Nuclear Physics B239 (1984) 477–507

CHIRAL ANOMALIES, HIGHER DIMENSIONS, AND DIFFERENTIAL GEOMETRY

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Received 6 June 1983
(Final version received 6 December 1983)

We determine the abelian and non-abelian chiral anomalies in $2n$-dimensional spacetime by a differential geometric method which enables us to obtain the anomalies without having to calculate Feynman diagrams, as has been done by Frampton and Kephart. The advantage of this method is that the construction automatically satisfies the Wess-Zumino consistency condition, a condition which has direct physical interpretation. We hope that our analysis sheds light on the mathematical structure associated with chiral anomalies. The mathematical analysis is self-contained and a brief review of differential forms and other mathematical tools is included.

1. Introduction

Chiral anomalies [1,2] have played a strikingly ubiquitous role⋆ in the development of particle theory ever since their discovery some fifteen years ago. Their central importance in particle theory could hardly have been anticipated by those who first calculated the triangle graphs [3]. To underline the importance of anomalies, we mention neutral pion decay, renormalizability of gauge theories, correlation between lepton and quark families, instantons and index theorems, anomaly match-

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⋆ Early calculations of π^0 decay and triangle graph [3], non-renormalization theorem [4], explicit calculation [5], physical applications of non-abelian anomaly [6], consistency condition and physical applications [7], short distance behaviour [8], renormalizability and freedom from anomalies [9], evaluation of anomalies [10], instantons [11], index theorems [12], anomaly matching in composite models [13], QCD effective lagrangian and meson-glueball mixing [14], monopole catalysis and fermion number violation [15], anomaly in curved space [16] (the last reference of which refers to anomaly and dynamical symmetry breaking), anomaly in supersymmetry [17], connection between anomaly and path integral [18].

ing in composite models, QCD effective Lagrangian and meson-glueball mixing, and monopole catalysis of proton decay.

Perhaps not surprisingly then, it turns out that chiral anomalies possess deep mathematical significance. Indeed, the mathematical underpinning of anomalies has been increasingly appreciated over the last few years [12, 19 (connection between anomaly and cohomology)]. In this paper, we calculate chiral anomalies in $2n$-dimensional spacetime by a differential geometric method. This enables us to determine the structure of the chiral anomalies *without ever calculating a Feynman diagram*. We hope that our calculation will illustrate, clarify, and expose some of the mathematical structures associated with chiral anomalies.

This paper is mathematically self-contained. It is not necessary to have prior knowledge of the method we use. A brief review of differential forms and other mathematical tools used in this paper is given in appendix A.

Some time ago, Frampton [20] had discussed chiral anomalies in higher dimensional spacetime. Recently, he and Kephart and others [21–24] have carried out an analysis of these anomalies. Related work has also been done by Townsend and Sierra [25].

We are motivated to calculate chiral anomalies in $2n$-dimensional spacetime partly because the mathematical structure is so elegant, but also because of growing interest in physical theories in higher dimensional spacetime [26, 27]. We expect that chiral anomalies should play an important role in the elucidation of these theories. We also understand that higher dimensional anomalies are important for the development of superstring theories (see the review of [28]). In addition, Frampton and Kephart [20–24] have proposed that the correct physical theory must be such that certain of these higher-dimensional anomalies vanish. At present, we do not see any physical reason underlying this very interesting supposition.

In our calculation there are no restrictions on the gauge group G and the fermion representation.

We emphasize that it is perfectly legitimate to talk of chiral anomalies in higher spacetime even though the relevant field theories are not renormalizable. For the purpose of this paper, we take a conservative view and consider the theory of a Dirac particle interacting with c-number external non-abelian gauge fields. The lagrangian is (in $D = 2n$ dimensions)

$$\mathcal{L} = \bar{\psi} i \gamma^{\mu} \left(\partial_{\mu} - i A^{i}_{\mu} \lambda^{i} \right) \psi , \tag{1.1}$$

with λ_i the generator of the gauge group G in the representation to which the fermions belong. Here $A^i_\mu = \mathcal{V}^i_\mu + \mathcal{A}^i_\mu \gamma_{D+1}$, $\gamma_{D+1} = -i^{n+1} \prod_{\mu=0}^{D-1} \gamma^\mu$. $\mathcal{V}^i_\mu$ and $\mathcal{A}^i_\mu$ are, respectively, vector and axial-vector gauge fields, while γ_{D+1} is the counterpart of the usual γ_5 in D-dimensions. The quantum action functional, $W[A_\mu]$, can be expressed by the path integral

$$e^{iW[A_\mu]} = \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi} \exp i \int d^D x\, \mathcal{L}. \tag{1.2}$$

This theory is without question renormalizable. There is no integration over $A_\mu(x)$ since it is an external field by assumption.

Our calculation is in flat space and the gauge potentials are introduced explicitly (and not as part of the metric as in the Kaluza-Klein theories). Since chiral anomalies are a manifestation of short-distance physics [8] we conjecture that our results, insofar as the gauge fields are concerned, continue to hold in curved space. We plan to treat the curvature tensor contribution [16] to chiral anomalies and its connection to the Kaluza-Klein program in a future paper.

We recall that there exist two distinct chiral anomalies. The $U_A(1)$ or abelian anomaly is associated with the noninvariance of the fermionic path-integral measure in eq. (1.2) under the local transformation

$$\psi(x) \to e^{i\theta(x)\gamma_{D+1}}\psi(x). \tag{1.3}$$

The (anomalous) Ward identities in this case can be derived from

$$\left.\frac{\delta W[A_\mu]}{\delta\theta(x)}\right|_{\theta(x)=0} = 0, \tag{1.4}$$

which expresses invariance of the quantum action functional under change of integration variables, eq. (1.3). On the other hand, the non-abelian anomaly is associated with the noninvariance of the quantum action functional under the gauge transformation of the gauge fields accompanied by the transformation

$$\psi(x) \to e^{i\theta^i(x)\lambda_i\gamma_{D+1}}\psi(x). \tag{1.5}$$

The (anomalous) Ward identities can be obtained from

$$\left.\frac{\delta W[A_\mu^\theta]}{\delta\theta^i(x)}\right|_{\theta^i(x)=0} = G_i(x), \tag{1.6}$$

where A_μ^θ is the gauge transformed A_μ corresponding to eq. (1.5). $G_i(x)$ is none other than the non-abelian anomaly.

If we turn to examine the conservation laws, then the $U_A(1)$ anomaly appears in the divergence of the axial U(1) current (to which no gauge bosons are coupled),

$$J_\mu^A = \bar{\psi}\gamma_\mu\gamma_{D+1}\psi, \tag{1.7}$$

while the non-abelian anomaly appears in the covariant divergence of the non-abelian axial currents,

$$J^A_{\mu i} = \bar{\psi}\gamma_\mu\gamma_{D+1}\lambda_i\psi . \tag{1.8}$$

In four-dimensional spacetime the $U_A(1)$ or abelian anomaly is given simply by

$$\partial^\mu J^A_\mu = \frac{-1}{16\pi^2}\varepsilon^{\mu\nu\lambda\sigma}\mathrm{tr}\left(F_{\mu\nu}F_{\lambda\sigma}\right), \tag{1.9}$$

(where our convention is such that $\varepsilon_{0123} = +1$, and $F_{\mu\nu} = -iF^i_{\mu\nu}\lambda_i$), while the non-abelian anomaly for, e.g., $SU(N)\times SU(N)$ is given by a rather complicated expression calculated by Bardeen [5],

$$\begin{aligned}\left(D^\mu J^A_\mu\right)_i = -\frac{1}{4\pi^2}\varepsilon^{\mu\nu\rho\sigma}\mathrm{tr}\Big\{\lambda^i\Big[&\tfrac{1}{4}V_{\mu\nu}V_{\rho\sigma} + \tfrac{1}{12}A_{\mu\nu}A_{\rho\sigma} \\ &+\tfrac{2}{3}\left(\mathcal{A}_\mu\mathcal{A}_\nu V_{\rho\sigma} + V_{\mu\nu}\mathcal{A}_\rho\mathcal{A}_\sigma - \mathcal{A}_\mu V_{\nu\rho}\mathcal{A}_\sigma\right) - \tfrac{8}{3}\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{A}_\rho\mathcal{A}_\sigma\Big]\Big\},\end{aligned} \tag{1.10}$$

where

$$\mathcal{V}_\mu = -i\lambda^i\mathcal{V}^i_\mu, \qquad \mathcal{A}_\mu = -i\lambda^i\mathcal{A}^i_\mu,$$

$$V_{\mu\nu} = \partial_\mu\mathcal{V}_\nu - \partial_\nu\mathcal{V}_\mu + \left[\mathcal{V}_\mu, \mathcal{V}_\nu\right] + \left[\mathcal{A}_\mu, \mathcal{A}_\nu\right],$$

$$A_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + \left[\mathcal{V}_\mu, \mathcal{A}_\nu\right] + \left[\mathcal{A}_\mu, \mathcal{V}_\nu\right].$$

Notice that the covariant divergence appears in eq. (1.10) as required by gauge covariance.

In writing down eq. (1.10) Bardeen has added suitable "counterterms" to the action to insure that the vector current

$$J^V_{\mu i} = \bar{\psi}\gamma_\mu\lambda_i\psi \tag{1.11}$$

is divergenceless,

$$D^\mu J^V_{\mu i} = 0 . \tag{1.12}$$

Nowadays, instead of this procedure, it is more customary to write the lagrangian in eq. (1.1) in terms of left- and right-handed fields:

$$\mathcal{L} = \bar{\psi}_L i\gamma^\mu\left(\partial_\mu - iA^i_{L\mu}\lambda_i\right)\psi_L + \bar{\psi}_R i\gamma^\mu\left(\partial_\mu - iA^i_{R\mu}\lambda_i\right)\psi_R . \tag{1.13}$$

One then treats the left and right handed currents,

$$J^{\mathrm{L}}_{\mu i} = \bar{\psi}_{\mathrm{L}} \gamma_\mu \lambda_i \psi_{\mathrm{L}}, \tag{1.14}$$

$$J^{\mathrm{R}}_{\mu i} = \bar{\psi}_{\mathrm{R}} \gamma_\mu \lambda_i \psi_{\mathrm{R}}, \tag{1.15}$$

symmetrically rather than impose eq. (1.12). The non-abelian anomalies are given by

$$D^\mu J^{\mathrm{L}}_{\mu i} = -\frac{1}{24\pi^2} \varepsilon^{\mu\nu\rho\sigma} \mathrm{tr}\Big\{ \lambda^i \big[\partial_\mu A_{\mathrm{L}\nu} \partial_\rho A_{\mathrm{L}\sigma} + \tfrac{1}{2} \partial_\mu A_{\mathrm{L}\nu} A_{\mathrm{L}\rho} A_{\mathrm{L}\sigma} - \tfrac{1}{2} A_{\mathrm{L}\mu} \partial_\nu A_{\mathrm{L}\rho} A_{\mathrm{L}\sigma} + \tfrac{1}{2} A_{\mathrm{L}\mu} A_{\mathrm{L}\nu} \partial_\rho A_{\mathrm{L}\sigma} \big] \Big\}, \tag{1.16}$$

$$D^\mu J^{\mathrm{R}}_{\mu i} = +\frac{1}{24\pi^2} \varepsilon^{\mu\nu\rho\sigma} \mathrm{tr}\Big\{ \lambda^i \big[\partial_\mu A_{\mathrm{R}\nu} \partial_\rho A_{\mathrm{R}\sigma} + \tfrac{1}{2} \partial_\mu A_{\mathrm{R}\nu} A_{\mathrm{R}\rho} A_{\mathrm{R}\sigma} - \tfrac{1}{2} A_{\mathrm{R}\mu} \partial_\nu A_{\mathrm{R}\rho} A_{\mathrm{R}\sigma} + \tfrac{1}{2} A_{\mathrm{R}\mu} A_{\mathrm{R}\nu} \partial_\rho A_{\mathrm{R}\sigma} \big] \Big\}, \tag{1.17}$$

with $A_{\mathrm{L}\mu} = -iA^i_{\mathrm{L}\mu}\lambda_i$, $A_{\mathrm{R}\mu} = -iA^i_{\mathrm{R}\mu}\lambda_i$. (In fact, these equations were also obtained by Bardeen on his way to eq. (1.10).) Since these equations are structurally identical we will only write one of them henceforth. Indeed, many theories of physical interest are constructed out of left-handed fields only. Thus, we will often only write eq. (1.16) and suppress the subscript L.

The right-hand side of eq. (1.9) is a total divergence,

$$\partial^\mu J^A_\mu = -\frac{1}{4\pi^2} \mathrm{tr}\big[\partial^\mu \varepsilon_{\mu\nu\rho\sigma} \big(A^\nu \partial^\rho A^\sigma + \tfrac{2}{3} A^\nu A^\rho A^\sigma \big) \big]. \tag{1.18}$$

The right-hand side of eq. (1.16) may also be rewritten as

$$D^\mu J_{\mu i} = -\frac{1}{24\pi^2} \mathrm{tr}\big[\lambda^i \partial^\mu \varepsilon_{\mu\nu\rho\sigma} \big(A^\nu \partial^\rho A^\sigma + \tfrac{1}{2} A^\nu A^\rho A^\sigma \big) \big]. \tag{1.19}$$

As promised, we have suppressed the subscript L. Notice the numerical coefficient $\frac{2}{3}$ in eq. (1.18) and $\frac{1}{2}$ in eq. (1.19).

The close resemblance between the two expressions on the right hand sides of eq. (1.18) and eq. (1.19) may be a potential source of confusion. It is our impression that, in conversations at least, some authors sometimes confuse these two expressions corresponding to the two types of chiral anomalies. The Abelian anomaly is gauge invariant, while the non-abelian anomaly is gauge covariant and given by the covariant divergence of a non-abelian current.

In this paper, we will show that there is an intimate but rather non-trivial connection between the abelian and the non-abelian anomalies. The non-abelian anomaly in $2n$-dimensional space may be formally obtained from the abelian anomaly in $(2n+2)$-dimensional space by a series of mathematical manipulations.

The outline of this paper is as follows. In sect. 2 we briefly review the Wess-Zumino consistency condition for anomalies, as well as its physical implications, which guides our subsequent construction of non-abelian anomalies. In sect. 3, we use differential geometric methods to construct both the $U_A(1)$ and non-abelian chiral anomalies and obtain quite compact formulae in the generic $2n$-dimensional case. Finally the normalization of anomalies is discussed in sect. 4. Some technical details and a brief review of differential forms are contained in three appendices.

The message of this paper is that one can determine all chiral anomalies, abelian and non-abelian, by a differential geometric method without having to evaluate a Feynman diagram. We derive compact integral representations for the abelian and non-abelian anomalies and show that they are always total divergences.

For $(n+1)$ even, such as that for the possibly physically interesting case* of $D=10$, the non-abelian chiral anomaly does not vanish for any fermion representation. In the symmeterized trace, one can take the $(n+1)$ generator matrices to be the same one, λ, and simply diagonalize λ. For spaces with dimension $D=4k$, many of the standard theorems [9] proved for $D=4$ continue to hold. For example, if the fermion representation is real, the non-abelian chiral anomaly vanishes. Frampton and Kephart [23, 24] have evaluated the group theoretic factor in the anomaly for the totally anti-symmetric representations of $SU(N)$. They have recently carried out an analysis of this factor and have also studied the impact of the anomaly on supersymmetric Yang-Mills theory in 6 and 10 dimensions and on superstring theory in 10 dimensions [29].

2. Chiral anomaly and consistency condition

The implication of the non-abelian anomaly for $\pi^0 \to 2\gamma$ is well known. This corresponds to the first term on the right-hand side of eq. (1.9) or eq. (1.10). In contrast, the physical significance of the other terms in eq. (1.10) is perhaps not as widely known. They determine the amplitude for the processes $\gamma \to 3\pi$ and $2\gamma \to 3\pi$, as was shown by Adler et al. and also by Terentiev [6].

Wess and Zumino [7] showed that Bardeen's expression, eq. (1.10), must satisfy a consistency condition. Following Wess and Zumino, we define

$$-X_i = \partial_\mu \frac{\delta}{\delta A_{\mu i}} + \left(A_\mu \times \frac{\delta}{\delta A_\mu} \right)_i . \tag{2.1}$$

(We use the purely left-handed chiral formalism here instead of the vector-axial

* We thank J. Schwarz for an instructive conversation on superstrings and 10-dimensional supersymmetric Yang-Mills theories.

formalism of ref. 7].) The X_i's generate chiral gauge transformations:

$$[X_i(x), X_j(y)] = f_{ijk} X_k(x)\, \delta(x-y). \tag{2.2}$$

In terms of X_i, eq. (1.6) can be written as

$$X_i(x) W = G_i(x), \tag{2.3}$$

with (as in eq. (1.19))

$$-G_i(A) \propto \mathrm{tr}\left[\lambda_i \partial_\mu \left(A_\nu \partial_\sigma A_\rho + \tfrac{1}{2} A_\nu A_\sigma A_\rho\right) \varepsilon^{\mu\nu\sigma\rho}\right]. \tag{2.4}$$

Here $A_\mu \equiv -iA^j_\mu \lambda_j$.

The Wess-Zumino condition follows simply from applying eq. (2.2) to W:

$$X_i(x) G_j(y) - X_j(y) G_i(x) = f_{ijk} G_k(x)\, \delta(x-y). \tag{2.5}$$

The importance of the consistency condition (eq. (2.5)) lies in the fact that since the operator X_i is non-linear in the gauge potential A_μ the condition completely determines $G_i(A)$ once the first term (on the right-hand side of eq. (2.4)) in $G_i(A)$ is given. In the vector-axial formalism, given the first term in Bardeen's expression (eq. (1.10)), $\varepsilon^{\mu\nu\sigma\rho}\mathrm{tr}[\lambda_a \partial_\mu \mathcal{V}_\nu \partial_\sigma \mathcal{V}_\rho]$, which one can perhaps argue must be present from knowing the abelian anomaly, one can determine the Bardeen expression in its entirety. The Wess-Zumino condition plays a crucial role in our analysis in sect. 3.

After the Wess-Zumino paper was published, it was realized that the analysis of Adler et al. [6] amounts to, in some sense, the consistency condition stated in physical terms. Adler et al. showed that, given the amplitude for $\pi^0 \to 2\gamma$, one can determine the amplitudes for $\gamma \to 3\pi$ and $2\gamma \to 3\pi$ by appealing to gauge and chiral invariances.

Given G_i, Wess and Zumino [7] showed that one can solve eq. (2.3) for the action functional W when Goldstone bosons are present. Recently, Witten [14] realized that the Wess-Zumino solution for W can be written in a remarkably compact form as an integral with topological significance over *five*-dimensional space, and which is closely related to mathematical objects appearing in our analysis. (See appendix A for further discussion.)

3. Differential geometric construction of anomalies

To carry out our differential geometric analysis [30] (a forthcoming paper by Zumino in which mathematical tools and concepts will be discussed from a more general viewpoint) of anomalies, we found it exceedingly convenient to use the language of differential forms. For our purposes, differential forms offer a compact index-free notation. The skeptical reader should contemplate what some of our expressions, such as eq. (3.37), would look like if one were to write out all the indices explicitly. Everything we will need is explained in appendix A to which the reader unfamiliar with this formalism may wish to turn now. We summarize some basic formulae here.

With A the potential 1-form, the gauge field 2-form is given by

$$F = \mathrm{d}A + A^2 . \tag{3.1}$$

Gauge transformations are described by

$$\delta_v A = -\mathrm{d}v - [A, v] \equiv -Dv , \tag{3.2}$$

$$\delta_v F = -[F, v] . \tag{3.3}$$

Here $v(x)$ is an infinitesimal 0-form taking values in the Lie algebra $\overline{\mathrm{G}}$. The Bianchi identity reads

$$DF \equiv \mathrm{d}F + [A, F] = 0 . \tag{3.4}$$

We re-write the expression for chiral anomalies in the notation of forms. The abelian anomaly reads (cf. eqs. (1.9) and (1.18))

$$d * J^A \propto \mathrm{tr}\, F^2 = \mathrm{tr}\left[\mathrm{d}\left(A\,\mathrm{d}A + \tfrac{2}{3}A^3\right)\right] , \tag{3.5}$$

while the non-abelian anomaly reads (cf. eq. (2.4))

$$D * J_i^A = -G_i(A) \propto \mathrm{tr}\left[\lambda_i \mathrm{d}\left(A\,\mathrm{d}A + \tfrac{1}{2}A^3\right)\right] . \tag{3.6}$$

(As explained in appendix A, we adopt a notational simplification of not writing the standard wedge product.) Now we turn to the differential geometric construction of anomalies.

3.1. $\mathrm{U_A}(1)$ ANOMALY

In $D = 2n$ dimensions, the $\mathrm{U_A}(1)$ anomaly is given by the $2n$-form

$$\Omega_{2n}(A) \equiv \mathrm{tr}\, F^n = \mathrm{Str}\, F^n . \tag{3.7}$$

This form is dictated by invariance considerations. The divergence $\partial^\mu J_\mu^A$ is gauge invariant, has dimension $2n$, and is odd under parity and time-reversal. Str denotes the symmetrized trace of a product of k matrices,

$$\mathrm{Str}(B_1, \ldots, B_k) = \frac{1}{k!} \sum_{(i_1, \ldots, i_k)} \mathrm{tr}\left(B_{i_1} B_{i_2} \ldots B_{i_k}\right) , \tag{3.8}$$

the sum being over all permutations $(i_1,\dots,i_k)$ of $(1,\dots,k)$. (In this subsection k is always n for $D=2n$.) In eq. (3.7), Str can clearly be replaced by the ordinary trace tr. But it turns out that the introduction of Str in this equation leads to some crucial simplifications in our subsequent manipulations. When some of the entries are the same, we write it in power form.

In mathematics, the object $\Omega_{2n}(A)$ is known as the nth Chern character. Its gauge invariance is easily checked by using eqs. (3.3) and

$$\delta_v\Omega_{2n}(A) = n\,\mathrm{Str}([v,F],F^{n-1}) = 0. \tag{3.9}$$

The Chern characters are closed due to the Bianchi identity:

$$\begin{aligned} \mathrm{d}\Omega_{2n} &= n\,\mathrm{Str}(\mathrm{d}F, F^{n-1}) \\ &= n\left\{\mathrm{Str}(DF,F^{n-1}) - \mathrm{Str}([A,F],F^{n-1})\right\} \\ &= 0. \end{aligned} \tag{3.10}$$

(For properties of symmetrized trace used above, see appendix A.)

According to Poincaré's lemma, eq. (3.10) implies that the Chern character can be locally written as an exterior differential of a certain $(2n-1)$-form:

$$\Omega_{2n}(A) = \mathrm{d}\omega^0_{2n-1}(A). \tag{3.11}$$

To determine ω^0_{2n-1}, let us vary A:

$$A \to A + \delta A. \tag{3.12}$$

Then

$$F \to F + \delta A A + A\delta A + \mathrm{d}(\delta A), \tag{3.13}$$

to first order and

$$\begin{aligned} \delta\Omega_{2n} &= n\,\mathrm{Str}(\mathrm{d}\,\delta A + \delta A A + A\delta A, F^{n-1}) \\ &= n\,\mathrm{tr}(\mathrm{d}\,\delta A F^{n-1} + \delta A A F^{n-1} - \delta A F^{n-1} A) \\ &= n\,\mathrm{tr}\big(\mathrm{d}\,\delta A F^{n-1} + \delta A[A,F]F^{n-2} \\ &\qquad + \delta A F[A,F]F^{n-3} + \cdots + \delta A F^{n-2}[A,F]\big) \\ &= n\,\mathrm{d}\,\mathrm{tr}\,\delta A F^{n-1}. \end{aligned} \tag{3.14}$$

In the last step we used the Bianchi identity. One may integrate eq. (3.14). In particular, letting $A_t = tA$, $F_t = t\,\mathrm{d}A + t^2A^2$, we find

$$\Omega_{2n} = n\,\mathrm{d}\int_0^1 \mathrm{d}t\, t^{n-1}\mathrm{tr}\{A(\mathrm{d}A + tA^2)^{n-1}\}. \tag{3.15}$$

We thus obtain a general formula for $\omega^0_{2n-1}(A)$:

$$\omega^0_{2n-1}(A) = n\int_0^1 \mathrm{d}t\, t^{n-1}\mathrm{Str}\big(A, (\mathrm{d}A + tA^2)^{n-1}\big). \tag{3.16}$$

(Eq. (3.15) defines ω^0_{2n-1} up to a gradient $\mathrm{d}p$, with p an arbitrary $(2n-2)$-form. We define ω^0_{2n-1} as in eq. (3.16). In eqs. (3.15) and (3.16) $\mathrm{d}t$ is an ordinary differential and commutes with $\mathrm{d}x$.)

Eq. (3.15) says that the $\mathrm{U_A}(1)$ anomaly can always be written as the total divergence of a certain current formed out of A.

Writing out the indices, we have

$$\begin{aligned} \partial^\mu J^A_\mu &= c\varepsilon^{\mu_1\mu_2\cdots\mu_{D-1}\mu_D}\mathrm{Str}\big(F_{\mu_1\mu_2},\dots,F_{\mu_{D-1}\mu_D}\big) \\ &= c\partial^\mu K_\mu(A), \end{aligned} \tag{3.17}$$

where c is an overall coefficient and

$$K_\mu(A) = \varepsilon_{\mu\nu\mu_3\dots\mu_D}\{\mathrm{Str}(A^\nu, F^{\mu_3\mu_4},\dots,F^{\mu_{D-1}\mu_D}) + \cdots\} \tag{3.18}$$

is a non-gauge-invariant current. In eq. (3.18) one can determine terms other than the first one from eq. (3.16).

The reader may wish to check eq. (3.16) for the familiar case $D=4$ or $n=2$,

$$\begin{aligned} \omega^0_3(A) &= 2\int_0^1 \mathrm{d}t\, t\,\mathrm{Str}(A, \mathrm{d}A + tA^2) \\ &= \mathrm{Str}(A,\mathrm{d}A) + \tfrac{2}{3}\mathrm{Str}(A, A^2) \\ &= \mathrm{tr}\big(A\,\mathrm{d}A + \tfrac{2}{3}A^3\big). \end{aligned} \tag{3.19a}$$

It gives the well-known result (eq. (1.18) in sect. 1)

$$K_\mu(A) = \varepsilon_{\mu\nu\rho\sigma}\mathrm{tr}\big(A^\nu\partial^\rho A^\sigma + \tfrac{2}{3}A^\nu A^\rho A^\sigma\big). \tag{3.19b}$$

For $D=6$ or $n=3$ we have

$$\begin{aligned} \omega^0_5(A) &= 3\int_0^1 \mathrm{d}t\, t^2\mathrm{Str}\big(A, (\mathrm{d}A + tA^2)^2\big) \\ &= \mathrm{Str}\big(A, (\mathrm{d}A)^2\big) + \tfrac{3}{2}\mathrm{Str}(A, A^2, \mathrm{d}A) + \tfrac{3}{5}\mathrm{Str}(A, A^2, A^2) \\ &= \mathrm{tr}\big(A(\mathrm{d}A)^2 + \tfrac{3}{2}A^3\,\mathrm{d}A + \tfrac{3}{5}A^5\big). \end{aligned} \tag{3.20a}$$

It corresponds to

$$K_\mu(A) = \varepsilon_{\mu\nu\rho\sigma\lambda\tau}\mathrm{tr}\{A^\nu\partial^\rho A^\sigma\partial^\lambda A^\tau + \tfrac{3}{2}A^\nu A^\rho A^\sigma\partial^\lambda A^\tau + \tfrac{3}{5}A^\nu A^\rho A^\sigma A^\lambda A^\tau\}\,. \quad (3.20\mathrm{b})$$

Historically, the fact that the $U_A(1)$ anomaly can be written as a total divergence was realized some years after the discovery of the anomaly. Eq. (3.10) and eq. (3.11) make it obvious that this holds in general.

3.2. NON-ABELIAN CHIRAL ANOMALIES

To determine the non-abelian anomalies in $D = 2n$ dimensions, our strategy is to find an object $G_i(A)$ which consists of only gauge fields and satisfies the Wess-Zumino condition. In form notation the condition (2.5) in the integrated form reads

$$\delta_u G(v, A) - \delta_v G(u, A) = G([u, v], A)\,, \quad (3.21)$$

where $\delta_v = \int \mathrm{d}^D x\, v^i(x) X_i(x)$ generates the gauge transformation (3.1) with $v(x) = v^i(x)\lambda_i$ as the gauge function

$$G(v, A) = \int_{S_D} v^i(x) G_i(A)\,. \quad (3.22)$$

Here the integration is over our D-dimensional flat space; it can be thought of as a D-dimensional sphere S_D if we consider only those gauge fields whose field strengths vanish at infinity sufficiently fast*.

Recall that Wess and Zumino [7] (also see Witten [14]) have found an effective action functional for $D = 4$, containing both scalar fields and gauge fields. They expressed it in terms of a 5-dimensional integral. Here our discussion is for pure gauge theories. Inspired by their work, we also go to a space one dimension higher, and consider in it the $(D+1)$-form $\omega^0_{2n+1}(A)$, which can be obtained formally from the $(D+2)$th Chern character $\Omega_{2n+2}(A)$ by (see eq. (3.16))

$$\Omega_{2n+2}(A) = \mathrm{d}\omega^0_{2n+1}(A)\,, \quad (3.23\mathrm{a})$$

$$\omega^0_{2n+1}(A) = (n+1)\int_0^1 \mathrm{d}t\, t^n \mathrm{Str}\left(A, (\mathrm{d}A + tA^2)^n\right). \quad (3.23\mathrm{b})$$

* Take the standard instanton discussion as an example. In the physics literature, one notes that the finiteness of the euclidean action requires that A_μ goes to a pure gauge $g^{-1}(x)\partial_\mu g(x)$ and thus defines a mapping of the sphere "at infinity" $S_3 = \partial E_4$ into the group G. In the mathematical literature, S_3 is taken to be a large but finite sphere. The portion of E_4 inside this sphere is identified as a disk D_4 = the northern hemisphere of S_4 with $\partial S_4 = S_3$ = the equator. The portion of E_4 outside the sphere S_3 is identified with the southern hemisphere of $S_4 \equiv \bar{D}_4$. Thus, E_4 is compactified to S_4. In other words, the mathematician identifies what the physicist loosely refers to as "infinity" as the southern hemisphere $\bar{D}_4$.

(Here we are dealing with Str for $n+1$ entries.) In the $(D+1)$-space $\Omega_{2n+2}(A)$, being a $(D+2)$-form, is actually zero, so that eq. (3.23) is formal. $\omega^0_{2n+1}(A)$ is called the Chern-Simons secondary topological invariant.

From the gauge invariance of Ω (eq. (3.4)) we know

$$\delta_v\left(\mathrm{d}\omega^0_{2n+1}\right)=\mathrm{d}\left(\delta_v\omega^0_{2n+1}\right)=0\,. \tag{3.24}$$

Thus, locally there must exist a certain $2n$-form $\omega^1_{2n}(v,A)$ such that

$$\delta_v\omega^0_{2n+1}(A)=\mathrm{d}\omega^1_{2n}(v,A)\,. \tag{3.25}$$

Here the subscript and superscript of $\omega^1_{2n}(v,A)$ indicate that it is a $2n$-form and of first order in v. Now we define

$$G(v,A)\equiv\int v^i(x)G_i(x)=\int_{\mathrm{S}_D}\omega^1_{2n}(v,A)\,, \tag{3.26}$$

where $v_i(x)\lambda^i=v(x)$. To prove that the so constructed $G_i(A)$ automatically satisfies eq. (3.21), we make the working hypothesis that there is no topological obstruction against extending smoothly the gauge fields from our D-dimensional space, S_D, to a $(D+1)$-dimensional ball, B_{D+1}, which has S_D as its boundary*. Thus it is meaningful to consider the following purely mathematical functional

$$U[A]=\int_{\mathrm{B}_{D+1}}\omega^0_{2n+1}(A)\,. \tag{3.27}$$

From eq. (3.25) we have the gauge variation of $U[A]$ as follows:

$$\delta_vU[A]=\int_{\mathrm{B}_{D+1}}\delta_v\omega^0_{2n+1}(A)=\int_{\mathrm{S}_{2n}}\omega^1_{2n}(v,A)\,. \tag{3.28}$$

Here we have used the Stokes theorem (see eq. (A.16)). Thus from

$$\delta_u\int_{\mathrm{S}_{2n}}\omega^1_{2n}(v,A)=\delta_u\delta_vU[A]\,, \tag{3.29}$$

it is easy to see that eq. (3.21) is satisfied for $G(v,A)$ defined by eq. (3.26), since $[\delta_u,\delta_v]=\delta_{[u,v]}$ acting on the gauge variant functional $U[A]$. This reflects the main advantage of our geometric procedure for constructing anomalies; namely, it gives

* We know that this hypothesis is true in many cases, e.g. [31] $\mathrm{G}=\mathrm{SU}(N)$ with $N>\frac{1}{2}D$ as $\pi_D(\mathrm{SU}(N))=0$. In the cases when it is not true, we can directly check the consistency condition in differential form, eq. (2.5). However, the integrated form has the advantage of avoiding δ-functions which may give rise to some subtleties while checking it.

directly the solution to the Wess-Zumino consistency condition. A derivation following the original treatment of Wess and Zumino will be given in ref. [30].

For physical applications we need to know the explicit expression for $\omega_{2n}^1(v, A)$ or $G^i(A)$ as determined by eqs. (3.23b) and (3.25). After some work, we have found the following general formula for computing $\omega_{2n}^1(v, A)$:

$$-\omega_{2n}^1(v, A) = (n+1)\int_0^1 \mathrm{d}t\,\{\mathrm{Str}(v, F_t^n) - t(t-1)n\,\mathrm{Str}(A,[v, A], F_t^{n-1})\} \tag{3.30}$$

where $F_t = t\,\mathrm{d}A + t^2A^2$. The proof will be given in appendix B.

To check their validity, we find, for $D=4$ or $n=2$, from eq. (3.30),

$$\begin{aligned}-\omega_4^1(v, A) &= \mathrm{Str}(v,\mathrm{d}A,\mathrm{d}A) + \tfrac{3}{2}\mathrm{Str}(v, A^2,\mathrm{d}A)\\ &\quad + \tfrac{3}{5}\mathrm{Str}(v, A^2, A^2) + \tfrac{1}{2}\mathrm{Str}(A, vA - Av,\mathrm{d}A)\\ &\quad + \tfrac{3}{10}\mathrm{Str}(A, vA - Av, A^2)\\ &= \mathrm{tr}\{v\,\mathrm{d}(A\,\mathrm{d}A + \tfrac{1}{2}A^3)\}\,. \end{aligned}\tag{3.31}$$

It leads to the well-known anomaly in eq. (1.19) for the chiral SU(N) gauge theory.

For $D=6$ or $n=3$ we have

$$\begin{aligned}-\omega_6^1(A) &= \mathrm{Str}(v,(\mathrm{d}A)^3) + \tfrac{12}{5}\mathrm{Str}(v, A^2,(\mathrm{d}A)^2) + 2\,\mathrm{Str}(v,\mathrm{d}A,(A^2)^2)\\ &\quad + \tfrac{4}{7}\mathrm{Str}(v,(A^2)^3) + \tfrac{3}{5}\mathrm{Str}(A, vA - Av,(\mathrm{d}A)^2)\\ &\quad + \tfrac{4}{5}\mathrm{Str}(A, vA - Av,\mathrm{d}A, A^2) + \tfrac{2}{7}\mathrm{Str}(A, vA - Av,(A^2)^2)\\ &= \mathrm{tr}\{v\,\mathrm{d}[A(\mathrm{d}A)^2 + \tfrac{2}{5}(A^3\mathrm{d}A + \mathrm{d}AA^3)\\ &\quad + \tfrac{1}{5}(A^2\mathrm{d}AA + A\,\mathrm{d}AA^2) + \tfrac{2}{5}A^5]\}\,. \end{aligned}\tag{3.32}$$

It corresponds to

$$\begin{aligned}-G_i(A) = \varepsilon_{\mu\nu\rho\sigma\lambda\tau}\partial^\mu \mathrm{tr}\{\lambda_i(&A^\nu\partial^\rho A^\sigma\partial^\lambda A^\tau + \tfrac{2}{5}A^\nu A^\rho A^\sigma A\partial^\lambda A^\tau\\ &+ \tfrac{2}{5}\partial^\nu A^\rho A^\sigma A^\lambda A^\tau + \tfrac{1}{5}A^\nu A^\rho\partial^\sigma A^\lambda A^\tau\\ &+ \tfrac{1}{5}A^\nu\partial^\rho A^\sigma A^\lambda A^\tau + \tfrac{2}{5}A^\nu A^\rho A^\sigma A^\lambda A^\tau)\}\,. \end{aligned}\tag{3.33}$$

We note that the non-abelian chiral anomalies $G^i(A)$ in both eqs. (1.19) and (3.33) are a total divergence like the $\mathrm{U_A}(1)$ anomaly. With the explicit representation of

$\omega_{2n}^{1}(v, A)$ derived in eq. (3.30) we can prove this statement in any even dimension. The details are given in appendix B. Introducing the symmetrized product of n matrices

$$P(\lambda_1,\ldots,\lambda_n)=\frac{1}{n!}\sum_{(i_1,\ldots,i_n)}\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_n}, \tag{3.34}$$

where the sum is taken over all permutations $(i_1,\ldots,i_n)$ of $(1,\ldots,n)$, we can express $\omega_{2n}^{1}(v, A)$ in a very compact form

$$-\omega_{2n}^{1}(v,A)=n(n+1)\int_0^1 \mathrm{d}t(1-t)\operatorname{tr}\left\{v\,\mathrm{d}P\left(A,F_t^{n-1}\right)\right\}, \tag{3.35}$$

where $F_t = t\,\mathrm{d}A + t^2A^2$.

Physically, $D=10$ dimensions may be of particular interest. Using eqs. (3.16) and (3.35) we obtain the $U_A(1)$ and non-abelian anomalies for $D=10$ as follows:

$$\begin{aligned}\omega_9^0(A)=\operatorname{tr}\Big\{&A(\mathrm{d}A)^4+\tfrac{5}{3}A^3(\mathrm{d}A)^3+\tfrac{5}{6}A\,\mathrm{d}AA^2(\mathrm{d}A)^2+\\&+\tfrac{5}{6}A(\mathrm{d}A)^2A^2\mathrm{d}A+\tfrac{15}{7}A^5(\mathrm{d}A)^2+\tfrac{10}{7}A^3\mathrm{d}AA^2\mathrm{d}A\\&+\tfrac{5}{7}A^4\mathrm{d}AA\,\mathrm{d}A+\tfrac{5}{2}A^7\mathrm{d}A+\tfrac{5}{9}A^9\Big\},\end{aligned} \tag{3.36}$$

$$\begin{aligned}-\omega_{10}^1(A)=\operatorname{tr}v\,\mathrm{d}\Big\{&A(\mathrm{d}A)^4+\tfrac{20}{7}P\big(A,A^2,(\mathrm{d}A)^3\big)\\&+\tfrac{45}{14}P\big(A,(A^2)^2,(\mathrm{d}A)^2\big)+\tfrac{5}{3}P\big(A,(A^2)^3,\mathrm{d}A\big)+\tfrac{1}{3}A^9\Big\}.\end{aligned} \tag{3.37}$$

The reader should realize how long the expression in eq. (3.37) is if written out explicitly without the P notation. Also, to appreciate how relatively simple the present derivation is, the reader may wish to recall the complexity of Bardeen's derivation of the four-dimensional expression (eq. (1.10)).

From eqs. (3.30) or (3.35) it can be seen that the non-abelian anomaly is proportional to the symmetric trace with $(n+1)$ entries, $\mathrm{Str}(\lambda_{i_1},\ldots,\lambda_{i_{n+1}})$, in $2n$ dimensions, as discussed in sect. 1.

4. Normalization of anomaly

It is clear from the form of the Wess-Zumino consistency condition (eq. (2.5)) that this condition alone can determine the anomaly only up to an overall constant. For the sake of completeness, we will fix the overall normalization here. We consider two possible ways: by generalizing Fujikawa's method [18] to higher dimensions and by looking at a Feynman diagram. The analysis will be given in appendix C. We find that the abelian anomaly is given by

$$\partial^{\mu} J_{\mu}^{A} = K_n \varepsilon_{\mu_1 \dots \mu_{2n}} \operatorname{Str} F^{\mu_1\mu_2} \cdots F^{\mu_{2n-1}\mu_{2n}}$$

$$\equiv A(x), \tag{4.1}$$

with

$$K_n = i^n / 2^{2n-1} \pi^n n! \tag{4.2}$$

in $D = 2n$ dimensional Minkowski space. The normalization of the abelian anomaly fixes the normalization of the non-abelian anomaly. (Compare eqs. (1.9), (1.10), (1.16) and (1.17).) The factor of i^n is merely due to our convention of $F_{\mu\nu}$. (See eq. (A.10) below.)

The normalization K_n possesses physical significance, as is well known [11]. Upon integration of eq. (4.1) over euclidean $2n$-space one relates the change in chirality $\Delta Q_A = \int \mathrm{d}^D x\, \partial^{\mu} J_{\mu}^{A}$ to the integral of the Chern character. Since the Chern characters are normalized in the mathematical literature so as to give integers when integrated over compact manifolds, K_n is normalized up to a multiplicative integer factor by purely mathematical reasons. A complete determination of K_n using pure mathematics may be found in ref. [30].

Note that eq. (4.1) is in fact the local version of the Atiyah-Singer index theorem. Therefore we can say that the Atiyah-Singer index theorem [12] anticipates the connection between topology and anomaly in the present case.

B. Zumino is grateful to Raymond Stora for introducing him to the more mathematical aspects of this subject. Y.-S. Wu thanks C. N. Yang for very stimulating discussions on this subject and Y. Jack Ng and H.-S. Tai for discussions on Chern classes during his stay at Stony Brook and at Princeton last year. A. Zee thanks D. Boulware for instructive discussions on differential forms.

This work was supported in part by the US Department of Energy under contract DE-AC06-81ER40048.

Appendix A

Differential forms are discussed in a number of standard references [32, 33]. Nevertheless, for the sake of pedagogical clarity and completeness, we will give a brief review here. For our purposes, differential forms simply provide an exceedingly compact *notation* which saves us the tedious task of writing out indices explicitly. It is akin to the introduction of the index notation for vectors and tensors which supplanted the practice, common in the physics literature before the turn of the century, of writing out all vectors and tensors component by component.

A scalar function $f(x)$ is called a 0-form. We define

$$\mathrm{d}f \equiv \frac{\partial f}{\partial x^\mu}\,\mathrm{d}x^\mu . \tag{A.1}$$

In D-dimensional space the index μ runs from 1 to D.

Given a vector function ϕ_μ we construct a 1-form ϕ,

$$\phi \equiv \phi_\mu\,\mathrm{d}x^\mu . \tag{A.2}$$

We define

$$\mathrm{d}\phi = \frac{\partial \phi_\mu}{\partial x^\nu}\,\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu . \tag{A.3}$$

The wedge product is defined so that

$$\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu . \tag{A.4}$$

Thus, $\mathrm{d}\phi$ gives the curl of ϕ.

In general, given an antisymmetric tensor with p indices $\phi_{\mu_1,\dots,\mu_p}$ we can construct a p-form

$$\phi = \phi_{\mu_1\dots\mu_p}\left(\frac{1}{p!}\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}\right). \tag{A.5}$$

Clearly, in D dimensions, we cannot have p-forms with $p > D$ which do not vanish identically. We define

$$\mathrm{d}\phi = \partial_\nu \phi_{\mu_1\dots\mu_p}\left(\frac{1}{p!}\mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}\right). \tag{A.6}$$

The advantage of writing ϕ and $\mathrm{d}\phi$ instead of the expressions in eqs. (A.5) and (A.6) should be clear.

In order to simplify the notation, we omit the wedge product symbol and simply regard $\mathrm{d}x^\mu$ as an anti-commuting Grassmann object.

We can multiply a p-form α and a q-form β together in the obvious way:

$$\begin{aligned}\alpha\beta &= \alpha_{\mu_1\dots\mu_p}\beta_{\nu_1\dots\nu_p}\frac{1}{p!q!}\mathrm{d}x^{\mu_1}\dots\mathrm{d}x^{\mu_p}\mathrm{d}x^{\nu_1}\dots\mathrm{d}x^{\nu_q}\\ &= (-1)^{pq}\beta\alpha .\end{aligned} \tag{A.7}$$

The rule for differentiating a product then reads

$$\mathrm{d}(\alpha\beta) = (\mathrm{d}\alpha)\beta + (-1)^p\alpha\,\mathrm{d}\beta . \tag{A.8}$$

In Yang-Mills theory, the potential is a 1-form,

$$A = A_\mu \mathrm{d}x^\mu . \tag{A.9}$$

Here $A_\mu = -iA^j_\mu \lambda_j$ and so A is at the same time a form and a matrix. In arithmetical manipulations, one must take care to keep this fact in mind.

The gauge field is

$$F = \mathrm{d}A + A^2 . \tag{A.10}$$

(Note that our definition of A_μ and $F_{\mu\nu} = -iF^j_{\mu\nu}\lambda^j$ differs by a factor of $(-i)$ from the one most often used in the physics literature. This is designed so that equations such as eq. (A.10) do not contain factors of i. Our λ^j matrices are hermitian and for SU(2), $\lambda^j = \frac{1}{2}\tau^j$. (Cf. eq. (1.1).)

Writing F out long-hand, we have

$$F = \left(\partial_\mu A_\nu - \partial_\nu A_\mu + \left[A_\mu, A_\nu\right]\right)\tfrac{1}{2}\,\mathrm{d}x^\mu \mathrm{d}x^\nu . \tag{A.11}$$

We illustrate the use of forms by deriving Bianchi's identity. We compute

$$\mathrm{d}F = \mathrm{d}A\,A - A\,\mathrm{d}A , \tag{A.12}$$

$$[A, F] = A\,\mathrm{d}A - \mathrm{d}A\,A . \tag{A.13}$$

Adding, we have

$$DF \equiv \mathrm{d}F + [A, F] = 0 . \tag{A.14}$$

As another exercise, we compute

$$\begin{aligned} \mathrm{d}\,\mathrm{tr}\,F^2 &= \mathrm{tr}(\mathrm{d}F\,F + F\,\mathrm{d}F) = 2\,\mathrm{tr}\,\mathrm{d}F F \\ &= -2\,\mathrm{tr}[A, F]F = 0 . \end{aligned} \tag{A.15}$$

Note that this holds in any dimension. In four dimensions, the statement is trivial since $\mathrm{tr}\,F^2$ is already a 4-form.

A p-form ϕ may be integrated over a p-dimensional surface M. If $\phi = \mathrm{d}\beta$, where β is some $(p-1)$-form, then Stokes theorem reads in the language of forms

$$\int_{\mathrm{M}} \phi = \int_{\partial \mathrm{M}} \beta . \tag{A.16}$$

∂M denotes the boundary of M. For example, in electromagnetism, $F = \mathrm{d}A$.

Integrating over M with M a two-dimensional surface we find

$$\int_{\mathrm{M}} F \equiv \int_{\mathrm{M}} \tfrac{1}{2} F_{\mu\nu}\,\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu}$$

$$= \int_{\partial\mathrm{M}} A_{\mu}\,\mathrm{d}x^{\mu} \equiv \int_{\partial\mathrm{M}} A\,.$$

A form α is said to be closed if

$$\mathrm{d}\alpha = 0\,. \tag{A.17}$$

It is said to be exact if

$$\alpha = \mathrm{d}\beta\,. \tag{A.18}$$

An exact form is obviously closed. Is a closed form exact? Poincaré's lemma states that a closed form is always *locally* exact. However, it may not be exact globally.

According to eq. (A.15), $\mathrm{tr}\, F^2$ is closed. Therefore, there must exist locally a 3-form γ so that

$$\mathrm{tr}\, F^2 = \mathrm{d}\gamma\,. \tag{A.19}$$

Topological quantization always involves a closed form which is not globally exact. Let $g(x)$ be elements of a simple group G. Consider the 1-form $v = g^{-1}\mathrm{d}g$. Then $\mathrm{tr}\, v^N$ is trivially closed on an N-dimensional manifold since it is already an N-form. Consider

$$Q = \int_{\mathrm{S}_N} \mathrm{tr}\, v^N\,, \tag{A.20}$$

where S_N denotes the N-dimensional sphere. If $\mathrm{tr}\, v^N$ is globally exact, i.e. $\mathrm{tr}\, v^N = \mathrm{d}\gamma$ with some $(N-1)$-form γ, then, by Stokes' theorem,

$$Q = \int_{\partial\mathrm{S}_N} \gamma = 0\,, \tag{A.21}$$

since $\partial\mathrm{S}_N$ = the boundary of $\mathrm{S}_N = 0$. On the other hand, if $\mathrm{tr}\, v^N$ is not globally exact, the above reasoning fails and Q may or may not be zero. In many cases, particularly when $\mathrm{G} = \mathrm{SU}(N)$, Q is recognizably just the integral describing the mapping of S_N onto the group G defined by $x \to g(x)$ for x a point on S_N. Thus, whenever $\mathrm{tr}\, v^N$ is globally exact, the homotopy group $\pi_N(\mathrm{G})$ is trivial. As an example, $\mathrm{G} = \mathrm{U}(1)$ we have $v = \partial_{\mu}\theta\,\mathrm{d}x^{\mu}$ which is exact locally but not globally. Choosing θ to go from 0 to

2π, v fails to be exact at $\theta = 0 = 2\pi$, and so

$$\int_{S_1} \frac{\partial \theta}{\partial x}\, dx = 2\pi . \tag{A.22}$$

This fact leads to the physical phenomenon of flux quantization.

For $N = 3$, Q is just the Skyrme [34] charge, up to an overall normalization factor. In particular, $\pi_3(SU(2)) = \pi_3(S^3) = Z$ corresponds to the mapping of $S_3 \to S_3$ and implies the existence of instantons.

In a recent paper [14], Witten showed that the action of the nonlinear sigma model contains the term

$$\int_{M^5} \operatorname{tr} v^5 , \tag{A.23}$$

where M^5 is a 5-dimensional manifold whose boundary ∂M^5 is 4-dimensional spacetime. The fact that $\operatorname{tr} v^5$ is not globally exact shows that the coefficient of this term is topologically quantized.

The preceding makes clear the topological significance of the abelian anomaly. According to eq. (3.11) the Chern characters

$$\operatorname{tr} F^n = d\omega^0_{2n-1} . \tag{A.24}$$

If the gauge potential A goes to a pure gauge at spatial infinity

$$A \to v = g^{-1} dg \tag{A.25}$$

then

$$dA \to -A^2 = -v^2 , \tag{A.26}$$

and so according to eq. (3.16)

$$\omega^0_{2n-1} \to \left[(-1)^{n-1} n!(n-1)!/(2n-1)!\right] \operatorname{tr} v^{2n-1} .$$

If E^{2n} denotes $2n$-dimensional euclidean space, then by Stokes' theorem

$$\int_{E^{2n}} \operatorname{tr} F^n = \int_{\partial E^{2n}} \omega^0_{2n-1} \propto \int_{S_{2n-1}} \operatorname{tr} v^{2n-1} . \tag{A.27}$$

Thus the integral of $\operatorname{tr} F^n$ over E^{2n} is associated with the homotopy group $\pi_{2n-1}(G)$. The case in which $n = 2$ and $G = SU(2)$ corresponds to the instanton.

In the text, we encounter current divergence. To write this in form language, we need to introduce the dual * operation. If ϕ is a p-form constructed of a rank-p

totally antisymmetric tensor in D-dimensional space, $\phi_{\mu_1\mu_2\ldots\mu_p}$, then $*\phi$ is the $(D-p)$-form constructed from the tensor $\varepsilon_{\mu_1\ldots\mu_{D-p}\ldots\mu_D}\phi^{\mu_{D-p+1}\ldots\mu_D}$. Notice the $*$ operation refers explicitly to the dimensionality D. (It requires the space to be equipped with a metric which can be used to lift the indices.)

Let J be a 1-form. Then the operation d^* evaluates the divergence:

$$d*J = d\left(\varepsilon_{\nu\mu_1\ldots\mu_{D-1}}J^{\nu}\frac{1}{(D-1)!}\mathrm{d}x^{\mu_1}\ldots\mathrm{d}x^{\mu_{D-1}}\right)$$

$$= \left(\partial_\lambda J^\lambda\right)\frac{1}{D!}\varepsilon_{\mu_1\ldots\mu_D}\mathrm{d}x^{\mu_1}\ldots\mathrm{d}x^{\mu_D}. \tag{A.28}$$

Appendix B

We will present the proof of eqs. (3.30) and (3.35) in detail. Before doing this we first give some useful formulae when we deal with symmetrized trace or product of a number of matrix forms.

Recall the definitions of the symmetrized trace and product of n matrices $\lambda_1,\ldots,\lambda_n$ belonging to the Lie algebra

$$\mathrm{Str}(\lambda_1,\lambda_2,\ldots,\lambda_n) = \frac{1}{n!}\sum_{(i_1,\ldots,i_n)}\mathrm{tr}(\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_n}), \tag{B.1}$$

$$\mathrm{P}(\lambda_1,\lambda_2,\ldots,\lambda_n) = \frac{1}{n!}\sum_{(i_1,\ldots,i_n)}\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_n}, \tag{B.2}$$

where the sum is over all permutations $(i_1,i_2,\ldots,i_n)$ of $(1,2,\ldots,n)$. Suppose g is an element of the Lie group. Then

$$\mathrm{Str}(g\lambda_1g^{-1},g\lambda_2g^{-1},\ldots,g\lambda_ng^{-1}) = \mathrm{Str}(\lambda_1,\lambda_2,\ldots,\lambda_n), \tag{B.3}$$

$$\mathrm{P}(g\lambda_1g^{-1},g\lambda_2g^{-1},\ldots,g\lambda_ng^{-1}) = g\mathrm{P}(\lambda_1,\lambda_2,\ldots,\lambda_n)g^{-1}. \tag{B.4}$$

If g is very close to the identity $g=1+\theta$, where θ takes values in the Lie algebra SU(N), then we have

$$\sum_{i=1}^{n}\mathrm{Str}(\lambda_1,\ldots,[\theta,\lambda_i],\ldots,\lambda_n) = 0, \tag{B.5}$$

$$\sum_{i=1}^{n}\mathrm{P}(\lambda_1,\ldots,[\theta,\lambda_i],\ldots,\lambda_n) = [\theta,\mathrm{P}(\lambda_1,\ldots,\lambda_n)]. \tag{B.6}$$

Now we suppose $\Lambda_i = a_i \otimes \lambda_i$ are matrix forms (a_i being forms of degree d_i). Then

$$\mathrm{Str}(\Lambda_1,\ldots,\Lambda_n) = a_1 \ldots a_n \otimes \mathrm{Str}(\lambda_1,\ldots,\lambda_n), \tag{B.7}$$

$$\mathrm{P}(\Lambda_1,\ldots,\Lambda_n) = a_1 \ldots a_n \otimes \mathrm{P}(\lambda_1,\ldots,\lambda_n). \tag{B.8}$$

If we want to have an expansion of $\mathrm{Str}(\Lambda_1,\ldots,\Lambda_n)$ or $\mathrm{P}(\Lambda_1,\ldots,\Lambda_n)$ like eq. (B.1) or (B.2), we have to introduce an extra sign in each term arising from the commutative properties of Λ_i as forms. In trying to generalize eqs. (B.5) and (B.6) to matrix forms, it is better to introduce the notation $[A, B]$ for two matrix forms A and B as follows:

$$[A, B] = AB - (-1)^{d_A d_B} BA \equiv a_1 a_2 \otimes [\lambda_1, \lambda_2], \tag{B.9}$$

where $A = a_1 \otimes \lambda_1$, $B = a_2 \otimes \lambda_2$ and d_A, d_B are degrees of the form A (or a_1) and B (or a_2) respectively. When we discuss the Lie algebraic properties of $[A, B]$, they are very similar to $[\lambda_1, \lambda_2]$. Now let us suppose Θ is a d-form, then from eqs. (B.5) and (B.6) we have

$$\sum_{i=1}^{n} (-1)^{(d_1 + \cdots + d_{i-1})d_\Theta} \mathrm{Str}(\Lambda_1,\ldots,[\Theta, \Lambda_i],\ldots,\Lambda_n) = 0, \tag{B.10}$$

$$\sum_{i=1}^{n} (-1)^{(d_1 + \cdots + d_{i-1})d_\Theta} \mathrm{P}(\Lambda_1,\ldots,[\Theta, \Lambda_i],\ldots,\Lambda_n) = [\Theta, \mathrm{P}(\Lambda_1,\ldots,\Lambda_n)]. \tag{B.11}$$

The extra sign in each term accounts for the exchange of Θ with $\Lambda_1 \ldots \Lambda_{i-1}$. In particular, if Θ is the potential 1-form A and the covariant derivative of the form Λ_i is defined as

$$D\Lambda_i = \mathrm{d}\Lambda_i + [A, \Lambda_i] \equiv \mathrm{d}\Lambda_i + A\Lambda_i - (-1)^{d_i} \Lambda_i A, \tag{B.12}$$

then we have

$$\mathrm{d}\,\mathrm{Str}(\Lambda_1,\ldots,\Lambda_n) = \sum_{i=1}^{n} (-1)^{d_1 + \cdots + d_{i-1}} \mathrm{Str}(\Lambda_1,\ldots,D\Lambda_i,\ldots,\Lambda_n), \tag{B.13}$$

$$D\mathrm{P}(\Lambda_1,\ldots,\Lambda_n) = \sum_{i=1}^{n} (-1)^{d_1 + \cdots + d_{i-1}} \mathrm{P}(\Lambda_1,\ldots,D\Lambda_i,\ldots,\Lambda_n). \tag{B.14}$$

Having been equipped with these formulae we turn to consider the nth Chern character,

$$\Omega_{2n}(A) = \mathrm{tr}\, F^n = \mathrm{Str}\, F^n, \tag{B.15}$$

where $F = \mathrm{d}A + A^2$ is the field strength 2-form. In order to find a $(2n-1)$-form $\omega_{2n-1}(A)$ such that $\Omega_{2n} = \mathrm{d}\omega_{2n-1}$, we use the following trick: introduce the following one-parameter family of potentials and strengths, $(0 \leqslant t \leqslant 1)$

$$A_t = tA\,, \qquad F_t = \mathrm{d}A_t + A_t^2 = t\,\mathrm{d}A + t^2A^2\,, \tag{B.16}$$

and consider, as in the text,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Omega_{2n}(A_t) &= n\,\mathrm{Str}\left\{\frac{\mathrm{d}}{\mathrm{d}t}F_t,\, F_t^{n-1}\right\}\\ &= n\,\mathrm{Str}\left\{\mathrm{d}A + 2tA^2,\, F_t^{n-1}\right\}\\ &= n\,\mathrm{Str}\left\{D_tA,\, F_t^{n-1}\right\},\end{aligned} \tag{B.17}$$

where $D_tA = \mathrm{d}A + [A_t, A]$ is the covariant derivative with respect to A_t. Since the Bianchi identity gives $D_tF_t = 0$, we obtain from eq. (B.17)

$$\frac{\mathrm{d}}{\mathrm{d}t}\Omega_{2n}(A_t) = n\,\mathrm{d}\,\mathrm{Str}\left\{A,\, F_t^{n-1}\right\}, \tag{B.18}$$

upon using eq. (B.13). Integrating from $t=0$ to $t=1$ we get

$$\begin{aligned}\Omega_{2n}(A) &= \mathrm{d}\omega^0_{2n-1}(A)\,,\\ \omega^0_{2n-1}(A) &= n\int_0^1 \mathrm{d}t\,\mathrm{Str}\left(A, (t\,\mathrm{d}A + t^2A^2)^{n-1}\right).\end{aligned} \tag{B.19}$$

Under the gauge transformation (with v 0-form valued in the Lie algebra)

$$\delta_v A = -Dv \equiv -\mathrm{d}v - [A, v]\,, \qquad \delta_v F = -[F, v]\,, \tag{B.20}$$

the nth Chern character is gauge invariant by eq. (B.10):

$$\delta_v\Omega_{2n}(A) = n\,\mathrm{Str}\left([v, F], F^{n-1}\right) = 0\,. \tag{B.21}$$

Therefore from eq. (B.19) it follows that

$$\delta_v\,\mathrm{d}\omega^0_{2n-1}(A) = \mathrm{d}\delta_v\,\omega^0_{2n-1}(A) = 0\,. \tag{B.22}$$

Poincaré's lemma leads to the conclusion that there exists locally a $(2n-2)$-form $\omega^1_{2n-2}(v, A)$ such that

$$\delta_v\omega^0_{2n-1}(A) = \mathrm{d}\omega^1_{2n-2}(v, A)\,. \tag{B.23}$$

To find the expression for $\omega^1_{2n-2}(v, A)$ we use

$$\delta_v \mathrm{d}A = -[\mathrm{d}A, v] + [A, \mathrm{d}v], \tag{B.24}$$

$$\delta_v F_t = -[F_t, v] - t(t-1)[A, \mathrm{d}v]. \tag{B.25}$$

Then from eq. (2.19),

$$\begin{aligned}\delta_v \omega^0_{2n-1}(v, A) &= n\int_0^1 \mathrm{d}t\left\{\mathrm{Str}\left(\delta_v A, F_t^{n-1}\right) + (n-1)\mathrm{Str}\left(A, \delta_v F_t, F_t^{n-2}\right)\right\}\\ &= -n\int_0^1 \mathrm{d}t\left\{\mathrm{Str}\left(\mathrm{d}v, F_t^{n-1}\right) + (n-1)t(t-1)\mathrm{Str}\left(A, [A, \mathrm{d}v], F_t^{n-2}\right)\right\}\\ &\quad -n\int_0^1 \mathrm{d}t\left\{\mathrm{Str}\left([A, v], F_t^{n-1}\right) + (n-1)\mathrm{Str}\left(A, [F_t, v], F_t^{n-2}\right)\right\}.\end{aligned} \tag{B.26}$$

Upon using eq. (B.10) we see that the second integral, namely the term proportional to v, vanishes. Since according to eq. (B.23) $\delta_v \omega^0_{2n-1}$ is locally exact we argue that in the first integral in eq. (B.26) we can immediately take the d operation outside the integral and obtain

$$\omega^1_{2n-2}(v, A) = -n\int_0^1 \mathrm{d}t\left\{\mathrm{Str}\left(v, F_t^{n-1}\right) + (n-1)t(t-1)\mathrm{Str}\left(A, [A, v], F_t^{n-2}\right)\right\} \tag{B.27a}$$

$$= -n\int_0^1 \mathrm{d}t\,\mathrm{tr}\left(v\left\{F_t^{n-1} + (n-1)t(t-1)\left[A, \mathrm{P}\left(A, F_t^{n-2}\right)\right]\right\}\right). \tag{B.27b}$$

We can also conclude that the form in the curved bracket in eq. (B.27b) is closed (as the action of d on it gives zero). Exploiting the latter statement, we can give a more compact formula for $\omega^1_{2n-2}(v, A)$ as follows: in fact, we have

$$\begin{aligned}\int_0^1 \mathrm{d}t\, F_t^{n-1} &= \int_0^1 \mathrm{d}t \sum_{k=0}^{n-1}\binom{n-1}{k} t^{n+k-1}\mathrm{P}\left((\mathrm{d}A)^{n-k-1}, (A^2)^k\right)\\ &= \sum_{k=0}^{n-1}\binom{n-1}{k}\frac{1}{n+k}\mathrm{P}\left((\mathrm{d}A)^{n-k-1}, (A^2)^k\right),\end{aligned} \tag{B.28}$$

$$\begin{aligned}&(n-1)\int_0^1 \mathrm{d}t\, t(t-1)\left[A, \mathrm{P}\left(A, F_t^{n-2}\right)\right]\\ &\quad = \sum_{k=0}^{n-2}(n-1)\int_0^1 \mathrm{d}t\binom{n-2}{k} t^{n+k-1}(t-1)\left[A, \mathrm{P}\left(A, (\mathrm{d}A)^{n-k-2}, (A^2)^k\right)\right]\\ &\quad = -\sum_{k=0}^{n-2}\binom{n-2}{k}\frac{n-1}{(n+k)(n+k+1)}\left[A, \mathrm{P}\left(A, (\mathrm{d}A)^{n-k-2}, (A^2)^k\right)\right].\end{aligned} \tag{B.29a}$$

By the formula (B.11) and $[A, A^2]=0$, we have

$$\left[A,\mathrm{P}\left(A,(\mathrm{d}A)^{n-k-2},(A^2)^k\right)\right]=2\mathrm{P}\left((\mathrm{d}A)^{n-k-2},(A^2)^{k+1}\right)$$
$$+(n-k-2)\mathrm{P}\left(A,\mathrm{d}A^2,(\mathrm{d}A)^{n-k-3},(A^2)^k\right). \tag{B.29b}$$

Using

$$\binom{n-1}{k+1}=\frac{n-1}{k+1}\binom{n-2}{k}, \tag{B.30}$$

the sum of the $(k+1)$th term in eq. (B.28) and the kth term in eq. (B.29) gives us

$$\frac{n-1}{(n+k)(n+k-1)}\binom{n-2}{k}\left\{\left(\frac{n+k}{k+1}-2\right)\mathrm{P}\left((\mathrm{d}A)^{n-k-2},(A^2)^{k+1}\right)\right.$$
$$\left.-(n-k-2)\mathrm{P}\left(A,\mathrm{d}A^2,(A^2)^k,(\mathrm{d}A)^{n-k-3}\right)\right\}$$
$$=\frac{n-k-2}{(n+k)(n+k+1)}\binom{n-1}{k+1}\mathrm{dP}\left(A,(\mathrm{d}A)^{n-k-3},(A^2)^{k+1}\right). \tag{B.31}$$

Therefore, the sum of eqs. (B.28) and (B.29) can be written as

$$\sum_{k=0}^{n-2}\binom{n-1}{k}\frac{n-k-1}{(n+k)(n+k-1)}\,\mathrm{dP}\left(A,(\mathrm{d}A)^{n-k-2},(A^2)^k\right)$$
$$=(n-1)\sum_{k=0}^{n-2}\binom{n-2}{k}\frac{1}{(n+k)(n+k-1)}\,\mathrm{dP}\left(A,(\mathrm{d}A)^{n-k-2},(A^2)^k\right)$$
$$=(n-1)d\int_0^1\mathrm{d}t(1-t)\mathrm{P}\left(A,F_t^{n-2}\right), \tag{B.32}$$

or

$$\omega^1_{2n-2}(v,A)=-n(n-1)\int_0^1\mathrm{d}t(1-t)\mathrm{tr}\left\{v\,\mathrm{dP}\left(A,F_t^{n-2}\right)\right\}. \tag{B.33}$$

Appendix C

C.1. FUJIKAWA'S PATH INTEGRAL METHOD

We assume the reader is familiar with Fujikawa's derivation of chiral anomaly [18]. His analysis may be generalized immediately to $D=2n$ dimensional space. Fujikawa observed that under the transformation

$$\begin{aligned}\psi(x) &\to \psi'(x) = e^{i\theta(x)\gamma_{D+1}}\psi(x),\\ \psi^\dagger(x) &\to \psi'^\dagger(x) = \psi^\dagger(x)e^{i\theta(x)\gamma_{D+1}},\end{aligned} \tag{C.1}$$

while the Dirac lagrangian transforms (for infinitesimal θ) as

$$\mathcal{L}' = \mathcal{L} - \partial_\mu\theta\bar{\psi}\gamma^\mu\gamma_{D+1}\psi - 2mi\theta\bar{\psi}\gamma_{D+1}\psi. \tag{C.2}$$

There is also a jacobian factor J for the transformation of the path-integral measure

$$J = \exp\left\{-2i\int d^Dx\,\theta(x)\sum_n \phi_n^\dagger(x)\gamma_{D+1}\phi_n(x)\right\}. \tag{C.3}$$

Here the basic functions $\phi_n(x)$ satisfy

$$i\gamma^\mu(\partial_\mu + A_\mu)\phi_n = \lambda_n\phi_n, \tag{C.4}$$

$$\int d^Dx\,\phi_n^\dagger(x)\phi_m(x) = \delta_{nm}. \tag{C.5}$$

The analysis is in compactified euclidean space. Regularizing the large eigenvalues by the factor $\exp(-\lambda_n^2/M^2)$ and changing to plane wave basis we evaluate the sum in the exponent in eq. (C.3) as follows:

$$\begin{aligned}\tfrac{1}{2}A(x) &= \sum_n \phi_n^\dagger(x)\gamma_{D+1}\phi_n(x) = \lim_{M\to\infty}\sum_n \phi_n^\dagger(x)\gamma_{D+1}e^{-(\lambda_n/M)^2}\phi_n(x)\\ &= \lim_{M\to\infty}\mathrm{Tr}\int\frac{d^Dk}{(2\pi)^D}\gamma_{D+1}e^{ikx}e^{-(\not{D}/M)^2}e^{-ikx}\\ &= \lim_{M\to\infty}\mathrm{Tr}\int\frac{d^Dk}{(2\pi)^D}\gamma_{D+1}\exp\left\{\frac{1}{2M^2}\left(2(-ik_\mu+A_\mu)^2 + \tfrac{1}{2}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\right)\right\}\\ &= \lim_{M\to\infty}\mathrm{Tr}\gamma_{D+1}\left\{\tfrac{1}{2}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\right\}^{D/2}\left(\frac{1}{2M^2}\right)^{D/2}\frac{1}{(D/2)!}\\ &\quad\times\int\frac{d^Dk}{(2\pi)^D}e^{-k_\mu k^\mu/M^2}\\ &= \frac{i^n}{(4\pi)^n n!}\varepsilon^{\mu_1\cdots\mu_{2n}}\mathrm{tr}\,F_{\mu_1\mu_2}\cdots F_{\mu_{2n-1}\mu_{2n}}.\end{aligned} \tag{C.6}$$

Note that in euclidean space we have $\gamma_{D+1} = (-i)^n \gamma_1\gamma_2 \dots \gamma_{D-1}\gamma_D$ $(D = 2n)$. The anomalous Ward identities can be represented by

$$\left.\frac{\delta W[A_\mu, \theta]}{\delta\theta(x)}\right|_{\theta(x)=0} = 0, \tag{C.7}$$

where $W[A_\mu, \theta]$ is the chirally rotated W. Combining eqs. (C.1), (C.6) and (C.7) we obtain the anomalous divergence for J_μ^A.

C.2. FEYNMAN DIAGRAMMATIC METHOD

We evaluate the abelian anomaly in a U(1) gauge theory. In our approach (which is of course not new (our approach here is similar in spirit to that of [35])) we insist that we deal at all times with well-defined finite quantities. We avoid manipulating potentially ill-defined (either in the ultraviolet or in the infrared) Feynman integrals and talking of surface terms which appear upon subtracting one integral from another. Furthermore, the arithmetic turns out to be rather simple in this approach and we can evaluate the anomaly at once for $D = 2n$ dimensional space. (Here, as elsewhere in the paper, we use the convention of Bjorken-Drell. In particular $\varepsilon_{01\dots 2n} = +1$.)

Denote by $T_{\mu_1\mu_2\dots\mu_n\lambda}(k_1, k_2, \dots, k_n)$ the Green function corresponding to $\langle 0|TV_{\mu_1}(x_1)\dots V_{\mu_n}(x_n)A_\lambda(0)|0\rangle$ (where V_μ and A_μ denote the vector and axial current $\bar\psi\gamma_\mu\psi$ and $\bar\psi\gamma_\mu\gamma_{D+1}\psi$ respectively). Let the (incoming) momentum "carried" by $V_{\mu_i}(x_i)$ be denoted by k_i. The anomalous Ward identity reads

$$\left(\sum_{i=1}^{n} k_i\right)^\lambda T_{\mu_1\dots\mu_n\lambda} = 2imP_{\mu_1\dots\mu_n} + c_n|k_1\mu_1\dots k_n\mu_n|\,. \tag{C.8}$$

We find it useful to introduce the notation

$$|p_1 p_2 \dots p_n| = \varepsilon_{\mu_1\mu_2\dots\mu_n} p_1^{\mu_1}\dots p_n^{\mu_n}, \tag{C.9}$$

$$|k_1\mu_1\dots k_n\mu_n| = \varepsilon_{\nu_1\mu_1\nu_2\mu_2\dots\nu_n\mu_n} k_1^{\nu_1}\dots k_n^{\nu_n}, \tag{C.10}$$

and so forth. In our approach we restore the fermion mass m. Thus, $P_{\mu_1\dots\mu_n}$ is the Fourier transform $\langle 0|TV_{\mu_1}(x_1)\dots V_{\mu_n}(x_n)P|0\rangle$ where P is the pseudoscalar operator $\bar\psi\gamma_{D+1}\psi$. The term with coefficient c_n in eq. (C.8) indicates the presence of the abelian chiral anomaly. The form of this term can be determined by general considerations [36].

Our strategy involves expanding eq. (C.8) in powers of the external momenta k_j. Noting that the anomaly is $O(k^n)$ we need only expand eq. (C.8) to $O(k^n)$ in order to determine c_n.

We write $T_{\mu_1\ldots\mu_n\lambda}$ as a sum of Lorentz covariants. There are two possible types of covariants:

$$|\mu_1\ldots\mu_n\lambda k_{a_1}k_{a_2}\ldots k_{a_{n-1}}| = K_{\mu_1\ldots\mu_n\lambda}, \tag{C.11}$$

$$|\mu_1\ldots\mu_{j-1}\mu_{j+1}\ldots\mu_n\lambda k_1\ldots k_n|k_{a_{\mu_j}} = L_{\mu_1\ldots\mu_n\lambda}. \tag{C.12}$$

(There are many distinct invariants corresponding to each type, of course. In eq. (C.11) the $(n-1)$ distinct momenta k_{aj} are taken from the set of n external momenta $k_1,\ldots,k_n$.) We denote generically the Lorentz scalar functions associated with $K_{\mu_1\ldots\mu_n\lambda}$ as A and with $L_{\mu_1\ldots\mu_n\lambda}$ as B. (Thus, for $D=2$, $T_{\mu\lambda} = A\varepsilon_{\mu\lambda} + B\varepsilon_{\mu\nu}k^\nu k_\lambda$.)

By power counting, the functions B are perfectly convergent and finite by two powers of momentum. On the other hand, the functions A appear to be logarithmically divergent. However, conservation of vector current requires that

$$k^{\mu_j}T_{\mu_1\ldots\mu_j\ldots\mu_n\lambda} = 0. \tag{C.13}$$

This tells us that any of the A functions are determined in terms of the B functions. Thus, $T_{\mu_1\ldots\mu_n\lambda}$ is perfectly finite.

The nice feature of this approach is that we can now forget about $T_{\mu_1\ldots\mu_n\lambda}$. Consider expanding eq. (C.8) in powers of k. We can safely Taylor expand $T_{\mu_1\ldots\mu_n\lambda}$ and $P_{\mu_1\ldots\mu_n}$ in powers of k. Since the fermion mass $m \neq 0$ there is no potential infrared difficulty. The preceding analysis indicates that $T_{\mu_1\ldots\mu_n\lambda}$ has a Taylor expansion with the first term of $O(k^{n+1})$. Thus the left-hand side of eq. (C.8) is of $O(k^{n+2})$ and is irrelevant for determining the coefficient c_n.

This entire discussion is to show that to calculate the anomaly we need only expand the perfectly convergent and finite quantity $P_{\mu_1\ldots\mu_n}$ to $O(k^n)$, which is in fact the order of its leading term. The quantity P is represented by the Feynman integral

$$P_{\mu_1\ldots\mu_n} = \frac{1}{(2\pi)^{2n}}(-1)i^{2n+1}\int \mathrm{d}^{2n}l\,\frac{N_{\mu_1\ldots\mu_n}}{D}$$

$$+(n!-1) \qquad \text{other terms by permutations.} \tag{C.14}$$

If we define $p_j \equiv \Sigma_{i=1}^{j}k_i$, we have the denominator

$$D = (l^2-m^2)\prod_{i=1}^{n}\left[(l+p_i)^2-m^2\right], \tag{C.15}$$

and the numerator

$$N_{\mu_1\ldots\mu_n} = \mathrm{tr}(\not l+\not p_n+m)\gamma_{\mu_n}\ldots(\not l+\not p_1+m)\gamma_{\mu_1}(\not l+m)\gamma_{D+1}. \tag{C.16}$$

Here

$$(\not{l}+\not{p})=\gamma_\nu(l+p)^\nu . \tag{C.17}$$

In our convention,

$$\gamma_{D+1} = -i^{n+1}\gamma^0\gamma^1\dots\gamma^{2n-1} = \gamma_{D+1}^\dagger ,$$

$$\gamma_{D+1}^2 = +1 . \tag{C.18}$$

By a simple chirality argument N has to be proportional to m. Taking a factor of m out, performing the trace, we find that N collapses to

$$N_{\mu_1\dots\mu_n} = 2^n(-i)^{n+1}|p_n\mu_n\dots p_1\mu_1|m$$

$$= 2^n(-i)^{n+1}|k_1\mu_1\dots k_n\mu_n|m . \tag{C.19}$$

Thus N can be taken out of the integral and the resulting integral may be interpreted as a loop graph with a boson running around it. To the required order we can replace D by $(l^2-m^2)^{n+1}$ with

$$\int\frac{\mathrm{d}^{2n}l}{(l^2-m^2+i\varepsilon)^{n+1}} = \frac{i(-1)^{n+1}\pi^n}{n!m^2} . \tag{C.20}$$

We obtain

$$K_n \equiv \frac{(-1)^{n+1}c_n}{2^n n!} = \frac{i^n}{2^{2n-1}\pi^n n!} . \tag{C.21}$$

The coefficient K_n is defined so that the divergence equation may be written as

$$\partial^\mu J^A_\mu = 2im\bar{\psi}\gamma_{D+1}\psi + K_n\varepsilon_{\mu_1\dots\mu_{2n}}F^{\mu_1\mu_2}\dots F^{\mu_{2n-1}\mu_{2n}} . \tag{C.22}$$

We emphasize the remarkable arithmetical brevity of this calculation. In particular, we feel that it is rather less tedious than the calculation in refs. [22, 24].

This calculation is closely related to a calculation using Pauli-Villars regularization. Suppose we regularize the one-loop diagram representing $T_{\mu_1\dots\mu_n\lambda}$ by a Pauli-Villars field with mass M. Then the regularized $T_{\mu_1\dots\mu_n\lambda}$ satisfies the "normal" Ward identity (compare with eq. (C.8))

$$\left(\sum_{i=1}^{n}k_i\right)^\lambda T_{\mu_1\dots\mu_n\lambda} = 2imP_{\mu_1\dots\mu_n} - 2iMP_{\mu_1\dots\mu_n}(M) . \tag{C.23}$$

As $M \to \infty$, the second term on the right-hand side of eq. (C.23) reproduces the anomaly. The calculation is equivalent to that leading to eq. (C.22) since upon expanding, we have

$$MP_{\mu_1 \ldots \mu_n}(M) \sim M^2 \left\{ \frac{1}{M^2} |k_1 \mu_1 \ldots k_n \mu_n| + \frac{1}{M^4} \mathrm{O}(k^{n+2}) + \cdots \right\}. \qquad \text{(C.24)}$$

We personally prefer the Pauli-Villars approach to chiral anomaly since the method particularly emphasizes the physical origin of the anomaly.

Our calculation was performed in an abelian gauge theory. In a non-abelian gauge theory we simply replace in eq. (C.22) $F_{\mu\nu} \to -i\lambda_j F^j_{\mu\nu}$ and take the symmetrized trace Str.

For completeness we mention that other methods for calculating anomalies, such as Schwinger's split point technique [3] or Crewther's short distance operator product expansion approach [8], should all be generalizable to higher dimensional spaces.

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