QUANTIZATION OF RELATIVISTIC SYSTEMS WITH CONSTRAINTS

E.S. FRADKIN
Physical Lebedev Institute, Academy of Sciences, Moscow, USSR

and

G.A. VILKOVISKY
State Committee of Standards, Moscow, USSR

Received 23 October 1974

The general dynamical system with constraints is quantized, and the S-matrix is constructed in the most general class of gauges including relativistic ones. In the case when constraints do not form a group a new type of additional diagrams arises securing unitarity of the theory: the four-fermion interaction of ghost fields.

Introduction. The present work solves the problem of S-matrix construction for general Hamiltonian systems with constraints [1] in a most general class of additional conditions which (in the case of gauge fields) includes relativistic gauges. Thus it proves to be possible to meet the requirements of relativistic covariance and unitarity in the physical space simultaneously. The difficulty of combination of these principles consisted in that the construction of the S-matrix unitary in the physical space naturally implies a reduction of a phase space, while the relativistic covariance requires its expansion because all the degrees of freedom become independent in relativistic gauges [2]. The problem of combination of dynamics and relativity principles is solved in the framework of a new approach to the Hamiltonian theory formulated in ref. [2] and differing from the Dirac's one.

Dirac's theory of systems with constraints is formulated in a phase space of canonical variables q^i, p_i, $i = 1, \dots n$ in terms of which the Hamiltonian $H_0(q^i, p_i)$ is given and which are subject to the constraint equations

$$T_\alpha(q^i, p_i) = 0, \qquad \alpha = 1, \dots m. \tag{1}$$

If the introduced quantities satisfy the following conditions in terms of Poisson brackets

$$\{T_\alpha, T_\beta\} = T_\gamma U^\gamma_{\alpha\beta}, \{H_0, T_\alpha\} = T_\beta V^\beta_\alpha, \tag{2}$$

where U and V are some functions (structure coefficients), then such a theory is equivalent to the dynamics of a fewer number $(n-m)$ of independent (physical) degrees of freedom q^*, p^* which may be singled out by imposition of additional conditions

$$\Phi^\alpha(q^i, p_i) = 0, \det\{\Phi^\alpha, T_\beta\} \neq 0. \tag{3}$$

Then

$$\int \mathrm{d}t(p_i\dot{q}^i - H_0)|_{T\,=\,\Phi=0} = \int \mathrm{d}t(p^*\dot{q}^* - H_{\mathrm{phys}}(q^*,p^*)),$$

the change of additional conditions being equivalent to the canonical transformation in the physical space of the variables q^*, p^*, and hence the choice of these conditions being immaterial.

Quantization of the described dynamical system consists in imposition of canonical commutation relations upon the variables q^*, p^*, which leads to the unitary S-matrix determined by the following functional integral

$$Z = \int \mathrm{d}q^*\,\mathrm{d}p^*\,\exp\left[\mathrm{i}\int \mathrm{d}t\,[p^*\dot{q}^* - H_{\mathrm{phys}}(q^*, p^*)]\right] \tag{4}$$

The problem is to rewrite the integral (4) in terms of explicitely known quantities and in a form general enough to meet the relativistic covariance requirements in the case of gauge fields. This problem is solved in the next section.

The general form of the unitary S-matrix. The brief outline of our approach is the following. Eqs. (1) and (3) together with the Hamilton equations may be obtained from a unique action

$$S = \int \mathrm{d}t(p_i \dot{q}^i - H_0 - \lambda^\mu T_\mu + \pi_\mu \Phi^\mu) \tag{5}$$

by means of introduction of Lagrange multipliers λ^μ and π_μ. In the case of field theories the set (q^i, λ^μ) forms a relativistic field and a relativistic (covariant) gauge condition necessarily contains λ^μ in the following general form [2]:

$$\Phi^\alpha_{\text{covar}} = \dot{\lambda}^\alpha + \chi^\alpha(q^i, p_i, \lambda). \tag{6}$$

We see now that under the substitution of (6) into (5), the action (5) yields no constraint equations, moreover Lagrange multipliers λ^μ become dynamically active, and Lagrange multipliers π_μ serve as their conjugate momenta. Thus we arrive at the canonical formalism in an enlarged phase space. The redeming fact is that the equivalence to the initial theory with constraints may be regained by introduction of a superlarge space with additional fermion degrees of freedom: $(\bar{\mathcal{P}}_\alpha, C^\alpha)$, $(\mathcal{P}^\alpha, \bar{C}_\alpha)$ [2]. The Hamiltonian of this complete system of independent canonical variables must be determined by the only requirement of unitarity in the physical subspace. This central problem is solved by the following theorem which is the heart of our approach.

Let there be a dynamical boson system (q^A, π_A), $A = 1, ..., n+m$, with a given Hamiltonian $H_0(q^A, \pi_A)$ and functions $G_a(q^A, \pi_A)$, $a = 1, ... 2m$, such that

$$\{G_a, G_b\} = G_c U^c_{ab}, \quad \{H_0, G_a\} = G_b V^b_a. \tag{7}$$

The only restrictions which we are forced to impose upon the appeared structure coefficients is that the absolutely antisymmetrical (with respect to any pair of equally placed indices parts of expressions

$$\{U^c_{ab}, U^e_{fg}\}, \ \{V^a_b, U^c_{fg}\}$$

vanish. Let us supplement this system with fermion canonical variables $(\eta^a, \mathcal{P}_a)$ consistently considered as elements of Grassman algebra. Finally let us introduce the arbitrary functions $\chi^a(q^A, \pi_A)$. Our theorem says: the following functional integral does not depend on the choice of $\chi^a(q^A, \pi_A)$:

$$Z = \int \mathrm{d}q\,\mathrm{d}\pi\,\mathrm{d}\eta\,\mathrm{d}\mathcal{P} \exp\left[i\int \mathrm{d}t(\pi_A \dot{q}^A + \mathcal{P}_a \dot{\eta}^a - H)\right], \tag{8}$$

$$\begin{aligned} H = H_0 &- G_a \chi^a - \mathcal{P}_a \{\chi^a, G_b\}\eta^b - \mathcal{P}_a U^a_{bc}\chi^b \eta^c \\ &+ \mathcal{P}_a V^a_b \eta^b - \tfrac{1}{2}\mathcal{P}_a \eta^b \{\chi^a, U^n_{bm}\}\mathcal{P}_n \eta^m. \end{aligned} \tag{9}$$

The proof of this main theorem is too complicated to be presented here. Application to systems with constraints is immediately attained by the following identifications

$$G_a = (\pi_\alpha, T_\alpha), \quad q^A = \begin{pmatrix} q^i \\ \lambda^\alpha \end{pmatrix}, \quad \pi_A = (p_i, \pi_\alpha),$$

$$\chi^a = \begin{pmatrix} \chi^\alpha \\ -\lambda^\alpha \end{pmatrix}, \quad \eta^a = \begin{pmatrix} \mathcal{P}^\alpha \\ C^\alpha \end{pmatrix}, \quad \mathcal{P}_a = (\bar{C}_\alpha, \bar{\mathcal{P}}_\alpha).$$

Then the purely boson (classical) part of the complete action in the exponential of eq. (8) takes just the form of eqs. (5–6), exhibiting the desired relativistic gauge. The rest (ghost) part of the complete action contains an utterly new type of additional diagrams: the four-fermion interaction term. The four-fermion vertex $\{\chi^\alpha, U^\gamma_{\beta\sigma}\}$ vanishes only in the case when constraints T_α form a group, i.e. when structure coefficients $U^\alpha_{\beta\gamma}$ of eq. (2) are the constants.

The unitarity of the S-matrix (8) turnes out to be a simple corollary of the main theorem: using this theorem it is possible to transform expression (8) identically to a noncovariant gauge (3) by a certain canonical displacement of variables. The resulting expression in the gauge (3) exactly coincides with that of ref. [3], and the latter is equal to expression (4) as shown in the cited reference. (Ref. [4] gives another example of canonical quantization in noncovariant gauges). Thus eq. (8) presents a most general expression of a unitary S-matrix in an arbitrary (including relativistic) gauge.

The functions χ^α may depend also on π_α, then eq. (8) gives the S-matrix in a nondegenerate gauge. Moreover the complete Hamiltonian (9) may be identically rewritten as

$$H = H_0 + \mathcal{P}_a V^a_b \eta^b - \{\mathcal{P}_a \chi^a, G_b \eta^b - \tfrac{1}{2} U^b_{mn}\,\mathcal{P}_b \eta^m \eta^n\} \tag{10}$$

in terms of the generalized boson – fermion brackets:

$$\{A, B\} = \frac{\delta A}{\delta q}\bigg|_{\mathrm{r}} \frac{\delta B}{\delta \pi}\bigg|_{\mathrm{l}} - (-1)^{\eta_A \eta_B} \frac{\delta B}{\delta q}\bigg|_{\mathrm{r}} \frac{\delta A}{\delta \pi}\bigg|_{\mathrm{l}},$$

η_A denoting the number of fermions in A, and subscripts "r" and "l" right and left derivatives. Eq. (8) with the Hamiltonian H in the form (10) is also correct for the gauge functions χ^α depending on η, $\mathcal{P}$.

Discussion. In the general case $U^\alpha_{\beta\gamma}$ are the functions of canonical variables and relations (2) are not the group relations. The Jacobi identity for group structure constants is then replaced by a more general relation:

$$U^{\alpha}_{\beta\mu} U^{\mu}_{\gamma\nu} + U^{\alpha}_{\nu\mu} U^{\mu}_{\beta\gamma} + U^{\alpha}_{\gamma\mu} U^{\mu}_{\nu\beta}$$
$$= \{U^{\alpha}_{\gamma\nu}, T_{\beta}\} + \{U^{\alpha}_{\beta\gamma}, T_{\nu}\} + \{U^{\alpha}_{\nu\beta}, T_{\gamma}\}$$

following from eq. (2). The classical Dirac's formulation and the quantum theory in noncovariant gauges (see ref. [3]) are indifferent to the presence or the absence of a group in eq. (2). We see that this circumstance becomes essential when a covariant gauge is employed: a new type of additional diagrams arises. The fact that constraints do not form a group does not however prevent the Lagrangian theory from being invariant under the gauge group (in the exact sense), which is just the case in the gravitational theory. This happens because in this case the gauge transformations cannot be presented as canonical transformations in Hamiltonian theory (contrary to the statement of ref. [5]), and thus they differ from transformations generated by constraints. The latter circumstance requires a certain nontrivial work in order to transform eq. (8) to the configuration space of Lagrangian theory. In gravidynamics this is necessary for verification of the correctness of the relativistic S-matrix of refs. [6–9], because all the other approaches failed to prove its unitarity (they used coordinate transformations which are not correct in quantum domain [2]), while in our approach the unitarity is explicit. Results of this verification will be reported in the nearest future.

Application of the present approach to the modern quark model (the theory of relativistic strings [10–11]) is also under investigation. Conditions of our main theorem are satisfied in both mentioned theories.

The detailed proof of all the statements of the present paper will be published.

References

[1] P.A.M. Dirac, Canad. Journ. Math., 2 (1950) 129; Proc. Roy. Soc., A246, (1958) 326.

[2] E.S. Fradkin, G.A. Vilkovisky, Phys. Rev. D8 (1973) 4241.

[3] L.D. Faddeev, Theor. Math. Phys. USSR, 1 (1969) 3.

[4] E.S. Fradkin, in Problems of theoretical physics, (Nauka, Moscow, 1969).

[5] J. Anderson, P. Bergmann, Phys. Rev., 83 (1951) 1018.

[6] B.S. DeWitt, Phys. Rev., 160 (1967) 1113; 162 (1967) 1195; 162 (1967) 1239.

[7] L.D. Faddeev, V.N. Popov, Phys. Lett., 25B (1967) 30.

[8] S. Mandelstam, Phys. Rev., 175 (1968) 1580; 175 (1968) 1604.

[9] E.S. Fradkin, I.V. Tyutin, Phys. Rev. D 2 (1970) 2841.

[10] I. Nambu, Proc. Intern. Conf. on symmetric and quark models, (Wayne State University, 1969).

[11] P. Goddard, I. Goldstone, C. Rebbi, C. Thorn, Nucl. Phys. B56 (1973) 109.