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Notes on harmonic superspace

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Abstract. An analysis of the use of additional commuting isospinor coordinates to formulate $N=2$ superspace and its Grassmann-analytic subspace is performed.

Supersymmetry is one of the few known theories which might solve the problems of unification and naturalness. Superspace is a general and systematic approach to formulating and fully exploiting supersymmetry by making this symmetry manifest at all stages. In the case of simple supersymmetry, superspace methods have proven much more powerful than more old-fashioned methods at the quantum level (in both explicit calculations and theorems of no-renormalisation), as well as allowing concise and clearer formulations of the classical theories. However, in the case of extended supersymmetry, even though the advantages of superspace at the quantum level are already evident in its use for proofs of finiteness, a full understanding of the formalism is still lacking.

A new approach to this problem has been taken by Galperin, Ivanov, Kalitzin, Ogievetsky and Sokatchev, who have introduced the concept of extending $N=2$ superspace to include the coordinates of the coset space SU(2)/U(1) [1]. This 'harmonic' superspace allows the extension of $N=2$ formalisms with only U(1) internal symmetry on the anticommuting coordinates to include a full SU(2) symmetry. A crucial ingredient is the 'analytic' superfield, which is a generalisation of the $N=1$ chiral superfield to an $N=2$ superfield chiral in half the anticommuting coordinates and antichiral in the other half [2]. Although these ideas bring a new understanding to some aspects of $N=2$ supersymmetry, we find below some disadvantages to constructing a complete $N=2$ superspace formalism around such superfields in the manner of reference [1] (restricted scalar-multiplet self-interactions, inability to expand supergravity about global superspace, non-locality in the coset-space coordinates). Nevertheless, harmonic superspace may be a useful tool for some parts of the usual $N=2$ formalism (e.g. solving constraints on covariant derivatives).

The coset space SU(2)/U(1) is described by using an isospinor u^a to parametrise the group SU(2) (u^a has unit modulus: $u^a\bar{u}_a=1$), while requiring a U(1) invariance $u^{a'}=e^{i\lambda}u^a$ (see reference [3] for notation). u^a, $\bar{u}_a$ form a zweibein for the SU(2) space. An analytic superfield is defined to satisfy [1]

$$u^a D_{a\alpha}F = u^a C_{ba}\bar{D}^b{}_{\dot\alpha}F = 0. \tag{1}$$

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(In six-dimensional notation [4], this is simply $u^a D_{a\alpha} F = 0$.) The solution to this constraint is

$$F = u^a u^b u^c u^d D^4{}_{abcd}\chi \tag{2a}$$

with gauge invariance

$$\delta\chi = u^a D_{a\alpha} K^\alpha + u^a C_{ba} \bar{D}^b{}_{\dot\alpha} K^{\dot\alpha} \tag{2b}$$

where

$$D^4{}_{abcd} = \frac{1}{4!} C_{e(a} D^2{}_{bc} C_{d)f} \bar{D}^{2ef} \rightarrow D_{(a\alpha} D^4{}_{bcde)} = \bar{D}^f{}_{\dot\alpha} C_{f(a} D^4{}_{bcde)} = 0. \tag{3}$$

(In 6D notation [4], $D^4{}_{abcd} = (1/4!) C^{\delta\gamma\beta\alpha} D_{a\alpha} D_{b\beta} D_{c\gamma} D_{d\delta}$.) Actions can be written as

$$\int \mathrm{d}u\, \mathrm{d}x (\bar{u}^4 D^4) F f(F) = \int \mathrm{d}u\, \mathrm{d}x\, \mathrm{d}^8\theta \chi f(F) \tag{4}$$

where $(\bar{u}^4 D^4) \equiv \bar{u}_a \bar{u}_b \bar{u}_c \bar{u}_d C^{ae} C^{bf} C^{cg} C^{dh} D^4{}_{efgh}$. The usefulness of the condition of analyticity (1) is equivalent to the importance of the operator $D^4{}_{abcd}$, which occurs frequently in actions and solutions of constraints in the usual $N = 2$ superspace formalism [4]. In harmonic superspace the operators

$$D_+ = u^a C_{ba}\, \partial/\partial \bar{u}_b \qquad D_- = \bar{u}_a C^{ba}\, \partial/\partial u^b \qquad D_3 = \tfrac{1}{2}(u^a\, \partial/\partial u^a - \bar{u}_a\, \partial/\partial \bar{u}_a) \tag{5}$$

generate a second SU(2) (in addition to the SU(2) which acts on the explicit isospin indices) which is broken down to the U(1) generated by D_3. (The zweibein u^a, $\bar{u}_a$ solders the two SU(2) together.) Analytic superfields are also chosen to be eigenvectors of D_3 (i.e. irreducible representations of this U(1)). Due to the form of the constraint (1) (or its solution (2)), D_+ acting on an analytic superfield gives another analytic superfield (but, being the raising operator of the second SU(2), raises the eigenvalue of D_3 by 1), and hence can be used freely in constructing actions (4) (e.g. defining $F_2 = D_+ F_1$, etc).

In reference [1] the free actions of scalar (hyper) multiplets were taken as

$$\int \mathrm{d}u\, \mathrm{d}x (\bar{u}^4 D^4) F (D_+)^{2-2t_3} F \qquad (t_3 = 0, \tfrac{1}{2}), \tag{6}$$

where t_3 is the eigenvalue of D_3. With F given by (2) (and $D_3\chi = (t_3 - 2)\chi$), these multiplets contain an infinite number of ordinary $N = 2$ superfields (from expanding χ in u and $\bar{u}$, where $t_3 - 2$ is half the number of u minus $\bar{u}$ in any term). Rewritten as in (4), after integrating du, each term has $D^4{}_{abcd}$ as a kinetic operator. The authors of reference [1] propose that this formulation is related to $N = 2$ scalar multiplets with finite numbers of ordinary superfields by imposing the constraint $(D_+)^{3-2t_3} F = 0$. However, this constraint has the solution

$$(D_+)^{3-2t_3} F = 0 \rightarrow \chi = \bar{u}_a \bar{u}_b (\bar{D}^{2ab} \psi + C^{ac} C^{db} D^2{}_{cd} \bar{\psi}) \tag{7}$$

where ψ is independent of u. It can easily be shown that the field equations resulting from varying ψ do not describe a scalar (or any other physical) multiplet. In fact, if we use (6) for the case $t_3 = 1$, it describes trivial dynamics before applying (7), but afterwards describes the tensor multiplet [5-7] (with $F = u^a u^b F_{ab}$ and F_{ab} the usual tensor-multiplet field strength). In reference [1] self-interactions for (6) were proposed, but the tensor multiplet allows more general ones [7, 8]. (For example, for $t_3 = 0$ with a single scalar multiplet, no self-interactions are allowed, whereas the tensor multiplet allows general hyper-Kähler manifolds with one Killing vector [9].) This is analogous to $N=1$, where chiral scalar multiplets allow more general self-interactions than scalar multiplets satisfying weaker constraints. Here, constraining to a finite number of ordinary $N=2$ superfields allows more general self-interactions. However, the $t_3=\frac{1}{2}$ case of (6) can be coupled to Yang-Mills in complex representations [1], unlike previous formulations. Perhaps it allows a finite truncation with both general Yang-Mills and self-interactions.

Yang-Mills itself, although described by a finite number of superfields [10, 6, 4], allows a simple description in harmonic superspace due to the fact that the usual constraints on the covariant derivatives $\nabla_{a\alpha}$ are equivalent to the statement that, for *any* given u, $u^a\nabla_{a\alpha}$ and $u^aC_{ba}\bar{\nabla}^b{}_{\dot\alpha}$ give vanishing anticommutators. This 'representation-preserving' constraint allows a consistent Yang-Mills covariantisation of the definition (1) of analyticity. (Analogously, in $N=1$ the corresponding Yang-Mills constraint allows the consistent covariantisation of chirality. In both cases, the only remaining constraint is the 'conventional' constraint which defines the vector covariant derivative.) The solution to these constraints is [1]

$$u^a\nabla_{a\alpha}=u^a\,\mathrm{e}^{-\Omega}D_{a\alpha}\,\mathrm{e}^{\Omega}\qquad u^aC_{ba}\bar{\nabla}^b{}_{\dot\alpha}=u^aC_{ba}\,\mathrm{e}^{-\Omega}\bar{D}^b{}_{\dot\alpha}\,\mathrm{e}^{\Omega}\tag{8a}$$

where Ω is a complex superfield belonging to the Lie algebra. The u independence of ∇ can be expressed as a further constraint since Ω may now depend on u:

$$\begin{aligned}&[\nabla_+,u^a\nabla_{a\alpha}]=[\nabla_+,u^aC_{ba}\bar{\nabla}^b{}_{\dot\alpha}]=0\\&[\nabla_3,u^a\nabla_{a\alpha}]=\tfrac{1}{2}u^a\nabla_{a\alpha}\qquad[\nabla_3,u^aC_{ba}\bar{\nabla}^b{}_{\dot\alpha}]=\tfrac{1}{2}u^aC_{ba}\bar{\nabla}^b{}_{\dot\alpha}\\&\nabla_+=D_+\qquad\nabla_3=D_3.\end{aligned}\tag{8b}$$

From the gauge invariance $\nabla'=\mathrm{e}^{\mathrm{i}K}\nabla\,\mathrm{e}^{-\mathrm{i}K}$, where K is u-independent, and the form of the solution (8a), we have

$$\mathrm{e}^{\Omega'}=\mathrm{e}^{\mathrm{i}\Lambda}\,\mathrm{e}^{\Omega}\,\mathrm{e}^{-\mathrm{i}K}\tag{9}$$

where Λ is analytic as in (1). As for $N=1$, we can transform from the 'real' representation of (8) to an 'analytic' representation by $\mathcal{O}'=\mathrm{e}^{\Omega}\mathcal{O}\,\mathrm{e}^{-\Omega}$ on all objects $\mathcal{O}$, to obtain [1]

$$\begin{aligned}&u^a\nabla_{a\alpha}=u^aD_{a\alpha}\qquad u^aC_{ba}\bar{\nabla}^b{}_{\dot\alpha}=u^aC_{ba}\bar{D}^b{}_{\dot\alpha}\\&[\nabla_+,u^aD_{a\alpha}]=[\nabla_+,u^aC_{ba}\bar{D}^b{}_{\dot\alpha}]=0\\&\nabla_+=\mathrm{e}^{\Omega}D_+\,\mathrm{e}^{-\Omega}=D_+-\mathrm{i}\Gamma_+.\end{aligned}\tag{10a}$$

In this representation the gauge transformations are (by (9)) $\nabla'=\mathrm{e}^{\mathrm{i}\Lambda}\nabla\,\mathrm{e}^{-\mathrm{i}\Lambda}$. Without loss of generality (and for U(1) covariance), we can also choose

$$D_3\Omega=0\rightarrow\nabla_3=D_3\qquad D_3\Gamma_+=\Gamma_+.\tag{10b}$$

Γ_+ is thus an analytic superfield, so we solve for Ω (and thus ∇) in terms of it

$$\mathrm{e}^{-\Omega}(D_+-\mathrm{i}\Gamma_+)\,\mathrm{e}^{\Omega}=D_+\tag{11a}$$

$$\rightarrow[(D_+-\mathrm{i}\Gamma_+)\,\mathrm{e}^{\Omega}]=0$$

$$\rightarrow(D_+)^{-1}(D_+-\mathrm{i}\Gamma_+)\,\mathrm{e}^{\Omega}=[1-(D_+)^{-1}\mathrm{i}\Gamma_+]\,\mathrm{e}^{\Omega}=f$$

$$\rightarrow\mathrm{e}^{\Omega}=[1-(D_+)^{-1}\mathrm{i}\Gamma_+]^{-1}f=f+(D_+)^{-1}\mathrm{i}\Gamma_+f+(D_+)^{-1}\mathrm{i}\Gamma_+(D_+)^{-1}\mathrm{i}\Gamma_+f+\dots\tag{11b}$$

where f is u-independent ($D_+ f = D_3 f = 0$) and $(D_+)^{-1}$ is some suitably defined inverse [1].

In reference [1] it was assumed that f could be written as e^{iV_0}, where V_0 is Lie algebra valued and real, and thus can be gauged to zero by K as in (9). This is not the case, as can be seen by solving (11a) explicitly for Ω to second order in Γ_+. Writing a perturbation expansion in terms of $\Omega = \Omega_0 + \Omega_1 + \ldots$ (and dropping pure K-gauge pieces)

$$\begin{aligned} &D_+\Omega_0 - i\Gamma_+ = 0 \to \Omega_0 = (D_+)^{-1} i\Gamma_+ \\ &D_+\Omega_1 - \tfrac{1}{2}[\Omega_0, D_+\Omega_0] + [\Omega_0, i\Gamma_+] = 0 \\ &\to \Omega_1 = \tfrac{1}{2}(D_+)^{-1}[i\Gamma_+, (D_+)^{-1} i\Gamma_+]. \end{aligned} \tag{12}$$

This differs from the solution obtained from (11b) when setting $f = 1$ (and taking the ln) by

$$\begin{aligned} &\tfrac{1}{2}(D_+)^{-1}[i\Gamma_+, (D_+)^{-1} i\Gamma_+] - \{(D_+)^{-1} i\Gamma_+ (D_+)^{-1} i\Gamma_+ - \tfrac{1}{2}[(D_+)^{-1} i\Gamma_+]^2\} \\ &\qquad = \tfrac{1}{2}[(D_+)^{-1} i\Gamma_+]^2 - \tfrac{1}{2}(D_+)^{-1}\{i\Gamma_+, (D_+)^{-1} i\Gamma_+\}. \end{aligned} \tag{13}$$

Although this expression is u-independent (D_+ is easily seen to annihilate it), it is not an element of the Lie algebra, and therefore cannot be gauged away by K transformations and must be included in f. However, although the 'solution' (11b) is not useful, solving (11a) and then using it to calculate the u-independent $\nabla_{a\alpha}$ in terms of ordinary superfields may be easier than solving the constraints without the aid of harmonic superspace. The appropriate Λ-gauge choice (at the linearised level; $\delta\Gamma_+ = D_+\Lambda$) is

$$(D_+)^3 i\Gamma_+ = 0 \to i\Gamma_+ = u^a u^b u^c u^d D^4{}_{abcd}\, \bar{u}_e \bar{u}_f V^{ef} \tag{14a}$$

where V^{ab} is the usual prepotential [10], with residual gauge parameter

$$(D_+)^4 \Lambda = 0 \to \Lambda = u^a u^b u^c u^d D^4{}_{abcd}\, \bar{u}_e \bar{u}_f \bar{u}_g \bar{u}_h (D^e{}_\alpha K^{fgh\alpha} + \mathrm{HC}). \tag{14b}$$

We thus see that V^{ab} will occur in ∇ (and in matter couplings) only as $D^4{}_{abcd} V^{ef}$ (and its derivatives), which is already a strong restriction. (Such matter couplings correspond to covariantising $D_+ \to D_+ - i\Gamma_+$ in the analytic representation rather than $D_{a\alpha} \to \nabla_{a\alpha}$ in the real representation. The analogue in $N = 1$ is using $\phi \to e^{\bar{\Omega}}\phi$, and thus $\bar{\phi}\phi \to \bar{\phi}\, e^V \phi$, rather than $\phi = \bar{D}^2\psi \to \bar{\nabla}^2\psi$.)

A similar treatment for supergravity is possible [1], but a manifestly globally supersymmetric formulation leads directly to ordinary superfields. In the analytic representation, the analogue to the analyticity conditions (10a) is

$$\begin{aligned} &[u^b(D_{b\beta}, C_{cb}\bar{D}^c{}_{\dot\beta}), \Gamma_+{}^A D_A] = (f_\beta{}^\alpha, f_{\dot\beta}{}^\alpha) u^a D_{a\alpha} + (f_\beta{}^{\dot\alpha}, f_{\dot\beta}{}^{\dot\alpha}) u^b C_{ab} \bar{D}^a{}_{\dot\alpha} \\ &\to u^b(D_{b\beta}, C_{cb}\bar{D}^c{}_{\dot\beta}) \Gamma_+{}^{a\alpha} = u^a (f_\beta{}^\alpha, f_{\dot\beta}{}^\alpha) \\ &\quad u^b(D_{b\beta}, C_{cb}\bar{D}^c{}_{\dot\beta}) \Gamma_{+a}{}^{\dot\alpha} = u^b C_{ab} (f_\beta{}^{\dot\alpha}, f_{\dot\beta}{}^{\dot\alpha}) \\ &\quad u^b(D_{b\beta}, C_{cb}\bar{D}^c{}_{\dot\beta}) \Gamma_+{}^{\alpha\dot\alpha} = i u^b (\delta_\beta{}^\alpha \Gamma_{+b}{}^{\dot\alpha}, \delta_{\dot\beta}{}^{\dot\alpha} C_{cb} \Gamma_+{}^{c\alpha}) \\ &\quad u^b(D_{b\beta}, C_{cb}\bar{D}^c{}_{\dot\beta}) \Gamma_+{}^5 = i u^b (C_{cb} \Gamma_+{}^c{}_\beta, -\Gamma_{+b\dot\beta}). \end{aligned} \tag{15}$$

(Now $D_A = (D_\alpha, D_{\dot\alpha}, \partial_{\alpha\dot\alpha}, \partial_5)$, where ∂_5 represents a central charge annihilating all fields, and $\{D_\alpha, D_\beta\} = C_{ab} C_{\alpha\beta} i\partial_5$.) The analogue in $N = 1$ is that the Λ^m, Λ^μ parts of the gauge parameter $\Lambda^M \partial_M$ are chiral when expanding covariant derivatives about ∂_M [11], but $\Lambda^A D_A$ satisfies $\bar{D}_{\dot\beta}\Lambda^\alpha = 0$, $\bar{D}_{\dot\beta}\Lambda^{\alpha\dot\alpha} = i\delta_{\dot\beta}{}^{\dot\alpha}\Lambda^\alpha$ when expanding about D_A [12].

The solution to (15) is

$$
\begin{aligned}
\Gamma_+{}^{\underline{\alpha}} &= u^b u^c u^d u^e D^4{}_{bcde}\chi^{\underline{\alpha}} + u^a \eta^{\alpha} \\
\Gamma_+{}^{\underline{\dot\alpha}} &= u^b u^c u^d u^e D^4{}_{bcde}\chi^{\underline{\dot\alpha}} + C_{ba} u^b \eta^{\dot\alpha} \\
\Gamma_+{}^{\alpha\dot\alpha} &= \mathrm{i} u^a u^b u^c u^d C_{ae} C_{fb} (\bar{D}^{e\dot\alpha} D^2{}_{cd}\chi^{f\alpha} + D_c{}^{\alpha}\bar{D}^{2ef}\chi_d{}^{\dot\alpha}) \\
\Gamma_+{}^5 &= \mathrm{i} u^a u^b u^c u^d (C_{be} C_{fc} C_{gd} D_{a\alpha}\bar{D}^{2ef}\chi^{g\alpha} + C_{ea}\bar{D}^e{}_{\dot\alpha} D^2{}_{bc}\chi_d{}^{\dot\alpha}).
\end{aligned}
\qquad (16a)
$$

In an appropriate gauge ($\delta\Gamma_+{}^{\underline{\alpha}} = D_+\Lambda^{\underline{\alpha}}$, with $\Lambda^{\underline{\alpha}}$ of the form of (16*a*); cf (14*a*)),

$$
\chi^{a\alpha} = C^{ab}\bar{u}_b\bar{u}_c\psi^{c\alpha} \qquad \chi_a{}^{\dot\alpha} = \bar{u}_a\bar{u}_b C^{cb}\bar{\psi}_c{}^{\dot\alpha} \qquad \eta^{\alpha} = \eta^{\dot\alpha} = 0 \qquad (16b)
$$

where ψ^{α} is the u-independent prepotential of the usual formalism [6]. Since explicit non-analytic superfields are necessary for a globally supersymmetric perturbation expansion, $N=2$ supergravity supergraphs are more conveniently done in ordinary superspace, although again the harmonic approach may be useful for solving constraints. (Another inconvenient feature of harmonic supergraphs is the occurrence of the operator $(D_+)^{-1}$, as in (12), and the fact that a separate u^a must be introduced for each vertex. This should be contrasted with the θ coordinates, for which all D appear in numerators and ultimately only one spinor θ is necessary [13].)

Harmonic superspace can also be used to solve constraints more general than just analyticity conditions. In the same way that the concept of chirality can be used to construct projection operators in four dimensions [14], it should be possible to use analyticity to construct projection operators in six dimensions as well as four. Instead of constructing projection operators as $\Box^{-2}D^{4-n}\bar{D}^4 D^n$ (since $\bar{D}^4 D^n$ picks out a chiral piece of a superfield), we instead consider an expression of the form $\Box^{-2}\int \mathrm{d}u\, (\bar{u}D)^{4-n}(uD)^4(\bar{u}D)^n$ (in six-dimensional notation). After u integration, the resulting operators are of the form $\mathcal{O}_1 = \Box^{-2}D^{4abcd}D^4{}_{abcd}$, $\mathcal{O}_2 = \Box^{-2}D^{3abc\alpha}D^4{}_{abcd}D^d{}_{\alpha}$, $\mathcal{O}_3 = \Box^{-2}D^{2ab\alpha\beta}D^4{}_{abcd}D^{2cd}{}_{\alpha\beta}$. (The D^n on either side of $D^4{}_{abcd}$ are totally symmetric in isospinor indices and totally antisymmetric in spinor indices.) The three projection operators for a real, scalar, six-dimensional superfield are given by $\mathcal{O}_1$, $\mathcal{O}_2$ and a linear combination of $\mathcal{O}_3$ and $\mathcal{O}_1$. (Four-dimensional projectors are obtained by Lorentz-reducing the D^n.)

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