The Index of Elliptic Operators: II

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By M. F. Atiyah and G. B. Segal

Introduction

The purpose of this paper is to show how the index theorem of [5] can be reformulated as a general "Lefschetz fixed-point theorem" on the lines of [2]. In this way we shall obtain the main theorem of [2], generalized to deal with arbitrary fixed-point sets, but only for transformations belonging to a compact group.

The content of this paper is essentially topological, and it should be viewed as a paper on the equivariant K-theory of manifolds. The analysis has all been done in [5], and what we do here is simply to express the topological index in terms of fixed-point sets. This is quite independent of the main theorem of [5] asserting the equality of the topological and analytical indices.

As in [5], we avoid cohomology and use only K-theory. In paper III of this series, we shall pass over to cohomology obtaining explicit formulas in terms of characteristic classes.

The basic result in K_G-theory which leads to the fixed-point formula is what we call the Localization Theorem. We review this in § 1. In § 2, this is then applied to the topological index. Some cases of special interest are then discussed in § 3.

1. The localization theorem

In [5], we reviewed some of the basic facts about the functor $K_G(X)$, defined for G a compact Lie group, and X a locally compact G-space. We did not however introduce the group $K_G^1(X)$ and the exact sequence for K_G-theory which we shall need to use here. Let us recall then [1] that one defines

$$K_G^{-n}(X) = K_G(\mathbf{R}^n \times X)$$

(where G acts trivially on $\mathbf{R}^n$), and that the periodicity theory gives natural isomorphisms

$$K_G^{-n} \cong K_G^{-n-2} .$$

Considering n as an integer mod 2, we then introduce

$$K_G^* = K_G^0 \oplus K_G^1 \qquad\qquad (K_G^0 = K_G) .$$

For a compact pair (X, Y), one then has an exact triangle

$$K_G^*(X)$$

$$K_G^*(X,\,Y) \xleftarrow{\ \delta\ } K_G^*(Y)\,,$$

where δ interchanges K_G^0 and K_G^1. More generally, for locally compact spaces with Y closed in X, we have an exact triangle

$$K_G^*(X)$$

$$K_G^*(X - Y) \xleftarrow{\ \delta\ } K_G^*(Y)\,.$$

This follows at once from the compact case on replacing X, Y by their one point compactifications X^+, Y^+, and observing that

$$K_G^*(X^+,\,Y^+) = K_G^*(X^+ - Y^+) = K_G^*(X - Y)\,.$$

We recall next that the tensor product induces a pairing

$$K_G(X) \otimes K_G(Y) \longrightarrow K_G(X \times Y)$$

for any two locally compact G-spaces X, Y. In particular, taking $Y = \text{point}$, we see that $K_G(X)$ is a module over $K_G(\text{point}) = R(G)$, the character ring of G. Replacing X by $\mathbf{R}^1 \times X$, the same is true for $K_G^1(X)$, and hence for $K_G^*(X)$. It is therefore possible to study $K_G^*(X)$ from the point of view of commutative algebra by analyzing it with respect to the prime ideals of $R(G)$. It is the purpose of this section to review the main results in this direction.

Let γ be a conjugacy class in G. Then it defines a prime ideal in $R(G)$, namely all characters which vanish on γ. For any $R(G)$-module M, we denote by M_γ the module obtained from M by *localizing* at this prime ideal. Thus M_γ is a module over the local ring $R(G)_\gamma$. An element of $R(G)_\gamma$ is a "fraction" u/s with $u, s \in R(G)$ and $s(\gamma) \neq 0$, but two fractions u/s and u'/s' represent the same element of $R(G)_\gamma$ if $\exists\, t \in R(G)$ with $t(\gamma) \neq 0$ and $tus' = tu's$. Elements of M_γ are "fractions" m/s $(m \in M, s \in R(G), s(\gamma) \neq 0)$ with a similar equivalence relation.

On the other hand if X is a G-space, we can consider the subspace

$$X^\gamma = \bigcup\nolimits_{g \in \gamma} X^g$$

where X^g denotes the fixed point set of g acting on X. Then X^γ is a *closed*[1] G-subspace of X. The main result is then the following.

LOCALIZATION THEOREM (1.1). *Let γ be a conjugacy class in G, $i\colon X^\gamma \to X$ the inclusion. Then*

$$i^*\colon K_G(X) \longrightarrow K_G(X^\gamma)$$

---

[1] γ is compact and X^γ is the image in X of a closed subspace of $\gamma \times X$ under the projection $\gamma \times X \to X$.

becomes an isomorphism

$$i_\gamma^*: K_G(X)_\gamma \longrightarrow K_G(X^\gamma)_\gamma$$

when localized at the prime ideal of $R(G)$ defined by γ.

The proof of this theorem is given in [7]. However, as it plays a key role in this paper, we shall review the proof. The first step is the following lemma about characters.

LEMMA (1.2). *Let H be a closed subgroup of a compact Lie group G, and let γ be a conjugacy class of G not meeting H. Then there exists $\chi \in R(G)$ such that*

(i) $\chi(\gamma) \neq 0$,

(ii) $\chi(h) = 0$ *for all $h \in H$.*

For general G, the proof of (1.2) is surprisingly difficult. However for the purposes of this paper, it is sufficient to consider *abelian* groups. In this case, (1.2) is a trivial consequence of the fact that characters separate points in the quotient group G/H.

When we localize at the prime ideal of $R(G)$ defined by γ, the element χ of (1.2) becomes (by (i)) a unit of $R(G)_\gamma$. By (ii), this unit annihilates $R(H)_\gamma$; this being viewed as an $R(G)_\gamma$-module in the obvious way. Thus we obtain

COROLLARY (1.3). *In the notation of (1.2), we have $R(H)_\gamma = 0$.*

Observe now that, for any conjugacy class γ of G, $R(H)_\gamma$ is itself a ring of fractions of $R(H)$. Also if M is an $R(H)$-module, and so an $R(G)$-module, M_γ is an $R(H)_\gamma$-module. Since our modules are always "unitary" (i.e. the identity of the ring acts as the identity on the module), it follows that $M_\gamma = 0$ whenever $R(H)_\gamma = 0$. This situation arises if γ satisfies the hypothesis of (1.2), and we take $M = K_G^*(X)$ where X is a compact G-space admitting a G-map onto G/H. The factorization $X \to G/H \to$ point gives rise to a factorization

$$K_G^*(X) \longleftarrow K_G^*(G/H) \longleftarrow K_G^*(\text{point})$$
$$\uparrow \qquad\qquad \uparrow$$
$$R(H) \longleftarrow R(G) ,$$

so that $K_G^*(X)$ is in fact an $R(H)$-module. Thus we deduce

COROLLARY (1.4). *Let γ, G, H be as in $(1, 2)$, and let X be a compact G-space admitting a G-map onto G/H. Then*

$$K_G^*(X)_\gamma = 0 .$$

Remark. If Y is any closed G-subspace of X, then it also admits a G-map $Y \to G/H$, and so $K_G^*(Y)_\gamma = 0$. The exact triangle of (X, Y), together with

the exactness of localization, then shows that $K_G^*(X, Y)_\gamma = 0$.

Suppose now that X is any locally compact G-space, and let $Y \subset X$ be an orbit with isotropy group H. Then *we can always find a closed G-neighborhood V of Y in X with a G-retraction onto Y*. This follows from the existence of a "slice" (cf. [7]); but for our present purposes, we only need to apply this when X is a differentiable G-manifold. In this case Y is a G-submanifold, and we can take V to be a closed tubular neighbourhood defined by a G-invariant riemannian metric. In any case, given that such neighbourhoods V of orbits exist, we can cover any compact G-subspace L of X by a finite number of sets $L_i = V_i \cap L$. Let H_i be the isotropy group connected with V_i, then we have G-maps $L_i \to G/H_i$. Assume now that γ is a conjugacy class of elements of G having *no fixed points* in X, then $\gamma \cap H_i = \varnothing$ for all i, and so $K_G^*(L_i)_\gamma = 0$ by (1.4). A simple induction on the number of L_i using exact sequences, and the remark following (1.5), then shows that $K_G^*(L)_\gamma = 0$. Replacing L by any compact G-subspace L', and then using the exact triangle for (L, L'), we deduce that $K_G^*(L, L')_\gamma = 0$. In particular, if U is an open relatively compact G-subspace of X, we have

$$K_G^*(U)_\gamma = K_G^*(\bar{U}, \partial \bar{U})_\gamma = 0 .$$
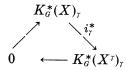
Since $K_G^*(X)$ is the direct limit of these $K_G^*(U)$, and since localization commutes with taking direct limits, we deduce

PROPOSITION (1.5). *Let X be a locally compact G-space, γ a conjugacy class of G having no fixed points in X. Then $K_G^*(X)_\gamma = 0$.*

Proposition (1.5) is the special case of (1.2) in which $X^\gamma = \varnothing$. To prove the localization theorem in general, we consider the exact triangle

$$
\begin{array}{ccc}
& K_G^*(X) & \\
\nearrow & & \searrow{\scriptstyle i^*} \\
K_G^*(X - X^\gamma) \longleftarrow & & K_G^*(X^\gamma) .
\end{array}
$$

We now localize this at the prime ideal defined by γ, and recall again that localization preserves exactness. Applying (1.5) to the space $X - X^\gamma$ (where γ has no fixed points), we obtain the exact triangle

$$
\begin{array}{ccc}
& K_G^*(X)_\gamma & \\
\nearrow & & \searrow{\scriptstyle i^*_\gamma} \\
0 \longleftarrow & & K_G^*(X^\gamma)_\gamma
\end{array}
$$

which establishes (1.1).

Remark. The proof of (1.1) which we have just given is complete for G *abelian* and X a *differentiable G-manifold* (note that we were careful to

avoid taking the one-point compactification of X). Only this case will be used in the rest of the paper.

2. The Lefschetz formula

Let G be a compact Lie group, X a compact differentiable G-manifold. The tangent bundle TX is then a differentiable G-manifold and we can consider the group $K_G(TX)$. In [5] we defined two $R(G)$-homomorphisms

$$\text{a-ind: } K_G(TX) \longrightarrow R(G)$$
$$\text{t-ind: } K_G(TX) \longrightarrow R(G) \ .$$

The main theorem of [5] asserts that these two homomorphisms coincide. In this section we shall show how the topological index t-ind can be computed in terms of fixed-point sets. Combined with the main theorem of [5], this will then give a "Lefschetz fixed-point formula" for elliptic operators. For simplicity of notation (and since in any case a-ind = t-ind), we shall omit the prefix and write ind for t-ind. When it is necessary to prevent confusion, we shall write $\text{ind}_G{}^X$ to indicate the space and group involved.

We recall first that in [5] we defined a functorial homomorphism

$$i_!: K_G(TX) \longrightarrow K_G(TY)$$

for any G-embedding $i: X \to Y$. In terms of this the topological index was defined as follows. Let $i: X \to E$ be a G-embedding of X in a real representation space of G, and let $j: P \to E$ be the inclusion of the origin. Then,

$$j_!: K_G(TP) \longrightarrow K_G(TE)$$

is an isomorphism and the topological index

$$\text{ind: } K_G(TX) \longrightarrow K_G(TP) = R(G)$$

is defined by

$$\text{ind} = (j_!)^{-1} \circ i_! \ .$$

It follows at once from this definition (and the functoriality of $i_!$) that, if $i: Z \to X$ is a G-embedding of the closed submanifold Z, the following diagram commutes

(2.1)
$$
\begin{array}{ccc}
K_G(TZ) & \xrightarrow{\ i_!\ } & K_G(TX) \\
 & {}_{\text{ind}^Z}\searrow & \downarrow{\scriptstyle \text{ind}^X} \\
 & & R(G) \ .
\end{array}
$$

In addition to the covariant map

$$i_!: K_G(TZ) \longrightarrow K_G(TX) \ ,$$

there is of course the usual contravariant restriction homomorphism

$$i^*: K_G(TX) \longrightarrow K_G(TZ) .$$

The relation between these is given by the following lemma [5; (3.1)].

LEMMA (2.2). *If $i: Z \to X$ is a G-embedding with normal bundle N, then*

$$i^*i_!: K_G(TZ) \longrightarrow K_G(TZ)$$

is multiplication by

$$\lambda_{-1}(N \otimes_{\mathbf{R}} \mathbf{C}) = \sum (-1)^i \lambda^i (N \otimes_{\mathbf{R}} \mathbf{C}) \in K_G(Z)$$

where the λ^i are the exterior powers, and $K_G(TZ)$ is regarded as a $K_G(Z)$-module in the usual way.

Suppose now that G is a topologically cyclic group, i.e., that it possesses an element g whose powers are dense in G. Then the fixed point set of g in X is fixed by the whole group, i.e.,

$$X^g = X^G .$$

Hence we have[2] (cf. [5; § 2]):

(2.3) $K_G(X^g) \cong K(X^g) \otimes R(G)$

and hence, localizing at the prime ideal of $R(G)$ defined by the conjugacy class $\{g\}$,

(2.4) $K_G(X^g)_g \cong K(X^g) \otimes R(G)_g .$

The following lemma (valid for any group G) characterizes the units in this ring.

LEMMA (2.5). *Let Y be a compact space on which G acts trivially, p any prime ideal of $R(G)$. Then an element $u \in K_G(Y)_p$ is a unit if and only if its restriction to each point $P \in Y$ is a unit in $K_G(P)_p = R(G)_p$.*

PROOF. Let[3] $H^0(Y; \mathbf{Z})$ denote the group of continuous maps $Y \to \mathbf{Z}$. Then by assigning to each vector bundle on Y its dimension (or rank), we obtain a homomorphism

$$\mathrm{rk}: K(Y) \longrightarrow H^0(Y; \mathbf{Z}) .$$

This splits so, if $K_1(Y) = \mathrm{Ker}\,(\mathrm{rk})$, we have a decomposition

$$K(Y) = K_1(Y) \oplus H^0(Y; \mathbf{Z}) .$$

In [1, (3.1.6)] it is shown that every element of $K_1(Y)$ is nilpotent. Hence an element of

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[2] Tensor products are taken over the integers $\mathbf{Z}$ unless othewise indicated.

[3] For our purposes $Y = \sum_{i=1}^n Y_i$ will be a finite sum of connected spaces Y_i so that $H^0(Y; \mathbf{Z})$ is a free abelian group with one generator for each component Y_i.

$$K_G(Y)_p \cong K(Y) \otimes R(G)_p \cong \left(K_1(Y) \otimes R(G)_p\right) \oplus \left(H^0(Y; \mathbf{Z}) \otimes R(G)_p\right)$$

is a unit if and only if its image in $H^0(Y; \mathbf{Z}) \otimes R(G)_p$ is a unit. But $H^0(Y; \mathbf{Z}) \otimes R(G)_p$ can be identified with the ring of continuous functions $Y \to R(G)_p$, and an element of this ring is a unit if and only if its value at every point $P \in Y$ is a unit.

Before proceeding further, we must make some remarks about the fixed point set X^g of an element $g \in G$. Since G is compact we can, by averaging over G, assume g is an isometry for some riemannian metric. Suppose $P \in X^g$, and let T_P denote the tangent space to X at P. Then g induces a linear transformation $g \mid T_P$ on T_P. If $\xi \in T_P$ is fixed by $g \mid T_P$, then so is the geodesic in the direction ξ. It follows easily from this that, in a neighbourhood of P, X^g is just the image under the exponential map[4] of the $(+1)$-eigenspace of $g \mid T_P$. Thus X^g is a submanifold of X, and the linear transformation $g \mid N_P$, induced by g on the normal N_P to X^g at P, has no eigenvalue $+1$. In other words

(2.6) $\qquad\qquad \det(1 - g \mid N_P) \neq 0 .$

We shall now apply Lemma (2.5) to deduce:

LEMMA (2.7). *Let G be topologically cyclic generated by g, and let X be a compact G-manifold. Let N^g denote the normal bundle[5] of X^g in X. Then*

$$\lambda_{-1}(N^g \otimes_{\mathbf{R}} \mathbf{C}) \in K_G(X^g)$$

becomes a unit in $K_G(X^g)_g$.

PROOF. By Lemma (2.5), it is sufficient to restrict to each point $P \in X^g$. Now an element χ of $R(G)$, i.e. a character, becomes a unit in $R(G)_g$ if and only if $\chi(g) \neq 0$. Taking χ to be the restriction of $\lambda_{-1}(N^g \otimes_{\mathbf{R}} \mathbf{C})$ to a point $P \in X^g$ we find[6]

$$\begin{aligned}
\chi(g) &= \sum (-1)^i \operatorname{Trace} \lambda^i(g \mid N_P \otimes_{\mathbf{R}} \mathbf{C}) \\
&= \det_{\mathbf{C}}(1 - g \mid N_P \otimes_{\mathbf{R}} \mathbf{C}) \\
&= \det_{\mathbf{R}}(1 - g \mid N_P) \\
&\neq 0 \qquad\qquad\qquad\qquad\qquad \text{by (2.6).}
\end{aligned}$$

This completes the proof.

We are now ready to prove our key result.

PROPOSITION (2.8). *Let G be topologically cyclic generated by g and let X be a compact G-manifold. Then*

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[4] defined by the riemannian metric.

[5] X^g may have components of different dimensions so that N^g will be a vector bundle with different dimensions over the different components of X^g.

[6] We write $\det_{\mathbf{R}}$ and $\det_{\mathbf{C}}$ when necessary to distinguish between determinants of real and complex linear transformations.

$$i_!: K_G(TX^g) \longrightarrow K_G(TX) \ ,$$

when localized at the prime ideal of $R(G)$ defined by g, becomes an isomorphism

$$(i_!)_g: K_G(TX^g)_g \longrightarrow K_G(TX)_g \ .$$

Its inverse is

$$\frac{i_g^*}{\lambda_{-1}(N^g \otimes_{\mathbf{R}} \mathbf{C})}$$

where i_g^* is obtained by localizing the restriction homomorphism

$$i^*: K_G(TX) \longrightarrow K_G(TX^g) \ .$$

PROOF. By the Localization Theorem (2.1), i_g^* is an isomorphism. We need only observe that by (2.6)

$$TX^g = (TX)^g \ .$$

In other words, the tangent vectors of X fixed by g are precisely the tangent vectors of X^g. By (2.2) the composition $i^* i_!$ is multiplication by $\lambda_{-1}(N^g \otimes_{\mathbf{R}} \mathbf{C})$. Since G acts trivially on TX^g, it follows that

$$K_G(TX^g) \cong K(TX^g) \otimes R(G)$$

and so (if 1 denotes the group with one element),

$$\operatorname{ind}_G^{X^g} \cong \operatorname{ind}_1^{X^g} \otimes \operatorname{Id} \ .$$

Thus the dependence of $\operatorname{ind}_G^{X^g}$ on G is rather trivial. On the other hand the dependence of ind_G^X on G is not so trivial. However, by localizing (2.1), we get the commutative diagram

(2.9)

$$
\begin{array}{ccc}
K_G(TX^g)_g & \xrightarrow{(i_!)_g} & K_G(TX)_g \\
 & {\scriptstyle (\operatorname{ind}_G^{X^g})}\searrow & \downarrow{\scriptstyle (\operatorname{ind}_G^X)_g} \\
 & R(G)_g &
\end{array}
$$

and we know, by (2.8), that $(i_!)_g$ is an isomorphism. This means that ind_G^X, when localized at g, can be computed in terms of $\operatorname{ind}_1^{X^g}$. In fact from (2.8) and (2.9), we get the following precise formula.

PROPOSITION (2.10). *Let G, g, X be as in (2.8), and let $u \in K_G(TX)$. Then we have*[7]

$$(\operatorname{ind}_G^X u)_g = (\operatorname{ind}_G^{X^g})_g \left[\frac{i^* u}{\lambda_{-1}(N \otimes_{\mathbf{R}} \mathbf{C})} \right] \ .$$

---

[7] The expression in square brackets is of course to be understood as an element in the localized ring $K_G(TX^g)_g$.

This formula computes the image of $\operatorname{ind}_G^x u$ in the localized ring $R(G)_g$. It is more usual to compute an element of $R(G)$ by regarding it as a function on G and evaluating it at all elements of G. To evaluate it on the generator g we observe that the evaluation map

$$R(G) \longrightarrow \mathbf{C}$$

given by $\chi \mapsto \chi(g)$ factors through the local ring $R(G)_g$: the evaluation map

$$R(G)_g \longrightarrow \mathbf{C} ,$$

given by $\chi/\psi \mapsto \chi(g)/\psi(g)$ being well-defined since $\psi(g) \neq 0$. Thus (2.10) will yield a formula for $(\operatorname{ind}_G^x u)(g)$. To write this in a convenient form, we shall, for any trivial G-space Y, introduce the evaluation maps

$$K_G(Y) \cong K(Y) \otimes R(G) \longrightarrow K(Y) \otimes \mathbf{C}$$
$$K_G(Y)_g \cong K(Y) \otimes R(G)_g \longrightarrow K(Y) \otimes \mathbf{C}$$

given respectively by

$$u \otimes \chi \longmapsto u \otimes \chi(g)$$
$$u \otimes \chi/\psi \longmapsto u \otimes \chi(g)/\psi(g)$$

for $u \in K(Y)$. Taking $Y = TX^g$ we can evaluate (2.10) on the element g, and we obtain the formula:

(2.11) $$(\operatorname{ind}_G^x u)(g) = (\operatorname{ind}_1^{x^g} \otimes \operatorname{Id})\left\{ \frac{i^* u(g)}{\lambda_{-1}(N^g \otimes_{\mathbf{R}} \mathbf{C})(g)} \right\} ,$$

where

$$\operatorname{ind}_1^{x^g} \otimes \operatorname{Id}: K(TX^g) \otimes \mathbf{C} \longrightarrow \mathbf{Z} \otimes \mathbf{C} = \mathbf{C}$$

is the natural extension of $\operatorname{ind}_1^{x^g}$ obtained by tensoring with $\mathbf{C}$.

In (2.11) we have assumed that G is topologically cyclic and generated by g. Now for any group G, and for any $g \in G$, let H be the closed subgroup generated by g. If $u \in K_G(TX)$, let u_H be the element of $K_H(TX)$ induced by u. By naturality of the topological index, we have

$$\operatorname{ind}_G^x u(g) = \operatorname{ind}_H^x u_H(g) .$$

Applying (2.11) with H intead of G, we then obtain an explicit formula for the topological character-index ind_G^x in terms of ordinary topological (integer) indices on various fixed point sets. In other words, the group G has been eliminated from the problem.

If we now combine this formula for the topological index with the main theorem of [5] we will obtain a general "Lefschetz formula". Thus let E be an elliptic complex on X invariant under G, and let $\sigma(E)$ be its symbol sequence. Then this defines an element

$$u = [\sigma(E)] \in K_G(TX) \, ,$$

and the analytical index of u evaluated at g is just the Lefschetz number

$$L(g, E) = \sum (-1)^i \operatorname{Trace} \left(g \mid H^i(E)\right)$$

where $H^i(E)$ are the homology groups of the elliptic complex.

Formula (2.11) then leads to the following general Lefschetz fixed-point theorem for G-invariant elliptic complexes.

THEOREM (2.12). *Let G be topologically cyclic generated by g, X a compact G-manifold, E an elliptic complex on X on which G acts. Let X^g denote the fixed-point set of g, N^g the normal bundle of X^g in X. Finally let $u = [\sigma(E)] \in K_G(TX)$ denote the class of the symbol of E, $i^*u \in K_G(TX^g)$ its restriction to X^g. Applying the evaluation map*

$$K_G(TX^g) \cong K(TX^g) \otimes R(G) \longrightarrow K(TX^g) \otimes \mathbf{C} \, ,$$

given by $a \otimes \chi \mapsto a \otimes \chi(g)$, we can then form $i^*u(g)$ and $\lambda_{-1}(N \otimes_{\mathbf{R}} \mathbf{C})(g)$ in $K(TX^g) \otimes \mathbf{C}$. The latter is invertible, and so

$$\frac{i^*u(g)}{\lambda_{-1}(N \otimes_{\mathbf{R}} \mathbf{C})(g)} \in K(TX^g) \otimes \mathbf{C}$$

is well-defined. Then the Lefschetz number $L(g, E)$ is given by

$$L(g, E) = \operatorname{index} \left\{ \frac{i^*u(g)}{\lambda_{-1}(N^g \otimes_{\mathbf{R}} \mathbf{C})(g)} \right\} \, ,$$

where index: $K(TX^g) \otimes \mathbf{C} \to \mathbf{C}$ *denotes the natural extension of the topological index $K(TX^g) \to \mathbf{Z}$.*

Remarks. 1. Theorem (2.12) reduces the problem of calculating an index in $R(G)$, or a Lefschetz number, to that of ordinary indices in $\mathbf{Z}$. In principle we could have used this to deduce the equality a-ind = t-ind for groups G from the corresponding equality without groups. However this would have been rather artificial, because the main step in either case is the commutative diagram (2.1) (for both a-ind and t-ind).

2. Applying the explicit cohomological formula for the index given in [4] to (2.12) we will of course obtain a corresponding expression for $L(g, E)$. This will be done in detail in paper III.

3. When the fixed point set X^g is finite, the topological index $K(TX^g) \to \mathbf{Z}$ is rather trivial, and (2.12) leads immediately to the explicit Lefschetz formula of [2]. This will be developed in the next section, but we should emphasize at this stage the exact relation of our Theorem (2.12) to the main result of [2]. In [2] general maps with simple fixed points are considered. These maps need not be invertible and, even if they are, they may not lie in

any compact group of automorphisms of the elliptic complex. The deformation methods which are available for compact groups (because of the discreteness of the characters) do not therefore apply, and the proof of [2] is by direct analysis. The case of higher dimensional fixed-point sets, however, requires more subtle analysis for direct treatment, and our present deformation method is therefore very advantageous, when applicable, i.e., where we have a compact group.

4. The right hand side of the formula in (2.12) is *a priori* only a complex number. The Lefschetz number $L(g, E)$, on the other hand, is the value on g of a character of G. For example, if G is finite these values must be algebraic integers. In this way (2.12) leads to "integrality theorems" for actions of groups on manifolds. These will be illustrated in the next section. It is perhaps worth pointing out that, for cyclic groups, these "integrality theorems" do not depend on the analysis in [5], since we can define the Lefschetz number topologically.

3. Special cases and applications

First, as indicated in Remark (3) above, we shall consider the case when the fixed point set X^g consists of a finite set of points. In this case, we have

$$K(TX^g) = \prod_P K(P) ,$$

where P runs over the fixed points of g. The topological index coincides on each factor $K(P)$ with the natural isomorphism $K(P) \cong \mathbf{Z}$. The normal bundle N^g at P is just the tangent space T_P to X at P, and

$$\lambda_{-1}(T_P \otimes_{\mathbf{R}} \mathbf{C})(g) = \det (1 - g \mid T_P) .$$

Finally if E_i are the bundles of the elliptic complex E, then the component $(i^*u)_P$ is just $\sum (-1)^i E_i$, and evaluated on g, this gives

$$\sum (-1)^i \operatorname{Trace} (g \mid E_i) .$$

Thus, as a special case of (2.12), we obtain

THEOREM (3.1). *Let G be a compact Lie group, X a compact G-manifold, E an elliptic complex on X on which G acts. Let $g \in G$ have a finite set of fixed points. Then the Lefschetz number $L(g, E)$ is given by*

$$L(g, E) = \sum_P \nu(P)$$

where the summation is taken over the fixed points of g, and

$$\nu(P) = \frac{\sum_i (-1)^i \operatorname{Trace} (g \mid E_{i,P})}{\det (1 - g \mid T_P)} .$$

We shall now show that this formula is essentially identical with the one

given in [2]. For this, we take $f: X \to X$ to be given by $f(x) = g^{-1}x$, and the maps

$$\varphi_{i,x}: E_{i,f(x)} \longrightarrow E_{i,x}$$

to be given by the action of g. The map T_i on sections of E_i induced by (f, φ_i) then coincides with the natural action of g: if $s \in \Gamma(E_i)$, we have

$$(gs)(gx) = g(s(x))$$

or

$$(gs)(y) = g(s(g^{-1}y)) = \varphi_{i,x}s(f(y)) = (T_i s)(y) .$$

Thus $L(g; E) = L(T)$ is the same Lefschetz number as in [2]. As for the terms $\nu(P)$ in [2], they are defined as

$$\frac{\sum (-1)^i \operatorname{Trace} \varphi_{i,P}}{|\det (1 - df_p)|} .$$

Now $\varphi_{i,P} = g \mid E_{i,P}$ and $df_P = g^{-1} \mid T_P$. Since $g \mid T_P$ is orthogonal, it follows that

$$\det (1 - g^{-1} \mid T_P) = \det (1 - g \mid T_P) > 0 .$$

Thus we have

$$\det (1 - g^{-1} \mid T_P) = |\det (1 - df_P)| ,$$

so that the terms $\nu(P)$ occurring in (3.1) are the same as those given in [2].

Theorem (3.1) has a number of interesting applications but, as these are dealt with fully in [3], we shall not pursue them here.

Another case of special interest is to take X a compact complex manifold, and G a finite group of complex analytic automorphisms of X. Suppose moreover that V is a holomorphic G-vector bundle over X. Then the Dolbeault complex $A(V)$

$$\longrightarrow A^{0,p}(V) \overset{\bar{\partial}}{\longrightarrow} A^{0,p+1}(V) \longrightarrow$$

(where $A^{0,p}(V)$ denotes the differential forms on X of type $(0, p)$ with coefficients in V) is an elliptic complex acted on by G. The homology groups of this complex may be identified with the sheaf cohomology groups $H^p(X, \mathcal{O}(V))$ where $\mathcal{O}(V)$ denotes the sheaf of germs of holomorphic sections of V. The Lefschetz number is thus

$$L(g, A(V)) = \sum_p (-1)^p \operatorname{Trace} (g \mid H^p(X, \mathcal{O}(V))) .$$

Consider now the fixed-point set X^g of g. It is a complex submanifold of X. If $a(X, V) \in K_G(TX)$ denotes the symbol class of the Dolbeault complex $A(V)$, then its restriction to $K_H(TX^g)$ (H the subgroup generated by g_1) is given by

$$i^*a(X, V) = a(X^g, V \mid X^g)\lambda_{-1}(\bar{N}^*) \,,$$

where N is the (complex) normal bundle of X^g in X. This follows from the usual decomposition of the exterior algebra of a direct sum. Now we have isomorphisms

$$N \otimes_R C \cong N \oplus \bar{N} \qquad \bar{N} \cong N^*$$

and so

$$\frac{\lambda_{-1}(\bar{N}^*)}{\lambda_{-1}(N \otimes_R C)} = \frac{\lambda_{-1}(N)}{\lambda_{-1}(N) \cdot \lambda_{-1}(N^*)} = \frac{1}{\lambda_{-1}(N^*)} \,.$$

Thus we get

(3.2)
$$\begin{aligned}
\frac{i^*a(X, V)}{\lambda_{-1}(N \otimes_R C)} &= \frac{a(X^g, V \mid X^g)}{\lambda_{-1}(N^*)} \\
&= \frac{a(X^g)[V \mid X^g]}{\lambda_{-1}(N^*)} \,,
\end{aligned}$$

where $a(X^g)$ is the symbol class of the Dolbeault complex of X^g.

If U is a holomorphic vector bundle over a compact complex manifold Y, one usually denotes by $\chi(Y, U)$ the Euler characteristic

$$\sum (-1)^p \dim H^p(Y, \mathcal{O}(U)) \,,$$

i.e., the index of the Dolbeault complex $A(Y, U)$. We extend this notation to any $u \in K(Y) \otimes C$. Thus

$$\chi(Y, u) = \text{index}\,(a(Y) \cdot u) \in C \,.$$

In this notation (3.2) yields

$$\text{index}\left\{\frac{i^*a(X, V)(g)}{\lambda_{-1}(N \otimes_R C)(g)}\right\} = \chi\left(X^g, \frac{[V \mid X^g](g)}{\lambda_{-1}(N^*)(g)}\right) \,.$$

Putting this into Theorem (2.12), we have therefore established

THEOREM (3.3). *Let X be a compact complex manifold, V a holomorphic vector bundle over X, G a finite group of automorphisms of the pair (X, V). For any $g \in G$, let X^g denote the fixed-point set in X of g, and let N^g denote the (complex) normal bundle of X^g in X. Then we have*

$$\sum (-1)^p \,\text{Trace}\,(g \mid H^p(X, \mathcal{O}(V))) = \chi\left(X^g, \frac{[V \mid X^g](g)}{\lambda_{-1}((N^g)^*)(g)}\right) \,.$$

Remark. In the case of a general elliptic complex E with symbol class u, we can restrict u to $K_G(TX^g)$ but not the actual complex. In the holomorphic case however there is, in some sense, a natural restriction, and so Theorem (3.3) can be expressed without reference to symbols.

We shall now show that Theorem (3.3) implies in effect a Riemann-Roch

theorem for the complex space X/G. For this we need the following lemma.

LEMMA (3.4). *Let X, V, G be as in (3.3), and let $f: X \to X/G$ be the projection of X onto the analytic space X/G. Then we have natural isomorphisms*

$$\left(H^p(X, \mathcal{O}(V))\right)^G \cong H^p\left(X/G, (f_*\mathcal{O}(V))^G\right)$$

where ( )^G denotes the invariant part under the action of G.

PROOF. We recall that the structure sheaf of X/G is by definition $f_*(\mathcal{O}_X)^G$. Now the higher direct images $R^q f_*(\mathcal{O}(V))$ $(q \geq 1)$ are all zero because any $y \in X/G$ has a base of neighbourhoods U with $f^{-1}(U)$ a disjoint finite union of complex balls. The Leray spectral sequence

$$H^p(X/G, R^q f_*(\mathcal{O}(V))) \Longrightarrow H^{p+q}(X, \mathcal{O}(V))$$

therefore collapses and gives rise to isomorphisms

$$H^p(X/G, f_*(\mathcal{O}(V))) \cong H^p(X, \mathcal{O}(V)) \, .$$

Taking invariant parts on both sides, and observing that, (since ( )^G is an exact functor for vector spaces over C),

$$H^p(X/G, (f_*(\mathcal{O}(V)))^G) \cong H^p(X/G, f_*(\mathcal{O}(V)))^G \, ,$$

the required result follows.

If W is a holomorphic vector bundle on $Y = X/G$, then $V = f^*W$ is a holomorphic G-vector bundle on X, and $(f_*(\mathcal{O}(V)))^G \cong \mathcal{O}(W)$. The Euler characteristic

$$\chi(Y, W) = \sum (-1)^p \dim H^p(Y, \mathcal{O}(W))$$

can then be computed by the following theorem.

THEOREM (3.5). *Let G be a finite group of automorphisms of a compact complex manifold X, and let W be a holomorphic vector bundle over the complex space $Y = X/G$. Then we have*

$$\chi(Y, W) = \frac{1}{|G|} \sum_{g \in G} \mu(g)$$

where

$$\mu(g) = \chi\left(X^g, \frac{[f^*W \mid X^g](g)}{\lambda_{-1}((N^g)^*)(g)}\right)$$

N^g is the normal bundle to X^g in X, and $|G|$ denotes the order of G.

PROOF. By (3.4) we have

$$\chi(Y, W) = \sum (-1)^p \dim H^p(X, \mathcal{O}(f^*W))^G \, .$$

The theorem follows by combining this with Theorem (3.3) and, recalling that

for any G-module M, we have

$$\dim M^G = \frac{1}{|G|} \sum_{g \in G} \text{Trace} \, (g \mid M) \, .$$

Remarks. 1. (3.5) reduces the Riemann-Roch theorem for the *singular space* X/G to the Riemann-Roch theorem for the *manifolds* X^g. Thus, if the terms $\mu(g)$ are expressed in explicit cohomological form, we shall obtain an explicit formula for $\chi(Y, W)$. This is of particular interest in the case of automorphic forms (cf. [6]).

2. It is probable that (3.3) and (3.5) continue to hold in abstract algebraic geometry provided $|G|$ is prime to the characteristic of the ground field.

Another interesting case of Theorem (2.12) is given by the Dirac operator of a spin-manifold, but we shall defer this example until paper III, (for isolated fixed points see [3; § 8]).

Theorem (2.12) becomes especially simple if the normal bundle N^g of X^g is G-trivial (i.e., isomorphic to $X^g \times M$ for some G-module M), or at least if it is G-trivial over each connected component. In this case, over each component, $\lambda_{-1}(N \otimes_{\mathbf{R}} \mathbf{C})$ is just an element of $R(G)$ and evaluated at g it gives

$$\lambda_{-1}(N \otimes_{\mathbf{R}} \mathbf{C})(g) = \det \, (1 - g \mid N_P) \, ,$$

where P is any point of the component. Thus (2.12) becomes

$$L(g, E) = \sum_j \sigma_j \, ,$$

where the summation is over the connected components X_j^g of X^g,

$$\sigma_j = \frac{1}{\det \, (1 - g \mid N_j)} \, \text{index} \, i_j^* u(g) \, ,$$

$i^* u_j \in K_G(TX_j^g)$, and N_j is the fibre of N at some point of X_j^g. Theorem (3.1), dealing with finite X^g, is of course a special case in which the normal bundle is G-trivial over each component.

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