THE ULTRA-VIOLET FINITENESS OF THE $N = 4$ YANG–MILLS THEORY ☆

Lars BRINK [1]
California Institute of Technology, Pasadena, CA 91125, USA

Olof LINDGREN
Institute of Theoretical Physics, Göteborg, S-41296 Göteborg, Sweden

and

Bengt E.W. NILSSON
Center for Theoretical Physics and Department of Physics, The University of Texas at Austin, Austin, TX 78712, USA

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The $N = 4$ supersymmetric Yang–Mills gauge theory was formulated by us in a previous paper in the light-cone gauge. It is described by a scalar complex superfield whose components are the propagating field components of the theory. In this paper we prove that the theory is ultra-violet finite to all orders in perturbation theory for a specific form of the light-cone gauge.

The Yang–Mills gauge theory with maximal extended supersymmetry [1] ($N = 4$) was conjectured by Gell-Mann and Schwarz to be an ultra-violet finite quantum field theory [2] and indeed it has been found to possess unique quantum properties [3]. The natural formalism to use to investigate these features is a superfield formalism implementing the supersymmetry in a manifest way. Although such formalisms do exist for the $N = 1$ and $N = 2$ supersymmetric Yang–Mills theories, a corresponding formalism for the $N = 4$ theory has not been found. In fact quite strong arguments have been advanced [4] which suggest that such a formalism may not exist for this model.

We introduced in a recent paper "the light-cone superfield formalism" [5] to overcome these problems. In the conventional treatment one tries to implement both the Poincaré and the supersymmetry algebra in a covariant manner. By choosing the light-cone gauge for the vector field $A^{+a} = 0$ [‡1], we give up the full manifest covariance under the Poincaré algebra. However, in this way we can formulate a quantum action entirely in terms of propagating field components. This action is in fact invariant under the whole supersymmetry algebra. By projecting the supersymmetry charges into its two light-cone spinor charges [which will be defined in eq. (3)] one can easily see that one of them is linearly realized while the other one (which was not discussed in ref. [5]) is non-linearly realized. In this light-cone gauge it is now possible to construct a scalar superfield, whose components are the pro-

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[1] Permanent address: Institute of Theoretical Physics, Göteborg, S-41296 Göteborg, Sweden.

‡1 We use the following conventions: $A^\mu B_\mu = A_i B_i + A_L B_L - A_0 B_0 \equiv A_i B_i - A_+ B_- - A_- B_+$, i, j etc. label transverse directions and L and 0 the longitudinal and time directions respectively. $A_\pm = 2^{-1/2}(A_0 \pm A_L)$, $A = 2^{-1/2}(A_1 + iA_2)$ and similarly for x^μ and p_μ. Gauge group indices are a, b, c etc. and internal SU(4) indices are m, n, p etc. Complex conjugation is denoted by a bar. (See also ref. [5].)

pagating modes of the theory and with this superfield at hand it is straightforward to construct the quantum action.

In this report we shall continue the program and formulate the supergraph Feynman rules from the action and show that by choosing a specific form of the light-cone gauge, all Green's functions can be proved to be ultraviolet finite to all order of perturbation theory.

In the light-cone gauge $A^{+a} = 0$ the propagating modes are (for each gauge degree of freedom):
$A(x)$: 1 complex field describing the vector particle,
$C^{mn}(x)$: 3 complex fields describing 6 scalar particles,
$\chi^m(x)$: 4 complex Grassmann fields describing 4 spin 1/2 particles,
where

$$\bar{C}_{mn} = \tfrac{1}{2}\epsilon_{mnpq}C^{pq} = (\overline{C^{mn}}) . \tag{1}$$

Thus C^{mn} transform as a 6 and $\bar{C}_{mn}$ as a $\bar{6}$, while χ^m and $\bar{\chi}_m$ transform as a 4 and $\bar{4}$ respectively, under SU(4).

Note that in the light-cone formalism a spinor satisfying the Weyl condition can be described by a complex Grassmann field without any Lorentz index.

The supersymmetry algebra reads

$$\{Q^m_\alpha, \bar{Q}^\beta_n\} = \delta^m_n(\gamma_\mu)_\alpha{}^\beta P^\mu , \tag{2}$$

where Q_α is a Majorana spinor. The light-cone generators are obtained through the projection

$$Q = \tfrac{1}{2}(\gamma_+\gamma_- + \gamma_-\gamma_+)Q \equiv Q_+ + Q_- . \tag{3}$$

In the representation of the γ-matrices chosen in ref. [5], the algebra reduces to (we denote the supersymmetry generators by q_+ and q_- in this representation)

$$\{q^m_+, \bar{q}_{+n}\} = -\sqrt{2}\delta^m_n p^+ , \quad \{q^m_-, \bar{q}_{-n}\} = -\sqrt{2}\delta^m_n p^- , \quad \{q^m_+, \bar{q}_{-n}\} = \sqrt{2}\delta^m_n p , \tag{4,5,6}$$

with $p = 2^{-1/2}(p_1 + ip_2)$.

If we interpret p^- as the hamiltonian which is standard in the light-cone frame, where $p^- = \mathrm{i}\partial/\partial x^+$, and x^+ is taken to be the evolution coordinate, the full algebra can be implemented on the fields A, C^{mn} and χ^m and their complex conjugate fields. Since q^m_- and $\bar{q}_{-n}$ close to give the hamiltonian this symmetry will be implemented in a non-linear manner in an interacting theory [6].

The q_+ algebra can be represented in a conventional way on a Grassmann parameter θ^m and its complex conjugate $\bar{\theta}_m$ as (the subscript + is dropped from now on)

$$q^m = -\partial/\partial\bar{\theta}_m + \mathrm{i}2^{-1/2}\theta^m\partial^+ , \quad \bar{q}_n = \partial/\partial\theta^n - \mathrm{i}2^{-1/2}\bar{\theta}_n\partial^+ . \tag{7,8}$$

A general representation in terms of a superfield will now be a function of θ^m and $\bar{\theta}_n$ [hence the SU(4) covariance is manifest]. However, such a superfield will not be an irreducible representation of the algebra (4). In fact we can also construct covariant derivatives d^m and $\bar{d}_n$

$$d^m = -\partial/\partial\bar{\theta}_m - \mathrm{i}2^{-1/2}\theta^m\partial^+ , \quad \bar{d}_n = \partial/\partial\theta^n + \mathrm{i}2^{-1/2}\bar{\theta}_n\partial^+ , \tag{9,10}$$

which anticommute with q^m and $\bar{q}_n$. To restrict the superfield ϕ we can impose a "chirality" condition

$$d^m\phi = 0 . \tag{11}$$

In the case of SU(4) one can impose a further constraint

$$d^m d^n\bar{\phi} = \tfrac{1}{2}\epsilon^{mnpq}\bar{d}_p\bar{d}_q\phi . \tag{12}$$

A scalar complex superfield satisfying (11) and (12) can be written as

$$\phi(x,\theta,\bar\theta) = (\partial^+)^{-1}A(y) + \mathrm{i}(\partial^+)^{-1}\theta^m\bar\chi_m(y) + \mathrm{i}\,2^{-1/2}\theta^m\theta^n\bar C_{mn}(y) + \tfrac{1}{6}\sqrt{2}\theta^m\theta^n\theta^p\epsilon_{mnpq}\chi^q(y)$$
$$+ \tfrac{1}{12}\theta^m\theta^n\theta^p\theta^q\epsilon_{mnpq}\partial^+\bar A(y)\,, \tag{13}$$

with $y = (x, \bar x, x^+, x^- - \mathrm{i}\,2^{-1/2}\theta^m\bar\theta_m)$ and where $(\partial^+)^{-1}$ is interpreted as [7]

$$\frac{1}{\partial^+}f(x^-) = \tfrac{1}{2}\int \mathrm{d}\xi\,\phi(\xi - x^-)f(\xi)\,. \tag{14}$$

We now see that the superfield ϕ is a representation of $N = 4$ supersymmetry in the light-cone gauge. In the previous paper [5] we constructed the action for this theory in terms of this superfield. The action was found to be [‡2]

$$S = 72\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathrm{d}^4\bar\theta\,\{-\bar\phi^a(\Box/\partial^{+2})\phi^a + \tfrac{4}{3}gf^{abc}[(\partial^+)^{-1}\bar\phi^a\phi^b\bar\partial\phi^c + \text{complex conjugate}]$$
$$- g^2 f^{abc}f^{ade}[(\partial^+)^{-1}(\phi^b\partial^+\phi^c)(\partial^+)^{-1}(\bar\phi^d\partial^+\bar\phi^e) + \tfrac{1}{2}\phi^b\bar\phi^c\phi^d\bar\phi^e]\}\,. \tag{15}$$

This action was constructed by comparing with the action written in component form. A more deductive way would be to implement the q_- supersymmetry. Such a procedure determines the interaction terms [6]. In our previous paper we erroneously said that this action is only Lorentz invariant on the mass shell. In fact the non-covariant Lorentz transformations (the ones involving the – directions) are implemented in a non-linear way and could also be used to construct the interaction terms.

In order to obtain the Feynman super graph rules, we need to know the functional derivatives with respect to a field satisfying (11) and (12). First we note that $\bar\phi$ is not an independent field. In fact we can use (11) and (12) to solve for $\bar\phi$:

$$\bar\phi(x,\theta,\bar\theta) = \tfrac{1}{48}(\bar d^4/\partial^{+2})\phi(x,\theta,\bar\theta)\,. \tag{16}$$

We then follow the standard route to find the functional derivatives of a chiral field [8] and get

$$\delta\phi^a(x,\theta,\bar\theta)/\delta\phi^b(x',\theta',\bar\theta') = (4!)^{-2}d^4\delta^4(x-x')\delta^4(\theta-\theta')\delta^4(\bar\theta-\bar\theta')\delta^a_b\,, \tag{17}$$

$$\delta\bar\phi^a(x,\theta,\theta)/\delta\phi^b(x',\theta',\bar\theta') = 12(4!)^{-4}(\bar d^4 d^4/\partial^{+2})\delta^4(x-x')\delta^4(\theta-\theta')\delta^4(\bar\theta-\bar\theta')\delta^a_b\,. \tag{18}$$

The generating functional is constructed by introducing "chiral" sources J^a satisfying (11) and (12) and can then be written as $[z = (x,\theta,\bar\theta)]$

$$W[J] = \int \mathcal{D}\phi^a(z)\exp\left(\mathrm{i}\int d^{12}z\,[\mathcal{L} + \tfrac{1}{4}\phi^a(\bar d^4/\partial^{+4})J^a]\right). \tag{19}$$

$W[J]$ is chosen so that

$$\delta^n W[J]/\delta J^{a_1}(z_1)\ldots\delta J^{a_n}(z_n)|_{J=0} = (\mathrm{i})^n\int\mathcal{D}\,\phi^a(z)\,\phi^{a_1}(z_1)\ldots\phi^{a_n}(z_n)\exp\left(\mathrm{i}\int \mathrm{d}^{12}z\,\mathcal{L}\right). \tag{20}$$

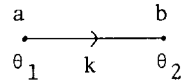
From this generating functional we can compute the various n-point Green's functions and derive the Feynman rules. The momentum space Feynman rules are given in table 1.

We shall begin by showing that all Green's functions are finite by naive power counting. The question of whether such a procedure is allowed will be addressed at the end of the paper, where it will be argued that naive power counting is in fact valid. The proof goes as follows.

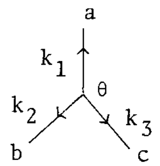
The first important observation is that a naive argument (to be explained below) shows that all supergraphs are

[‡2] We use the convention $\theta^4 \equiv \epsilon_{mnpq}\theta^m\theta^n\theta^p\theta^q$ and similarly for the other fermionic operators and parameters.

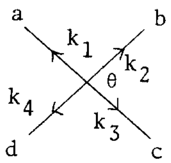
Table 1
Momentum space Feynman rules [a)].



$$= \frac{2i}{(4!)^3}\,\delta^{ab}\,\frac{d_1^4}{k^2}\,\delta^8(\theta_1-\theta_2)$$



$$= 2ig f^{abc}\int d^8\theta\left(\frac{1}{48}\frac{k_3}{k_1^+}\frac{\bar{d}_2^4}{k_2^{+2}}\frac{\bar{d}_3^4}{k_3^{+2}} - \frac{\bar{k}_3}{k_1^+}\frac{\bar{d}_1^4}{k_1^{+2}} + \text{permutations}\right)$$

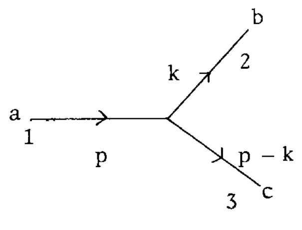


$$= -\frac{ig^2}{32}\int d^8\theta\left[f^{eab}f^{ecd}\left(\frac{k_2^+k_4^+}{(k_1^++k_2^+)(k_3^++k_4^+)}\frac{\bar{d}_3^4}{k_3^{+2}}\frac{\bar{d}_4^4}{k_4^{+2}} + \frac{1}{2}\frac{\bar{d}_2^4}{k_2^{+2}}\frac{\bar{d}_4^4}{k_4^{+2}}\right) + \text{permutations}\right]$$

a) Explanatory comments: $\bar{d}_i^4 \equiv \bar{d}^4(k_i^+,\theta)$ where k_i is pointing away from θ and in the expression for the 3-point function $k \equiv 2^{-1/2}(k_1 + ik_2)$.

of superficial degree of divergence equal to zero. The power counting rules employed here are the ones appropriate for supergraphs [8] and it is convenient to do this analysis in the context of a computation of the superspace effective action (see ref. [8]). The δ-functions and the θ-integrals appearing in the propagator and in the vertex function (see table 1) are then not to be taken into account in computing the dimensionality of these quantities. This fact is due to the property of supergraphs that they can always be reduced to a local expression in θ [8]. The stepwise elimination of δ-functions and θ-integrals giving rise to this property also shows that each time a superloop is contracted to a point in θ-space a factor of $d^4\bar{d}^4$ is absorbed, cancelling the dimension of d^4k giving the loop zero dimensionality. Therefore the propagator, the vertex function (see table 1) and loops *all* have zero dimensionality and hence any supergraph will have superficial degree of divergence equal to zero. The naive aspect of this argument is that we have so far been blind to the *exact* form of the vertex function which contains inverse powers of momenta. Scrutinizing this function we find that some terms in it will make the divergence properties worse when one leg is external (for instance the term $(k_3/k_1^+)(\bar{d}_2^4/k_2^{+2})\bar{d}_3^4/k_3^{+2}$ when the leg labelled 1 is external). However, we will show below that not only can this be remedied by manipulating the vertex function but it will also turn out that any supergraph is finite. We emphasize at this point that the internal structure of a diagram is completely irrelevant and does not effect the arguments below [9].

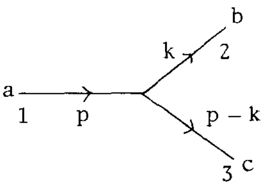
In order to see that all supergraphs are in fact finite we start by considering the part of the 3-point vertex function consisting of the first term and its permutations in the expression given in table 1. Fixing leg "1" to be external we get in the limit $k \gg p$:



$$\approx \frac{1}{24}\,ig f^{abc}\int d^8\theta\left[-\left(\frac{p-2k}{p^+} - 4\frac{k}{k^+}\right)\frac{\bar{d}_2^4}{k^{+2}}\frac{\bar{d}_3^4}{k^{+2}} + \frac{k}{k^+}\frac{\bar{d}_1^4}{p^{+2}}\frac{\bar{d}_3^4}{k^{+2}} - \frac{k}{k^+}\frac{\bar{d}_1^4}{p^{+2}}\frac{\bar{d}_2^4}{k^{+2}}\right]. \tag{21}$$

Any superspace Feynman diagram having *all* external legs attached to any term in (21) will be finite. In the case of the first term this follows from the fact that a partial integration of $\bar{d}_2^4$ (or $\bar{d}_3^4$) will cause it to act on the external leg (since $\bar{d}^n = 0$ for $n > 4$) removing *two* powers of momenta from the inside of the diagram. Thus, as claimed above, this procedure does not only eliminate the problem caused by $1/p^+$ but it does in fact also prove that all diagrams having only this kind of external vertex are finite (irrespective of the number of loops). The rest of the contribution to the 3-point vertex function in (21) will also generate finite graphs since a partial integration of $\bar{d}_2^4$ in the last term in (21) onto the third leg will make the worst divergent part in this term cancel the corresponding part in the second term.

Writing out the remaining terms of the 3-point vertex function in table 1 we find in the limit $k \gg p$:

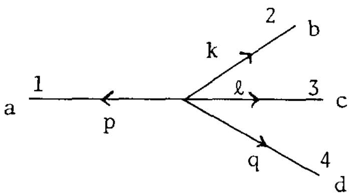


$$\approx -2ig f^{abc}\int \mathrm{d}^8\theta\left(-\frac{\bar{p}-2\bar{k}}{p^+}\frac{\bar{d}_1^4}{p^{+2}}+\frac{\bar{k}}{k^+}\frac{\bar{d}_2^4}{k^{+2}}-\frac{\bar{k}}{k^+}\frac{\bar{d}_3^4}{k^{+2}}\right). \tag{22}$$

The terms in (22) are treated in the same way as the terms in (21) except that the partial integration needed in order to "save" the first term in this case is performed using the d^4 coming from the internal propagators attached to the vertex. Thus all diagrams built with any combination of external vertices all coming from the 3-point vertex function are ultra-violet finite. Note that the partial integration of $\bar{d}^4$ in the third term in (22) generates terms which have the different $\bar{d}$'s in $\bar{d}^4$ divided between the external leg and the internal leg. These terms will, however, give rise to only ultra-violet finite diagrams.

We now turn to the 4-point vertex function. Consider first the case when two legs are external. If this vertex function is written in the limit of large internal momenta one sees directly that the manipulations employed above work also in this case. Thus this kind of vertex can also be used for the external vertices without ever generating an infinite diagram.

The last case to consider is the 4-point vertex function with one external leg. Fixing the leg labelled 1 as external with momentum p and the internal momenta being k, ℓ and $q = -(k + \ell + p)$ we get in the limit $k, \ell \gg p$:



$$\approx -\tfrac{1}{32}\,\mathrm{i}g^2\int \mathrm{d}^8\theta$$

$$\times\Bigg\{f^{eab}f^{ecd}\left[\frac{q^+-\ell^+}{q^++\ell^+}\left(\frac{\bar{d}_1^4}{p^{+2}}\frac{\bar{d}_2^4}{k^{+2}}+\frac{\bar{d}_3^4}{\ell^{+2}}\frac{\bar{d}_4^4}{q^{+2}}\right)+\frac{\bar{d}_2^4}{k^{+2}}\frac{\bar{d}_4^4}{q^{+2}}-\frac{\bar{d}_2^4}{k^{+2}}\frac{\bar{d}_3^4}{\ell^{+2}}+\frac{\bar{d}_1^4}{p^{+2}}\frac{\bar{d}_3^4}{\ell^{+2}}-\frac{\bar{d}_1^4}{p^{+2}}\frac{\bar{d}_4^4}{q^{+2}}\right]$$

$$+f^{eac}f^{ebd}\left[\frac{q^+-k^+}{q^++k^+}\left(\frac{\bar{d}_1^4}{p^{+2}}\frac{\bar{d}_3^4}{\ell^{+2}}+\frac{\bar{d}_2^4}{k^{+2}}\frac{\bar{d}_4^4}{q^{+2}}\right)+\frac{\bar{d}_1^4}{p^{+2}}\frac{\bar{d}_2^4}{k^{+2}}-\frac{\bar{d}_1^4}{p^{+2}}\frac{\bar{d}_4^4}{q^{+2}}+\frac{\bar{d}_3^4}{\ell^{+2}}\frac{\bar{d}_4^4}{q^{+2}}-\frac{\bar{d}_2^4}{k^{+2}}\frac{\bar{d}_3^4}{\ell^{+2}}\right]$$

$$+f^{ead}f^{ebc}\left[\frac{\ell^+-k^+}{\ell^++k^+}\left(\frac{\bar{d}_1^4}{p^{+2}}\frac{\bar{d}_4^4}{q^{+2}}+\frac{\bar{d}_2^4}{k^{+2}}\frac{\bar{d}_3^4}{\ell^{+2}}\right)+\frac{\bar{d}_3^4}{\ell^{+2}}\frac{\bar{d}_4^4}{q^{+2}}-\frac{\bar{d}_2^4}{k^{+2}}\frac{\bar{d}_4^4}{q^{+2}}+\frac{\bar{d}_1^4}{p^{+2}}\frac{\bar{d}_2^4}{k^{+2}}-\frac{\bar{d}_1^4}{p^{+2}}\frac{\bar{d}_3^4}{\ell^{+2}}\right]\Bigg\}. \tag{23}$$

The terms in (23) divide into two distinct sets of terms: those with both $\bar{d}^4$'s acting on internal legs and the rest (having one $\bar{d}^4$ acting on the external leg). The former set of terms give rise to finite diagrams when used as external vertices due to the following simple calculation. Collecting the terms from, for instance, the first bracket (i.e. with the common factor $f^{eab}f^{ecd}$) in (23) having a factor $\bar{d}_3^4\bar{d}_4^4$ (two of these terms need a partial integration in order to be cast into this form) we find

$$\bar{d}_3^4\bar{d}_4^4 = \left(\frac{q^+ - \ell^+}{q^+ + \ell^+}\frac{1}{\ell^{+2}}\frac{1}{q^{+2}} + \frac{1}{k^{+2}}\frac{1}{q^{+2}} - \frac{1}{k^{+2}}\frac{1}{\ell^{+2}}\right) = \mathrm{O}(p)\,,$$

where we used momentum conservation. Note that $\mathrm{O}(p)$ terms lead to only finite diagrams.

The final observation that establishes the finiteness is that the terms in the full vertex function that we so far have not proved to generate finite super Feynman diagrams *all* have one $\bar{d}^4$ acting on the external leg. Now, in order for a diagram to be infinite *all* its external vertices must be of this last kind and thus *all* external legs are of the same chirality. However, this implies that the diagram will not give a contribution to the superspace effective action (since its measure is over the *full* superspace) unless a factor d^4 is partially integrated onto external legs in which case the diagram is finite. This completes the proof that there are no infinite superspace Feynman diagrams in this theory in the light-cone gauge.

We have so far studied the model with $(\partial^+)^{-1}$ defined in (14). In fact this specific definition prevents us from from making a Wick rotation to implement Weinberg's theorem [9]. However, it is known that there is still some gauge freedom left when we have chosen the condition $A^{+a} = 0$. This freedom essentially amounts to a choice of the pole structure in p^+ [10]. In a recent report Mandelstam [11] treated the same problem as above in a light-cone formalism similar to ours, although only with θ^m's and no $\bar{\theta}_m$'s, giving up the possibility of using covariant derivatives. In his proof he chose a gauge with the pole structure $(p^+ + \mathrm{i}\epsilon p^-)^{-1}$. This choice is appropriate also in our case enabling us to perform the Wick rotation. Weinberg's theorem is then valid and we conclude that for this specific gauge choice the perturbation expansion is completely finite. This implies that the charge renormalization function ψ (sometimes called β) vanishes to all orders of perturbation theory in any gauge.

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