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A 1/n EXPANDABLE SERIES OF NON-LINEAR σ MODELS WITH INSTANTONS

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We formulate and discuss in detail the recently discovered $\mathbb{C}P^{n-1}$ non-linear σ models in two dimensions. We find that the fundamental particles in these theories are confined by a topological Coulomb force.

1. Introduction

The most attractive feature of two-dimensional non-linear σ models is their similarity with Yang-Mills theories in four space-time dimensions, a parallelism, which has already inspired some exciting conjectures on the latter. For example, the Pohlmeyer (or "dual") symmetry [1] of the σ models feeds the hope that such a symmetry might exist in four dimensions [2], too. A somewhat disappointing aspect of the analogy between Yang-Mills and σ models is the absence of stable instantons in the O(n) σ models ($n \geqslant 4$). In particluar, effects in the O(3) σ model due to instantons [3] *** were not accessible to the powerful $1/n$ expansion and could therefore be explored only by the infrared-divergent dilute-gas approximation [4]. In this paper we define and analyze a new series of SU(n) non-linear σ models, which are $1/n$ expandable and whose members are all topologically non-trivial. These new models have first been proposed by Eichenherr [5], who also showed that they have the dual symmetry and that the n = 2 case is equivalent to the familiar O(3) model.

In sect. 2 we outline the construction of a general non-linear σ model and then specialize to the $\mathbb{C}P^{n-1}$ models, the new series announced above. The instanton

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*** Refs. [3a,b] are recent reviews of instanton physics.

structure of the $\mathbb{C}P^{n-1}$ models is made explicit in sect. 3 and the quantization *via* the $1/n$ expansion is undertaken in sect. 4. The physical interpretation of the results including a discussion of θ-vacua is contained in sect. 5. In sect. 6, we draw conclusions and indicate some interesting possibilities to extend our work.

2. Definition of the $\mathbb{C}P^{n-1}$ non-linear σ models *

The definition of a general non-linear σ model proceeds along the following steps. Let G be a compact Lie group and H some closed subgroup of G. The G/H non-linear σ model is a theory of fields $\phi(x)$ in space-time, which take values in the coset space G/H. The group G acts on such fields according to the transformation law

$$\phi'(x) = g\phi(x)\,, \qquad g \in \mathrm{G}\,. \tag{1}$$

To define an action S for the fields $\phi(x)$ which is invariant under the transformation (1), we take any G-invariant metric on G/H and set

$$S = \tfrac{1}{2}\int \mathrm{d}^2x \sum_{\mu=1}^{2} \langle \partial_\mu\phi(x), \partial_\mu\phi(x)\rangle\,. \tag{2}$$

The notation is as follows. $\partial_\mu\phi(x)$ is the tangent vector to the curve

$$p(t) = \phi(x + te_\mu)\,, \qquad e_\mu\text{: unit vector in direction } \mu\,,$$

in G/H at $p(0) = \phi(x)$. The bracket $\langle\partial_\mu\phi, \partial_\mu\phi\rangle$ then denotes the length squared of $\partial_\mu\phi$ with respect to the chosen metric on G/H. Here and in what follows space-time is taken two-dimensional and Euclidean; but, of course, classical σ models can be defined in any dimension and with any metric.

Fields $\phi(x)$, which approach a consatnt value $\phi_\infty \in \mathrm{G/H}$ as $|x| \to \infty$, need not be continuously deformable into each other, i.e., they fall into homotopy classes, the collection of which is called the second homotopy group $\pi_2(\mathrm{G/H})$ of G/H (see e.g., ref. [6]). The topologically non-trivial σ models are now easily singled out with the help of the following theorem **

$$\pi_2(\mathrm{G/H}) = \pi_1(\mathrm{H})_\mathrm{G}\,, \tag{3}$$

where $\pi_1(\mathrm{H})_\mathrm{G}$ is the subset of $\pi_1(\mathrm{H})$ formed from those closed paths in H, which could be contracted to a point in G. Eq. (3) reveals that whether or not the G/H model has instantons depends not so much on G, but on the stability group H. For example, if we choose G = SO(n) and H = SO(n − 1), the G/H σ model is simply the ordinary O(n) σ model, which is topologically trivial for $n \geqslant 4$. On the other

* The first, abstract part of this section is not needed later and can be skipped for a first reading.

** We are indebted to P. Goddard for having drawn our attention to this remarkable result.

hand, if we take instead $H = SO(2) \times SO(n-2)$, then $\pi_1(H)_G = \mathbb{Z}$ and the associated σ model has instantons for all n.

More exotic σ models are obtained when H has several Abelian factors. In that case instantons are labelled by more than one integer topological number. In this paper, however, we concentrate on what seems to be the simplest series of topologically non-trivial σ models. This series arises from the choice

$$G = SU(n) , \qquad H = S(U(1) \times U(n-1)) \simeq U(n-1) . \tag{4}$$

G/H can then be identified with the $n-1$ dimensional complex projective space $\mathbb{C}P^{n-1}$, i.e., the space of all equivalence classes $[z]$ of complex vectors $(z_1, ..., z_n) \neq 0$, two of which z and z' being equivalent, if

$$z' = \lambda z , \qquad \lambda \in \mathbb{C} \tag{5}$$

(see e.g., ref. [7], p. 159). The group $SU(n)$ acts on $\mathbb{C}P^{n-1}$ by

$$g \cdot [z] = [gz] , \qquad (gz)_\alpha = g_{\alpha\beta} z_\beta , \qquad g \in SU(n) . \tag{6}$$

There is only one $SU(n)$ invariant metric on $\mathbb{C}P^{n-1}$, which, by the general formula eq. (2), yields an action S for fields $[z](x)$. To write it down explicitly, it is convenient to simplify our notation as follows. Instead of fields $[z](x)$, we may equally well consider fields of complex unit vectors

$$(z_1(x), ..., z_n(x)) , \qquad |z_1|^2 + ... + |z_n|^2 = 1 , \tag{7}$$

keeping in mind, however, that fields related by a gauge transformation

$$z'_\alpha(x) = e^{i\Lambda(x)} z_\alpha(x) \tag{8}$$

should be considered equivalent. In other words, $z(x)$ is one of the unit length representatives of the class $[z](x)$ and eq. (8) simply reflects the ambiguity involved, when choosing $z(x)$ out of $[z](x)$.

Under a gauge transformation, eq. (8), the composite field,

$$A_\mu = \tfrac{1}{2} i \bar{z} \cdot \overleftrightarrow{\partial}_\mu z = \tfrac{1}{2} i \{\bar{z}_\alpha \partial_\mu z_\alpha - (\partial_\mu \bar{z}_\alpha) z_\alpha\} , \tag{9}$$

transforms like an Abelian gauge field:

$$A'_\mu = A_\mu - \partial_\mu \Lambda . \tag{10}$$

The action

$$S = \frac{n}{2f} \int d^2x \overline{D_\mu z} \cdot D_\mu z , \qquad D_\mu = \partial_\mu + i A_\mu \tag{11}$$

is therefore gauge invariant and it can be shown to be equal to the one defined by the general formula, eq. (2). A factor n and a dimensionless coupling constant $f > 0$ have been included here for later convenience.

As can be inferred from the theorem eq. (3), the $\mathbb{C}P^{n-1}$ models defined above are topologically interesting. Indeed, if

$$[z](x) \to [z^\infty] \qquad \text{as} \qquad |x| \to \infty \,, \tag{12}$$

it follows that

$$z_\alpha(x) \to g\left(\frac{x}{|x|}\right) z_\alpha^\infty \,, \qquad |g| = 1 \,, \qquad |x| \to \infty \,. \tag{13}$$

The direction-dependent phase $g(x/|x|)$ defines a mapping from the circle at infinity into U(1). Its winding number is

$$Q = \frac{1}{2\pi} \int d^2x \epsilon_{\mu\nu} \partial_\mu A_\nu \,, \qquad \epsilon_{12} = +1 \,. \tag{14}$$

Thus, for any field $z(x)$ satisfying the boundary condition (12), the topological charge Q is an integer. One can show that Q labels the homotopy classes of fields $z(x)$ in a one-to-one fashion, i.e., any two fields with equal charge can be continuously deformed into each other and, furthermore, for any integer p there exist fields such that $Q = p$.

We finally remark that the $\mathbb{C}P^1$ model is nothing else than the O(3) σ model. This latter model describes spin fields $q^a(x)$, $a = 1, 2, 3$, of unit length: $q^a q^a = 1$. The relation between the O(3) and the $\mathbb{C}P^1$ model is

$$q^a = \bar{z}_\alpha \sigma^a_{\alpha\beta} z_\beta \,, \qquad \sigma^a\text{: Pauli matrices} \,, \tag{15}$$

which maps spin fields $q^a(x)$ onto gauge-equivalence classes of $\mathbb{C}P^1$ fields $z_\alpha(x)$ and *vice versa.* Briefly, z_α is the spinor representation of q^a. When the action eq. (11) is written in terms of the q^a variables, one recovers the usual Heisenberg spin field action, thus proving the complete equivalence of the two theories.

3. Instantons in $\mathbb{C}P^{n-1}$ models

To derive the instanton equations, we first rewrite the topological density $q(x)$ as follows:

$$q = \frac{1}{2\pi} \epsilon_{\mu\nu} \partial_\mu A_\nu = \frac{i}{2\pi} \epsilon_{\mu\nu} \overline{D_\mu z} \cdot D_\nu z \,. \tag{16}$$

Applying the Cauchy-Schwartz inequality, we then conclude that

$$S \geqslant \frac{n\pi}{f} |Q| \,, \tag{17}$$

the equality sign holding if and only if

$$D_\mu z = \underset{(-)}{+} i \epsilon_{\mu\nu} D_\nu z \,. \tag{18}$$

Finite action solutions of these first-order equations are called instantons (anti-instantons). Instantons are absolute minima of the action in a sector with definite topological charge and are therefore automatically solutions of the second-order field equations

$$D_\mu D_\mu z + (\overline{D_\mu z} \cdot D_\mu z)\, z = 0 \,, \qquad |z|^2 = 1 \,. \tag{19}$$

The space $\mathbb{C}P^{n-1}$ has a remarkable (and for the existence of instantons perhaps crucial) property that makes it possible to completely solve the self-duality equations (18): it is a *complex* manifold. An atlas of holomorphic charts (U_j, φ_j), $j = 1, ..., n$, is given by

$$U_j = \{[z] \in \mathbb{C}P^{n-1} | z_j \neq 0\} \,, \tag{20}$$

$$\varphi_j \colon U_j \to \mathbb{C}^{n-1} \,,$$

$$\varphi_j([z]) = \frac{1}{z_j}(z_1, ..., z_n) \equiv (w_1^{(j)}, ..., w_n^{(j)}) \,, \tag{21}$$

(note that $w_j^{(j)} = 1$, i.e., the coordinates are $w_\alpha^{(j)}$, $\alpha \neq j$). If $[z] \in U_j \cap U_k$, the respective coordinates are related by

$$(w_1^{(j)}, ..., w_n^{(j)}) = \frac{1}{w_j^{(k)}} (w_1^{(k)}, ..., w_n^{(k)}) \,. \tag{22}$$

To any field $z(x)$ eq. (21) associates a field $w^{(j)}(x)$ of coordinates, which, of course, does not depend on the gauge chosen for $z(x)$. Conversely,

$$z(x) = e^{i\Lambda(x)} \frac{w^{(j)}(x)}{|w^{(j)}(x)|} \,, \tag{23}$$

for some gauge function $\Lambda(x)$.

When we now insert the representation (23) into the self-duality equations (18), they become linear:

$$\partial_\mu w^{(j)} = \underset{(-)}{+}\, i\epsilon_{\mu\nu}\partial_\nu w^{(j)} \,. \tag{24}$$

Defining a complex variable $s = x_1 - ix_2$ we recognize eq. (24) as the Cauchy-Riemann equations with respect to s. In other words, the smooth solutions of the self-duality equations (18) are the holomorphic (anti-holomorphic) mappings from the complex s-plane into $\mathbb{C}P^{n-1}$. This does not imply that the $w_\alpha^{(j)}(x)$ are entire functions of s (of $\bar{s}$), but any singularity should be removable by an appropriate change of charts. The transition law (22) then requires the possible singularities of $w_\alpha^{(j)}$ to be isolated poles. Summarizing, we have:

Theorem. The most general smooth solution of the self-duality equation

$$D_\mu z = i\epsilon_{\mu\nu} D_\nu z \,, \qquad |z|^2 = 1 \,,$$

is given by

$$z_\alpha(x) = e^{i\Lambda(x)} \frac{w_\alpha(x)}{|w(x)|}, \tag{25}$$

where $\Lambda(x)$ is a real function, w_α a meromorphic function of $s = x_1 - ix_2$ and $w_j = 1$ for some j.

Note that in general $\Lambda(x)$ must be discontinuous to insure continuity of $z_\alpha(x)$. In order that the solution (25) has finite action, it is necessary and sufficient that the w_α's are *rational* functions of s. We do not prove this result here, because the argument, though elementary, is rather long and tricky.

The topological charge Q of an instanton solution is equal to the number of poles of w (including those at $s = \infty$). Thus, choosing w to have only one pole, we obtain the one instanton solution. It can conveniently be written in the form

$$z_\alpha(x) = \frac{\lambda u_\alpha + [(x_1 - a_1) - i(x_2 - a_2)]\, v_\alpha}{(\lambda^2 + (x-a)^2)^{1/2}}. \tag{26}$$

Here, a_μ is the position of the instanton, $\lambda > 0$ is its scale size and the two constant vectors u_α, v_α subject to the constraints,

$$|u|^2 = |v|^2 = 1\,, \qquad \bar{u} \cdot v = 0\,, \tag{27}$$

characterize the orientation of the instanton in color space. As expected, the solution (26) approaches a pure gauge with unit winding number when $|x| \to \infty$:

$$z_\alpha(x) = \frac{x_1 - ix_2}{|x|} v_\alpha + O\left(\frac{1}{|x|}\right). \tag{28}$$

The vacuum far away from the instanton is hence described by the parameters v_α. The remaining $2n$ real parameters a_μ, λ and u_α can then be interpreted as the independent degrees of freedom of the instanton relative to the asymptotic vacuum.

4. $1/n$ expansion of the quantum $\mathbb{C}P^{n-1}$ models

The generating functional for the Euclidean Green functions of the quantum $\mathbb{C}P^{n-1}$ model is

$$Z(J, \bar{J}, K_\mu) = \int \mathcal{D}z\mathcal{D}\bar{z} \prod_x \delta\left(|z(x)|^2 - \frac{n}{2f}\right)$$

$$\times \exp\{-S + \int d^2x [\bar{J}(x) \cdot z(x) + \bar{z}(x) \cdot J(x) + K_\mu(x) A_\mu(x)]\}\,. \tag{29}$$

Here, we have rescaled z_α by a factor $(n/2f)^{1/2}$ so that the action (11) becomes

$$S = \int d^2x \left\{ \partial_\mu \bar{z} \cdot \partial_\mu z + \frac{f}{2n} (\bar{z} \cdot \overleftrightarrow{\partial}_\mu z)(\bar{z} \cdot \overleftrightarrow{\partial}_\mu z) \right\}. \tag{30}$$

For later convenience, we have furthermore introduced a source K_μ for the topological gauge field, which now reads *

$$A_\mu = \frac{f}{n} i\bar{z} \cdot \overleftrightarrow{\partial}_\mu z \,. \tag{31}$$

Also, we do not yet fix the gauge but first work out the effective action for the $1/n$ expansion.

The $1/n$ expansion has already been discussed for a number of similar theories, in particular for the Gross-Neveu model [8], the O(n) σ models [9] and ϕ^4 field theory [10]. It is also well-known to work for the Heisenberg ferromagnet [11], i.e., for the lattice O(n) σ model. In what follows, we shall therefore be rather brief and refer the reader to the literature for more details.

One starts by introducing Lagrange multiplier fields $\alpha(x)$ and $\lambda_\mu(x)$ to make the action quadratic in z **:

$$\prod_x \delta\left(|z(x)|^2 - \frac{n}{2f}\right) \exp\left[\frac{f}{n}\int d^2x\{-\tfrac{1}{2}(\bar{z}\cdot\overleftrightarrow{\partial}_\mu z)(\bar{z}\cdot\overleftrightarrow{\partial}_\mu z) + iK_\mu(\bar{z}\cdot\overleftrightarrow{\partial}_\mu z)\}\right]$$

$$= \int \mathcal{D}\alpha\mathcal{D}\lambda_\mu \exp\left[\int d^2x\left\{\frac{i}{\sqrt{n}}\alpha\left(|z|^2 - \frac{n}{2f}\right)\right.\right.$$

$$\left.\left. - \left(m^2 + \frac{1}{n}\lambda_\mu\lambda_\mu\right)|z|^2 + \frac{1}{\sqrt{n}}\lambda_\mu[i(\bar{z}\cdot\overleftrightarrow{\partial}_\mu z) + K_\mu] - \frac{f}{2n}\, K_\mu K_\mu\right\}\right]. \tag{32}$$

Here, we have introduced a new parameter $m^2 > 0$, which is completely irrelevant at this stage, but will be used later. Inserting eq. (32) into the generating functional (29), we get

$$Z(J, \bar{J}, K_\mu) = \int \mathcal{D}z\mathcal{D}\bar{z}\mathcal{D}\alpha\mathcal{D}\lambda_\mu \exp\left\{-S'\right.$$

$$\left. + \int d^2x\left[\bar{J}\cdot z + \bar{z}\cdot J + \frac{1}{\sqrt{n}}K_\mu\lambda_\mu - \frac{f}{2n}K_\mu K_\mu\right]\right\}, \tag{33}$$

where

$$S' = \int d^2x\left\{\bar{z}\cdot\Delta z + \frac{i\sqrt{n}}{2f}\alpha\right\}, \tag{34}$$

and the differential operator Δ is given by

$$\Delta = -D_\mu D_\mu + m^2 - \frac{i}{\sqrt{n}}\alpha\,, \qquad D_\mu = \partial_\mu + \frac{i}{\sqrt{n}}\lambda_\mu\,. \tag{35}$$

* i.e., with this normalization, $Q = (1/2\pi)\int d^2x\epsilon_{\mu\nu}\partial_\mu A_\nu$ is integral.

** We do not keep track of constant factors in front of $Z(J, \bar{J}, K_\mu)$.

The gauge invariance of the theory is now reflected by the invariance of S' under the transformation

$$z'(x) = \mathrm{e}^{i\Lambda(x)} z(x)\,, \tag{36a}$$

$$\alpha'(x) = \alpha(x)\,, \qquad \lambda'_\mu(x) = \lambda_\mu(x) - \sqrt{n}\partial_\mu\Lambda(x)\,. \tag{36b}$$

Next, performing the Gaussian z integral, we arrive at

$$Z(J, \bar{J}, K_\mu) = \int \mathcal{D}\alpha\mathcal{D}\lambda_\mu \exp\left\{-S_{\mathrm{eff}} + \int \mathrm{d}^2x \left[\bar{J}\cdot\Delta^{-1}J + \frac{1}{\sqrt{n}}K_\mu\lambda_\mu - \frac{f}{2n}K_\mu K_\mu\right]\right\}, \tag{37}$$

with an effective action

$$S_{\mathrm{eff}} = n\ \mathrm{Tr}\log\Delta + \frac{i\sqrt{n}}{2f}\int \mathrm{d}^2x\alpha(x)\,. \tag{38}$$

It can be expanded in a power series of $1/\sqrt{n}$:

$$S_{\mathrm{eff}} = \sum_{\nu=1}^{\infty} (n)^{1-\nu/2}\, S^{(\nu)} + \text{constant}\,. \tag{39}$$

The lowest term

$$S^{(1)} = \frac{i}{2f}\int \mathrm{d}^2x\alpha(x) - i\ \mathrm{Tr}\{(-\Box + m^2)^{-1}\,\alpha\}$$

$$= i\tilde{\alpha}(0)\left\{\frac{1}{2f} - \int\frac{\mathrm{d}^2q}{(2\pi)^2}\,(m^2+q^2)^{-1}\right\}, \tag{40}$$

where

$$\tilde{\alpha}(p) = \int \mathrm{d}^2x\ \mathrm{e}^{-ipx}\,\alpha(x)$$

is the Fourier transform of α, contains a divergent one-loop integral. When regularized with a Pauli-Villars cutoff Λ, it comes out to be $(4\pi)^{-1}\log(\Lambda^2/m^2)$. Thus, if we let the bare coupling f vary with respect to the cutoff according to

$$\frac{2\pi}{f} = \log\frac{\Lambda^2}{\mu^2} + \frac{2\pi}{f_{\mathrm{R}}(\mu)}\,. \tag{41}$$

(μ is the normalization point and $f_{\mathrm{R}}(\mu)$ the renormalized coupling), the divergencies cancel in $S^{(1)}$. Eq. (41) shows that the $\mathbb{C}\mathrm{P}^{n-1}$ models are asymptotically free: when $\Lambda \to \infty$, the bare coupling goes to zero.

As $n \to \infty$ the term $\sqrt{n}S^{(1)}$ gives a rapidly oscillating contribution to the integrand in eq. (37). However, we are completely free to impose the saddlepoint condition

$S^{(1)} = 0$ by choosing the arbitrary parameter m^2 such that

$$m^2 = \mu^2 \exp\left(-\frac{2\pi}{f_R(\mu)}\right) . \tag{42}$$

The rest of the integrand in eq. (37) can then the expanded in a power series of $1/\sqrt{n}$ and integrated term by term. Before that we complete our discussion of the effective action S_{eff}.

The quadratic part of S_{eff} can be written in the form

$$S^{(2)} = \tfrac{1}{2} \int d^2x\, d^2y \{\alpha(x)\, \Gamma^\alpha(x-y)\, \alpha(y) + \lambda_\mu(x)\, \Gamma^\lambda_{\mu\nu}(x-y)\, \lambda_\nu(y)\} , \tag{43}$$

where the Fourier transforms of Γ^α and $\Gamma^\lambda_{\mu\nu}$ are

$$\widetilde{\Gamma}^\alpha(p) = \int \frac{d^2q}{(2\pi)^2} \{(m^2+q^2)(m^2+(p+q)^2)\}^{-1} , \tag{44}$$

$$\widetilde{\Gamma}^\lambda_{\mu\nu}(p) = 2\delta_{\mu\nu} \int \frac{d^2q}{(2\pi)^2} (m^2+q^2)^{-1} - \int \frac{d^2q}{(2\pi)^2} \frac{(p_\mu + 2q_\mu)(p_\nu + 2q_\nu)}{(m^2+q^2)(m^2+(p+q)^2)} . \tag{45}$$

The integral (44) is easily evaluated:

$$\Gamma^\alpha(p) = A(p) \equiv \frac{1}{2\pi} [p^2(p^2+4m^2)]^{-1/2} \log \frac{\sqrt{p^2+4m^2} + \sqrt{p^2}}{\sqrt{p^2+4m^2} - \sqrt{p^2}} . \tag{46}$$

The one-loop integrals in eq. (45) are both divergent. However, with a Pauli-Villars regulator the divergent parts are seen to cancel and we are left with

$$\widetilde{\Gamma}^\lambda_{\mu\nu}(p) = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\left\{(p^2+4m^2)\, A(p) - \frac{1}{\pi}\right\} . \tag{47}$$

A priori we would have to add a finite renormalization term $a\delta_{\mu\nu}$ to this expression, because we are using a graph by graph regularization. However, the effective action must be invariant under the gauge transformation (36b) and this is the case if and only if we put $a = 0$.

The contributions $S^{(\nu)}$, $\nu \geqslant 3$, to the effective action are convergent one-loop integrals and do not cause any problems. Thus any Green function of z, $\bar{z}$ and A_μ *

$$\langle z_{\alpha_1}(x_1) \ldots A_{\mu_l}(x_l)\rangle = Z(0,0,0)^{-1} \int \mathcal{D}\alpha \mathcal{D}\lambda_\mu\, e^{-S^{(2)}}$$

$$\times \exp\Big(-\sum_{\nu=3}^{\infty} n^{1-\nu/2} S^{(\nu)}\Big) \frac{\delta}{\delta \bar{J}_{\alpha_1}(x_1)} \cdots \frac{\delta}{\delta K_{\mu_l}(x_l)}$$

* We drop the contact term $(f/2n)\, K_\mu K_\mu$ which is zero, because $f(\Lambda) \to 0$ as $\Lambda \to \infty$.

$$\times \exp\left[\int d^2x \quad \bar{J}\cdot\Delta^{-1}J + \frac{1}{\sqrt{n}}K_\mu\lambda_\mu\right]\Bigg|_{\substack{J=0\\ \bar{J}=0\\ K_\mu=0}} , \tag{48}$$

can be $1/n$ expanded by writing the integrand of the functional integral in a power series of $1/\sqrt{n}$ and performing the Gaussian integrals on α and λ_μ. Of course, at this stage we cannot do without fixing the gauge. For definiteness we choose the Lorentz gauge

$$\partial_\mu\lambda_\mu = 0 . \tag{49}$$

The outcome of the calculation above can then conveniently be represented in terms of Feynman diagrams built from z-, α- and λ_μ-lines (fig. 1) and the vertices displayed in fig. 2. The propagators are

$$G(p) = (m^2 + p^2)^{-1} , \tag{50}$$

$$D^\alpha(p) = A(p)^{-1} , \tag{51}$$

$$D^\lambda(p) = \left\{(p^2 + 4m^2)A(p) - \frac{1}{\pi}\right\}^{-1} . \tag{52}$$

Internal lines are integrated with measure $d^2q/(2\pi)^2$ and the vacuum polarization diagrams in fig. 3 should not be drawn. Green functions involving the topological

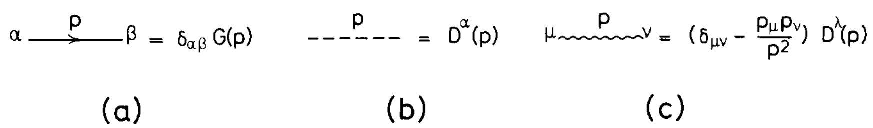

Fig. 1. Graphical representation of (a) the z propagator, (b) the α propagator, and (c) the λ_μ propagator.

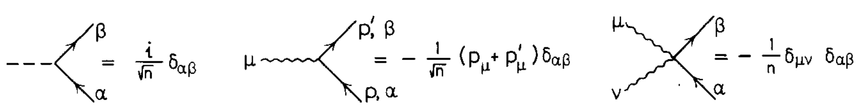

Fig. 2. Vertices for the $1/n$ expansion.

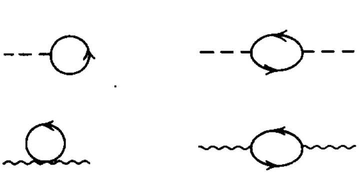

Fig. 3. Forbidden (sub) diagrams.

field A_μ correspond to Feynman diagrams with external λ_μ-lines since, by eq. (48)

$$A_\mu(x) = \frac{1}{\sqrt{n}} \lambda_\mu(x) . \tag{53}$$

Higher-order diagrams in the $1/n$ expansion are ultraviolet divergent and must be renormalized. Presumably this amounts to a renormalization of the mass m and the z- and α-field normalizations as is the case in the O(n) σ model *, but we did not verify this statement.

5. The physical interpretation of $\mathbb{C}P^{n-1}$ models

The Feyman rules for the $1/n$ expansion derived in sect. 4 show that the quantum $\mathbb{C}P^{n-1}$ models describe an SU(n) vector of charged, interacting particles with mass m. For reasons to become clear soon, we call them "partons". *Classical* $\mathbb{C}P^{n-1}$ models are conformally invariant so that the non-zero parton mass is entirely due to quantum fluctuations. Furthermore, by eq. (42), m depends non-perturbatively on the renormalized coupling constant.

Partons interact by exchanging α and λ_μ quanta. Since $A(p)$ is analytic for $\mathrm{Re}\, p^2 > -4m^2$ and

$$A(p) > 0 \qquad \text{for} \qquad p^2 \geqslant 0 , \qquad A(0) = (4\pi m^2)^{-1} , \tag{54}$$

the α-interaction is short ranged. It does not correspond to an exchange of a physical particle, because $A(p) \neq 0$ for any p^2. On the other hand, the gauge field propagator (52) has a pole at $p^2 = 0$, i.e.,

$$D^\lambda(p) = 12\pi m^2/p^2 + \mathrm{O}(1) , \qquad (p^2 \to 0) , \tag{55}$$

thus giving rise to a long-range force between partons. In the non-relativistic limit, the λ_μ exchange has the same effects as a linear Coulomb potential so that the partons are confined. In other words, the physical Hilbert space contains only states invariant under the center Z_n of SU(n), i.e., states of zero n-ality. Confinement in $\mathbb{C}P^{n-1}$ models is strictly quantum mechanical: because of scale invariance the classical field gives an energy inversely proportional to the distance between two point charges.

One can prove that the pole in $D^\lambda(p)$ persists to any finite order of $1/n$. Also, the infrared divergencies caused by internal λ_μ-lines in higher-order diagrams cancel in Green functions of local, gauge-invariant fields such as $\bar{z}_\alpha z_\beta$.

Due to the gauge character of λ_μ, the pole of $D^\lambda(p)$ does not imply that there is a zero-mass particle: the two-point function of the gauge-invariant field $\epsilon_{\mu\nu}\, \partial_\mu \lambda_\nu$ is analytic in momentum space for $\mathrm{Re}\, p^2 > -4m^2$ and consequently falls off exponentially in position space. Also, there is no other particle associated with λ_μ,

* We thank K. Symanzik for a clarifying letter concerning this question.

because

$$D^{\lambda}(p)^{-1} > 0 \qquad (p^2 > 0)\,, \qquad D^{\lambda}(p)^{-1} \neq 0\,, \qquad (\text{any } p^2 \neq 0)\,. \tag{56}$$

The relation (15) between the $\mathbb{C}P^1$ model and the O(3) σ model now becomes physically meaningful. From the $1/n$ expansion of the O(n) σ model we know that the spin field q^a is the interpolating field for a triplet of massive mesons. By eq. (15), they can be thought of as two-particle bound states of the $\mathbb{C}P^1$ partons. For the general $\mathbb{C}P^{n-1}$ model the meson spectrum is not known, although for large n one should be able to calculate it within a non-relativistic or perhaps a semiclassical approximation.

So far we have been discussing the $\mathbb{C}P^{n-1}$ models in the $\theta = 0$ vacuum. The $\theta \neq 0$ vacuum is defined by the modified action [12]

$$S^{\theta} = S - i\theta Q\,, \tag{57}$$

S being the ordinary action (30) and Q the topological charge. By eq. (53) this amounts to a change of the effective action S_{eff} according to

$$S^{\theta}_{\text{eff}} = S_{\text{eff}} - i\frac{\theta}{2\pi\sqrt{n}}\int \mathrm{d}^2x\,\epsilon_{\mu\nu}\partial_{\mu}\lambda_{\nu}(x)\,. \tag{58}$$

Correspondingly, the Feynman rules for the $1/n$ expansion are supplemented by a new vertex where a single λ_{μ}-line ends in the vacuum picking up a factor $-(\theta/2\pi\sqrt{n})$ $\epsilon_{\mu\nu}p_{\nu}, p \to 0$. For example, the topological density $q(x) = (2\pi)^{-1}\,\epsilon_{\mu\nu}\partial_{\mu}A_{\nu}(x)$ now has a non-zero vacuum expectation value

$$\langle q(x)\rangle_{\theta} = iF(\theta)\,, \qquad F(\theta) = \frac{3m^2}{n\pi}\,\theta + \mathrm{O}\left(\frac{1}{n^2}\right)\,. \tag{59}$$

Thus, in a θ vacuum, there is a constant background topological density.

That the addition of the "boundary term" $i\theta Q$ to the action has an influence on the correlation functions, is due to the pole (55) of the propagator of the topological field. If that pole was absent, the zeros $\epsilon_{\mu\nu}p_{\nu}, p \to 0$, from the θ vertices would not be cancelled and there would be no θ dependence at all. Thus, we see that the confining Coulomb force is linked to the contribution of topologically non-trivial field configurations to the functional integral. It has been argued [13] that this connection is a very general one, but it is not clear at present, what the implications of the analogous Coulomb force in higher dimensions are.

Because the topological charge Q of smooth fields takes on integral values only, one naively expects that the physics of $\mathbb{C}P^{n-1}$ models is periodic in θ with period 2π. At first sight, the result (59) seems to contradict this conclusion. The discrepancy can be explained as follows. The phenomenology of θ vacua in our model is very much the same as in the *massive* Schwinger model [14]. We thus expect that if $|\theta| > \pi$ the vacuum breaks down by pair production of partons until the strong topological background fields has decreased to a value corresponding to $|\theta| < \pi$. With respect to the $1/n$ expansion, this is a non-perturbative effect so that one should trust eq. (59) only for $|\theta| \leqslant \pi$. Although the argument given here is perfectly plausible, we do not consider it rigorous enough to settle the important question of whether or not field configurations with fractional topological charge [15] contribute to the functional integral.

6. Conclusions and outlook

Classical $\mathbb{C}P^{n-1}$ non-linear σ models are conformally invariant and have topologically stabilized instantons, whereas their quantum versions are asymptotically free with a vacuum that could be interpreted, in a semiclassical approximation, as a Bloch wave [12]. All of these properties are shared by the four-dimensional Yang-Mills theory, but the quantum $\mathbb{C}P^{n-1}$ models have the additional advantage to be $1/n$ expandable. Exploiting this fact, we were able to show that the physical particles in $\mathbb{C}P^{n-1}$ models are bound states of massive partons, which are permanently confined by a topological Coulomb force. In particular, it follows that topologically non-trivial field configurations make a very important contribution to the low-energy physics in these theories.

Our analysis can be amplified in several directions. First of all, the structure of $\mathbb{C}P^{n-1}$ models can be enriched by adding massless fermions in a chirally symmetric way. For example, a supersymmetric action is

$$S = \frac{n}{2f}\int d^2x\{\overline{D_\mu z} \cdot D_\mu z + \bar{\psi}\not{D}\psi + \tfrac{1}{4}[(\bar{\psi}\gamma_\mu\psi)^2 - (\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2]\} , \tag{60}$$

supplemented by the constraints

$$\bar{z} \cdot z = 1 , \qquad \bar{z} \cdot \psi = 0 , \qquad \bar{\psi} \cdot z = 0 . \tag{61}$$

We shall come back to this model in a future publication [16] and show that it exhibits chiral symmetry breaking without a Goldstone boson [17].

Another interesting possibility is to test the reliability of the dilute-gas approximation by comparing it with the results of the $1/n$ expansion. The necessary one-loop calculations to do this seem straightforward, but there is the problem of zero-mass Gaussian fluctuations, which produce typically two-dimensional infrared divergencies, and the disturbing fact that the calculations [4] published so far give *ultraviolet* divergent results.

Finally, the presence of higher conservation laws [1,5] in $\mathbb{C}P^{n-1}$ models leads us to suspect that the meson spectrum and the corresponding S-matrix could perhaps be calculated exactly. Although this problem has not yet been solved, some candidate factorized S-matrices have already been constructed [18].

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Note added

Cremmer and Scherk kindly informed us that they constructed the supersymmetric $\mathbb{C}P^{n-1}$ model in the context of supergravity in four dimensions [19].

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