Nuclear Physics B221 (1983) 331-348

SUPERSYMMETRY IN SIX DIMENSIONS

P.S. HOWE

Department of Mathematics, Kings College, London, UK

G. SIERRA and P.K. TOWNSEND

Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, Paris, France*

Received 31 January 1983

We present off-shell formulations of six-dimensional supersymmetric matter and gauge theories in superspace. We construct the maximally supersymmetric YM theory in terms of $d = 6$ superfields. From various on-shell superfields we construct several versions of the irreducible 40 + 40 component spin-2 supercurrent, including the superconformal supercurrent. We also discuss extended supersymmetry in terms of six-dimensional extended superfields.

1. Introduction

The study of extended supersymmetry in four dimensions leads naturally to the study of simple supersymmetry in higher dimensional spacetimes, as dimensional reduction of the latter yields automatically the former. In particular, reduction of a supersymmetric field theory in $d = 6$ yields an $N = 2$ supersymmetric theory in $d = 4$. It has recently been shown that the maximally supersymmetric $N = 4$ super-Yang-Mills (YM) theory can be written in terms of $N = 2$ superfields [1] and that the non-renormalization theorems of $N = 2$ superfield perturbation theory [2] imply that it is ultraviolet finite [1,3]. The details of $N = 2$ superfields and $N = 2$ superfield perturbation theory are complicated and considerable simplification is achieved by passing to the $d = 6$ superfield formulation. In this article we shall derive and study some of the more useful supermultiplets of six-dimensional supersymmetry and develop some superfield techniques.

Six-dimensional superfields have been studied previously [4,5] principally in connection with $d = 6$ super YM theory. The constraints on the superspace field strengths F_{AB}, have been found, the Bianchi identities "solved", and a form of the action found as a sub-superspace integral [5]. As things stand, however, the results

* Laboratoire Propre du Centre National de la Recherche Scientifique, associé à l'Ecole Normale Supérieure et à l'Université de Paris Sud. Postal address: 24, rue Lhomond, 75231 Paris CEDEX 05, France.

for $d=6$ super YM theory are scarcely simpler than those for $N=2$ super YM in $d=4$. Considerable simplification can be achieved by use of $d=6$ SU(2) Majorana-Weyl spinors [6]. These are four-component complex spinors transforming also as a SU(2) doublet ($i=1,2$) and satisfying the "SU(2)-Majorana" condition

$$\left(\psi_\alpha^i\right)^* \stackrel{\text{def}}{=} \bar{\psi}_{\dot{\alpha} i} = \varepsilon_{ij} B_{\dot{\alpha}}{}^{\beta} \psi_\beta^j . \tag{1.1}$$

The matrix B is unitary and satisfies $B^*B = -1$. The use of these spinors leads to a manifestly SU(2) covariant formulation of $d=6$ supersymmetric field theories. In addition, the existence of the B matrix means that we need never consider dotted SU*(4) indices; undotted indices suffice. The supersymmetry algebra is

$$\left\{Q_\alpha^i, Q_\beta^j\right\} = \varepsilon^{ij} P_{\alpha\beta} , \tag{1.2}$$

where $P_{\alpha\beta} = (\Sigma^a)_{\alpha\beta} P_a$. The $(\Sigma^a)_{\alpha\beta}$, $a=1,2,\dots 6$, are the six 4×4 antisymmetric matrices. In practice one needs only the algebra of supercovariant derivatives D_α^i, which is

$$\left\{D_\alpha^i, D_\beta^j\right\} = i\varepsilon^{ij} \partial_{\alpha\beta} . \tag{1.3}$$

Our convention can be found in appendix A.

For some purposes we shall want to break SU(2) to U(1). In this case we make the following identifications

$$D_\alpha^1 = D_\alpha , \qquad D_\alpha^2 = \bar{D}_\alpha . \tag{1.4}$$

From the relation (1.1) we see that $\bar{D}_\alpha$ is related to the complex conjugate of D_α^1, $(D_\alpha^1)^* \stackrel{\text{def}}{=} \bar{D}_{\dot{\alpha}}$, by

$$\bar{D}_\alpha = -B^*{}_\alpha{}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}} . \tag{1.5}$$

The breaking of SU(2) allows the definition of the $d=6$ analogue of a chiral superfield, C, which satisfies

$$\bar{D}_\alpha C = 0 \Rightarrow D_\alpha \bar{C} = 0 . \tag{1.6}$$

As the complex conjugation of a spinor does not change its chirality in $d=6$, the condition (1.6) is really a Grassman analyticity condition [7], rather than a chirality condition. However, for simplicity, we will continue to refer to a field C satisfying (1.6) as a chiral superfield.

There are three principal off-shell multiplets that we shall discuss in this article. The first is the $d=6$ analogue of the "$N=2$ linear multiplet". It is an $8+8$

component multiplet that can describe super-matter (propagating spins $\leqslant \frac{1}{2}$). In SU(2) covariant notation it is described by a SU(2) **3**, scalar superfield L^{ij} while in SU(2) *non*-covariant notation it is described by a chiral superfield, C, both with additional constraints. The second multiplet is the $d=6$ super YM multiplet. In SU(2) non-covariant notation this was given in ref. [5]. We extend this work by giving the explicit solution to the superspace constraints in the linearized case. In SU(2) covariant notation the multiplet is described by an SU(2)-Majorana-anti-Weyl spinor superfield W_i^{α}, which can be expressed in terms of an SU(2) triplet prepotential V_{ij}, just as in $d=4$ [8]. The third multiplet we give in SU(2) covariant form only; it is a 24 + 24 component multiplet described by an SU(2) **5**, scalar, superfield L^{ijkl} with an additional constraint. As for the analogous $d=4N=2$ multiplet it may be used to construct a "relaxed hypermultiplet" which in turn is the basis for a $d=6$ supersymmetric form of the maximally supersymmetric YM theory. In fact, the SU(2) covariant formulation of $d=6$ supersymmetry is quite close to the SU(2) covariant $N=2$ supersymmetry in $d=4$, although considerably simpler. This is of use in the development of $d=6$ superfield perturbation theory, and the derivation of the non-renormalization theorems, as will be shown in a future publication. We do not discuss supergravity in this paper, although we do make a step in that direction by constructing various forms of the 40 + 40 irreducible spin-2 supercurrent that can be found as bilinears of on-shell $d=6$ superfields. In particular we construct a $d=6$ conformal supercurrent using an on-shell conformal antisymmetric tensor gauge multiplet.

In sect. 6 we discuss the extension of some of our results to extended $d=6$ supersymmetry.

2. The SU(2) non-covariant formalism and $d=6$ chiral fields

The chiral multiplet C is a 24 + 24 component multiplet, but is not irreducible. It can be further reduced by the condition

$$D^{3\alpha}C + i\partial^{\alpha\beta}\overline{D}_{\beta}\overline{C} = 0\,. \tag{2.1}$$

The derivative $D^{3\alpha}$ is the product of $3D$'s. We introduce the following notation for the products of D's:

$$D_{\alpha}D_{\beta} = D_{\alpha\beta}\,, \qquad D_{\alpha}D_{\beta\gamma} = -\varepsilon_{\alpha\beta\gamma\delta}D^{3\delta}\,, \qquad D_{\alpha}D^{3\beta} = \delta_{\alpha}^{\beta}D^{4}\,. \tag{2.2}$$

The components of the reduced chiral multiplet are

$$C\,, \qquad D_{\alpha}C\,, \qquad D_{\alpha\beta}C = \partial_{\alpha\beta}u + iV_{\alpha\beta}\,, \tag{2.3}$$

with $V_{\alpha\beta}$ further constrained by

$$\partial^{\alpha\beta} V_{\alpha\beta} = 0\,. \tag{2.4}$$

This constraint can be solved in terms of a fourth rank antisymmetric tensor gauge field. A geometrical formulation of this multiplet exists but we defer a discussion of this to the SU(2) covariant formulation in sect. 3.

From the reduced chiral superfield C we can construct an action for $d=6$ super-matter as follows

$$I = \int \mathrm{d}^6x\, \mathrm{d}^4\theta \{C^2\}\,. \tag{2.5}$$

Here, $\mathrm{d}^4\theta$ is the integration measure for the chiral superspace.

The $d=6$ YM multiplet and action in SU(2) non-covariant form has been discussed elsewhere [5], so we will be brief. The superspace field strengths $F_{AB} = (F_{\alpha\beta}, F_{\alpha\dot{\beta}}, F_{a\alpha}, F_{a\dot{\alpha}}, F_{ab})$ are constrained by $F_{\alpha\beta} = F_{\alpha\dot{\beta}} = 0$. (Here we need to keep the dotted index notation to avoid confusion.) From the Bianchi identities one finds $(F_{a\gamma} \sim F_{\alpha\beta,\gamma})$

$$F_{\alpha\beta,\gamma} = \varepsilon_{\alpha\beta\gamma\delta} \overline{W}^{\delta} \tag{2.6}$$

and the field strength superfield $\overline{W}^{\delta}$ has the components

$$\begin{aligned} &\mathfrak{D}_{\alpha} \overline{W}^{\beta} = \delta_{\alpha}^{\beta} \overline{X}\,, \\ &\mathfrak{D}_{\alpha} W^{\beta} = \delta_{\alpha}^{\beta} Y + M_{\alpha}{}^{\beta}\,, \qquad Y = \overline{Y}\,. \end{aligned} \tag{2.7}$$

$M_{\alpha}{}^{\beta}$ (equivalent to F_{ab}) is the YM field strength, while X, $\overline{X}$ and Y are auxiliary fields.

The constraint on $\overline{W}^{\alpha}$ is easily solved at the linearized level, in terms of a chiral prepotential superfield A, as

$$\overline{W}^{\alpha} = \tfrac{1}{2} i \left(D^{3\alpha} A + i \partial^{\alpha\beta} \overline{D}_{\beta} \overline{A} \right), \qquad \overline{D}_{\alpha} A = 0\,. \tag{2.8}$$

The last component of A is the auxiliary field X, while Y is actually the divergence of the vector $\frac{1}{2}(D_{\alpha\beta} A + \overline{D}_{\alpha\beta} \overline{A})$. The gauge transformation of A,

$$\delta A = B\,, \qquad \overline{D}_{\alpha} B = D^{3\alpha} B + i \partial^{\alpha\beta} \overline{D}_{\beta} B = 0\,, \tag{2.9}$$

removes the components of a reduced chiral multiplet.

The linearized action in terms of A is

$$I = \int \mathrm{d}^6x\, \mathrm{d}^4\theta \{A X(A)\}\,. \tag{2.10}$$

The equation of motion is $X=0$ and this implies $\partial_a Y=0$ ($D_a Y=0$ in the non-abelian case). This allows $Y=$ constant in the abelian case and if this constant is non-zero supersymmetry is spontaneously broken. (This is a variant of the Fayet-Illiopoulos mechanism which will be discussed later.)

It would be of interest to investigate whether the quantization of $d=6$ super YM theory can be set up in terms of the chiral prepotential A, but we shall not pursue this further here.

3. SU(2) covariant superfields in $d=6$

Spinor indices now come with an SU(2) index. We introduce the notation

$$
\begin{aligned}
D^i_\alpha D^j_\beta &= \tfrac{1}{2} i\varepsilon^{ij}\partial_{\alpha\beta} + \varepsilon^{ij} D_{\alpha\beta} + D^{ij}_{\alpha\beta}\,, \\
D_{\alpha\beta} &= \tfrac{1}{2} D_{\alpha i} D^{\ i}_{\beta} = D_{\beta\alpha}\,, \\
D^{ij}_{\alpha\beta} &= D^{(i}_{[\alpha} D^{j)}_{\beta]}\,,
\end{aligned} \tag{3.1}
$$

for the products of 2 D's, where vector indices are represented by a pair of antisymmetric spinor indices. We shall also use

$$
D^{ijkl} = D^{(i}_\alpha D^j_\beta D^k_\gamma D^{l)}_\delta \varepsilon^{\alpha\beta\gamma\delta}\,, \tag{3.2}
$$

for the Lorentz scalar product of four D's.

We shall first discuss the $d=6$ "linear multiplet". We introduce a superspace four-form and its five-form field strength, F. The constraints on F are

$$
F^{ijklm}_{\alpha\beta\gamma\delta\varepsilon} = F_{\alpha\alpha',\beta\gamma\delta\varepsilon}^{\quad\ jklm} = F_{\alpha\alpha',\beta\beta',\gamma\delta\varepsilon}^{\qquad\ ijk} = 0\,. \tag{3.3}
$$

The Bianchi identities for F now imply that

$$
F_{\alpha\alpha',\beta\beta',\gamma\gamma',\delta\delta'}^{\qquad\qquad\ \ ij} = \left(\varepsilon_{\alpha\alpha'\gamma\delta}\varepsilon_{\beta\beta'\gamma'\delta'} + \varepsilon_{\beta\beta'\alpha\delta}\varepsilon_{\gamma\gamma'\alpha'\delta'} + \varepsilon_{\gamma\gamma'\beta\delta}\varepsilon_{\alpha\alpha'\beta'\delta'}\right) L^{ij}\,, \tag{3.4}
$$

and that L^{ij} satisfies

$$
D^{(i}_\alpha L^{jk)} = 0\,. \tag{3.5}
$$

The notation in (3.4) indicates that the indices connected by an upper bracket are symmetrized, while those connected by a lower bracket are antisymmetrized. The components of L^{ij}, and their supersymmetry transformations are

$$D^i_\alpha L^{jk} = \varepsilon^{i(j}\lambda_\alpha{}^{k)}, \qquad \lambda^k_\alpha = \tfrac{2}{3} D_{\alpha i} L^{ik},$$

$$D^i_\beta \lambda^j_\alpha = \varepsilon^{ij} G_{\beta\alpha} - i\partial_{\beta\alpha} L^{ij}, \qquad G_{\beta\alpha} = \tfrac{1}{2} D_{[\beta i}\lambda_{\alpha]}{}^i,$$

$$D^k_\gamma G_{\beta\alpha} = -2i\partial_{\gamma[\beta}\lambda_{\alpha]}{}^k - \tfrac{1}{2}\partial_{\beta\alpha}\lambda_\gamma{}^k. \tag{3.6}$$

The component $G_{\alpha\beta}$ is conserved, $\partial^{\alpha\beta} G_{\alpha\beta} = 0$, and is the dual of the field strength F_{abcde}. The action for L^{ij} is the sub-superspace integral, (recall that $D \sim \partial/\partial\theta$ under a $\mathrm{d}^6 x$ integral),

$$I = \int \mathrm{d}^6 x \, D^{ijkl}\{L_{ij} L_{kl}\}. \tag{3.7}$$

For the super YM theory, we introduce the superspace gauge potential $\mathcal{A}_A$ via the covariant derivatives $\mathcal{D}_A = D_A + i[\mathcal{A}_A, \;]$, $A = (\alpha\alpha', {}^i_\alpha)$. $\mathcal{A}_A$ is in the adjoint representation of some compact group, G. The field strengths F_{AB} are given by the (anti) commutator

$$[\mathcal{D}_A, \mathcal{D}_B\} = iF_{AB} - t_{AB}{}^C \mathcal{D}_C. \tag{3.8}$$

$t_{AB}{}^C$ is the (flat space) torsion tensor. Its only non-vanishing component is

$$t^{ij}_{\alpha\beta}{}^{\gamma\delta} = -i\varepsilon^{ij}\delta^\gamma_{[\alpha}\delta^\delta_{\beta]}. \tag{3.9}$$

We impose the following constraint on F_{AB};

$$F^{ij}_{\alpha\beta} = 0. \tag{3.10}$$

From the Bianchi identities we then deduce that

$$F_{\alpha\beta,}{}^k_\gamma = \varepsilon_{\alpha\beta\gamma\delta} W^{\delta k}, \tag{3.11}$$

and that $W^{\delta k}$ is constrained by

$$\mathcal{D}^i_\alpha W^{\beta j} = M_\alpha{}^\beta \varepsilon^{ij} + \delta_\alpha{}^\beta X^{ij}, \qquad M_\alpha{}^\alpha = X^i_i = 0. \tag{3.12}$$

$M_\alpha{}^\beta$ can be identified as the ordinary YM field strength through

$$F_{\beta\beta',\gamma\gamma'} = 2i\varepsilon_{\gamma\gamma'\delta[\beta} M_{\beta']}{}^\delta, \tag{3.13}$$

(which is in fact antisymmetric in $\beta\beta' \leftrightarrow \gamma\gamma'$). The constraint (3.12) implies further that

$$\mathcal{D}^i_\alpha X^{jk} = i\varepsilon^{i(j}\mathfrak{d}_{\alpha\beta}W^{\beta k)},$$

$$\mathcal{D}^k_\gamma M_\alpha{}^\beta = -\tfrac{1}{2}i\delta^\beta_{(\gamma}\mathfrak{d}_{\alpha)\delta}W^{\delta k} - \tfrac{3}{2}i\delta^\beta_{[\gamma}\mathfrak{d}_{\alpha]\delta}W^{\delta k} - i\mathfrak{d}_{\gamma\alpha}W^{\beta k} \tag{3.14}$$

and that $M_\alpha{}^\beta$ satisfies its Bianchi identity,

$$\mathfrak{d}_{\beta(\alpha}M_{\gamma)}{}^\beta = 0. \tag{3.15}$$

Notice that we use the notation $\mathfrak{d}_{\beta\alpha}$ for a YM covariantized $\partial_{\beta\alpha}$ so as not to confuse it with $\mathcal{D}_{\beta\alpha} = \mathcal{D}_{\alpha\beta}$, a product of two supercovariant derivatives. The superspace action for W_i^α is

$$I = \int \mathrm{d}^6x\, D^{ij}_{\alpha\beta}\{\mathrm{Tr}(W_i^\alpha W_j^\beta)\}, \tag{3.16}$$

with the trace over YM indices. The SU(2) triplet X_{ij} occurs in the action as $X_{ij}X^{ij}$ and is the auxiliary field.

To find the prepotential form of the action, which is a necessary step in the development of superfield perturbation theory, we must examine the constraints on F_{AB} in more detail. The constraint $F^{ij}_{\alpha\beta} = 0$ reads

$$D^i_\alpha \mathcal{A}^j_\beta + D^j_\beta \mathcal{A}^i_\alpha + i\{\mathcal{A}^i_\alpha, \mathcal{A}^j_\beta\} - i\varepsilon^{ij}\mathcal{A}_{\alpha\beta} = 0. \tag{3.17}$$

The antisymmetric part in ij is a conventional constraint determining $\mathcal{A}_{\alpha\beta}$;

$$\mathcal{A}_{\alpha\beta} = -iD_{[\alpha j}\mathcal{A}_{\beta]}{}^j + \tfrac{1}{2}\{\mathcal{A}_{\alpha j}, \mathcal{A}_\beta{}^j\}, \tag{3.18}$$

but the symmetric part is a differential constraint on $\mathcal{A}^i_\alpha$:

$$D^{(i}_{(\alpha}\mathcal{A}^{j)}_{\beta)} + \tfrac{1}{2}i\{\mathcal{A}^{(i}_\alpha, \mathcal{A}^{j)}_\beta\} = 0. \tag{3.19}$$

It will be shown in a forthcoming article how this constraint may be solved [3]. The solution is based on that of the linearized constraint, which we give here. The constraint is

$$D^{(i}_{(\alpha}\mathcal{A}^{j)}_{\beta)} = 0, \tag{3.20}$$

and its solution is

$$\mathcal{A}^i_\alpha = D^i_\alpha u + D_{\alpha j}D^{ijkl}V_{kl}, \tag{3.21}$$

which can be established by using the identity

$$D_{\alpha\beta} D^{ijkl} = D^{ijkl} D_{\alpha\beta} = 0 . \tag{3.22}$$

The U superfield is a pure gauge transformation and may be set to zero. The SU(2) triplet V_{ij} is the prepotential and is subject to the pregauge transformation

$$\delta V_{ij} = D_\alpha{}^k \Lambda^\alpha{}_{kij}, \qquad \Lambda^\alpha{}_{kij} = \Lambda^\alpha{}_{(kij)}, \tag{3.23}$$

which is very similar to the $d=4$ case [8]. One can, by breaking SU(2) and suitably fixing gauges, recover the non-covariant chiral formalism presented in sect. 2.

The linearized action in terms of the prepotential is

$$I_{\text{lin}} = \int \mathrm{d}^6 x\, \mathrm{d}^8 \theta \left\{ V_{ij} X^{ij}(V) \right\}, \tag{3.24}$$

now a full superspace integral. The field equation is $X^{ij} = 0$. For the abelian case one can add to the action an SU(2)-violating $N=2$ Fayet-Illiopoulos term

$$I_{\text{FI}} = \int \mathrm{d}^6 x\, \mathrm{d}^8 \theta \left\{ V_{ij} \xi^{ij} \right\}, \qquad \xi^{ij} = \text{constant}. \tag{3.25}$$

To conclude this section we discuss the SU(2) **5** superfield L^{ijkl} $(= L^{(ijkl)})$ with the constraint

$$D_\alpha{}^{(i} L^{jklm)} = 0 . \tag{3.26}$$

Its components and transformation laws are

$$D_\alpha{}^p L^{ijkl} = \tfrac{4}{5} \varepsilon^{p(i} \psi_\alpha{}^{jkl)}, \qquad \psi_\alpha{}^{jkl} = D_{\alpha p} L^{pjkl},$$

$$D_\beta{}^m \psi_\alpha{}^{jkl} = \tfrac{3}{4} \varepsilon^{m(j} G^{kl)}_{\beta\alpha} - \tfrac{5}{4} i \partial_{\beta\alpha} L^{mjkl}, \qquad G^{kl}_{\beta\alpha} = D_{\beta p} \psi_\alpha{}^{klp},$$

$$D_\gamma{}^i G^{kl}_{\beta\alpha} = \tfrac{1}{9} \varepsilon_{\gamma\beta\alpha\delta} \varepsilon^{i(k} \xi^{\delta l)} - \tfrac{1}{3} i \partial_{\beta\alpha} \psi^{ikl}_\gamma - \tfrac{8}{3} i \partial_{\gamma[\beta} \psi_{\alpha]}{}^{ikl}, \qquad \xi^{\delta l} = \varepsilon^{\beta\alpha\gamma\delta} D_{\gamma p} G^{pl}_{\beta\alpha},$$

$$D_\varepsilon{}^m \xi^{\delta l} = \tfrac{1}{8} \varepsilon^{ml} \delta_\varepsilon{}^\delta C + \tfrac{9}{2} i \partial^{\beta\alpha} G^{ml}_{\beta\alpha} \delta^\delta_\varepsilon - 6 i \partial^{\gamma\delta} G^{ml}_{\gamma\varepsilon}, \qquad C = D_{\sigma p} \xi^{\sigma p},$$

$$D_\alpha{}^i C = -4 i \partial_{\alpha\beta} \xi^{\beta i}. \tag{3.27}$$

4. The $d=6$ relaxed hypermultiplet and YM coupling

The coupling of L^{ij} and L^{ijkl} to $d=6$ YM is straightforward; the constraints are simply replaced by gauge covariant ones. We shall proceed directly to the combined "relaxed hypermultiplet" [1], for which the $d=6$ constraints are

$$\mathcal{D}_\alpha^{(i} L^{jk)} = \mathcal{D}_{\alpha m} L^{ijkm}, \qquad \mathcal{D}_\alpha^{(m} L^{ijkl)} = 0. \tag{4.1}$$

In ref. [1], in order to construct the $N=4$ YM theory in $N=2$, $d=4$, superfields, another Lorentz and SU(2) scalar superfield, was needed. In $d=6$ the analogous superfield, S, satisfies

$$\mathcal{D}_{\alpha\beta} S = 0, \tag{4.2}$$

which is simpler than the $d=4$ constraint. The products of two $\mathcal{D}$'s are given by the covariantization of (3.1)

$$\mathcal{D}_\alpha^i \mathcal{D}_\beta^j = \tfrac{1}{2} i \mathfrak{D}_{\alpha\beta} \varepsilon^{ij} + \mathcal{D}_{\alpha\beta} \varepsilon^{ij} + \mathcal{D}_{\alpha\beta}^{ij}, \tag{4.3}$$

hence

$$\mathcal{D}_{\alpha\beta} = \tfrac{1}{2} \mathcal{D}_{(\alpha i} \mathcal{D}_{\beta)}^{i}.$$

Because of the absence of scalar fields in the $d=6$ YM multiplet the solutions of the constraints (4.1) and (4.2) are particularly simple:

$$\begin{aligned} L^{ijkl} &= -\tfrac{3}{8} \mathcal{D}^{ijkl} \left(\mathcal{D}_\alpha{}^m \rho^\alpha{}_m \right), \\ L^{ij} &= \mathcal{D}^{ijkl} \mathcal{D}_{\alpha k} \rho^\alpha{}_l, \\ S &= \mathcal{D}^{ijkl} X_{ijkl}. \end{aligned} \tag{4.4}$$

To establish this result one uses the YM covariant analogue of (3.22)

$$\begin{aligned} \mathcal{D}_{\alpha\beta} \mathcal{D}^{ijkl} &= \mathcal{D}^{ijkl} \mathcal{D}_{\alpha\beta} = 0, \\ \mathcal{D}^{ijkl} &= \mathcal{D}_\alpha^{(i} \mathcal{D}_\beta^j \mathcal{D}_\gamma^k \mathcal{D}_\delta^{l)} \varepsilon^{\alpha\beta\gamma\delta}. \end{aligned} \tag{4.5}$$

The maximally supersymmetric YM theory in $d=6$ (that which reduces to the $N=4$ theory in $d=4$) is now given by the sum of three independent $d=6$ superinvariants I_0, I_1, I_2. I_0 is the $d=6$ super YM action (3.16), while I_1 and I_2 are

$$\begin{aligned} I_1 &= \int \mathrm{d}^8\theta\, \mathrm{d}^6 x \left\{ \lambda_\alpha{}^i(\rho) \rho^\alpha{}_i \right\}, \qquad \lambda_\alpha{}^i = \mathcal{D}_{\alpha j} L^{ij}, \\ I_2 &= \int \mathrm{d}^8\theta\, \mathrm{d}^6 x \left\{ L^{ijkl}(\rho) X_{ijkl} \right\}. \end{aligned} \tag{4.6}$$

Thus all the results of ref. [1] can be recast in $d=6$.

To conclude this section we give the complete transformation rules for the combined L^{ij}, L^{ijkl}, "relaxed hypermultiplet":

$$\mathcal{D}^i_\alpha L^{jk} = \varepsilon^{i(j}\lambda^{k)}_\alpha + \psi^{ijk}_\alpha ,$$

$$\mathcal{D}^i_\beta \lambda^j_\alpha = \varepsilon^{ij} G_{\beta\alpha} - i\eth_{\beta\alpha} L^{ij} + G^{ij}_{\beta\alpha} ,$$

$$\mathcal{D}^k_\gamma G_{\beta\alpha} = -2i\eth_{\gamma[\beta}\lambda_{\alpha]}{}^k - \frac{i}{2}\eth_{\beta\alpha}\lambda^k_\gamma - \varepsilon_{\gamma\beta\alpha\delta}\left[W^\delta_i, L^{ik}\right] + \tfrac{1}{6}\varepsilon_{\gamma\beta\alpha\delta}\xi^{\delta k} ,$$

$$\mathcal{D}^p_\alpha L^{ijkl} = \tfrac{4}{5}\varepsilon^{p(i}\psi_\alpha{}^{jkl)} ,$$

$$\mathcal{D}^m_\beta \psi^{jkl}_\alpha = \tfrac{3}{4}\varepsilon^{m(j}G^{kl)}_{\beta\alpha} - \tfrac{5}{4}i\eth_{\beta\alpha}L^{mjkl} ,$$

$$\mathcal{D}^i_\gamma G^{kl}_{\beta\alpha} = \tfrac{1}{9}\varepsilon_{\gamma\beta\alpha\delta}\varepsilon^{i(k}\xi^{\delta l)} - \tfrac{1}{3}i\eth_{\beta\alpha}\psi^{ikl}_\gamma - \tfrac{8}{3}i\eth_{\gamma[\beta}\psi^{kli}_{\alpha]}$$
$$- \tfrac{5}{3}\varepsilon_{\gamma\beta\alpha\delta}\left[W^\delta{}_p, L^{iklp}\right] ,$$

$$\mathcal{D}^m_\varepsilon \xi^{\delta l} = \tfrac{1}{8}\varepsilon^{ml}\delta^\delta_\varepsilon C + \tfrac{9}{2}i\delta^\delta_\varepsilon \eth^{\beta\alpha}G^{ml}_{\beta\alpha} - 6i\eth^{\gamma\delta}G^{ml}_{\gamma\varepsilon}$$
$$-6\left\{W^\delta_p, \psi^{pml}_\varepsilon\right\} + 12\delta^\delta_\varepsilon\left\{W^\sigma_p, \psi^{pml}_\sigma\right\} - 15\delta^\delta_\varepsilon\left[X_{pq}, L^{mlpq}\right] ,$$

$$C = -12i\eth^{\beta\alpha}G_{\beta\alpha} - 24\left\{W^\alpha_i, \lambda^i_\alpha\right\} + 24\left[X_{ij}, L^{ij}\right]. \tag{4.7}$$

5. The supercurrent and superconformal invariance

The lagrangian superfield,

$$J^{\alpha\beta}_{ij} = \mathrm{Tr}\left\{W^{[\alpha}_{(i}W^{\beta]}_{j)}\right\} , \tag{5.1}$$

used in the construction of the YM action in (3.16) is also the spin-2 supercurrent superfield. As a consequence of the YM superfield equations, $J^{\alpha\beta}_{ij}$ satisfies

$$D_{\gamma(k}J^{\alpha\beta}_{ij)} = 0 . \tag{5.2}$$

The components and transformation rules are

$$D_\gamma{}^k J^{\alpha\beta}_{ij} = \varepsilon^k{}_{(i}S_\gamma{}^{\alpha\beta}{}_{j)} ,$$

$$D_{\delta i}S_\gamma{}^{\alpha\beta}{}_j = -i\partial_{\delta\gamma}J^{\alpha\beta}_{ij} + \varepsilon_{ij}T_{\gamma\delta}{}^{\alpha\beta} + \varepsilon_{ij}\delta^{[\alpha}_{[\delta}K_{\gamma]}{}^{\beta]} ,$$

$$D_{\varepsilon i}T_{\gamma\delta}{}^{\alpha\beta} = -\tfrac{1}{2}\left(4i\partial_{\varepsilon[\delta}S_{\gamma]}{}^{\alpha\beta}{}_i + i\partial_{\delta\gamma}S_\varepsilon{}^{\alpha\beta}{}_i\right)$$
$$-\delta^{[\alpha}_{[\delta}\left(2i\partial_{\gamma]\varepsilon}S_\tau{}^{\beta]\tau}{}_i + 2i\partial_{|\varepsilon\tau|}S_{\gamma]}{}^{\beta]\tau}{}_i - i\partial_{\gamma]\tau}S_\varepsilon{}^{\beta]\tau}{}_i\right)$$
$$-i\delta^{[\alpha}_{[\delta}\delta^{\beta]}_{\gamma]}\left(\partial_{\varepsilon\sigma}S_\tau{}^{\sigma\tau}{}_i\right) ,$$

$$D_{\alpha i}K_\gamma{}^\beta = -\left(4i\partial_{\alpha[\gamma}S_{\tau]}{}^{\beta\tau}{}_i + i\partial_{\gamma\tau}S_\alpha{}^{\beta\tau}{}_i\right) + i\delta_\gamma{}^\beta\left(\partial_{\alpha\sigma}S_\tau{}^{\sigma\tau}{}_i\right). \tag{5.3}$$

The specific realization of this supercurrent multiplet as a bilinear of $d=6$ super Maxwell fields is,

$$
\begin{aligned}
J^{\alpha\beta}_{ij} &= W^{[\alpha}_{(i}W^{\beta]}_{j)}\,,\\
S_{\gamma}{}^{\alpha\beta}{}_{j} &= 2M_{\gamma}{}^{[\alpha}W^{\beta]}_{j}\,,\\
T_{\gamma\delta}{}^{\alpha\beta} &= 2\Big(M^{[\alpha}_{[\gamma}M_{\delta]}{}^{\beta]} + \delta^{[\alpha}_{[\gamma}M_{|\varepsilon|}{}^{\beta]}M_{\delta]}{}^{\varepsilon} - \tfrac{1}{4}\delta^{[\alpha}_{[\gamma}\delta^{\beta]}_{\delta]}M_{\varepsilon}{}^{\sigma}M_{\sigma}{}^{\varepsilon}\\
&\quad + \tfrac{1}{2}i\Big[W^{[\alpha k}\partial_{\delta\gamma}W^{\beta]}{}_{k} - \delta^{[\alpha}_{[\gamma}W^{|\varepsilon|k}\partial_{\delta]\varepsilon}W^{\beta]}{}_{k}\Big]\Big)\,,\\
K_{\gamma}{}^{\beta} &= 2\Big(M_{\varepsilon}{}^{\beta}M_{\gamma}{}^{\varepsilon} - \tfrac{1}{4}\delta^{\beta}_{\gamma}M_{\varepsilon}{}^{\sigma}M_{\sigma}{}^{\varepsilon} - \tfrac{1}{2}i\partial_{\gamma\varepsilon}\big[W^{\varepsilon k}W^{\beta}{}_{k}\big]\Big)\,.
\end{aligned}
\tag{5.4}
$$

The component $J^{\alpha\beta}_{ij}$ is the triplet of conserved SU(2) currents. $s_{\gamma}{}^{\alpha\beta}{}_{j} \sim s_{\gamma}{}^{a}{}_{j}$ is the supersymmetry current, $T_{\gamma\delta}{}^{\alpha\beta} \sim T_{ab}$ is the energy-momentum tensor, and $K_{\alpha}{}^{\beta} \sim K_{[ab]}$ is an identically conserved antisymmetric tensor current. Since the super-Maxwell theory in $d=6$ is not superconformal invariant the γ trace of the supersymmetry current, $s_{[\gamma\alpha\beta]j}$, and the trace of the energy-momentum tensor, $T_{\alpha\beta}{}^{\alpha\beta}$, do not vanish. The multiplet has $40+40$ components. One might suppose that $s_{[\gamma\alpha\beta]j}$ and $T_{\alpha\beta}{}^{\alpha\beta}$ would form a smaller submultiplet, the "trace" or "anomaly" supermultiplet, as in $d=4$. In this case one would have a smaller superconformal spin-2 supercurrent multiplet for which $\gamma \cdot s = T_{a}{}^{a} = 0$ with *fewer* than $40+40$ components. However, it is known that the non-vanishing of $T_{a}{}^{a}$ does not imply reducibility [9] and indeed the multiplet of (5.3) is irreducible. This is borne out by the fact that the superconformal spin-2 supercovariant multiplet constructed below *also* has $40+40$ components.

None of the supersymmetric field theories in $d=6$ considered so far is superconformally invariant. An example of a superconformally invariant theory is provided by a real scalar superfield, ϕ, satisfying

$$D^{ij}_{\alpha\beta}\phi = 0\,. \tag{5.5}$$

The components and transformation rules of ϕ are

$$
\begin{aligned}
D^{i}_{\alpha}\phi &= \lambda^{i}{}_{\alpha}\,,\\
D^{j}_{\beta}\lambda^{i}{}_{\alpha} &= \tfrac{1}{2}i\varepsilon^{ji}\partial_{\beta\alpha}\phi + \varepsilon^{ji}F_{\beta\alpha}\,, \qquad F_{\beta\alpha} = \tfrac{1}{2}D_{(\beta k}\lambda_{\alpha)}{}^{k}\,,\\
D^{i}_{\alpha}F_{\beta\gamma} &= -i\partial_{\alpha(\beta}\lambda^{i}{}_{\gamma)}\,.
\end{aligned}
\tag{5.6}
$$

$F_{\alpha\beta}$ is the self-dual field strength for a two-index antisymmetric tensor gauge potential. The constraint (5.5) implies the field equations for ϕ, $\lambda^{i}{}_{\alpha}$, and $F_{\alpha\beta}$. There is no Lorentz invariant off-shell extension of this model because there is no way to implement the self-duality constraint in an action without propagating additional modes [10]. For present purposes this fact is of no consequence as we shall use only the on-shell multiplet ϕ to construct a superconformal spin-2 supercurrent.

In abstract form the superconformal spin-2 supercurrent is a real scalar superfield J with the constraint

$$D^{3\alpha ijk}J = 0\,. \tag{5.7}$$

Its components and transformation rules are

$$\begin{aligned}
&D_\beta^j J = \psi_\beta^j\,,\\
&D_\beta^j \psi_\alpha^i = \tfrac{1}{2} i\varepsilon^{ij}\partial_{\beta\alpha}J + V_{\beta\alpha}^{ji} + \varepsilon^{ji}C_{\beta\alpha}\,, \qquad \begin{pmatrix} V_{\beta\alpha}^{ji} = V_{[\beta\alpha]}^{(ji)} \\ C_{\beta\alpha} = C_{(\beta\alpha)} \end{pmatrix},\\
&D_\gamma{}^k V_{\beta\alpha}^{ij} = \tfrac{4}{5} i\varepsilon^{k(i}\partial_{\gamma[\beta}\psi_{\alpha]}^{j)} + \tfrac{1}{5} i\varepsilon^{k(i}\partial_{\beta\alpha}\psi_\gamma{}^{j)} + \varepsilon^{k(i}S_{\gamma\beta\alpha}{}^{j)}\,,\\
&D_\gamma{}^i C_{\beta\alpha} = -\tfrac{4}{5} i\partial_{\gamma(\beta}\psi_{\alpha)}{}^i + S_{(\alpha\beta)\gamma}{}^i\,, \qquad \begin{pmatrix} S_{\gamma\beta\alpha}{}^j = S_{\gamma[\beta\alpha]}{}^j \\ S_{[\gamma\beta\alpha]}{}^j = 0 \end{pmatrix},\\
&D_\delta^j S_{\gamma\beta\alpha}{}^i = \tfrac{2}{3} i\partial_{\delta\gamma}V_{\alpha\beta}^{ij} + \tfrac{2}{15} i\partial_{\beta\alpha}V_{\delta\gamma}^{ij} + \tfrac{2}{15} i\partial_{\gamma[\beta}V_{\alpha]\delta}^{ji} + \tfrac{2}{3} i\partial_{\delta[\beta}V_{\alpha]\gamma}^{ji}\\
&\qquad + \varepsilon^{ji}\Big[i\partial_{\delta[\alpha}C_{\beta]\gamma} - \tfrac{1}{5} i\partial_{\gamma[\alpha}C_{\beta]\delta} - \tfrac{1}{5} i\partial_{\beta\alpha}C_{\delta\gamma}\Big]\\
&\qquad + \tfrac{1}{2}\varepsilon^{ji}T_{\delta\gamma\beta\alpha}\,, \qquad \big(T_{\delta\gamma\beta\alpha} = T_{[\delta\gamma][\beta\alpha]} = T_{\beta\alpha\delta\gamma}\,, \qquad T_{[\delta\gamma\beta]\alpha} = 0\big)\,,\\
&D_\varepsilon{}^i T_{\delta\gamma\beta\alpha} = -\tfrac{2}{3} i\partial_{\delta\alpha}S_{\varepsilon\beta\gamma}{}^i - 2i\partial_{\varepsilon[\delta}S_{\gamma]\beta\alpha}{}^i - 2i\partial_{\varepsilon[\beta}S_{\alpha]\delta\gamma}{}^i\\
&\qquad - \tfrac{1}{3} i\partial_{\delta\gamma}S_{\varepsilon\beta\alpha}{}^i - \tfrac{1}{3} i\partial_{\beta\alpha}S_{\varepsilon\delta\gamma}{}^i\,.
\end{aligned} \tag{5.8}$$

$V_{\alpha\beta}^{ij}$ are the conserved SU(2) currents; $S_{\gamma\beta\alpha}{}^i$ is the γ traceless conserved spinor current and $T_{\alpha\beta\gamma\delta}$ is the traceless and conserved energy-momentum tensor.

The specific realization of this multiplet as a bilinear in ϕ of (5.5) is as follows:

$$\begin{aligned}
&J = \tfrac{1}{2}\phi^2\,,\\
&K_\alpha^i = \phi\lambda^i{}_\alpha\,,\\
&V_{\beta\alpha}^{ji} = \lambda^{(j}_{[\beta}\lambda^{i)}_{\alpha]}\,,\\
&C_{\beta\alpha} = \tfrac{1}{2}\lambda_{(\beta k}\lambda_{\alpha)}{}^k + \phi F_{\beta\alpha}\,,\\
&S_{\gamma\beta\alpha}{}^i = 2F_{\gamma[\beta}\lambda_{\alpha]}^i + \tfrac{1}{5} i\big(\partial_{\gamma[\beta}\phi\lambda^i{}_{\alpha]} - \partial_{\beta\alpha}\phi\lambda^i{}_\gamma + \phi\partial_{\beta\alpha}\lambda^i{}_\gamma\big)\,,\\
&T_{\delta\gamma\beta\alpha} = 4F_{\gamma[\beta}F_{\alpha]\delta} + i\big(\lambda^i{}_{[\alpha}\partial_{|\delta\gamma|}\lambda_{\beta]i} + \lambda^i{}_{[\delta}\partial_{|\alpha\beta|}\lambda_{\gamma]i}\big)\,,\\
&\qquad + \tfrac{1}{5}\big[(\partial_{\gamma[\beta}\phi)(\partial_{\alpha]\delta}\phi) + (\partial_{\beta\alpha}\phi)(\partial_{\delta\gamma}\phi) - \phi\partial_{\beta\alpha}\partial_{\delta\gamma}\phi\big]\,.
\end{aligned} \tag{5.9}$$

The contragredient multiplet to J is a gauge multiplet H with gauge transformation $\delta H = D^{3\alpha ijk}\Lambda_{\alpha ijk}$. Its fields are those of linearized $d=6$ conformal supergravity.

When gauge fields are present, as in the $d=6$ super-Maxwell theory and the above model described by (5.5), the form of the supercurrent is fixed, viz. $J_{ij}^{\alpha\beta}$ in the former case and J in the latter. There is a $d=6$ on-shell superfield description of the hypermultiplet, which consists of only scalars and spinors, and for which either supercurrent is possible. The hypermultiplet superfield is a scalar SU(2) doublet transforming as a doublet under an additional "external" SU(2), ϕ^{Ii}. It satisfies the constraint

$$D_\alpha^j\phi^{Ii} = \varepsilon^{ji}\psi_\alpha{}^I . \tag{5.10}$$

The transformation rules are given by (5.10) and

$$D_\beta^j\psi_\alpha^I = -i\partial_{\beta\alpha}\phi^{Ij}, \tag{5.11}$$

and the constraint (5.10) also implies the field equations for ϕ^{Ii} and $\psi_\alpha{}^I$. The interacting version of this model is the $d=6$ hyper-Kähler σ model and will be described elsewhere [11]. Both of the above two supercurrents may be constructed as a bilinear in ϕ^{Ii}. They are

$$J_{ij}^{\alpha\beta} = i\phi^I{}_{(i}\partial^{\alpha\beta}\phi_{Ij)}, \qquad J = \tfrac{1}{2}\phi^{Ii}\phi_{Ii} . \tag{5.12}$$

Indeed, given the superconformal multiplet J of (5.12) one can rearrange it to give the non-superconformal multiplet $J_{ij}^{\alpha\beta}$.

Both of the above versions of the irreducible $40+40$ component supercurrent contain the currents of SU(2). From the non-SU(2) covariant version of the YM multiplet we can construct yet another version of the supercurrent which contains only a U(1) conserved current. The supercurrent superfield is

$$J^{\alpha\beta} = W^\alpha\overline{W}^\beta + \overline{W}^\alpha W^\beta . \tag{5.13}$$

Its components are $J^{(\alpha\beta)}$, $J^{[\alpha\beta]}$ (conserved), $S_{\gamma[\alpha\beta]}$ (the conserved, but not traceless supersymmetry current), $T_{\alpha\beta\gamma\delta}$ (the energy-momentum tensor), and the identically conserved antisymmetric tensor current $K_\alpha{}^\beta$.

We mention that a useful tool in working out the components of a superfield is a variant of the tableau calculus [13], which we outline in appendix B.

6. Extended supersymmetry in $d = 6$

There are several possible extensions of the $d=6$ superalgebra from the simple ($N=2$) algebra of (1.2) to an $N=4$ algebra. The four supercharges may be taken to have the same chirality or the extra two may be taken to have the opposite chirality.

In the first case we have an USp(4) invariant algebra

$$\{Q_\alpha{}^i, Q_\beta{}^j\} = \Omega^{ij} P_{\alpha\beta}, \qquad i, j = 1,2,3,4, \tag{6.1}$$

with Ω^{ij} the invariant "metric" of USp(4). In the second case we have the $SU(2) \times SU(2)$ invariant algebra

$$\begin{aligned}
&\{Q_\alpha{}^i, Q_\beta{}^j\} = \varepsilon^{ij} P_{\alpha\beta}, \qquad && i, j = 1,2, \\
&\{Q_\alpha{}^i, Q^\beta{}_{i'}\} = 0, && \\
&\{Q^\alpha{}_{i'}, Q^\beta{}_{j'}\} = \varepsilon_{i'j'} P^{\alpha\beta}, && i'j' = 1,2.
\end{aligned} \tag{6.2}$$

If one identifies the two SU(2)'s in (6.2) a central charge would be allowed in the algebra through $\{Q_\alpha{}^i, Q^\beta{}_j\} = \delta_\alpha{}^\beta \delta^i{}_j Z$, and if this SU(2) is also dropped one can include four central charges. We shall not consider algebras with central charges further here.

In order to describe $N=4$ super YM theory in $d=6$ extended superfields one is forced to the algebra (6.2) rather than (6.1). The constraints on the superspace field strengths F_{AB} are

$$F_{\alpha\beta}^{ij} = F_{i'j'}^{\alpha\beta} = 0, \qquad F_{\alpha j'}^{i\beta} = \delta_\alpha{}^\beta W^i{}_{j'} \tag{6.3}$$

and by the Bianchi identities $W^i{}_{j'}$ satisfy

$$\mathfrak{D}_\beta^{(j} W^{i)}{}_{j'} = \mathfrak{D}_{(i'}^\alpha W^i{}_{j')} = 0 \tag{6.4}$$

and the theory is on-shell. The components are $W^i{}_{j'}$, $\lambda_{\alpha j}$, $\lambda^\alpha{}_i$, and the YM field strength $M_\alpha{}^\beta$.

On the other hand the $N=4$ superconformal antisymmetric tensor model requires the algebra (6.1). It is described by a Lorentz scalar, USp(4) **5**, superfield $\phi^{ij} = -\phi^{ji}$ satisfying the constraint

$$D_\alpha{}^i \phi^{jk} = \Omega^{i[j} \lambda_\alpha{}^{k]} + \tfrac{1}{4} \Omega^{jk} \lambda^i_\alpha, \tag{6.5}$$

and has components ϕ^{ij}, λ^i_α and $F_{\alpha\beta} = \Omega_{ij} D_{(\alpha}{}^i \lambda_{\beta)}{}^j = F_{\beta\alpha}$.

From these two on-shell multiplets we can construct two versions of the $N=4$ 128 + 128 irreducible spin-2 supercurrent [12, 9]. From the YM superfield $W^i{}_{i'}$ we construct the supercurrent superfield

$$J_{i'j'}^{ij} = W_{(i'}^{(i} W_{j')}^{j)}. \tag{6.6}$$

Its components are

$$J^{(ij)}_{(i'j')}, \quad \chi^{(ij)}_{\alpha\ k'}, \quad \chi^{\alpha}{}_{(i'j')}{}^{k}, \quad V^{(ij)}_{[\alpha\beta]}, \quad V^{[i'j']}_{[\alpha\beta]}, \quad N^{i\beta}_{\alpha j'},$$

$$M^{i}{}_{j'}, \quad S^{i}_{\alpha\beta\gamma}, \quad S^{\alpha}_{i'\beta\gamma}, \quad T_{\alpha\beta\gamma\delta}, \quad K_{\alpha}{}^{\beta}. \tag{6.7}$$

The V's are the SU(2)$\times$SU(2) currents, N is traceless and non-conserved, the S's are the conserved, but not γ-traceless supersymmetry currents, T is the energy-momentum tensor and K an identically conserved antisymmetric tensor.

The conformal supercurrent superfield is constructed from the ϕ^{ij} superfield and is

$$J_{ijkl} = \left(\phi_{ij}\phi_{kl}\right)_{\mathbf{14}}. \tag{6.8}$$

Its components are

$$J_{ijkl}, \quad \chi_{\alpha i,\, jk}(\mathbf{16}), \quad G^{[ij]}_{(\alpha\beta)}, \quad V^{(ij)}_{[\alpha\beta]}, \quad S_{\alpha\beta\gamma}{}^{j}, \quad T_{\alpha\beta\gamma\delta}, \tag{6.9}$$

where $V^{ij}_{\alpha\beta}$ are the OSp(4) conserved currents, S is the γ traceless supersymmetry current, and T is the energy-momentum tensor. The G's are five self-dual third rank tensors.

Note added

After completion of this work we received a preprint, ref. [14], containing many of the results presented in sects. 3 and 4.

Appendix A

CONVENTIONS

SU(2) indices are raised and lowered as follows:

$$V^{i} = \varepsilon^{ij} V_{j}, \qquad V_{i} = V^{j} \varepsilon_{ji}. \tag{A.1}$$

Pairs of antisymmetric SU(4) indices are raised and lowered with the ε tensor:

$$V^{\alpha\beta} = \tfrac{1}{2}\varepsilon^{\alpha\beta\gamma\delta} V_{\gamma\delta}, \qquad V_{\alpha\beta} = \tfrac{1}{2}\varepsilon_{\alpha\beta\gamma\delta} V^{\gamma\delta}. \tag{A.2}$$

The Γ matrices are

$$\Gamma^a = \begin{pmatrix} 0 & (\Sigma^a)_{\alpha\dot\beta} \\ (\tilde\Sigma^a)^{\dot\alpha\beta} & 0 \end{pmatrix}, \tag{A.3}$$

$$(\Sigma^a)_{\alpha\dot\beta} = (1, \Sigma^i), \qquad \Sigma^i \text{ hermitian},$$

$$(\tilde\Sigma^a)^{\dot\alpha\beta} = (1, -\Sigma^i), \tag{A.4}$$

$$(\Sigma^a)_{\alpha\beta} = -B_\beta{}^{\dot\beta}(\Sigma^a)_{\alpha\dot\beta} = -(\Sigma^a)_{\beta\alpha}. \tag{A.5}$$

The B matrix can be chosen to be real, $B^2 = -1$. The Σ's satisfy

$$(\Sigma^a)_{\alpha\beta}(\Sigma^b)^{\alpha\beta} = -4\mu^{ab}, \qquad \mu = \mathrm{diag}(+,-,-,-,-,-),$$

$$(\Sigma^a)_{\alpha\beta}(\Sigma_a)_{\gamma\delta} = -2\varepsilon_{\alpha\beta\gamma\delta}. \tag{A.6}$$

Vector indices may be converted to spinor indices by means of the Σ matrix:

$$V_{\alpha\beta} = (\Sigma^a)_{\alpha\beta}V_a. \tag{A.7}$$

We have the following correspondences between tensors and multispinors:

$$F_{ab} = F_{[ab]} \leftrightarrow F_\alpha{}^\beta, \qquad F_\alpha{}^\alpha = 0,$$

$$F_{abc} = F_{[abc]} = {}^*F_{[abc]} \leftrightarrow F_{\alpha\beta} = F_{\beta\alpha},$$

$$F_{abc} = F_{[abc]} = -{}^*F_{[abc]} \leftrightarrow F^{\alpha\beta} = F^{\beta\alpha},$$

$$T_{ab} = T_{ba}, \qquad T_a{}^a = 0 \leftrightarrow T_{\alpha\beta,\gamma\delta} = T_{[\alpha\beta][\gamma\delta]},$$

$$T_{[\alpha\beta\gamma\delta]} = 0. \tag{A.8}$$

Appendix B

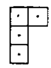
TABLEAU CALCULUS

We represent a supercovariant derivative $D_\alpha{}^i$ graphically by a box with a dot

$$D_\alpha{}^i \sim \boxed{\cdot}\,. \tag{B.1}$$

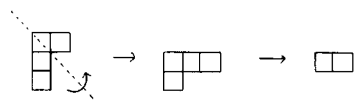
The box is a Young tableau that simultaneously records its SU*(4) representation, **4**, and its SU(2) representation, **2**. The dot is there to remind us that the indices α and i

are associated with a derivative. Products of D's are represented by a Young tableaux obtained from products of boxes. The relation $D_{(\alpha}{}^{i}D_{\beta}{}^{j}D_{\gamma)}{}^{k}=0$ becomes the rule that *no more than two boxes with dots may appear in a row*. For a product of D's such as



(B.2)

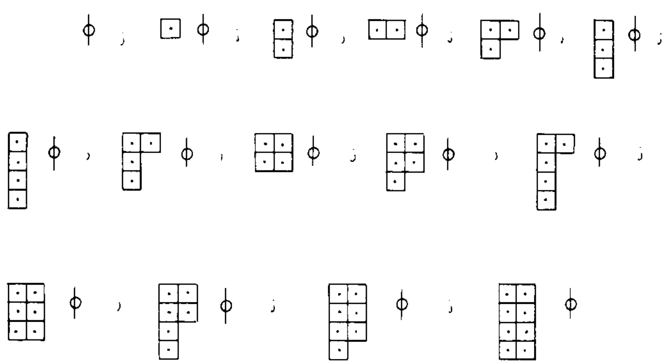
the SU(4) representation is that represented directly by the tableau, i.e. **15**. The SU(2) representation is obtained by reflection about the main diagonal



(B.3)

i.e. **2**. The fact that no more than two $\boxdot$'s may occur in a row is obviously consistent with the fact that, for SU(2), no more than two boxes may occur in a column.

The components of a scalar superfield, ϕ, are now readily found as follows:



(B.4)

and the SU*(4) and SU(2) representations read off according to the above rules.

References

[1] P.S. Howe, K.S. Stelle and P.K. Townsend, Nucl. Phys. B214 (1983) 519

[2] M.T. Grisaru and W. Siegel, Nucl. Phys. B201 (1982) 292

[3] P.S. Howe, K.S. Stelle and P.K. Townsend, in preparation; reported by K.S. Stelle in Group theoretical methods in physics, Zvenigorod (1982)

[4] W. Siegel, Phys. Lett. B80 (1979) 220

[5] L.N. de Matos Pimentao, diplomarbeit, Institut für Theoretische Physik, Universität Karlsruhe (1980);
B. Nilsson, Nucl. Phys. B174 (1980) 335;
J. Wess, Proc. 5th Johns Hopkins Workshop on current problems in particle theory, Baltimore, May 1981

[6] T. Kugo and P.K. Townsend, Nucl. Phys. B221 (1983) 357
[7] A.S. Galperin, E.A. Ivanov and J.I. Ogievetsky, Dubna preprint E2 80 750
[8] L. Mesinçescu, JINR report P2-12572 (1979)
[9] P.S. Howe and O. Lindström, Phys. Lett. B103 (1981) 422
[10] N. Marcus and J.H. Schwartz, preprint CALT 68 910 (1982)
[11] G. Sierra and P.K. Townsend, preprint LPTENS 83/7, to appear in Phys. Lett. B
[12] R. Bergshoeff, M. de Roo and B. de Wit, Nucl. Phys. B182 (1981) 173
[13] P.S. Howe, K.S. Stelle and P.K. Townsend, Nucl. Phys. B191 (1981) 445; B192 (1981) 332
[14] J. Koller, preprint CALT-68-975