (up to an H transformation) by our convention that the spontaneous breakdown of $\text{U}(2) \otimes \text{U}(2)$ to H gives a mass to h , not q .

In order to settle this question, we must examine the "potential" term in the effective Lagrangian. In general, this has the form (6.8):

$$V(g) = - \sum_{\alpha\beta} e_{\alpha A}(g) e_{\beta B}(g) I_{AB}, \quad (9.14)$$

where I_{AB} is some unknown H invariant of order M_h^4 , and $e_{\alpha A}(g)$ are the coefficients which give the G_W generators as linear combinations of the $\text{U}(2) \otimes \text{U}(2)$ generators

$$\begin{aligned} e_{\alpha, BL}(g) &= e(g_L)_{\alpha\beta}, \\ e_{\alpha, BR}(g) &= e(g_R)_{\alpha\beta}. \end{aligned} \quad (9.15)$$

Thus V here takes the form

$$\begin{aligned} V(g) &= -e^2 \sum_{\alpha\beta\gamma} [(g_L)_{\alpha\beta} (g_L)_{\alpha\gamma} I_{LB, L\gamma} \\ &\quad + 2(g_L)_{\alpha\beta} (g_R)_{\alpha\gamma} I_{LB, R\gamma} \\ &\quad + (g_R)_{\alpha\beta} (g_R)_{\alpha\gamma} I_{RB, R\gamma}]. \end{aligned}$$

But the g matrices are orthogonal, so this immediately simplifies to

$$V(g) = -2e^2 \sum_{\beta\gamma} (g_L^{-1} g_R)_{\beta\gamma} I_{LB, R\gamma} + \text{constant}.$$

In addition, the H invariance of I_{AB} requires that it be invariant under *independent* rotations around the 3 axis on either the $L\alpha$ and/or $R\alpha$ indices; thus, in particular,

$$I_{LB, R\gamma} = I n_\beta n_\gamma, \quad (9.16)$$

where \vec{n} is a unit vector pointing in the 3-direction

$$\vec{n} = (0, 0, 1),$$

and I is some unknown constant of order M_h^4 . The potential has now become simply

$$V(g) = -2e^2 I (n, g_L^{-1} g_R n). \quad (9.17)$$

This is to be minimized over the whole range of orthogonal matrices g_L, g_R . The location of such a minimum is quite obvious:

$$(A) \text{ For } I > 0, \quad g_L^{-1} g_R n = +n. \quad (9.18a)$$

$$(B) \text{ For } I < 0, \quad g_L^{-1} g_R n = -n. \quad (9.18b)$$

This does not, of course, entirely determine g_L and g_R ; given any solution, we can find another of the form

$$g'_L = g_1 g_L g_2, \quad g'_R = g_1 g_R g_3,$$

where g_1 is an arbitrary orthogonal matrix, and g_2 and g_3 are orthogonal matrices representing arbitrary independent rotations around the 3 axis. But g_1 represents a redefinition of the weak gauge couplings by an $\text{SU}(2)$ transformation belonging to the gauge group G_W , while g_2 and g_3 represent a redefinition of the fermion fields by a transformation belonging to that subgroup H of $\text{U}(2) \otimes \text{U}(2)$ which is not spontaneously broken. Clearly, there is no way that this remaining ambiguity in the g 's could ever be resolved, nor is there any reason why we would wish to do so. Thus, we can freely choose *any* orthogonal g_L and g_R matrices which satisfy the condition for a minimum, Eq. (9.18).

We will now need to consider the two cases separately.

A. $I > 0$

Here it is convenient to choose g_L and g_R as unit matrices

$$g_L = g_R = 1. \quad (9.19)$$

The generators of the weak gauge group are then given by (9.13) as

$$\vec{w} = e(\vec{\lambda}_L + \vec{\lambda}_R) = e\sqrt{2} \vec{\tau}, \quad (9.20)$$

with $\vec{\tau}$ the usual 2×2 Pauli matrices. There is a single 2×2 matrix which is a generator of *both* a G_W gauge transformation and an unbroken H transformation

$$e\tau_3 = \frac{1}{\sqrt{2}} w_3 = \frac{e}{2} (t_L + t_R) - \frac{e}{\sqrt{2}} t_0. \quad (9.21)$$

This corresponds to a “photon,” which keeps zero mass despite the spontaneous symmetry breaking. The gauge bosons corresponding to the other two generators of G_W , w_1 , and w_2 , acquire a mass by the Higgs mechanism, given by (7.10) is

$$\mu_1^2 = \mu_2^2 = e^2 (F_L^2 + F_R^2). \quad (9.22)$$

These massive gauge bosons have w_3 charges $+1$ and -1 . Also, q has w_3 charge $+1$, while the Goldstone bosons $\Pi_{1L} \pm i\Pi_{2L}$, $\Pi_{1R} \pm i\Pi_{2R}$, and Π_0 have w_3 charges ± 1 , ± 1 , and 0 , respectively.

In order to classify the Goldstone bosons, we note first that there are two independent linear combinations of the x_a that can be expressed as linear combinations of generators of H and G_W :

$$e(x_{1L} + x_{1R}) = w_1, \quad (9.23)$$

$$e(x_{2L} + x_{2R}) = w_2, \quad (9.24)$$

and there is also one other linear combination of the x_a that can be expressed as a linear combination of a generator of H and the generator of the exact global symmetry $\psi \rightarrow \exp(i\gamma_5 \epsilon) \psi$ of the whole Lagrangian

$$x_0 = \sqrt{2} \gamma_5 1 - \frac{1}{\sqrt{2}} (t_L + t_R). \quad (9.25)$$

In accordance with the conclusions of Sec. VIII, we must therefore expect that this theory has two fictitious Goldstone bosons, one true Goldstone boson, and $5 - 2 - 1 = 2$ pseudo-Goldstone bosons. Their fields are of the form

$$\Pi^n = \sum_a u_a^n \Pi_a,$$

with u^n a set of *orthonormal* vectors subject to certain conditions: For the fictitious Goldstone

bosons $\sum_a u_a^n x_a / F_a$ must be a linear combination of (9.23) and (9.24), so the fields are

$$\Pi^I \equiv (F_L^2 + F_R^2)^{-1/2} (F_L \Pi_{1L} + F_R \Pi_{1R}), \quad (9.26)$$

$$\Pi^{II} \equiv (F_L^2 + F_R^2)^{-1/2} (F_L \Pi_{2L} + F_R \Pi_{2R}). \quad (9.27)$$

For the true Goldstone boson, $\sum_a u_a^n x_a / F_a$ must be proportional to (9.25), so the field is simply

$$\Pi^0 \equiv \Pi_0. \quad (9.28)$$

For the pseudo-Goldstone bosons the u_a^n need only be orthogonal to all the others, so the fields are

$$\Pi^I \equiv (F_L^2 + F_R^2)^{-1/2} (-F_R \Pi_{1L} + F_L \Pi_{1R}), \quad (9.29)$$

$$\Pi^2 \equiv (F_L^2 + F_R^2)^{-1/2} (-F_R \Pi_{2L} + F_L \Pi_{2R}). \quad (9.30)$$

The masses of Π^I , Π^{II} , and Π^0 are zero, while second-order weak (and “electromagnetic”) effects will give Π^I and Π^2 masses that are equal (because w_3 invariance is unbroken) and of order eM_h . These masses are proportional to the constant I in Eq. (9.16), but I is unknown, so the calculation is not worthwhile. The effect of the transformation to unitarity gauge is just to eliminate the fields of the fictitious Goldstone bosons:

$$\Pi^I = \Pi^{II} = 0.$$

The five “old” fields Π'_a may then be expressed in terms of the three “new” fields Π'' as

$$\Pi'_{1L} = -\frac{F_R}{(F_L^2 + F_R^2)^{1/2}} \Pi^{1'},$$

$$\Pi'_{1R} = \frac{F_L}{(F_L^2 + F_R^2)^{1/2}} \Pi^{1'},$$

$$\Pi'_{2L} = -\frac{F_R}{(F_L^2 + F_R^2)^{1/2}} \Pi^{2'},$$

$$\Pi'_{2R} = \frac{F_L}{(F_L^2 + F_R^2)^{1/2}} \Pi^{2'},$$

and, of course

$$\Pi'_0 = \Pi^{0'}.$$

In particular, the bilinear interaction (9.9) of the nonfictitious Goldstone bosons with the quarks is (now dropping primes)

$$(F_L^2 + F_R^2)^{-1/2} \left[-\frac{F_R}{F_L^2} \bar{q} \gamma^\mu (1 + \gamma_5) q - \frac{F_L}{F_R^2} \bar{q} \gamma^\mu (1 - \gamma_5) q \right] (\Pi^1 \partial_\mu \Pi^2 - \Pi^2 \partial_\mu \Pi^1), \quad (9.31)$$

and their trilinear self-interaction (9.10) is

$$2^{-1/2} (F_L^2 + F_R^2)^{-1/2} F_0^{-1} \left[F_R \left(1 - \frac{F_0^2}{F_L^2} \right) + F_L \left(1 - \frac{F_0^2}{F_R^2} \right) \right] \partial_\mu \Pi^0 (\Pi^1 \partial^\mu \Pi^2 - \Pi^2 \partial^\mu \Pi^1). \quad (9.32)$$

The quark mass matrix is given to order $e^2 M_h$ by Eq. (5.29), which here becomes

$$M_q = e^2 \sum_{\alpha} (Q_{\alpha L, \alpha L} + 2Q_{\alpha L, \alpha R} + Q_{\alpha R, \alpha R}),$$

where Q_{AB} is some constant matrix of order M_h which transforms as a tensor under H , in the sense of Eq. (C4). (Again, we let α and β run over the values 1, 2, 3.) It is straightforward to show that the most general such tensor has

$$Q_{\alpha L, \beta L} = Q_{\alpha R, \beta R} = 0,$$

and also (with a suitable choice of relative phase for the left- and right-handed quark fields)

$$Q_{\alpha L, \beta R} = Q[\frac{1}{2}(1 + \gamma_5)n_{\alpha}^{\dagger}n_{\beta} + \frac{1}{2}(1 - \gamma_5)n_{\alpha}n_{\beta}^{\dagger}],$$

where $Q \approx M_h$ is some unknown constant, and

$$\hat{n}^{\pm} \equiv \frac{1}{\sqrt{2}}(1, \pm i, 0).$$

Thus, the quark here does acquire a mass of order $e^2 M_h$

$$M_q = 2e^2 Q. \quad (9.33)$$

Also, using Eq. (5.28), the Yukawa coupling here is

$$\frac{4iM_q}{\sqrt{2}F_0} \bar{q} \gamma_5 q \Pi^0 \quad (9.34)$$

as required by a Goldberger-Treiman relation.

It happens that in this model either the massive vector bosons or the pseudo-Goldstone bosons are absolutely stable. This is just because they have w_3 charges ± 1 ; the only lighter states into which they could decay consist of "photons," true Goldstone bosons, and quark-antiquark pairs, all of which are w_3 neutral.

B. $I < 0$

Here it is convenient to choose g_L and g_R as equal and opposite rotations of 90° about the 1 axis. The generators of the weak gauge group are then

$$w_1 = e(\lambda_{1L} + \lambda_{1R}) = e\sqrt{2} \tau_1,$$

$$w_2 = e(\lambda_{3L} - \lambda_{3R}) = e\sqrt{2} \gamma_5 \tau_3,$$

$$w_3 = e(-\lambda_{2L} + \lambda_{2R}) = -e\sqrt{2} \gamma_5 \tau_2,$$

with $\hat{\tau}$ again the 2×2 Pauli matrices. No linear combination of these generators is a generator of the unbroken subgroup H , so there is no "photon" here—every vector boson gets a mass. From (7.11), we find that the masses are

$$\mu_1^2 = \mu_3^2 = e^2(F_L^2 + F_R^2),$$

$$\mu_2^2 = F_0^2.$$

In order to classify the Goldstone bosons, we note first that there are three independent linear combinations of the x_a that can be expressed as linear combinations of the generators of H and G_W :

$$x_{1L} + x_{1R} = w_1/e, \quad (9.35)$$

$$x_0 = \frac{1}{\sqrt{2}}(t_L - t_R) - w_2/e, \quad (9.36)$$

$$x_{2L} - x_{2R} = -w_3/e, \quad (9.37)$$

while there is no other linear combination of the x_a that can be expressed as a linear combination of a generator of H and the generator of any exact global symmetry of the whole Lagrangian. According to Sec. VIII, we must now expect that this theory has three fictitious Goldstone bosons, no true Goldstone bosons, and $5 - 3 = 2$ pseudo-Goldstone bosons. Their fields are of the form

$$\Pi^n = \sum_a u_a^n \Pi_a, \quad (9.38)$$

with u_a^n a set of orthonormal vectors subject to certain conditions: For the fictitious Goldstone bosons $\sum_a u_a^n x_a/F_a$ must be a linear combination of (9.35)–(9.37), so the fields are

$$\Pi^1 = (F_L^2 + F_R^2)^{-1/2}(F_L \Pi_{1L} + F_R \Pi_{1R}), \quad (9.39)$$

$$\Pi^0 = \Pi_0, \quad (9.40)$$

$$\Pi^1 = (F_L^2 + F_R^2)^{-1/2}(F_L \Pi_{2L} - F_R \Pi_{2R}). \quad (9.41)$$

For the pseudo-Goldstone bosons the u^n must simply be orthogonal to the others, so the fields are

$$\Pi^1 = (F_L^2 + F_R^2)^{-1/2}(-F_R \Pi_{1L} + F_L \Pi_{1R}), \quad (9.42)$$

$$\Pi^2 = (F_L^2 + F_R^2)^{-1/2}(F_R \Pi_{2L} + F_L \Pi_{2R}). \quad (9.43)$$

The masses of Π^1 , Π^1 , and Π^0 are zero, while second-order weak effects give Π^1 and Π^2 masses proportional to I and of order eM_h . These latter masses are equal, because the whole Lagrangian has an exact global symmetry which is not spontaneously broken, generated by

$$\lambda_{0L} - \lambda_{0R} + \lambda_{3L} - \lambda_{3R} = t_L - t_R = \lambda_{0L} - \lambda_{0R} + w_2/e, \quad (9.44)$$

and this rotates Π^1 and Π^2 into each other.

The effect of the transformation to unitarity gauge is to eliminate the fields of the fictitious Goldstone bosons

$$\Pi'^1 = \Pi'^2 = \Pi'^3 = 0.$$

The five fields Π'_a may be expressed in terms of the two remaining Π^n fields, as

$$\begin{aligned}\Pi'_{1L} &= -(F_L^2 + F_R^2)^{-1/2} F_R \Pi^{1'}, \\ \Pi'_{1R} &= (F_L^2 + F_R^2)^{-1/2} F_L \Pi^{1'}, \\ \Pi'_{2L} &= +(F_L^2 + F_R^2)^{-1/2} F_R \Pi^{2'}, \\ \Pi'_{2R} &= (F_L^2 + F_R^2)^{-1/2} F_L \Pi^{2'},\end{aligned}$$

$$\Pi'_0 = 0.$$

In particular, the bilinear interaction (9.9) of the nonfictitious Goldstone bosons with the quarks is (now dropping primes)

$$\left[\frac{F_R}{F_L^2(F_L^2 + F_R^2)^{1/2}} \bar{q} \gamma^\mu (1 + \gamma_5) q - \frac{F_L}{F_R^2(F_L^2 + F_R^2)^{1/2}} \bar{q} \gamma^\mu (1 - \gamma_5) q \right] (\Pi^1 \partial_\mu \Pi^2 - \Pi^2 \partial_\mu \Pi^1). \quad (9.45)$$

The trilinear coupling (9.10) now vanishes. Also, as a consequence of the exact unbroken symmetry generated by (9.44), the quark mass remains zero to all orders in e . The Yukawa $\Pi \bar{q} q$ coupling vanishes, as required by a Goldberger-Treiman relation, even though it is not forbidden by the symmetry generated by (9.44).

The moral of this analysis is twofold. First, a theory with a given group-theoretic character and a given field content can have enormously different physical consequences depending on how the subgroups G_w and H line up with each other. The differences between the two cases found in this section are summarized in Table I. In addition, although these theories do not have the predictive power of a theory in which the spontaneous symmetry breaking is due to vacuum expectation values of weakly coupled scalar fields, the predictive power of theories with dynamical symmetry breaking is by no means negligible.

X. IMPLICATIONS

The foregoing analysis has been chiefly concerned with mathematical formalism rather than physical applications. We close with some remarks about the implications of this analysis for real particles and interactions.

The weak interactions in this class of theories arise both from the exchange of intermediate vector bosons and also from a direct Fermi interaction in the effective Lagrangian. Both are of the same order of magnitude; the direct Fermi interaction has a coupling constant of order M^{-2}

(where M is the scale associated with the dynamical symmetry breaking), while the exchange of vector bosons of mass $\mu \approx eM$ produces an effective Fermi coupling of order $e^2/\mu^2 \approx M^{-2}$. Either way, we are led to the estimate that $M \approx 300$ GeV.

The two kinds of weak interactions can of course be distinguished by their energy dependence at energies of order eM or greater. They may also be distinguished even at lower energies by their symmetry properties; the direct Fermi interaction is invariant under the unbroken subgroup H of $U(N) \otimes U(N)$, while the vector-boson exchange interaction is not.

Some of the fermions of these theories may get masses of order M from the dynamical symmetry breaking. However, the ordinary quarks $\mathcal{P}, \mathcal{N}, \lambda, \mathcal{P}'$ (etc.?) can hardly be this heavy,²⁵ so we must suppose that the unbroken subgroup H must be large enough to prevent the appearance of masses of order M . The masses of the ordinary quarks would then have to arise from higher-order corrections, which would presumably give them values of order $e^2 M$. This is a gratifying result, for it offers at least a qualitative explanation of the mysterious fact that the ratio of the mass scale of the hadrons (say, 1 GeV) to that of the Fermi interaction (300 GeV) is roughly of order α .

Of course, in order to produce quark masses of order $e^2 M$, the unbroken symmetry group H must not be too large. Specifically, there must be no chiral symmetries in H which are also symmetries of the weak and electromagnetic interactions. The breakdown of $U(N) \otimes U(N)$ to H may be signaled by the appearance of fermion masses of order M , but

TABLE I. Summary of the properties of the model discussed in Section IX, in two cases corresponding to the two different possible minima of the potential $V(g)$.

	A	B
Massive vector bosons	2 degenerate	3 (2 degenerate)
Massless vector bosons	1	0
Unbroken global symmetries (including fermion conservation)	1	2
True Goldstone bosons	1	0
Pseudo-Goldstone bosons	2 degenerate	2 degenerate
Quark mass	$\approx e^2 M_h$	0

it is also possible that H forbids all fermion masses of order M while allowing other nonchiral interactions, such as scalar, tensor, or pseudoscalar Fermi interactions of order M^{-2} .

In all cases that I have examined, H must be sufficiently small so that there are some broken symmetries, not in H , that are also not in the weak and electromagnetic gauge group G_w . In particular, in a theory with N' heavy fermions and $N-N'$ ordinary quarks, the broken symmetry generators include all Hermitian matrices, chiral or nonchiral, which connect ordinary quark and heavy fermion fields. If these were all generators of the weak and electromagnetic gauge group, their multiple commutators would be also gauge generators. But these commutators span the algebra of $SU(N) \otimes SU(N)$. This is not possible because then all the unbroken chiral symmetries in $SU(N) \otimes SU(N)$, which keep the ordinary quarks from getting masses of order M , would also be symmetries of the weak and electromagnetic interactions, so that the ordinary quarks could not get masses of any order in e . Also, the weak and electromagnetic gauge group cannot include $SU(N) \otimes SU(N)$; triangle anomalies would make such a theory nonrenormalizable.²⁶

For every broken symmetry which is not a symmetry of the weak and electromagnetic interactions, there is a pseudo-Goldstone boson that is not eliminated by the Higgs phenomenon. These particles have masses of order $e \times 300$ GeV, and do not interact strongly at ordinary energies, but the charged pseudo-Goldstone bosons could of course be pair-produced by the electromagnetic interactions. Thus, it will be important to distinguish carefully between pseudo-Goldstone bosons and intermediate vector bosons when colliding beams reach energies adequate to produce such particles in pairs.

It makes a great difference in the description of the decay modes and interactions of pseudo-Goldstone bosons whether they can interact with ordinary hadrons and leptons singly, or only in pairs. The broken symmetry generator corresponding to a given pseudo-Goldstone boson might have no matrix elements between quark or lepton states, but only between states of which one is a heavy (≈ 300 GeV) fermion. The pseudo-Goldstone bosons would still interact in pairs with ordinary hadrons, as in Eqs. (9.31) and (9.45), but they could only decay into each other. On the other hand, the quarks might not be entirely neutral under the various broken symmetry generators in $U(N) \otimes U(N)$, in which case some of the pseudo-Goldstone bosons would be able to decay into ordinary hadrons. In the absence of a candidate for a realistic model, it is not worth pursuing

these various possibilities in great detail.

In addition to pseudo-Goldstone bosons, such theories will usually have true Goldstone bosons of zero mass which also are not eliminated by the Higgs mechanism. This is because there are always some broken global symmetries of the weak and electromagnetic interactions which are not themselves elements of the weak and electromagnetic gauge group. For instance, one such global symmetry is the $U(1)$ chiral transformation which multiplies all fermion fields with a common factor $\exp(i\gamma_5\theta)$; this must not be a member of the weak and electromagnetic gauge group, because if it were then triangle anomalies would make the theory nonrenormalizable, and it must be spontaneously broken, because otherwise none of the fermions in the theory could pick up any mass. If the generator x of any such broken exact global symmetry could be written as a sum of a gauge symmetry generator w and a nonspontaneously broken symmetry generator h , then the corresponding Goldstone boson would be eliminated by the Higgs mechanism; however, in this case the theory would have an extra exact nonspontaneously broken symmetry (apart from fermion conservation) generated by $x - w = h$. This is what happens in case B of the model discussed in Sec. IX; the extra symmetry there keeps the quark massless to all orders in e . If we do not want to allow such extra exact symmetries then the generators of broken nongauge symmetries of the weak and electromagnetic interactions must not be linear combinations of gauge and unbroken symmetry generators, and the corresponding Goldstone bosons cannot be eliminated by the Higgs mechanism.

In the particular case of the chiral $U(1)$ symmetry mentioned above, it is possible that the massless true Goldstone boson, although not eliminated by the Higgs mechanism, would nevertheless not be observable as a free particle. There is a triangle anomaly connecting one $\gamma_\mu\gamma_5$ vertex to two colored gluons; this anomaly forces us to include gluon terms in the conserved chiral current, which make it not gluon-gauge invariant. Since the corresponding true Goldstone boson cannot be proved to appear as a pole in any gluon-gauge-invariant operator, there is at least a chance that it is a trapped particle, like the unobserved ninth pseudoscalar meson²⁷ with mass $< \sqrt{3}m_\pi$.

In a variety of models there are also true Goldstone bosons which *could* be observed as free particles. For instance, in the familiar four-quark version²⁸ of the $SU(2) \otimes U(1)$ model there is (in the absence of elementary spin-zero fields) an exact global symmetry of the weak and electromagnetic as well as the strong interactions, of the form

$$\begin{aligned}\mathcal{N}_R &\rightarrow \cos\phi \mathcal{N}_R + \sin\phi \lambda_R, \\ \lambda_R &\rightarrow -\sin\phi \mathcal{N}_R + \cos\phi \lambda_R, \\ \mathcal{P}_R, \mathcal{P}'_R, \mathcal{P}_L, \mathcal{P}'_L, \mathcal{N}_L, \lambda_L &\text{ invariant.}\end{aligned}$$

[As usual a subscript L or R denotes multiplication with $(1+\gamma_5)$ or $(1-\gamma_5)$, respectively.] This must be spontaneously broken, for otherwise λ and \mathcal{N} quarks could not have any mass, and for the same reason it cannot be expressed as a sum of a gauge and an unbroken generator. Also, in this case the symmetry current is both conserved and gluon-gauge-invariant. Thus, a four-quark $SU(2) \otimes U(1)$ model with purely dynamical symmetry breaking will have an untrapped massless true Goldstone boson.

I do not know whether present experiments rule out the possibility of electrically neutral and weakly interacting spin-zero bosons of zero mass. However, if we assume (as seems reasonable) that such particles do not in fact exist, then their absence puts a strong constraint on theories of dynamical symmetry breaking. In particular, it is probably necessary to have weak interactions that connect the ordinary quarks with heavy (~ 300 -GeV) fermions, not only as a means of giving masses of order $e^2 \times 300$ GeV to the ordinary quarks, but also to avoid the unwanted anomaly-free global symmetries of the weak interactions.

We now come to one of the most puzzling and unsatisfactory features of dynamical symmetry breaking. In the currently popular gauge theories of strong interactions, the strong gauge coupling constant is fairly small at a renormalization point of order 2–3 GeV, and decreases further with increasing energy.³ How then can the strong interaction produce a spontaneous symmetry breaking characterized by parameters F_a of order 300 GeV? Indeed, we believe that the strong interactions do induce a spontaneous symmetry breakdown, with the pion octet playing the role of Goldstone bosons, but the parameter F_π is 190 MeV, not 300 GeV.

Another difficulty arises when we try to include the leptons. If it is the ordinary strong interactions that produce the dynamical symmetry breaking discussed in this article, then can the color-neutral leptons get a mass in any order of e ?

One way to approach these problems is to suppose that in addition to the color $SU(3)$ associated with the observed strong interactions, there is another gauge group whose generators commute with color $SU(3)$, associated with a new class of “extra-strong” interactions, which act on leptons as well as other fermions. If the gauge coupling constant of the extra-strong interactions reaches a value of order unity at a renormalization point of scale 300 GeV, then the extra-strong interac-

tions could produce the dynamical symmetry breaking discussed in this article. Also, we would not observe direct effects of the extra-strong interactions at ordinary energies, provided that this dynamical symmetry breaking left no subgroup of the extra-strong gauge group unbroken, so that all vector bosons of the extra strong interactions got masses of order 300 GeV.

At first sight this possibility seems quite natural in the framework of the unified simple gauge theories of weak electromagnetic, and strong interactions discussed in the Introduction.⁶ The spontaneous superstrong breakdown of the original simple gauge group can leave any number of subgroups unbroken, and some of these may have gauge couplings which grow faster with decreasing renormalization-point energy than the coupling constant of the ordinary strong interactions. However, this naturalness disappears on closer examination. Within the realm of validity of perturbation theory, the gauge couplings g_i of the various simple subgroups of the original unified simple group are given by

$$g_i^2(\mu) = \frac{g_i^2(\bar{M})}{1 + 2b_i g_i^2(\bar{M}) \ln(\bar{M}/\mu)},$$

where μ is the scale of a variable renormalization point; \bar{M} is the superlarge mass at which all the $g_i(\mu)$ become equal; and b_i is the coefficient of g_i^3 in the Gell-Mann–Low function $\beta_i(g)$. Suppose we identify the onset of strong coupling for any simple subgroup as the point μ_i at which $g_i(\mu)$ reaches some definite value, of order unity, but taken sufficiently small so that perturbation theory is still valid. Then the ratio of the μ 's for two subgroups will be given by

$$\mu_j/\mu_i = (\mu_i/\bar{M})^{b_i/b_j-1}.$$

But \bar{M} is likely to be enormous,⁶ perhaps as large as 10^{19} GeV. Thus, unless b_i and b_j are unreasonably close, the onset of strong interaction will differ by many orders of magnitude for different simple subgroups. From this point of view, it is hard to understand how the onset of strong coupling for the ordinary strong interactions (a few hundred MeV) and the extra-strong interactions (300 GeV) could be so close.

Another possibility is that the color $SU(3)$ gauge group is a subgroup of a larger gauge group which acts on leptons as well as on other fermions, and whose coupling constant reaches a value of order unity at a renormalization point of order 300 GeV. This could produce a dynamical symmetry breakdown of the larger group to color $SU(3)$. In the effective field theory⁹ which describes physics below 300 GeV, there could be a color $SU(3)$ gauge symmetry, but since perturbation theory breaks

down at 300 GeV, the strong gauge coupling in this effective theory would have no simple relation to the strong gauge coupling above 300 GeV, and might well be somewhat smaller. It would rise very slowly with decreasing renormalization-point energy, and even if it started just under 300 GeV at a value only a little less than its value above 300 GeV, it would not regain this value until much smaller renormalization scales were reached.

In either case, it is not the ordinary color SU(3) gluons that could produce the dynamical symmetry breaking which gives masses to the intermediate vector bosons. These gluons presumably *do* produce the dynamical breakdown of the previously unbroken subgroup H to SU(3) or SU(4) at energies of order $F_\pi = 190$ MeV, with the pion octet as Goldstone bosons, and with the quark masses of order $e^2 \times 300$ GeV furnishing the intrinsic H breaking which gives masses to the pion octet.

In closing, it is interesting to compare the conclusions of this article with the results obtained in theories with elementary spin-zero fields. In order to give the weak interactions the right strength, the vacuum expectation values of some of these fields must be of order 300 GeV. However, we can still distinguish between three kinds of theory:

I. It may be that the spin-zero fields have weak [say, $O(e^2)$] couplings to themselves and to the fermions. This is the case originally considered,¹ and it is the sort of theory with by far the greatest predictive power. The quark and lepton masses in such theories could arise directly from vacuum expectation values of the spin-zero fields, so there would be no need for heavy fermions. A characteristic feature of these theories is the appearance of Higgs scalars with masses that are less than 300 GeV by a factor of order \sqrt{f} , where f is the coupling constant of the quartic interaction.

II. It may be that the spin-zero fields have weak couplings to fermions, but strong interactions to themselves. In this case much of the gauge-theory phenomenology would survive, but it would be impossible to relate the vector-boson mass ratio to mixing angles, or to say anything at all about the existence of Higgs scalars. Again, there would be no need for heavy fermions.

III. It may be that the spin-zero fields have strong couplings both to fermions and to themselves. In general, such a theory would have very little predictive power; we would not even be able to say that weak processes like β decay arise from exchange of single vector bosons rather than from complicated higher-order effects.

Viewed in this way, gauge theories with dynamical symmetry breaking seem hardly distinguishable from theories of type III. The one significant

difference, which gives theories of the type discussed in this article much greater predictive power than theories of type III, is the occurrence of a natural accidental symmetry, $U(N) \otimes U(N)$.

APPENDIX A: STRUCTURE OF COVARIANT DERIVATIVES

First, we note that under the requirements (1) and (2) of Sec. V, the effective Lagrangian \mathcal{L}_1 can be made formally invariant under $U(N) \otimes U(N)$ by introducing the G_W gauge fields

$$W_{A\mu} = \sum_{\alpha} e_{\alpha A} W_{\alpha\mu}, \quad (A1)$$

and imagining that these fields transform under $U(N) \otimes U(N)$ like λ_A . That is, we give $W_{A\mu}$ the formal transformation rule

$$W_{A\mu} \rightarrow W'_{A\mu} = \sum_B R_{AB}(g) W_{B\mu}, \quad (A2)$$

where g is an arbitrary element of $U(N) \otimes U(N)$, and $R_{AB}(g)$ is the corresponding orthogonal matrix in the regular representation of $U(N) \otimes U(N)$:

$$g^{-1} \lambda_A g = \sum_B R_{AB}(g) \lambda_B. \quad (A3)$$

[This is just like the well-known trick of introducing a fictitious octet of "photons" in order to study the SU(3) properties of electromagnetic corrections.]

Equation (A2) is a linear transformation rule, while \mathcal{L}_1 also contains quark and Goldstone boson fields which transform according to the nonlinear rules (5.2) and (5.3). It is therefore convenient, in order to make the whole of \mathcal{L}_1 invariant under $U(N) \otimes U(N)$, to replace $W_{A\mu}$ with a field which belongs to the same sort of nonlinear realization of $U(N) \otimes U(N)$ as does q :

$$\tilde{W}_{A\mu} \equiv \sum_B \Lambda_{BA}(\Pi) W_{B\mu} = \sum_{B,\alpha} \Lambda_{BA}(\Pi) e_{\alpha B} W_{\alpha\mu}, \quad (A4)$$

where Λ_{AB} is an orthogonal matrix defined by

$$\Lambda_{AB}(\Pi) = R_{AB} \left(\exp \left(i \sum_a \frac{\Pi_a x_a}{F_a} \right) \right). \quad (A5)$$

It is straightforward, using the transformation rules (A2) and (5.2)–(5.4), to check that $\tilde{W}_{A\mu}$ undergoes the $U(N) \otimes U(N)$ transformation

$$\tilde{W}_{A\mu} \rightarrow \tilde{W}'_{A\mu} = \sum_B R_{AB} \left(\exp \left(i \sum_i \mu_i(\Pi, g) t_i \right) \right) \tilde{W}_{B\mu}. \quad (A6)$$

The field $\tilde{W}_{A\mu}$ thus behaves under $U(N) \otimes U(N)$ transformations just like q , except of course that it belongs to a different linear representation of the unbroken subgroup H .

The Lagrangian \mathcal{L}_1 will thus be globally $U(N)$

$\otimes U(N)$ invariant if it is algebraically invariant under the unbroken subgroup H and if it is composed of just the following ingredients: quark fields q , their $U(N) \otimes U(N)$ covariant derivatives

$$\partial_\mu q - i \sum_{i,a} t_i q E_{ai}(\Pi) \partial_\mu \Pi, \quad (\text{A7})$$

together with Goldstone boson fields Π_a and their covariant derivatives

$$D_\mu \Pi_a \equiv F_a \sum_b D_{ba}(\Pi) \partial_\mu \Pi_b F_b^{-1}, \quad (\text{A8})$$

and $\tilde{W}_{A\mu}$ fields and their covariant derivatives

$$\partial_\mu \tilde{W}_{A\nu} - i \sum_{i,B} (t_i)_{AB} \tilde{W}_{B\nu} E_{Ai}(\Pi) \partial_\mu \Pi_a,$$

plus gluon fields and their derivatives. [The functions D_{ba} and E_{ai} are defined by Eq. (5.9).] Aside from terms which are separately $U(N) \otimes U(N)$ invariant and hence may be dropped, the covariant derivative of the gauge field may also be written

$$D_\mu \tilde{W}_{A\nu} = \sum_B \Lambda_{BA}(\Pi) \partial_\mu W_{B\nu}. \quad (\text{A9})$$

We now must impose requirement (3) of Sec. V, that the Lagrangian be locally as well as globally invariant under G_W and H_S . For a space-time dependent $U(N) \otimes U(N)$ transformation $g(x)$, the gauge field $W_{\alpha\mu}$ transforms according to the rule

$$w_\alpha W'_{\alpha\mu} = g(w_\alpha W_{\alpha\mu})g^{-1} - (\partial_\mu g)g^{-1}. \quad (\text{A10})$$

Also, derivatives of g appear in the G_W transformation rules for the quantities (A7)–(A9). By using the derivative of Eq. (5.4) to evaluate these g derivatives, we can easily see that in order to cancel them, we must add gauge field terms to (A7) and (A8), so that those derivatives become G_W -covariant quantities¹⁸:

$$\partial_\mu q - i \sum_{i,a} t_i q E_{ai}(\Pi) \partial_\mu \Pi_a - i \sum_{A,i} t_i q C_{iA} \tilde{W}_{A\mu}, \quad (\text{A11})$$

$$\sum_b D_{ba}(\Pi) \partial_\mu \Pi_b + F_a \sum_a C_{aA} \tilde{W}_{A\mu}. \quad (\text{A12})$$

Note that these quantities are still formally locally $U(N) \otimes U(N)$ covariant, as well as globally G_W covariant.

Also, G_W invariance requires that derivatives of the G_W gauge field only appear in the Yang-Mills curl²⁰

$$F_{\alpha\mu\nu} \equiv \partial_\mu W_{\alpha\nu} - \partial_\nu W_{\alpha\mu} - \sum_{\beta\gamma} C_{\alpha\beta\gamma} W_{\beta\mu} W_{\gamma\nu}, \quad (\text{A13})$$

where $C_{\alpha\beta\gamma}$ are the structure constants of G_W . It is elementary to show that

$$\sum_{B\alpha} \Lambda_{BA}(\Pi) e_{\alpha B} F_{\alpha\mu\nu} = D_\mu \tilde{W}_{A\nu} - D_\nu \tilde{W}_{A\mu} - \sum_{BC} f_{ABC} \tilde{W}_{B\mu} \tilde{W}_{C\nu}, \quad (\text{A14})$$

where f_{ABC} are the $U(N) \otimes U(N)$ structure constants. Thus, this quantity is both locally G_W covariant and (formally) globally $U(N) \otimes U(N)$ covariant.

Finally, we must impose invariance under the unbroken strong gauge group H_S . According to our assumptions, it is only q that transforms non-trivially under H_S , so we must simply add a gluon term in (A11). This, together with (A12) and (A14), comprise the three sorts of fully covariant derivatives allowed in the effective Lagrangian \mathcal{L}_1 .

APPENDIX B: STRUCTURE OF \mathcal{L}_2

The quantity $J_{AB}(\Pi)$ in Eq. (5.10) must be an $U(N) \otimes U(N)$ tensor in the sense that for any element g of $U(N) \otimes U(N)$, we have

$$J_{AB}(\Pi') = \sum_{CD} R_{AC}(g) R_{BD}(g) J_{CD}(\Pi), \quad (\text{B1})$$

with Π' and R defined by Eqs. (5.4) and (A3). But it follows from (A4) and (A5) that

$$R(g) \Lambda(\Pi) = \Lambda(\Pi') R \left(\exp \left(i \sum_i \mu_i t_i \right) \right), \quad (\text{B2})$$

so contracting Eq. (B1) on the left-hand side with $\Lambda^{-1}(\Pi')$ gives

$$I_{AB}(\Pi') = \sum_{CD} R_{AC} \left(\exp \left(i \sum_i \mu_i t_i \right) \right) \times R_{BD} \left(\exp \left(i \sum_i \mu_i t_i \right) \right) I_{CD}(\Pi), \quad (\text{B3})$$

where

$$I_{AB}(\Pi) \equiv \sum_{CD} \Lambda_{AC}^{-1}(\Pi) \Lambda_{BD}^{-1}(\Pi) J_{CD}(\Pi). \quad (\text{B4})$$

If we now choose $\Pi_a = 0$ and

$$g = \exp \left(i \sum_a x_a \pi_a / F_a \right),$$

we find from (5.4) that

$$\Pi'_a = \pi_a, \quad \mu'_i = 0,$$

so Eq. (B3) reads here

$$I_{AB}(\pi) = I_{AB}(0) \equiv I_{AB}. \quad (\text{B5})$$

Hence I_{AB} is a constant. Equation (B3) then says that it is a constant tensor under H . That is,

$$I_{AB} = \sum_{CD} R_{AC}(h) R_{BD}(h) I_{CD} \quad (\text{B6})$$

for arbitrary elements $h \in H$.

APPENDIX C: STRUCTURE OF \mathcal{L}_m

The quantity $N_{AB}(\Pi)$ in Eq. (5.26) is an $U(N) \otimes U(N)$ tensor, in the sense that for an arbitrary element g of $U(N) \otimes U(N)$, we have

$$\exp\left(-i \sum_i t_i \mu_i(\Pi, g)\right) N_{AB}(\Pi') \exp\left(i \sum_i t_i \mu_i(\Pi, g)\right) = \sum_{CD} R_{AC}(g) R_{BD}(g) N_{CD}(\Pi), \quad (C1)$$

with μ_i , Π' , and R defined by Eqs. (5.4) and (A3). Using Eq. (B2) and contracting Eq. (C1) on the left with $\Lambda^{-1}(\Pi')$ gives

$$\exp\left(-i \sum_i t_i \mu_i\right) Q_{AB}(\Pi') \exp\left(i \sum_i t_i \mu_i\right) = \sum_{CD} R_{AC}\left(\exp\left(i \sum_i t_i \mu_i\right)\right) R_{BD}\left(\exp\left(i \sum_i t_i \mu_i\right)\right) Q_{CD}(\Pi'). \quad (C2)$$

If we choose $\Pi_a = 0$ and

$$g = \exp\left(i \sum_a x_a \pi_a / F_a\right),$$

Then Eq. (5.4) gives

$$\Pi'_a = \pi_a, \quad \mu_i = 0,$$

so Eq. (C2) in this case just tells us that $Q(\pi)$ is π independent,

$$Q_{AB}(\pi) = Q_{AB}(0) \equiv Q_{AB}. \quad (C3)$$

Equation (C2) may thus be written

$$h^{-1} Q_{AB} h = \sum_{CD} R_{AC}(h) R_{BD}(h) Q_{CD} \quad (C4)$$

for arbitrary elements $h \in H$.

APPENDIX D: CONSTRUCTION OF UNITARITY GAUGE

Recall first that the x_a and t_i are defined to span the algebra of $U(N) \otimes U(N)$, so that any element λ of this algebra may be written

$$\lambda = \sum_i \mu_i^0 t_i + \sum_a \phi_a^0 x_a,$$

with μ_i^0 and ϕ_a^0 so far unconstrained. We then define

$$\phi_a(\theta) \equiv \phi_a^0 + \sum_{\alpha A} \theta_{\alpha} e_{\alpha A} B_{aA},$$

and choose θ to minimize the positive continuous function

$$\sum_a F_a^2 \phi_a^2(\theta).$$

Denoting the value of $\phi_a(\theta)$ at this minimum as ϕ_a , we then have

$$0 = 2 \sum_a F_a^2 \phi_a \sum_A e_{\alpha A} B_{aA},$$

so ϕ_a satisfies the constraint (7.2). Also, we can rewrite λ as

$$\begin{aligned} \lambda &= \sum_i \mu_i^0 t_i + \sum_a \phi_a x_a - \sum_{\alpha A} \theta_{\alpha} e_{\alpha A} B_{aA} x_a \\ &= \sum_i \mu_i^0 t_i + \sum_a \phi_a x_a - \sum_{\alpha} \theta_{\alpha} w_{\alpha} + \sum_{i \alpha A} \theta_{\alpha} e_{\alpha A} C_{iA} t_i, \end{aligned}$$

so that λ may be decomposed as in Eq. (7.1), with

$$\mu_i = \mu_i^0 + \sum_{\alpha A} \theta_{\alpha} e_{\alpha A} C_{iA}.$$

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