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AN EFFECTIVE ACTION FOR THE SUPERSYMMETRIC CP^{n-1} MODEL

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We explicitly construct the low-energy effective action for the "composite" fields in the supersymmetric CP^{n-1} model. It contains two chiral superfields S and $\bar{S}$. We find that the anomaly structure is exactly reproduced by the term $S \log S$ first proposed by Veneziano and Yankielowicz. In addition, the effective action contains additional terms involving derivatives of the superfields S and $\bar{S}$. These terms are invariant under the anomalous transformations and, in general, cannot be neglected in the low-energy limit.

1. Introduction

One of the main goals of particle physics today is the unification of the strong and electroweak interactions up to a mass scale of about 10^{14}–10^{17} GeV [1]. To construct a renormalizable theory that reproduces the observed low-energy phenomenology one is naturally led to a spontaneously broken gauge theory. Conventionally, the symmetry breaking is achieved by introduction of the scalar Higgs fields. By suitable adjustment of parameters these fields produce the breaking of the original symmetry in at least two different steps, with a large energy gap. However, increasingly it has

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been realized that the presence of such elementary scalar fields is unnatural [2] and gives rise to the gauge hierarchy problem [3].

It has recently been suggested [4] that a neat way to solve these problems is to have a supersymmetric theory as the fundamental, unifying theory of the strong and electroweak interactions. Starting from such a unified, supersymmetric gauge theory the main task is to show that the observed low-energy phenomenology can be reproduced as a consequence of symmetry breaking of the original theory. Obviously, this requires supersymmetry as well as gauge symmetry breaking. The observed low-energy physics should be described by an effective lagrangian containing the low-mass particles. However, it is not a simple task to construct such effective lagrangians starting from the original supersymmetric gauge theory.

A first step in this direction was taken by Veneziano and Yankielowicz [5]. Starting from an $N=1$ supersymmetric gauge theory they constructed an effective lagrangian in terms of a chiral, scalar superfield. Their effective lagrangian reproduced all the anomalous Ward identities (conformal, axial and γ trace anomaly) of the original theory. In their construction they assumed that the effective lagrangian was only a function of the chiral superfields S and $\bar{S}$. This means that they excluded the possibility of having terms involving the supersymmetric covariant derivative of these chiral superfields. In the low-energy approximation such terms are not in general negligible.

In order to elucidate such problems, we have considered the supersymmetric CP^{n-1} model. After integration over the fundamental fields we explicitly construct a low-energy effective lagrangian for the "composite" fields. As in the four-dimensional case treated in ref. [5], our effective action involves the two chiral superfields S and $\bar{S}$. It can be written as follows:

$$S_{\mathrm{eff}}=\frac{n}{4\pi}\int \mathrm{d}^2x\left\{\int \mathrm{d}\theta_{\mathrm{R}}\,\mathrm{d}\bar{\theta}_{\mathrm{L}}\,S\left(\log\left(\frac{S}{\mu}\right)-1\right)\right.$$
$$+\int \mathrm{d}\theta_{\mathrm{L}}\,\mathrm{d}\bar{\theta}_{\mathrm{R}}\,\bar{S}\left(\log\left(\frac{\bar{S}}{\mu}\right)-1\right)-\tfrac{1}{2}\int \mathrm{d}\theta_{\mathrm{R}}\,\mathrm{d}\bar{\theta}_{\mathrm{L}}\,\mathrm{d}\theta_{\mathrm{L}}\,\mathrm{d}\bar{\theta}_{\mathrm{R}}\log S\log\bar{S}$$
$$\left.+\int \mathrm{d}\theta_{\mathrm{R}}\,\mathrm{d}\bar{\theta}_{\mathrm{L}}\,\mathrm{d}\theta_{\mathrm{L}}\,\mathrm{d}\bar{\theta}_{\mathrm{R}}\,Z(S,\bar{S},\Delta,\bar{\Delta})\right\}. \tag{1.1}$$

The first two terms are exactly those guessed by Veneziano and Yankielowicz to reproduce the correct anomalous structure. The third term is the kinetic term for the various fields. It has a different structure than in ref. [5] as a consequence of the different dimensionality of space-time. Finally, we obtain an additional term, $Z(S,\bar{S},\Delta,\bar{\Delta})$, explicitly given in eq. (4.17). This term involves trilinear and higher couplings that cannot be written purely in terms of S and $\bar{S}$, but involve the superfields Δ and $\bar{\Delta}$, obtained by applying two derivatives to $\log S$ and $\log\bar{S}$, as given in eq. (4.7). Unlike the other terms this additional term cannot be guessed purely from symmetry arguments. It is invariant under the anomalous transformations, depending entirely on the dynamics of the specific model. However, it is interesting that the low-energy effective lagrangian contains such terms involving covariant derivatives of the original chiral superfield. In fact, in the low-energy approximation there is no reason to neglect such terms obtained by applying up to two covariant derivatives to the original chiral superfields.

Being written entirely in terms of superfields, our effective action displayed in (1.1) is manifestly supersymmetric. In our model, as in ref. [5], supersymmetry is not spontaneously broken.

This paper is organized as follows. In sect. 2, we discuss the supersymmetric CP^{n-1} model in some detail. Sect. 3 is devoted to the construction of the effective action, using Schwinger's method [6]* for slowly varying fields. In sect. 4 we give the construction of the effective action in terms of superfields. Sect. 5 contains a discussion and summary of the relevant properties of the effective action. Finally, in appendix A we give some detail on the method used to construct the effective action, in appendix B we collect some useful formulas for the explicit computation of the effective action and in appendix C we give the transformations of the various fields under the superconformal symmetry.

2. The supersymmetric CP^{n-1} model

After the discovery of the purely bosonic CP^{n-1} model it was straightforward to construct its supersymmetric generalization. Initially, the supersymmetric model was constructed in terms of $N = 1$ superfields [8, 9]. It was subsequently realized that the lagrangian was invariant under an extended $N = 2$ supersymmetry. In $N = 2$ superspace formulation, as given in ref. [8], the lagrangian has the simple form

$$\mathfrak{L} = \bar{\phi} e^{-V} \phi + \frac{n}{2f} V, \tag{2.1}$$

where ϕ and $\bar{\phi}$ are chiral superfields, V is the vector $N = 2$ superfield and f is a dimensionless coupling constant. The factor n has been extracted from f, as in ref. [8], in a way such that f is a constant when $n \to \infty$. Under an $N = 2$ supersymmetry transformation ϕ transforms as follows:

$$\delta\phi = \left\{ \bar{\alpha}_L \left(\frac{\partial}{\partial \bar{\theta}_L} - \tfrac{1}{2}\theta_L \partial_- \right) + \bar{\alpha}_R \left(\frac{\partial}{\partial \bar{\theta}_R} - \tfrac{1}{2}\theta_R \partial_+ \right) \right.$$
$$\left. + \alpha_R \left(\frac{\partial}{\partial \theta_R} - \tfrac{1}{2}\bar{\theta}_R \partial_+ \right) + \alpha_L \left(\frac{\partial}{\partial \theta_L} - \tfrac{1}{2}\bar{\theta}_L \partial_- \right) \right\} \phi. \tag{2.2}$$

* The Schwinger method has been already applied to the non-supersymmetric CP^{n-1} model, see ref. [7].

It is useful to define the following covariant derivatives

$$D_{\mathrm{L}}=\frac{\partial}{\partial\theta_{\mathrm{L}}}+\tfrac{1}{2}\bar{\theta}_{\mathrm{L}}\partial_-,\qquad \bar{D}_{\mathrm{L}}=\frac{\partial}{\partial\bar{\theta}_{\mathrm{L}}}+\tfrac{1}{2}\theta_{\mathrm{L}}\partial_-,$$

$$D_{\mathrm{R}}=\frac{\partial}{\partial\theta_{\mathrm{R}}}+\tfrac{1}{2}\bar{\theta}_{\mathrm{R}}\partial_+,\qquad \bar{D}_{\mathrm{R}}=\frac{\partial}{\partial\bar{\theta}_{\mathrm{R}}}+\tfrac{1}{2}\theta_{\mathrm{R}}\partial_+.\tag{2.3}$$

The chiral superfields have the following expansion in terms of superfields

$$\phi=\mathrm{e}^{-u}\left[Z+\bar{\theta}_{\mathrm{L}}\psi_{\mathrm{R}}+\bar{\theta}_{\mathrm{R}}\psi_{\mathrm{L}}+\bar{\theta}_{\mathrm{L}}\bar{\theta}_{\mathrm{R}}G\right],$$

$$\bar{\phi}=\mathrm{e}^{u}\left[\bar{Z}+\bar{\psi}_{\mathrm{R}}\theta_{\mathrm{L}}+\bar{\psi}_{\mathrm{L}}\theta_{\mathrm{R}}+\theta_{\mathrm{R}}\theta_{\mathrm{L}}\bar{G}\right],\tag{2.4}$$

and, in Wess-Zumino gauge, the vector superfield is given by

$$\begin{aligned}V={}&i\bar{\theta}_{\mathrm{L}}\theta_{\mathrm{L}}B_-+i\bar{\theta}_{\mathrm{R}}\theta_{\mathrm{R}}B_++\bar{\theta}_{\mathrm{L}}\theta_{\mathrm{R}}\bar{\varphi}+\bar{\theta}_{\mathrm{R}}\theta_{\mathrm{L}}\varphi+\bar{\theta}_{\mathrm{L}}\theta_{\mathrm{R}}\bar{\theta}_{\mathrm{R}}\chi_{\mathrm{L}}\\&+\bar{\theta}_{\mathrm{R}}\theta_{\mathrm{L}}\bar{\theta}_{\mathrm{L}}\chi_{\mathrm{R}}+\theta_{\mathrm{L}}\bar{\theta}_{\mathrm{R}}\theta_{\mathrm{R}}\bar{\chi}_{\mathrm{L}}+\theta_{\mathrm{R}}\bar{\theta}_{\mathrm{L}}\theta_{\mathrm{L}}\bar{\chi}_{\mathrm{R}}\\&+\bar{\theta}_{\mathrm{R}}\theta_{\mathrm{L}}\bar{\theta}_{\mathrm{L}}\theta_{\mathrm{R}}D,\end{aligned}\tag{2.5}$$

where

$$u=\tfrac{1}{2}\left(\bar{\theta}_{\mathrm{L}}\theta_{\mathrm{L}}\partial_-+\bar{\theta}_{\mathrm{R}}\theta_{\mathrm{R}}\partial_+\right),\tag{2.6}$$

$$B_\pm=B_1\pm iB_2,\qquad \partial_\pm=\partial_1\pm i\partial_2.\tag{2.7}$$

We have written the various superfields using light cone variables in euclidean superspace. Everything can be rewritten in a manifestly Lorentz invariant way by using the definition of right and left components of a spinor:

$$\theta_{\binom{\mathrm{L}}{\mathrm{R}}}=\tfrac{1}{2}(1\pm i\gamma_5)\theta,\qquad \bar{\theta}_{\binom{\mathrm{R}}{\mathrm{L}}}=\bar{\theta}(1\pm i\gamma_5).\tag{2.8}$$

In the representation for the γ matrices

$$\gamma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad \gamma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},$$

$$\gamma_5=\gamma_1\gamma_2=i\begin{pmatrix}1&0\\0&-1\end{pmatrix},\tag{2.9}$$

the spinors have the following form

$$\theta=\begin{pmatrix}\theta_{\mathrm{R}}\\\theta_{\mathrm{L}}\end{pmatrix},\qquad \bar{\theta}=\left(\bar{\theta}_{\mathrm{L}},\bar{\theta}_{\mathrm{R}}\right).\tag{2.10}$$

Note that the notation we are using here is slightly different from that used in ref. [8].

The gauge superfield V has no kinetic term and therefore, as in the case of the purely bosonic model, can be eliminated from (2.1) using the algebraic equation of motion

$$V=\log\left(\frac{2f}{n}\bar{\phi}\phi\right). \tag{2.11}$$

Notice that, in the $N=2$ formalism, one does not need to add any constraint to the lagrangian (2.1). The constraints are automatically obtained when we vary (2.1) with respect to the auxiliary field V, and are given by (2.11) by using (2.5).

Performing the integration over θ and $\bar{\theta}$ in (2.1) we obtain the lagrangian in terms of component fields:

$$\begin{aligned}\mathcal{L}=\bar{Z}&\left[-\tfrac{1}{2}(\partial-iB)_+(\partial-iB)_- -\tfrac{1}{2}(\partial-iB)_-(\partial-iB)_+ +\bar{\varphi}\varphi-D\right]Z\\&+\bar{\psi}_L(\partial-iB)_-\psi_L+\bar{\psi}_R(\partial-iB)_+\psi_R\\&+\bar{\psi}_L\varphi\psi_R+\bar{\psi}_R\bar{\varphi}\psi_L+\bar{Z}(\bar{\chi}_L\psi_R+\bar{\chi}_R\psi_L)\\&+Z(\bar{\psi}_L\chi_R+\bar{\psi}_R\chi_L)-G\bar{G}+\frac{n}{2f}D.\end{aligned} \tag{2.12}$$

In order to remain in the Wess-Zumino gauge, supersymmetry transformations have to be accompanied by suitable gauge transformations. The resulting modified supersymmetry transformations, which leave the classical action corresponding to (2.12) invariant, are given by

$$\begin{aligned}&\delta Z=\bar{\alpha}_L\psi_R+\bar{\alpha}_R\psi_L,\qquad \delta\bar{Z}=\bar{\psi}_R\alpha_L+\bar{\psi}_L\alpha_R,\\&\delta\psi_R=\bar{\alpha}_R G+\left[(\partial-iB)_-\alpha_L-\bar{\varphi}\alpha_R\right]Z,\\&\delta\bar{\psi}_R=\alpha_R\bar{G}+\left[-(\partial+iB)_-\bar{\alpha}_L-\varphi\bar{\alpha}_R\right]\bar{Z},\\&\delta\psi_L=-\bar{\alpha}_L G+\left[(\partial-iB)_+\alpha_R-\varphi\alpha_L\right]Z,\\&\delta\bar{\psi}_L=-\alpha_L\bar{G}+\left[-(\partial+iB)_+\alpha_R-\bar{\varphi}\bar{\alpha}_L\right]\bar{Z},\\&\delta G=(\alpha_R\chi_L-\alpha_L\chi_R)Z+\alpha_R(\partial-iB)_+\psi_R-\alpha_L(\partial-iB)_-\psi_L\\&\qquad+\alpha_R\bar{\varphi}\psi_L-\alpha_L\varphi\psi_R,\\&\delta\bar{G}=(\bar{\chi}_L\bar{\alpha}_R-\bar{\chi}_R\bar{\alpha}_L)\bar{Z}+\bar{\alpha}_R(\partial+iB)_+\psi_R-\bar{\alpha}_L(\partial+iB)_-\bar{\psi}_L\\&\qquad-\bar{\alpha}_R\varphi\bar{\psi}_L+\bar{\alpha}_L\bar{\varphi}\bar{\psi}_R,\end{aligned} \tag{2.13}$$

for the matter multiplet and

$$
\begin{aligned}
\delta B_{\pm} &= i\left(\bar{\chi}_{\binom{\mathrm{L}}{\mathrm{R}}}\alpha_{\binom{\mathrm{L}}{\mathrm{R}}} + \bar{\alpha}_{\binom{\mathrm{L}}{\mathrm{R}}}\chi_{\binom{\mathrm{L}}{\mathrm{R}}}\right),\\
\delta\varphi &= \bar{\chi}_{\mathrm{L}}\alpha_{\mathrm{R}} + \bar{\alpha}_{\mathrm{L}}\chi_{\mathrm{R}},\\
\delta\bar{\varphi} &= \bar{\chi}_{\mathrm{R}}\alpha_{\mathrm{L}} + \bar{\alpha}_{\mathrm{R}}\chi_{\mathrm{L}},\\
\delta\chi_{\binom{\mathrm{R}}{\mathrm{L}}} &= (D \mp F)\alpha_{\binom{\mathrm{R}}{\mathrm{L}}} + \partial_{\mp}\begin{pmatrix}\varphi\\ \bar{\varphi}\end{pmatrix}\alpha_{\binom{\mathrm{L}}{\mathrm{R}}},\\
\delta\bar{\chi}_{\binom{\mathrm{R}}{\mathrm{L}}} &= (D \pm F)\bar{\alpha}_{\binom{\mathrm{R}}{\mathrm{L}}} - \partial_{\mp}\begin{pmatrix}\bar{\varphi}\\ \varphi\end{pmatrix}\bar{\alpha}_{\binom{\mathrm{L}}{\mathrm{R}}},\\
\delta D &= \tfrac{1}{2}\left(\bar{\alpha}_{\mathrm{R}}\partial_{+}\chi_{\mathrm{R}} + \bar{\alpha}_{\mathrm{L}}\partial_{-}\chi_{\mathrm{L}} + \alpha_{\mathrm{L}}\partial_{-}\bar{\chi}_{\mathrm{L}} + \alpha_{\mathrm{R}}\partial_{+}\bar{\chi}_{\mathrm{R}}\right),
\end{aligned}
\tag{2.14}
$$

for the vector multiplet, where

$$F = \partial_1 B_2 - \partial_2 B_1 .$$

Finally, using (2.8) and (2.10) we can write the lagrangian (2.12) in a manifestly Lorentz invariant form:

$$
\begin{aligned}
\mathcal{L} = \bar{Z}&\left[-(\partial - iB)_{\mu}(\partial - iB)^{\mu} + \bar{\varphi}\varphi - D\right]Z\\
&+\bar{\psi}\left[\gamma^{\mu}(\partial - iB)_{\mu} + \tfrac{1}{2}(1 + i\gamma_5)\bar{\varphi} + \tfrac{1}{2}(1 - i\gamma_5)\varphi\right]\psi\\
&-\bar{G}G + \bar{Z}\bar{\chi}\psi + Z\bar{\psi}\chi + \frac{n}{2f}D .
\end{aligned}
\tag{2.15}
$$

At the end of this section we want to discuss the invariance properties of the classical supersymmetric CP^{n-1} model (2.1). It is easy to check that the action (2.1) is invariant under conformal, superconformal and chiral transformations. Under these transformations the two sets of variables $x_{+}, \theta_{\mathrm{R}}, \bar{\theta}_{\mathrm{R}}$ and $x_{-}, \theta_{\mathrm{L}}, \bar{\theta}_{\mathrm{L}}$ transform independently from each other as follows:

$$
\begin{aligned}
\delta x_{(\pm)} &= \tfrac{1}{2}\left[\bar{\theta}_{\binom{\mathrm{R}}{\mathrm{L}}}\alpha_{\binom{\mathrm{R}}{\mathrm{L}}} - \bar{\alpha}_{\binom{\mathrm{R}}{\mathrm{L}}}\theta_{\binom{\mathrm{R}}{\mathrm{L}}}\right] + A_{(\pm)},\\
\delta\theta_{\binom{\mathrm{R}}{\mathrm{L}}} &= \alpha_{\binom{\mathrm{R}}{\mathrm{L}}} - \tfrac{1}{2}\dot{\alpha}_{\binom{\mathrm{R}}{\mathrm{L}}}\bar{\theta}_{\binom{\mathrm{R}}{\mathrm{L}}}\theta_{\binom{\mathrm{R}}{\mathrm{L}}} + \tfrac{1}{2}\dot{A}_{(\pm)}\theta_{\binom{\mathrm{R}}{\mathrm{L}}} + ia_{(\pm)}\theta_{\binom{\mathrm{R}}{\mathrm{L}}},\\
\delta\bar{\theta}_{\binom{\mathrm{R}}{\mathrm{L}}} &= \bar{\alpha}_{\binom{\mathrm{R}}{\mathrm{L}}} + \tfrac{1}{2}\dot{\bar{\alpha}}_{\binom{\mathrm{R}}{\mathrm{L}}}\bar{\theta}_{\binom{\mathrm{R}}{\mathrm{L}}}\theta_{\binom{\mathrm{R}}{\mathrm{L}}} + \tfrac{1}{2}\dot{A}_{(\pm)}\bar{\theta}_{\binom{\mathrm{R}}{\mathrm{L}}} - ia_{(\pm)}\bar{\theta}_{\binom{\mathrm{R}}{\mathrm{L}}},
\end{aligned}
\tag{2.16}
$$

where

$$x_{(\pm)} = \tfrac{1}{2}(x_1 \mp ix_2), \qquad \partial_{(\pm)} = \partial_1 \pm i\partial_2 .$$

The parameters of the transformations α_R, A_+, a_+ [α_L, A_-, a_-] are arbitrary functions of $x_+[x_-]$ and the dot denotes the derivative with respect to the argument: for instance, $\dot{A}_+ \equiv (\partial/\partial x_+)A_+(x_+)$. For simplicity, we will restrict ourselves to those transformations (2.16) with $A_- = a_- = \alpha_L = 0$.

Under the infinitesimal transformations (2.16) the fields V, ϕ and $\bar{\phi}$ transform as follows:

$$\delta_A \begin{pmatrix} V \\ \phi \\ \bar{\phi} \end{pmatrix} = \{\partial_+ A_+ - \tfrac{1}{2}(\partial_R \theta_R + \partial_{\bar{R}} \bar{\theta}_R)\dot{A}_+\} \begin{pmatrix} V \\ \phi \\ \bar{\phi} \end{pmatrix},$$

$$\delta_\alpha \begin{pmatrix} V \\ \phi \\ \bar{\phi} \end{pmatrix} = \{-\partial_{\bar{R}}[\bar{\alpha}_R + \tfrac{1}{2}\dot{\bar{\alpha}}_R \bar{\theta}_R \theta_R] - \partial_R[\alpha_R - \tfrac{1}{2}\dot{\alpha}_R \bar{\theta}_R \theta_R] + \tfrac{1}{2}\partial_+(\bar{\theta}_R \alpha_R - \bar{\alpha}_R \theta_R)\} \begin{pmatrix} V \\ \phi \\ \bar{\phi} \end{pmatrix},$$

$$\delta_a \begin{pmatrix} V \\ \phi \\ \bar{\phi} \end{pmatrix} = i[\partial_{\bar{R}} \bar{\theta}_R - \partial_R \theta_R] a_+ \begin{pmatrix} V \\ \phi \\ \bar{\phi} \end{pmatrix}, \tag{2.17}$$

where the derivatives act on any function to the right. The fact that the variations (2.17) of V, ϕ and $\bar{\phi}$ under the transformations (2.16) are given by total derivatives ensures that the action constructed from (2.1) is invariant under conformal, superconformal and conformal chiral transformations. Of course in the quantum CP^{n-1} model, these invariances cannot be maintained because of the appearance of anomalies.

3. Construction of the effective action

In this section we derive an expression for the supersymmetric effective action in the large-n limit. We employ the Schwinger method [6] for slowly varying fields. Here we quote the main results, leaving full details to appendix A.

To obtain the effective action we first integrate over the fundamental Z and ψ fields in the path integral. From (2.12) this gives

$$S_{\text{eff}} = n \operatorname{Tr} \log(\mathcal{M}_B - \bar{\chi}\mathcal{M}_F^{-1}\chi) - n \operatorname{Tr}\log(\mathcal{M}_F) + \int d^2x \frac{nD}{2f}, \tag{3.1}$$

where

$$\mathcal{M}_B = -\tfrac{1}{2}(\partial - iB)_+(\partial - iB)_- - \tfrac{1}{2}(\partial - iB)_-(\partial - iB)_+ + \bar{\varphi}\varphi - D,$$

$$\mathcal{M}_F = \begin{pmatrix} \varphi & (\partial - iB)_- \\ (\partial - iB)_+ & \bar{\varphi} \end{pmatrix}, \tag{3.2}$$

$$\chi = \begin{pmatrix} \chi_R \\ \chi_L \end{pmatrix}, \qquad \bar{\chi} = (\bar{\chi}_L, \bar{\chi}_R). \tag{3.3}$$

As shown by Schwinger [6, 7], it is possible to obtain explicit expressions for such effective actions in the limit of slowly varying fields, including non-zero $F_{\mu\nu}$. This means that φ, $\bar{\varphi}$ and D are effectively made constant in the calculation and A_μ a linear function of x. From (3.1) and appendix A we obtain

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}^{\text{bos}} + \mathcal{L}_{\text{eff}}^{\text{fer}}, \tag{3.4}$$

where

$$\mathcal{L}_{\text{eff}}^{\text{bos}} = n\frac{F}{4\pi}\left[\log\left(\frac{\bar{\varphi}}{\varphi}\right) + \int_\varepsilon^\infty \frac{\mathrm{d}\tau}{\tau}\frac{\mathrm{e}^{-\bar{\varphi}\varphi\tau}}{\sinh(F\tau)}\left(\coth(F\tau) - \mathrm{e}^{D\tau}\right)\right] + n\frac{D}{2f}, \tag{3.5}$$

$$\mathcal{L}_{\text{eff}}^{\text{ferm}} = n\frac{F}{4\pi}\int_0^\infty \mathrm{d}\tau\,\frac{\mathrm{e}^{-\bar{\varphi}\varphi\tau}}{\sinh(F\tau)}$$

$$\times\left[\frac{\bar{\chi}_{\mathrm{R}}\varphi\chi_{\mathrm{L}}}{(D+F)}\left(\mathrm{e}^{-F\tau} - \mathrm{e}^{D\tau}\right) + \frac{\bar{\chi}_{\mathrm{L}}\bar{\varphi}\chi_{\mathrm{R}}}{D-F}\left(\mathrm{e}^{F\tau} - \mathrm{e}^{D\tau}\right) + \frac{\bar{\chi}_{\mathrm{R}}\chi_{\mathrm{L}}\bar{\chi}_{\mathrm{L}}\chi_{\mathrm{R}}}{D^2 - F^2}\right.$$

$$\left.\times(1 - \bar{\varphi}\varphi\tau)\left(\mathrm{e}^{D\tau} - \mathrm{e}^{\pm F\tau}\right)\right]. \tag{3.6}$$

Here, we have split the effective lagrangian into two pieces. All the χ and $\bar{\chi}$ dependences are contained in $\mathcal{L}_{\text{eff}}^{\text{fer}}$, whilst $\mathcal{L}_{\text{eff}}^{\text{bos}}$ contains those terms remaining in $\mathcal{L}_{\text{eff}}$ when χ and $\bar{\chi}$ are set to zero. $\mathcal{L}_{\text{eff}}^{\text{fer}}$ is independent of the choice of sign in the last exponential. The dependence on the vector potential is through the magnetic field

$$F = \partial_1 B_2 - \partial_2 B_1. \tag{3.7}$$

Finally, ε is an ultra-violet cut-off parameter, introduced in order to regularize the usual logarithmic divergence in the linear D term.

In order to construct a renormalized effective lagrangian we rewrite (3.5) as follows:

$$\mathcal{L}_{\text{eff}}^{\text{bos}} = \frac{n}{4\pi}\left\{F\log\frac{\bar{\varphi}}{\varphi} + \int_\varepsilon^\infty \frac{\mathrm{d}\tau}{\tau}\left[\mathrm{e}^{-\bar{\varphi}\varphi\tau}\left(F\coth(F\tau) - \frac{1}{\tau}\right) - \mathrm{e}^{-(\bar{\varphi}\varphi - D)\tau}\left(\frac{F}{\sinh(F\tau)} - \frac{1}{\tau}\right)\right.\right.$$

$$\left.\left. + \frac{\mathrm{e}^{-\bar{\varphi}\varphi\tau}}{\tau}\left(1 - \mathrm{e}^{D\tau} + D\tau\right)\right]\right\}$$

$$+ nD\left[\frac{1}{2f} - \frac{1}{4\pi}\int_\varepsilon^\infty \frac{\mathrm{d}\tau}{\tau}\mathrm{e}^{-\bar{\varphi}\varphi\tau}\right]. \tag{3.8}$$

The first term of (3.8) is convergent and the limit $\varepsilon \to 0$ can be taken safely. To compute the last divergent term in the limit $\varepsilon \to 0$, a dimensional constant μ is introduced, satisfying the relation

$$\frac{1}{2f(\varepsilon)} + \frac{1}{4\pi}\left(\log\frac{\varepsilon}{\mu^2} + \text{constant}\right) = 0\,. \tag{3.9}$$

We are then left with the finite term $(n/4\pi)D\log(\bar{\varphi}\varphi/\mu^2)$.

Thus $\mathcal{L}_{\text{eff}}^{\text{bos}}$ becomes

$$\mathcal{L}_{\text{eff}}^{\text{bos}} = \frac{n}{4\pi}\left\{(D+F)\log\frac{\bar{\varphi}}{\mu} + (D-F)\log\frac{\varphi}{\mu} + \bar{\varphi}\varphi\mathcal{F}\left(\frac{D+F}{\bar{\varphi}\varphi}, \frac{D-F}{\bar{\varphi}\varphi}\right)\right\}, \tag{3.10}$$

where

$$\mathcal{F}\left(\frac{D+F}{\bar{\varphi}\varphi}, \frac{D-F}{\bar{\varphi}\varphi}\right) = \frac{1}{\bar{\varphi}\varphi}\int_0^\infty \frac{d\tau}{\tau}\left[e^{-\bar{\varphi}\varphi\tau}\left(F\coth(F\tau) - \frac{1}{\tau}\right) + \frac{e^{-\bar{\varphi}\varphi\tau}}{\tau}\left(1 - e^{D\tau} + D\tau\right)\right.$$
$$\left. - e^{-(\bar{\varphi}\varphi - D)\tau}\left(\frac{F}{\sinh(F\tau)} - \frac{1}{\tau}\right)\right]. \tag{3.11}$$

Our form (3.10) for the bosonic part of the effective action agrees with that obtained by Riva [7] for the non-supersymmetric CP^{n-1} model, up to a minor printing error in ref. [7].

For non-constant fields, the effective action of (3.4)–(3.6) is not invariant under the supersymmetry transformations (2.14). It is only invariant under the following transformations

$$\delta D = \delta F = 0\,, \qquad \delta\chi_{\binom{R}{L}} = (D \mp F)\alpha_{\binom{R}{L}}\,, \qquad \delta\bar{\chi}_{\binom{R}{L}} = (D \pm F)\bar{\alpha}_{\binom{R}{L}}\,,$$
$$\delta\varphi = \bar{\chi}_L\alpha_R + \bar{\alpha}_L\chi_R\,, \qquad \delta\bar{\varphi} = \bar{\chi}_R\alpha_L + \bar{\alpha}_R\chi_L\,. \tag{3.12}$$

These can be obtained from (2.14) in the constant field approximation. These transformations leave the following field combinations invariant

$$\hat{\varphi} = \varphi - \frac{\bar{\chi}_L\chi_R}{D-F}\,, \qquad \bar{\hat{\varphi}} = \bar{\varphi} - \frac{\bar{\chi}_R\chi_L}{D+F}\,, \qquad \delta\hat{\varphi} = \delta\bar{\hat{\varphi}} = 0\,. \tag{3.13}$$

The complete effective lagrangian of (3.4)–(3.6) is reproduced in a manifestly supersymmetric manner by substituting $\hat{\varphi}$ and $\bar{\hat{\varphi}}$ for φ and $\bar{\varphi}$ in (3.10). Thus, we obtain the following low-energy effective lagrangian:

$$\mathcal{L}_{\text{eff}} = \frac{n}{4\pi}\left\{(D+F)\log\frac{\bar{\hat{\varphi}}}{\mu} + (D-F)\log\frac{\hat{\varphi}}{\mu} + \mathcal{F}\left(\frac{D+F}{\hat{\varphi}\bar{\hat{\varphi}}}, \frac{D-F}{\hat{\varphi}\bar{\hat{\varphi}}}\right)\hat{\varphi}\bar{\hat{\varphi}}\right\}, \tag{3.14}$$

$\mathcal{F}$ being given by (3.11).

Using the formulas of appendix B the function $\mathcal{F}$ can be computed explicitly. It is given by

$$\mathcal{F}(x;\,y)=(x-y)\log\frac{\Gamma\left(\dfrac{1}{x-y}\right)}{\Gamma\left(\dfrac{1}{x-y}+\dfrac{1}{2}-\dfrac{x+y}{2(x-y)}\right)}+y\log(x-y)\,, \tag{3.15}$$

$\mathcal{F}$ can be expanded in a power series as shown in appendix B. The lowest term in the expansion gives the following contribution to the effective lagrangian:

$$\frac{n}{4\pi}\left[\frac{1}{2}\,\frac{F^2-D^2}{\bar{\varphi}\varphi}\right]. \tag{3.16}$$

This is just the dynamically generated kinetic term for the vector multiplet.

Note that $\mathcal{F}$ vanishes when one of its arguments vanishes; i.e.,

$$\mathcal{F}(x;0)=\mathcal{F}(0,\,y)=0\,. \tag{3.17}$$

This property can also be proved in the original definition of $\mathcal{F}$ (3.11).

4. Superfield formulation of the effective lagrangian

In sect. 3 we have computed the effective action for the "composite" fields in the approximation of slowly varying fields. The effective action we have derived is invariant under the modified supersymmetry transformation of (2.14) when derivatives of the fields are neglected. This is, of course, a consequence of the fact that the low-energy approximation is not a supersymmetric approximation.

The minimal way to restore the supersymmetry of the effective action for arbitrary space-time dependent fields is to rewrite (3.14) in terms of superfields and covariant derivatives. In fact, this procedure will introduce derivative terms in the effective lagrangian for some components of the superfield as required by the supersymmetry transformations (2.14). An important requirement that we impose on our construction is that we recover (3.14) when all derivative terms are eliminated.

Let us start from the renormalized effective lagrangian (3.14), using the definitions (3.13) for φ and $\bar{\varphi}$. As shown in appendix B, the function $\mathcal{F}$ contains only quadratic or higher terms of $D+F/\bar{\varphi}\varphi$ and $D-F/\bar{\varphi}\varphi$.

In order to write (3.14) in superspace we could use the vector superfield (2.5). However, unlike (3.14), V is not gauge invariant. It is more convenient to use a superfield that is gauge invariant by itself. Starting from V we can construct the following gauge invariant, chiral superfields

$$S=D_{\mathrm{L}}\bar{D}_{\mathrm{R}}V\,,\qquad \bar{S}=D_{\mathrm{R}}\bar{D}_{\mathrm{L}}V\,. \tag{4.1}$$

They satisfy

$$D_{\rm L}S = D_{\rm R}\bar{S} = 0\,, \qquad \bar{D}_{\rm R}S = \bar{D}_{\rm L}\bar{S} = 0\,, \tag{4.2}$$

as a consequence of

$$D_{\rm L}^2 = D_{\rm R}^2 = 0\,.$$

Note that these chirality conditions of S and $\bar{S}$ are not the same as those on ϕ and $\bar{\phi}$ of (2.4). The expansion of S and $\bar{S}$ in terms of component fields is given by

$$S = {\rm e}^{-v}\tilde{S}\,, \qquad \bar{S} = {\rm e}^{v}\tilde{\bar{S}}\,, \tag{4.3}$$

where

$$\tilde{S} = \varphi + \bar{\theta}_{\rm L}\chi_{\rm R} + \bar{\chi}_{\rm L}\theta_{\rm R} + \bar{\theta}_{\rm L}\theta_{\rm R}(D-F)\,,$$

$$\tilde{\bar{S}} = \bar{\varphi} + \bar{\chi}_{\rm R}\theta_{\rm L} + \bar{\theta}_{\rm R}\chi_{\rm L} + \bar{\theta}_{\rm R}\theta_{\rm L}(D+F)\,,$$

$$v = \tfrac{1}{2}\left(\bar{\theta}_{\rm R}\theta_{\rm R}\partial_{+} - \bar{\theta}_{\rm L}\theta_{\rm L}\partial_{-}\right).$$

Using these chiral superfields it is easy to write the first two terms of (3.14) in superspace. This can be done by means of the following identities

$$\int {\rm d}\theta_{\rm R}\,{\rm d}\bar{\theta}_{\rm L}\,\tilde{S}\left[\log\left(\frac{\tilde{S}}{\mu}\right) - 1\right] = (D-F)\log\left(\frac{\varphi - \bar{\chi}_{\rm L}\chi_{\rm R}/(D-F)}{\mu}\right), \tag{4.4}$$

$$\int {\rm d}\theta_{\rm L}\,{\rm d}\bar{\theta}_{\rm R}\,\tilde{\bar{S}}\left[\log\left(\frac{\tilde{\bar{S}}}{\mu}\right) - 1\right] = (D+F)\log\left(\frac{\bar{\varphi} - \bar{\chi}_{\rm R}\chi_{\rm L}/(D+F)}{\mu}\right). \tag{4.5}$$

By integrating over θ and $\bar{\theta}$ we obtain exactly the first two terms of (3.14).

The last term in (3.14) can be expanded in a power series

$$\frac{n}{4\pi}\hat{\varphi}\bar{\hat{\varphi}}\mathcal{F}\left(\frac{D+F}{\hat{\varphi}\bar{\hat{\varphi}}};\frac{D-F}{\hat{\varphi}\bar{\hat{\varphi}}}\right) = \frac{n}{4\pi}\hat{\varphi}\bar{\hat{\varphi}}\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty}C_{\mu\nu}\left(\frac{D+F}{\hat{\varphi}\bar{\hat{\varphi}}}\right)^{\mu}\left(\frac{D-F}{\hat{\varphi}\bar{\hat{\varphi}}}\right)^{\nu}. \tag{4.6}$$

Now an arbitrary function of the superfields S and $\bar{S}$ does not contain higher than linear powers of $(D+F)$ and $(D-F)$. In order to reproduce (4.6) we must introduce additional superfields. This can certainly be done in many different ways. We find it convenient to use the following superfields

$$\Delta = SD_{\rm R}\bar{D}_{\rm L}\log S\,, \qquad \bar{\Delta} = \bar{S}D_{\rm L}\bar{D}_{\rm R}\log\bar{S}\,. \tag{4.7}$$

They have the following expansion in terms of component fields

$$\frac{\Delta}{S}=e^{v}\left[\frac{D-F}{\hat{\varphi}}+\theta_{L}\partial_{-}\left(\frac{\bar{\chi}_{L}}{\varphi}\right)+\bar{\theta}_{R}\partial_{+}\left(\frac{\chi_{R}}{\varphi}\right)+\bar{\theta}_{R}\theta_{L}\partial_{+}\partial_{-}\log\varphi\right],$$

$$\frac{\bar{\Delta}}{\bar{S}}=e^{-v}\left[\frac{D+F}{\hat{\bar{\varphi}}}+\theta_{R}\partial_{+}\left(\frac{\bar{\chi}_{R}}{\bar{\varphi}}\right)+\bar{\theta}_{L}\partial_{-}\left(\frac{\chi_{L}}{\bar{\varphi}}\right)+\bar{\theta}_{L}\theta_{R}\partial_{+}\partial_{-}\log\bar{\varphi}\right]. \tag{4.8}$$

We impose the following conditions on the desired supersymmetrization of (4.6):
(i) it should reduce to (4.6) when derivatives of the component fields are dropped;
(ii) it should be invariant under the transformations of the superconformal group;
(iii) when expanded in a power series, no negative powers of $D\pm F$ or their derivatives must occur.

The second requirement is motivated by the superconformal invariance of the classical action. The non-invariance of the first two terms in (3.14) can be ascribed to divergences which appear in the process of calculating them. No such divergences appear in the calculation of the two and higher point Green functions summarized by (4.6). Hence we expect the superconformal symmetry to remain.

The third condition ensures that the effective action is well defined in the limits $D\pm F\to 0$. It is a natural condition since an actual calculation of the effective action by Feynman diagram expansion will give only positive powers of $D\pm F$.

By limiting ourselves to functions of $\Delta,\bar{\Delta},S$ and $\bar{S}$ we now proceed to find a unique superfield action satisfying the above three conditions.

According to appendix C, our superfields transform under right superconformal transformations as

$$\delta S=\left(\Gamma_{\alpha}-\dot{\bar{\alpha}}_{R}\theta_{R}\right)S,\qquad \delta\bar{S}=\left(\Gamma_{\alpha}-\dot{\alpha}_{R}\bar{\theta}_{R}\right)\bar{S},$$

$$\delta\left(\frac{\Delta}{S\bar{S}}\right)=\Gamma_{\alpha}\left(\frac{\Delta}{S\bar{S}}\right),\qquad \delta\left(\frac{\bar{\Delta}}{S\bar{S}}\right)=\Gamma_{\alpha}\left(\frac{\bar{\Delta}}{S\bar{S}}\right), \tag{4.9}$$

where the linear differential operator Γ_{α} is given in (C.3). In (4.9) Γ_{α} gives a transport term and the other terms represent extra rotations. It is the simple transformation laws (4.9) which motivate our choice of Δ and $\bar{\Delta}$ according to (4.7) instead of, e.g., $D_{R}\bar{D}_{L}S$ and $D_{L}\bar{D}_{R}S$.

An arbitrary monomial in S, $\bar{S}$, Δ and $\bar{\Delta}$ will rotate by the sum of the rotations of its factors in addition to being transported by Γ_{α} under superconformal transformations. Since according to the first expression for Γ_{α} in (C.3), the transport term is a total derivative, the superspace integral of the monomial will be invariant apart from the rotations. Hence

$$\int d^{2}x\, d^{4}\theta\, \frac{\Delta^{m}\bar{\Delta}^{n}}{(S\bar{S})^{m+n}}, \tag{4.10}$$

is superconformally invariant. For constant fields it reduces to

$$mn\int d^2x\,\frac{(D-F)^{m+1}(D+F)^{n+1}}{(\bar{\hat{\varphi}}\hat{\varphi})^{m+n}}\,. \tag{4.11}$$

In this way we can reproduce all terms in (4.6) except those for which $\mu=1$ or $\nu=1$. To reproduce, e.g., a term which is linear in $D-F$ one would at first thought try a term with a factor $\ln\Delta$. This indeed reduces to the desired form for constant fields. But when derivative terms are retained, there appear such terms with inverse powers of $(D-F)$. They appear because the θ independent term in Δ is precisely $D-F$. Condition (iii) requires another choice. The alternative choice is to use $\ln S$ instead of $\ln\Delta$. It gives the desired expressions for constant fields

$$\int d^2x\,d^4\theta\ln S\left(\frac{\bar\Delta}{S\bar S}\right)^n=n\int d^2x\,(D-F)\left(\frac{D+F}{\bar{\hat{\varphi}}\hat{\varphi}}\right)^{n+1}. \tag{4.12}$$

But the effect of the rotation term in the superconformal transformation (4.9) must be checked. We have

$$\begin{aligned}\delta_\alpha\ln S\left(\frac{\bar\Delta}{S\bar S}\right)^n&=\Gamma_\alpha\left(\ln S\left(\frac{\bar\Delta}{S\bar S}\right)^n\right)-(\dot\alpha_R\theta_R S)\left(\frac{\partial}{\partial S}\ln S\right)\left(\frac{\bar\Delta}{S\bar S}\right)^n\\&=\Gamma_\alpha\left(\ln S\left(\frac{\bar\Delta}{S\bar S}\right)^n\right)-\dot\alpha_R\theta_R\left(\frac{\bar\Delta}{\bar S S}\right)^n.\end{aligned} \tag{4.13}$$

The integral of this over superspace is actually zero. The reason is that $(\bar\Delta/\bar S S)$ is antichiral. Therefore all θ_L dependence in the second term resides in a factor e^v and vanishes when integrated over x_-. The same is not true when there is also a chiral factor present.

Our strategy for constructing a superspace action in terms of $\Delta,\bar\Delta,S$ and $\bar S$ only which fulfills the three above conditions should now be clear. For each term in (4.7) one constructs a term with a factor $(\Delta/\bar S S)^{\mu-1}$ if $\mu\geqslant 2$, a factor $\ln S$ if $\mu=1$, a factor $(\bar\Delta/\bar S S)^\nu$ if $\nu\geqslant 2$ and a factor $\ln\bar S$ if $\nu=1$. The explicit expression for the superspace action, which we obtain by this method, is

$$\frac{n}{4\pi}\int d^4\theta\int d^2x\left\{W\left(\frac{\Delta}{S\bar S},\frac{\bar\Delta}{S\bar S}\right)+V\left(\frac{\bar\Delta}{S\bar S}\right)\ln S+U\left(\frac{\Delta}{S\bar S}\right)\ln\bar S-\tfrac12\ln S\ln\bar S\right\}, \tag{4.14}$$

where

$$W(x,y)=\int_0^x\frac{\mathrm{d}u}{u}\int_0^y\frac{\mathrm{d}v}{v}\left[G(u,v)-G(u,0)-G(0,v)+G(0,0)\right],$$

$$V(y)=\int_0^y\frac{\mathrm{d}v}{v}\left[G(0,v)-G(0,0)\right],$$

$$U(x)=\int_0^x\frac{\mathrm{d}u}{u}\left[G(u,0)-G(0,0)\right],$$

$$G(u,v)=\frac{1}{uv}\mathcal{F}(u,v), \tag{4.15}$$

where $\mathcal{F}(u,v)$ is given by (3.11) or (3.15).

The last term in (4.14) comes from $G(0,0)=-\frac{1}{2}$. It contains the dynamically generated kinetic terms for the vector multiplet.

Collecting all terms we get the following final expression for the supersymmetrized effective action

$$S_{\mathrm{eff}}=\frac{n}{4\pi}\int\mathrm{d}^2x\left\{\int\mathrm{d}\theta_{\mathrm{R}}\,\mathrm{d}\bar{\theta}_{\mathrm{L}}\,\tilde{S}\left(\log\left(\frac{\tilde{S}}{\mu}\right)-1\right)+\int\mathrm{d}\theta_{\mathrm{L}}\,\mathrm{d}\bar{\theta}_{\mathrm{R}}\,\bar{\tilde{S}}\left(\log\left(\frac{\bar{\tilde{S}}}{\mu}\right)-1\right)\right.$$
$$\left.+\int\mathrm{d}\theta_{\mathrm{R}}\,\mathrm{d}\bar{\theta}_{\mathrm{L}}\,\mathrm{d}\theta_{\mathrm{L}}\,\mathrm{d}\bar{\theta}_{\mathrm{R}}\left[-\tfrac{1}{2}\log S\log\bar{S}+Z(S,\bar{S},\Delta,\bar{\Delta})\right]\right\}, \tag{4.16}$$

where

$$Z(S,\bar{S},\Delta,\bar{\Delta})=W\left(\frac{\Delta}{S\bar{S}},\frac{\bar{\Delta}}{S\bar{S}}\right)+\ln(S)V\left(\frac{\bar{\Delta}}{S\bar{S}}\right)+\ln(\bar{S})U\left(\frac{\Delta}{S\bar{S}}\right), \tag{4.17}$$

and W, V and U are given in (4.15).

5. Properties of the effective action

In this section we will analyze the physical meaning of the various terms in the effective action and discuss the positivity of the effective lagrangian in (3.10).

As is the case for the $N=1$ four-dimensional supersymmetric Yang-Mills theory, the $N=2$ supersymmetric CP^{n-1} model has an axial, conformal and γ trace anomaly. The first two terms in (4.16) are required to reproduce the correct anomaly structure of our supersymmetric model. They were first proposed by Veneziano and Yankielowicz [5] to have an effective lagrangian containing the correct anomalous

Ward identities. Here, they have been confirmed by means of a purely dynamical calculation.

Let us check the transformation properties of the first two terms in (4.16)

$$S_{\text{anomalous}} = \frac{n}{4\pi}\int d^2x \left\{ \int d\theta_R\, d\bar{\theta}_L\, \tilde{S}\left(\log\frac{\tilde{S}}{\mu} - 1\right) + \int d\theta_L\, d\bar{\theta}_R\, \tilde{\bar{S}}\left(\log\frac{\tilde{\bar{S}}}{\mu} - 1\right)\right\}, \tag{5.1}$$

under the anomalous transformations given by (2.16) and (2.17). Using the definition (4.1) of S and $\bar{S}$ and the transformation (2.17) of V it is easy to check that the variation of $\tilde{S}$ and $\tilde{\bar{S}}$ is given by the following total derivatives:

$$\begin{aligned}
\delta_A \tilde{S} &= \left(\partial_+ A_+ - \tfrac{1}{2}\partial_R \theta_R \dot{A}_+\right)\tilde{S},\\
\delta_\alpha \tilde{S} &= \left(-\partial_R \alpha_R - \partial_+ \bar{\alpha}_R \theta_R\right)\tilde{S},\\
\delta_a \tilde{S} &= \left(-i\partial_R \theta_R a_+ \tilde{S}\right),\\
\delta_A \tilde{\bar{S}} &= \left(\partial_+ A_+ - \tfrac{1}{2}\bar{\partial}_R \bar{\theta}_R \dot{A}_+\right)\tilde{\bar{S}},\\
\delta_\alpha \tilde{\bar{S}} &= \left(-\bar{\partial}_R \bar{\alpha}_R - \partial_+ \alpha_R \bar{\theta}_R\right)\tilde{\bar{S}},\\
\delta_a \tilde{\bar{S}} &= i\bar{\partial}_R \bar{\theta}_R a_+ \tilde{\bar{S}}.
\end{aligned} \tag{5.2}$$

These transformations imply that $\int d^2x \int d\theta_R\, d\bar{\theta}_L\, \tilde{S}$ and $\int d^2x \int d\theta_L\, d\bar{\theta}_L\, \tilde{\bar{S}}$ are invariant under conformal, superconformal and chiral transformations. However, the variation of the terms with $\tilde{S}\log(\tilde{S}/\mu)$ and $\tilde{\bar{S}}\log(\tilde{\bar{S}}/\mu)$ gives the correct anomalous transformations of the effective action. In fact, we obtain

$$\delta S_{\text{anomalous}} = \frac{n}{4\pi}\int d^2x \left\{ \int d\theta_R\, d\bar{\theta}_L \left[-ia_+ + \tfrac{1}{2}\dot{A}_+ - \dot{\bar{\alpha}}_R \theta_R\right]\tilde{S} + \int d\theta_L\, d\bar{\theta}_R \left[+ia_+ + \tfrac{1}{2}\dot{A}_+ - \dot{\alpha}_R \bar{\theta}_R\right]\tilde{\bar{S}}\,. \right. \tag{5.3}$$

Integrating over θ we finally obtain

$$\delta S_{\text{anomalous}} = \frac{n}{2\pi}\int d^2x \left\{ ia_+ F + \tfrac{1}{2}\dot{A}_+ D + \tfrac{1}{2}\left[\dot{\bar{\alpha}}_R \chi_R + \dot{\alpha}_R \bar{\chi}_R\right]\right\}. \tag{5.4}$$

Thus, we have checked that the terms (5.1) reproduce the correct anomalous transformations. Following the considerations of sect. 4 and appendix C, it is then also straightforward to show that the $Z(S, \bar{S}, \Delta, \bar{\Delta})$ in (4.16) is invariant under the anomalous transformations.

In conclusion we have shown that the anomalous Ward identities of the supersymmetric CP^{n-1} model are exactly reproduced in the effective lagrangian by a term $\tilde{S}\log\tilde{S}$ and $\bar{\tilde{S}}\log\bar{\tilde{S}}$ as first proposed in ref. [5].

By means of our dynamical calculation we obtain additional terms involving Δ and $\bar{\Delta}$ which contain two covariant derivatives of S and $\bar{S}$. In general, these terms cannot be neglected at low energy. However, they are strictly dependent on the dynamics of the specific model and cannot be predicted by symmetry considerations alone.

Finally, to end this section we wish to discuss the problem of the positivity of the effective action. The problem is that some terms in (3.14) involving the auxiliary field D appear with negative sign. It is well known that auxiliary fields appear in the euclidean lagrangian with a negative sign, see for example the term $G\bar{G}$ in (2.12). However, in the case of a renormalizable theory the auxiliary fields appear only up to second power in the classical lagrangian and no derivative term is allowed. Thus, they can be eliminated using the equation of motion to give a positive definite lagrangian. The situation is more complicated in the case of an effective lagrangian as (3.14) where one can have any power of the auxiliary field D and also derivative terms as in the supersymmetrized expression (4.16). Unfortunately, we have not been able to prove in general that, if one eliminates the auxiliary field D by means of its equation of motion, the corresponding lagrangian is positive definite. However, we can explicitly prove it for the case $F=0$. Then we have a special case of the effective actions considered in ref. [10]. In this case, (3.12) becomes:

$$\mathcal{L}^{\mathrm{eff}}(F=0)=\frac{n}{4\pi}\left\{\varphi\bar{\varphi}\log\frac{\varphi\bar{\varphi}}{\mu^2}-D-(\varphi\bar{\varphi}-D)\log\frac{\varphi\bar{\varphi}-D}{\mu^2}\right\}. \tag{5.5}$$

The equation of motion for D gives

$$D=\varphi\bar{\varphi}-\mu^2, \tag{5.6}$$

and the effective lagrangian (5.5) becomes

$$\mathcal{L}^{\mathrm{eff}}(F=0)=\frac{n}{4\pi}\{\varphi\bar{\varphi}(\log(\varphi\bar{\varphi})-1)+\mu^2\}, \tag{5.7}$$

which is positive definite with a minimum when $\varphi\bar{\varphi}=\mu^2$.

It should also be noted that the term $F\log(\bar{\varphi}/\varphi)$ in (3.8) is not real. However, it becomes real in Minkowski space due to the substitution $F\to iF$.

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Appendix A

Here we give the details of our calculation leading from eq. (3.1) to eqs. (3.4)–(3.6).

The first term in the expression (3.2) for the effective action is expanded in the fermion fields. Since these are constant and anticommute, the expansion ends after a finite number of terms, so we get:

$$\frac{1}{n}S_{\text{eff}} = \operatorname{Tr}\ln \mathcal{M}_{\mathrm{B}} - \operatorname{Tr}\ln \mathcal{M}_{\mathrm{F}} - \operatorname{Tr}\left(\mathcal{M}_{\mathrm{B}}^{-1}\bar{\chi}\mathcal{M}_{\mathrm{F}}^{-1}\chi\right)$$

$$-\tfrac{1}{2}\operatorname{Tr}\left(\mathcal{M}_{\mathrm{B}}^{-1}\bar{\chi}\mathcal{M}_{\mathrm{F}}^{-1}\chi\mathcal{M}_{\mathrm{B}}^{-1}\bar{\chi}\mathcal{M}_{\mathrm{F}}^{-1}\chi\right)V + \int \mathrm{d}^2x\frac{nD}{2f}. \tag{A.1}$$

In order to obtain expressions for the propagators $\mathcal{M}_{\mathrm{B}}^{-1}$ and $\mathcal{M}_{\mathrm{F}}^{-1}$ we proceed to calculate eigenvectors and eigenvalues of $\mathcal{M}_{\mathrm{B}}$ and $\mathcal{M}_{\mathrm{F}}$.

Because of gauge invariance, a specific choice of vector potential

$$A_\mu = (0, x, F), \tag{A.2}$$

is sufficient to reproduce an arbitrary constant magnetic field F. For definiteness and without loss of generality in the final expression we assume that

$$F > 0, \qquad \operatorname{Im}\varphi > 0. \tag{A.3}$$

The matrix $\mathcal{M}_{\mathrm{F}}$ is now

$$\mathcal{M}_{\mathrm{F}} = \begin{pmatrix} \varphi & \partial_1 - i(\partial_2 - ix_1F) \\ \partial_1 + i(\partial_2 - ix_1F) & \bar{\varphi} \end{pmatrix}. \tag{A.4}$$

It is diagonalizable, for its hermitian and antihermitian parts commute. In order to have normalizable eigenvectors, we put the system in a box of length l in the x_2 direction, and require periodic boundary conditions. It is not necessary to require a finite extension of space in the x_1 direction. The eigenvectors of (A.4) corresponding to an eigenvalue ω have the form

$$\psi(x_1, x_2) = \begin{pmatrix} \psi_1(x_1) \\ \psi_2(x_1) \end{pmatrix} \mathrm{e}^{ipx_2}, \tag{A.5}$$

where

$$p = 2\pi\nu/l, \qquad \nu = 0, \pm 1, \pm 2, \dots,$$

$$(\varphi - \omega)\psi_1 + (\partial_1 + p - Fx_1)\psi_2 = 0,$$

$$(\bar{\varphi} - \omega)\psi_2 + (\partial_1 - p + Fx_1)\psi_1 = 0. \tag{A.6}$$

The solutions to these equations are harmonic oscillator functions. With normalization and sign conventions for these functions such that

$$(\partial + x)H_n(x) = \sqrt{2n}\, H_{n-1}(x),$$

$$(\partial - x)H_{n-1}(x) = -\sqrt{2n}\, H_n(x),$$

$$\int H_n^2(x)\, \mathrm{d}x = 1. \tag{A.7}$$

The normalized eigenvectors read

$$\psi_{n\nu\pm} = \sqrt{\frac{\sqrt{F}}{l}} \begin{pmatrix} C_{1n\pm} H_n\left(\dfrac{Fx_1 - p_\nu}{\sqrt{F}}\right) \\ C_{2n\pm} H_{n-1}\left(\dfrac{Fx_1 - p_\nu}{\sqrt{F}}\right) \end{pmatrix} e^{ip_\nu x_2},$$

$$\sqrt{2}\, C_{1n\pm} = \left(1 \pm \frac{\mathrm{Im}\,\varphi}{\sqrt{2nF + (\mathrm{Im}\,\varphi)^2}}\right)^{1/2},$$

$$\sqrt{2}\, C_{2n\pm} = \left(1 \mp \frac{\mathrm{Im}\,\varphi}{\sqrt{2nF + (\mathrm{Im}\,\varphi)^2}}\right)^{1/2},$$

$$\omega_{n\pm} = \mathrm{Re}\varphi \pm i\sqrt{2nF + (\mathrm{Im}\,\varphi)^2},$$

$$p_\nu = 2\pi\nu/l,$$

$$\nu = \cdots -2, -1, 0, 1, 2, \dots, \qquad n = 0, 1, 2, \dots. \tag{A.8}$$

For each value of n and ν there are two eigenvectors corresponding to the upper and lower sign, respectively, except for $n = 0$. For $n = 0$ only the upper sign gives an acceptable eigenvector.

The fermion propagator may now be written as

$$\Delta_F(x, y) = \sum_{n=0}^{\infty} \sum_{\nu=-\infty}^{+\infty} \sum_{\pm}{}' \psi_{n,\nu\pm}(x) \frac{1}{\omega_{n\pm}} \psi^{+}_{n,\nu\pm}(y), \tag{A.9}$$

where the prime on the third summation indicates that the lower sign has to be omitted when $n = 0$. Since the eigenvectors form a complete set, $\Delta_F(x, y)$ is the inverse of $\mathcal{M}_F$ in the sense that

$$\mathcal{M}_F(x)\Delta_F(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta^2(x-y).$$

The matrix $\mathcal{M}_B$ is

$$\mathcal{M}_B = -\partial_1^2 - (\partial_2 - ix_1 F)^2 + \bar{\varphi}\varphi - D. \tag{A.10}$$

A similar calculation as for the fermion matrix gives the boson propagator

$$\Delta_B(x, y) = \sum_{n=0}^{\infty} \sum_{\nu=-\infty}^{+\infty} \chi_{n,\nu}(x) \frac{1}{\omega_n^2} \chi^{+}_{n,\nu}(y), \tag{A.11}$$

where

$$\omega_n^2 = (2n+1)F + \varphi^*\varphi - D,$$

$$\chi_{n\nu}(x) = \sqrt{\frac{\sqrt{F}}{l}}\, H_n\left(\frac{Fx_1 - p_2}{\sqrt{F}}\right) e^{ip_\nu x_2},$$

$$p_\nu = 2\pi\nu/l. \tag{A.12}$$

We proceed to calculate the various terms in eq. (A.1) using eqs. (A.8), (A.9), (A.11), (A.12) and the formula

$$-\int_0^\infty \frac{d\tau}{\tau} e^{-\omega\tau} = \ln\omega + \text{const}. \tag{A.13}$$

The constant is divergent. Whenever necessary we regularize this divergence by introducing a small cut-off $\varepsilon \to 0$ at the lower end of the integral.

The first term in eq. (A.2) becomes

$$\operatorname{Tr}\log \mathcal{M}_B = \text{const} - \int_0^\infty \frac{d\tau}{\tau} \sum_{n,\nu} e^{-\omega_n \tau}.$$

Since ω_n only depends on n, the summation over ν is divergent. This divergence is merely a harmless volume factor, however. The function $\psi_{n\nu}$ is centered around

$x_1 = p/F = (2\pi\nu/lF)$. Therefore, summation over ν corresponds to integration over x_1. Since the fields are constant, this is a divergence which must necessarily appear in the expression for the action unless the total volume of space is finite. We use the notation

$$l\int \mathrm{d}x_1 = \int \mathrm{d}^2x = V, \tag{A.14}$$

and obtain

$$\sum_{\nu} = \frac{lF}{2\pi}\int \mathrm{d}x_1 = \frac{F}{2\pi}\int \mathrm{d}^2x = V\frac{F}{2\pi}. \tag{A.15}$$

Neglecting the divergent constant in (A.13), we obtain

$$\begin{aligned}\operatorname{Tr}\log \mathfrak{M}_{\mathrm{B}} &= -\frac{VF}{2\pi}\int_0^{\infty}\frac{\mathrm{d}\tau}{\tau}\sum_{n=0}^{\infty}\mathrm{e}^{-[(2n+1)B+\bar{\varphi}\varphi-D]\tau} \\ &= -\frac{VF}{4\pi}\int_0^{\infty}\frac{\mathrm{d}\tau}{\tau}\frac{\mathrm{e}^{-\tau[\bar{\varphi}\varphi-D]}}{\sinh(F\tau)}.\end{aligned} \tag{A.16}$$

A similar calculation for the second term gives

$$\begin{aligned}-\operatorname{Tr}\log \mathfrak{M}_{\mathrm{F}} &= -\frac{VF}{2\pi}\left\{\ln\omega_0 + \sum_{n=1}^{\infty}\ln\omega_{n_+}\omega_{n_-}\right\} \\ &= -\frac{VF}{2\pi}\left\{\ln\varphi - \int\frac{\mathrm{d}\tau}{\tau}\sum_{n=1}^{\infty}\mathrm{e}^{-\tau(\bar{\varphi}\varphi+2nF)}\right\} \\ &= -\frac{VF}{2\pi}\left\{\ln\varphi - \tfrac{1}{2}\int\frac{\mathrm{d}\tau}{\tau}\frac{\mathrm{e}^{-\tau(\bar{\varphi}\varphi+F)}}{\sinh(F\tau)}\right\} \\ &= \frac{VF}{4\pi}\left\{\ln\frac{\bar{\varphi}}{\varphi} + \int\frac{\mathrm{d}\tau}{\tau}\frac{\mathrm{e}^{-\tau\bar{\varphi}\varphi}}{\sinh(F\tau)}\coth(F\tau)\right\}.\end{aligned} \tag{A.17}$$

The eigenfunctions (A.8) and (A.12) enter in the calculation of the two remaining terms in eq. (A.1). These terms correspond to one-loop graphs with external χ and $\bar{\chi}$ lines. Since the external lines carry zero momentum there is a single momentum p running through the loop. As in the calculation of the previous two terms, summation over this momentum gives a factor $VF/2\pi$. The x_1 integrations at the vertices are trivial to perform using the orthonormality of the functions $H_n((x_1F-p)/VF)$. We are then left with sums over the indices $\pm$ for each fermion propagator, which are of the form

$$\sum_{\pm}{}' \frac{C_{1n\pm}^2}{\omega_{n\pm}} = \frac{\bar{\varphi}}{2nF+\bar{\varphi}\varphi},$$

$$\sum_{\pm} \frac{C_{2n\pm}^2}{\omega_{n\pm}} = \frac{\varphi}{2nF+\bar{\varphi}\varphi},$$

$$\sum_{\pm} \frac{C_{1n\pm}C_{2n\pm}}{\omega_{n\pm}} = \frac{-i\sqrt{2nF}}{2nF+\bar{\varphi}\varphi}, \tag{A.18}$$

and a single sum over n. After eq. (A.18) has been used, we are left with a rational function of n. This we decompose into partial fractions and perform the sum over n using formulas obtained by differentiation of eq. (A.13).

Application of these steps to the third term in (A.1) gives

$$\begin{aligned}
-\mathrm{Tr}(\Delta_{\mathrm{B}}\bar{\chi}\Delta_{\mathrm{F}}\chi) &= -\int \mathrm{d}^2x\,\mathrm{d}^2y\,\Delta_{\mathrm{B}}(x,y)\bar{\chi}\Delta_{\mathrm{F}}(x,y)\chi \\
&= -\frac{VF}{2\pi}\sum_{n=0}^{\infty}\frac{1}{\omega_n^2}\sum_{\pm}{}'\Big\{\bar{\chi}_{\mathrm{L}}C_{1n\pm}\frac{1}{\omega_{n\pm}}C_{1n\pm}\chi_{\mathrm{R}} \\
&\quad +\bar{\chi}_{\mathrm{R}}C_{2(n+1)\pm}\frac{1}{\omega_{(n+1)\pm}}C_{2(n+1)\pm}\chi_{\mathrm{L}}\Big\} \\
&= -\frac{VF}{2\pi}\sum_{0}^{\infty}\frac{1}{\omega_n^2}\left\{\frac{\bar{\chi}_{\mathrm{L}}\bar{\varphi}\chi_{\mathrm{R}}}{2nF+\bar{\varphi}\varphi}+\frac{\bar{\chi}_{\mathrm{R}}\varphi\chi_{\mathrm{L}}}{2(n+1)F+\bar{\varphi}\varphi}\right\} \\
&= -\frac{VF}{2\pi}\sum_{n=0}^{\infty}\left\{\frac{\bar{\chi}_{\mathrm{L}}\bar{\varphi}\chi_{\mathrm{R}}}{D-F}\left(\frac{1}{(2n+1)F+\bar{\varphi}\varphi-D}-\frac{1}{(2n+1)F+\bar{\varphi}\varphi-F}\right)\right. \\
&\quad \left.+\frac{\bar{\chi}_{\mathrm{R}}\varphi\chi_{\mathrm{L}}}{D+F}\left(\frac{1}{(2n+1)F+\bar{\varphi}\varphi-D}-\frac{1}{(2n+1)F+\bar{\varphi}\varphi+F}\right)\right\} \\
&= -\frac{VF}{4\pi}\int_0^{\infty}\mathrm{d}\tau\,\frac{\mathrm{e}^{-\bar{\varphi}\varphi\tau}}{\sinh(nF)}\left\{\frac{\bar{\chi}_{\mathrm{L}}\bar{\varphi}\chi_{\mathrm{R}}}{D-F}(\mathrm{e}^{-D\tau}-\mathrm{e}^{-F\tau})\right. \\
&\quad \left.+\frac{\bar{\chi}_{\mathrm{R}}\varphi\chi_{\mathrm{L}}}{D+F}(\mathrm{e}^{-D\tau}-\mathrm{e}^{F\tau})\right\}.
\end{aligned}$$

An analogous calculation of the last term in (A.1) goes as follows:

$$-\tfrac{1}{2}\mathrm{Tr}(\Delta_B\bar{\chi}\Delta_F\chi\Delta_B\bar{\chi}\Delta_F\chi)$$

$$= -\mathrm{Tr}(\Delta_B\bar{\chi}_R\Delta_F\chi_L\Delta_B\bar{\chi}_L\Delta_F\chi_R) - \mathrm{Tr}(\Delta_B\bar{\chi}_R\Delta_F\chi_R\Delta_B\bar{\chi}_L\Delta_F\chi_L)$$

$$= -\frac{VF}{2\pi}\bar{\chi}_R\chi_L\bar{\chi}_L\chi_R\sum_{n=0}^{\infty}$$

$$\times\left\{\frac{1}{\omega_n^2}\left(\sum_{\pm}C_{2(n+1)\pm}\frac{1}{\omega_{(n+1)\pm}}C_{2(n+1)\pm}\right)\frac{1}{\omega_n^2}\left(\sum_{\pm}{}'C_{1n\pm}\frac{1}{\omega_{n\pm}}C_{1n\pm}\right)\right.$$

$$\left.-\frac{1}{\omega_n^2}\left(\sum_{\pm}C_{2(n+1)\pm}\frac{1}{\omega_{(n+1)\pm}}C_{1(n+1)\pm}\right)\frac{1}{\omega_{n+1}^2}\left(\sum_{\pm}C_{1(n+1)\pm}\frac{1}{\omega_{(n+1)\pm}}C_{2(n+1)\pm}\right)\right\}$$

$$= -\frac{VF}{2\pi}\bar{\chi}_R\chi_L\bar{\chi}_L\chi_R\sum_{n=0}^{\infty}\left\{\frac{1}{\omega_n^2}\frac{\varphi}{2(n+1)F+\bar{\varphi}\varphi}\frac{1}{\omega_n^2}\frac{\bar{\varphi}}{2nF+\bar{\varphi}\varphi}\right.$$

$$\left.-\frac{1}{\omega_n^2}\frac{-i\sqrt{2(n+1)F}}{2(n+1)F+\bar{\varphi}\varphi}\frac{1}{\omega_{n+1}^2}\frac{-i\sqrt{2(n+1)F}}{2(n+1)F+\bar{\varphi}\varphi}\right\}.$$

After some algebra this can be cast into the form

$$-\frac{VF}{2\pi}\bar{\chi}_R\chi_L\bar{\chi}_L\chi_R\sum_{n=0}^{\infty}\frac{1}{D^2-F^2}\left(1+\bar{\varphi}\varphi\frac{\mathrm{d}}{\mathrm{d}(\bar{\varphi}\varphi)}\right)\sum_{n=0}^{\infty}\left(\frac{1}{2nF+\bar{\varphi}\varphi}-\frac{1}{2nF+\bar{\varphi}\varphi-D}\right)$$

$$= -\frac{VF}{2\pi}\bar{\chi}_R\chi_L\bar{\chi}_L\chi_R\frac{1}{2(D^2-F^2)}\int_0^{\infty}\frac{\mathrm{d}\tau\,\mathrm{e}^{-\bar{\varphi}\varphi\tau}}{\sinh(\tau F)}(1-\tau\bar{\varphi}\varphi)(\mathrm{e}^{\tau F}-\mathrm{e}^{\tau D}).$$

This is the last term in (3.4).

Appendix B

This appendix is devoted to the explicit evaluation of the function $\mathcal{F}$ defined in (3.11). $\mathcal{F}$ can be written in the following form

$$\mathcal{F} = \sum_{i=1}^{3} I_i, \tag{B.1}$$

where

$$I_1 = \frac{F}{\bar{\varphi}\varphi}\int_0^\infty \frac{\mathrm{d}t}{t} \mathrm{e}^{-(\bar{\varphi}\varphi/F)t}\left(\coth t - \frac{1}{t}\right),$$

$$I_2 = -\frac{F}{\bar{\varphi}\varphi}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-(\bar{\varphi}\varphi - D/F)t}\left(\frac{1}{\sinh t} - \frac{1}{t}\right),$$

$$I_3 = \frac{D}{\bar{\varphi}\varphi}\int_0^\infty \mathrm{d}t \frac{\mathrm{e}^{-t(\varphi\bar{\varphi}/D)}}{t^2}\left(1 - \mathrm{e}^t + t\right). \tag{B.2}$$

I_3 can easily be computed and one gets:

$$I_3 = \log\frac{\bar{\varphi}\varphi}{\bar{\varphi}\varphi - D} + \frac{D}{\bar{\varphi}\varphi}\log\frac{\bar{\varphi}\varphi - D}{\bar{\varphi}\varphi} - \frac{D}{\bar{\varphi}\varphi}. \tag{B.3}$$

In order to compute I_2 and I_3 we need the following formulas:

$$\int_0^\infty \frac{\mathrm{d}t}{t}\mathrm{e}^{-at}\left(\coth t - \frac{1}{t}\right) = 2\int_0^\infty \frac{\mathrm{d}t}{t}\mathrm{e}^{-(1/2)at}\left[\frac{1}{\mathrm{e}^t - 1} - \frac{1}{t} + \tfrac{1}{2}\right], \tag{B.4}$$

$$\int_0^\infty \frac{\mathrm{d}t}{t}\mathrm{e}^{-at}\left(\frac{1}{\sinh t} - \frac{1}{t}\right) = 2\int_0^\infty \frac{\mathrm{d}t}{t}\mathrm{e}^{-at}\left[\frac{1}{\mathrm{e}^t - 1} - \frac{1}{t} + \tfrac{1}{2}\right]$$
$$-2\int_0^\infty \frac{\mathrm{d}t}{t}\mathrm{e}^{-(1/2)at}\left[\frac{1}{\mathrm{e}^t - 1} - \frac{1}{t} + \tfrac{1}{2}\right]. \tag{B.5}$$

The right-hand side of (B.4) and (B.5) can be explicitly computed by means of the formula:

$$\int_0^\infty \frac{\mathrm{d}t}{t}\mathrm{e}^{-at}\left[\frac{1}{\mathrm{e}^t - 1} - \frac{1}{t} + \tfrac{1}{2}\right] = \log\Gamma(a) - \left(a - \tfrac{1}{2}\right)\log a + a - \tfrac{1}{2}\log 2\pi. \tag{B.6}$$

Using (B.4), (B.5) and (B.6) we can compute $I_2 + I_3$ and we get:

$$I_2 + I_3 = 2\frac{F}{\varphi\bar{\varphi}}\log\frac{\Gamma\left(\frac{\varphi\bar{\varphi}}{2F}\right)\Gamma\left(\frac{\varphi\bar{\varphi} - D}{2F}\right)}{\Gamma\left(\frac{\varphi\bar{\varphi} - D}{F}\right)} + \log 4\frac{\varphi\bar{\varphi} - D}{\varphi\bar{\varphi}}$$
$$+ \frac{F}{\varphi\bar{\varphi}}\left[\log\frac{\varphi\bar{\varphi}}{8\pi F}\right] - D\left[\log\left(2\frac{\varphi\bar{\varphi} - D}{F}\right) - 1\right]. \tag{B.7}$$

Adding (B.3) and (B.7) and using the well-known formula

$$2^{2z-1}\Gamma(z)\Gamma(z+\tfrac{1}{2})=\sqrt{\pi}\,\Gamma(2z)\,, \tag{B.8}$$

we finally get

$$\mathcal{F}\left(\frac{D+F}{\varphi\bar{\varphi}},\frac{D-F}{\varphi\bar{\varphi}}\right)=2\frac{F}{\varphi\bar{\varphi}}\log\frac{\Gamma\left(\frac{\varphi\bar{\varphi}}{2F}\right)}{\Gamma\left(\frac{\varphi\bar{\varphi}-D+F}{2F}\right)}+\frac{D-F}{\varphi\bar{\varphi}}\log\frac{2F}{\varphi\bar{\varphi}}\,, \tag{B.9}$$

which is equal to the expression (3.15).

In order to expand $\mathcal{F}$ in power series, the following two formulas are useful:

$$\coth t-\frac{1}{t}=\sum_{k=1}^{\infty}\frac{2^{2k}B_{2k}}{(2k)!}t^{2k-1}\,,$$

$$\frac{1}{\sinh t}-\frac{1}{t}=-\sum_{k=1}^{\infty}\frac{2(2^{2k-1}-1)}{(2k)!}B_{2k}t^{2k-1}\,. \tag{B.10}$$

Using (B.10) in (B.2) we get the following expansion for $\mathcal{F}$:

$$\begin{aligned}\mathcal{F}\left(\frac{D+F}{\varphi\bar{\varphi}},\frac{D-F}{\varphi\bar{\varphi}}\right)=&\frac{F}{\varphi\bar{\varphi}}\sum_{k=1}^{\infty}\frac{B_{2k}}{k(2k-1)}\\&\times\left[2^{2k-1}\left(\frac{F}{\bar{\varphi}\varphi}\right)^{2k-1}+(2^{2k-1}-1)\left(\frac{F}{\bar{\varphi}\varphi-D}\right)^{2k-1}\right]\\&-\frac{D}{\bar{\varphi}\varphi}\sum_{k=2}^{\infty}\frac{1}{k(k-1)}\left(\frac{D}{\bar{\varphi}\varphi}\right)^{k-1}\,,\end{aligned} \tag{B.11}$$

B_{2k} are the Bernoulli numbers. We list a few of them:

$$B_0=1\,,\qquad B_2=\tfrac{1}{6}\,,\qquad B_4=-\tfrac{1}{30}\,,\qquad B_6=\tfrac{1}{42}\,. \tag{B.12}$$

The first three terms of the expansion (B.11) are given by:

$$\mathcal{F}\left(\frac{D+F}{\bar{\varphi}\varphi},\frac{D-F}{\bar{\varphi}\varphi}\right)=-\frac{1}{2}\frac{D^2-F^2}{(\bar{\varphi}\varphi)^2}-\frac{1}{6}\frac{D}{\bar{\varphi}\varphi}\left(\frac{D^2-F^2}{(\varphi\bar{\varphi})^2}\right)-\frac{1}{12}\left(\frac{D^2-F^2}{(\varphi\bar{\varphi})^2}\right)^2+\cdots\,. \tag{B.13}$$

Appendix C

In this appendix we give details of the behaviour of the various superfields under the transformations of the superconformal algebra. This algebra consists of conformal, chiral and superconformal transformations, which we denote by δ_A, δ_a and δ_α respectively. We also use $A, B, \ldots, a, b, \ldots$ and $\alpha, \beta, \ldots,$ to denote the infinitesimal parameters of these transformations. $\alpha, \beta, \ldots,$ are of anticommuting type, the others are of commuting type. In two dimensions, the superconformal algebra reduces to the sum of two algebras. One acts on $x_+\bar{\theta}_R$ and θ_R and its parameters are arbitrary functions of x_+. The other acts on $x_-\bar{\theta}_L$ and θ_L and its parameters are arbitrary functions of x_-. In the following, we concentrate on the first subalgebra.

The commutation relations of this algebra are

$$[\delta_A, \delta_B] = \delta_C, \qquad C = B\dot{A} - A\dot{B},$$

$$[\delta_A, \delta_a] = \delta_b, \qquad b = -A\dot{a},$$

$$[\delta_A, \delta_\alpha] = \delta_\beta, \qquad \begin{cases} \beta = -\dot{\alpha}A + \frac{1}{2}\alpha\dot{A}, \\ \bar{\beta} = -\dot{\bar{\alpha}}A + \frac{1}{2}\bar{\alpha}\dot{A}, \end{cases}$$

$$[\delta_a, \delta_b] = 0,$$

$$[\delta_a, \delta_\alpha] = \delta_\beta, \qquad \begin{cases} \beta = i\alpha a, \\ \bar{\beta} = -i\bar{\alpha}a, \end{cases}$$

$$[\delta_\alpha, \delta_\beta] = \delta_A + \delta_a, \qquad \begin{cases} A = -\bar{\alpha}\beta + \bar{\beta}\alpha, \\ a = \frac{1}{2}i\left(-\dot{\bar{\alpha}}\beta + \bar{\alpha}\dot{\beta} + \dot{\bar{\beta}}\alpha - \bar{\beta}\dot{\alpha}\right). \end{cases} \tag{C.1}$$

Here and in the following, we omit indices +, R and $\overline{\mathrm{R}}$ when this does not cause any ambiguity. Since the δ_A and δ_a transformations can be obtained by commuting two δ_α type transformations, we focus our attention on the latter transformations. From (2.17) V transforms under an infinitesimal, superconformal transformation as

$$\delta_\alpha V = \Gamma_\alpha V, \tag{C.2}$$

where Γ_α is the linear differential operator

$$\begin{aligned}\Gamma_\alpha &= \tfrac{1}{2}\partial_+\left(\bar{\theta}_R\alpha_R + \theta_R\bar{\alpha}_R\right) - \partial_{\bar{R}}\left(\bar{\alpha}_R + \tfrac{1}{2}\dot{\bar{\alpha}}_R\bar{\theta}_R\theta_R\right) - \partial_R\left(\alpha_R - \tfrac{1}{2}\dot{\alpha}_R\bar{\theta}_R\theta_R\right) \\ &= \tfrac{1}{2}\left(\bar{\theta}_R\alpha_R + \theta_R\bar{\alpha}_R\right)\partial_+ + \left(\bar{\alpha}_R + \tfrac{1}{2}\dot{\bar{\alpha}}_R\bar{\theta}_R\theta_R\right)\partial_{\bar{R}} + \left(\alpha_R - \tfrac{1}{2}\dot{\alpha}_R\bar{\theta}_R\theta_R\right)\partial_R .\end{aligned} \tag{C.3}$$

To compute the variations of the covariant derivatives of V we require the commutators

$$[D_{\bar{R}}, \Gamma_\alpha] = \theta_R \dot{\bar{\alpha}}_R D_{\bar{R}},$$
$$[D_R, \Gamma_\alpha] = \bar{\theta}_R \dot{\alpha}_R D_R,$$
$$[D_L, \Gamma_\alpha] = [D_{\bar{L}}, \Gamma_\alpha] = 0. \tag{C.4}$$

Using (C.2)–(C.4) and the definitions of S and $\bar{S}$ in (4.1), we obtain

$$\delta_\alpha S = (\Gamma_\alpha - \dot{\bar{\alpha}}_R \theta_R) S,$$
$$\delta_\alpha \bar{S} = (\Gamma_\alpha - \dot{\alpha}_R \bar{\theta}_R) \bar{S}. \tag{C.5}$$

From these equations we see that transformations of S and $\bar{S}$ involve rotations in addition to the always present transport term. Now, Δ and $\bar{\Delta}$, defined in (4.6) and (4.8), transform as follows:

$$\delta_\alpha \Delta = (\Gamma_\alpha - \dot{\bar{\alpha}}_R \theta_R - \dot{\alpha}_R \bar{\theta}_R) \Delta,$$
$$\delta_\alpha \bar{\Delta} = (\Gamma_\alpha - \dot{\alpha}_R \bar{\theta}_R - \dot{\bar{\alpha}}_R \theta_R) \bar{\Delta}. \tag{C.6}$$

Consider now the variables $\Delta/S\bar{S}$ and $\bar{\Delta}/S\bar{S}$. Since the extra rotation phase in the transformation law for a product is the sum of the phases of the respective factors, it is trivial to see that any function of $\Delta/S\bar{S}$ and $\bar{\Delta}/S\bar{S}$ transforms as a total derivative. This is true also for conformal and chiral transformations since these can be obtained by commuting two superconformal transformations. Thus, $\Delta/S\bar{S}$ and $\bar{\Delta}/S\bar{S}$ are suitable variables to use in building a superconformal, superspace lagrangian.

References

[1] J. Ellis, Proc 21st Scottish Universities Summer School in Physics (1980);
P. Langacker, Phys. Reports 72 (1981) 185
[2] G. 't Hooft, Proc. of Cargèse Summer Institute (1979)
[3] E. Gildener, Phys. Rev. D14 (1976) 1667
[4] L. Maiani, Proc. of the Ecole d'Eté de physique des particules, Gif-sur-Yvette (1979) p. 3;
E. Witten, Nucl. Phys. B186 (1981) 513;
S. Dimopoulos and H. Georgi, Nucl. Phys. B193 (1981) 150;
N. Sakai, Z. Phys. C11 (1982) 153
[5] G. Veneziano and S. Yankielowicz, Phys. Lett. 113B (1982) 231
[6] J. Schwinger, Phys. Rev. 82 (1951) 664;
J. Schwinger, Particles, sources and fields, vol. 2, Ch. 4.8
[7] F. Riva, Nuovo Cim. 61A (1981) 69
[8] A. D'Adda, P. Di Vecchia and M. Lüscher, Nucl. Phys. B152 (1979) 125
[9] E. Witten, Nucl. Phys. B144 (1979) 285
[10] A. Davis, P. Salomonson and J.W. Van Holten, Nucl. Phys. B208 (1982) 484