From two-dimensional conformal to four-dimensional self-dual theories: Quaternionic analyticity

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It is shown that self-dual theories generalize to four dimensions both the conformal and analytic aspects of two-dimensional conformal field theories. In the harmonic space language there appear several ways to extend complex analyticity (natural in two dimensions) to quaternionic analyticity (natural in four dimensions). To be analytic, conformal transformations should be realized on CP^3, which appears as the coset of the complexified conformal group modulo its maximal parabolic subgroup. In this language one visualizes the twistor correspondence of Penrose and Ward and consistently formulates the analyticity of Fueter.

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I. INTRODUCTION

A. From 2D complex to 4D quaternionic analyticity

The two *real* coordinates of two-dimensional (2D) Euclidena space are quite naturally combined into *a single complex number*:

$$x^{\mu}=\{x^1,x^2\} \longrightarrow z=x^1+ix^2 \ . \tag{1}$$

As is well known, the most general conformal coordinate transformation in two (*and only in two*) dimensions is *analytic* in this complex coordinate:

$$z'=f(z) \ , \quad \bar{z}'=\bar{f}(\bar{z}) \ . \tag{2}$$

Owing to the Cauchy-Riemann condition, its d'Alembertian vanishes:

$$\frac{\partial}{\partial\bar{z}}f(z)=0 \longrightarrow \Box f(z)=\frac{\partial}{\partial z}\frac{\partial}{\partial\bar{z}}f(z)=0 \ . \tag{3}$$

In any higher dimension, conformal transformations depend on a finite number of parameters, and the d'Alembertian of infinitesimal conformal boosts *does not vanish*.

In four dimensions, coordinates are well known to be unified into a quaternion as naturally as, in two dimensions, they are unified into a complex number. Specifically, in the spinor formalism we have

$$x^m=\{x^0,x^1,x^2,x^3\} \longrightarrow z=x^{\alpha\dot{\alpha}}=\begin{bmatrix} x^0-ix^3 & -ix^1-x^2 \\ -ix^1+x^2 & x^0+ix^3 \end{bmatrix}=x^0I-i\sigma_a x^a=x^0+e_a x^a \tag{4}$$

and the Pauli matrices represent the algebra of the quaternionic units, $e_a=-i\sigma_a$:

$$e_a e_b=-\delta_{ab}+\epsilon_{abc}e_c \ . \tag{5}$$

The analytic transformation (2) are fundamental to 2D conformal field theories. It is natural to wonder whether there exist 4D theories in which some form of quaternionic analyticity would play a corresponding role [1,2]. However, the notion of quaternionic analyticity proves to be rather delicate, and we shall see that there are several potential forms, only some of which will prove interesting (see, e.g., [3]).

B. Difficulties with quaternionic analyticity

A straightforward extension of the Cauchy-Riemann condition would be

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$$\frac{\partial}{\partial \bar{z}} f = \frac{1}{2}\left[\frac{\partial}{\partial x^0} + \frac{1}{3} e_a \frac{\partial}{\partial x^a}\right] f = 0 , \qquad (6)$$

where $\partial/\partial\bar{z}$ has been defined in such a way that $(\partial/\partial\bar{z})z=0$ and $(\partial/\partial\bar{z})\bar{z}=1$. It is well known, however, (see, e.g., [3]) that the only solution to Eq. (6) in the form of a power series in z is $f=a+zb$, with constant quaternions a and b, owing to the noncommutativity of quaternions. Even $(\partial/\partial\bar{z})z^2=\frac{1}{3}(z-\bar{z})$.

Heretofore, *Fueter quaternion analyticity* [4–6] was the only successful attempt to find something less restrictive than (6). An analytic function of a quaternion, z, is defined by a Weierstrass-like series

$$f(z)=\sum a_n z^n , \qquad (7)$$

where the coefficients a_n are real or complex numbers [or quaternions, but standing only to one side of z^n; say, left as in (7)]. Such a function can be shown to obey some Cauchy-Riemann-like condition, but of *the third order* in derivatives instead of the first. The equation $\Box f(z)=0$ does not hold in general; however, the equation $\Box^2 f(z)=0$ is preserved. It was emphasized in [6] that the above definition may hold in some SO(4) frames of reference, but not in others.[1]

Despite these difficulties, in the self-dual theories [7] and in the $N=2$ supersymmetric theories [8,9], there arise manifolds of a quaternionic character, namely, quaternionic-Kähler and hyper-Kähler manifolds. Indeed, the problems associated with quaternionic analyticity are very reminiscent of the difficulties involved in finding a nontrivial notion of quaternionic geometry. It is tempting to speculate that the paucity of solutions to Eq. (6) is the analytic manifestation of the fact that only flat metrics are hyper Kähler with respect to integrable almost quaternionic structures.

Therefore, the fact that interesting quaternionic geometries *do* exist suggests that it should be possible to find a useful notion of quaternionic analyticity. It is realized within the harmonic [10] (a kind of twistor [11–15]) space approach which has proven to be effective in theories possessing quaternionic structures [10,16–20] as well as for properly improving [5,6] the above Fueter definition (see also Sec. V).

C. Plan and results

The aim of the present note is to show that the harmonic space approach opens new horizons in a search for useful definitions of quaternionic analyticity, including those with Cauchy-Riemann conditions of the first order in derivatives, and with analytic functions with vanishing d'Alembertians. This approach helps to achieve these goals.

There are several ways to implement a harmonic version of quaternionic analyticity. One way leads just to the self-dual Yang-Mills and Einstein theories, which thus appear as 4D counterparts to 2D conformal field theories, in the sense that both are analytic in the dimensionally appropriate sense. We shall deal in this paper with a presentation of the self-dual Yang-Mills theories in harmonic space and with their analytic structure there.

Another way leads to a "covariantization" of the Fueter definition [5] and the corresponding coset space will be discussed also.

The tempting problem of finding a four-dimensional counterpart to the two-dimensional conformal field theories has been attacked by a number of authors; see, in particular, recent papers such as [21], [22], and [23]. The 4D self-dual gauge and gravity theories were considered very promising candidates. It must also be mentioned that intimate connections of these theories to various one- and two-dimensional integrable systems were discussed more than ten years ago already, e.g., [24–26]. A suggestion was even made [27,28] that all integrable systems might be deduced by dimensional reduction from the 4D self-dual theories, inheriting their remarkable properties. A number of recent papers [29–34] provide strong support for this suggestion, revealing also the importance of both signatures, (4,0) and (2,2).

The harmonic space approach allows a systematic study of the self-dual theories and their symmetries based on their quaternionic analyticity (in the harmonic sense), which replaces the standard complex one of the 2D conformal theories. The conformal invariance of self-duality also plays an essential role: (a) It puts space and harmonic coordinates on an equal footing; (b) the requirement that conformal transformations must be analytic leads to the remarkable phenomenon of *complexification*—they have to be realized as a real Spin(5,1) group acting on a five-dimensional compact coset of its complexification, $\mathrm{Spin}(6,C)\sim \mathrm{SL}(4,C)$. This coset is just CP^3. This manifold is complex with respect to the usual complex conjugation. However, it turns out to be *real* with respect to some combined conjugation, which is the product of the complex and antipodal conjugations.

In this paper we restrict ourselves to a discussion of the self-dual Yang-Mills theories in Euclidean space. An outline of the paper is as follows. In Sec. II we recall some basic properties of harmonics and their relation to quaternions. The necessity of complexification is discussed on the simplest level in Sec. II. In Sec. III the simplest notion of harmonic analyticity is introduced. Its role in self-dual gauge theories is demonstrated in Sec. IV, where also some other facts [16,18,19] concerning the harmonic space treatment of self-duality are collected, and simple examples of quaternionic analytic transformations are given. Some other cosets of the 4D rotation group (other than S^2) are discussed briefly in Sec. V, including the one needed for the Fueter approach. Section VI is devoted to a thorough examination of the cosets of the conformal group SO(5,1), or, more precisely, of its universal cover Spin(5,1), since we have to deal with spi-

[1] Indeed, four-dimensional rotations $\mathrm{SO}(4)\simeq \mathrm{SU}(2)_L\times \mathrm{SU}(2)_R$ are known to have the quaternionic form [1,6] $z'=mz\bar{n}$, $m\bar{m}=n\bar{n}=1$, where $m\in \mathrm{SU}(2)_L$ and $n\in \mathrm{SU}(2)_R$ are unit quaternions representing these groups. There are problems already with the z^2 term. It is transformed into $mz\bar{n}mz\bar{n}$, which cannot be expressed in the initial analytic form owing to the noncommutativity of quaternions.

norial harmonics. Again we are compelled to consider its action on a coset CP^3 of its complexified form $\mathrm{Spin}(6,C)\sim \mathrm{SL}(4,C)$. In this section necessary techniques are worked out, and we show how to calculate efficiently the form of transformations on such cosets. Some mathematical definitions and statements are given in Appendix B. Appendix A presents a five-parameter family of quaternionic complex structures in 4D.

II. HARMONICS

A. SU(2)/U(1)

Before discussing these problems it will be worth recalling some basics concerning harmonics [16,10]. As in any realization of twistor program [11,12], the harmonic space approach [10] proceeds by considering an enlarged space, which is the price paid to enable us to define appropriate analyticities. In our case this space includes the two-dimensional sphere S^2. We begin by considering it as a coset of the rotation group $\mathrm{Spin}(4)=\mathrm{SU}(2)_L\times \mathrm{SU}(2)_R$ modulo its subgroup $\mathrm{U}(1)_L\times \mathrm{SU}(2)_R$. In other words, we present this sphere as a coset SU(2)/U(1). Of course, one could choose polar (θ,ϕ) or stereographic $(z,\bar{z})$ coordinates to describe this sphere. However, it turns out to be much more convenient to use just harmonics instead of any specific coordinates, because harmonics are defined on the sphere *globally*. We shall deal with a 2×2 matrix [10]

$$U=(u_\alpha^-,u_\alpha^+)=\begin{pmatrix} u_1^- & u_1^+ \\ u_2^- & u_2^+ \end{pmatrix} . \tag{8}$$

Harmonics have SU(2) indices α and U(1) charges $+,-$. They transform under SU(2) as spinors, thus for $M\in \mathrm{SU}(2)$ $(M^\dagger M=1)$, we have

$$u_\alpha^{\pm\prime}=M_\alpha^\beta u_\beta^\pm , \quad U'=\mathrm{M\,U} . \tag{9}$$

In accordance with their coset nature, harmonics are defined modulo U(1) transformations, implemented by a matrix P:

$$U \longrightarrow UP , \quad P=\begin{pmatrix} e^{-i\lambda} & 0 \\ 0 & e^{+i\lambda} \end{pmatrix} \in \mathrm{U}(1) , \tag{10}$$

or

$$u_\alpha^{+\prime}\simeq e^{i\lambda}u^{\alpha +} , \quad u_\alpha^{-\prime}\simeq e^{-i\lambda}u_\alpha^- . \tag{11}$$

Owing to this freedom, one can take transformations of the matrix U in a form

$$U'=MUP \quad (P^\dagger P=1) . \tag{12}$$

Sometimes it is convenient to fix this matrix P in a certain way in order to pass to some specific coordinates, etc. However, the global description of a quotient manifold is then lost, and there arises the well-known Riemann-Hilbert problem.

Finally, in SU(2)/U(1) description of S^2, all matrices U, M, and P are unitary, and harmonics of opposite U(1) charges are *complex conjugates*:

$$u_\alpha^-=\overline{u^{+\alpha}} , \tag{13}$$

where SU(2) indices are raised in the usual way, $u^{+\alpha}=\epsilon^{\alpha\beta}u_\beta^+$. Harmonics have to obey the constraint

$$\det U=u^{+\alpha}u_\alpha^-=1 \tag{14}$$

and the completeness relation

$$u^{+\alpha}u_\beta^- -u^{-\alpha}u_\beta^+=\delta_\beta^\alpha \tag{15}$$

holds. This relation makes useful projection possible,

$$f_\beta=(u^{+\alpha}u_\beta^- -u^{-\alpha}u_\beta^+)f_\alpha=(u^{+\alpha}f_\alpha)u_\beta^- -(u^{-\alpha}f_\alpha)u_\beta^+ , \tag{16}$$

so all free undotted indices can be ascribed to harmonics only. We shall often use this technique.

Note that since 2×2 matrices are unitary, both M and U can be thought of as quaternions of unit norm. Since harmonics are defined only up to U(1), *the* U(1) *phase must not enter any formulas*. This means that the U(1) charge has to be conserved and that "functions" on the sphere must possess a *definite* U(1) charge, q. In other words, all terms in their decomposition have to contain products of harmonics u^+,u^- of the same charge q. For instance, for $q=+1$

$$f^+(u)=f^\alpha u_\alpha^+ +f^{\{\alpha\beta\gamma\}}u_\alpha^+u_\beta^+u_\gamma^- + \cdots . \tag{17}$$

Such quantities will acquire an overall U(1) phase; however, this is unimportant owing to the presupposed preservation of the U(1) charge. Of course, complete symmetrization in indices $\alpha,\beta,\gamma,\ldots$ is implied in each term of a harmonic decomposition such as (17), which otherwise could be reduced to lower-order terms by using the constraint (14).

In fact, u_α^+,u_α^- are the fundamental spherical harmonics of spin $\frac{1}{2}$, familiar to everyone from quantum mechanics,[2] while (16) is an example of the harmonic decomposition on S^2. This is why we shall refer to u_i^+, u_i^- in what follows simply as harmonics.

It is convenient to perform both differentiation and integration on the two-sphere directly in terms of harmonics. The action of the harmonic derivative D^{++} on the harmonics themselves is defined according to a simple rule:

$$D^{++}u_\alpha^+=0 , \quad D^{++}u_\alpha^-=u_\alpha^+ . \tag{18}$$

B. Complexification of SU(2)

The following important note is in order. When considering conformal invariance of the self-dual equations

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[2] $$U=\begin{pmatrix} \cos\frac{\theta}{2}e^{-i\phi/2} & i\sin\frac{\theta}{2}e^{-i\phi/2} \\ i\sin\frac{\theta}{2}e^{i\phi/2} & \cos\frac{\theta}{2}e^{i\phi/2} \end{pmatrix}$$

in Euler angles, the third one is irrelevant, being the phase from Eq. (12).

(as well as in some other cases) it proves necessary to *complexify* the above treatment. The reason is as follows. The parameters of conformal boosts have dimension $(\text{length})^{-1}$. So conformal transformations of harmonics will be linear in the space coordinate. If u^-, u^+ were complex conjugates, as in (13), and if the u^+ transformation were analytic [see Secs. III, IV and Eqs. (38), (40)], then transformations of u^- would be unavoidably nonanalytic. Considering the action of SU(2) on a coset of its complexification, one can have both u^+ and u^- transforming analytically (see Secs. III, IV), because in this case they cease to be complex conjugates of one another.

Before, we presented the two-dimensional sphere S^2 as a coset SU(2)/U(1). Now we are going to consider the SU(2) group action in the S^2 coset of its complexification SL(2,C). The latter can be represented by 2×2 unimodular matrices. It is noncompact and has a unimodular triangular subgroup, which is its maximal parabolic subgroup [35,15,36] (see Appendix B for mathematical definitions and techniques). It is known that a two-sphere can also be considered as a coset of the complexified group. Making use of the Iwasawa decomposition, taking the parabolic subgroup to be $P=\mathrm{U}(1)\times AN$, A and N being subgroups of SL(2,C), we have

$$\frac{\mathrm{SL}(2,C)}{P}=\frac{\mathrm{SU}(2)\times AN}{\mathrm{U}(1)\times AN}=\frac{\mathrm{SU}(2)}{\mathrm{U}(1)}=S^2 ,$$

or

$$S^2=\frac{\mathrm{SL}(2,C)}{P}=\frac{\begin{bmatrix} a & b \\ c & d\end{bmatrix}}{\begin{bmatrix} \rho & 0 \\ z^{--} & \rho^{-1}\end{bmatrix}} , \quad ad-bc=1 , \tag{19}$$

where a, b, c, d, ρ, and z^{--} are complex numbers. According to this presentation, harmonics are defined up to the parabolic group transformations

$$u_\alpha^{+\prime}=\rho^{-1}u_\alpha^+ , \quad u_\alpha^{-\prime}=\rho u_\alpha^- + z^{--}u_\alpha^+ , \tag{20}$$

which are quite general for the u^- harmonics. Consequently, we come to the crucial conclusion that *any transformation of harmonics can be reduced to*

$$\delta u_\alpha^+=\lambda^{++}u_\alpha^- , \quad \delta u_\alpha^-=0 , \tag{21}$$

with some parameters λ^{++} (choosing the appropriate compensating parabolic group transformation).

We shall refer in what follows to this gauge fixing as the u^- or "normal" form, because in all gauge theories, including gravity, there exists a normal gauge in which all prepotentials depend only on u^- and are independent of u^+ harmonics. The normal form simplifies reasonings and calculations considerably.

A general procedure for finding such a form follows from the above rule (12). For infinitesimal transformations δM it reads

$$\delta U=\delta M U + U\Delta P . \tag{22}$$

One can always find such compensating ΔP such that $\delta u_\alpha^-=0$. For instance, for rotations one has

$$\delta u_\alpha^\pm=\delta l_\alpha^\beta u_\beta^\pm . \tag{23}$$

Passing to the u^--form, we write

$$(0,\delta u_\alpha^+)=(\delta l_\alpha^\beta u_\beta^- , \delta l_\alpha^\beta u_\beta^+) + (u_\alpha^-\Delta\rho+\Delta z^{--}u_\alpha^+ , -\Delta\rho u_\alpha^+) . \tag{24}$$

Then, projecting on harmonics [see (16)], we obtain, for the parabolic transformation parameters,

$$\Delta\rho=-u^{+\gamma}\delta l_\gamma^\beta u_\beta^- , \quad \Delta z^{--}=u^{-\gamma}\delta l_\gamma^\beta u_\beta^- , \tag{25}$$

while transformations of harmonics acquired the u^- form

$$\delta u_\alpha^-=0 , \quad \delta u_\alpha^+=(u^{+\gamma}\delta l_\gamma^\beta u_\beta^+)u_\alpha^- . \tag{26}$$

With these new rules of the game, *harmonics u^+ and u^- are no longer complex conjugates.* Nevertheless a new combined conjugation can be defined [10], which is a product of the complex conjugation and the antipodal map (just a map of a point on one end of diameter to one on the other end):

$$\hat{u}_\alpha^\pm=u^{\pm\alpha} , \quad \hat{u}^{\pm\alpha}=-u_\alpha^\pm . \tag{27}$$

The reality properties are discussed in terms of this newly defined conjugation.

An important comment is that these reality properties of harmonics are preserved by the action of SU(2) on S^2, but not by the complete SL(2,C).

C. Harmonics as square roots of quaternions

We have mentioned above that harmonics are deeply related to quaternions. In fact, in a general reference system, quaternions can be considered to be bilinear combinations of harmonics.

To see this we unify U(1) charges into one index i:

$$u_\alpha^\pm=u_\alpha^i , \quad i=(+,-) . \tag{28}$$

Then the defining constraint (14) and completeness relation (15) acquire a symmetric form

$$u_i^\alpha u_\alpha^j=\delta_j^i , \quad u_i^\alpha u_\beta^i=\delta_\beta^\alpha . \tag{29}$$

Now the whole two-parameter family of quaternionic units that are arbitrarily oriented in three-dimensional space is given by

$$(e_a)_\alpha^\beta=-iu_\alpha^i(\sigma_a)_i^j u_j^\beta . \tag{30}$$

This can be easily checked with the help of (29). The above representation (4) corresponds to a special gauge fixing

$$u_1^i=\begin{bmatrix}1\\0\end{bmatrix} , \quad u_2^i=\begin{bmatrix}0\\1\end{bmatrix} . \tag{31}$$

This harmonic nature of quaternions explains why they are needed in all problems where manifolds have quaternionic structures: in the $N=2$ supersymmetric theories [10,20] and in the self-dual ones [16,18,19].

III. HARMONIC QUATERNIONIC ANALYTICITY

We shall begin our discussion of quaternionic analyticity by recalling some arguments of [16]. First of all, speaking of the coordinate $x^{\alpha\dot\alpha}$ as a quaternion z, one has in mind the 4D rotation group in the form $\mathrm{Spin}(4)=\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$. It is natural to realize it on some of its coset spaces. The simplest possibility is to choose the two-sphere

$$\mathrm{SU}(2)_R/\mathrm{U}(1)=\{u_\alpha^\pm\}\ . \tag{32}$$

It is convenient to pass to space coordinates

$$x^{\pm\dot\alpha}=x^{\alpha\dot\alpha}u_\alpha^\pm\ ,\quad x^{\alpha\dot\alpha}=-x^{+\dot\alpha}u^{-\alpha}+x^{-\dot\alpha}u^{+\alpha}\ . \tag{33}$$

As one can recognize, this is the Penrose twistor transformation, written in the language of harmonic space. The usage of x^+ and x^- coordinates permits the introduction of a new kind of analytic function, which is dependent on x^+ and harmonics, but is independent of x^-. The corresponding Cauchy-Riemann conditions will be of first order in derivatives:

$$\frac{\partial}{\partial x^{-\dot\alpha}}f^A(x,u)=0\ , \tag{34}$$

where A symbolizes the spinor indices and the U(1) charges. As a consequence of this condition we have

$$\Box f^A=\frac{\partial}{\partial x^m}\frac{\partial}{\partial x^m}f^A=0 \tag{35}$$

because one can check that

$$\Box=\frac{\partial}{\partial x^{+\dot\alpha}}\frac{\partial}{\partial x^-_{\dot\alpha}}$$

is the usual d'Alembertian in four dimensions. So the d'Alembertian of a quaternionic analytic function vanishes for reasons completely analogous to those acting in the case of the customary complex analyticity.

It is evident that the property of analyticity is preserved by the general quaternionic analytic transformations mixing coordinates $x^{+\dot\alpha},u_\alpha^+,u_\alpha^-$:

$$\delta x^{+\dot\alpha}=f^{+\dot\alpha}(x^+,u^\pm)\ , \tag{36}$$

$$\delta u_\alpha^+=w^{++}(x^+,u^\pm)u_\alpha^-\ ,$$

$$\delta u_\alpha^-=0$$

with arbitrary quaternionic analytic functions $f^{+\dot\alpha}$ and w^{++} as parameters. When writing down these transformations, we have effectively taken into account that harmonics are defined modulo transformations (20). Note also that no assumptions were made concerning a form of transformations of coordinates $x^{-\dot\alpha}$,

$$\delta x^{-\dot\alpha}=\phi^{-\dot\alpha}(x^+,x^-,u^\pm)\ , \tag{37}$$

where local parameters ϕ are nonanalytic and can depend on x^- in any manner. To be more concrete, we shall give some examples.

(1) The Poincaré group (left and right rotations, δl_β^α and $\delta r_{\dot\beta}^{\dot\alpha}$, respectively, $\delta l_\beta^\beta=\delta r_{\dot\alpha}^{\dot\alpha}=0$, and translations $\delta b^{\alpha\dot\alpha}$) is represented by analytic quaternionic transformations:

$$\delta x^{+\dot\alpha}=-\delta r_{\dot\beta}^{\dot\alpha}x^{+\dot\beta}+\delta b^{\alpha\dot\alpha}u_\alpha^+\ ,\quad \delta u_\alpha^\pm=+\delta l_\alpha^\beta u_\beta^\pm\ ,$$

$$(\delta x^{\alpha\dot\alpha}=-\delta l_\beta^\alpha x^{\beta\dot\alpha}-\delta r_{\dot\beta}^{\dot\alpha}x^{\alpha\dot\beta}+\delta b^{\alpha\dot\alpha}\ ,$$

$$\delta x^{-\dot\alpha}=-\delta r_{\dot\beta}^{\dot\alpha}x^{-\dot\beta}+\delta b^{\alpha\dot\alpha}u_\alpha^-)\ .$$

Note that rotations of harmonics can be also represented as in (23).

(2) The same for dilatations

$$\delta x^{+\dot\alpha}=\delta d x^{+\dot\alpha}\ ,\quad \delta u_\alpha^\pm=0$$

and $\delta x^{\alpha\dot\alpha}=\delta d x^{\alpha\dot\alpha}$, $\delta x^{-\dot\alpha}=\delta d x^{-\dot\alpha}$. [It must be remembered that harmonics are defined modulo transformation (20).] The above transformations exhaust all quaternionic analytic transformations that are linear in $x^{+\dot\alpha}$ and do not lead to an explicit appearance of harmonics in $\delta x^{\alpha\dot\alpha}$.

(3) The affine transformations

$$\delta x^{\alpha\dot\alpha}=a_{\beta\dot\beta}^{\alpha\dot\alpha}x^{\beta\dot\beta}\ ,\quad a_{\alpha\dot\beta}^{\alpha\dot\alpha}=a_{\beta\dot\alpha}^{\alpha\dot\alpha}=0$$

are definitely nonanalytic for any choice of reparametrizations of harmonics. For instance, if $\delta u_\alpha^\pm=0$, then

$$\delta x^{+\dot\alpha}=u_\alpha^+a_{\beta\dot\beta}^{\alpha\dot\alpha}(u^{+\beta}x^{-\dot\beta}-u^{-\beta}x^{+\dot\beta})$$

contains both $x^{+\dot\alpha}$ and $x^{-\dot\alpha}$.

(4) Passing to transformations bilinear in $x^{+\dot\alpha}$ we observe, first of all, that *the conformal boosts belong to the quaternionic analytic transformations*. Explicitly, conformal boosts are

$$\delta x^{\alpha\dot\alpha}=x^{\alpha\dot\beta}\delta k_{\dot\beta\beta}x^{\beta\dot\alpha} \tag{38}$$

($\delta k_{\dot\beta\beta}$ are parameters), so that

$$\delta x^{+\dot\alpha}=-x^{+\dot\beta}\delta k_{\dot\beta\beta}u^{-\beta}x^{+\dot\alpha}\ , \tag{39}$$

$$\delta u_\alpha^+=-x^{+\dot\beta}\delta k_{\dot\beta\beta}u^{+\beta}u_\alpha^-\ ,\quad \delta u_\alpha^-=0\ , \tag{40}$$

$$(\delta x^{-\dot\alpha}=-x^{+\dot\beta}\delta k_{\dot\beta\beta}u^{-\beta}x^{-\dot\alpha})\ .$$

The conformal boost parameter has dimension $(\text{length})^{-1}$. So, conformal transformations for harmonics contain space coordinates x linearly. It is worth stressing once again that it becomes possible to avoid an appearance of $x^{-\dot\alpha}$ (nonanalyticity) in the transformations of u_α^- harmonics only because any transformation of u^- can be compensated by an appropriate parabolic group one (passing to the normal form).

Note that, under combined conjugation (27),

$$\widehat{x}^{+\dot\alpha}=-x_{\dot\alpha}^+\ , \tag{41}$$

and this is consistent with *real* Poincaré and conformal transformations.

Note also that reality properties of the above general transformations (36) have to be consistent with the combined conjugation (27), (41) as well.

We shall restrict ourselves to these examples.

It is remarkable that *just this kind of analyticity is inherent* in the important theories discussed in the next section.

IV. SELF-DUAL GAUGE THEORIES IN FOUR DIMENSIONS

To recover the quaternionic analytic nature of the theories quoted in the title, we shall recall here the Ward procedure [12,11] for dealing with the self-dual gauge equations in R^4. (See also [23] and [37], where self-dual equations on Kählerian and quaternionic manifolds are discussed.) The usage of the harmonic space language [16,18,19] makes the situation completely understandable, showing a transparent correspondence between a quaternionic analytic (in the sense of the preceding section) double U(1) charged function $V^{++}(x^+,u)$ and solutions of the self-dual equations. Note that consideration of the self-dual Einstein theory goes along similar lines.

The commutator of covariant derivatives $D_{\alpha\dot\alpha}=\partial_{\alpha\dot\alpha}+iA(x)_{\alpha\dot\alpha}$ (connection A takes values in the Lie algebra of the gauge group for the Yang-Mills theory and in the tangent Lorentz group for the Einstein theory),

$$[D_{\alpha\dot\alpha},D_{\beta\dot\beta}]=\epsilon_{\alpha\beta}F_{\dot\alpha\dot\beta}(x)+\epsilon_{\dot\alpha\dot\beta}F_{\alpha\beta}(x)\ , \tag{42}$$

defines the Yang-Mills field strengths $F_{\alpha\beta},F_{\dot\alpha\dot\beta}$. In the spinor formalism, the self-duality equation is simply $F_{\dot\alpha\dot\beta}(x)=0$. Taking into account the definition (42), it is evident that this equation is *equivalent* to

$$[D_{\alpha\dot\alpha},D_{\beta\dot\beta}]=\epsilon_{\dot\alpha\dot\beta}F_{\alpha\beta}(x)\ . \tag{43}$$

Having harmonics at our disposal, we can disentangle this equation in the following way. Multiplying Eq. (43) by $u^{+\dot\alpha}u^{+\dot\beta}$, and defining

$$D^+_\alpha=u^{+\dot\alpha}D_{\alpha\dot\alpha}\ , \tag{44}$$

we obtain

$$[D^+_\alpha,D^+_\beta]=0\ . \tag{45}$$

It is more convenient to replace (44) with the equivalent [taking into account U(1) charge conservation] commutator relation [cf. (18)]

$$[D^{++},D^+_\alpha]=0\ . \tag{46}$$

The pair of equations (45) and (46) is equivalent to the self-duality conduction (43). However, this pair is considerably simpler. The first of them states that the covariant derivatives D^+ commute. So its solution is "pure gauge":

$$D^+_\alpha=h\partial^+_\alpha h^{-1}=\partial^+_\alpha+h(\partial^+_\alpha h^{-1})\ , \tag{47}$$

where derivative ∂^+_α does not contain any connection, and a "bridge" $h=h(x,u)$ takes values in the gauge group. By choosing coordinates (33) one has

$$D^+_\alpha=u^{+\dot\alpha}\partial_{\alpha\dot\alpha}=\frac{\partial}{\partial x^{-\alpha}}\ . \tag{48}$$

In the gauge where D^+_α becomes short, the harmonic derivative D^{++} grows long (because the bridge h generally depends on harmonics),

$$\begin{aligned}D^{++}\to\mathcal{D}^{++}&=h^{-1}D^{++}h=D^{++}+h^{-1}(D^{++}h)\\&=D^{++}+iV^{++}\ ,\end{aligned} \tag{49}$$

acquiring a harmonic connection that is globally defined on S^2 (gauge algebra valued indeed):

$$V^{++}=-ih^{-1}(D^{++}h)\ . \tag{50}$$

Now the second equation of the pair becomes the *Cauchy-Riemann condition for the harmonic analyticity*,

$$\frac{\partial}{\partial x^{-\alpha}}V^{++}=0\ , \tag{51}$$

stating that for the self-dual case the harmonic connection is analytic; i.e., it is independent of $x^{-\alpha}$, $V^{++}=V^{++}(x^{+\alpha},u^{\pm})$. Vice versa, if V^{++} is analytic, it encodes a solution of the self-dual equations.

Moreover, one can get rid of positively charged harmonics u^+, since gauge potentials V^{++} are defined only up to gauge transformations

$$\delta V^{++}(x^+,u)=[D^{++}+iV^{++}(x^+,u)]\lambda(x^+,u)\ .$$

There exists a *normal gauge* [16] where V^{++} contains in its harmonic decomposition only negatively charged harmonics u^-:

$$V^{++}=V^{++}(x^+,u^-)\ .$$

So, an analytic V^{++} encodes a solution of the self-dual equation. This is a transparent manifestation of the "twistor correspondence." However, we are more familiar with the usual space R^4 than with the harmonic one. To pass to the former, one has to determine the bridge h from Eq. (50) for a given analytic V^{++}, and then substitute the bridge h into the expression for the usual vector connection

$$A_a(x)=-ih\frac{\partial}{\partial x^a}h^{-1}\ .$$

Of importance is that (50) has a solution for almost any V^{++} [18].

Thus, there is no problem in solving the self-duality equations in the harmonic space, while a solution of (50) on the two-sphere is needed to pass to the ordinary space. Instantons and monopoles are special solutions of the self-duality equation having finite action and finite energy, respectively. They have been completely described in the harmonic space language, including ADHM construction, etc. [18,19].

Of course, any change of the harmonic analytic connection

$$V'(x^+,u)=V^{++}(x^+,u)+g^{++}(x^+,u) \tag{52}$$

results in passing from one solution of the self-dual equation to another. So the most general Bäcklund transformation is encoded again by the double U(1) charged analytic object $g^{++}(x^+,u)$. An important geometric class of them consists of the general analytic diffeomorphisms (29) accompanied with a "similarity" transformation, defined by a general analytic weight $c(x^+,u)$ that takes values in the gauge algebra:

$$V'(x^{+'},u')=e^{c(x^+,u)}V^{++}(x^+,u)e^{-c(x^+,u)} . \tag{53}$$

This class includes a great many Bäcklund transformations. Speaking of diffeomorphisms, we can restrict ourselves to those that are realized in the normal form.

So, a kind of quaternionic analyticity, the harmonic one, is inherent in the self-dual 4D gauge theories. Of course, they are also conformally invariant. However, now conformal transformations form a finite-dimensional subgroup Spin(5,1) of analytic transformations; those given by (39) and (40). Therefore these theories can be naturally considered as 4D extensions of the 2D conformal field theories in both their conformal and analytic aspects.

V. EXAMPLES OF OTHER QUATERNIONIC ANALYTICITIES

Only one coset space was considered above: the two-sphere S^2. There are other possibilities, however, some of which we shall now discuss briefly.

A. A product of two S^2

The first example is

$$\frac{SU(2)_L}{U(1)_L}\times\frac{SU(2)_R}{U(1)_R} \tag{54}$$

with harmonization of both the left and right SU(2) groups and with two distinct U(1) charges. In this case there are both left ($v_\alpha^{\oplus,\ominus}$) and right ($u_{\dot\alpha}^{+,-}$) harmonics having left ($\oplus,\ominus$) or right ($+,-$) U(1) charges, respectively. The definition of the corresponding quaternionic analyticity is rather obvious. One has to split $x^{\alpha\dot\alpha}$ into four pieces

$$\begin{aligned} &x^{+\oplus}=x^{\alpha\dot\alpha}v_\alpha^\oplus u_{\dot\alpha}^+ \ , \quad x^{+\ominus}=x^{\alpha\dot\alpha}v_\alpha^\ominus u_{\dot\alpha}^+ \ , \\ &x^{-\oplus}=x^{\alpha\dot\alpha}v_\alpha^\oplus u_{\dot\alpha}^- \ , \quad x^{-\ominus}=x^{\alpha\dot\alpha}v_\alpha^\ominus u_{\dot\alpha}^- \ . \end{aligned} \tag{55}$$

It is easy to arrange quaternion conjugations that transform these variables among themselves. To combine them into the ordinary four-coordinate is also straightforward:

$$\begin{aligned} x^{\alpha\dot\alpha}=&v^{\oplus\alpha}u^{+\dot\alpha}x^{-\ominus}-v^{\oplus\alpha}u^{-\dot\alpha}x^{+\ominus}-v^{\ominus\alpha}u^{+\dot\alpha}x^{-\oplus} \\ &+v^{\ominus\alpha}u^{-\dot\alpha}x^{+\oplus} \ . \end{aligned} \tag{56}$$

One can define analytic functions to depend on only one of the four coordinates (55), say on $x^{-\oplus}$ (and in some way on harmonics). Then there will be three Cauchy-Riemann conditions of the first order in derivatives:

$$\partial^{+\oplus}f=\partial^{-\ominus}f=\partial^{-\oplus}f=0 \ . \tag{57}$$

Again, the consequence is that the d'Alembertian of the analytic function vanishes: $\Box f\equiv\partial^{+\oplus}\partial^{-\ominus}f-\partial^{-\oplus}\partial^{+\ominus}f=0$. At present we do not know a field-theoretical model connected with such quaternionic analyticity.

B. A diagonal U(1) case

Further, one can *identify* two U(1) groups and consider the coset space

$$\frac{SU(2)\times SU(2)}{U(1)} \tag{58}$$

that is connected with the same harmonics $v_\alpha^\pm, u_{\dot\alpha}^\pm$ as in the above example, both having, however, charges of the same diagonal U(1) subgroup. This circumstance, as we shall see, will help. As in the previous case, the four-coordinate is split up into four separate variables:

$$\begin{aligned} &x^{++}=x^{\alpha\dot\alpha}v_\alpha^+u_{\dot\alpha}^+ \ , \quad x^1=x^{\alpha\dot\alpha}v_\alpha^-u_{\dot\alpha}^+ \ , \\ &x^2=x^{\alpha\dot\alpha}v_\alpha^+u_{\dot\alpha}^- \ , \quad x^{--}=x^{\alpha\dot\alpha}v_\alpha^-u_{\dot\alpha}^- \ , \end{aligned} \tag{59}$$

while

$$\begin{aligned} x^{\alpha\dot\alpha}=&v^{+\alpha}u^{+\dot\alpha}x^{--}-v^{+\alpha}u^{-\dot\alpha}x^1-v^{-\alpha}u^{+\dot\alpha}x^2 \\ &+v^{-\alpha}u^{-\dot\alpha}x^{++} \ . \end{aligned} \tag{60}$$

The Cauchy-Riemann conditions are again of the first order in derivatives; for a function that has to depend only on, say, x^1, they are

$$\partial^{++}f=\partial^2 f=\partial^{--}f=0 \ , \tag{61}$$

and any analytic function satisfying (61) will obey

$$\Box f\equiv\partial^{++}\partial^{--}f-\partial^1\partial^2 f=0 \ , \tag{62}$$

where ∂^1 and ∂^2 differentiate with respect to x^1 and x^2, respectively.

C. Fueter quaternionic analyticity revisited

As was stated in the Introduction, a Fueter analytic function does not satisfy the Cauchy-Riemann condition or the equation $\Box f=0$. Instead the Cauchy-Riemann-like condition holds for $\Box f$, leading to an equation of fourth order, $\Box^2 f=0$. To demonstrate these statements it is worth emphasizing that *Fueter analyticity is connected connected just to the harmonic approach of Sec. V B.* Indeed, in [6] it was shown that to make the Fueter decomposition (7) formally covariant one has to introduce another quaternion p^{-1}, with the transformation properties inverse to those of z:

$$z'=mz\bar n \ , \quad p^{-1'}=np^{-1}\bar m \ , \quad m\bar m=n\bar n=1 \ , \tag{63}$$

where $m\in SU(2)_L$ and $n\in SU(2)_R$ are unit quaternions representing these groups, respectively (cf. footnote 1).

Now a new variable, termed a left quator in [6,5],

$$y=zp^{-1} \ , \tag{64}$$

will have a "more convenient," purely left transformation law:

$$y'=my\bar m \ . \tag{65}$$

Correspondingly, the modified Fueter definition [5]

$$f(y)=\sum a_n y^n \tag{66}$$

will be consistent with four-dimensional rotations, with y belonging to the $(1,0)\oplus(0,0)$ representation of SO(4). It is easy to see that this newly introduced auxiliary quaternion p *may be taken to be a vector harmonic*, composed of the spinor harmonics introduced in Sec. V B according to

$$p^{-1}_{\alpha\dot{\alpha}}=(v^+_\alpha u^-_{\dot{\alpha}}+v^-_\alpha u^+_{\dot{\alpha}}) . \tag{67}$$

[There is some freedom in defining the coefficients on the right-hand side of (67).] Note that p^{-1} becomes the unit matrix in the special reference system (31) for both types of harmonics. Thus the Fueter analytic function is a power series of a left quator (in the terminology of [5]):

$$\begin{aligned} y^\alpha_\beta &= x^{\alpha\dot{\alpha}}(v^+_\beta u^-_{\dot{\alpha}}+v^-_\beta u^+_{\dot{\alpha}}) \\ &= x^{++}(L^{--})^\alpha_\beta + x^{--}(L^{++})^\alpha_\beta + x^1(P^1)^\alpha_\beta + x^2(P^2)^\alpha_\beta , \end{aligned} \tag{68}$$

where definitions (59) were used, and we have introduced operators

$$\begin{aligned} &(L^{++})^\alpha_\beta = v^{+\alpha}v^+_\beta , \quad (L^{--})^\alpha_\beta = -v^{-\alpha}v^-_\beta , \\ &(P^1)^\alpha_\beta = v^{+\alpha}v^-_\beta , \quad (P^2)^\alpha_\beta = -v^{-\alpha}v^+_\beta . \end{aligned} \tag{69}$$

They satisfy the following algebra (with indices suppressed for brevity): the operators P are projectors,

$$\begin{aligned} &P^1P^1=P^1 , \quad P^2P^2=P^2 , \\ &P^1+P^2=1 , \quad P^1P^2=P^2P^1=0 , \end{aligned} \tag{70}$$

while for the L's we have

$$\begin{aligned} &L^{--}L^{++}=P^1 , \quad L^{++}L^{--}=P^2 , \\ &L^{--}L^{--}=L^{++}L^{++}=0 \end{aligned} \tag{71}$$

and the remaining products are

$$\begin{aligned} &L^{++}P^1=P^2L^{++}=L^{++} , \quad L^{--}P^2=P^1L^{--}=L^{--} , \\ &L^{++}P^2=P^1L^{++}=L^{--}P^1=P^2L^{--}=0 . \end{aligned} \tag{72}$$

We now wish to show that the general Fueter-analytic function of Eq. (66) is biharmonic, and satisfies some equation of third order in derivatives. To this end, consider an integral representation for f,

$$f(y)=\frac{1}{2\pi i}\oint dz \frac{f(z)}{z-y} \tag{73}$$

which follows from the fact that it has a Weierstrass-like decomposition (66). Thus we may restrict our attention to the function $(z-y)^{-1}$, or, simpler yet, y^{-1}. Using the above algebra we have

$$y^{-1}=(x^{++}x^{--}-x^1x^2)^{-1}z , \tag{74}$$

where

$$z=x^{++}(L^{--})+x^{--}(L^{++})-x^1(P^2)-x^2(P^1) . \tag{75}$$

It is now helpful to consider the differential operators

$$\begin{aligned} &V=L^{--}\partial^{++}+L^{++}\partial^{--}-P^1\partial^1-P^2\partial^2 , \\ &T=L^{--}\partial^{++}+L^{++}\partial^{--}+P^1\partial^2+P^2\partial^1 , \end{aligned} \tag{76}$$

which satisfy

$$VT=TV=\partial^{++}\partial^{--}-\partial^1\partial^2=\Box , \tag{77}$$

$$Tz=0 . \tag{78}$$

Taking into account the fact that

$$\Box(x^{++}x^{--}-x^1x^2)^{-1}=\Box(1/x^2)=0 \tag{79}$$

we see from (77)–(79) that

$$V^2Ty^{-1}=0 . \tag{80}$$

Thus we have proven that any Fueter-analytic function $f(y)$ satisfies the third order, "Cauchy-Riemann," condition

$$(L^{--}\partial^{++}+L^{++}\partial^{--}-P^1\partial^1-P^2\partial^2)\Box f=0 \tag{81}$$

and hence the biharmonic equation

$$\Box^2 f=0 . \tag{82}$$

It is worth recalling that the conformal group of Euclidean four-dimensional space can be represented by Fueter-type transformations [6]. They are realized on z nonlinearly as quaternionic-Möbius transformations [1] constructed in the usual way from the quaternionic entries of the two-by-two matrix belonging to SL$(2,H)$:

$$z'=(az+b)(cz+d)^{-1} , \tag{83}$$

where a, b, c, and d are constant quaternions, satisfying

$$\det(a-bd^{-1}c)\det d=|ad-bd^{-1}cd|^2=1 \tag{84}$$

(the unimodularity condition for a 2×2 matrix with quaternionic entries[3]). Indeed, if $c\neq0$, $d\neq0$, then z' can be written as

$$z'=ac^{-1}+(bd^{-1}-ac^{-1})(1+czd^{-1})^{-1}$$

and it is a sum of a constant quaternion and a Fueter analytic function of a composite argument (involving transformation parameters other than the coordinate itself) $y=czd^{-1}$ multiplied from the left by another constant quaternion. If $c=0$, then $d\neq0$ and z' is a linear function of $y=azd^{-1}$. Finally, for $d=0$ and $c\neq0$, z' is a linear function of $t=ba^{-1}c^{-1}$. Four-dimensional rotations correspond to (83) with $b=c=0$, $a=m$ and $d=n$, $m\bar{m}=n\bar{n}=1$, cf. (8); dilatation is generated when a is a real parameter, $d=1$, $b=c=0$; translations have parameter b while $a=d=1$, $c=0$; for conformal boosts c is a parameter and $a=d=1$, $b=0$.

In [6] infinite-dimensional quasiconformal groups are considered that generalize (83), being subgroups of the four-dimensional group of diffeomorphisms.

---

[3]A 2×2 matrix with quaternionic (or again 2×2 matrix) entries can be decomposed into a product

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} I & bd^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} a-bd^{-1}c & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} I & 0 \\ d^{-1}c & I \end{bmatrix}$$

of matrices having evident determinants [6].

VI. UNIFYING SPACE AND TWO-SPHERE: COMPLEXIFYING THE CONFORMAL GROUP

Above we harmonized the rotation group SO(4), and harmonics came out without a visible connection to the space coordinates x^m that are coordinates of the coset of the Poincaré group modulo its rotation subgroup SO(4). It would be desirable to have space and harmonic (twistor) coordinates treated on an equal footing. Conformal symmetry helps us achieve this goal.

The conformal group for the Euclidean four-dimensional space is well known to be SO(5,1); however, since we are dealing with harmonics in spinorial representations of the Lorentz group, we are really dealing with its universal cover, Spin(5,1). One could start by considering its cosets to find out whether there is a suitable six-dimensional one. In the previous section the quaternion-Möbius form of SO(5,1) was mentioned. In spinor form it is represented by a matrix

$$M=\begin{bmatrix} l^{\alpha}_{\beta} & b^{\dot\beta}_{\alpha} \\ c^{\beta}_{\dot\alpha} & r^{\dot\beta}_{\dot\alpha} \end{bmatrix} \tag{85}$$

with unit determinant

$$\det[l^{\beta}_{\alpha}-b^{\dot\alpha}_{\alpha}(r^{-1})^{\dot\beta}_{\dot\alpha}c^{\alpha}_{\dot\beta}]\times\det r^{\dot\beta}_{\dot\alpha}=1 \tag{86}$$

(see footnote 3). It has the same entries as in Sec. V: l^{α}_{β} and $r^{\dot\alpha}_{\dot\beta}$ present left and right rotations, respectively, and dilatations, while $b^{\dot\beta}_{\alpha}$ and $c^{\beta}_{\dot\alpha}$ are translations and conformal boosts, respectively. Now using the Iwasawa decomposition

$$\mathrm{Spin}(5,1)=\mathrm{Spin}(5)\times AN \tag{87}$$

(see Appendix B) we really can find a six-dimensional coset. To this end, one has to choose $\mathrm{Spin}(3)\times\mathrm{SO}(2)\times AN$ as a parabolic subgroup P [the same A, N as in (87)]. Then the Grassmanian

$$\frac{\mathrm{Spin}(5,1)}{P}=\frac{\mathrm{SO}(5)}{\mathrm{SO}(3)\times\mathrm{SO}(2)} \tag{88}$$

will be the only six-dimensional coset. However, the left rotation group $\mathrm{SU}(2)_L$ comes out in the original *noncomplexified form*. By the same argument as in Sec. II, this will lead to nonalalytic conformal transformations.

This again suggests *complexification*, now of the *conformal group*. We are therefore led to consider the action of Spin(5,1) on cosets of $\mathrm{Spin}(6,C)\sim\mathrm{Sl}(4,\mathrm{C})$. Indeed, this works. Starting with the Iwasawa decomposition (see Appendix B) of the latter,

$$\mathrm{SL}(4,C)=\mathrm{SU}(4)\times AN , \tag{89}$$

and choosing the parabolic subgroup to be

$$P=\mathrm{SU}(3)\times\mathrm{U}(1)\times AN \tag{90}$$

[with the same AN as in (89)] we come to the coset

$$\frac{\mathrm{SL}(4,C)}{P}=\frac{\mathrm{SU}(4)}{\mathrm{SU}(3)\times\mathrm{U}(1)}=\mathrm{CP}^3 , \tag{91}$$

the AN in the numerator and denominator being "canceled." Note that the appearance of the three-dimensional complex projective manifold agrees with the twistorial literature [11,14,12], etc. Coordinates of this manifold are two space coordinates $x^{+\dot\alpha}$ and harmonics that can be represented by one complex coordinate, as we shall now see.

The reader can consult Appendix B for some definitions and techniques. Using them, we shall give here a direct derivation in brief of the form of the Spin(5,1) transformations realized on this CP^3 coset. We shall proceed in the same manner as we did in Sec. II, where we dealt with the appropriate coset of the complexified $\mathrm{SU}(2)_L$ group.

Generally speaking, it is better to work with the full set of harmonics forming a Spin(5,1) matrix

$$U=\begin{bmatrix} u^{s}_{\alpha} & u^{\dot s}_{\alpha} \\ u^{s}_{\dot\alpha} & u^{\dot s}_{\dot\alpha} \end{bmatrix}$$

and identify them under the action of the subgroup P. This would be a *global* definition of G/P, and within this framework the Riemann-Hilbert problem would be completely avoided.

However, to show how to work just with six ordinary coordinates of the six-dimensional manifold, we shall use here the subgroup P to eliminate locally redundant degrees of freedom in U. These local coordinates of our coset can be written as the entries of a triangular matrix:

$$U=\begin{bmatrix} u^{s}_{\alpha} & -u^{-}_{\alpha}x^{+\dot s} \\ 0 & \delta^{\dot s}_{\alpha} \end{bmatrix} . \tag{92}$$

The conformal group acts on U by multiplication from the left by a matrix $M\in\mathrm{Spin}(5,1)$, Eq. (85). To preserve the form (92) we fix a gauge by using appropriate compensating parabolic group transformations P [cf. (11) and (22)]. For infinitesimal transformations we have

$$\delta U=M\times U\times P-U\approx\delta M\times U+U\times\Delta P . \tag{93}$$

As in Sec. II, harmonics are defined only up to a transformation (20) belonging to the parabolic group. So we can take as a starting point that a gauge (21) is fixed; i.e., we shall work with transformations in the normal form:

$$\delta u^{-}_{\alpha}=0 , \quad \delta u^{+}_{\alpha}=\lambda^{++}u^{-}_{\alpha} .$$

We shall now calculate explicitly the transformations of $x^{+\dot\alpha}$ and u^{+}_{α}, as well as the compensating transformations belonging to the parabolic group, using Eq. (93) together with the requirement (21). The ingredients are

$$(A)\quad \delta U=\begin{bmatrix}(0,\lambda^{++}u_\alpha^-) & -u_\alpha^-\delta x^{+\dot p}\\ 0 & 0\end{bmatrix}, \tag{94}$$

$$(B)\quad \delta M\times U=\begin{bmatrix}\delta\tilde l_\alpha^\beta u_\beta^s+\delta d\delta_\alpha^s & -\delta\tilde l_\alpha^\beta u_\beta^- x^{+\dot s}-\delta d u_\alpha^- x^{+\dot s}+\delta b_\alpha^{\dot s}\\ -\delta c_{\dot\alpha}^\beta u_\beta^s & -\delta c_{\dot\alpha}^\beta u_\beta^- x^{+\dot s}+\delta\tilde r_{\dot\alpha}^{\dot s}-\delta d\delta_{\dot\alpha}^{\dot s}\end{bmatrix}, \tag{95}$$

where we singled out dilatations δd: now $\delta\tilde l_s^s=\delta\tilde r_{\dot s}^{\dot s}=0$. The induced parabolic group transformations form a matrix

$$\Delta P=\begin{bmatrix}\begin{bmatrix}\Delta a+\Delta d & 0\\ \Delta z^{--} & -\Delta a+\Delta d\end{bmatrix}_p^s & \begin{bmatrix}0\\ \Delta a^{-\dot s}\end{bmatrix}_p\\ \Delta c_{\dot p}^s & -\Delta d\delta_{\dot p}^{\dot s}+\Delta\tilde r_{\dot p}^{\dot s}\end{bmatrix}. \tag{96}$$

For the last ingredient, the matrix $U\times\Delta P$, we shall write the entries separately. The upper left corner:

$$(u_\alpha^-(\Delta a+\Delta d-x^{+\dot p}\Delta c_{\dot p}^-)+u_\alpha^+\Delta z^{--},$$
$$u_\alpha^+(-\Delta a+\Delta d)-u_\alpha^- x^{+\dot p}\Delta c_{\dot p}^+)\ . \tag{97}$$

The lower left corner:

$$\Delta c_{\dot\alpha}^s\ . \tag{98}$$

The upper right corner:

$$u_\alpha^+\Delta a^{-\dot s}+u_\alpha^-\Delta d x^{+\dot s}-u_\alpha^- x^{+\dot p}\Delta\tilde r_{\dot p}^{\dot s}\ . \tag{99}$$

The lower right corner:

$$-\Delta d\delta_{\dot\alpha}^{\dot s}+\Delta\tilde r_{\dot\alpha}^{\dot s}\ . \tag{100}$$

Now we have to substitute ingredients (92), (95), (97), (98), (99), and (100) into Eq. (93). Then projecting all entries on $u_\alpha^\pm$, we get from the resulting equations explicit expressions for the infinitesimal transformations of coset coordinates

$$\delta x^{+\dot\alpha}=(2\delta d+u^{+\gamma}\delta\tilde l_\gamma^\beta u_\beta^-+x^{+\dot p}\delta c_{\dot p}^\beta u_\beta^-)x^{+\dot\alpha}$$
$$-u^{+\gamma}\delta b_\gamma^{\dot\alpha}-x^{+\dot s}\delta\tilde r_{\dot s}^{\dot\alpha}\ , \tag{101}$$
$$\delta u_\alpha^+=(u^{+\gamma}\delta\tilde l_\gamma^\beta u_\beta^++x^{+\dot p}\delta c_{\dot p}^\beta u_\beta^+)u_\alpha^-\ ,$$

and indeed

$$\delta u_\alpha^-=0\ . \tag{102}$$

One recognizes in (101), (102) transformations of coordinates and harmonics obtained already in Sec. II B. For the accompanying compensating transformations from the parabolic group we get

$$\Delta z^{--}=u^{-\gamma}\delta\tilde l_\gamma^\beta u_\beta^-\ ,$$
$$\Delta a=-\tfrac12 x^{+\dot s}\delta c_{\dot s}^\beta u_\beta^- -u^{+\gamma}\delta\tilde l_\gamma^\beta u_\beta^-\ , \tag{103}$$
$$\Delta\tilde r_{\dot\alpha}^{\dot\beta}=-\delta\tilde r_{\dot\alpha}^{\dot\beta}+\delta c_{\dot\alpha}^\beta u_\beta^- x^{+\dot\beta}-\tfrac12 x^{+\dot s}\delta c_{\dot s}^\beta u_\beta^-\delta_{\dot\alpha}^{\dot\beta}\ , \tag{104}$$
$$\Delta d=-\delta d-\tfrac12 x^{+\dot s}\delta c_{\dot s}^\beta u_\beta^-\ , \tag{105}$$

$$\Delta c_{\dot\alpha}^s=-\delta c_{\dot\alpha}^s\ , \tag{106}$$

and finally

$$\Delta a^{-\dot s}=-\delta b^{\beta\dot s}u_\beta^-+u^{-\gamma}\delta\tilde l_\gamma^\beta u_\beta^- x^{+\dot s}\ . \tag{107}$$

Of great importance is that all these transformations and manipulations are consistent with the combined conjugation discussed in Sec. II B, which is realized on harmonics and coordinates by (27) and (41).

In this form we may easily identify the complex coordinates for our coset. The first two are $x^{+\dot\alpha}$ and the third, z, may be obtained by setting

$$u_\alpha^-=(1,0)\ ,\quad u_\alpha^+=(z,1)\ . \tag{108}$$

The transformation law for z follows from Eq. (101).

An important lesson is that the *complex* (in the common sense) manifold is *real* with respect to the combined conjugation. All transformations must be consistent with this fact. In particular, this compatibility condition picks out the Spin(5,1) subgroup of Spin(6,C).

Remark. It is rather easy to find the finite transformations in the normal form. For instance, conformal transformations are written

$$\tilde x^{+\dot\alpha}=\frac{x^{+\dot\alpha}}{1-x^{+\dot\beta}c_{\dot\beta}^\sigma u_\sigma^-}\ ,$$
$$\tilde u_\alpha^+=u_\alpha^++\frac{x^{+\dot\beta}c_{\dot\beta}^\sigma u_\sigma^+}{1-x^{+\dot\beta}c_{\dot\beta}^\sigma u_\sigma^-}u_\alpha^-\ ,\quad \tilde u_\alpha^-=u_\alpha^-\ . \tag{109}$$

They are singular at some finite value of the conformal parameter because of gauge fixing with only one set of coordinates (i.e., one chart) for the whole manifold.[4] As mentioned above, the global description of this coset can be achieved by using 32 harmonics (instead of these 6 coordinates) defined modulo the parabolic group transformations and obeying the unimodularity constraint.

VII. CONCLUSIONS

An enlargement of space variables through the addition of some harmonic (or twistorial) variables is known to admit a new kind of analyticity, the harmonic (or twistorial) analyticity. We have observed in the present paper that it is a facet of a quaternionic analyticity and that there are several ways to define it. In the 4D self-dual Yang-Mills and Einstein theories, an analyticity of this

[4]The authors are indebted to A. Galperin, who stimulated this comment.

kind replaces the standard complex analyticity of the 2D conformal theories.

The self-dual equations are conformally invariant. A remarkable consequence of harmonic analyticity is that the 4D conformal group has to be realized on the coset CP^3 of the *complexification*, $SL(4,C)$. The reasoning is quite general: in any coset of the real group it is impossible to have the x-dependent transformations of u^- and u^+ simultaneously analytic. It has to be emphasized, however, that we deal only with the "Euclidean" conformal group Spin(5,1). As a consequence, a combined conjugation can be defined (instead of a complex one), in the framework of which CP^3 is a real manifold. It is worth mentioning that the same phenomenon of complexification appears also in the $N=2$ and $N=3$ supersymmetric theories.

These topics will be discussed elsewhere, as well as a more complete analysis and a classification of "analytic" symmetries of the self-dual equations. It is performed most effectively in the normal gauge, taking the transformations in the normal form. The consideration in parallel of the symmetries of the self-dual equations and of those of the lower-dimensional integrable systems seems to be attractive and could elucidate many subtleties. We postpone also to future publications an investigation of the harmonic analytic features of the self-dual gauge theories in signature (2,2) that are expected to have intriguing peculiarities due to a different "noncompactness" of the corresponding rotation and conformal groups.

Note finally the $(SU(2)\times SU(2))/U(1)$ harmonics turn out to underlie Fueter analyticity.

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APPENDIX A: QUATERNIONIC STRUCTURES

An almost quaternionic structure is a set of three tensors of type (1,1), $(J_a)^n_m$, acting on the tangent bundle of a manifold, that represent a basis of the quaternionic algebra (5):

$$J_a J_b = -\delta_{ab} + \epsilon_{abc} J_c \ . \tag{A1}$$

Because of the noncommutativity of quaternions one has to distinguish between left (L_a) and right (R_a) quaternionic structures. We saw above that the right quaternionic structures form a two-parameter family (30):

$$R_{a\,\alpha\dot\alpha}^{\;\beta\dot\beta} = -iu^i_{\dot\alpha}\sigma^j_{a_i}u^{\dot\beta}_j\delta^\beta_\alpha \ . \tag{A2}$$

An analogous statement is valid with respect to the left quaternionic structures L_a.

An interesting observation: Given two mutually commuting quaternionic structures, e.g., L_a and R_a, one can construct a one-parameter family of quaternionic structures that interpolate between them. To do this we construct the operator

$$\eta = \tfrac{1}{2}(1 - L_a R_a) \ . \tag{A3}$$

It has the properties (that follow from quaternionic algebras for L_a, R_a and because they commute mutually)

$$\eta^2 = 1 \ , \quad \eta L_a \eta = R_a \ , \quad \eta R_a \eta = L_a \ . \tag{A4}$$

Now it becomes obvious that the quaternionic algebra (5) will be satisfied with a "mixed" quaternionic structure

$$J_a = e^{c\eta} L_a e^{-c\eta} = L_a \cosh^2 c - R_a \sinh^2 c + \epsilon_{abc} R_b L_c \cosh c \sinh c \tag{A5}$$

and commuting with this quaternionic structure

$$J'_a = e^{c\eta} R_a e^{-c\eta} = R_a \cosh^2 c - L_a \sinh^2 c - \epsilon_{abc} R_b L_c \cosh c \sinh c \ , \tag{A6}$$

where c is a (real) parameter. Therefore, in 4D space there is a five-parameter system of quaternionic structures (two parameters in the choice of L_a, two in that of R_a, and the parameter c).

APPENDIX B: COMPACT COSETS OF NONCOMPACT GROUPS

Here, some mathematical definitions and statements are presented in a form convenient for us, together with illustrations drawn from the paper.

The Iwasawa decomposition for a noncompact semisimple group G is (see textbooks [35], [15])

$$G = KAN \ . \tag{B1}$$

Here K, A, and N are subgroups of G having the following meaning: K is the maximal compact subgroup of G. Denote generators of G by $\wp$ and those of K by κ. Let υ be remaining generators of G and $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ is a maximal Abelian subalgebra in υ. $A = e^\alpha$ is a commutative subgroup of G generated by α. Finally, all generators of G are decomposed in a direct sum of eigenspaces under an adjoint action of α:

$$[\alpha_k, \wp_\gamma] = \gamma(\alpha_k)\wp_\gamma \ , \qquad \wp = \sum_\gamma \wp_\gamma \ , \quad \gamma = \{\gamma(\alpha_1), \ldots, \gamma(\alpha_n)\} \ . \tag{B2}$$

It is said that γ is positive, $\gamma > 0$, if its first nonvanishing component is positive. A space $n = \{n_\gamma\}$ of generators $\wp_\gamma$ with positive γ is a maximal nilpotent subalgebra of $\wp$. $N = e^n$ is a corresponding maximal nilpotent subgroup of G.

Now, the maximal solvable subgroup of G is the product AN. The Borel parabolic subgroup of G is

$$B = MAN \ , \tag{B3}$$

where M is the centralizer of the subgroup A in K, i.e., a subgroup of K commuting with A.

The parabolic subgroups P of G are defined as those that contain the Borel one as their subgroup. In other words,

$$P = LAN , \tag{B4}$$

where L is a subgroup of the maximal compact subgroup K above, containing in turn M as a subgroup. The Borel subgroup is a minimal parabolic subgroup. It is a "gist" of noncompactness, as can be seen through the remarkable Borel theorem [36]:

A coset of a noncompact group G modulo any of its parabolic subgroups P is a compact space.

Moreover, parabolic subgroups P can be defined as just those such that the cosets G/P are compact. The Borel subgroup is the smallest parabolic group.

An intuitive demonstration of this theorem is quite transparent:

$$\frac{G}{P} = \frac{KAN}{LAN} = \frac{K}{L} , \tag{B5}$$

K and L being compact. Despite being oversimplified, this consideration is effective in that it shows explicitly which compact manifold has been derived.

We shall now give some examples from the paper:

$$(1)\quad G = \mathrm{SL}(2,C) = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} ,$$

$$\alpha\delta - \beta\gamma = 1 \text{ (see Sec. II B)} ,$$

$$K = \mathrm{SU}(2) = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} , \quad |a|^2 + |b|^2 = 1 ,$$

$$\alpha = \begin{bmatrix} -\phi & 0 \\ 0 & \phi \end{bmatrix} , \quad A = e^{\alpha} ,$$

$$n = \begin{bmatrix} 0 & 0 \\ z & 0 \end{bmatrix} , \quad [\alpha, n] = +\phi n , \quad N = e^{n} .$$

The parabolic group used is

$$P = \mathrm{U}(1) \times AN , \tag{B6}$$

where U(1) is a subgroup of K with $a = e^{-i\phi}, b = 0$. We see that

$$\frac{G}{P} = \frac{\mathrm{SU}(2)}{\mathrm{U}(1)} = S^2 .$$

(2) $G = \mathrm{Spin}(5,1)$, represented by matrix (85):

$$\begin{bmatrix} l^{\alpha}_{\beta} & b^{\dot\beta}_{\alpha} \\ c^{\beta}_{\dot\alpha} & r^{\dot\beta}_{\dot\alpha} \end{bmatrix} . \tag{B7}$$

Its maximal compact subgroup $K = \mathrm{Spin}(5)$ is given by the same matrix with identification $b^{\dot\alpha}_{\alpha} = \epsilon^{\dot\alpha\dot\beta}\epsilon_{\alpha\beta}c^{\beta}_{\dot\beta}$ and with unimodular l^{α}_{β} and $r^{\dot\alpha}_{\dot\beta}$.

For this case

$$A = \exp \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$$

is the dilatation group. Finally,

$$n = \begin{bmatrix} 0 & 0 \\ c^{\alpha}_{\dot\alpha} & O \end{bmatrix}$$

are conformal boosts; $N = e^{n}$.

The parabolic subgroup $P = \mathrm{Spin}(3) \times \mathrm{SO}(2) \times AN$ leads to a six-dimensional coset:

$$\frac{\mathrm{Spin}(5,1)}{P} = \frac{\mathrm{Spin}(5)}{\mathrm{Spin}(3) \times \mathrm{SO}(2)} = \frac{\mathrm{SO}(5)}{\mathrm{SO}(3) \times \mathrm{SO}(2)} . \tag{B8}$$

With this coset, however, conformal transformations would be nonanalytic, as explained in Sec. VI.

(3) $\mathrm{Spin}(6,C) \sim \mathrm{SL}(4,C)$. It is convenient to represent it again by a matrix (B7), however now with complexified entries.

Now the maximal compact group is $K = \mathrm{Spin}(6)$ $[\sim \mathrm{SU}(4)]$ given by the unitarized matrix (B7).

The group $A = e^{a_i \alpha_i}$ has three generators:

$$\alpha = \begin{bmatrix} -\sigma_3 & 0 \\ 0 & 0 \end{bmatrix} , \quad \begin{bmatrix} 0 & 0 \\ 0 & -\sigma_3 \end{bmatrix} , \quad \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} .$$

Indices $\gamma = \{\gamma(\alpha_1), \gamma(\alpha_2), \gamma(\alpha_3)\}$ can be shown in a matrix form:

$$\begin{bmatrix} 000 & 0-20 & --+ & --- \\ 020 & 000 & -++ & -+- \\ ++- & +-- & 000 & 00-2 \\ +++ & +-+ & 002 & 000 \end{bmatrix} ,$$

where triples of indices in each entry are its indices γ.

So, there are six *complex* (equivalent to twelve real) generators n with positive indices. They are arranged below the main diagonal. According to our general rule the maximal nilpotent group is $N = e^{n}$ and $B = AN$.

For a parabolic subgroup

$$P = \mathrm{SU}(3) \times \mathrm{U}(1) \times AN , \tag{B9}$$

one gets a six-dimensional coset

$$\frac{\mathrm{SL}(4,C)}{P} = \frac{\mathrm{SU}(4) \times AN}{\mathrm{SU}(3) \times \mathrm{U}(1) \times AN} = \frac{\mathrm{SU}(4)}{\mathrm{SU}(3) \times \mathrm{U}(1)} , \tag{B10}$$

i.e., just CP^3 projective space.

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