# BOUND FERMION STATES ON A VORTEX LINE IN A TYPE II SUPERCONDUCTOR

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This note discusses the excitations of low energy $\epsilon \ll \Delta_\infty$ (where $\Delta_\infty$ is the gap in zero field) which exist near an Abrikosov vortex line [1] in a pure superconductor of type II. The energy gap $\epsilon_0$ for these excitations is very small $\epsilon_0 \sim \Delta_\infty^2/E_F$ where $E_F$ is the Fermi energy. Above $\epsilon_0$ the density of states is finite and comparable to that of a cylinder of normal metal of radius $\xi$ (the coherence length). These low lying states will play a major role in the discussion of transport and relaxation phenomena in type II superconductors at low temperatures.

We restrict our attention to materials where a) the Landau-Ginzburg parameter $\kappa = \lambda/\xi$ is much larger than unity, b) the field $H$ is much smaller than the upper critical field $H_{c2}$. Then the distance $d$ between lines is much larger than $\xi$, and most of the excitation (with energies $\epsilon \sim \Delta_\infty$) can be obtained as follows: at each point $r$ one computes the superfluid velocity $v_s(r)$ and writes that locally the excitations have a shifted BCS spectrum

$$\epsilon(k,r) = \{\Delta_\infty{}^2 + \xi_k{}^2\}^{\frac{1}{2}} + \hbar v_s \cdot k \,,$$

(where $\xi_k = \hbar^2(k^2 - k_F{}^2)/2m$. Since the minimum spatial extension of wave packets made with these excitations is of order $\xi$, while the fields and the velocities $v_s$ are modulated on a scale $d \gg \xi$, this procedure is in general correct [2]).

It fails however for the low lying excited states ($\epsilon \ll \Delta_\infty$) which are localised very near one line where $v_s$ and the order parameter $\Delta$ vary rapidly. From now on, we consider only these particular excitations in the vicinity of one single line. The excitation energies are the positive eigenvalues of the system [3]):

$$\begin{aligned} \epsilon\, u(r) &= \left[\frac{1}{2m}(p - \frac{eA}{c})^2 - E_F\right] u(r) + W(r)\, v(r) \,, \\ \epsilon\, v(r) &= \left[-\frac{1}{2m}(p + \frac{eA}{c})^2 + E_F\right] v(r) + W(r)\, u(r) \,. \end{aligned} \tag{1}$$

Here the pair potential $W(r) = V\langle\psi_\uparrow(r)\psi_\downarrow(r)\rangle$ is strongly $r$-dependent. We choose a gauge such that:

$$W(r) = \Delta(r)\, e^{-i\theta} \qquad (\Delta \text{ real}) \,,$$

where $(r,\theta,z)$ are cylindrical coordinates around the line axis *. $\Delta(r)$ vanishes for $r = 0$, increases linearly at small $r$, and goes to a constant limit $\Delta_\infty$ for $r \gtrsim \xi$. We rewrite (1) in a condensed spinor notation $\hat{\varphi} = \binom{u}{v}$ and eliminate the phase in $W$ by setting $\hat{\varphi} = e^{-\frac{1}{2}i\sigma_z\theta}\,\hat{\psi}$ ($\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices). Eq. (1) becomes

$$\sigma_z\{\frac{1}{2m}(p - \sigma_z\frac{eA}{c} - \tfrac{1}{2}\sigma_z\hbar\nabla\theta)^2 - E_F\}\hat{\psi} + \sigma_x\Delta\hat{\psi} = \epsilon\hat{\psi} \,. \tag{2}$$

Note that

$$A \sim Hr \text{ and } eA/c\,\hbar\nabla\theta \sim (H/\Phi_0)r^2 \,,$$

(where $\Phi_0 = ch/2e$ is the flux quantum). For the excitations of interest $r \lesssim \xi$ and

$$eA/c\hbar\,\nabla\theta \sim H\xi^2/\Phi_0 \sim H/H_{c2} \ll 1.$$

* Note that $u$ and $v$ are not invariant by a $2\pi$ rotation around the line axis, but only by a $4\pi$ rotation.

Thus we can neglect all magnetic field effects. We then look for solutions of the form

$$\hat{\psi} = \exp(\mathrm{i}k_F z \cos\alpha)\exp(\mathrm{i}\mu\theta)\hat{f}(r) ,$$

where $k_F$ is the Fermi wave vector, $\alpha$ an arbitrary angle, and $\mu$ a positive or negative integer *.

Dropping the A term in (2) we obtain

$$\sigma_z \frac{\hbar^2}{2m}\{-\frac{\mathrm{d}^2\hat{f}}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}\hat{f}}{\mathrm{d}r} + (\mu - \tfrac{1}{2}\sigma_z)^2\frac{\hat{f}}{r^2} - k_F{}^2\sin^2\alpha\hat{f}\} + \sigma_x\Delta(r)\hat{f} = \epsilon\hat{f} . \tag{3}$$

It is possible to solve (3) completely in the region $0 < \mu \ll k_F\xi$ which turns out to be the important one. Consider a radius $r_c$ such that

$$(\mu + \tfrac{1}{2})k_F{}^{-1} \ll r_c \ll \xi .$$

For $r < r_c$, the $\Delta$ term in (3) can be neglected and $\hat{f} = \binom{f_+}{f_-}$ is given by:

$$f_\pm(r) = A_\pm J_{\mu\mp\frac{1}{2}}\{k_F\sin\alpha \pm q)r\} , \tag{4}$$

where $J$ is a Bessel function, $A_\pm$ are arbitrary coefficients, and $q = \epsilon/v_F\sin\alpha$.

For $r > r_c$, we put

$$\hat{f} = \hat{g}(r)H_m(k_F r\sin\alpha) + \mathrm{cc} \qquad (m = \sqrt{\mu^2 + \tfrac{1}{4}}) , \tag{5}$$

where $H$ is a Hankel function and $\hat{g}$ a slowly varying envelope. The equation for $g$ may be reduced to

$$-\mathrm{i}\sigma_z\hbar v_F\sin\alpha\frac{\mathrm{d}\hat{g}}{\mathrm{d}r} + \Delta\sigma_x\hat{g} = (\epsilon + \frac{\mu\hbar^2}{2mr^2})\hat{g} \qquad (v_F = hk_F/m) . \tag{6}$$

For $\epsilon \ll \Delta_\infty$ and $k_F r \gg \mu$, the right hand side is a small perturbation, Treating it to first order, we get

$$\hat{g} = \mathrm{const}\times\binom{\mathrm{e}^{\frac{1}{2}\mathrm{i}\psi}}{-\mathrm{i}\mathrm{e}^{-\frac{1}{2}\mathrm{i}\psi}}\mathrm{e}^{-K} , \qquad K(r) = (\hbar v_F\sin\alpha)^{-1}\int_0^r \Delta(r)\,\mathrm{d}r , \tag{7}$$

$$\psi(r) = -\int_r^\infty \exp\{2K(r) - 2K(r')\}(2q + \frac{\mu}{k_F r'^2\sin\alpha})\mathrm{d}r' ,$$

$$\psi(r_c) \cong -\mu(k_F r_c\sin\alpha)^{-1} + 2qr_c - 2\int_0^\infty \mathrm{d}r'\,\mathrm{e}^{-2K(r')}(q - \frac{\mu\Delta(r')}{k_F v_F\sin^2\alpha}) . \tag{8}$$

Finally we match the solutions (4) and (5) at $r = r_c$, making use of the asymptotic forms

$$J_m(z) = \mathrm{const}\,z^{-\frac{1}{2}}\sin\{z + \frac{m^2}{2z} - \tfrac{1}{2}\pi(m - \tfrac{1}{2})\}\ \text{etc.} ,$$

and obtain the condition (for $\mu \neq 0$) **

$$\psi(r_c) = 2qr_c - \mu(k_F r_c\sin\alpha)^{-1} . \tag{9}$$

Comparing with (8) we see that all the $r_c$ dependent terms cancel out and find the eigenvalue

$$\epsilon_{\mu\alpha} = qv_F\sin\alpha = \mu(k_F\sin\alpha)^{-1}\frac{\int_0^\infty \frac{\Delta(r)}{r}\mathrm{e}^{-2K(r)}\mathrm{d}r}{\int_0^\infty \mathrm{e}^{-2K(r)}\mathrm{d}r}$$

$$= \mu(k_F\sin\alpha)^{-1}(\mathrm{d}\Delta/\mathrm{d}r)_{r=0}\,g(\alpha) \qquad (\mu \neq 0,\ \mu \ll k_F\xi) . \tag{10}$$

The dimensionless function $g(\alpha)$ defined by (10) depends on the exact shape assumed for $\Delta(r)$ but is al-

* See footnote preceding page.

** For $\mu = 0$, the eigenvalue is large and the approximation (7) is not valid.

ways close to 1. In particular $g(0) = g(\pi) = 1$, If we choose for $\Delta(r)$ the form derived from the Landau-Ginzburg equation (near $T_c$) we get $g(\frac{1}{2}\pi) = 0.79$. Thus the eigenvalues (10) are of order $\mu\Delta_\infty/k_F\xi \sim \mu\Delta_\infty^2/E_F$, the lowest one corresponding to $\mu = 1$. The density of states $N_l$ associated with the levels (10) is (for one spin direction)

$$N_l(\epsilon) = \frac{1}{2\pi}\{(\mathrm{d}\Delta/\mathrm{d}r)_{r=0}\}^{-1} k_F^2 \int_0^\pi \mathrm{d}\alpha \frac{\sin^2\alpha}{g(\alpha)} \quad (\frac{\Delta_\infty}{E_F} < \epsilon \ll \Delta_\infty) . \tag{11}$$

With the above choice of $\Delta(r)$, the integral $\int \mathrm{d}\alpha$ is equal to 1.92.

We have made separate studies 1) of higher excited states with the same $\mu$ and $\alpha$, 2) of the region $\mu \sim k_F\xi$, 3) of various special cases ($\mu = 0$), ($\alpha = 0$) ($\alpha \to 0$). None of these contribute significantly to the density of states for $\epsilon \ll \Delta_\infty$. Finally, eq. (11) shows that $N_l(\epsilon) \sim N(0)\xi^2$, i.e., each line is equivalent to a normal region of radius $\sim \xi$. The low lying states occupy only a fraction $\sim (\xi/d)^2 \sim (H/H_{c2})^2$ of the volume. They will be of importance mainly at low temperatures $T$ (roughly when $(\xi/d)\exp 2(\Delta_\infty/T) > 1$). Then the specific heat will be linear in $T$, the thermal conductivity will be anisotropic (maximum along the lines) and the nuclear relaxation may, in some cases, become limited by the spin diffusion rate. It would also be of great interest to detect the threshold energy by ultrasonic absorption.

1) A.A.Abrikosov, J.Exptl.Theoret.Phys.(USSR) 32 (1957) 1442; translation: Soviet Phys.JETP 5 (1957) 1174.
2) M.Cyrot, to be published.
3) N.N.Bogoliubov, V.V.Tolmachev, D.V.Shirkov. A new method in the theory of superconductivity (Consultants Bureau Inc., New York, 1959).

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# A MODEL FOR THE MAGNETIC BEHAVIOUR AND THE $I$-$H$ CURVE OF SUPERCONDUCTORS WITH AN ANCHORED FILAMENT STRUCTURE

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In this note a model is described for the calculation of the magnetisation curve of:

a) a cylindrical superconducting rod in a field parallel to the rod axis;
b) the $I$-$H$ curves for fields either parallel or perpendicular to the axis.

In this model anchored superconducting filaments are assumed embedded either in a soft superconductor or in an insulator. If $H > H_c$ the filaments are considered to behave as superconducting wires in vacuum. Then, according to the London theory the maximum super current through a very thin circularly cylindrical filament is given by

$$i_m = A[-2H_\perp + \sqrt{H_f^2 - H_{/\!/}^2}] , \tag{1}$$

where $A$ is a constant, $H_f$ is the critical magnetic field of the filament while $H_\perp$ and $H_{/\!/}$ are the components of the magnetic field being perpendicular or parallel to the filament respectively. For the calculation of the magnetisation curve we assume according to the model of Bean [1] and of Shaw and Mapother [2] that inside the body of the cylindrical rod ringshaped filaments are arranged concentrically with the axis of the rod. Furthermore, we suppose the density of the filaments to be uniform in this rod. By definition the magnetisation of a cylindrical rod in a longitudinal magnetic field is *:

$$M = \frac{\int (H(r) - H) r \, \mathrm{d}r}{\int r \, \mathrm{d}r} . \tag{2}$$

Here $H$ is the external magnetic field while $H(r)$ is the magnetic field in the interior of the cylindrical rod at a distance $r$ from the axis.

Calculations of the magnetisation are possible after having determined the magnetic field as a function of $r$. The field distribution over the rod

* Giorgi units are used.