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ZERO MODES OF THE VORTEX–FERMION SYSTEM

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Electric charge of a fermion that is coupled only to the gauge field of an abelian vortex need not be quantized, in contrast the situation for a magnetic monopole field. Correspondingly, zero-energy modes of the Dirac equation are not determined topologically. On the basis of mathematical and physical arguments we suggest that a complete description of the fermion–vortex system should include a fermion–scalar field interaction. This quantizes the charge and leads to a non-linear Dirac equation which possesses $|n|$ zero-eigenvalue modes in the n-vortex background field. The result suggests that an index theorem exists for this Dirac equation on the non-compact space R^2. When regularity requirements on the wave function are relaxed, additional normalizable zero-energy modes are present, even in the vacuum sector, $n = 0$, and fermions are converted to bosons.

1. Introduction

The occurrence of zero-eigenvalue modes for the Dirac equation, on a non-compact space in a topologically interesting background field, is by now a familiar and well-studied phenomenon. Examples exist in one dimension [1], where kinks [solitons] provide the topologically non-trivial setting; in three dimensions, where 't Hooft–Polyakov monopoles [2], or Julia–Zee dyons [3], are used as the background field [1]; and in four dimensions, where instanton configurations [4] couple to Dirac spinors*. In this paper, we fill the dimensional gap, and analyze the problem for an infinite two-dimensional space**, where the topological structure is found in the abelian gauge theory coupled to charged scalars – viz. Ginzburg–Landau–Abrikosov "vortices" or Nielsen–Olesen "strings" [7].

This problem has already been studied by Nohl and de Vega [8]. Both authors consider massive fermions coupled only to the gauge field. No zero-energy eigenvalues are found, but for configurations with $|n|$ vortices there are $|n|-1$ states at threshold. [In the massless limit, these would be unisolated zero-energy modes.]

Our treatment differs from theirs in that we include a coupling to the scalar field. Also we set the fermion mass to zero; nevertheless a mass is generated dynamically by the scalar–fermion interaction. We find $|n|$ isolated zero-energy states.

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* Owing to the conformal invariance of the four-dimensional Yang–Mills theory, one may equivalently consider the problem on non-compact R^4 or on compact S^4; see Jackiw and Rebbi, ref. [5].

** Two-dimensional Dirac equations on a closed and compact space have been studied; see ref. [6]. Here we are concerned with the non-compact space R^2.

The motivation for including a scalar–fermion coupling is the following. Unlike with the magnetic monopole, an interaction with the gauge field of an abelian vortex does not force quantization of the fermionic charge; this is, however, achieved by coupling fermions to scalars. Also our experience with the monopole shows that binding for topologically interesting zero-eigenvalue modes is provided by the scalar Higgs field [1]. Finally, one may mention that in the physical setting of superconductivity, the equations governing electronic motion include a coupling to the charged scalar pairing field, as well as to the electromagnetic field [9].

While none of our results are surprising – indeed they complement naturally the one-, three- and four-dimensional examples – there are some unexpected aspects to the investigation. Since the scalar field is charge bearing, the fermion–scalar interaction must involve charged bilinears. As a consequence, the Dirac equation is unconventional: it is mildly nonlinear, relating wave functions to their complex conjugates. Solutions cannot be arranged into angular momentum eigenstates, even when the background field is spherically symmetric. In the presence of the vortex, charge conjugation no longer connects the positive-energy solutions with the negative-energy ones, and the zero-energy modes are not charge-conjugation eigenstates. Nevertheless, another conjugation symmetry can be identified – we call it "particle conjugation" – which has the requisite property of taking positive-energy solutions to negative-energy solutions, and a zero-energy solution is self-conjugate.

In sect. 2, the equations are formulated; in sect. 3, they are solved; while sect. 4 is devoted to concluding remarks, where it is also observed that relaxing regularity requirements allows for further normalizable, zero-energy modes, even in the vacuum sector, $n = 0$, and fermions are converted to bosons.

2. Dirac equations

An n-vortex configuration of the abelian gauge-scalar field theory may be taken in the following form [8], which is spherically symmetric in the sense that a rotation supplemented by a gauge transformation leaves it invariant:

$$\phi(\boldsymbol{r}) = \mathrm{e}^{in\theta} f(r)\,, \tag{2.1a}$$

$$\begin{aligned} gA(\boldsymbol{r}) &= \varepsilon^{ij}\hat{r}^j A(r)\,, \\ \boldsymbol{r} &= (r\cos\theta, r\sin\theta)\,. \end{aligned} \tag{2.1b}$$

Here q is the charge of the scalar field, and the asymptotic values for the scalar and vector fields are

$$f(r) \xrightarrow[r\to 0]{} f_0 r^{|n|}\,, \qquad f(r) \xrightarrow[r\to\infty]{} f_\infty\,, \tag{2.2a}$$

$$A(r) \xrightarrow[r\to 0]{} 0\,, \qquad A(r) \xrightarrow[r\to\infty]{} -\frac{n}{r}\,. \tag{2.2b}$$

The asymptotes at the origin insure that no singularity occurs there; while those at infinite r set the scalar field at its vacuum value and allow the vector field to carry topological flux n/q:

$$-\frac{1}{4\pi}\int \mathrm{d}^2 r\varepsilon^{ij}F_{ij} = \frac{1}{2\pi}\int \mathrm{d}\boldsymbol{r}\cdot\boldsymbol{A} = \frac{n}{q}. \tag{2.3}$$

We shall not need the explicit profiles; in particular we do not require that ϕ and $\boldsymbol{A}$ solve the non-linear field equations of the theory*.

For the Dirac lagrangian we take

$$\mathscr{L} = \bar{\Psi}(\gamma^\mu[i\partial_\mu - eA_\mu])\Psi - \tfrac{1}{2}ig\phi\bar{\Psi}\Gamma\Psi^{\mathrm{c}} + \tfrac{1}{2}ig^*\phi^*\bar{\Psi}^{\mathrm{c}}\bar{\Gamma}\Psi. \tag{2.4}$$

The fermion's electric charge is e; since the fermions couple bilinearly to the scalar field, $q=2e$**. The scalar coupling constant is g; the coupling matrix Γ and its Dirac adjoint $\bar{\Gamma}$ will be presently specified. Ψ^{c} is the charge conjugate spinor, related to $\bar{\Psi}$ by the charge conjugation matrix C.

$$\Psi_i^{\mathrm{c}} = C_{ij}\bar{\Psi}_j. \tag{2.5}$$

The Dirac spinors and matrices are four-component objects, when the model is formulated in four-dimensional space–time. Owing to the cylindrical symmetry of the vortex [its z-independence], there occurs an effective reduction to a two-space, one-time theory, and the Dirac spinors and matrices could be taken with only two components. This choice, however, leads in general to a more restricted physical content than the four-component realization [11], so we remain for a moment with the larger possibility. But, as will be presently demonstrated, absence of a mass term in the lagrangian permits a reduction.

There are two charged bilinears that are Lorentz scalars, and therefore candidates for the scalar–fermion interaction.

$$\Gamma = \gamma_5 = \bar{\gamma}_5, \qquad \text{(scalar)} \tag{2.6a}$$

$$\Gamma = I, \qquad \text{(pseudoscalar)}. \tag{2.6b}$$

Observe further that in the absence of mass, a discrete γ_5 transformation leaves $\mathscr{L}$ unchanged:

$$\Psi \to i\gamma_5\Psi, \qquad \bar{\Psi} \to -i\bar{\Psi}\gamma_5. \tag{2.7a}$$

Since γ_5 commutes with C and its square is $-I$, one verifies the invariance of the langrangian:

$$\mathscr{L} \to \mathscr{L}. \tag{2.7b}$$

* Such profiles have been constructed numerically; see ref. [10].

** The charge in the Nohl–de Vega treatment [8] is set [arbitrarily] at $q=e$; the number of their bound states would be different for different choices.

As a consequence, we may use two-component spinors, and the two possible spinor–scalar interactions (2.6) coincide. The Dirac equation becomes

$$i\partial_t \Psi = \boldsymbol{\alpha} \cdot (\boldsymbol{p} - e\boldsymbol{A})\Psi - g\phi\sigma^2\Psi^* , \qquad \boldsymbol{p} = \frac{1}{i}\boldsymbol{\nabla} , \tag{2.8}$$

where $\boldsymbol{\alpha}$ is the pair of Pauli matrices (σ^1, σ^2).

Separation of the time variable requires a two-phase ansatz:

$$\Psi = \mathrm{e}^{-iEt}\psi^{(+)}(\boldsymbol{r}) + \mathrm{e}^{iEt}\psi^{(-)}(\boldsymbol{r}) . \tag{2.9}$$

Then the static functions $\psi^{(\pm)}$ satisfy

$$E\psi^{(+)} = \boldsymbol{\alpha} \cdot (\boldsymbol{p} - e\boldsymbol{A})\psi^{(+)} - g\phi\sigma^2\psi^{(-)*} , \tag{2.10a}$$

$$-E\psi^{(-)} = \boldsymbol{\alpha} \cdot (\boldsymbol{p} - e\boldsymbol{A})\psi^{(-)} - g\phi\sigma^2\psi^{(+)*} . \tag{2.10b}$$

The transformation

$$\psi^{(+)} \to \sigma^3\psi^{(+)} , \qquad \psi^{(-)} \to \sigma^3\psi^{(-)} , \tag{2.11}$$

takes a solution with eigenvalue E into one with eigenvalue $-E$. We call the operation (2.11) "particle conjugation".

For zero energy, the two equations (2.10) collapse into one,

$$0 = \boldsymbol{\alpha} \cdot (\boldsymbol{p} - e\boldsymbol{A})\psi - g\phi\sigma^2\psi^* , \tag{2.12}$$

and the solution ψ can be chosen to be an eigenmode of particle conjugation.

We shall solve (2.10) in the vacuum sector $[\phi(\boldsymbol{r}) = f_\infty,\ q\boldsymbol{A}(\boldsymbol{r}) = 0]$ to demonstrate that the fermions acquire a mass $gf_\infty = \mu$ – a quantity which may be taken positive, without loss of generality. The zero-energy equation (2.12), residing on R^2, will be solved for the vortex $[\phi(\boldsymbol{r}) = f(r)\,\mathrm{e}^{in\theta},\ q\boldsymbol{A}(\boldsymbol{r}) = -\hat{\boldsymbol{e}}_\theta A(r)]$. The corresponding equation with an antivortex $[\phi^*(\boldsymbol{r}),\ -q\boldsymbol{A}(\boldsymbol{r})]$ is solved by $\sigma^2\psi^*$ – the charge conjugate of the vortex solution ψ.

3. Solutions of the Dirac equation

3.1. VACUUM SECTOR

In the vacuum sector, ϕ is the constant f_∞, and $\boldsymbol{A}$ vanishes. With the ansatz

$$\Psi = \mathrm{e}^{-ikx}\psi_0^{(+)} + \mathrm{e}^{ikx}\psi_0^{(-)} \tag{3.1}$$

the constant spinors $\psi_0^{(\pm)}$ reduce (2.8) to

$$\gamma^\mu k_\mu \psi_0^{(+)} = i\mu\sigma^1\psi_0^{(-)*} , \tag{3.2a}$$

$$\gamma^\mu k_\mu \psi_0^{(-)} = -i\mu\sigma^1\psi_0^{(+)*} , \tag{3.2b}$$

$$\mu = gf_\infty ,$$

where γ^μ is the triplet of Dirac (Pauli) matrices $(\sigma^3, i\sigma^2, -i\sigma^1)$. Since (3.2b) implies that

$$\sigma^1 \psi_0^{(-)*} = -i\mu \frac{\gamma^\mu k_\mu}{k^2} \psi_0^{(+)} \tag{3.3}$$

it follows from (3.2a) that $k^2 = \mu^2$; i.e. the fermions acquire mass μ.

3.2. VORTEX SECTOR

With the vortex configuration (2.1) and the definition for the spinor

$$\psi = \begin{pmatrix} e^{(1/2)\int_0^r d\rho A(\rho)} \psi_U \\ e^{-(1/2)\int_0^r d\rho A(\rho)} \psi_L \end{pmatrix}, \tag{3.4}$$

the gauge potential disappears from eqs. (2.12), and they decouple:

$$e^{i\theta}\left(\partial_r + \frac{i}{r}\partial_\theta\right)\psi_U + gf\, e^{in\theta}\psi_U^* = 0\,, \tag{3.5a}$$

$$e^{-i\theta}\left(\partial_r - \frac{i}{r}\partial_\theta\right)\psi_L - gf\, e^{in\theta}\psi_L^* = 0\,. \tag{3.5b}$$

A two-phase ansatz separates the angular dependence:

$$\psi_U = U_U(r)\, e^{im\theta} + V_U(r)\, e^{i(n-1-m)\theta}\,, \tag{3.6a}$$

$$\psi_L = U_L(r)\, e^{-im\theta} - V_L(r)\, e^{i(n+1+m)\theta}\,, \tag{3.6b}$$

$$m = \text{integer} \neq \pm\tfrac{1}{2}n - \tfrac{1}{2}\,.$$

[The cases $m = \pm\frac{1}{2}n - \frac{1}{2}$ are special and will be discussed separately.]

$$\begin{aligned} \left(\partial_r - \frac{m}{r}\right) U_U + gfV_U^* &= 0\,, \\ \left(\partial_r - \frac{n-1-m}{r}\right) V_U + gfU_U^* &= 0\,; \end{aligned} \tag{3.7a}$$

$$\begin{aligned} \left(\partial_r - \frac{m}{r}\right) U_L + gfV_L^* &= 0\,, \\ \left(\partial_r + \frac{n+1+m}{r}\right) V_L + gfU_L^* &= 0\,. \end{aligned} \tag{3.7b}$$

Since the second set of equations in (3.7) [lower components of ψ] is obtained from the first [upper components of ψ] by the replacement $n \to -n$, we analyze only the former, and supress the subscript on U and V.

Eqs. (3.7a) are not linear; they involve both functions and their complex conjugates. Linear combinations of solutions are again solutions only if they are superposed with real coefficients. Eqs. (3.7a) are linearized by decomposing into real and imaginary parts.

$$\begin{aligned} U &= U^{(+)} + iU^{(-)} , \\ V &= V^{(+)} + iV^{(-)} . \end{aligned} \tag{3.8}$$

Thus real, linear equations are derived:

$$\begin{aligned} \left(\partial_r - \frac{m}{r}\right) U^{(\pm)} \pm gfV^{(\pm)} &= 0 , \\ \left(\partial_r - \frac{n-1-m}{r}\right) V^{(\pm)} \pm gfU^{(\pm)} &= 0 . \end{aligned} \tag{3.9}$$

To recognize the general solution, let us define a pair of real, normalizable functions (u_m, v_m) that satisfy

$$\begin{aligned} \left(\partial_r - \frac{m}{r}\right) u_m + gfv_m &= 0 , \\ \left(\partial_r - \frac{n-1-m}{r}\right) v_m + gfu_m &= 0 . \end{aligned} \tag{3.10}$$

General solutions to eqs. (3.9) involve real constants $a_m^{(\pm)}$:

$$\begin{aligned} &U^{(\pm)} = a_m^{(\pm)} u_m , \qquad V^{(\pm)} = \pm a_m^{(\pm)} v_m , \qquad a_m \text{ real} , \\ &U = a_m u_m , \qquad V = a_m^* v_m , \qquad a_m = a_m^{(+)} + i a_m^{(-)} . \end{aligned} \tag{3.11}$$

Since an overall real constant is irrelevant, we conclude that (3.7a) is solved by

$$\begin{gathered} U = \mathrm{e}^{i\alpha_m} u_m , \qquad V = \mathrm{e}^{-i\alpha_m} v_m \\ \psi_U = \mathrm{e}^{i\alpha_m} u_m(r)\, \mathrm{e}^{im\theta} + \mathrm{e}^{-i\alpha_m} v_m(r)\, \mathrm{e}^{i(n-1-m)\theta} . \end{gathered} \tag{3.12}$$

Superposing ψ_U's with different phases α_m does not produce a different type of solution.

$$\begin{aligned} &\sum_{\alpha_m} c(\alpha_m) \{ \mathrm{e}^{i\alpha_m} u_m(r)\, \mathrm{e}^{im\theta} + \mathrm{e}^{-i\alpha_m} v_m(r)\, \mathrm{e}^{i(n-1-m)\theta} \} \\ &\quad = |c'_m| \{ \mathrm{e}^{i\alpha'_m} u_m(r)\, \mathrm{e}^{im\theta} + \mathrm{e}^{-i\alpha'_m} v_m(r)\, \mathrm{e}^{i(n-1-m)\theta} \} . \end{aligned}$$

Here $|c'_m|\, \mathrm{e}^{\pm i\alpha'_m} = \sum_{\alpha_m} c_m(\alpha)\, \mathrm{e}^{\pm i\alpha_m}$, since $c(\alpha_m)$ must be real. [We shall show later that the phase α_m does not correspond to a new degree of freedom.]

To determine the mode functions u_m and v_m, note first that there are two linearly independent solutions to the linear system (3.10). The behavior at large r is, apart from powers of r, $\mathrm{e}^{\pm\mu r}$. Therefore only one is acceptable, since ψ in (3.4) must be normalizable. At the origin, the asymptotes which follow from (3.10), with $f \sim f_0 r^{|n|}$, are

$$
\begin{aligned}
u_m &\to r^m, r^{|n|+n-m}, \\
v_m &\to r^{|n|+1+m}, r^{n-1-m}.
\end{aligned}
\tag{3.13}
$$

The exponentiation of $\int_0^r \mathrm{d}\rho A(\rho)$ does not affect behavior at the origin; A vanishes there. To match the single well-behaved solution at infinity, both solutions must be acceptable at the origin. Consequently we infer from (3.13) that regularity of the Dirac wave functions will be assured only when

$$
n-1 \geqslant m \geqslant 0, \qquad m = \text{integer}. \tag{3.14}
$$

Thus n solutions for ψ_{U} are obtained with positive vortex number, and ψ_{L} vanishes. Correspondingly for $n<0$, ψ_{L} is present, but ψ_{U} vanishes, and condition (3.14) holds again with n replaced by $|n|$.

Returning through the definitions (3.12) and (3.4), we recover the spinor zero-energy modes:

$$
\begin{aligned}
\psi \underset{n>0}{=} & \exp\left[\tfrac{1}{2}\int_0^r \mathrm{d}\rho A(\rho) + \tfrac{1}{2}i(|n|-1)\right] \\
& \times \begin{pmatrix} \mathrm{e}^{i\alpha_m} u_m(r)\, \mathrm{e}^{-i[|n|/2-1/2-m]\theta} + \mathrm{e}^{-i\alpha_m} u_m(r)\, \mathrm{e}^{i[|n|/2-1/2-m]\theta} \\ 0 \end{pmatrix}, \\
\psi \underset{n<0}{=} & \exp\left[-\tfrac{1}{2}\int_0^r \mathrm{d}\rho A(\rho) - \tfrac{1}{2}i(|n|-1)\right] \\
& \times \begin{pmatrix} 0 \\ \mathrm{e}^{i\tilde{\alpha}_m} u_m(r)\, \mathrm{e}^{i[|n|/2-1/2-m]\theta} - \mathrm{e}^{-i\tilde{\alpha}_m} v_m(r)\, \mathrm{e}^{-i[|n|/2-1/2-m]\theta} \end{pmatrix}, \\
& |n|-1 \geqslant m \geqslant 0, \qquad m = \text{integer}.
\end{aligned}
\tag{3.15}
$$

It is seen that ψ is an eigenstate of particle conjugation

$$
\sigma^3 \psi = \frac{n}{|n|}\psi. \tag{3.16}
$$

Although the phases α_m and $\tilde{\alpha}_m$ are arbitrary, we may relate the latter to the former by requiring that the $n<0$ solution be the charge conjugate of the $n>0$ solution. This gives $\mathrm{e}^{i\tilde{\alpha}_m} = i\,\mathrm{e}^{-i\alpha_m}$.

It remains to discuss the cases $m = \pm\frac{1}{2}n - \frac{1}{2}$, which are available only for odd n. For $m = \frac{1}{2}n - \frac{1}{2}$ the analysis of (3.6b) is unchanged, but the reduction (3.6a) involves only one function:

$$
\begin{aligned}
\psi_{\mathrm{U}} &= U(r)\, \mathrm{e}^{(i/2)(n-1)\theta}, \\
\left(\partial_r - \frac{n-1}{2r}\right) & U + gfU^* = 0.
\end{aligned}
\tag{3.17}
$$

Separation into real and imaginary parts as in (3.8) gives

$$\left(\partial_r - \frac{n-1}{2r}\right) U^{(\pm)} \pm gfU^{(\pm)} = 0\,, \tag{3.18}$$

which may be solved by quadrature:

$$U^{(\pm)} = r^{n/2-1/2}\, \mathrm{e}^{\mp\int_0^r \mathrm{d}\rho g f(\rho)}\,. \tag{3.19}$$

This is non-singular at the origin only when $n>0$, and at infinity only with the upper sign. The result fits into the general formula (3.15) with $u_{|n|/2-1/2} = v_{|n|/2-1/2} = r^{|n|/2-1/2}\,\mathrm{e}^{-\int_0^r \mathrm{d}\rho g f(\rho)}$, $m=\frac{1}{2}|n|-\frac{1}{2}$. Similarly, $m=-\frac{1}{2}n-\frac{1}{2}$ is acceptable only for $n<0$ and (3.15) gives the correct result with $u_{|n|/2-1/2} = v_{|n|/2-1/2} = r^{|n|/2-1/2}\,\mathrm{e}^{-\int_0^r \mathrm{d}\rho g f(\rho)}$, $m=\frac{1}{2}|n|-\frac{1}{2}$.

To understand that the phase α_m does not correspond to a new degree of freedom, and to count the number of independent solutions let us consider an arbitrary solution to (3.5), obtained as a superposition [with real coefficients] of the $|n|$ modes.

$$\psi_{\mathrm{U}} = \sum_{m=0}^{n-1} b_m \{\mathrm{e}^{i\alpha_m} u_m(r)\,\mathrm{e}^{im\theta} + \mathrm{e}^{-i\alpha_m} v_m(r)\,\mathrm{e}^{i(n-1-m)\theta}\}\,. \tag{3.20a}$$

[We have taken $n>0$.] Since the b_m are real, the above may also be written as

$$\psi_{\mathrm{U}} = \sum_{m=0}^{n-1} \{d_m u_m(r)\,\mathrm{e}^{im\theta} + d_m^* v_m(r)\,\mathrm{e}^{i(n-1-m)\theta}\}\,. \tag{3.20b}$$

It is thus seen that the phases are associated with the complex expansion coefficients of ψ_{U}.

The expansion may be rewritten to exhibit explicitly that only n arbitrary, real constants occur, and not $2n$. Note first that the defining equations (3.10) imply that the mode functions satisfy a recurrence relation:

$$u_m = \varepsilon_{n-1-m} v_{n-1-m}\,, \qquad u_m = \varepsilon_{n-1-m} u_{n-1-m}$$
$$\varepsilon_{n-1-m}\ \text{real}\,, \qquad \varepsilon_m \varepsilon_{n-1-m} = 1\,. \tag{3.21}$$

[This covariance is seen already in the asymptotes (3.13).] Eqs. (3.21) may be used to express the late terms in (3.20b) in terms of the early ones. For example, for even n

$$\psi_{\mathrm{U}} = \sum_{m=0}^{n/2-1} \{d_m u_m(r)\,\mathrm{e}^{im\theta} + d_m^* v_m(r)\,\mathrm{e}^{i(n-1-m)\theta}\}$$
$$+ \sum_{m=n/2}^{n-1} \{d_m u_m(r)\,\mathrm{e}^{im\theta} + d_m^* v_m(r)\,\mathrm{e}^{i(n-1-m)\theta}\}\,. \tag{3.22a}$$

In the second sum we use (3.21), replace m by $n-1-m$, and combine with the first sum:

$$\psi_U = \sum_{m=0}^{n/2-1} \{(d_m + \varepsilon_m d^*_{n-1-m})u_m(r)\,e^{im\theta} + (d^*_m + \varepsilon_m d_{n-1-m})v_m(r)\,e^{i(n-1-m)\theta}\}\,. \tag{3.22b}$$

Upon recalling that ε_m is real, this may be rewritten in terms of n arbitrary, real constants:

$$\psi_U = e^{(i/2)(n-1)\theta} \sum_{m=0}^{n/2-1} \{e_m u_m(r)\,e^{-i[n/2-1/2-m]\theta} + e^*_m v_m(r)\,e^{i[n/2-1/2-m]\theta}\}\,,$$

$$e_m = d_m + \varepsilon_m d^*_{n-1-m}\,. \tag{3.23a}$$

A similar manipulation for odd n gives

$$\psi_U = e^{(i/2)(n-1)\theta} \sum_{m=0}^{n/2-3/2} \{e_m u_m(r)\,e^{-i[n/2-1/2-m]\theta} + e^*_m v_m(r)\,e^{i[n/2-1/2-m]\theta}\}$$

$$+ e^{(i/2)(n-1)\theta} e_{n/2-1/2} u_{n/2-1/2}(r)\,, \tag{3.23b}$$

$$e_{n/2-1/2} = d_{n/2-1/2} + d^*_{n/2-1/2}\,.$$

Thus we conclude that in both cases there are exactly n linearly independent, zero-energy modes.

The solutions cannot be chosen as angular momentum eigenstates, save when $m = \frac{1}{2}|n| - \frac{1}{2}$. This is not because the background fields lack rotational symmetry; on the contrary a rotation, supplemented by a gauge transformation, leaves them invariant. Rather it is because the problem is not linear, and linear combinations cannot be taken to isolate angular momentum eigenstates. [The equations are a *real* linear system, but not a *complex* linear system. But real functions do not give rise to angular momentum eigenstates.] Nevertheless a rotation on the general solution (3.23) transforms it into another solution of the same form: rotating by the angle ω; i.e. adding ω to θ, has the effect of changing e_m to $e_m\,e^{-i(n/2-1/2-m)\omega}$, and multiplying ψ_U by $e^{i(n/2-1/2)\omega}$, which is just a gauge transformation. This is exactly the rotational covariance of (3.5a).

4. Conclusion

Fermions in the field of an abelian vortex have their charge quantized only through an interaction with the charged scalar field, which is the source for the vortex. The resulting Dirac equation is mildly non-linear. Massless fermions acquire a mass through the scalar interaction, and in the n-vortex sector there are $|n|$ isolated, linearly independent, zero-energy bound states, which are eigenstates of a particle conjugation transformation, but not, in general, of angular momentum. Presumably the finite energy states become continuous for $|E| \geq \mu$; we do not know if there are other bound states for $|E| \leq \mu$.

Index theorems on non-compact spaces of one, three [12] and four dimensions [13] have been constructed to count the Dirac equation zero-modes. The result in the present two-dimensional example suggests that here too an *a priori* argument can be given★. It would be most interesting, from the mathematical point of view, to construct such a theorem. We conjecture that the index is★★

$$\operatorname{Im}\frac{1}{2\pi}\int \mathrm{d}^2 r\varepsilon^{ij}\,\partial_i\,\partial_j \ln\phi = \operatorname{Im}\frac{1}{2\pi}[\ln\phi|_{\theta=2\pi}-\ln\phi|_{\theta=0}]\,.$$

The occurrence of zero-modes of the Dirac equation can sometimes be related to a supersymmetry of the theory [15, 16]. Whether such considerations are applicable here is an open question★★★, though some obstacles are to be noted: Fermions [charge e] and scalars [charge $2e$] belong to different representations of SO(2); the back reaction on the scalars, $\psi^*\sigma^2\psi^*$ and $\psi\sigma^2\psi$, vanishes, but the back reaction on the gauge fields, $\bar{\psi}\gamma^\mu\psi$, is non-zero in the time component.

Finally let us call attention to the fact that more zero-energy modes are obtained if we relax the regularity conditions. We have demanded that the spinors be single valued $\psi(\theta=0)=\psi(\theta=2\pi)$, and regular at the origin. If we allow an integrable $r^{-1/2}$ singularity at the origin, and angular double valuedness $\psi(\theta=0)=\pm\psi(\theta=2\pi)$, which still keeps all bilinears single valued, then (3.14) may be relaxed to

$$\begin{aligned} n &\geq m+\tfrac{1}{2}\geq 0\,, &&\text{(upper components)}\,,\\ -n &\geq m+\tfrac{1}{2}\geq 0\,, &&\text{(lower components)}\,,\\ 2m &= \text{integer}\,. && \end{aligned} \tag{4.1}$$

In particular even in the vacuum sector there exist normalizable zero-energy modes. This is seen explicitly from (3.4), (3.5) and (3.17)–(3.19), with $gf=\mu$, $\mathbf{A}=0$, $n=0$. Two solutions, related by charge conjugation, are found.

$$\psi_1=\begin{pmatrix} r^{-1/2}\,\mathrm{e}^{-\mu r}\,\mathrm{e}^{-i\theta/2} \\ 0\end{pmatrix}=-\sigma^2\psi_2^*\,, \tag{4.2a}$$

$$\psi_2=\begin{pmatrix} 0 \\ ir^{-1/2}\,\mathrm{e}^{-\mu r}\,\mathrm{e}^{i\theta/2}\end{pmatrix}=\sigma^2\psi_1^*\,. \tag{4.2b}$$

It is seen that the above states are eigenvectors of particle conjugation, as well as

★ Of course, for compact, closed S^2, and in the absence of scalar interactions, the Atiyah-Singer index theorem is applicable; see ref. [6]. Here we need a theorem for non-compact R^2, and the presence of the scalar fields is essential, just as it is for the R^1 and R^3 examples studies in ref. [12]. An index theorem on R^2, but not for the Dirac equation, has been considered by Weinberg [14]. However, he expresses the index in terms of the Pontryagin invariant, while we believe the proper formula on a non-compact space should involve scalar fields; see ref. [12].

★★ Note added in proof: Such an index theorem has now been derived by E. Weinberg, Columbia University preprint (1981) CU-TP-202, unpublished.

★★★Supersymmetric generalizations of the abelian Higgs model have been constructed; see ref. [16].

of angular momentum, $J=(1/i)\,\partial_\theta+\frac{1}{2}\sigma^3$, with zero eigenvalue:

$$J\psi_{1,2}=0\,. \tag{4.3}$$

Thus fermions are converted to bosons*.

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* This is analogous to the conversion of fermions to bosons in the field of the monopole; see ref. [1].