THE FEYNMAN INTEGRAL FOR SINGULAR LAGRANGIANS

L. D. Faddeev

A generalization is obtained for the continual Feynman integral, adapted for the quantization of a mechanical system describable by a singular Lagrangian. As an example, the quantization of the Yang — Mills field is considered.

The quantization of classical mechanical systems using the Feynman integral [1] has not, up to now, enjoyed the popularity which it deserves. In our opinion it is the most convenient of the known methods of quantization and can be applied to situations in which the generally accepted canonical quantization encounters difficulties. In this paper we try to show this by means of the example of a mechanical system, the Lagrangian of which, $L(q, \dot{q})$, is singular, in the sense that the usual equations

$$p_i = \frac{\partial L(q, \dot{q})}{\partial \dot{q}^i}$$

cannot be solved for the $\dot{q}$. Such systems were investigated in general form by Dirac [2, 3]. For them he developed a generalized Hamiltonian formulation and discussed its application to quantization. But in his approach, in particular, in the application to a gravitational field [4, 5], a number of difficulties occur. Thus, the problem of ordering the operator factors, which is essential for the verification of the coordination of the links [6], is nontrivial. Moreover, the phase space is nonlinear, and the spectrum of the generalized coordinates does not coincide with the ordinary spectrum of the canonical variables (compare [7]). Finally, the formulation, in the case of field theory, is not relativistically covariant, which complicates the control of infinities.

We intend to show that the difficulties enumerated above do not occur in quantization by the method of the Feynman integral and do not have the same central character. Of course, the infinities which are characteristic for relativistic field theory remain even in this approach, so that the problem of renormalization still remains.

The fundamental proposition of the paper is formulated in terms of a mechanical system with a finite number of degrees of freedom. This makes the discussions and conclusions shorter and more apparent. In addition their general nature becomes clearer. The transition to field theory is made in the usual way.

Field theory gives the most interesting examples of systems with singular Lagrangians. All fields having some kind of geometrical sense, e.g., an electromagnetic field, the Yang — Mills field, and the Einstein gravitational field,* are described by singular Lagrangians. The general expressions obtained in this paper are illustrated by means of the example of the Yang — Mills field. The application to the gravitational field will be described separately.

1. The Hamiltonian Form of the Feynman Integral

In his first paper in 1948 [1] Feynman introduced and studied the continual integral along trajectories in the configuration space of a mechanical system. A more convenient and more general form is given by

* In talking about the "geometrical sense" of these fields we have in mind that they are all cohesions in some vector stratification (cf., for example, the general definition in [8]).

Leningrad Branch of the V. A. Steklov Mathematics Institute, Academy of Sciences of the USSR.
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the expression obtained by Feynman in 1951 [9] for the continual integral in which integration is along trajectories in phase space. That paper [9] has received little attention and subsequently the form of the continual integral described in it has been criticized several times (cf., for example, [10] and [11]).

The Feynman integral gives an expression for the transition matrix element which is constructed in terms of the classical characteristics of the system. Let $q = (q^1, \ldots, q^n)$, $p = (p_1, \ldots, p_n)$ be the canonical coordinates and momenta; Let $H(q, p)$ be the Hamiltonian function. Then

$$\left\langle q'' \middle| \exp\left\{-\frac{i}{h} H(t''-t')\right\} \middle| q' \right\rangle = \int \exp\left\{\frac{i}{h}\int_{t'}^{t''}\left(\sum_{i=1}^{n} p_i\dot{q}^i - H(q,p)\right)dt\right\}\prod_{i,t}\frac{dp_i(t)\,dq^i(t)}{(2\pi h)^n}. \tag{1}$$

The exponential function on the right side contains the classical action (cf., for example, [12]), computed for the trajectory $q(t)$, $p(t)$, $t' \leq t \leq t''$, joining the surfaces $q = q'$ and $q = q''$ in phase space.

Unfortunately, up to the present time there has not been developed a general internal definition of integrals of the form (1). In the literature a limiting procedure is referred to in which the trajectories $q(t)$ and $p(t)$ are replaced by piecewise linear and piecewise constant functions respectively and the integral becomes finite-dimensional. This approach was described in detail in [11], where it is also possible to find a proof of the unitarity and a deduction of Schrödinger's equation.

For applications to field theory, another use of the continual integral is interesting. Firstly, the whole expression for the S-matrix is of prime importance here, when the interval of time in (1) is infinite, $-\infty < t < \infty$, and the trajectories $q(t)$ have a particular asymptotic form for $t \to \pm\infty$. Secondly, the Hamiltonian function $H(q, p)$ has the form

$$H = H_0 + gH_{\text{int}},$$

where H_0 is a quadratic form and H_{int} is a form of degree higher than second in the field variables; g is a small parameter. If we expand in powers of g we obtain integrals of the form

$$\int \exp\left\{\frac{i}{h}\left[\sum_i p_i\dot{q}^i - \sum_{i,k}\left(A^{ik}p_ip_k + B_i^k q^i p_k + C_{ik}q^iq^k\right)\right]dt\right\} Q[q(t), p(t)]\prod_{i,t}\frac{dp_i(t)\,dq^i(t)}{(2\pi h)^n},$$

where A^{ik}, $B_i^{\ k}$, and C_{ik} are constant matrices; Q is a polynomial in $q(t)$ and $p(t)$. These integrals are Gaussian and are taken in explicit form. The expressions which are thus obtained can naturally be described by Feynman diagrams. Thus the continual Feynman integral is a generating expression for Feynman diagrams.

To conclude this section we make some observations on the canonical invariance of (1). Under the canonical transformations

$$\delta q^i = \frac{\partial\Phi}{\partial p_i}, \quad \delta p_i = -\frac{\partial\Phi}{\partial q^i},$$

the measure $\Pi dp\, dq$ is not changed, the Hamiltonian function suffers a change of variables and

$$\delta\int_{t'}^{t''}\sum_i p_i dq^i = \left(\sum_i p_i\frac{\partial\Phi}{\partial p_i} - \Phi\right)\Bigg|_{t=t'}^{t=t''}.$$

Thus, the only change in the integrand in the new variables is in t at the ends of the interval. In the case of a linear canonical transformation, when Φ is a quadratic form, we can show that the change is equivalent to a unitary transformation. The author does not know how to deduce a similar property for the general canonical transformation, although it is quite possible that it exists. The difficulty is, of course, connected with the very definition of the integral (1).

In field theory the interesting canonical transformations are linearized asymptotically as $t \to \pm\infty$. In this sense we can assert that the representation of the elements of the S-matrix as continual integrals of the form (1) is canonically invariant.

2. The Phase Space of a System with Singular Lagrangian

We give here the necessary details on the Hamiltonian formulation of the equations of motion for a system with singular Lagrangian. Apart from the above-mentioned fundamental paper by Dirac [2], much of the detail can be found in the papers of Bergman and his colleagues (cf., for example, [13], [14], and the literature cited therein). We shall give the results in a form which is convenient for us and clarify certain aspects, which, we hope, will make it unnecessary to refer to the original articles.

In the case of interest to us the canonical variables $q = (q^1, \ldots, q^n)$, $p = (p_1, \ldots, p_n)$ do not vary throughout the phase space Γ, but satisfy the equations

$$\varphi^a(q, p) = 0, \quad a = 1, \ldots, m. \tag{2}$$

The functions $\varphi^a(q, p)$ are called links. It is natural to assume that they are independent and "irreducible" in the sense that Eqs. (2) define a surface M of dimension $2n - m$ in Γ, an arbitrary function f vanishing on M, being a linear combination of the links

$$f = \sum_a c_a(q, p)\varphi^a(q, p) \tag{3}$$

with coefficients c_a which are, in general, variable.

We shall consider the case, which at first sight is special, in which the links φ^a and the Hamiltonians H satisfy the additional conditions

$$\{\varphi^a, \varphi^b\} = \sum_c c_c{}^{ab}\varphi^c; \tag{4}$$

$$\{H, \varphi^a\} = \sum_b c_b{}^a\varphi^b. \tag{5}$$

Here $\{f, g\}$ is the usual Poisson bracket in Γ, $c_c{}^{ab}$ and $c_b{}^a$ are certain functions. In other words, we assume that the Poisson brackets of the links with themselves and of the links with the Hamiltonian vanish on M. In the theory of gauge invariant fields links satisfy the enumerated conditions. We note that for (4) to hold it is necessary that m does not exceed n.

The equations of motion can be obtained from the variational principle with generalized action of the form

$$S[q, p, \lambda] = \int \Big(\sum_i p_i\dot{q}^i - H - \sum_a \lambda_a\varphi^a \Big) dt, \tag{6}$$

so that, apart from the canonical variables, they contain as independent unknown functions the $\lambda_a(t)$, which act as Lagrangian multipliers. Thus, the equations of motion consist of canonical equations of the form

$$\dot{q}^i = \frac{\partial H}{\partial p_i} + \sum_a \lambda_a \frac{\partial \varphi^a}{\partial p_i}; \quad \dot{p}_i = -\frac{\partial H}{\partial q^i} - \sum_a \lambda_a \frac{\partial \varphi^a}{\partial q^i} \tag{7}$$

and conditions (2). Using (4) and (5) it is easy to see that conditions (2) hold for arbitrary functions $\lambda_a(t)$ if they hold for the initial conditions. In other words, trajectories beginning on M do not leave that surface, so that Eqs. (7) define a coordinate transformation on M, containing not only the Hamiltonian, but also m arbitrary functions $\lambda_a(t)$, $a = 1, \ldots, m$.

Naturally not all the functions on M can be assumed to be observable quantities, but only those for which arbitrariness in the choice of $\lambda_a(t)$ does not affect their variation in time. This requirement is satisfied by the functions f for which

$$\{f, \varphi^a\} = \sum_b d_b{}^a\varphi^b. \tag{8}$$

Indeed, in the equations of motion for such functions

$$\dot{f}=\{H, f\}+\sum_a \lambda_a\{\varphi^a, f\} \tag{9}$$

terms depending on λ_a vanish on M.

Equations (8) and (9) require clarification. The function f which occurs in them is an arbitrary continuation in Γ of a function defined on M. Since the links (3) are irreducible, any two such continuations differ by a linear combination of links and, by (4), condition (8) is independent of the choice of the continuation. In what follows we shall make use of such a method of describing functions and equations on M several times. We can say that we are thus considering classes of functions on Γ as functions on M. We take in one class all functions which differ by a linear combination of links.

The function f which is defined on M and satisfies (8) does not depend essentially on all its variables. Indeed, (8) can be considered as a set of m first-order differential equations on M with (4) serving as the integrability condition. Hence the function f is uniquely defined by its values in the submanifold of the initial conditions of the equations which is of dimension $(2n-m)-m=2(n-m)$.

As such a submanifold we can take the surface Γ^*, defined by the equations

$$\chi_a(q, p)=0, \quad a=1, \ldots, m, \tag{10}$$

which are called the additional conditions. The functions χ_a must satisfy the condition

$$\det \|\{\chi_a, \varphi^b\}\| \neq 0, \tag{11}$$

since only in this case can Γ^* be the initial surface for Eqs. (8). It is convenient to assume also that the χ_a commute with each other

$$\{\chi_a, \chi_b\}=0. \tag{12}$$

In this case we can simply introduce canonical variables on Γ^*.

Indeed, if (12) holds, then by a canonical transformation in Γ we can pass to new variables in which the χ_a take the simple form

$$\chi_a(q, p)=p_a,$$

where the p_a, $a=1, \ldots, m$ are part of the canonical momenta in the new system of variables. Let q^a denote the corresponding conjugate coordinates and let q^*, p^* be the remaining canonical variables. In the new variables condition (11) is as follows:

$$\det\left\|\frac{\partial \varphi^a}{\partial q^b}\right\| \neq 0,$$

so that Eqs. (2) can be solved for the q^a. As a result, the surface Γ^* is defined in Γ by the equations

$$p_a=0, \quad q^a=q^a(q^*, p^*),$$

and q^* and p^* act as independent variables on Γ^*. It appears that these are canonical variables; the Poisson bracket of any functions f and g satisfying (8) can be computed as follows:

$$\{f, g\}|_M=\sum\left(\frac{\partial f^*}{\partial p^*} \frac{\partial g^*}{\partial q^*}-\frac{\partial f^*}{\partial q^*} \frac{\partial g^*}{\partial p^*}\right), \tag{13}$$

where

$$f^*=f(q^a(q^*, p^*), q^*, 0, p^*), \tag{14}$$

and g^* is defined similarly, the left side of (13) containing arbitrary continuations of f and g from M to Γ. To verify (13) it is convenient to compute the Poisson bracket $\{f, g\}$ in the noncanonical coordinates $\eta=(\varphi^a, q^*, p_a, p^*)$. Then

$$\{f, g\} = \sum_{\alpha, \beta} \Omega^{\alpha\beta} \frac{\partial f}{\partial \eta^\alpha} \frac{\partial g}{\partial \eta^\beta}, \tag{15}$$

where

$$\Omega^{\alpha\beta} = \{\eta^\alpha, \eta^\beta\}.$$

By (4) and (8) the series of terms on the right side of (15) vanishes and as a result it coincides with the right side of (13), where

$$f^* = f(\eta)\,|_{p_a = \varphi^a = 0},$$

which is equivalent to (14). We emphasize again that the deduction that q^* and p^* are canonical is essentially linked with condition (12).

Thus, we have two methods of describing observable quantities on our system. In the first of these the observables are functions on M (more exactly, classes of functions on Γ), satisfying (8). The Poisson bracket is defined as the value on M of the Poisson bracket on Γ. In the second method the observables are arbitrary functions on Γ^*. To pass to the second method we should choose the additional conditions χ_a to solve Eqs. (2) and (10) and construct f^* according to (14). We can show that this procedure is independent of the choice of the additional conditions and the change in the χ_a when (11) and (12) hold is reduced to a canonical transformation in Γ^*.

In practice it is not simple to solve the links, so it is preferable to learn how to work with the first method of describing observables. On the other hand, in using the second method of description, we operate in ordinary phase space and can use the usual expressions of mechanics, in particular, Eq. (1) for the continual integral. Thus, to verify the validity of some expression in the first method of description of the observables, it is sufficient to verify that it passes over to the usual expression as we pass to the second method as described above. It is in this way that we begin the next section for the case of the continual integral.

To conclude this section we note that the case in which the links do not satisfy condition (4) can be considered in a similar manner. In this case the phase space Γ^* is of dimension $2(n + k - m)$, where $2k$ is the rank of the matrix $\{\varphi^a, \varphi^b\}$ in M. Conditions (8) in this case are modified. In place of the Poisson bracket we have to use the Dirac bracket [2, 14]. Such cases obviously do not occur in examples from field theory.

Discussion of the geometrical meaning of the constructions developed in this section is relegated to a special appendix.

3. The Modified Feynman Integral

In this section we show the form of the continual Feynman integral for the case of the mechanical system described in the previous section. Let such a system be defined by the canonical variables $q = (q^1, \ldots, q^n)$ and $p = (p_1, \ldots, p_n)$, the Hamilton function $H(q, p)$, and the links $\varphi^a(q, p)$ which satisfy conditions (4) and (5). We choose additional conditions $\chi_a(q, p)$ for which (11) and (12) hold. Then the expression for the matrix element of the S-matrix is

$$\langle \text{out}|S|\text{in}\rangle = \int \exp\left\{\frac{i}{h}\int_{-\infty}^{\infty}\left(\sum p_i \dot{q}^i - H\right)dt\right\} \prod_t d\mu(q(t), p(t)), \tag{16}$$

where the measure of the integration is defined as follows:

$$d\mu(q, p) = \prod_a \delta(\chi_a)\,\delta(\varphi^a)\det\|\{\chi_a, \varphi^b\}\| \prod_i \frac{dp_i dq^i}{(2\pi h)^{n-m}}. \tag{17}$$

Here the trajectories $q(t)$ coincide as $t \to \pm\infty$ with the solutions $q_{in}(t)$ and $q_{out}(t)$ of the equations describing the asymptotic motion and are uniquely defined by the states $|\text{in}\rangle$ and $\langle\text{out}|$ respectively, and the linearized conditions.

To prove this we reduce the expression to an integral of the form (1) in which integration is along trajectories in the phase space Γ^*. To do this we pass to the description in the second section using the coordinates q^a, q^*, p_a, and p^*. The integral (16), to within nonessential boundary terms, referred to at the end of the first section, has a form similar to (16), but with a different measure:

$$d\bar{\mu}(q,p)=\prod_a \delta(p_a)\delta(\varphi^a)\det\left\|\frac{\partial\varphi^b}{\partial q^a}\right\|\prod_i \frac{dp_i dq^i}{(2\pi h)^{n-m}},$$

which, obviously, can be rewritten as

$$\prod_a \delta(p_a)\delta(q^a-q^a(q^*,p^*))\,dp_a dq^a \prod \frac{dp^*\,dq^*}{(2\pi h)^{n-m}}.$$

Integration with respect to q^a and p_a is eliminated by the delta functions. As a result, the integral takes the form

$$\int \exp\left\{\frac{i}{h}\int_{-\infty}^{\infty}\left(\sum p^*\dot{q}^*-H^*\right)dt\right\}\prod_t \frac{dp^*\,dq^*}{(2\pi h)^{n-m}},$$

which coincides literally with (1), and we can consider that we have proved the validity of (16).

We note now that the integral in this equation can be rewritten

$$\int \exp\left\{\frac{i}{h}\int_{-\infty}^{\infty}\left(\sum_i p_i\dot{q}^i-H-\sum_a \lambda_a\varphi^a\right)dt\right\}\prod_t\left(\prod_a \delta(\chi_a)\det\|\{\chi_a,\varphi^b\}\|\prod_i\frac{dp_i\,dq^i}{(2\pi h)^n}\prod_b d\lambda_b\right), \tag{18}$$

since it is obvious that integration with respect to λ can be carried out explicitly and (18) becomes of the form of (16). In (18) we integrate the functional $\exp\{(i/h)S\}$, where S is the generalized action (6) with respect to some measure in the space of the trajectories $q(t)$, $p(t)$, $\lambda(t)$ lying on the surface $\chi_a(q,p)=0$. The explicit expressions for the measure in (18) are the fundamental result of the paper. In the next section we shall see that the general expression (18) can be particularized to the example of the Yang – Mills field.

To conclude this section we show that the integral (16) is independent of the choice of the additional conditions $\chi_a(q,p)$. Let $\delta\chi_a$ be an infinitely small change in these conditions. To within a linear combination of the links we can represent $\delta\chi_a$ as the result of an infinitesimal canonical transformation in Γ, the generator of which is also a linear combination of the links. Indeed,

$$\delta\chi_a=\{\Phi,\chi_a\}+\sum_b c_{ab}\varphi^b,\quad \Phi=\sum_a h_a\varphi^a,$$

where as h_a we can take the solution of the system of equations

$$\sum_b \{\chi_a,\varphi^b\}h_b=-\delta\chi_a,$$

which, from (11), has a unique solution. In the canonical transformation the links are replaced by linear combinations of themselves

$$\delta\varphi^a=\sum_b A_b{}^a\varphi^b,\quad \delta\varphi=A\varphi,$$

and, thus, the quantities in (16) are changed in the following manner:

$$\chi\to\chi+\delta\chi,\quad \varphi\to(I+A)\varphi,\quad H\to H,$$
$$\prod_a \delta(\varphi^a)\to(1+\operatorname{tr}A)^{-1}\prod_a\delta(\varphi^a),$$
$$\det\|\chi_a,\varphi^b\}\|\to(1+\operatorname{tr}A)\det\|\{\chi_a+\delta\chi_a,\varphi^b\}\|.$$

We use here an obvious abbreviated notation and omit linear combinations of the links which vanish on integration. As a result of the canonical transformation the integral takes the same form as (16), but with χ replaced by $\chi + \delta\chi$, which proves that it is independent of the choice of χ.

4. The Yang – Mills Field

The simplest nontrivial application of the continual integral introduced in the third section is the quantization of the Yang – Mills field [15]. We shall show how to obtain from (18) an expression for the modified Feynman integral for this field, as previously obtained by V. N. Popov and the author [16] from different considerations. As shown in [17], from this expression there follows a modified diagram technique in perturbation theory, the necessity of which was first remarked by Feynman [18] (for other conjectures on this subject see [19-21]).

The Yang – Mills vector field is constructed from any compact group G (cf. [22] and also [8]). Let t_α be generators of the group, normed by the condition $\mathrm{tr}\,(t_\alpha t_\beta) = -2\delta_{\alpha\beta}$; $f^{\alpha\beta\gamma}$ the corresponding structural constants which are antisymmetrical in all three indices. The Yang – Mills field is conveniently described by the vector† A_μ and the antisymmetric tensor $F_{\mu\nu}$, where these quantities are matrices in the Lie algebra of the group G. We can put

$$A_\mu = A_\mu{}^\alpha t_\alpha, \quad F_{\mu\nu} = F_{\mu\nu}{}^\alpha t_\alpha,$$

where $A_\mu{}^\alpha$ and $F_{\mu\nu}{}^\alpha$ form a vector in the combined representation of G. Both notations are convenient below.

The Lagrange function has the form

$$L = \frac{1}{4}\mathrm{tr}\left\{(\partial_\mu A_\nu - \partial_\nu A_\mu + \varepsilon[A_\mu, A_\nu])F_{\mu\nu} - \frac{1}{2}F_{\mu\nu}F_{\mu\nu}\right\}$$

and does not contain the time derivatives of the variables A_0 and F_{ik}. The latter occur in L quadratically and can be eliminated by putting

$$F_{ik} = \partial_i A_k - \partial_k A_i + \varepsilon[A_i, A_k]. \tag{19}$$

After the substitution (19), the action takes the form

$$S = -\frac{1}{2}\int \mathrm{tr}\left\{E_k\partial_0 A_k - \frac{1}{2}(E_kE_k + G_kG_k) - A_0 C\right\}dx, \tag{20}$$

where we have introduced the notation

$$E_i = F_{0k}, \quad G_i = \varepsilon_{ikl}F_{kl}$$

and C is given by the expression

$$C = \partial_k E_k + \varepsilon[A_k, E_k].$$

Comparing (20) and (6), we see that E_k and A_k act as canonical variables, A_0 is the Lagrangian multiplier and C is a link. If we introduce the fundamental Poisson bracket

$$\{E_k{}^\alpha(\mathbf{x}), A_l{}^\beta(\mathbf{y})\} = \delta_{kl}\delta_{\alpha\beta}\delta^{(3)}(\mathbf{x}-\mathbf{y}),$$

it is easy to verify that equations of the type of (4) and (5) hold:

$$\{C^\alpha(\mathbf{x}), C^\beta(\mathbf{y})\} = f^{\alpha\beta\gamma}\delta^{(3)}(\mathbf{x}-\mathbf{y})C^\gamma(\mathbf{x})$$

and

$$\left\{\int \mathrm{tr}(E_kE_k + G_kG_k)\,d^3x, \quad C(\mathbf{y})\right\} = 0.$$

†In this section we put h = 1, c = 1 and use the normal relativistic notation $\mu, \nu = 0, 1, 2, 3, 4$; $i, k = 1, 2, 3$; $a_\mu b_\mu = a_0 b_0 - a_k b_k$.

Further, the equation

$$C(x) = 0 \tag{21}$$

makes it possible to express the axial part of the vector E_k in terms of the vector A_k and the polar part of E_k, and thus reduce the dimensionality of the phase space (six functions of three variables) by one such function. Indeed, if we put

$$E_k = E_k{}^L + E_k{}^T, \quad \partial_k E_k{}^T = 0, \quad E_k{}^L = \partial_k \varphi,$$

we can rewrite (21) as

$$M\varphi = \Delta\varphi + \varepsilon[A_k, \varphi] = -\varepsilon[A_k, E_k{}^T], \tag{22}$$

which can be solved uniquely for φ if we assume natural conditions as $r \to \infty$.

Thus, all the techniques developed in the second and third sections can be applied to the system under consideration. For the additional condition we naturally take

$$X = \partial_k A_k = 0,$$

since the corresponding Poisson bracket

$$\{X^\alpha(\mathbf{x}),\, C^\beta(\mathbf{y})\} = -[\delta^{\alpha\beta}\Delta + \varepsilon f^{\alpha\gamma\beta} A_k{}^\gamma \partial_k]\,\delta^{(3)}(\mathbf{x}-\mathbf{y})$$

coincides with the kernel introduced in (22) of the operator M, which has a unique inverse.

With such a choice of the additional condition, the continual integral (18) takes the form

$$\int \exp\{iS\} \prod_t \det{}' M \prod_{x,\,k} \delta(\partial_i A_i)\, dA_k\, dE_k\, dA_0. \tag{23}$$

Here S is the action (20) and by det' M we denote the regularized determinant of the operator

$$\det{}' M = \det(M M_0{}^{-1}), \quad M_0\varphi = \Delta\varphi,$$

which only differs from det M by an uninteresting constant (and infinite) factor.

The continual integral (23) is a generating expression for Feynman diagrams in the Coulomb gauge. Its essential disadvantage is that it is not relativistically covariant. However, we can rewrite the integral in explicit covariant form. To do this we integrate firstly with respect to E_k, having taken the Gaussian integral. Then we obtain an integral of the form

$$\int \exp\{iS[A]\} \prod_t \det{}' M \prod_{x,\,\mu} \delta(\partial_i A_i)\, dA_\mu, \tag{24}$$

where S[A] is the covariant action of the Yang — Mills field, expressed in terms of the potentials A_μ

$$S[A] = \frac{1}{8} \int \operatorname{tr}(\partial_\mu A_\nu - \partial_\nu A_\mu + \varepsilon[A_\mu, A_\nu])^2\, dx. \tag{25}$$

Only the factor

$$\prod_t \det{}' M \prod_x \delta(\partial_i A_i), \tag{26}$$

defining the measure of the integration, is not covariant in (24). It turns out that the value of the integral (24) is not changed if in place of this factor we use the covariant expression

$$\det{}' N \prod_x \delta(\partial_\nu A_\nu), \tag{27}$$

where N is the four-dimensional generalization of the operator M

$$N\varphi = \Box\varphi + \varepsilon[A_\mu, \partial_\mu\varphi]$$

and det′ again denotes the regularized determinant. As a result (24) can be rewritten as

$$\int \exp\{iS[A]\}\det{}' N \prod_{x,\mu} \delta(\partial_\nu A_\nu)\, dA_\mu. \tag{28}$$

The diagram technique, occurring in the computation of this integral in perturbation theory, has been described in detail in [17]. Its characteristic feature is the phenomenon of additional diagrams generated by the expansion of $\ln\det' N$.

Other approaches to the problem of the quantization of the Yang – Mills field, developed by De Witt [19], Mandelstam [20], and Veltman [21], have led to similar results. We note that, as De Witt [19] showed, perturbation theory is based on the integral

$$\int \exp\left\{iS[A] + i\beta\int (\partial_\mu A_\mu)^2 dx\right\}\det{}' N \prod_{x,\mu} dA_\mu(x)$$

with arbitrary $\beta \neq 0$. In diagrams generated by this integral the arbitrary part of the propagator of the vector field is $(1/\beta) - 1$, while in computing (28) we can use only the Landau gauge.

To prove that it is possible to replace (26) with (27), it is pertinent to clarify its geometrical meaning. The action (25) is invariant with respect to the gauge transformation

$$A_\mu \to A_\mu^\Omega = \Omega^{-1}A_\mu\Omega + \frac{1}{\varepsilon}\Omega^{-1}\partial_\mu\Omega,$$

where $\Omega(x)$ is an arbitrary matrix function of G. We can say that $S[A]$ is a function on the manifold $\mathfrak{A}$ of classes of the fields A_μ, where fields which differ only by a gauge transformation form one class. The integral (24) can be considered as an integral with respect to this manifold, since the condition

$$\partial_k A_k = 0 \tag{29}$$

selects one representative in each class. The condition

$$\partial_\mu A_\mu = 0 \tag{30}$$

has a similar property, so that we can look on (29) and (30) as two different means of uniquely parameterizing the manifold $\mathfrak{A}$ of classes.

We note now that every measure on $\mathfrak{A}$ is obtained from a gauge invariant measure on the set of all fields A_μ. These measures have the form

$$m[A]\prod_{\mu,x} dA_\mu(x), \tag{31}$$

where $m[A]$ is a gauge invariant function. If the parameterization of $\mathfrak{A}$ is defined by the equation $f[A] = 0$, the corresponding measure on $\mathfrak{A}$ is given by the expression

$$m[A]\Delta_f[A]\delta(f[A])\prod_{\mu,x} dA_\mu(x), \tag{32}$$

where $\Delta_f[A]$ is obtained by averaging over the group

$$(\Delta_f[A])^{-1} = \int \delta(f[A^\Omega])\prod_x d\Omega(x) \tag{33}$$

and $d\Omega$ is an invariant measure on G. We can verify that (26) and (27) are obtained if we apply this recipe to an invariant measure of the form (31) with $m = 1$ for the cases $f = \partial_k A_k$ and $f = \partial_\mu A_\mu$, respectively. To do this, repeating the considerations in [17], we note that (32) contains values of $\Delta_f[A]$ only for fields A_μ satisfying the equation of $f[A] = 0$. For such fields every contribution of (33) gives a neighborhood of a unique element of G, so that the integral can be rewritten as

$$\int \prod_x \delta\left(\frac{\delta f[A]}{\delta A_\mu(x)}(\partial_\mu u + \varepsilon [A_\mu, u])\right) \prod_x du(x),$$

where we have put $\Omega = 1 + u$, and u is a matrix of the Lie algebra of G, and we have linearized the argument of the delta-function. If we apply this expression to the case of conditions (29) and (30), we arrive at (26) and (27), respectively, thus proving the equivalence of the integrals (24) and (28).

APPENDIX

This appendix is intended for readers who are disappointed by the superficial noninvariance of the description of the second section. We shall clarify here the geometrical meaning of the conditions and considerations formulated there. At the same time we shall sharpen the conditions under which these considerations are actually valid. We shall, naturally, use the invariant language of differential geometry (cf., for example, [8]), which is the most adequate for general problems in mechanics (cf., for example, [23, 24]).

The phase space of a mechanical system with n degrees of freedom is a smooth manifold Γ of 2n dimensions with a nondegenerate closed differential 2-form Ω

$$d\Omega = 0, \quad \Omega^n \neq 0.$$

Using this form we can introduce the structure of the Lie algebra into the algebra of functions $\mathfrak{A}$ on Γ, i.e., we can define the antisymmetric operation $\{f, g\}$, $f \in \mathfrak{A}$, $g \in \mathfrak{A}$, satisfying Jacobi's identity and called the Poisson bracket. We find the following variant of the definition convenient. From the definition of the function f, the equation

$$\Omega(X_f, Y) = df(Y) = Yf \tag{A.1}$$

uniquely defines the vector field X_f. The Poisson bracket is defined as

$$\{f, g\} = \Omega(X_f, X_g) = X_g f = -X_f g$$

and, obviously, is linear and antisymmetric. Jacobi's identity is a corollary of the closed nature of Ω.

Thus, in the linear space of functions on the phase space, we have, apart from the usual product fg, one additional binary operation $\{f, g\}$. The second operation is differentiation with respect to the first

$$\{f, gh\} = \{f, g\}h + \{f, h\}g. \tag{A.2}$$

The algebra $\mathfrak{A}$ of functions on Γ, with the operations described, is an algebra of observable (quantities) of the mechanical system under consideration.

In local coordinates $\xi = (\xi^1, \ldots, \xi^{2n})$ on Γ the form Ω is specified by the antisymmetric nonsingular matrix $\Omega_{\alpha\beta}$, satisfying the condition

$$\frac{\partial}{\partial \xi^\alpha} \Omega_{\beta\gamma} + \text{cyclic permutation} = 0.$$

The vector field X_f has the following components in a natural basis:

$$X_f^\alpha = \sum_\beta \Omega^{\alpha\beta} \frac{\partial f}{\partial \xi^\beta},$$

where $\Omega^{\alpha\beta}$ is the inverse of $\Omega_{\alpha\beta}$. The Poisson bracket is specified by the equation

$$\{f, g\} = \sum_{\alpha,\beta} \Omega^{\alpha\beta} \frac{\partial f}{\partial \xi^\alpha} \frac{\partial g}{\partial \xi^\beta}$$

[compare with (15)] and has the usual form

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} \right)$$

in the canonical variables $\xi = (q, p)$, $q = (q^1, \ldots, q^n)$, $p = (p_1, \ldots, p_n)$, in which Ω is given by

$$\Omega = 2 \sum_i dp_i \wedge dq^i.$$

The fundamental object introduced in the second section is the phase space Γ^*. Its description involves: 1) the phase space Γ with canonical coordinates (q, p) and canonical form Ω; 2) links $\varphi^a(q, p)$, $a = 1, \ldots, m$, i.e., functions on Γ, subject to the conditions described at the beginning of the second section. The concepts and notation introduced above help us to describe Γ^* in an invariant manner.

We denote by M the submanifold defined by the equation

$$\varphi^a(q, p) = 0, \quad a = 1, \ldots, m.$$

Let $\tilde{\Omega} = \Omega|_M$ be the value of Ω on M. The form Ω is closed, but singular. We can show that, by (4), its rank is $2(n - m)$. It is not less than this number, since the rank of a regular matrix cannot be lowered by more than 2m when it is restricted to a submanifold of codimensionality m. On the other hand, we can select m linearly independent zero vectors from this form. We note that vector fields tangential to M are restrictions to M of vector fields Y on Γ satisfying

$$d\varphi^a(Y) = 0, \quad a = 1, \ldots, m,$$

which, by (A.1), we can rewrite as

$$\Omega(X_{\varphi^a}, Y)|_M = 0, \quad a = 1, \ldots, m. \tag{A.3}$$

By (4), we have

$$\{\varphi^a, \varphi^b\}|_M = \Omega(X_{\varphi^a}, X_{\varphi^b})|_M = 0, \tag{A.4}$$

and, comparing this with (A.3), we see that the vectors X_{φ^a} are tangential to M and are zero vectors of the form $\tilde{\Omega}$, which proves our assertion.

Every singular closed 2-form Ω, defined on some manifold, generates its foliation. Layers are maximal integral manifolds of an involutive distribution (cf., for example, [25]), formed by the zero vectors of a form. The set of such zero vectors is denoted by P and we shall show that since $X \in P$ and $Y \in P$, we have $[X, Y] \in P$. We have

$$0 = 3d\Omega(X, Y, Z) = \Omega([X, Y], Z) + X\Omega(Y, Z) + \text{cyclic permutation},$$

and, by the condition on X and Y, the single term which does not vanish by definition is the first term on the right hand side. Thus, it also vanishes and this proves that P is involutive.

We note that in the case under consideration, we have described explicitly the basis for the zero space of the form Ω and the involutive nature of this distribution can be verified directly using (A.4).

We now assume that foliation is indeed a stratification and let Γ^* be its basis. The form Ω^* of the inverse image of Ω on Γ^* is closed and nondegenerate. Thus, Γ^* is a phase space which is the object we require. The condition, formulated in this paragraph, for the existence of a stratification imposes very strong restrictions on the links φ^α which we could not formulate in the main text.

Arbitrary smooth functions on Γ^*, forming the algebra of observables $\mathfrak{A}^*$ of our system, can be considered as functions on M, which are constant on the layers. This condition can be described as

$$X_{\varphi^a} f|_M = \{\varphi^a, f\}|_M = 0,$$

which coincides with (8) and at the same time clarifies that condition.

We now give an alternative algebraic definition of the algebra $\mathfrak{A}^*$. Let $\mathfrak{A}$ be the algebra of observables on Γ with operations $f + g$, fg, and $\{f, g\}$. We denote by Φ the set of linear combinations of the links φ^a, so that $\varphi \in \Phi$ implies that $\varphi = \Sigma c_a \varphi^a$, where the c_a are arbitrary functions on Γ. The set Φ is an ideal in $\mathfrak{A}$ with respect to ordinary multiplication. Moreover, Φ is a subalgebra of $\mathfrak{A}$ with respect to the operation $\{f, g\}$, since, by (4), $\varphi_1 \in \Phi$ and $\varphi_2 \in \Phi$ imply $[\varphi_1, \varphi_2] \in \Phi$. Consider now the set $\tilde{\mathfrak{A}}$ of functions f such that for all $\varphi \in \Phi$

$$\{f, \varphi\} \in \Phi$$

and which can be termed the normalizator of Φ in $\mathfrak{A}$ with respect to the Lie operation. The existence of such a nontrivial (distinct from Φ) set is a strong condition on the links φ^a, comparable with the above-mentioned condition on stratification.

The set $\tilde{\mathfrak{A}}$ is a subalgebra of the algebra $\mathfrak{A}$. Indeed, if $f \in \tilde{\mathfrak{A}}$ and $g \in \tilde{\mathfrak{A}}$, then, by (A.2), and the Jacobi identity

$$\{fg, \varphi\} = f\{g, \varphi\} + \{f, \varphi\}g \in \Phi,$$
$$\{\{f, g\}\varphi\} = -\{\{g, \varphi\}f\} + \{\{f, \varphi\}g\} \in \Phi.$$

Further, Φ is, by definition, an ideal in $\tilde{\mathfrak{A}}$. The algebra $\mathfrak{A}^* = \tilde{\mathfrak{A}}/\Phi$ is one further realization of the algebra of observables of the system under consideration.

In the actual description of the space Γ^* in the second section we introduced the additional condition (10). Equations (10) define a submanifold Γ^* of the space M and condition (11) implies that Γ^* is a transversal integral manifold of the distribution P. We impose on φ^a and χ_b the condition that each integral manifold intersects Γ^* at only one point. In this case we have actually constructed a stratification and Γ^* can be used as the realization of its basis.

Condition (12) has only technical value and simply makes it easier to choose canonical coordinates on Γ^*. Indeed, if it holds, then in some canonical coordinates on Γ the manifold Γ^* is defined by the equation

$$p_a = 0, \quad q^a(q^*, p^*) = 0.$$

The inverse image of the form $\sum dp_i \wedge dq^i$ is $\sum dp^* \wedge dq^*$, so that the variables q^* and p^* are canonical in Γ^*.

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