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Electric and Magnetic Form Factors of the Nucleon*

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INTRODUCTION

While preparing for experiments with the Cambridge electron accelerator, we reviewed the field of nucleon form factors as derived from electron-scattering experiments. We haev concluded that a more suitable basis for discussion of the experiments is formed by the charge and magnetic form factors $G_E(q^2)$ and $G_M(q^2)$ to be discussed below, rather than the more conventional Dirac and Pauli form factors $F_1(q^2)$ and $F_2(q^2)$. The simplification afforded by use of the former in the quantitative treatment of the errors and error correlations has had several interesting consequences with regard both to critical evaluation of the existing experimental data and to the interpretation of nucleon structure in terms of the vector mesons, ρ and ω_0.

The possible use of G_E and G_M as alternatives to F_1 and F_2 was first mentioned by Yennie *et al.*,[1] who also pointed out that these linear combinations of F_1 and F_2 correspond to zero and one unit of angular momentum transferred along the direction of the virtual photon exchanged in the scattering process. Sachs and his collaborators[2] conjectured that $G_E(q^2)$ and $G_M(q^2)$ might prove more physically meaningful than F_1 and F_2 after noticing that the rms radii of the charge and magnetic moment spatial distributions were given by $[6dG_E/dq^2]^{\frac{1}{2}}$ and $[6dG_M/dq^2,]^{\frac{1}{2}}$ respectively, at $q^2 = 0$. The descriptive terminology "charge" and "magnetic" is derived from this fact. We wish to emphasize our belief, however, that the usefulness of the combinations is fully apparent in momentum space and is independent of possible ambiguities arising in interpretation of the spatial Fourier transform.[1] Hand[3] has pointed out the noninterference of G_E and G_M in the Rosenbluth formula for elastic scattering and the consequent advantages of this type of analysis over the method of intersecting ellipses previously used. He also observed the physical interpretation in terms of transverse and longitudinal photons and the complete generality of this type of angular distribution in the case of single photon exchange for arbitrary inelastic reactions as well. The form of the most general possible angular distribution was known previously and discussed by several authors, but we know of no attempt at either applications or interpretation. A preliminary discussion of the implications of this type of analysis was made by Hand, Miller, and Wilson[4] in a brief communication. This paper extends these arguments and summarizes existing data analyzed in this way.

The theoretical discussion of electron–nucleon interactions can be traced to early discussions of the neutron–electron interaction by Foldy[5,6] and by Zemach[7] (see also Yennie[1] and Drell[8]). The particular formula used for electron–proton scattering is due to Rosenbluth.[9] The elements of the calculation are made clear by the Feynmann diagram of Fig. 1. It

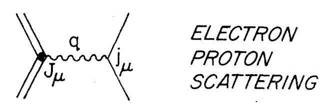

Fig. 1. Feymann diagram for electron–proton scattering with one photon exchanged.

is assumed that only one photon is exchanged between the electron and photon. The matrix element for scattering then contains the term

$$-4\pi ie\langle -\bar{u}_{p'}|J_\mu|u_p\rangle(1/q^2)\langle\bar{u}_{e'}|j_\mu|u_e\rangle\,, \tag{1}$$

where J_μ, j_μ are operators describing the proton and electron current, respectively, and q^2 is the square of the invariant 4-momentum transfer given in terms of the initial and final electron or proton momenta by

$$q_\mu = P'_\mu - P_\mu = k_\mu - k'_\mu\,. \tag{2}$$

If we assume no structure of the electron we replace j_μ by

$$j_\mu = -ie\gamma_\mu\,. \tag{3}$$

We always need to calculate the expectation value of j_μ, $\langle\bar{u}_{e'}|j_\mu|u_e\rangle$ between the Dirac spinors $\bar{u},u$. Since we anticipate structure for the nucleon, we

* Supported by the U. S. Atomic Energy Commission.

[1] D. R. Yennie, M. M. Levy, and D. G. Ravenhall, Rev. Mod. Phys. **29**, 144 (1957).

[2] F. J. Ernst, R. G. Sachs, and K. C. Wali, Phys. Rev. **119**, 1105 (1960); R. G. Sachs, *ibid.* **126**, 2256 (1962).

[3] L. N. Hand, thesis, Stanford University, 1961 (unpublished); Phys. Rev. (to be published).

[4] L. N. Hand, D. G. Miller, and R. Wilson, Phys. Rev. Letters **8**, 110 (1962).

[5] L. L. Foldy, Phys. Rev. **87**, 688 (1952).

[6] L. L. Foldy, Rev. Mod. Phys. **30**, 471 (1958).

[7] A. C. Zemach, Phys. Rev. **104**, 1771 (1957).

[8] S. D. Drell and F. Zachariasen, *Electromagnetic Structure of the Nucleons* (Oxford University Press, New York, 1961).

[9] M. Rosenbluth, Phys. Rev. **79**, 615 (1950).

must replace J_μ by a more general form than that for j_μ. Foldy has shown that the most general form for relativistic covariance, current conservation, and a spin 1/2 particle is

$$J_\mu = e[\gamma_\mu F_1(q^2) + i\kappa(\gamma_\mu\gamma_\nu - \gamma_\mu\gamma_\nu)q_\nu F_2(q^2)] , \quad (4)$$

where κ is the anomalous magnetic moment in nuclear magnetons. $F_1(q^2)$ and $F_2(q^2)$ are arbitrary functions of q^2 and are here normalized to unity at $q^2 = 0$.

We note that a difference of $F_1(q^2)$ and $F_2(q^2)$ would imply a different spatial extent of the Dirac particle and the anomalous moment subsequently introduced by the Pauli term. The form factors F_1 and F_2 are conveniently called the Dirac and Pauli form factors. (Sometimes they are carelessly called the electric and magnetic form factors. Such usage leads to confusion with other form factors to be discussed.)

Although the form (4) for J_μ is the most general possible form, it is not necessarily the only simple one. Others may be formed involving linear combinations of F_1 and F_2. We may show, for example, by direct use of the Dirac equation, that the following formula for J_μ gives the same expectation value $\langle \bar{u}_{fp} | J_\mu | u_{ip} \rangle$

$$J_\mu = \left(\frac{e}{1 + q^2/4M^2}\right) [P_\mu G_E(q^2) + ir_\mu G_M(q^2)] \quad (5)$$

where P_μ is the mean of the initial and final proton momenta and

$$r_\mu = -i/2[\gamma_\mu(\mathbf{p}\cdot\mathbf{q}) - (\mathbf{q}\cdot\mathbf{p})\gamma_\mu] .$$

where $\quad (\mathbf{p}\cdot\mathbf{q}) = (\mathbf{p}_\mu\gamma_\nu\mathbf{q}_\mu\gamma_\nu) . \quad (6)$

Then

$$G_E(q^2) = F_1(q^2) - (q^2/4M^2)\kappa F_2(q^2) \quad (7)$$

$$G_M(q^2) = F_1(q^2) + \kappa F_2(q^2) . \quad (8)$$

[We here adopt the metric $q^2 = q\cdot q - q_0^2$ so that q^2 is positive for electron scattering.] We use G instead of F because G_M does not tend to unity as q^2t ends to zero. Others have used $F_{\text{mag}} = G_M/\mu$; $F_{ch} = G_E$. The current operator was first written explicitly in the form (5) by Barnes[10] who independently arrived at the most important conclusion of this paper.

The matrix element (5) of the current operator takes on a particularly simple form in the Breit frame [defined by $(q_0)_B = 0$] reached by performing a Lorentz transformation along $\mathbf{q}$. In this frame the statement of charge conservation, as expressed in momentum space, becomes: $\mathbf{J}_B\cdot\mathbf{q}_B = 0$. In other words, the current operator must consist of a scalar and a transverse vector:

$$\mathbf{J}_B^0 = eG_E(q^2) \quad (9)$$

$$\mathbf{J}_B = ieG_M(q^2)(\boldsymbol{\sigma} \times \mathbf{q})/M . \quad (10)$$

This form is identical to that obtained in nonrelativistic theory from a Coulomb field and an interaction $\mu\, \boldsymbol{\sigma}\cdot\mathbf{B}$ with $\mu(q^2) = (e/2M)\, G_M(q^2)$. The Breit frame plays a similar role for spacelike vectors to that played by the center-of-mass frame for timelike vectors in that a considerable algebraic simplification results if cross sections are expressed in this frame. The factor $[1 + (q_0^2)_{\text{lab}}/q^2]^{-1}$, which appears in the Rosenbluth formula, and its generalizations arises from the relation:

$$(\cot^2 \theta/2)_B = (\cot^2 \theta/2)_{\text{lab}}/[1 + (q_0^2)_{\text{lab}}/q^2] . \quad (11)$$

Interaction with the convection current leaves the spin unchanged while interaction with the spin current flips the nucleon spin with respect to $\mathbf{q}$.

CROSS-SECTION CALCULATIONS

It transpires that if polarizations are not measured, the terms corresponding to G_E and G_M contribute separately to the cross section.

If we complete the cross-section calculation, we arrive at a formula for the cross section for the scattering of an electron by the proton:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha r_e m}{2E \sin(\theta/2)}\right)^2 \frac{E'}{E} \{\cot^2 \theta/2$$
$$\times [F_1^2 + (q^2/4M^2)\kappa F_2^2] + 2(q^2/4M^2)(F_1 + \kappa F_2)^2\}$$

or (12)

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha r_e m}{2E \sin(\theta/2)}\right)^2 \frac{E'}{E} \left\{\frac{\cot^2 \theta/2}{1 + q^2/4M^2} \times [G_E^2 + (q^2/4M^2)G_M^2] + 2(q^2/4M^2)G_M^2\right\} \quad (13)$$

where α, r, m are the fine-structure constant, the electron Compton wavelength (3.86×10^{-11}) cm, and the electron mass. E and E' are the incident and scattered electron energies, and θ is the scattering angle.

Formulas (12) and (13) hold in the first Born approximation and in the extreme relativistic limit for the electron. Errors of the order of α and m^2/q^2 can appear.

Equations (12) and (13) show that in order to separate the form factors F_1 and F_2 or G_E and G_M (which we remember are functions only of the square of the 4-momentum transfer q^2), we must measure the cross section at two values of $\cot \theta/2$ for the same

[10] K. J. Barnes, Phys. Letters 1, 166 (1962).

q^2. Equation (12) shows that to separate F_1 and F_2 two simultaneous quadratic equations must be solved. The standard method of solving a quadratic equation is to "complete the square." When we do this, we are led automatically to Eq. (13) as an intermediate step.

To separate G_M and G_E we may plot

$$\left(\frac{2E\sin\theta/2}{\alpha r_e m}\right)^2\frac{E}{E'}\frac{d\sigma}{d\Omega}$$

against $\cot^2\theta/2$. The intercept at $\cot\theta/2=0$ ($\theta=180°$) gives $G_M{}^2$, whence the slope gives $G_E{}^2$. The ambiguity of sign of G_M and G_E is a demonstration of the inability of scattering experiments in first Born approximation to measure the sign of either the charge or the magnetic moment. We know from other measurements $G_M(0)=\mu$ (the total magnetic moment) and $G_E(0)=1$ for the proton and 0 for the neutron. The usual requirement that $G(q^2)$ be a smooth function gives the sign of all but G_{En}. Once the sign is determined, F_1 and F_2 may be found from Eqs. (7) and (8). It is easy to see that the errors on F_1 and F_2 will usually be more correlated than those of G_E and G_M and larger. An expression of the cross section in terms of G_E and G_M is thus to be preferred.

It is, of course, possible to consider F_1 and F_2 and to use the full error matrix in all calculations. We find it hard to think about simple errors and to think about the error matrix is harder. At large momentum transfers, for example, we will show that $G_M(q^2)$ is well known and $G_E(q^2)$ hardly at all. F_1 and F_2 are, therefore, not known separately and the error correlation must always be considered.

A graphical method has sometimes been used in the past to solve the simultaneous quadratic equations (the method of intersecting ellipses). The procedure is, however, the geometrical analog of the algebraic procedure just described. With such a procedure it has not proved easy to obtain useful error estimates. The algebraic procedure is more useful. We note that much of the previous data is analyzed by a plot against $\tan^2\theta/2$ instead of $\cot^2\theta/2$. The latter procedure is preferable because it enables points for 180° electron–proton scattering (proton angle equal to 0°) to be included in the plot.

EXTENSION TO INELASTIC PROCESSES

For inelastic processes the current operator may still be separated in a way analogous to (5) for inelastic electron-scattering processes. This fact has been slowly realized over a period of years. We write the general formula for scattering of electrons by nucleons or by a nucleus with one photon exchange term only:

$$\frac{d\sigma}{d\Omega}=\Gamma_l(q^2,q_0)\frac{\cot^2\theta/2}{1+q_0^2/q^2}+\Gamma_t(q^2,q_0)\left[2+\frac{\cot^2\theta/2}{1+q_0^2/q^2}\right], \tag{14}$$

where $q^2=\mathbf{q}\cdot\mathbf{q}-q_0^2$ is the invariant square of the 4-momentum transfer, q_0 its 4th component, and Γ_t and Γ_l are functions to be determined.

Equation (13) appears as a special case for elastic scattering when $q_0=q^2/2M$.

Bjorken[11] first showed that the cross section could be expressed in the form $A+B\cot^2\theta/2$ using Eq. (14). However, we wish to go further. The separation of the constants Γ_l and Γ_t in Eq. (14) appears more complex than that of Bjorken, but it has a physical meaning, of exchange of longitudinal and transverse photons, respectively. Thus, $\Gamma_t(q^2)$ is just the total photon absorption cross section. We remember here, that the photon is transversely polarized and a longitudinal field cannot be produced by real photons. This distinction has been made before[12] for pion production. Its generality was observed by Hand[3] and by Gourdin.[13]

We may, for example, apply Eq. (14) to nuclear physics. The term $\Gamma_t(q^2q_0)$ tends to the photon cross section as q^2 tends to zero. The term $\Gamma_l(q^2,q_0)$ can excite $0\to 0$ transitions inaccessible to photons. In this connection we emphasize that the separation of nuclear transitions into electric and magnetic multipoles is not the same as the separation we have here into G_E and G_M (or Γ_l and Γ_t), only electric multipoles enter into Γ_l but both electric and magnetic multipoles enter into Γ_t.

PROTON–ANTIPROTON ANNIHILATION

The cross section for proton–antiproton annihilation leading to an electron–proton pair also involves the current operator; in this case, we need the expectation value $\langle\bar{u}_{fp}\,u_{ip}|J_\mu|0\rangle$. Again we may define two form factors in the region of negative q^2 by equations analogous to (4) or (5). The cross section becomes

$$\begin{aligned}d\sigma/d\Omega&=\{\pi\alpha^2/2q[-(q^2+4M^2)]^{1/2}\}\\&\quad\times[(4M^2/q^2)G_E^2\sin^2\theta+G_M^2(1+\cos^2\theta)]\\&=(\pi/2)\{\alpha^2/q[-(q^2+4M^2)]^{1/2}\}\\&\quad\times\{[F_1+(q^2/4M^2)\kappa F_2]^2(4M^2/q^2)\sin^2\theta\\&\quad+(F_1+\kappa F_2)^2(1+\cos^2\theta)\}\,.\end{aligned} \tag{15}$$

[11] J. D. Bjorken (private communication).
[12] R. H. Dalitz and D. R. Yennie, Phys. Rev. **105**, 1598 (1957).
[13] M. Gourdin, Nuovo Cimento **21**, 1094 (1961).

Here q^2 is negative and therefore represents the energy transferred by the photon. It must clearly be numerically greater than $4M^2$. Also, from their definition the form factors are, in general, complex. It is obviously simpler to use G_M and G_E than F_1 and F_2 in this context also. They are, so far, from the positive q^2 region and separately defined, but it will later be assumed that there is an analytic continuation between the two regions.

For example, Frazer and Fulco[14] have discussed the reaction $N+\overline{N}\rightarrow\pi+\pi$. This can be regarded as an intermediate state in the reaction $N+\overline{N}\rightarrow\pi+\pi\rightarrow e+\bar{e}$, as shown in the diagram of Fig. 2.

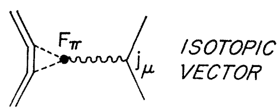

FIG. 2. Feynmann diagram for isotopic vector electron–nucleon scattering with two-pion intermediate state.

Frazer and Fulco show that the nucleon form factors can be expressed in terms of the amplitude for the $N+\overline{N}\rightarrow\pi+\pi$ process multiplied by the pion form factor. We can then (formally) continue this relation for the nucleon form factors to the unphysical region to $q^2<-4m_\pi^2$ in terms of this process. We further assume that an analytic continuation is possible through the rest of the nonphysical region.

We note, here, that while for electron scattering ($q^2>0$) it can be shown that G_E and G_M are wholly real, they can be complex for the annihilation reaction $q^2<-4M^2$ and in the nonphysical region up to $q^2<-4m_\pi^2$.

The close relation of the form factors to the reaction $\pi+\pi\rightarrow N+\overline{N}$ is formally written by expressing them in terms of the pion form factor and the amplitude T for the $\pi+\pi\rightarrow N+\overline{N}$ reaction as in the expression

$$\langle e^+e^-|J_\mu|N\overline{N}\rangle = \sum\langle e^+e^-|J_\mu|\pi\pi\rangle\langle\pi\pi TN\overline{N}\rangle\,. \quad (16)$$

The pion, being spinless, has only one form factor. The difference between the two nucleon form factors is thus seen to be a consequence of the differing states in the second term of (16).

We note here that two pions and even pion numbers generally can only couple to a photon in an isovector ($T=1$) state.[8] An odd number of pions couple in an isoscalar ($T=0$) state. A single pion cannot couple. Thus the contributions of the $\pi+\pi\rightarrow N+\overline{N}$ reaction to the proton and neutron form factors are related and it is convenient to express this by forming the isotopic vector and scalar combinations

$$G_v = \tfrac{1}{2}\,(G_p - G_N)$$
$$G_s = \tfrac{1}{2}\,(G_p + G_N)\,. \quad (17)$$

G_v and G_s can couple solely through states of an even and odd number of pions, respectively.

Helicity Representation

The helicity of a particle is defined as the angular momentum of a particle about its direction of motion. For a spinless particle the helicity is always zero. For a spin 1/2 electron or nucleon the helicity is $\pm\frac{1}{2}$; for spin 1 photons it is 0 for longitudinal photons or ± 1 for transverse photons. Real photons of mass zero can only have helicity ± 1.

The helicity concept has been most useful for physical problems where one particle is extremely relativistic. Calculations of neutrino reactions are simplified by the definite neutrino helicity. The spin of a relativistic electron follows the direction of motion as the electron is scattered.

Jacob and Wick[15] discussed the helicity representation for two-body reactions. Durand *et al.*[16] even calculated J_μ in this representation. Of course, they encountered the abovementioned separation into convection current and spin current for the virtual photon is either longitudinal (corresponding to G_E) or transverse (corresponding to G_M).

FORM FACTORS IN THE NONPHYSICAL REGION

Equation (13) for electron–proton scattering defines the form factors for the region $q^2>0$ and Eq. (12) for proton–antiproton annihilation defines them for the region $q^2<-4M^2$. There is no physical process which defines them in the intervening region. We can give a meaning to a definition only in terms of an analytic continuation.

For the production of two nucleons in the state of angular momentum J, Frazer and Fulco[14] used helicity amplitudes. f_+^J refers to both nucleons having the same helicity and f_-^J denotes opposite nuclear helicities. Of course, $J=1$ for the form-factor problem so a longitudinal photon corresponds to f_+ and a transverse photon to f_-. Miller[17] expressed the form factors for electron–nucleon scattering directly in terms of these helicity amplitudes for the nucleons. In particular, the connection can be seen [from

[14] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1603, 1609, (1960).

[15] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).

[16] L. Durand, P. C DeCalles, and R. B. Marr, Phys. Rev. **126**, 1882 (1962).

[17] D. G. Miller, Phys. Rev. **127**, 1365 (1962).

Eqs. (2.3a) and (2.3b) of reference 14 and Eqs. (7) and (8) above] to be

$$\mathrm{Im}\,\{G_{MV}\} = (e/2)F_\pi^*(f_-/\sqrt{2})(k^3/E) \quad (18)$$

$$\mathrm{Im}\,\{G_{EV}\} = (e/2)F_\pi^*(f_+/M)(k^3/E)\,, \quad (19)$$

where k is the barycentric pion momentum.

Bowcock *et al.*[18] and Hamilton *et al.*[19] related pion–nucleon scattering to isovector electron–nucleon scattering through the amplitudes for $\pi + \pi \to N + \bar{N}$. Unfortunately, they followed Frazer and Fulco in using F_1 and F_2 to compare with experiment.

From our point of view, pion–nucleon scattering in which there is nucleon spin flip, is very closely related to magnetic isovector electron–nucleon scattering. Non-spin-flip pion–nucleon scattering is akin to electric isovector electron–nucleon scattering. For example, the reaction $\pi^- + p \to \pi^\circ + n$ (which must be $T = 1$ for the two pions) in the state $L = 0$ (no spin flip) can be related directly to the form factor G_{EV}. The details of this relationship are discussed elsewhere.[17]

Now, another reason for our preference for electric and magnetic form factors becomes clear. The decomposition of J_μ into convection current and spin current focusses attention on the relative nucleon spin. This decomposition can assist detailed comparisons between electron–nucleon and pion–nucleon scattering—presently our two most precise ways to study the pion–pion interaction.

Relation of Positive and Negative q^2—Dispersion Relations

The processes of electron–nucleon scattering and of proton–antiproton annihilation may be related in the first Born approximation or, more generally, by dispersion relations. We must plausibly assume that there are no discontinuities or poles in the nonphysical region, except for a branch cut on the real axis from $-4m_\pi^2$ to $-\infty$.

In reference 14, for example, it is shown that with certain plausible assumptions, any of the form factors, F_1, F_2, G_E, G_M obey dispersion relations. The question arises, does $F(z) \to 0$ as $z \to \infty$, in which case $F(z)$ obeys an unsubtracted dispersion relation giving for spacelike q^2

$$F(q^2) = \frac{1}{\pi}\int \frac{\mathrm{Im}\,F(q'^2)}{q'^2 - q^2 - i\epsilon}\,dq'^2\,, \quad (20)$$

or do we have to make the weaker assumption that $F(z)/z \to 0$ as $z \to \infty$, giving a subtracted dispersion relation

$$F(q^2) = F(0) + \frac{q^2}{\pi}\int \frac{\mathrm{Im}\,F(q'^2)}{q'^2(q'^2 - q^2 - i\epsilon)}\,dq'^2\,. \quad (21)$$

Perhaps even a second or a third subtraction is necessary. Experiment must decide and theory can only give a guide. As we shall see the different form factors may need different subtractions.

The dispersion relations (20) or (21) become simpler if we assume that the integral is dominated by the pion–pion resonances recently discovered. It is, therefore, now customary to replace the integral by a sum over resonances. Equations (18) and (19) then become, respectively,

$$F(q^2) = \frac{1}{\pi}\sum_R \frac{\mathrm{Im}\,F(q_R)^2}{q_R^2 - q^2} \quad (22)$$

$$F(q^2) = F(0) + \frac{1}{\pi}\sum_R \frac{q^2}{q_R^2}\left[\frac{\mathrm{Im}\,F(q_R)^2}{q_R^2 - q^2}\right]. \quad (23)$$

We will follow this custom but will not discuss its validity.

Phenomenological Fits to Form Factors

The extensive use of the helicity amplitudes f_+ and f_- suggest that we should express G_E and G_M in the form (20) or (21). If we wish to discuss the nucleon form factors given by low-energy electron scattering in terms of nearby resonances, we have no way of distinguishing between a resonance at $q_R^2 \gg |q^2|$ and a necessity for subtraction as given by $F(0)$.

Thus, we assume that we can express G_E and G_M in the form

$$G_E = 1 - \sum_R \alpha_R + \sum_R \frac{\alpha_R}{1 + q^2/-q_R^2} \quad (24)$$

$$G_M = \mu\left(1 - \sum_R \beta_R + \sum_R \frac{\beta_R}{1 + q^2/-q_R^2}\right). \quad (25)$$

What does this assumption imply for F_1 and F_2? We note that at large q^2, $G_E(q^2) \to 1 - \Sigma\alpha$. From Eq. (6) we must have $F_2(q^2) \to 0$. We thus reach the conclusion that F_2 must either have no components from resonances at large q^2 or, if we take the point of view that the constant in G_E corresponds to a real limit as $q^2 \to \infty$ and therefore G_E obeys a subtracted dispersion relation, F_2 must obey an unsubtracted relation. In general, F_2 must have one less subtraction than G_E. This conclusion agrees with that from perturbation theory.

For example, if we allow F_2 to obey a subtracted relation, or equivalently, to have a constant core,

[18] J. Bowcock, N. Cottingham, and D. Lurie, Nuovo Cimento **16**, 918 (1960) **19**, 142 (1961).
[19] J. Hamilton, T. D. Spearman, and W. S. Woolcock, Ann. Phys. (N. Y.) **17**, 1 (1962).

and one resonance

$$F_1 = 1 - \gamma + \frac{\gamma}{1 + q^2/(-q_R^2)}$$

$$F_2 = 1 - \delta + \frac{\delta}{1 + q^2/(-q_R^2)}, \qquad (26)$$

then

$$G_E = 1 - \gamma + \frac{\kappa q^2}{4M^2}(1 - \delta) + \frac{\gamma + \kappa q^2/4M^2}{1 + q^2/(-q_R^2)}. \qquad (27)$$

This can reduce to the simple form (22) if and only if $\delta = 1$.

We note that, in general, it is not possible to express G_E and G_M in the form of resonances plus a hard core and simultaneously to express F_1 and F_2 in this form. This is possible only if there is no core for F_2.

We should not be surprised at this conclusion. The form (24) and (25) is an approximation and only experiment can tell us the behavior of G and F as $q^2 \to \infty$. α and β in Eqs. (24) and (25) are not expected to be constants and will have variations of the order of $q^2/4M^2$. Sachs[2] has speculated that there is a hard core of a Dirac particle for which $F_2(\infty) = 0$ and $G_M(\infty) = G_E(\infty)$ and $F_1(\infty)$. Our fits which will be derived later show that at low q^2 $G_{Mp}(q^2) = \mu_p(G_{Ep}(q^2)$, which is far removed from Sachs' postulate. Above $q^2 = 10$ F^{-2} errors increase. The most recent Cornell data show that Sachs' relation holds at $q^2 = 50$ F^{-2}.

Numbers for the Proton Form Factors

We will here include the data of 8 experiments. That of Bumiller *et al.* at Stanford,[20] of Littauer *et al.* at Cornell,[21] of Janssens *et al.* at Stanford,[22] of Lehmann *et al.* at Paris,[23,24] of Yount and Pine at Stanford,[25] Berkelman *et al.* at Cornell,[26] Drickey and Hand at Stanford,[27] and Gram at Stanford.[28] For the first two (and most extensive) sets of data, Bumiller and Littauer, the data were not taken at the same momentum transfer at different angles. This, combined with the problem of solving the coupled quadratic equations, has complicated attempts at a proper error analysis in terms of F_1 and F_2. We here use the same data and analyze in terms of G_E and G_M.

The first step in the analysis is to remove the major part of the angular dependence of Eq. (3) by defining and evaluating

$$R(\theta,q^2) = \cot^2(\theta/2)[G_E^2 + (q^2/4M^2)G_M^2] + (q^2/4M^2)(1 + q^2/4M^2)G_M^2. \qquad (28)$$

We plot $R(\theta,q^2)$ as a function of q^2 at constant θ. A smooth curve is drawn through the points. At this stage our procedure differs from the "smoothing" procedure of Bumiller. We do not read points from the smooth curve. Instead, we draw a line parallel to the smooth curve through each experimental point $R(\theta,q^2)$ and construct a new "interpolated" point at even values of $q^2[q^2 = 4$F^{-2}, 6F^{-2}, 8F^{-2}, etc.]. We assign to this interpolated point the error of the original point. Thus, if the original point lies above the smooth curve, so does the constructed point. The procedure is quite insensitive to the detailed method of drawing the curve, as shown in Fig. 3.

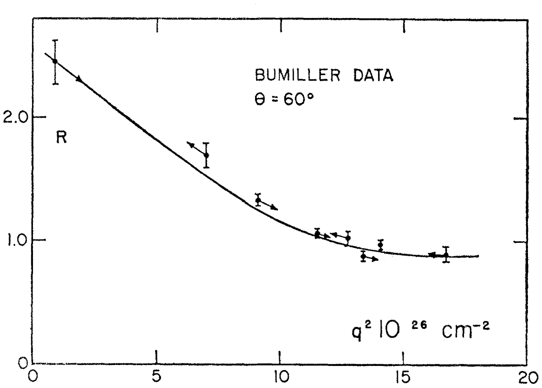

FIG. 3. The function $R(\theta,q^2)$ for $\theta = 60°$ showing the procedure for interpolation to even values of q^2.

We then evaluated $G_E(q^2)$ and $G_M(q^2)$ at each value of q^2 by a least-squares fit to Eq. (28). Each measure of R (R_i) at the same momentum transfer is assigned a weight equal to the inverse square of the quoted statistical error $(1/\sigma_i)^2$. Then we obtain the best values of the quantities X_1 and X_2, given by

$$X_1 = (q^2/4M^2)(1 + q^2/4M^2)G_M^2 \qquad (29)$$

$$X_2 = G_E^2 + (q^2/4M^2)G_M^2 \qquad (30)$$

by solving the simultaneous equations

$$k_1 = \sum_i R_i/\sigma_i^2 = X_1 b_{11} + X_2 b_{12} \qquad (31)$$

[20] F. Bumiller, M. Croissiaux, E. Dally, and R. Hofstadter, Phys. Rev. **124**, 1623 (1961).
[21] R. M. Littauer, H. F. Schopper, and R. R. Wilson, Phys. Rev. Letters **7**, 141 (1961).
[22] T. Janssens, R. Hofstadter, E. B. Hughes, and M. Yearian, Bull. Am. Phys. Soc. **7**, 488 (Q2) (1962).
[23] P. Lehmann, R. Taylor, and Richard Wilson, Phys. Rev. **126**, 1183 (1962).
[24] P. Lehmann, R. Dudelzak, *Proceedings of the International Conference on High-Energy Physics CERN, 1962*).
[25] D. Yount and J. Pine, Phys. Rev. **128**, 1842 (1962).
[26] K. Berkelman, R. M. Littauer and G. Rouse, *Proceedings of the 1962 International Conference on High-Energy Physics, CERN* (Centre d'Études Recherches Nucléaires, Geneva, Switzerland, 1962); Phys. Rev. (to be published).
[27] D. J. Drickey and L. N. Hand, Phys. Rev. (to be published).
[28] P. A. M. Gram and E. B. Dally, Bull. Am. Phys. Soc. **7**, 489 (1962).

$$k_2 = \sum_i (R_i \cot^2 \theta_i/2)/\sigma_i^2 = X_1 b_{21} + X_2 b_{22}\,, \quad (32)$$

where

$$b_{11} = \sum_i 1/\sigma_i^2$$
$$b_{12} = b_{21} = \sum_i (\cot^2 \theta/2)/\sigma_i^2$$
$$b_{22} = \sum_i (\cot^4 \theta/2)/\sigma_i^2\,.$$

The error matrix is defined by the inverse matrix

$$\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}^{-1}. \quad (33)$$

Then,

$$X_1 = d_{11} \sum_i R_i/\sigma_i^2 + d_{12} \sum_i R_i \cot^2 (\theta/2)/\sigma_i^2 \quad (34)$$

$$X_2 = d_{21} \sum_i R_i/\sigma_i^2 + d_{22} \sum_i R_i \cot^2 (\theta/2)/\sigma_i^2 \quad (35)$$

and G_E and G_M are deduced. The error on any combination of G_E and G_M, and hence, on the adjusted value of the cross section at any point, is determined by the error matrix. Thus, the error on X_1 is $d_{11}^{1/2}$ and on G_M is $d_{11}^{1/2}/(q^2/4M^2)(1 + q^2/4M^2)$. The error on G_E is more complex and is

$$\left(d_{22} - \frac{d_{12}}{(1 + q^2/4M^2)} + \frac{d_{11}}{(1 + q^2/4M^2)}\right)^{1/2}.$$

These errors take no account of the fact that the points do not always fit the straight line [Eq. (26)] well. The errors are called errors by internal consistency.[29] The goodness of fit is found by evaluating

$$\chi^2 = \sum_i (R - R_i)^2/\sigma_i^2\,. \quad (36)$$

The expected value of χ^2 is the number of degrees of freedom of the system which is $N - 2$, where N is the number of pieces of data. If we multiply the error by $(\chi^2 + 2 - N)^{1/2}$ we arrive at the error by external consistency. In most cases, this was greater than that by internal consistency.

Figure 4 shows such a fit for Bumiller's data at $q^2 = 10$ F^{-2}, $\chi^2 = 13$ with an expected value of 4. Visually, the data appear to fit quite well. This shows the importance of correct quantitative fits.

This procedure has been followed separately for the Littauer and Bumiller data for $q^2 \geqslant 4$ F^{-2}. We have *raised* the form factors of Bumiller by 3% at large momentum transfers to allow for better radiative corrections, the better calculations of Tsai,[30] and a slight nonlinearity of this correction due to the small value of $\Delta E/E$ used in the evaluation. We note at once that the errors are larger than the errors for F_1 and F_2 of Bumiller. This is because we have refrained from smoothing. Our errors on each point are truly independent, whereas Bumiller presents errors which in some way represent errors on the curve as a whole.

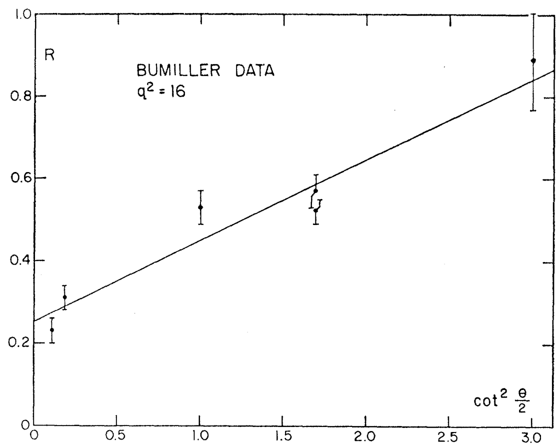

FIG. 4. A least-squares fit to Eq. (28) for $q^2 = 10$ using Bumiller's data.

The scatter of both the Bumiller points and the Littauer points is greater than the error. This would be reduced if we increased the error to be that of external consistency. This would increase the errors by an average factor of 1.5 for the Bumiller data and 2.0 for the Littauer data. We prefer to leave the errors as they stand to show, in the one plot, the reproducibility of the data. Moreover, the systematic errors are not clear and are omitted.

For $q^2 < 2$, using the points of Yount, Bumiller and Drickey, it is not possible to separately derive G_E and G_M, so they have been analyzed assuming $G_M = \mu G_E$ and values of G_E deduced. Likewise, the value of G_M of Yount at $q^2 = 4.6$ F^{-2} assumes a value of G_E.

The new data of Lehmann *et al.* and Berkelman *et al.* already are analyzed directly in terms of G_E and G_M and we quote directly. The new data of Aitken *et al.* are at constant q^2, in terms of F_1 and F_2, and it is trivial to extend to G_E and G_M.

The data of 180° electron–proton scattering[28] measure G_{Mp} directly.

The data for G_{Ep} and G_{Mp} are tabulated in Table I and are plotted in Figs. 5–7. Two lines are drawn on the plots. The solid line is the authors' guess as to the truth. This line satisfies $G_M = \mu G_E$ for all q^2; this is an amusing relation for which we can find no meaning. The dotted line is an attempt, to be described later, to fit the data with dispersion relations using Eqs. (24) and (25).

[29] E. R. Cohen, K. M. Crowe, and J. W. M. DuMond, *Fundamental Constants of Physics* (Interscience Publishers, Inc., New York, 1957), Chap. VII.
[30] P. Tsai, Phys. Rev. **122**, 1898 (1961).

TABLE I. Proton form factors.[a]

q^2 (F^{-2})	q^2 $(\text{BeV}/c)^2$	$\hbar^2q^2/4M^2c^2$	$t = \hbar^2q^2/M_\pi^2c^2$	G_{Ep}	dG_{Ep}	G_{Mp}	dG_{Mp}	Author
0.28	0.0109	0.0031	0.56	0.973	0.014			Yount
0.30	0.0116	0.0033	0.60	0.959	0.010			Lehmann
0.30	0.0116	0.0033	0.60	0.974	0.006			Drickey
0.36	0.0140	0.0040	0.70	0.967	0.040			Bumiller
0.49	0.0190	0.0054	0.98	0.933	0.009			Lehmann
0.57	0.0221	0.0063	1.14	0.915	0.037			Bumiller
0.60	0.0233	0.0066	1.20	0.940	0.007			Drickey
0.62	0.0241	0.0069	1.24	0.922	0.010			Yount
0.79	0.0307	0.0087	1.58	0.920	0.037			Bumiller
0.93	0.0361	0.0103	1.86	0.848	0.034			Bumiller
1.00	0.0388	0.0111	2.00	0.881	0.009	2.508	0.036	Lehmann
1.05	0.0408	0.0116	2.10	0.884	0.009			Lehmann
1.30	0.0505	0.0144	2.60	0.867	0.025			Yount
1.38	0.0536	0.0153	2.75	0.873	0.036	2.437	0.100	Bumiller
1.60	0.0621	0.0177	3.19	0.849	0.004			Drickey
2.00	0.0776	0.0221	3.99	0.810	0.024			Bumiller
2.00	0.0776	0.0221	3.99	0.825	0.022	1.703	0.302	Littauer
2.00	0.0776	0.0221	3.99	0.784	0.013	2.234	0.036	Lehmann
2.20	0.0854	0.0243	4.38	0.790	0.006			Drickey
2.98	0.116	0.0329	5.95	0.725	0.021	2.034	0.016	Lehmann
4.00	0.155	0.0442	7.99	0.650	0.034	1.776	0.119	Bumiller
4.00	0.155	0.0442	7.99	0.749	0.173	1.366	1.567	Littauer
5.20	0.202	0.0575	10.38			1.794	0.049	Yount
5.64	0.219	0.0624	11.26			1.493	0.017	Gram
6.00	0.233	0.0663	11.98	0.557	0.122	1.551	0.400	Bumiller
6.00	0.233	0.0663	11.98	0.654	0.186	1.284	0.056	Littauer
8.00	0.310	0.0885	15.97	0.400	0.109	1.375	0.251	Bumiller
8.00	0.310	0.0885	15.97	0.473	0.624	1.217	0.031	Littauer
8.87	0.344	0.0981	17.71			1.202	0.023	Gram
10.0	0.388	0.111	20.0	0.419	0.033	0.989	0.030	Bumiller
10.0	0.388	0.111	20.0	0.558	0.021	0.821	0.034	Littauer
10.0	0.388	0.111	20.0	0.417	0.02	1.119	0.045	Janssen
12.0	0.466	0.133	24.0	0.30	0.044	1.07	0.051	Bumiller
12.0	0.466	0.133	24.0	0.466	0.032	0.957	0.050	Littauer
12.43	0.482	0.148	24.81			0.937	0.014	Gram
14.0	0.543	0.155	27.9	0.256	0.054	0.916	0.048	Bumiller
15.0	0.582	0.166	29.9	0.416	0.022	0.717	0.019	Littauer
16.0	0.621	0.177	31.9	0.281	0.050	0.782	0.038	Bumiller
16.23	0.630	0.179	32.4			0.726	0.011	Gram
18.0	0.699	0.199	35.9	0.350	0.024	0.662	0.016	Bumiller
20.0	0.776	0.221	39.9	0.316	0.045	0.524	0.025	Littauer
20.2	0.784	0.223	40.3			0.554	0.024	Gram
22.0	0.854	0.243	43.9	0.295	0.046	0.563	0.015	Bumiller
24.0	0.931	0.265	47.9	0.249	0.109	0.416	0.041	Littauer
24.0	0.931	0.265	47.9	0.204	0.545	0.521	0.037	Bumiller
25.0	0.970	0.276	49.9	0.396	0.037	0.447	0.016	Berkelman
30.0	1.164	0.332	59.9	0	0.2	0.430	0.026	Littauer
30.0	1.164	0.332	59.9	0.359	0.037	0.382	0.014	Berkelman
35.0	1.358	0.387	69.9	0.258	0.044	0.314	0.012	Berkelman
40.0	1.552	0.442	79.9	0.436	0.073	0.232	0.018	Berkelman
45.0	1.746	0.498	89.9	0.00	0.250	0.238	0.022	Berkelman

[a] Errors by internal consistency only.

Several features emerge. In the region $4 < q^2 < 15$ F^{-2} Littauer *et al.* find larger values for G_E than does Bumiller. This may be directly attributed to high values of the cross section at 45°. The Bumiller data in general lies closer to the smooth curves than other data and is to be preferred.

At $15 \lesssim q^2 < 20$ F^{-2} Bumiller finds that G_E flattens off at about $G_E = 0.25$. For preliminary data this presumably be due to strongly interacting particles and would show up in high-energy nucleon–nucleon scattering. It doesn't.[31] The Berkelman data suggest also that G_E flattens off at $G_E = 0.25$.

If we assume $\mu G_{Ep} = G_{Mp}$, early data of Hand[32] and more of the Bumiller[20] data may be used to measure G_{Mp}. They agree well with the recent data[26] which can measure G_{Mp} independent of G_{Ep}.

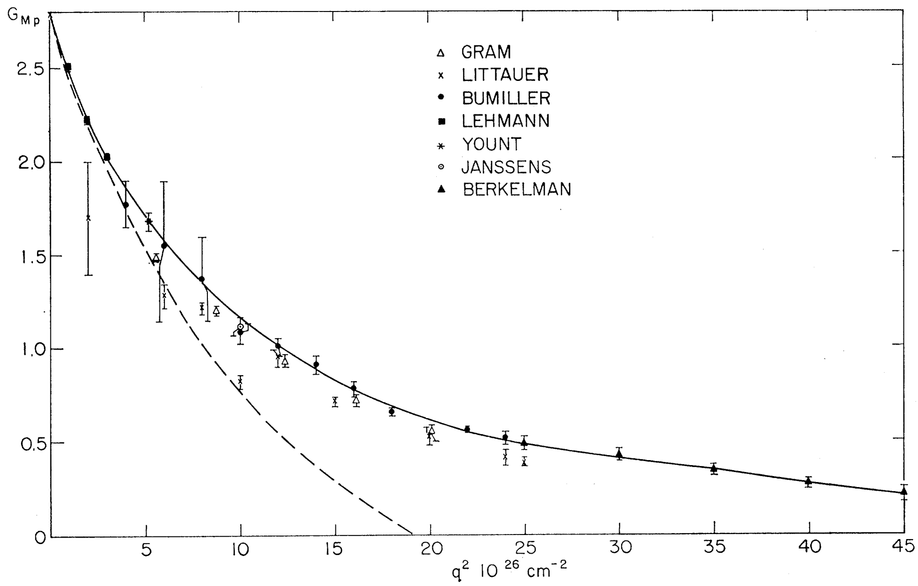

FIG. 5. G_{Mp} vs q^2. The solid line is given by Eq. (47), and (48). The dotted line is an attempt to fit with a hard core.

We may ask, why does G_M seem to be so much better determined than G_E? First, because it is entirely separated in Eq. (10) (given by $\cot^2 \theta/2 = 0$), whereas for large $\cot^2 \theta/2$ we measure $G_E^2 + (q^2/4M^2)G_M^2$ and at $q^2 = 20$ F^{-2} G_M dominates. Second, the term in G_E is very sensitive to the measurement of angle and energy as is easily seen from Eq. (10). By rearrangement we see that the term for $\cot^2 \theta/2 = 0$ varies as

$$d\sigma/d\Omega = \tfrac{1}{2}\,\alpha^2 r_0^2 (E'/E)^2 G_M^2\,. \quad (37)$$

The only energy dependence is through the factor E'/E and there is no angular dependence. Thus, a systematic error in energy or angle measurement by Littauer *et al.* would not affect G_M values appreciably, but would lead to inaccurate G_E values. (If this error were in all points, it would be proportional and could not make $G_E = 0$ when it is really finite.)

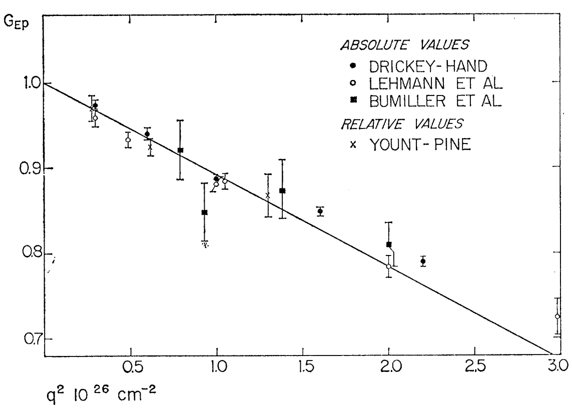

FIG. 6. G_{Ep} vs q^2. The line is given by Eq. (43).

Second Born Approximation

So far the analysis has been based entirely on the first Born approximation (Fig. 1). It is desirable to

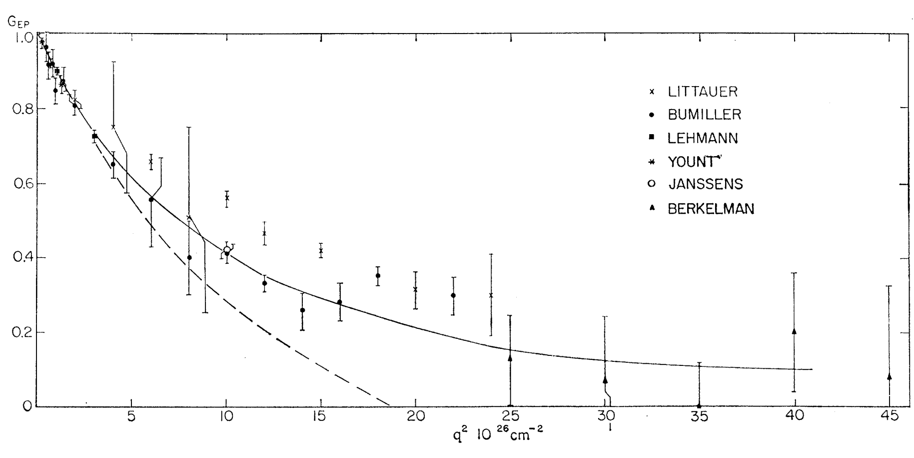

FIG. 7. G_{Ep} vs q^2 up to $q^2 = 50$ F^{-2}. The solid line is given by Eqs. (51) and (52). The dotted line is an attempt to fit with a hard core. Note that the recent Cornell data, not on the graph, give values for G_{E_p} larger than those shown.

check this. There are several arguments why second Born approximation terms are small. A typical term is shown in Fig. 8(a) and may be seen to be related to a Compton-effect term [Fig. 8(b)].

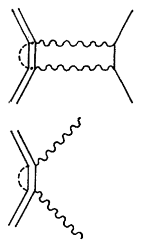

FIG. 8. (a) A Feymann diagram for electron-proton scattering with exchange of two photons. (b) A Feynmann diagram for the proton Compton effect.

The first Born approximation gives a real scattering amplitude and the first effect on the cross section is expected to be an interference between this and the real part of the second Born term. This interference term changes sign between electron–proton and positron–proton scattering. Yount and Pine[24] have compared these cross sections to 1% up to $q^2 = 1.5\ \mathrm{F}^{-2}$ and 3% at $q^2 = 5\ \mathrm{F}^{-2}$ and find no effect.

The Compton effect shows a peak in cross section at an energy corresponding to the $J = 3/2$, $T = 3/2$ resonance. At first sight it might be thought that this gives a large effect. However, this is associated with an absorption and the related scattering amplitude is imaginary giving no interference. Above the 3/2, 3/2 resonance preliminary data of Deutsch up to 0.8 BeV shows that the Compton cross section becomes quite small.

Third, we may note that the second Born approximation term corresponds to a virtual excitation followed by a de-excitation of the nucleon. We expect it to be large only when inelastic electron–proton scattering is large. This has been studied by Hand[3] who finds that the ratio of the cross section for exciting the 3/2, 3/2 resonance, integrated over this resonance, to the elastic scattering cross section, is approximately constant as a function of momentum transfer and is from about 7% of the elastic scattering at forward angles to about 20% at backward angles.

Fourth, we may appeal to the fact that formula (26) is well obeyed. Early data from Littauer *et al.* and Bumiller *et al.* suggested a deviation from this formula. The present data gives no such deviation. Gourdin and Martin[33] have shown the general form to be expected if two photon contributions appear.

[33] M. Gourdin and Martin, Nuovo Cimento (to be published); see also D. Flamm and W. Kummer, *Proceedings of the 1962 Annual International Conference on High-Energy Physics* (Centre d'Études Recherches Scientifiques, Geneva, Switzerland, 1962).

If they add to form a state of angular momentum and parity 0^-, 0^+, or 1^+ compared with 1^- for the single photon, the angular distribution remains the same, but the interpretation of the constants is, of course, different. A 1+ term will add to the forward scattering. Thus, the angular distribution is a poor way of picking two photon effects.

There are recent speculations that the photon might be a Regge pole. Then an energy dependent term would be expected to multiply Eqs. (9), (10), and (26) and the forward scattering would be *reduced* at high momentum transfers. The energy spread over which data has been taken is sufficiently small so that this would not yet be noticeable.

Numbers for the Neutron Form Factors

There are three sources of information on neutron form factors. First, the neutron–electron interaction has been extensively studied with thermal neutrons. The data and analysis have been extensively reviewed[6] and will not be repeated here (See Table II). Three experiments agree on the value which is a measure of $(\partial/\partial q^2)\ (G_{En})$ at $q^2 = 0$. In the past this has been expressed as

$$(\partial/\partial q^2)(G_{En}) = (\partial/\partial q^2)(F_{1n}) + \frac{\kappa}{4M^2}\,. \qquad (38)$$

The second of these terms is called the Foldy term and accounts for almost all the experimental effect, so that $(\partial/\partial q^2)\ F_{1n} \simeq 0$ [$F_{1n}\ (q^2 = 0) = 0$ also, because the neutron has no charge].

No meaning for the zero value of F_{1n}/q^2 has been found and from our point of view it is an accident.

We quote

$$(\partial/\partial q^2)G_{En} = (0.021 \pm 0.001)F^2\,, \qquad (39)$$

which is probably the best determined of all nucleon form factor data.

The next method is the measurement of inelastic electron scattering from the deuteron. Intuitively, this should give us a value for $\sigma_{ep} + \sigma_{en}$ multiplied by a factor giving the probability of finding the low momentum in the deuteron. This factor has been evaluated relativistically by Durand.[34] There are various small corrections which have been estimated by Durand and by Bosco.[35] Of these corrections the most important is the interaction in the final state of the two nucleons.

We note here that the general equation (14) applies and therefore the corrections separate into those applicable to $G_{En}^2 + G_{Ep}^2$ and those applicable to

[34] L. Durand, III, Phys. Rev. **123**, 1393 (1961).
[35] B. Bosco and R. B. De Bar (to be published).

TABLE II. Neutron form factors.[a]

q^2 (F^{-2})	q^2 $(\mathrm{BeV}/c)^2$	$\hbar^2q^2/4M^2c^2$	$t=\hbar^2q^2/m_\pi{}^2$	G_{En}^2	dG_{En}^2	G_{En}	dG_{En}	$-G_{Mn}$	dG_{Mn}	Author
0.96	0.0373	0.0106	1.92			−0.076	0.06			Friedman
1.00	0.0388	0.0111	2.00			+0.014	0.009	1.44	0.20	Lehmann
1.14	0.0442	0.0126	2.28			−0.028	0.06			Friedman
1.88	0.0730	0.0208	3.75			+0.039	0.06			Friedman
2.00	0.0776	0.0221	3.99	0.00	0.07	0.0	0.3	1.3	0.4	DeVries
2.28	0.0885	0.0252	4.55			−0.009	0.06			Friedman
2.56	0.0994	0.0283	5.11					1.31	0.13	Friedman
3.20	0.1242	0.0354	6.39			+0.054	0.05			Friedman
3.24	0.126	0.0358	6.47					1.34	0.13	Friedman
4.00	0.155	0.0442	7.99			−0.013	0.05			Friedman
4.00	0.155	0.0442	7.99					0.99	0.10	Friedman
4.00	0.155	0.0442	7.99	−0.03	0.08	0.0	0.3	1.3	0.3	DeVries
4.93	0.191	0.0545	9.84			+0.095	0.05			Friedman
5.06	0.196	0.0559	10.10					1.09	0.11	Friedman
5.86	0.227	0.0648	11.70			−0.013	0.04			Friedman
6.00	0.233	0.0663	11.98	0.087	0.18	±0.3	0.2	1.2	0.6	DeVries
6.86	0.266	0.0759	13.69			+0.006	0.04			Friedman
7.95	0.309	0.0879	15.87			+0.015	0.04			Friedman
8.00	0.310	0.0885	15.97	0.01	0.16	±0.1	0.3	1.1	0.4	DeVries
10.0	0.388	0.111	19.96	0.01	0.06	±0.1	0.25	0.9	0.15	DeVries
12.0	0.466	0.133	23.96	−0.01	0.05	±0.1	0.23	0.8	0.1	DeVries
14.0	0.543	0.155	27.95	−0.09	±0.05	0.0	0.23	0.9	0.1	DeVries
16.0	0.621	0.177	31.94	−0.012	0.06	0.0	0.23	0.6	0.1	DeVries
18.0	0.699	0.199	35.93	0.036	0.09	±0.19	0.15	0.5	0.1	DeVries
22.0	0.854	0.243	43.92	0.11	0.12	±0.33	0.14	0.4	0.15	DeVries

[a] Errors by internal consistency.

$G_{Mn}^2+G_{Mp}^2$. For example, the final-state interactions for the magnetic terms are in the ${}^1S\,{}^3P\,{}^1D$ states of the two nucleons, and for the electric terms the ${}^3S\,{}^1P\,{}^3D$ states. Durand only calculates for $q^2=6.8$ F^{-2} and 11.6 F^{-2} corresponding to the nucleon–nucleon interactions at 150 MeV and 240 MeV lab energies. He finds 2% effects with the S and P states canceling. Bosco calculates S states only and finds 5% effects which for S states only are consistent. For $q^2<6$ the phase will predominate and be larger, so the correction can be expected to rise. Bosco finds 10% to the electric terms at $q^2=3$ and 2% to the magnetic terms.

We use the factor Durand quotes in a form convenient for our units. The electron–deuteron scattering cross section at the peak of the distribution is given by

$$\frac{d^2\sigma}{d\Omega dE'}=\left[\left(\frac{d\sigma}{d\Omega}\right)_{en}+\left(\frac{d\sigma}{d\Omega}\right)_{ep}\right]\frac{4.57\times10^{-3}}{(q^2/4M^2)^{1/2}}\times\left(1+\frac{q^2}{4M^2}\right)^{-1/2}\frac{1}{[1+2E/M\sin^2\theta/2]},\quad(40)$$

where dE' is in MeV and q^2 is in F^{-2}. The estimated error is 5%. DeVries[36] has measured the ratio $S(q^2,\theta)=\sigma_p/(d^2\sigma/d\Omega dE')$ over a wide range of angles and energies. From his data we interpolate S to the values of q^2 for which we have derived G_{Mp} and G_{Ep}. Figure 9 illustrates this procedure. Using the error matrix previously evaluated we find an adjusted value for σ_p with its equation analogous to Eq. (28).

We repeat the analysis altering the constant in Eq. (37) by 8% to cover a five-percent calculation error by Durand and a five-percent systematic error by DeVries. We thus derive G_{Mn}, with its error, and G_{En}^2. At low q^2, G_{En}^2 is found to be negative. This is contrary to the findings of DeVries (using the same data) who did not use our interpolation procedure but who forced a fit to a straight line. This we believe is a mistake of presmoothing the data.

The negative value of G_{En}^2 is within the error of zero and would be raised if a larger final state inter-

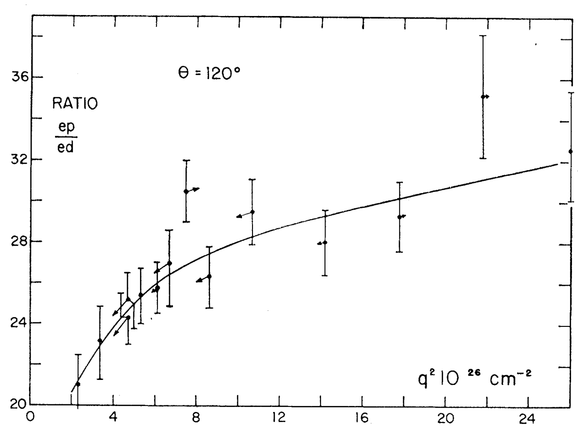

FIG. 9. $S(q^2,\theta)$ at $\theta=135°$ showing the procedure for interpolation to even values of q^2.

[36] C. DeVries, R. Hofstadter, and R. Herman, Phys. Rev. Letters 8, 381 (1962).

action in the ${}^3S$ state were included, as Bosco suggests.

We do not include the earlier data of Yearian and Sobottka which we regard as superseded by, though not inconsistent with, that of DeVries, nor the deuteron data of Littauer *et al.*, which is probably of lower precision.

FIG. 10. G_{Mn} vs q^2. The solid line is given by Eqs. (51) and (52).

G_{Mn} and G_{En} are plotted in Figs. 10 and 11. The large error in G_{En} at low q^2 is due to the fact that it is close to zero and $G_{En}^2 + G_{Ep}^2$ is what is measured.

Recent data by Yearian *et al.*[37] give slightly smaller values for G_{En}^2 at low q^2 but agree at large q^2.

The Durand–Bosco theory has not been exten-

where $2G_{Es} = G_{Ep} + G_{En}$ is the scalar charge form factor of the nucleon and F_d represents deuteron structure and will be the Fourier transform of the deuteron ground-state wavefunction. A recent dispersion theory calculation[40] confirms this intuition for small q^2. [Some previous ideas were that G_{Es} should be replaced by F_{1s} in (32), which is actually a negligible change.]

Kramer and Glendenning[41] have used nucleon–nucleon interaction information to evaluate F_d. They need the binding energy of the deuteron, the np triplet scattering length, the deuteron quadrupole moment, and also they use the fact that the distant parts of the nuclear potential are given by one pion exchange. [In fact, the triplet scattering length is more precisely known than they assumed $a_t = 5.40 + 0.01$ F.]

Using this they have derived G_{Es} from the data of Friedman.[42] There is also one recent precise point of

we derive G_{En} from this and plot it in Fig. 11.

[*Note added in proof.* More precise numbers have been obtained by Drickey[27] since this compilation. These numbers suggest $G_{En} = 0$ in contradiction to Eq. (39). This is probably due to a breakdown of Eq. (42).]

The intuitive formula for G_{Md} in terms of G_{Ms} is not given by dispersion theory. Nevertheless, we plot G_{Mn} so derived with the other data in Fig. 10.

related for these, so they were not calculated immediately.

We note that G_{Mv} is numerically large and quite well known while G_{Ms} is small and poorly known. G_{Ev} and G_{Es} are of the same order of magnitude and poorly known because of the lack of knowledge of G_{En}.

However, for tentative use we suggest the vector and scalar form factors in Figs. 12–15. Any theoretical conclusion using these figures should be checked against the proton form factors, or better still, against the cross sections directly.

Summary of Experimental Numbers for the Proton

From the above discussion it is possible to summarize the salient features of the data and the degree to which the $T = 1$, $J = 1^-\rho$ meson and $T = 0$, $J = 1^-\omega$ meson can be said to contribute to nucleon structure. These will contribute "poles" in the form factor at $q_R^2 = -14.5$ and $-15.8\ \text{F}^{-2}$, respectively, corresponding to mass values of 750 MeV and 785 MeV for the ρ and ω. The near equality of these masses and the present precision in determination of the nucleon form factors does not justify treating α_ρ and α_ω or β_ρ and β_ω [see formulas (22) and (23)] as independent variables at present. This allows us the

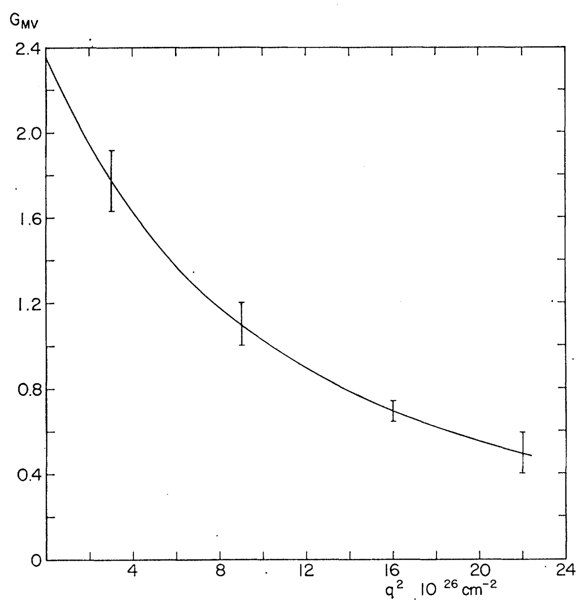

FIG. 12. G_{Mv} vs q^2, Eq. (51).

great simplification of treating the combined effects of both resonances as a single mass $q_R^2 \approx 15\ \text{F}^{-2}$ and considering the fit to the proton data alone, without the necessary increase in errors if the uncertainties in the neutron data were included. It is our belief that the masses of the contributing resonances are better determined by direct observation of the resonances in production experiments and not by a fit to the electron scattering data as was historically the case. We begin by noting several features of the proton data.

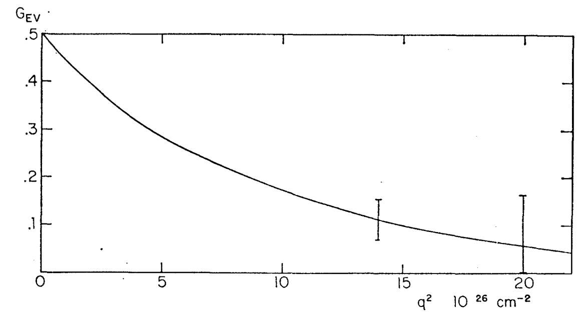

FIG. 13. G_{Ev} vs q^2, Eq. (47).

The accurate values of G_{Ep} at low q^2 enable us to state the derivative (dG_{Ep}/dq^2) at $q^2 = 0$ accurately. We use the data in Table I $q^2 = 1.05\ \text{F}^{-2}$. It is necessary to use a quadratic fit to $G_{Ep} = 1 - 1/6\langle r^2\rangle q^2 + Aq^4$, using values of q^2 up to 3 F^{-2} to determine the parameter A. The data are quite consistent and we deduce

$$(dG_{Ep}/dq^2)_{q^2=0} = -0.108 \pm 0.003\ \text{F}^{-2}\,, \quad (43)$$

where the rms radius of the proton is given by

$$\langle r^2\rangle^{1/2} = 0.805 \pm 0.011\ \text{F}\,. \quad (44)$$

Validity of the Relation $G_{Mp} = \mu_p G_{Ep}$

In deriving some of the values of G_{Ep} for the above determination of $\langle r^2\rangle$ it was necessary to make a small correction for the magnetic scattering even at low q^2 and forward-scattering angles. The best data on G_{Ep} and G_{Mp} as independent variables come from the work of Lehmann *et al.* who find $G_{Ep} = G_{Mp}/\mu_p$ to better than 2% for $q^2 \leqslant 3.0\ \text{F}^{-2}$. Above $q^2 = 3.0$, the existing data, if the errors were to be taken literally, would show oscillations of G_{Ep} about G_{Mp}/μ_p (see Fig. 6). There are, however, clear inconsistencies in the data and it is better to await future precise work in this region. For example, the latest point of Janssen at $q^2 = 10\ \text{F}^{-2}$ indicates $G_{Mp}/\mu_p G_{Ep} = 0.96 \pm 0.06$ and is a change from previous values for the same quantity of 0.53 ± 0.04 (Littauer) and 0.85 ± 0.08 (Bumiller). Above $q^2 = 15$, the recent data suggest that the relation breaks down.

We do not wish to assert any particular significance to the apparent similarity of the "charge" and "magnetic" distributions and have made use of this relation only in determining the slope of G_{Mp} at $q^2 = 0$, in a region where it is well verified by experiment.

Fit to One-Pole Term

We select G_{Mp} as the most accurately known of the nucleon form factors and attempt to fit in the one-pole approximation ($q_R^2 \approx -15$ F^{-2}) to the formula (25). From the accurate slope determination, we can determine the relative amounts of (ρ,ω) meson and

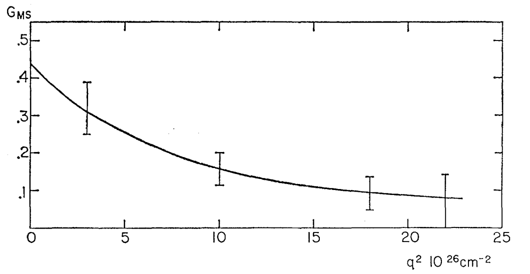

FIG. 14. G_{Ms} vs q^2, Eq. (52).

core. The steep slope in comparison with the $q_R^2 \approx -15$ F^{-2} forces the vector meson contribution to be larger and of opposite sign than that of the core. G_{Mp} thus necessarily passes through zero between 20 and 25 F^{-2}, in contradiction with experiment. On Figs. 5–7, 10, and 11 are plotted, in dotted lines, an attempt at such a fit to the proton and neutron data. There are several suggested ways to overcome this difficulty. First, Devries and Hofstadter prefer to introduce an "effective mass" in the one-pole approximation, which is, of necessity, lower than the ρ and ω masses. Attempts to fit the one-pole approximation with a second mass lower than the ρ,ω meson masses give results much too high at high q^2, unless a canceling core is added. If we vary the one-pole mass and the amount of core it is possible to obtain a fit to the data as DeVries and Hofstadter have shown.

Second, we can retain the $(\rho,\ \omega)$ pole term if one allows the replacement of the constant core term by a "soft" core possessing a mass squared in the range 20–40 F^{-2}. The amount of (ρ,ω) coupling is strongly dependent on the particular value chosen and varies by a factor 1.7 for M^2_{core} between 20 and 40 F^{-2}. The fits obtained are quite similar and it is not possible to decide between them by appealing to the accurate data available for G_{Mp}. The true momentum dependence of the "core" remains a mystery and thus shields us from other than a rough determination of the vector–meson coupling using electron-scattering data alone. It might be mentioned in passing that the "exponential model" formula $G_{Mp} = \mu_p/(1 + q^2/18.5)^2$ is consistent with all known data to $q^2 = 45$ F^{-2}. This formula differs only slightly from that obtained using a soft core of mass 30 F^{-2} over this q^2 range.

Third, another distinct possibility is that some part of the steep form factor slope is actually caused by the breakdown of quantum electrodynamics. It is well known that electron–proton scattering poses one of the most severe tests of electrodynamics. The great similarity between the shapes of proton and neutron form factors and the fact that $G_{En} \approx 0$ suggest a single over-all multiplicative function, possibly arising from a modification of the photon propagator or electron vortex. Were this to be the case, effects of the vector meson coupling would still be expected to be present, further complicating the situation.

We wish to emphasize here that we can see no meaning to the coupling constants derived using a constant core and ignoring the high momentum transfer data or using an effective mass different from the known ρ and ω masses. If we use the fit using a "soft" core, which fits the high-energy data, we can hope the coupling constants can have some meaning whether or not the core mass represents a further vector boson or bosons or a breakdown of quantum electrodynamics.

Coupling Constants

We now have, from Eqs. (39) and (43), accurate values for the slopes of G_{Ep} and G_{En} at low momentum transfer. We then derive for the isotopic form factors:

$$(dG_{Ev}/dq^2)_{q^2=0} = -0.0645 \pm 0.0017 \text{ F}^{-2} \quad (45)$$

$$(dG_{Es}/dq^2)_{q^2=0} = -0.0435 \pm 0.0017 \text{ F}^{-2}\,. \quad (46)$$

If we assume that there is a mass for the soft core

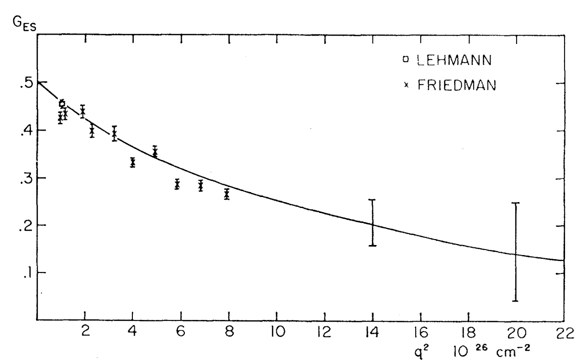

FIG. 15. G_{Es} vs q^2, Eq. (48).

which is the same in both scalar and vector states $M_c^2 = 30$ F^{-2}, we find

$$G_{EV} = 0.5\left[\frac{-1.68}{1 + q^2/30} + \frac{2.68}{1 + q^2/14.5}\right] \quad (47)$$

$$G_{ES} = 0.5\left[\frac{-0.80}{1 + q^2/30} + \frac{1.80}{1 + q^2/15.8}\right]. \quad (48)$$

From the data on G_{Mp} and G_{Mn} we find

$$(dG_{Mp}/dq^2)_{q^2=0} = -0.30 \pm 0.02 \text{ F}^{-2} \tag{49}$$

$$(dG_{Mn}/dq^2)_{q^2=0} = +0.20 \pm 0.08 \text{ F}^{-2}\,. \tag{50}$$

Assuming the same soft core mass $M_c^2 = 30$ F^{-2}, we find

$$G_{MV} = 2.35\left[\frac{-1.0}{1+q^2/30} + \frac{2.0}{1+q^2/14.5}\right] \tag{51}$$

$$G_{MS} = 0.44\left[\frac{-1.7}{1+q^2/30} + \frac{2.7}{1+q^2/15.8}\right]. \tag{52}$$

The fits of Eqs. (47), (48), (51), and (52) are shown as solid lines in the figures. They fit quite well.

The values for α_ρ, $\beta_\rho\alpha_\omega$, β_ω depend upon the soft core masses as well as the experimental data. We thus find that α_ρ, β_ρ, α_ω, are known to no better than 30%, and β_ω hardly at all.

[*Note added in proof.* These fits to the electric form factors ignore the recent Cornell data above $q^2 = 25$ F^{-2} which were only available in proof.]

The Pion–Pion Interaction

The relationship of the coupling constants just derived to the pion–pion interaction can now be reexamined. In the work of one of us described earlier,[17] a fit to the electric isotopic vector form factor was discussed using a constant core. This, we now believe, is unrealistic. The coupling constant is 1.5 times greater than that previously assumed. In order to obtain agreement with the pion–nucleon charge exchange scattering, we must now take a smaller width of 20 MeV for the ρ meson.

The isotopic vector magnetic form factor may be related to the pion–nucleon scattering P phase shifts by the method of Ball and Wong.[43] The most recent attempt to do this is by Singh and Udgaonkar[44] who also fit the isotopic vector electric form factor and claim agreement with a width of 120 MeV for the ρ meson, which is the experimental width. Since they also assume hard cores, which is palpably wrong, it is hard to understand their claim that agreement has been achieved.

[43] J. S. Ball and D. V. Wong, Phys. Rev. Letters **6**, 29 (1961).

[44] V. Singh and B. M. Udgaonkar, Phys. Rev. **128**, 1820 (1962).

Electropion Production

The latest work on electropion production is by Hand[1] who measures the inelastically scattered electron up to $q^2 = 20$ F^{-2} and compares with the Fubini–Nambu–Wataghin dispersion theory. He notes that only Γ_t in Eq. (4) is important. The dominant term in this is the 3/2, 3/2 resonance which is proportional to G_{Mv} (the photoproduction goes as $\mu_p - \mu_n = G_{Mv}$ for $q^2 = 0$, as is well known). There are smaller terms in G_{Ev}.

Since the spin separation of the current operator J_μ is again useful, G_E and G_M should again appear naturally. The theory predicts the total photoproduction cross section quite well. The agreement is good if it is assumed that $G_{En} = -0.2$. This evidence for a negative G_{En} must be rejected in the face of good data from elastic electron deuteron scattering where the interpretation is more certain.

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[45] J. S. Levinger, Nuovo Cimento **26**, 813 1962; M. W. Kirson and J. S. Levinger, Phys. Rev. (to be published).