COMPACT GAUGE FIELDS AND THE INFRARED CATASTROPHE

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It is shown that infrared phenomena in the gauge theories are guided by certain classical solutions of the Yang-Mills equations. The existence of such solutions can lead to a finite correlation length which stops infrared catastrophe. In the present paper we deal only with theories with a compact but abelian gauge group. In this case the problems of correlation length and charge confinement are completely solved.

It was pointed out by different authors [1] several years ago that the infrared phenomena, occurring with a gauge field, might provide a natural explanation for the confinement of quarks. At the same time there exist no methods for analyzing the interaction of gauge fields in the deep infrared region. It is the purpose of the present paper to work out a formalism which permits, at least partly, to take into account the infrared effects in gauge-field interactions. Our main idea is that the system of gauge fields acquires a finite correlation length through the following phenomenon.

Imagine that we are calculating a certain correlation function in the euclidean formulation of the gauge theory. This means averaging over all possible fields A_μ with the weight equal to:

$$\exp\{-S(A)\} = \exp\left\{-\frac{1}{4g^2}\,\mathrm{Sp}\int F_{\mu\nu}^2\,\mathrm{d}^4x\right\} \tag{1}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\,.$$

Assume that the charge $g^2 \ll 1$; then the leading role in the averaging will be played by the fields close to that defined by the equation:

$$\frac{\delta S}{\delta \bar{A}_\mu(x)} = 0;\quad S[\bar{A}] < \infty. \tag{2}$$

Usually one takes into account only the trivial minima of S, i.e. $A_\mu = 0$, and developes the perturbation theory as a small deviation from this. For the correlation function with the distance R the parameter of the perturbation expansion is $g^2 \log R/a$ where a is the inverse cut-off. Hence, for very large R, perturbation theory is not applicable and another $\bar{A}$ might become essential. Indeed, though the weight with which non-trivial minima enters the averaging is small being proportional to

$$\exp\{-S(\bar{A})\} = \exp(-E/g^2) \tag{3}$$

(where E is certain constant) their influence on the cor relation is large if the classical field $\bar{A}$ is long ranged. (In fact, the contribution to the correlation will be shown to be proportional to $\exp\{-E/g^2\}R^4$).

Now assume that the fields $\bar{A}_\mu$ are such as if they were produced by certain "particles" in the four-dimensional euclidean space. In other words there exist the "one-particle" minima of S, the "two-particle" and so on. Of course, the "energy" E depends on the number of the above mentioned pseudo-particles. The average density of pseudo-particles in our system is very small, being proportional to $\exp(-E/g^2)$. However, their existence creates long range random fields in our system. Due to these random fields, the correlation length becomes finite. This is precisely the phenomena we are going to investigate.

The above discussion was based on the crucial assumption that there exists pseudo-particle solutions of the gauge field equations. It will be proved in the second paper of this series that such solutions indeed exist for every compact nonabelian gauge group.

In this first paper we confine ourselves to the problem of realizing the above program in the case of compact but abelian gauge fields. The purpose of this consideration is two fold. First, it is a good and simple model for trying our program on. Second, the compactness of quantum electrodynamics seems to be an attractive hypothesis and our results may have physical applications. For example we shall prove the existence of a certain critical charge in QED.

The definition of the theory is as follows. Let us introduce a lattice in the four dimensional space, necessary in the definition of functional integrals. Generally, the action should have the form:

$$S = \sum_{x,\mu,\nu} f(F_{x,\mu\nu}) \tag{3}$$

$$F_{x,\mu\nu} = A_{x,\mu} + A_{x+a_\mu,\nu} - A_{x+a_\nu,\mu} - A_{x,\nu}$$

where a_μ is lattice vector,

$$f(x) \underset{x\to 0}{\approx} \frac{1}{4g^2} x^2.$$

The hypothesis of the compactness of the gauge group means that $A_{x,\mu}$ are the angular variables, and the group is the circle and not the line. This is equivalent to the hypothesis that:

$$f(x+2\pi) = f(x). \tag{4}$$

Gauge theories on the lattice have been considered earlier by Wilson [2] and the present author (unpublished). See also [3].

The immediate consequence of the periodicity of $f(x)$ is that the nearest neighbours $A_{x+a_\lambda,\mu}$ and $A_{x,\mu}$ can be different by $2\pi N$ (where N is integer) without producing large action. Hence, in the continuous limit $F_{\mu\nu}$ may have the following singularities:

$$F_{\mu\nu}(x) = F_{\mu\nu}^{\text{reg}} + 2\pi \sum_i N_{i\mu\nu}\, \delta^{(S_i)}(x) \tag{5}$$

where $\delta^{(S)}(x)$ is the surface δ-function. The second term in (5) will not contribute to the action, due to the periodicity.

It will be convenient for us to analyze first the three dimensional theory. In this case there exists quasiparticle solutions of Maxwell equations which simply coincide with the Dirac monopole solution. If we introduce the field:

$$F_\alpha \equiv \tfrac{1}{2}\epsilon_{\alpha\beta\gamma} F_{\beta\gamma} \tag{6}$$

then the general pseudo-particle solution will be given by:

$$F_\alpha = \sum_a \frac{q_a}{2} \cdot \frac{(x-x_a)_\alpha}{|x-x_a|^3} - 2\pi\delta_{\alpha 3} \sum q_a \theta(x_3 - x_{3a})\,\delta(x-x_{1a})\,\delta(x-x_{2a}). \tag{7}$$

If $\{q_a\}$ are integers then the singularities in (7) are just of the permitted type.

The action is given by:

$$S(\bar{A}) = E/g^2$$
$$E = \frac{\pi}{2} \sum_{a\neq b} \frac{q_a q_b}{|x_a - x_b|} + \epsilon \sum q_a^2 \tag{8}$$

(the value of the constant ϵ depends on the lattice type and is not essential for us).

Now let us analyze the correlation function introduced in [2] which is most convenient in the confinement problem:

$$F(C) \equiv \exp\{-W(C)\} = \langle \exp\{\mathrm{i} \oint_C A_\mu\, \mathrm{d}x_\mu\}\rangle \tag{9}$$

(here C is some large contour).

For the evaluation of (9) let us substitute $A_\mu = \bar{A}_\mu + a_\mu$. Since the integral over a_μ is gaussian we get:

$$F(C) = F_0(C) \frac{\Sigma \exp\{-S(\bar{A})\} \exp\{\mathrm{i}\oint \bar{A}_\mu\, \mathrm{d}x_\mu\}}{\Sigma \exp\{-S(\bar{A})\}} \tag{10}$$

(Here F_0 is the contribution of $\bar{A} = 0$).

The sum in (10) goes over all possible configurations of pseudo-particles. Now, let us use the formula:

$$\exp\{\mathrm{i}\oint A_\mu\, \mathrm{d}x_\mu\} = \exp\{\mathrm{i}\int F_\alpha\, \mathrm{d}\sigma_\alpha\} \tag{11}$$

in which, due to the periodicity of the exponent, only the first term from (7) should be substituted.

The problem is reduced now to the calculation of the free energy of the monopoles plasma with the "temperature" g^2 in the external field:

$$\varphi^{\mathrm{e}}(x) = \mathrm{i}\, \frac{\partial}{\partial x_\alpha} \int \frac{\mathrm{d}\sigma_\alpha}{|x-y|}. \tag{12}$$

This problem was solved by using Debye method which is correct for sufficiently small g^2. The result is two fold. First, there exist the Debye correlation length and the corresponding photon mass m equal to:

$$m^2 = \exp\{-\epsilon/g^2\} \tag{13}$$

(in the units of the inverse lattice length).

Secondly:

$$W[C] = \text{const}(g^2 m A) \tag{14}$$

where A is the area of the contour C. Eq. (14) was derived for arbitrary planar contour. According to

Wilson [1] this result means "charge confinement" in the three dimensional QED with the compact gauge group.

In the case of the four dimensional QED it can be shown that the only classical solutions with finite action are closed rings. This follows from the fact that singular points in this case should form lines. To prove this, assume that it is not so, and consider the pseudo particle solution with $x = 0$. Consider the cube K with $x_4 = 0$. Then it should be:

$$\oint F_{\mu\nu}\, \mathrm{d}\sigma_{\mu\nu} = 2\pi q \tag{15}$$

But, after small variation of the x_4, our pseudo-particle will be outside the cube, and this contradicts (15).

Since the closed rings produce only dipole forces their influence on the correlation are rather weak. We showed that in this case the correlation length remains infinite and that

$$W[C] = \mathrm{const}\, \exp(-B/g^2)\cdot L \tag{16}$$

where L is the length of the contour C, and B is some constant. This result means the absence of the charge confinement for small g^2. Since it was proved in [2] that for large g^2 the charge confinement exist there are some critical charge g_c^2 at which the phase transition occurs. It is not clear now whether this critical charge is connected with the fine structure constant.

The extension of the above ideas on the nonabelian theory will be presented in the other papers of this series.

References

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[3] R. Balian et al., Phys. Rev. D10 (1974) 3376.