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Gauge Invariance and Mass. II*

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The possibility that a vector gauge field can imply a nonzero mass particle is illustrated by the exact solution of a one-dimensional model.

It has been remarked[1] that the gauge invariance of a vector field does not necessarily require the existence of a massless physical particle. In this note we shall add a few related comments and give a specific model for which an exact solution affirms this logical possibility. The model is the physical, if unworldly situation of electrodynamics in one spatial dimension, where the charge-bearing Dirac field has no associated mass constant. This example is rather unique since it is a simple model for which there is an exact divergence-free solution.[2]

GENERAL DISCUSSION

The Green's function of an Abelian vector gauge field has the structure

$$\mathcal{G}_{\mu\nu}(x,x')=\pi_{\mu\nu}(-i\partial)\mathcal{G}(-i\partial)\delta(x-x'),$$

where $\pi_{\mu\nu}(p)$ is a gauge-dependent projection matrix and

$$\mathcal{G}(p)=\int_0^\infty dm^2\,\frac{B(m^2)}{p^2+m^2-i\epsilon},$$

which is subject to the sum rule

$$1=\int_0^\infty dm^2\,B(m^2).$$

An alternative form of $\mathcal{G}(p)$ is

$$\mathcal{G}(p)=\left[p^2+\lambda^2-i\epsilon+(p^2-i\epsilon)\int_0^\infty dm^2\,\frac{s(m^2)}{p^2+m^2-i\epsilon}\right]^{-1},$$

where the function $s(m^2)$ and the constant λ^2 are nonnegative. The latter has been derived[3] with the understanding that the pole at $z=0$ of the expression

$$-\frac{\lambda^2}{z}+\int_0^\infty dm^2\,\frac{s(m^2)}{m^2-z}=\int_0^\infty \frac{dm^2}{m^2-z}[s(m^2)+\lambda^2\delta(m^2)]$$

is completely described by the parameter λ. Accordingly,

$$\mathcal{G}(0)=\frac{1}{\lambda^2}=\int_0^\infty dm^2\,\frac{B(m^2)}{m^2},$$

and $\lambda^2>0$ unless $m=0$ is contained in the spectrum. Thus, it is necessary that λ vanish if $m=0$ is to appear as an isolated mass value in the physical spectum. But it is also necessary that

$$s(0)=0,$$

such that

$$\int_{\to 0}^\infty \frac{dm^2}{m^2}s(m^2)<\infty,$$

for only then do we have a pole at $p^2=0$,

$$p^2\sim 0:\quad \mathcal{G}(p)\sim B_0/(p^2-i\epsilon),\quad 0<B_0<1.$$

Under these conditions,

$$B(m^2)=B_0\delta(m^2)+B_1(m^2),$$

where

$$B_0=\left(1+\int_0^\infty\frac{dm^2}{m^2}s(m^2)\right)^{-1}$$

and

$$B_1(m^2)=[s(m^2)/m^2]\Big/\left[1+P\int_0^\infty dm'^2\frac{s(m'^2)}{m'^2-m^2}\right]+[\pi s(m^2)]^2.$$

The physical interpretation of $s(m^2)$ derives from the relation of the Green's function to the vacuum transformation function in the presence of sources. For sufficiently weak external currents $J_\mu(x)$,

$$\langle 0|0\rangle^J=\exp\left[\tfrac{1}{2}i\int(dx)(dx')J^\mu(x)\mathcal{G}_{\mu\nu}(x,x')J^\nu(x')\right]$$
$$=\exp\left[\tfrac{1}{2}i\int(dp)J^\mu(p)^*\mathcal{G}(p)J_\mu(p)\right],$$

which involves the reduction of the projection matrix $\pi_{\mu\nu}(p)$ to $g_{\mu\nu}$ for a conserved current, or equivalently

$$p_\mu J^\mu(p)=0.$$

We shall present this transformation function as a measure of the response to the external vector potential

$$A_\mu(p)=\mathcal{G}(p)J_\mu(p),$$

namely,

$$\langle 0|0\rangle^J=\exp\left[\tfrac{1}{2}i\int(dp)A^\mu(p)^*\mathcal{G}(p)^{*-1}A_\mu(p)\right].$$

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[1] J. Schwinger, Phys. Rev. **125**, 397 (1962).
[2] There is a divergence in the so-called Thirring model [W. E. Thirring, Ann. Phys. (New York) **3**, 91 (1958)], which uses local current interactions rather than a Bose field.
[3] J. Schwinger, Ann. Phys. (New York) **9**, 169 (1960).

The probability that the vacuum state shall persist despite the disturbance is

$$|\langle 0|0\rangle^J|^2=\exp\left[-\int(dp)A_\mu(p)^*A_\mu(p)\ \mathrm{Im}\mathcal{G}(p)^{*-1}\right]$$
$$=\exp\left[-\pi\int(dp)dm^2\delta(p^2+m^2)\times s(m^2)(-\tfrac{1}{2})F^{\mu\nu}(p)^*F_{\mu\nu}(p)\right],$$

which exhibits $s(m^2)$ as a measure of the probability that an external field $F_{\mu\nu}$ will produce a vacuum excitation involving an energy-momentum transfer measured by the mass m.

The vanishing of $s(m^2)$ at $m=0$ is normal threshold behavior for an excitation function. If a zero-mass particle is not to exist, $m=0$ must be an abnormal threshold. Two possibilities can be distinguished. In the first of these, $s(m^2)$ is finite or possibly singular at $m=0$, but in such a way that

$$\lim_{z\to 0} z\int_0^\infty dm^2\frac{s(m^2)}{m^2-z}=0.$$

Then the physical mass spectrum begins at $m=0$ but there is no recognizable zero-mass particle. For the second situation, $s(m^2)$ has a delta-function singularity at $m^2=0$,

$$s(m^2)=\lambda^2\delta(m^2)+s_1(m^2),$$

and

$$s_1(m^2)=0,\quad m^2<m_0^2.$$

If the threshold mass m_0 is zero, the restriction of the previous situation applies to the function $s_1(m^2)$. Now, $m=0$ is not contained in the spectrum at all. This statement is true even if $m_0=0$ for, according to the structure of $B_1(m^2)=B(m^2)$,

$$B(m^2)=\frac{m^2s_1(m^2)}{[R(m^2)]^2+[\pi m^2 s_1(m^2)]^2},$$

in which

$$R(m^2)=m^2-\lambda^2+m^2P\int_{m_0^2}^\infty dm'^2\frac{s_1(m^2)}{m'^2-m^2},$$

we have

$$\lim_{m^2\to 0}B(m^2)=\lim_{m^2\to 0}\frac{m^2s_1(m^2)}{\lambda^4}=0.$$

Let us suppose that m_0 is the threshold of a continuous spectrum. A stable particle of mass $m<m_0$ will exist if $R(m_0^2)>0$. Should both $R(m_0^2)$ and $s_1(m_0^2)$ be zero there would be a stable particle of mass m_0. No stable particle exists if $R(m_0^2)<0$. But there is always an unstable particle, in a certain sense. By this we mean that $R(m^2)$ vanishes at some mass value $m_1>m_0$, under the general restrictions required for the continuity of the function $R(m^2)$, as a consequence of this function's asymptotic approach to $+\infty$ with increasing m^2. The mass m_1 will be physically recognizable as the mass of an unstable particle if the mass width

$$\gamma=\frac{\pi m_1 s_1(m_1^2)}{[dR(m_1^2)/dm_1^2]}$$

is sufficiently small. [We take the derivative of $R(m_1^2)$ to be positive, which is appropriate for the simplifying assumption that only one zero occurs.] The contribution of such a fairly sharp resonance to the sum rule for $B(m^2)$ is given by

$$\int_{m\sim m_1}dm^2B(m^2)=[dR(m_1^2)/dm_1^2]^{-1}<1.$$

SIMPLE MODELS

Some of these possibilities can be illustrated in very simple physical contexts. We consider the linear approximation to the problem of electromagnetic vacuum polarization for spaces of dimensionality $n=2$ and 1. A modification of a technique[4] previously applied to three-dimensional space yields for $m>m_0$:

$$s(m^2)=\int_0^{(1-m_0^2/m^2)^{1/2}}dv(1-v^2)(e^2/8\pi^2)\quad\text{for}\quad n=3$$
$$=\int_0^{(1-m_0^2/m^2)^{1/2}}dv(1-v^2)(e^2/4\pi^2)\times[m^2(1-v^2)-m_0^2]^{-1/2}\quad\text{for}\quad n=2$$
$$=\int_0^{(1-m_0^2/m^2)^{1/2}}dv(1-v^2)(e^2/\pi)\delta[m^2(1-v^2)-m_0^2]\quad\text{for}\quad n=1;$$

for $m<m_0$:

$$s(m^2)=0,$$

where the known result for $n=3$ has been included for comparison. The threshold mass m_0 is that for single pair creation. It should be noted that the coupling constant e^2 of electrodynamics in n-dimensional space has the dimensions of a mass raised to the power $3-n$. For $n<3$ this single pair approximation does not lead to difficulties concerning the existence of such integrals as

$$B_0^{-1}-1=\int_0^{\to\infty}\frac{dm^2}{m^2}B(m^2),$$

since, for $m\gg m_0$:

$$s(m^2)\sim(e^2/12\pi^2)\quad\text{for}\quad n=3,$$
$$\sim(e^2/16\pi)(1/m)\quad\text{for}\quad n=2,$$
$$\sim(e^2/2\pi)(m_0^2/m^4)\quad\text{for}\quad n=1.$$

The particular situation in which we are interested appears at the limit $m_0\to 0$. Then we have

$$s(m^2)=(e^2/16\pi)(1/m)\quad\text{for}\quad n=2,$$
$$=(e^2/\pi)\delta(m^2)\quad\text{for}\quad n=1.$$

[4] *Selected Papers on Quantum Electrodynamics* (Dover Publications, New York, 1958), p. 209.

Two-dimensional electrodynamics illustrates the first of the two possibilities for an anomalous threshold at $m=0$. The spectral function $B(m^2)$ describes a purely continuous spectrum,

$$dm^2\,B(m^2)=\frac{2}{\pi}\frac{e^2}{16}\frac{dm}{m^2+(e^2/16)^2},$$

and an m integration from 0 to ∞ satisfies the sum rule. In one-dimensional electrodynamics we meet a special case of the second possibility, with

$$\lambda^2=e^2/\pi,\quad s_1(m^2)=0.$$

Accordingly,

$$B(m^2)=\delta(m^2-(e^2/\pi))$$

and the mass spectrum is localized at one point, describing a stable particle of mass $e/\pi^{1/2}$.

The basis indicated for the latter conclusion will not be very convincing, but it is an exact result. To prove this we first compute for one spatial dimension the electric current induced by an arbitrary external potential in the vacuum state of a massless charged Dirac field. The appropriate gauge-invariant expression for the current[5] is

$$j_\mu(x)=-\tfrac{1}{2}e\,\mathrm{tr}q\alpha_\mu G(x,x')\exp\left[-ieq\int_{x'}^{x}d\xi^\mu\,A_\mu(\xi)\right]\Bigg|_{x'\to x},$$

in which the approach of x' to x is performed from a spatial direction in order to maintain time locality. The Green's function is defined by the differential equation

$$\alpha^\mu[\partial_\mu-ieqA_\mu(x)]G(x,x')=\delta(x-x'),$$

together with the outgoing wave boundary condition, in the absence of the potential. Only two Dirac matrices appear here, $\alpha^0=-\alpha_0=1$ and $\alpha^1=\alpha_1$, which has the eigenvalues ± 1. Those are also the eigenvalues of the independent charge matrix q. The Green's function equation can be satisfied by writing

$$G(x,x')=G^0(x,x')\exp\{ieq[\phi(x)-\phi(x')]\},$$

where

$$\alpha^\mu\partial_\mu\phi=\alpha^\mu A_\mu(x)$$

and

$$\alpha^\mu\partial_\mu G^0(x,x')=\delta(x-x').$$

The latter defines the free Green's function, which is given explicitly by

$$G^0(x,x')=\int_0^\infty\frac{dp}{2\pi}\exp[ip\alpha^\mu(x_\mu-x_\mu')]\quad\text{for}\quad x^0>x^{0\prime},$$

$$=-\int_{-\infty}^0\frac{dp}{2\pi}\exp[ip\alpha^\mu(x_\mu-x_\mu')]\quad\text{for}\quad x^0<x^{0\prime}.$$

[5] The necessity for the line integral factor has been noted before [J. Schwinger, Phys. Rev. Letters 3, 296 (1959)].

At equal times, and for sufficiently small x_1-x_1', we have

$$G(x,x')\exp\left[-ieq\int_{x'}^{x}d\xi^\mu\,A_\mu(\xi)\right]$$

$$\simeq\frac{i}{2\pi}\frac{\alpha_1}{x_1-x_1'}-\frac{eq}{2\pi}\alpha_1[\partial_1\phi(x)-A_1(x)].$$

The first term does not contribute to the vacuum current when the limit $x_1'\to x_1$ is performed symmetrically. On utilizing the relation

$$\alpha_1(\partial_1\phi-A_1)=-(\partial_0\phi-A_0),$$

we find that

$$j_\mu(x)=-\frac{e^2}{\pi}A_\mu(x)+\partial_\mu\left[\frac{e^2}{4\pi}\,\mathrm{tr}\phi(x)\right].$$

This expression for the induced current is Lorentz covariant, gauge invariant, and obeys the equation of conservation. It is also a linear function of the external field. To verify these statements we construct a differential equation for $\mathrm{tr}\phi(x)$ by multiplying the ϕ equation with $\partial_0-\alpha_1\partial_1$ and evaluating the trace. The result is

$$\partial^2\tfrac{1}{4}\,\mathrm{tr}\phi(x)=\partial_\mu A^\mu(x),$$

and therefore

$$\tfrac{1}{4}\,\mathrm{tr}\phi(x)=-\int(dx')D(x,x')\partial_\mu'A^\mu(x'),$$

in which D is the outgoing-wave Green's function defined by

$$-\partial^2D(x,x')=\delta(x-x').$$

By using a symbolic matrix notation for coordinates and vector indices, we can write

$$j=-(e^2/\pi)(1+\partial D\partial)A,$$

which exhibits the symmetrical projection matrix

$$\pi=1+\partial D\partial,$$
$$\partial\pi=\pi\partial=0,$$

that guarantees gauge invariance and current conservation.

We shall insert this result in the functional differential equation obeyed by the Green's functional $G[J]$, the vacuum transformation function in the presence of external currents. It is convenient to use the particular system of equations that refer to the Lorentz gauge,

$$\left\{(\partial\partial-\partial^2)\frac{1}{i}\frac{\delta}{\delta J}-(1+\partial D\partial)\left[J+j\left(\frac{1}{i}\frac{\delta}{\delta J}\right)\right]\right\}G[J]=0,$$

$$\partial\frac{\delta}{\delta J}G[J]=0,$$

which also utilize a symbolic notation for vectorial coordinate functions. We have written $j(-i\delta/\delta J)$ to indicate the conversion of $j(A)$ into a functional differential operator by the substitution $A\rightarrow -i\delta/\delta J$. The functional differential equation implied by the known structure of this operator is

$$\pi\left[\left(-\partial^2+\frac{e^2}{\pi}\right)\frac{1}{i}\frac{\delta}{\delta J}-J\right]G[J]=0,$$

or, on uniting the two defining properties of the functional,

$$\left(\frac{1}{i}\frac{\delta}{\delta J}-\pi\mathcal{G}J\right)G[J]=0,$$

in which

$$[-\partial^2+(e^2/\pi)]\mathcal{G}(x,x')=\delta(x-x').$$

The Green's functional $G[J]$ is therefore given exactly by

$$G[J]=\exp\left[\tfrac{1}{2}i\int(dx)(dx')J^\mu(x)\mathcal{G}_{\mu\nu}(x,x')J^\nu(x')\right],$$

with

$$\mathcal{G}_{\mu\nu}(x,x')=\pi_{\mu\nu}(-i\partial)\mathcal{G}(-i\partial)\delta(x-x')$$

and

$$\mathcal{G}(p)=\frac{1}{p^2+(e^2/\pi)-i\epsilon}.$$

Thus, all states that can be excited by vector currents are fully described as noninteracting ensembles of Bose particles with the mass $e/\pi^{1/2}$.

Concerning the complete Green's functional including Fermi sources, $G[\eta J]$, we shall only remark that

$$G[\eta J]=\exp\left[-\frac{1}{2}\int(dx)(dx')\eta(x)\times G\left(x,x',\frac{1}{i}\frac{\delta}{\delta J}\right)\eta(x')\right]G[J],$$

in which the Green's function can be presented as

$$G(x,x',A)=G^0(x,x')\exp\left[i\int(d\xi)j^\mu(\xi,x,x')A_\mu(\xi)\right]$$

with

$$j^\mu(\xi,x,x')=eq\alpha^\mu\left(\alpha^1\frac{\partial}{\partial\xi^1}-\frac{\partial}{\partial\xi^0}\right)[D(\xi,x)-D(\xi,x')].$$

On expanding the Green's functional in even powers of the Fermi source, we encounter functional differential operators that are contained in one or more factors of the type

$$\exp\left[\int(d\xi)j^\mu(\xi,x,x')\delta/\delta J^\mu(\xi)\right],$$

the effect of which is simply to produce the translation $J\rightarrow J+j$ in $G[J]$. The first Fermi Green's function is

$$G(x,x')=G(x,x',-i\delta/\delta J)G[J]|_{J=0}$$
$$=G^0(x,x')\exp\left[\tfrac{1}{2}i\int(d\xi)(d\xi')\times j^\mu(\xi,x,x')\mathcal{G}_{\mu\nu}(\xi,\xi')j^\nu(\xi',x,x')\right].$$

The latter exponential factor is given by

$$\exp\left[-\frac{i}{4\pi}\int(dp)\left(\frac{1}{p^2-i\epsilon}-\frac{1}{p^2+(e^2/\pi)-i\epsilon}\right)\times(1-e^{ip(x-x')})\right].$$

We shall be content to note that this integral and the similar integrals encountered in more general Green's functions are completely convergent. The detailed physical interpretation of the Green's functions is rather special and apart from our main purpose.

These simple examples are quite uninformative in one important respect. They do not exhibit a critical dependence upon the coupling constant. As we have discussed previously, one can view the electromagnetic field as undercoupled and the hypothetical vector field that relates to nucleonic charge as overcoupled, in the sense of a critical value at which the massless Bose particle ceases to exist. The corresponding appearance of an anomalous zero-mass threshold must be attributed to a dynamical mechanism. We can supply an artificial mathematical model that illustrates the situation. Let the following be a contributory term in $s(m^2)$:

$$s_0(m^2)=\frac{\lambda^2}{\pi}\frac{m\gamma}{(m^2-m_0^2\kappa)^2+(m\gamma)^2},$$

in which m_0 is a characteristic physical fermion mass, and λ/m_0, γ/m_0, and κ are positive functions of the (dimensionless) coupling constant. In electrodynamics the near-resonant contributions of such a term can be identified with the creation of a unit angular momentum positronium state, while the values far below resonance refer to the creation of three-photon states (the model falsifies the latter, which should vary as m^8 for $m\ll m_0$). It is reasonable to suppose that κ decreases with increasing strength of the coupling, and we can imagine that a critical value exists for which both κ and γ reach zero, with finite λ. In that circumstance,

$$s_0(m^2)=\lambda^2\delta(m^2),$$

and the null-mass particle disappears from the spectrum. Since this argument requires that one type of excitation move down to zero mass at the critical coupling strength, it is plausible that some other types of excitation will then be located at fairly small fractions of m_0. Thus, one could anticipate that the known spin-0 bosons, for example, are secondary dynamical manifestations of strongly coupled primary fermion fields and vector gauge fields. This line of thought emphasizes that the question "Which particles are fundamental?" is incorrectly formulated. One should ask "What are the fundamental fields?"

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Scattering of Electromagnetic Waves in Saxon-Schiff Theory

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We calculate the diffraction of electromagnetic waves by weak scatterers with complex dielectric constant and permeability using the Saxon-Schiff theory of potential scattering. Boundary conditions, polarizations, and the optical theorem are discussed to some extent. Our results for the scattering amplitude contain certain special cases obtained previously by other authors. In an Appendix, we compare the results for the scattering by a homogeneous dielectric sphere with those of the exact Mie theory. It is seen that the Saxon-Schiff theory gives a good qualitative agreement insofar as it reproduces the diffraction maxima and minima, in vast superiority to the Born approximation. In the asymptotic limit $kR\rightarrow\infty$, the radar cross section is shown to agree with the exact result for a not too large index of refraction.

THE theory of Saxon and Schiff,[1] originally developed for high-energy scalar potential scattering, has been applied to the scattering of electromagnetic waves by dielectric bodies.[2] Schiff[3] has also considered scattering of vector waves using an earlier version of the theory, valid for either small or large angles only. In this note, we derive the scattering amplitude of electromagnetic waves for a general weak scatterer with complex dielectric constant and permeability, and demonstrate that the results can be made to reduce to the large- and small-angle expressions of Schiff[3] in the respective limits.

Maxwell's equations, setting $c=1$ and assuming a harmonic time dependence of the fields,

$$\sim\exp(-ikt),$$

become

$$\nabla\times\mathbf{E}=ik\mu\mathbf{H},\quad \nabla\times\mathbf{H}=(\sigma-ik\epsilon)\mathbf{E}. \tag{1}$$

No free charges are assumed to be present; σ is the conductivity, and ϵ, μ are dielectric constant and permeability, respectively (we shall use Gaussian units, $\epsilon_0=\mu_0=1$). Taking the divergence of the second equation, we get

$$\nabla\cdot\epsilon'\mathbf{E}=0, \tag{2}$$

where we have introduced the complex dielectric constant,

$$\epsilon'=\epsilon(1+i\nu),$$

with

$$\nu=\sigma/k\epsilon.$$

Elimination of $\mathbf{H}$ from (1) gives the wave equation

$$\nabla^2\mathbf{E}+K^2\mathbf{E}=\nabla\nabla\cdot\mathbf{E}-\mu^{-1}\nabla\mu\times(\nabla\times\mathbf{E}), \tag{3}$$

with the squared propagation constant

$$K^2=k^2\mu\epsilon'. \tag{4}$$

Equation (2) can again be obtained by taking the divergence of the wave equation.

Following reference (1), a Green's function

$$F(\mathbf{r},\mathbf{r}')=F(\mathbf{r}',\mathbf{r})=-(4\pi\rho)^{-1}e^{iS(\mathbf{r},\mathbf{r}')} \tag{5}$$

will be considered, where

$$\rho=|\mathbf{r}-\mathbf{r}'|;$$

the phase is assumed to have the limits

$$\lim_{\mathbf{r}'\rightarrow\mathbf{r}}\rho^{-1}S(\mathbf{r},\mathbf{r}')=C(\mathbf{r}),\qquad \lim_{r\rightarrow\infty}\nabla S=k\mathbf{n}+O(r^{-1});\quad \mathbf{r}=\mathbf{n}r. \tag{6}$$

This Green's function satisfies the differential equation

$$\nabla^2F+(\nabla S)^2F=\delta(\mathbf{r}-\mathbf{r}')+iF\rho^2\nabla\cdot(\rho^{-2}\nabla S). \tag{7}$$

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