ANNALS OF PHYSICS **135**, 373–415 (1981)

Semiclassical Functional Integrals for Self-Dual Gauge Fields

H. OSBORN

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 9EW, United Kingdom

Received February 3, 1981; revised March 26, 1981

The semiclassical approximation to the functional integral for four-dimensional Euclidean gauge theories is discussed in detail for general stationary points of the action. It is shown how to take the limit from a compact space to flat space, and the zero modes corresponding to global gauge transformations are carefully discussed. The results are specialised to general self-dual multi-instanton gauge fields given by the general construction of Atiyah *et al.* It is shown how the normalization matrix of the zero modes can be determined and the complete expression for the functional measure is given for the two instanton case. This is shown to factorise for well-separated instantons. Some technical matters are discussed in an appendix and a resumé of results for multi-instanton functional determinants is included.

1. INTRODUCTION

The dynamical role of instantons in nonabelian gauge theories has become a matter of some controversy. The most unambiguous result which stemmed from the discovery of the nonsingular single instanton solution for four-dimensional Euclidean gauge theories by Belavin *et al.* [1] is that the corresponding quantum field theories, such as QCD, depend on an additional parameter θ [2] (at least in the absence of any coupled massless fermion fields or scalar fields which realise a $U(1)$ chiral symmetry). The parameter θ can be introduced as a term θq in the Lagrangian density where

$$q = -(1/32\pi^2)\, \varepsilon^{\alpha\beta\gamma\delta} \operatorname{tr}(F_{\alpha\beta} F_{\gamma\delta}) \tag{1.1}$$

(detailed conventions are deferred to Section 2). However, present theoretical techniques are insufficient to allow a calculation of the consequential θ dependence of quantities of interest such as the vacuum energy density $\mathscr{E}(\theta)$ even for pure nonabelian gauge theories without any additional interactions. $\mathscr{E}(\theta)$ can be regarded as defined by

$$\langle e^{i\theta \int_{x\in V} d^4x\, q(x)} \rangle \underset{V\to\infty}{\sim} e^{-V\mathscr{E}(\theta)}, \qquad V = \int_{x\in V} d^4x, \tag{1.2}$$

where $\langle\cdot\rangle$ denotes the expectation value according to the Euclidean gauge theory

0003-4916/81/100373–43\$05.00/0

functional integration measure. Even though for QCD θ is presumably zero, and coupled quark fields are involved as well as gluon gauge fields, $\mathscr{E}(\theta)$ contains information of physical interest. $\mathscr{E}''(0)$ can be related to the mass of the $U(1)$ singlet pseudoscalar Goldstone boson, at least to the extent that the $1/N$ expansion provides a good approximation [3]. Indeed the fact that $\nu = \int d^4x\, q(x)$ can be nonzero for topologically nontrivial gauge fields, and that this leads to a resolution of the $U(1)$ problem associated with this supposed Goldstone boson, is the main phenomenological consequence of instanton ideas so far [4].

Since conventional perturbative procedures essentially correspond to topologically trivial gauge fields in the functional integration θ dependence must be nonperturbative in origin. Also, since for lattice gauge theories the introduction of θ is problematic [5] and unexplored in calculation, the only present methods capable of discussing θ dependence are semiclassical calculations. The functional integral is approximated by expansion about a set of nontrivial gauge field configurations, $\{A^c_\mu\}$. For continuous (up to gauge transformations) gauge fields defined on a compact four-dimensional manifold, such as S^4, or for solutions of the classical gauge field equations on flat Euclidean space R^4 [6], ν is an integer, the Pontryagin index or instanton number k.

The most elementary such background gauge field, apart from the trivial perturbative vacuum $A^c_\mu = 0$, which is a solution, is the Belavin *et al.* [1] instanton with $k = \pm 1$. The corresponding semiclassical contribution to the functional integration measure in terms of integrals over the instanton parameters, its position and scale, was first obtained by 't Hooft for the gauge group $SU(2)$ [7]. This calculation has been further analysed [8] and straightforwardly extended to $SU(n)$ [9]. These results have been used for a variety of calculations [10] generally in the context of the dilute-gas approximation [11]. This assumes that the set $\{A^c_\mu\}$ consists of gauge fields corresponding to arbitrary superpositions of arbitrary numbers of single instanton and anti-instanton fields. The resulting field configurations are supposed to be an approximate solution of the classical equations and for each such A^c_μ the corresponding contribution to the functional measure is taken, initially at least, to be just an appropriate product of single instanton measures together with a statistical weight factor $1/n_+!\, n_-!$, where n_+, n_- are the numbers of instantons or anti-instantons, $k = n_+ - n_-$. In this form the functional measure corresponds exactly in statistical mechanical terms to a free gas of two species of bosons, although interactions need to be subsequently introduced between instantons and anti-instantons as mixed configurations are not stationary points of the action [11, 12]. Neglecting these effects it is easy to derive [4, 11]

$$\mathscr{E}(\theta) = K(1 - \cos\theta), \tag{1.3}$$

but K is infinite in terms of the semiclassical approximation due to a badly divergent integral over the instanton scale sizes. Such divergent scale integrations are the standard bugbear of this approach (of course a cutoff is generally inserted by hand to reflect our ignorance of QCD as the distance scale increases which results from asymptotic freedom, and this may even be partially justified self-consistently). The result that the formalism emphasises large-scale sizes reflects a basic problem with the dilute-gas approximation; if the instanton scales are comparable with their separations the initial configurations of superposed instantons and anti-instantons are no longer even approximate stationary points of the action. There is then a crucial distinction as compared with other model scale noninvariant theories, which possess instantons of fixed size allowing an instanton gas approximation to be carefully justified in appropriate regimes, and where the instantons play an important dynamical role [13].

There are also more basic difficulties connected with the choice of the initial set of gauge field configurations $\{A_\mu^c\}$ in this approach. It is unclear to what extent introducing superpositions of overlapping instantons and anti-instantons may involve overcounting in the functional measure as compared with the contribution of just the purely self-dual instanton or anti-self-dual anti-instanton fields with the same values of k (double counting may be partially corrected for by the interactions derived from the classical action but this has not been demonstated). Further the statistical weight factors that are absolutely crucial to regarding the system as a statistical mechanical gas can only be justified at all when the initial field configuration is well separated.

The whole approach has been questioned by Witten [14] arguing from the $1/N$ expansion that ν, for typical gauge fields in the functional integration measure, should not be restricted to integer values (implying presumably that dependence on θ need not be periodic of period 2π) rendering instantons irrelevant. Nevertheless his arguments apply most directly, in our view, to the dilute gas approximation. For the closely related two-dimensional CP^N model theory, where the large-N limit can actually be evaluated, the formal analysis of Jevicki [15] and the finite-temperature calculations of Affleck [16] (at finite temperature there is a natural cutoff on the instanton scale integrations, ensuring that the dilute-gas scheme is much better justified) suggest that there is no inherent conflict between the $1/N$ expansion and instanton methods. The fact that at large N we expect expansions in inverse powers of N may be reconciled with semiclassical instanton methods by supposing that the sum over all instanton configurations is nonanalytic, as in two-dimensional Coulomb gas which corresponds to the statistical mechanics of instantons in the two-dimensional $O(3)$ σ model [17].

To apply the semiclassical procedure systematically to gauge theories, allowing for dense configurations of instantons, needs a well-defined complete set of classical solutions $\{A_\mu^c\}$ about which to expand the functional integration measure. The only known solutions with finite action are the self- or anti-self-dual multi-instanton gauge fields for which we now have a general construction for any gauge group and arbitrary k [18]. The determinants of the operators that arise in the quadratic expansion of the action about these multi-instanton solutions, and also the Faddeev–Popov gauge-fixing determinant, have been extensively investigated [19], even if completely explicit results have not been obtained. Here we wish to report on the extent to which the functional integral can be reduced to finite-dimensional integrals over multi-instanton parameters; some results have been given elsewhere [20].

The immediate problem in realising this aim is that the ADHM construction [18], while seductively elegant and quite sufficient with the known Green functions to discuss multi-instanton determinants, does not provide an unconstrained parameterisation for the multi-instanton solutions with the full quota [21] of parameters, save for $|k| = 1, 2$, as exemplified in 't Hooft's multi-instanton solution in the extended form obtained by Jackiw *et al.* [22] (referred to henceforth as the HJNR solution), and with some complications for $k = 3$ [23]. We are thus restricted to giving the overall framework of the calculation for general k and explicit results for $k = 2$ and gauge group $SU(2)$.

It is to be hoped that general unconstrained parameterisations of multi-instantons exist which, if expressed in terms of the ADHM construction, may then be incorporated within our formalism. If such a scheme corresponds to the very attractive instanton quark suggestion of Belavin *et al.* [24], based on two-dimensional intuitions [17], then the $4Ck$ instanton parameters (C is an integer depending on the gauge group, $C = n$ for $SU(n)$; this count includes those corresponding to global gauge transformations) can be represented as $C \times k$ freely varying Euclidean position coordinates in the four-dimensional space, corresponding to C species of instanton quarks with multiplicity k. The symmetry group of the parameterisation is supose to be $(\mathscr{S}_k)^C$, leading to a statistical weight in the functional integral $1/(k!)^C$. The scale for an individual instanton depends on the magnitude of the relative dispersion of its constituent quarks. In the functional integral the large-scale divergences are now absorbed into the standard large-volume limit (as of course is the integral over instanton positions in the dilute-gas approach). Further two-dimensional models suggest that the statistical mechanical instanton quark system should be in a dense plasma phase with the typical instanton configurations strongly overlapping, but with the calculational framework still justified. With just a set $\{A_\mu^c\}$ consisting of pure instanton and anti-instanton configurations $\mathscr{E}(\theta)$, for instance, should then no longer involve divergent parameter integrations at the semiclassical level without any extraneous cutoffs.

Even if this goal can be achieved, it is still unclear to what extent this would give information about the nonperturbative structure of nonabelian gauge theories. The problem is that taking the starting point of the semiclassical expansion to be the restricted set $\{A_\mu^c\}$ of pure instanton and anti-instanton configurations above, even though they sample all topological sectors of gauge equivalence classes of gauge fields labelled by the Pontryagin index k, is almost certainly not adequate for a sensible field theory. In particular, the restriction seems likely to lead to the violation of cluster decomposition of correlation functions for local gauge invariant functions of the fields at widely separated points in Euclidean space [25]. This can be seen by considering the positive action density

$$\rho = -(1/16\pi^2)\,\mathrm{tr}(F_{\mu\nu}F_{\mu\nu}). \tag{1.4}$$

Phenomenologically $\langle\rho\rangle$ is nonzero [26] and this may even be realised in semiclassical calculations for suitably dense configurations of background fields, as in the dilute-gas approximation when $\langle \rho \rangle$ involves the usual divergent scale integration. However, to the extent that the gauge theory functional integral is dominated by self-dual or anti-self-dual field configurations, $q = \pm\rho$ and then, for example

$$\langle q(x)\, q(0) \rangle = \langle \rho(x)\, \rho(0) \rangle, \tag{1.5}$$

which would suggest that in this framework $\langle q(x)\, q(0) \rangle$ is nonzero for $|x| \to \infty$. This would imply that $\mathscr{E}(\theta)$ fails to exist in the infinite-volume limit and furthermore, since $\langle q \rangle = 0$, a lack of cluster decomposition. It might be possible to decompose the resultant theory into two irreducible phases with $\langle q \rangle = \pm \langle q \rangle$ but this would imply unacceptable parity violation and other undesirable consequences.

It remains unclear what additional field configurations, if the dilute instanton anti-instanton gas scheme is regarded as unsatisfactory, need to be included in the set $\{A^c_\mu\}$ to provide a valid semiclassical approximation of the functional integral, both in terms of topology and measure, and thus avoid results like (1.5). Some suggestions have been made [27] but they are difficult to evaluate.

Despite the various difficulties outlined above it still seems worthwhile to pursue multi-instanton calculations as far as possible. If a more general scheme exists they will certainly be part of it. In Section 2 we discuss the overall framework for semiclassical approximation of the functional integral for Euclidean gauge theories defined on a compact four-dimensional manifold to avoid infrared difficulties. Careful attention is paid to the noncompact flat space limit and the treatment of the gauge zero modes which then arise. The formalism is further specialised to the case of self-dual gauge fields and the reduction of the various determinants to that of the covariant Laplacian acting on scalars is briefly recounted. In conjunction with this in Appendix A we discuss the variation of the metric and show how the relationships derived formally on flat space can be checked by considering a very special instanton configuration on S^4. In Section 3 we briefly review the ADHM multi-instanton construction and consider the corresponding form for the set of normalisable gauge field zero modes in the background gauge. It is shown how the normalisation matrix of these zero modes, the determinant of which forms the Jacobian in the transformation of the functional integral to one over instanton parameters, can be evaluated. Corresponding to this section, in Appendix B we give a summary of results [19] on multi-instanton determinants. In Section 4, after briefly considering $k = 1$, we consider in detail the evaluation of the Jacobian for $k = 2$ as the determinant of a 16×16 matrix. This result is then specialised to the case of the HJNR parameterisation and the reduction of the parameter set from 16 to 14, required for the conformally extended HJNR solution, is discussed. In Appendix C the appropriate form of the matrices in the ADHM formalism corresponding to the $k = 2$ extended HJNR solution is obtained. The end result is as has been presented earlier [20]. In Section 5 it is shown how the measure and the determinants factorise for $k = 2$ into products of the associated single instanton measures and determinants for well-separated instantons. This is demonstrated, by carefully choosing appropriate variables for each individual instanton, to the extent that there are no corrections to factorisation even up to terms falling as the inverse square of the instanton separation. Finally, a few brief remarks on the relevance of the results are offered in the Conclusion.

2. Semiclassical Approximation for Gauge Theories

Initially we restrict our attention to gauge fields A_μ, for gauge group $\mathscr{G}$, defined over a compact four-dimensional manifold with positive definite metric $g_{\alpha\beta}$. In the fundamental representation the elements of $\mathscr{G}$ are given by matrices satisfying $g^\dagger g = 1_{\mathscr{G}}$. The gauge fields are then represented by antihermitian matrices acting on the fundamental representation vector space for the gauge group, $A_\mu = A_\mu^a t_a$. t_a, $a = 1 \cdots d_{\mathscr{G}}$, are a set of generators for the Lie algebra $\mathscr{L}_{\mathscr{G}}$ in this representation normalised by $\mathrm{tr}(t_a t_b) = -\frac{1}{2}\delta_{ab}$. The Euclidean gauge theory action is then

$$I[A] = -(1/2g^2) \int dv\, \mathrm{tr}(F_{\mu\nu} F^{\mu\nu}), \qquad dv = d^4x \sqrt{g},$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{2.1}$$

For any field strength $F_{\mu\nu}$ the corresponding dual is

$$*F^{\mu\nu} = \tfrac{1}{2}(1/\sqrt{g})\, \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \tag{2.2}$$

and the Pontryagin index is given by

$$k = -(1/16\pi^2) \int dv\, \mathrm{tr}(F_{\mu\nu} {*F^{\mu\nu}}). \tag{2.3}$$

This is therefore identical with ν, as introduced in Section 1, although we now consider only continuous gauge fields so that k is an integer. For self-dual fields $F^c_{\mu\nu} = {*F^c_{\mu\nu}}$ and $k > 0$ is interpreted as the instanton number, $I[A^c] = 8\pi^2 k/g^2$.

The Euclidean quantum fields theory is formally defined in terms of the functional integration measure

$$(1/Z_0)\, d[A]_{\mathrm{g.i.}}\, e^{-I[A]}, \tag{2.4}$$

for Z_0 a normalisation factor and $d[A]_{\mathrm{g.i.}}$ the standard Faddeev–Popov measure constructed heuristically on gauge equivalence classes $A_\mu \underset{g}{\sim} A_\mu^g$,

$$A_\mu^g = g^{-1} A_\mu g + g^{-1} \partial_\mu g \tag{2.5}$$

for $g(x) \in \mathscr{G}$. Thus if the gauge is fixed by a condition $F(A) = 0$ we have [28]

$$d[A]_{\mathrm{g.i.}} = d[A]\, G[A]\, \delta[F(A)], \tag{2.6}$$

where

$$1 = G[A] \int d[g]\, \delta[F(A^g)], \tag{2.7}$$

with $d[A]$ the translation invariant measure on $A_\mu(x)$ and $d[g]$ the invariant measure on $g(x)$, $d[g]=d[g\bar{g}]$ for any $\bar{g}(x)$. If A_μ satisfies the gauge condition, $F(A)=0$, and in a gauge equivalent neighbourhood $F(A^g)=O(A)\lambda$ for $g\to 1_{\mathscr{G}}+\lambda$ when $d[g]\to d[\lambda]$ then from (2.7)

$$G[A]=\det O(A), \tag{2.8}$$

neglecting any Gribov ambiguities arising to the extent that the gauge is not uniquely specified by $F(A^g)=0$.

So much is a rehearsal of standard lore, but is repeated in order that each step may now be made precise in the semiclassical context. We assume there is a background field A_μ^c for which the action (2.1) is stationary and consider approximating the measure (2.4) in its neighbourhood by letting

$$A_\mu=A_\mu^c+a_\mu \tag{2.9}$$

and keeping just the contribution from the lowest nontrivial order in an expansion in the fluctuation a_μ. It is convenient to introduce, besides the standard covariant derivative $\mathscr{D}_\mu$ in the adjoint representation

$$\mathscr{D}_\mu X=\partial_\mu X+[A_\mu^c,X], \tag{2.10}$$

a dual $\mathscr{D}^{*\mu}$ given by

$$\mathscr{D}^{*\mu}V_\mu=-(1/\sqrt{g})\,\mathscr{D}_\mu(g^{\mu\nu}\sqrt{g}\,V_\nu). \tag{2.11}$$

Then the action can be expanded as

$$I[A]\approx I[A^c]+(1/2g^2)(a,\Delta a), \tag{2.12}$$

where

$$\begin{aligned}(\Delta a)_\mu&=(\mathscr{D}_1^*(\mathscr{D}a))_\mu-[F_\mu^{c\nu},a_\nu],\\ (\mathscr{D}_1^*T)_\mu&=-(1/\sqrt{g})\,g_{\mu\rho}\mathscr{D}_\sigma(g^{\sigma\alpha}g^{\rho\beta}\sqrt{g}\,T_{\alpha\beta}),\qquad T_{\alpha\beta}=-T_{\beta\alpha},\\ (\mathscr{D}a)_{\alpha\beta}&=\mathscr{D}_\alpha a_\beta-\mathscr{D}_\beta a_\alpha;\end{aligned} \tag{2.13}$$

and the scalar product on the fluctuation vector fields a_μ is

$$(a,a')=-2\int dv g^{\mu\nu}\,\mathrm{tr}(a_\mu a'_\nu). \tag{2.14}$$

For infinitesimal gauge transformations $A_\mu^{cg}\to A_\mu^c+\mathscr{D}_\mu\lambda$ for $g\to 1_{\mathscr{G}}+\lambda$ giving $a_\mu^g=\mathscr{D}_\mu\lambda$. Since the initial action is gauge invariant then manifestly Δ must annihilate such gauge fluctuations, which is achieved by

$$\Delta\mathscr{D}=0,\qquad \mathscr{D}^*\Delta=0. \tag{2.15}$$

To define the measure we may translate $d[A]$ in (2.6) to $d[a]$, where in terms of a complete orthonormal set of vector fields $\{a_n\}$ in $\mathscr{L}_{\mathscr{G}}$, $(a_n, a_m) = \delta_{nm}$

$$d[a] = \prod_n (\mu/(2\pi)^{1/2} g)\, d\alpha_n, \qquad a_\mu = \sum_n \alpha_n a_{n\mu}. \tag{2.16}$$

Here μ is an arbitrary scale mass present in the functional measure reflecting the cutoff scale arising from the regularisation of ultraviolet divergences in perturbative expansions. The natural gauge choice in the semiclassical context is to take the fluctuation field a_μ orthogonal to infinitesimal gauge transformations or $\mathscr{D}^*a = 0$. In this background gauge $F(A) = \mathscr{D}^*a$ and $O(A) \approx \Delta^0 = \mathscr{D}^*\mathscr{D}$. This specifies the gauge uniquely in the neighbourhood of A^c_μ if Δ^0 is invertible which requires there are no covariantly constant fields σ such that $\mathscr{D}_\mu \sigma = 0$. However, if the background field A^c_μ has a nontrivial stability group $\mathscr{H}$

$$h^{-1} A^c_\mu h = A^c_\mu, \qquad h \in \mathscr{H} \subset \mathscr{G}, \tag{2.17}$$

and if $h \to 1_{\mathscr{G}} + \sigma$ then $[A^c_\mu, \sigma] = 0$ so that $\sigma \in \mathscr{L}_{\mathscr{H}}$ is both constant and covariantly constant. We assume that the overall gauge for A^c_μ is chosen so that $\mathscr{H}$ is maximal, independent of x, and all covariantly constant scalars are of this form. The background gauge then does not determine the gauge up to $g(x) \to hg(x)$ for $h \in \mathscr{H}$ but this can be easily coped with by limiting the condition $\mathscr{D}^*a = 0$ to apply only to scalar fields in $\mathscr{L}_{\mathscr{G}}$ orthogonal to constant elements of $\mathscr{L}_{\mathscr{H}}$. Thus if a suitable orthonormal set of scalar fields in $\mathscr{L}_{\mathscr{G}}$, $\{\omega_n\}$, is chosen, $(\omega_n, \omega_m) = -2 \int dv\, \mathrm{tr}(\omega_n \omega_m) = \delta_{nm}$, we may identify

$$\omega_i = (1/V^{1/2})\, t_i, \qquad \mathrm{tr}(t_i t_j) = -\tfrac{1}{2}\delta_{ij}, \qquad V = \int dv, \tag{2.18}$$

for t_i, $i = 1 \cdots d_{\mathscr{H}}$, a basis of $\mathscr{L}_{\mathscr{H}}$ and the gauge-fixing delta functional is taken to be $\delta'[\mathscr{D}^*a]$, where

$$\delta'[\omega] = \prod_{n \neq i} (2\pi)^{1/2} g \delta(\beta_n), \qquad \omega = \sum_n \beta_n \omega_n. \tag{2.19}$$

The gauge invariant measure (2.6) involves from (2.7) evaluation of

$$1 = G[A^c] \int d[g]\, \delta'[\mathscr{D}^*a^g], \qquad A^g_\mu = A^c_\mu + a^g_\mu. \tag{2.20}$$

The measure $d[g]$ can be specified by requiring for $g(x) \to 1_{\mathscr{G}} + \lambda(x)$, $d[g] \to d[\lambda]$, where

$$d[\lambda] = \prod_n (\mu^2/(2\pi)^{1/2} g)\, d\lambda_n, \qquad \lambda = \sum_n \lambda_n \omega_n. \tag{2.21}$$

To evaluate (2.20) it is first necessary to decompose $d[g]$ according to

$$d[g] = K\, d\mu_{\mathscr{H}}(h)\, d'[g], \tag{2.22}$$

where $d'[g]$ is the measure on equivalence classes $g'(x) \sim hg(x)$, $h \in \mathscr{H}$, and $d\mu_{\mathscr{H}}$ is the $d_{\mathscr{H}}$ dimensional measure on $\mathscr{H}$. For $g \to 1_{\mathscr{G}} + \lambda$, $d'[g] \to d'[\lambda]$, defined as in (2.21) but restricted to a measure over $\{\lambda_n\}$, $n \neq i$. In (2.22) K is fixed by the normalisation chosen for $d\mu_{\mathscr{H}}$. If conveniently for $h \to 1_{\mathscr{G}} + \sigma^i t_i$, $d\mu_{\mathscr{H}}(h) \to \prod_i d\sigma^i$ then from (2.18) and (2.21) $K = (\mu^2 V^{1/2}/(2\pi)^{1/2} g)^{d_{\mathscr{H}}}$. Hence inserting (2.22) in (2.20) and as $\mathscr{D}^* a^g \to \Delta^0 \lambda$ for $g \to 1_{\mathscr{G}} + \lambda$, identifying $\{\omega_n\}$ with the eigenfunctions of $\Delta^0 = \mathscr{D}^*\mathscr{D}$, we find, formally but without any difficulty from vanishing eigenvalues,

$$G[A^c] = \left(\frac{(2\pi)^{1/2} g}{\mu^2 V^{1/2}}\right)^{d_{\mathscr{H}}} \frac{1}{V_{\mathscr{H}}} \det{}' \frac{\Delta^0}{\mu^2}, \tag{2.23}$$

where $V_{\mathscr{H}} = \int d\mu_{\mathscr{H}}(h)$ the volume of $\mathscr{H}$ (for $SU(2)$ in our conventions $V_{SU(2)} = 16\pi^2$) and det′ denotes the restriction to nonzero eigenvalues.

The gauge-fixing delta functional in $d[A]_{g.i.}$ may now be eliminated if the fluctuation field a_μ is decomposed accoridng to

$$a_\mu = a_\mu^t + a_\mu^l, \qquad \mathscr{D}^* a^t = 0, \qquad a^l = \mathscr{D}\omega \tag{2.24}$$

which is unique for ω orthogonal to any covariantly constant field σ. Thus in (2.16) $d[a] = d[a^t]\, d[a^l]$ and for ω expanded as in (2.19) in terms of the eigenfunctions of Δ^0, $\Delta^0 \omega_n = \lambda_n \omega_n$, we have $a_{n\mu}^l = \mathscr{D}_\mu \omega_n \lambda_n^{-1/2}$, $\alpha_n^l = \beta_n / \lambda_n^{1/2}$, $n \neq i$. Hence the gauge fluctuations a_μ^l can be easily integrated out

$$\int d[a^l]\, \delta'[\mathscr{D}^* a^l] = [\det{}'(\Delta^0/\mu^2)]^{-1/2} \tag{2.25}$$

(the factors in (2.19) and (2.21) were chosen to achieve this result with (2.23)). The approximation to the measure deriving from (2.4), (2.6), (2.12), (2.23), and (2.25) now becomes

$$\frac{1}{Z_0} e^{-I[A^c]} \left(\frac{(2\pi)^{1/2} g}{\mu^2 V^{1/2}}\right)^{d_{\mathscr{H}}} \frac{1}{V_{\mathscr{H}}} d[a^t]\, e^{-1/2(a^t, \Delta a^t)} \left(\det{}' \frac{\Delta^0}{\mu^2}\right)^{1/2}. \tag{2.26}$$

To eliminate the functional integration over a_μ^t we suppose that the orthonormal set $\{a_\mu^t\}$ in (2.16) are now eigenfunctions of Δ. We assume there are N independent zero modes solving $\Delta Z = 0$ which arise from infinitesimal variation of the parameters $\{\gamma_r\}$ of the initial stationary point of the action A^c,

$$\frac{\partial A_\mu^c}{\partial \gamma_r} = Z_{r\mu} + \mathscr{D}_\mu \Lambda_r, \qquad \mathscr{D}^* Z_r = 0, \qquad r = 1,\ldots, N. \tag{2.27}$$

The integration over the corresponding expansion parameters may be converted to an integration over the parameters $\{\gamma_r\}$ by a Jacobian factor

$$\prod_1^N d\alpha_r^t = (1/S) \prod_1^N d\gamma_r (\det J)^{1/2}, \qquad J_{rs} = (Z_r, Z_s), \tag{2.28}$$

where S denotes an appropriate statistical weight factor corresponding to the number of times each gauge equivalence class $A^c_\mu(\gamma) \underset{g}{\sim} A^c_\mu(\gamma')^g$ is obtained as the parameters $\{\gamma_r\}$ vary over their natural range given by the particular parameterisation of the solution $A^c_\mu(\gamma)$. Since the action is stationary at $A^c_\mu(\gamma)$, for any γ_r, $I[A^c]$ will be independent of γ_r. For $a'^{\,t}_\mu$ orthogonal to these zero modes we have

$$\int d[a'^{t}]\, e^{-(a'^{t},\Delta a'^{t})/2} = \left(\det{}' \frac{\Delta}{\mu^2}\right)^{-1/2} \tag{2.29}$$

at least if the remaining eigenvalues are positive, corresponding to A^c_μ being a minimum of the action, as for self-dual configurations. The appropriate treatment for a finite number of negative eigenvalues is unclear, possibly requiring analytic continuation of the functional integration. In the absence of such problems the functional integral in the neighbourhood of $A^c_\mu(\gamma)$ is thus reduced to a finite dimensional integral over the parameters γ_r. It is convenient now to choose the normalisation scale Z_0 so that for trivial vacuum $A^c_\mu = 0$, for which $\mathscr{H} = \mathscr{G}$, the measure is 1. Hence from (2.26), (2.28) and (2.29) the result is

$$\begin{aligned} & e^{-I[A^c]}(\mu V^{1/2})^{d_\mathscr{G} - d_\mathscr{H}} \frac{V_\mathscr{G}}{V_\mathscr{H}} \left(\frac{\mu}{(2\pi)^{1/2} g}\right)^{N + d_\mathscr{G} - d_\mathscr{H}} \frac{1}{S} \prod_1^N d\gamma_r \\ & \times (\det J)^{1/2} \left[\frac{\det'(\Delta^0/\mu^2)}{\det'(\Delta^0_0/\mu^2)}\right]^{1/2} \left[\frac{\det'(\Delta/\mu^2)}{\det(\Delta_0/\mu^2)}\right]^{-1/2}, \end{aligned} \tag{2.30}$$

where Δ^0_0, Δ_0 denote corresponding operators to Δ^0, Δ acting on scalars, vectors for $A^c_\mu = 0$. An expression akin (2.30) has been obtained by different methods by Schwarz [29].

Our primary interest is to apply (2.30) to flat space but the presence of an explicit dependence on the volume V in (2.30) indicates some care has to be taken about the limit $g_{\mu\nu} \to \delta_{\mu\nu}$. On flat space there exist $d_\mathscr{G} - d_\mathscr{H}$ normalisable vector zero modes $Z^g_{k\mu}$ which are pure gauges but satisfy the background gauge condition, in addition to the N zero modes $Z_{r\mu}$ resulting from variation of parameters as in (2.27). Thus we have

$$Z^g_{k\mu} = \mathscr{D}_\mu \psi_k, \qquad \mathscr{D}^2 \psi_k = 0, \qquad k = 1,\dots, d_\mathscr{G} - d_\mathscr{H}. \tag{2.31}$$

These gauge zero modes are related to global gauge rotations of A^c_μ. If we parameterise $g \in \mathscr{G}$ according to $g(\tau,\sigma) = hg(\tau,\sigma')$ for all $h \in \mathscr{H}$ so $\{\tau_k\}$ labels elements of $\mathscr{G}/\mathscr{H}$ then

$$\begin{aligned} & \frac{\partial}{\partial \tau_k}(g^{-1} A^c_\mu g) = g^{-1} \mathscr{D}_\mu \omega_k g = g^{-1}(\mathscr{D}_\mu \psi_k + \mathscr{D}_\mu \Lambda_k) g, \\ & \frac{\partial}{\partial \sigma_i}(g^{-1} A^c_\mu g) = 0, \end{aligned} \tag{2.32}$$

where if, for $|x| \to \infty$, $A^c_\mu(x) \to g(\hat{x})^{-1} \partial_\mu g(x)$ ($g(\hat{x})$ may be chosen so that $g(\hat{x})\, h = hg(\hat{x})$ for all $h \in \mathscr{H}$) the gauge zero modes ψ_k are determined by

$$\psi_k(x) \to g(\hat{x})^{-1} \omega_k\, g(\hat{x}), \qquad \omega_k = \frac{\partial g}{\partial \tau_k} g^{-1} \in \mathscr{L}_{\mathscr{G}} \diagdown \mathscr{L}_{\mathscr{H}} \tag{2.33}$$

for an appropriate A_k in (2.32).

These scalar fields ψ_k can be exhibited directly in terms of the Green function for $-\mathscr{D}^2$ on flat space. In the basis for $\mathscr{L}_{\mathscr{G}}$ given by $\{t_a\}$ this is $G_{ab}(x, y)$, where

$$-\mathscr{D}^2_{ac} G_{cb}(x, y) = \delta_{ab} \delta^4(x - y) \tag{2.34}$$

and it may be expressed as

$$G_{ab}(x, y) = \frac{U_{ab}(x, y)}{4\pi^2 |x - y|^2}. \tag{2.35}$$

For $\{t_i\}$ forming a basis of $\mathscr{L}_{\mathscr{H}}$ then, since $[A^c_\mu, t_i] = 0$, we have $U_{ia}(x, y) = \delta_{ia}$ and for $|y| \to \infty$ (2.34) requires

$$U_{ab}(x, y) \to -2\, \mathrm{tr}(t_a u(x)^\dagger g(\hat{y})\, t_b\, g(\hat{y})^{-1} u(x)), \tag{2.36}$$

with $u(x)$ acting on the fundamental representation space of $\mathscr{G}$. Hence from (2.34), (2.35) and (2.36) the solutions of $-\mathscr{D}^2 \psi = 0$ are given by

$$\begin{aligned} \psi(x) &= u(x)^\dagger\, \omega u(x), \qquad && \omega \in \mathscr{L}_{\mathscr{G}}, \\ &= \omega, && \omega \in \mathscr{L}_{\mathscr{H}}. \end{aligned} \tag{2.37}$$

The asymptotic form of the solution (2.37) corresponds exactly to (2.33) since $u(x) \to g(\hat{x})$ for $|x| \to \infty$ and clearly there are $d_{\mathscr{G}} - d_{\mathscr{H}}$ such independent $\mathscr{D}_\mu \psi$.

The effect of these normaliseable gauge vector modes is that Δ^0 has $d_{\mathscr{G}} - d_{\mathscr{H}}$ eigenvalues which are proportional to $1/V$ as $V \to \infty$. This may be seen by diagonalising Δ^0 in the space spanned by the $\{\psi_k\}$. Defining

$$\begin{gathered} I_{kl} = (\psi_k, \Delta^0 \psi_l) = (Z^g_k, Z^g_l), \\ (\psi_k, \psi_l) \xrightarrow[V \to \infty]{} V b_{kl}, \qquad b_{kl} = -2\, \mathrm{tr}(\omega_k \omega_l), \end{gathered} \tag{2.38}$$

using the asymptotics (2.33), the relevant eigenvalues are then approximately given by solving $\det(I - \lambda V b) = 0$. These eigenvalues become exact to leading order in $1/V$ as $V \to \infty$ and as I remains finite in the flat-space limit then necessarily the solutions $\lambda_m \sim 1/V$. The eigenvalues $\{\lambda_m\}$ disappear from the spectrum of Δ^0 as the discrete spectrum for a compact space becomes continuous for $V \to \infty$ so we suppose

$$\frac{\det'(\Delta^0/\mu^2)}{\det'(\Delta^0_0/\mu^2)} \to \frac{\det I}{\det b} (\mu^2 V)^{-d_{\mathscr{G}} + d_{\mathscr{H}}} \frac{\det(-\mathscr{D}^2/\mu^2)}{\det(-\mathscr{D}^2_0/\mu^2)} \tag{2.39}$$

in the flat-space limit. This argument is clearly far from complete, although a similar result is obtained by 't Hooft [7]. On the right-hand side of (2.39) $\det -\mathscr{D}^2$ does not exist, at least with zeta function regularisation, on flat space due to infrared divergences (the ultraviolet divergences are straightforwardly regularised) but the ratio to the corresponding determinant of $-\mathscr{D}_0^2 = -\partial^2 1_{\mathscr{G}}$ should be finite (the infrared divergences are independent of A^c_μ). Further discussion is given in Appendix A, but if (2.39) is inserted in (2.30) the volume dependence is cancelled and the flat-space limit with $V \to \infty$ can be directly taken. With ω_k given in (2.33) we also have

$$\int \prod_1^{d_{\mathscr{G}}-d_{\mathscr{H}}} d\tau_k (\det b)^{1/2} = V_{\mathscr{G}}/V_{\mathscr{H}} \tag{2.40}$$

since this is just the integral over the group invariant measure on the coset space $\mathscr{G}/\mathscr{H}$. Hence adjoining the additional variables $\{\tau_k\}$ corresponding to integrations over global gauge transformations we obtain from (2.30) on flat space

$$\begin{aligned} & e^{-I[A^c]} \left(\frac{\mu}{(2\pi)^{1/2} g}\right)^{N+d_{\mathscr{G}}-d_{\mathscr{H}}} \frac{1}{S} \prod_1^{d_{\mathscr{G}}-d_{\mathscr{H}}} d\tau_k \prod_1^N d\gamma_r (\det I \det J)^{1/2} \\ & \times \left[\frac{\det(-\mathscr{D}^2/\mu^2)}{\det(-\mathscr{D}_0^2/\mu^2)}\right]^{1/2} \left[\frac{\det'(\Delta/\mu^2)}{\det'(\Delta_0/\mu^2)}\right]^{1/2}, \end{aligned} \tag{2.41}$$

for

$$\begin{gathered} \Delta_{\mu\nu} = -\delta_{\mu\nu}\mathscr{D}^2 + \mathscr{D}_\nu \mathscr{D}_\mu - F^{ad}_{\mu\nu}, \\ (X^{ad} Y = [X, Y]) \end{gathered} \tag{2.42}$$

acting on vectors satisfying $\mathscr{D}_\mu a_\mu = 0$. In general there is no necessity for separating the gauge parameters $\{\tau_k\}$ from the physical parameters $\{\gamma_r\}$ If we consider a transformation to $\alpha_a(\gamma, \tau)$, $a = 1 \cdots N + d_{\mathscr{G}} - d_{\mathscr{H}}$ then since $(Z^g_k, Z_r) = 0$

$$\prod_1^{d_{\mathscr{G}}-d_{\mathscr{H}}} d\tau_k \prod_1^N d\gamma_r (\det I \det J)^{1/2} \to \prod_1^{N+d_{\mathscr{G}}-d_{\mathscr{H}}} d\alpha_a (\det \bar{J})^{1/2}, \tag{2.43}$$

for $g(\sigma,\tau)^{-1} A^c_\mu(\gamma)\, g(\sigma,\tau) \underset{\bar{g}}{\sim} \bar{A}^c_\mu(\alpha)$ and where

$$\begin{gathered} \bar{J}_{ab} = (\bar{Z}_a, \bar{Z}_b), \\ \frac{\partial \bar{A}^c_\mu}{\partial \alpha_a} = \bar{Z}_{a\mu} + \bar{\mathscr{D}}_\mu \bar{\Lambda}_{a\mu}, \qquad \bar{\mathscr{D}}_\mu \bar{Z}_{a\mu} = 0. \end{gathered} \tag{2.44}$$

$\bar{\Lambda}_a$ must be restricted so that the global gauge transformations are realised in the $\{Z_{a\mu}\}$. If $\bar{g}(x)$ above is chosen nonsingular and asymptotically $\bar{g}(x) \to g(\sigma,\tau)^{-1} g(\hat{x})^{-1} g(\sigma,\tau)\, g(\hat{x})$ for $|x| \to \infty$, this is achieved if $\bar{\Lambda}_a(x) \to 0$. The replacement (2.43) also assumes that the mapping $\{\gamma_r, \tau_k\} \to \{\alpha_a\}$ is one–one, otherwise the weight factor S should be correspondingly changed.

The result (2.41), with (2.43), is now in the form that would be obtained by a collective coordinate approach on flat space *ab initio* but with hopefully here a better justification of the treatment of the gauge modes.

Further simplification can be realised if A^c_μ corresponds to a self-dual ($F^c_{\mu\nu} = {}^*F^c_{\mu\nu}$) multi-instanton field as assumed henceforth. From the standard analysis [21] of the number of parameters specifying a general multi-instanton solution

$$N + d_{\mathscr{G}} - d_{\mathscr{F}} = 4C_2(\mathscr{G})\, k \tag{2.45}$$

for $C_2(\mathscr{G})$ the quadratic Casimir of $\mathscr{G}$ ($C_2(SU(n)) = n$, $C_2(Sp(n)) = n + 1$, $C_2(O(n)) = n - 2$, $n \geqslant 5$).

The spectrum of Δ can also be related for $F^c_{\mu\nu} = {}^*F^c_{\mu\nu}$ to that of $-\mathscr{D}^2$ on flat space [30] since if $T_{a\mu} = \bar{\eta}_{a\mu\nu}\mathscr{D}_\nu$, $a = 1, 2, 3$ and $\bar{\eta}$ is 't Hooft's anti-self-dual symbol [7], then with Δ given in (2.42) and the properties of $\bar{\eta}$,

$$T_{a\mu} T_{a\nu} = -\Delta_{\mu\nu}, \qquad T_{a\mu} T_{b\mu} = \delta_{ab}\mathscr{D}^2. \tag{2.46}$$

Hence for any eigenfunction ϕ of $-\mathscr{D}^2$, with nonzero eigenvalue λ, we can obtain three independent eigenfunctions $T_{a\nu}\phi$ (satisfying $\mathscr{D}_\nu T_{a\nu}\phi = 0$) for $\Delta_{\mu\nu}$ with the same eigenvalue λ, and also conversely for an eigenfunction a_μ of $\Delta_{\mu\nu}$ there is an eigenfunction $T_{a\nu}a_\nu$ of $-\mathscr{D}^2$. A proper eigenvalue problem in this flat-space case, however, requires restriction to a finite domain D of R^4 with some definite boundary conditions on the boundary ∂D so that the resultant $-\mathscr{D}^2_D$ and Δ_D are self-adjoint operators. To preserve the equality of eigenvalues, the boundary condition must respect the relation between the eigenfunctions. A possible choice is $\phi = 0$ on ∂D and hence $T_{a\nu}a_\nu = 0$ or $(\mathscr{D}a)_{\alpha\beta} = {}^*(\mathscr{D}a)_{\alpha\beta}$ on ∂D, which also ensures the zero modes of Δ_D are the same as those of Δ. Thus we would obtain for the associated zeta function regularised determinants

$$\det{}'\Delta_D = (\det - \mathscr{D}^2_D)^3. \tag{2.47}$$

This result is not valid when the radius of the boundary surface $\partial D \sim S^3$ becomes infinite bcause of the infrared divergences mentioned earlier. The corresponding relation between the determinants defined on a compact space such as S^4 of course fails. For our purposes we nevertheless assume

$$\det{}'\Delta/\det{}'\Delta_0 = (\det - \mathscr{D}^2/\det - \mathscr{D}^2_0)^3, \tag{2.48}$$

which would follow from (2.47) and is also valid in the infinite volume limit on flat space since the infrared divergences are cancelled on both sides. In addition, (2.48) is supposed to be true for the determinants defined on S^4 as the radius $a \to \infty$. Further consideration of this result is given in Appendix A. The boundary conditions described above to achieve (2.47) are connected to those necessary to preserve supersymmetry [31] and may be related to spectral boundary conditions [32].

With (2.47) the functional measure in the neighbourhood of a self-dual field A^c_μ of instanton number k becomes

$$e^{-8\pi^2 k/g^2}\left(\frac{\mu}{(2\pi)^{1/2}g}\right)^{4C_2(\mathscr{G})k}\frac{1}{S}\prod_1^{4C_2(\mathscr{G})k} d\alpha_a(\det \bar{J})^{1/2}\left[\frac{\det(-\mathscr{D}_0^2/\mu^2)}{\det(-\mathscr{D}^2/\mu^2)}\right], \tag{2.48}$$

where the statistical weight S_k of course depends on k.

The dependence on the scale mass μ is easily isolated since [19]

$$\frac{\det(-\mathscr{D}_0^2/\mu^2)}{\det(-\mathscr{D}^2/\mu^2)} \propto e^{-C_2(\mathscr{G})k \ln \mu^2/6}, \tag{2.49}$$

and the arbitrariness in μ can be removed in favour of the invariant renormalisation group scale Λ by writing

$$e^{-8\pi^2/g^2}\mu^{11C_2(\mathscr{G})/3} = \Lambda^{11C_2(\mathscr{G})/3} \tag{2.50}$$

in accord with expectation from the single-loop β function.

3. Zero Modes for General Multi-Instanton Fields

A necessary part of the semiclassical approximation to the functional integral discussed in the previous section is the Jacobian formed by the determinant of the normalisation matrix of the zero modes corresponding to a variation of the parameters of the minima of the action A^c_μ. For general self-dual multi-instanton gauge fields these can be exhibited in the context of the formalism for the ADHM construction [18], which we briefly recapitulate.

The general self-dual gauge field is expressed as [33, 23]

$$A^c_\mu = v^\dagger \partial_\mu v, \qquad v^\dagger v = 1_{\mathscr{G}}, \tag{3.1}$$

where $v(x)^\dagger$ is a matrix acting from a space V to the fundamental representation space of the gauge group $\mathscr{G}$. $v^\dagger$ is determined by equations

$$v^\dagger \Delta = 0, \qquad \Delta(x) = a + bx, \tag{3.2}$$

for a, b matrices acting from $W \times S$ to V with W k dimensional (or $2k$ dimensional for $\mathscr{G} = O(n)$ regarding quaternions as two dimensional) and S, a two-dimensional quaternionic space on which the Euclidean space coordinate, encoded as a quaternion, $x = x_\alpha e_\alpha$, acts (e_α denotes a quaternion basis, $\bar{e}_\alpha$ its conjugate). The basic constraint of the ADHM construction is

$$\Delta^\dagger \Delta = f^{-1} 1_2 \tag{3.3}$$

with f: $W^* \to W$ nonsingular and 1_2 the identity on S. With f a natural projection operator P acting on V and projecting out a $N_{\mathscr{G}}(=\operatorname{tr}(1_{\mathscr{G}}))$ dimensional space can be constructed,

$$P = 1_V - \Delta f \Delta^\dagger = vv^\dagger. \tag{3.4}$$

The constaint (3.3) entails in terms of a and b

$$a^\dagger a = \mu 1_2, \qquad b^\dagger b = \nu 1_2, \qquad (a^\dagger b)^T = a^\dagger b, \tag{3.5}$$

where for matrices X: $W \times S \to W \times S$

$$---X^{T}--- \;=\; ---\epsilon X^{T} \epsilon--- \tag{3.6}$$

with dashed lines representing quaternionic indices and ε the two-dimensional antisymmetric symbol on S. Useful relations are

$$(X^T)^T = X, \qquad X + (X^\dagger)^T = 1_2 \operatorname{tr}_2(X), \qquad e_\alpha^T = e_\alpha.$$

For $\mathscr{G} = Sp(n)$ when a, b can be regarded as matrices of quaternions the operation T when applied to matrices acting on $W \times S \to W \times S$ is equivalent to taking the transpose of the matrix of quaternions.

With this formulation of the ADHM construction it is easy to verify that $F^c_{\mu\nu} = {}^*F^c_{\mu\nu}$ using

$$e_\alpha \bar{e}_\beta = \delta_{\alpha\beta} + \eta_{\alpha\beta}, \qquad \eta_{\alpha\beta} = {}^*\eta_{\alpha\beta} \tag{3.7}$$

and also that k is the instanton number or Pontryagin index. A deeper result |18| is that there is a unique correspondence between gauge equivalence classes of self-dual gauge fields $A^c_\mu \sim A^{cg}_\mu$ and the matrices a, b, satisfying (3.5), up to a transformation

$$a \to QaR, \qquad b \to QbR \tag{3.8}$$

for $Q^\dagger Q = 1_V$, $Q \in \mathscr{G}_V$ and R belonging to the general linear group $\mathscr{G}_W$ acting on W. Manifestly the transformation (3.8) preserves the constraints (3.5). There is also a stability group $\mathscr{H}_V \subset \mathscr{G}_V$ defined by those elements $Q \in \mathscr{G}_V$ such that

$$a = QaR, \qquad b = QbR \tag{3.9}$$

for some appropriate $R \in \mathscr{H}_W \subset \mathscr{G}_W$.

It is useful to identity the subgroup $\mathscr{G}_{V_\infty} \subset \mathscr{G}_V$, which is projected out by $P_\infty = \lim_{|x| \to \infty} P = 1_V = bv^{-1}b^\dagger$ generated on $P_\infty V$ by those elements of $\mathscr{G}_V$ satisfying

$$\begin{aligned} &Q = qP_\infty + Q_\perp, \qquad P_\infty Q_\perp = Q_\perp P_\infty = 0, \\ &qP_\infty = P_\infty q, \quad q^\dagger q P_\infty = P_\infty, \qquad q \in \mathscr{G}_{V_\infty}. \end{aligned} \tag{3.10}$$

Since for $|x| \to \infty$ $v(x)$ $(= P_\infty v(x))$ corresponds to the element $g(\hat{x})$ of $\mathscr{G}$ defined by

TABLE I

Spaces for the ADHM Construction Corresponding to Differing Gauge Groups and the Respective Groups Acting on Them

$\mathscr{G}$	$Sp(n)$	$U(n)$	$O(n), \quad n \geqslant 5$
V	H^{k+n}	C^{2k+n}	R^{4k+n}
W	R^k	C^k	H^k
$\mathscr{G}_V$	$Sp(k+n)$	$U(2k+n)$	$O(4k+n)$
$\mathscr{G}_W$	$Gl(k, R)$	$Gl(k, C)$	$Gl(k, H)$
$\mathscr{H} \simeq \mathscr{H}_{V_\infty}$	$Sp(n-k), \quad n > k$	$U(1), \quad n \leqslant 2k$ $U(n-2k) \times U(1), n > 2k$	$O(n-4k), \quad n > 4k$ $O(n-4) \times Sp(1), k = 1$
$\mathscr{G}_{W_0}$	$O(k)$	$U(k)$	$Sp(k)$

Note. H denotes quaternions. The case of generic a, b only is considered.

$A^c_\mu \sim g(\hat{x})^{-1}\,\partial_\mu\, g(\hat{x})$, it is clear that $\mathscr{G}_{V_\infty}$ is isomorphic to $\mathscr{G}$. Furthermore the stability group $\mathscr{H} \subset \mathscr{G}$ of A^c_μ, defined in (2.19), is similarly isomorphic with the correspondingly defined, as in (3.10), $\mathscr{H}_{V_\infty} \subset \mathscr{H}_V$. The detailed nature of the spaces V, W and the groups acting on them $\mathscr{G}_V$, $\mathscr{G}_W$, $\mathscr{H}$ and also $\mathscr{G}_{W_0}$, to be defined later, for differing gauge groups is specified in Table I. With this information it is straightforward [23, 33, 34] to count the number of parameters specifying a, b subject to the constraints (3.5) modulo transformations (3.8) giving a complete set [21] of parameters for the consequential self-dual A^c_μ modulo gauge transformations for every gauge group $\mathscr{G}$ and general k.

The zero modes, corresponding to a self-dual field A^c_μ, in the background gauge for flat space satisfy

$$\mathscr{D}_{[\mu} Z_{\nu]} = {}^*\mathscr{D}_{[\mu} Z_{\nu]}, \qquad \mathscr{D}_\mu Z_\mu = 0, \tag{3.11}$$

with $\mathscr{D}_\mu = \partial_\mu + A^{\text{cad}}_\mu$. In the ADHM formalism $Z_\mu \in \mathscr{L}_{\mathscr{G}}$ can be constructed as [35]

$$Z_\mu = v^\dagger C \bar{e}_\mu f b^\dagger v - v^\dagger b f e_\mu C^\dagger v, \tag{3.12}$$

with $C\colon W \times S \to V$ satisfying linear conditions to ensure (3.11). To determine these it is convenient to first introduce

$$a_\mu = v^\dagger b f e_\mu \tag{3.13}$$

and obtain, with $D_\mu = \partial_\mu + A_\mu$,

$$D_\mu a_\nu = v^\dagger b f c_\alpha f (e_\mu \bar{e}_\nu e_\alpha - 2\delta_{\alpha\mu} e_\nu - 2\delta_{\alpha\nu} e_\mu), \qquad \Delta^\dagger b = c_\alpha \bar{e}_\alpha$$

so that, using (3.7)

$$D_{[\mu}a_{\nu]} = {}^*D_{[\mu}a_{\nu]}, \qquad D_\mu a_\mu = 0, \tag{3.14}$$

demonstrating that a_μ is a vector zero mode in the fundamental representation. Using now $D_\mu v^\dagger = -v^\dagger b e_\mu f \Delta^\dagger$ in conjunction with (3.13), (3.14) then the conditions for Z_μ in (3.12), with C independent of x, to obey (3.11) can be expressed as

$$\begin{aligned} &K_{[\mu\nu]} = {}^*K_{[\mu\nu]}, \qquad K_{\mu\mu} = 0, \\ &K_{\mu\nu} = e_\mu \Delta^\dagger C \bar{e}_\nu - e_\nu C^\dagger \Delta \bar{e}_\mu . \end{aligned} \tag{3.15}$$

It is then not difficult to see that $\Delta^\dagger C = (\Delta^\dagger C)^T$ is sufficient, and also necessary, for (3.15) to hold, using the properties of the operation T. Since $\Delta(x) = a + bx$ with x arbitrary the basic conditions on C are

$$a^\dagger C = (a^\dagger C)^T, \qquad b^\dagger C = (b^\dagger C)^T. \tag{3.16a, b}$$

For each gauge group $\mathscr{G}$ there are $4C_2(\mathscr{G})\,k$ independent such C_r (up to linear combinations with real coefficients) providing a basis $\{Z_{r\mu}\}$ of zero modes (for $Sp(n)$ C has $(k+n)\,k$ quaternion elements, (3.16) provides $4k(k-1)$ conditions, for $U(n)$ C has $2(2k+n)\,k$ complex elements, (3.16) gives $8k^2$ conditions and for $O(n)$ C has $(4k+n)\,k$ quaternion elements while (3.16) gives $8(2k^2+k)$ conditions).

With the explicit form for the zero modes provided by (3.12) and (3.16) the normalisation matrix can be determined using the result [36]

$$-\mathrm{tr}(Z_{r\mu} Z_{s\mu}) = -\tfrac{1}{2}\partial^2\, \mathrm{tr}(C_r^\dagger P C_s f + f C_r^\dagger C_s). \tag{3.17}$$

This can be verified directly, using conditions (3.16) for C_r, C_s, but most conveniently can be obtained by virtue of the formalism for tensor products [35, 37] from the corresponding result for the zero modes in the fundamental representation [19], $a_\mu^\dagger a_\mu = -\tfrac{1}{2}\partial^2 f \times 1_2$. From (3.17) the flat-space version of the scalar product (2.14) for these zero modes can be evaluated as

$$\begin{aligned} &(Z_r, Z_s) = 4\pi^2 \langle C_r, C_s \rangle, \\ &\langle C_r, C_s \rangle = \mathrm{tr}(C_r^\dagger P_\infty C_s v^{-1} + C_s^\dagger C_r v^{-1}). \end{aligned} \tag{3.18}$$

The symmetry of $\langle C_r, C_s \rangle$ can be easily demonstrated using the condition (3.16b) once more.

To apply this to the Jacobian in the semiclassical functional integration it is necessary to relate the variation of the self-dual gauge field A_μ^c, induced by a variation in the parameters on which a, b depend, to this basis of zero modes $\{Z_{r\mu}\}$. From the conditions $\Delta^\dagger v = 0$, $v^\dagger v = 1_{\mathscr{G}}$ we may obtain [19], for arbitrary variations $\delta\Delta$ preserving the constraints $\Delta^\dagger \Delta \propto 1_2$, $\delta v = -f\delta\Delta^\dagger v + v\delta g$, $\delta g = -\delta g^\dagger \in \mathscr{L}_{\mathscr{G}}$, and hence

$$\delta A_\mu^c = v^\dagger \delta\Delta f \bar{e}_\mu b^\dagger v - v^\dagger b e_\mu f \delta\Delta^\dagger v + \mathscr{D}_\mu \delta g, \tag{3.19}$$

where $\mathscr{D}_\mu \delta g$ of course represents an infinitesimal gauge variation in A_μ^c resulting from $v \to vg$. For δA_μ^c given by (3.19) there is a corresponding zero mode Z_μ in the form (3.12) determined by

$$\delta A_\mu^c = Z_\mu + \mathscr{D}_\mu \Lambda, \tag{3.20}$$

with Λ chosen so that C, determined by δa, δb, satisfies (3.16). Since

$$\mathscr{D}_\mu(v^\dagger \delta Q v) = -v^\dagger \delta Q \Delta f \bar{e}_\mu b^\dagger v - v^\dagger b e_\mu f \Delta^\dagger \delta Q v, \tag{3.21}$$

if we represent Λ in (3.20) as $v^\dagger \delta Q v$ with $\delta Q = -\delta Q^\dagger \in \mathscr{L}_{\mathscr{G}_V}$ we can from (3.12) and (3.19) obtain the relation

$$C = \delta\Delta + \delta Q \Delta + \Delta \delta R \tag{3.22}$$

using also $v^\dagger \Delta = 0$ and where $\delta R \in \mathscr{L}_{\mathscr{G}_W}$. Since $\Delta = a + bx$, (3.22) reduces to

$$\begin{aligned} C &= \delta a + \delta Q a + a \delta R, \\ 0 &= \delta b + \delta Q b + b \delta R. \end{aligned} \tag{3.23}$$

The terms involving δQ, δR in (3.23) correspond to an infinitesimal transformation of the form (3.8). Thus the arbitrariness in a, b due to (3.8) is restricted for their infinitesimal variations to form C satisfying (3.16).

The set of zero modes $\{Z_{r\mu}\}$ determined by (3.12) and (3.16) includes the $d_{\mathscr{G}} - d_{\mathscr{K}}$ gauge zero modes $Z_{k\mu}^g = \mathscr{D}_\mu \psi_k$ resulting from global, x-independent gauge transformations of A_μ^c as in (2.32). By comparing (2.33) and (3.21), taking $\psi = -v^\dagger \delta q P_\infty v$ with δq corresponding to an element of $\mathscr{L}_{\mathscr{G}_{V_\infty}}$, the gauge zero mode $Z_\mu^g = \mathscr{D}_\mu \psi$ can be expressed as in (3.12), where

$$C^g = \delta q P_\infty a, \qquad \delta q P_\infty = P_\infty \delta q. \tag{3.24}$$

It is easy to verify that this C^g satisfies the conditions (3.16). As discussed in Section 2 it is natural on flat space to extend the set of parameters describing A_μ^c to include those corresponding to such global gauge transformations. For self dual A_μ^c in the ADHM construction this is achieved if the parameters $\{\alpha_a\}$ specifying a, b are such that

$$\begin{aligned} a(\alpha') &= q P_\infty a(\alpha) R + Q_\perp a(\alpha) R, \\ b(\alpha') &= Q_\perp b(\alpha) R \end{aligned} \tag{3.25}$$

for all $q \in \mathscr{G}_{V_\infty}/\mathscr{H}_{V_\infty}$ for some appropriate $Q_\perp$, R. With such $a(\alpha)$, $b(\alpha)$ this then implies that there are $4C_2(\mathscr{G})\,k$ linearly independent δa, δb matching the basis of zero modes provided by the solutions of (3.16). To ensure that the global gauge variation is realised by the gauge zero modes Z_μ^g it is necessary that the gauge term $\mathscr{D}_\mu \Lambda$ in (3.20) should be orthogonal to the Z_μ^g. This necessitates $\Lambda(x) \to 0$ for $|x| \to \infty$ or that in (3.22) and (3.23)

$$P_\infty \delta Q P_\infty = 0. \tag{3.26}$$

To discuss the solution of (3.23) and (3.16) it is simplest, and indeed natural on flat space when $|x| \to \infty$ is no longer regarded as a single point and plays a privileged role, to take b as fixed, independent of the parameters $\{\alpha_a\}$. This involves no loss of generality since any specific b can always be achieved by a transformation of the form (3.8). The remaining arbitrariness is then reduced since Q, R are then restricted by $b = QbR$, implying $R^\dagger \nu R = \nu$. With $\delta b = 0$ in (3.23), and also (3.26), δQ can be determined in terms of δR giving then

$$C = \delta a - b\delta R' b^\dagger a + a\delta R' \nu, \qquad \delta R' = \delta R \nu^{-1} = -\delta R'^\dagger. \tag{3.27}$$

Also with b fixed $b^\dagger \delta a = (b^\dagger \delta a)^T$ follows from the constraint condition on $b^\dagger a$ in (3.5) and this also ensures (3.16b) for C, with δR otherwise arbitrary in (3.27). Thus $\delta R'$ in (3.27) is to be determined by the requirement (3.16a) $a^\dagger C = (a^\dagger C)^T$, although K given by

$$a^\dagger \delta a - (a^\dagger \delta a)^T = 1_2 K, \qquad K = -K^\dagger, \tag{3.28}$$

(which follows from $a^\dagger a = 1_2 \mu$) is nonzero. Thus the equation for $\delta R'$ becomes

$$K - a^\dagger b \, \delta R' \, b^\dagger a + \mu \delta R' \nu + \nu \delta R' \mu = 0. \tag{3.29}$$

This can be solved by introducing the $k^2 \times k^2$ matrix M^a which acts on $W \times W^*$ (W^* here designates the complex conjugate of the space W) and is defined through its inverse

$$M^{a-1} = \mu \times \nu^t + \nu \times \mu^t - a^\dagger b \times (b^\dagger a)^t, \tag{3.30}$$

where t denotes the transpose for matrices defined on W. (If A is a generic matrix acting on W then for $\mathscr{G} = Sp(n)$ $A^t = A$ so that $M^a = M$ with M defined on $W \times W$ as in (3.30) but without the transposes. For $\mathscr{G} = O(n)$ $A^t = -\hat{\jmath} A^\dagger \hat{\jmath}$, with $1, \hat{\imath}, \hat{\jmath}, \hat{k}$ a quaternion basis for W in this case, so that $M^a = (1 \times -\hat{\jmath})\, M(1 \times \hat{\jmath})$.) M^a, which is nonsingular except when the instanton configuration degenerates to one of lower k, arises naturally when the tensor product formalism is applied to give the ADHM construction for the adjoint representation from the tensor product of the fundamental representation and its complex conjugate [35].

The solution of (3.29) with M^a defined by (3.30) is then

$$\delta R'_{ij} = -M^a_{ij,i'j'} K_{i'j'}, \tag{3.31}$$

with the indices defined on W, $1_{ij,i'j'} = \delta_{ii'}\delta_{jj'}$, $M^a_{ij,i'j'} = M^a_{j'i',ji} = M^{a*}_{ji,j'i'}$. The effect of inserting the $\delta R'$ in (3.29) is that variations in δa due to transformations

$$a \to QaR, \qquad R^\dagger \nu R = \nu, \qquad Q = bR^{-1}\nu^{-1}b^\dagger + P_\infty, \tag{3.32}$$

are projected out to form C. The result of (3.27) and (3.31) is now to allow the normalisation matrix for the zero modes in (3.20) to be related to the variations $\delta_r a$ by using (3.18) and

$$\langle C_r, C_s\rangle = \langle \delta_r a, \delta_s a\rangle + 2K_{rji}M^a_{ij,i'j'}K_{si'j'}, \tag{3.33}$$

with K_r given in terms of $\delta_r a$ by (3.28). (For $\mathscr{G} = Sp(n)$ $K^t = -K$ so (3.33) involves only the antisymmetric part M_A, acting on the $\frac{1}{2}k(k-1)$-dimensional space $(W \times W)_A$, of the matrix M referred to above. For $\mathscr{G} = O(n)$ $(Kf)^t = Kf$, $f^t = -f$ so (3.33) then involves only the symmetric part, acting on the space $(W \times W)_S$). Since $\delta_r a$ satisfies (3.16b) both terms on the right-hand side of (3.33) are symmetric in r, s.

If b is not fixed under variation of the parameters then this can be reduced to the previous case with only δa by defining, as in (3.23)

$$\begin{aligned} \overline{\delta a} &= \delta a + \overline{\delta Q} a + a \overline{\delta R}, \\ \overline{\delta R} &= -\tfrac{1}{2} v^{-1}\,\delta v, \\ \overline{\delta Q} &= -\delta b v^{-1} b^\dagger + b v^{-1} \delta b^\dagger + \tfrac{1}{2} b v^{-1}(b^\dagger \delta b - \delta b^\dagger b)\, v^{-1} b^\dagger, \end{aligned} \tag{3.34}$$

which has $\overline{\delta Q}$ obeying (2.26). With $\delta a \to \overline{\delta a}$ it is now possible to proceed as before since $\overline{\delta a}$ satisfies (3.16b).

For a given parameterisation of the matrices $a(\alpha)$, $b(\alpha)$, where the parameters $\{\alpha_a\}$ vary over a space M^k_α of dimension $4C_2(\mathscr{G})\,k$, the associated statistical weight is the number of times each gauge equivalence class of the self-dual gauge field $A^c_\mu(\alpha)$ is obtained as the parameters $\{\alpha_a\}$ vary over M^k_α. By virtue of the unique correspondence between the gauge equivalence class of A^c_μ, and a, b up to transformations (3.8), S_k is thus the number of solutions of

$$a(\alpha') = Qa(\alpha)\,R, \qquad b(\alpha') = Qb(\alpha)\,R \tag{3.35}$$

modulo Q, R leaving a, b invariant as in (3.9). Since the parameters $\{\alpha_a\}$ are assumed to be extended to include those corresponding to the global gauge group $\mathscr{G}/\mathscr{H} \simeq \mathscr{G}_{V_\infty}/\mathscr{H}_{V_\infty}$, as introduced in (3.25), we require in (3.35)

$$\begin{gathered} P_\infty Q P_\infty = P_\infty, \qquad SR \sim R, \qquad S \in \mathscr{H}_W, \\ \{R\} \in \mathscr{G}_W/\mathscr{H}_W. \end{gathered} \tag{3.36}$$

If b is fixed, independent of $\{\alpha_a\}$, the Q, R in (3.35) can be restricted as in (3.32). For a sensible parameterisation the solutions of (3.35) and (3.36) should be discrete and S_k finite, independent of the specific $\{\alpha_a\} \in M^k_\alpha$. Further the invariance groups $\mathscr{H}_V$, $\mathscr{H}_W$ in (3.9), and also $\mathscr{H}_{V_\infty}$ should be isomorphic for all $a(\alpha)$, $b(\alpha)$ with $\{\alpha_a\} \in M^k_\alpha$.

If b is fixed there is a natural canonical form to assume for a, b which is

$$a = \begin{pmatrix} v \\ B \end{pmatrix}, \qquad b = -\begin{pmatrix} 0 \\ 1_k \times 1_2 \end{pmatrix}, \tag{3.37}$$

with $v = 1_k$, the identity on W. With this canonical form the associated Q, R in the transformation (3.8) which leave b invariant take the form

$$Q = \begin{pmatrix} q & 0 \\ 0 & R^{\dagger} \end{pmatrix}, \qquad R^{\dagger} R = 1_k \tag{3.38}$$

for $q \in \mathscr{G}$ and $R \in \mathscr{G}_{W0}$, listed in Table I. The stability group $\mathscr{H} \subset \mathscr{G}$ is further generated by elements q such that

$$qvR = v, \qquad R^{\dagger} BR = B \tag{3.39}$$

for some $R \in \mathscr{H}_W \subset \mathscr{G}_{W0}$ and the parameters corresponding to global gauge transformations can be incorporated as in (3.25) by taking

$$v(\alpha') = qv(\alpha) R, \qquad B(\alpha') = R^{\dagger} B(\alpha) R \tag{3.40}$$

for all $\{q\} \in \mathscr{G}/\mathscr{H}$ and some appropriate $R \in \mathscr{G}_{W0}$. The statistical weight S_k for a given parameterisation is then identified, as in (3.35), with the number of solutions of

$$\begin{gathered} v(\alpha') = v(\alpha) R, \qquad B(\alpha') = R^{\dagger} B(\alpha) R, \\ \{R\} \in \mathscr{G}_{W0}/\mathscr{H}_W, \end{gathered} \tag{3.41}$$

so that $\{\alpha_a\}$, $\{\alpha'_a\}$ then lead to $A^c_\mu(\alpha)$, $A^c_\mu(\alpha')$ in the same gauge equivalence class.

4. Semiclassical Functional Integral for $SU(2)$ $k = 1, 2$

To apply the results of Section 3 we need a general unconstrained parameterisation of the matrix a, b satisfying the constraints (3.5). This can only be achieved in a transparent fashion for $k = 1, 2$. For simplicity we consider only $\mathscr{G} = Sp(1) \simeq SU(2)$ here, although embeddings in higher groups should present no real problems.

To demonstrate the effectiveness of these general methods we first regain the well-known results [7, 8] for $k = 1$. In this case we may take

$$a = \begin{pmatrix} r \\ y \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ -1_2 \end{pmatrix} \tag{4.1}$$

so that the quaternion y represents the instanton position and $\rho = |r|$ its scale. The global $Sp(1)$ gauge variable are contained, as in (3.25) or (3.40), by realisation of the transformation $r \to r' = qr$, $|q| = 1$. If r, y are allowed to vary over the whole of $H \cong R^4$ the statistical weight $S_1 = 2$ corresponding to $R = \pm 1$ in (3.40) (this is just a reflection of the fact that the gauge group is really $SU(2)/Z_2$). Also the variations δa induced by varying r, y directly satisfy (3.16) so the basic measure in (2.43), (2.44) is

$$d^4y \, d^4r (\det \bar{J})^{1/2}, \tag{4.2}$$

where from (3.18) in a Cartesian 4-vector basis for r, y

$$\tilde{J} = 8\pi^2 \begin{pmatrix} 2 \times 1_4 & 0 \\ 0 & 1_4 \end{pmatrix}, \qquad (\det \tilde{J})^{1/2} = (8\pi^2)^4\, 4. \tag{4.3}$$

From Appendix B, Eq. (B.13)

$$\ln \left\{ \frac{\det(-\mathscr{D}_0^2/\mu^2)}{\det(-\mathscr{D}^2/\mu^2)} \right\} = -\frac{1}{3} \ln \mu^2 |r|^2 - \alpha(1) \tag{4.4}$$

so combining (4.2), (4.3) and (4.4) with (2.48) and (2.50) ($S_1 = 2$) the basic one instanton $SU(2)$ functional measure is

$$(\Lambda^2)^{11/3} \left(\frac{4\pi}{g^2} \right)^4 2e^{-\alpha(1)} \frac{d^4y\, d^4r}{|r|^{2/3}}. \tag{4.5}$$

This result is usually presented with the invariant measure of the global $SU(2) \simeq S^3$ gauge group integrated out, giving an appropriate volume factor, realised in (4.5) by $d^4r \to 2\pi^2 |r|^3\, d|r|$.

For $k = 2$ and $\mathscr{G} = Sp(1)$ we take a, b to be of the form (3.37) and then parameterise v, B introduced there in terms of quaternions as

$$v = (v_1, v_2) = (\pi\sigma_0, 2\bar{\pi}^{-1}\tau), \qquad B = \begin{pmatrix} \rho + \tau & \sigma \\ \sigma & \rho - \tau \end{pmatrix}, \tag{4.6}$$

where B is required to be symmetric as a matrix of quaternions since the constraints (3.5) imply $B^T = B$. The only non linear constraints from (3.5) is now $a^\dagger a = \mu 1_2$ which here gives

$$\bar{v}_1 v_2 - \bar{v}_2 v_1 = 2(\bar{\sigma}\tau - \bar{\tau}\sigma). \tag{4.7}$$

This constraint can be solved in general by

$$\bar{v}_2 v_1 - 2\bar{\tau}\sigma = 2\lambda |\tau|^2, \qquad \sigma = \sigma_0 + \lambda\tau, \tag{4.8}$$

for λ arbitrary and real. The global $Sp(1)$ gauge transformations are realised in this case by $v \to qv$ or $\pi \to q\pi$ for q a quaternion, $|q| = 1$. The arbitrariness up to transformations (3.9) correspond now to taking $R \in O(2)$ with, for $\det R = 1$, a transformation depending continuously on an angle θ, $a \to a_\theta$ where

$$v_\theta = vR_\theta, \qquad B_\theta = R_\theta^{-1} B R_\theta, \qquad R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{4.9}$$

or in terms of $u = (\tau, \sigma)$

$$u_\theta = uR_{2\theta}, \qquad \rho_\theta = \rho \tag{4.10}$$

plus the discrete transformation for $\det R = -1$, taking $R_r = \sigma_3$

$$v_r = (v_1, -v_2), \qquad u_r = (\tau, -\sigma), \qquad \rho_r = \rho. \tag{4.11}$$

These transformations, (4.9), (4.10), (4.12), necessarily preserve the constraints (4.8). If λ is freely varying in (4.8) then the transformation in (4.9), (4.10) can be continuously implemented on v, B, given in (4.6), by letting $\lambda \to \lambda_\theta$ with infinitesimally

$$\left.\frac{d\lambda_\theta}{d\theta}\right|_{\theta=0} = \frac{P}{2\,|\tau|^2},$$
$$P = 4\,|\tau|^2\,(1+\lambda^2) - 4\,|\sigma_0|^2 - |v_1|^2 + |v_2|^2. \tag{4.12}$$

We may identify 16 possible parameters describing a, b in this case by choosing as the basic independent variables the four quaternions $X = (\rho, \tau, \sigma, \pi)$ or $X_0 = (\rho, \tau, \sigma_0, \pi)$, equivalently (ρ, τ, v_1, v_2). The parameterisation can then be completed by choosing for λ in (4.8) some definite functions $\lambda(X)$ or $\lambda(X_0)$. The variation under $O(2)$ transformations in λ thus induced by this dependence is infinitesimally

$$\left.\frac{d\lambda(X_\theta)}{d\theta}\right|_{\theta=0} \tag{4.13}$$

or $X \to X_0$, where $X_{\delta\theta}$, or $X_{0\delta\theta}$, is easily given by (4.9) and (4.10) to first order in $\delta\theta$. To the extent that (4.12) and (4.13) are not equal the $O(2)$ transformations cannot be implemented infinitesimally in this parameterisation and the 16 real parameters given by X or X_0 fully describe the $k=2$ multi-instanton configuration, at least in some open region of the parameter space M_α^2.

With the parameter set X_0 and $\lambda(X_0)$ given it is now possible to calculate the normalisation matrix of the zero modes induced by variations δX_0 using, from (3.33) and (3.18), in this case

$$\begin{aligned} \langle C', C\rangle &= \langle \delta' a, \delta a\rangle - 4k'kN_A^{-1}, \\ \langle \delta' a, \delta a\rangle &= \operatorname{tr}(\delta' a^\dagger (1 + P_\infty)\,\delta a), \end{aligned} \tag{4.14}$$

where k, k' are elements of the 2×2 matrix K, K' defined in (3.28)

$$K = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}, \qquad k = -2\,\operatorname{tr}_2(\bar\sigma_0\,\delta\bar\pi\bar\pi^{-1}\tau) + 2\,|\tau|^2\,\delta\lambda \tag{4.15}$$

and N_A corresponds to the 1×1 antisymmetric part of the matrix M_A^{-1}

$$N_A = |v_1|^2 + |v_2|^2 + 4(|\tau|^2 + |\sigma|^2). \tag{4.16}$$

In (4.15) and henceforth $\frac{1}{2}\operatorname{tr}_2$ denotes the real part of the associated quaternion; i.e., $q + \bar q = 1_2\,\operatorname{tr}_2(q)$. It is easy to check that for infinitesimal $O(2)$ transformations, taking $\delta_\theta a = a_{\delta\theta} - a$ and the corresponding result for k, in (4.14) gives zero showing that in the scalar product $\langle C', C\rangle$ these transformations are projected out, as required by its construction. This allows the removal of the $\delta\lambda$ terms in (4.15), and also those present in $\delta\sigma$ when δa is formed, by performing such an infinitesimal $O(2)$ transformation using (4.12)

$$\delta_t X_0 = \delta X_0 - \frac{2\,|\tau|^2}{P}\,\delta\lambda\,\frac{d}{d\theta}X_{0\theta}\bigg|_{\theta=0} \tag{4.17}$$

so that $\delta_t\lambda(X_0) = 0$. In (4.17) we can take $\delta\lambda = \lambda^T_{X_0}\delta X_0$ regarding δX_0 as a 16-dimensional real column vector, and hence

$$\delta_t X_0 = C_0\delta X_0, \qquad C_0 = 1 - \frac{2\,|\tau|^2}{P}\frac{d}{d\theta}X_{0\theta}\bigg|_{\theta=0}\lambda^T_{X_0}. \tag{4.18}$$

In terms of $\delta_t X_0$, (4.14) becomes

$$\begin{aligned}\langle C', C\rangle &= \langle\delta'_t a, \delta_t a\rangle - 4k'_t k_t N_A^{-1},\\ k_t &= -2\mathrm{tr}_2(\delta_t\bar\pi\phi), \qquad \phi = \bar\pi^{-1}\tau\bar\sigma_0,\end{aligned} \tag{4.19}$$

with $\langle\delta'_t a, \delta_t a\rangle$ the quadratic form

$$\begin{aligned}\langle\delta'_t a, \delta_t a\rangle &= 2\mathrm{tr}_2(\bar W'^T A W) + 2\mathrm{tr}_2(\bar Z'^T W + \bar W'^T Z)\\ &\quad + 2\left(|\sigma_0|^2 + 4\frac{|\tau|^2}{|\pi|^4}\right)\mathrm{tr}_2(\delta'_t\bar\pi\delta_t\pi),\end{aligned} \tag{4.20}$$

where

$$W = \begin{pmatrix}\delta_t\rho\\ \delta_t\tau\\ \delta_t\sigma_0\end{pmatrix}, \qquad Z = \begin{pmatrix}0\\ -(4/|\pi|^2)\,\delta_t\bar\pi\bar\pi^{-1}\tau\\ \bar\pi\delta_t\pi\sigma_0\end{pmatrix},$$
$$A = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 + 4/|\pi|^2 + \lambda^2 & \lambda\\ 0 & \lambda & 1 + |\pi|^2\end{pmatrix}. \tag{4.21}$$

If $Y = W + A^{-1}Z$ (4.20) reduces to

$$\begin{aligned}\langle\delta'_t a, \delta_t a\rangle &= 2\mathrm{tr}_2(\bar Y'^T A Y) + \frac{2N_A}{|\pi|^2\det A}\mathrm{tr}_2(\delta'_t\bar\pi\delta_t\pi)\\ &\quad - \frac{8\lambda}{|\pi|^2\det A}\mathrm{tr}_2(\bar\phi(\delta'_t\pi\bar\pi + \pi\delta'_t\bar\pi)\,\delta_t\pi\\ &\quad + \delta'_t\bar\pi(\pi\delta_t\bar\pi + \delta_t\pi\bar\pi)\,\phi).\end{aligned} \tag{4.22}$$

The final result for $\langle C', C\rangle$ is most conveniently expressed by changing from a quaternion q to a Cartesian basis q_α with q now regarded as a real four-dimensional column vector, so that using $k_t = -4\phi^T\delta_t\pi$ (4.19) is now

$$\tfrac{1}{4}\langle C', C\rangle = \delta'_t\pi^T B\delta_t\pi + \delta'_t Y^T A \times 1_4\delta_t Y,$$
$$B = 1_4\frac{N_A}{|\pi|^2 \det A} - \frac{8\lambda}{\det A}(\phi\pi^T + \pi\phi^T) - 16N_A^{-1}\phi\phi^T. \tag{4.23}$$

For the parameter set X_0 it is now possible to determine the corresponding measure as in (2.43)

$$d^4\rho\, d^4\tau\, d^4\sigma_0\, d^4\pi(\det \bar{J})^{1/2}, \tag{4.24}$$

where $\bar{J}$ is here a 16×16 matrix related to $\langle C', C\rangle$ according to (3.18)

$$4\pi^2\langle C', C\rangle = \delta' X_0^T\bar{J}\delta X_0 \tag{4.25}$$

so that its determinant can be read off from (4.23) as

$$\begin{aligned} \det \bar{J} &= (16\pi^2)^{16} \det B(\det A)^4 (\det C_0)^2 \\ &= (16\pi^2)^{16}(N_A^2|\pi|^8)\, I_0^2, \\ I_0 &= P \det C_0. \end{aligned} \tag{4.26}$$

Inserting (4.26) into (4.24) gives the resultant measure

$$(16\pi^2)^8\, d^4\rho\, d^4\tau\, d^4\sigma_0(d^4\pi/|\pi|^4)\, N_A\,|I_0|, \tag{4.27}$$

or equivalently

$$\frac{1}{16}(16\pi^2)^8\, d^4\rho\,\frac{d^4\tau}{|\tau|^4}\, d^4v_1\, d^4v_2 N_A\,|I_0|. \tag{4.28}$$

From (4.18) $\det C_0$ is easily evaluated

$$\det C_0 = 1 - 2\frac{|\tau|^2}{P}\lambda_{X_0}^T\frac{d}{d\theta}X_{0\theta}\bigg|_{\theta=0} = 1 - 2\frac{|\tau|^2}{P}\frac{d}{d\theta}\lambda(X_{0\theta})\bigg|_{\theta=0} \tag{4.29}$$

and with P given in (4.12) I_0 can be expressed as

$$I_0 = 2|\tau|^2\frac{d}{d\theta}(\lambda_\theta - \lambda(X_{0\theta}))\bigg|_{\theta=0} = 4|\tau|^2 - 2\bar{\tau}\frac{d}{d\theta}\sigma(X_{0\theta})\bigg|_{\theta=0} \tag{4.30}$$

which, despite appearances, is nevertheless real. From (4.10) $\delta_\theta\sigma = 2\tau\delta\theta$ so (4.30) demonstrates that I_0 vanishes just when the $O(2)$ transformation can be implemented infinitesimally on a with the parameter set X_0 and the given $\lambda(X_0)$.

For the set X and assuming $\lambda(X)$ is given the same procedure can be followed but instead of (4.18) we may take

$$\delta_t X_0 = C\delta X, \qquad C = S\left(1 - \frac{2|\tau|^2}{P}\frac{d}{d\theta}X_\theta\bigg|_{\theta=0}\lambda_X^T\right),$$

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times 1_4. \tag{4.31}$$

The result for det $\bar{J}$ is as above but now with $I = P \det C$ instead of I_0, giving the measure

$$(16\pi^2)^8\, d^4\rho\, d^4\tau\, d^4\sigma \frac{d^4\pi}{|\pi|^4} N_A\, |I| \qquad \text{or} \qquad \frac{d^4\pi}{|\pi|^4} \to \frac{d^4 v_2}{|v_2|^4}, \tag{4.32}$$

where, in a similar fashion to (4.30),

$$I = -|v_2|^2 - \bar{v}_2 \frac{d}{d\theta} v_1(X_\theta)\bigg|_{\theta=0}. \tag{4.33}$$

Since $\delta_\theta v_1 = -v_2\delta\theta$, I vanishes, as before, when the $O(2)$ transformations can be implemented infinitesimally, this time with parameters X and $\lambda(X)$.

The results (4.27) and (4.28) with (4.29) or (4.23) with (4.33) are not in themselves sufficient for constructing the $k=2$ semiclassical functional integral since the parameter sets X_0 or X are by no means natural ones for characterising A_μ^c for any obvious choices $\lambda(X_0)$ or $\lambda(X)$. In the absence of better alternatives we choose to exhibit the functional integral in terms of the parameters of the conformally extended HJNR solution [22] which expresses $A_\mu^c(x)$ for $k=2$, in a suitable gauge, in terms of the scalar field

$$\varphi(x) = \sum_0^2 \frac{\lambda_i^2}{|x - y_i|^2}. \tag{4.34}$$

The parameters are then three quaternions, or Euclidean 4 vectors, $y_i \in H \simeq R^4$ and three positive scale parameters $\lambda_i \in R_+$. The overall scale of ϕ is irrelevant so only two independent ratios of the λ_i are true parameters. Altogether the parameter space is 14 dimensional in this case which allows for the realisation of a $U(1)$ subgroup of the global $SU(2)$ group of gauge transformations [22]. To determine the relevant measure over this parameter space it is necessary to include appropriate volume factor corresponding to the integration over the coset space $SU(2)/U(1)$. This necessitates carefully identifying the $U(1)$ subgroup in terms of the given parameterisation of the matrices a, b as functions of the 't Hooft parameters y_i, λ_i (for a, b expressed as in (3.37) with $k=2$ the $U(1)$ subgroup is given by $\{q_\theta\}$ for which $v(y_{i\theta}, \lambda_{i\theta}) = q_\theta v(y_i, \lambda_i)_\theta$, $B(y_{i\theta}, \lambda_{i\theta}) = B(y_i, \lambda_i)_\theta$; different parameterisations of v, B are given by $v' = qvR$, $B' = R^{-1}BR$ for any $R \in O(2)$, $q \in Sp(1)$ but the associated $U(1)$ subgroups are related by similarity transformations).

To discuss this it is convenient to introduce an intermediate parameterisation between the 't Hooft set and the set $X = (\rho, \tau, \sigma, v_2)$ considered above. This is obtained by representing $v = (v_1, v_2)$ in terms of $u = (\tau, \sigma)$ according to

$$v = uH, \qquad H = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{4.35}$$

where H is real. The constraint (4.7) takes the form

$$\det H = -2 \tag{4.36}$$

and under $O(2)$ transformations from (4.9), (4.10) and (4.11) we have

$$H_\theta = R_{2\theta}^{-1} H R_\theta, \qquad H_r = \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix}. \tag{4.37}$$

Alternatively the elements of H can be used to form two $O(2)$ vectors

$$x_\pm = \begin{pmatrix} \alpha \mp \delta \\ \gamma \pm \beta \end{pmatrix}, \qquad x_{+\theta} = R_{3\theta}^{-1} x_+, \qquad x_{-\theta} = R_\theta^{-1} x_-, \tag{4.38}$$

where from (4.36) $x_+^2 - x_-^2 = 8$. Since H_θ is equivalent to H and H obeys the single constraint (4.36) H is taken to be a function of two independent real parameters λ_1, λ_2 varying over a space M_λ.

With $v_2 = \beta\tau + \delta\sigma$ from (4.35) manifestly v_2 spans a two-dimensional plane in $H \simeq R^4$, for τ, σ independent and assuming (β, δ) range over R^2 for $(\lambda_1, \lambda_2) \in M_\lambda$. For the set $X = (\rho, \tau, \sigma, v_2)$ the global gauge group acts only on v_2 by $v_2 \to qv_2$, $|q| = 1$, $q \in Sp(1) \simeq S^3$. The $U(1) \simeq S^1$ subgroup which can obviously be implemented on $(\beta, \delta) \in R^2$ corresponds to those q which rotate v_2 in the plane spanned by τ, σ. The appropriate volume for the coset space $Sp(1)/U(1)$ is easily identified in this case by taking in (4.32)

$$\frac{d^4 v_2}{|v_2|^4} \to \pi \frac{d^2 v_2}{|v_2|^2}$$
$$d^2 v_2 = d\beta\, d\delta (|\sigma|^2 |\tau|^2 - \sigma \cdot \tau^2)^{1/2}, \qquad \sigma \cdot \tau = \tfrac{1}{2}\mathrm{tr}_2(\bar{\sigma}\tau). \tag{4.39}$$

Since $\delta_\theta \tau = -2\sigma\delta\theta$, $\delta_\theta \sigma = 2\tau\delta\theta$ I in (4.33) becomes

$$I = -(f+1)|v_2|^2, \qquad f = \frac{1}{\delta}\left(\frac{d\gamma}{d\theta} - 2\alpha\right) = \frac{1}{\beta}\left(\frac{d\alpha}{d\theta} + 2\gamma\right) \tag{4.40}$$

using (4.36), and where the derivatives are determined by $\delta_\theta v_2 = v_1 \delta\theta$ or

$$\frac{d\delta}{d\theta} = \gamma + 2\beta, \qquad \frac{d\beta}{d\theta} = \alpha - 2\delta \tag{4.41}$$

which gives $d\lambda_1/d\theta$, $d\lambda_2/d\theta$. Combining (4.32) with (4.39), (4.40) and (4.41) the measure becomes

$$
\begin{aligned}
&(16\pi^2)^8\,\pi N_A\,d^4\rho\,d^4\tau\,d^4\sigma(|\sigma|^2\,|\tau|^2-\sigma\cdot\tau^2)^{1/2}\,d\mu(\lambda_1,\lambda_2),\\
&d\mu(\lambda_1,\lambda_2)=d\beta\,d\delta\,|f+1|\\
&\qquad=\frac{1}{2}\left|\varepsilon_{ij}\left(3\frac{\partial x_{-i}}{\partial\lambda_1}\frac{\partial x_{-j}}{\partial\lambda_2}-\frac{\partial x_{+i}}{\partial\lambda_1}\frac{\partial x_{+j}}{\partial\lambda_2}\right)\right|\,d\lambda_1\,d\lambda_2.
\end{aligned}
\tag{4.42}
$$

The final form (4.42) is manifestly invariant under $O(2)$ transformations, even with $\theta(\lambda_1,\lambda_2)$.

This result can be applied to the HJNR case by using the results of Appendix C (λ_0 may be set equal to 1 but it is convenient to keep it arbitrary)

$$
\begin{aligned}
&\rho=\frac{1}{2}z^2\{(\lambda_1^2+\lambda_2^2)y_0+(\lambda_0^2+\lambda_2^2)\,y_1+(\lambda_0^2+\lambda_1^2)\,y_2\},\\
&\sigma=z\,\frac{\lambda_0\lambda_1\lambda_2}{\lambda_0^2+\lambda_1^2}\,(y_0-y_1),\qquad z^2=\frac{1}{\lambda_0^2+\lambda_1^2+\lambda_2^2},\\
&\tau=\frac{1}{2}z^2\{\lambda_0^2(y_0-y_2)+\lambda_1^2(y_1-y_2)\}-\frac{1}{2}\frac{\lambda_0^2-\lambda_1^2}{\lambda_0^2+\lambda_1^2}\,(y_0-y_1)
\end{aligned}
\tag{4.43}
$$

and

$$
\begin{aligned}
\alpha=0,\qquad \tfrac{1}{2}\beta=1/\gamma=\lambda_2/(\lambda_0^2+\lambda_1^2)^{1/2},\\
\delta=(\lambda_0^2-\lambda_1^2)/z\lambda_0\lambda_1(\lambda_0^2+\lambda_1^2)^{1/2}.
\end{aligned}
\tag{4.44}
$$

In (4.44) $\beta\geqslant 0$ (with $\lambda_2\geqslant 0$) so v_2 spans only a half plane. To compensate for this in (4.39) we supply a factor 2 corresponding to summing over $q=\pm 1\in Sp(1)$ giving

$$
\begin{aligned}
&d\mu(\lambda_1,\lambda_2)=\frac{4}{W}\frac{d\lambda_1}{\lambda_1}\frac{d\lambda_2}{\lambda_2},\qquad W=z^3\lambda_0\lambda_1\lambda_2\\
&d^4\rho\,d^4\tau\,d^4\sigma=(W^4/16)\,d^4y_0d^4y_1\,d^4y_2,\\
&(|\sigma|^2\,|\tau|^2-\sigma\cdot\tau^2)^{1/2}=(W/4)\,\Gamma(|y_0-y_1|^2,|y_1-y_2|^2,|y_2-y_0|^2)^{1/2},
\end{aligned}
\tag{4.45}
$$

where Γ is the triangle function defined by

$$
\Gamma(a,b,c)=2ab+2bc+2ca-a^2-b^2-c^2 \tag{4.46}
$$

and N_A is expressible as

$$
N_A=z^2(\lambda_2^2\,|y_0-y_1|^2+\lambda_0^2\,|y_1-y_2|^2+\lambda_1^2\,|y_2-y_0|^2). \tag{4.47}
$$

It remains to determine the statistical weight S_2 for the $k=2$ $SU(2)$ self-dual gauge field in the HJNR parameterisation. As discussed in Section 3 this is the number of solutions of (3.41) which with v, B given as in (4.6) and (4.35) and, since σ, τ as expressed in (4.43) can vary freely, is readily seen in this case to be the number of solutions of

$$H(\lambda'_0, \lambda'_1, \lambda'_2) = H(\lambda_0, \lambda_1, \lambda_2)_\theta \qquad \text{or} \qquad H(\lambda'_0, \lambda'_1, \lambda'_2) = H(\lambda_0, \lambda_1, \lambda_2)_{r,\theta}. \tag{4.48}$$

By analysing (4.37) with $\alpha' = \alpha = 0$, as in (4.44), the solutions of (4.48) just realise the obvious permutation symmetry group $\mathscr{S}^3$ of the HJNR solution as exemplified in (4.34). Thus

$$\begin{aligned} &H(\lambda_1, \lambda_0, \lambda_2) = H(\lambda_0, \lambda_1, \lambda_2)_{r,\pi} \\ &\left.\begin{aligned} H(\lambda_1, \lambda_2, \lambda_0) = H(\lambda_0, \lambda_1, \lambda_2)_{\theta_1}, \qquad \tan\theta_1 = \lambda_1/z\lambda_0\lambda_2 \\ H(\lambda_2, \lambda_0, \lambda_1) = H(\lambda_0, \lambda_1, \lambda_2)_{\theta_2}, \qquad \tan\theta_2 = -\lambda_0/z\lambda_1\lambda_2 \end{aligned}\right\} \quad \pi/2 < \theta_{1,2} < 3\pi/2. \end{aligned} \tag{4.49}$$

In addition, there are $O(2)$ rotations with $\theta = \pi$ in (4.48) which must be included by virtue of the compensating factor 2 introduced after (4.44) which ensures $H \simeq -H$ so that *in toto* $S_2 = 12$. With the factor S_2^{-1} the measure from (4.42) and (4.45) becomes

$$(8\pi^2)^8 \frac{8\pi}{6} W^4 \frac{d\lambda_1}{\lambda_1} \frac{d\lambda_2}{\lambda_2} d^4y_0\, d^4y_1\, d^4y_2\, N_A \Gamma^{1/2}, \tag{4.50}$$

which is manifestly symmetric under permutations. Hence we have ultimately derived the expression reported earlier [20].

The complete expression (2.48) for the functional measure involves $\det(-\mathscr{D}_0^2/\mu^2)/\det(-\mathscr{D}^2/\mu^2)$ which even for $k = 2$ cannot yet be evaluated exactly. However, an approximate form has been obtained [19] which is correct in all limits when it becomes singular and is quoted in (B. 12). With this and (2.50) the result is

$$(\Lambda^2)^{22/3} \left(\frac{4\pi}{g^2}\right)^8 e^{-2\alpha(1)} \frac{8\pi}{6} W^4 \frac{d\lambda_1}{\lambda_1} \frac{d\lambda_2}{\lambda_2} d^4y_0\, d^4y_1\, d^4y_2 \frac{N_A^{4/3}}{N_S^{1/3}} \Gamma^{1/2}, \tag{4.51}$$

where with the $k = 2$ HJNR parameterisation

$$N_S = W^2 \,|y_0 - y_1|^2\, |y_1 - y_2|^2\, |y_2 - y_0|^2. \tag{4.52}$$

The singularities of the resultant measure (4.51) arise for $N_S = 0$, which correspond to the gauge field configuration degenerating to one of lower topological index given by $y_i \to y_j$, $i \neq j$ in (4.52), and also $\Gamma = 0$, which with Γ defined by (4.45) and (4.46) corresponds to a collinear gauge field with y_0, y_1, y_2 on a line. This singularity is specific to the HJNR parameterisation since in the collinear case there is a gauge degree of freedom in the parameterisation representing non compact translations along the line [22]. This has the counterpart in our formalisms that the $O(2)$

transformations (4.9) can be directly implemented on the HJNR parameters y_0, y_1, y_2, λ_0, λ_1, λ_2 instead of a $U(1)$ subgroup of the global $Sp(1)$ gauge group.

5. Factorisation

For well-separated instantons the $k=2$ multi-instanton determinants and the associated semiclassical functional integration measure are expected to decompose into two factors corresponding to the appropriate single instanton determinants and measure. This was demonstrated for the HJNR case, (4.50) or (4.51), in Ref. [20]. However, previous analysis suggests [38] that the factorisation should be valid without any $O(|d|^{-2})$ corrections, where $|d|$ is the separation of the two instantons. This can now be shown explicitly using our exact results.

To achieve this it is crucial to find the correct variables to describe the position, scale, and group orientation of the separated individual instantons. Modifying slightly our previous expressions (3.37) and (4.6) for our present purpose gives

$$\Delta(x)=\begin{pmatrix} v_1 & v_2 \\ y_1-x & \sigma \\ \sigma & y_2-x \end{pmatrix}, \tag{5.1}$$

and hence from (3.3)

$$\begin{gathered} f(x)^{-1}=\begin{pmatrix} |y_1-x|^2+|v_1|^2+|\sigma|^2 & e(x) \\ e(x) & |y_2-x|^2+|v_2|^2+|\sigma|^2 \end{pmatrix}, \\ e(x)=\bar{v}_1 v_2+(\bar{y}_1-\bar{x})\sigma+\bar{\sigma}(y_2-x)=\bar{v}_2 v_1+(\bar{y}_2-\bar{x})\sigma+\bar{\sigma}(y_1-x). \end{gathered} \tag{5.2}$$

Now if $d=y_1-y_2$ for $|d|\to\infty$ we take

$$x=y_1+O(1), \qquad v_1, v_2=O(1), \qquad \sigma=O(|d|^{-1}) \tag{5.3}$$

and then it is easy to see that $f(x)^{-1}$ has an eigenvalue

$$|y_1-x|^2+\rho_1^2+O(|d|^{-3}) \tag{5.4}$$

for

$$\rho_1^2=|v_1|^2-\frac{e_1^2}{|d|^2}+|\sigma|^2, \qquad e_1=e(y_1). \tag{5.5}$$

In view of the relation [19]

$$\operatorname{tr}(F_{\mu\nu}F_{\mu\nu})=\partial^2\partial^2 \ln \det f^{-1} \tag{5.6}$$

it is clear that y_1 can be naturally interpreted as the instanton position, ρ_1 the scale in this limit. Similarly the other instanton is located at y_2 with scale ρ_2 given by

$$\rho_2^2 = |v_1|^2 - \frac{e_2^2}{|d|^2} + |\sigma|^2, \qquad e_2 = e(y_2). \tag{5.7}$$

With (5.5) and (5.7) it is now possible to show the factorisation of the determinants since in the approximations given in (B.12) they depend only on N_S/N_A, and N_S, N_A from their definitions (B.7) and (B.10) (or (4.16) for N_A) are given in this case (with $\nu = 1$) by

$$\begin{aligned} N_S &= |d|^2 |v_1|^2 |v_2|^2 + |v_1|^2 |v_2|^2 (|v_1|^2 + |v_2|^2) \\ &\quad + |d|^2 |\sigma|^2 (|v_1|^2 + |v_2|^2) - |v_1|^2 e_2^2 - |v_2|^2 e_1^2 + O(|d|^{-2}) \\ N_A &= |d|^2 + |v_1|^2 + |v_2|^2 + O(|d|^{-2}) \end{aligned} \tag{5.8}$$

so that

$$N_S/N_A = \rho_1^2 \rho_2^2 + O(|d|^{-4}) \tag{5.9}$$

To similarly discuss the measure we use the form (4.28) which now becomes

$$\begin{aligned} &(8\pi^2)^8\, 16 d^4 v_1\, d^4 v_2\, d^4 y_1\, d^4 y_2\, (N_A |I_0|)/|d|^4, \\ &I_0 = P - \frac{1}{2} |d|^2 \frac{d}{d\theta} \lambda(v_{1\theta}, v_{2\theta}, d_\theta) \Big|_{\theta = 0}, \\ &P = |d|^2 (1 + \lambda^2) - |v_1|^2 + |v_2|^2 + |v_1|^2 |v_2|^2/|d|^2, \end{aligned} \tag{5.10}$$

where from (5.3) λ is assumed $O(|d|^{-2})$ as $|d| \to \infty$. To leading order $N_A \sim I_0 \sim |d|^2$ so (5.10) is directly a product of single instanton meaures as displayed in (4.2) and (4.3). To go beyond this to include $O(|d|^{-2})$ corrections necessitates more care as discussed above for the functional determinants. To this end we consider a transformation of the form (3.9) applied to Δ in (5.1) with

$$Q_1 = \begin{pmatrix} 1 - \frac{1}{2}|v_2|^2/|d|^2 & 0 & v_2 d^{-1} \\ 0 & 1 & \sigma d^{-1} \\ -\bar{d}^{-1} \bar{v}_2 & -\bar{d}^{-1}\bar{\sigma} & 1 \end{pmatrix}, \quad R_1 = \begin{pmatrix} 1 & e_1/|d|^2 \\ -e_1/|d|^2 & 1 \end{pmatrix}, \tag{5.11}$$

so that in the limiting situation (5.3)

$$Q_1 \Delta(x) R_1 = \begin{pmatrix} r_1 & O(|d|^{-2}) \\ y_1 - x & O(|d|^{-2}) \\ O(|d|^{-2}) & -d \end{pmatrix}, \tag{5.12}$$

where

$$r_1 = (1 - \tfrac{1}{2}|v_2|^2/|d|^2)\, v_1 + v_2 d^{-1} \sigma \tag{5.13}$$

so that $|r_1| = \rho_1 + O(|d|^{-4})$. Thus, comparing with the single instanton expression (4.1), we see that r_1 describes the scale and the group orientation of the instanton. Similarly for the other instanton at y_2 the scale and group orientation are given by

$$r_2 = (1 - \tfrac{1}{2}|v_1|^2/|d|^2)\, v_2 - v_1 d^{-1}\sigma. \tag{5.14}$$

The variables r_1, y_1, r_2, y_2 are now the appropriate ones to discuss the factorisation of the measure (5.10) including corrections $O(|d|^{-2})$. Writing

$$\delta\lambda = \lambda_{v_1}^T \delta v_1 + \lambda_{v_2}^T \delta v_2 + \lambda_d^T \delta d$$

I_0 becomes from (5.10)

$$I_0 = |d|^2 - |v_1|^2 + |v_2|^2 - \tfrac{1}{2}|d|^2 (\lambda_{v_2}^T v_1 - \lambda_{v_1}^T v_2) + O(|d|^{-2}). \tag{5.15}$$

Furthermore since, again going over to a Cartesian 4-vector basis,

$$r_1 = (1 + \tfrac{1}{2}|v_2|^2/|d|^2)\, v_1 + \tfrac{1}{2}\lambda v_2,$$
$$r_2 = (1 + \tfrac{1}{2}|v_1|^2/|d|^2 - 2/|d|^2\, v_1 v_1^T)\, v_2 - \tfrac{1}{2}\lambda v_1$$

we have

$$\begin{aligned} d^4 r_1 &\approx (1 + 2|v_2|^2/|d|^2 + \tfrac{1}{2}\lambda_{v_1}^T v_2)\, d^4 v_1, \\ d^4 r_2 &\approx (1 - \tfrac{1}{2}\lambda_{v_2}^T v_1)\, d^4 v_2. \end{aligned} \tag{5.16}$$

Hence from (5.15) and (5.16) with N_A given in (5.8) the measure becomes

$$(8\pi^2)^8\, 16 d^4 y_1\, d^4 r_1\, d^4 y_2\, d^4 r_2 (1 + O(|d|^{-4})). \tag{5.17}$$

The appropriate statistical weight S_2 of course depends on the details of the parameterisation but in this case it can be written as $S_2 = 8n$, since in the extreme factorised configuration with

$$B \to \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}$$

there are four gauge equivalence transformations leaving B invariant $B = R^{-1}BR$ (corresponding as in (4.9) and (4.11) to the transformations generated by R_π, R_r) and also the transformation $R_{\pi/2}^{-1} B R_{\pi/2}$ interchanges y_1 and y_2. In each case $v' = vR$ can be implemented in terms of the variables v_1, v_2. Thus n is the number of distinct ways in which the $k = 2$ self-dual gauge field can be decomposed into two well-separated instantons for appropriate limits of its parameters. The resultant complete functional measure in this limit is then a product of two single instanton measures of the form (4.5), using now the factorisation of the determinants given by (5.9), with a statistical weight $1/2n$.

6. Conclusion

The results for the semiclassical approximation applied to nonabelian gauge theories are still very far from complete. Even when applied to self-dual gauge fields which are strict minima of the Euclidean action, so that the method is well defined, the main problem is the lack of a natural parameterisation for such general multi-instanton gauge fields. The intricacy of the expressions (4.50) and (4.51) for the $k=2$ case in terms of the HJNR parameterisation is probably a reflection of the lack of a good description of the relevant degrees of freedom even for the two instanton case.

Ultimately, the programme of attempting to compute the semiclassical approximation exactly as far as possible is only of real significance if the resulting statistical mechanical system is in a different phase from that describing well-separated instantons. The factorisation results show the absence of any long-range Coulombic-type force between instantons but with other descriptions such as in terms of hypothetical instanton quarks, the picture could well be entirely different. The well-known problems with the large N limit, if they are to be reconciled, in fact require the expansion over differing field configurations to be nonanalytic, as may be expected if the statistical mechanics are those appropriate to a plasma phase.

Appendix A

We discuss here initially the behaviour of the determinants introduced in Section 2 under conformal variations of the metric $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$, following procedures described by Schwarz [29]. The determinants are defined in the context of zeta function regularisation [19]. Thus for a positive self-adjoint operator θ this method defines

$$\ln \det{}' \theta/\mu = -\ln \mu \zeta_\theta(0) - \zeta'_\theta(0),$$

$$\zeta_\theta(s) = [1/\Gamma(s)] \int_0^\infty dt t^{s-1} \operatorname{tr}(e^{-t\theta} - P_\theta), \tag{A.1}$$

where P_θ is the projector on the zero modes of θ, $\theta P_\theta = 0$ and $\zeta_\theta(s)$ for $s \approx 0$ is defined by analytic continuation. Since $\Gamma(s) \sim s$ as $s \to 0$

$$\zeta_\theta(0) = \operatorname{tr}(e^{-t\theta})|_{t^0} - \operatorname{tr}(P_\theta), \tag{A.2}$$

where the notation on the r.h.s. of (A.2) indicates that only terms $O(t^0)$ are kept in a Laurent series expansion in t.

To apply this formalism to Δ, Δ^0 it is convenient to first introduce the elliptic operator

$$\Delta^{\mathrm{I}} = \Delta + \mathscr{D}\mathscr{D}^* \tag{A.3}$$

acting on vector fields a_μ in $\mathscr{L}_{\mathscr{C}}$. Using the decomposition (2.24) and (2.15) it is easy to see that

$$\operatorname{tr}_{a_\mu}(e^{-t\Delta^1}) = \operatorname{tr}_{a_\mu^t}(e^{-t\Delta}) + \operatorname{tr}_\omega(e^{-t\Delta^0})$$
$$\ln \det{}' \Delta/\mu^2 = \ln \det{}' \Delta^1/\mu^2 - \ln \det{}' \Delta^0/\mu^2. \tag{A.4}$$

Under an infinitesimal conformal change $\Omega^2 = 1 + \delta\sigma$ we find

$$\delta\Delta^1 = -\delta\sigma\Delta^1 + \delta\sigma\mathscr{D}\mathscr{D}^* + \mathscr{D}\mathscr{D}^*\delta\sigma - 2\mathscr{D}\delta\sigma\mathscr{D}^*,$$
$$\delta\Delta^0 = -2\delta\sigma\Delta^0 + \mathscr{D}^*\delta\sigma\mathscr{D}, \tag{A.5}$$

and with $\Delta^1\mathscr{D} = \mathscr{D}\Delta^0$, $\mathscr{D}^*\Delta^1 = \Delta^0\mathscr{D}^*$ following from (2.15),

$$\delta\{\operatorname{tr}(e^{-t\Delta^1}) - 2\operatorname{tr}(e^{-t\Delta^0})\} = -t\frac{d}{dt}\{\operatorname{tr}(e^{-t\Delta^1}\delta\sigma) - 2\operatorname{tr}(e^{-t\Delta^0}\delta\sigma)\}. \tag{A.6}$$

Hence using (A.4)

$$-\delta(\ln\det{}'\Delta - \ln\det{}'\Delta^0)$$
$$= \operatorname{tr}((e^{-t\Delta^1} - P_{\Delta^1})\,\delta\sigma)|_{t^0} - 2\operatorname{tr}((e^{-t\Delta^0} - P_{\Delta^0})\,\delta\sigma)|_{t^0}. \tag{A.7}$$

This expression, along with $\zeta_{\Delta^1}(0)$, $\zeta_{\Delta^0}(0)$ given by (A.2), may be evaluated by considering the usual expression as $t \to 0$ of the heat kernels $\mathscr{G}^{1\nu}_\mu$, $\mathscr{G}^0$ which correspond to the operators $e^{-t\Delta^1}$, $e^{-t\Delta^0}$

$$\operatorname{tr}\mathscr{G}^i(x,x;t) \sim \sum_{n=0} B^i_{2n}(x)\,t^{n-2}, \qquad i = 0, 1, \tag{A.8}$$

where the tr here denotes a trace over both group and Lorentz indices. Applying (A.8) to (A.2) and (A.7)

$$\zeta_{\Delta^i}(0) = \int dv B^i_4 - \operatorname{tr}(P_{\Delta^i}),$$
$$-\delta(\ln\det{}'\Delta - \ln\det{}'\Delta^0)$$
$$= \int dv\delta\sigma(B^1_4 - 2B^0_4) - \operatorname{tr}(P_{\Delta^1}\delta\sigma) + 2\operatorname{tr}(P_{\Delta^0}\delta\sigma). \tag{A.9}$$

By standard calculations [39]

$$16\pi^2 B^0_4 = \frac{1}{12}\operatorname{tr}(F^{cad}_{\mu\nu}F^{cad\mu\nu}) + \frac{d_{\mathscr{G}}}{180}\left(C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} + R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{3}R^2\right)$$
$$+ \frac{d_{\mathscr{G}}}{72}R^2 + \frac{d_{\mathscr{G}}}{30}\nabla^2 R,$$

$$16\pi^2 B^1_4 = -\frac{5}{3}\operatorname{tr}(F^{cad}_{\mu\nu}F^{cad\mu\nu}) - \frac{11 d_{\mathscr{G}}}{180}C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}$$
$$+ \frac{16 d_{\mathscr{G}}}{45}\left(R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{3}R^2\right) + \frac{d_{\mathscr{G}}}{36}R^2 - \frac{d_{\mathscr{G}}}{30}\nabla^2 R. \tag{A.10}$$

where $C_{\mu\nu\alpha\beta}$ is the Weyl conformal tensor, $R_{\alpha\beta}$ the Ricci tensor, R the curvature scalar and $\mathrm{tr}(X^{ad}Y^{ad}) = 2C_2(\mathscr{G})\,\mathrm{tr}(XY)$. The crucial part of B_4^1 for present purposes may be verified by considering the calculation of B_4^1 just on flat space as was done by Corrigan *et al.* [19] for B_4^0. In this case, working with the fundamental representation for simplicity,

$$\begin{gathered} \Delta^1_{\mu\nu} = -\delta_{\mu\nu} D^2 - 2F_{\mu\nu}, \qquad D_\mu = \partial_\mu + A_\mu, \\ \mathscr{G}^1(x,y;t)_{\mu\nu} = (1/16\pi^2 t^2)\, e^{-|x-y|^2/4t} \sum_{n=0} a_n^1(x,y)_{\mu\nu} t^n, \\ a_0^1(x,x)_{\mu\nu} = \delta_{\mu\nu}. \end{gathered} \tag{A.11}$$

It is easy to derive

$$\begin{gathered} n a_n^1(x,y) + (x-y)\cdot D_x a_n^1(x,y) = -\Delta^1 a_{n-1}^1(x,y), \\ (x-y)\cdot D_x a_0^1(x,y) = 0 \end{gathered}$$

and solve these iteratively to obtain

$$\begin{aligned} &a_1^1(x,x)_{\mu\nu} = 2F_{\mu\nu}(x), \\ &a_2^1(x,x)_{\mu\nu} = \tfrac{1}{12} F_{\alpha\beta}(x) F_{\alpha\beta}(x)\,\delta_{\mu\nu} + 2F_{\mu\alpha}(x) F_{\alpha\nu}(x) + \tfrac{1}{3}\mathscr{D}^2 F_{\mu\nu}(x). \end{aligned} \tag{A.12}$$

Hence $\mathrm{tr}\, a_2^1 = -\frac{5}{3}\mathrm{tr}(F_{\mu\nu}F_{\mu\nu})$ in accord with (A.10). Applying this with (A.9)

$$\begin{aligned} &-\delta(\ln \det{}' \Delta/\det \Delta_0 - \ln \det{}' \Delta^0/\det{}' \Delta^0_0) \\ &\quad = -\frac{11}{3} C_2(\mathscr{G}) \frac{1}{16\pi^2} \int dv\,\delta\sigma\, \mathrm{tr}(F^c_{\mu\nu} F^{c\mu\nu}) - \mathrm{tr}(P_{\Delta^1}\delta\sigma) \\ &\quad\quad + 2\,\mathrm{tr}((P_{\Delta^0} - P_{\Delta^0_0})\,\delta\sigma). \end{aligned} \tag{A.13}$$

Since we have taken the zero modes of Δ^0 to be constant elements in $\mathscr{L}_{\mathscr{F}}$, (2.18), $\mathrm{tr}(P_{\Delta^0}) = d_{\mathscr{F}}/V$ and hence as $\delta\sqrt{g} = 2\delta\sigma\sqrt{g}$

$$\mathrm{tr}((P_{\Delta^0} - P_{\Delta^0_0})\,\delta\sigma) = \tfrac{1}{2}(d_{\mathscr{F}} - d_{\mathscr{G}})\,\delta V/V. \tag{A.14}$$

For P_{Δ^1} we assume that the only solutions of $\Delta^1 a = 0$ correspond to $\Delta a' = 0$ (this is certainly true for $F^c_{\mu\nu} = \pm {}^*F^c_{\mu\nu}$ by considering $(a, \Delta^1 a)$ and noting $(a, \Delta a) = (T^\pm, T^\pm) \geqslant 0$ for $T^\pm_{\alpha\beta} = (\mathscr{D}a)_{\alpha\beta} \pm {}^*(\mathscr{D}a)_{\alpha\beta}$). Thus for N zero modes $\{Z_{r\mu}\}$ and $J_{rs} = (Z_r, Z_s)$

$$\mathrm{tr}\, P_{\Delta^1}(x,y)_{\mu\nu} = -2\,\mathrm{tr}(Z_{r\mu}(x) Z_{s\nu}(y))\, J^{-1}_{sr}. \tag{A.15}$$

Under a conformal change, as $\mathscr{D}^* Z_r = 0$, $\delta Z_r = \mathscr{D}\delta A_r$, where δA_r can be determined in terms of $\delta\sigma$, but since $(\mathscr{D}\delta A_r, Z_s) = 0$

$$\mathrm{tr}(P_{\Delta^1}\delta\sigma) = \delta J_{rs} J^{-1}_{sr}. \tag{A.16}$$

From (A.13), (A.14) and (A.16) in conjunction with $\zeta_{\Delta^1}(0)$, $\zeta_{\Delta^0}(0)$ we have as far as their dependence on the conformal scale Ω^2 and scale mass μ is concerned ($V=\int dv\Omega^4$)

$$-[\ln\{\det'(\Delta/\mu^2)/\det(\Delta_0/\mu^2)\} - \ln\{\det'(\Delta^0/\mu^2)/\det'(\Delta^0_0/\mu^2)\}] = \tilde{\mathscr{R}}$$

$$\sim -\frac{11}{3}C_2(\mathscr{G})\frac{1}{16\pi^2}\int dv \ln\mu^2\Omega^2 \operatorname{tr}(F^c_{\mu\nu}F^{c\mu\nu}) - \ln\det J\mu^2$$

$$-(d_{\mathscr{G}}-d_{\mathscr{H}})\ln V\mu^4. \tag{A.17}$$

This combination of $\det'\Delta$ and $\det'\Delta^0$ is exactly what enters in (2.30) and we can see how the explicit dependence on V cancels between (2.30) and (A.17) in support of the weak argument based on the limit $V\to 0$ and (2.39).

A similar result can be obtained for the conformal scalar operator $M=\Delta^0+\frac{1}{6}R$ for which $\delta M=\frac{1}{2}(M\delta\sigma-3\delta\sigma M)$ and as in Corrigan *et al.* [19]

$$-\ln\{\det(M/\mu^2)/\det(M_0/\mu^2)\} = \tilde{\mathscr{D}}$$

$$\sim \frac{1}{6}C_2(\mathscr{G})\frac{1}{16\pi^2}\int dv \ln\mu^2\Omega^2 \operatorname{tr}(F^c_{\mu\nu}F^{c\mu\nu}). \tag{A.18}$$

In the flat space limit $M\to -\mathscr{D}^2$, as does Δ^0, but here there are no problems with eigenvalues behaving like $1/V$ due to the presence of the R term. In the conformally flat case, $g_{\mu\nu}=\Omega^2\delta_{\mu\nu}$, (A.17) and (A.18) give the entire dependence on the metric and for $F^c_{\mu\nu}={}^*F^c_{\mu\nu}$ using (2.39) and (2.48) gives a relation which should be valid on any conformally flat manifold

$$\tilde{\mathscr{R}} = 2\tilde{\mathscr{D}} - 4C_2(\mathscr{G})\frac{1}{16\pi^2}\int d^4x \ln\Omega^2 \operatorname{tr}(F^c_{\mu\nu}F^c_{\mu\nu})$$

$$-\ln\det J/\det J_0 + \ln\det I_0/\det b - (d_{\mathscr{G}}-d_{\mathscr{H}})\ln V\mu^2, \tag{A.19}$$

where J_0, I_0 refer to normalisation matrices on flat space $\Omega^2=1$.

We now proceed to verify (A.19) for arbitrary k on S^4 for which $\Omega(x)=2(1+x^2/a^2)^{-1}$, $V=\frac{8}{3}\pi^2a^4$ by letting $\mathscr{G}=U(2k)$, and considering the special configuration of k commuting $SU(2)$ instantons centred at $x=0$ with scale a, $A^{\text{diag}}_\mu = A^1_\mu\times 1_k$, as suggested by Berg and Luscher [19]. In this case the eigenvalues for M, Δ, Δ^0 are the same as for $k=1$ but with degeneracy $\times k^2$ so that $\tilde{\mathscr{D}}=k^2\tilde{\mathscr{D}}_1$, $\tilde{\mathscr{R}}=k^2\tilde{\mathscr{R}}_1$ and using $O(5)$ symmetry the eigenvalues and their degeneracies can be explicitly computed. The zeta function determinants were determined by Chadha *et al.* [8], who found

$$\tilde{\mathscr{D}}_1 = -\tfrac{1}{3}\ln\mu^2a^2 + 8\zeta'_R(-1) - \ln 2 + \tfrac{5}{3},$$

$$\tilde{\mathscr{R}}_1 = -\tfrac{11}{3}\ln\mu^2a^2 + 16\zeta'_R(-1) + 2\ln 2 + 3\ln 3 + 5\ln 5 - 10, \tag{A.20}$$

where $\zeta_R(s)$ is the standard Riemann zeta function. In this case we have $\mathscr{H} = U(k)$, corresponding to elements of $U(2k)$ of the form $g = 1_2 \times h$, $h \in U(k)$, and $d_{U(2k)} = 4k^2$, $d_{U(k)} = k^2$, $C_2(U(2k)) = 2k$. Also as $\operatorname{tr}(F_{\mu\nu}F_{\mu\nu}) = -96ka^4/(x^2+a^2)^4$ in this special case then

$$-(1/16\pi^2)\int d^4x \ln \Omega^2 \operatorname{tr}(F_{\mu\nu}F_{\mu\nu}) = k(2\ln 2 - 5). \tag{A.21}$$

To verify (A.19) it now remains only to determine J, J_0, I_0 and b as defined in (2.28) and (2.38). In terms of the ADHM construction with the notation of Section 3 for this special case

$$\underset{\sim}{a} = a\begin{pmatrix} 1_{2k} \\ 0 \end{pmatrix}, \qquad \underset{\sim}{b} = \begin{pmatrix} 0 \\ 1_{2k} \end{pmatrix}, \tag{A.22}$$

where the matrices $\underset{\sim}{a}$, $\underset{\sim}{b}$ introduced in (3.2) are so distinguished to avoid subsequent confusion. The gauge modes Z^g_μ correspond to the $3k^2$ generators of $U(2k)/U(k)$ which can be represented as $\sigma_i \times \lambda_l$ $i = 1, 2, 3$, $l = 1 \cdots k^2$, λ_l being generators of $U(k)$, with $\operatorname{tr}(\lambda_l\lambda_m) = \delta_{lm}$ (in (3.39) we can take $q = 1_2 \times R^\dagger$, $R \in U(k)$ which identifies $\mathscr{H} = U(k)$ once more). Thus as in (3.24) we may take

$$C^g_{li} = a\begin{pmatrix} \omega_{li} \\ 0 \end{pmatrix}, \qquad \omega_{li} = \tfrac{1}{2} i\lambda_l \sigma_i,$$

which gives, using (3.18),

$$\begin{gathered} b = 1_{k^2} \times 1_3, \qquad I_0 = 4\pi^2 a^2\, 1_{k^2} \times 1_3, \\ \det I_0 / \det b = (4\pi^2 a^2)^{3k^2}. \end{gathered} \tag{A.23}$$

The nongauge modes $Z_{r\mu}$ correspond in this special case to infinitesimal translations and dilations of the individual $SU(2)$ instantons. These can be represented as in (3.12) with

$$C^t_{li} = \begin{pmatrix} 0 \\ \lambda_l e_i \end{pmatrix} = \underset{\sim}{b}\lambda_l e_i, \qquad \begin{pmatrix} \lambda_l \\ 0 \end{pmatrix} = (1/a)\,\underset{\sim}{a}\lambda_l, \tag{A.24}$$

which from (3.18) again leads to

$$J_0 = 8\pi^2\, 1_{k^2} \times \begin{pmatrix} 1_4 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \det J_0 = (8\pi^2)^{5k^2} 2^{k^2}. \tag{A.25}$$

On S^4, $J_{rs} = (\tilde{Z}_r, \tilde{Z}_s)$, where the gauge condition is now $\mathscr{D}_\mu(\Omega^2 \tilde{Z}_{r\mu}) = 0$ and $\tilde{Z}_{r\mu} = Z_{r\mu} + \mathscr{D}_\mu \Lambda_r$. Since $x_\mu Z^d_{l\mu} = 0$, using (A.24) in (3.12), we have trivially $\tilde{Z}^d_{l\mu} = Z^d_{l\mu}$. For $\tilde{Z}^t_{li\mu}$, if we define

$$\tilde{C}^t_{li} = C^t_{li} - (1/2a)\, G_{li}\underset{\sim}{\Delta}, \qquad G_{li} = \begin{pmatrix} 0 & -\bar{e}_i \\ e_i & 0 \end{pmatrix} \times \lambda_l$$

which since G_{ll} is a generator of $U(4k)$ corresponds from (3.20), (3.21) and (3.22), when $\tilde{C}^t_{li}$ is inserted in (3.12) to give $\tilde{Z}^t_{li\mu}$, to the required relation $\tilde{Z}^t_{li\mu} = Z^t_{li\mu} + \mathscr{D}_\mu A_{li}$. Elementary calculations can now verify $\mathscr{D}_\mu(\Omega^2 \tilde{Z}^t_{li\mu}) = 0$ in this case using $f(x) = 1_k(x^2+a^2)^{-1}$. To calculate the normalisation matrix we have to perform the integrals by hand in this case using

$$-2\,\mathrm{tr}(Z^d_{l\mu} Z^d_{m\mu}) = [48x^2a^2/(x^2+a^2)^2]\,\delta_{lm},$$
$$-2\,\mathrm{tr}(Z^t_{li\mu} Z^t_{mj\mu}) = [12/(x^2+a^2)^4]\,\delta_{lm}(\delta_{ij}(x^2+a^2)^2 - 4x_ix_ja^2)$$

and find

$$J = \frac{32\pi^2}{5} 1_{k^2} \times 1_5, \qquad \det J = (32\pi^2/5)^{5k^2} \tag{A.26}$$

Hence $\det J/\det J_0 = (2^9/5^5)^{k^2}$ and with (A.20), (A.21), and (A.23) this verifies (A.19) confirming the relations between the determinants on which it was based. This relation is also implicit in previous work [7–9] for $k=1$ insofar as the same result was obtained ultimately independent of whether the calculation was conducted on S^4, using an $O(5)$ formalism, or on flat space.

Appendix B

We attempt here to summarise the results [19] of previous investigations concerning multi-instanton functional determinants. For use in semiclassical functional integrals it is only necessary, by virtue of Section 2 and Appendix A, to discuss the determinant of the conformal covariant Laplacian $M = \Delta^0 + \frac{1}{6}R$ on conformally flat spaces $g_{\mu\nu} = \Omega^2\delta_{\mu\nu}$ (the determinant of the Dirac operator can be similarly reduced to this case). For M acting on the fundamental representation of the gauge group $\mathscr{G}$, so that on flat space $M = -D^2$, $D_\mu = \partial_\mu + A_\mu$, the basic formula is

$$\mathscr{D}_k = -\ln\left\{\frac{\det(M/\mu^2)}{\det(M_0/\mu^2)}\right\}$$
$$= \frac{1}{12}\left(I + \theta + \frac{1}{16\pi^2}\int d^4x \ln\Omega^2\,\mathrm{tr}(F_{\mu\nu}F_{\mu\nu}) - k\left(\ln\mu^2 + \frac{7}{3}\right)\right) - \alpha\left(\frac{1}{2}\right)k, \tag{B.1}$$

where $\alpha(\frac{1}{2}) = -2\zeta'(-1) - \frac{1}{6}\ln 2 - \frac{5}{72}$ is a constant introduced and tabulated by 't Hooft [7], and I, θ are integral expressions in terms of the basic matrices a, b

$$I = \lim_{R^2\to\infty}\left\{\frac{1}{\pi^2}\int_{x^2<R^2} d^4x(5\,\mathrm{tr}(fvfv) - 4\,\mathrm{tr}(fc_\alpha fc_\alpha fc_\beta fc_\beta)) - k\ln R^2\right\},$$

$$c_\alpha = \frac{1}{2}\partial_\alpha f^{-1},$$

$$\theta = -(1/4\pi^2)\frac{1}{5}\int d^4x \int_0^1 dx_5\, \varepsilon_{\alpha\beta\gamma\delta\rho}\, \mathrm{tr}(m_\alpha m_\beta m_\gamma m_\delta m_\rho),$$

$$m_\alpha = l^{-1}\partial_\alpha l, \qquad l(x, x_5) = f(x)^{-1} x_5 + (1+x^2)\, 1_k(1-x_5). \tag{B.2}$$

Equations (B.1) and (B.2) are obtained by considering the variation of the determinant under variations δa, δb and then fixing possible additive constants by restricting to the special case of $\mathscr{G} = Sp(k)$ on S^4 with $A_\mu \to A_\mu^{\mathrm{diag}} = A_\mu^1 \times 1_k$, as discussed in Appendix A, so that $\mathscr{D}_k$ takes the value $k\mathscr{D}_1(\frac{1}{2})$ with, as given by Chadha *et al.* [8] with zeta function regularisation,

$$\mathscr{D}_1(\tfrac{1}{2}) = -\tfrac{1}{12}\ln \mu^2 a^2 - (\alpha(\tfrac{1}{2}) + \tfrac{1}{6}\ln 2 - \tfrac{5}{36}) \tag{B.3}$$

the value of the $k=1$ determinant on S^4 for the 2 dimensional representation of $Sp(1)$.

For the gauge field obtained by taking the direct product of the fundamental representation with itself, $A_\mu^{\mathrm{d.p.}} = 1_{\mathscr{G}} \times A_\mu + A_\mu \times 1_{\mathscr{G}}$, Jack [19] was able to show, in an exactly analogous fashion,

$$\mathscr{D}^{\mathrm{d.p.}}_{2N_{\mathscr{G}}k} = 2N_{\mathscr{G}}\mathscr{D}_k + \ln\det\{M(\nu\times\nu)\} + J,$$

$$J = -\frac{1}{16\pi^2}\int d^4x \ln\det f\nu\, \partial^2\partial^2 \ln\det f\nu,$$

$$-\partial^2\partial^2 \ln\det f\nu = \mathrm{tr}(F_{\mu\nu}F_{\mu\nu}) \tag{B.4}$$

with $N_{\mathscr{G}}$ the dimension of the fundamental representation for $\mathscr{G}$ and M the matrix acting on $W\times W$ which plays a crucial role in the ADHM formalism for tensor products and was relevant in Section 3

$$M^{-1} = \nu\times\mu + \mu\times\nu - a^\dagger b \times b^\dagger a \tag{B.5}$$

Further, for the associated gauge field A_μ^A pertaining to the representation of $\mathscr{G}$ defined on the antisymmetric tensor product space the result is

$$\mathscr{D}^A_{(N_{\mathscr{G}}-2)k} = (N_{\mathscr{G}}-2)\,\mathscr{D}_k + \tfrac{1}{2}(J - 2\ln N_S - I - \theta + \tfrac{3}{2}k), \tag{B.6}$$

with

$$\ln N_S = -\ln\det\{M_S(\nu\times\nu)\} - k\ln 2, \tag{B.7}$$

where M_S, M_A act on the symmetric, antisymmetric product spaces $(W\times W)_{S,A}$, $\det M = \det M_S \det M_A$. The constants in (B.4) and (B.6) are determined in a similar manner as above, since in the special case considered there $\mathscr{D}^{\mathrm{d.p.}}_{4k^2} = k^2\mathscr{D}_1(1)$, $\mathscr{D}^A_{2k(k-1)} = \frac{1}{2}k(k-1)\,\mathscr{D}_1(1)$, where $\mathscr{D}_1(1)$ is the single $Sp(1)\simeq SU(2)$ instanton determinant on S^4 for the three-dimensional representation [8].

$$\mathscr{D}_1(1) = -\tfrac{1}{3}\ln\mu^2 a^2 - (\alpha(1) + \tfrac{2}{3}\ln 2 - \tfrac{5}{9}),$$
$$\alpha(1) = -8\zeta'(-1) + \tfrac{1}{3}\ln 2 - \tfrac{5}{9}, \tag{B.8}$$

with $\alpha(1)$ also given by 't Hooft [7].

For $\mathscr{G} = Sp(1)$ these results provide further constraints since then $(2\times 2)_A = 1$, $\mathscr{D}_0^A = 0$ and the corresponding determinant for the adjoint representation is given by $\tilde{\mathscr{D}}_k = \mathscr{D}_{4k}^{\mathrm{d.p.}}$. Thus on flat space $\Omega^2 = 1$,

$$\mathscr{D}_k = \tfrac{1}{12}(J - 2\ln N_S - k(\ln\mu^2 + \tfrac{5}{6})) - \alpha(\tfrac{1}{2})\,k,$$
$$\tilde{\mathscr{D}}_k = \tfrac{1}{3}(4J - 5\ln N_S - 3\ln N_A - k(\ln\mu^2 + \tfrac{10}{3})) - \alpha(1)\,k, \tag{B.9}$$

defining as in (B.7)

$$\ln N_A = -\ln\det\{M_A(\nu\times\nu)\}. \tag{B.10}$$

It remains to determine J, but this has not yet been achieved explicitly even for $k=2$. However, an approximate form has been given by Osborn and Moody [19] which is correct in all limits when the determinant becomes singular, becoming exact for $k=1$

$$J \approx \ln N_S + \ln N_A + \tfrac{5}{6}k. \tag{B.11}$$

Hence finally for gauge group $Sp(1)$

$$\mathscr{D}_k \approx -\tfrac{1}{12}(\ln N_S - \ln N_A + k\ln\mu^2) - \alpha(\tfrac{1}{2})\,k,$$
$$\tilde{\mathscr{D}}_k \approx -\tfrac{1}{3}(\ln N_S - \ln N_A + k\ln\mu^2) - \alpha(1)\,k. \tag{B.12}$$

For the 't Hooft solution expressions for N_A and N_S in terms of the corresponding parameters have been given for arbitrary k [19]; for $k=2$ the results are given in (4.47) and (4.52). For $k=1$ (B.12) takes the form

$$\mathscr{D}_1 = -\tfrac{1}{12}\ln\mu^2\rho^2 - \alpha(\tfrac{1}{2}),$$
$$\tilde{\mathscr{D}}_1 = = \tfrac{1}{3}\ln\mu^2\rho^2 - \alpha(1), \tag{B.13}$$

with ρ the instanton scale parameter.

Appendix C

The conformally extended HJNR solution [22] has been expressed in terms of the ADHM construction for $\mathscr{G} = Sp(1)$ by taking a, b of the form [33]

$$a = \begin{pmatrix} \lambda_1 y_0 & \cdots & \lambda_k y_0 \\ \lambda_0 y_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_0 y_k \end{pmatrix}, \qquad b = -\begin{pmatrix} \lambda_1 & \cdots & \lambda_k \\ \lambda_0 & & 0 \\ & \ddots & \\ 0 & & \lambda_0 \end{pmatrix} \times 1_2 \tag{C.1}$$

This can be reexpressed in the canonical form (3.37) by a transformation (3.8). For $k = 1$ we can take, with $w = (\lambda_0^2 + \lambda_1^2)^{-1/2}$

$$Q = w \begin{pmatrix} \lambda_0 & -\lambda_1 \\ \lambda_1 & \lambda_0 \end{pmatrix}, \qquad Qbw = -\begin{pmatrix} 0 \\ 1_2 \end{pmatrix} \tag{C.2}$$

and then

$$Qaw = w^2 \begin{pmatrix} \lambda_0 \lambda_1 (y_0 - y_1) \\ \lambda_1^2 y_0 + \lambda_0^2 y_1 \end{pmatrix} \tag{C.3}$$

identifying the instanton position and scale, as in (4.1), in this example.

For $k = 2$, in which case the parameterisation (C.1) is still general, then based on (C.2) a fairly natural form for $Q \in O(3)$ emerges

$$Q = \begin{pmatrix} z\lambda_0 & -z\lambda_1 & -z\lambda_2 \\ w\lambda_1 & w\lambda_0 & 0 \\ wz\lambda_0\lambda_2 & -wz\lambda_1\lambda_2 & z/w \end{pmatrix}, \qquad R = \begin{pmatrix} w & -wz\lambda_1\lambda_2/\lambda_0 \\ 0 & z/w\lambda_0 \end{pmatrix} \tag{C.4}$$

for $z = (\lambda_0^2 + \lambda_1^2 + \lambda_2^2)^{-1/2}$. With the Q, R in (C.4) then

$$QaR = \begin{pmatrix} v \\ B \end{pmatrix}, \qquad QbR = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} \times 1_2,$$

where

$$\begin{aligned} v &= w(z\lambda_0\lambda_1(y_0 - y_1), z^2\lambda_2\{\lambda_0^2(y_0 - y_2) + \lambda_1^2(y_1 - y_2)\}), \\ B &= w^2 \begin{pmatrix} \lambda_1^2 y_0 + \lambda_0^2 y_1 & z\lambda_0\lambda_1\lambda_2(y_0 - y_1) \\ z\lambda_0\lambda_1\lambda_2(y_0 - y_1) & z^2\lambda_2^2(\lambda_0^2 y_0 + \lambda_1^2 y_1) + (z^2/w^4) y_2 \end{pmatrix}. \end{aligned} \tag{C.5}$$

From (C.5), (4.43) and (4.44) can be straightforwardly read off.

ACKNOWLEDGMENTS

I am grateful to P. Goddard and P. Mansfield for many helpful conversations over a long period. Various results were developed in parallel by differing methods. Also, I have benefited from written communications from E. Corrigan.

Note added in proof. For $\mathscr{G} = O(n)$ the counting given in the text has to be slightly modified. On the left-hand side of (2.3) we should take $2k$ and with $C_2(O(n)) = n - 2$, $n \geqslant 5$ we should have in this case $\mathrm{tr}(X^{\mathrm{ad}} Y^{\mathrm{ad}} = C_2(O(n))\ \mathrm{tr}(XY)$.

The topological charge for gauge theories on a lattice has recently been discussed by P. di Vecchia, K. Fabricius, G. C. Rossi and G. Veneziano, CERN preprint TH 3091 and also M. Luscher, Bern preprint BUTP 1981-10.

REFERENCES

1. A. A. BELAVIN, A. M. POLYAKOV, A. S. SCHWARZ, AND YU. S. TYUPKIN, *Phys. Lett. B* **59** (1975), 85.
2. R. JACKIW AND C. REBBI, *Phys. Rev. Lett.* **37** (1976), 172. C. CALLAN, R. DASHEN, AND D. GROSS, *Phys. Lett. B* **63** (1976), 334.
3. E. WITTEN, *Nucl. Phys. B* **56** (1979), 269.
4. S. COLEMAN, The uses of instantons, *in* "The Why's of Subnuclear Physics: Proceedings of the 1977 International Enrico Fermi School of Physics" (A. Zichichi, Ed.), Plenum, New York, 1979.
5. P. COTTA-RAMUSINO AND G. DELL'ANTONIO, *Comm. Math. Phys.* **70** (1979), 75.
6. K. K. UHLENBECK, *Bull. Amer. Math. Soc.* **1** (1979), 579.
7. G. 'T HOOFT, *Phys. Rev. D* **14** (1976), 3432; *Phys. Rev. D* **18**(1978), 2199(E).
8. A. A. BELAVIN AND A. M. POLYAKOV, *Nucl. Phys. B* **123** (1977), 429. F. F. ORE, *Phys. Rev. D* **16** (1977), 2577. S. CHADHA, P. DI VECCHIA, A. D'ADDA, AND F. NICODEMI, *Phys. Lett. B* **72** (1977), 103. G. M. SHORE, *Ann. Phys. (N.Y.)* **122** ((1979), 321.
9. YU. A. BASHILOV AND S. V. POKROVSKY, *Nucl. Phys. B* **143** (1978), 431. C. BERNARD, *Phys. Rev. D* **19** (1979), 3013.
10. N. ANDREI AND D. GROSS, *Phys. Rev. D* **18** (1978), 468. L. BAULIEU, J. ELLIS, M. K. GAILLARD, AND W. J. ZAKREWSKI, *Phys. Lett. B* **77** (1978), 290; **81** (1979), 224: **81** (1979), 41. M. SUZUKI, *Phys. Lett. B* **76** (1978, 466. S.-S. SHEI, *Nucl. Phys. B* **156** (1979), 225. J. HIETARINTA, W. F. PALMER, AND S. S. PINSKY, *Phys. Rev. Lett.* **38** (1977), 103; **39** (1977), 850(E). J. HIETARINA AND W. F. PALMER, *Phys. Lett. B* **71** (1977), 415. E. MOTTOLA, *Phys. Rev. D* **17** (1978), 1103. R. D. CARLITZ, *Phys. Rev. D* **17** (1978), 3225. R. D. CARLITZ AND C. LEE, *Phys. Rev. D* **17** (1978), 3238. R. D. CARLITZ AND D. B. CREAMER, *Ann. Phys. (N.Y.)* **118** (1979), 118. C. LEE AND W. A. BARDEEN, *Nucl. Phys. B* **153** (1979), 210. W. F. PALMER AND S. S. PINSKY, *Particles and Fields* **3** (1980, 209.
11. C. G. CALLAN, R. DASHEN, AND D. J. GROSS, *Phys. Rev. D* **17** (1978), 2717.
12. C. G. CALLAN, R. F. DASHEN, AND D. J. GROSS, *Phys. Lett. B* **78** (1978), 307; *Phys. Rev. D* **20** (1979), 1826. H. LEVINE AND L. G. YAFFE, *Phys. Rev. D* **19** (1979), 1225. L. G. YAFFE, *Nucl. Phys. B* **151** (1979), 247. H. LEVINE, *Nucl. Phys. B* **157** (1979), 237. A. JEVICKI, *Phys. Rev. D* **21** (1980), 992. E.-M. ILGENFRITZ, D. I. KAZAKOV, AND M. MUELLER-PREUSSKER, *Phys. Lett. B* **87** (1979), 242. Y. HOSOTANI AND H. YAMAGISHI, preprint EF1 79/59, Y. HOSOTANI, preprint EF1 80/14.
13. A. M. POLYAKOV, *Nucl. Phys. B* **120** (1977), 429.
14. E. WITTEN, *Nucl. Phys. B* **149** (1979), 285.
15. A. JEVICKI, *Phys. Rev. D* **20** (1979), 3331.
16. I. AFFLECK, *Phys. Lett. B* **92** (1980), 149; *Nucl. Phys. B* **162** (1979), 461; **171** (1980), 420.
17. V. A. FATEEV, I. V. FROLOV, AND A. S. SCHWARZ, *Nucl. Phys. B* **154** (1979),1. B. BERG AND M. LUSCHER, *Comm. Math. Phys.* **69** (1979), 57.
18. M. F. ATIYAH, N. J. HITCHIN, V. G. DRINFELD, AND YU. I. MANIN, *Phys. Lett. A* **65** (1978), 185. V. G. DRINFELD AND YU. I. MANIN, *Comm. Math. Phys.* **63**(1978), 177.
19. E. CORRIGAN, P. GODDARD, H. OSBORN, AND S. TEMPLETON, *Nucl. Phys. B* **159** (1979), 469. B. BERG AND M. LUSCHER, *Nucl. Phys. B* **160** (1979), 281. H. OSBORN, *Nucl. Phys. B* **159** (1979), 497. I. JACK, *Nucl. Phys. B* **174** (1980), 526. H. OSBORN AND G. P. MOODY, *Nucl. Phys. B* **173** (1980), 422. B. BERG AND J. STEHR, *Nucl. Phys. B* **173** (1980), 422. B. BERG AND J. STEHR, *Nucl. Phys. B* **175** (1980), 293.
20. P. GODDARD, P. MANSFIELD, AND H. OSBORN, *Phys. Lett. B* **98** (1981), 59.
21. A. S. SCHWARZ, *Phys. Lett. B* **67** (1977), 172. R. JACKIW AND C. REBBI, *Phys. Lett. B* **67** (1977), 189. M. F. ATIYAH, N. J. HITCHIN, AND I. M. SINGER, *Proc. Roy. Soc. A* **362** (1978), 425. C. W. BERNARD, N. H. CHRIST, A. H. GUTH, AND E. WEINBERG, *Phys. Rev. D* **16** (1977), 2967.
22. R. JACKIW, C. NOHL, AND C. REBBI, *Phys. Rev. D* **15** (1977), 1642.
23. N. H. CHRIST, E. WEINBERG, AND N. K. STANTON, *Phys. Rev. D* **18** (1978), 2013.

24. A. A. BELAVIN, V. A. FATEEV, A. S. SCHWARZ, AND YU. S. TYUPKIN, *Phys. Lett. B* **83** (1979), 317.
25. B. BERG, DESY preprint 79/58. H. LEUTWYLER, *Nucl. Phys. B* **179** (1981), 129. M. LUSCHER, *in* "Recent Developments in Gauge Theories" (G. 't Hooft, Ed.), Proceedings, 1979 Nato Advanced Study Institute in Cargèse, Plenum, New York, 1980.
26. M. A. SHIFMAN, A. I. VAINSHTEIN, AND V.I. ZAKHAROV, *Nucl. Phys. B* **147** (1979), 385.
27. K. BARDAKCI, D. G. CALDI, AND H. NEUBERGER, *Nucl. Phys. B* **177** (1981), 282. W. NAHM, *Phys. Lett. B* **96** (1980), 323. M. F. ATIYAH, unpublished.
28. E. S. ABERS AND B. W. LEE, *Phys. Rep. C* **9** (1973), 1.
29. A. S. SCHWARZ, *Comm. Math. Phys.* **64** (1979), 233.
30. A. D'ADDA AND P. DI VECCHIA, *Phys. Lett. B* **73** (1978), 162.
31. S. ROUHANI, *Nucl. Phys. B* **182** (1981), 462.
32. M. HORTACSU, K. D. ROTH, AND B. SCHROER, *Nucl. Phys. B* **171** (1980), 530.
33. E. CORRIGAN, D. B. FAIRLIE, P. GODDARD, AND S. TEMPLETON, *Nucl. Phys. B* **140** (1978), 31.
34. H. OSBORN, *Nucl. Phys. B* **140** (1978), 45.
35. E. CORRIGAN, P. GODDARD, AND S. TEMPLETON, *Nucl. Phys. B* **151** (1979), 93.
36. E. CORRIGAN, unpublished.
37. E. CORRIGAN AND P. GODDARD, "Some Aspects of Instantons," DAMTP 79/80.
38. C. BERNARD, *Phys. Rev. D* **18** (1978), 2026.
39. S. M. CHRISTENSON AND M. J. DUFF, *Nucl. Phys. B* **154** (1979), 301. M. DANIEL, *Comm. Math. Phys.* **73** (1980), 95.