to $(1,8)_R+(8,1)_L$ or

$$B_1 \to e^{i\gamma_5\alpha}B_1e^{-i\gamma_5\alpha}.$$

Alternatively, the baryon matrix B_2 could transform by $(\bar{3},3)_R+(3,\bar{3})_L$ or

$$B_2 \to e^{i\gamma_5\alpha}B_2e^{i\gamma_5\alpha}.$$

In the first case the trace of the baryon matrix is invariant and can be set equal to zero. In the second case it is not and we have nine bayons instead of eight. It is easy to verify that the matrix $B_2e^{-2i\gamma_5\xi}$ transforms exactly like B_1. Furthermore the matrices $e^{-i\gamma_5\xi}B_2e^{-i\gamma_5\xi}$ and $e^{-i\gamma_5\xi}B_1e^{i\gamma_5\xi}$ have the same nonlinear transformation law as the matrix B, which, in finite form, is

$$B \to e^{-iu'}Be^{iu'}.$$

These examples are in agreement with the general theorems of Sec. 5.

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Structure of Phenomenological Lagrangians. II*

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The general method for constructing invariant phenomenological Lagrangians is described. The fields are assumed to transform according to (nonlinear) realizations of an internal symmetry group, given in standard form. The construction proceeds through the introduction of covariant derivatives, which are standard forms for the field gradients. The case of gauge fields is also discussed.

1. INTRODUCTION

THE most convenient way of deriving the physical consequences of the assumptions of (broken) chiral $SU(2)\times SU(2)$ [or $SU(3)\times SU(3)$] is by the method of phenomenological Lagrangians. These Lagrangians consist of a part which is invariant under the field transformations which realize the group and of a symmetry-breaking part which is usually assumed to transform simply under the group. The transformation laws of the fields under the group are in general nonlinear, but they become linear when restricted to the parity conserving $SU(2)$ [or $SU(3)$] subgroup. In the preceding paper,[1] the general form of the field transformation law is given for the general case of a compact, connected, semisimple Lie group. In the present paper, we give the general method for the construction of the invariant part of the Lagrangian. The symmetry-breaking terms in the Lagrangian are usualy assumed to belong to a linear representation of the group. In this case, one can easily construct them as functions of the fields by using the results of Sec. 5 of the preceding paper.

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[1] S. Coleman, J. Wess, and Bruno Zumino, preceding paper, Phys. Rev. **177**, 2239 (1969). In this paper one can find references to other work, in particular to papers which describe in detail the Lagrangian method as applied to chiral groups.

2. COVARIENT DERIVATIVES AND INVARIANT LAGRANGIANS

Our starting point is the analysis of nonlinear realizations of a compact Lie group given by Coleman, Wess, and Zumino. We dispense here with all proofs and definitions and quote only their final result. Let G be a compact, connected, semisimple Lie group and H a continuous subgroup of G. Let V_i and A_l be a complete orthonormal set of generators of G such that V_i are the generators of H. Any element g of G may be decomposed uniquely as a product of the form

$$g=e^{\xi\cdot A}e^{u\cdot V}.$$

A nonlinear realization of G which becomes a linear representation when restricted to the subgroup H is given on coordinates (ξ,ψ) by

$$(\xi,\psi) \to (\xi',\psi')=g(\xi,\psi), \tag{1}$$

where

$$ge^{\xi\cdot A}=e^{\xi'\cdot A}e^{u'\cdot V} \tag{2}$$

and

$$\psi'=D(e^{u'\cdot V})\psi. \tag{3}$$

Here $D(h)$ is any linear representation of the subgroup H which, if it is reducible, we assume to be written in fully reduced form. The main result of Coleman, Wess, and Zumino is that, by a suitable redefinition of coordinates, any nonlinear realization of G which is linear on H can be brought into the above standard form. Equation (3) can also be written as

$$\psi' = e^{u'\cdot T}\psi\,, \tag{4}$$

where T_i are the matrices which represent the generators V_i in the representation $D(h)$. If g is an element of the subgroup H (which we call h), then the transformation given by Eqs. (1)–(3) is linear:

$$(\xi,\psi) \rightarrow (\xi',\psi') = (D^{(b)}(h)\xi, D(h)\psi)\,, \tag{5}$$

where $D^{(b)}(h)$ is the linear representation of H induced on ξ by

$$he^{\xi\cdot A}h^{-1} = e^{\xi'\cdot A}.$$

We also recall briefly how the standard coordinates (ξ,ψ) can be introduced in the manifold on which the group operates. One first finds a set of coordinates (ξ,Ψ) which, by the subgroup H, transform linearly as in Eq. (5), i.e.,

$$h:\quad (\xi,\Psi) \rightarrow (D^{(b)}(h)\xi, D(h)\Psi).$$

The standard coordinates of the point (ξ,Ψ), are then defined as (ξ,ψ), where ψ is defined by

$$e^{-\xi\cdot A}(\xi,\Psi) = (0,\psi)\,. \tag{6}$$

These standard coordinates can be shown to transform under G according to the standard form given in Eqs. (1)–(3). In particular, they satisfy Eq. (6):

$$e^{-\xi\cdot A}(\xi,\psi) = (0,\psi)\,.$$

A Lagrangian density is a function of the fields and their gradients. The transformation properties of the gradients $\partial_\mu\xi$ and $\partial_\mu\psi$ are, of course, determined by those of the fields; therefore, the group can be realized by transformations on the manifold $(\xi,\psi,\partial_\mu\xi,\partial_\mu\psi)$. These transformations are not in standard form but, since the gradients transform linearly by the subgroup H, they can be brought into standard form by a change of coordinates. Let us denote the new coordinates, which transform in the standard way, by $(\xi,\psi,D_\mu\xi,D_\mu\psi)$. According to Eq. (6), they are given by

$$e^{-\xi\cdot A}(\xi,\psi,\partial_\mu\xi,\partial_\mu\psi) = (0,\psi,D_\mu\xi,D_\mu\psi)\,. \tag{7}$$

The new coordinates $D_\mu\xi$ and $D_\mu\psi$ are a sort of "covariant derivatives." They have been constructed so that their transformation under G is analogous to Eq. (3), i.e.,

$$(D_\mu\xi)' = D^{(b)}(e^{u'\cdot V})(D_\mu\xi)\,, \tag{8}$$

$$(D_\mu\psi)' = D(e^{u'\cdot V})(D_\mu\psi)\,. \tag{9}$$

We seek now explicit expressions for the covariant derivatives.

We first need to find how the field gradients transform. Let us observe that, corresponding to an infinitesimal displacement dx_μ in space-time we have, from Eq. (2),

$$g(de^{\xi\cdot A}) = (de^{\xi'\cdot A})e^{u'\cdot V} + e^{\xi'\cdot A}d(e^{u'\cdot V}) \tag{10}$$

and, according to Eq. (4),

$$d\psi' = e^{u'\cdot T}d\psi + (de^{u'\cdot T})\psi\,. \tag{11}$$

These equations could be used to work out explicitly the transformation laws of the field gradients. According to Eq. (7), we now take for g the ξ-dependent transformation

$$g = e^{-\xi\cdot A}\,, \tag{12}$$

which has the effect that

$$\xi' = 0\,,\quad u' = 0\,,\quad \psi' = \psi\,.$$

In this case

$$de^{\xi'\cdot A} \equiv e^{(\xi'+d\xi')\cdot A} - e^{\xi'\cdot A} = e^{d\xi'\cdot A} - 1 = d\xi'\cdot A$$

and similarly

$$de^{u'\cdot V} = du'\cdot V\,.$$

Equation (10) now takes the form

$$e^{-\xi\cdot A}de^{\xi\cdot A} = du'\cdot V + d\xi'\cdot A\,,$$

which can be used to compute du' and $d\xi'$, and Eq. (11) becomes

$$d\psi' = d\psi + du'\cdot T\psi\,.$$

Therefore the covariant derivatives are

$$D_\mu\xi = p_\mu \tag{13}$$

and

$$D_\mu\psi = \partial_\mu\psi + v_\mu\cdot T\psi\,, \tag{14}$$

where p_μ and v_μ are defined from

$$e^{-\xi\cdot A}\partial_\mu e^{\xi\cdot A} = v_\mu\cdot V + p_\mu\cdot A\,. \tag{15}$$

Clearly $D_\mu\xi$ and $D_\mu\psi$ are not themselves gradients.

It is perhaps instructive to verify, using the explicit formulas (13)–(15), the transformation properties of the covariant derivatives, given in (8) and (9). If we eliminate g between Eq. (2) and the equation obtained from it by differentiation,

$$g\partial_\mu e^{\xi\cdot A} = (\partial_\mu e^{\xi'\cdot A})e^{u'\cdot V} + e^{\xi'\cdot A}(\partial_\mu e^{u'\cdot V})\,,$$

we obtain

$$e^{-\xi'\cdot A}\partial_\mu e^{\xi'\cdot A} = e^{u'\cdot V}e^{-\xi\cdot A}(\partial_\mu e^{\xi\cdot A})e^{-u'\cdot V} + e^{u'\cdot V}(\partial_\mu e^{-u'\cdot V})\,.$$

Therefore

$$p_\mu'\cdot A = e^{u'\cdot V}p_\mu\cdot A e^{-u'\cdot V} \tag{16}$$

and

$$v_\mu'\cdot V = e^{u'\cdot V}v_\mu\cdot V e^{-u'\cdot V} - (\partial_\mu e^{u'\cdot V})e^{-u'\cdot V}\,. \tag{17}$$

Equation (16) can also be written

$$p_\mu' = D^{(b)}(e^{u'\cdot V})p_\mu$$

and Eq. (17) implies

$$(D_\mu\psi)' = e^{u'\cdot T}D_\mu\psi\,.$$

We are now able to find the general form for an invariant Lagrangian.[2] First we observe, using the group element (12), that an invariant Lagrangian must satisfy

$$L(\xi,\psi,\partial_\mu\xi,\partial_\mu\psi)=L(0,\psi,D_\mu\xi,D_\mu\psi).$$

On the other hand, from the transformation laws for the field ψ and for the covariant derivatives it follows immediately that a function of ψ, $D_\mu\xi$, and $D_\mu\psi$ is invariant under G if and only if it is constructed to be superficially invariant under the subgroup H.

3. EXPLICIT FORMULAS AND EXAMPLES

Equations (13)–(15) for the covariant derivates can be made more explicit. To compute v_μ and p_μ one can use the formula

$$e^{-\xi\cdot A}\partial_\mu e^{\xi\cdot A}=[(1-e^{-\Delta_{\xi\cdot A}})/\Delta_{\xi\cdot A}]\partial_\mu\xi\cdot A\,, \tag{18}$$

where we define

$$\Delta_{\xi\cdot A}X=[\xi\cdot A,X]$$

and let functions of the operator $\Delta_{\xi\cdot A}$ be defined by their power-series expansion. One sees immediately that

$$D_\mu\xi=\partial_\mu\xi+\cdots,$$

where the dots denote nonlinear terms.

For some groups (for instance, for the chiral groups) the commutator of two generators of the type A_l is equal to one of the generators V_i. Whenever this happens, in the right-hand side of Eq. (18), one can separate the odd from the even multiple commutators. One obtains in this way

$$v_\mu\cdot V=[(1-\cosh\Delta_{\xi\cdot A})/\Delta_{\xi\cdot A}]\partial_\mu\xi\cdot A$$

and

$$p_\mu\cdot A=(\sinh\Delta_{\xi\cdot A}/\Delta_{\xi\cdot A})\partial_\mu\xi\cdot A\,.$$

If we consider, in particular, the group $G=SU(n)\times SU(n)$ and identify A_l with the n-axial generators and V_i with the n-vector generators, the above equations may be further transformed. Let f_{ijk} be the totally antisymmetric structure constants for $SU(n)$ in a canonical basis where the Cartan inner product is given by $g_{ij}=-c\delta_{ij}$, with $c>0$. The matrices $(t_i)_{jk}=-f_{ijk}$ satisfy the commutation relations of the group algebra (adjoint representation)

$$[t_i,t_j]=f_{ijk}t_k\,.$$

Treating ξ, $\partial_\mu\xi$, v_μ, and p_μ as n-component vectors, we can write

$$D_\mu\xi=p_\mu=[\sinh(\xi\cdot t)/\xi\cdot t]\partial_\mu\xi\,,$$
$$v_\mu=[(1-\cosh(\xi\cdot t))/\xi\cdot t]\partial_\mu\xi\,,$$

and

$$D_\mu\psi=\partial_\mu\psi+T\cdot\left(\frac{1-\cosh(\xi\cdot t)}{\xi\cdot t}\partial_\mu\xi\right)\psi\,.$$

[2] We consider here Lagrangians containing only the fields and their first derivatives, but the generalization to Lagrangians with higher derivatives is straightforward.

In the case of $SU(2)\times SU(2)$, these formulas give the transparent series

$$D_\mu\xi=\partial_\mu\xi+\frac{1}{3!}(\partial_\mu\xi\times\xi)\times\xi+\frac{1}{5!}\cdots,$$

$$D_\mu N=\partial_\mu N-\tfrac{1}{2}i\tau\cdot\left(\frac{1}{2!}\partial_\mu\xi\times\xi+\frac{1}{4!}[(\partial_\mu\xi\times\xi)\times\xi]\times\xi+\cdots\right)N\,,$$

where N is the two-component nucleon field ($T=-\frac{1}{2}i\tau$).

The above methods can be used to construct a Lagrangian, invariant under $SU(3)\times SU(3)$, describing the interaction between the pseudoscalar octet and the baryon octet. We use the familiar 3×3 matrix notation and denote with B the traceless baryon matrix. The matrix

$$\xi=\tfrac{1}{2}\sum_{i=1}^{8}\xi_i\lambda_i$$

is proportional to the pseudoscalar matrix; similarly, we write

$$p_\mu=\tfrac{1}{2}\sum_{i=1}^{8}p_{\mu i}\lambda_i$$

and

$$v_\mu=\tfrac{1}{2}\sum_{i=1}^{8}v_{\mu i}\lambda_i\,.$$

In this notation, Eqs. (13)–(15) take the form

$$D_\mu\xi=p_\mu=\partial_\mu\xi+\cdots,$$
$$D_\mu B=\partial_\mu B-i[v_\mu,B]\,,$$

and

$$e^{i\gamma_5\xi}\partial_\mu e^{-i\gamma_5\xi}=-iv_\mu-i\gamma_5 p_\mu\,,$$

where now

$$v_\mu=\tfrac{1}{2}i[\xi,\partial_\mu\xi]+\cdots.$$

A simple invariant Lagrangian is

$$L=\mathrm{Tr}\{-a^{-2}p_\mu{}^2+i\bar{B}(\gamma_\mu\partial_\mu+M)B+\bar{B}\gamma_\mu[v_\mu,B]+\bar{B}\gamma_\mu\gamma_5(b_1p_\mu B+b_2Bp_\mu)\}\,.$$

The first term indicates that the normalized pseudoscalar matrix is

$$\pi=(1/a)\xi\,.$$

If we express L in terms of the matrix π, and neglect higher nonlinearities, we find a pseudovector meson-baryon interaction with independent F and D coupling

$$a\,\mathrm{Tr}\{\bar{B}\gamma_\mu\gamma_5(b_1\partial_\mu\pi B+b_2B\partial_\mu\pi)\}$$

as well as a "current-current" coupling

$$\tfrac{1}{2}ia^2\,\mathrm{Tr}\{B\gamma_\mu[[\pi,\partial_\mu\pi],B]\}\,.$$

In a meson-baryon scattering calculation one must include the second-order effect of the trilinear pseudovector interaction as well as the first-order effect of the quadrilinear current-current interaction. Only this total contribution has an invariant dynamical meaning (independent of the particular choice of fields).

4. GAUGE FIELDS

The construction of a Lagrangian invariant under coordinate-dependent group transformations requires the introduction of a set of gauge fields $\rho_{\mu i}$ and $a_{\mu l}$, associated respectively with the generators V_i and A_l. Let their transformation law be given by

$$\rho_\mu'\cdot V+a_\mu'\cdot A=g(\rho_\mu\cdot V+a_\mu\cdot A)g^{-1}-f^{-1}(\partial_\mu g)g^{-1},$$

where f is a constant which, as it turns out, gives the strength of the universal coupling of the gauge fields to all other fields. Instead of defining $v_{\mu i}$ and $p_{\mu l}$ by Eq. (15), we now define them by

$$e^{-\xi\cdot A}[\partial_\mu+f(\rho_\mu\cdot V+a_\mu\cdot A)]e^{\xi\cdot A}=v_\mu\cdot V+p_\mu\cdot A\,. \quad (19)$$

This equation can be used to compute v_μ and p_μ as functions of ξ, $\partial_\mu\xi$, ρ_μ, and a_μ. It is easy to verify that the transformation formulas for p_μ and v_μ given by Eqs. (16) and (17) are now valid also if the group element g is a function of the space-time variable x_μ, since the additional terms which arise from the differentiation of g are compensated by corresponding terms in the transformation law of the gauge fields. The "covariant derivatives" can therefore be defined, as before, by

$$D_\mu\xi=p_\mu\,,\quad D_\mu\psi=\partial_\mu\psi+v_\mu\cdot T\psi\,,$$

with the new meaning of p_μ and v_μ. Clearly

$$D_\mu\xi=\partial_\mu\xi+fa_\mu+\cdots,$$

where the dots denote nonlinear terms.

The most general Lagrangian invariant under coordinate-dependent group transformations can be obtained by adding to the generalized Yang-Mills Lagrangian for the fields ρ_μ and a_μ any local function of ψ, ξ, v_μ and their derivatives which is superficially invariant under the coordinate-dependent subgroup H. If we further add to this Lagrangian a mass term for the gauge fields

$$-\tfrac{1}{2}m^2[\sum_i(\rho_{\mu i})^2+\sum_l(a_{\mu l})^2],$$

the invariance is restricted to coordinate-independent group transformations. Observe that, without this mass term, the field ξ_l, which appears in Eq. (19) as a gauge parameter, could be completely eliminated from the Lagrangian. With the mass term, the gauge fields satisfy the conservation equations

$$\partial_\mu\rho_{\mu i}=0\,,\quad \partial_\mu a_{\mu l}=0$$

(or corresponding partial conservation equations if one also adds to the Lagrangian terms which break the coordinate-independent group invariance). Observe also that, while the fields $\rho_{\mu i}$ have (bare) mass m, the mass of the fields $a_{\mu l}$ differs from m. This follows from the fact that the invariant kinetic term for the field ξ has the form

$$-\tfrac{1}{2}\eta^2\sum_l(D_\mu\xi_l)^2=-\tfrac{1}{2}\eta^2\sum_l(\partial_\mu\xi_l+fa_{\mu l}+\cdots)^2,$$

where η is a normalization parameter. This introduces in the Lagrangian an additional term proportional to $\sum_l(a_{\mu l})^2$ as well as a term proportional to $\sum_l\partial_\mu\xi_l a_{\mu l}$. When this bilinear coupling is transformed away[3] by introducing the field

$$\hat{a}_{\mu l}=a_{\mu l}+[\eta^2f/(\eta^2f^2+m^2)]\partial_\mu\xi_l\,,$$

the associated mass is seen to be given by

$$\hat{m}^2=m^2+\eta^2f^2\,.$$

For completeness we recall here the form of the generalized Yang-Mills Lagrangian.[4] Let us denote the entire set of gauge fields $\rho_{\mu i}$ and $a_{\mu l}$ by $\phi_{\mu a}$ and the entire set of generators V_i and A_l by Z_a. Let c_{abc} be the totally antisymmetric structure constants of the group in a canonical basis where the Cartan metric tensor of the group is $g_{ab}=-c\delta_{ab}$, $c>0$. The structure equations of the group are then

$$[Z_a,Z_b]=c_{abc}Z_c$$

(sum over repeated indices). The generalized Yang-Mills Lagrangian is given simply by

$$L_{\rm YM}=-\tfrac{1}{4}(\phi_{\mu\nu a})^2\,,$$

where

$$\phi_{\mu\nu a}=\partial_\mu\phi_{\nu a}-\partial_\nu\phi_{\mu a}+fc_{abc}\phi_{\mu b}\phi_{\nu c}\,.$$

[3] For the case of chiral groups see J. Schwinger, Phys. Rev. Letters 24B, 473 (1967); J. Wess and Bruno Zumino, Phys. Rev. 163, 1727 (1967).

[4] C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954); R. Shaw, dissertation, Cambridge University, 1954 (unpublished); R. Utiyama, Phys. Rev. 101, 1597 (1956). After these early papers, the subject of non-Abelian gauge groups has been extensively treated by many authors. We quote only a recent paper which is more directly relevant to the present approach: T. D. Lee, S. Weinberg, and Bruno Zumino, Phys. Rev. Letters, 18, 1029 (1967).