[10]There are other ways of obtaining a small v_-, e.g., introducing an S_3 singlet field φ_0 which does not couple to fermions, but for technical reasons we have chosen soft breaking. Further details are contained in G. Segrè and H. A. Weldon, University of Pennsylvania Report No. UPR-0113T (to be published).

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Path-Integral Measure for Gauge-Invariant Fermion Theories

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It is shown that the path-integral measure for gauge-invariant fermion theories is not invariant under the γ_5 transformation and the Jacobian gives rise to an extra phase factor corresponding to the Adler-Bell-Jackiw anomaly. The derivation of "anomalous" Ward-Takahashi identities by means of the variational derivative can thus be made consistent in the path-integral formalism.

The derivation of the "anomalous" chiral Ward-Takahashi (W-T) identities[1] in the path-integral formalism has not been transparent in the past, as the "anomaly" term, which is seen only after the one-loop renormalization, had to be added to the action by hand. We here show that the path-integral measure for gauge-invariant fermion theories is not invariant under the γ_5 transformation and it gives rise to an extra phase factor corresponding to the anomaly. The derivation of W-T identities can thus be made consistent in the path-integral formalism.

We start with the gauge-invariant Lagrangian

$$\mathcal{L}=\bar{\psi}(i\not{D}-m)\psi+(\tfrac{1}{2}g^2)\operatorname{Tr}F_{\mu\nu}F^{\mu\nu} \tag{1}$$

suitably continued to Euclidean space. The operator $\not{D}\equiv\gamma^\mu(\partial_\mu+A_\mu)$ after the Wick rotation $x^0\to -ix^4$ and $A_0\to iA_4$ becomes a Hermitian operator

$$\not{D}=i\gamma^0D_4+\gamma^kD_k\equiv\gamma^4D_4+\gamma^kD_k. \tag{2}$$

We consider the fermions in the n-dimensional representation of the gauge group SU(n):

$$iA_\mu\equiv gA_\mu{}^aT^a \tag{3}$$

with

$$[T^a,T^b]=if_{abc}T^c,\quad \operatorname{Tr}(T^aT^b)=(\tfrac{1}{2})\delta^{ab},$$

and

$$F_{\mu\nu}\equiv\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu].$$

The variation of (1) under the γ_5 transformation

$$\begin{aligned}\psi(x)&\to\exp[i\alpha(x)\gamma_5]\psi(x),\\ \bar{\psi}(x)&\to\bar{\psi}(x)\exp[i\alpha(x)\gamma_5]\end{aligned} \tag{4}$$

gives rise to

$$\mathcal{L}\to\mathcal{L}-\partial_\mu\alpha(x)\bar{\psi}\gamma^\mu\gamma_5\psi-2mi\alpha(x)\bar{\psi}\gamma_5\psi \tag{5}$$

for an infinitesimal parameter $\alpha(x)$.

To define the functional integral precisely, we first expand $\psi(x)$ and $\bar{\psi}(x)$ as

$$\begin{aligned}\psi(x)&=\textstyle\sum_n a_n\varphi_n(x),\\ \bar{\psi}(x)&=\textstyle\sum_n\varphi_n(x)^\dagger\bar{b}_n\end{aligned} \tag{6}$$

in terms of a complete set of eigenfunctions of the Hermitian operator $\not{D}$, (2), in Euclidean space:

$$\begin{aligned}\not{D}\varphi_n(x)&=\lambda_n\varphi_n(x),\\ \int\varphi_n(x)^\dagger\varphi_m(x)d^4x&=\delta_{n,m}.\end{aligned} \tag{7}$$

The coefficients a_n and $\bar{b}_n$ in (6) are the elements of the Grassmann algebra. We note that $\psi(x)$ and $\bar{\psi}(x)$ are independent quantities in the classical level. (In the chiral form, ψ_L and $\bar{\psi}_L$ are expanded in φ_L and $\varphi_R{}^\dagger$, respectively.) The functional-integral measure is then defined by

$$\begin{aligned}d\mu&\equiv\textstyle\prod_x[\mathfrak{D}A_\mu(x)]\,\mathfrak{D}\bar{\psi}(x)\,\mathfrak{D}\psi(x)\\ &=\textstyle\prod_x[\mathfrak{D}A_\mu(x)]\prod_{n,m}d\bar{b}_m\,da_n\end{aligned} \tag{8}$$

by choosing the arbitrary normalization factor to be unity. We include the Faddeev-Popov factor into $[\mathfrak{D}A_\mu]$ whenever necessary.

Under the chiral transformation (4), the coefficients in (6) are transformed as

$$\psi(x) \to \psi'(x) \equiv \exp[i\alpha(x)\gamma_5]\psi(x) \equiv \sum_n a_n'\varphi_n(x) \tag{9}$$

with

$$a_n' = \sum_m \int \varphi_n(x)^\dagger \exp[i\alpha(x)\gamma_5]\varphi_m(x)\,dx\,a_m \equiv \sum_m C_{n,m}a_m. \tag{10}$$

The integral over the elements of the Grassmann algebra in (8) is identical to the left derivative,[2] $\prod da_n = \prod \partial/\partial a_n$. By noting that left derivatives anticommute with each other, we obtain from (10)

$$\prod da_n' = (\det C_{k,l})^{-1}\prod da_n. \tag{11}$$

The Jacobian factor becomes

$$\begin{aligned}(\det C_{k,l})^{-1} &= \det[\delta_{k,l} + i\int \alpha(x)\varphi_k(x)^\dagger\gamma_5\varphi_l(x)dx]^{-1}\\ &= \exp[-i\int\alpha(x)\sum_k \varphi_k(x)^\dagger\gamma_5\varphi(x)dx]\end{aligned} \tag{12}$$

for infinitesimal $\alpha(x)$. The summation in the exponent of (12) is an ill-defined quantity and we evaluate it by introducing a cutoff M (i.e., $|\lambda_k| \lesssim M$)

$$\begin{aligned}\sum_k \varphi_k(x)^\dagger\gamma_5\varphi_k(x) &= \lim_{M\to\infty}\sum_k \varphi_k{}^\dagger(x)\gamma_5\exp[-(\lambda_k/M)^2]\varphi_k(x)\\ &= \lim_{M\to\infty,\, y\to x} \operatorname{Tr}\gamma_5\exp[-(\not{D}/M)^2]\,\delta(x-y)\\ &= \lim_{M\to\infty,\, y\to x}\int\frac{d^4k}{(2\pi)^4}\operatorname{Tr}\gamma_5\exp(-\{D^\mu D_\mu + (\tfrac{1}{4})[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\}/M^2)e^{ik(x-y)}\\ &= \lim_{M\to\infty}\frac{1}{16}\operatorname{Tr}\gamma_5([\gamma^\mu,\gamma^\nu]F_{\mu\nu})^2\frac{1}{2!M^4}\int\frac{d^4k}{(2\pi)^4}\exp(-k^\mu k^\mu/M^2).\end{aligned} \tag{13}$$

The result is[3]

$$\sum_k \varphi_k(x)^\dagger\gamma_5\varphi_k(x) = \tfrac{1}{2}(-1/8\pi^2)\operatorname{Tr}{}^*F^{\mu\nu}F_{\mu\nu} \tag{14}$$

with ${}^*F^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ ($\epsilon^{1234} = \epsilon^{1230} = 1$), which is valid for the Abelian gauge theory also. As the Jacobian for $\mathfrak{D}\bar{\psi}$ gives rise to an identical factor, we finally obtain the transformation law of the functional measure (8) under the γ_5 transformation (4):

$$d\mu \to d\mu\exp[i\int\alpha(x)(1/8\pi^2)\operatorname{Tr}{}^*F^{\mu\nu}F_{\mu\nu}\,dx]. \tag{15}$$

By definition of the generating functional of Green's functions $Z(\eta,\bar{\eta},J)$ in Euclidean space

$$Z(\eta,\bar{\eta},J) \equiv N^{-1}\int d\mu\exp[\int dx(\mathfrak{L} + \bar{\psi}\eta + \bar{\eta}\psi - A_\mu^a J_a{}^\mu)], \tag{16}$$

the W-T identities are collectively represented by the variational derivative

$$[\delta/\delta\alpha(x)]Z(\eta,\bar{\eta},J)|_{\alpha=0} = 0. \tag{17}$$

The normalization factor N in (16) is fixed as $Z(0,0,0) = 1$. From (4), (5), and (15) we have, for example,

$$\begin{aligned}\partial_\mu\langle[j_5{}^\mu(x)\psi(y)\bar{\psi}(z)]_+\rangle &= 2mi\langle[j_5(x)\psi(y)\bar{\psi}(z)]_+\rangle\\ &\quad - i\delta(x-y)\langle[\gamma_5\psi(y)\bar{\psi}(z)]_+\rangle - i\delta(x-z)\langle[\psi(y)\bar{\psi}(z)\gamma_5]_+\rangle\\ &\quad - (i/8\pi^2)\langle[\operatorname{Tr}{}^*F^{\mu\nu}F_{\mu\nu}(x)\psi(y)\bar{\psi}(z)]_+\rangle,\end{aligned} \tag{18}$$

which is the ordinary W-T identity.[1] [The Minkowski version is obtained by a Wick rotation which removes the imaginary factor i from the last three terms.] The gauge-invariant currents are defined from (5) as $j_5{}^\mu(x) = \bar{\psi}\gamma^\mu\gamma_5\psi$ and $j_5(x) = \bar{\psi}\gamma_5\psi$.

The Adler-Bardeen theorem[4] in the present formulation is reduced to finding a regularization scheme which preserves the "naive" W-T identity (18). Our derivation of (18) suggests that similar identities are valid for multiple insertions of $j_5{}^\mu$.

We note that the evaluation of (13) is basically at the classical level and, in fact, (14) corresponds to a local version of the Atiyah-Singer index theorem[5,6]:

$$n_+ - n_- = \nu, \tag{19}$$

where n_+ and n_- stand for the number of zero-"energy" eigenvectors in (7) with positive and negative chirality,[3] respectively, and ν is the Pontryagin index $\nu \equiv (-1/16\pi^2)\mathrm{Tr}\int {}^*F^{\mu\nu}F_{\mu\nu}dx$. Equation (19) follows from (14) by noting that γ_5 and $\not{D}$ anticommute. The regularization employed in (13) preserves the index theorem for arbitrary M. In the path integral all the quantities appearing in the integrand are regarded as *classical fields*. The path integral therefore provides an ideal means to connect the semiclassical index theorem (14) with the quantum mechanical W-T identity (18).

The functional measure $\mathfrak{D}\bar{\psi}\mathfrak{D}\psi$ in (8) can formally be shown to be independent of the choice of basis vectors in (6). For a general basis $\xi_n(x)$, the exponential factor in the Jacobian (12) is naively given by

$$\sum_n \xi_n(x)^\dagger \gamma_5 \xi_n(x) = \sum_{k,l,n} \langle \varphi_l | \xi_n \rangle \langle \xi_n | \varphi_k \rangle \varphi_k(x)^\dagger \gamma_5 \varphi_l(x) = \sum_k \varphi_k(x)^\dagger \gamma_5 \varphi_k(x), \tag{20}$$

where $\varphi_k(x)$ is defined in (7). The left-hand side of (20) is, however, an ill-defined quantity for a general basis, and it vanishes for the "plane wave" basis with $A_\mu = 0$. The important implication of the index theorem[5,6] is then that the "energy" operator $\not{D}$, (2), when it comes to the chiral properties, does not allow a naive "unitary" transformation such as (20) among the basis vectors belonging to different (local) indices. The origin of this failure of the naive unitary transformation is seen in the evaluation of (13), which is essentially a transformation of basis vectors from (7) to plane waves to evaluate $\mathrm{Tr}\gamma_5 \exp[-(\not{D}/M)^2]$. It shows that the chirality asymmetry (19) appears in the sector of well-defined zero eigenvalues for the basis (7), whereas the asymmetry is transferred to the sector of infinite frequencies for plane waves.

This consideration indicates that the "standard" basis (7), which diagonalizes the operator $\not{D}$, correctly represents the topological structure of the gauge theory, and it should be used to derive the *exact relations* such as W-T identities.

The fermion sector of the functional measure is thus strongly constrained by the gauge field. If one uses the plane-wave basis in perturbation theory, the nonunitary transformation in (20) induces the "index defect," and the missing "anomaly" term in W-T identities after the infinite-frequency sector is correctly taken into account in the gauge-invariant loop expansion.[1]

For the case of N flavors of fermions in quantum chromodynamics, the functional measure is transformed as

$$d\mu \to d\mu \exp[iN(1/8\pi^2)\int \alpha(x)\,\mathrm{Tr}\,{}^*F^{\mu\nu}F_{\mu\nu}dx] \tag{21}$$

for chiral U(1), and the measure is invariant under any chiral SU(N) transformation, as can be seen by diagonalizing the traceless SU(N) generators.

In the presence of instantons,[7] (21) splits into a sum of terms for, for example, *global* α:

$$d\mu \to \sum_\nu d\mu_{(\nu)} \exp(-2iN\nu\alpha) \tag{22}$$

with ν the Pontryagin index, thus formally leading to the θ vacuum.[8] Equation (22) gives rise to the chirality selection rule specified by the index theorem.[3] Our derivation of (18) shows that W-T identifies hold without modification in the θ vacuum.

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Measurement of the Elastic Scattering of Neutrinos and Antineutrinos by Protons

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We have observed 217 (66) events of the process $\nu p \to \nu p$ ($\bar{\nu}p \to \bar{\nu}p$) with an estimated background of 82 (28). The neutral-to-charged–current ratios are $\sigma(\nu p \to \nu p)/\sigma(\nu n \to \mu^- p) = 0.11 \pm 0.02$ and $\sigma(\bar{\nu}p \to \bar{\nu}p)/\sigma(\bar{\nu}p \to \mu^+ n) = 0.19 \pm 0.05$ for $0.40 < Q^2 < 0.90$ (GeV/c)^2, where $-Q^2$ is the square of the momentum transfer to the nucleon. These yield $\sigma(\bar{\nu}p \to \bar{\nu}p)/\sigma(\nu p \to \nu p) = 0.53 \pm 0.17$. The neutral-current form factors at $Q^2 = 0$ are $G_E = 0.5^{+0.25}_{-0.5}$, $G_M = 1.0^{+0.35}_{-0.04}$, and $g_A = 0.5^{+0.2}_{-0.15}$.

We report a measurement of $\nu p \to \nu p$ and $\bar{\nu}p \to \bar{\nu}p$, in an experiment characterized by high statistics and the virtual absence of neutron background. Previous studies[1,2] of this reaction have been hampered by low statistics or large background from the reaction $np \to np$.

The experiment was performed at Brookhaven National Laboratory in a "wide-band," horn-focused ν beam. The detector[3] is a large ionization calorimeter containing 30 tons of liquid scintillator and subdivided into 216 cells viewed at each end by photomultipliers. Total charge and timing information from each photomultiplier yield the energy deposited and position of energy deposition along the length of the cell. Drift chambers provide additional information on the position and angle of particle trajectories in approximately 40% of the detector. Compared with our previous experiment,[1] neutron background has been largely eliminated by improved shielding, and the electronics has been improved to allow the detection of muon decay.

The selection criterion for neutral-current events is containment, i.e., no energy deposition occurs in the most upstream 40 cm or most downstream 20 cm of the calorimeter, nor within 46 cm of the edge of any module. From this sample, events are selected in which the pattern of cells fired (a minimum of three are required) and drift-chamber hits is consistent with having been produced by a single, unscattered track.

Single-prong events are identified as proton or pion by comparison of the observed and calculated patterns of energy deposition. For kinetic energies greater than 210 MeV, approximately 10% of the events are ambiguous between proton and pion; however, this ambiguity is further reduced by observation of the decay chain $\pi \to \mu\nu$, $\mu \to e\nu\bar{\nu}$. The final sample consists of 217 $\nu p \to \nu p$ and 66 $\bar{\nu}p \to \bar{\nu}p$ candidates.

The neutrino beam consists of 12 bunches of 35 ns duration (full width at half maximum) occurring every 222 ns. Figures 1(a) and 1(b) compare the event time (modulo 222 ns) for the selected proton events with that for $\nu n \to \mu^- p$ and $\bar{\nu}p \to \mu^+ n$ events. The neutral- and charged-current distributions are essentially identical; at most 1% (3.5%) of the in-time neutrino (antineutrino) events could have been induced by slow neutrons originating far upstream. Background from "prompt" neutrons, i.e., neutrons produced by ν interactions in the shielding close to the detector, is also inconsistent with the observed timing distribution. Figure 1(c) compares the mean event time, observed as a function of distance from the upstream shielding wall, with that expected for (1) ν-induced events and (2) events resulting from neutrons produced in the front and side shielding walls. The latter are characterized by a mean velocity $\langle\beta\rangle \approx 0.6$. The data indicate no significant neutron background. Furthermore, the observed distribution of events

