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Quantum deformations of $D=4$ Poincaré and Weyl algebra from q-deformed $D=4$ conformal algebra ☆

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We describe the Cartan–Weyl basis of the quantum Lie algebra $U_q(Sl(4;\mathbb{C}))$ and consider two choices of its real forms describing two different q-deformations $U_q^{(i)}(O(4,2))$ $(i=1,2)$ of the $D=4$ conformal algebra. The first choice $(i=1)$ contains as quantum Lie subalgebras (Hopf subalgebras) the q-deformed Lorentz algebra as well as the q-deformed Weyl algebra (Poincaré algebra+dilatations). The second real form $(i=2)$ leads after a particular contraction $\left[{}^{R}_{q}\,{}^{\to\infty}_{\to 1}\right]$ to a new κ-deformation of the Poincaré algebra, which is embedded in the 11-dimensional κ-deformed Hopf algebra, containing besides Poincaré generators an additional abelian central factor.

1. Introduction

Recently the quantum deformations of space–time symmetries in four dimensions were considered. In particular quantum deformations of the $D=4$ Lorentz group [1–5] and the $D=4$ Lorentz algebra [5–7] were obtained as well as quantum deformations of the $D=4$ Poincaré algebra [8,9]. Let us recall, however, that all space–time symmetries are contained in "master" conformal symmetries, describing the geometry of massless particles and fields. The aim of this paper is to study the quantum deformations of the $D=4$ conformal algebra $SU(2,2)\simeq O(4,2)$ and further deduce from it the quantum deformations of the Poincaré algebra. We shall present two schemes:

(i) By considering quantum subalgebras of the q-deformed real $D=4$ conformal algebra $U_q(O(4,2))$. It appears that one can choose real conformal $O(4,2)$ generators in such a way that the q-deformation of its Weyl (Poincaré+dilatations) subalgebra remains a quantum subalgebra, with Hopf algebra structure.

(ii) By considering the contractions of the q-deformed real $D=4$ conformal algebra $U_q(O(4,2))$. It appears that the Hopf algebra structure can be obtained for the quantum algebra $\mathscr{P}\oplus\tilde{D}$, where the central generator $\tilde{D}$ is obtained by a particular contraction of the dilatation generator.

The plan of our paper is the following one. Firstly, in section 2 we shall describe the Cartan–Weyl basis of the quantum complexified $D=4$ conformal algebra $U_q(SL(4,\mathbb{C}))$, and introduce its two real forms $U_q^{(i)}(O(4,2))$ $(i=1,2)$ (for $i=1$ q is real and for $i=2$ we have $|q|=1$). In section 3 we shall consider in more detail $U_q^{(1)}(O(4,2))$ as the q-deformation of the $D=4$ conformal algebra. It appears that the Lorentz sector forms a

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quantum subalgebra and it is described by the q-deformation of the Lorentz algebra firstly introduced in ref. [6] and further studied in ref. [9]. Moreover, as it was mentioned above, the quantum algebra $U_q^{(1)}(O(4,2))$ contains also as its quantum subalgebra the quantum Weyl Hopf algebra. In section 4 we shall consider the other real form $U_q^{(2)}(O(4,2))$, with the Lorentz sector not forming a quantum subalgebra. It appears that by performing the limit

$$[{}^{R\to\infty}_{q\to 1}]:\ \mathrm{i}R^2\ln q \underset{R\to\infty}{\rightarrow} \kappa^{-2}\,, \tag{1.1}$$

we obtain a κ-deformation of the Poincaré algebra (κ is a mass-like fixed parameter), different from the one presented earlier by contracting $U_q(O(3,2))$, via the limit (see ref. [8]) [#1]

$$\{{}^{R\to\infty}_{q\to 1}\}:\ \mathrm{i}R\ln q \underset{R\to\infty}{\rightarrow} \kappa^{-1}\,. \tag{1.2}$$

When our calculations were ready we found in ref. [9] a general scheme describing real forms of quantum noncompact algebras and superalgebras as well as some examples. We would like to mention that $U_q^{(1)}(O(4,2))$ corresponds to the noncompact choice of two Cartan generators of a real form (maximally noncompact case), and the real algebra $U_q^{(2)}(O(4,2))$ corresponds to the case with all three Cartan generators compact (maximally compact case). The explicit example of the q-deformed conformal algebra, presented in ref. [9] is closely related with our choice of the $U_q^{(1)}(O(4,2))$ algebra in section 3. The main difference consists in the choice of a real structure, reducing $U_q(SL(4;\mathbb{C}))$ to $U_q(O(4,2))$. In ref. [9] only the Cartan involutions $\Delta_\pm\to\Delta_\mp$ (Δ_+-positive roots, Δ_--negative roots) were considered, and in our case we shall use in section 3 the involution which maps $\Delta^\pm\to\Delta^\pm$. We would like to point out that the choice of reality conditions determines the explicit formulae relating the "root" and "physical" real conformal generators.

2. Cartan–Weyl basis for $U_q(SL(4;\mathbb{C}))$

In order to describe the q-deformation of the real $D=4$ conformal algebra $SU(2,2)\simeq O(4,2)$ we introduce firstly the Cartan–Chevalley basis for $U_q(SL(4;\mathbb{C}))$ ($i,j=1,2,3$; $[x]_q=(q-q^{-1})^{-1}(q^x-q^{-x})$) describing the quantum complexified $D=4$ conformal algebra

$$[h_i,h_j]=0\,,\quad [h_i,e_{\pm j}]=\pm a_{ij}e_{\pm j}\,,\quad [e_i,e_{-j}]=\delta_{ij}[h_i]_q\,,\quad a_{ij}=\begin{pmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{pmatrix}, \tag{2.1}$$

where h_i describe the Cartan subalgebra, and e_i, e_{-i} ($i=1,2,3$) the generators corresponding to simple roots. The generators corresponding to nonsimple roots are defined as follows (for the general scheme see refs. [12–14]):

$$e_4=[e_1,e_2]_q\,,\quad e_{-4}=[e_{-2},e_{-1}]_{q^{-1}}\,,\quad e_5=[e_2,e_3]_q\,,\quad e_{-5}=[e_{-3},e_{-2}]_{q^{-1}}\,,$$
$$e_6=[e_1,e_5]_q\,,\quad e_{-6}=[e_{-5},e_{-1}]_{q^{-1}}\,, \tag{2.2}$$

where $[A,B]_x\equiv AB-xBA$.

The relations (2.1) are extended to the generators (2.2) in the following way:

$$[e_4,e_{-4}]=[h_1+h_2]_q\equiv[h_4]_q\,,\quad [e_5,e_{-5}]=[h_2+h_3]\equiv[h_5]_q\,,\quad [e_6,e_{-6}]=[h_1+h_2+h_3]_q\equiv[h_6]_q\,, \tag{2.3}$$

and $h_4=h_1+h_2$, $h_5=h_2+h_3$, $h_6=h_1+h_2+h_3$, as well as ($\alpha=4,5,6$)

$$[h_i,e_{\pm\alpha}]=\pm a_{i\alpha}e_{\pm\alpha}\,, \tag{2.4a}$$

[#1] The limit (1.2) was firstly proposed for $D=3$ quantum geometries by the Firenze group (see refs. [10,11]).

where

$$a_{i\alpha}=\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}. \tag{2.4b}$$

The q-Serre relations produce the following collection of bilinear formulae:

$$[e_1,e_2]_q=e_4\,,\quad [e_1,e_3]=0\,,\quad [e_1,e_4]_{q^{-1}}=0\,,\quad [e_1,e_5]_q=e_6\,,\quad [e_1,e_6]_{q^{-1}}=0\,,$$
$$[e_2,e_3]_q=e_5\,,\quad [e_2,e_5]_{q^{-1}}=0\,,\quad [e_2,e_4]_q=0\,,\quad [e_2,e_6]=0\,,$$
$$[e_3,e_5]_q=0\,,\quad [e_3,e_6]_q=0\,,\quad [e_4,e_3]_q=e_6\,, \tag{2.5}$$

which can be supplemented by

$$[e_4,e_5]=-(q-q^{-1})e_6e_2\,,\quad [e_4,e_6]_{q^{-1}}=0\,,\quad [e_5,e_6]_q=0\,. \tag{2.6}$$

Further we obtain

$$[e_1,e_{-5}]=0\,,\quad [e_2,e_{-4}]=e_{-1}q^{h_2}\,,\quad [e_2,e_{-6}]=0\,,\quad [e_3,e_{-5}]=e_{-2}q^{h_3}\,,\quad [e_3,e_{-6}]=e_{-4}q^{h_3}\,,$$
$$[e_4,e_{-1}]=-e_2q^{h_1}\,,\quad [e_4,e_{-3}]=0\,,\quad [e_4,e_{-5}]=(q-q^{-1})q^{-h_2}e_{-3}e_1\,,$$
$$[e_5,e_{-2}]=-e_3q^{h_2}\,,\quad [e_5,e_{-6}]=e_{-1}q^{h_2+h_3}\,,\quad [e_6,e_{-1}]=-e_5q^{h_1}\,,\quad [e_6,e_{-4}]=-e_3q^{h_1+h_2}\,. \tag{2.7}$$

If we add to the relations (2.5)–(2.7) the conjugated ones ($h_i\to h_i$, $e_{\pm i}\to e_{\mp i}$, $q\to q^{-1}$) we obtain the q-deformation of the complete Cartan–Weyl basis of $U_q(SL(4;\mathbb{C}))$.

In order to describe $U_q(SL(4;\mathbb{C}))$ as a quantum bialgebra we introduce the formulae for the coproduct ($i=1$, 2, 3):

$$\Delta(e_{\pm i})=e_{\pm i}\otimes k_i+k_i^{-1}\otimes e_{\pm i}\,,\quad \Delta(k_i^{\pm 1})=k_i^{\pm 1}\otimes k_i^{\pm 1}\,, \tag{2.8a}$$

and further one gets

$$\Delta(e_4)=e_4\otimes k_4+k_4^{-1}\otimes e_4+(q^{-1}-q)k_2^{-1}e_1\otimes e_2k_1\,,$$
$$\Delta(e_{-4})=e_{-4}\otimes k_4+k_4^{-1}\otimes e_{-4}+(q-q^{-1})k_1^{-1}e_{-2}\otimes e_{-1}k_2\,,$$
$$\Delta(e_5)=e_5\otimes k_5+k_5^{-1}\otimes e_5+(q^{-1}-q)k_3^{-1}e_2\otimes e_3k_2\,,$$
$$\Delta(e_{-5})=e_{-5}\otimes k_5+k_5^{-1}\otimes e_{-5}+(q-q^{-1})k_2^{-1}e_{-3}\otimes e_{-2}k_3\,,$$
$$\Delta(e_6)=e_6\otimes k_6+k_6^{-1}\otimes e_6+(q^{-1}-q)(k_5^{-1}e_1\otimes e_5k_1+k_3^{-1}e_4\otimes e_3k_4)\,,$$
$$\Delta(e_{-6})=e_{-6}\otimes k_6+k_6^{-1}\otimes e_{-6}+(q-q^{-1})(k_1^{-1}e_{-5}\otimes e_{-1}k_5+k_4^{-1}e_{-3}\otimes e_{-4}k_3)\,, \tag{2.8b}$$

where $k_A=q^{h_A/2}$ ($A=1, ..., 6$). The formulae for antipodes of the Cartan–Chevalley basis

$$S(e_{\pm i})=-q^{\pm 1}e_{\pm i}\,,\quad S(k_i^{\pm 1})=k_i^{\mp 1}\,, \tag{2.9a}$$

are extended to the generators (2.2) as follows:

$$S(e_{\pm 4})=q^{\pm 2}\tilde{e}_{\pm 4}\,,\quad S(e_{\pm 5})=q^{\pm 2}\tilde{e}_{\pm 5}\,,\quad S(e_{\pm 6})=-q^{\pm 3}\tilde{e}_{\pm 6}\,, \tag{2.9b}$$

where

$$\tilde{e}_4=[e_2,e_1]_q\,,\quad \tilde{e}_{-4}=[e_{-1},e_{-2}]_{q^{-1}}\,,\quad \tilde{e}_5=[e_3,e_2]_q\,,\quad \tilde{e}_{-5}=[e_{-2},e_{-3}]_{q^{-1}}\,,$$
$$\tilde{e}_6=[e_3,\tilde{e}_4]_q\,,\quad \tilde{e}_{-6}=[\tilde{e}_{-4},e_{-3}]_{q^{-1}}\,. \tag{2.10}$$

We see therefore that antipodes describe an outer automorphism of the Cartan–Weyl basis.

In order to describe the real quantum conformal algebra $U_q(O(4, 2))$ we should restrict the Cartan–Weyl basis of $U_q(SL(4;\mathbb{C}))$ by the reality conditions. We shall consider here the following two ⊕-involutions [15], describing an antiautomorphism in both algebra and coalgebra sectors:

(i) First ⊕-involution, q real:

$$h_1^{\oplus}=-h_3\,,\quad h_2^{\oplus}=-h_2\,,\quad e_{\pm1}^{\oplus}=e_{\pm3}\,,\quad e_{\mp2}^{\oplus}=e_{\pm2}\,,\quad e_{\mp4}^{\oplus}=e_{\pm5}\,,\quad e_{\pm6}^{\oplus}=e_{\pm6}\,. \tag{2.11}$$

The q-deformation $U_q^{(1)}(O(4, 2))$ corresponding to the choice (2.11) of reality conditions will be described in section 3.

(ii) Second ⊕-involution, $|q|=1$:

$$h_i^{\oplus}=h_i\,,\quad i=1,2,3\,,$$

$$e_1^{\oplus}=e_{-1}\,,\quad e_2^{\oplus}=-e_{-2}\,,\quad e_3^{\oplus}=e_{-3}\,,\quad e_4^{\oplus}=-e_{-4}\,,\quad e_5^{\oplus}=-e_{-5}\,,\quad e_6^{\oplus}=-e_{-6}\,. \tag{2.12}$$

The q-deformation $U_q^{(2)}(O(4, 2))$ corresponding to the choice (2.12) will be contracted in section 4 in order to obtain the quantum deformation of the Poincaré algebra with the mass-like deformation parameter κ.

Unfortunately, we were not able to find a genuine $*$-operation (+-involution, which is the antiautomorphism in the algebra sector and the automorphism in the coalgebra sector) which provides a quantum deformation of $O(4,2)\simeq SU(2,2)$.

3. $U_q^{(1)}(O(4, 2))$ as a q-deformation of $D=4$ conformal algebra

3.1. Lorentz quantum subalgebra

Let us introduce the generators of the Lorentz group as follows:

$$M_+=M_{23}+iM_{31}=e_1+e_{-3}\,,\quad M_-=M_{23}-iM_{31}=-(e_3+e_{-1})\,,\quad M_3=\tfrac{1}{2}i(h_1-h_3)\,, \tag{3.1a}$$

$$L_+=M_{20}+iM_{01}=e_1-e_{-3}\,,\quad L_-=M_{20}-iM_{01}=e_{-1}-e_3\,,\quad L_3=M_{03}=\tfrac{1}{2}(h_1+h_3)\,. \tag{3.1b}$$

We obtain the following commutation relations:

$$[M_+,M_-]=[L_3+iM_3]_q-[L_3-iM_3]_q\,,\quad [M_3,M_\pm]=\pm iM_\pm\,, \tag{3.2a}$$

$$[L_+,L_-]=[L_3-iM_3]_q-[L_3+iM_3]_q\,,\quad [L_3,L_\pm]=M_\pm\,, \tag{3.2b}$$

$$[M_\pm,L_\mp]=[L_3-iM_3]_q+[L_3+iM_3]_q\,,\quad [M_\pm,L_3]=-L_\pm\,,\quad [M_3,L_\pm]=\pm iL_\pm\,,$$

$$[M_\pm,L_\pm]=0\,,\quad [M_3,L_3]=0\,. \tag{3.2c}$$

Using the reality conditions (2.11), we obtain that $M_{\mu\nu}^{\oplus}=-M_{\mu\nu}$ and the relations (3.2a)–(3.2c) describe the q-deformation of the Lorentz algebra. The coproduct formulae take the form

$$\Delta(M_\pm)=M_\pm\otimes q^{L_3}\cos(\eta M_3)+q^{-L_3}\cos(\eta M_3)\otimes M_\pm\pm iL_\pm\otimes q^{L_3}\sin(\eta M_3)\pm iq^{-L_3}\sin(\eta M_3)\otimes L_\pm\,,$$

$$\Delta(M_3)=M_3\otimes 1+1\otimes M_3\,,$$

$$\Delta(L_\pm)=L_\pm\otimes q^{L_3}\cos(\eta M_3)+q^{L_3}\cos(\eta M_3)\otimes L_\pm\pm iM_\pm\otimes q^{L_3}\sin(\eta M_3)\pm iq^{-L_3}\sin(\eta M_3)\otimes M_\pm\,,$$

$$\Delta(L_3)=L_3\otimes 1+1\otimes L_3\,, \tag{3.3}$$

where $q=e^{\eta}$, and

$$S(M_\pm)=-\tfrac{1}{2}(q+q^{-1})M_\pm+\tfrac{1}{2}(q^{-1}-q)L_\pm\,,\quad S(M_3)=-M_3\,, \tag{3.4}$$

$$S(L_\pm)=-\tfrac{1}{2}(q+q^{-1})L_\pm+\tfrac{1}{2}(q^{-1}-q)M_\pm\,,\quad S(L_3)=-L_3\,. \tag{3.4 cont'd}$$

We see, therefore, that the q-deformation of the Lorentz subalgebra describes the Hopf bialgebra, i.e., it is a genuine quantum algebra. The Lorentz quantum algebra presented in this subsection was firstly considered in ref. [6], and further discussed in ref. [9].

3.2. Weyl quantum subalgebra (see also ref. [9])

Let us introduce the four-momenta as follows:

$$P_0=-\mathrm{i}(e_2+e_6)\,,\quad P_1=e_5-e_4\,,\quad P_2=\mathrm{i}(e_5+e_4)\,,\quad P_3=\mathrm{i}(e_2-e_6)\,, \tag{3.5}$$

where $P_\mu^\oplus=-P_\mu$ [see (2.11)]. The q-deformed algebra in the four-momentum sector looks as follows:

$$[P_0,P_1]=\mathrm{i}\frac{q-1}{q+1}\{P_0,P_2\}\,,\quad [P_0,P_2]=\mathrm{i}\frac{1-q}{1+q}\{P_0,P_1\}\,,\quad [P_0,P_3]=0\,,$$

$$[P_1,P_2]=-\tfrac{1}{2}\mathrm{i}(q^{-1}-q)(P_3^2-P_0^2)\,,\quad [P_3,P_1]=\mathrm{i}\frac{q-1}{q+1}\{P_3,P_2\}\,,\quad [P_3,P_2]=\mathrm{i}\frac{1-q}{1+q}\{P_3,P_1\}\,. \tag{3.6}$$

Further, the deformation of the covariance relation (for $q=1$: $[M_{\mu\nu},P_\lambda]=g_{\nu\lambda}P_\mu-g_{\mu\lambda}P_\nu$) takes the form

$$[M_3,P_0]=0\,,\quad [M_3,P_1]=-P_2\,,\quad [M_3,P_2]=P_1\,,\quad [M_3,P_3]=0\,,$$

$$[L_3,P_0]=P_3\,,\quad [L_3,P_1]=0\,,\quad [L_3,P_2]=0\,,\quad [L_3,P_3]=P_0\,, \tag{3.7a}$$

and further ($P_\pm=P_1\pm\mathrm{i}P_2$)

$$[L_+,P_0]-[M_+,P_0]=\mathrm{i}q^{-L_3-\mathrm{i}M_3}P_+\,,\quad [L_-,P_0]-[M_-,P_0]=\mathrm{i}P_-q^{L_3-\mathrm{i}M_3}\,,$$

$$[M_\pm,P_0]+[L_\pm,P_0]=\frac{2\mathrm{i}}{1+q}P_\pm\pm\frac{1-q}{1+q}(\{M_\pm,P_3\}+\{L_\pm,P_3\})\,, \tag{3.7b}$$

$$[M_+,P_3]-[L_+,P_3]=-\mathrm{i}q^{-L_3-\mathrm{i}M_3}P_+\,,\quad [M_-,P_3]-[L_-,P_3]=-\mathrm{i}P_-q^{L_3-\mathrm{i}M_3}\,,$$

$$[M_\pm,P_3]+[L_\pm,P_3]=-\frac{2\mathrm{i}}{1+q}P_\pm\pm\frac{1-q}{1+q}(\{M_\pm,P_0\}+\{L_\pm,P_0\})\,, \tag{3.7c}$$

$$[M_+,P_2]-[L_+,P_2]=q^{-L_3-\mathrm{i}M_3}(P_0-P_3)\,,\quad [M_-,P_2]-[L_-,P_2]=(P_3-P_0)q^{L_3-\mathrm{i}M_3}\,,$$

$$[M_\pm,P_2]+[L_\pm,P_2]=\mp\frac{2}{1+q}(P_0+P_3)+\mathrm{i}\frac{q-1}{q+1}(\{M_\pm,P_1\}+\{L_\pm,P_1\})\,, \tag{3.7d}$$

$$[M_+,P_1]-[L_+,P_1]=\mathrm{i}q^{-L_3-\mathrm{i}M_3}(P_3-P_0)\,,\quad [M_-,P_1]-[L_-,P_1]=\mathrm{i}(P_3-P_0)q^{L_3-\mathrm{i}M_3}\,,$$

$$[M_\pm,P_1]+[L_\pm,P_1]=\frac{2\mathrm{i}}{1+q}(P_0+P_3)+\mathrm{i}\frac{1-q}{1+q}(\{M_\pm,P_2\}+\{L_\pm,P_2\})\,. \tag{3.7e}$$

If we introduce the dilatation generator

$$D=\tfrac{1}{2}(h_1+h_3+2h_2)\,, \tag{3.8}$$

which due to (2.11) satisfies the relation $D^\oplus=-D$, one checks easily that it enters the q-deformed Weyl algebra in an undeformed way:

$$[D,M_{\mu\nu}]=0\,,\quad [D,P_\mu]=P_\mu\,. \tag{3.9}$$

We see that in the algebra sector the relations (3.2a), (3.2b), (3.6) and (3.7a)–(3.7e) describe the q-deformed Poincaré algebra which is, however, not closed in the coalgebra sector:

$$\begin{aligned}
&\Delta(P_+)=P_+\otimes q^{(D-\mathrm{i}M_3)/2}+q^{-(D-\mathrm{i}M_3)/2}\otimes P_+ +\tfrac{1}{2}\mathrm{i}(q-q^{-1})q^{(L_3-D)/2}(M_+ +L_+)\otimes(P_0-P_3)q^{(L_3-\mathrm{i}M_3)/2}\,,\\
&\Delta(P_-)=P_-\otimes q^{(D+\mathrm{i}M_3)/2}+q^{-(D+\mathrm{i}M_3)/2}\otimes P_- +\tfrac{1}{2}\mathrm{i}(q^{-1}-q)q^{-(L_3+\mathrm{i}M_3)/2}(P_0-P_3)\otimes(M_- +L_-)q^{(D-L_3)/2}\,,\\
&\Delta(P_0+P_3)=(P_0+P_3)\otimes q^{(L_3+D)/2}+q^{-(L_3+D)/2}\otimes(P_0+P_3)\\
&\quad+\tfrac{1}{2}\mathrm{i}(q-q^{-1})[q^{-(D+\mathrm{i}M_3)/2}(M_+ +L_+)\otimes P_-q^{(L_3-\mathrm{i}M_3)/2}-q^{-(L_3+\mathrm{i}M_3)/2}P_+\otimes(M_- +L_-)q^{(D-\mathrm{i}M_3)/2}]\,,\\
&\Delta(P_0-P_3)=(P_0-P_3)\otimes q^{(D-L_3)/2}+q^{(L_3-D)/2}\otimes(P_0-P_3)\,,\\
&\Delta(D)=D\otimes\mathbf{1}+\mathbf{1}\otimes D\,.
\end{aligned}\tag{3.10}$$

Further we have the following formulae for antipodes:

$$\begin{aligned}
&S(P_+)=-qP_+ +\tfrac{1}{2}\mathrm{i}q(q^2-1)(M_+ +L_+)(P_0-P_3)\,,\\
&S(P_-)=-qP_- +\tfrac{1}{2}\mathrm{i}q(1-q^2)(P_0-P_3)(M_- +L_-)\,,\\
&S(P_0-P_3)=-q(P_0-P_3)\,,\\
&S(P_0+P_3)=-q(P_0+P_3)+\tfrac{1}{2}\mathrm{i}q(q^2-1)(M_+ +L_+)P_- +\tfrac{1}{4}q^2(q^2-1)[P_0-P_3,M_+ +L_+]_q(M_- +L_-)\,,\\
&S(D)=-D\,.
\end{aligned}\tag{3.11}$$

We see therefore from (3.10), (3.11) that the q-deformation of the 11-dimensional Weyl algebra forms a Hopf algebra. We obtain the following sequence of quantum Hopf algebras:

$$\mathrm{U}_q(\mathrm{O}(3,1))\subset\mathrm{U}_q(\mathscr{P}_4\oplus D)\subset\mathrm{U}_q^{(1)}(\mathrm{O}(4,2))\,,\tag{3.12}$$

where $\mathscr{P}_4\oplus D$ denotes the Weyl algebra ($\mathscr{P}_4$ = Poincaré algebra).

4. κ-deformation of the Poincaré algebra from the contraction of $\mathrm{U}_q^{(2)}(\mathrm{O}(4,2))$

Following the techniques presented in ref. [8], we shall consider the real form (2.12) of $\mathrm{U}_q(\mathrm{SL}(4;\mathbb{C}))$ as an intermediate step in the derivation of the κ-deformation of the $D=4$ Poincaré algebra where κ is a mass-like parameter.

We assume the generators of the $D=4$ conformal algebra in the following way:

$$\begin{aligned}
&M_{12}=\tfrac{1}{2}\mathrm{i}(h_1+h_3)\,,\quad M_{23}=\tfrac{1}{2}\mathrm{i}(e_1+e_{-1}+e_3+e_{-3})\,,\quad M_{31}=\tfrac{1}{2}(e_1-e_{-1}+e_3-e_{-3})\,,\\
&M_{01}=\tfrac{1}{2}\mathrm{i}(e_6-e_{-6}+e_2-e_{-2})\,,\quad M_{02}=\tfrac{1}{2}(e_6+e_{-6}-e_2-e_{-2})\,,\quad M_{03}=\tfrac{1}{2}\mathrm{i}(e_4-e_{-4}-e_5+e_{-5})\,,
\end{aligned}\tag{4.1}$$

$$\begin{aligned}
&M_{40}=\tfrac{1}{2}(e_4+e_{-4}+e_5+e_{-5})\,,\quad M_{41}=\tfrac{1}{2}\mathrm{i}(e_1+e_{-1}-e_3-e_{-3})\,,\\
&M_{42}=\tfrac{1}{2}(e_1-e_{-1}-e_3+e_{-3})\,,\quad M_{43}=\tfrac{1}{2}\mathrm{i}(h_1-h_3)\,,\\
&M_{50}=\tfrac{1}{2}\mathrm{i}(h_1+h_3+2h_2)\,,\quad M_{51}=\tfrac{1}{2}(e_2+e_{-2}+e_6+e_{-6})\,,\\
&M_{52}=\tfrac{1}{2}\mathrm{i}(e_2-e_{-2}-e_6+e_{-6})\,,\quad M_{53}=\tfrac{1}{2}(e_4+e_{-4}-e_5-e_{-5})\,,\quad M_{54}=-\tfrac{1}{2}\mathrm{i}(e_4-e_{-4}+e_5-e_{-5})\,,
\end{aligned}\tag{4.2}$$

which due to relations (2.12) satisfy the reality condition $M_{AB}^{\ominus}=-M_{AB}$. The $\mathrm{O}(4,2)$ q-deformed commutation relations correspond to the following assignment of the signature $g_{AB}=\mathrm{diag}(-++++-)$ ($A,B=0,1,2,3,4,5$) and the physical basis is given by the Lorentz generators $M_{\mu\nu}$ ($\mu,\nu=0,1,2,3$) and $P_\mu=M_{4\mu}+M_{5\mu}$, $K_\mu=M_{5\mu}-M_{4\mu}$, $D=M_{45}$. The Cartan subalgebra is described by the following three commuting generators: (M_3, P_0+K_0, P_3-K_3).

In order to obtain the κ-deformation of the $D=4$ Poincaré algebra we introduce the limit (1.1) with the following rescaling of the generators:

$$\tilde{M}_{\mu\nu}=M_{\mu\nu}\,,\quad \tilde{K}_\mu=K_\mu\,,\quad \tilde{P}_\mu=\frac{1}{R}P_\mu\,,\quad \tilde{D}=\frac{1}{R}D\,. \tag{4.3}$$

The rescaling (4.3) corresponds to the contraction of the nonsymmetric and nonreductive coset K=G/H, where G is a conformal group (O(4, 2)) and H is the Poincaré group ($O(3,1)\oplus T_4$) formed by the Lorentz group extended by conformal accelerations. Substituting in the formulae of section 2 the relations (4.1), (4.2), or more explicitly

$$e_{\pm 1}=\frac{1}{2i}\left[M_\pm+\tfrac{1}{2}(R\tilde{P}_1-K_1)\pm\tfrac{1}{2}i(R\tilde{P}_2-K_2)\right],\quad e_{\pm 2}=\frac{1}{2i}\left[\pm L_\mp\pm\tfrac{1}{2}(R\tilde{P}_2+K_2)+\tfrac{1}{2}i(R\tilde{P}_1+K_1)\right],$$

$$e_{\pm 3}=\frac{1}{2i}\left[M_\pm+\tfrac{1}{2}(K_1-R\tilde{P}_1)\mp\tfrac{1}{2}i(R\tilde{P}_2-K_2)\right],\quad e_{\pm 4}=\frac{1}{2i}\left[\pm(L_3+R\tilde{D})+\tfrac{1}{2}i(R\tilde{P}_0-K_0)+\tfrac{1}{2}i(R\tilde{P}_3+K_3)\right],$$

$$e_{\pm 5}=\frac{1}{2i}\left[\pm(R\tilde{D}-L_3)+\tfrac{1}{2}i(R\tilde{P}_0-K_0)-\tfrac{1}{2}i(R\tilde{P}_3+K_3)\right],\quad e_{\pm 6}=\frac{1}{2i}\left[\pm L_\pm\mp\tfrac{1}{2}(R\tilde{P}_2+K_2)+\tfrac{1}{2}i(R\tilde{P}_1+K_1)\right], \tag{4.4}$$

$$h_1=-iM_3-\tfrac{1}{2}i(R\tilde{P}_3-K_3)\,,\quad h_2=iM_3-\tfrac{1}{2}i(R\tilde{P}_0+K_0)\,,\quad h_3=-iM_3+\tfrac{1}{2}i(R\tilde{P}_3-K_3)\,, \tag{4.5}$$

we obtain in the limit $[{}^{R\to\infty}_{q\to 1}]$ the following relations:

(a) κ-deformation of the Lorentz sector (M_i, L_i).

$$[M_+,M_-]=2iM_3\,,\quad [M_3,M_+]=iM_+\,,\quad [M_3,M_-]=-iM_-\,,$$

$$[L_+,L_-]=-2iM_3\,,\quad [L_3,L_+]=-iM_+-\frac{3}{8\kappa^2}\tilde{P}_0(\tilde{P}_2-i\tilde{P}_1)\,,\quad [L_3,L_-]=iM_-+\frac{3}{8\kappa^2}\tilde{P}_0(\tilde{P}_2+i\tilde{P}_1)\,,$$

$$[M_3,L_3]=0\,,\quad [M_3,L_+]=iL_+\,,\quad [M_3,L_-]=-iL_-\,,$$

$$[M_+,L_3]=-iL_+-\frac{1}{8\kappa^2}\tilde{P}_3(\tilde{P}_1+i\tilde{P}_2)\,,\quad [M_-,L_3]=iL_-+\frac{1}{8\kappa^2}\tilde{P}_3(\tilde{P}_1-i\tilde{P}_2)\,,$$

$$[M_+,L_+]=\frac{1}{8\kappa^2}(\tilde{P}_1+i\tilde{P}_2)^2\,,\quad [M_-,L_-]=-\frac{1}{8\kappa^2}(\tilde{P}_1-i\tilde{P}_2)^2\,,$$

$$[M_\mp,L_\pm]=\mp 2iL_3\pm\frac{1}{8\kappa^2}(2\tilde{P}_3^2-\tilde{P}_1^2-\tilde{P}_2^2)\,. \tag{4.6}$$

(b) In the limit $[{}^{R\to\infty}_{q\to 1}]$ we obtain the following coproduct formulae:

$$\Delta M_3=M_3\otimes\mathbf{1}+\mathbf{1}\otimes M_3\,,\quad \Delta M_\pm=M_\pm\otimes\mathbf{1}+\mathbf{1}\otimes M_\pm+\frac{1}{8\kappa^2}(\tilde{P}_\pm\otimes\tilde{P}_3-\tilde{P}_3\otimes\tilde{P}_\pm)\,,$$

$$\Delta L_3=L_3\otimes\mathbf{1}+\mathbf{1}\otimes L_3-\frac{i}{4\kappa^2}(\tilde{P}_+\otimes\tilde{P}_-+\tilde{P}_-\otimes\tilde{P}_+)\,,$$

$$\Delta L_\pm=L_\pm\otimes\mathbf{1}+\mathbf{1}\otimes L_\pm+\frac{1}{4\kappa^2}(\tilde{D}\otimes\tilde{P}_\pm-\tilde{P}_\pm\otimes\tilde{D})\pm\frac{i}{8\kappa^2}\left[(\tilde{P}_0\pm\tilde{P}_3)\otimes\tilde{P}_\pm-\tilde{P}_\pm\otimes(\tilde{P}_0\mp\tilde{P}_3)\right], \tag{4.7}$$

and the formulae for antipodes:

$$S(M_3)=-M_3\,,\quad S(M_\pm)=-M_\pm\,,$$

$$S(L_3)=-L_3-\frac{i}{2\kappa}\tilde{P}_+\tilde{P}_-\,,\quad S(L_\pm)=-L_\pm\pm\frac{i}{4\kappa}\tilde{P}_\pm\tilde{P}_3\,. \tag{4.8}$$

We see that in order to obtain the closed bialgebra one has to add the central generator $\tilde{D}$, which extends the Lorentz κ-algebra (4.6) in the following trivial way:

$$[M_i,\tilde{D}]=[L_i,\tilde{D}]=0\,, \tag{4.9}$$

and further

$$\Delta\tilde{D}=\tilde{D}\otimes 1+1\otimes\tilde{D}+\frac{i}{4\kappa^2}(\tilde{P}_+\otimes\tilde{P}_- -\tilde{P}_-\otimes\tilde{P}_+)\,, \tag{4.10a}$$

$$S(\tilde{D})=-\tilde{D}\,. \tag{4.10b}$$

We see that the generators $(M_i, L_i, \tilde{D})$ form a Hopf algebra.

(c) κ-deformation of the centrally extended Poincaré algebra (Poincaré $\oplus\tilde{D}$).

The limit $[{}^{R\to\infty}_{q\to 1}]$ implies supplementing of the algebra (4.6) and (4.9) by the "classical" relations

$$[\tilde{P}_\mu,\tilde{P}_\nu]=0\,,\quad [M_{\mu\nu},\tilde{P}_\lambda]=g_{\nu\lambda}\tilde{P}_\mu-g_{\mu\lambda}\tilde{P}_\nu\,, \tag{4.11}$$

and the "classical" formulae for the coproduct and antipode

$$\Delta\tilde{P}_\mu=\tilde{P}_\mu\otimes 1+1\otimes\tilde{P}_\mu\,,\quad S(\tilde{P}_\mu)=-\tilde{P}_\mu\,. \tag{4.12a,b}$$

The relations (4.6)–(4.12) describe the κ-deformed Poincaré Hopf algebra centrally extended in the coalgebra sector by the abelian generator $\tilde{D}$.

5. Final remarks

Standard Drinfeld–Jimbo deformation of the $D=4$ conformal algebra permits to deduce two different quantum deformations of the Poincaré algebra:

(a) The q-deformation of the Poincaré algebra, discussed in section 3. In this case:

(i) Quantum deformation of the Lorentz algebra forms a quantum subgroup, which is a Hopf algebra [see formulae (3.4)].

(ii) In order to introduce the q-deformed Poincaré algebra as a Hopf algebra one has to add the eleventh dilatation generator. In such a way one obtains the q-deformed Weyl algebra as a quantum group.

(iii) The four-momenta are nonabelian and form the quadratic relations (3.6), describing a closed subalgebra. It could be interesting to present the q-deformation of the theory of induced representations of the Poincaré group, with the noncommutative nature of four-momenta taken into consideration.

(b) The κ-deformation of the Poincaré algebra, discussed in section 4. This case is analogous to the one discussed in ref. [8], and it is obtained by a quantum generalization of the de Sitter contraction, with deformation parameter q approaching the value $q=1$ in a way correlated with the limit $R\to\infty$ [see (1.1)]. The κ-deformations have the following common features:

(i) The κ-deformation of the Lorentz algebra ceases to be a quantum subalgebra of the κ-deformation of the Poincaré algebra.

(ii) The κ-deformed Poincaré algebra can be extended to a Hopf algebra without adding new generators (see ref. [9]). In section 4 we present another κ-deformation, with a Hopf bialgebra structure requiring the addition of the eleventh central generator.

(iii) The four-momenta stay abelian [see (4.11)].

Both types of quantum deformation of the Poincaré algebra (q-deformation and κ-deformation) have their advantages and disadvantages and should be studied more in detail in the future.

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Note added in proof

The statement at the end of section 2 should be weakened. Recently we found two different + -involutions (antiautomorphism in the algebra sector and automorphism in the coalgebra sector), one with $|q|=1$ and the second with q real, but for both cases one can not obtain the closed algebra of deformed four-momenta generators.

A more detailed discussion of different involutions for the conformal algebra, including the q-deformed O(5, 1) euclidean conformal algebra and a discussion of some Casimir operators will be given in a subsequent paper, written by the present authors and J. Sobczyk.

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