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ON THE PHASE TRANSITION TOWARDS PERMANENT QUARK CONFINEMENT

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In quantized gauge field theories one can introduce sets of operators that modify the gauge-topological structure of the fields but whose physical effect is essentially local. In 2 + 1 dimensional non-Abelian gauge theories these operators form scalar fields and it is argued that when the local gauge symmetry is not spontaneously broken then these topological fields develop a vacuum expectation value and their mutual symmetry breaks spontaneously. It is shown that quarks are then permanently confined. In 3 + 1 dimensional non-Abelian gauge theories one finds that the topological operators and the gauge field operators form a closed algebra from which it is deduced that this system can be in one of the four different phases: (i) spontaneous breakdown *via* an explicit or composite Higgs field, (ii) no Higgs field but permanent confinement of gauge quantum numbers, (iii) Higgs effect and still confinement, presumably only if there is an unbroken subgroup, and (iv) an intermediate phase (critical point?) with massless particles. Finally, the algebra can be realized in a simple model where phases (i) and (ii) can be obtained from each other by a dual transformation.

1. Introduction

At the beginning of this century physicists made an extremely important step in their quest to understand the laws of physics. They learned to rely only on hard measurements and observations when they investigate exotic realms of Nature such as very high speeds or very small distances. Preconceived notions from everyday experiences may have to be rejected if they are not confirmed by observations. Thus, the ether theory was rejected because the ether was not observed, and the question whether photons are particles or waves was considered irrelevant because in the new quantum theory both interpretations are not in contradiction. Any theory is acceptable as soon as it combines different observations into one self-consistent mathematical scheme.

In the light of this philosophy many physicists nowadays are reluctant to accept the new quark theory of hadrons. The concept that hadrons are bound states of more elementary constituents called quarks is not directly confirmed by experiment because quarks have never been isolated. It could well be that quarks are merely products of our limited imagination; the real world could be something much more subtle. The idea that an infinite potential well keeps those quarks permanently together may seem a little *ad hoc*; and many physicists phrase their criticism by citing N. Bohr: "Your theory is crazy, but not crazy enough". In our opinion, that is the worst way to extrapolate from past experiences.

In this paper we wish to argue that the picture of permanently confined quarks is correct, not because we couldn't imagine anything more fancy, but because it results from a mathematically precise theory of phase transitions. It is by now generally accepted that the small-distance behavior of quarks is accurately described by a conventional quantum field theory with a non-Abelian local gauge invariance [1]. It is also well-established that the long-distance behavior of this theory cannot be derived with any of the conventional perturbative methods. Many sophisticated, but unconventional, procedures have been invented to derive that indeed the cumulative forces of the gauge gluons keep the quarks permanently together [1,2].

The absoluteness of this confinement law seems puzzling but is not new in physics. In low-temperature physics the discovery of superconductors caused similar bafflement, but the absolute disappearance of any electrical resistance could be established so easily experimentally that any remaining skepticism could soon be discarded. What we will establish in this paper is that absolute confinement is realized in a phase which is in many respects similar to the superconducting phase. In a certain sense it is the extreme opposite ("superinsulator"). The operators we work with are not only the conventional field operators (which we will call "order parameters") but also a certain set of topologically defined operators (operators that create or destroy topological quantum numbers, to be called "disorder parameters" [3]). In the superconductor it is the order parameter; in the confinement phase it is the disorder parameter that will characterize the new phase.

In 2 + 1 dimensions we will describe this phase by simply postulating the disorder parameter to have a non-vanishing vacuum expectation value. The existence of a non-trivial center $Z(N)$ of a gauge group $SU(N)$ is crucial for this description.

We will then proceed to the more relevant 3 + 1 dimensional case very carefully. The commutator algebra of our operator fields is remarkably elegant and with far-reaching implications. We deduce from this algebra, without considering the details of the dynamics, that the system can only be in one of three or four phases:

(i) spontaneous breakdown of the local symmetry by explicit or composite Higgs fields, where quarks can always come free;

(ii) permanent and absolute quark confinement;

(iii) partial Higgs mechanism and still quark confinement (presumably only possible if there is a non-Abelian subgroup, i.e. $N \geqslant 3$);

(iv) an intermediate phase (critical point?) which must contain massless particles.

Although this result has been obtained by many other authors [2] we think that our derivation is particularly elegant and the formulation in terms of disorder parameters is new. We also find two other things. First the choice between phases (i), (ii), (iii) and (iv) is purely determined by the details of the dynamics and the transition from (i) to (ii) or (iii) through (iv) is presumably a genuine phase transition, so confinement may or may not occur in a gauge theory. To establish which phase is realized may be a long-lasting numerical and technical affair. The other result is that rather than instantons or "merons" it is the field configurations with non-trivial $Z(N)$ topological charge that should be considered responsible for the long-range confinement of quarks.

Our topological operator algebra gets a more profound meaning in some models in 3 or 4 dimensions with gauge fields on a lattice, although the lattice is only introduced here to obtain a finite cut-off and not to enable us to perform large coupling-constant expansions. The point is that in these models the conventional and the topological field operators (order and disorder parameters) are related to each other through the dual transformation (transition from the lattice to the dual lattice) and one of the more interesting models is self-dual. These models are known or very likely expected to possess a finite critical point corresponding to the phase transition between the two elementary phases and this implies that for these models we can definitely establish a range of values for the coupling parameter within which absolute quark confinement is a fact ⋆. In four dimensions the Higgs phase (i) and the confinement phase (ii) are obtained from each other by this dual transformation and are therefore mathematically very much alike.

The redaction of the paper is as follows. We start by considering gauge theories in 2 + 1 dimensions because formulation of the confinement phase is so much easier there. In sect. 2 the topological operator fields (disorder parameters) are defined and it is shown how to compute their Green functions in conventional perturbation theory. In sect. 3 we concentrate on this field and consider the case where its vacuum expectation value does not vanish and show how quarks are confined in this mode. Then, in sect. 4 we set up a precisely analogous formalism for the theory in 3 + 1 dimensions. We define order operators $A(\mathrm{C}, t)$ and disorder operators $B(\mathrm{C}, t)$ where C is any closed oriented curve in 3-space. Either the theory contains massless particles, or either the A- or the B-type operators (or both) must create closed vortex lines at C. If B does so, then we have the Higgs mode. If A does so, we have permanent confinement. In sect. 5 we consider the representations of our operator algebra and in sect. 6 we discuss briefly the simplest models which contain such an algebra and where the replacement $A \leftrightarrow B$ can be easily done, so that the Higgs mode, after a dual transformation, becomes a confinement mode.

Much of what we say is not new or revolutionary but our aim is to give a formulation of the confinement phenomenon that takes away some of its mysteriousness and suspiciousness.

⋆ Apart from the (to our mind extremely plausible) assumption that this phase transition occurs only once and not three times or more, in which case confinement would hold for different values of the coupling parameter.

2. Quantum Chromodynamics in 2 + 1 dimensions: topological operators and their Green functions

To set up our arguments step by step we begin by considering a simple and well understood gauge model in two space, one time dimensions. The gauge group is SU(N) and the gauge symmetry is spontaneously and completely broken by scalar Higgs fields [4]. These Higgs fields, H, must be a set of unique representations of SU(N)/Z(N) such as the octet and decuplet representations of SU(3)/Z(3). Thus all (vector and scalar) fields are invariant under the center Z(N) of the gauge group SU(N). (This is the subgroup of matrices $e^{2\pi i n/N} I$, where n is integer.) Quarks, which are not invariant under Z(N), are not yet introduced at this stage.

Besides the massive photons and Higgs particle(s) this model contains one other class of particles: extended soliton solutions that are stable because of a topological conservation law *. Consider, therefore, a region R in two-dimensional space surrounded by another region B where the energy density is zero (vacuum). In B, the Higgs field $H(x)$ satisfies

$$|\langle H(x)\rangle| = F\,, \tag{2.1}$$

where F is a fixed number. There must be a gauge rotation $\Omega(x)$ so that

$$\Omega(x)\,H(x) = H_0\,, \tag{2.2}$$

where H_0 is fixed. $\Omega(x)$ is determined up to elements of Z(N). We also require absence of singularities so $\Omega(x)$ is continuous. Consider a closed contour C(θ) in B parametrized by an angle θ with $0 \leqslant \theta \leqslant 2\pi$, and C(0) = C($2\pi$). Consider the case that C goes clockwise around R. Since B is not simply connected we may have

$$\Omega(2\pi) = e^{2\pi i n/N}\,\Omega(0)\,, \tag{2.3}$$

with $0 \leqslant n < N$, n integer. Because of continuity, n is conserved. If $n \neq 0$ and if we require absence of singularities in R then the field configuration in R cannot be that of the vacuum or a gauge rotation thereof, so there must be some finite amount of energy in R. The field configuration with lowest energy E, in the case $n = 1$, describes a stable soliton with mass $M = E$. If $N > 2$ then solitons differ from antisolitons (which correspond to $n = N - 1$) and the number of solitons minus antisolitons is conserved modulo N.

An alternative way to represent the fields corresponding to a soliton configuration is to extend $\Omega(x)$ to be also within R (which, however, may be possible only if we admit a singularity for Ω at some point x_0 in R). We then apply Ω^{-1} to the above field configuration. This has the advantage that everywhere in B we keep $H = H_0$, regardless of the number n of solitons in R, but the price of that is to allow for

* For a general introduction to solitons see e.g. Coleman [5]. For an introduction to this procedure see 't Hooft [6].

a singularity x_0 in R. We will refer to this as the "second representation" of the soliton, for later use.

A set of operators $\phi(x)$ is now defined as follows [3,7]. Let $|A_i(x), H(x)\rangle$ be a state in Hilbert space which is an eigenstate of the space components of the vector fields and the Higgs fields, with $A_i(x)$ and $H(x)$ as given eigenvalues. Then

$$\phi(x_0)\,|A_i(x)\,,H(x)\rangle = |A_i^{\Omega^{[x_0]}}(x)\,,H^{\Omega^{[x_0]}}(x)\rangle\,, \tag{2.4}$$

where $\Omega^{[x_0]}$ is a gauge rotation with the property that for every closed curve $C(\theta)$ that encloses x_0 once we have

$$\Omega^{[x_0]}\,(\theta = 2\pi) = e^{\pm 2\pi i/N}\Omega^{[x_0]}\,(\theta = 0)\,, \tag{2.5}$$

where the minus sign holds for clockwise and the + sign for anticlockwise C. When x_0 is outside C, then

$$\Omega^{[x_0]}(\theta = 2\pi) = \Omega^{[x_0]}(\theta = 0)\,. \tag{2.6}$$

The singularity of $A^{\Omega^{[x_0]}}$ at $x = x_0$ must be smeared over an infinitesimal region around x_0 but we will not consider this "renormalization" problem in this paper. In what sense is $\phi(x)$ a *local* operator? The operator formalism in gauge theories is most conveniently formulated in the gauge $A_0(x, t) = 0$. Then, time-independent continuous gauge rotations $\Omega(x)$ still form an invariance group. For physical states $|\psi\rangle$ however,

$$\langle AH|\psi\rangle = \langle A^\Omega H^\Omega|\psi\rangle\,, \tag{2.7}$$

where Ω is any single-valued gauge rotation. So $\phi(x)$ would have been trivial were it not that $\Omega^{[x_0]}$ has a singularity at x_0.

The operator $\phi(x)$ leads physical states into physical states, and the details of Ω apart from (2.5) and (2.6) are irrelevant. It is now easy to verify that

$$\begin{aligned}
&\phi(x)\,\phi(y)\,|\psi\rangle = \phi(y)\,\phi(x)\,|\psi\rangle\,,\\
&\phi^\dagger(x)\,\phi(y)\,|\psi\rangle = \phi(y)\,\phi^\dagger(x)\,|\psi\rangle\,,
\end{aligned} \tag{2.8}$$

if $|\psi\rangle$ is a physical state, because both the left- and the right-hand sides of (2.8) are completely defined by the singularities alone of the combined gauge rotations. Also, when $R(x)$ is a conventional gauge-invariant field operator composed of fields at x then obviously

$$[R(x)\,,\phi(y)] = 0 \qquad \text{for } x \neq y\,, \qquad \text{but not necessarily for } x = y\,. \tag{2.9}$$

Because of (2.8) and (2.9) ϕ is considered to be a local field operator when it acts on physical states.

From its definition it must be clear that $\phi(x)$ absorbs one topological unit, so we say that $\phi(x)$ is the annihilation (creation) operator for one "bare" soliton (antisoliton) at x and $\phi^\dagger(x)$ is the creation (annihilation) operator for one "bare" soliton (antisoliton).

Let us now illustrate how one can compute Green functions involving $\phi(x)$ by ordinary saddle-point techniques in a functional integral. Let us consider $\langle \mathrm{T}(\phi(\mathbf{0}, t_1)\, \phi^\dagger(\mathbf{0}, 0))\rangle = f(t_1)$ by computing the corresponding functional-integral expression:

$$f(t_1) = \frac{\int_{\mathrm{C}} \mathrm{D}A\, \mathrm{D}H \exp S(A, H)}{\int \mathrm{D}A\, \mathrm{D}H \exp S(A, H)}\,, \tag{2.10}$$

where C is the set of field configurations where the fields make a sudden gauge jump at $t = 0$ described by $\Omega^{[\mathbf{0}]}$ (see (2.5) and (2.6)) and at $t = t_1$ they jump back by a transformation $\Omega^{[\mathbf{0}]\dagger}$. The fields must be continuous everywhere else.

This was how $f(t_1)$ follows from the definitions but it is more elegant to transform a little further. By gauge transforming back in the region $0 < t < t_1$ we get that the fields are continuous everywhere except at $\boldsymbol{x} = 0$, $0 < t < t_1$. So there is a Dirac string [8] going from $(\mathbf{0}, 0)$ to $(\mathbf{0}, t)$. At $0 < t < t_1$ we obtain in this way the soliton in its "second representation": a non-trivial field configuration with a singularity at the origin. In short: $\langle \phi(x)\, \phi^\dagger(0)\rangle$ is obtained by integrating over field configurations with a Dirac string in space-time from 0 to x.

Let us compute $f(t_1)$ for Euclidean time $t_1 = i\tau$, τ real. Let the field theory be pure SU(N) without Higgs scalars. We must find the least negative action configuration with the given Dirac string. The conditions (2.5), (2.6) can be realized for an Abelian subgroup of gauge rotations Ω, so let us take

$$\Omega^{[\mathbf{0}]\dagger}(\theta) = \begin{bmatrix} e^{-i\theta/N} & & \emptyset \\ & \ddots\; e^{-i\theta/N} & \\ \emptyset & & e^{i\theta(1-N^{-1})} \end{bmatrix}, \tag{2.11}$$

the last diagonal element being different from the others because we must have $\det \Omega = 1$, for all θ. Here θ is the angle around the time axis (remember space-time is here three dimensional). The singularity at the Dirac string must be the one obtained when $\Omega^{[\mathbf{0}]\dagger}$ acts on the vacuum.

The transformations (2.11) form an Abelian subgroup, so, as an ansatz, the field configuration with this string singularity may be chosen to be in an Abelian subset of fields corresponding to this subgroup:

$$\tfrac{1}{2} \lambda^a_{ij} A^a_\mu(x) = a_\mu(x)\, \lambda^{ij}_{(N)}\,;$$

$$\lambda^{ij}_{(N)} = \begin{bmatrix} 1 & & & & & \emptyset \\ & 1 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & 1 & \\ \emptyset & & & & & 1-N \end{bmatrix}, \tag{2.12}$$

$$\mathcal{L} = -\tfrac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} = -\tfrac{1}{2}N(N-1)\,F_{\mu\nu}F_{\mu\nu}\,,$$

with

$$F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu\,. \tag{2.13}$$

Here λ^a_{ij} are the conventional Gell-Mann matrices extended to SU(N). Within this set of fields we just have the linear Maxwell equations, and the Dirac string is the one corresponding to two oppositely charged Dirac monopoles, one at 0 and one at t_1. Their magnetic charges are $\pm 2\pi/gN$.

The total action of this configuration is

$$S = -\tfrac{1}{2}N(N-1)\int F_{\mu\nu}F_{\mu\nu}\,\mathrm{d}^3x = -\frac{N-1}{2N}\left(\frac{4\pi^2}{g^2}\right)\left[Z_1 + Z_2 - \frac{1}{4\pi|\tau|}\right], \tag{2.14}$$

where Z_1 and Z_2 are the self-energies of the monopoles, which diverge but can be subtracted, leaving a space-time independent renormalization constant.

We now assume that (2.14) is indeed an absolute extremum for the total action of all field configurations with the given string singularity. We think that this assumption is plausible but present no proof. We thus obtain a first approximation to (2.10):

$$f(t_1) = A\,\exp\left(\frac{(N-1)\,\pi}{2g^2\,|\tau|N} + \mathrm{O}(\log(g^2\tau))\right), \tag{2.15}$$

$$t_1 = i\tau, \qquad \tau \text{ real}\,.$$

Here A is a fixed constant obtained after subtraction of the infinite self-action at the sources. Note the plus sign in our exponent due to the attractive force between the monopole-antimonopole pair. The terms of higher order in $g^2\tau$ can in principle be computed in a quantum perturbation expansion but of course at large τ we expect no convergence [9]. Of course, when the Higgs mechanism is turned on then the soliton acquires a finite mass, say M, and then at large τ we expect

$$f(t_1) \to A\,\exp(-M|\tau|)\,,$$

$$\tau \to \infty\,,$$

$$t_1 = i\tau\,,$$

$$\tau = \text{real}\,, \tag{2.16}$$

just as any ordinary (dressed) propagator.

However, there are also interactions, in particular the N-soliton processes, described essentially by

$$f(x_1, ..., x_N) = \langle \mathrm{T}(\phi(x_1)\,\phi(x_2)\,...\,(x_N))\rangle\,, \tag{2.17}$$

where N is the group parameter. Again we choose $\{x_k\}$ to be Euclidean. There is some freedom in choosing the Dirac strings, for instance we can let one string leave at each point $x_1, ..., x_{N-1}$ and let these all assemble at x_N. Because of the modulo N conservation law, the $N-1$ quanta coming in at x_N are equivalent to one quantum leaving at x_N.

To find the field configuration with least negative action we could try again the ansatz (2.12) but then the result is that $x_1, ..., x_{N-1}$ repel each other and all attract x_N, an unsymmetric and therefore unlikely result. Obviously, any field configuration where the signs in the exponents such as (2.15) are positive, corresponding to attraction, will give much larger, therefore dominating, contributions to the amplitude. So let us try to produce such a field configuration now.

Observe that pure permutations of the N spinor components are good elements of SU(N), so by pure gauge rotations we are allowed to move the unequal diagonal element of (2.11) up or down the diagonal. So let us now again choose one Dirac string leaving at each of $x_1, ..., x_{N-1}$ and all entering at x_N, but this time all these strings have their unequal diagonal element (see (2.11)) in a different position along the diagonal. At x_N the combined rotation is again of type (2.11) as one can easily verify, so this is a more symmetric configuration. Since the gauge transformations we performed so far are all diagonal elements of SU(N) they actually form the subgroup $[\mathrm{U}(1)]^{N-1}$ of SU(N) which is Abelian still. Let us define

$$\lambda^{ij}_{(k)} = \delta_{ij} - N\,\delta_{ik}\delta_{jk}\,, \qquad k = 1, ..., N\,, \tag{2.18}$$

with

$$\sum_{k=1}^{N} \lambda_{(k)} = 0\,,$$

so one of these λ matrices is actually redundant. Our present ansatz is

$$\tfrac{1}{2}\lambda^a_{ij} A^a_\mu(x) = \sum_{k=1}^{N} A^{(k)}_\mu(x)\,\lambda^{ij}_{(k)}\,, \tag{2.19}$$

without bothering about the invariance

$$A^{(k)}_\mu(x) \to A^{(k)}_\mu + B_\mu\,, \qquad \text{all } k\,. \tag{2.20}$$

The Lagrangian is

$$\begin{aligned}\mathcal{L} &= -\tfrac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} = -\tfrac{1}{2} \sum_{k,l} \mathrm{Tr}\,\lambda_{(k)}\lambda_{(l)} F^{(k)}_{\mu\nu} F^{(l)}_{\mu\nu} \\ &= -\tfrac{1}{2}N(N-1) \sum_k F^{(k)}_{\mu\nu} F^{(k)}_{\mu\nu} + \tfrac{1}{2}N \sum_{k\neq l} F^{(k)}_{\mu\nu} F^{(l)}_{\mu\nu}\,,\end{aligned} \tag{2.21}$$

with

$$F^{(k)}_{\mu\nu} = \partial_\mu A^{(k)}_\nu - \partial_\nu A^{(k)}_\mu\,. \tag{2.22}$$

Notice the change of sign in (2.21) when fields of different index overlap. We now choose a Dirac monopole corresponding to the kth subgroup U(1) at x_k, so $A^{(k)}(x)$ is just the field of one monopole at x_k (remember that, because we stay in the Abelian subclass of fields, linear superpositions are allowed). The diagonal terms in (2.21) then only contribute to the monopole "self-energy" (more precisely: self-action) and only contribute to the already established Z factor that must be subtracted. Only the cross terms give non-trivial effects:

$$S = -\frac{1}{2}\frac{4\pi^2}{g^2 N}\left[(N-1)\sum_{k=1}^{N} Z_k - \frac{1}{4\pi}\sum_{k>l}\frac{1}{|x_k - x_l|}\right]. \tag{2.23}$$

Again we assume that this represents the least negative action configuration without presenting the complete proof. A good check of this equation is that if $N-1$ points come close and the Nth stays far away then we recover (2.14) apart from overall normalization, exactly as one should expect. So we conclude:

$$f(x_1, ..., x_N) = A' \exp\left[\frac{\pi}{2g^2 N}\sum_{k>l}\frac{1}{|x_k - x_l|} + \mathrm{O}\left(\log(g^2\tau)\right)\right], \tag{2.24}$$

for Euclidean $\{x_k\}$.

The above was mainly to illustrate how to compute in lowest-order perturbation expansion the Green functions that we will discuss further in sect. 3. We did not prove that the field configurations we choose to expand about are really the minimal ones (which we do expect) but there is no need to elaborate on that point further here.

3. Spontaneous symmetry breaking and confinement

In sect. 2 we found that some SU(N) gauge theories in 2 + 1 dimensions possess a topological quantum number, conserved modulo N, and that Green functions corresponding to exchange of this quantum may be computed. Our field ϕ behaves as a local complex scalar field (real for SU(2)) in all respects. We expect that these Green functions satisfy all the usual Wightman axioms [10] except that they are more singular than usual at small distances. In fact, the Green functions as computed in sect. 2 can be exactly reproduced in a theory with free scalar particles and non-polynomial sources using superpropagator techniques [11]. We leave that as an exercise for the reader. In pure SU(N) the trouble is that only the quantum corrections give some interesting structure to these Green functions. The terms of higher orders in g^2 in (2.15) and (2.24) correspond to quantum loop corrections. They have not yet been computed.

However, let us assume that the resulting Green functions can be computed and are roughly generated by some effective Lagrangian with necessarily strong couplings, for instance:

$$\mathcal{L}(\phi, \phi^*) = -\partial_\mu\phi^*\partial_\mu\phi - M^2\phi^*\phi - \frac{\lambda_1}{N!}\left(\phi^N + (\phi^*)^N\right) - \tfrac{1}{2}\lambda_2(\phi^*\phi)^2\ . \tag{3.1}$$

Here we have assumed that the Higgs mechanism (either by some explicit or by some dynamical Higgs field) makes all fields massive, generating also a large soliton mass M. Of course we expect many possible interaction terms but only the most important ones are added in (3.1): the λ_1 term produces the N-soliton interaction and is responsible for the non-vanishing result (2.24) for the expression (2.17), the N-point function. The λ_2 term is of course also expected and is included in (3.1) for reasons that will become clear later.

Observe now a Z(N) global symmetry that leaves the Lagrangian (3.1), and the Green functions considered in sect. 2, invariant:

$$\phi \to e^{2\pi i/N}\phi \,,$$
$$\phi^* \to e^{-2\pi i/N}\phi^* \,. \tag{3.2}$$

This is simply the symmetry associated with the topologically conserved soliton quantum number. Modulo N conservation laws correspond to Z(N) global symmetries.

As we saw, the mass term is essentially due to the Higgs mechanism. Roughly, $M^2 \propto \mu_{\mathrm{H}}^2$, where μ_{H}^2 is the second derivative of the Higgs potential at the origin. We can now consider either switching off the Higgs field, or just changing the sign of μ_{H}^2 so that $\langle H\rangle \to 0$. What happens to M^2 (the soliton mass)? If it stays positive then the physical soliton has not gone away and the topological conservation law remains valid. The symmetry of the theory is as in the Higgs mode and we define this mode to be a "dynamical Higgs mode". However, a very good possibility is that M^2 also switches its sign, so that $\langle\phi\rangle \to F \neq 0$. The topological global Z(N) symmetry may get spontaneously broken. This mode can be recognized directly from the Green functions of sect. 2. The criterion is

$$|F|^2 = \lim_{|\tau|\to\infty} f(t_1) \,, \qquad t_1 = i\tau \,. \tag{3.3}$$

Just for amusement we might note that (2.15) indeed seems to give $F \neq 0$, but of course the quantum corrections may not be neglected and we do not know at present how to actually compute the limit (3.3).

Spontaneous breakdown of the topological Z(N) symmetry is a new phase the system may choose, depending on the dynamics. We may compare a bunch of molecules that chooses to be in a gaseous, liquid or solid phase depending on the dynamics and on the values of certain intensive parameters.

Let us study this new phase more closely. The vacuum will now have a Z(N) degeneracy, that is, an N-fold degeneracy. Labeling these vacua by an index from 1 to N we have

$$\langle\phi_1\rangle = e^{2\pi i/N}\langle\phi_2\rangle = \dots \,. \tag{3.4}$$

Since the symmetry is discrete there are no Goldstone particles; all physical particles have some finite mass. Again we are able to construct a set of topologically stable objects: the Bloch walls that separate two different vacua. These vortex-like structures are stable because the vacua that surround them are stable (Bloch walls are vortex-like because our model is in 2 + 1 dimensions). The width of the Bloch wall or vortex is roughly proportional to the inverse of the lowest mass of all physical particles, and therefore finite. The Bloch wall carries a definite amount of energy per unit of length.

We will now show the relevance of the operator

$$A(\mathrm{C}) = \mathrm{Tr}\ \mathrm{P} \exp \oint_{\mathrm{C}} ig\, A_k(x)\, \mathrm{d}x^k\ , \tag{3.5}$$

for these Bloch walls. Here C is an arbitrary oriented contour in 2-dim. space and P stands for the path ordering of the integral. A_k^{ij} are the space components of the gauge vector field in the matrix notation. $A(\mathrm{C})$ is a non-local gauge-invariant operator that does not commute with ϕ. Let us explain that. As is well known, if C' is an open contour, then the operator

$$A(\mathrm{C}', x_1, x_2) = \mathrm{P} \exp \int_{x_1, \mathrm{C}'}^{x_2} ig\, A_k(x)\, \mathrm{d}x^k$$

transforms under a gauge rotation Ω as

$$A^{\Omega}(\mathrm{C}', x_1, x_2) = \Omega(x_1)\, A(\mathrm{C}', x_1, x_2)\, \Omega^{-1}(x_2)\ , \tag{3.6}$$

Now the operator $\phi(x_0)$ was defined by a gauge transformation Ω that is multivalued when followed over a contour that encloses x_0. So when we close C' to obtain C, and C encloses x_0 once,

$$A(\mathrm{C}) = \mathrm{Tr}\, A(\mathrm{C}, x_1, x_1)\ , \tag{3.7}$$

then the value of A makes a jump by a factor $\exp(\pm 2\pi i/N)$ when the operator $\phi(x_0)$ acts, so

$$A(\mathrm{C})\, \phi(x_0) = \phi(x_0)\, A(\mathrm{C}) \exp(2\pi i n/N)\ , \tag{3.8}$$

where n counts the number of times that C winds around x_0 in a clockwise fashion minus the number of times it winds around x_0 anticlockwise. Eq. (3.8) is an extension of eq. (2.9) for non-local operators $A(\mathrm{C})$. As we will see, (3.8) can be generalized to 3 + 1 dimensions. Now let us interpret (3.8) in a framework where $\phi(x)$ is diagonalized. Then, as we see, $A(\mathrm{C})$ is an operator that causes a jump by a factor $\mathrm{e}^{2\pi i n/N}$ of $\phi(x)$ for all x inside C. So $A(\mathrm{C})$ causes a switch from one vacuum to another vacuum within C in the case that $Z(N)$ is spontaneously broken. In other words $A(\mathrm{C})$ creates a "bare" Bloch wall or vortex exactly at the curve C.

Our model does not yet include quarks. Quarks are not invariant under the center $Z(N)$, so they do not admit a direct definition of $\phi(x)$. This difficulty is to be expected when one considers the physics of the system. The vortices that were locally stable without quarks may now become locally unstable due to virtual quark-antiquark pair creation. Most authors therefore consider quark confinement to be a basic property of the glue surrounding the quarks, in which quarks must be inserted perturbatively. Such a procedure is justified by the experimental evidence; all hadrons can be labeled according to the number and types of quarks they contain; none of them is said to be composed out of an unspecifiable or infinite number of quarks. The number of gluons on the other hand cannot easily be given.

Thus, we assume that perturbation expansions with respect to the number of fermion lines are acceptable, and to amplitudes with a given number of fermion lines we can still apply our previously described formalism. Consider a state now described by a given gauge field configuration in which one quark moves with a wave function $\psi(\boldsymbol{x})$. Again we wish to apply to this state the "operator" $\phi(\boldsymbol{x}_0)$. The result is not only a gauge singularity of the gluon fields at $\boldsymbol{x}_0$ but now the quark field ψ becomes multivalued. Obviously such an object $\phi(\boldsymbol{x}_0)$ cannot be called an operator in the usual sense because it produces something that is not in ordinary Hilbert space. Only in an extended Hilbert space including multivalued quark fields is $\phi(\boldsymbol{x})$ an operator. In this extended Hilbert space the annihilation operator $\psi(\boldsymbol{x})$ is also multivalued. However, the product

$$\psi(\boldsymbol{x}_1)\,\phi(\boldsymbol{x}_0) \tag{3.9}$$

removes the crazy multivalued quark from the scene and therefore this combination acts on one-quark states as an operator in the usual sense, transforming a state with one quark into a state with no quarks. The operator is still multivalued: if we rotate $\boldsymbol{x}$ around $\boldsymbol{x}_0$ by 360° then a phase factor

$$\mathrm{e}^{2\pi i/N}$$

results.

Before proceeding, we note that we prefer to limit ourselves to discussion of physical, that is, gauge-invariant operators only. (3.9) is not yet gauge-invariant because the quark operator is not. Therefore we will consider, for instance,

$$[\mathrm{P}\exp\int_{\infty}^{\boldsymbol{x}} ig\,A_k(\boldsymbol{x}')\,\mathrm{d}x'^k]\;\psi(\boldsymbol{x}_0)\,\phi(\boldsymbol{x}_0) \tag{3.10}$$

instead of (3.9). The integration is along some given contour. The factor in front makes (3.10) gauge-invariant and is itself single-valued. So we still find the phase factor when $\boldsymbol{x}$ rotates around $\boldsymbol{x}_0$. We could define the phase by chosing the discontinuity at the point where $\boldsymbol{x}_0$ coincides with the integration contour. Of course, instead of letting the integral in (3.10) start at ∞, one could consider only finite contours ending at points where quarks are created or destroyed.

We would conclude that one can still define amplitudes such as

$$\langle|\phi^\dagger(x_4,t_4)\,A(C'',\infty,x_3,t_3)\,\psi(x_3,t_3)\,\phi(x_2,t_2)\,\bar{\psi}(x_1,t_1)\,A(\mathrm{C}',x_1,\infty,t_1)|\rangle\,, \tag{3.11}$$

but either they are multivalued under rotations of $\boldsymbol{x}_3$ around $\boldsymbol{x}_2$ or they must have discontinuities for instance when x_2 coincides with C″. The initial and final states may not contain any quarks. We must be careful, however, for at times t between t_2 and t_3 a state in an unphysical Hilbert space (multivalued quark field) has to propagate according to some new equation of motion. We think that that is not a serious difficulty since we can simply postulate the usual Dirac equation here.

Now let us interpret the above-mentioned properties of this amplitude (3.11) in terms of a representation where $\phi(x)$ is diagonalized. Choose $x_1 = x_3$ and $C' = C''$. Also choose t_1, t_2 and t_3 close together. Our initial state had some value for ϕ and at $t = t_1$ a quark is added to the system. At $t = t_2$ we measure $\phi(x_2)$. From the above we know that the amplitude now has to be multivalued. $\phi(x_2)$ picks up a phase factor $e^{2\pi i/N}$ when x_2 rotates around x_1, the position of the quark, or, equivalently, ϕ has a discontinuity at the curve C′. Thus, we see that not only has a quark been created at t, but that also a Bloch wall is attached to it. The quark is the end point of a vortex. In short, the conventional operator

$$\bar{\psi}(x_1)\,[\mathrm{P}\exp\int_{x_1}^{x_2} igA_k(x)\,\mathrm{d}x^k]\;\psi(x_2) \tag{3.12}$$

creates not only a quark pair but also a vortex in between them. This vortex is topologically stable if $\langle\phi\rangle = F \neq 0$. If we have a configuration with N quarks then ϕ makes a full rotation over 2π when it follows a closed contour around. This is why a "baryon" consisting of N quarks is not confined to anything else. Evidently, for real baryons N must be 3.

Our conclusion is as follows. In SU(N) gauge theories where all scalar fields are in representations that are invariant under the center Z(N) of SU(N) (such as octet or decuplet representations of SU(3)), there exists a non-trivial topological global Z(N) invariance. If the Higgs mechanism breaks SU(N) completely then the vacuum is Z(N) invariant. However, we can also have spontaneous breakdown of Z(N) symmetry. If that breakdown is complete then we can have no Higgs mechanism for SU(N), because in that mode "colored" objects are permanently and completely confined by the infinitely rising linear potentials due to the Bloch-wall-vortices. We can also envisage the intermediate modes where a Higgs mechanism breaks SU(N) partly, and Z(N) is partly broken. Finally, if neither Higgs' effect, nor spontaneous breakdown of Z(N) take place, then there must be massless particles causing complicated long-range interactions, as we will show more explicitly for the 3 + 1 dimensional case. Presumably that corresponds to the critical point where the phase transition takes place (quantum electrodynamics could be placed in that last category but it is not an SU(N) gauge theory, to which we limited ourselves).

Eqs. (2.8) and (3.8) are the basic commutation relations satisfied by our topological fields ϕ. They suggest a dual relationship between A and ϕ. Indeed, one could start with a scalar theory exhibiting global Z(N) invariance and then define the topological operator A(C) through eq. (3.8), but it is impossible to see this way that A can be written as the ordered exponent of an integral of a vector potential, and also the gauge group SU(N) cannot be recovered. As we will explain later, the center Z(N) is more basic to this all than the complete group SU(N).

A good name for the field ϕ is the "disorder parameter" [3] since it does not commute with the other, usual, fields which have been called order parameters in solid-state physics. The fact that in the quark confinement phase the degenerate vacuum states are eigenstates of this disorder parameter shows the close analogy with the superconductor where the vacuum state is an eigenstate of the order parameter. Later we will elaborate on the dual relationship between order and disorder parameters.

4. SU(N) gauge theories in 3 + 1 dimensions

In the previous sections the construction of a scalar field and the successive formulation of the spontaneous breakdown of the topological Z(N) symmetry were only possible because the model was in 2 space, 1 time dimensions. Also the boundary between different but equivalent vacua can only serve as a vortex in 2 + 1 dimensions. It would have the topology of a sheet in 3 + 1 dimensions and therefore not be useful as a vortex of conserved electric flux. So in 3 + 1 dimensions the formulation of quark confinement must be considerably different from the 2 + 1 dimensional case. Nevertheless extension of our ideas to 3 + 1 dimensions is not only possible, as we will show now, but even straightforward, when one examines the following arguments more closely.

Again we take an SU(N) theory where all scalar fields are invariant under the center Z(N) of SU(N), now in 3 + 1 dimensions. The order parameters are all conventional gauge-invariant field combinations. A useful set is the non-local ones defined on each closed oriented curve C in 3-space:

$$A(\mathrm{C}, t) = \mathrm{Tr\ P\ exp} \oint_{\mathrm{C}} gi\, A_k(x, t)\, \mathrm{d}x^k \,. \tag{4.1}$$

At equal times all $A(\mathrm{C}, t)$ commute:

$$[A(\mathrm{C}_1, t), A(\mathrm{C}_2, t)] = 0 \,. \tag{4.2}$$

The analogue of the scalar disorder parameter field of the 2 + 1 dimensional theory is here a set of operators $B(\mathrm{C}, t)$ that are also defined on closed oriented curves C. The effect of $B(\mathrm{C}, t)$ on an eigenstate of the vector and Higgs field operators is a gauge transformation $\Omega^{[\mathrm{C}]}$ which is singular on the curve C. If another closed curve C$'$ winds through C with n windings in a certain direction and C$'$ is parametrized by $0 \leqslant \theta \leqslant 2\pi$, then we require

$$\Omega^{[\mathrm{C}]}(2\pi) = \Omega^{[\mathrm{C}]}(0)\, \mathrm{e}^{2\pi i n/N} \,. \tag{4.3}$$

Although this Ω is multivalued the fields remain single-valued. As we did in sect. 2

we can now consider the action of $B(\mathrm{C}, t)$ on physical states ψ only * and observe that $B(\mathrm{C}, t)$ has no direct physical effect away from C. Computation of Green functions such as

$$\langle B(\mathrm{C}, t)\rangle$$

or

$$\langle \mathrm{T}(B(\mathrm{C}_1, t_1)\, B(\mathrm{C}_2, t_2))\rangle ,$$

is now very similar to the computation of the topological Green functions we discussed before. The Dirac string is here replaced by a Dirac surface, usually chosen such that it closes the curves C, C_1, C_2, etc. If the usual Higgs mechanism is switched on then $B(\mathrm{C})$ creates a vortex that is the three-dimensional analogue of the soliton in 2 dimensions. The vortex is very similar to the Nielsen-Olesen vortex [12]. It is locally stable by a topological conservation law, but of course closed strings are unstable against collapse. Consequently we expect $\langle B(\mathrm{C}, t)\rangle \neq 0$ even if the Higgs mechanism works. This is different from the 2 + 1 dimensional case. We have no global $\mathrm{Z}(N)$ symmetry.

It is illustrative to compare the qualitative behavior of $\langle A(\mathrm{C}, t)\rangle$ and $\langle B(\mathrm{C}, t)\rangle$ for large C in a theory with complete Higgs effect, and small coupling constants, where one can rely on perturbation theory. Because the gauge photons are short-range their propagators drop exponentially at large distances, so $\langle A(\mathrm{C}, t)\rangle$ gets its main contributions from gauge-photon propagators that connect only closely separated points on C. For large smooth C we therefore expect to a good approximation that

$$\langle A(\mathrm{C}, t)\rangle = \alpha_1 \exp(-\alpha_2 L(\mathrm{C})) , \tag{4.4}$$

where $L(\mathrm{C})$ is the total length of C and $\alpha_{1,2}$ are fixed constants. This holds for all large smooth curves in Euclidean space.

To estimate $\langle B(\mathrm{C}, t)\rangle$ we observe that its definition is invariant under rotations in Euclidean space. We take for C a rectangle which we then rotate 90° in Euclidean space. Then C is a square with two sides of length l parallel to the z axis and two sides of length τ parallel to the Euclidean time axis. Thus a stretch of Nielsen-Olesen string of length l is created, evolves for a Euclidean time of length τ and is then annihilated. As is well known in Euclidean space the amplitude for such a process approaches, for large τ,

$$\mathrm{e}^{-E\tau} , \tag{4.5}$$

where E is the energy of the created object in its ground state. The Nielsen-Olesen vortex has total energy Ml, where M is exactly the mass of the soliton of the 2 + 1

* Our arguments do not change if we consider the various possible θ vacua defined by states that rotate by an angle $k\theta$ under gauge transformations with a Pontryagin winding number k. This we say because $\Omega^{[\mathrm{C}]}$ can be chosen to be completely in a U(1) subgroup of SU(N), as we saw in sect. 2.

dimensional theory, as it occurs in eq.(2.16). Thus we find

$$\langle B(\mathrm{C})\rangle \simeq \beta_1 \exp(-Ml\tau) = \beta_1 \exp(-M\Sigma(\mathrm{C})) , \tag{4.6}$$

where $\Sigma(\mathrm{C})$ is the area enclosed by C. Thus, we find that $\langle A(\mathrm{C}, t)\rangle$ and $\langle B(\mathrm{C}, t)\rangle$ behave qualitatively differently at large curves C.

The commutation relations between the A- and the B-type operators are as beautiful as in the 2 + 1 dimensional case. By arguments totally analogous to those preceeding eq. (3.8) we find

$$A(\mathrm{C})B(\mathrm{C}') = B(\mathrm{C}')A(\mathrm{C}) \exp(2\pi i n/N) , \tag{4.7}$$

where n is the number of times that the closed curve C$'$ winds around the closed curve C, in a prescribed direction. Also, analogous to eq. (2.8),

$$[B(\mathrm{C}), B(\mathrm{C}')] \,|\psi\rangle = 0 \tag{4.8}$$

if $|\psi\rangle$ is a physical state.

So we observe that the commutation rules between the A- and B-type operators (eqs. (4.2), (4.7), (4.8)) are totally symmetric under interchange $A \leftrightarrow B$, apart from a parity reflection (the sign of n). The commutation rules are universal for any SU(N) gauge theory with scalars and spinors invariant under the center Z(N) of SU(N).

The central point of this paper is now the following: from these commutation relations alone, one can derive that the system may be in one of four different phase configurations.

(i) The conventional complete Higgs mode. The Higgs field is either explicit or dynamical. There are no massless particles. The mode is further characterized by

$$\langle A(\mathrm{C})\rangle \to \alpha_1 \exp(-\alpha_2 \, \mathrm{L}(\mathrm{C})) ,$$

$$\langle B(\mathrm{C})\rangle \to \beta_1 \exp(-\beta_2 \, \Sigma(\mathrm{C})) ,$$

as we have already seen.

(ii) Absolute and permanent confinement of all colored sources. Again no massless particles. We have the opposite of the previous phase:

$$\langle A(\mathrm{C})\rangle \to \alpha_1 \exp(-\alpha_2 \Sigma(\mathrm{C})) ,$$

$$\langle B(\mathrm{C})\rangle \to \beta_1 \exp(-\beta_2 L(\mathrm{C})) .$$

The first of these relations is Wilson's criterion for quark confinement.

(iii) Partial Higgs mechanism and still confinement, which is when SU(N) is broken down spontaneously to a smaller non-Abelian subgroup:

$$\langle A(\mathrm{C})\rangle \to \alpha_1 \exp(-\alpha_2 \Sigma(\mathrm{C})) ,$$

$$\langle B(\mathrm{C})\rangle \to \beta_1 \exp(-\beta_2 \Sigma(\mathrm{C})) .$$

Finally, one can have that,

(iv) there are massless particles, presumably of vector type. Indeed, the operator algebra can be realized in a set of operators acting on free Maxwell fields. In SU(N) theories we think this configuration corresponds to the phase transition point. We will argue why this configuration will not be realized in general in SU(N) gauge theories.

So, in particular, from the commutation rules we can exclude rigorously the possibility that no massless particles occur and that both

$$\langle A(\mathrm{C})\rangle \to \alpha_1 \exp(-\alpha_2 L(\mathrm{C}))\,,$$
$$\langle B(\mathrm{C})\rangle \to \beta_1 \exp(-\beta_2 L(\mathrm{C}))\,.$$

The first of these relations would tell us that there is no confinement and the second would imply that we are not in a Higgs mode. Further, phases (i) and (ii) are each other's mirror image. Since phase (i), the Higgs mode, is well understood qualitatively, phase (ii) the confinement mode is now understood to be the dually opposite phase configuration. Let us now derive this result.

We first have to understand what the commutation rule (4.7) implies for Green functions in 4-dimensional Euclidean space. The A and B fields are still defined on closed curves C and C$'$, but in 4 dimensions the winding number n is not well-defined topologically. The situation is to be compared with fermionic Green functions which, due to the fact that Fermi fields anticommute, are only well-defined up to a sign. Whatever the convention will be, we must require that exchange of two external fermion lines corresponds to a change of sign in the amplitude.

By careful topological considerations one can now convince oneself of the following requirement for the phase factor of a Green function

$$\langle A(\mathrm{C}), B(\mathrm{C}'), \ldots \rangle$$

as a function of the position of the closed oriented contours C and C$'$ in Euclidean 4-dimensional space. For simplicity we consider C and C$'$ both to be flat, that is, they both fit in different flat planes. In fig. 1 we draw a 3-dimensional cross section of 4-dim. space, containing the full plane of the C curve. The C$'$ plane does not fit in the 3-dimensional picture, with which it only has a line L in common. L does not lie in the C plane. We consider the case where C$'$ goes through L at the point a and back at b, see fig. 1. Now we continuously vary C$'$, by moving the point b, not a. If b follows a closed curve C$''$ in 3-dim. space, winding through C exactly once, then when it reaches its original position we obtain the same Green function multiplied by a phase factor $\exp(\pm 2\pi i/N)$, where the sign depends on the relative orientations of C, C$'$ and C$''$. The best way to see this property is to identify the direction orthogonal to the C plane in fig. 1 with time. As soon as we move the point b through the C plane, inside C then the time of C$'$ switches from before to after C so we pick up the phase factor due to (4.7). We do not get this phase factor back when b moves back through the C plane outside the C curve. So, like the familiar fermionic Green function, these vacuum expectation values are really well-defined up to a common phase factor.

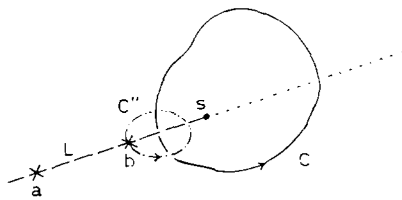

Fig. 1. Three-dimensional cross section of 4-dimensional Euclidean space, exhibiting the curve C and intersecting the curve C′ at two points a and b. We let b run over the curve C″. We see one straight line L of the plane through C′, which intersects the plane through C at s.

We now proceed to prove our statements on the allowed phase configurations. Let us assume that massless particles are absent, (thus excluding case (iv)), and take two macroscopic curves C and C′. Consider $\langle A(\mathrm{C})\, B(\mathrm{C}')\rangle$, and choose also a macroscopic curve C″ as in fig. 1. The curves C and C′ then never come close to each other; only their planes have one point s in common, but s is both far away from C and C′ (i.e. the points a and b). So it is not so unreasonable to expect declustering:

$$\langle A(\mathrm{C})\, B(\mathrm{C}')\rangle \overset{?}{=} \langle A(\mathrm{C})\rangle\langle B(\mathrm{C}')\rangle + \text{exponentially vanishing interaction terms.} \tag{4.7}$$

However, clearly, this cannot be, because $\langle A(\mathrm{C})\rangle$ is independent of C′ and *vice versa* so we can never satisfy the phase-factor requirement just derived. So we must have

$$\langle A(\mathrm{C})\, B(\mathrm{C}')\rangle \simeq \langle A(\mathrm{C})\rangle\langle B(\mathrm{C}')\rangle\, \mathrm{e}^{i\alpha(\mathrm{C},\mathrm{C}')} . \tag{4.8}$$

Here $\alpha(\mathrm{C}, \mathrm{C}')$ is an observable angle, apart from multiples of $2\pi/N$. If there are no massless particles $\alpha(\mathrm{C}, \mathrm{C}')$ cannot be a smoothly varying function. So $\alpha(\mathrm{C}, \mathrm{C}')$ jumps during the process pictured in fig. 1. Where will the jump take place? Either when s hits b (the curve C′), but then s is still far away from C; or when s hits C, but then s is still far away from b and a (the curve C′). So, either a physical change must take place when b crosses the plane inside C, or when C crosses the plane inside C′ (the line ab in fig. 1). This implies that a physically observable sheet must occur either inside C, or inside C′ (or both). The sheet determines the special subset of curves C, C′ for which $\alpha(\mathrm{C}, \mathrm{C}')$ is not equal to a multiple of $2\pi/N$, which can only be if whatever objects are created by $A(\mathrm{C})$ and $B(\mathrm{C}')$ strongly interact. Such a physically observable sheet must carry a finite action-density, so if it occurs in C′, then $\langle B(\mathrm{C}')\rangle \to \beta_1 \exp(-\beta_2 \Sigma(\mathrm{C}'))$ (case (i)); if it occurs in C then $\langle A(\mathrm{C})\rangle \to \alpha_1 \exp(-\alpha_2 \Sigma(\mathrm{C}))$ (case (ii)). If it occurs in both we have case (iii).

When massless particles occur, the long-distance communication between C and C′ cannot be neglected, so in that case $\alpha(\mathrm{C}, \mathrm{C}')$ may vary smoothly to recover the phase factor (case (iv)).

In spite of the symmetry of the A and B commutation rules, this symmetry between A and B is entirely lost in the modes (i) and (ii). Since the dynamics of the theory must determine which mode is realized we expect that when we vary the input parameters of the theory the transitions from one phase into another must be sudden ones, characteristic of phase transitions. Whether phase (iv) (massless particles) only occurs at critical points, or can be a more generic configuration we do not know but we shall give arguments in favor of the critical point.

Our observation can be brought in connection with a result by Corrigan and Olive [13] in the following way. These authors considered Dirac-type monopoles as being end points of a string of singularities created by an operator of the type $B(\mathrm{C})$ as constructed in our paper, where C is now an open contour. Let us now add a U(1) gauge system. If quarks have a U(1) electric charge $Q = e/N$ then these Dirac monopoles must carry U(1) magnetic charge $M = 2\pi/e$, as explained in ref. [13].

Now assume that there is a mass gap in the SU(N) part of the theory (the only massless particles must be the U(1) photons). Then at macroscopic distances we only see the U(1) electric charges Q and the U(1) magnetic charges M which now must satisfy Dirac's original rule

$$QM = 2\pi n\ , \qquad n \text{ integer}\ .$$

This can only be if either the Dirac monopoles or the quarks are confined by strings, modulo grouplets of N. There must be either A-type, or B-type strings (or, but unlikely, both).

5. Representation of the operator algebra

We have a set of operators, $A(\mathrm{C}, t)$ and $B(\mathrm{C}, t)$ and their complete equal-time commutation rules, which are non-trivial. Can we find a simple representation of these operators as matrices in a Hilbert space satisfying these rules? Can one then find a Hamiltonian corresponding to a Lorentz covariant theory?

The first observation one can make is that in pure electrodynamics (without charged particles, so without any interactions) operators may be constructed that satisfy the commutation rules. We can choose

$$A(\mathrm{C}) = \exp(i \oint_{\mathrm{C}} A_\mu^{\mathrm{em}}(x)\, \mathrm{d}x^\mu)\ , \tag{5.1}$$

and $B(\mathrm{C})$ is an operator defined by an (Abelian) gauge transformation $\Lambda^{[\mathrm{C}]}$, with the requirement that following an oriented C$'$ that winds through C with n windings and is parametrized by $0 \leqslant \theta \leqslant 2\pi$, we must get

$$\Lambda(2\pi) = \Lambda(0) + n\Delta\ , \tag{5.2}$$

where Δ is some fixed number. One finds

$$A(\mathrm{C}')\, B(\mathrm{C}) = B(\mathrm{C}) A(\mathrm{C}')\, \exp(in\Delta)\ , \tag{5.3}$$

Fig. 2. A curve with allowed N-prong vertices.

so choosing $\Delta = 2\pi/N$ we get the desired algebra. Clearly, this is an example of a realization of the commutation rules in a system with massless particles (photons).

However, one also notes that we have a much larger algebra than required. We could choose for Δ anything we like instead of $2\pi/N$. So it seems that the Hilbert space of photons is larger than necessary. Another delicate point in this representation is that in SU(N) gauge theories with N finite, $A(\mathrm{C})$ and $B(\mathrm{C})$ are also defined on oriented curves C with vertices consisting of N lines all going out or all going in (in the case of A because of the invariant $\epsilon_{i_1 \ldots i_N}$, and in the case of B because each element Z of Z(N) satisfies $Z^N = 1$, see fig. 2). Extensions of our Abelian operators on these curves is however only possible if A and B are identically equal to zero on such curves.

So let us make a new attempt, being as economic as possible without producing null operators.

Consider first two curves C_1 and C_2 winding through each other. The commutation rules are

$$
\begin{aligned}
&A_1 B_2 = B_2 A_1 \exp(2\pi i/N)\,,\\
&A_2 B_1 = B_1 A_2 \exp(2\pi i/N)\,,\\
&\text{all other products commute}\,.
\end{aligned}
\tag{5.4}
$$

Here, A_i stands for $A(\mathrm{C}_i)$. Of course, $A_i = 0$ or $B_i = 0$ is a solution. However, let us assume that A_i and B_i can be diagonalized, and that all eigenvalues are different from zero. Let

$$A_1 |n\rangle = \lambda_n |n\rangle\,. \tag{5.5}$$

Then

$$A_1 B_2 |n\rangle = \lambda_n \mathrm{e}^{2\pi i/N} B_2 |n\rangle\,. \tag{5.6}$$

Thus we find

$$
\begin{aligned}
&B_2 |n\rangle = \mu_n |n+1\rangle\,,\\
&\lambda_{n+1} = \lambda_n \mathrm{e}^{2\pi i/N}\,.
\end{aligned}
\tag{5.7}
$$

Apart from some arbitrariness in normalization we obtain

$$
A_1 = \begin{bmatrix} \mathrm{e}^{2\pi i/N} & & & & \\ & \mathrm{e}^{4\pi i/N} & & \emptyset & \\ & & \cdot & & \\ & & & \cdot & \\ \emptyset & & & \cdot & \mathrm{e}^{-2\pi i/N} \end{bmatrix},
$$

$$B_2 = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \emptyset \\ & & \ddots & \ddots & \\ & \emptyset & & & 1 \\ 1 & & & & 0 \end{bmatrix} . \tag{5.8}$$

A_2 and B_1 are also $N \times N$ matrices in a Hilbert space orthogonal to this one. As we see, the matrices A and B can be chosen to be unitary.

Extension to larger sets of curves is straightforward. Let us take as the next step a finite grid or lattice and consider all possible closed contours on this lattice.

For simplicity, our lattice is taken to be cubic. Let C_1, C_2 and C_3 be three closed oriented contours such that C_3 spans the same area as C_1 and C_2 together. Then the product $A(\mathrm{C}_1)\,A(\mathrm{C}_2)$ satisfies the same commutation rules as $A(\mathrm{C}_3)$. Since we want an economic representation we impose the restriction that in such cases

$$A(\mathrm{C}_3) = A(\mathrm{C}_1)\,A(\mathrm{C}_2)\,, \tag{5.9}$$

and similarly

$$B(\mathrm{C}_3) = B(\mathrm{C}_1)\,B(\mathrm{C}_2)\,. \tag{5.10}$$

It will then be sufficient to define A and B on the smallest elementary surfaces of the lattice only. We make one refinement: in view of the nature of our commutation rules we define the A operators only on elementary surfaces of the lattice, and the B operators only on elementary surfaces of the dual of that lattice (the elementary boxes of a lattice define the points of its dual lattice and *vice versa*).

Since a curve C_3 may be decomposed into different sets of C_1 and C_2 we must take care to impose (5.9) and (5.10) on each of these sets. The best way to guarantee these identities is to write

$$A(\mathrm{C}) = \prod_{b \in \mathrm{C}} R_b\,, \tag{5.11}$$

$$B(\mathrm{C}') = \prod_{l \in \mathrm{C}'} S_l\,, \tag{5.12}$$

where b labels the elementary lattice bonds that make the curve C and l are the bonds of the dual lattice that form C'. First we require

$$[R_b, R_{b'}] = 0\,, \qquad [S_l, S_{l'}] = 0\,. \tag{5.13}$$

The bonds b and l are taken to be oriented: we write $\bar{b}$ and $\bar{l}$ for the oppositely oriented bonds. Further,

$$R_{\bar{b}} = (R_b)^\dagger\,, \qquad S_{\bar{l}} = (S_l)^\dagger\,. \tag{5.14}$$

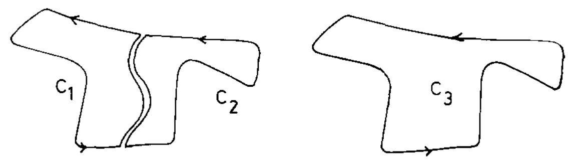

Fig. 3. If C_1 and C_2 together span the same area as the single curve C_3 the restrictions $A(C_1)$ $A(C_2) = A(C_3)$ and $B(C_1)\,B(C_2) = B(C_3)$ are consistent with the commutation rules although the first do not follow from the definition of A.

We will restrict ourselves to unitary R and S matrices. There is a local gauge invariance, both for R and for S. At each point x of the lattice we may transform

$$R_b \to \Omega R_b \tag{5.15}$$

if b connects to x and is oriented towards x, and

$$R_b \to R_b \tag{5.15b}$$

if b is not connected to x. At each point p of the dual lattice a similar transformation is defined for the S_l. These transformations leave A and B invariant, if Ω is unitary.

Let us choose Ω by requiring all R_{b3} and S_{l3} to be equal to the identity for vertical bonds $b3$ and $l3$. That fixes Ω up to boundary effects. The remaining bonds will be labeled as $b1$, $l1$, for the x direction and $b2$, $l2$ for the y direction. Up to boundary effects,

$$\begin{aligned} R_{b1}S_{l2} &= S_{l2}R_{b1}\ \exp(2\pi i/N)\,, \\ R_{b2}S_{\bar{l}1} &= S_{\bar{l}1}R_{b2}\ \exp(2\pi i/N)\,, \end{aligned} \tag{5.16}$$

only if $l1$ lies directly but at arbitrary distance above $b2$, and $l2$ directly but ar arbitrary distance above $b1$. For all other pairs (b, l) R_b and S_l commute. Define Z_b by

$$R_b Z_{b'} = Z_{b'} R_b\ \exp(2\pi\delta_{bb'}/N)\,. \tag{5.17}$$

Then a solution of (5.16) is

$$S_{l2} = \prod_{b1>l2} Z_{b1}\,, \qquad S_{l1} = \prod_{b2>l1} Z^{\dagger}_{b2}\,, \tag{5.18}$$

where $b1 > l2$ is shorthand for: $b1$ lies directly but at arbitrary distance above $l2$. One can easily verify the validity of the elementary commutation rules between the resulting matrices A and B.

Eq. (5.17) has as a solution the two $N \times N$ matrices of eq. (5.8). Hence our total Hilbert space has dimension NL, where L is the number of bonds of the lattice (or of the dual lattice; the difference is a negligible boundary effect). Note that our system of operators can immediately be extended to curves of the type depicted in fig. 2. In fact, what we have is nothing but a $Z(N)$ gauge theory on the lattice. This may be the simplest but certainly not the only representation of our commutation algebra. We may multiply A with anything that commutes with B or *vice versa.*

The representation is essentially self-dual. If the symmetry allows a Higgs mode to be realized then the confinement mode can also occur.

6. A simple model: conclusions

Sect. 4 tells us that the existence of a phenomenon that we call "quark confinement" as a new phase of gauge field theories follows solely from the commutation rules between A- and B-type operators. Whether the phase is actually realized in a given gauge theory depends on the dynamics. Attempts to "explain" confinement as they occur in the literature make use of large field configurations with non-trivial topological properties [2]. True, large field fluctuations will tend to cause the disorder parameter rather than the order parameter to develop large vacuum expectation values. Perhaps that is why these attempts sometimes show encouraging results [2]. But instantons, "merons" and the like also occur in Higgs field theories. It will always be the competition between ordering effects (Higgs scalar fields) and disordering effects (fluctuations with large topological quantum numbers) that in the end determines which phase will be realized. Thus it is not correct to suggest that confinement takes place formally also in spontaneously broken gauge theories at super-large distances because these too have instantons etc. There, simply the other phase has been realized.

What does our theory now have in common with the BCS theory of the superconductor? The BCS theory tells us that due to a complicated dynamics involving phonons, a two-electron bound state, called a Cooper pair, may develop a non-vanishing vacuum expectation value (order parameter). Whether or not this happens depends on the relative strength of certain intrinsic parameters. To translate that picture to our theory, replace "phonons" by "instantons", "merons" or other objects. Replace "Cooper pairs" by $Z(N)$ solitons (in the 2 + 1 dimensional case) or by the $Z(N)$ variant of the Nielsen-Olesen flux tube (in the 3 + 1 dimensional case). The order parameter is replaced by the "disorder parameter" $\phi(x)$ or $B(C)$.

Thus, in 2 + 1 dimensions we can replace Wilson's criterion for permanent quark confinement by $\langle\phi(x)\rangle \neq 0$, which may be considerably easier to verify in numerical computations than Wilson's area law. In 3 + 1 dimensions we can now replace Wilson's area law by

$$\langle B(C)\rangle \to \beta_1 \mathrm{e}^{-\beta_2 L(C)} \tag{6.1}$$

for large curves C, as our new condition for permanent quark confinement. In other words, $\langle B(C)\rangle$ goes to zero much slower than an exp(−area) law and this might again be easier to verify numerically than Wilson's criterion, for which delicate cancellations are crucial.

Sect. 5 tells us that the relevant commutation rules are already realized in the

$Z(N)$ gauge theory on a lattice. The group $Z(N)$ is only a subgroup of U(1). This is why we stated in the beginning of sect. 5 that the U(1) representation that we found first was probably too large. Now let us combine the results of sects. 4 and 5. We conclude then that confinement may be studied most easily in the $Z(N)$ gauge theory on a lattice. Consider for simplicity $N = 2$ and extend the lattice to Euclidean space so that Loretnz invariance in the continuum limit is guaranteed. This model, both in 3 and in 4 dimensions, has been studied extensively in ref. [14]. The 3-dimensional case is dual to the 3-dimensional Ising model. Extensive numerical analysis shows a phase transition here at some finite value of the coupling constant. When the Ising model is in the condensed (ordered) phase then the gauge model has a non-vanishing vacuum expectation value for the disorder parameter and hence confinement is a fact. (Quarks are to be coupled to the $Z(N)$ gauge fields as if the $Z(N)$ gauge operators were just the center of a larger color $SU(N)$ gauge set.)

The Z(2) gauge theory on a 4-dimensional lattice is known to be self-dual: see ref. [14], where also numerical studies are carried out. These calculations seem to confirm that there is a phase transition at the symmetry value of the coupling constant. Ref. [14] also finds a similar phase transition for the SU(2) model on a lattice.

The phase transition may also occur as a function of temperature. Field theory at finite temperature leads to a Euclidean space with periodic boundary conditions in the imaginary time direction. The infrared structure is then as in 3-dimensional Euclidean space and after a dual transformation one obtains a scalar disorder parameter $\phi(x)$. As has been noted by Polyakov [2,15] one may therefore expect a phase transition with critical exponents as in 3-dimensional scalar theories.

It would be tempting to abolish the SU(3) color theory for hadrons altogether, replacing it by a Z(3) theory on a Euclidean lattice and taking the continuum limit close to the critical point. This is presumably not justifiable. At the critical point we have a finite coupling constant. As far as we understand the theory, in the scaling region (defined by a distance scale large compared to the lattice size and small compared to hadronic sizes) there would be no small effective coupling but a strongly interacting soup full with anomalous dimensions, contrary to experimental observations.

Nevertheless, analytic and numerical study of $Z(N)$ theories on a lattice in 4 dimensions should be carried out. We expect the Z(2) theory for instance to be numerically much better accessible than the complicated SU(2) gauge theory, so it may become a valuable toy to obtain our so desperately wanted intuitive understanding of the mechanism that causes quarks to be permanently confined.

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