CHIRAL ACTIONS FOR $N = 2$ SUPERSYMMETRIC TENSOR MULTIPLETS ☆

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Actions for self-interacting $N = 2$ tensor multiplets are written as integrals over general or chiral superspace. The general form for SU(2)-invariant actions is given.

The application of functional techniques in superspace (e.g., to formulate supergraph rules) requires expressing actions as integrals over general ($\int d^4x \times d^{2N}\theta\, d^{2N}\bar{\theta}$) or chiral ($\int d^4x\, d^{2N}\theta$) superspace of functions of unconstrained superfields. No other measures allow the usual manipulations (integration by parts of spinor derivatives, etc.). This is related to the fact that functional derivatives can be defined only with respect to general or chiral superfields (if manifest Lorentz and SU(N) covariance are to be maintained). Sometimes the action cannot be expressed completely in terms of field strengths, and requires explicit potentials, but this is also true for nonsupersymmetric theories (e.g., the 3D gauge-invariant vector mass term is of the form $\int d^3x\, \epsilon^{abc} \times A_a \partial_b A_c$ [1]).

The tensor multiplet is the simplest multiplet in extended ($N = 2$) supersymmetry. It consists of an antisymmetric tensor gauge field, a scalar, an isospinor spinor, and an auxiliary isotriplet of scalars. On shell it describes the lowest-spin representation of $N = 2$ supersymmetry, the $N = 2$ scalar (hyper)multiplet. Thus, a first step in the understanding of the superspace formulation of extended supersymmetry is to obtain general (self-interacting) actions for this multiplet over general or chiral superspace.

Karlhede, Lindström and Roček have recently proposed a general action for self-interacting $N = 2$ tensor multiplets by using a subintegration which breaks manifest SU(2) covariance [2]. We show how to reexpress their result in chiral (or general) superspace. Their formulation also involves the use of contour integration over a subsidiary variable. Upon reduction from $N = 2$ superspace to $N = 1$, this contour integration represents a general solution to a certain restriction on the lagrangian required for $N = 2$ supersymmetry. We derive the corresponding restrictions in $N=2$ superspace by requiring chirality or gauge invariance, solve them by Fourier transformation, and obtain the contour-integral representation as a result.

The multiplet is described by a real isovector field strength [3]

$$F_a{}^b = \mathrm{i}(C^{bc} \mathrm{D}^2_{ac} \Phi - C_{ac} \bar{\mathrm{D}}^{2bc} \bar{\Phi}) , \qquad (1)$$

in terms of a chiral scalar gauge field. (See ref. [4] for notation.) To construct an action in superspace we first break SU(2) by introducing a "zweibein" u^a, v^a for the group space ($C_{ba} u^a v^b \neq 0$). We then define the variable

$$\eta = u^a u^b C_{ca} F_b{}^c , \qquad (2)$$

which satisfies

$$u^a \mathrm{D}_{a\alpha} \eta = u^a C_{ba} \bar{\mathrm{D}}^b{}_{\dot{\alpha}} \eta = 0 , \qquad (3)$$

as a consequence of (1). As a result, the expression

$$v^a v^b v^c v^d \mathrm{D}^2_{ab} C_{ce} C_{fd} \bar{\mathrm{D}}^{2ef} f(\eta)$$

gives a total spacetime derivative when acted upon by any spinor derivative. We can therefore write an action as [2]

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$$S = \int du (C_{ba} u^a v^b)^{-4}$$

$$\times \int d^4x\, v^a v^b v^c v^d D^2_{ab} C_{ce} C_{fd} \bar{D}^{2ef} g(u, \eta) , \qquad (4)$$

where the factor $(C_{ba} u^a v^b)^{-4}$ has been chosen to cancel all v dependence, and $\int du$ is some integration over the u variables. In particular, Karlhede, Lindström and Roček choose

$$u^a = (1, \zeta) , \quad \int du = \frac{1}{2\pi i} \oint d\zeta , \qquad (5)$$

for some appropriate contour in the complex ζ-plane. In general, S will be SU(2)-invariant only for specific choices of g and the contour.

To write S as an integral over chiral superspace, we first define

$$h = g/\eta \quad \text{or} \quad \partial g/\partial\eta , \qquad (6)$$

so that $S = \int g$ is replaced by $S = \int h\eta$ (for deriving actions) or $\delta S = \int h \delta\eta$ (for deriving field equations), respectively. Expressing this one explicit η in terms of Φ by (1),(2), we note (3) allows us to pull the derivatives off Φ so they act on the entire integrand (since they vanish on h), and combine them with the explicit derivatives in (4). In particular, the D^4 ($\bar{D}^4$) obtained for the Φ ($\bar{\Phi}$) term can be converted into an $\int d^4\theta$ ($\int d^4\bar{\theta}$). The remaining two derivatives $\bar{D}^2$(D^2) can then be pushed past the Φ ($\bar{\Phi}$) and onto h. In summary (schematically), for the Φ term of h,

$$S = \int (v^2 D^2)(v^2 \bar{D}^2) g = \int (v^2 D^2)(v^2 \bar{D}^2) \eta h$$

$$= \int (v^2 D^2)(v^2 \bar{D}^2)(u^2 D^2 \Phi) h$$

$$= \int (v^2 D^2)(v^2 \bar{D}^2)(u^2 D^2)(\Phi h)$$

$$= \int (u \cdot v)^2 D^4 (v^2 \bar{D}^2)(\Phi h)$$

$$= (u \cdot v)^2 \int d^4\theta (v^2 \bar{D}^2)(\Phi h)$$

$$= (u \cdot v)^2 \int d^4\theta\, \Phi (v^2 \bar{D}^2 h) .$$

(We can also write $\Phi = \bar{D}^4 \psi$ and turn the $\bar{D}^4$ into an $\int d^4\bar{\theta}$ to produce an integral over general superspace. Such a procedure is necessary in $D = 6$, where $F \sim D^7 \chi$ in terms of a spinor isospinor χ.) We thus obtain

$$S = -\int du (C_{ba} u^a v^b)^{-2}$$

$$\times \int d^4x \left(\int d^4\theta\, i\Phi v^a v^b C_{ac} C_{db} \bar{D}^{2cd} \right.$$

$$\left. - \int d^4\bar{\theta}\, i\bar{\Phi} v^a v^b D^2_{ab} \right) h(u, \eta) . \qquad (7)$$

Finally, we take the spinor derivatives of h, and find

$$S = i \int d^4x\, d^4\theta\, \phi W - i \int d^4x\, d^4\bar{\theta}\, \bar{\Phi} \tilde{W} , \qquad (8a)$$

$$W = A(F) W_0 - B^{ab}(F) C_{ac} C_{db} \psi^{2cd} ,$$

$$\tilde{W} = A(F) \bar{W}_0 - B^{ab}(F) \bar{\psi}^2_{ab} ,$$

$$\bar{D}^a{}_{\dot\alpha} F_b{}^c = \tfrac{1}{2} \psi^{(c}{}_{\dot\alpha} \delta_{b)}{}^a , \quad \bar{D}^a{}_{\dot\alpha} \psi^b{}_{\dot\beta} = C_{\dot\alpha\dot\beta} C^{ab} W_0 ,$$

$$\bar{D}^a{}_{\dot\alpha} W_0 = 0 , \quad \psi^{2ab} = \tfrac{1}{2} \psi^{a\dot\alpha} \psi^b{}_{\dot\alpha} , \qquad (8b)$$

$$A(F) = \int du\, h' , \quad B^{ab}(F) = \int du\, u^a u^b h'' , \qquad (8c)$$

where $' = \partial/\partial\eta$.

To obtain a specific action, we choose a particular function $h(u, \eta)$ and perform the $u(\zeta)$-integration which defines A and B. For example [2],

$$g = \eta^2/2\zeta \to A = \tfrac{1}{2} , \quad B^{ab} = 0 , \qquad (9a)$$

$$g = \eta \ln \eta \to A = \tfrac{1}{2}(-\det F)^{-1/2} ,$$
$$B^{ab} = \tfrac{1}{4}(-\det F)^{-3/2} C^{ac} F_c{}^b , \qquad (9b)$$

giving the $N = 2$ superspace actions for the free [3] and improved [5] multiplets, where the contour for the former is counterclockwise about the origin, and for the latter counterclockwise about the more positive pole of $\eta(\zeta, F) = 0$ (or clockwise about the more negative one, or the average; $h' = 1/\eta$, so that are no cuts).

An alternative approach to introducing u^a is to search directly for the most general action satisfying the condition of gauge invariance. It must therefore be of the form (8a), where W depends only on F, and

$$\bar{D}^a{}_{\dot\alpha} W = 0\,, \quad D^2_{ab} W = C_{ac} C_{db} \bar{D}^{2cd} \bar{W}\,, \tag{10}$$

in order that the integrand be chiral and for invariance under the gauge transformation

$$\delta\Phi = \bar{D}^4 D^2_{ab} C^{cb} K_c{}^a\,, \tag{11}$$

for real K. By dimensional analysis, W can contain only two explicit $\bar{D}$'s, and we therefore obtain (8b). If we then solve just the chirality condition on W, we find

$$\begin{aligned} 0 &= \bar{D}^a{}_{\dot\alpha} W \\ &= A_{,b}{}^a \psi^b{}_{\dot\alpha} W_0 + B^{ac} C_{cb} \psi^b{}_{\dot\alpha} W_0 + B^{bc}{}_{,b}{}^a C_{cd} \psi^{3d}{}_{\dot\alpha}\,, \end{aligned} \tag{12}$$

where $\psi^a{}_{\dot\alpha} \psi^{2bc} = \frac{1}{2} C^{a(b} \psi^{3c)}{}_{\dot\alpha}$ and ${}_{,b}{}^a = \partial/\partial F_a{}^b$. We thus have

$$B^{ab} = -C^{ac} A_{,c}{}^b\,, \quad A_{,a}{}^b{}_{,b}{}^a = 0\,. \tag{13a,b}$$

(13a) gives B in terms of A, whereas (13b) is the 3D Laplace equation for A in terms of its argument $F_a{}^b$. The integral representation for A in (8c) is just a general solution of this equation, in analogy to $N = 1$ [2], and substitution into (13a) gives the expression in (8c) for B. This integral representation can also be obtained by Fourier transformation: The Laplace equation says that the momentum $p_a{}^b$ conjugate to $F_a{}^b$ has vanishing determinant, which means that (with appropriate analytic continuation) it can be written as $p_a{}^b = u^c u^b C_{ac}$, leading to a solution of the form $A(F) = \int du\, \hat{A}(u) e^{i\eta}$. Setting $u^a = \lambda(1, \zeta)$ and integrating over λ, we obtain (8c) with the choice (5).

We can also look directly for solutions of (13b) which are SU(2) covariant. This means that A is a function of only the "radial" coordinate $(-\det F)^{1/2}$. The most general "spherically symmetric" solution (away from the "origin" $\det F = 0$) is thus proportional to a constant plus the "Green function" $(-\det F)^{-1/2}$: i.e., a linear combination of the free and improved multiplets. (This result can also be derived in the $N = 1$ formulation.)

Generalization to an arbitrary number n of tensor multiplets is straightforward. We then find, from the u-analysis,

$$\begin{aligned} S &= i \int d^4x\, d^4\theta\, \Phi^i W_i - i \int d^4x\, d^4\bar\theta\, \bar\Phi^i \bar{W}_i\,, \\ W_i &= A_{ij} W_0{}^j - B^{ab}{}_{ijk} C_{ac} C_{db} \psi^{2cdjk}\,, \\ \bar{W}_i &= A_{ij} \bar{W}_0{}^j - B^{ab}{}_{ijk} \bar\psi^2{}_{ab}{}^{jk}\,, \end{aligned} \tag{14}$$

$$\begin{aligned} A_{ij}(F^k) &= \int du\, h_{i,j}(u, \eta^k)\,, \\ B^{ab}{}_{ijk} &= \int du\, u^a u^b h_{i,jk}\,. \end{aligned} \tag{14 cont'd}$$

On the other hand, from the chirality-analysis we find

$$B^{ab}{}_{ijk} = -C^{ac} A_{ij,c}{}^b{}_k\,, \quad A_{ij,a}{}^b{}_{k,b}{}^a{}_l = 0\,, \tag{15a,b}$$

$$A_{i[j,a}{}^b{}_{k]} = 0\,. \tag{15c}$$

The generalized Laplace equations (15b) are again analogous to the $N = 1$ case [6]. The new condition (15c) follows from the symmetry of B in its explicit solution (15a). (In (14), $\psi^{2abij} = \psi^{2baji}$.) Again, the general solution to (15) is given by the integral representation in (14). (In particular, (15c) requires that h appears with a derivative $\partial/\partial\eta^j$ in (14), whereas in (8c) the derivative was superfluous.) To obtain this result by Fourier transformation, we note that (15b) implies $p_a{}^b{}_i = \lambda_i u^c u^b C_{ac}$, and integrating over the λ produces (14). In the SU(2) covariant case, it is more convenient to keep the Fourier form:

$$A_{ij} = \int d^2u\, d^n\lambda\, \hat{A}_i(u, \lambda) \lambda_j \exp(i\lambda_k u^c u^b C_{ac} F_b{}^{ak})\,. \tag{16}$$

SU(2) covariance of A_{ij} then demands a linear combination of

$$\hat{A}_i(u, \lambda) = \delta^2(u) \hat{A}_i(\lambda) \to A_{ij} = \text{constant}\,,$$

$$\hat{A}_i(u,\lambda) = \hat{A}_i(\lambda) \to$$

$$A_{ij} = \int d^n\lambda\, \hat{A}_i(\lambda) \lambda_j (-\det \lambda \cdot F)^{-1/2}\,, \tag{17}$$

corresponding to free multiplets and generalized improved (conformal) multiplets, respectively. Furthermore, we can derive a very simple form for the action in the general case by noting that W_i, inside the $\int d^n\lambda$, depends on Φ^i only as $\lambda_i \Phi^i$. the symmetry of $\delta^2 S/\delta\Phi^i \delta\Phi^j$ then implies $W_i \sim \lambda_i$, so that the general n-multiplet action can be expressed explicitly in terms of the general one-multiplet action:

$$S_n(\Phi^i) = \int d^n\lambda\, S_1(\lambda, \lambda \cdot \Phi)\,, \tag{18}$$

corresponding to the replacement $A(F) \to A(\lambda, \lambda \cdot F)$. Eqs. (8a), (8b), (13), (18) then completely specify S_n.

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