The cross section can therefore be written

$$\frac{d^5\sigma_L}{d\epsilon_2 d\Omega_2 d\Omega_q^*/4\pi}=\left[\frac{\alpha^2\cos^2(\frac{1}{2}\theta)}{\epsilon_1^2\sin^4(\frac{1}{2}\theta)}\right]\frac{mq}{W}\{(k^4/k^{*4})|\mathfrak{J}_c|^2$$
$$+[k^2/2k^{*2}+(W^2/m^2)\tan^2(\tfrac{1}{2}\theta)](|\mathfrak{J}^{+1}|^2+|\mathfrak{J}^{-1}|^2)$$
$$+(k^2/2k^{*2})2\,\mathrm{Re}(\mathfrak{J}^{+1})^*(\mathfrak{J}^{-1})+(k^2/k^{*2})$$
$$\times[(W^2/m^2)\tan^2(\tfrac{1}{2}\theta)+k^2/k^{*2}]^{1/2}\sqrt{2}$$
$$\times\mathrm{Im}\mathfrak{J}_c^*(\mathfrak{J}^{+1}+\mathfrak{J}^{-1})\}\,. \quad \text{(C26)}$$

Now from I,

$$\mathfrak{J}_{\lambda_2\lambda_1}{}^{\lambda_k}=(4k^*q)^{-1/2}\sum_J(2J+1)\mathfrak{D}_{\lambda_1-\lambda_k,\lambda_2}{}^J(-\phi_p-\theta_p\phi_p)^*$$
$$\times\langle\lambda_2|T^J(W,k^2)|\lambda_1\lambda_k\rangle\,, \quad \text{(C27)}$$

$$(\mathfrak{J}_c)_{\lambda_2\lambda_1}=(k^*/k_0)(4k^*q)^{-1/2}\sum_J(2J+1)$$
$$\times\mathfrak{D}_{\lambda_1\lambda_2}{}^J(-\phi_p-\theta_p\phi_p)^*\langle\lambda_2|T^J(W,k^2)|\lambda_1 0\rangle\,. \quad \text{(C28)}$$

The relation between the angles is obtained by comparing Fig. 4 of I with Fig. 14:

$$\theta_p=\theta_{kq}\,,\quad \phi_p=2\pi-\phi_{kq}\,. \quad \text{(C29)}$$

If only the electron is detected, one must integrate over the final pion direction, and using the orthogonality of the $\mathfrak{D}$ functions the interference terms in (C26) go out. Finally, summing and averaging over nucleon helicities gives

$$\frac{1}{2}\sum_{\lambda_1}\sum_{\lambda_2}\int\frac{d\Omega_q^*}{4\pi}|(\mathfrak{J}_c)_{\lambda_2\lambda_1}|^2$$
$$=\frac{1}{2}\sum_{\lambda_1}\sum_{\lambda_2}\left(\frac{k^*}{k_0}\right)^2(4k^*q)^{-1}\sum_J(2J+1)$$
$$\times|\langle\lambda_2|T^J(W,k^2)|\lambda_1 0\rangle|^2$$
$$=\sum_{J^\pi}(J+\tfrac{1}{2})|C_{l\pm}|^2\,, \quad \text{(C30)}$$

$$\frac{1}{2}\sum_{\lambda_1}\sum_{\lambda_2}\int\frac{d\Omega_q^*}{4\pi}[|\mathfrak{J}_{\lambda_2\lambda_1}{}^{+1}|^2+|\mathfrak{J}_{\lambda_2\lambda_1}{}^{-1}|^2]$$
$$=\frac{1}{2}\sum_{\lambda_1}\sum_{\lambda_2}(4k^*q)^{-1}\sum_J(2J+1)[|\langle\lambda_2|T^J(W,k^2)|\lambda_1 1\rangle|^2$$
$$+|\langle\lambda_2|T^J(W,k^2)|\lambda_1-1\rangle|^2]$$
$$=\sum_{J^\pi}(J+\tfrac{1}{2})[|T_{3/2}{}^{l\pm}|^2+|T_{1/2}{}^{l\pm}|^2]\,, \quad \text{(C31)}$$

where the eigenstates of parity have been introduced. The resulting cross section for detection of only the electron is that given in Eq. (3.31) of I. The resonant amplitudes for producing the $|\pi N\rangle$ channel in the coupled-channel calculations are obtained by multiplying Eq. (5.31) by $\cos\varepsilon$.

PHYSICAL REVIEW VOLUME 184, NUMBER 5 25 AUGUST 1969

Anomalous Ward Identities in Spinor Field Theories

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(Received 24 February 1969)

We consider the model of a spinor field with arbitrary internal degrees of freedom having arbitrary nonderivative coupling to external scalar, pseudoscalar, vector, and axial-vector fields. By carefully defining the S matrix in the interaction picture, the vector and axial-vector currents associated with the external vector and axial-vector fields are found to satisfy anomalous Ward identities. If we require that the vector currents satisfy the usual Ward identities, the divergence of the axial-vector current contains well-defined anomalous terms. These terms are explicitly calculated.

I. INTRODUCTION

THE presence of anomalous terms in the Ward identities for currents defined in a number of spinor field theories has been noted by several authors.[1–3] The existence of these terms may be traced to the local products of field operators which are so singular as to prohibit the naive use of the field equations. In a version of the σ model, the anomalous terms in the Ward identity for the neutral isospin current have led to a low-energy theorem for the decay $\pi^0\to\gamma\gamma$.[2]

In this paper, we consider a theory of a spinor field with an arbitrary number of internal degrees of freedom coupled to external scalar, pseudoscalar, vector, and

* Research sponsored by the Air Force Office of Scientific Research, U. S. Air Force, under AFOSR Grant No. 68-1365.

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[1] J. Steinberger, Phys. Rev. **76**, 1180 (1949); J. Schwinger, *ibid.* **82**, 664 (1951).

[2] S. L. Adler, Phys. Rev. **177**, 2426 (1969).

[3] C. R. Hagen, Phys. Rev. **176**, 2622 (1969); R. Jackiw and K. Johnson, *ibid.* **182**, 1457 (1969); R. Brandt, *ibid.* **180**, 1490 (1969); K. Wilson, *ibid.* **181**, 1909 (1969); J. S. Bell and R. Jackiw, Nuovo Cimento **60**, 47 (1969).

axial-vector fields. In the interaction picture, the S matrix is carefully defined using a symmetric ϵ separation on the spinor loops. The terms which would be singular in the limit as ϵ goes to zero are isolated. The currents are defined by variation of the S matrix with respect to the external vector and axial-vector fields. By examining the Ward identities for these currents, anomalous terms are found to arise from the smaller loops which are, at least, linearly divergent.

A renormalized S matrix for the loops is defined by the subtraction of contact terms which remove the divergent terms and many of the anomalous terms in the Ward identities. However, not all of the anomalous terms can be removed in this way. In particular, the choice of counter term which preserves the usual Ward identities for the vector currents provides well-defined anomalous terms in the axial-vector current Ward identities. The result of Adler[2] is obtained for the neutral isospin current coupled to external photons. When charged currents are present, additional terms are found.

We indicate the following plan for this paper. In Sec. II, we define the theory and discuss some of its general properties. The terms which are singular in the limit as ϵ goes to zero are found in Sec. III. In Sec. IV, the Ward identities are examined and the anomalous terms isolated. The renormalized S matrix is defined in Sec. V, where the resultant anomalous terms in the Ward identities are also discussed.

II. DEFINITIONS

We propose to consider a theory consisting of a quantized spinor field with arbitrary internal degrees of freedom having arbitrary nonderivative couplings to external scalar, pseudoscalar, vector, and axial-vector fields. This theory is described by the Lagrangian density

$$\mathcal{L}(z)=\bar{\psi}(z)[i\gamma\cdot\partial+\tilde{\Gamma}(z)]\psi(z)\,, \tag{1}$$

where

$$\gamma\cdot\partial\equiv\gamma^\mu\partial_\mu\,,\quad \gamma_5^\dagger=\gamma_5.$$

The field $\psi(z)$ is a Dirac spinor field and is a column vector in the internal space. The function $\tilde{\Gamma}(z)$ is given by

$$\tilde{\Gamma}(z)=-G_0P_+(z)+\gamma_\mu V_+{}^\mu(z)\,, \tag{2}$$

where

$$\begin{aligned}P_+(z)&=F+\Sigma(z)+i\gamma_5\Pi(z)\,,\\ V_+{}^\mu(z)&=V^\mu(z)+\gamma_5A^\mu(z)\,.\end{aligned} \tag{3}$$

F is a constant matrix in the internal space and serves to give the spinor fields arbitrary masses. The fields $\Sigma(z)$, $\Pi(z)$, $V^\mu(z)$, and $A^\mu(z)$ are all matrices in the internal space and may be written

$$\begin{aligned}\Sigma(z)&=\lambda_\Sigma{}^\alpha\Sigma_\alpha(z)\,,\quad \Pi(z)=\lambda_\Pi{}^\alpha\Pi_\alpha(z)\,,\\ V^\mu(z)&=\lambda_V{}^\alpha V_\alpha(z)\,,\quad A^\mu(z)=\lambda_A{}^\alpha A_\alpha(z)\,.\end{aligned} \tag{4}$$

The fields $\Sigma_\alpha(z)$, $\Pi_\alpha(z)$, $V_\alpha(z)$, and $A_\alpha(z)$ are the external scalar, pseudoscalar, vector, and axial-vector fields, respectively, and $\lambda_\Sigma{}^\alpha$, $\lambda_\Pi{}^\alpha$, $\lambda_V{}^\alpha$, and $\lambda_A{}^\alpha$ are their respective coupling matrices.

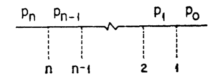

FIG. 1. Feynman diagram for the external fields coupling to an external spinor line.

We will discuss this theory in the interaction picture where the free Lagrangian density is given by

$$\mathcal{L}_0(z)=\bar{\psi}(z)(i\gamma\cdot\partial-M_0)\psi(z)\,, \tag{5}$$

and the interaction Lagrangian density is given by

$$\mathcal{L}_I(z)=\bar{\psi}(z)\Gamma(z)\psi(z)\,. \tag{6}$$

The mass term M_0 in the free Lagrangian is chosen to be proportional to the unit matrix in the internal space, and $\Gamma(z)$ is defined by

$$\Gamma(z)=\tilde{\Gamma}(z)+M_0.$$

In addition to the external fields, $\Gamma(z)$ contains the mass-splitting counter term M_0-G_0F.

Using the conventional transformation to the interaction picture,[4] the S matrix is formally given by

$$S=T\exp\left[i\int dz\,\mathcal{L}_I(z)\right], \tag{7}$$

where T is the time-ordering operator. There are two types of Feynman diagrams for this S matrix. The diagram where the external fields couple to a free spinor line is shown in Fig. 1. These diagrams are perfectly well defined by the S matrix in Eq. (7). The external fields may also couple to a spinor loop, as shown in Fig. 2. These diagrams are not all well defined, since the loops with four or fewer internal spinor lines have formal divergences.

To define the loops, we use a symmetric ϵ separation at the vertices, which is effected by the replacement of $\mathcal{L}_I(z)$ by

$$\mathcal{L}_I{}^\epsilon(z)=\bar{\psi}\left(z+\frac{\epsilon}{2n}\right)\Gamma(z)\psi\left(z-\frac{\epsilon}{2n}\right), \tag{8}$$

where ϵ is a spacelike four-vector, and n is the number of external lines on the loop. In addition, a symmetric average of ϵ over the spacelike directions is to be performed after the loop is calculated. The loop is then defined by the limit as $\epsilon^2\to 0$, if it exists. For the loops which are singular in the limit as ϵ^2 goes to zero, we must first isolate and remove the singular terms before

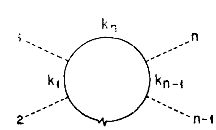

FIG. 2. Feynman diagram for the external fields coupling to a spinor loop.

[4] S. G. Gasiorowicz, *Elementary Particle Physics* (John Wiley & Sons, Inc., New York, 1966).

passing to the limit. The procedure used to isolate the singular terms is given in Sec. III. This prescription serves to define all of the loops. The expression for the loop in Fig. 2 is given by

$$S_\epsilon^n(\Gamma)=(-1)\frac{1}{n}(i)^n\int dz_1\cdots\int dz_n[-i(2\pi)^{-4}]^n$$
$$\times\int dk_1 e^{-ik_1\cdot(z_2-z_1)}\cdots\int dk_n e^{-ik_n\cdot(z_1-z_n)}e^{i\epsilon\cdot(k_1+\cdots k_n)/n}$$
$$\times\mathrm{tr}\{(M_0-\boldsymbol{k}_n)^{-1}\Gamma(z_n)\cdots(M_0-\boldsymbol{k}_1)^{-1}\Gamma(z_1)\}\,. \quad (9)$$

The use of $1/n$ in the definition of the ϵ separation is necessary so that loops with different numbers of vertices may be simply related, as needed for the Ward identities.

This definition of the loops is not the only one possible, but it has the advantage of preserving the explicit symmetry of the loops. This property will be useful in our discussions in Secs. III and IV. However, this definition of the loops will give rise to many anomalous terms in the Ward identities, as will be seen in Sec. IV. Hence, it will be necessary to redefine the loops in order to preserve as many of the Ward identities as possible. In Sec. V, this redefinition of the loops is performed through the additions of explicit counterterms.

The vector and axial-vector currents are defined by the variation of the S matrix with respect to the external vector and axial-vector fields:

$$\begin{aligned}J_\mu{}^\alpha(x)&=-i[\delta/\delta V_\alpha{}^\mu(x)]\{S\}\\&=T\left\{\bar\psi(x)\gamma_\mu\lambda_V{}^\alpha\psi(x)\exp\left[i\int dz\,\mathcal{L}_I(z)\right]\right\}_\epsilon,\\J_{5\mu}{}^\alpha(x)&=-i[\delta/\delta A_\alpha{}^\mu(x)]\{S\}\\&=T\left\{\bar\psi(x)\gamma_\mu\gamma_5\lambda_A{}^\alpha\psi(x)\exp\left[i\int dz\,\mathcal{L}_I(z)\right]\right\}_\epsilon.\end{aligned}\quad (10)$$

As above, when the current couples into a loop, the current vertex in Eq. (10) must be replaced by the ϵ-separated vertex as given in Eq. (8).

In this section, we have defined, in the interaction picture, a theory of a quantized spinor field interacting with arbitrary external fields. In Sec. III, we isolate the terms which remain ϵ-dependent in the limit as ϵ goes to zero.

III. DIVERGENCES

In Sec. II, the loops were defined by a symmetric average on ϵ and by then taking the limit as ϵ^2 goes to zero. For loops with more than four vertices, this limit exists and may be taken inside the loop integral. This limit does not exist for the smaller loops which contain terms which remain ϵ-dependent. In this section, we isolate the divergent terms in the four-, three-, two-, and one-vertex loops.

A. Four-Vertex Loop

The four-vertex loop is logarithmically divergent. From Eq. (9), we write the ϵ-separated loop as

$$S_\epsilon^4(\Gamma)=(-1)\tfrac{1}{4}(i)^4\int dz_1\int dz_2\int dz_3\int dz_4[-i(2\pi)^{-4}]^4\int dk_1e^{-ik_1\cdot(z_2-z_1)}\int dk_2e^{-ik_2\cdot(z_3-z_2)}\int dk_3e^{-ik_3\cdot(z_4-z_3)}$$
$$\times\int dk_4e^{-ik_4\cdot(z_1-z_4)}e^{i\epsilon\cdot(k_1+k_2+k_3+k_4)/4}\,\mathrm{tr}\{(M_0-\boldsymbol{k}_4)^{-1}\Gamma(z_4)(M_0-\boldsymbol{k}_3)^{-1}\Gamma(z_3)(M_0-\boldsymbol{k}_2)^{-1}\Gamma(z_2)(M_0-\boldsymbol{k}_1)^{-1}\Gamma(z_1)\}\,. \quad (11)$$

We introduce the loop integration variable $P=\frac{1}{4}(k_1+k_2+k_3+k_4)$ and make the change of variables

$$k_1=P+q_1,\quad k_2=P+q_2,\quad k_3=P+q_3,\quad k_4=P+q_4. \quad (12)$$

The momenta q_1, q_2, q_3, and q_4 in the trace are expanded in a Taylor series keeping only those terms which would be singular in the limit as ϵ goes to zero. As we are to perform a symmetric average on ϵ, we may also perform a symmetric average on the loop momentum P. Using these transformations, the singular part of the four-vertex loop becomes

$$S_\epsilon^4(\Gamma)=(-1)\tfrac{1}{4}(2\pi)^{-16}\int dz_1\int dz_2\int dz_3\int dz_4\int dP\,e^{iP\cdot\epsilon}\int dq_1e^{-iq_1\cdot(z_2-z_1)}\int dq_2e^{-iq_2\cdot(z_3-z_2)}\int dq_3e^{-iq_3\cdot(z_4-z_3)}$$
$$\times\int dq_4e^{-iq_4\cdot(z_1-z_4)}\delta\left(\frac{q_1+q_2+q_3+q_4}{4}\right)$$
$$\times\mathrm{tr}\{(M_0-\boldsymbol{P}-\boldsymbol{q}_4)^{-1}\Gamma(z_4)(M_0-\boldsymbol{P}-\boldsymbol{q}_3)^{-1}\Gamma(z_3)(M_0-\boldsymbol{P}-\boldsymbol{q}_2)^{-1}\Gamma(z_2)(M_0-\boldsymbol{P}-\boldsymbol{q}_1)^{-1}\Gamma(z_1)\}$$
$$=(-1)\tfrac{1}{4}(2\pi)^{-16}\int dz_1\cdots\int dz_4\int dP\,e^{iP\cdot\epsilon}\int dq_1e^{-iq_1\cdot(z_2-z_1)}\cdots\int dq_4e^{-iq_4\cdot(z_1-z_4)}\delta\left(\frac{q_1+q_2+q_3+q_4}{4}\right)$$
$$\times(M_0{}^2-P^2)^{-4}\,\mathrm{tr}\{\boldsymbol{P}\Gamma(z_4)\boldsymbol{P}\Gamma(z_3)\boldsymbol{P}\Gamma(x_2)\boldsymbol{P}\Gamma(z_1)\}$$

$$=(-1)\tfrac{1}{4}(2\pi)^{-4}\int dz\int dP\, e^{iP\cdot\epsilon}(M_0{}^2-P^2)^{-4}(P^2)^2$$
$$\times\{\tfrac{1}{12}\operatorname{tr}\gamma_\mu\Gamma(z)\gamma^\mu\Gamma(z)\gamma_\nu\Gamma(z)\gamma^\nu\Gamma(z)+(1/24)\operatorname{tr}\gamma_\mu\Gamma(z)\gamma_\nu\Gamma(z)\gamma^\mu\Gamma(z)\gamma^\nu\Gamma(z)\}$$
$$=-iC_1(\epsilon)\int dz\{(1/48)\operatorname{tr}\gamma_\mu\Gamma(z)\gamma^\mu\Gamma(z)\gamma_\nu\Gamma(z)\gamma^\nu\Gamma(z)+(1/96)\operatorname{tr}\gamma_\mu\Gamma(z)\gamma_\nu\Gamma(z)\gamma^\mu\Gamma(z)\gamma^\nu\Gamma(z)\}\,, \tag{13}$$

where we define the following ϵ-dependent functions:

$$C_1(\epsilon)=-i(2\pi)^{-4}\int dP\, e^{iP\cdot\epsilon}(M_0{}^2-P^2)^{-2}\,,\quad C_2(\epsilon)=-i(2\pi)^{-4}\int dP\, e^{iP\cdot\epsilon}(M_0{}^2-P^2)^{-1}. \tag{14}$$

B. Three-Vertex Loops

Using transformations similar to those for four-vertex loop and taking advantage of the cyclic permutation symmetry in the indices, the linearly divergent three-vertex loop may be written as

$$S_\epsilon{}^3(\Gamma)=(-1)\tfrac{1}{3}(i)^3\int dz_1\int dz_2\int dz_3[-i(2\pi)^{-4}]^3\int dk_1 e^{-ik_1\cdot(z_2-z_1)}\int dk_2 e^{-ik_2\cdot(z_3-z_2)}\int dk_3 e^{-ik_3\cdot(z_1-z_3)}$$
$$\times e^{i\epsilon\cdot(k_1+k_2+k_3)/3}\operatorname{tr}\{(M_0-\boldsymbol{k}_3)^{-1}\Gamma(z_3)(M_0-\boldsymbol{k}_2)^{-1}\Gamma(z_2)(M_0-\boldsymbol{k}_1)^{-1}\Gamma(z_1)\}$$
$$=(-1)\tfrac{1}{3}(2\pi)^{-12}\int dz_1\int dz_2\int dz_3\int dP\, e^{iP\cdot\epsilon}\int dq_1 e^{-iq_1\cdot(z_2-z_1)}\int dq_2 e^{-iq_2\cdot(z_3-z_2)}\int dq_3 e^{-iq_3\cdot(z_1-z_3)}$$
$$\times\delta\left(\frac{q_1+q_2+q_3}{3}\right)[3(M_0{}^2-P^2)^{-3}\operatorname{tr}\{\boldsymbol{P}\Gamma(z_3)\boldsymbol{P}\Gamma(z_2)M_0\Gamma(z_1)\}$$
$$+3(M_0{}^2-P^2)^{-4}\operatorname{tr}\{\boldsymbol{P}\boldsymbol{q}_3\boldsymbol{P}\Gamma(z_3)\boldsymbol{P}\Gamma(z_2)\boldsymbol{P}\Gamma(z_1)\}]$$
$$=iC_1(\epsilon)\int dz\{\tfrac{1}{4}\operatorname{tr}\gamma_\mu\Gamma(z)\gamma^\mu\Gamma(z)M_0\Gamma(z)+\tfrac{1}{12}\operatorname{tr}\gamma_\mu\Gamma(z)\gamma^\mu\Gamma(z)i\gamma\cdot\overleftrightarrow{\partial}\Gamma(z)\}\,, \tag{15}$$

where

$$A\overleftrightarrow{\partial}_\mu B=A(\partial_\mu B)-(\partial_\mu A)B.$$

C. Two-Vertex Loops

The singular part of the quadratically divergent two-vertex loop is determined to be

$$S_\epsilon{}^2(\Gamma)=(-1)\tfrac{1}{2}(i)^2\int dz_1\int dz_2[-i(2\pi)^{-4}]^2\int dk_1 e^{-ik_1\cdot(z_2-z_1)}\int dk_2 e^{-ik_2\cdot(z_1-z_2)}e^{i\epsilon\cdot(k_1+k_2)/2}$$
$$\times\operatorname{tr}\{(M_0-\boldsymbol{k}_2)^{-1}\Gamma(z_2)(M_0-\boldsymbol{k}_1)\Gamma(z_1)\}$$
$$=(-1)\tfrac{1}{2}(2\pi)^{-8}\int dz_1\int dz_2\int dP\, e^{iP\cdot\epsilon}\int dq_1 e^{-iq_1\cdot(z_2-z_1)}\int dq_2 e^{-iq_2\cdot(z_1-z_2)}\delta\left(\frac{q_1+q_2}{2}\right)$$
$$\times[(M_0{}^2-P^2)^{-2}\operatorname{tr}\{\boldsymbol{P}\Gamma(z_2)\boldsymbol{P}\Gamma(z_1)+M_0\Gamma(z_2)M_0\Gamma(z_1)\}$$
$$+2(M_0{}^2-P^2)^{-3}\operatorname{tr}\{\boldsymbol{P}\boldsymbol{q}_2\boldsymbol{P}\Gamma(z_2)M_0\Gamma(z_1)+\boldsymbol{P}\boldsymbol{q}_2M_0\Gamma(z_2)\boldsymbol{P}\Gamma(z_1)+M_0\boldsymbol{q}_2\boldsymbol{P}\Gamma(z_2)\boldsymbol{P}\Gamma(z_1)\}$$
$$+(M_0{}^2-P^2)^{-4}\operatorname{tr}\{2\boldsymbol{P}\boldsymbol{q}_2\boldsymbol{P}\boldsymbol{q}_2\boldsymbol{P}\Gamma(z_2)\boldsymbol{P}\Gamma(z_1)+\boldsymbol{P}\boldsymbol{q}_2\boldsymbol{P}\Gamma(z_2)\boldsymbol{P}\boldsymbol{q}_1\boldsymbol{P}\Gamma(z_1)\}]$$
$$=i\tfrac{1}{8}[C_2(\epsilon)-M_0{}^2C_1(\epsilon)]\int dz\{\operatorname{tr}\gamma_\mu\Gamma(z)\gamma^\mu\Gamma(z)\}-iC_1(\epsilon)\int dz[\tfrac{1}{2}M_0{}^2\operatorname{tr}\Gamma(z)\Gamma(z)$$
$$+\tfrac{1}{4}M_0\operatorname{tr}\Gamma(z)i\gamma\cdot\overleftrightarrow{\partial}\Gamma(z)-\tfrac{1}{12}\operatorname{tr}\gamma\cdot\partial\Gamma(z)\gamma\cdot\partial\Gamma(z)+(1/24)\operatorname{tr}\gamma_\mu\partial_\nu\Gamma(z)\gamma^\mu\partial^\nu\Gamma(z)]. \tag{16}$$

D. One-Vertex Loop

The one-vertex loop is given by

$$S_\epsilon{}^1(\Gamma)=(-1)i\int dz[-i(2\pi)^{-4}]\int dk\, e^{i\epsilon\cdot k}\operatorname{tr}(M_0-\boldsymbol{k})^{-1}\Gamma(z)=-iM_0C_2(\epsilon)\int dz\operatorname{tr}\Gamma(z). \tag{17}$$

By directly evaluating the singular part of the smaller loops, we have isolated the terms which remain ϵ-dependent in the limit as ϵ goes to zero. We define the sum of all single loop diagrams by

$$S_\epsilon(\tilde{\Gamma})=S_\epsilon(\Gamma-M_0)=\sum_{n=1}^{\infty} S_\epsilon{}^n(\Gamma). \tag{18}$$

The ϵ-dependent part of $S_\epsilon(\tilde{\Gamma})$ is given by the sum of the contributions from Eqs. (13) and (15)–(17). The sum of these terms is given in Eq. (19), where we have, for convenience, replaced $\Gamma(z)$ by $\tilde{\Gamma}+M_0$ in these expressions. We have

$$\begin{aligned}S_\epsilon(\tilde{\Gamma})=\tfrac{1}{8}[C_2(\epsilon)+M_0{}^2C_1(\epsilon)]i\int dz\ \mathrm{tr}\{\gamma_\mu\tilde{\Gamma}(z)\gamma^\mu\tilde{\Gamma}(z)\}-C_1(\epsilon)i\int dz[(1/48)\ \mathrm{tr}\{\gamma_\mu\tilde{\Gamma}(z)\gamma^\mu\tilde{\Gamma}(z)\gamma_\nu\tilde{\Gamma}(z)\gamma^\nu\tilde{\Gamma}(z)\}\\ +(1/96)\ \mathrm{tr}\{\gamma_\mu\tilde{\Gamma}(z)\gamma_\nu\tilde{\Gamma}(z)\gamma^\mu\tilde{\Gamma}(z)\gamma^\nu\tilde{\Gamma}(z)\}-\tfrac{1}{12}\ \mathrm{tr}\{\gamma_\mu\tilde{\Gamma}(z)\gamma^\mu\tilde{\Gamma}(z)i\gamma\cdot\overleftrightarrow{\partial}\tilde{\Gamma}(z)\}\\ -(1/24)\ \mathrm{tr}\{\gamma_\mu\partial_\nu\tilde{\Gamma}(z)\gamma^\mu\partial^\nu\tilde{\Gamma}(z)\}-\tfrac{1}{12}\ \mathrm{tr}\{\gamma\cdot\partial\tilde{\Gamma}(z)\gamma\cdot\partial\tilde{\Gamma}(z)\}]+\text{finite terms}.\end{aligned} \tag{19}$$

IV. WARD IDENTITIES

In this section, we examine the Ward identities for the theory defined in Sec. II. As noted in the Introduction, we expect the existence of additional terms in the Ward identities for the loops because of the singular nature of the smaller loops. The additional terms which do arise are, of course, dependent upon the precise definition of the singular loops. We will calculate the anomalous terms in the Ward identities for currents attached to the loops defined by symmetric ϵ separation. We need not consider the Ward identities for currents attached to a free spinor line, since these can contain no abnormal terms.[5]

The vector and axial-vector currents were defined in Eq. (10) by variation of the S matrix with respect to the external vector and axial-vector fields, respectively. For the currents attached to the single-loop diagrams [see Fig. 2 and Eq. (9)], it is convenient to make the substitution

$$V_+{}^\mu(x)\rightarrow V_+{}^\mu(x)+\partial^\mu\Lambda_+(x),$$

where

$$\Lambda_\pm(x)=\lambda_V{}^\alpha\Lambda^\alpha(x)\pm\lambda_A{}^\alpha\gamma_5\Lambda_5{}^\alpha(x). \tag{20}$$

The S matrix for the n-vertex loop becomes

$$S_\epsilon{}^n(\Gamma+\gamma\cdot\partial\Lambda_+)=S_\epsilon{}^n(\Gamma)+S_\epsilon{}^n(\gamma\cdot\partial\Lambda_+,\Gamma)+O(\Lambda_+{}^2). \tag{21}$$

The currents are given by

$$\begin{aligned}J_\mu{}^\alpha(x)&=-i[\delta/\delta\partial^\mu\Lambda^\alpha(x)]S_\epsilon{}^n(\Gamma+\gamma\cdot\partial\Lambda_+)|_{\Lambda_+=0}\\ &=-i[\delta/\delta\partial^\mu\Lambda^\alpha(x)]S_\epsilon{}^n(\gamma\cdot\partial\Lambda_+,\Gamma),\\ J_{5\mu}{}^\alpha(x)&=-i[\delta/\delta\partial^\mu\Lambda_5{}^\alpha(x)]S_\epsilon{}^n(\gamma\cdot\partial\Lambda_+,\Gamma).\end{aligned} \tag{22}$$

The divergences of the currents are given by

$$\begin{aligned}\partial^\mu J_\mu{}^\alpha(x)&=i[\delta/\delta\Lambda^\alpha(x)]S_\epsilon{}^n(\gamma\cdot\partial\Lambda_+,\Gamma),\\ \partial^\mu J_{5\mu}(x)&=i[\delta/\delta\Lambda_5{}^\alpha(x)]S_\epsilon{}^n(\gamma\cdot\partial\Lambda_+,\Gamma).\end{aligned} \tag{23}$$

We will evaluate $S_\epsilon{}^n(\gamma\cdot\partial\Lambda_+,\Gamma)$ using the expression for the n-vertex loop as given in Eq. (9). We have

$$\begin{aligned}S_\epsilon{}^n(\gamma\cdot\partial\Lambda_+,\Gamma)=(-1)(i)^n\int dz_1\cdots\int dz_n[-i(2\pi)^{-4}]^n\int dk_1e^{-ik_1\cdot(z_2-z_1)}\cdots\int dk_ne^{-ik_n\cdot(z_1-z_n)}\\ \times e^{i\epsilon\cdot(k_1+\cdots+k_n)/n}\ \mathrm{tr}\{(M_0-\boldsymbol{k}_n)^{-1}\gamma\cdot\partial\Lambda_+(z_n)(M_0-\boldsymbol{k}_{n-1})^{-1}\cdots(M_0-\boldsymbol{k}_1)^{-1}\Gamma(z_1)\}.\end{aligned} \tag{24}$$

Using an integration by parts and a cyclic permutation of the indices, we obtain

$$\begin{aligned}S_\epsilon{}^n(\gamma\cdot\partial\Lambda_+,\Gamma)&=(-1)(i)^n\int dz_1\cdots\int dz_n[-i(2\pi)^{-4}]^n\int dk_1e^{-ik_1\cdot(z_2-z_1)}\cdots\int dk_ne^{-ik_n\cdot(z_1-z_n)}e^{i\epsilon\cdot(k_1+\cdots+k_n)/n}\\ &\qquad\times\mathrm{tr}\{(M_0-\boldsymbol{k}_n)^{-1}i(\boldsymbol{k}_{n-1}-\boldsymbol{k}_n)\Lambda_+(z_n)(M_0-\boldsymbol{k}_{n-1})^{-1}\cdots(M_0-\boldsymbol{k}_1)^{-1}\Gamma(z_1)\}\\ &=(-1)i^n\int dz_1\cdots\int dz_n[-i(2\pi)^{-4}]^n\int dk_1e^{ik_1\cdot(z_2-z_1)}\cdots\int dk_ne^{-ik_n\cdot(z_1-z_n)}e^{i\epsilon\cdot(k_1+\cdots+k_n)/n}\\ &\qquad\times\mathrm{tr}\{(M_0-\boldsymbol{k}_n)^{-1}[iM_0\Lambda_-(z_n)-iM_0\Lambda_+(z_n)]\cdots(M_0-\boldsymbol{k}_1)^{-1}\Gamma(z_1)\}\\ &\quad+(-1)i^n\int dz_1\cdots\int dz_n[-i(2\pi)^{-4}]^n\int dk_1e^{-ik_1\cdot(z_2-z_1)}\cdots\int dk_ne^{-ik_n\cdot(z_1-z_n)}e^{i\epsilon\cdot(k_1+\cdots+k_{n-1})/(n-1)}\\ &\qquad\times\mathrm{tr}\{(M_0-\boldsymbol{k}_{n-1})^{-1}\Gamma(z_{n-1})\cdots(M_0-\boldsymbol{k}_1)^{-1}[\Gamma(z_1)i\Lambda_+(z_n)-i\Lambda_-(z_1)\Gamma(z_n)]\}\end{aligned}$$

[5] V. Takahashi, Nuovo Cimento 6, 370 (1957).

$$+(-1)i^n\int dz_1\cdots\int dz_n[-i(2\pi)^{-4}]^n\int dk_1 e^{-ik_1\cdot(z_2-z_1)}\cdots\int dk_n e^{-ik_n\cdot(z_1-z_n)}$$
$$\times[e^{i\epsilon\cdot(k_1+\cdots+k_n)/n}-e^{i\epsilon\cdot(k_1+\cdots+k_{n-1})/(n-1)}]$$
$$\times\mathrm{tr}\{(M_0-\boldsymbol{k}_{n-1})^{-1}\Gamma(z_{n-1})\cdots(M_0-\boldsymbol{k}_1)^{-1}[\Gamma(z_1)i\Lambda_+(z_n)-i\Lambda_-(z_1)\Gamma(z_n)]\}. \quad (25)$$

In the second term above, we perform the k_n and z_n integrations. In the third term, we make the following replacements:

$$k_n=P\,,\quad k_m=P+q_m\,,\quad m=1,\ldots,n-1$$

and use an integration by parts in P to replace $i\epsilon\cdot q$ by $-q\partial_P$. Equation (25) becomes

$$S_\epsilon^n(\gamma\cdot\partial\Lambda,\Gamma)=(-1)i^n\int dz_1\cdots\int dz_n[-i(2\pi)^4]^n\int dk_1 e^{-ik_1\cdot(z_2-z_1)}\cdots\int dk_n e^{-ik_n\cdot(z_1-z_n)}e^{i\epsilon\cdot(k_1+\cdots+k_n)/n}$$
$$\times\mathrm{tr}\{(M_0-\boldsymbol{k}_n)^{-1}[iM_0\Lambda_-(z_n)-iM_0\Lambda_+(z_n)]\cdots(M_0-\boldsymbol{k}_1)^{-1}\Gamma(z_1)\}$$
$$+(-1)i^{n-1}\int dz_1\cdots\int dz_{n-1}[-i(2\pi)^{-4}]^{n-1}\int dk_1 e^{-ik_1\cdot(z_2-z_1)}\cdots\int dk_{n-1}e^{-ik_{n-1}\cdot(z_1-z_{n-1})}$$
$$\times e^{i\epsilon\cdot(k_1+\cdots+k_{n-1})/(n-1)}\,\mathrm{tr}\{(M_0-\boldsymbol{k}_{n-1})^{-1}\Gamma(z_{n-1})\cdots(M_0-\boldsymbol{k}_1)^{-1}[\Gamma(z_1)i\Lambda_+(z_1)-i\Lambda_-(z_1)\Gamma(z_1)]\}$$
$$+(-1)(2\pi)^{-4n}\int dz_1\cdots\int dz_n\int dq_1 e^{-iq_1\cdot(z_2-z_1)}\cdots\int dq_{n-1}e^{-iq_{n-1}\cdot(z_n-z_{n-1})}\int dP\, e^{iP\cdot\epsilon}$$
$$\times[e^{-(q_1+\cdots+q_{n-1})\cdot\partial_P/n}-e^{-(q_1+\cdots+q_{n-1})\cdot\partial_P/(n-1)}]$$
$$\times\mathrm{tr}\{(M_0-\boldsymbol{P}-\boldsymbol{q}_{n-1})^{-1}\Gamma(z_{n-1})\cdots(M_0-\boldsymbol{P}-\boldsymbol{q}_1)^{-1}[\Gamma(z_1)i\Lambda_+(z_n)-i\Lambda_-(z_1)\Gamma(z_n)]\}$$
$$=S_\epsilon^n(iM_0(\Lambda_--\Lambda_+),\,\Gamma)+S_\epsilon^{n-1}(\Gamma i\Lambda_+-i\Lambda_-\Gamma,\,\Gamma)+D_\epsilon^n(\Lambda_+,\Gamma). \quad (26)$$

The first term on the right-hand side of Eq. (26) corresponds to the normal divergence in the interaction picture, the second term corresponds to the normal equal-time commutator, while the third term gives the anomalous divergence. From Eq. (26), the anomalous term in the n-vertex Ward identity is given by

$$D_\epsilon^n(\Lambda_+,\Gamma)=(-1)(2\pi)^{-4n}\int dz_1\cdots\int dz_n\int dq_1 e^{-iq_1\cdot(z_2-z_1)}\cdots\int dq_{n-1}e^{-iq_{n-1}\cdot(z_n-z_{n-1})}\int dP\, e^{iP\cdot\epsilon}$$
$$\times[e^{-(q_1+\cdots+q_{n-1})\cdot\partial_P/n}-e^{-(q_1+\cdots+q_{n-1})\cdot\partial_P/(n-1)}]$$
$$\times\mathrm{tr}\{(M_0-\boldsymbol{P}-\boldsymbol{q}_{n-1})^{-1}\Gamma(z_{n-1})\cdots(M_0-\boldsymbol{P}-\boldsymbol{q}_1)^{-1}[\Gamma(z_1)i\Lambda_+(z_n)-i\Lambda_-(z_1)\Gamma(z_n)]\}. \quad (27)$$

The expression for D_ϵ^n in Eq. (27) may be evaluated by noting that the term in square brackets is of order ϵ. Therefore, only those terms in the trace which are linearly, or more, divergent will survive in the limit as ϵ goes to zero. We calculate D_ϵ^n by first expanding the trace in a Taylor series in the momenta q, keeping only those terms which would be at least linearly divergent. The derivatives are then allowed to act; each derivative lowers the degree of divergence by one power. We retain only those terms which would survive in the limit as ϵ goes to zero. For $n>4$, the loops are sufficiently convergent, and D_ϵ^n is zero in the limit. We make an explicit calculation of the smaller loops and obtain

(a) $n=4$:

$$D_\epsilon^4(\Lambda_+,\Gamma)=\frac{1}{36}\frac{\pi^2}{(2\pi)^4}i\int dz\,\mathrm{tr}\{\partial^\mu\Lambda_+(z)\gamma_\mu\Gamma(z)\gamma_\nu\Gamma(z)\gamma^\nu(z)+\partial^\mu\Lambda_+(z)\gamma_\nu\Gamma(z)\gamma_\mu\Gamma(z)\gamma^\nu\Gamma(z)+\partial^\mu\Lambda_+(z)\gamma_\nu\Gamma(z)\gamma^\nu\Gamma(z)\gamma_\mu\Gamma(z)\};$$

(b) $n=3$:

$$D_\epsilon^3(\Lambda_+,\Gamma)=-\frac{\pi^2}{(2\pi)^4}i\int dz\ \mathrm{tr}\{\tfrac{1}{4}M_0\partial^\mu\Lambda_+(z)\gamma_\mu\Gamma(z)\Gamma(z)+\tfrac{1}{8}M_0\partial^\mu\Lambda_+(z)\Gamma(z)\gamma_\mu\Gamma(z)+\tfrac{1}{8}M_0\partial^\mu\Lambda_-(z)\Gamma(z)\gamma_\mu\Gamma(z)$$
$$+(1/24)M_0[\Lambda_-(z)-\Lambda_+(z)]\Gamma(z)\gamma\cdot\overleftrightarrow{\partial}\Gamma(z)+\tfrac{1}{12}i\partial^\mu\Lambda_+(z)\gamma_\mu\Gamma(z)\gamma\cdot\overleftrightarrow{\partial}(z)-\tfrac{1}{12}i\partial^\mu\Lambda_+(z)\gamma^\nu\Gamma(z)\gamma_\mu\overleftrightarrow{\partial}_\nu\Gamma(z)$$
$$-3\times6^{-3}i\partial^\mu\Lambda_+(z)\gamma_\nu\Gamma(z)\gamma^\nu\overleftrightarrow{\partial}_\mu\Gamma(z)+\tfrac{3}{2}\times6^{-3}i\partial^\mu\Lambda_+(z)\gamma_\mu\Gamma(z)\gamma\cdot\overleftrightarrow{\partial}\Gamma(z)+\tfrac{3}{2}\times6^{-3}i\partial^\mu\Lambda_+(z)\gamma^\nu\Gamma(z)\gamma_\mu\overleftrightarrow{\partial}_\nu\Gamma(z)\};\quad(28)$$

(c) $n=2$:

$$D_\epsilon^2(\Lambda_+,\Gamma)=-\tfrac{1}{2}[C_2(\epsilon)+M_0^2C_1(\epsilon)]i\int dz\ \mathrm{tr}\{\partial^\mu\Lambda_+(z)\gamma_\mu\Gamma(z)\}$$
$$+\frac{\pi^2}{(2\pi)^4}i\int dz\ \mathrm{tr}\{\tfrac{1}{8}M_0i[\Lambda_+(z)-\Lambda_-(z)]\partial^2\Gamma(z)+(1/24)\partial^\mu\Lambda_+(z)\gamma_\mu\partial^2\Gamma(z)\};$$

(d) $n=1$:

$D_\epsilon^1(\Lambda_+,\Gamma)=0$.

We now consider the Ward identity satisfied by the sum of all single loop diagrams. Using the S matrix defined in Eq. (18) and noting that

$$S_\epsilon^1(\gamma\cdot\partial\Lambda_+,\Gamma)=S_\epsilon^1(iM_0(\Lambda_--\Lambda_+),\ \Gamma)=D_\epsilon^1(\Lambda_+,\Gamma)=0\,,$$

we find

$$S_\epsilon(\gamma\cdot\partial\Lambda_+,\tilde\Gamma)=\sum_{n=1}^\infty S_\epsilon^n(\gamma\cdot\partial\Lambda_+,\Gamma)=\sum_{n=2}^\infty\{S_\epsilon^n(iM_0(\Lambda_--\Lambda_+),\ \Gamma)+S_\epsilon^{n-1}(\Gamma i\Lambda_+-i\Lambda_-\Gamma,\ \Gamma)+D_\epsilon^n(\Lambda_+,\Gamma)\}$$
$$=\sum_{n=1}^\infty\{S_\epsilon^n((\Gamma-M_0)i\Lambda_+-i\Lambda_-(\Gamma-M_0),\ \Gamma)+D_\epsilon^n(\Lambda_+,\Gamma)\}$$
$$=S_\epsilon(\tilde\Gamma i\Lambda_+-i\Lambda_-\tilde\Gamma,\ \tilde\Gamma)+D_\epsilon(\Lambda_+,\tilde\Gamma)\,,\quad(29)$$

where

$$D_\epsilon(\Lambda_+,\tilde\Gamma)=\sum_{n=1}^\infty D_\epsilon^n(\Lambda_+,\Gamma)=\sum_{n=2}^4 D_\epsilon^n(\Lambda_+,\Gamma)\,.\quad(30)$$

The first term on the right-hand side of Eq. (29) corresponds to the usual operator divergence, while the second term gives the anomalous divergence for the loops as defined in Sec. II. The anomalous divergence D_ϵ is given by the sum of the contributions in Eq. (28): We obtain the result

$$D_\epsilon(\Lambda_+,\tilde\Gamma)=\sum_{n=2}^4 D_\epsilon^n(\Lambda_+,\Gamma)=\frac{\pi^2}{(2\pi)^4}i\int dz\ \mathrm{tr}\{(1/36)\partial^\mu\Lambda_+(z)\gamma_\mu\tilde\Gamma(z)\gamma_\nu\tilde\Gamma(z)\gamma^\nu\tilde\Gamma(z)+(1/36)\partial^\mu\Lambda_+(z)\gamma_\nu\tilde\Gamma(z)\gamma_\mu\tilde\Gamma(z)\gamma^\nu\tilde\Gamma(z)$$
$$+(1/36)\partial^\mu\Lambda_+(z)\gamma_\nu\tilde\Gamma(z)\gamma^\nu\tilde\Gamma(z)\gamma_\mu\tilde\Gamma(z)+\tfrac{1}{6}i\Lambda_+(z)\gamma\cdot\partial\tilde\Gamma(z)\gamma\cdot\partial\tilde\Gamma(z)-\tfrac{1}{6}i\Lambda_-(z)\partial^\mu\tilde\Gamma(z)\gamma_\mu\partial^\nu\tilde\Gamma(z)\gamma_\nu$$
$$-(1/72)\partial_\mu\gamma_\nu\tilde\Gamma(z)\partial^\mu\gamma^\nu[\tilde\Gamma(z)i\Lambda_+(z)-i\Lambda_-(z)\tilde\Gamma(z)-\gamma\cdot\partial\Lambda_+(z)]$$
$$+(1/72)\gamma\cdot\partial\tilde\Gamma(z)\gamma\cdot\partial[\tilde\Gamma(z)i\Lambda_+(z)-i\Lambda_-(z)\tilde\Gamma(z)-\gamma\cdot\partial\Lambda_+(z)]\}$$
$$+\frac{\pi^2}{(2\pi)^4}M_0i\int dz\ \mathrm{tr}\{-\tfrac{1}{12}\tilde\Gamma(z)i\gamma\cdot\partial[\tilde\Gamma(z)i\Lambda_+(z)-i\Lambda_-(z)\tilde\Gamma(z)-\gamma\cdot\partial\Lambda_+(z)]$$
$$-\tfrac{1}{12}[\tilde\Gamma(z)i\Lambda_+(z)-i\Lambda_-(z)\tilde\Gamma(z)-\gamma\cdot\partial\Lambda_+(z)]i\gamma\cdot\partial\tilde\Gamma(z)\}$$
$$+\frac{1}{4}\left[C_2(\epsilon)+M_0^2C_1(\epsilon)+\frac{\pi^2}{(2\pi)^4}M_0^2\right]i\int dz\ \mathrm{tr}\{\gamma_\mu\gamma\cdot\partial\Lambda_+(z)\gamma^\mu\tilde\Gamma(z)\}\,,\quad(31)$$

where we have replaced Γ by $\tilde\Gamma+M_0$ and used integrations by parts in z to combine some of the terms.

This completes the discussion of the Ward identities for the loops defined with symmetric ϵ separation. As was noted in the beginning of this section, the Ward identities for the currents attached to external spinor lines contain no anomalous terms. Since these two types of diagrams are the only ones present in this theory, we may use the result in Eq. (29) to evaluate the divergences of the operator currents defined in Eq. (10). The anomalous terms will be given in terms of D_ϵ, using relations of the form given in Eq. (23). In the interaction picture, we find

$$\begin{aligned}\partial^\mu J_\mu{}^\alpha(x) &= \partial^\mu \left\{ \bar\psi(x)\gamma_\mu \lambda_V{}^\alpha \psi(x) \exp\left[i\int dz\, \mathcal{L}_I(z)\right]\right\}_\epsilon \\ &= \left\{\left[\bar\psi(x)[\lambda_V{}^\alpha, i\tilde\Gamma(x)]\psi(x) + i\frac{\delta}{\delta\Lambda^\alpha(x)} D_\epsilon(\Lambda_+,\tilde\Gamma)\right]\exp\left[i\int dz\,\mathcal{L}_I(z)\right]\right\}_\epsilon , \\ \partial^\mu J_{5\mu}{}^\alpha(x) &= \partial^\mu \left\{ \bar\psi(x)\gamma_\mu\gamma_5 \lambda_A{}^\alpha \psi(x) \exp\left[i\int dz\, \mathcal{L}_I(z)\right]\right\}_\epsilon \\ &= \left\{\left[\bar\psi(x)\{\lambda_A{}^\alpha\gamma_5, -i\tilde\Gamma(x)\}\psi(x) + i\frac{\delta}{\delta\Lambda_5{}^\alpha(x)} D_\epsilon(\Lambda_+,\tilde\Gamma)\right]\exp\left[i\int dz\,\mathcal{L}_I(z)\right]\right\}_\epsilon .\end{aligned} \tag{32}$$

The equivalent form for the divergence equations, written in the "Heisenberg picture," would be

$$\begin{aligned}\partial^\mu J_\mu{}^\alpha(x) &= \partial^\mu\{\bar\psi(x)\gamma_\mu\lambda_V{}^\alpha\psi(x)\}_\epsilon \\ &= \{\bar\psi(x)[\lambda_V{}^\alpha, i\tilde\Gamma(x)]\psi(x)\}_\epsilon \\ &\qquad + i[\delta/\delta\Lambda^\alpha(x)]D_\epsilon(\Lambda_+,\tilde\Gamma), \\ \partial^\mu J_{5\mu}{}^\alpha(x) &= \partial^\mu\{\bar\psi(x)\gamma_\mu\gamma_5\lambda_A{}^\alpha\psi(x)\}_\epsilon \\ &= \{\bar\psi(x)\{\lambda_A{}^\alpha\gamma_5, -i\tilde\Gamma(x)\}\psi(x)\}_\epsilon \\ &\qquad + i[\delta/i\delta\Lambda_5{}^\alpha(x)]D_\epsilon(\Lambda_+,\tilde\Gamma),\end{aligned} \tag{33}$$

where $\psi(x)$ are the Heisenberg spinor field operators, and the local product of the spinor fields is defined to correspond to symmetric ϵ separation in the interaction picture.

In this section, we have shown how anomalous terms arise in the Ward identities and divergence equations for the currents in a theory defined by symmetric ϵ separation. These terms arose from the singular nature of the smaller loop diagrams. Because of the way we have defined the loops, the anomalous divergence contains many terms and may be seen in Eq. (31). In Sec. V, we will redefine the loops in order to determine a "minimal" anomalous divergence.

V. RENORMALIZATION

The definition for the spinor loops given in Sec. II is not the only one we could have used. The absorptive parts of the loops are unaffected by the addition of local polynomials in the external fields and their derivatives. In this section we define renormalized loops by the addition of counterterms to the single-loop S matrix given in Eq. (18). The counterterms will serve two purposes. They should remove the terms, isolated in Sec. III, which were singular in the limit as ϵ^2 goes to zero. In addition, the counterterms will be chosen to remove many of the anomalous terms in Ward identities found in Sec. IV.

The renormalized single-loop S matrix is given by

$$S_R(\tilde\Gamma) = S_\epsilon(\tilde\Gamma) - R(\tilde\Gamma), \tag{34}$$

where $S_\epsilon(\tilde\Gamma)$ is defined in Eq. (18) to be the sum of all single-loop diagrams with symmetric ϵ separation, and $R(\tilde\Gamma)$ is a local polynomial in the external fields and their derivatives. The renormalized S matrix satisfies the following Ward identity, corresponding to the one satisfied by the ϵ-separated S matrix in Eq. (29):

$$\begin{aligned} S_R(\gamma\cdot\partial\Lambda_+,\tilde\Gamma) &= S_\epsilon(\gamma\cdot\partial\Lambda_+,\tilde\Gamma) - R(\gamma\cdot\partial\Lambda_+,\tilde\Gamma) \\ &= S_\epsilon(\tilde\Gamma i\Lambda_+ - i\Lambda_-\tilde\Gamma,\,\tilde\Gamma) + D_\epsilon(\Lambda_+,\tilde\Gamma) - R(\gamma\cdot\partial\Lambda_+,\tilde\Gamma) \\ &= S_R(\tilde\Gamma i\Lambda_+ - i\Lambda_-\tilde\Gamma,\,\tilde\Gamma) + R(\tilde\Gamma i\Lambda_+ - i\Lambda_-\tilde\Gamma - \gamma\cdot\partial\Lambda_+,\,\tilde\Gamma) \\ &\qquad + D_\epsilon(\Lambda_+,\tilde\Gamma) \\ &= S_R(\tilde\Gamma i\Lambda_+ - i\Lambda_-\tilde\Gamma,\,\tilde\Gamma) + D_R(\Lambda_+,\tilde\Gamma), \end{aligned} \tag{35}$$

where

$$D_R(\Lambda_+,\tilde\Gamma) = R(\tilde\Gamma i\Lambda_+ - i\Lambda_-\tilde\Gamma - \gamma\cdot\partial\Lambda_+,\,\tilde\Gamma) + D_\epsilon(\Lambda_+,\tilde\Gamma). \tag{36}$$

$D_R(\Lambda_+,\tilde\Gamma)$ gives the anomalous divergence for the loops defined by the subtraction of the counterterm $R(\tilde\Gamma)$.

In order that the loops be finite in the limit as ϵ^2 goes to zero, we must include counterterms which remove the singular terms given in Eq. (19). We define

$$\begin{aligned} R_1(\tilde\Gamma) = \frac{1}{8}\left[C_2(\epsilon) + M_0{}^2C_1(\epsilon) + \frac{\pi^2}{(2\pi)^4}M_0{}^2\right] i\int dz\, \mathrm{tr}\{\gamma_\mu\tilde\Gamma(z)\gamma^\mu\tilde\Gamma(z)\} - C_1(\epsilon) i\int dz\, \mathrm{tr}\{(1/48)\gamma_\mu\tilde\Gamma(z)\gamma^\mu\tilde\Gamma(z)\gamma_\nu\tilde\Gamma(z)\gamma^\nu\tilde\Gamma(z) \\ + (1/96)\gamma_\mu\tilde\Gamma(z)\gamma_\nu\tilde\Gamma(z)\gamma^\mu\tilde\Gamma(z)\gamma^\nu\tilde\Gamma(z) - \tfrac{1}{12}\gamma_\mu\tilde\Gamma(z)\gamma^\mu\tilde\Gamma(z) i\gamma\cdot\overleftrightarrow{\partial}\tilde\Gamma(z) \\ - (1/24)\gamma_\mu\partial_\nu\tilde\Gamma(z)\gamma^\mu\partial^\nu\tilde\Gamma(z) - \tfrac{1}{12}\,\mathrm{tr}\gamma\cdot\partial\tilde\Gamma(z)\gamma\cdot\partial\tilde\Gamma(z)\}. \end{aligned} \tag{37}$$

By comparing Eq. (37) with Eq. (19), we see that $S_{R_1}(\tilde\Gamma) = S_\epsilon(\tilde\Gamma) - R_1(\tilde\Gamma)$ is finite in the limit as ϵ goes to zero. The contribution of R_1 to the anomalous divergence may be calculated using Eq. (36). The second term in Eq. (37)

does not contribute, while the first term gives

$$R_1(\tilde{\Gamma}i\Lambda_+ - i\Lambda_-\tilde{\Gamma} - \gamma\cdot\partial\Lambda_+, \tilde{\Gamma}) = -\frac{1}{4}\left[C_2(\epsilon)+M_0^2C_1(\epsilon)+\frac{\pi^2}{(2\pi)^4}M_0^2\right]i\int dz\, \mathrm{tr}\{\gamma_\mu\gamma\cdot\partial\Lambda_+(z)\gamma^\mu\tilde{\Gamma}(z)\}\,. \tag{38}$$

This contribution exactly cancels the third term in the expression for the anomalous divergence D_ϵ in Eq. (31).

We now consider additional counterterms which will be used to remove some of the terms remaining in the anomalous divergence D_{R_1}. We define the counterterm $R_2(\tilde{\Gamma})$ by

$$\begin{aligned}R_2(\tilde{\Gamma}) = {}&\frac{1}{24}\frac{\pi^2}{(2\pi)^4}M_0 i\int dz\,\mathrm{tr}\{\tilde{\Gamma}(z)i\gamma\cdot\overleftrightarrow{\partial}\tilde{\Gamma}(z)\}+\frac{1}{144}\frac{\pi^2}{(2\pi)^4}i\int dz\,\mathrm{tr}\{\gamma_\mu\partial_\nu\tilde{\Gamma}(z)\gamma^\mu\partial^\nu\tilde{\Gamma}(z)-\gamma\cdot\partial\tilde{\Gamma}(z)\gamma\cdot\partial\tilde{\Gamma}(z)\\ &+\frac{1}{12}\frac{\pi^2}{(2\pi)^4}G_0^2 i\int dz\,\mathrm{tr}\{\gamma\cdot V_-(z)P_+(z)\gamma\cdot V_-(z)P_+(z)+2\gamma\cdot V_-(z)\gamma\cdot V_+(z)P_-(z)P_+(z)\}\\ &-\frac{1}{72}\frac{\pi^2}{(2\pi)^4}i\int dz\,\mathrm{tr}\{2V_{+\mu}(z)V_+^{\mu}(z)V_{+\nu}(z)V_+^{\nu}(z)+V_{+\mu}(z)V_{+\nu}(z)V_+^{\mu}(z)V_+^{\nu}(z)\}\,,\end{aligned} \tag{39}$$

where $P_+(z)$ and $V_+^\mu(z)$ are given in terms of $\tilde{\Gamma}(z)$ in Eq. (2). The contribution of these counterterms to the anomalous divergence may be calculated using Eq. (36). For the S matrix defined by

$$S_{R_1+R_2}(\tilde{\Gamma}) = S_\epsilon(\tilde{\Gamma}) - R_1(\tilde{\Gamma}) - R_2(\tilde{\Gamma})\,, \tag{40}$$

we find the associated anomalous divergence for the loops has the form

$$\begin{aligned}D_{R_1+R_2}(\Lambda_+,\tilde{\Gamma}) &= R_1(\tilde{\Gamma}i\Lambda_+ - i\Lambda_-\tilde{\Gamma}-\gamma\cdot\partial\Lambda_+,\tilde{\Gamma})+R_2(\tilde{\Gamma}i\Lambda_+ - i\Lambda_-\tilde{\Gamma}-\gamma\cdot\partial\Lambda_+,\tilde{\Gamma})+D_\epsilon(\Lambda_+,\tilde{\Gamma})\\ &= -\frac{1}{6}\frac{\pi^2}{(2\pi)^4}i\int dz\, i\epsilon_{\mu\nu\sigma\tau}\,\mathrm{tr}\gamma_5\{2i\Lambda_+(z)\partial^\mu V_+^\nu(z)\partial^\sigma V_+^\tau(z)-\partial^\mu\Lambda_+(z)V_+^\nu(z)V_+^\sigma(z)V_+^\tau(z)\}\,,\end{aligned} \tag{41}$$

where $\epsilon_{\mu\nu\sigma\tau}$ is the totally antisymmetric tensor, with $\epsilon_{0123}=1$.

The loops, as defined in Eq. (40), have a number of interesting properties. They are all finite in the limit as ϵ^2 goes to zero; therefore, we may pass to this limit. The additional, finite counterterm R_2 allows the anomalous divergence to take on the particularly simple form given in Eq. (41). We note that all of the terms containing scalar and pseudoscalar fields have been removed from the anomalous divergence. In a theory with scalar and pseudoscalar external fields, only "matrix elements" or three or more currents are affected by the anomalous divergence. In addition, we note that the anomalous divergence is proportional to the pseudotensor $\epsilon_{\mu\nu\sigma\tau}$ so that only matrix elements having an abnormal parity relation are affected.

We briefly mention the construction of the full S matrix and the operator divergence equations with the loops defined by Eq. (40). In the interaction picture, the full S matrix is given by

$$S_{R_1+R_2} = T\left\{\exp\left[i\int dz\,\mathcal{L}_I(z)\right]\right\}_\epsilon \exp[-R_1(\tilde{\Gamma})-R_2(\tilde{\Gamma})]\,. \tag{42}$$

The currents are obtained from the S matrix in Eq. (42) by variation with respect to the external vector and axial-vector fields, as done in Eq. (10). We obtain

$$\begin{aligned}J_\mu{}^\alpha(x) &= \left\{\left[\bar{\psi}(x)\gamma_\mu\lambda_V{}^\alpha\psi(x)+i\frac{\delta}{\delta\partial^\mu\Lambda^\alpha(x)}[R_1(\gamma\cdot\partial\Lambda_+,\tilde{\Gamma})+R_2(\gamma\cdot\partial\Lambda_+,\tilde{\Gamma})]\right]\exp\left[i\int dz\,\mathcal{L}_I(z)\right]\right\}_\epsilon\\ &\qquad\times\exp[-R_1(\tilde{\Gamma})-R_2(\tilde{\Gamma})]\\ &= \{\bar{\psi}(x)\gamma_\mu\lambda_V{}^\alpha\psi(x)\}_{R_1+R_2}\,,\\ J_{5\mu}{}^\alpha(x) &= \left\{\left[\bar{\psi}(x)\gamma_\mu\lambda_A{}^\alpha\psi(x)+i\frac{\delta}{\delta\partial^\mu\Lambda_5{}^\alpha(x)}[R_1(\gamma\cdot\partial\Lambda_+,\tilde{\Gamma})+R_2(\gamma\cdot\partial\Lambda_+,\tilde{\Gamma})]\right]\exp\left[i\int dz\,\mathcal{L}_I(z)\right]\right\}_\epsilon\\ &\qquad\times\exp[-R_1(\tilde{\Gamma})-R_2(\tilde{\Gamma})]\\ &= \{\bar{\psi}(x)\gamma_\mu\gamma_5\lambda_A{}^\alpha\psi(x)\}_{R_1+R_2}\,.\end{aligned} \tag{43}$$

The divergence equations, corresponding to those in Eq. (33), are given by

$$\begin{aligned}
\partial^\mu J_\mu{}^\alpha(x) &= \partial^\mu\{\bar\psi(x)\gamma_\mu\lambda_V{}^\alpha\psi(x)\}_{R_1+R_2} \\
&= \{\bar\psi(x)[\lambda_V{}^\alpha, i\tilde\Gamma(x)]\psi(x)\}_{R_1+R_2} + i[\delta/\delta\Lambda^\alpha(x)]D_{R_1+R_2}(\Lambda_+,\tilde\Gamma) \\
&= \{\bar\psi(x)[\lambda_V{}^\alpha, i\tilde\Gamma(x)]\psi(x)\}_{R_1+R_2} - \tfrac{1}{6}[\pi^2/(2\pi)^4]i\epsilon_{\mu\nu\sigma\tau}\,\mathrm{tr}\{2i\lambda_V{}^\alpha\gamma_5\partial^\mu V_+{}^\nu(x)\partial^\sigma V_+{}^\tau(x) \\
&\qquad + \lambda_V{}^\alpha\gamma_5[\partial^\mu V_+{}^\nu(x)V_+{}^\sigma(x)V_+{}^\tau(x) - V_+{}^\mu(x)\partial^\nu V_+{}^\sigma(x)V_+{}^\tau(x) + V_+{}^\mu(x)V_+{}^\nu(x)\partial^\sigma V_+{}^\tau(x)]\}, \\
\partial^\mu J_{5\mu}{}^\alpha(x) &= \partial^\mu\{\bar\psi(x)\gamma_\mu\gamma_5\lambda_A{}^\alpha\psi(x)\}_{R_1+R_2} \\
&= \{\bar\psi(x)\{\lambda_A{}^\alpha\gamma_5, -i\tilde\Gamma(x)\}\psi(x)\}_{R_1+R_2} + i[\delta/\delta\Lambda_5{}^\alpha(x)]D_{R_1+R_2}(\Lambda_+,\tilde\Gamma) \\
&= \{\bar\psi(x)\{\lambda_A{}^\alpha\gamma_5, -i\tilde\Gamma(x)\}\psi(x)\}_{R_1+R_2} - \tfrac{1}{6}[\pi^2/(2\pi)^4]i\epsilon_{\mu\nu\sigma\tau}\,\mathrm{tr}\{2i\lambda_A{}^\alpha\partial^\mu V_+{}^\nu(x)\partial^\sigma V_+{}^\tau(x) \\
&\qquad + \lambda_A{}^\alpha[\partial^\mu V_+{}^\nu(x)V_+{}^\sigma(x)V_+{}^\tau(x) - V_+{}^\mu(x)\partial^\nu V_+{}^\sigma(x)V_+{}^\tau(x) + V_+{}^\mu(x)V_+{}^\nu(x)\partial^\sigma V_+{}^\tau(x)]\}.
\end{aligned} \tag{44}$$

In the derivation of the renormalized loops in Eq. (40), we have treated the vector and axial-vector fields symmetrically. Therefore, both the vector and axial-vector currents contain anomalous divergences. In some theories, such as electrodynamics or the gluon model, it would be useful to define the loops so that the vector currents obey the usual divergence equations, while the axial-vector currents have the only anomalous divergences. We are able to do this by again redefining the loops by the addition of the following counter term:

$$\begin{aligned}
R_3(\tilde\Gamma) = -\frac{1}{6}\frac{\pi^2}{(2\pi)^4}i\int dz\, i\epsilon_{\mu\nu\sigma\tau}\,\mathrm{tr}\{&-iV_-{}^\mu(z)V_+{}^\nu(z)i\overleftrightarrow{\partial}{}^\sigma V_+{}^\tau(z)\gamma_5 \\
&- V_-{}^\mu(z)V_+{}^\nu(z)V_+{}^\sigma(z)V_+{}^\tau(z)\gamma_5 + \tfrac{1}{4}V_+{}^\mu(z)V_-{}^\nu(z)V_+{}^\sigma(z)V_+{}^\tau(z)\gamma_5\}.
\end{aligned} \tag{45}$$

The loops are then defined by

$$S_{R_1+R_2+R_3}(\tilde\Gamma) = S_\epsilon(\tilde\Gamma) - R_1(\tilde\Gamma) - R_2(\tilde\Gamma) - R_3(\tilde\Gamma), \tag{46}$$

and the associated anomalous divergence is given by

$$\begin{aligned}
D_{R_1+R_2+R_3}(\Lambda_+,\tilde\Gamma) &= R_3(\tilde\Gamma i\Lambda_+ - i\Lambda_-\tilde\Gamma - \gamma\cdot\partial\Lambda_+,\ \tilde\Gamma) + D_{R_1+R_2}(\Lambda_+,\tilde\Gamma) \\
&= -\frac{1}{4\pi^2}i\int dz\,\epsilon_{\mu\nu\sigma\tau}\,\mathrm{tr}_I\Lambda_5(z)\{\tfrac{1}{4}F_V{}^{\mu\nu}(z)F_V{}^{\sigma\tau}(z) + \tfrac{1}{12}F_A{}^{\mu\nu}(z)F_A{}^{\sigma\tau}(z) + \tfrac{2}{3}iA^\mu(z)A^\nu(z)F_V{}^{\sigma\tau}(z) + \tfrac{2}{3}iF_V{}^{\mu\nu}(z)A^\sigma(z)A^\tau(z) \\
&\qquad + \tfrac{2}{3}iA^\mu(z)F_V{}^{\nu\sigma}(z)A^\tau(z) - (8/3)A^\mu(z)A^\nu(z)A^\sigma(z)A^\tau(z)\},
\end{aligned} \tag{47}$$

where tr_I means the trace only over the internal degrees of freedom. The field-strength tensors $F_V{}^{\mu\nu}(z)$ and $F_A{}^{\mu\nu}(z)$ are of the Yang-Mills type[6] and are invariant under vector and axial-vector gauge transformations. They are given by

$$\begin{aligned}
F_V{}^{\mu\nu}(z) &= \partial^\mu V^\nu(z) - \partial^\nu V^\mu(z) - i[V^\mu(z),V^\nu(z)] - i[A^\mu(z),A^\nu(z)], \\
F_A{}^{\mu\nu}(z) &= \partial^\mu A^\nu(z) - \partial^\nu A^\mu(z) - i[V^\mu(z),A^\nu(z)] - i[A^\mu(z),V^\nu(z)].
\end{aligned} \tag{48}$$

The full S matrix and the operator currents are defined by the addition of $R_3(\tilde\Gamma)$ to the expressions in Eqs. (42) and (43), respectively. The operator divergence equations for the vector and axial-vector currents corresponding to those in Eq. (44) may be determined using the expression for the anomalous divergence in Eq. (47). We obtain the result

$$\begin{aligned}
\partial^\mu J_\mu{}^\alpha(x) &= \partial^\mu\{\bar\psi(x)\gamma_\mu\lambda_V{}^\alpha\psi(x)\}_{R_1+R_2+R_3} \\
&= \{\bar\psi(x)[\lambda_V{}^\alpha, i\tilde\Gamma(x)]\psi(x)\}_{R_1+R_2+R_3} + i[\delta/\delta\Lambda^\alpha(x)]D_{R_1+R_2+R_3}(\Lambda_+,\tilde\Gamma) \\
&= \{\bar\psi(x)[\lambda_V{}^\alpha, i\tilde\Gamma(x)]\psi(x)\}_{R_1+R_2+R_3}, \\
\partial^\mu J_{5\mu}{}^\alpha(x) &= \partial^\mu\{\bar\psi(x)\gamma_\mu\gamma_5\gamma_A{}^\alpha\psi(x)\}_{R_1+R_2+R_3} \\
&= \{\bar\psi(x)\{\lambda_A{}^\alpha\gamma_5, -i\tilde\Gamma(x)\}\psi(x)\}_{R_1+R_2+R_3} + i[\delta/\delta\Lambda_5{}^\alpha(x)]D_{R_1+R_2+R_3}(\Lambda_+,\tilde\Gamma) \\
&= \{\bar\psi(x)\{\lambda_A{}^\alpha\gamma_5, -i\tilde\Gamma(x)\}\psi(x)\}_{R_1+R_2+R_3} + (1/4\pi^2)\epsilon_{\mu\nu\sigma\tau}\,\mathrm{tr}_I[\lambda_A{}^\alpha\{\tfrac{1}{4}F_V{}^{\mu\nu}(x)F_V{}^{\sigma\tau}(x) + \tfrac{1}{12}F_A{}^{\mu\nu}(x)F_A{}^{\sigma\tau}(x) \\
&\qquad + \tfrac{2}{3}iA^\mu(x)A^\nu(x)F_V{}^{\sigma\tau}(x) + \tfrac{2}{3}iF_V{}^{\mu\nu}(x)A^\sigma(x)A^\tau(x) + \tfrac{2}{3}iA^\mu(x)F_V{}^{\nu\sigma}(x)A^\tau(x) - (8/3)A^\mu(x)A^\nu(x)A^\sigma(x)A^\tau(x)\}].
\end{aligned} \tag{49}$$

The divergence equations given in Eq. (49) represent the end product of our entire calculation. The anomalous term in the divergence of axial-vector current is, indeed, the minimal anomalous divergence. The addition of further counterterms would either destroy the normal divergence of the vector current or give additional terms in the axial-vector-current divergence equations. To see this fact, we examine the possible counterterms that could be added to the S matrix in Eq. (46). In order to preserve the vector-current divergence equation, the vector fields can only enter the counterterm through the field-strength tensors in Eq. (48). To have any effect on the anomalous diver-

[6] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1959).

gence in Eq. (49), the counterterm must be proportional to the pseudotensor $\epsilon_{\mu\nu\sigma\tau}$; therefore, the product of the fields must be odd under parity and even under charge conjugation. Under these operations, the fields transform according to

$$\begin{aligned} P:&\quad V\to V,\ A\to -A,\ F_V\to F_V,\ F_A\to -F_A\,,\\ C:&\quad V\to -V^T,\ A\to A^T,\ F_V\to -F_V{}^T,\ F_A\to F_A{}^T\,, \end{aligned} \tag{50}$$

where V^T means the transpose of the matrix indices in the internal space.

The possible counterterms are given by

$$\begin{aligned} R_4(V,A)=i\int dz\ \epsilon_{\mu\nu\sigma\tau}[&R_{41}\,\mathrm{tr}_I\{F_V{}^{\mu\nu}(z)F_A{}^{\sigma\tau}(z)\}\\ &+R_{42}\,\mathrm{tr}_I\{F_V{}^{\mu\nu}(z)\}\ \mathrm{tr}_I\{F_A{}^{\sigma\tau}(z)\}\\ &+iR_{43}\,\mathrm{tr}_I\{F_A{}^{\mu\nu}(z)A^\sigma(z)A^\tau(z)\}\\ &+R_{44}\,\mathrm{tr}_I\{F_A{}^{\mu\nu}(z)A^\sigma(z)\}\ \mathrm{tr}_I A^\tau(z)]\\ &+\text{terms that are higher order in the fields or their derivatives.} \end{aligned} \tag{51}$$

The first two terms in Eq. (51) may be present, but they do not give any contribution to the anomalous divergence. The third term can be eliminated, since it is odd under charge conjugation. Note that we have not included terms in Eq. (51) which would violate parity. The fourth term may be present and can contribute terms to the anomalous divergence. However, this term does not have the coupling structure of a spinor loop and cannot be used to cancel completely any of the anomalous terms in Eq. (49). For completeness, we give the form of the contribution that this counterterm would give to the anomalous axial-vector-current divergence:

$$\begin{aligned} D_{R_4}&(\Lambda_5,V,A)\\ &=R_4([A,i\Lambda_5],[V,i\Lambda_5]-\partial\Lambda_5,V,A)\\ &=R_{44}i\int dz\ \epsilon_{\mu\nu\sigma\tau}[2\,\mathrm{tr}_I\{i\Lambda_5(z)[A^\sigma(z),F_V{}^{\mu\nu}(z)]\}\\ &\quad\times\mathrm{tr}_I\{A^\tau(z)\}+2\,\mathrm{tr}_I\{i\Lambda_5(z)\}\ \mathrm{tr}_I\{F_V{}^{\mu\nu}(z)A^\sigma(z)A^\tau(z)\}\\ &\quad-\tfrac{1}{2}\,\mathrm{tr}_I\{\Lambda_5(z)\}\ \mathrm{tr}_I\{F_A{}^{\mu\nu}(z)F_A{}^{\sigma\tau}(z)\}]. \end{aligned} \tag{52}$$

The terms which are higher order in the fields or their derivatives in Eq. (51) will not give contributions to the anomalous divergence of the form given in Eq. (49) and would simply add higher-order terms to the anomalous divergence.

We should note that all of the counterterms that we have used to define the loops are consistent with Weinberg's theorem[7] for the asymptotic behavior of the vertex functions. Therefore, if these definitions of the loops were used in an otherwise renormalizable, interacting field theory, the addition of these counterterms would not destroy the renormalizability.

The result that we have obtained for the anomalous axial-vector current in Eq. (49) may be directly related to the recent result of Adler[2] concerning the anomalous divergence of the neutral axial-vector isospin current. In these theories, the external axial-vector field is not present, and the external vector field is the photon field. The anomalous divergence is given in Eq. (49) by making the following substitutions:

$$F_V{}^{\mu\nu}(x)=e_0\lambda^q F_\gamma{}^{\mu\nu}(x)\,,\quad A^\mu(x)=0\,, \tag{53}$$

where λ^q is the charge matrix for the spinor field. The anomalous divergence becomes

$$\begin{aligned} \partial^\mu J_{5\mu}{}^0(x)=J_5{}^0(x)+(1/4\pi)\alpha_0&\\ \times\mathrm{tr}_I\lambda_A{}^0\lambda^q\lambda^q\epsilon_{\mu\nu\sigma\tau}&F_\gamma{}^{\mu\nu}(x)F_\gamma{}^{\sigma\tau}(x)\,, \end{aligned} \tag{54}$$

where $\lambda_A{}^0$ is the coupling matrix for the neutral axial-vector isospin current to the spinor field, and $\alpha_0=e_0{}^2/4\pi$. This is the result quoted by Adler.

VI. CONCLUSION

In this paper, we have studied the theory of a quantized spinor field with arbitrary internal degrees of freedom having arbitrary coupling to external scalar, pseudoscalar, vector, and axial-vector fields. By going into the interaction picture, we were able to carefully define and make finite all S-matrix elements. The vector and axial-vector currents were defined by a variation of the S matrix with respect to the external vector and axial-vector fields. Because of the singular nature of the smaller spinor loops, the Ward identities satisfied by these currents were found to contain anomalous terms. By considering all possible S-matrix elements, we were able to write the divergence equations for the operator currents defined in this theory, as given in Eq. (33).

The anomalous terms in the divergence equations for the currents could be cast in a particularly simple from by a redefinition of the spinor loops. By treating the vector and axial-vector currents symmetrically, the currents were found to satisfy the divergence equations in Eq. (44). If we required that the vector currents have the normal divergences, the divergences of the axial-vector currents contained the minimal anomalous terms given in Eq. (49). These anomalous terms were minimal in the sense that any further redefinition of the S matrix would either destroy the normal vector-current divergences or simply give additional terms in the anomalous axial-vector-current divergences.

The result that we have obtained for the minimal anomalous divergence of axial-vector current is not actually dependent upon our original definition of the S matrix. Any other definition which makes the S matrix well defined and finite could be used as a starting

[7] S. Weinberg, Phys. Rev. **118**, 838 (1960). The asymptotic behaviors of the n-point functions as determined by Weinberg are that the two-point function may be quadratically divergent, the three-point function may be linearly divergent, and the four-point function may be logarithmically divergent. The counter terms R_1, R_2, and R_3 in Eqs. (37), (39), and (45) do obey these asymptotic bounds.

point. The Ward identities could then be calculated, and we would, in general, expect to find many anomalous terms, as we did in the case of the S matrix defined by symmetric ϵ separation. However, by again adding the appropriate counterterms, we would find the same minimal anomalous divergence given in Eq. (49). Therefore, while the form of counterterms R_1, R_2, and R_3 does depend upon the original definition of the S matrix, the form of the minimal anomalous divergence does not.

We wish to comment briefly on the form of the minimal anomalous divergence obtained in Eq. (49). We note that all of the anomalous terms dependent upon the scalar and pseudoscalar fields have been removed. Therefore, in a theory having only scalar and pseudoscalar external fields, the Ward identities for the matrix elements of three or more currents are the only ones affected by the presence of the anomalous terms. In addition, the anomalous terms are all proportional to the pseudotensor $\epsilon_{\mu\nu\sigma\tau}$, and hence will affect only those matrix elements of currents having an abnormal parity relation.

Anomalous terms in the axial-vector-current Ward identities can arise in two ways. In Fig. 3, we illustrate the types of diagrams which were found to have anomalous Ward identities. The Ward identities for the triangle and box diagrams involve terms which are linearly, or more highly, divergent leading to the existence of anomalous terms when the loop integration variable is translated. Anomalous terms may also arise when the smaller loops are redefined to satisfy the correct vector-current Ward identities. Therefore, while the Ward identities for the pentagons contain no anomalous terms arising directly from the linear divergences, they do contain anomalous terms when the box diagrams are defined to have the correct vector-current Ward identities. The diagrams which do have anomalous Ward identities may be seen by referring to the expression for the anomalous divergence in Eq. (49). For neutral currents, only the AVV and AAA triangle diagrams will have anomalous Ward identities. For charged currents, anomalous Ward identities also arise for the $VVVA$ and $VAAA$ box diagrams and for the $VVVVA$, $VVAAA$, and $AAAAA$ pentagon diagrams.

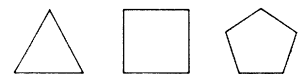

FIG. 3. Loop diagrams which may have anomalous Ward identities.

The anomalous terms in the divergence of the axial-vector current can contribute to low-energy theorems for off-shell matrix elements of the naive divergence operator. If these low-energy theorems are combined with a smoothness assumption for matrix elements of the naive divergence operator, the existence of the anomalous terms may lead to physical consequences. For these cases, we note that the full divergence operator will not be, in general, a smooth operator.[2,3] For example, the anomalous terms in the Ward identity for the $AAAAA$ pentagon may add anomalous terms to the low-energy theorems used in the (partially conserved axial-vector current) calculation of five-pion scattering.[8] Adler[2] has shown that the anomalous Ward identity for the AVV triangle yields a low-energy theorem for the decay $\pi^0 \rightarrow \gamma\gamma$ in a truncated version of the σ model. We note that this low-energy theorem has been shown to be exact to any finite order in perturbation theory for the truncated σ model by Adler and Bardeen.[9] Using the results of this paper, the calculation may be extended to the full σ model which includes charged mesons.

ACKNOWLEDGMENTS

The author would like to thank Dr. S. L. Adler for introducing him to this problem and for many stimulating discussions. The author would also like to thank Dr. I. C. Kaysen for the hospitality of the Institute for Advanced Study, where the work was performed. After the completion of this manuscript, the author learned that I. Gerstein and R. Jackiw have made an independent study of anomalous Ward identities for triangle diagrams with mesons having $SU(3)\times SU(3)$ couplings. However, we disagree with their conclusions concerning the box and pentagon diagrams.

[8] The author would like to thank R. F. Dashen for raising this point.

[9] S. L. Adler and W. A. Bardeen, Phys. Rev. **182**, 1515 (1969).