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AN EFFECTIVE LAGRANGIAN FOR SOLITONS

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An effective Lagrangian is proposed for deriving the properties of solitons. The Lagrangian has only local interactions, and involves a new local field to represent the soliton. The soliton form factor and the long-distance force between solitons are calculated.

1. Introduction

Many field theories possess a static soliton solution ϕ_c, of finite energy, which signals the existence of a new particle in the quantum theory. In the original paper of Goldstone and Jackiw [1], and in subsequent work [2], powerful methods have been developed for calculating the physical properties of these solitons in detail. These methods treat the soliton quite differently from the mesons, the particles associated with the elementary fields of the theory. The exact classical soliton solution is required, involving the complete Lagrangian, whereas the mesons are built up perturbatively from plane-wave solutions of the linearized theory.

It would be desirable to find a unifying scheme in which the soliton itself were built up perturbatively from a local interaction of the meson field with a local soliton field. We show that this can be done *via* an effective Lagrangian, which reproduces known properties of the soliton, at least semiclassically. The use of a local soliton field is not new, for in the well-known case of the sine-Gordon model, the equivalent Thirring model reformulation involves a local Fermi field representing the soliton [3]. Another example is the Prasad-Sommerfield monopole [4]. Montonen and Olive [5] have conjectured that the physics of monopoles is contained in a dual theory, which is identical to the original SO(3) Higgs model which has the monopole solution, except for a reversal of the roles of electricity and magnetism, and of

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the corresponding charges. So the gauge fields now represent the monopoles and the vector bosons are classical solutions.

Three particular requirements suggest that an effective Lagrangian involving a soliton field might be desirable for a wide class of theories. The first is that of calculating inter-soliton forces. In ref. [5], it was pointed out that the asymptotic force between Prasad-Sommerfield monopoles, calculated classically from the field equations [6], could be reproduced in the dual theory from the Born approximation to monopole-monopole scattering, involving the exchange of the elementary mesons. A similar result also holds for sine-Gordon solitons. The force is there proportional to $\exp(-ms)$, where m is the elementary meson mass and s the separation [7]. This is a one-dimensional Yukawa force, again suggesting meson exchange.

Secondly, there is the analytic structure of the soliton form factor. which is the matrix element $\langle p'|\phi|p\rangle$ of the elementary field ϕ between soliton states, and is just the Fourier transform of ϕ_c, to leading order in $\hbar$, with poles at $(p'_\mu - p_\mu)^2 = (nm)^2$ for integer n [1]. Goldstone and Jackiw conjecture that these poles correspond to multi-meson thresholds. The leading pole would be reproduced by the bare meson-soliton vertex that occurs in the Born diagram for soliton-soliton scattering *via* meson exchange, with the same coupling constant. Thirdly, the meson interactions of the original theory, together with unitarity considerations, suggest that further diagrams which include multi-meson vertices should contribute to the soliton form factor, and it is clear that such diagrams should have multi-meson threshold singularities.

We therefore propose an effective Lagrangian consisting of the original interacting meson Lagrangian and a free-field soliton Lagrangian, with an additional Yukawa interaction between the meson field and the new field operator for the soliton. We can calculate processes involving arbitrary numbers of solitons and mesons with simple Feynman rules in a manifestly covariant formalism. Unfortunately, we have no proof that all such calculations give correct answers, nor that the quantum corrections to the leading semiclassical results are correct.

In this paper the discussion is restricted to one-dimensional solitons. In sect. 2, we derive properties of ϕ_c for a general potential. Then, in sect. 3, we introduce the effective Lagrangian and calculate the soliton form factor to leading order, comparing the result with ϕ_c. We also relate the form-factor poles to anomalous thresholds of the Feynman diagrams which contribute. In sect. 4, we calculate the inter-soliton asymptotic force and compare the soliton-soliton scattering amplitude obtained from the effective Lagrangian.

2. The soliton solution

It is well-known that certain one-dimensional scalar field theories possess static solutions of finite energy, called solitons [8]. Consider the general Lagrangian

$$\mathcal{L}_0(\phi) = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - V(\phi)\,. \tag{2.1}$$

V is a symmetric potential ($V(\phi) = V(-\phi)$) with neighbouring minima at $\phi = \pm c$ and $V(\pm c)$ vanishes.

The field equation for time-independent ϕ is

$$\frac{d^2\phi}{dx^2} = V'(\phi) , \tag{2.2}$$

which can be integrated once to give

$$\left(\frac{d\phi}{dx}\right)^2 = 2V(\phi) . \tag{2.3}$$

The constant of integration is chosen to allow the boundary condition $\phi(\pm\infty) = \pm c$. A further integration leads to the solution

$$x = \int_0^{\phi} [2V(\phi')]^{-1/2} d\phi' . \tag{2.4}$$

This is the implicit form of the soliton $\phi_c(x)$, whose location is chosen so that ϕ_c is antisymmetric in x and interpolates between the values $\mp c$. The antisoliton $\phi_c(-x)$ is obtained by changing the sign of the square root, and either solution can be translated by changing the lower limit of integration. The mass of the soliton is given by the field energy

$$M = \int_{-\infty}^{\infty} \left[\frac{1}{2}\left(\frac{d\phi_c}{dx}\right)^2 + V(\phi_c)\right] dx \tag{2.5}$$

$$= \sqrt{2} \int_{-c}^{c} [V(\phi)]^{1/2} d\phi , \tag{2.6}$$

using (2.3). This is a classical mass, independent of $\hbar$.

To investigate the analytic properties of ϕ_c, we assume that $V(\phi)$ is analytically continued to complex ϕ. By regarding (2.4) as a contour integral, we can now continue $\phi_c(x)$ to complex x. Assuming that in the neighbourhood of $\phi = \pm c$, V has the approximate quadratic behaviour

$$V(\phi) \simeq \tfrac{1}{2} m^2 (\phi \pm c)^2 , \tag{2.7}$$

with $m\hbar$ the elementary meson mass after quantization, we see that $[2V]^{-1/2}$ has a pole with residue m^{-1} at $\phi = \pm c$. The integral (2.4) depends on the number of times the contour encircles these poles (fig. 1), making x a multivalued function of ϕ, whose values differ by multiples of $2\pi i m^{-1}$. Consequently ϕ_c is periodic in $\mathrm{Im}(x)$. For integer n,

$$\phi_c(x + 2\pi i n m^{-1}) = \phi_c(x) , \tag{2.8}$$

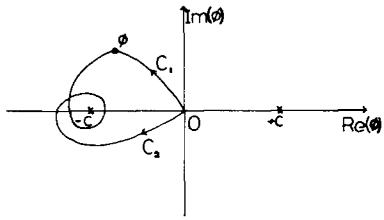

Fig. 1.

so the Fourier transform $\tilde{\phi}_c(q)$ has poles at $q = inm$, whose possible physical significance was remarked on in ref. [1].

We now discuss an alternative method for solving (2.2), which is directly related to the diagrammatic techniques of sect. 3. Expand V as a power series about its minimum at $\phi = -c$. Let $\chi = \phi + c$ and write

$$V(\chi) = \tfrac{1}{2}m^2\chi^2 + \sum_{n=3}^{\infty} \frac{V_{(n)}}{n!}\chi^n \,. \tag{2.9}$$

$V_{(n)}$ is the n-point meson vertex in the vacuum sector of the theory. The field equation (2.2) is now

$$\frac{d^2\chi}{dx^2} - m^2\chi = \sum_{n=2}^{\infty} \frac{V_{(n+1)}}{n!}\chi^n \,, \tag{2.10}$$

with a solution, in the neighbourhood of $x = -\infty$, which can be written as a power series in $\exp(mx)$,

$$\chi(x) = \sum_{n=1}^{\infty} a_n \exp(nmx) \,. \tag{2.11}$$

We can find the coefficient a_n by substituting (2.11) into (2.10) and comparing the terms proportional to $\exp(nmx)$. Clearly, to calculate a_n we first need to know a_i for all $i < n$. It is consistent for a_1 to take an arbitrary value and then all other coefficients are fixed in terms of a_1. It can be checked inductively that

$$a_n = P_{n-1}(V_{(n+1)}, V_{(n)}, \dots, V_{(3)}, a_1, m^2) \tag{2.12}$$

a polynomial of degree $n - 1$ in the vertices shown. The first three coefficients are

$$a_1 \,, \quad a_2 = a_1^2 \frac{V_{(3)}}{6m^2} \,, \qquad a_3 = a_1^3\left(\frac{V_{(3)}^2}{48m^4} + \frac{V_{(4)}}{48m^2}\right) \,, \tag{2.13}$$

and for all n, the linear term in the polynomial (2.12) is

$$a_n = a_1^n \frac{V_{(n+1)}}{(n^2 - 1)\,n!\,m^2} + O(V_{(i)}^2) \,. \tag{2.14}$$

To obtain the antisymmetric soliton we require $\chi(0) = c$, so now fix $a_1 = A$ to satisfy this boundary condition. Then $\phi(x) = \chi(x) - c$ is the soliton solution for negative x, assuming the series converges. Using the antisymmetric property again, the full solution is

$$\phi_c(x) = c\epsilon(x) - \sum_{n=1}^{\infty} a_n \epsilon(x) \exp(-nm|x|)\,, \tag{2.15}$$

whose Fourier transform is

$$\tilde{\phi}_c(q) = -\frac{2ic}{q} + \sum_{n=1}^{\infty} a_n \frac{2iq}{q^2 + (nm)^2}\,, \tag{2.16}$$

with poles at $q = inm$, as before.

Now it was pointed out in ref. [1] that, unless the potential $V(\phi)$ is periodic, it seems necessary to identify the soliton and antisoliton as the same particle. This is because, classically, solitons and antisolitons must alternate along the x direction, which is inconsistent with a localized particle interpretation. The simplest way to make the identification is to consider the displacement of the soliton field from the closest vacuum value. Denoting this new field by χ_c, we have

$$\chi_c(x) = c - |\phi_c(x)| \tag{2.17}$$

$$= \sum_{n=1}^{\infty} a_n \exp(-nm|x|)\,, \tag{2.18}$$

which is the same, whether ϕ_c is the soliton or antisoliton. We shall refer to χ_c as the symmetrized soliton.

Suppressing the sign of ϕ_c is similar to what happens in the no soliton sector of the quantum theory. There, spontaneous symmetry breaking occurs, but the physics is independent of whether the vacuum expectation value $\langle\phi_0\rangle$ is $-c$ or $+c$, and the sign of ϕ is unobservable. In fact, the real reason for introducing χ_c will only be apparent when we calculate the soliton form factor using the effective Lagrangian.

For the moment, we just note some properties of χ_c. Since the potential is symmetric, $\chi_c(x)$ is symmetric in x and satisfies (2.10) for positive and negative x. At the origin, however, χ_c has a discontinuous derivative which implies a source term $\kappa\delta(x)$ on the r.h.s. of (2.10), with

$$\kappa = \mathrm{disc}\left.\frac{\mathrm{d}\chi_c}{\mathrm{d}x}\right|_{x=0} \tag{2.19}$$

$$= -2m \sum_{n=1}^{\infty} n a_n \tag{2.20}$$

$$= -2m\left[A + \sum_{n=2}^{\infty} \frac{V_{(n+1)}A^n}{(n^2-1)(n-1)!m^2} + \mathrm{O}(V_{(i)}^2)\right]. \tag{2.21}$$

We shall later be able to identify κ with the soliton-meson vertex of the effective Lagrangian.

The Fourier transform of the symmetrized soliton is

$$\tilde{\chi}_c(q) = \sum_{n=1}^{\infty} a_n \frac{2nm}{q^2 + (nm)^2} \ . \tag{2.22}$$

The pole at $q = 0$, associated with the topological charge is now absent, and the signs of the pole residues in the lower half plane are reversed. It is this expression which is reproduced by the Feynman diagrams of the effective Lagrangian in sect. 3.

The general results of this section are illustrated by the ϕ^4 and sine-Gordon solitons. The ϕ^4 theory potential,

$$V(\phi) = \frac{1}{2\lambda}(\tfrac{1}{4}m^2 - \lambda\phi^2)^2 \ , \tag{2.23}$$

gives a soliton solution

$$\phi_c(x) = \frac{m}{2\lambda^{1/2}} \tanh(\tfrac{1}{2}mx) \ , \tag{2.24}$$

which has periodicity $2\pi i m^{-1}$ in $\mathrm{Im}(x)$ as expected. ϕ_c can be written as a power series like (2.15) whose region of convergence is exactly $(0 \leqslant |x| \leqslant \infty)$, and the early coefficients are those expected from the only vertices $V_{(3)}$ and $V_{(4)}$ which occur.

The sine-Gordon potential

$$V(\phi) = \frac{m^2}{\beta^2}(1 + \cos\beta\phi) \ , \tag{2.25}$$

yields the soliton

$$\phi_c(x) = \frac{4}{\beta} \tan^{-1}(\exp(mx)) - \frac{\pi}{\beta} \ , \tag{2.26}$$

with similar properties. The Fourier transforms, which can be written in closed form in each case, have the expected poles.

3. The effective Lagrangian

Based on the motivating remarks of the introduction, we propose for one-dimensional scalar solitons the effective Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\chi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\chi}{\partial x}\right)^2 - \frac{1}{2}m^2\chi^2 - \sum_{n=3}^{\infty} \frac{V_{(n)}}{n!}\chi^n$$
$$+ \frac{1}{2}\left(\frac{\partial\psi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\psi}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{M}{\hbar}\right)^2 \psi^2 + \tfrac{1}{2}G\psi^2\chi \ . \tag{3.1}$$

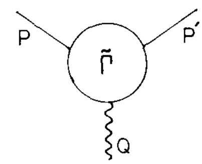

Fig. 2.

The first four terms are just the Lagrangian $\mathcal{L}_0$ written in terms of the meson field $\chi = c + \phi$. The scalar field ψ, assumed to commute with ϕ, represents the soliton with physical mass M. The coupling constant G will be fixed later. The Lagrangian is not equivalent to the original one, because of the extra terms involving the soliton field. However, we shall only do perturbative calculations with the effective Lagrangian hoping to reproduce from Feynman diagrams the non-perturbative soliton physics of the original theory. The soliton cannot reappear as a classical solution of the effective Lagrangian if we work perturbatively, and so we avoid the possibility of double counting the soliton.

The soliton form factor $\tilde{\Gamma}(q) = \langle p'|\chi|p\rangle$ is calculated using conventional Feynman rules for the vertex function (fig. 2). We shall evaluate the form factor to leading order in $\hbar$, but anticipating that G is of order $\hbar^{-2}$, diagrams with loop integrals will contribute. The leading-order diagrams are those where the external meson is the base of meson tree whose top branches are attached to a single soliton line passing through the diagram. Some examples are shown in fig. 3. The soliton-meson vertices can be reordered in any topologically distinct manner. We demonstrate that these graphs correspond to the Fourier transform of terms in the series expansion of $\chi_c(x)$, the symmetrized soliton.

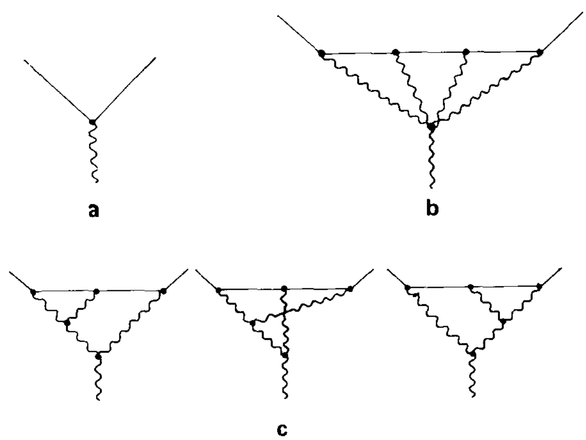

Fig. 3.

The soliton kinematics to leading order is non-relativistic, as discussed by Gervais et al. [2]. We are calculating the form factor as a function of q, the external meson wave number, so the momentum transfer is $O(\hbar)$. Since M is $O(1)$, in the frame of a slow incoming soliton, the outgoing soliton is also slow. Consequently, we write the incoming and outgoing soliton 2-momenta as

$$P_\mu = \hbar p_\mu = (M, \hbar p), \qquad P'_\mu = \hbar p'_\mu = (M, \hbar p') , \tag{3.2}$$

respectively, and the momentum transfer as

$$Q_\mu = (0, \hbar(p' - p)) = (0, \hbar q) . \tag{3.3}$$

Similar considerations apply to the internal soliton propagators. Because of the meson propagators, the dominant region of integration is where the loop wave vectors are $O(1)$. Each soliton propagator is of the form

$$i\hbar \left[(p_\mu + k_\mu)^2 - \left(\frac{M}{\hbar}\right)^2 + i\epsilon \right]^{-1} , \tag{3.4}$$

where k_μ is a linear combination of loop wave vectors. The leading approximation to (3.4) is therefore

$$i\hbar^2 [2Mk^0 + i\epsilon]^{-1} , \tag{3.5}$$

which is the eikonal approximation for a slow, heavy particle interacting with light, relativistic particles.

The elementary vertex in fig. 3a gives

$$\tilde{\Gamma}_0(q) = \frac{G}{q^2 + m^2} , \tag{3.6}$$

where $\tilde{\Gamma}_n$ denotes the sum of diagrams with n meson vertices, or less.

Now consider the diagrams with one meson vertex as in fig. 3b, and in particular the diagram with the $(n + 2)$ point vertex. Ordering the soliton and meson propagators from the left to the right of the diagram, and enforcing momentum conservation on the $(r + 1)$th meson propagator, the internal meson momenta are

$$\hbar [k_1, k_2, ..., k_r, q - \sum_{i=1}^{n} k_i, k_{r+1}, ..., k_n] , \tag{3.7}$$

and the internal soliton momenta are

$$\hbar [p + k_1, p + k_1 + k_2, ..., p + \sum_{i=1}^{r} k_r, p' - \sum_{i=r+1}^{n} k_i, ..., p' - k_n] . \tag{3.8}$$

To evaluate the diagram in the eikonal approximation we use the combinatorial tricks discussed by Lévy and Sucher [9]. We find a contribution to $\tilde{\Gamma}_1$ of

$$\left(\frac{-i\hbar^2}{4\pi^2}\right)^n \frac{V_{(n+2)} G^{n+1}}{q^2 + m^2} \, I , \tag{3.9}$$

where

$$I = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \mathrm{d}^2 k_i \left\{ \frac{1}{[2Mk_1^0 + i\epsilon] \dots [2M(k_1^0 + \dots + k_r^0) + i\epsilon]} \right.$$

$$\times \frac{1}{[-2M(k_{r+1}^0 + \dots + k_n^0) + i\epsilon] \dots [-2Mk_n^0 + i\epsilon]}$$

$$\left. \times \prod_{i=1}^{n} \frac{1}{k_i^2 - m^2 + i\epsilon} \times \frac{1}{(q - \sum_{i=1}^{n} k_i)^2 - m^2 + i\epsilon} \right\}. \tag{3.10}$$

Now, permuting the labels k_i cannot change I, so we sum over all permutations, giving $n!I$. Suppose $\{k_i'\}$ is a permutation of $\{k_i\}$, and denote by S_1, S_2 the sets

$$S_1 = \{k_1', \dots, k_r'\}, \qquad S_2 = \{k_{r+1}', \dots, k_n'\}. \tag{3.11}$$

Summing over the permutations which leave S_1, S_2 unchanged, and using the identity

$$\sum_{\text{Perms}\{c_i\}} c_1^{-1}(c_1 + c_2)^{-1} \dots (c_1 + c_2 + \dots + c_r)^{-1} = (c_1 c_2 \dots c_r)^{-1}, \tag{3.12}$$

we obtain for the soliton part of the integrand

$$\sum_{S_1, S_2} \left(\prod_{i=1}^{r} \frac{1}{2Mk_i'^0 + i\epsilon} \right) \left(\prod_{i=r+1}^{n} \frac{1}{-2Mk_i'^0 + i\epsilon} \right). \tag{3.13}$$

The choice of r does not change I, so we sum over r as well. The soliton part is now

$$\prod_{i=1}^{n} \left(\frac{1}{2Mk_i^0 + i\epsilon} + \frac{1}{-2Mk_i^0 + i\epsilon} \right), \tag{3.14}$$

which is the same as

$$\prod_{i=1}^{n} \left(-\frac{2\pi i}{2M} \delta(k_i^0) \right). \tag{3.15}$$

The meson part of the integrand is invariant through these combinatorial operations, so the integral I reduces to

$$I = \frac{1}{(n+1)!} \left(\frac{-2\pi i}{2M} \right)^n \int_{-\infty}^{\infty} \prod_{i=1}^{n} \mathrm{d}k_i' \left(\prod_{i=1}^{n} \frac{-1}{(k_i')^2 + m^2} \right) \frac{-1}{(q - \sum_{i=1}^{n} k_i')^2 + m^2}. \tag{3.16}$$

This is an n-fold convolution, whose Fourier transform is easily evaluated and then inverted to obtain the result

$$I = \frac{1}{n!}\left(\frac{4\pi^2 i}{4mM}\right)^n \frac{1}{q^2 + (n+1)^2 m^2} \,. \tag{3.17}$$

Summing all diagrams with one meson vertex, and including the bare vertex diagram, we obtain finally

$$\widetilde{\Gamma}_1(q) = \frac{G}{q^2 + m^2} - \sum_{n=2}^{\infty} \frac{G V_{(n+1)}}{(n-1)!}\left(\frac{G\hbar^2}{4mM}\right)^{n-1} \frac{1}{(q^2 + m^2)(q^2 + (nm)^2)} \,. \tag{3.18}$$

Writing $G = 4mM\hbar^{-2}A_0$, we find that $\widetilde{\Gamma}_1(q)$ is the Fourier transform of

$$\Gamma_1(x) = 2M\hbar^{-2}\left[A_0 - \sum_{n=2}^{\infty} \frac{V_{(n+1)}A_0^n}{(n^2-1)(n-1)!m^2}\right] \exp(-m|x|)$$

$$+ 2M\hbar^{-2} \sum_{n=2}^{\infty} \frac{V_{(n+1)}A_0^n}{(n^2-1)\,n!\,m^2} \exp(-nm|x|) \,. \tag{3.19}$$

Now we fix A_0, and hence G, so that the coefficient of $\exp(-m|x|)$ in $\Gamma_1(x)$ agrees with the corresponding coefficient, A, in the classical symmetrized soliton, χ_c. Thus,

$$A = A_0 - \sum_{n=2}^{\infty} \frac{V_{(n+1)}A_0^n}{(n^2-1)(n-1)!m^2} + \mathrm{O}(V_{(i)}^2) \,, \tag{3.20}$$

which we can invert to give A_0 as a power series in $V_{(i)}$

$$A_0 = A + \sum_{n=2}^{\infty} \frac{V_{(n+1)}A^n}{(n^2-1)(n-1)!m^2} + \mathrm{O}(V_{(i)}^2) \,. \tag{3.21}$$

We then find that $\Gamma_1(x) = 2M\hbar^{-2}\chi_c(x)$ correct to $\mathrm{O}(V_{(i)})$. Furthermore, comparing (3.21) with (2.21), we see that the bare vertex G represents the source required to obtain the symmetrized soliton. The physical interpretation is that the bare soliton field in the effective Lagrangian represents a point particle, and the meson field self-interactions give the extended particle, with a non-trivial form factor. Since the scalar soliton field represents a single particle, we should not be surprised that $\widetilde{\Gamma}(q)$ is neither the Fourier transform of the classical soliton or antisoliton, but rather of the symmetrized soliton.

Now, we have no general method for studying further diagrams, but we have calculated the contribution to Γ_2 from the diagrams with two 3-point meson vertices, as shown in fig. 3c. We use the same combinatorial methods as above, applied to the sum of the three diagrams. Again, we permute the loop momenta and sum over the three ways of imposing momentum conservation in the meson tree. The meson part of the integrand is invariant under these operations when the diagrams are combined. The resulting contribution to $\tilde{\Gamma}_2(q)$ is

$$\frac{\hbar^4 G^3 V_{(3)}^2}{8M^2 m^4 (q^2 + m^2)} \left[\frac{1}{3}\frac{1}{q^2 + (2m)^2} - \frac{1}{4}\frac{1}{q^2 + (3m)^2}\right], \tag{3.22}$$

which is the Fourier transform of

$$2M\hbar^{-2} \frac{V_{(3)}^2 A_0^3}{m^4} \left[\tfrac{23}{144} \exp(-m|x|) - \tfrac{1}{9} \exp(-2m|x|) + \tfrac{1}{48} \exp(-3m|x|)\right] \tag{3.23}$$

These are just the terms we expect, if

$$\Gamma(x) = 2M\hbar^{-2}\chi_c(x) \tag{3.24}$$

to all orders in $\{V_{(i)}\}$. We conclude that to leading order in $\hbar$, the form factor is the Fourier transform of the symmetrized soliton.

The Fourier transform of the soliton (2.22) has poles at $q = \pm inm$, where n is an integer and m is the meson mass. Goldstone and Jackiw suggested that these poles may be identified with multi-meson thresholds in the crossed soliton, antisoliton annihilation channel. On the basis of the diagrams we have calculated, it appears that these poles arise from anomalous thresholds.

As is well-known, the singularities of a Feynman diagram are associated with a number of internal momenta (N say) being on-mass-shell [10]. A simple pole occurs if the number of loop integrations equals $N-1$. For diagrams of the kind in fig. 3b. there is an anomalous threshold pole when all internal momenta are on-mass-shell. The dual diagram (fig. 4) shows that for $M >> m\hbar$, the soliton momenta are physical and time-like, while the meson momenta are all approximately space-like, and hence of form $k_\mu = (0, \pm im)$. The anomalous thresholds occur therefore when $q = inm$, and are poles. It is not clear whether the leading behaviour of the diagram is simply this pole, without checking the residue. We mention, in conclusion, that Coleman [11] has shown that anomalous thresholds are also responsible for poles in the exact S-matrix of the sine-Gordon theory.

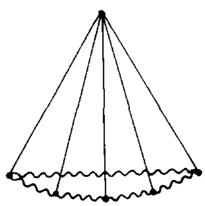

Fig. 4.

4. The force between solitons

Having constructed an effective Lagrangian to reproduce the soliton form factor, we now use it to calculate the simplest multi-soliton property, namely the long-distance force between two solitons.

In their seminal paper, Perring and Skyrme [7] calculated this force classically in the sine-Gordon model. We rederive the result using the exact soliton-antisoliton scattering solution. The field equation

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = -\frac{m^2}{\beta}\sin\beta\phi\,, \tag{4.1}$$

has an exact solution [8]

$$\tan\tfrac{1}{4}\beta\phi = \frac{1}{u}\,\frac{\sinh(\gamma umt)}{\cosh(\gamma ux)}\,, \tag{4.2}$$

representing, for $t \gg 0$, a separating soliton and antisoliton. The soliton interpolates between vacua $\phi = 0,\ 2\pi\beta^{-1}$. Defining the soliton position by $\phi = \pi\beta^{-1}$ its equation of motion is

$$u\cosh(\gamma ux) = \sinh(\gamma umt)\,. \tag{4.3}$$

For non-relativistic asymptotic velocities, where $u \ll 1$, (4.3) has the approximate solution

$$x = -\frac{1}{m}\log u + ut - \frac{1}{m}\exp(-2umt)\,. \tag{4.4}$$

The acceleration is therefore

$$\ddot{x} = -4m\exp(-2mx)\,. \tag{4.5}$$

The separation of the soliton and antisoliton is $s = 2x$, and the soliton mass, $M = 8m\beta^{-2}$, so there is an attractive potential

$$U(s) = -\frac{32m}{\beta^2}\exp(-ms)\,. \tag{4.6}$$

We know that the asymptotic form of the soliton solution is $A\exp(-mx)$ where $A = 4\beta^{-1}$, so the potential is

$$U(s) = -2mA^2\exp(-ms)\,. \tag{4.7}$$

We now show that (4.7) is a universal result for any one-dimensional solitons. Since there is no exact multi-soliton solution in general, we must employ a different technique. Following Goldberg et al. [12], who studied the force between monopoles, we use the energy-momentum tensor. Consider the Lagrangian (2.1). The field momentum on the interval (a, b) is

$$P = -\int_a^b \frac{\partial \phi}{\partial t}\frac{\partial \phi}{\partial x}\,\mathrm{d}x\,, \tag{4.8}$$

whose time rate of change is

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -\int_a^b \left(\frac{\partial^2 \phi}{\partial t^2}\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial t}\frac{\partial^2 \phi}{\partial x \partial t}\right)\mathrm{d}x\,. \tag{4.9}$$

We can use the field equation and integrate to obtain

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \left[-\frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + V(\phi)\right]_a^b\,, \tag{4.10}$$

so that the force on the interval can be identified with the pressure at the endpoints.

To calculate the force between solitons, we consider the initial rate of change of momentum of a static field configuration which is a linear superposition of an anti-soliton on the left and a soliton on the right with large separation s. Thus

$$\phi(x) = \phi_1(x) + \phi_2(x) + c\,, \tag{4.11}$$

where

$$\phi_1(x) = \phi_c(-x)\,, \qquad \phi_2(x) = \phi_c(x - s)\,. \tag{4.12}$$

Choose $a \ll 0$ and $0 \ll b \ll s$, so that the soliton fields take their asymptotic forms at a and b, and $\phi_2 \ll \phi_1$. Initially, to leading order in ϕ_2,

$$\frac{\mathrm{d}P}{\mathrm{d}t} \simeq \left[-\frac{1}{2}\left(\frac{\partial \phi_1}{\partial x}\right)^2 - \frac{\partial \phi_1}{\partial x}\frac{\partial \phi_2}{\partial x} + V(\phi_1) + V'(\phi_1)\,\phi_2\right]_a^b \tag{4.13}$$

$$= \left[-\frac{\partial \phi_1}{\partial x}\frac{\partial \phi_2}{\partial x} + \frac{\partial^2 \phi_1}{\partial x^2}\phi_2\right]_a^b\,, \tag{4.14}$$

using the field equation (2.3). In the region between the solitons

$$\phi_1 \simeq -c + A\exp(-mx)\,, \tag{4.15}$$

$$\phi_2 \simeq -c + A\exp(m(x - s))\,, \tag{4.16}$$

so the endpoint b contribution to $\dot{P}$ is $2A^2m^2\exp(-ms)$, independently of b. The endpoint a contribution is smaller than this and vanishes as $a \to -\infty$. The intersoliton potential is therefore

$$U(s) = -2A^2 m\exp(-ms)\,, \tag{4.17}$$

as obtained previously in the sine-Gordon case, and in agreement with Rajaraman [13] in the case of ϕ^4 theory.

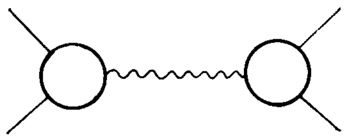

Fig. 5.

It is now a simple matter to show that this classical calculation is consistent with the amplitude for soliton-soliton scattering in the effective field theory. In the non-relativistic limit, and using units where $\hbar = 1$, the cross section in field theory is

$$\sigma = \frac{1}{64p^2M^2}|T(q)|^2 . \tag{4.18}$$

T is the scattering amplitude as a function of the momentum transfer q, and p is the incoming soliton momentum in the c.m. frame.

We compare this result with the cross section for non-relativistic potential scattering. Using the Golden Rule, the cross section is

$$\sigma = \frac{M^2}{4p^2}|\tilde{U}(q)|^2 , \tag{4.19}$$

where $\tilde{U}$ is the Fourier transform of the potential. The intersoliton potential we have calculated gives

$$|\tilde{U}(q)| = \frac{4m^2A^2}{q^2+m^2} . \tag{4.20}$$

On the other hand, the leading contribution to the field theory T-matrix, from the Born-like diagram (fig. 5) is

$$|T(q)| \simeq (\tilde{\Gamma}(q))^2(q^2+m^2) . \tag{4.21}$$

The asymptotic force is given by the pole at $q^2 = -m^2$. Since $\tilde{\Gamma}$ has a pole term

$$\tilde{\Gamma} \sim \frac{4mMA}{q^2+m^2}, \tag{4.22}$$

we see that

$$|T(q)| \sim \frac{16m^2M^2A^2}{q^2+m^2} , \tag{4.23}$$

so that the cross sections (4.18) and (4.19) agree. We have therefore established that the effective Lagrangian reproduces the asymptotic force law, *via* the Born term scattering amplitude. Note that it is essential in this calculation that $\tilde{\Gamma}(q)$ has no pole at $q = 0$, for otherwise there would appear to be a linear potential. This is a justification for dealing with the symmetrized soliton. We remark, finally, on the corresponding results in the case of monopoles. Montonen and Olive [5] showed that their dual theory reproduced the correct monopole force. On the basis of the above calculation, we expect that their theory would also give the correct asymptotic behaviour of the classical monopole, *via* the form factor. This is indeed the case, for the potential due to either photon or Higgs particle exchange, both being massless, is

$$\widetilde{U}(q) = \pm \frac{g^2}{q^2}, \tag{4.24}$$

implying that the form factor has a pole term

$$\widetilde{\Gamma}(q) = \pm 2M \frac{g}{q^2} . \tag{4.25}$$

This is consistent with the asymptotic behaviour of the Higgs field and magnetic scalar potential, which are respectively

$$\phi(r) = c - \frac{1}{er}, \qquad A(r) = \pm \frac{1}{er}, \tag{4.26}$$

since $g = (4\pi e)^{-1}$.

5. Conclusions

The effective Lagrangian proposed here reproduces known semiclassical soliton properties. The methods are covariant and perturbative, and require summing infinite classes of diagrams. We can imagine two cases where this would be an advantage over conventional techniques. Firstly, if there were no exact soliton solution known, but only a series solution, and secondly, if the quantum corrections were more important than knowing the exact soliton solution. In cases of physical interest, where the soliton mass may be similar to the mass of the elementary mesons, it is a possible assumption that the best order for summing diagrams is according to the number of loop integrals, so that simple meson loop diagrams are more important than a complicated meson tree diagram which contributes semiclassically. Whether, in fact, the diagrams with meson loops give the correct quantum corrections is a matter of speculation.

Finally, we suggest that the best application of this work might be in reverse. Theories of physical interest may involve an interacting Lagrangian plus the free Lagrangian of another field, together with a Yukawa coupling of the fields. It is perhaps possible to work with just the interacting Lagrangian and with a classical soliton solution for the absent field. A candidate for this treatment is QCD, with quarks being treated as solitons.

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