SUPERGAUGE MULTIPLETS AND SUPERFIELDS

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Superfields are defined as functions of the space-time variable x and of anticommuting two-component spinors θ and $\bar{\theta}$. They have definite transformation properties under supergauge transformations. Their expansion in θ and $\bar{\theta}$ generates a finite number of ordinary fields forming a multiplet. A number of operations are defined which allow the construction of new superfields from given ones. The corresponding set of multiplets is complete, in the sense that the product of any two can be decomposed as a sum of multiplets belonging to the same set.

Supergauge multiplets and superfields. Supergauge transformations in four dimensions have been defined on field multiplets composed of spinor and tensor fields [1] and examples of renormalizable field theories invariant under supergauge transformations have been given [2, 3]. The study of the supergauge group and its representations has been continued with different techniques [4]. In particular, Salam and Strathdee [5] have introduced the interesting concept of super field and have described the supergauge transformations as operations on superfields. The superfields of Salam and Strathdee are functions $\Phi(x, \theta)$ of the space-time variable x and of a totally anticommuting Majorana spinor θ. Since the square of each component of θ vanishes, a power series expansion in θ for the function $\Phi(x, \theta)$ terminates after a finite number of terms. The superfield is therefore equivalent to a finite number of ordinary fields, a multiplet of fields. The transformation properties of superfields imply transformation properties for the multiplet. In ref. [5], the superfield equivalent to the vector multiplet of ref. [1] is described and a law of composition of two vector multiplets is obtained simply and elegantly as resulting from the multiplication of the corresponding superfields. The scalar multiplet of ref. [1] is obtained as a vector multiplet satisfying an invariant constraint. The fact that the scalar multiplet cannot be obtained directly is due to the use, in ref. [5], of Majorana spinors θ and to the essentially complex nature of the scalar multiplet. In this letter we extend the concept of superfield by using a two-component complex spinor θ (and its conjugate $\bar{\theta}$) and define a superfield for the scalar multiplet, which we believe to be the building block of all field representations of the supergauge[1]. We also define the operations of "shift" and invariant differentiation which, together with complex conjugation and the multiplication of two superfields, are the basic operations for the construction of new representations from given ones.

The supergauge charges can be taken as two component complex spinors[2] Q_α satisfying, together with their Hermitian conjugates $\bar{Q}_{\dot{\alpha}}$, the anticommutation relations

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma_\mu)_{\alpha\dot{\beta}}\, P^\mu, \tag{1}$$

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0, \tag{2}$$

where P^μ is the four-momentum operator. The commutation relations

$$[Q_\alpha, P_\mu] = [P_\lambda, P_\mu] = 0 \tag{3}$$

complete the algebra. Using totally anticommuting parameters ζ^α and their conjugates $\bar{\zeta}^{\dot{\alpha}}$, one can write

[1] The multiplication of scalar multiplets is an essential tool in the construction of the renormalizable field theories studied in the papers of refs. [2, 3].

[2] We use the spinor notation of Van der Waerden, with $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha} = -\epsilon_{\alpha\beta}$, $\epsilon^{12} = 1$ (same for dotted indices) and $(\sigma_\mu)_{\alpha\dot{\beta}} \equiv (1, \sigma_i)$. For a two-component spinor we write $(\psi_\alpha)^* = \bar{\psi}_{\dot{\alpha}}$, $(\chi_{\dot{\alpha}})^* = \bar{\chi}_\alpha$ (same for upper indices).

(1) as a commutation relation

$$[\zeta_1 Q, \bar{Q}\bar{\zeta}_2] = 2\zeta_1\sigma_\mu\bar{\zeta}_2 P^\mu, \tag{4}$$

where $\zeta_1 Q = \zeta_1^\alpha Q_\alpha$, $\bar{Q}\bar{\zeta}_2 = \bar{Q}_{\dot\alpha}\bar{\zeta}_2^{\dot\alpha}$, etc.

A group element can be written in three different ways

$$\Phi(x,\theta,\bar\theta) = \exp(\mathrm{i}\bar{Q}\bar\theta + \mathrm{i}\theta Q - \mathrm{i}x\cdot P), \tag{5}$$

$$\Phi_1(x,\theta,\bar\theta) = \exp(\mathrm{i}\theta Q - \mathrm{i}x\cdot P)\exp(\mathrm{i}\bar{Q}\bar\theta), \tag{6}$$

$$\Phi_2(x,\theta,\bar\theta) = \exp(\mathrm{i}\bar{Q}\bar\theta - \mathrm{i}x\cdot P)\exp(\mathrm{i}\theta Q). \tag{7}$$

These three forms are connected by the relations

$$\Phi(x_\mu,\theta,\bar\theta) = \Phi_1(x_\mu + \mathrm{i}\theta\sigma_\mu\bar\theta,\theta,\bar\theta) = \Phi_2(x_\mu - \mathrm{i}\theta\sigma_\mu\bar\theta,\theta,\bar\theta). \tag{8}$$

Operating from the left with the group element

$$G = \exp(\mathrm{i}\zeta Q + \mathrm{i}\bar{Q}\bar\zeta) \tag{9}$$

one finds, respectively,

$$G\Phi(x_\mu,\theta,\bar\theta) = \Phi(x_\mu + \mathrm{i}\theta\sigma_\mu\bar\zeta - \mathrm{i}\zeta\sigma_\mu\bar\theta, \theta+\zeta, \bar\theta+\bar\zeta), \tag{10}$$

$$G\Phi_1(x_\mu,\theta,\bar\theta) = \Phi_1(x_\mu + 2\mathrm{i}\theta\sigma_\mu\bar\zeta + \mathrm{i}\zeta\sigma_\mu\bar\zeta, \theta+\zeta, \bar\theta+\bar\zeta), \tag{11}$$

$$G\Phi_2(x_\mu,\theta,\bar\theta) = \Phi_2(x_\mu - 2\mathrm{i}\zeta\sigma_\mu\bar\theta - \mathrm{i}\zeta\sigma_\mu\bar\zeta, \theta+\zeta, \bar\theta+\bar\zeta), \tag{12}$$

or, in infinitesimal form,

$$\delta\Phi = \left[\zeta\frac{\partial}{\partial\theta} + \bar\zeta\frac{\partial}{\partial\bar\theta} + \mathrm{i}(\theta\sigma_\mu\bar\zeta - \zeta\sigma_\mu\bar\theta)\frac{\partial}{\partial x_\mu}\right] \tag{13}$$

$$\delta\Phi_1 = \left[\zeta\frac{\partial}{\partial\theta} + \bar\zeta\frac{\partial}{\partial\bar\theta} + 2\mathrm{i}\theta\sigma_\mu\bar\zeta\frac{\partial}{\partial x_\mu}\right]\Phi_1, \tag{14}$$

$$\delta\Phi_2 = \left[\zeta\frac{\partial}{\partial\theta} + \bar\zeta\frac{\partial}{\partial\bar\theta} - 2\mathrm{i}\zeta\sigma_\mu\bar\theta\frac{\partial}{\partial x_\mu}\right]\Phi_2. \tag{15}$$

We shall now abstract and take (10)–(12) or equivalently (13)–(15) as the basic transformation properties of superfields. Clearly, from a superfield transforming according to any one of these three laws, one can construct a superfield transforming according to any other one by "shifting" the variable x as indicated in (8). Observe that the shift $\mathrm{i}\theta\sigma_\mu\bar\theta$ is pure imaginary. One can require Φ to be real, since by (10) or (13) it will stay real, but Φ_1 and Φ_2 are essentially complex and transform inequivalently as the complex conjugate of each other. The product of two superfields of the same "type" (transforming in the same way) is again a superfield of that type. If one wants to multiply two superfields of different types, one must first reduce them to the same type of shifting one (or both).

Given a superfield Φ_1 transforming as in (14), one can construct another one by taking its derivative with respect to $\bar\theta$, since $\bar\theta$ does not enter in the differential operator (14). Similarly, the differentiation with respect to θ is an invariant operation on a superfield Φ_2 transforming as in (15). Since Φ_1 and Φ_2 are related by (8) one sees that there are two invariant differentiation operators on each type of field. They are

$$\text{on } \Phi\text{:}\quad \frac{\partial}{\partial\theta} + \mathrm{i}\sigma_\mu\bar\theta\frac{\partial}{\partial x_\mu} \quad\text{and}\quad -\frac{\partial}{\partial\bar\theta} - \mathrm{i}\theta\sigma_\mu\frac{\partial}{\partial x_\mu} \tag{16}$$

$$\text{on } \Phi_1\text{:}\quad \frac{\partial}{\partial\theta} + 2\mathrm{i}\sigma_\mu\bar\theta\frac{\partial}{\partial x_\mu} \quad\text{and}\quad -\frac{\partial}{\partial\bar\theta} \tag{17}$$

$$\text{on } \Phi_2\text{:}\quad \frac{\partial}{\partial\theta} \quad\text{and}\quad -\frac{\partial}{\partial\bar\theta} - 2\mathrm{i}\theta\sigma_\mu\frac{\partial}{\partial x_\mu}. \tag{18}$$

In particular, on a field of the type Φ_1 one can impose the invariant constraint that it be independent of $\bar\theta$, or linear in $\bar\theta$ or a general function of $\bar\theta$ (which means quadratic in $\bar\theta$, since the square of each component of $\bar\theta$ vanishes). These three cases correspond to three types of multiplets, of which the first (together with its complex conjugate) gives the scalar multiplet of ref. [1] and the third (which by a shift can be changed to type Φ) is the vector multiplet of ref. [1]. The second case corresponds to a new type of multiplet, which we may call the "linear" multiplet. It contains two scalars, three spinors and one vector, all complex. This new kind of multiplet, like the scalar multiplet, is inequivalent to its complex conjugate, while the vector multiplet is equivalent to its complex conjugate. Superfields do not need to be scalar functions of x, θ and $\bar\theta$ but can have dotted and undotted spinor indices. The invariant operations of differentiation given by (16)–(18) endow the superfield with one additional spinor index.

For the sake of clarity we give now a few examples. Let Φ_1 be independent of $\bar\theta$

$$\Phi_1(x,\theta) = A(x) + \theta^\alpha\psi_\alpha(x) + \theta^\alpha\epsilon_{\alpha\beta}\theta^\beta F(x), \tag{19}$$

then (14) implies

$$\delta A = \zeta\psi, \qquad \delta\psi = 2\mathrm{i}\sigma_\mu\bar\zeta\frac{\partial}{\partial x_\mu}A + 2\zeta F,$$

$$\delta F = -\mathrm{i}\frac{\partial\psi}{\partial x_\mu}\sigma_\mu\bar\zeta, \tag{20}$$

the transformation law for a scalar multiplet[3].

From (19) one can obtain another superfield of the same type by complex conjugation, shift and double differentiation

$$\Phi_1'(x_\mu,\theta) = -\tfrac{1}{4}\frac{\partial}{\partial\bar\theta}\epsilon\frac{\partial}{\partial\bar\theta}\Phi_1^*(x_\mu - 2\mathrm{i}\theta\sigma_\mu\bar\theta,\bar\theta). \tag{21}$$

It is easy to check that its components are

$$A' = F^*, \qquad \psi' = -\mathrm{i}\sigma_\mu\frac{\partial}{\partial x_\mu}\bar\psi, \qquad F' = \Box A^*. \tag{22}$$

The equations of motion for a self-interacting scalar multiplet investigated in the papers of ref. [2] can be written in a manifestly supergauge invariant way as

$$-\tfrac{1}{4}\frac{\partial}{\partial\bar\theta}\epsilon\frac{\partial}{\partial\bar\theta}\Phi_1^*(x_\mu - 2\mathrm{i}\theta\sigma_\mu\bar\theta,\bar\theta) + m\Phi_1(x_\mu,\theta) + 2g\Phi_1^2(x_\mu,\theta) = 0. \tag{23}$$

The components of this superfield equation (plus those of the complex conjugate) give the equations of motion of the individual fields of the scalar multiplet.

According to (8), $\Phi_1(x_\mu + \mathrm{i}\theta\sigma_\mu\bar\theta,\bar\theta)$ is a vector multiplet constructed from the scalar multiplet $\Phi_1(x,\theta)$. It has been called previously the gradient of the scalar multiplet because its vector component is the gradient of the scalar A occurring in (19). Another example is the construction of a vector multiplet from two scalar multiplets $\Phi_1(x,\theta)$ and $\Phi_2(x,\bar\theta)$ of types one and two

$$\Phi_1(x_\mu + \mathrm{i}\theta\sigma_\mu\bar\theta,\theta)\,\Phi_2(x_\mu - \mathrm{i}\theta\sigma_\mu\bar\theta,\bar\theta). \tag{24}$$

As a special case one can take $\Phi_2 = \Phi_1^*$ and then (24) is real. This construction, together with the gradient operation described above, has been used extensively in ref. [3]. It is quite striking how the relations (7) and (8) of that paper can be proved almost trivially in the superfield formalism, while their proof using explicitly the fields of the multiplets is rather lengthy.

As further interesting examples we mention the scalar, the linear and the vector multiplet each with an undotted spinor index. Together with their complex conjugates, these superfields correspond to the following real field multiplets. The first, $\Phi_\alpha(x,\theta)$, contains two Majorana spinors, a scalar, a pseudoscalar and an antisymmetric tensor. One can impose on it the supergauge invariant constraint

$$-\tfrac{1}{4}\frac{\partial}{\partial\theta}\epsilon\frac{\partial}{\partial\theta}\Phi_\alpha(x_\mu + 2\mathrm{i}\theta\sigma_\mu\bar\theta,\theta) = \mathrm{i}(\sigma_\mu)_{\alpha\dot\beta}\frac{\partial}{\partial x_\mu}\bar\Phi^{\dot\beta}(x_\mu,\bar\theta), \tag{25}$$

which implies that the scalar is constant, the two spinors are related by the Dirac operator and the tensor is the curl of a vector. The product $\Phi_\alpha(x,\theta)\,\Phi^\alpha(x,\theta)$ of two such representations contains the free Lagrangian for a vector and for a Majorana spinor (see eq. (14) of ref. [1]). The second, $\Phi_\alpha(x,\theta,\bar\theta)$ (linear in $\bar\theta$) contains two Majorana spinors, a scalar, a pseudoscalar, two vectors, two pseudovectors, an antisymmetric tensor and a Majorana vector-spinor. This multiplet is contained in the decomposition of the product of two scalar multiplets. Finally, the third, $\Phi_\alpha(x,\theta,\bar\theta)$ contains four Majorana spinors, two scalars, two pseudoscalars, two vectors, two pseudovectors, two antisymmetric tensors and a Majorana vector-spinor. It is contained in the product of two vector multiplets.

The compactness of the superfield notation simplifies considerably calculations and proofs. We leave it to the reader to show that, given any two multiplets constructed with the operations described above, the product of any field of one by any field of the other can be expanded as a linear combination of fields belonging to some multiplets. The proof would be extremely tedious and almost unmanageable if one used the field multiplets explicitly.

[3] In the notation of ref. [1], the fields A and F would be called $(A - \mathrm{i}B)/2$ and $(F + \mathrm{i}G)/2$, respectively. The two-component spinor ψ consists of the upper components of the spinor ψ of ref. [1], in a representation in which γ_5 is diagonal

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