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EXACT GELL-MANN-LOW FUNCTION OF SUPERSYMMETRIC YANG-MILLS THEORIES FROM INSTANTON CALCULUS

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The instanton contribution to the vacuum energy in supersymmetric Yang-Mills theories is considered. Using the renormalizability of the theory we find the exact beta function for n-extended supersymmetry ($n=1,2,4$). The coefficients of the beta function have a geometrical meaning: they are associated with the number of boson and fermion zero modes in the instanton field. If extra matter superfields are added our method allows one to fix the first two coefficients. We prove a non-renormalization theorem which extends the cancellation of vacuum loops to the case of the external instanton field.

1. Introduction

In this paper we shall determine the Gell-Mann-Low function to all orders in the coupling constant in a class of supersymmetric Yang-Mills (SYM) theories. Our derivation is based only on instanton calculus [1, 2] and such general features of the theory as renormalizability and supersymmetry. Moreover, for arbitrary SYM theory with any number of matter multiplets we can fix in this way the first two coefficients of the beta function.

Although our method relies on some specific features of instanton calculus, after these elements are established theoretically the procedure reduces to a simple counting of fermionic and bosonic zero modes in the instanton field. Thus, instead of computing multiloop graphs we deal with a set of integer numbers. Coefficients of the beta function acquire in this language a geometrical meaning.

Our starting point is the description [3] of a generalized instanton solution containing both boson and fermion classical fields. The complete set of relevant collective coordinates was analysed in ref. [3]. It was shown in this paper that the (differential) instanton contribution to the vacuum-vacuum transition, obtained at the one-loop level, respects supersymmetry. Since the latter is a true symmetry of the action, including quantum corrections, we shall accept that the same situation is realized in all orders. This will allow us to find the *exact* answer for the one-instanton contribution. (We shall sometimes use another term for the same effect: the instanton-induced effective interaction. Use of the other language will help us to establish a few important relations.)

The possibility of obtaining the exact result is due to a non-renormalization theorem, which states that the result of the lowest-order calculation is not modified in higher orders. The theorem generalizes on the instanton case the well-known assertion [4] according to which the vacuum energy in supersymmetric theories vanishes order by order. The standard proof refers, so to say, to an "empty" space. As we shall see, in the presence of external fields of definite chirality (and the intanton (super)field is indeed chiral) the result stays valid.

It is important that the effects of non-zero modes, fermionic and bosonic, totally cancel each other and thus the problem reduces to consideration of zero modes alone.

In this way we specify the instanton effective interaction in terms of the collective coordinates (x_0, ρ, etc), ultraviolet regulator mass M and the bare coupling constant α_{s0}.

On the other hand, renormalizability of the theory implies that explicit M dependence may be absorbed in α_{s0}. By requiring the overall independence on M we get the law according to which $\alpha_{s0} \equiv \alpha_s(M)$ "runs". Moreover, using the following definition of the Gell-Mann–Low function

$$\beta(\alpha_{s0}) = \mathrm{d}\alpha_{s0}/\mathrm{d}\ln M,$$

we get for n-extended supersymmetry with SU(N) as the gauge group

$$\beta(\alpha_s) = \frac{\alpha_s^2}{2\pi} \frac{(n-4)N}{1-\left(\frac{2-n}{2\pi}\right)N\alpha_s}. \tag{1}$$

Eq. (1) reproduces, for instance, the vanishing of the Gell-Mann–Low function for $n=4$ [5, 6] as well as some other known results. We shall come back to discussion of this point in sect. 5.

Actually the last step requires reservations. The point is that the explicit M dependence is absorbed in $\alpha_s(M)$ only in theories with a single renormalization factor Z. Just this situation takes place in the case of extended supersymmetry: one and the same factor renormalizes all fermion and boson fields and the only coupling constant present in the theory. Including, say, additional matter multiplets introduces new coupling constants and/or new factors $Z_i \neq Z$. Now part of the M dependence is absorbed in Z_i, and the method outlined above no more predicts $Z(M)$ unambiguously; instead, it yields a relation between Z and Z_i. This is the reason why we can not calculate the exact beta function for arbitrary supersymmetric theory. However, invoking very simple additional arguments we are always able to determine the first two coefficients.

The organization of the paper is as follows. In sect. 2 we review the description of instantons in supersymmetric theories and find the differential instanton contribution to the vacuum energy. Sect. 3 presents the proof of the non-renormalization theorem. The β function of n-extended SYM theories is found here. In sect. 4 we discuss additional matter supermultiplets. In sects. 5, 6 we pursue the relation between our results and those known previously and discuss some implications.

2. Starting elements of the method

To begin with we shall discuss the simplest SYM theory: $n=1$, $N=2$. The corresponding lagrangian contains a triplet of gluons and a triplet of gluinos and looks as follows (for a review see ref. [7]):

$$\mathscr{L}=-\tfrac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu}+i\varphi^{a+}\sigma^\mu\mathscr{D}_\mu\varphi^a+\cdots,\tag{2}$$

$$a=1,2,3\,,\qquad \sigma^\mu=(1,\boldsymbol{\sigma})\,.$$

Here $G^a_{\mu\nu}$ is the gluon field strength tensor, φ^a is the right-handed (two-component) fermion field in the adjoint representation of the $SU(2)_{\text{color}}$ group, and dots denote gauge fixing and ghost terms which are inessential here.

The BPST instanton $(v^a_\mu)_{\text{cl}}$ is parametrized by two collective coordinates, the instanton centre x_0 and its size ρ:

$$(v^a_\mu)_{\text{cl}}=\frac{2}{g}\frac{\eta_{a\mu\nu}(x-x_0)_\nu}{[(x-x_0)^2+\rho^2]},\tag{3}$$

where $\eta_{a\mu\nu}$ are the 't Hooft symbols. Because of the supersymmetry this bosonic solution of the classical Yang–Mills equation is accompanied by fermionic solutions as well [8]. The latter actually represent the fermion zero modes. There are four such zero modes in the model considered, and they bring in new collective coordinates, α_α and $\bar\beta_{\dot\alpha}$ $(\alpha,\dot\alpha=1,2)$ where α, $\bar\beta$ are the Grassmann numbers. In order to introduce the fermionic collective coordinates in a supersymmetric way we use the following expression (for details see ref. [3]):

$$\begin{aligned}V(x_0,\rho,\alpha,\bar\beta)&=\mathrm{e}^{iPx_0}\,\mathrm{e}^{-i(\alpha Q+\bar S\bar\beta)}\,V_{\text{inst}}(x_0=0,\rho)\\&\quad\times\mathrm{e}^{i(\alpha Q+\bar S\bar\beta)}\,\mathrm{e}^{-iPx_0}\,.\end{aligned}\tag{4}$$

Here $V_{\text{inst}}\,(x_0=0,\rho)$ is the "initial" superfield containing only the bosonic solution (2), with no fermion components; moreover, Q and $\bar S$ are the generators of supersymmetric and superconformal transformations, respectively. Notice that $\bar Q$ and S do not act on V_{inst}. As usual, P denotes the generator of translations.

Substituting the classical field in the action and accounting for quantum fluctuations in the bilinear approximation we get, to lowest order, the one-instanton contribution to the vacuum energy:

$$I(x_0,\rho,\alpha,\bar\beta)=C\left(\frac{2\pi}{\alpha_{\text{s0}}}\right)^2\exp\left(-\frac{2\pi}{\alpha_{\text{s0}}}\right)(M\rho)^6\,\mathrm{d}^4x_0\frac{\mathrm{d}\rho}{\rho^5}\,\mathrm{d}^2\alpha\,\mathrm{d}^2\bar\beta\,,\tag{5}$$

where C is a numerical constant and M is the Pauli–Villars regulator mass.

A few explanatory remarks are in order here. First of all, calculation of I is quite standard [2]. The supersymmetry manifests itself in cancellation of all non-zero modes [9]. Indeed, for non-zero eigenvalues the bosonic and fermionic modes are degenerate. Thus, we are left with the zero modes alone. In particular, all M dependence in eq. (5) comes from the zero modes, or, more exactly, from the corresponding modes of the Pauli–Villars regulator fields (for details see ref. [10]). The only point which might deserve comment is the pre-exponential factor $(2\pi/\alpha_{s0})^2$. In QCD we get used to the fact that each bosonic zero mode results in a factor $(2\pi/\alpha_{s0})^{1/2}$ while the fermion zero modes generate $(2\pi/\alpha_{s0})^0$ (see ref. [2] or the review paper [10]). There are eight bosonic modes in the model considered, therefore, at first sight, one might expect occurrence of $(2\pi/\alpha_{s0})^4$.

Actually, our definition of the collective coordinates α and $\bar{\beta}$, dictated by supersymmetry (see eq. (4)), does not correspond to the standard normalization of the fermion zero modes. Instead of unity the normalization integral is equal to $1/g^2$. Recalling that the zero modes enter only via regulator fields we conclude that each of them yields a $(M/g^2)^{-1}$ factor in the regularized fermion determinant [10, 11]. Since the vacuum energy in the case at hand is proportional to $(\det \mathscr{D}_\mu \gamma_\mu)^{1/2}$ [11] the effect of fermions is proportional to g^4/M^2, while the standard normalization would yield $1/M^2$.

This does not mean, of course, that previous calculations (see e.g. [11]) were erroneous. Consistent use of supersymmetric coordinates α and $\bar{\beta}$ would all the same generate conventional expressions of the type $(2\pi/\alpha_{s0})^4 \exp(-2\pi/\alpha_{s0})$ in the effective four-fermion lagrangian or in the correlation function $\langle \varphi^a_\alpha(x)\varphi^{a\alpha}(x), \varphi^b_\beta(0)\varphi^{b\beta}(0)\rangle$ (see e.g. [12]).

The origin of the extra g factor per each fermion zero mode in (5) can be explained in the following way. $I(x_0, \rho, \alpha, \bar{\beta})$ may be considered as an effective interaction with four fermion legs (fig. 1). *Eq.* (5) *includes the wave-function renormalization.* Moreover, this is the only renormalization existing for $I(x_0, \rho, \alpha, \bar{\beta})$ since all vertex corrections vanish (see sect. 3). The fact that the fermion wave-function renormalization reduces to g becomes obvious if we redefine the fields v_μ, φ,

$$\tilde{v}^a_\mu = g v^a_\mu\,, \qquad \tilde{\varphi}^a = g\varphi^a\,,$$

and rewrite the lagrangian in the following form

$$S = \int d^4x\, d^2\theta \frac{1}{g^2} W^\alpha W_\alpha\,, \tag{6}$$

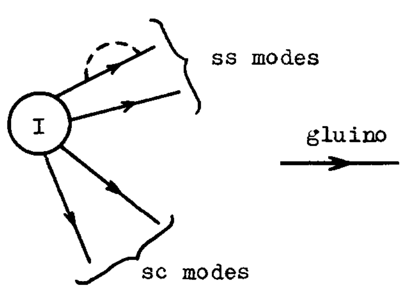

Fig. 1. Instanton-induced effective interaction in the simplest SYM theory ($n = 1$, $N = 2$). The dashed line shows the external line renormalization which is included in eqs. (5), (9) and others of that type. Proper vertex corrections are absent (sect. 3).

where the spinor superfield W_α incorporates both $(g\varphi^a)$ and $(gG^a_{\mu\nu})$. From eq. (6) it is clear that the combinations $(g\varphi)$ and $(gG_{\mu\nu})$ stay unrenormalized. In more general theories with additional matter supermultiplets we shall introduce different renormalization factors for gluino and quark fields respectively.

Let us also mention that the integral one-instanton contribution to the vacuum energy naturally vanishes, according to the standard rules of integration over the Grassmann variables $\int \mathrm{d}^2\alpha\, \mathrm{d}^2\bar{\beta} = (0)^4$*. The vanishing of the vacuum energy is in full accordance with supersymmetry. The differential expression (5) is also supersymmetric: and this is a much stronger result. Under the supersymmetry transformations the instanton collective coordinates change in the following way [3]

(a) εQ generator:

$$\alpha \to \alpha + \varepsilon\,, \qquad x_0 \to x_0\,, \qquad \bar{\beta} \to \bar{\beta}\,, \qquad \rho \to \rho\,; \tag{7}$$

(b) $\bar{Q}\bar{\varepsilon}$ generator:

$$\alpha \to \alpha\,, \qquad (x_0)_{\alpha\dot{\beta}} \to (x_0)_{\alpha\dot{\beta}} - 4i\alpha_\alpha \bar{\varepsilon}_{\dot{\beta}}\,, \qquad \bar{\beta} \to \bar{\beta}(1 + 4i(\bar{\beta}\bar{\varepsilon}))\,,$$
$$\rho \to \rho(1 + 2i(\bar{\beta}\bar{\varepsilon}))\,. \tag{8}$$

One can readily convince oneself that both transformations (7), (8) leave the product $\mathrm{d}^4x_0\, \mathrm{d}\rho^2\, \mathrm{d}^2\alpha\, \mathrm{d}^2\bar{\beta}$ intact [3]. In this way we explicitly check the supersymmetry of $I(x_0, \rho, \alpha, \bar{\beta})$ in the lowest (one-loop) approximation.

To finish with preliminaries it is useful to rewrite eq. (5) in the following identical form

$$I(x_0, \rho, \alpha, \bar{\beta}) = C\left\{\left(\frac{2\pi}{\alpha_{s0}}\right)^{(n_b - n_f)/2} \mathrm{e}^{-2\pi/\alpha_{s0}} (M\rho)^{n_b - n_f/2}\right\} \mathrm{d}^4x_0 \frac{\mathrm{d}\rho}{\rho^5} \mathrm{d}^2\alpha\, \mathrm{d}^2\bar{\beta}\,, \tag{9}$$

where n_b (n_f) denotes the number of boson (fermion) zero modes. First of all, eq. (9) explicitly demonstrates the geometrical nature of the exponents in eq. (5). Moreover, the expression in braces, obtained in the simplest SYM theory ($n = 1, N = 2$), stays valid for any N. It is also valid for extended supersymmetry ($n = 2$ and $n = 4$). Although in the latter case the lagrangian contains matter supermultiplets, they are renormalized by the same Z factor. The absence of new renormalization constants is ensured by additional conserved supercharges.

The numbers n_b, n_f are trivially calculable (see e.g. [10]):

$$n_b = 4N\,, \qquad n_f = 2Nn\,. \tag{10}$$

Now, even leaving aside for a while the non-renormalization theorem (sect. 3) we immediately get from eq. (9) the first two coefficients of the beta function. It is sufficient to know that the explicit M dependence is exhausted by $M^{n_b - n_f/2}$ and the correction factor which might appear in higher orders should have the form $(1 + \text{const.}\ \alpha_{s0} + \cdots)$. The explicit M dependence in eq. (9) must be compensated for

* The zero of the fourth order reflects the existence of four fermion zero modes.

by implicit M dependence of the coupling constant α_{s0}: the general feature of all renormalizable theories. With this information in hand we take the logarithm of the product in braces, differentiate it with respect to ln M, and arrive at the following relation

$$-N(2-n)\frac{1}{\alpha_{s0}}\beta(\alpha_{s0})+\frac{2\pi}{\alpha_{s0}^2}\beta(\alpha_{s0})+\text{const}\,\beta(\alpha_{s0})+\cdots=N(n-4)\,. \qquad (11)$$

As we see, it unambiguously fixes the first two coefficients:

$$\beta(\alpha_s)=\frac{(n-4)N\alpha_s^2}{2\pi}\left(1+\frac{(2-n)N}{2\pi}\alpha_s+O(\alpha_s^2)\right). \qquad (12)$$

Recall that just these coefficients are universal in the sense that they do not depend on the renormalization scheme. As is well-known, the third and further coefficients can vary, and in principle one can find such a definition of the coupling constant under which the third and all further coefficients vanish [13].

3. Non-renormalization theorem

We shall prove in this section that multiloop diagrams do not change the results quoted in eqs. (5) or (9). The situation is absolutely analogous to the well-known assertion according to which induced F-terms in effective lagrangians stay unrenormalized* provided that loop corrections are calculated in a supersymmetric way (see e.g. [7]).

The key point is that the instanton interaction under discussion represents an F-term in the space of collective coordinates, not a D-term. Indeed, let us compare the chiral realization of the supergroup in the $(x, \theta, \bar\theta)$ superspace and the transformation law of (x_0, α) (see eqs. (7), (8)). Supertransformations corresponding to parameters ε, $\bar\varepsilon$ respectively reduce to

$$\text{(a)}\qquad \theta\to\theta+\varepsilon\,,\qquad \bar\theta\to\bar\theta\,,\qquad x^{\text{ch}}_{\alpha\dot\beta}\to x^{\text{ch}}_{\alpha\dot\beta}\,,\qquad x^{\text{ach}}_{\alpha\dot\beta}\to x^{\text{ach}}_{\alpha\dot\beta}-4i\varepsilon_\alpha\bar\theta_{\dot\beta}\,;$$

$$\text{(b)}\qquad \theta\to\theta\,,\qquad \bar\theta\to\bar\theta+\bar\varepsilon\,,\qquad x^{\text{ch}}_{\alpha\dot\beta}\to x^{\text{ch}}_{\alpha\dot\beta}+4i\theta_\alpha\bar\varepsilon_{\dot\beta}\,,\qquad x^{\text{ach}}_{\alpha\dot\beta}\to x^{\text{ach}}_{\alpha\dot\beta}\,,$$

(ch = chiral, ach = antichiral). At the same time, the collective coordinates are changed in the following way:

$$\text{(a)}\qquad \alpha\to\alpha+\varepsilon\,,\qquad (-x_0)\to(-x_0)\,,$$

$$\text{(b)}\qquad \alpha\to\alpha\,,\qquad (-x_0)_{\alpha\dot\beta}\to(-x_0)_{\alpha\dot\beta}+4i\alpha_\alpha\bar\varepsilon_{\dot\beta}\,.$$

Thus, $(-x_0)$ plays the role of x_{chiral} and α plays the role of θ_0, the right-handed fermion coordinate of the instanton centre [3].

Another pair of coordinates, ρ and $\bar\beta$, must be considered as a (super)parameter. Their transformation laws correspond to no realization of the supergroup in linear

* More exactly, this assertion refers to amputated vertices. External lines acquire non-trivial Z factors, and we have accounted for this effect above.

space. Moreover, one cannot introduce, instead of $\bar{\beta}$, another left-handed collective coordinate with a normal transformation law. It is impossible to trade $\bar{\beta}$ for $\bar{\varepsilon}$ because the jacobian turns out to be zero. Indeed,

$$d^2\bar{\varepsilon} = |\partial\bar{\beta}_{\dot{\alpha}}/\partial\bar{\varepsilon}_{\dot{\beta}}|\, d^2\bar{\beta}\,, \tag{13}$$

where $|\partial\bar{\beta}/\partial\bar{\varepsilon}|$ is the jacobian, and the Grassmann nature of the parameters is accounted for by inverting the jacobian as compared with the case of usual numbers.

However, using eq. (8) we get

$$|\partial\bar{\beta}_{\dot{\alpha}}/\partial\bar{\varepsilon}_{\dot{\beta}}| = -4\bar{\beta}^2\bar{\beta}^2\,, \tag{14}$$

and the right-hand side vanishes since it is proportional to the fourth power of the Grassmann parameter.

Now it is almost evident that, as for any F-term, there should exist a non-renormalization theorem for the instanton-induced interaction. This is indeed the case.

We shall consider the instanton field with given $(x_0, \rho, \alpha, \bar{\beta})$. We shall analyse diagrams for the vacuum energy with two or more loops in the presence of the external field of definite chirality (fig. 2). But first let us recall the standard derivation of the theorem establishing the fact that vacuum loops in SYM theories cancel each other in perturbation theory.

In any theory the vacuum energy is proportional to*

$$E_{\text{vac}} = \text{const} \int d^4x\,,$$

where we use only the uniformity of space–time. Similarly, in SYM theories one has

$$E_{\text{vac}} = \text{const} \int d^4x\, d^4\theta\,. \tag{15}$$

Here we exploit the fact that the interaction vertices are of the D-term type. For instance, the graph of fig. 2 would reduce to such a form after integration over d^4y

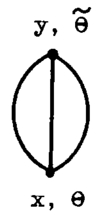

Fig. 2. Two-loop vacuum diagram. Superfield propagators are denoted by solid lines. If the external instanton field is imposed, free propagators are to be substituted by those in the external field.

* This expression (as well as eq. (15) and others of that type) assume that the symmetry is not broken spontaneously. As we have shown previously this is true for instantons.

and $\mathrm{d}^4\tilde{\theta}$. The structure of eq. (15) is fixed by the uniformity of the superspace. Integration over $\mathrm{d}^4\theta$ in eq. (15) ensures the vanishing of E_{vac}.

Furthermore, in the presence of an external field centred at x_0 the vacuum energy becomes

$$E_{\mathrm{vac}} = \mathrm{const} \int \mathrm{d}^4 x\, f((x-x_0)^2)\,,$$

where f is some function which depends on the external field. Generalization of this statement in the supersymmetric case with a chiral external field is

$$E_{\mathrm{vac}} = \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, \mathrm{d}^4 x\, f(x_{\mathrm{ch}} - x_0,\, \theta - \theta_0)\,. \tag{16}$$

Here $(x_{\mathrm{ch}})_\mu = x_\mu + i\theta\sigma_\mu\bar{\theta}$ and (x_0, θ_0) are the superspace coordinates of the centre of the external field ($\theta_0 = -\alpha$). The most crucial point is that explicit $\bar{\theta}$ dependence cannot enter since there is no way to cancel the shift of $\bar{\theta}$ by redefinition of the collective coordinates.

In particular, the structure of the integral for E_{vac} in the instanton field is as follows

$$\begin{aligned} E_{\mathrm{vac}} = \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, \mathrm{d}^4 x\, \{ & f((x_{\mathrm{ch}} - x_0)^2) \\ & \times [\theta + \alpha + (x_{\mathrm{ch}} - x_0)_\mu \sigma_\mu \bar{\beta}]^2 + \mathrm{const.} \}\,. \end{aligned} \tag{17}$$

Integration over $\mathrm{d}^2\theta$ gives a non-trivial result, generally speaking, but the expression (17) still vanishes because of $\int \mathrm{d}^2\bar{\theta}$. This completes the proof of the absence of corrections in eqs. (5), (9).

The last step is already known. We want the renormalized coupling constant to absorb all M dependence of $I(x_0, \rho, \alpha, \bar{\beta})$. Within this renormalization prescription we arrive at eq. (11) with the first two terms in the left-hand side. The corresponding beta function is given in eq. (1).

4. Additional matter multiplets

What will happen if we introduce some matter superfields in the lagrangian (2) without extending supersymmetry? For instance, with three supermultiplets we can preserve the number of zero modes the same as in $n = 4$ theory. However, switching off the self-interaction of matter superfields (characteristic for $n = 4$) we, certainly, change the beta function, and the question may arise why and how our derivation is modified in this case.

There is one apparent reason for modification. Namely, only n-extended supersymmetry is a theory with a single coupling constant g. In other cases there are additional renormalization constants. We shall come back to a more detailed discussion of the issue below. Let us mention now another possible reason why the

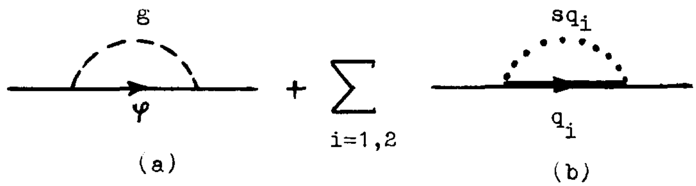

Fig. 3. Gluino wave function renormalization in a one-loop approximation. We denote the quark field by q_i where i is the flavor index; the corresponding scalar quark is denoted by sq_i.

exact Gell-Mann–Low function might be calculable only in the n-extended supersymmetry. The point is that only in this case the collective coordinates have clear group-theoretical meaning, and we deal with an F-term for sure (see above). If we just add matter fields without extending the supersymmetry of the lagrangian, perhaps, it is possible to redefine some coordinates and make out of them a "normal" left-handed parameter. Then our non-renormalization theorem will be inapplicable in such theories. Although this seems unlikely, the issue requires further investigation. Recall, however, that the first two coefficients were found in sect. 2 with no reference to the non-renormalization theorem. We shall demonstrate that in this point the procedure works even in the presence of extra matter fields.

As an example, consider the following problem. Start with the lagrangian (2) and introduce two chiral superfields S^a and T^a $(a = 1, 2, 3)$ coupled only to the gauge superfield. (With a single superfield S^a we would automatically get $n = 2$ supersymmetry.) Let us show first that renormalizations of gluino and quark wave functions are different.

One-loop graphs determining the corresponding Z factors are depicted on figs. 3, 4. Diagrams 3a and 4a are identical, while there is an essential difference between diagrams 3b, 4b. Indeed, in the former case we are dealing with the sum over two flavors in the intermediate state, while in the latter one the doubling is absent: a quark of each flavor is coupled to the scalar quark of the same flavor.

These diagrams are very simple, of course. Still, one should treat them with caution. Straightforward calculation, say, in the Landau or Feynman gauge would violate supersymmetry, and this would spoil our derivation. Besides that, we do not want to deal with Feynman graphs at all. Our aim is to find the coefficients without digressing in quantum debris, from purely classical considerations. Actually, we shall use graphs of figs. 3, 4 with a very limited purpose: they will help us to establish a certain relation between the Z factors.

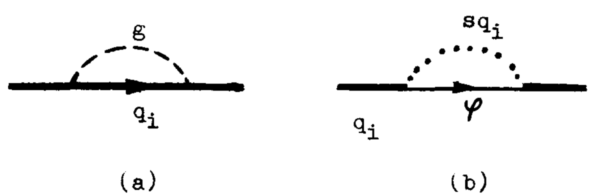

Fig. 4. Quark wave-function renormalization in the same approximation.

Notice that the gluino wave-function renormalization, as previously, coincides with that of g (see eq. (6)). Thus, as in the previous calculation of I we insert a g factor per each gluino zero mode. At the same time, the situation with quark zero modes becomes a bit more complicated. In the case at hand the anomalous dimension γ of the quark field turns out to be larger than that of the gluino field by a factor of 2 (at the one-loop level), and, correspondingly, each quark zero mode requires g^2.

The proof is based again on simple counting of integer numbers. With no matter fields the only surviving graph (3a) yields $\gamma_\varphi = b = 6$ $(= n_\mathrm{b} - \frac{1}{2}n_\mathrm{f})$ (see eqs. (9), (6); b stands for the first coefficient of the Gell-Mann–Low function). Introducing the matter supermultiplet S^a increases n_f by four units and, hence, decreases γ_φ and b by two units. The presence of two matter supermultiplets means that $\gamma_\varphi = b = 6 - 2\times 2 = 2$.

Thus, the relative weight of the diagrams depicted in fig. 3b is -2×2. The diagram of fig. 4b is just the same, but we have a single graph, not two. Therefore, the quark field anomalous dimension is, evidently, $6-2=4$. As a result,

$$\gamma_\mathrm{q} = 2\gamma_\varphi .$$

With this information we return to the calculation of I. In the model considered the vacuum energy turns out to be proportional to

$$I \sim \left(\frac{2\pi}{\alpha_{\mathrm{s}0}}\right)^2 \left(\frac{\alpha_{\mathrm{s}0}}{2\pi}\right)^8 M^2\, \mathrm{e}^{-2\pi/\alpha_{\mathrm{s}0}} . \tag{18}$$

The factor $(\alpha_{\mathrm{s}0}/2\pi)^8$ is due to eight quark zero modes. Notice that this factor is established only at the one-loop level. However, the one-loop approximation in the pre-exponential factor is evidently sufficient in order to fix α_s in the exponential up to two loops.

The N-colored analogue of eq. (18) looks as follows:

$$I \sim \left(\frac{2\pi}{\alpha_{\mathrm{s}0}}\right)^N \left(\frac{\alpha_{\mathrm{s}0}}{2\pi}\right)^{4N} M^N\, \mathrm{e}^{-2\pi/\alpha_{\mathrm{s}0}} , \tag{19}$$

which results, in turn, in the following beta function:

$$\beta(\alpha_\mathrm{s}) = -\frac{N\alpha_\mathrm{s}^2}{2\pi}\left(1 - \frac{3N\alpha_\mathrm{s}}{2\pi} + \mathrm{O}(\alpha_\mathrm{s}^2)\right) . \tag{20}$$

Eq. (20) exactly reproduces the result obtained by conventional methods [14, 7].

5. Comments on the literature

Let us now discuss how eq. (1) agrees with earlier results. It predicts the vanishing of the β function for $n=4$. This has been advocated previously on different grounds [6, 5]. For $n=2$ the β function quoted in eq. (1) reduces to a single term, and indeed, the vanishing of the two- and three-loop contributions to $\beta(\alpha_s)$ has been checked explicitly [15].

For $n=1$ eq. (1) predicts the whole series in α_s. Since higher-order terms depend, generally speaking, on the definition of $\beta(\alpha_s)$ it is worth specifying our definition of the Gell-Mann–Low function.

Our definition holds within the effective lagrangian approach. Indeed, $\alpha_s(\rho)$, according to eq. (5), determines the action for the instanton of size ρ. It means that we integrate over fluctuations whose size is less than ρ and reduce their effect to the effective lagrangian of the form

$$\mathcal{L}_{\text{eff}} \sim \frac{1}{g^2(\rho)} W^2 .$$

Thus, the definition we use is at least gauge invariant. Moreover, it is also supersymmetric.

Eq. (1) for $n=1$ agrees with the explicit calculations at the two-loop level and disagrees with them for three loops [15]. However, at the three-loop level the renormalization prescription for α_s becomes important, and the authors of ref. [15] use a different scheme.

Finally, eq. (1) at $n=1$ agrees with the beta function implicit in the recent paper by Jones [16]. Moreover, comparison with this paper brings us to a problem, and to explain this it is worthwhile first rephrasing Jones' result.

Consider the chiral superfield $\alpha_s^{-1}\beta(\alpha_s)W^2$. Its F and G components are of the form (in the Majorana notation)

$$F = \alpha_s^{-1}\beta(\alpha_s)[-\tfrac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} + \tfrac{1}{2}i\bar{\lambda}^a\hat{\mathscr{D}}\lambda^a + \tfrac{1}{2}\mathscr{D}^2] ,$$
$$G = \alpha_s^{-1}\beta(\alpha_s)[-\tfrac{1}{4}G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} + \tfrac{1}{2}\partial_\mu a_\mu] , \tag{21}$$

where $\varphi^a = (2)^{-1/2}(1-\gamma_5)\lambda^a$ and a_μ is the axial-vector current, $a_\mu = \frac{1}{2}\bar{\lambda}^a\gamma_\mu\gamma_5\lambda^a$. Substituting the Adler-Bardeen relation [17]

$$\partial_\mu a_\mu = \frac{N\alpha_s}{4\pi} G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} , \tag{22}$$

one finds for the G component

$$G = -\frac{\pi}{N\alpha_s^2}\beta(\alpha_s)\left[1-\frac{N\alpha_s}{2\pi}\right]\partial_\mu a_\mu . \tag{23}$$

Moreover, all the components of the superfield are to be renormalized by a common factor provided that the renormalization procedure respects supersymmetry. It is known that

$$\theta_{\mu\mu} = \frac{\beta(\alpha_s)}{4\alpha_s} G^a_{\mu\nu}G^a_{\mu\nu} ,$$

and this combination is renorm-invariant. Moreover, it concides with the F com-

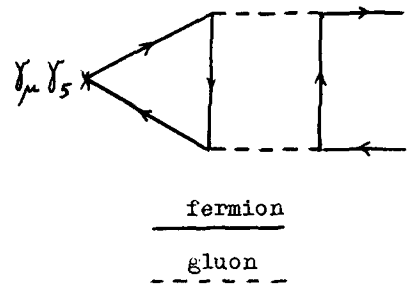

Fig. 5. Two-loop graphs which result in a non-vanishing anomalous dimension of the axial-vector current.

ponent of $\alpha_s^{-1}\beta(\alpha_s)W^2$ (see eq. (21)). Therefore

$$\frac{\beta(\alpha_s)}{N\alpha_s^2}\left(1-\frac{N\alpha_s}{2\pi}\right)\partial_\mu a_\mu\,, \tag{24}$$

must also be renorm-invariant. Substituting $\beta(\alpha_s)$ given by eq. (1) (with $n=1$) we come to the conclusion that $\partial_\mu a_\mu$ is renorm-invariant. But apparently it is not! Indeed, the anomalous dimension of the axial-vector current a_μ* in QED has been calculated some time ago [18]. It is associated with the two-loop graph of fig. 5. The calculation stays valid in supersymmetric gluodynamics as well. At the moment we are not aware of any resolution of the puzzle, although it seems that several points must be checked before we can assess how serious the difficulty is.

6. Concluding remarks

In conclusion, let us comment on the implications of eq. (1). For $n=4$ the coupling constant remains a constant at any scale. For $n=2$ the asymptotic freedom regime (pure logarithm) extends to all momenta. For $n=1$ the coupling constant can not be larger than

$$(\alpha_s)_{\max}=2\pi/N\,.$$

This value of α_s is the branch point in the M plane, and α_s becomes non-analytic.

In a more general aspect, we have shown that the Gell-Mann–Low function in SYM theories is determined from a purely *classical* consideration; without direct computation of quantum loops. Such a peculiar situation (when the coupling constant renormalization is fixed essentially at the classical level, and all relevant coefficients have a geometrical meaning) seems to emerge for the first time in the four-dimensional field theory. This might indicate that the relative roles of classical and quantum effects in supersymmetric dynamics are quite unusual.

* Renorm-invariance of a_μ is also vital in order to put it in the same supermultiplet as J_μ and $\theta_{\mu\nu}$. The issue of anomalies is central in ref. [16]. To formulate our problem, however, we can avoid mentioning the anomalies altogether.

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