SUPERSYMMETRY

P. FAYET and S. FERRARA

Laboratoire de Physique Théorique de l'Ecole Normale Supérieure,
24 rue Lhomond, 75231 Paris Cedex 05, France

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P. FAYET* and S. FERRARA**

Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, 24 rue Lhomond, 75231 Paris Cedex 05, France

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Contents:

Abstract:

Supersymmetry transformations turn bosons into fermions and conversely. We discuss the algebraic aspects of the new structure, its role in relativistic quantum field theory and its possible applications to particle physics.

* Address after October 1977: California Institute of Technology, Pasadena, California 91125, U.S.A.
** On leave of absence from Laboratori Nazionali di Frascati, I.N.F.N., Frascati, Italy. Address after September 1977: Theory Division, CERN, 1211 Genève 23, Switzerland.

1. Introduction

In the past three years great progress has been made in the extension of invariance considerations in the physics of particles and fields, to symmetries in which fermions and bosons play a common role. This has been achieved by introducing the concept of supersymmetry (Fermi–Bose symmetry) as an extension of the Lie Algebra of the Poincaré group [65, 124, 130]. Supersymmetry transformations can be obtained as a generalization to four-dimensional space–time of the supergauge transformations in dual models [91, 102, 60]. We first explain what this new structure is and shall examine later possible applications to physics.

The supersymmetry algebra, in its simplest form, involves the generators of space–time rotations and translations, $M^{\mu\nu}$ and P^{μ}, together with a self-conjugate spin $\frac{1}{2}$ generator Q_{α} (which turns boson fields into fermion fields, and vice-versa); they satisfy the following (anti) commutation rules:

$$
\begin{aligned}
&[Q_{\alpha}, M^{\mu\nu}] = \mathrm{i}(\sigma^{\mu\nu}Q)_{\alpha}\\
&[Q_{\alpha}, P^{\mu}] \;\;= 0\\
&\{Q_{\alpha}, \bar{Q}_{\beta}\} \;\;= -2(\gamma_{\mu})_{\alpha\beta}P^{\mu}.
\end{aligned}
\tag{1.1}
$$

Because of the spinorial character of the generator Q_{α}, the algebra involves both commutation and anticommutation relations. It is not an ordinary Lie Algebra, but what is called in the mathematical literature a Graded Lie Algebra (GLA). The spinorial generator Q_{α} is a grading representation of the Poincaré Lie Algebra. Supersymmetry extends this algebra in a non-trivial way; one can associate in irreducible representations a finite number of bosons and fermions; we have now a genuine relativistic spin-containing symmetry. If the spinorial generators belong themselves to some representation of an internal symmetry group, the resulting algebra provides also a fusion between space–time and internal symmetry [107, 32, 138, 129, 68] overcoming previous no-go theorems [19]. Irreducible multiplets combine in this case fermions and bosons with different internal quantum numbers.

Interest in supersymmetry started with the work of Wess and Zumino [130, 131, 132], and investigations concerned mainly the simplest algebra (1.1). A powerful tool for constructing tensor products of representations and invariant interactions is the superfield formalism introduced by Salam and Strathdee [106, 48]; a single superfield describes a finite number of ordinary local fields, including both fermion and boson fields.

Gauge invariance is consistent with supersymmetry; spin $\frac{1}{2}$ fermions as well as vector bosons can be gauge particles; vector fields are associated with (self-conjugate) spinor fields, both belonging to the adjoint representation of the internal symmetry group [132, 49, 113].

Supersymmetry has to be broken, since no fermions and bosons having the same mass are known, with the possible exception of massless particles (neutrino, photon, graviton); spontaneous supersymmetry breaking yields a massless spin $\frac{1}{2}$ Goldstone field [41], and one hoped originally that it could represent the neutrino [124].

There have been several attempts to use supersymmetry for the description of fundamental interactions. There exist now semi-realistic models of weak and electromagnetic interactions [35], describing the electron and its neutrino, the photon, massive vector bosons $W_{\pm}$ and Z, and other heavy particles. The photon and the neutrino belong to the same multiplet, and leptonic number is carried by the spinorial generator in the supersymmetry algebra (as is charge in ordinary gauge theories). Supersymmetry transformations relate vector and Higgs bosons with leptons (in particular the SU (2) × U (1) mixing angle is also a neutrino–heavy neutrino mixing angle), and the potential of Higgs scalars is strongly constrained.

In the above model the neutrino was the Goldstone particle; however this idea does not seem tenable in a realistic theory, owing to low-energy theorems [7, 30]; possible reasons for which Goldstone particles have not been observed will be discussed in the text.

Applications of supersymmetry in hadron physics seem farther from reality than in lepton physics; new states are generated when supersymmetry transformations are applied to quark states; possibly supersymmetry generators carry also flavour and colour indices: however, these attempts led to irreducible multiplets containing a large number of unwanted states [107, 108, 140].

Another possible application to hadron physics concerns the string picture. Hadrons are described as string excitations, and may be related to some non-local field theory or to a strong coupling limit of a local field theory. In the quantum theory of strings supersymmetry is not a visible (flavour) symmetry, but is related to the gauge (reparametrization) group of the string. All possible supergauge algebras of a string system have been classified; a sensible quantum theory of a string with fermionic degrees of freedom has been shown to be possible for spacetime dimension $d = 10$ or 2; $d = 2$ is a limiting situation for which the string Lagrangian collapses into a two-dimensional local field theory describing the self-interaction of a massless boson [91, 102, 2].

Applications of supersymmetry in the framework of local quantum field theory has been one of its greater successes. Invariance under supersymmetry implies, via the Noether theorem, the existence of a conserved local vector-spinor current $J^{\mu}_{\alpha}(x)$; the spinorial generator Q_α is given by

$$Q_\alpha = \int J^0_\alpha(x)\,\mathrm{d}^3x. \tag{1.2}$$

Ward identities following from this invariance are consistent with renormalizable perturbation theory. Moreover the number of counterterms in the Lagrangian density may be smaller than expected from simple symmetry considerations [74].

An interesting result concerns asymptotic freedom in supersymmetric non-abelian gauge theories. The quartic interaction of scalar particles is not fixed by an independent parameter, but by the square of the gauge coupling constant. Thus one can accommodate a relatively large number of scalar fields without destroying asymptotic freedom [49, 76].

Building a higher symmetry which makes softer the quantum corrections is interesting with respect to the renormalizability problem of quantum gravity. The present status is that only pure (self-coupled) gravity has some hope to be renormalizable. Gravity coupled to any matter system (scalar, Dirac and photon fields) is one-loop non-renormalizable, also in the case of variants of Einstein theory of general relativity [123].

A supersymmetric version of Einstein pure gravity has been constructed recently (supergravity). It involves a massless spin $\frac{3}{2}$ field minimally coupled to gravitation, with an additional quartic contact term required for invariance under local spinorial transformations [54, 27, 55]. Products of such transformations generate general coordinate transformations [55] in the same way as products of (global) supersymmetry transformations generate four-dimensional translations. Whether this will lead to a renormalizable theory of quantum gravity is at present an open problem.

The lesson we have learned from the impact of supersymmetry in the physics of particles and fields is that a richer algebraic structure enlarging the concept of Lie Algebra can be helpful in the search of unified theories of fundamental interactions, in which possibly internal and space–time symmetries are also unified. In addition, these higher symmetries soften and hopefully eliminate the divergences of relativistic quantum field theory.

The article is organized as follows:

In section 2 we give the algebraic foundations of supersymmetries, we discuss the concept of Graded Lie Algebra and we give the most relevant examples of supersymmetry algebras and their representations.

In section 3 we consider supersymmetric Lagrangian field theories. The tensor calculus based on the notion of superfield is developed and renormalization properties are studied. Theories with combined supersymmetry and local gauge symmetries are discussed in detail. The problem of asymptotic freedom and infrared convergence is also considered.

In section 4 we study spontaneous breaking for internal symmetry, gauge symmetry, and supersymmetry. We show how, though supersymmetry often makes use of Majorana spinors, it is possible to define a conserved quantum number, extending the notion of fermionic number. Field theory models invariant under larger algebras involving more than one spinorial generator are considered, and the vacuum degeneracy which may exist at lowest order is studied.

In section 5 we present a tentative model of weak and electromagnetic interactions for leptons of the electron sector. Consequences of low-energy theorems for a Goldstone spin $\frac{1}{2}$ particle are discussed, together with possible ways to describe also the muon sector.

The final section (section 6) is mainly devoted to geometrical aspects of supersymmetry, closely related to the concept of superspace, i.e. the enlargement of Minkowski space with anticommuting variables. This is the natural framework to study combined supersymmetric and local gauge theories. Of particular interest is the case in which the gauge group is the Einstein group of general coordinate transformations. Gauge spin $\frac{3}{2}$ particles emerge and supersymmetry is realized in a local way. Finally the supergauge groups of the string systems are discussed in detail and the supersymmetric aspects of two string models based on a fermionic gauge symmetry are considered.

2. Supersymmetries and Graded Lie Algebras

2.1. Graded Lie Algebras

In the present paragraph we give some mathematical preliminaries concerning the basic algebraic structure which underlies the concept of supersymmetry [138, 129, 20, 95, 44].

The word supersymmetry, as commonly used in current literature, denotes a particular class of those algebraic structures which mathematicians call Graded Lie Algebras (GLA) [10, 20, 77, 98, 86].

Graded Lie Algebras are extensions of usual Lie Algebras where a grading is introduced, namely a distinction between even and odd elements: even elements belong to an ordinary Lie Algebra, and in fact obey usual commutation relations; odd elements, responsible for the grading, obey anti-commutation relations among themselves and commutation relations with the (even) elements of the Lie Algebra. These latter relations indeed specify that the odd elements are a representation of the Lie Algebra, the grading representation.

If one denotes by the symbol A_m the set of elements of a D-dimensional Lie Algebra ($m = 1, \dots D$) and Q_α the set of elements of a d-dimensional ($\alpha = 1, \dots d$) grading representation then one gets the following basic system of commutation rules for the ($D + d$)-dimensional Graded Lie Algebra,

$$\begin{aligned} [A_m, A_n] &= f^l_{mn} A_l \\ [A_m, Q_\alpha] &= S^\beta_{m\alpha} Q_\beta \\ \{Q_\alpha, Q_\beta\} &= F^m_{\alpha\beta} A_m \end{aligned} \tag{2.1}$$

and moreover the set of Jacobi identities

$$[A_m,[A_n,Q_\alpha]] + [Q_\alpha,[A_m,A_n]] + [A_n,[Q_\alpha,A_m]] = 0$$
$$[A_m,\{Q_\alpha,Q_\beta\}] + \{[Q_\alpha,A_m],Q_\beta\} + \{[Q_\beta,A_m],Q_\alpha\} = 0$$
$$[Q_\alpha,\{Q_\beta,Q_\gamma\}] + [Q_\gamma,\{Q_\alpha,Q_\beta\}] + [Q_\beta,\{Q_\gamma,Q_\alpha\}] = 0 \tag{2.2}$$

which in their turn imply the following constraints for the "structure constants" of the Graded Lie Algebra:

$$f^l_{mn}S^\delta_{l\alpha} - S^\gamma_{n\alpha}S^\delta_{m\gamma} + S^\gamma_{m\alpha}S^\delta_{n\gamma} = 0$$
$$F^l_{\beta\gamma}f^n_{ml} - S^\delta_{m\gamma}F^n_{\beta\delta} - S^\delta_{m\beta}F^n_{\gamma\delta} = 0$$
$$F^l_{\beta\gamma}S^\delta_{l\alpha} + F^l_{\gamma\alpha}S^\delta_{l\beta} + F^l_{\alpha\beta}S^\delta_{l\gamma} = 0. \tag{2.3}$$

We remark that the first of these relations is nothing but the representation property of the odd generators Q_α. Furthermore we note that these relations are empty when the GLA is abelian i.e. $f^l_{mn} = S^\delta_{l\alpha} = 0$. This is precisely the case of the usual supersymmetry algebra of space–time where the bosonic part is identified with the four-dimensional translation generators [65, 130].

However, these relations are meaningful for the more complicated supersymmetry Algebra, where the bosonic part includes the Poincaré Algebra or the conformal Algebra of space–time [130].

It is worth mentioning that the problem of classifying all possible Graded Lie Algebras is of great interest with regard to possible applications of these algebraic structures in Physics. In particular both mathematicians and physicists [10, 20, 77, 98, 86] worked on the problem of classifying all possible semisimple Graded Lie Algebras, which is analogous to the problem solved by Cartan for ordinary Lie Algebras.

In particular, quite recently [77, 86], a complete classification of all strictly semisimple GLA has been found, i.e. those semisimple GLA for which a non-degenerate metric matrix can be introduced. Moreover it turns out that for this particular class of GLA the corresponding Lie Algebra admits one and only one possible grading, i.e. the grading representation (odd generators) is unique. The classification given by Haag, Łopuszański, Sohnius [68] concerning all possible gradings of the conformal algebra is a particular example of this class of GLA. Here the (compactified) Lie Algebra is SU (4) × U (N) and its only possible grading representation is just the fundamental 8 N-dimensional representation.

For non semisimple Lie Algebras however the grading is certainly not unique.

For example the usual supersymmetry algebra of space–time [130], used to build up field theory models, whose Lie Algebra is just the Poincaré Algebra, admits in fact an infinite set of possible gradings. All possible grading representations of the Poincaré Lie Algebras have been classified by the authors of ref. [68].

2.2. Supersymmetry algebra in four-dimensions

The simplest and most useful (up to now) supersymmetry algebra in physical 3 space + 1 time dimension Minkowski space is a particularly simple example of a Graded Lie Algebra. It is obtained by adding to the Poincaré Algebra a Majorana (self-conjugate) spinor charge Q_α, which acts as a grading representation, as follows [65, 130]:

$$\{Q_\alpha,\bar{Q}_\beta\} = -2\gamma^\mu_{\alpha\beta}P_\mu$$
$$[P_\mu,Q_\alpha] = 0$$
$$[Q_\alpha,M^{\mu\nu}] = \mathrm{i}\,\sigma^{\mu\nu}_{\alpha\beta}Q_\beta \tag{2.4}$$

in which $\sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu,\gamma^\nu]$ and $\bar{Q}_\alpha = Q^{\mathrm{T}}\gamma^0$.

The Q_α are four hermitian operators and we use a Majorana representation for Dirac matrices*. P^μ and $M^{\mu\nu}$ are the usual generators of displacements and homogeneous Lorentz transformations of space–time. We observe that the homogeneous Lorentz generators act as isomorphisms of the GLA generated by the charges Q_α and P_μ, just reflecting their own properties under Lorentz transformations.

Moreover we note that the square mass operator $P^\mu P_\mu$ is a Casimir invariant for the above GLA while the spin operator $W^\mu W_\mu$ (W_μ being the Pauli–Lubansky–Bargman vector) is not.

This fact has the immediate consequence that (linear) representations of the Poincaré Algebra with the same mass but different spin are contained in the same irreducible representation of the GLA defined by eqs. (2.4).

The previous result shows that Graded Lie Algebras play a perhaps unique role in particle physics, because they realize truly relativistic spin-containing symmetries in which particles of different spin belong to the same supermultiplet.

Surprisingly enough the fact that Graded Lie Algebras contain fermionic charges which obey anticommutation relations also ensures that supermultiplets contain only a finite number of states (particles) with different spin and statistics. This fact is extremely important if one wants to construct conventional renormalizable quantum field theories invariant under this symmetry and satisfying the usual Wightman axioms.

Although all main applications of supersymmetry in the physics of particles and fields deal with the supersymmetry algebra we have previously introduced (namely a graded version of the Poincaré Algebra) a larger space–time supersymmetry algebra has been introduced in the original work of Wess and Zumino [130].

In fact the GLA which naturally arises in generalizing dual model supergauge symmetries [91, 102, 60, 137] is a grading of the 15-dimensional conformal algebra of space–time (locally isomorphic to SU(2,2)) where the grading representation is given by a couple of spinor charges Q_α, S_α. These charges are called restricted and special supersymmetry generators, and admit the following commutation relations with the conformal and chiral charges [42, 68]:

$$[Q_\alpha,D] = \tfrac{1}{2}\mathrm{i}Q_\alpha, \qquad [S_\alpha,D] = -\tfrac{1}{2}\mathrm{i}S_\alpha$$
$$[Q_\alpha,M_{\mu\nu}] = \mathrm{i}(\sigma_{\mu\nu}Q)_\alpha, \qquad [S_\alpha,M_{\mu\nu}] = \mathrm{i}(\sigma_{\mu\nu}S)_\alpha$$
$$[Q_\alpha,P_\mu] = 0, \qquad [S_\alpha,K_\mu] = 0$$
$$[Q_\alpha,K_\mu] = -\mathrm{i}(\gamma_\mu S)_\alpha, \qquad [S_\alpha,P_\mu] = \mathrm{i}(\gamma_\mu Q)_\alpha$$
$$[Q_\alpha,\Pi] = -\tfrac{3}{4}\mathrm{i}(\gamma_5 Q)_\alpha, \qquad [S_\alpha,\Pi] = \tfrac{3}{4}\mathrm{i}(\gamma_5 S)_\alpha, \tag{2.5}$$

and the following anticommutation relations among themselves

$$\{Q_\alpha,\bar{Q}_\beta\} = -2\gamma^\mu_{\alpha\beta}P_\mu, \qquad \{S_\alpha,\bar{S}_\beta\} = 2\gamma^\mu_{\alpha\beta}K_\mu$$
$$\{Q_\alpha,\bar{S}_\beta\} = 2(\sigma^{\mu\nu}M_{\mu\nu} - D + 2\gamma_5\Pi)_{\alpha\beta}. \tag{2.6}$$

*For conventions and notations see p. 327.

Note in particular the presence, in the Graded Lie Algebra, of the additional generator Π (the chiral charge), which shows that the Lie Algebra which can be graded is actually U (2,2) = SU (2,2) $\otimes$ U (1). This result agrees with the analysis carried out in ref. [86], which shows that U (4) rather than SU (4) can be graded with the vector representation.

It is in the more general form given by eqs. (2.5) and (2.6) that the space–time supersymmetry algebra in four-dimension was introduced by Wess and Zumino [130].

If one introduces now parameters α's which are totally anticommuting spinors (they can be regarded as odd elements of a Grassmann Algebra)

$$\alpha_i\alpha_j = -\alpha_j\alpha_i, \qquad i,j = 1, \ldots 4, \tag{2.7}$$

the previous anticommutation relations in (2.6) can be written as commutation relations among the charges

$$\delta^0 = \bar{\alpha}^0 Q, \qquad \delta^1 = \bar{\alpha}^1 S \tag{2.8}$$

induced by the odd generators of the algebra.

This fact is of course related to the property that a GLA over complex numbers can actually be converted into an ordinary Lie Algebra over a Grassmann Algebra [62, 98].

For example the commutator of two infinitesimal (restricted) supersymmetry transformations gives

$$[\delta_1,\delta_2] = [\bar{\alpha}_1 Q, \bar{\alpha}_2 Q] = -2\bar{\alpha}_1\gamma^\mu\alpha_2 P_\mu, \tag{2.9}$$

i.e. a four-dimensional translation of (real) parameter $a_\mu = 2\mathrm{i}\bar{\alpha}_1\gamma_\mu\alpha_2$, which is an even element of the Grassmann Algebra.

We observe further that a general infinitesimal supersymmetry transformation is an eight (anti-commuting) parameter transformation because we have to associate two independent spinor parameters α^0, α^1 to the two generators Q_α and S_α. These parameters can be embedded in a linearly x-dependent spinor [130]

$$\alpha(x) = \alpha^0 + \gamma_\mu x^\mu \alpha^1 \tag{2.10}$$

which is the most general solution of the differential equation

$$(\gamma_\mu\partial_\nu + \gamma_\nu\partial_\mu - \tfrac{1}{2} g_{\mu\nu}\gamma\cdot\partial)\,\alpha(x) = 0. \tag{2.11}$$

This ensures that the product of two such transformations is indeed a general conformal transformation of the space–time point x_μ, with parameter

$$\xi_\mu(x) = 2\mathrm{i}\bar{\alpha}_1(x)\,\gamma_\mu\alpha_2(x). \tag{2.12}$$

It satisfies the differential equation

$$\partial_\mu\xi_\nu(x) + \partial_\nu\xi_\mu(x) - \tfrac{1}{2} g_{\mu\nu}\,\partial\cdot\xi(x) = 0. \tag{2.13}$$

We know that the general solution of eqs. (2.13) is

$$\xi_\mu(x) = c_\mu + \omega_{\mu\nu}x^\nu + \epsilon x_\mu + a_\mu x^2 - 2x_\mu a\cdot x, \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}. \tag{2.14}$$

Comparing (2.14) and (2.12) we see that the 15 infinitesimal parameters of the conformal transformation (2.14) are given by the following 15 bilinear expressions in the α's

$$c_\mu = 2\mathrm{i}\bar{\alpha}_1^0\gamma_\mu\alpha_2^0$$
$$a_\mu = 2\mathrm{i}\bar{\alpha}_1^1\gamma_\mu\alpha_2^1$$
$$\epsilon = 2\mathrm{i}(\bar{\alpha}_1^0\alpha_2^1 - \bar{\alpha}_2^0\alpha_1^1)$$
$$\omega_{\mu\nu} = \mathrm{i}(\bar{\alpha}_1^0[\gamma_\mu,\gamma_\nu]\,\alpha_2^1 - \bar{\alpha}_2^0[\gamma_\mu,\gamma_\nu]\,\alpha_1^1). \tag{2.15}$$

The remaining 16th independent bilinear combination in the α's is related to a chiral transformation of parameter

$$\eta = 4\mathrm{i}(\bar{\alpha}_1^1\gamma_5\alpha_2^0 - \bar{\alpha}_2^1\gamma_5\alpha_1^0) \tag{2.16}$$

which, on the other hand, is not involved in the transformation of the space–time point x_μ.

The above algebraic system given by eqs. (2.5), (2.6), can be written [42] in a much more compact form if we embed the two spinor charges Q_α, S_α in an eight-dimensional Majorana spinor χ_a. It transforms according to the Dirac representation of the group SU (2,2) locally isomorphic to O (4,2). The conformal charges can be written as a 6 X 6 skew-symmetric matrix J_{AB}, generating pseudo rotations in six-dimensions [83]. Then we get

$$[\chi_a, J_{AB}] = \mathrm{i}\gamma_{ABa}{}^b\chi_b, \qquad [\chi_a,\Pi] = -\tfrac{3}{4}\mathrm{i}\gamma_{7a}{}^b\chi_b,$$
$$\{\chi_a,\bar{\chi}_b\} = 2(\gamma_{a\,b}^{AB}J_{AB} + 2\gamma_{7ab}\Pi) \tag{2.17}$$

where $\gamma_{AB} = \frac{1}{4}[\gamma_A,\gamma_B]$, $\qquad \gamma_7 = \dfrac{1}{6!}\epsilon^{ABCDEF}\gamma_A\gamma_B\gamma_C\gamma_D\gamma_E\gamma_F$

and the six matrices γ_A satisfy the Clifford algebra in six-dimensions

$$\{\gamma_A,\gamma_B\} = 2g_{AB}, \qquad \begin{array}{l} g_{AA} = (1,1,1,-1,-1,1) \\ g_{AB} = 0 \qquad (A \neq B). \end{array} \tag{2.18}$$

The basic space–time supersymmetry algebra given by the commutation relations in (2.17), although it has been introduced first in the literature as a four-dimensional extension of the supergauge algebra of dual models [75, 53, 116], has not found any substantial application in particle physics. This is due to the fact that this algebra is only compatible with massless systems, because of the presence of scale and conformal transformations. In fact it is well known that symmetries of strictly massless theories are usually spoilt by the renormalization procedure (perturbative renormalization) and are generally plagued with anomalies.

For these reasons and because, at the classical level, this larger algebra is already implied by the restricted algebra given by eqs. (2.4) together with conformal algebra, all physical applications of supersymmetry have been devoted to the Graded Lie Algebra defined by eqs. (2.4).

This algebra is expected to be free of anomalies and to be preserved by the renormalization procedure. We will show in the forthcoming sections that this is indeed the case.

2.3. Supersymmetry with internal symmetries

The introduction of a Majorana spinor charge, as given by eq. (2.4), provides the simplest example of a Graded Lie Algebra in 4-dimensional space–time. Salam and Strathdee [107], Dondi and Sohnius [32] and Wess [129], Zumino [138] have shown that the previous algebra can be generalized in order to incorporate an internal symmetry in a non trivial way, by assuming that the spinor charges belong to some representations of a compact Lie group $\mathcal{G}$.

In this case it is more convenient to rewrite the relevant commutation relations given by eq. (2.4) in terms of two-component Weyl spinors $Q_\alpha, \bar{Q}_{\dot\alpha}$ as follows [48]

$$\begin{aligned} &\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0 \\ &\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu, \qquad [Q_\alpha, P_\mu] = [\bar{Q}_{\dot\alpha}, P_\mu] = 0 \end{aligned} \tag{2.19}$$

in which the dotted (undotted) indices take the values $\alpha, \dot\alpha = 1, 2$ and refer to $(0,\frac{1}{2})$, $(\frac{1}{2},0)$ representations of the spinor group $SL(2,C)$.

Suppose now that we have a set of Q_α's (labelled by an index L) which transform according to some representation of $\mathcal{G}$ and the $\bar{Q}^L_{\dot\beta} = (Q^L_\beta)^*$ which transform according to the complex conjugate representation.

The GLA given by (2.19) generalizes as follows

$$\begin{aligned} &\{Q^L_\alpha, Q^M_\beta\} = \{\bar{Q}^L_{\dot\alpha}, \bar{Q}^M_{\dot\beta}\} = 0 \\ &\{Q^L_\alpha, \bar{Q}^M_{\dot\beta}\} = 2\delta^{LM}\sigma^\mu_{\alpha\dot\beta} P_\mu \\ &[Q^L_\alpha, P_\mu] = [\bar{Q}^L_{\dot\alpha}, P_\mu] = 0 \\ &[Q^L_\alpha, B_l] = \mathrm{i} S^{LM}_l Q^M_\alpha \\ &[B_l, B_m] = \mathrm{i} f_{lmk} B_k \end{aligned} \tag{2.20}$$

where the S^{LM}_l are the hermitian matrices of the representation containing the Q^L_α, and the B_l are the generators of the internal symmetry group $\mathcal{G}$.

As it is already clear from the structure of the commutators, the GLA given by eqs. (2.20) actually gives an example of a genuine relativistic symmetry of the SU(6) type [108] in the sense that irreducible supermultiplets will accommodate states of different spin and different (internal symmetry) quantum numbers as well.

The possibility of achieving relativistic symmetries, in which a complete fusion between space–time and internal symmetry is obtained, seems very attractive; however we must admit, to be honest, that the algebraic structure considered in this section has not yet found any substantial application for two reasons. First the particle contents of the supermultiplets seems unrealistic for any given representation of the internal symmetry group [107, 108]. For instance we could assume [129, 140] that the spinor charge transforms according to the fundamental representation of the SU(3) flavour and SU(3) colour group. Then we would find that in the same supermultiplet there are colour singlet states with the desired quantum numbers but also non-singlet colour states with unwanted quantum numbers, which are degenerate in mass with the colour singlet states in the limit of exact symmetry.

As a second point, one does not know renormalizable field theory models invariant under algebras mixing in a non-trivial way supersymmetry and internal symmetry [15, 51, 145, 140], with the exception of the class of models discussed in ref. [38]; we shall come back to this point in subsection 4.7.

2.4. Particle supermultiplets

The irreducible representations of the supersymmetry algebras introduced in the previous paragraph can be worked out, as shown by Salam and Strathdee [107, 108], by means of the Wigner method of induced representations.

Consider for instance the basic system of commutation and anticommutation relations given by

$$[Q^L_\alpha, P_\mu] = 0, \qquad [Q^L_\alpha, M_{\mu\nu}] = \mathrm{i}(\sigma_{\mu\nu}Q^L)_\alpha$$
$$\{Q^L_\alpha, Q^M_\beta\} = 0, \qquad \{Q^L_\alpha, \bar{Q}^M_{\dot\beta}\} = 2\delta^{LM}\sigma^\mu_{\alpha\dot\beta}P_\mu \tag{2.21}$$

where the Q^L_α's belong to some representation of a compact symmetry group $\mathscr{G}$.

In order to construct unitary representations of the GLA given by (2.21) one starts with the observation that the spinor charges Q's leave invariant the manifold of states with given momentum P_μ. On this manifold the anticommutators of the Q's become fixed numbers and generate a Clifford algebra.

We consider first the simplest case when there is no internal symmetry; so we refer to the algebra given by (2.19). In this case the spinorial charge is a (self-adjoint) Majorana spinor.[1]

For non-vanishing mass we can always assume that the four-momentum has the form $P^\mu = (M, \mathbf{O})$ (one particle state at rest). The little algebra of the GLA is therefore generated by the Q's and by the angular momentum J, the O(3) group being the little group of a time-like vector.

In terms of Weyl two-component spinors the anticommutation relations become in the rest-frame

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0$$
$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = \delta_{\alpha\dot\beta} \tag{2.22}$$

when a suitable normalization factor has been chosen.

As a consequence of (2.22) the spinor charges Q's satisfy the algebra of creation and annihilation operators and can be used to build up, in the usual way, a 4-dimensional Fock space with positive metric.

Indeed one can start with the state of one particle at rest $|M,J,J_3\rangle$ considered as a Clifford vacuum, i.e.

$$\bar{Q}_{\dot\alpha}|M,J,J_3\rangle = 0 \tag{2.23}$$

and construct the new states

$$|M,J,J_3,n_1,n_2\rangle = Q_1^{n_1}Q_2^{n_2}|M,J,J_3\rangle \tag{2.24}$$

in which the couple (n_1, n_2) takes the possible values (0,0), (0,1), (1,0), (1,1). These states span an irreducible $4(2J+1)$ dimensional representation of the little algebra given by eqs. (2.22).

The spin-parity contents of such representation is

$$(J-\tfrac{1}{2})^\eta\, J^{\mathrm{i}\eta} J^{-\mathrm{i}\eta}\, (J+\tfrac{1}{2})^{-\eta} \tag{2.25}$$

where $\eta = \pm\mathrm{i}, \pm 1$ respectively for J integer of half-integer.

This irreducible representation of the supersymmetry algebra, when reduced with respect to the

[1] A parity operation can be defined by assuming that, under space reflections, the spinorial charge Q_α undergoes the following transformation $(Q_\alpha)_P = (\gamma^0 Q)_\alpha$.

Poincaré group, is therefore the direct sum of four irreducible non-equivalent representations

$$(M,J-\tfrac{1}{2})^{\eta} \oplus (M,J)^{i\eta} \oplus (M,J)^{-i\eta} \oplus (M,J+\tfrac{1}{2})^{-\eta}\,. \tag{2.26}$$

We now consider some examples [107, 129, 108]. The representations of lowest dimensionality are those with $J = 0, \frac{1}{2}, 1, \frac{3}{2}$.

Their particle contents is respectively:
a scalar, a pseudoscalar and a spin $\frac{1}{2}$ particle; a pseudoscalar (scalar), a vector (pseudovector) and two spin $\frac{1}{2}$ particles; a vector, a pseudovector, a spin $\frac{1}{2}$ and a spin $\frac{3}{2}$; a pseudovector (vector), two spin $\frac{3}{2}$ and a tensor (pseudotensor).

For zero-mass the classification of particle supermultiplets changes due to the fact that the little algebra of a light-like vector is different. In fact if P_μ is light-like ($P^\mu P_\mu = 0$) we can choose P_μ of the form (1,0,0,1). The little algebra now becomes

$$\{Q_1,\bar{Q}_1\} = 1, \qquad \{Q_2,\bar{Q}_2\} = 0, \qquad \{Q_1,\bar{Q}_2\} = \{Q_2,\bar{Q}_1\} = 0. \tag{2.27}$$

The (positive norm) physical states are now $|\lambda\rangle, Q_1|\lambda\rangle$ while $Q_2|\lambda\rangle, Q_1Q_2|\lambda\rangle$ are zero-norm states. The Clifford vacuum $|\lambda\rangle$ is a representation of the little group E_2 of a light-like vector, i.e. a massless state of given helicity λ. The representation of the little algebra contains four states of helicities $\pm\lambda$, $\pm(\lambda+\frac{1}{2})$, if parity is included.

To give some examples let us consider the massless multiplets corresponding to $\lambda = \frac{1}{2}$ and $\lambda = \frac{3}{2}$.

They describe massless particles of helicities given by $(\pm\frac{1}{2}, \pm 1)$, $(\pm\frac{3}{2}, \pm 2)$ respectively, so they might be regarded as supermultiplets containing the photon and the graviton. These two particle supermultiplets will play an important role in the construction of supersymmetric theories with gauge fields.

We now consider the algebra mixing in a non-trivial way supersymmetry with an internal symmetry group $\mathscr{G}$, and we study its rest-frame representations.

The rest-frame anticommutation relations are given by the following expressions

$$\{Q^L_\alpha, Q^M_\beta\} = \{\bar{Q}^L_{\dot\alpha}, \bar{Q}^M_{\dot\beta}\} = 0$$

$$\{Q^L_\alpha, Q^M_{\dot\beta}\} = \delta^{LM}\delta_{\alpha\dot\beta}\,. \tag{2.28}$$

If the Q^L_α ($L = 1, \ldots n$) belong to a real n-dimensional representation of the internal symmetry group $\mathscr{G}$, they will give 2^{2n} independent states when repeatedly applied to a one-particle state $|\Omega\rangle$ which behaves as a Clifford ground state:

$$\bar{Q}^L_{\dot\alpha}|\Omega\rangle = 0. \tag{2.29}$$

When the representation is complex, if one wants to define charge-conjugation and parity in a consistent way, one must double the spinor charges, i.e. consider Dirac spinors rather than Majorana spinors. These charges, applied to the ground state, will erect 2^{4n} independent particle states.

In order to specify the representation of the little algebra we have only to give the quantum numbers of the ground state, i.e. the spin, mass and the representation of the internal symmetry group to which this state belongs.

The lowest dimensionality representation is obtained by demanding that the ground state be a singlet. Its dimension is 2^{2n} (2^{4n}) for a real (complex) n-dimensional representation of $\mathscr{G}$.

All other unitary representations are obtained by composition of this lowest representation with the representation of the ground state. Their dimensionality is given by the expression:

$$D = 2^{2n} \times (2J + 1) \times d$$

where J and d are respectively the spin and the dimension of the representation defining the ground state.

To be more explicit we shall consider some examples. First we take the Q^L_α in the fundamental representation of SU(2); then we have $n = 2$ and the spinor charges give 16 states, when applied to a Clifford vacuum $|J,J_3;I,I_3\rangle$ defined by the equation

$$\bar{Q}^L_{\dot{\alpha}}|J,J_3;I,I_3\rangle = 0. \tag{2.30}$$

We consider the lowest representation, i.e. $J = I = 0$. The spin, isospin, parity contents of the states of the little algebra will be

$$(0,0)^+ \oplus (\tfrac{1}{2},\tfrac{1}{2})^{\mathrm{i}} \oplus (1,0)^- \oplus (0,1)^- \oplus (\tfrac{1}{2},\tfrac{1}{2})^{-\mathrm{i}} \oplus (0,0)^+. \tag{2.31}$$

As explained before, all other irreducible representations are obtained by composition of this representation with an SU(2) ⊗ SU(2) multiplet (J,I) which behaves like a Clifford ground state.

It will be noticed that the rest frame states given by (2.31) can be classified in SU (4) multiplets, namely

$$16 = 1 \oplus 4 \oplus 6 \oplus \bar{4} \oplus 1. \tag{2.32}$$

The SU(4) generators are explicitly given by the following bilinear expressions

$$\tfrac{1}{2}\bar{Q}^L_{\dot{\alpha}}\sigma^{\dot{\alpha}\beta}Q^L_\beta, \quad \tfrac{1}{2}\bar{Q}^L_{\dot{\alpha}}\delta^{\dot{\alpha}\beta}\tau^i_{LM}Q^M_\beta, \quad \tfrac{1}{2}\bar{Q}^L_{\dot{\alpha}}\sigma^{\dot{\alpha}\beta}\tau^i_{LM}Q^M_\beta \tag{2.33}$$

in which $\boldsymbol{\sigma},\tau^i$ are the spin–isospin Pauli matrices.

This result can be generalized as follows [107, 108] : if we start with $n\,(2n)$ Weyl charges Q^L_α ($L = 1, \ldots n$) the resulting 2^{2n} (2^{4n}) states can be uniquely decomposed into a direct sum of irreducible representations of the SU($2n$) (SU($4n$)) algebra. The spin of these states runs from 0 up to $\tfrac{1}{2}n\,(n)$.

As another example let us take the Q^L_α transforming according to the vector representation of O(3). In this case $2^{2n} = 64$ and the resulting states can be classified according to SU (6) representations

$$64 = 1 \oplus 6 \oplus 15 \oplus 20 \oplus \overline{15} \oplus \bar{6} \oplus 1. \tag{2.34}$$

The spin–isospin–parity contents $(J,I)^P$ of these states is

$$\begin{aligned}&(0,0)^+ \oplus (\tfrac{1}{2},1)^{\mathrm{i}} \oplus (0,0)^- \oplus (1,1)^- \oplus (0,2)^- \oplus (\tfrac{3}{2},0)^{-\mathrm{i}} \oplus \\ &\quad \oplus (\tfrac{1}{2},1)^{-\mathrm{i}} \oplus (\tfrac{1}{2},2)^{-\mathrm{i}} \oplus (0,0)^+ \oplus (1,1)^+ \oplus (0,2)^+ \oplus (\tfrac{1}{2},1)^{\mathrm{i}} \oplus (0,0)^-.\end{aligned} \tag{2.35}$$

As a final example we consider the case in which the charges are in the quark representation, i.e. they are SU(3) triplets. In this case the lowest representation is of dimension $2^{12} = 4096$ and the particle states can be classified according to SU(12) as follows

$$4096 = 1 \oplus 12 \oplus 66 \oplus 220 \oplus 495 \oplus 792 \oplus 924 \oplus \overline{792} \oplus \overline{495} \oplus \overline{220} \oplus \overline{66} \oplus \overline{12} \oplus 1. \tag{2.36}$$

The maximum spin in the above representation is $J = 3$ (SU(12) singlet).

2.5. All possible symmetries of the S-matrix

We have seen in the previous sections that the original supersymmetry algebra given by eqs. (2.4) can be generalized to include, in a non-trivial way, an internal symmetry. This provides a first example for a genuine relativistic symmetry where particle states of different spin and isospin-like quantum numbers lie in the same irreducible multiplet: We have also seen that no restrictions on the internal symmetry group and its representations are required to get a consistent supersymmetry algebra.

This last fact is due to the non-semisimple structure of the bosonic part of the space–time GLA's.

On the other hand it is well known that, before the introduction of supersymmetry, existing no-go theorems [19] destroyed the hope of having a fusion between internal symmetries and space–time symmetries.

Of course the apparent contradiction was overcome by noticing that supersymmetries require GLA's rather than ordinary Lie Algebras; indeed such a structure was one of the basic assumptions for the proof of no-go theorems. In fact previous authors never contemplated the possibility of enlarging the Poincaré algebra with spinor charges obeying anticommutation relations.

In this respect a remarkable theorem has been derived by Haag, Łopuszański and Sohnius [68]; it can be regarded as the natural extension of the Coleman and Mandula [19] theorem to the case of GLA's. This theorem states that the maximal symmetry of the S matrix is the direct product of an internal symmetry with the supersymmetry algebra given by eqs. (2.20); the only allowed extension is the possible appearance of central charges [67] in the anticommutator of two undotted (dotted) spinors. To be more precise, let Q^L_α be a set of Weyl spinors transforming according to some representation of a compact Lie group $\mathcal{G}$. Then the only possible modification of eqs. (2.20) is a non-vanishing anticommutator among the Q_α's, i.e.

$$\{Q^L_\alpha, Q^M_\beta\} = \epsilon_{\alpha\beta} Z^{LM} \tag{2.37}$$

where $[Z^{LM}, G] = 0$ for any element G in the GLA.

We remark, *en passant*, that the generator in the anticommutation relation modifies the particle contents of the supermultiplets described in the previous section just because it changes the structure of the rest-frame supersymmetry algebra.

The proof of the above results is based on the hypothesis that a generator of a symmetry (Lie Algebra) or of a supersymmetry (Graded Lie Algebra) obeys some simple but fundamental properties:

(1) It commutes with the S operator.

(2) It acts additively on the states of several incoming particles.

(3) It connects different particle types which have the same mass.

In addition to these properties one must further require the absence of massless particles to avoid the usual infrared difficulties (long range forces).

The justification of requirements (1), (2), (3) can be given in several ways. Note for instance that in the framework of local quantum field theory these properties are a consequence of the more fundamental requirement that the infinitesimal change, under the action of a symmetry generator, of a local field is again local.

Finally we would like to mention a further remarkable algebraic result found by these authors.

As far as the zero-mass situation is concerned, a much more stringent result can be obtained, disregarding infrared problems and symmetry breaking difficulties. Namely the only possible extension of the conformal algebra to include an internal symmetry is to have Weyl spinor charges which transform according to the vector representation of the unitary group $U(N)$ (for $N = 4$ one can equally have $U(4)$ or $SU(4)$) where the $U(1)$ part is just the usual chiral generator. The only remaining freedom is just the number N of spinorial charges. The construction of the algebra using the six-dimensional-formalism has been discussed in ref. [144].

3. Superfields and Lagrangian models

3.1. Field representations and superfields

In order to apply the concept of supersymmetry in local quantum field theory, representations of this symmetry on field operators must be obtained.

The first non-trivial linear representation of the supersymmetry algebra was found by Wess and Zumino [130] in the following form:

$$\begin{aligned}
\delta A(x) &= \mathrm{i}\bar{\alpha}\psi(x)\\
\delta B(x) &= \mathrm{i}\bar{\alpha}\gamma_5\psi(x)\\
\delta\psi(x) &= \partial_\mu(A(x)-\gamma_5 B(x))\gamma^\mu\alpha + (F(x)+\gamma_5 G(x))\alpha\\
\delta F(x) &= \mathrm{i}\bar{\alpha}\gamma^\mu\partial_\mu\psi(x)\\
\delta G(x) &= \mathrm{i}\bar{\alpha}\gamma_5\gamma^\mu\partial_\mu\psi(x)
\end{aligned}\tag{3.1}$$

where $A(x)$, $F(x)$ are scalars, $B(x)$ and $G(x)$ pseudo-scalars and $\psi(x)$ is a Majorana spinro;

$\delta A(x)$ stands for $[\bar{\alpha}Q, A(x)]$ etc.

The commutator of two such variations acts on any field of the multiplet as a four-dimensional translation.

For instance

$$(\delta_1\delta_2-\delta_2\delta_1)A(x) = [\delta_1,\delta_2]\,A(x) = -2\mathrm{i}\bar{\alpha}_1\gamma^\mu\alpha_2\partial_\mu A(x) \text{ etc.} \tag{3.2}$$

We remark the important point that only a finite number of local fields are needed to close such a representation. We observe further that, on the mass shell, the following relations hold

$$(\Box - m^2)A(x) = (\Box - m^2)B(x) = (\gamma\cdot\partial + m)\psi(x) = 0 \tag{3.3}$$

which imply that

$$\delta(mA(x)+F(x)) = 0,\qquad \delta(mB(x)+G(x)) = 0 \tag{3.4}$$

and therefore one can take $F(x) = -mA(x)$, $G(x) = -mB(x)$. This shows that the field multiplet in (3.1) describes the off-shell particle multiplet which corresponds to $J = 0$ according to the classification of supermultiplets given in subsection 2.4.

A non-linear representation of the same algebra has been obtained previously by Volkov and Akulov [124] in terms of a single Majorana spinor ψ, as follows

$$\delta\psi = \frac{1}{a}\alpha + \mathrm{i}a(\bar{\alpha}\gamma^{\mu}\psi)\,\partial_{\mu}\psi \tag{3.5}$$

in which a is a (universal) constant.

These authors also gave an invariant (non-linear) action describing the self-interaction of the spinor ψ and suggested to consider it as a theory for the neutrino.

Their interaction was not renormalizable.

A different non-linear realization has been obtained by Zumino [138] in the following form:

$$\delta\psi = \frac{1}{a}\alpha + \mathrm{i}a(\bar{\alpha}\gamma^{\mu}\psi)\partial_{\mu}\psi + \mathrm{i}a(\bar{\alpha}\gamma_5\gamma^{\mu}\psi)\gamma_5\partial_{\mu}\psi. \tag{3.6}$$

Although non-linear realizations of supersymmetry could be interesting in principle, as for ordinary chiral theories, we will not consider them further, and we will mainly deal with the general theory of linear representations and of their field realizations.

The multiplet defined through the transformations (3.1) contains only spin 0 and $\frac{1}{2}$ fields. It was called originally the scalar multiplet. From now on it will be called chiral multiplet according to the Salam–Strathdee [114] nomenclature. This definition, as it will be explained later, is connected with the superfield realization of this multiplet.

A second multiplet, which contains a vector field, was also introduced by the authors of ref. [130]. Under an infinitesimal supersymmetry transformation the variation of its fields is given by

$$\begin{aligned}
\delta C(x) &= \mathrm{i}\bar{\alpha}\gamma_5\chi(x)\\
\delta\chi(x) &= \gamma^{\mu}\mathcal{V}_{\mu}(x)\alpha - \partial_{\mu}C(x)\gamma_5\gamma^{\mu}\alpha + (M(x) + \gamma_5 N(x))\alpha\\
\delta M(x) &= \mathrm{i}\bar{\alpha}\lambda(x) + \mathrm{i}\bar{\alpha}\gamma^{\mu}\partial_{\mu}\chi(x)\\
\delta N(x) &= \mathrm{i}\bar{\alpha}\gamma_5\lambda(x) + \mathrm{i}\bar{\alpha}\gamma_5\gamma^{\mu}\partial_{\mu}\chi(x)\\
\delta\mathcal{V}_{\mu}(x) &= \mathrm{i}\bar{\alpha}\gamma_{\mu}\lambda(x) + \mathrm{i}\bar{\alpha}\partial_{\mu}\chi(x)\\
\delta\lambda(x) &= -\tfrac{1}{2}(\partial_{\mu}\mathcal{V}_{\nu}(x) - \partial_{\nu}\mathcal{V}_{\mu}(x))\gamma^{\mu}\gamma^{\nu}\alpha + \gamma_5 D(x)\alpha\\
\delta D(x) &= \mathrm{i}\bar{\alpha}\gamma_5\gamma^{\mu}\partial_{\mu}\lambda(x).
\end{aligned} \tag{3.7}$$

This multiplet consists of three pseudoscalars D,C,N, a scalar M, a vector $\mathcal{V}_{\mu}$ and two Majorana spinors χ and λ.

It has been called the vector multiplet because it is the smallest multiplet of fields which includes a vector field.

The multiplets defined by the transformation laws (3.1) and (3.7) have been obtained first by looking at the minimal set of local fields needed to realize the supersymmetry algebra.

Salam and Strathdee [106] have been able, afterwards, to find the key for developing the tensor calculus of supersymmetric field theories, by introducing the concept of superfield.

This technique proved very powerful to treat supersymmetric theories in a simple way, and later it allowed to give a precise geometrical meaning of supersymmetry transformations by the introduction of a superspace whose points are labelled by Minkowski and anticommuting coordinates (Grassmann coordinates). We will describe the geometrical picture of supersymmetry in some detail in section 6.

Following Salam and Strathdee [106], we consider the group action on the space of left cosets with respect to the subgroup of homogeneous Lorentz transformations. This space is parametrized in terms of a space–time coordinate x_μ and of an anticommuting spinor θ_α.

To obtain the group-action over this space we define the unitary operator

$$L(x,\theta) = \exp(-\mathrm{i}x\cdot P)\exp(-\bar\theta Q) \tag{3.8}$$

and consider what happens when any operator representing a transformation of the Graded Lie Group (obtained by exponentiation of the Graded Lie Algebra given by eqs. (2.4)) acts on its left.

We find

$$\exp(-\bar\alpha Q)\, L(x,\theta) = L(x + \mathrm{i}\bar\alpha\gamma\theta, \theta + \alpha) \tag{3.9}$$

$$\exp(-\mathrm{i}c\cdot P)\, L(x,\theta) = L(x + c, \theta) \tag{3.10}$$

$$\exp(-\tfrac{1}{2}\mathrm{i}\omega_{\mu\nu}J^{\mu\nu})\, L(x,\theta) = L(\Lambda x, a(\Lambda)\theta)\exp(-\tfrac{1}{2}\mathrm{i}\omega_{\mu\nu}J^{\mu\nu}) \tag{3.11}$$

respectively for supersymmetry, translations and homogeneous Lorentz transformations.

Here $a(\Lambda) = \exp(-\frac{1}{2}\mathrm{i}\omega_{\mu\nu}\sigma^{\mu\nu})$ denotes the usual Lorentz transformation on a 4-component spinor.

Eqs. (3.9), (3.10), (3.11) define the action of the supersymmetry group on functions defined on the parameter-space (x_μ, θ_α) (superspace) and indicate how a field operator over this space should transform.

Thus, for example, for a Lorentz invariant superfield we have

$$\phi(x,\theta) = L(x,\theta)\,\phi L^{-1}(x,\theta) \tag{3.12}$$

and one gets the following supersymmetry transformation property

$$\exp(-\bar\alpha Q)\,\phi(x,\theta)\exp(\bar\alpha Q) = \phi(x + \mathrm{i}\bar\alpha\gamma\theta, \theta + \alpha). \tag{3.13}$$

The truly remarkable property of the previously defined superfield is that, because of the anticommuting properties of the spinor parameters θ_α, it is equivalent, as it can be seen by Taylor-expanding in the θ_α variable, to a 16-component set of ordinary local fields defined by Minkowski space. This is due to the fact that the monomials $\theta_{\alpha_1}\ldots\theta_{\alpha_n}$ must be completely antisymmetric in their indices and therefore vanish for $n > 4$.

As a consequence we have

$$\begin{aligned}\phi(x,\theta) = {}& \phi(x) + \phi^\alpha(x)\,\theta_\alpha + \tfrac{1}{2}\phi^{[\alpha\beta]}(x)\,\theta_\alpha\theta_\beta \\ & + \tfrac{1}{6}\phi^{[\alpha\beta\gamma]}(x)\,\theta_\alpha\theta_\beta\theta_\gamma + \tfrac{1}{24}\phi^{[\alpha\beta\gamma\delta]}\theta_\alpha\theta_\beta\theta_\gamma\theta_\delta\end{aligned} \tag{3.14}$$

so that a superfield is a short way to denote a finite multiplet of fields

$$\phi(x,\theta)\colon (\phi(x), \phi^\alpha(x), \phi^{[\alpha\beta]}(x), \phi^{[\alpha\beta\gamma]}(x), \phi^{[\alpha\beta\gamma\delta]}(x)). \tag{3.15}$$

In quantum field theory it is sometimes convenient to write the action of the algebra on a quantum field-operator, instead of the action of the integrated group.

This action can be read by inspection of the previously defined finite transformations; so one has

$$[\phi(x,\theta), Q_\alpha] = \left(-\frac{\partial}{\partial\bar\theta} + \mathrm{i}\gamma^\mu\theta\partial_\mu\right)\phi(x,\theta) \tag{3.16}$$

$$[\phi(x,\theta), P_\mu] = -\mathrm{i}\,\partial_\mu\phi(x,\theta) \tag{3.17}$$

$$[\phi(x,\theta), J_{\mu\nu}] = -\mathrm{i}\left(x_\mu\partial_\nu - x_\nu\partial_\mu + \bar\theta\sigma_{\mu\nu}\frac{\partial}{\partial\bar\theta}\right)\phi(x,\theta) \tag{3.18}$$

respectively for supersymmetry, translations and Lorentz transformations.

Following Ferrara, Wess and Zumino [48] we find convenient to use Weyl spinors θ_α, $\theta_{\dot\alpha}$ related to the (4-component) Majorana spinor θ_α by the relations

$$\theta_\alpha = \tfrac{1}{2}(1 - i\gamma_5)\theta, \qquad \bar\theta_{\dot\alpha} = (\theta_\alpha)^* = \tfrac{1}{2}(1 + i\gamma_5)\theta. \tag{3.19}$$

It follows that the superfield defined by eqs. (3.12) can be rewritten as

$$\phi(x,\theta_\alpha,\bar\theta_{\dot\alpha}) = L(x,\theta_\alpha,\bar\theta_{\dot\alpha})\,\phi L^{-1}(x,\theta_\alpha,\bar\theta_{\dot\alpha}) \tag{3.20}$$

where

$$L(x,\theta_\alpha,\bar\theta_{\dot\alpha}) = \exp(-ix\cdot P + i\theta Q + i\bar\theta\bar Q);$$

in terms of component fields the superfield (3.20) admits the general expansion

$$\begin{aligned}\phi(x,\theta,\bar\theta) = {} & C + i\theta^\alpha\chi_\alpha - i\bar\theta_{\dot\alpha}\bar\chi^{\dot\alpha} + \theta^\alpha\theta_\alpha\tfrac{1}{2}i(M + iN) \\ & - \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}\tfrac{1}{2}i(M - iN) - \theta\sigma_\mu\bar\theta\mathscr{V}^\mu + i\theta^\alpha\theta_\alpha\bar\theta_{\dot\beta}(\bar\lambda - \tfrac{1}{2}i\partial_\mu\chi\sigma^\mu)^{\dot\beta} \\ & - i\bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}\theta^\beta(\lambda + \tfrac{1}{2}i\sigma^\mu\partial_\mu\bar\chi)_\beta + \theta^\alpha\theta_\alpha\bar\theta_{\dot\beta}\bar\theta^{\dot\beta}\tfrac{1}{2}(D + \tfrac{1}{2}\Box C)\end{aligned} \tag{3.21}$$

where, without loss of generality, ϕ can be assumed to be real. The set of (real) fields defined by eqs. (3.21) is exactly the vector multiplet, the (real) component fields of which transform according to eqs. (3.7). Because of the transformation property (3.16) one easily realizes that the component fields in (3.21) transform into

$$\begin{aligned} & C \to \chi \\ & \chi \to \partial C, M, N, \mathscr{V}_\mu \\ & M, N \to \partial\chi, \lambda \\ & \lambda \to \partial_\mu\mathscr{V}_\nu, D \\ & D \to \partial\lambda \end{aligned} \tag{3.22}$$

under the supersymmetry transformation.

In order to develop further the tensor calculus of supersymmetry we observe that the group manifold can be parametrized in three different but equivalent ways, by means of the following unitary operators:

$$\begin{aligned} L(x,\theta,\bar\theta) &= \exp(-ix\cdot P + i\theta Q + i\bar\theta\bar Q) \\ L_1(x,\theta,\bar\theta) &= \exp(-ix\cdot P + i\theta Q)\exp(i\bar\theta\bar Q) \\ L_2(x,\theta,\bar\theta) &= \exp(-ix\cdot P + i\bar\theta\bar Q)\exp(i\theta Q). \end{aligned} \tag{3.23}$$

The three expressions in (3.23) correspond to three different (but equivalent) realizations of supersymmetry transformations, i.e. to three different definitions of the superfield, related by the following identities

$$\phi(x,\theta,\bar\theta) = \phi_1(x + i\theta\sigma\bar\theta,\theta,\bar\theta) = \phi_2(x - i\theta\sigma\bar\theta,\theta,\bar\theta). \tag{3.24}$$

Note that, because the shift on the space–time coordinate is purely imaginary, the fields $\phi_1(x,\theta,\bar\theta)$ and $\phi_2(x,\theta,\bar\theta)$ transform as complex representations while the field $\phi(x,\theta,\bar\theta)$ is real as already implied by eqs. (3.13).

Under a finite transformation the superfields undergo the following transformation

$$\begin{aligned}
T_\alpha\phi(x,\theta,\bar\theta) &= \phi(x + \mathrm{i}\theta\sigma\bar\alpha - \mathrm{i}\alpha\sigma\bar\theta, \theta + \alpha, \bar\theta + \bar\alpha)\\
T_\alpha\phi_1(x,\theta,\bar\theta) &= \phi(x + 2\mathrm{i}\theta\sigma\bar\alpha + \mathrm{i}\alpha\sigma\bar\alpha, \theta + \alpha, \bar\theta + \bar\alpha)\\
T_\alpha\phi_2(x,\theta,\bar\theta) &= \phi(x - 2\mathrm{i}\alpha\sigma\bar\theta - \mathrm{i}\alpha\sigma\bar\alpha, \theta + \alpha, \bar\theta + \bar\alpha)
\end{aligned}\tag{3.25}$$

or in infinitesimal form:

$$\begin{aligned}
\delta\phi &= \left(\alpha\frac{\partial}{\partial\theta} + \bar\alpha\frac{\partial}{\partial\bar\theta} + \mathrm{i}(\theta\sigma_\mu\bar\alpha - \alpha\sigma_\mu\bar\theta)\frac{\partial}{\partial x_\mu}\right)\phi\\
\delta\phi_1 &= \left(\alpha\frac{\partial}{\partial\theta} + \bar\alpha\frac{\partial}{\partial\bar\theta} + 2\mathrm{i}\theta\sigma_\mu\bar\alpha\frac{\partial}{\partial x_\mu}\right)\phi_1\\
\delta\phi_2 &= \left(\alpha\frac{\partial}{\partial\theta} + \bar\alpha\frac{\partial}{\partial\bar\theta} - 2\mathrm{i}\alpha\sigma_\mu\bar\theta\frac{\partial}{\partial x_\mu}\right)\phi_2.
\end{aligned}\tag{3.26}$$

From (3.25), (3.26) it follows immediately that $\partial/\partial\theta$, $\partial/\partial\bar\theta$ are covariant differentiations for type 1 and type 2 superfields respectively. Therefore for any superfield ϕ (for example in the real basis) there exist two types of covariant derivatives [48, 114] namely

$$\mathrm{D}_\alpha = \frac{\partial}{\partial\theta} + \mathrm{i}\sigma_\mu\bar\theta\partial^\mu, \qquad \bar{\mathrm{D}}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta} - \mathrm{i}\theta\sigma_\mu\partial^\mu. \tag{3.27}$$

For type 1 or 2 fields they obviously become

$$\mathrm{D}_\alpha = \frac{\partial}{\partial\theta}, \qquad \bar{\mathrm{D}}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta} - 2\mathrm{i}\theta\sigma_\mu\partial^\mu \tag{3.28}$$

$$\mathrm{D}_\alpha = \frac{\partial}{\partial\theta} + 2\mathrm{i}\sigma_\mu\bar\theta\partial^\mu, \qquad \bar{\mathrm{D}}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta}. \tag{3.29}$$

We shall call them covariant derivatives (they transform as spinors with respect to Lorentz transformations and are invariant with respect to supersymmetry transformations and translations).

The covariant derivatives obey the following algebra

$$\{\mathrm{D}_\alpha, \bar{\mathrm{D}}_{\dot\alpha}\} = -2\mathrm{i}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu, \qquad \{\mathrm{D}_\alpha, \mathrm{D}_\beta\} = \{\bar{\mathrm{D}}_{\dot\alpha}, \bar{\mathrm{D}}_{\dot\beta}\} = 0 \tag{3.30}$$

and satisfy the obvious relations

$$\mathrm{D}_\alpha\mathrm{D}_\beta\mathrm{D}_\gamma = \bar{\mathrm{D}}_{\dot\alpha}\bar{\mathrm{D}}_{\dot\beta}\bar{\mathrm{D}}_{\dot\gamma} = 0. \tag{3.31}$$

Because of the existence of the operations given by eqs. (3.27, 28, 29) a (complex) superfield ϕ can be reduced by imposing the constraint

$$\mathrm{D}_\alpha\phi = 0 \qquad \text{or} \qquad \bar{\mathrm{D}}_{\dot\alpha}\phi = 0. \tag{3.32}$$

A superfield satisfying one of the constraints (3.32) $\bar{\mathrm{D}}_{\dot\alpha}\phi = 0$ ($\mathrm{D}_\alpha\phi = 0$) is called chiral, left-handed ($\bar{\mathrm{D}}_{\dot\alpha}\phi = 0$) or right-handed ($\mathrm{D}_\alpha\phi = 0$) respectively.

The nomenclature, due to Salam and Strathdee [114], arises from the fact that, for a chiral (left-handed) field one has

$$\phi(x,\theta,\bar\theta) = \phi_1(x + \mathrm{i}\theta\sigma\bar\theta, \theta) \tag{3.33}$$

with

$$\phi_1(x,\theta) = \tfrac{1}{2}(A - \mathrm{i}B) + \theta^\alpha\psi_\alpha + \theta^\alpha\theta_\alpha\tfrac{1}{2}(F + \mathrm{i}G). \tag{3.34}$$

Therefore ϕ_1 depends only on the chiral (left-handed) two-component spinor θ_α.

The expansion of the superfield (3.34) involves two complex scalars (two real scalars and two real pseudoscalars) and a complex Weyl spinor (real four-component Majorana spinor). These fields are exactly the components of the scalar multiplet of Wess and Zumino [130], with transformation laws defined by eqs. (3.1).

This multiplet is the lowest (non-trivial) representation of the supersymmetry algebra on field operators.

Strictly speaking there are (Lorentz invariant) superfields which are neither chiral nor vector [103], i.e. superfields which satisfy the relation

$$\mathrm{D}^\alpha\mathrm{D}_\alpha\phi = 0 \qquad \text{or} \qquad \bar{\mathrm{D}}_{\dot\alpha}\bar{\mathrm{D}}^{\dot\alpha}\phi = 0. \tag{3.35}$$

They were called linear superfields in ref. [48]. However, as they have not been used in any interesting application, we will omit them from our following discussion.

We have so far confined our considerations to superfields without additional (external) Lorentz index. Such an index is completely irrelevant as far as supersymmetry transformations are concerned. It can be added easily [94], without modifications of the previously established results [104, 92, 120]. The only transformation affected by this generalization is the homogeneous Lorentz transformation given by eqs. (3.18), which now reads:

$$[\phi_{\{\alpha\}}(x,\theta), J_{\mu\nu}] = -\mathrm{i}\left[\left(x_\mu\partial_\nu - x_\nu\partial_\mu + \bar\theta\sigma_{\mu\nu}\frac{\partial}{\partial\theta}\right)\delta^{\{\beta\}}_{\{\alpha\}} + \Sigma^{\{\beta\}}_{\{\alpha\}}\right]\phi_{\{\beta\}}(x,\theta). \tag{3.36}$$

The indices $\{\alpha\}$ denote the spinor character of the field ϕ and $\Sigma_{\mu\nu}$ is the matrix (generator) of the corresponding representation of the Lorentz group $SL(2,C)$.

The superfield formalism is mostly convenient for working out tensor products of supersymmetry representations. In fact one has simply to deal with ordinary multiplication for different superfields which is a well-defined operation. Such a multiplication will be an essential ingredient to build up supersymmetric field theories, as well as to derive Feynman graph techniques in superspace. If we label a chiral left-handed (right-handed) superfield with a subscript, we easily verify the following properties

$$\begin{aligned} &\phi_\mathrm{L}\psi_\mathrm{L} = (\phi\psi)_\mathrm{L}, \qquad \phi_\mathrm{R}\psi_\mathrm{R} = (\phi\psi)_\mathrm{R} \\ &(\phi_\mathrm{L}\psi_\mathrm{R} \pm \phi_\mathrm{R}\psi_\mathrm{L}) = (\phi\psi)_{\pm V}, \qquad \phi_V\psi_V = (\phi\psi)_V \end{aligned} \tag{3.37}$$

in which $\phi_\mathrm{R} = \bar\phi_\mathrm{L}$, $\psi_\mathrm{L} = \bar\psi_\mathrm{R}$ and the subscript V is relative to a vector superfield.

Moreover, we observe the chiral character of the following operation

$$DD\phi = (DD\phi)_\mathrm{R} \tag{3.38}$$

for any superfield ϕ. This follows from the algebra of covariant derivatives (eqs. (3.31)). In particular, if ϕ_L has (complex) components $\tfrac{1}{2}(A - \mathrm{i}B)$, ψ_α, $\tfrac{1}{2}(F + \mathrm{i}G)$, given by eqs. (3.34), the new chiral field

$$\phi'_\mathrm{L} = \tfrac{1}{4}\bar{\mathrm{D}}_{\dot\alpha}\bar{\mathrm{D}}^{\dot\alpha}\bar\phi_\mathrm{L} \tag{3.39}$$

has components respectively given by $\tfrac{1}{2}(F - \mathrm{i}G)$, $-\mathrm{i}\sigma\cdot\partial\bar\psi$, $\tfrac{1}{2}\Box(A + \mathrm{i}B)$. In terms of real fields this means that the multiplet $F, G, \gamma\cdot\partial\psi$, $\Box A$, $\Box B$ behaves exactly like the multiplet A, B, ψ, F, G (see eqs. (3.1)) under supersymmetry transformations.

3.2. The supersymmetric spinor–scalar interaction (super-S^3 theory)

In the present section we will discuss how to construct supersymmetric Lagrangian field theories and will describe in some detail the simplest of them, namely the self-interaction of a chiral multiplet.

Let us consider the transformation law of a general superfield ϕ under the action of the supersymmetry generators, given by eqs. (3.16). The variation of the last component field (i.e. the field which, in the superfield expansion, corresponds to the monomial of highest power in the anticommuting variables θ_α's) is always a total derivative:

$$\delta\phi_{\text{LAST}} = \mathrm{i}\,\not{\partial}\,\chi \tag{3.40}$$

in which χ is some Fermi field. It follows that the quantity

$$\int \mathrm{d}^4x\ \phi_{\text{LAST}} \tag{3.41}$$

is invariant under the transformation (3.40). The fundamental property (3.41) gives us the criterion for constructing supersymmetric Lagrangian densities. Namely in order to construct a supersymmetric coupling of several supermultiplets one simply considers the resulting superfield obtained by multiplication (with possible insertion of covariant derivatives) and takes its last component field.

The first supersymmetric field theory model which has been studied [131] is the self-interaction of a chiral multiplet, namely the interaction of the real fields A, B, ψ, F, G the transformation properties of which are given by eqs. (3.1).

In order to derive the corresponding Lagrangian density we will use the superfield techniques described in the previous subsection extensively. In particular we will show that the kinetic term, the mass term and the interaction terms are separately supersymmetric.

The kinetic and mass terms are given by the only (real) bilinear products which can be built from the chiral superfield $S(\bar{\mathrm{D}}_{\dot\alpha}S = 0)$.

They are respectively

$$S\bar{S} \qquad \text{and} \qquad S^2. \tag{3.42}$$

Kinetic term: the supersymmetric (kinetic) Lagrangian density is given by the D component of the real vector superfield $S\bar{S}$:

$$[S\bar{S}]_D = -\tfrac{1}{2}(\partial_\mu A)^2 - \tfrac{1}{2}(\partial_\mu B)^2 - \tfrac{1}{2}\mathrm{i}\bar{\psi}\not{\partial}\psi + \tfrac{1}{2}F^2 + \tfrac{1}{2}G^2 \tag{3.43}$$

which is equal, up to a four-divergence, to the F component of the chiral bilinear $S\bar{D}\bar{D}\bar{S}$.

Mass term: it corresponds to the F component of the chiral multiplet S^2

$$[mS^2]_F = m(FA + GB - \tfrac{1}{2}\mathrm{i}\bar{\psi}\psi). \tag{3.44}$$

(Cubic) Interaction term: it corresponds to the F component of the chiral multiplet S^3

$$[\tfrac{4}{3}gS^3]_F = g(FA^2 - FB^2 + 2GAB - \mathrm{i}\bar{\psi}\psi A + \mathrm{i}\bar{\psi}\gamma_5\psi B). \tag{3.45}$$

Therefore the total Lagrangian density is

$$\mathscr{L} = \mathscr{L}_{\mathrm{k}} + \mathscr{L}_{\mathrm{m}} + \mathscr{L}_{\mathrm{I}}. \tag{3.46}$$

We observe that the fields F and G are not dynamical fields. For instance if we use their own equations of motion[2]

[2] La seule raison d'être of the fields F and G is to give a linear form to the supersymmetry transformations on fields (they are always linearly acting on particle states).

$$F + mA + g(A^2 - B^2) = 0$$
$$G + mB + 2gAB = 0, \tag{3.47}$$

the Lagrangian (3.46) takes the conventional form

$$\mathscr{L} = -\tfrac{1}{2}(\partial_\mu A)^2 - \tfrac{1}{2}(\partial_\mu B) - \tfrac{1}{2}\mathrm{i}\bar{\psi}\partial\!\!\!/\psi - \tfrac{1}{2}m^2A^2 - \tfrac{1}{2}m^2B^2$$
$$- \tfrac{1}{2}\mathrm{i}m\bar{\psi}\psi - gmA(A^2 + B^2) - \tfrac{1}{2}g^2(A^2 + B^2)^2 - \mathrm{i}g\bar{\psi}(A - \gamma_5 B)\psi, \tag{3.48}$$

i.e. a combination of cubic, quartic and Yukawa couplings with well-defined relations among masses and coupling constants.[3]

A vector-spinor current can be derived via the usual Noether procedure

$$J^\mu = \gamma^\lambda\partial_\lambda(A - \gamma_5 B)\gamma^\mu\psi - (F + \gamma_5 G)\gamma^\mu\psi. \tag{3.49}$$

It is conserved, as a consequence of field equations.

Wess and Zumino [131], in the one-loop approximation, and, later, Iliopoulos and Zumino [74], to all orders, have shown that the spinor-scalar theory can be renormalized in a way consistent with supersymmetry. Moreover they showed that this theory is less divergent than one would have naïvely expected from simple symmetry considerations [87].

Indeed only one kind of divergence is present in this theory, a logarithmic infinity leading to a common wave function renormalization of the fields A, B and ψ. For instance the renormalized mass and coupling constant are given, to all orders, by the following relations

$$m_{\mathrm{r}} = Zm_B, \qquad g_{\mathrm{r}} = Z^{3/2}g_B, \tag{3.50}$$

Z being the wave function renormalization constant.

Tsao [122] and Piguet and Schweda [101] have later shown that relations (3.50) can be derived from the softly broken chiral Ward identities of the model.

Ferrara, Iliopoulos and Zumino [45] have studied the renormalization group equations of the super-S^3 theory. They have shown that the renormalized one-particle irreducible n-point functions $\Gamma_{\phi_1\ldots\phi_n}$, in which external lines stand for any field of the chiral multiplet, satisfy the following simple Callan–Symanzik equations at any order of perturbation theory

$$\left(m\frac{\partial}{\partial m} + \beta(g)\frac{\partial}{\partial g} - n\gamma(g)\right)\Gamma_{\phi_1\ldots\phi_n}(p_i;m,g) = \frac{m}{2g}\,\delta(g)\,\Gamma_{A,\phi_1\ldots\phi_n}(0,p_i;m,g). \tag{3.51}$$

The renormalization group functions $\beta(g)$, $\gamma(g)$ and $\delta(g)$ are expressed in terms of a single function $f(g)$ as follows:

$$\beta(g) = \tfrac{3}{2}\frac{gf(g)}{1+f(g)}, \quad \gamma(g) = \tfrac{1}{2}\frac{f(g)}{1+f(g)}, \quad \delta(g) = \frac{1}{1+f(g)}, \tag{3.52}$$

$f(g)$ being defined by

$$f(g) = m_B\frac{\partial \operatorname{Log} Z}{\partial m_B} = \frac{g^2}{4\pi^2} + \ldots\,. \tag{3.53}$$

[3] Interestingly enough the authors of ref. [21] have shown that the supersymmetric relation among the dimensionless Yukawa and scalar couplings separate two classes of theories which are different with respect to their vacuum properties.

We note, *en passant*, the peculiar fact that the inhomogeneous term in the Callan–Symanzik equations (3.51) corresponds to an (elementary field) A insertion.

Moreover from (3.52), as well as from (3.50), we have the important relation

$$\beta(g) = 3g\gamma(g) \tag{3.54}$$

which teaches us that no fixed point exists which can be reached continuously from the origin.

The result (3.54) can equally be derived by showing that, once the supersymmetry Ward identities have been imposed, there are no conformal invariant solutions for the three-point functions with anomalous dimensions of the fields.

Iliopoulos and Zumino [74] considered also the renormalization of the previous theory with a very soft explicit symmetry breaking; namely they added to the Lagrangian density (3.48) the term

$$-cA. \tag{3.55}$$

In this case the Noether vector-spinor current given by eqs. (3.49) is no longer conserved and one has the partial conservation law

$$\partial_\mu J^\mu(x) = c\,\psi(x). \tag{3.56}$$

Nevertheless the term (3.55) breaks the original symmetry so smoothly that the renormalization procedure of the supersymmetric case can still be applied. Due to the symmetry breaking the masses of the fields are no longer equal, but satisfy the mass formula

$$m_A^2 + m_B^2 = 2m_\psi^2. \tag{3.57}$$

At higher orders such a relation is changed only by finite corrections. The super-S^3 model has also been used to illustrate a possible formulation of supersymmetry on a space–time lattice [146].

3.3. Supergraph techniques and renormalization

In the previous subsection we have seen that, for the simplest renormalizable field theory model invariant under supersymmetry, quantum corrections are less divergent than one would have expected from symmetry considerations only.

Tsao [122], Piguet and Schweda [101] have clarified later why such a phenomenon occurs, namely that only a common wave function renormalization has to be performed. They have shown that this is due to the combined Ward identities of supersymmetry and of softly broken chiral invariance.

However, superfield techniques offer a very powerful method to handle Feynman graphs and their ultraviolet properties in supersymmetric theories.

In fact Delbourgo [22], Salam and Strathdee [109] and other authors [14, 70, 151, 57] observed that manifestly covariant Feynman rules can be defined for superfields in such a way that one can easily control cancellations among usual Feynman graphs due to the supersymmetry Ward identities.

Supergraphs, i.e. Feynman rules for superfields, correspond to collections of ordinary Feynman graphs in which all the fields of a supermultiplet are simultaneously considered. Cancellations among these graphs show in a different (improved) ultraviolet behaviour of a single supergraph.

In order to apply the supergraph techniques to the Lagrangian considered in the previous subsection we have first to rewrite the invariant action as an integral of a Lagrangian density in superspace (x_μ, θ_α).

To do this let us use, according to Berezin [9], integration over anticommuting (Grassmann) variables. Such an operation is well-defined. For instance one makes use of the following fundamental property

$$\int \mathrm{d}\theta_i \theta_j = \delta_{ij}. \tag{3.58}$$

If we use the definition (3.58) the action of the super-S^3 theory (in the presence of external sources) can be written as

$$\int \mathrm{d}^4x \left(\int \mathrm{d}^2\theta [\tfrac{1}{16} S\bar{D}\bar{D}\bar{S} + \tfrac{1}{4} mS^2 + \tfrac{1}{3} gS^3] + \text{h.c.} \right) + \int \mathrm{d}^4x \left(\int \mathrm{d}^2\theta JS + \text{h.c.} \right). \tag{3.59}$$

In fact, because of (3.58) we have the obvious property

$$\begin{aligned} &\tfrac{1}{2} \int \mathrm{d}^4\theta \phi_V(x,\theta) = D(x) \\ &\tfrac{1}{2} \int \mathrm{d}^2\theta\, S(x,\theta) = (F(x) + \mathrm{i}G(x)) \end{aligned} \tag{3.60}$$

for a vector and a chiral superfield respectively.

The free propagator of the chiral field is obtained by inverting the field equations for $g = 0$

$$S(1) = \frac{1}{16(m^2 - \Box_1)} (\bar{D}\bar{D}\bar{J}(1) - mJ(1)) \tag{3.61}$$

in which the argument (1) refers to a point in superspace $(x_1\theta_1)$.

Using the functional differential properties

$$\frac{\delta S(1)}{\delta S(2)} = \delta_s(1,2) = \delta^4(x_1 - x_2 + \mathrm{i}\bar{\theta}_1\gamma\theta_2)\delta^2(\theta_{12})$$

$$\frac{\delta \bar{S}(1)}{\delta \bar{S}(2)} = \delta_{\bar{s}}(1,2) = \delta^4(x_1 - x_2 + \mathrm{i}\bar{\theta}_1\gamma\theta_2)\,\delta^2(\bar{\theta}_{12})$$

$$\frac{\delta S(1)}{\delta \bar{S}(2)} = \frac{\delta \bar{S}(1)}{\delta S(2)} = 0; \qquad \delta^2(\theta_{12}) = \tfrac{1}{4}(\theta_1 - \theta_2)^\alpha\,(\theta_1 - \theta_2)_\alpha \tag{3.62}$$

which are straightforward adaptations to superspace of the usual relation $\delta\phi(x_1)/\delta\phi(x_2) = \delta^4(x_1 - x_2)$, one obtains the superfield propagators in momentum space

$$\langle S(1)\,\bar{S}(2)\rangle = \frac{\delta S(1)}{\mathrm{i}\delta\bar{J}(2)} = \frac{-\mathrm{i}}{16(m^2 + k^2)} \exp(\bar{\theta}_1\gamma\theta_2 k - \theta_{12}\sigma\bar{\theta}_{12}k)$$

$$\langle S(1)\,S(2)\rangle = \frac{\delta S(1)}{\mathrm{i}\delta J(2)} = \frac{\mathrm{i}m}{16(m^2 + k^2)} \exp(\bar{\theta}_1\gamma\theta_2 k) \cdot \theta_{12}\theta_{12}. \tag{3.63}$$

As first observed by Delbourgo [22], the absence of mass and coupling constant renormalization at the one-loop level has a very simple explanation in terms of power-counting for supergraphs.

Consider the one-loop diagrams for self-energy. They are

$$\begin{aligned} &\Sigma_{++}(1,2) = \Delta_{++}(1,2)\,\Delta_{++}(2,1) \\ &\Sigma_{+-}(1,2) = \Delta_{+-}(1,2)\,\Delta_{-+}(2,1). \end{aligned} \tag{3.64}$$

Because $\Delta^2_{++}(1,2) = 0$ then $\Sigma_{++} = 0$. For Σ_{+-} we get

$$\sum_{+-}(q,\theta_1,\theta_2) \cong \int \mathrm{d}^4k \, \frac{1}{k^2+m^2} \cdot \frac{1}{(k+q)^2+m^2}$$

$$\times \exp(\bar{\theta}_1\gamma\theta_2 k - \theta_{12}\sigma\bar{\theta}_{12}k) \cdot \exp(-\bar{\theta}_1\gamma\theta_2(k+q) + \theta_{12}\sigma\bar{\theta}_{12}(k+q))$$

$$= \exp(-\bar{\theta}_1\gamma\theta_2 q + \theta_{12}\sigma\bar{\theta}_{12}q) \int \mathrm{d}^4k \, \frac{1}{k^2+m^2} \frac{1}{(k+q)^2+m^2}, \tag{3.65}$$

i.e. logarithmic divergence.

Actually this divergence can be absorbed in a counterterm of the form $S\bar{S}$, i.e. a wave function renormalization.

Because $\Sigma_{++} = 0$ no mass counterterm S^2 is needed.

We consider now the vertex correction. There are only two basic graphs which can be written

$$V_{+++}(1,2,3) = \Delta_{++}(1,2)\, \Delta_{++}(2,3)\, \Delta_{++}(3,1)$$

$$V_{++-}(1,2,3) = \Delta_{++}(1,2)\, \Delta_{+-}(2,3)\, \Delta_{-+}(3,1). \tag{3.66}$$

Due to the anticommuting properties of the θ's one has $\theta_{12}\theta_{12}\theta_{13}\theta_{13}\theta_{23}\theta_{23} = 0$, therefore $V_{+++} = 0$. For the V_{++-} vertex graph we get (dropping the dependent part in the exponential factors)

$$\int \mathrm{d}^4k \, \frac{1}{k^2+m^2} \frac{1}{(k+q)^2+m^2} \frac{1}{(k+t)^2+m^2} \exp(\bar{\theta}_1\gamma\theta_2 k)$$

$$\times\, \theta_{12}\theta_{12} \cdot \exp(\bar{\theta}_2\gamma\theta_3 k - \theta_{23}\sigma\theta_{23}k) \exp(\bar{\theta}_3\gamma\theta_1 k + \theta_{31}\sigma\bar{\theta}_{31}k)$$

$$= \int \mathrm{d}^4k \, \frac{1}{k^2+m^2} \frac{1}{(k+q)^2+m^2} \frac{1}{(k+t)^2+m^2} \theta_{12}\theta_{12}, \tag{3.67}$$

i.e. a finite result. Therefore, there is no independent coupling constant renormalization.

In the same way one can show that all higher n-point functions are one-loop finite.

Capper, Leibbrandt [14], Fujikawa and Lang [57] extended the previous analysis to all orders.

They established the following formula for the superficial degree of divergence of any supergraph in the super-S^3 theory

$$d = 4L - 2I + 2\mu \tag{3.68}$$

in which $\mu = \min(n_+ - 1, n_- - 1)$ and L, I, n_+, n_- are the numbers of loops, propagators and vertices respectively.

Using the topological relations

$$I + E_+ = 3n_+, \qquad L = I - (n-1)$$

$$I + E_- = 3n_-, \qquad n = n_+ + n_- \tag{3.69}$$

one finally gets

$$d = 2 - 2E_- - 4N \tag{3.70}$$

in which $E_\pm$ are the numbers of external lines of $+(S)$ or $-(\bar{S})$ type, and $N = \frac{1}{3}(E_+ - E_-)$. N can be assumed to be non-negative.

The power-counting formula (3.70) shows that the only divergent graphs are those with $N = 0$, $E_+ = E_- = 1$, i.e. the self-energy graphs Σ_{+-} (tadpole graphs vanish identically).

We remark that the previous power counting formula can be generalized to any S^n chiral theory leading to the result that these theories are no longer renormalizable for $n > 3$. This is in agreement with the results of Lang and Wess [79], who first showed the one-loop non-renormalizability of the super-S^4 theory.

The regularization of the super-S^3 theory has been performed in three different ways, namely using Pauli–Villars regularization by Wess and Zumino [131], the higher derivative method by Iliopoulos, Zumino [74], and in the B P H Z scheme by Piguet and Schweda [101].

All these methods are consistent with supersymmetry. A slight modification of the second method (higher covariant derivatives) has been used by Ferrara and Piguet [46] to perform the regularization of supersymmetric gauge theories. The B P H Z renormalization procedure has been applied to gauge theories by Piguet and Rouet [99, 100].

3.4. Supersymmetric gauge theories

In the previous subsection we have described in detail the self-interaction of a chiral supermultiplet, namely the supersymmetric interaction of (real) scalar, pseudoscalar and spinor fields.

The trilinear chiral coupling is the only possible interaction allowed once the restriction of renormalizability has been imposed [133]. A possible generalization is to endow the chiral multiplet of some internal symmetry indices.

Indeed this situation has been investigated in detail by Fayet and O'Raifeartaigh to provide models which undergo a spontaneous breaking of internal symmetry as well as of supersymmetry [36, 96].

Certainly the most promising models which can be built in the framework of supersymmetry are the gauge theories both of the Yang–Mills type [49, 113] (local internal symmetry) and of the Einstein type (local space–time symmetry) [27, 54]. While the supersymmetric extension and properties of the latter has not yet been fully understood, the former has been investigated in a large extent and we shall confine ourselves to this type of gauge theories in the present subsection.

As it is well-known, gauge theories require gauge fields, namely local vector boson fields.

From the analysis carried out in the previous subsections we know that the simplest superfield which contains, among the component fields, a vector boson is just the one which has been called the vector superfield. The transformation properties of its components are given by eqs. (3.7).

The main problem one has to deal with is to extend the usual (Abelian) gauge transformation

$$\mathscr{V}_\mu \to \mathscr{V}_\mu + \partial_\mu A \tag{3.71}$$

in a way consistent with supersymmetry. A possible way is to enlarge the gauge function A to an entire supermultiplet such that the change of a vector superfield still transforms as a vector superfield under supersymmetry [132]. This is the most appropriate way in the superfield formulation of supersymmetric gauge theories.

An alternative possibility, considered by Freedman and De Wit [31], would be that of modifying the supersymmetry transformation laws, in the presence of a gauge group, replacing ordinary derivative by covariant derivatives, the algebra still remaining the same for gauge invariant quantities (for example the Lagrangian density).

We will mainly follow the first procedure, which is also the one which fits naturally with the superfield formalism. According to Wess and Zumino [132], to define the gauge transformations in a way consistent with supersymmetry one must extend the gauge function A to a (be a gauge) chiral superfield $\Lambda(x,\theta)$ with components A, B, ψ, F, G. Under an Abelian gauge transformation the (real) vector superfield $V(x,\theta,\bar{\theta})$ describing the gauge vector multiplet, undergoes the following transformation:

$$\delta V = \mathrm{i}(\Lambda - \bar{\Lambda}). \tag{3.72}$$

In terms of component fields, (3.72) reads as follows

$$\delta C = B, \qquad \delta\chi = \psi, \qquad \delta M = F, \qquad \delta N = G, \qquad \delta \mathscr{V}_\mu = \partial_\mu A, \qquad \delta\lambda = 0, \qquad \delta D = 0. \tag{3.73}$$

Because of (3.73) one realizes immediately that C, χ, M, N are not physical degrees of freedom (they are submitted to arbitrary translations) while the fields λ and D are gauge invariant. In fact the fields C, χ, M, N can be gauged away for a particular choice of gauge, such that (Wess–Zumino gauges) [132]

$$\psi = -\chi, \quad B = -C, \quad F = -M, \quad G = -N \tag{3.74}$$

while A remains arbitrary.

In these (non-covariant) gauges the superfield takes the form

$$V_{\mathrm{WZ}}(x,\theta,\bar{\theta}) = -\theta\sigma_\mu\bar{\theta}V^\mu + \mathrm{i}\theta\theta\bar{\theta}\bar{\lambda} - \mathrm{i}\bar{\theta}\bar{\theta}\theta\lambda + \tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D$$

and one still has the gauge symmetry of conventional gauge theories.

We note, in particular, that in this class of gauges, all powers of the superfield V with $n > 2$ vanish, i.e.

$$V^n_{\mathrm{WZ}}(x,\theta,\bar{\theta}) = 0 \qquad \text{for } n > 2. \tag{3.75}$$

This property will be important in deriving supersymmetric Lagrangians in the Wess–Zumino gauge.

The supersymmetric generalization of the electromagnetic field strength corresponds to the following (spinor) chiral superfield

$$W_\alpha = \bar{D}\bar{D}D_\alpha V \tag{3.76}$$

which contains the gauge invariant component fields

$$\lambda, D, \partial_\mu V_\nu - \partial_\nu V_\mu, \gamma\cdot\partial\lambda. \tag{3.77}$$

Under the transformation in (3.72), we have

$$\delta W_\alpha = 0.$$

The free Lagrangian of the (Abelian) vector multiplet is given by the F (last) component of the chiral multiplet $W^\alpha W_\alpha$

$$\frac{1}{2^5} W^\alpha W_{\alpha_F} = -\tfrac{1}{4}V^2_{\mu\nu} - \tfrac{1}{2}\mathrm{i}\bar{\lambda}\partial\!\!\!/\lambda + \tfrac{1}{2}D^2. \tag{3.78}$$

The (charged) supersymmetric matter can be identified with a complex chiral multiplet

$$S = (S_1 + \mathrm{i}S_2)/\sqrt{2} \tag{3.79}$$

defining $T = (\bar{S}_1 + \mathrm{i}\bar{S}_2)/\sqrt{2}$.

We have

$$\delta S = -2\mathrm{i}g\Lambda S, \qquad \delta T = -2\mathrm{i}g\bar{\Lambda}T. \tag{3.80}$$

The supersymmetric extension of Q.E.D. is obtained [132] by writing the supersymmetric minimal coupling, i.e. by modifying the kinetic terms for the chiral fields S and T in such a way as to preserve the local symmetry (3.80).

This gives the superfield Lagrangian density

$$\bar{S}e^{2gV}S + \bar{T}e^{-2gV}T \tag{3.81}$$

to which one can add the mass term of the chiral multiplet

$$2m\,\bar{T}S. \tag{3.82}$$

The complete Lagrangian density for the Abelian theory is given by the following expression[4]

$$\frac{1}{2^5}\,W^\alpha W_{\alpha_F} + (\bar{S}e^{2gV}S + \bar{T}e^{-2gV}T)_D + (2m\,\bar{T}S)_F. \tag{3.83}$$

Using (3.76) and (3.79) the Lagrangian (3.83) can be written in a manifestly real form as follows

$$-\left(\frac{1}{2^3}\right)VD^\alpha\bar{D}\bar{D}D_\alpha V_D + [(\bar{S}_1S_1 + \bar{S}_2S_2)\,\mathrm{ch}\,2gV + \mathrm{i}(\bar{S}_1S_2 - \bar{S}_2S_1)\,\mathrm{sh}\,2gV]_D + m(S_1^2 + S_2^2)_F. \tag{3.84}$$

Note that in the Wess–Zumino gauges $V = V_{WZ}$ (eqs. (3.74)) the interaction part becomes indeed polynomial

$$2g^2(\bar{S}_1S_1 + \bar{S}_2S_2)\,V_{WZ}^2 + 2\mathrm{i}g(\bar{S}_1S_2 - \bar{S}_2S_1)\,V_{WZ} \tag{3.85}$$

and, in terms of superfields, takes a form which is reminiscent of scalar Q.E.D.

The Lagrangian density $\mathscr{L}$, which corresponds to (3.85) becomes

$$\begin{aligned}\mathscr{L} = &-\tfrac{1}{2}(\partial_\mu A_1)^2 - \tfrac{1}{2}(\partial_\mu A_2)^2 - \tfrac{1}{2}(\partial_\mu B_1)^2 - \tfrac{1}{2}(\partial_\mu B_2)^2\\ &-\tfrac{1}{2}\mathrm{i}\bar{\psi}_1\not{\partial}\psi_1 - \tfrac{1}{2}\mathrm{i}\bar{\psi}_2\not{\partial}\psi_2 - \tfrac{1}{2}m^2(A_1^2 + A_2^2 + B_1^2 + B_2^2)\\ &-\tfrac{1}{2}\mathrm{i}m(\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2) - \tfrac{1}{4}V_{\mu\nu}^2 - \tfrac{1}{2}\mathrm{i}\bar{\lambda}\not{\partial}\lambda\\ &-gV_\mu(A_1\overleftrightarrow{\partial}^\mu A_2 + B_1\overleftrightarrow{\partial}^\mu B_2 - \mathrm{i}\bar{\psi}_1\gamma^\mu\psi_2) - \mathrm{i}g\bar{\lambda}[(A_1 + \gamma_5B_1)\psi_2\\ &-(A_2 + \gamma_5B_2)\psi_1] - \tfrac{1}{2}g^2V_\mu^2(A_1^2 + A_2^2 + B_1^2 + B_2^2) - \tfrac{1}{2}g^2(A_1B_2 - A_2B_1)^2\end{aligned} \tag{3.86}$$

in which we have used the field equations in order to eliminate the auxiliary fields F_i, G_i, D

$$F_i + mA_i = 0, \qquad G_i + mB_i = 0, \qquad i = 1,2$$

$$D + g(A_1B_2 - A_2B_1) = 0. \tag{3.87}$$

Ferrara, Zumino [49], Salam and Strathdee [113] extended the gauge transformation of the vector supermultiplet to the case of a non-Abelian (Yang–Mills) local gauge group $\mathscr{G}$.

For simplicity we shall assume $\mathscr{G}$ to be semisimple. It is convenient to use matrix-notation for fields or functions which transform according to the adjoint representation of the group. For example the gauge-superfield V as well as the gauge function Λ are matrix superfields.

$$V = V_a t^a, \qquad \Lambda = \Lambda_a t^a \tag{3.88}$$

[4]The normalization factors in eqs. (3.83, 84) are chosen in order to reproduce the Lagrangian in (3.86) with the same normalization conventions.

where the matrices t^a are the group-generators in the vector (fundamental) representation.

For a (finite) gauge transformation of (chiral) parameter $\Lambda(\bar{\mathrm{D}}_{\dot\alpha}\Lambda = 0)$ V transforms as

$$\exp(gV) \to \exp(-\mathrm{i}g\Lambda^+)\exp(gV)\exp(\mathrm{i}g\Lambda) \tag{3.89}$$

where matrix multiplication is understood.

The finite transformation (3.89) corresponds to a complicated (non-linear) transformation for the gauge superfield $V_a(x,\theta,\bar\theta)$

$$\delta V_a = \Lambda^b C_{ba}(V) + \bar\Lambda^b \bar C_{ba}(V) \tag{3.90}$$

where $C_{ba}(V) = \mathrm{i}C^1_{ba}(V) + C^2_{ba}(V)$ is an infinite power series in V_a. However, in the Wess–Zumino gauges one has

$$V_a V_b V_c = 0 \tag{3.91}$$

so that $C^1_{ba} = \delta_{ba}$, $C^2_{ba} = -f_{bac}V^c$.

In this case the infinitesimal (in Λ) transformation (3.89) reduces to

$$\delta V = \mathrm{i}(\Lambda - \Lambda^+) - \tfrac{1}{2}\mathrm{i}[\Lambda + \Lambda^+, V] \tag{3.92}$$

and one recovers the usual local Yang–Mills transformations of the component V^a_μ, λ^a, D^a.

The non-Abelian field strength is given by the following (matrix) chiral superfield

$$W_\alpha = \bar D\bar D(\mathrm{e}^{-gV}\mathrm{D}_\alpha \mathrm{e}^{gV}) \tag{3.93}$$

which becomes, under the transformation (3.89),

$$W_\alpha \to \mathrm{e}^{-\mathrm{i}g\Lambda}W_\alpha \mathrm{e}^{\mathrm{i}g\Lambda}. \tag{3.94}$$

The quantity[5]

$$\frac{1}{g^2}\,\mathrm{Tr}(W^\alpha W_\alpha) \tag{3.95}$$

is gauge invariant and provides the supersymmetric Lagrangian density in superspace.

The F component of the supermultiplet in (3.95) reduces, in the Wess–Zumino gauge, to the following expression

$$\mathrm{Tr}(-\tfrac{1}{4}V^2_{\mu\nu} - \tfrac{1}{2}\mathrm{i}\bar\lambda \not{D}\lambda) \tag{3.96}$$

where

$$V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu + \mathrm{i}g[V_\mu, V_\nu]$$
$$\mathrm{D}_\mu\lambda = \partial_\mu\lambda + \mathrm{i}g[V_\mu,\lambda]$$

and the equations of motion of the auxiliary fields $D^a = 0$ have been used.

The Lagrangian (3.96) implies the quite striking result that a Yang–Mills interaction of a (massless) fermion in the adjoint representation is automatically supersymmetric. To be convinced of this result it is sufficient to observe that, in such a case, a vector-spinor current exists

$$J^\mu(x) = -\tfrac{1}{2}\mathrm{Tr}(V_{\rho\sigma}\gamma^\rho\gamma^\sigma\gamma^\mu\lambda), \tag{3.97}$$

[5]From now on the superfield Lagrangians are written up to overall normalizing factors.

which is conserved as a consequence of the field equations. It is interesting to stress the fact that the fermion λ^a, which, in conventional gauge theories, would be considered as a matter particle, appears in fact as a gauge-particle associated with the fermionic degree of freedom of the (generalized) gauge transformation. As a consequence, supersymmetry (or a generalization of it) offers the possibility of unification of radiation (gauge particles) and matter in a single gauge principle where the basic geometrical entities are Bose and Fermi coordinates (Grassmann coordinates).

We will come back to these speculations in the sixth section where attempts and justifications for constructing supersymmetric gravitational theories will be described.

The gauge vector multiplet can be put in interaction with a supersymmetric matter system (chiral multiplet) using a straightforward adaptation of the supersymmetric minimal coupling of the Abelian theory (extended Q.E.D.).

For definiteness let us confine our discussion to the unitary group $\mathscr{G} = \mathrm{SU}(N)$. Then if S_i is a (column vector) chiral superfield ($\bar{D}_{\dot\alpha} S_i = 0$) which transforms according to some unitary representation of SU (N), under a (finite) gauge transformation, it undergoes the following change

$$S \to e^{-ig\Lambda} S, \qquad \bar{S} \to \bar{S} e^{ig\Lambda^+}. \tag{3.98}$$

Here $\Lambda_{ij} = \Lambda_a R^a_{ij}$, R^a_{ij} being the generators of SU (N) in the representation of the matter field.
The gauge vector superfield can be written in the similar form

$$V_{ij} = V_a R^a_{ij}. \tag{3.99}$$

It is obvious that the expression

$$\bar{S} e^{gV} S \tag{3.100}$$

is invariant under local SU (N) transformations.

Similarly, if T is a chiral superfield which transforms according to the complex conjugate representation of S ($D_\alpha T_i = 0$) then

$$T \to e^{-ig\Lambda^+} T, \qquad \bar{T} \to \bar{T} e^{ig\Lambda} \tag{3.101}$$

and the quantity

$$\bar{T} e^{-gV} T \tag{3.102}$$

is gauge invariant.

Note that, if we want our theory to conserve parity, we must assume that complex conjugate representations (S, T) are also parity-conjugate namely that parity exchanges S and T.
That it must be so is clear from the fact that the gauge superfield V is assumed to be odd under parity.
The parity preserving gauge coupling of the matter is given by the following superfield

$$\bar{S} e^{gV} S + \bar{T} e^{-gV} T. \tag{3.103}$$

A simplification occurs if the matter field belongs to some real representation of SU (N) (for example the adjoint representation). Then

$$\Lambda^{\mathrm{T}} = -\Lambda, \; V^{\mathrm{T}} = -V \tag{3.104}$$

and one can make the identification

$$\bar{T} = S. \tag{3.105}$$

The Yang–Mills coupling becomes

$$\bar{S}e^{gV}S \tag{3.106}$$

and the mass-term

$$S^2. \tag{3.107}$$

Suppose now that S belongs to the adjoint representation. In the Wess–Zumino gauges the complete Lagrangian density reduces to the form

$$\begin{aligned}\mathrm{Tr}\{&-\tfrac{1}{4}V_{\mu\nu}^2 - \tfrac{1}{2}\mathrm{i}\bar{\lambda}\not{D}\lambda - \tfrac{1}{2}(\mathrm{D}_\mu A)^2 - \tfrac{1}{2}(\mathrm{D}_\mu B)^2 - \tfrac{1}{2}\mathrm{i}\bar{\psi}\not{D}\psi \\ &- \tfrac{1}{2}m^2A^2 - \tfrac{1}{2}m^2B^2 - \tfrac{1}{2}\mathrm{i}m\bar{\psi}\psi + g\bar{\lambda}[A + \gamma_5 B, \psi] + \tfrac{1}{2}g^2[A,B]^2\}\end{aligned} \tag{3.108}$$

in which we used a matrix notation $S = S_a t^a$ for the chiral superfield S.

The supersymmetric Lagrangian density (3.108) with one chiral multiplet in the adjoint representation, manifests an unexpected additional symmetry [49, 113] for $m = 0$.

In fact, if one defines the Dirac spinor

$$\varphi = (\lambda + \mathrm{i}\psi)/\sqrt{2} \tag{3.109}$$

in terms of the Majorana spinors λ and ψ belonging respectively to the vector and chiral multiplet, the complete Lagrangian density takes the form

$$\mathrm{Tr}(-\tfrac{1}{4}V_{\mu\nu}^2 - \tfrac{1}{2}(\mathrm{D}_\mu A)^2 - \tfrac{1}{2}(\mathrm{D}_\mu B)^2 - \tfrac{1}{2}\mathrm{i}\bar{\varphi}\overleftrightarrow{\not{D}}\varphi - \mathrm{i}g\bar{\varphi}[A + \gamma_5 B, \varphi] + \tfrac{1}{2}g^2[A,B]^2) \tag{3.110}$$

and therefore is invariant under the global U(1) group which corresponds to phase transformations

$$\varphi \to \mathrm{e}^{\mathrm{i}\alpha}\varphi, \qquad \varphi^* \to \mathrm{e}^{-\mathrm{i}\alpha}\varphi^* \tag{3.111}$$

on the complex spinor field φ.

Moreover in this case the (complex) Dirac vector spinor current [138]

$$J_\mu^\mu = \mathrm{Tr}(-\tfrac{1}{2}V_{\rho\sigma}\gamma^\rho\gamma^\sigma\gamma^\mu\varphi + \mathrm{i}g\,[A,B]\,\gamma_5\gamma^\mu\varphi - \mathrm{i}\gamma^\lambda \mathrm{D}_\lambda(A - \gamma_5 B)\gamma^\mu\varphi) \tag{3.112}$$

is conserved as a consequence of the field equations. This implies that, in this situation, a bigger supersymmetry algebra holds, in which the Fermi charges are the components of a Dirac spinor instead of a Majorana spinor. We shall come back to this point in subsection 4.7, [38].

3.5. Perturbation theory and renormalization of supersymmetric gauge theories

The problem of renormalization in supersymmetric gauge theories presents some additional difficulties compared to the super-S^3 theory. This is because the Lagrangian density takes a familiar form only in a class of non-covariant gauges (Wess–Zumino gauges) in which the interaction term in (3.83), (3.103) becomes polynomial. The advantage of having a polynomial Lagrangian density has its drawback in so far as explicit supersymmetry covariance is lost. This makes an enormous complication in studying the Ward identities coming from supersymmetry in perturbation theory.

However, one-loop renormalizability of the Abelian model has been proved by Wess and Zumino [132] in this class of non-covariant gauges.

The renormalization program in covariant gauges (supersymmetric gauges) has been carried out by many authors using different procedures. Slavnov [117, 118] and De Wit [29] used the Slavnov-Taylor identities combined with the supersymmetry Ward identities for Feynman diagrams of the individual component fields of the multiplets. Honerkamp et al. [71] used background field methods and supergraph techniques.

The B.P.H.Z. renormalization scheme has been applied to supersymmetric gauge theories, superfields being used directly [99, 100, 143].[6] Ferrara and Piguet [46] used the Becchi–Rouet–Stora [8] transformations (Slavnov transformations) and the Slavnov identities for superdiagrams. These last authors used, as regularizing procedure, the higher covariant derivative method. This method can be regarded as a generalization to gauge theories of the regularization used by Iliopoulos and Zumino [74] in the chiral model. They were also able to show the absence of mass renormalization for the chiral multiplet, as it was in the model without the gauge multiplet.

Ferrara and Piguet [46] and later Delbourgo [23] showed the renormalizability of the abelian gauge theory with softly broken gauge invariance (mass term for the vector multiplet). In the massive vector theory [65] the vector multiplet, being massive, contains new physical degrees of freedom, namely a pseudoscalar and a spinor with the same mass M.[7]

The main point, in the problem of quantization of supersymmetric gauge theories, is the introduction of a gauge breaking term consistent with supersymmetry. It is straightforward to realize that the supersymmetric extension of the usual Fermi type gauge-breaking term

$$\frac{1}{\alpha}(\partial_\mu V^\mu)^2 \tag{3.113}$$

is provided by the last component of the following vector multiplet

$$\frac{2}{\alpha} DDV\bar{D}\bar{D}V. \tag{3.114}$$

The free part of the Lagrangian superfield density for the vector field becomes (in presence of external sources)

$$2\mathrm{D}^\alpha V\bar{D}\bar{D}\mathrm{D}_\alpha V + \frac{2}{\alpha} DDV\bar{D}\bar{D}V + JV. \tag{3.115}$$

The free-field equations are

$$\left[\mathrm{D}^\alpha\bar{D}\bar{D}\mathrm{D}_\alpha - \frac{1}{2\alpha}(DD + \bar{D}\bar{D})^2\right] V = \tfrac{1}{4}J. \tag{3.116}$$

From (3.116) we get

$$V(1) = \frac{1}{4^4}\,\frac{1}{\Box_1{}^2}\left[\mathrm{D}_1^\alpha\bar{D}_1\bar{D}_1\mathrm{D}_{1\alpha} - \frac{\alpha}{2}(D_1D_1 + \bar{D}_1\bar{D}_1)^2\right] J(1) \tag{3.117}$$

and for the free propagator of the vector field

$$\Delta(1,2) = \frac{\delta V(1)}{\mathrm{i}\delta J(2)}. \tag{3.118}$$

After Fourier-transformation, (3.118) leads to the expression:

[6] Recently the authors of ref. [25] used a supersymmetric dimensional regularization procedure [24] in order to prove the formal Becchi–Rouet–Stora [8] Ward-identities in supersymmetric gauge theories.

[7] The Lagrangian for the supersymmetric massive Q.E.D. has been given by Gol'fand and Likhtman [65].

$$\Delta(k,\theta_1,\theta_2) \cong \frac{1}{K^4} \exp(\bar{\theta}_1\gamma\theta_2 k)\, [4(1-\alpha) + (1+\alpha)\,\theta_{12}\theta_{12}\bar{\theta}_{12}\bar{\theta}_{12}k^2]. \tag{3.119}$$

For $\alpha = 1$ we get the Feynman gauge in which the propagator has the particularly simple form

$$\Delta_{\mathrm{FEYN}}(k,\theta_1\theta_2) \simeq \frac{1}{K^2}\,\theta_{12}\theta_{12}\bar{\theta}_{12}\bar{\theta}_{12}. \tag{3.120}$$

For a massive-gauge field (massive QED) the propagator becomes

$$\Delta(k,\theta_1,\theta_2) \cong \frac{1}{M^2}\exp(\bar{\theta}_1\gamma\theta_2 k)\left(\frac{1}{k^2+\alpha M^2} - \frac{1}{k^2+M^2}\right) + \frac{1}{4M^2}\,\theta_{12}\theta_{12}\bar{\theta}_{12}\bar{\theta}_{12}\left(2 - \frac{M^2}{k^2+\alpha M^2} - \frac{K^2}{k^2+M^2}\right). \tag{3.121}$$

It is easily seen that this propagator describes a particle multiplet of mass M together with a ghost multiplet of mass $\alpha^{1/2}M$. Such ghost particles decouple from S matrix elements because of the gauge Ward identities, but provide the same ultraviolet behaviour of (3.121) as in the zero-mass theory (3.119).

In non-abelian gauge theories one is faced with the problem of introducing a Faddeev–Popov term in the complete Lagrangian density.

The Faddeev–Popov superfields C_+, C_- must have the same supersymmetry properties as the gauge function Λ and as the gauge-superfield $\bar{D}\bar{D}V$, so they are (anticommuting) chiral superfields. In terms of component fields they describe anticommuting scalar fields (the usual F.P. ghosts) as well as commuting spin $\frac{1}{2}$ fields. The latter are the supersymmetric counterparts of the usual ghost fields and are a new feature of supersymmetric gauge theories. They decouple in the Wess–Zumino gauges but are needed in general supersymmetric gauges.

The power counting formulae for the superficial degree of divergence of supergraphs (Slavnov [117, 118], Ferrara and Piguet [46]), combined with the Slavnov identities, show that the supersymmetric gauge theory can be renormalized by performing a charge and wave function renormalization for the gauge multiplet, and a wave function renormalization for the matter and ghost chiral multiplets.

3.6. *Asymptotic freedom and infrared convergence*

It is well-known that conventional (pure) non-abelian gauge theories are asymptotically free, i.e. the effective coupling constant goes to zero in the ultraviolet region. We know also, through the previous subsections, that pure supersymmetric gauge theories contain unavoidably elementary fermions which, as it is well-known, generally tend to destabilize the origin; thus it appears important to see first whether supersymmetric pure gauge theories are also asymptotically free.

Ferrara and Zumino [49] computed the Callan–Symanzik function at the one-loop level, for a gauge multiplet interacting with n (chiral) matter multiplets belonging to the adjoint representation of SU (N). The result is

$$\beta(g) = \frac{A}{16\pi^2}\, g^3 \tag{3.122}$$

with

$$A = N(n-3)$$

so that asymptotic freedom holds for $n < 3$.[8] In particular pure supersymmetric Yang–Mills theories ($n = 0$) are asymptotically free as well as the model ($n = 1$) with U(1) global symmetry discussed in the previous section (eqs. (3.110)). For $n = 3$, $\beta = 0$ and a second-loop calculation is required. Jones [76] performed the calculation of the β-function at the second-loop level, for any group $\mathcal{G}$ and with n chiral multiplets belonging to an arbitrary representation of the gauge group.

The result is the following

$$\beta(g) = \frac{A}{16\pi^2} g^3 + \frac{B}{(16\pi^2)^2} g^5 \tag{3.123}$$

with

$$A = nT(R) - 3C_2(G) \tag{3.124}$$

$$B = 4nC_2(R)T(R) + 2nC_2(G)T(R) - 6C_2^2(G). \tag{3.125}$$

The general condition for asymptotic freedom is

$$nT(R) - 3C_2(G) < 0. \tag{3.126}$$

Note that for $A = 0$ the theory is never asymptotically free because

$$B(A = 0) = 12C_2(G)C_2(R) > 0.$$

In particular, for $\mathcal{G} = \mathrm{SU}(N)$ and the matter in the adjoint representation, we have

$$B = 6N^2(n - 1). \tag{3.127}$$

The theory is infrared-free for $n = 3$.

For $n = 2$ we have asymptotic freedom with a perturbative infrared fixed point at $g^2/16\pi^2 = 1/6N$.

Supersymmetric gauge theories which are asymptotically free offer a perhaps unique example of the possibility of incorporating scalar mesons without destroying the ultraviolet stability of the origin. This is due to the fact that all the couplings, the scalar potential included, are controlled by the same coupling constant g, the gauge coupling constant.

This result is particularly important in view of the possibility of having a criterion to fix the Higgs potential. The presence of Higgs bosons is in fact needed in non-abelian gauge theories (modulo infrared slavery's dreams) to provide their infrared convergence.

The problem of infrared convergence [52] in non-abelian supersymmetric gauge theories has been considered by O'Raifeartaigh et al. [12] and more recently by Slavnov [119] in a different context.

O'Raifeartaigh et al. [12] considered the supersymmetric gauge theory with an additional trilinear matter–matter interaction and studied under what circumstances the resulting theory can be both asymptotically-free and infrared-convergent. (This coupling cannot be generated by the "minimal" coupling because of its odd property under the discrete symmetry $S \to -S$.)

Note that in this case there is at least one additional coupling constant in the game due to the cubic self-interaction of the (chiral) matter multiplet. This interaction is needed because the scalar

[8] Suzuki [121] and Gross [66] have pointed out that, although the exactly supersymmetric Yang–Mills theory is asymptotically free, the symmetry limit cannot be obtained as a local minimum in the parameter space of independent coupling constants, starting with an approximate supersymmetry. This indicates that only soft or spontaneous breaking does not ruin the asymptotic freedom of supersymmetric theories.

potential induced by the Yang–Mills coupling is not sufficient to trigger spontaneous symmetry breaking and therefore possibly infrared convergence.

The very important point (very different from conventional Yang–Mills theories) is that, as far as asymptotic freedom is concerned (at least if the matter belongs to the adjoint representation) the condition for asymptotic freedom is still $n < 3$, independently of the matter coupling.

The condition to achieve infrared convergence is for the little group of the symmetry breaking direction to be abelian.

As a model which verifies the above requirements (therefore providing an asymptotically free and infrared convergent gauge theory) the previous authors considered two chiral multiplets in the adjoint representation of SU (3) leading to a breaking along two different directions with $U(1) \otimes U(1)$ as little group.

A different mechanism has been exhibited by Slavnov [119]; he considered the $SU(2) \otimes U(1)$ supersymmetric gauge model with an allowed U (1) linear term à la Fayet–Iliopoulos [41].

The Lagrangian superfield is

$$\bar{S} \exp(g\tau V + g'V')S + \bar{T} \exp\{-(g\tau V + g'V')\} T - \frac{\mu^2}{g'} V'. \tag{3.128}$$

Making a suitable shift, to eliminate the linear term, the above superfield gives rise to the following Lagrangian density for $g' = 0$

$$\mathscr{L} = \ldots + (\bar{S} \exp(g\tau V)S + \bar{T} \exp(-g\tau V)T)_D + \mu^2 (\bar{S} \exp(g\tau V)S - \bar{T} \exp(-g\tau V)T)_C \tag{3.129}$$

where D and C denote the D and C components of the corresponding vector superfields.

Note for instance that the last term (in the Wess–Zumino gauges) is nothing but mass terms for scalar fields. The limiting Lagrangian (3.129) is locally Yang–Mills (SU (2)) invariant but only globally U (1) invariant (no abelian vector field remains). The Lagrangian (3.128) can be considered as a regularization of the theory defined by (3.129). The important point is that the regularizing Lagrangian is invariant under broken supersymmetry transformations (where the shifted superfield has been used) and for any g' (including $g' = 0$), satisfies the broken supersymmetry Ward identities necessary to prove the renormalizability of the theory. However, the g'-independent Lagrangian (3.129) is no longer invariant under any limiting transformation (which becomes singular for $g' \to 0$).

The method suggested by this limiting procedure allows to construct models which are both asymptotically-free and infrared convergent, without the introduction of new couplings. The spectrum of the simplest of these models (which is however, parity-violating) consists of one massive vector triplet, two massive Dirac spinors, one massive real scalar and one massless Weyl spinor.

Let us note finally the problem of infrared anomalies in supersymmetric models involving scalar particles; this occurs in particular for spontaneously broken supersymmetric models with chiral superfields only [142].

4. Spontaneous symmetry breaking and conservation laws

4.1. Generalities about spontaneous symmetry breaking

In order to study spontaneous (super) symmetry breaking, one has to determine the vacuum state: among stationary solutions of equations of motion one looks for those of lowest energy. The possibility that supersymmetry could be spontaneously broken has been considered by

Iliopoulos and Zumino [74], Salam and Strathdee [112]. Then a massless spin $\frac{1}{2}$ particle (Goldstone fermion) must appear; the first example was found by Fayet and Iliopoulos [41].

We shall be interested in supersymmetric gauge theories, describing vector and chiral multiplets. Let D_K, F_l and G_l be the auxiliary scalar fields, and φ stand for the physical scalar fields. All of them may take non-vanishing vacuum expectation values. To see if they respect supersymmetry we have to study the transformation properties of spinor fields. Taking vacuum expectation values we find:

$$\langle\{Q_\alpha, \bar{\lambda}_{K\beta}\}\rangle = \langle\gamma_5 D_K\rangle_{\alpha\beta}$$
$$\langle\{Q_\alpha, \bar{\psi}_{l\beta}\}\rangle = \langle F_l + \gamma_5 G_l\rangle_{\alpha\beta}. \tag{4.1}$$

Therefore the conditions under which spontaneous supersymmetry breaking occurs are that one at least of the auxiliary fields D_K, F_l and G_l takes a non-vanishing vacuum expectation value,[9] whatever those taken by the physical scalar fields φ.

If such theories exist, spontaneous supersymmetry breaking implies the appearance of a massless Goldstone fermion. Let us first suppose, for simplicity, that F is the only auxiliary field taking a non-vanishing vacuum expectation value, and let ψ be the Majorana spinor associated to it. Formula (4.1) can be written in terms of the conserved vector-spinor current $J^\mu_\alpha(x)$:

$$\left\langle\left\{\int J^0_\alpha(x)\,\mathrm{d}^3x,\ \bar{\psi}_\beta(x')\right\}\right\rangle = \langle F(x')\rangle\,\delta_{\alpha\beta}. \tag{4.2}$$

Using current conservation one can deduce that

$$\int \mathrm{d}^4x\, \mathrm{e}^{ikx}\langle T^* J^\mu_\alpha(x)\,\bar{\psi}_\beta(0)\rangle = M(k^2)k^\mu\delta_{\alpha\beta} + \ldots \tag{4.3}$$

in which $M(k^2)$ has a simple pole at $k^2 = 0$, with residue proportional to $\langle F\rangle$ [112]. This indicates that the intermediate states which contribute must include a massless spin $\frac{1}{2}$ particle: the Goldstone fermion.

In models with combined supersymmetry and gauge invariance, technical complications in the proof of the Goldstone theorem have been discussed by de Wit and Freedman [31].

More simply the Goldstone field is the one for which the variation has a non-vanishing vacuum expectation value. An example is the well-known Goldstone model describing two real fields A and B, invariant under U(1) rotations generated by a scalar operator Y [63]:

$$[Y, A] = B, \qquad [Y, B] = -A. \tag{4.4}$$

If B takes a non-vanishing vacuum expectation value, invariance under rotation is spontaneously broken and A is the scalar Goldstone field.

We have a similar situation for supersymmetry: the commutator of Y with the Goldstone scalar field A is replaced by the anticommutator of the supersymmetry generators Q_α with the Goldstone spinor field; the field B is replaced by the auxiliary scalar field taking a non-vanishing vacuum expectation value.

Using these arguments it is very easy to find the Goldstone spinor in any particular situation (a general formula has been written in ref. [115]). Before showing that spontaneous supersymmetry breaking is indeed possible and leads to mass splittings inside multiplets, we shall exhibit some of its very particular features.

[9] These conditions are not always sufficient to ensure that spontaneous supersymmetry breaking has some physical consequences; for example the Lagrangian density: $\mathscr{L} = -\frac{1}{2}\mathrm{i}\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}\partial^\mu A\partial_\mu A - \frac{1}{2}\partial^\mu B\partial_\mu B + \frac{1}{2}(F^2 + G^2) + sF$ describes a free massless chiral multiplet. We have both the features of a model in which supersymmetry is spontaneously broken ($\langle F\rangle = -s$, ψ being the Goldstone spinor), or conserved. The same thing occurs for a free massless vector multiplet.

The terms in the Lagrangian density which are relevant in order to determine the potential of scalar fields at the tree approximation are:

$$\mathscr{L} = \ldots\ldots + \Sigma\ \tfrac{1}{2}(D_K^2 + F_l^2 + G_l^2) - \ \Sigma\,(D_K D_K(\varphi) + F_l F_l(\varphi) + G_l G_l(\varphi)) \tag{4.5}$$

where D_K, F_l and G_l are the auxiliary scalar fields, and $D_K(\varphi)$, $F_l(\varphi)$ and $G_l(\varphi)$ are polynomial functions of degree two in the physical scalar fields, denoted by φ.

Auxiliary fields can be expressed in terms of physical ones by means of the equations of motion:

$$D_K = D_K(\varphi), \qquad F_l = F_l(\varphi), \qquad G_l = G_l(\varphi). \tag{4.6}$$

These can be used to eliminate auxiliary fields from the Lagrangian density. We immediately obtain the potential of the physical scalar fields φ at the tree approximation:

$$V(\varphi) = \tfrac{1}{2}\ \Sigma\,(D_K(\varphi)^2 + F_l(\varphi)^2 + G_l(\varphi)^2). \tag{4.7}$$

Now we study spontaneous supersymmetry breaking at the classical level. Suppose we know two stationary solutions for equations of motion, corresponding to two different vacuum states, $|\Omega\rangle$ supersymmetric and $|\Omega'\rangle$ which is not. In the supersymmetric vacuum state $|\Omega\rangle$, all auxiliary fields have vanishing vacuum expectation values, and $V(\varphi)$ vanishes: there is no vacuum energy. On the contrary in the state $|\Omega'\rangle$ which is not supersymmetric, at least one of the auxiliary fields has a non-vanishing vacuum expectation value, and $V(\varphi)$ is strictly positive. Therefore, if one has to choose between possible vacuum states among which some are supersymmetric, as $|\Omega\rangle$, and some are not, as $|\Omega'\rangle$, the former are always stable.

This makes the problem of spontaneous breaking more difficult for supersymmetry than for an ordinary symmetry, such as internal or gauge invariance: then spontaneous breaking is obtained by the Goldstone [63, 64] or Higgs–Kibble [33, 69, 78] mechanisms; there exist a symmetric state, together with non-symmetric ones. The former, which has a higher energy, is unstable.

For supersymmetry it is not possible to proceed in this way, since a supersymmetric state is always stable. However, this does not mean that spontaneous supersymmetry breaking is impossible: there exist models [41, 36, 96] for which equations of motion have no supersymmetric stationary solution. Then the vacuum state is not supersymmetric, and supersymmetry is spontaneously broken. This can be realized if vacuum expectation values of all auxiliary fields D_K, F_l and G_l cannot vanish simultaneously. At the tree approximation, this means that the system

$$D_K(\varphi) = 0, \qquad F_l(\varphi) = 0, \qquad G_l(\varphi) = 0 \tag{4.8}$$

has no solution.

To sum up, spontaneous supersymmetry breaking is determined by vacuum expectation values of auxiliary scalar fields, whereas spontaneous breaking of other (internal, or gauge, ...) symmetries are determined, as usual, by vacuum expectation values of physical scalar fields.

In order to illustrate the peculiarity of spontaneous supersymmetry breaking, we show how $V(\varphi)$ looks like in the model of refs. [84, 37] where φ denotes a complex scalar field (see fig. 1).

Because spontaneous breaking is not obtained for supersymmetry as easily as for internal symmetries, we shall first study how the latter can be spontaneously broken, the former being conserved.

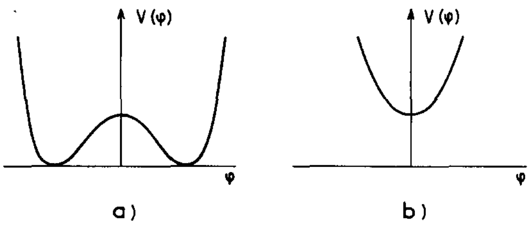

Fig. 1. Representation of the potential $V(\varphi)$ in a supersymmetric gauge theory, in terms of the complex scalar field φ. (a) Supersymmetry is conserved (for $\langle\varphi\rangle = 0$ it would be spontaneously broken, but the corresponding vacuum state is unstable). (b) Supersymmetry is spontaneously broken.

4.2. Spontaneous breaking of internal symmetry[10]

The models studied in this subsection are generalizations of the original model of Wess and Zumino [131] to a set of n left-handed chiral superfields S_a (one can use as well their conjugates, the right-handed chiral superfields $S_a{}^*$) [74, 114, 93, 35, 36, 96]. The most general Lagrangian density compatible with supersymmetry and renormalizability requirements reads:

$$\mathcal{L} = [S_a{}^* S_a]_D + [\tfrac{4}{3} g_{abc} S_a S_b S_c + m_{ab} S_a S_b + s_a S_a]_F. \tag{4.9}$$

We shall use in this subsection the following complex notations:

$$\mathcal{A}_a = A_a - \mathrm{i} B_a, \qquad \mathcal{F}_a = F_a + \mathrm{i} G_a. \tag{4.10}$$

Auxiliary fields are eliminated by means of the equations of motion

$$\mathcal{F}_a{}^* + g_{abc}\, \mathcal{A}_b \mathcal{A}_c + m_{ab}\, \mathcal{A}_b + s_a = 0 \tag{4.11}$$

and formula (4.7) gives the potential of scalar fields at the tree approximation:

$$V(\mathcal{A}_a) = \tfrac{1}{2}\, \mathcal{F}_a{}^* \mathcal{F}_a = \tfrac{1}{2} |g_{abc}\, \mathcal{A}_b \mathcal{A}_c + m_{ab} \mathcal{A}_b + s_a|^2. \tag{4.12}$$

Let us denote by z_a the vacuum expectation values for the n complex fields $\mathcal{A}_a$. The potential has a vanishing minimum if the z_a's satisfy

$$g_{abc} z_b z_c + m_{ab} z_b + s_a = 0. \tag{4.13}$$

This is a system of n equations quadratic in the n unknown quantities z_a. In general it has solutions: all auxiliary fields have vanishing vacuum expectation values and supersymmetry is conserved [114, 93] (however, there exist exceptional situations, studied in subsection 4.6, for which the system (4.13) has no solution; then spontaneous supersymmetry breaking occurs [36, 96]).

We translate the physical scalar fields $\mathcal{A}_a$, i.e. the superfields S_a (this is compatible with supersymmetry):

$$\mathcal{A}_a = \tilde{\mathcal{A}}_a + z_a, \qquad S_a = \tilde{S}_a + \tfrac{1}{2} z_a. \tag{4.14}$$

Substituting in (4.9) we obtain immediately the Lagrangian density relative to the translated superfields $\tilde{S}_a$:

$$\mathcal{L} = [\tilde{S}_a{}^* \tilde{S}_a]_D + [\tfrac{4}{3} g_{abc} \tilde{S}_a \tilde{S}_b \tilde{S}_c + (2 g_{abc} z_c + m_{ab})\, \tilde{S}_a \tilde{S}_b]_F. \tag{4.15}$$

[10] Or "supersymmetric extension of the Goldstone mechanism".

The mass matrix for the translated superfields is (cf. ref. [93])

$$\tilde{m}_{ab} = 2g_{abc}z_c + m_{ab}. \tag{4.16}$$

This result can be verified on component fields. For spinors we have to look at mass terms and Yukawa couplings in the Lagrangian density generalizing (3.48), while for scalars we find:

$$\tilde{m}_{ab} = -\left\langle \frac{\partial \mathcal{F}_a^*}{\partial \mathcal{A}_b} \right\rangle = \left\langle \frac{\partial}{\partial \mathcal{A}_b} (g_{abc}\mathcal{A}_b\mathcal{A}_c + m_{ab}\mathcal{A}_b + s_a) \right\rangle = 2\, g_{abc}z_c + m_{ab}. \tag{4.17}$$

It is easy to demand that the Lagrangian density (4.9) be also invariant under some internal symmetry group. Whenever (4.13) has solutions, supersymmetry is conserved: the Lagrangian density still describes n chiral multiplets, each of them containing a Majorana spinor and a complex scalar of equal masses. But internal symmetry may be spontaneously broken, depending on the values taken by z_a; in this case multiplets inside an irreducible representation of the internal symmetry group may have different masses.

Some of these multiplets will be massless, since they contain massless Goldstone bosons arising from spontaneous internal symmetry. Let us verify this; suppose that

$$\mathcal{A}_b = z_b + \delta\mathcal{A}_b \tag{4.18}$$

is a solution of (4.13), which can be deduced from z_b by means of an internal symmetry transformation. By variation of (4.13) we find

$$(2\, g_{abc}z_c + m_{ab})\, \delta\mathcal{A}_b = 0 \tag{4.19}$$

which means that $\delta\mathcal{A}_b$ is an eigenvector of the mass matrix $\tilde{m}_{ab}$ (given by (4.16)) with eigenvalue zero. Thus the chiral multiplets corresponding to the Goldstone directions defined by $\delta\mathcal{A}_b$ have vanishing masses (this realizes the supersymmetric extension of the Goldstone mechanism).[11]

We come now to the case of a single irreducible representation. The example has been given by Salam and Strathdee with SU(2) × SU(2) as internal symmetry group [114], and generalized by O'Raifeartaigh [93]. Let us consider n chiral superfields belonging to an irreducible representation which is real and self-coupling in third order; this is true for the adjoint representation of SU(N) for $N \geqslant 3$, and for the adjoint × adjoint representation of $G \times G$, where G is a simple compact group (since the direct product of the antisymmetric structure constants forms a symmetric d-coupling). We have

$$g_{abc} = g\, d_{abc}, \qquad m_{ab} = m\, \delta_{ab}, \qquad s_a = 0 \tag{4.20}$$

and equations (4.13) read:

$$g\, d_{abc}z_bz_c + mz_a = 0. \tag{4.21}$$

Besides the trivial solution $z_a = 0$, which preserves both supersymmetry and internal symmetry, these equations have in general non-trivial solutions, for which supersymmetry is conserved but internal symmetry spontaneously broken, and massless chiral multiplets appear.

[11] The number of massless chiral multiplets obtained in this way may be smaller than the number of spontaneously broken symmetry generators, since two Goldstone bosons may belong to the same chiral multiplet. Consider for example SU (2) × U (1) as the internal symmetry group, and let $S = \binom{S_0}{S_-}$ be a doublet of massless chiral superfields describing the physical scalar fields $\binom{\varphi_0}{\varphi_-}$. If φ_0 takes a (real) non-vanishing vacuum expectation value, SU (2) × U (1) global symmetry is reduced down to an U (1) subgroup. $\sqrt{2}$ Im φ_0 and φ_- are the three Goldstone bosons. The model still describes two massless chiral multiplets; the one relative to S_0 contains one Goldstone boson ($\sqrt{2}$ Im φ_0), whereas the one relative to S_- contains two Goldstone bosons (φ_-).

All masses are proportional to m and can be evaluated using the single coupling constant g; one-parameter-mass-formulae are obtained.

One finds at the same time solutions where internal symmetry is conserved and solutions where it is spontaneously broken. They have the same energy (at least at the classical level; see also subsection 4.8) and we are not obliged to choose the latter.

We can avoid this by using more than one representation of the internal symmetry group, and obtain spontaneous breaking of internal symmetry in a necessary way. We describe now a method due to Fayet [35, 36], using a left-handed chiral superfield N invariant under the internal symmetry group, and chiral superfields S (left-handed) and T (right-handed) with the same transformation properties under internal symmetry.[12] We can restrict the Lagrangian density to be (see subsections 4.5 and 4.6):

$$\mathscr{L} = [S^\dagger S + T^\dagger T + N^* N]_D + [2hT^\dagger SN + sN]_F \tag{4.22}$$

in which h and s are real, and can be taken of opposite signs.

The system (4.13) which sets to zero the vacuum expectation values of auxiliary scalar fields, reads:

$$\begin{aligned} &\langle 2hT^\dagger S + s\rangle = 0 \\ &\langle 2hT^\dagger N\rangle = 0 \\ &\langle 2hNS\rangle = 0 \end{aligned} \tag{4.23}$$

in which $\langle\ \ \rangle$ denotes the vacuum expectation value of the $\frac{1}{2}(A - \mathrm{i}B)$ component of a superfield, at the classical level.

The system is soluble, and supersymmetry is conserved. We find (with h and s non-zero) that $\langle N\rangle = 0$, but $\langle S\rangle$ and $\langle T\rangle$ do not vanish; internal symmetry is spontaneously broken and massless chiral multiplets appear.

The method relies heavily on the absence in the Lagrangian density of terms proportional to the F or G components of N^2 or N^3 (otherwise we could choose $\langle S\rangle = \langle T\rangle = 0$ and internal symmetry would be conserved). We shall see in subsections 4.5 and 4.6 that this can be achieved by means of a new symmetry (R-invariance) demanded to the Lagrangian density; this invariance will be related with the conservation of a quantum number, sometimes interpreted as a lepton number.

Equation (4.23) determines only $\langle T^\dagger S\rangle$; we still have some freedom to fix $\langle S\rangle$ and $\langle T\rangle$ (such models contain massless non-Goldstone bosons; cf. subsection 4.8). Let us define

$$|\langle S\rangle| = \tfrac{1}{2}v'', \qquad |\langle T\rangle| = \tfrac{1}{2}v' \tag{4.24}$$

and

$$\eta = \frac{|\langle T^\dagger S\rangle|}{|\langle T^\dagger\rangle||\langle S\rangle|}, \qquad \mathrm{tg}\,\delta = \frac{|\langle T\rangle|}{|\langle S\rangle|} = \frac{v'}{v''}. \tag{4.25}$$

Equation (4.23) gives:

$$\tfrac{1}{2}\eta h v' v'' + s = 0. \tag{4.26}$$

[12] We can take $T^\dagger = S$ if S is in a real representation of the internal symmetry group SU (N). In this section we shall generally denote the conjugate of a chiral superfield by * when it is a singlet of the internal symmetry group, and by † when it can be a multiplet of this group. We do not consider here the fourth superfield N introduced in ref. [36] to obtain spontaneous supersymmetry breaking.

We see on (4.22) that the (left-handed) chiral superfield N acquires a mass by joining a (right-handed) chiral superfield (obtained by combination, using the mixing angle δ, of two superfields: one member of the multiplet $S^\dagger$ and one member of the multiplet T); for this pair of chiral superfields we find the mass:

$$\tfrac{1}{2} h (v'^2 + v''^2)^{1/2} \qquad (4.27)$$

while other multiplets remain massless.

The simplest example (with U(1) as the gauge group) describes a pair of massive chiral multiplets together with a massless one containing the Goldstone boson.

An important example concerns SU(2) × U(1). Then S and T are doublets. We obtain a pair of massive chiral superfields, and three massless ones containing the Goldstone bosons arising from the spontaneous breaking of SU(2) × U(1). These are three when SU(2) × U(1) is broken except for an U(1) subgroup. If we realize locally the internal symmetry, we obtain first a supersymmetric version of spontaneously broken SU(2) × U(1) gauge theories, and later a semi-realistic model of weak and electromagnetic interactions (cf. subsections 5.1 and 5.2). The pair of massive chiral multiplets describes the physical particles denoted $(e_0; \omega, \varphi)$; the three massless chiral multiplets containing the Goldstone particles join the three vector multiplets containing the gauge particles Z and $W_\pm$, and mass terms are obtained; the vacuum degeneracy observed earlier is lifted by the gauge interaction.

4.3. Spontaneous breaking of gauge invariance[13]

In a supersymmetric theory spontaneous breaking of internal symmetry leads to the appearance of massless chiral multiplets including Goldstone bosons. The situation is much more interesting when the internal symmetry group is realized locally: when this gauge invariance is spontaneously broken, Goldstone bosons are eliminated by the Higgs–Kibble mechanism [33, 69, 78], whereas gauge bosons acquire masses [114, 41].

We first discuss the features of spontaneous breaking of gauge invariance in a supersymmetric theory. We recall that a chiral multiplet describes a Majorana spinor and two real scalars; a massless vector multiplet, a vector and a Majorana spinor; a massive vector multiplet, a vector, a Dirac spinor and a real scalar.

The original Lagrangian density describes gauge-invariant interactions of massless vector multiplets, and chiral multiplets. Whenever a vector boson acquires a mass as a consequence of spontaneous breaking of gauge symmetry, a Goldstone boson is eliminated by the Higgs–Kibble mechanism. If supersymmetry is conserved, a whole vector multiplet becomes massive, melting into the chiral multiplet generated from the Goldstone boson under supersymmetry transformations; Majorana spinors of both multiplets join together and form the Dirac spinor of the massive vector multiplet.

The simplest model describes nothing else than this massive vector multiplet. Thus we are interested in the U(1) gauge invariant interaction of a massless vector multiplet with a single chiral multiplet [84, 37]; in the latter reference it is shown also how to derive directly the self-interaction for a massive vector multiplet, and how it is connected with spontaneously broken gauge invariance (see also refs. [80, 81]).

[13] Or "supersymmetric extension of the Higgs mechanism".

Let V and S be a vector and a left-handed chiral superfield respectively. The former describes a vector multiplet of generalized gauge fields, the latter a chiral multiplet of matter fields. The generalized gauge transformations [130] are parametrized by a left-handed chiral superfunction Λ:

$$V \to V + \mathrm{i}(\Lambda - \Lambda^*), \qquad S \to \exp(-2\mathrm{i}e\Lambda)S. \tag{4.28}$$

The Lagrangian density reads

$$\mathscr{L} = \mathscr{L}_0 + [S^* \exp(2eV)S + \xi V]_D \tag{4.29}$$

in which

$$\mathscr{L}_0 = -\tfrac{1}{4} V^{\mu\nu} V_{\mu\nu} - \tfrac{1}{2} \mathrm{i}\bar{\lambda}\partial\!\!\!/\lambda + \tfrac{1}{2} D^2 \tag{4.30}$$

is the Lagrangian density of the free massless vector multiplet.

$\mathscr{L}$ becomes polynomial in the Wess–Zumino gauge; V^μ, λ and D are the only non-vanishing components of V, while the components of S are denoted by A, B, ψ, F, G as usual. We define the complex field

$$\varphi = -\mathrm{i}(A - \mathrm{i}B)/\sqrt{2} \tag{4.31}$$

and use the equations of motion

$$F = G = 0, \qquad D + \xi + e\varphi^\dagger\varphi = 0 \tag{4.32}$$

to eliminate auxiliary fields. We obtain:

$$\begin{aligned}\mathscr{L} = &-\tfrac{1}{4} V^{\mu\nu} V_{\mu\nu} - \tfrac{1}{2} \mathrm{i}\bar{\lambda}\partial\!\!\!/\lambda - \mathrm{i}\bar{\psi}_{\mathrm{L}} D\!\!\!\!/\,\psi_{\mathrm{L}} - \mathrm{D}^\mu\varphi^\dagger \mathrm{D}_\mu\varphi \\ &- \tfrac{1}{2}|\xi + e\varphi^\dagger\varphi|^2 + \mathrm{i}e\sqrt{2}(\bar{\psi}_{\mathrm{L}}\lambda\varphi + \varphi^\dagger\bar{\lambda}\psi_{\mathrm{L}})\end{aligned} \tag{4.33}$$

in which the covariant derivative D_μ is defined by

$$\mathrm{i}\mathrm{D}_\mu = \mathrm{i}\partial_\mu - eV_\mu. \tag{4.34}$$

Spontaneous symmetry breaking is governed by the potential

$$V(\varphi) = \tfrac{1}{2}|\xi + e\varphi^\dagger\varphi|^2. \tag{4.35}$$

There are two different situations, already discussed in subsection 4.1 (see fig. 1):
– if $\xi e < 0$, auxiliary fields have vanishing vacuum expectation values but φ has not; supersymmetry is conserved and gauge invariance spontaneously broken.
– if $\xi e > 0$, φ has a vanishing vacuum expectation value but the auxiliary field D has not; gauge invariance is conserved and supersymmetry spontaneously broken (see next subsection).

Here we suppose $\xi e < 0$. The complex field φ has a non-vanishing vacuum expectation value. Using the remaining gauge freedom we can choose

$$\langle A \rangle = 0, \qquad \langle B \rangle = v \tag{4.36}$$

where v is a positive real number satisfying

$$\tfrac{1}{2} ev^2 + \xi = 0. \tag{4.37}$$

Gauge invariance is spontaneously broken. The Goldstone boson is eliminated by the Higgs mechanism, whereas the gauge boson becomes massive. We choose a gauge where $A = 0$; translating B

$$\tilde{B} = B - v \tag{4.38}$$

we obtain a mass term for the Dirac spinor[14]

$$E = \psi_L + \lambda_R \tag{4.39}$$

and the Lagrangian density reads:

$$\begin{aligned}\mathscr{L} = &-\tfrac{1}{4} W^{\mu\nu}W_{\mu\nu} - \mathrm{i}\bar{E}\partial\!\!\!/ E - \tfrac{1}{2}\,\partial^\mu\tilde{B}\partial_\mu\tilde{B}\\ &- \tfrac{1}{2}(m + e\tilde{B})^2\, W^\mu W_\mu - \tfrac{1}{2}(m\tilde{B} + \tfrac{1}{2}e\tilde{B}^2)^2 - \mathrm{i}(m + e\tilde{B})\,\bar{E}E + e\bar{E}_L W\!\!\!\!/ E_L.\end{aligned} \tag{4.40}$$

It describes a vector W^μ, a Dirac spinor E and a real scalar $\tilde{B}$, which have the same mass

$$m = ev. \tag{4.41}$$

On this example, we exhibit the following features of spontaneous breaking of gauge invariance in a supersymmetric theory:

– a vector (V^μ) becomes massive, eliminating the Goldstone boson (A),

– a real scalar field (B) is translated but remains a physical field, massive,

– the Majorana spinors belonging to the vector and chiral multiplets (λ and ψ respectively) join together to form a massive Dirac spinor (E).

Finally the massless vector multiplet containing the gauge vector boson melts with the chiral multiplet containing the would-be Goldstone boson, to form a single massive vector multiplet.

Now we give a few remarks.

By adding a Dirac spinor E, we have made the Higgs model supersymmetric; the potential $V(\varphi)$ is given by (4.35); supersymmetry fixes the coefficient of $(\varphi^\dagger\varphi)^2$ in $V(\varphi)$ to be $\frac{1}{2}e^2$. The model depends of one mass, and one coupling constant only. Supersymmetry restricts considerably the possible Higgs sector in a gauge theory. Note also that the potential has a vanishing vacuum expectation value; no cosmological term is induced when coupling with gravitation is considered [139]. Renormalizability is spoilt by the anomaly due to the V–A coupling of the vector particle. This is not a problem since we present the model for its simplicity; similar ones can be constructed, where the current is a pure vector and there is no anomaly [41, 38].

The Lagrangian density (4.40) is obviously invariant under phase transformations

$$E \to \mathrm{e}^{-\mathrm{i}\alpha}E. \tag{4.42}$$

We can define a conserved "fermionic number" carried by the Dirac spinor E. In terms of superfields, it means that the Lagrangian density (4.29) is invariant under the set of transformations

$$\begin{cases} V(x,\theta,\bar{\theta}) \to V(x,\theta\mathrm{e}^{-\mathrm{i}\alpha}, \bar{\theta}\mathrm{e}^{\mathrm{i}\alpha}) \\ S(x,\theta,\bar{\theta}) \to S(x,\theta\,\mathrm{e}^{-\mathrm{i}\alpha}, \bar{\theta}\mathrm{e}^{\mathrm{i}\alpha}). \end{cases} \tag{4.43}$$

This gives for physical components

$$\begin{cases} V^\mu \to V^\mu \\ \lambda \to \mathrm{e}^{\gamma_5\alpha}\lambda \end{cases} \qquad \begin{cases} \varphi \to \varphi \\ \psi \to \mathrm{e}^{-\gamma_5\alpha}\psi. \end{cases} \tag{4.44}$$

We recover (4.42) immediately. The question of fermion number conservation will be studied in a general way in subsection 4.5.

To conclude this subsection we show how (4.40) can be derived directly as a supersymmetric

[14] $\frac{1}{2}(1 - \mathrm{i}\gamma_5)$ and $\frac{1}{2}(1 + \mathrm{i}\gamma_5)$ are the left-handed and right-handed projectors, respectively.

self-interaction for a massive vector multiplet described by the superfield W [37]. We first write the free Lagrangian density:

$$\mathscr{L}=\mathscr{L}_0+\tfrac{1}{2}m^2[W^2]_D=-\tfrac{1}{4}W^{\mu\nu}W_{\mu\nu}-\tfrac{1}{2}\mathrm{i}\bar{\lambda}\not{\partial}\lambda+\tfrac{1}{2}D^2$$
$$+\tfrac{1}{2}m^2(2CD-\partial^\mu C\partial_\mu C-W^\mu W_\mu+M^2+N^2-2\mathrm{i}\bar{\chi}\lambda-\mathrm{i}\bar{\chi}\not{\partial}\chi). \tag{4.45}$$

While only W^μ and λ are physical in a massless vector multiplet, W^μ, λ, χ and C are physical in the massive one: adding the term $\frac{1}{2}m^2[W^2]_D$ to the Lagrangian density breaks explicitly the generalized gauge invariance of the massless Lagrangian density $\mathscr{L}_0$, and doubles the number of degrees of freedom by making W^μ massive, χ and C physical. Eliminating auxiliary fields D, M, N we obtain, for the physical fields

$$\begin{array}{rl} W^\mu & \text{vector} \\ E=m\,\chi_{\mathrm{L}}+\lambda_{\mathrm{R}} & \text{Dirac spinor} \\ \tilde{B}=mC & \text{real scalar} \end{array} \tag{4.46}$$

the usual expression of the Lagrangian density.

In order to define a supersymmetric interaction we study the Lagrangian density

$$\mathscr{L}=\mathscr{L}_0+[f(W)]_D. \tag{4.47}$$

It is particularly simple when the second derivative of f is an exponential function. Besides the trivial case of the free massive vector multiplet we find:

$$\mathscr{L}=\mathscr{L}_0+[\tfrac{1}{4}v^2\exp(2eW)+\xi W]_D. \tag{4.48}$$

ξ and e are the only parameters since v can be redefined by translating the scalar component C. For $\xi e<0$ the potential has a vanishing minimum and supersymmetry is conserved. We can choose v in order for $\langle C\rangle$ to vanish; it satisfies

$$\tfrac{1}{2}ev^2+\xi=0. \tag{4.49}$$

Eliminating auxiliary fields D, M, N, and defining the physical fields

$$\begin{cases} W^\mu \\ E=ev\exp(eC)\,\chi_{\mathrm{L}}+\lambda_{\mathrm{R}} \\ \tilde{B}=v(\exp(eC)-1) \end{cases} \tag{4.50}$$

we obtain for the non-polynomial Lagrangian density (4.48) the polynomial expression (4.40), which is the Lagrangian density of a spontaneously broken gauge theory. Formulas (4.45) and (4.46) are recovered when expanding (4.48) and (4.50) in the limit $e\to 0^+$, $m=ev$ fixed (i.e. $\xi\to-\infty$).

Now we give the connection between the two possible ways to obtain the model. Instead of studying (4.29) in the Wess–Zumino gauge, one can choose a different gauge condition, which preserves supersymmetry: the chiral superfield S is completely gauged-away. Performing on (4.29) the generalized gauge transformation

$$\begin{cases} S\to-\tfrac{1}{2}\mathrm{i}v \\ V\to W \end{cases} \tag{4.51}$$

we recover expression (4.48) of the Lagrangian density.

We have described the supersymmetric extension of the Higgs model. Similar mechanisms work for non-Abelian gauge groups; massless vector multiplets become massive, melting with chiral multiplets containing the would-be Goldstone bosons. The example of SU (2) × U (1) will be described in subsection 5.1. We shall now study a method to obtain spontaneous supersymmetry breaking.

4.4. Spontaneous supersymmetry breaking: the U(1) method

In order to lift the mass-degeneracy inside a multiplet it is necessary to obtain a breaking for supersymmetry. We have already seen that the problem of spontaneous supersymmetry breaking has very special features; these are connected with the presence of the Hamiltonian in the supersymmetry algebra, and with the special form of the potential of the scalar fields, at the tree approximation.

Nevertheless spontaneous supersymmetry breaking is indeed possible, and the first example has been given by Fayet and Iliopoulos [41], using the supersymmetric U (1) gauge invariant model of Wess and Zumino [132]. A vector superfield V interacts with two chiral superfields S and T, left-handed and right-handed respectively. However, the method can be explained most simply when only the superfields V and S are present [84, 37].

The Lagrangian density has already been written in the previous subsection, and we can use again formulas (4.29) to (4.35). Here we suppose $\xi e > 0$. The physical scalar field φ has a vanishing vacuum expectation value and gauge invariance is preserved.

The vacuum expectation values for auxiliary scalar fields are

$$\langle F\rangle = \langle G\rangle = 0, \qquad \langle D\rangle = -\,\xi. \tag{4.52}$$

Equations of motion have no supersymmetric stationary solution, and the vacuum state is not supersymmetric.

If Q_α's are the supersymmetry generators, we have, according to (4.1)

$$\langle\{Q_\alpha, \bar{\lambda}_\beta\}\rangle = -\,\xi(\gamma_5)_{\alpha\beta} \tag{4.53}$$

while the similar expression involving ψ vanishes. It follows from subsection 4.1 that supersymmetry is spontaneously broken, and λ is the massless Goldstone spinor.

Spontaneous supersymmetry breaking has – as a consequence – the existence of mass splittings inside multiplets: V describe the gauge vector boson V^μ and the Majorana spinor λ, both massless. But S describes the left-handed Dirac spinor $\psi_{\rm L}$ massless, and the complex scalar φ, with mass $\mu = \sqrt{\xi e}$.

Now we consider the case of two chiral superfields S left-handed, and T right-handed. They describe as physical fields $(\psi_{\rm L}, \varphi'')$ and $(\psi_{\rm R}, \varphi')$, where ψ is a Dirac spinor, φ'' and φ' two complex scalars exchanged under parity. We start from the parity-invariant Lagrangian density (3.83); the auxiliary field D appears as a pseudoscalar. If we add to the Lagrangian density the term ξD, which preserves both supersymmetry and gauge invariance, parity is broken explicitly but softly: by terms of dimension two at most in the Lagrangian density [41].

The latter reads:

$$\mathcal{L} = \mathcal{L}_0 + [S^* \exp(2eV)\, S + T^* \exp(-2eV)\, T]_D + 2m[T^*S]_F + \xi D \tag{4.54}$$

and the potential is (with notations of ref. [35])

$$V(\varphi'', \varphi') = m^2(\varphi''^\dagger\varphi'' + \varphi'^\dagger\varphi') + \tfrac{1}{2}|\xi + e(\varphi''^\dagger\varphi'' - \varphi'^\dagger\varphi')|^2. \tag{4.55}$$

If $m = 0$, V has a vanishing minimum; gauge invariance is spontaneously broken and supersymmetry conserved. The vacuum state is not completely determined and there are massless particles (see subsection 4.8).

If $0 < \xi e < m^2$, gauge invariance is conserved, supersymmetry spontaneously broken, λ is the Goldstone spinor.

If $0 < m^2 < \xi e$, φ' has to be translated; both gauge invariance and supersymmetry are spontaneously broken; the Goldstone spinor is a combination of λ_L and ψ_L.

This method of spontaneous supersymmetry breaking assumes that the Lagrangian density contains a term ξD, i.e. that the gauge group contains an invariant factor U (1). A different method will be given in subsection 4.6.

4.5. What about fermion number?

Supersymmetry naturally involves Majorana spinors, whereas nature seems to prefer Dirac spinors, in association with the conservation of quantum numbers such as lepton or baryon numbers. This problem has been raised very early, and appears to be more difficult when mass-terms are to be obtained [49, 113, 26].

4.5.1. Fermion number

γ_5 transformations play an important role in the definition of such a quantum number, as pointed out in refs. [110, 35]. In subsection 4.3 we have studied the self-interaction of a massive vector multiplet. The Lagrangian density is invariant under phase transformations for the Dirac spinor, generated by a γ_5 transformation on the Majorana spinor argument of the superfields [37].

In a more general way we consider a gauge-invariant supersymmetric theory describing vector superfields V_K and (left-handed) chiral superfields S_a. We define a set of transformations (denoted by R) which generalize (4.43) in the simplest way:

$$\begin{cases} V_K(x,\theta,\bar\theta) \to V_K(x,\theta e^{-i\alpha}, \bar\theta e^{i\alpha}) \\ S_a(x,\theta,\bar\theta) \to S_a(x,\theta e^{-i\alpha}, \bar\theta e^{i\alpha}) \,. \end{cases} \tag{4.56}$$

D-components of vector superfields are R-invariant, but F- or G-components of chiral superfields are not. The following terms in a Lagrangian density are R-invariant:

– terms describing free vector multiplets, or self-interacting Yang–Mills vector multiplets,

– linear combinations of auxiliary components D_K (but only those relative to invariant Abelian gauge subgroups can appear in the Lagrangian density),

– terms describing, for chiral multiplets, kinetic energy and interactions with vector multiplets.

But terms proportional to F- or G-components of chiral superfields $S_aS_bS_c$, S_aS_b or S_a, are not R-invariant. Demanding R-invariance to the Lagrangian density yields the following constraints on the parameters g_{abc}, m_{ab} and s_a (introduced in subsection 4.2)

$$g_{abc} = m_{ab} = s_a = 0. \tag{4.57}$$

Transformation (4.56) gives, for physical components,

$$\begin{cases} V_K^\mu \to V_K^\mu \\ \lambda_K \to \exp(\gamma_5\alpha)\,\lambda_K \end{cases} \qquad \begin{cases} A_a - iB_a \to A_a - iB_a \\ \psi_a \to \exp(-\gamma_5\alpha)\,\psi_a. \end{cases} \tag{4.58}$$

Majorana spinors can be written as two-component Dirac spinors, for which we have

$$\lambda_{K\mathrm{R}} \to \mathrm{e}^{-\mathrm{i}\alpha}\lambda_{K\mathrm{R}}, \qquad \psi_{a\mathrm{L}} \to \mathrm{e}^{-\mathrm{i}\alpha}\,\psi_{a\mathrm{L}}. \tag{4.59}$$

We can define a conserved "fermion number", which vanishes for bosons and equals (-1) for right-handed Dirac spinors coming from gauge multiplets and left-handed Dirac spinors coming from matter multiplets.

Mass terms are obtained if some of the physical scalar fields acquire non-vanishing vacuum expectation values, by spontaneous breaking of gauge invariance. As an example (in subsection 4.3) we have obtained a massive Dirac spinor $E = \psi_{\mathrm{L}} + \lambda_{\mathrm{R}}$ with fermion number -1. We shall see other examples in section 5.

4.5.2. A new quantum number

Using formulas (4.56) we succeeded in defining a conserved fermion number. But all direct mass terms or interaction terms for chiral multiplets only have been forbidden, and this may be too strong a restriction. It is useful to define R-invariance in a more flexible way, allowing to select only the desired terms in the Lagrangian density (we have already used this possibility in subsection 4.2). Thus a generalized definition of R-invariance was introduced in refs. [110, 35], and the conserved quantum number associated with it was interpreted as the leptonic number in the semi-realistic model described in the latter reference. The new feature is that the leptonic number is not carried by fermions only, but also by some of the scalar bosons.

We define now R-invariance by combining a γ_5 transformation on the argument of superfields with global phase transformations on chiral superfields:

$$\begin{cases} V_K(x,\theta,\bar\theta) \to V_K(x,\theta\mathrm{e}^{-\mathrm{i}\alpha},\bar\theta\mathrm{e}^{\mathrm{i}\alpha}) \\ S_a(x,\theta,\bar\theta) \to \exp(\mathrm{i}n_a\alpha)\,S_a(x,\theta\mathrm{e}^{-\mathrm{i}\alpha},\bar\theta\mathrm{e}^{\mathrm{i}\alpha}). \end{cases} \tag{4.60}$$

A product of left-handed chiral superfields transforms according to

$$\prod_a S_a(x,\theta,\bar\theta) \to \exp\left\{\mathrm{i}\left(\sum_a n_a\right)\alpha\right\}\prod_a S_a(x,\theta,\bar\theta). \tag{4.61}$$

Its F- or G-components are R-invariant if and only if

$$\sum_a n_a = 2. \tag{4.62}$$

The new definition of R-invariance is less restrictive for the Lagrangian density: constraints (4.57) are replaced by:

$$\begin{cases} g_{abc} = 0 \ \text{unless}\ n_a + n_b + n_c = 2 \\ m_{ab} = 0 \ \text{unless}\ n_a + n_b = 2 \\ s_a \ \ = 0 \ \text{unless}\ n_a = 2. \end{cases} \tag{4.63}$$

R-transformations act on the component fields as follows:

$$\begin{cases} V_K^\mu \to V_K^\mu \\ \lambda_K \to \exp(\gamma_5\alpha)\,\lambda_K \\ D_K \to D_K \end{cases} \qquad \begin{cases} A_a - \mathrm{i}B_a \to \exp(\mathrm{i}n_a\alpha)\,(A_a - \mathrm{i}B_a) \\ \psi_a \to \exp\{\gamma_5(n_a - 1)\alpha\}\,\psi_a \\ F_a + \mathrm{i}G_a \to \exp\{\mathrm{i}(n_a - 2)\alpha\}\,(F_a + \mathrm{i}G_a) \end{cases} \tag{4.64}$$

in which the last line can be omitted, since it follows from equations of motion for auxiliary fields.

R will be also an invariance of the vacuum, provided physical scalar fields $(A_a - \mathrm{i}B_a)$ have vanishing vacuum expectation values if $n_a \neq 0$. Then we can define a conserved quantum number, which is n_a for $(A_a - \mathrm{i}B_a)$, $(n_a - 1)$ for the left-handed Dirac spinor $\psi_{a\mathrm{L}}$ (or as well $(1 - n_a)$ for its conjugate $\psi_{a\mathrm{R}}$); as before, it is 0 for V_K^μ, and (-1) for the right-handed Dirac spinor $\lambda_{K\mathrm{R}}$ (or as well $+1$ for its conjugate $\lambda_{K\mathrm{L}}$).

For a *gauge invariant* theory[15] in which all spinor fields have quantum number ± 1, n_a has the only possible values 0 and 2. If the Lagrangian density contains at least one *direct* mass term, or interaction term, for chiral superfields, we must have both superfields with $n_a = 0$ and superfields with $n_a = 2$. The model describes:

$$\begin{cases} \text{vectors with quantum number } 0 \\ \text{spinors with quantum number } \pm 1 \\ \text{scalars with quantum number } 0 \text{ and } 2. \end{cases}$$

The appearance of bosons carrying two units of quantum number is rather general in this type of model (but it can be avoided when all mass terms are obtained from spontaneous breaking of gauge invariance). This original feature does not appear as a problem. In the example described in section 5, the (complex) scalar neutral heavy lepton ω has leptonic number 2; it couples electron to heavy electron, which have lepton number 1 and (-1) respectively.

4.5.3. A toy-model to play with R-invariance and parity

R-invariance is an essential tool in order to obtain restrictions for Lagrangian densities, and to construct Dirac spinors. It has already been used in subsection 4.2 to obtain a spontaneous breaking of internal symmetry, and will be used in the next subsection to obtain a new method for spontaneous supersymmetry breaking. We explain now, on a very simple example, how it works.

Let S_1 and S_2 be two left-handed superfields, which transform under R according to (4.60) with $n_1 = \frac{2}{3}$, $n_2 = \frac{4}{3}$. The most general Lagrangian density compatible with supersymmetry, renormalizability and R-invariance reads:

$$\mathscr{L} = [S_1{}^*S_1 + S_2{}^*S_2]_D + [2m\, S_1 S_2 + \tfrac{4}{3} g S_1^3]_F. \tag{4.65}$$

Parity, defined in the usual way, is conserved. In terms of the physical fields we have:

$$\begin{aligned}\mathscr{L} = &-\tfrac{1}{2}\mathrm{i}(\bar\psi_1 \not\partial \psi_1 + \bar\psi_2 \not\partial \psi_2) - \tfrac{1}{2}(\partial^\mu A_1 \partial_\mu A_1 + \partial^\mu A_2 \partial_\mu A_2 + \partial^\mu B_1 \partial_\mu B_1 + \partial^\mu B_2 \partial_\mu B_2) \\ &- \mathrm{i}m\bar\psi_1\psi_2 - \mathrm{i}g\bar\psi_1(A_1 - \gamma_5 B_1)\psi_1 - \tfrac{1}{2}m^2(A_1^2 + B_1^2) - \tfrac{1}{2}|m(A_2 - \mathrm{i}B_2) + g(A_1 - \mathrm{i}B_1)^2|^2. \end{aligned} \tag{4.66}$$

All particles have mass m. The model describes a Dirac spinor $\psi = \psi_{2\mathrm{L}} + \psi_{1\mathrm{R}}$, and two complex scalars $(A_1 - \mathrm{i}B_1)/\sqrt{2}$ and $(A_2 - \mathrm{i}B_2)/\sqrt{2}$ with quantum numbers 1, 2 and 4 respectively (after rescaling by a factor three).

A conserved parity operator can be defined (with A_1, A_2 scalars, B_1, B_2 pseudoscalars, ψ_1 and ψ_2 of the same parity), but it does not commute with the conserved quantum number. R-invariance is appropriate to construct easily Dirac spinors carrying a quantum number ± 1; mass terms can be obtained, and we have all desirable properties for lepton number. On the other hand, it is not

[15] When only chiral superfields are involved, a rescaling can be performed on the conserved quantum number, and different assignments are possible; we shall give such an example soon.

possible, in general, to construct a parity operator compatible with the assignment of quantum number; we have not yet all desirable properties for baryon-number. We shall come back to this question in subsection 4.7.

4.6. Spontaneous supersymmetry breaking: a second method

4.6.1. Spontaneous supersymmetry breaking for chiral superfields

Let us consider a system of chiral superfields, as in subsection 4.2. The Lagrangian density, and the potential of scalar fields at the tree approximation, are given by formulas (4.9) and (4.12). We have seen that, generally, the system

$$g_{abc} z_b z_c + m_{ab} z_b + s_a = 0 \tag{4.67}$$

has solutions, and supersymmetry is conserved.

However, there exist exceptional situations for which the system has no solution [36, 96]; the minimum of the potential is strictly positive, and one at least of the auxiliary fields has a non-vanishing vacuum expectation value. Supersymmetry is spontaneously broken and mass splittings inside multiplets are obtained.[16] We shall find such situations when some of the free parameters g_{abc}, m_{ab} and s_a are taken equal to zero; this is achieved by means of the R-invariance described in the previous subsection. A general study has been made by O'Raifeartaigh [96].

We give now the example of Fayet [36], which is also invariant under global SU(2) × U(1) transformations. S and T are doublets of left-handed and right-handed superfields, N and N' are a triplet and a singlet of left-handed superfields. R-transformations are defined by (4.60), with $n = 0$ for S and $T^\dagger$, $n = 2$ for N and N'. The most general Lagrangian density compatible with supersymmetry, SU(2) × U(1) and R-invariance is:

$$\mathscr{L} = [S^\dagger S + T^\dagger T + N^\dagger N + N'^* N']_D + [2\,T^\dagger (h\tau N + h'N')S + sN']_F. \tag{4.68}$$

The system (4.67) which sets the vacuum expectation values of auxiliary scalar fields to zero, reads:

$$\begin{aligned} &\langle 2hT^\dagger \tau S\rangle = 0 \\ &\langle 2h'T^\dagger S + s\rangle = 0 \\ &\langle 2T^\dagger (h\tau N + h'N')\rangle = 0 \\ &\langle 2(h\tau N + h'N')S\rangle = 0 \end{aligned} \tag{4.69}$$

in which $\langle\ \ \rangle$ denotes the vacuum expectation value of the $\frac{1}{2}(A - \mathrm{i}B)$ component of a superfield, at the classical level.

The first two equations are not compatible, and the system has no solution. Spontaneous supersymmetry breaking is obtained. SU(2) × U(1) also is spontaneously broken, and reduced to an U(1) subgroup.

Let θ be the SU(2) × U(1) mixing angle:

$$\operatorname{tg}\theta = h'/h. \tag{4.70}$$

Exactly as the vector boson fields Z^μ, A^μ and $W^\mu_\pm$ are defined from W^μ and W'^μ in SU(2) × U(1)

[16] Unless one superfield is completely decoupled, and responsible alone for the non-vanishing minimum of the potential [96].

gauge theories, here we define the superfields:

$$\begin{cases} N_Z = \cos\theta\, N_3 + \sin\theta\, N' \\ N_\gamma = -\sin\theta\, N_3 + \cos\theta\, N' \end{cases} \qquad N_- = (N_1 + \mathrm{i}N_2)/\sqrt{2} \tag{4.71}$$

together with

$$S = \begin{pmatrix} S_0 \\ S_- \end{pmatrix}, \quad T = \begin{pmatrix} T_0 \\ T_- \end{pmatrix}. \tag{4.72}$$

The only auxiliary component taking a non-vanishing vacuum expectation value is the F-component of the superfield N_γ; thus the corresponding Majorana spinor ζ_γ is the Goldstone spinor.[17] Since this auxiliary component f_γ is coupled only to the scalar components of the superfields S_- and T_- the zeroth-order mass matrix is supersymmetric except for a term proportional to the A-component of the superfield $T_-^\dagger S_-$. As a result all particles in a multiplet have equal zeroth-order masses, excepted those described by the "charged" superfields S_- and T_-; for these, linear relations between mass^2 are obtained.

The vacuum state is not completely determined at the tree approximation: $\langle N_\gamma\rangle$ remains undetermined, together with the ratio

$$\operatorname{tg}\delta = |\langle T_0\rangle|/|\langle S_0\rangle| \tag{4.73}$$

which defines δ, the mixing angle between the "neutral" superfields S_0^* and T_0. Higher order calculations have to be performed [72]. There exist non-Goldstone scalar particles massless at zeroth order.

This can easily be understood: let $\tilde{w}_-$ and $\tilde{z}$ be the Goldstone bosons and ζ_γ the Goldstone spinor, arising from spontaneous breaking of SU(2) × U(1) and supersymmetry, respectively. Zeroth order mass degeneracy remains for "neutral" multiplets (i.e. those which are not coupled to N_γ) and supersymmetry associates to $\tilde{z}$ and ζ_γ the bosons z (real) and ω_γ (complex), both massless at zeroth order.

4.6.2. Application to gauge theories

At this point we know that spontaneous supersymmetry breaking can occur:

– for chiral superfields only (using R-invariance),

– for a gauge theory, when the gauge group contains an invariant Abelian factor U(1).

An example of spontaneous supersymmetry breaking in a theory with a semi-simple gauge group can now be given easily: in the previous model we realize SU(2) gauge invariance locally, adding a SU(2) Yang–Mills triplet of vector superfields V. Equations written in (4.69) remain unchanged, thus non compatible. Supersymmetry is spontaneously broken, and ζ_γ is still the Goldstone spinor.

The most interesting situation is obtained when both invariances SU(2) and U(1) are realized locally, by adding the vector multiplet V, triplet, and V', singlet; the model is particularly remarkable when h and h', which fix the coupling of S and T with N and N', are equal to the gauge

[17] In papers [36, 38], we now denote the singlet (triplet) chiral superfield by N' (N), and its Majorana component by ζ' ($\boldsymbol{\zeta}$); h' and h are the coupling constants with S and T.

coupling constants g and g' which fix the coupling of S and T with V and V' (together with self-couplings of V, and couplings of V with N). Then a single angle θ, given by:

$$\mathrm{tg}\,\theta = h'/h = g'/g \tag{4.74}$$

determines the mixing for both vector and chiral multiplets, and the model describes the following superfields:

$$\begin{cases} V_Z = \cos\theta\, V_3 + \sin\theta\, V' \\ V_\gamma = -\sin\theta\, V_3 + \cos\theta\, V' \end{cases} \qquad V_- = \frac{V_1 + \mathrm{i} V_2}{\sqrt{2}}$$

$$\begin{cases} N_Z = \cos\theta\, N_3 + \sin\theta\, N' \\ N_\gamma = -\sin\theta\, N_3 + \cos\theta\, N' \end{cases} \qquad N_- = \frac{N_1 + \mathrm{i} N_2}{\sqrt{2}}$$

$$S = \begin{pmatrix} S_0 \\ S_- \end{pmatrix}, \qquad T = \begin{pmatrix} T_0 \\ T_- \end{pmatrix}. \tag{4.75}$$

Supersymmetry is spontaneously broken, together with SU(2) X U(1) gauge invariance, reduced to an U(1) subgroup. We shall now see that this model is in fact invariant under a larger algebra.

4.7. A larger algebra

In subsection 4.5 we have seen how to define a conserved quantum number in supersymmetric theories. But something more is necessary in order to describe the real world: for hadrons we have to define commuting baryon number and parity operators; for leptons we need both an electronic and a muonic number. If fermions and bosons in an irreducible representation have different conserved quantum numbers (baryonic, or electronic and muonic) these have to be carried by Fermi-type generators of the algebra. In both cases, this will lead us to considering larger algebras involving two Majorana charges, instead of a single one. This fact was realized by Salam and Strathdee for baryon number and parity [111], and by Fayet for the two leptonic numbers [38]. In the first case the algebra ("complex supersymmetry") contains one Dirac spinor charge; in the second case ("hypersymmetry") it includes two left-handed Dirac charges, often appearing as a doublet of a global internal symmetry group SU(2) X U(1), which might be relevant to describe electron–muon universality.

In fact a very simple supersymmetric model, studied by Ferrara and Zumino [49] and Salam and Strathdee [113], appears to have the above properties, and is automatically invariant under the larger algebra. It describes the supersymmetric interaction of Yang–Mills vector superfields V_K with massless left-handed chiral superfields N_K, both in the adjoint representation of the non-abelian gauge group SU(N); V_K describes a vector V_K^μ and a Majorana spinor λ_K, and N_K a Majorana spinor ζ_K and two real scalars a_K and b_K. One can define a conserved baryon number, commuting with parity; it is carried by the Dirac spinors $(\lambda_K + \mathrm{i}\zeta_K)/\sqrt{2}$. A complex conserved vector spinor current has been written by Bardeen and Zumino [138] (formula (3.112)).

The model is also invariant under global SU(2) X U(1) transformations [38]. V_K^μ, a_K and b_K are SU(2) isosinglets whereas $\binom{\zeta_{KL}}{\lambda_{KL}}$ are SU(2) isodoublets; the U(1) factor is associated with R-invariance (N_K having R-character $n = 2$).

It follows that the model is invariant under a set of two two-component Dirac charges transforming as an SU(2) isodoublet. Using an Abelian subgroup U(1) X U(1) of SU(2) X U(1) we can define two conserved quantum numbers. The two spinorial charges carry (1,0) and (0,1) units of these quantum numbers, respectively.[18]

This example describes only massless particles: masses can be introduced for models derived from those given by Lagrangian densities (4.22) and (4.68). Let us give the general idea. We start from a gauge and R-invariant supersymmetric theory, and associate[19] with every gauge vector supermultiplet $\mathscr{V}_K = (V_K^\mu; \lambda_K)$ a massless chiral supermultiplet $\mathscr{N}_K = (\zeta_K; a_K, b_K)$ (for Yang–Mills factors of the gauge group, the index K labels the adjoint representation); the set of fields described by $\mathscr{V}_K$ and $\mathscr{N}_K$ (one vector, two Majorana spinors and one complex scalar) is called a "gauge hypermultiplet". Remaining chiral multiplets are associated in pairs: S left-handed, and T right-handed, with the same gauge transformation properties; in this way we obtain "matter hypermultiplets", each of them describing a Dirac spinor and two complex scalars with the same gauge transformation properties.

When equalities between coupling constants defining supersymmetric interactions are satisfied, models become invariant under a larger algebra involving two Majorana spinors [111, 38]. One of the simplest examples is relative to the U(1) gauge invariant interaction of a gauge hypermultiplet with a matter hypermultiplet. The former is described by the vector superfield V and the neutral left-handed superfield N; the latter by the charged chiral superfields S and T, which are left-handed and right-handed respectively. The Lagrangian density is obtained from (4.22), with U(1) internal symmetry realized locally:

$$\mathscr{L} = \mathscr{L}_0 + [N^*N]_D + [S^* \exp(2eV)\, S + T^* \exp(-2eV)\, T]_D + [4eT^*SN + sN]_F \qquad (4.76)$$

in which $\mathscr{L}_0$ is the free Lagrangian density for the vector superfield V. The choice of a single coupling constant e ensures that the model has a richer symmetry.

Thus we can describe interactions of gauge hypermultiplets (with one vector, two Majorana spinors and one complex scalar) and matter hypermultiplets (with one Dirac spinor and two complex scalars). As we have obtained the supersymmetric extension of the Higgs mechanism in subsection 4.3, we can obtain here an hypersymmetric extension of the Higgs mechanism: when gauge invariance is spontaneously broken, a gauge hypermultiplet becomes massive, melting with the matter hypermultiplet containing the Goldstone boson. Such a massive gauge hypermultiplet consists of one real vector, two Dirac spinors and five real scalars, all with the same mass.[20]

It is possible to obtain a spontaneous hypersymmetry breaking, which yields two Goldstone spinors. This occurs in the locally invariant SU(2) × U(1) model [36, 38] already considered in the previous subsection. Hypersymmetry is realized if the SU(2) × U(1) gauge coupling constants g and g' are taken equal to h and h' respectively. Both SU(2) × U(1) and hypersymmetry are spontaneously broken. The former is reduced to an Abelian subgroup. Two Goldstone spinors appear, λ_γ and ζ_γ, respectively components of the superfields denoted by V_γ and N_γ in formula (4.75).

4.8. Beyond the tree approximation

We have already noted that the minimum of the potential of scalar fields is not always uniquely determined at the tree approximation (up to a gauge or internal symmetry transformation); thus we

[18] This theory can be obtained by dimensional-reduction of a six-dimensional one [141].

[19] Exactly as supersymmetry associates with every gauge vector field V_K^μ a Majorana spinor λ_K.

[20] If the Goldstone boson belongs to a vector hypermultiplet we find a different type of massive vector hypermultiplet. There is a charged massive vector hypermultiplet, describing a vector, two Dirac spinors and a scalar (all charged) [40].

have often different candidates for the vacuum state, which have the same energy at the tree approximation but are physically non-equivalent, even at the classical level [114, 93]; when such states can be continuously connected, (non-Goldstone) bosons, massless in zeroth order, appear [135, 36, 96]. It is worthwhile asking if the degeneracy is removed when higher-orders in perturbation theory are considered, and if these non-Goldstone bosons remain massless.

Higher order computations have been performed for several examples [58, 135, 97]; they show that, at a point which preserves supersymmetry at the classical level, the effective potential vanishes; this has been verified for the general case of a gauge-invariant supersymmetric theory at the one-loop level [16, 128]. Then supersymmetry is expected to be conserved, and this has been shown at first-order in perturbation theory [149].

Zumino has shown that, when supersymmetry is conserved, the sum of all vacuum diagrams vanishes identically [139]. It follows that the effective potential of scalar fields vanishes identically [97, 128]. This may be considered, naïvely, as a consequence of the equality $H = \frac{1}{4}\Sigma_\alpha Q_\alpha^2$ [74]. A derivation using superpropagators has been given for any supersymmetric gauge theory [134, 16].

The main consequence is that, for points preserving supersymmetry, higher order considerations do not remove the vacuum degeneracy which can exist at lowest order. Moreover, when such possible non-equivalent vacuum states can be continuously connected, the spectrum includes non-Goldstone scalar bosons, which are exactly massless.[21]

These may appear even if the above hypothesis were not satisfied: we only have to assume the existence of one point preserving supersymmetry: all non-Goldstone bosons belonging to the same multiplet as a true Goldstone boson, or a massless Majorana spinor,[22] or a gauge vector boson [38], are exactly massless.

One of these bosons has a simple interpretation when the Lagrangian density involves only dimensionless parameters (however, such theories are scale-invariant only at the classical level). The overall scale of the set of vacuum expectation values is completely undetermined, at least at lowest order; one of the bosons considered above is the dilation [82, 83], pseudo-Goldstone boson associated with the breaking of scale invariance. The dilaton remains obviously massless, if all vacuum expectation values can be scaled, supersymmetry being preserved [128].

Let us denote by $(a, b; \zeta)$ the chiral supermultiplet containing the dilaton. These particles can be associated with the breaking of scale, chiral and special supersymmetry transformations, respectively [40] (any supersymmetric theory formally invariant under the conformal group is also invariant under the supersymmetry algebra (2.5), involving the chiral generator Π and the special supersymmetry generator S_α; ζ appears as a "pseudo-Goldstone spinor", associated with the breaking of S_α). In the hypersymmetric extension of Q.E.D. (Lagrangian density (4.76) with $s = 0$), the dilaton belongs to a massless hypermultiplet, describing the photon, a neutral Dirac spinor and two real scalars a and b [38]. It is also the case in hypersymmetric pure Yang–Mills theories, if the gauge group is spontaneously broken, for example SU(2) down to U(1).

Now we come to the case where spontaneous supersymmetry breaking occurs at the tree approximation.[23] The degeneracy existing at lowest order (see subsection 4.6) can be removed;

[21] It is sometimes easy to relate the existence of these scalar bosons with a larger symmetry group for the potential of scalar fields, considered at the classical level [16]. We reserve the name "pseudo-Goldstone bosons" [127] for scalar bosons associated with a larger symmetry group for both the kinetic term and the classical potential of scalar fields. In non-supersymmetric theories these particles are generally massless only in zeroth order.

[22] The mass being forbidden by a phase-and-γ_5 transformation, provided this is compatible with renormalizability.

[23] In two dimensions, supersymmetry can be spontaneously broken at the tree approximation, the corresponding solution being unstable at the one loop level [11].

non-Goldstone scalar particles acquire masses at higher orders. This has been verified for some explicit examples; it is suggested in ref. [72] that phase-and-γ_5 invariances (such as R-invariance) will be respected at higher orders.

Note that the zeroth order mass spectrum of scalar particles often depends of the vacuum state selected (and not only by an overall scale factor) [36], whereas it would not do for pseudo-Goldstone bosons associated with an internal pseudosymmetry group.

5. Models of weak and electromagnetic interactions

We have seen how to construct supersymmetric gauge theories admitting a conserved quantum number, which has the desirable properties for lepton number, and we are able to obtain spontaneous breaking for both supersymmetry and gauge invariance. Now, we attempt to apply supersymmetry to weak and electromagnetic interactions.

Among the motivations to search for a supersymmetric theory of the physical world is the possibility that a massless neutrino field appears naturally: this might occur in a natural way for one or (and) the other of two reasons:

(i) spontaneous supersymmetry breaking generates a massless spin $\frac{1}{2}$ Goldstone particle (Volkov and Akulov [124]),

(ii) in a gauge invariant theory, supersymmetry associates in a massless vector supermultiplet a Majorana spinor to each vector field; the neutrino field could be the Majorana spinor associated with the photon field (Wess and Zumino [132]).

A difficulty is that the physical neutrino field appears as a two-component Dirac spinor rather than a Majorana spinor; this is associated with lepton-number conservation, and we have seen in subsection 4.5 that the problem can be solved by means of R-invariance, which eventually acts as a constraint on the Lagrangian density.

It is now possible to construct superunified gauge theories of weak and electromagnetic interactions, which could be considered as realistic as far as we are concerned only with the leptons of the electron sector. The $SU(2) \times U(1)$ model of Fayet [35] describes a photon γ, heavy vector bosons $W_\pm$ and Z; the electron e, its neutrino ν, and heavy leptons, together with heavy scalar bosons. In this example, the neutrino is both: (i) the Goldstone particle; (ii) the companion of the photon.

But what happens if we want to construct a realistic model? The first step is to include the muon sector. Thus we would like to extend motivations (i) and (ii), in order they provide two neutrinos in a natural way:

(i) the two neutrinos are two Goldstone particles; a larger algebra including two Majorana charges among its generators has to be considered; two Goldstone spinors are indeed obtained in ref. [38],

(ii) the two neutrinos are associated with the photon and another boson under supersymmetry transformations; the latter could be scalar, with an internal invariance group $SU(2)$ corresponding to the symmetry electron–muon [37, 38].

Bardeen [7], de Wit and Freedman [30] have shown that hypothesis (i) has by itself testable consequences; experimental results on β-decays spectrum lead to abandon the Goldstone neutrino hypothesis. But we may keep hypothesis (ii) among the motivations for searching supersymmetric models of weak and electromagnetic interactions. An attractive scheme might be obtained by use of a larger algebra, which associates in a representation the photon, the two neutrinos and a (neutral) complex scalar particle [38].

5.1. Spontaneous breaking of SU(2) × U(1) gauge invariance

In this subsection we describe one possible way[24] to obtain spontaneous breaking of SU (2) × U (1) gauge invariance, supersymmetry being conserved [35]. We demand also:

– invariance under R-transformations, in order to obtain lepton-number conservation,

– invariance under parity; all spin-one particles are coupled to pure-vector currents; in the next section where parity will be broken, we shall obtain a vector-like theory of weak and electromagnetic interactions, reminiscent of the one of ref. [34]; this approach ensures automatically that models are anomaly-free.[25]

We have seen in subsection 4.4 how, for each vector multiplet acquiring a mass, a chiral multiplet is eliminated by the supersymmetric extension of the Higgs mechanism. If SU (2) × U (1) gauge invariance is reduced down to an U (1) subgroup, three vector multiplets (containing the vector bosons $W_\pm$ and Z) become massive, whereas the one containing the photon stays massless; three chiral multiplets are eliminated.

We shall study the SU (2) × U (1) gauge invariant interaction of the vector superfields V and V' with the SU (2) doublets of chiral superfields S and T, left-handed and right-handed respectively. We add also a fifth chiral superfield, which is gauge invariant but couples S and T, as a direct mass term would do; let N be this (left-handed) superfield, and N^* its (right-handed) conjugate.[26] R-invariance is defined by (4.60), in which the index n vanishes for the superfields S and $T^\dagger$, and equals two for the superfield N. The most general Lagrangian density compatible with supersymmetry, SU (2) × U (1), R and parity invariance reads:

$$\mathscr{L} = \mathscr{L}_0 + [N^*N]_D + [S^\dagger \exp(g\tau V + g'V')\, S + T^\dagger \exp\{-(g\tau V + g'V')\}\, T]_D + [2hT^\dagger SN + sN]_F \tag{5.1}$$

in which $\mathscr{L}_0$ stands for the Lagrangian density of the free and self-interacting vector multiplets V and V'. $\mathscr{L}$ depends on the three coupling constants g, g' and h, and the mass scale is fixed by the single parameter s.[27]

An explicit study of the Lagrangian density when the soft parity breaking term $\xi D'$ is added, will be given in the next subsection. Here we indicate the main features of the model in the absence of such a term.

All auxiliary fields have vanishing vacuum expectation values. Supersymmetry is conserved and all particles in a multiplet have equal masses.

The left-handed superfield S and the right-handed superfield T describe respectively the physical

[24] This is not the only way to proceed. For example we can omit the superfield N in the following study (just take $h = 0$); the Lagrangian density depends only on two parameters, which are the dimensionless coupling constants g and g'. The vacuum state is no more determined at the tree approximation. The model can describe three massive and one massless vector multiplet, together with a massless chiral one: the dilaton multiplet (cf. subsection 4.8). Note that we can choose $g' = 0$; then the "photon" multiplet disappears.

[25] As remarked in subsection 4.5, R-transformations and parity do not commute; parity maps the set of R-transformations into itself; in the final model R-invariance, and thus lepton-number, are conserved, but not parity. In the gauge theory with spontaneous parity breaking of ref. [34], we had a similar situation for the so-called Q-transformations; the method introduced there to restrict a parity-conserving Lagrangian density and obtain a vector-like theory has a natural expression in terms of superfields, and leads to (4.64), with $n = 1$ [35].

[26] For simplicity we call this superfield N, and the corresponding coupling constant h, although notations N', h' should be preferable for comparison with (4.68).

[27] We can choose s/h negative, without restriction.

scalar fields φ'' and φ'. These, which are SU (2) doublets exchanged under parity, take equal vacuum expectation values; we have, in the convenient gauge:

$$\langle\varphi''\rangle = \begin{pmatrix} v/\sqrt{2} \\ 0 \end{pmatrix} \qquad \langle\varphi'\rangle = \begin{pmatrix} v/\sqrt{2} \\ 0 \end{pmatrix} \tag{5.2}$$

in which v is a positive real fixed by

$$\tfrac{1}{2} hv^2 + s = 0. \tag{5.3}$$

The superfield N describes the physical scalar field

$$\omega = (a - \mathrm{i}b)/\sqrt{2} \tag{5.4}$$

which has a vanishing vacuum expectation value. Since φ'' and φ' are R-invariant, this symmetry is conserved; so is parity.

SU (2) × U (1) gauge invariance is spontaneously broken down is an U (1) subgroup, as in the Weinberg–Salam [126, 105] and other models. For the SU (2) doublets S and T, the covariant derivative is defined by:

$$\mathrm{i}D_\mu = \mathrm{i}\partial_\mu - \tfrac{1}{2}(g\boldsymbol{\tau} W_\mu + g' W'_\mu) \tag{5.5}$$

and the SU (2) × U (1) mixing angle by

$$\mathrm{tg}\,\theta = g'/g. \tag{5.6}$$

The mass term for vector bosons is:

$$\mathscr{L}_{\mathrm{m}} = -2[v/\sqrt{2},0]\,[\tfrac{1}{2}(g\boldsymbol{\tau} W^\mu + g' W'^\mu)]^2 \begin{bmatrix} v/\sqrt{2} \\ 0 \end{bmatrix}. \tag{5.7}$$

Let us define the physical fields:

$$\begin{cases} Z^\mu = \cos\theta\; W_3^\mu + \sin\theta\; W'^\mu \\ A^\mu = -\sin\theta\; W_3^\mu + \cos\theta\; W'^\mu \end{cases} \qquad W_-^\mu = \frac{W_1^\mu + \mathrm{i}W_2^\mu}{\sqrt{2}}. \tag{5.8}$$

A^μ is the massless "photon" field; Z^μ and W_-^μ are the fields of neutral and charged heavy vector bosons, with masses:

$$m_Z = \left(\frac{g^2 + g'^2}{2}\right)^{1/2} v, \qquad m_W = \frac{gv}{\sqrt{2}}. \tag{5.9}$$

Since supersymmetry is conserved, Z^μ, W_-^μ belong to massive vector multiplets, and A^μ to a massless one.

– The vector multiplet $\mathscr{V}_Z$ contains a vector Z^μ, a Dirac spinor E_0 and a real scalar z.

– The (charged) vector multiplet $\mathscr{V}_-$ contains a vector W_-^μ, two Dirac spinors E_- and e_- and a complex scalar w_-.

– The vector multiplet $\mathscr{V}_\gamma$ contains a vector A^μ and a Majorana spinor λ_γ (we can use also the left-handed Dirac spinor $\lambda_{\gamma L} = \frac{1}{2}(1 - \mathrm{i}\gamma_5)\lambda_\gamma$, or as well its conjugate, the right-handed Dirac spinor $\lambda_{\gamma R} = \frac{1}{2}(1 + \mathrm{i}\gamma_5)\lambda_\gamma$).

From our initial five chiral multiplets, three have been eliminated by the supersymmetric extension of the Higgs mechanism, and two, neutral, remain; they describe:

– A Dirac spinor e_0, two complex scalars ω and φ with mass

$$m_{e_0} = hv/\sqrt{2}. \tag{5.10}$$

This mass is arbitrary; the multiplet plays the role of the physical Higgs scalar in the Weinberg–Salam model.

R is still an invariance for the spontaneously broken theory. From formulas (4.64) we deduce the action of R on physical fields:

$$\begin{aligned} e_- &\to \exp(\mathrm{i}\alpha)\, e_- & E_- &\to \exp(-\mathrm{i}\alpha)\, E_- \\ e_0 &\to \exp(\mathrm{i}\alpha)\, e_0 & E_0 &\to \exp(-\mathrm{i}\alpha)\, E_0 \\ \lambda_{\gamma\mathrm{L}} &\to \exp(\mathrm{i}\alpha)\, \lambda_{\gamma\mathrm{L}} & \omega &\to \exp(2\mathrm{i}\alpha)\, \omega \end{aligned} \tag{5.11}$$

all other fields being unchanged.

We obtain "lepton-number" conservation; this one is carried by the Dirac spinors e_-, e_0, $\lambda_{\gamma\mathrm{L}}$ (+1), and E_-, $E_0(-1)$, but also by the complex scalar field $\omega(+2)$.

Let ψ denote the doublet of Dirac spinors described by the superfields S and T, and λ, λ' and ζ the Majorana spinors described by V, V' and N. We have

$$e_- = -\lambda_{-\mathrm{L}} + \psi_{-\mathrm{R}}, \qquad E_- = \psi_{-\mathrm{L}} - \lambda_{-\mathrm{R}}, \tag{5.12}$$

$$\lambda_\gamma = -\sin\theta\, \lambda_3 + \cos\theta\, \lambda'. \tag{5.13}$$

We note that the Majorana spinor λ_γ has V − A couplings with e_-, and V + A with E_-. In the next section we shall obtain a mass splitting between e_- and E_-, respectively interpreted as electron and heavy electron fields, $\nu_\mathrm{L} = \frac{1}{2}(1 - \mathrm{i}\gamma_5)\lambda_\gamma$ being the neutrino field. Thus a V − A current couples ν_L to e_- and a V + A current couples $\nu_\mathrm{R}^\mathrm{C}$ to E_-; we can speak as well of V − A charged currents only, coupling the neutrino field ν_L to the electron field e_-, or the heavy antielectron field E_+.

5.2. A semi-realistic model of weak and electromagnetic interactions

5.2.1. Lagrangian density and symmetry breaking

Now we relax the parity constraint on the model of the previous section [35]. The auxiliary component D' of the vector multiplet V' is a pseudoscalar, which is gauge invariant and supersymmetric up to a four-derivative. We add to the Lagrangian density (5.1) the parity-breaking term $\xi D'$.[28] Now parity is broken explicitly, but only in a soft way: by terms of dimension 2 at most in the Lagrangian density. We write the complete expression of $\mathscr{L}$:

$$\begin{aligned} \mathscr{L} = \mathscr{L}_0 &+ [N^*N]_D + [S^\dagger \exp(g\boldsymbol{\tau}V + g'V')S + T^\dagger \exp\{-(g\boldsymbol{\tau}V + g'V')\}T]_D \\ &+ [2hT^\dagger SN + sN]_F + \xi D'. \end{aligned} \tag{5.14}$$

We shall express this Lagrangian density in terms of the physical fields:

– The vector superfields V and V' describe the vector fields W^μ and W'^μ, the Majorana spinors λ and λ'.

– The chiral (left-handed and right-handed) superfields S and T describe the spinor fields ψ_L and ψ_R, the scalar fields φ'' and φ' respectively; all are SU(2) doublets.

– The chiral superfield N describes the Majorana spinor ζ, the scalar fields a and b.

[28] In principle we should also use G-components of chiral superfields $T^\dagger SN$ and N to construct the Lagrangian density. However, these can be eliminated by means of global phase transformations on chiral superfields.

The Lagrangian density reads:

$$\begin{aligned}\mathscr{L} = & -\tfrac{1}{4} W^{\mu\nu}W_{\mu\nu} - \tfrac{1}{4} W'^{\mu\nu}W'_{\mu\nu} - \tfrac{1}{2}\,\mathrm{i}\bar{\lambda}\not{D}\lambda - \tfrac{1}{2}\,\mathrm{i}\bar{\lambda}'\not{\partial}\lambda' \\ & - \tfrac{1}{2}\,\mathrm{i}\bar{\zeta}\not{\partial}\zeta - \tfrac{1}{2}\,\partial^\mu a\partial_\mu a - \tfrac{1}{2}\,\partial^\mu b\partial_\mu b - \mathrm{i}\bar{\psi}\not{D}\psi - \mathrm{D}^\mu\varphi'^\dagger \mathrm{D}_\mu\varphi' - \mathrm{D}^\mu\varphi''^\dagger \mathrm{D}_\mu\varphi'' \\ & + \frac{\mathrm{i}}{\sqrt{2}}\,[\bar{\psi}_\mathrm{R}(g\tau\lambda + g'\lambda')\varphi' + \varphi'^\dagger(g\tau\bar{\lambda} + g'\bar{\lambda}')\psi_\mathrm{R} + \bar{\psi}_\mathrm{L}(g\tau\lambda + g'\lambda')\varphi'' + \varphi''^\dagger(g\tau\bar{\lambda} + g'\bar{\lambda}')\psi_\mathrm{L}] \\ & + \frac{1}{\sqrt{2}}\,[\bar{\psi}_\mathrm{R}(h\zeta)\varphi'' - \varphi''^\dagger(h\bar{\zeta})\psi_\mathrm{R} + \bar{\psi}_\mathrm{L}(h\zeta)\varphi' - \varphi'^\dagger(h\bar{\zeta})\psi_\mathrm{L}] \\ & - \tfrac{1}{2}\,\mathrm{i}h\bar{\psi}(a - \gamma_5 b)\psi - V(\varphi',\varphi'',a,b) \end{aligned} \tag{5.15}$$

in which $V(\varphi',\varphi'',a,b)$ is the potential of the scalar fields:

$$\begin{aligned} V(\varphi',\varphi'',a,b) = & \tfrac{1}{2} g^2(\varphi'^\dagger\varphi'\varphi''^\dagger\varphi'' - \varphi'^\dagger\varphi''\varphi''^\dagger\varphi') + \tfrac{1}{8}(g^2 + g'^2)(\varphi''^\dagger\varphi'' - \varphi'^\dagger\varphi')^2 \\ & + \tfrac{1}{2}\xi g'(\varphi''^\dagger\varphi'' - \varphi'^\dagger\varphi') + \tfrac{1}{2}\xi^2 + \tfrac{1}{4}h^2(a^2 + b^2)(\varphi'^\dagger\varphi' + \varphi''^\dagger\varphi'') + \tfrac{1}{2}\,|h\varphi''^\dagger\varphi' + s|^2. \end{aligned} \tag{5.16}$$

In the convenient gauge the physical fields have the vacuum expectation values:

$$\langle a\rangle = \langle b\rangle = 0$$

$$\langle\varphi''\rangle = \begin{pmatrix} v''/\sqrt{2} \\ 0 \end{pmatrix}, \qquad \langle\varphi'\rangle = \begin{pmatrix} v'/\sqrt{2} \\ 0 \end{pmatrix} \tag{5.17}$$

in which the real v'' and v' are fixed by

$$\tfrac{1}{8}(g^2 + g'^2)(v''^2 - v'^2) = -\tfrac{1}{2}\xi g' \qquad \tfrac{1}{2}hv'v'' + s = 0. \tag{5.18}$$

R-invariance is preserved by spontaneous breaking, and lepton number, still defined by (5.11), is conserved. SU(2) × U(1) gauge invariance is reduced down to the usual electromagnetic one. The potential has no more a vanishing minimum: the auxiliary field

$$D_\gamma = -\sin\theta\, D_3 + \cos\theta\, D' \tag{5.19}$$

takes a non-vanishing vacuum expectation value

$$\langle D_\gamma\rangle = -\xi\cos\theta \tag{5.20}$$

provided ξ and g are not zero. Supersymmetry is spontaneously broken if ξ, g and g' are not zero (for $g' = 0$ the superfield V', decoupled, is alone responsible for the non-vanishing vacuum expectation value of the potential). It follows from (4.1) that the Goldstone spinor is

$$\lambda_\gamma = -\sin\theta\,\lambda_3 + \cos\theta\,\lambda'. \tag{5.21}$$

The Goldstone spinor λ_γ is associated with the photon field A^μ in the massless vector multiplet $\mathscr{V}_\gamma$, corresponding to the electromagnetic U(1) gauge invariance.

5.2.2. Spectrum

Particles are still associated in supermultiplets, but the mass degeneracy is removed for charged ones: the masses of the Dirac spinors e_- and E_- are split on each side of the mass of the bosons W_- and w_-, which remain equal. The new zeroth-order mass spectrum is represented on fig. 2.

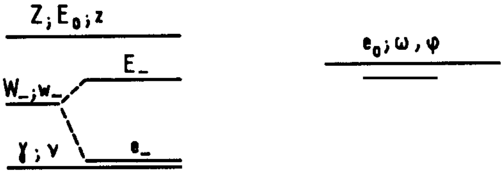

Fig. 2. Zeroth order mass spectrum.

To diagonalize the mass matrix we introduce the mixing angle δ, defined by tg $\delta = v'/v''$ (cf. subsection 4.2), which is expected to be small. The physical vector fields are given by (5.8); the physical spinor fields by:

$$\begin{cases} e_- = -\lambda_{-\mathrm{L}} + \psi_{-\mathrm{R}} \\ E_- = \psi_{-\mathrm{L}} - \lambda_{-\mathrm{R}} \\ e_0 = \mathrm{i}\zeta_{\mathrm{L}} + (\psi_0 \cos\delta - \psi_0^* \sin\delta)_{\mathrm{R}} \\ E_0 = (\psi_0 \cos\delta + \psi_0^* \sin\delta)_{\mathrm{L}} - \lambda_{Z\mathrm{R}} \\ \nu_{\mathrm{L}} = \lambda_{\gamma\mathrm{L}} \end{cases} \tag{5.22}$$

and the physical scalar fields by

$$\begin{cases} w_- = \varphi'_- \cos\delta - \varphi''_- \sin\delta \\ z = \sqrt{2}\,\mathrm{Re}\,(-\varphi'_0 \sin\delta + \varphi''^*_0 \cos\delta) \\ \varphi = \varphi'_0 \cos\delta + \varphi''^*_0 \sin\delta \\ \omega = (a - \mathrm{i}b)/\sqrt{2}. \end{cases} \tag{5.23}$$

They have zeroth order masses:

$$m_Z = m_{E_0} = m_z = \tfrac{1}{2}(g^2 + g'^2)^{1/2}\,(v'^2 + v''^2)^{1/2}$$
$$m_{e_0} = m_\omega = m_\varphi = \tfrac{1}{2}h\,(v'^2 + v''^2)^{1/2}$$
$$\begin{cases} m_{W_-} = m_{w_-} = \left(\dfrac{m_{E_-}{}^2 + m_{e_-}{}^2}{2}\right)^{1/2} = \tfrac{1}{2}g(v'^2 + v''^2)^{1/2} \\ m_{e_-} = \dfrac{gv'}{\sqrt{2}} \\ m_{E_-} = \dfrac{gv''}{\sqrt{2}} \end{cases}$$
$$m_\gamma = m_{\lambda_\gamma} = 0. \tag{5.24}$$

Note that the ratio:

$$m_{e_-}/m_{E_-} = v'/v'' = \mathrm{tg}\,\delta \tag{5.25}$$

vanishes with s.

e_- and E_- are interpreted as electron and heavy electron field respectively, and $\nu_{\mathrm{L}} = \frac{1}{2}(1 - \mathrm{i}\gamma_5)\lambda_\gamma$ as a neutrino field; in the currents, originally purely vector, V–A and V+A parts have been split, as explained at the end of the previous subsection. We obtain now V − A expressions for all charged currents, which couple ν_L to e_{-L} or E_{+L}. e_0 and E_0 are interpreted as heavy neutrino fields.

Couplings of ν_L with vector particles are given by:

$$\lambda_L = -\begin{bmatrix} E_{+L} \\ E^*_{0L}\cos\theta + \nu_L \sin\theta \\ e_{-L} \end{bmatrix} . \tag{5.26}$$

θ is both the SU(2) × U(1) mixing angle, as in Weinberg–Salam-type models, and the neutrino–heavy-neutrino mixing angle, as in Georgi–Glashow-type models.

5.2.3. Electromagnetic interactions

The U(1) gauge invariance of quantum electrodynamics remains unbroken. The photon, massless, interacts with the electron, light, and with heavy particles: the intermediate bosons W_- and w_-, and the heavy electron E_- (with $m_{W_-} = m_{w_-} \simeq m_{E_-}/\sqrt{2}$). The fact that electromagnetism conserves parity has been obtained in a natural way, since we constructed a vector-like theory generalizing the one of ref. [34].

In supersymmetric theories there is a problem concerning anomalous magnetic moments of charged Dirac spinors: Ferrara and Remiddi have shown that they vanish identically, and this is related with the mass degeneracy between fermions and bosons [47]. The problem is solved here by spontaneous supersymmetry breaking; mass equalities are not preserved: the electron and the photon are the only light particles, together with the neutrino, which has no electromagnetic interaction. The anomalous magnetic moment of the electron has its usual value (i.e. $\frac{1}{2}(g-2) = \alpha/2\pi$ at the one loop level), apart from negligible corrections due to weak interactions.

In brief, interactions between electron and photon are correctly described by this spontaneously broken supersymmetric model, together with other (weak) interactions. The Goldstone spinor, companion of the photon, is a massless particle free from electromagnetic interactions.

5.2.4. Weak interactions

The heavy vector boson W_- has V − A couplings between electron and neutrino fields. But the heavy scalar boson w_- also couples electron and neutrino fields. Both W_- and w_- can be exchanged in a reaction such as νe or $\bar{\nu} e$ scattering (a theory including the muon sector has to avoid the exchange of scalar particles in processes such as μ-decay); see fig. 3.

The relevant part in the Lagrangian density is

$$\mathscr{L} = g \sin\theta \, [W^\mu_- \bar{e}_{-L}\gamma_\mu \nu_L + \mathrm{i}\sqrt{2}\cos\delta \, w_- \bar{e}_{-R}\, \nu_L] + \text{h.c.} \tag{5.27}$$

Using a Fierz transformation we find that the effective Lagrangian density, in the local limit, is proportional to

$$\bar{e}\gamma^\mu e \; \bar{\nu}_L \gamma_\mu \nu_L . \tag{5.28}$$

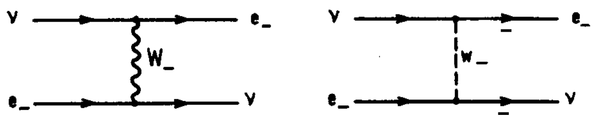

Fig. 3. Diagrams contributing to neutrino–electron scattering.

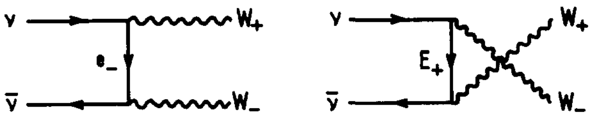

Fig. 4. Diagrams contributing to $\nu + \bar{\nu} \to W_+ + W_-$ scattering.

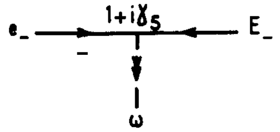

Fig. 5. Coupling of the scalar particle ω with charged leptons (the arrows show lepton number conservation).

It involves the product of two effective neutral currents; the one associated with the electron is a pure vector. It follows that

$$\sigma_{\nu e} = \sigma_{\bar{\nu} e}. \tag{5.29}$$

The heavy vector boson Z has no direct coupling to the neutrino field (this is true for all SU(2) X U(1) models in which photon and neutrino are in the same multiplet). However, contributions to νe or $\bar{\nu}$e scattering amplitudes due to the exchange of other particles, vector or scalar, rearrange in order to simulate the exchange of a neutral vector boson.

The existence of heavy leptons follows necessarily from the fact that the neutral vector boson Z does not couple to the neutrino, according to a well-known argument [1]: for example, in the process $\nu + \bar{\nu} \to W_+ + W_-$, cancellations occurs between the two amplitudes represented in fig. 4. Since no neutral vector boson is exchanged in the s-channel, there must exist a heavy electron E_- with leptonic number (-1), as in the Georgi–Glashow model [59].

The model describes also e_0 and E_0, which are heavy neutrinos carrying the same leptonic numbers as e_- and E_-, i.e. 1 and (-1), respectively. Moreover there exist a heavy neutral scalar boson ω, with leptonic number 2; whereas the neutral vector boson Z cannot couple electron to heavy electron, the scalar particle ω can do (see fig. 5).

5.2.5. Electron mass

The model depends on two parameters with dimension mass^2, ξ and s. If s vanishes, so does the mixing angle δ, and the zeroth order electron mass (cf. formula (5.25)). This can be realized by requiring the Lagrangian density (5.14) to be invariant under X-transformations,[29] defined by (4.60) with $n = 0$ for the superfields S and N, and $n = 2$ for the superfield $T^\dagger$. Under a X-transformation we have

$$\begin{cases} \varphi'' \to \varphi'' \\ \varphi' \to \exp(-2i\alpha)\varphi' \\ e_- \to \exp(\gamma_5\alpha)e_- . \end{cases} \tag{5.30}$$

At the tree approximation φ'' only has a non-vanishing vacuum expectation value; X-invariance is preserved and forbids a zeroth-order mass term for the electron. A non-vanishing zeroth order mass for the electron appears as a result of an explicit breaking of X-invariance, obtained by choosing s small but non-zero.

[29] We could also use Q-transformations, corresponding to $n = 1$ for S and $T^\dagger$, $n = 0$ for N. They generalize those of ref. [34]; in particular we have for SU(2) doublets $\psi \to \psi$, $\varphi' \to \exp(-i\alpha)\varphi'$, $\varphi'' \to \exp(i\alpha)\varphi''$ and for the singlet $\lambda \to \exp(\gamma_5\alpha)\lambda$.

5.2.6. Another model

We now compare the previous model with a similar one constructed by Mainland and Tanaka [85]. The latter includes a smaller number of fields; but it is not vector-like as before, and anomalies spoil the renormalizability; at the classical level the electron is massless.

Instead of five chiral superfields, three only are used: S is still a doublet of left-handed superfields, but T is now a singlet right-handed superfield. The Lagrangian density

$$\mathscr{L} = \mathscr{L}_0 + [S^\dagger \exp(g\tau V + g'V')\, S + T^\dagger \exp(-2g'V')\, T]_D + \xi D' \tag{5.31}$$

depends only on three parameters ξ, g and g'.

S and T describe as physical scalar fields the doublet $\binom{\varphi''_0}{\varphi''_-}$ and the singlet φ'_-, respectively. When $\xi g' < 0$, φ''_0 has a non-vanishing vacuum expectation value $v''/\sqrt{2}$ given by:

$$\tfrac{1}{8}(g^2 + g'^2)\, v''^2 = -\tfrac{1}{2}\xi g'. \tag{5.32}$$

Thus the model can be compared very easily to the previous one, in the case where the mixing angle δ vanishes (i.e. when X-invariance fixes $\delta = 0$). Here the fields e_0, ω, φ do not appear. For the remaining fields, the diagonalization of the mass matrix yields the same results for both models (this is obvious for all fields except the scalar $w_- = \varphi'_-$, for which a special argument is needed), and formulas (5.22) to (5.26) can still be applied, with of course $\delta = 0$. Again λ_γ is the Goldstone spinor and represents the neutrino, in the same multiplet as the photon.

As in the previous model both W_- and w_- can be exchanged in a νe or $\bar{\nu} e$ scattering. Note that the Majorana spinor λ_γ couples e_{-R} to w_- in the same way in both models (exactly as the photon γ has the same couplings with e_{-R}); formula (5.27) can still be applied, with $\delta = 0$. Again νe and $\bar{\nu} e$ scattering can be interpreted in terms of effective neutral currents (the one associated with the electron being a pure vector), while Z is not coupled to the neutrino field. Lepton number is attributed as before, using R-invariance, but carried only by fermions: it is 1 for e_- and ν_L, (-1) for E_- and E_0. The model is invariant under X-transformations ((5.30) is still valid) which forbid a mass term for the electron at the classical level; now X-invariance cannot be broken explicitly.

In this subsection we have described an example of spontaneously broken supersymmetric theory of weak and electromagnetic interactions limited to the electron sector. The neutrino is at the same time: (i) the Goldstone spinor, (ii) the companion of the photon. Photon and neutrino appear as two states of the same particle, with different spins.

Such a "superunified gauge theory" of weak and electromagnetic interactions shows how one more step towards the unification of the description of particles may be possible: we understand now that photon and neutrino may appear as two aspects of a single object (exactly as photon and intermediate vector boson, or electron and neutrino, in unified gauge theories), leptonic number being carried by some generators in the supersymmetry algebra (as is charge in unified gauge theories). However, we do not know yet how these features can be included in a realistic theory.

5.3. The problem of the neutrino as a Goldstone particle

In the previous model of weak and electromagnetic interactions (for leptons of the electron sector) the neutrino appeared as the Goldstone particle.

This idea that the neutrino could be a Goldstone particle was suggested by Volkov and Akulov [124]. It has very important consequences, as pointed out by Bardeen [7], de Wit and Freedman [30]. Low-energy theorems of Adler type can be applied to a Goldstone neutrino,[30] and one finds

[30] Here we call Goldstone neutrino any massless spin $\frac{1}{2}$ Goldstone particle.

that the low-energy behaviour of the amplitude for emitting such a neutrino is inconsistent with experimental information on end-of-lepton-spectrum behaviour in β-decays. Thus it seems that the known neutrino cannot be a Goldstone particle.

We indicate how to derive these results (using notations of ref. [30]); we are interested in the Goldstone particle ν_G arising from the spontaneous breaking of some supersymmetry (not necessarily the one defined by algebra (2.4)). Let $S^\mu(x)$ be the corresponding supersymmetry current, which transforms under the Lorentz group as a combined 4-vector and Majorana spinor; it can be derived from the supersymmetry generator Q_α, which transforms as a Majorana spinor. The matrix elements of $S^\mu(x)$ between physical states are gauge invariant and conserved:

$$\partial_\mu \langle \mathrm{B}|S^\mu(x)|\mathrm{A}\rangle = 0. \tag{5.33}$$

If the symmetry generated by Q is spontaneously broken, the supersymmetry current $S^\mu(x)$ contains a term proportional to the field of the Goldstone spinor ν_G:

$$S^\mu(x) = -\,\mathrm{i} f_\nu \gamma^\mu \nu_G(x) + \dots . \tag{5.34}$$

We study the amplitude for a low-energy Goldstone neutrino to be emitted in a physical process $\mathrm{A} \to \mathrm{B} + \nu_G$. We have:

$$\begin{aligned} 0 &= \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}qx}\, \partial_\mu \langle \mathrm{B}|S^\mu(x)|\mathrm{A}\rangle \\ &= (2\pi)^4 \delta^4 (q + p_\mathrm{B} - p_\mathrm{A})\, [f_\nu M(q) + q^\mu R_\mu(q)] \end{aligned} \tag{5.35}$$

in which $M(q)$ is the neutrino pole term and $q^\mu R_\mu(q)$ comes from other terms in the current. In the absence of singular contributions to $R_\mu(q)$, the second term in (5.35) vanishes with q, and we find that the amplitude to emit a low-energy Goldstone neutrino in the process $\mathrm{A} \to \mathrm{B} + \nu_G$ vanishes with its four-momentum. This is not the case for the neutrino emitted in a β-decay, which cannot be a Goldstone particle.

De Wit and Freedman derived also a very interesting result concerning neutral current interactions of a Goldstone neutrino [30]. Using two-currents Ward identities, they have shown that the scattering amplitude for the process $\mathrm{A} + \nu_G \to \mathrm{B} + \nu_G$, in the limit of vanishing four-momentum for both the incoming and outcoming neutrinos, is proportional to the matrix element of the electromagnetic current $\langle \mathrm{B}|J^\mu_{\mathrm{em}}|\mathrm{A}\rangle$. Thus we have, in this limit, an effective neutral current, which is proportional to the electromagnetic current.

This fact has been noted already in the model of the previous subsection [35] for the process $\mathrm{e} + \nu \to \mathrm{e} + \nu$. An extension of this model, due to Capper [13] has been used by Capper, Salam and Strathdee to verify low-energy theorems [17].

We discuss now the consequences of these results. It seems that the idea that the electron neutrino, and probably also the muon neutrino, could be Goldstone particles, has to be abandoned. Thus we are led to the question: where is (are) the Goldstone particle(s)? We shall attempt to give some possible answers.

The simplest one is just the direct consequence of low-energy theorems themselves, which prevent Goldstone neutrinos, if they exist in nature as physical particles, to appear in known processes. It has been suggested by Iliopoulos that they might appear as right-handed components for neutrino fields [73].[31]

[31] We have been told that similar considerations have been made by Salam. Note that a four-component Dirac neutrino field $\nu = \psi_{0\mathrm{L}} + \lambda_{\gamma\mathrm{R}}$ already appeared in the model described in section 4 of ref. [35], for $m = 0$; the Goldstone spinor λ_γ gives the right-handed component for the Dirac neutrino field ν. Under supersymmetry transformations, λ_γ is associated with the photon, and $\psi_{0\mathrm{L}}$ with a complex scalar field φ_0'', which remains massless at zeroth order. Two conserved quantum numbers can be defined; for γ, $\lambda_{\gamma\mathrm{R}}$, $\psi_{0\mathrm{L}}$, φ_0'', they are (0,0), (−1,0), (0,1) and (1,1) respectively.

Another possibility has been discussed by Fayet [39]. One can consider limits of supersymmetric theories (along the lines of refs. [37, 119] for which Goldstone spinors have very weak coupling constants, or even are completely decoupled. To understand the mechanism we return to the model of subsections 4.3 and 4.4, describing the gauge invariant interaction of the vector superfield V with the chiral superfield S. The physical fields are, for V, the vector V^μ and the Majorana spinor λ, both massless; for S, the left-handed Dirac spinor ψ_L, massless, and the complex scalar φ with mass $\sqrt{\xi e}$. In the limit $e \to 0$, $\xi \to \infty$, with $\xi e = \mu^2$ fixed, the vector superfield V, which describes the Majorana spinor λ, decouples, but mass splitting inside the chiral multiplet remains. This method, or a similar one, can be applied to more complicated models, and allows to get rid of the Goldstone spinor(s) in a supersymmetric (hypersymmetric) theory. Note that supersymmetry transformations become singular in the above limit.

A third possibility is that the spin $\frac{1}{2}$ Goldstone particles are eliminated by a generalized Higgs mechanism involving spin $\frac{3}{2}$ particles, as proposed by Volkov, Akulov and Soroka [124, 125]. This may occur in theories involving also gravitation, along the lines exposed in section 6; but one has to deal with the difficulties due to the introduction of high spin fields in quantum field theory.

In consequence the fact that Goldstone particles are not observed is not an argument against supersymmetric theories. Nevertheless one of the initial motivations, namely that known neutrinos might be Goldstone particles, disappeared. But we can still consider the attractive idea that the photon and a neutrino are in the same supermultiplet. We shall discuss in the following section possible ways to obtain both an electronic and a muonic neutrino.

5.4. How to include the muon sector?

In a supersymmetric theory any spin $\frac{1}{2}$ particle has to be associated with a boson, which has spin either 0 or 1. The two neutrinos ν_e and ν_μ might be associated with two complex scalar bosons (with, possibly, the Goldstone particle associated with the photon), or with two vector bosons (one combination of these might be the photon).

We shall discuss in more detail the situation in which one neutrino is associated with the photon, the other with a complex scalar boson [37]. Such a scheme involves both a vector superfield V and a chiral superfield S, with physical components $(V^\mu;\lambda)$ and $(\zeta;\omega)$ respectively. Suppose for example they describe the particles $(\gamma;\nu_\mu)$ and $(\nu_e^*;\omega)$. All particles in a multiplet will have a common electronic number, the supersymmetry generator carrying one unit of muonic number. Electronic and muonic number conservation are obtained from the set of transformations

$$\begin{cases} V(x,\theta,\bar\theta) \to V(x,\theta \mathrm{e}^{-\mathrm{i}\alpha}, \bar\theta \mathrm{e}^{\mathrm{i}\alpha}) \\ N(x,\theta,\bar\theta) \to \mathrm{e}^{\mathrm{i}(\alpha+\beta)}\, N(x,\theta \mathrm{e}^{-\mathrm{i}\alpha}, \bar\theta \mathrm{e}^{\mathrm{i}\alpha}). \end{cases} \tag{5.36}$$

For the physical components we find:

$$\begin{cases} V^\mu \to V^\mu \\ \lambda_L \to \mathrm{e}^{\mathrm{i}\alpha}\lambda_L \\ \zeta_L \to \mathrm{e}^{\mathrm{i}\beta}\,\zeta_L \\ \omega \to \mathrm{e}^{\mathrm{i}(\alpha+\beta)}\,\omega. \end{cases} \tag{5.37}$$

This means that γ, ν_μ, ν_e and ω have muonic and electronic number (0,0), (1,0), (0,1) and (1,1) respectively. Thus a (neutral) complex scalar particle is naturally introduced, with electronic number one and muonic number one. Such quantum numbers prevent the ω-particle to contribute in processes such as μ-decay.[32] The ω-particle is massless when supersymmetry is conserved. After spontaneous breaking it may acquire a mass, but not necessarily at lowest order.

The above considerations are somewhat unsatisfactory since both neutrinos are treated in a different way. It has been proposed to introduce an internal invariance group $SU(2)_I$, associated with the symmetry electron-muon, in such a way that $\binom{\nu_e}{\nu_\mu}$ is an SU(2) isodoublet, whereas γ and ω are isosinglets [38]. We explained already in subsection 4.7 that the corresponding algebra has a doublet of two-component Dirac spinors among its generators. One carries one unit of muonic number, and the other one unit of electronic number. In this framework not only spin $\frac{1}{2}$ fields but also spin 0 fields appear as gauge fields. $(\gamma;\nu_\mu,\ \nu_e;\omega)$ is a hypermultiplet of U(1) gauge particles associated with quantum electrodynamics.

6. Geometry of superspace, supergravity and strings

6.1. Gauge supersymmetries

We have seen, in the previous sections, that supersymmetry can be introduced by regarding the field operators as functions of points $Z = (x,\theta)$ in a superspace; the bosonic part x_μ is relative to the usual Minkowski space (more precisely, it is an even element of a Grassmann Algebra) and the fermionic part θ_α is an anticommuting Majorana spinor θ_α (odd element of a Grassmann Algebra).

Then the usual supersymmetry transformation on a Lorentz invariant superfield is simply a (generalized) translation on point Z

$$T_\alpha\phi(Z) = \phi(Z_\alpha) \tag{6.1}$$

where

$$Z_\alpha = (x,\theta)_\alpha = (x_\mu + \mathrm{i}\bar\theta\gamma_\mu\alpha,\ \theta+\alpha),$$

α being an anticommuting (Majorana) spinorial parameter.

This geometrical interpretation of supersymmetry transformations is not only useful to construct manifestly covariant Feynman rules in quantum field theory models but also, and more interestingly, for its further possible extension to the case of a gauge symmetry, namely to the case in which the transformation parameters are themselves point-dependent.

The simplest realizations of theories involving both a gauge group and supersymmetry have already been considered in previous sections with the study of supersymmetric gauge theories. In these theories the gauge function $\Lambda(Z)$ is a parameter which is a (chiral) superfunction i.e. a function defined on the superspace (x_μ,θ_α); it corresponds to a local transformation of some internal symmetry group $\mathscr{G}$ which commutes with space–time symmetries.

In this framework usual spin 1 gauge fields $V_\mu(x)$ are accompanied by fermionic spin $\frac{1}{2}$ gauge fields $\lambda_\alpha(x)$, belonging to the adjoint representation of $\mathscr{G}$.

The need for Fermi type gauge fields is merely due to the fact that space–time has been enlarged to include additional Fermi-type coordinates θ_α. The beautiful and unifying role of a local

[32] Couplings of ω may exist with electron and heavy muon, or muon and heavy electron.

supersymmetry is therefore already evident at this stage, because spin $\frac{1}{2}$ fields, commonly regarded as matter fields, become here gauge fields with respect to the bigger gauge group present in supersymmetric theories.

There are other types of spin $\frac{1}{2}$ fields in supersymmetric gauge theories which have not a similar interpretation. They belong to matter supermultiplets, which contain also scalar and pseudo-scalar fields.

These multiplets can belong to any representation of the gauge group $\mathcal{G}$.

Nevertheless minimally coupled theories of gauge and matter supermultiplets are renormalizable in terms of a single coupling constant g, the gauge coupling constant. This is a real improvement with respect to conventional gauge theories where additional self-interactions of (scalar) matter fields are needed to make the theory consistent with the renormalization procedure.

These arguments have given the hope that combined gauge Einstein symmetry (gravitation) and supersymmetry could provide a convenient framework to build up a theory of gravity (which might be renormalizable) together with a possible unification of different kinds of fundamental interactions.

Several approaches and powerful mathematical techniques have been developed recently in order to investigate supergravity theories.

The geometry of superspace and superfield techniques have been used by Arnowitt and Nath [89, 4, 5, 90, 88] and by Zumino [140] in apparently different contexts.

Freedman, Van Nieuwenhuizen and Ferrara [54] and Deser and Zumino [27] constructed the (minimal) supersymmetric extension of pure Einstein gravity using only spin 2 and $\frac{3}{2}$ fields. Their theory will be described in the next subsection.

Arnowitt and Nath used a Riemannian geometry in superspace by enlarging in a standard way the usual four-dimensional Riemann space to a $4 + 4N$ dimensional Riemann space in which points are labelled by $4 + 4N$ coordinates

$$Z_{\mathrm{A}} = (x_\mu, \theta^a_\alpha), \qquad a = 1, \dots N, \tag{6.2}$$

θ^a_α being a N-plet of Majorana spinors.

In this space a metric tensor $g_{\mathrm{AB}}(Z)$ can be introduced together with the Christoffel affinity

$$\Gamma^{\mathrm{C}}_{\mathrm{AB}} = (-1)^{bc}\, \tfrac{1}{2}[(-1)^{bd} g_{\mathrm{AD,B}} + \eta_{ab}(-1)^{ad} g_{\mathrm{BD,A}} - g_{\mathrm{AB,D}}]\, g^{\mathrm{DC}} \tag{6.3}$$

in which $\eta_{ab} = (-1)^{a+b+ab}$ and the indices $a,b, \dots$ take values 0 and 1 for bosonic and fermionic labels respectively. g^{BC} is the inverse metric, i.e.

$$g^{\mathrm{AC}} g_{\mathrm{CB}} = \delta^{\mathrm{A}}{}_{\mathrm{B}}. \tag{6.4}$$

The parallel transport of a vector around an infinitesimal closed loop allows one to determine the curvature tensor

$$R^{\mathrm{D}}{}_{\mathrm{ABC}} = -\Gamma^{\mathrm{D}}{}_{\mathrm{AC,B}} + (-1)^{bc}\Gamma^{\mathrm{D}}{}_{\mathrm{AB,C}} - (-1)^{c(d+e)}\Gamma^{\mathrm{E}}{}_{\mathrm{AC}}\Gamma_{\mathrm{EB}}{}^{\mathrm{D}} + (-1)^{b(c+d+e)}\Gamma^{\mathrm{E}}{}_{\mathrm{AC}}\Gamma^{\mathrm{D}}{}_{\mathrm{EB}}. \tag{6.5}$$

The only allowed (covariant) contractions are

$$R_{\mathrm{AB}} = (-1)^c R^{\mathrm{C}}{}_{\mathrm{ABC}} \tag{6.6}$$

$$R = (-1)^b g^{\mathrm{BA}} R_{\mathrm{AB}}, \tag{6.7}$$

i.e. respectively the contracted curvature tensor and the curvature scalar.

The fundamental gauge group which unifies gravitation with other types of interactions is nothing but the group of arbitrary coordinate transformations in superspace

$$Z^{A'} = Z^{A'}(Z) \tag{6.8}$$

which leaves the line element

$$ds^2 = dz^A g_{AB}(z) dz^B \tag{6.9}$$

invariant.

The gauge group defined through equations (6.8, 6.9), contains, as subgroups, the usual Einstein gauge group as well as the local group corresponding to the internal symmetry structure of the Fermi coordinates θ_α^a. For a 2N-plet of Majorana spinors this group turns out to be O(2N).

The most general (second order) differential field equations are

$$R_{AB} = \lambda g_{AB}. \tag{6.10}$$

They can be derived from an action principle in superspace

$$\delta A = 0 \qquad \text{with} \qquad A = \int dz \sqrt{-g}\,(R + 2\lambda) \tag{6.11}$$

in which $\sqrt{-g} = (-\det g_{AB})^{1/2}$ and the definition of the determinant in superspace can be found in ref. [4].

According to Arnowitt and Nath [89, 5, 90, 88] no additional matter stress tensor is needed in eqs. (6.10) because the usual matter fields are already contained in the Riemann tensor in superspace.

More precisely the usual matter stress tensor is already included in the left-hand side of eqs (6.10) in supergravity theory. In the above sense supersymmetric gauge theories of this type ("super-gauge" theories) are self-sourced, as much as conventional pure Yang–Mills theories are interacting self-sourced theories. For instance, if θ_α is a Dirac spinor, then the super-gauge group (6.8) contains, as subgroups, the Einstein gauge group and the abelian U(1) gauge group of electromagnetism, and eq. (6.10) is viewed as a theory of gravity coupled to electromagnetism.

The physics in such theories must be triggered by a spontaneous breaking of the super-gauge group (6.8) in such a way that, in the flat limit, one recovers the usual supersymmetry transformations.

Namely one looks for solutions of the field equations of the type [88, 6]

$$R_{AB}(g^0_{AB}) = \lambda g^0_{AB} \tag{6.12}$$

where

$$g^0_{AB} = \langle 0 | g_{AB} | 0 \rangle$$

and

$$g^0_{AB} : g^0_{\mu\nu} = \eta_{\mu\nu}, \quad g^0_{\mu\alpha} = -\,i\beta(\bar\theta\gamma_\mu)_\alpha, \quad g^0_{\alpha\beta} = \eta_{\alpha\beta} + \beta^2(\bar\theta\gamma_\mu)_\alpha\,(\bar\theta\gamma^\mu)_\beta.$$

Of course spontaneous symmetry breaking must occur in some particular directions such that the subgroups of Einstein and e.m. gauge transformations remain unbroken, in order for the photon and the graviton to remain massless. After spontaneous symmetry breaking there exists a self-contained Higgs mechanism responsible for a mass growth in the broken sector of the supergauge group. As a consequence, the Einstein gravitational constant G_E and the electric charge e are related by an equation of the form

$$G_{\mathrm{E}} = e^2/M_{\mathrm{G}}^2 \tag{6.13}$$

implying the existence of a superheavy boson of mass $M_{\mathrm{G}} \simeq 10^{18}\,\mathrm{GeV}$.

Arnowitt and Nath [88, 6] have observed further that the nature of spontaneous symmetry breaking of the super-gauge group (eqs. (6.12)) is so stringent that it fixes almost uniquely the nature of the internal symmetry group. For instance spontaneously broken solutions with $\lambda \neq 0$ in (6.12) can only occur for U(1) ($N = 2$), while for $\lambda = 0$ internal symmetries with bigger groups are implied.

For a couple of Dirac spinors ($N = 4$) one gets $\mathrm{U}(2) = \mathrm{SU}(2) \otimes \mathrm{U}(1)$ and one can think of a theory which unifies gravitational weak and electromagnetic interactions.

In order to incorporate strong interactions in this scheme one may think to provide the spinor coordinates θ_α^a with an extra colour (SU(3)) index θ_α^{ac}. In this way an additional colour octet of massless gluons could give rise to strong interactions.

The scheme previously discussed seems promising because of its qualitative features; however, it is not at all clear whether it would be entirely consistent with a Lagrangian quantum field theory, in particular because of the dimensionality of the fields, if it could give a ghost-free and renormalizable theory [136]. It is worthwhile mentioning that these difficulties are of the same nature as similar difficulties encountered in usual supersymmetric theories in flat space–time, when spinorial charges with internal symmetry quantum numbers are introduced.

A different scheme has been suggested by Zumino [140]. This scheme seems more closely connected with conventional supersymmetry transformations in Minkowski space. The great advantage here is that only Majorana spinors are used from the start; this will automatically ensure that, in the absence of gravitational interactions, the usual renormalizable supersymmetrical field theories will be recovered in flat Minkowski space.

In this framework a curved superspace $Z^M = (x^\mu, \theta^m)$ is described by a non-Riemannian geometry, which is related more closely to the flat superspace geometry. The most general affine geometry of this superspace is described by a manifold of points

$$Z^M = (x^\mu, \theta^m) \tag{6.14}$$

and by a set of canonical forms

$$\omega^A = \mathrm{d}z^M E_M{}^A(z) \tag{6.15}$$

and connection forms

$$\varphi_A{}^B = \mathrm{d}z^M \hat{\varphi}_{MA}{}^B(z) = \omega^C \varphi_{CA}{}^B(z). \tag{6.16}$$

$E_M{}^A(z)$ is a superfield which generalizes the usual Vierbein field and plays the role of the metric tensor g_{AB} in the Nath–Arnowitt Riemannian geometry.

In flat space the canonical forms (6.15) reduce to

$$\omega^A: \begin{array}{l} \omega^\alpha = \mathrm{d}x^\alpha + \mathrm{i}\,\mathrm{d}\bar{\theta}\gamma^\alpha\theta \\ \omega^a = \mathrm{d}\theta^a \end{array} \tag{6.17}$$

and the Vierbein field to

$$E_M^0{}^A(z): \begin{array}{ll} E_\mu^{0\alpha} = \delta_\mu{}^\alpha, & E_m^{0\alpha} = \mathrm{i}(\gamma^\alpha)_{mn}\theta^n \\ E_\mu^a = 0, & E_m^{0a} = \delta_m{}^a. \end{array} \tag{6.18}$$

We observe that

$$E_A^{0M}\partial_M = (\mathrm{D}_a, \partial_\alpha) \tag{6.19}$$

in which $\partial_\alpha = \partial/\partial x^\alpha$ and $\mathrm{D}_a = \partial_a - \mathrm{i}(\bar{\theta}\gamma^\alpha)_a\partial_\alpha$ are the usual covariant derivatives of supersymmetric theories.

Covariant differentiation of canonical forms

$$\mathrm{D}\omega^A = \mathrm{d}\omega^A - \omega^B\varphi_B{}^A = \Omega^A = \tfrac{1}{2}\,\omega^C\omega^B\Omega_{BC}{}^A \tag{6.20}$$

defines the torsion form Ω^A and the torsion tensor $\Omega_{BC}{}^A$.

The curvature form $R_A{}^B$ and the curvature tensor $R_{CDA}{}^B$ are defined by

$$\mathrm{d}\varphi_A{}^B - \varphi_A{}^C\varphi_C{}^B = R_A{}^B = \tfrac{1}{2}\,\omega^D\omega^C R_{CDA}{}^B. \tag{6.21}$$

Of course the torsion tensor can be expressed in terms of Vierbein and connection forms.

Usually the geometry of an affine space is specified by the group structure of the tangent space. For example in usual 4-dimensional space–time the tangent space is a Minkowski space and the group-structure is just the Lorentz group O(3,1). In superspace the analogue of local Lorentz transformations will be the group of linear transformations of the canonical forms (6.15) with infinitesimal transformations written as:

$$\delta\omega^A = \omega^B X_B{}^A(z) \tag{6.22}$$

the matrices $X_A{}^B$ at every point of superspace form a Graded Lie Algebra.

If one assumes that $X_B{}^A$ are just the superspace generalization of Lorentz transformations, the tangent space is characterized by a numerically invariant tensor η_{AB} which generalizes the usual metric tensor $\eta_{\mu\nu}$:

$$\eta_{AB} = (\eta_{\mu\nu} = \eta_{\mu\nu},\, \eta_{ab} = \eta_{ab},\, \eta_{\mu a} = \eta_{m\alpha} = 0)$$

moreover the torsion forms vanish. In this case the usual Arnowitt, Nath geometry is recovered (in the Vierbein formulation).

However, according to Zumino, the matrices $X_A{}^B$ are not the above generalized Lorentz transformations in superspace but they have a form of the type

$$X_A{}^B = \begin{pmatrix} \omega_\alpha{}^\beta & 0 \\ 0 & \tfrac{1}{4}\,\omega_{\mu\nu}\sigma^{\mu\nu}{}_a{}^b \end{pmatrix} \tag{6.23}$$

where $\omega_\alpha{}^\beta(z)$ is a local Lorentz transformation.

We stress that the group structure specified by (6.23) is not the same as in the Nath–Arnowitt geometry [89] (nor a subalgebra) but rather a particular contraction [136].

The group given by (6.23) has several numerically invariant tensors, for example:

$$\eta_{AB} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \eta^{AB} = \begin{pmatrix} 0 & 0 \\ 0 & \eta^{ab} \end{pmatrix} \qquad \gamma_A{}^{BC} = \gamma_\alpha{}^{bc} \qquad \text{(0 otherwise)} \tag{6.24}$$

The Vierbein superfield $E_M{}^A(z)$ is the basic gauge field of this theory. It can be shown that proper field equations can be written in such a way that they describe two massless particles with spin 2 and spin $\frac{3}{2}$. No other particle is associated with the Vierbein field so that the present formulation of the gauge structure of superspace should give a supergravity theory non-equivalent to the Nath–Arnowitt proposal, in which no solution [88, 6] was obtained when a single Majorana spinor θ_α was used.

6.2. Supergravity theory

Recently Ferrara, Freedman and Van Nieuwenhuizen [54], Deser and Zumino [27] found a Lagrangian which is the supersymmetric version of pure Einstein gravity. They used supersymmetry transformations involving anticommuting parameters $\epsilon(x)$ which are Majorana spinors with an arbitrary x-dependence.[33]

In this sense this theory should be equivalent to the theory that can be derived from the geometrical non-Riemannian structure introduced by Zumino [140].

The first authors used, from the start, only physical fields i.e. fields which correspond to particles.

It is known, from the representation theory, that the multiplet which contains a massless spin 2 particle (graviton) contains also a massless spin $\frac{3}{2}$ particle [107, 50]. No other particle exists in the same multiplet. It is therefore natural to use Vierbein and Rarita–Schwinger fields only, in order to build up the supersymmetric extension of Einstein theory.[34]

The price one has to pay, if one restricts to particle fields only, is that the transformation law of these fields is non-linear and also field-dependent but nevertheless it is known that one can study supersymmetric theories using only particle fields.

It turns out that the supersymmetric version of Einstein theory of gravity is given by the Lagrangian

$$\mathscr{L}(x) = \mathscr{L}_{\mathrm{E}}(x) + \mathscr{L}_{3/2}(x) + \mathscr{L}_4(x) \tag{6.25}$$

where

$$\mathscr{L}_{\mathrm{E}} = \tfrac{1}{4} k^{-2} \sqrt{-g} R \tag{6.26}$$

is the usual Einstein Lagrangian

$$\mathscr{L}_{3/2} = -\tfrac{1}{2} \epsilon^{\lambda\rho\mu\nu} \bar{\psi}_\lambda \gamma_5 \gamma_\mu \mathscr{D}_\nu \psi_\rho \tag{6.27}$$

is the Rarita–Schwinger Lagrangian minimally coupled to gravitation, and

$$\mathscr{L}_4 = -\frac{k^2}{32} (-g)^{-1/2} [\epsilon^{\tau\alpha\beta\nu} \epsilon_\tau{}^{\gamma\delta\mu} + \epsilon^{\tau\alpha\mu\nu} \epsilon_\tau{}^{\gamma\delta\beta} - \epsilon^{\tau\beta\mu\nu} \epsilon_\tau{}^{\gamma\delta\alpha}] (\bar{\psi}_\alpha \gamma_\mu \psi_\beta)(\bar{\psi}_\gamma \gamma_\nu \psi_\delta) \tag{6.28}$$

is a quartic self-interaction of the spin $\frac{3}{2}$ field. The spin $\frac{3}{2}$ field, which, from the point of view of conventional theories, would play the role of a matter field, is interpreted here as a gauge field, exactly as $g_{\mu\nu}(x)$. This is very similar to the situation encountered in supersymmetric Yang–Mills theories where gauge vector fields are accompanied by gauge spin $\frac{1}{2}$ fields [49, 113].

Deser and Zumino [27] have shown that supergravity is nothing but the spin $\frac{3}{2}$ Rarita–Schwinger Lagrangian minimally coupled to Cartan first-order formulation of general relativity.[35]

In their approach, the spin $\frac{3}{2}$ contact term $\mathscr{L}_4$ arises from equations of motion for the torsion tensor $C_{\mu\rho}{}^\tau(x)$ which takes the form

[33] It is worthwhile mentioning that in supergravity theories Einstein gauge invariance is not the primary symmetry. For instance the product of two local supersymmetry transformations gives rise to a general coordinate change [54, 55].

[34] It has been recently shown [61] that a massless spin $\frac{3}{2}$ particle, "minimally" coupled to gravitation, exists also in the closed string sector of the Neveu–Schwarz–Ramond model.

[35] It has been further shown by these authors that supergravity is free from usual higher spin inconsistencies and that propagation is causal.

$$C_{\mu\rho}{}^{\tau} = \tfrac{1}{2}\,\mathrm{i}\bar{\psi}_\mu \gamma^\tau \psi_\rho. \tag{6.29}$$

Note that the torsion tensor,[36] in first order formulation, is a function of the Vierbein and connection coefficient treated as independent fields

$$C_{\mu\nu}{}^{a} = \mathrm{D}_\mu V_\nu{}^a - \mathrm{D}_\nu V_\mu{}^a \tag{6.30}$$

in which $\mathrm{D}_\mu = \partial_\mu - \frac{1}{2}\,\omega_{\mu,ab}\Sigma^{ab}$ is the covariant derivative and $\omega_{\mu,ab}$ are the connection coefficients.

The advantage of the first order formulation is that the gravitational supermultiplet, in which the Vierbein field $V_\mu{}^a$ and the connections $\omega_{\mu,ab}(x)$ are treated as independent dynamical variables, transforms linearly under supersymmetry:

$$\begin{aligned}
\delta V_\mu{}^a(x) &= \mathrm{i}\bar{\epsilon}(x)\gamma^a\psi_\mu(x)\\
\delta\psi_\mu(x) &= 2\mathrm{D}_\mu\epsilon(x)\\
\delta\omega_\mu{}^{ab}(x) &= B_\mu{}^{ab}(x) - \tfrac{1}{2}\,V_\mu{}^b(x)\,B_c{}^{ac}(x) + \tfrac{1}{2}\,V_\mu{}^a(x)\,B_c{}^{bc}(x)
\end{aligned} \tag{6.31}$$

in which $\sqrt{-g}B_a^{\lambda\mu}(x) = \mathrm{i}\bar{\epsilon}(x)\gamma_5\gamma_a\mathscr{D}_\nu\psi_\rho(x)\epsilon^{\lambda\mu\nu\rho}$ and $\epsilon(x)$ is an arbitrary x-dependent (Majorana) spinorial anticommuting parameter.

Moreover the original Lagrangian itself takes the simpler interpretation of a minimally coupled spin 2, $\frac{3}{2}$ theory.

Like in Yang–Mills theories, one could envisage a coupling of supergravity with matter superfields. There are two types of these multiplets, gauge vector multiplets $(V_\mu(x), \lambda_\alpha(x))$ and chiral multiplets $(A(x), B(x), \psi_\alpha(x))$. For example if one considers the neutral gauge multiplet given by $(V_\mu(x), \lambda_\alpha(x))$ then one would obtain the supersymmetric version of gravity coupled with electromagnetism. This theory describes the interaction of the gravitational and photon fields with additional massless spin $\frac{3}{2}$ and $\frac{1}{2}$ fields.

Scalar multiplets of the type $(A(x), B(x), \psi_\alpha(x))$ could be added as well. We note that an explicit mass term in the Lagrangian can only be associated with these multiplets. They could have self-interactions as well as Yang–Mills interactions. Coupled to supergravity theory they would provide the most general Lagrangian consistent with supersymmetry as well as Yang–Mills and Einstein gauge symmetries. The construction of such Lagrangian has not yet been carried out but it would be extremely interesting if the implementation of Einstein gauge symmetry with a local spinorial group (local supersymmetry) could provide a renormalizable or at least less divergent theory of quantum gravity.

6.3. Local Graded Lie Algebras in two-dimensions

From a historical point of view the introduction of supersymmetries in particle physics was suggested by the observation that in some two-dimensional quantum field theories, the so-called dual strings, it is indeed possible to enlarge the conformal algebra (gauge group of the string) to a whole Graded Lie Algebra with commutation and anticommutation relations given by [91, 102].

[36] The presence of torsion in the supersymmetric variant of general relativity has been discussed in ref. [3].

$$
\begin{aligned}
[L_n, L_m] &= (n-m)\, L_{n+m} \\
[L_n, G_r] &= (\tfrac{1}{2} n - r)\, G_{n+r} \\
\{G_r, G_s\} &= 2\, L_{r+s}
\end{aligned}
\tag{6.32}
$$

where n, r run respectively over all integers and half-integers.

The Lie Algebra contained in (6.32) is just the infinite dimensional conformal algebra of two-dimensional space–time. G_r is an anticommuting spinor which acts as grading representation. Note that there is an infinite set of odd generators G_r in 2-dimensions, this fact is related to the properties of conformal transformations in 2-dimensional space–time. The latter transformations involve, in fact, arbitrary functions of the light-cone variables $\tau + \sigma$, $\tau - \sigma$, with $\tau = x^0$, $\sigma = x^1$.

The Lie Algebra contained in (6.32) as well as the complete Graded Lie Algebra (6.32) are the basic gauge algebras of two consistent dual string models [75, 53, 116], namely the Veneziano model (Nambu-Goto string) and the Neveu–Schwarz–Ramond [91, 102] model.

Consistency requirements of quantum theory imply that these strings move in a Minkowski space–time of dimensions 26 and 10 respectively. These values are usually called critical dimensions of the model.

The fact that in the Neveu–Schwarz–Ramond model the critical dimension is lower, is closely related to the property that its gauge group (local symmetry of the 2-dimensional theory) is bigger than in the Veneziano model. The string interpretation of the fermionic model was recently completed by Zumino [140] who showed that the dynamic of the above system is entirely fixed by demanding invariance under general coordinate transformations on a "superspace" the points of which are labelled by the string coordinates (σ, τ) and anticommuting (Majorana) spinor θ_α connected to the fermionic degree of freedom. The absence of ghosts in the richer spectrum of states of the Neveu–Schwarz–Ramond model is just due to the bigger symmetry of the system.

The previous considerations suggested that the right way to lower the critical dimension of string models and therefore to construct more interesting models was connected to the possibility of enlarging the graded Lie algebra (6.32) by adding new degrees of freedom to the string.[37] On the other hand, from the analysis carried out in section 2, it is known that, in 4 space–time dimensions it is indeed possible to generalize the supersymmetry algebra to the case in which the spinor charges are not Majorana spinors but rather spinors belonging to some representation of an internal symmetry compact Lie group G. When the Lie algebra is the whole conformal algebra, the internal symmetry group is fixed to be SU(N) with the spinorial charge in the fundamental (N-dimensional) representation [68]. This analysis recently had a drawback in two-dimensional field theories because of the reasons previously discussed. In fact, Ademollo et al. [2] solved the problem of constructing local supersymmetry algebras in two-dimensions with internal symmetry.

Here, the additional difficulty with respect to the 4-dimensional case is connected to the exceptional properties of the two-dimensional conformal group. As a consequence, the graded Lie algebras, which grade the conformal algebra, are inevitably infinite dimensional.

The construction of local supersymmetry algebras in two-dimensional space–time, to include an internal symmetry, can be obtained by studying the composition law in the parameter space $(\sigma, \tau, \theta^i_\alpha)$ where θ^i_α are N (Majorana) spinors, which are odd elements of a Grassmann algebra, i.e. $\theta^i_\alpha \theta^j_\beta = -\theta^j_\beta \theta^i_\alpha$. Because of the exceptional properties of conformal transformations in 2-dimensions, one can define light-cone variables

[37] A different connection among supersymmetries and strings has been considered in ref. [18].

$$\xi = \tfrac{1}{2}(\tau + \sigma), \qquad \zeta = \tfrac{1}{2}(\tau - \sigma)$$
$$\theta^i_\alpha = \tfrac{1}{2}(1 - \gamma_5)_{\alpha\beta}\theta^i_\beta, \qquad \chi^i_\alpha = \tfrac{1}{2}(1 + \gamma_5)_{\alpha\beta}\theta^i_\beta \tag{6.33}$$

and confine oneself to the ξ, θ^i part only.

Under an infinitesimal (restricted) supersymmetry change (constant parameter α^i) ξ, θ^i undergo the following transformations

$$\delta\xi = \mathrm{i} \sum_{i=1}^{N} \alpha^i \theta^i$$
$$\delta\theta^i = \alpha^i \tag{6.34}$$

while under a conformal transformation of parameter $u(\xi)$

$$\delta\xi = u(\xi)$$
$$\delta\theta^i = \tfrac{1}{2}\dot{u}(\xi)\theta^i. \tag{6.35}$$

The most general transformation of the algebra one gets combining (6.34) and (6.35) is given by

$$\delta\xi = \mathrm{i}(2 - n)\alpha_{i_1\ldots i_n}(\xi)\theta^{i_1}\ldots\theta^{i_n}, \qquad n = 0, \ldots N$$
$$\delta\theta^i = n\,\alpha_{i_1\ldots i_{n-1}}{}^i(\xi)\theta^{i_1\cdots i_{n-1}} + \mathrm{i}\dot{\alpha}_{i_1\ldots i_{n-1}}(\xi)\theta^{i_1}\ldots\theta^{i_{n-1}}\theta^i \tag{6.36}$$

where $\alpha_{i_1\ldots i_n}(\xi)$ are commuting (anticommuting) parameters for n even (odd) and are completely antisymmetric in their indices.

The overall number of independent transformations is therefore

$$\sum_{n=0}^{N} \binom{N}{n} = 2^N \tag{6.37}$$

where $2^N/2$ have commuting parameters and $2^N/2$ anticommuting ones.

From (6.36) one gets the following abstract graded Lie algebra [2]

$$[G_n^{i_1\cdots i_R}, G_m^{j_1\cdots j_S}]_{(-)^{RS+1}} = \mathrm{i}^{-RS}\Big\{[n(2-S) - m(2-R)]\, G_{n+m}^{i_1\cdots i_R j_1\cdots j_S}$$
$$- \mathrm{i} \sum_{h=1}^{R}\sum_{k=1}^{S} (-1)^{h+k+S}\delta^{i_h, j_k} G_{n+m}^{i_1\cdots \hat{i}_h \cdots i_R j_1 \cdots \hat{j}_k \cdots j_S}\Big\} \tag{6.38}$$

where the square-bracket on the left-hand side means a commutator (anticommutator) if RS is even (odd). $G_n^{i_1\cdots i_R}$ are 2^N local (n-independent) completely antisymmetric tensors with respect to the O(N) group and are bosonic (fermionic) for R even (odd). The index n runs over all integers (half-integers) for bosonic (fermionic operators).

It is worthwhile mentioning that the previous formulae contain the Virasoro and Neveu–Schwarz algebras as particular cases (respectively $N = 0, 1$). Moreover the O(N) algebras with $N > 2$ contain $2^N - (\tfrac{1}{2}N(N+1) + 1)$ local subcanonical charges in addition to the $\tfrac{1}{2}N(N+1) + 1$ canonical charges which correspond to local conformal, supersymmetry and internal symmetry transformations. One important exception is a subalgebra SU(2) of the O(4) local algebra which does not involve any subcanonical charge. In this case, the spinor charges are complex isodoublets. Interestingly enough, the graded Lie algebras with a local internal symmetry O(2) or SU(2), which close without any additional subcanonical charge, are just the only graded Lie algebras which admit a Fock space realization, besides the Neveu–Schwarz and Virasoro algebras.

The O (2) (respectively SU (2)) algebra is realized in terms of 2(4) sets of bosonic oscillators and 2(4) sets of fermionic ones. In terms of U (1) complex notations, one gets for the O (2) algebra [2]

$$
\begin{aligned}
&[L_n, L_m] = (n-m) L_{n+m} + \tfrac{1}{4} D\, n(n^2-1)\, \delta_{n,-m} \\
&\{G_r, G_s\} = \{\bar{G}_r, \bar{G}_s\} = 0 \\
&\{G_r, \bar{G}_s\} = 2L_{r+s} + 2(r-s)\, T_{r+s} + \tfrac{1}{4} D\,(4r^2-1)\, \delta_{r,-s} \\
&[L_n, T_m] = -\, m\, T_{n+m} \qquad [T_n, T_m] = \tfrac{1}{4} D\, n \delta_{n,-m} \\
&[L_n, G_r] = (\tfrac{1}{2} n - r)\, G_{n+r} \qquad [L_n, \bar{G}_r] = (\tfrac{1}{2} n - r)\, \bar{G}_{n+r} \\
&[T_n, G_r] = \tfrac{1}{2} G_{n+r} \qquad [T_n, \bar{G}_r] = -\tfrac{1}{2} \bar{G}_{n+r}
\end{aligned}
\tag{6.39}
$$

where $G = (G^1 + \mathrm{i} G^2)/\sqrt{2}$ and G^i is an O(2) doublet.

For SU (2) one gets the following graded Lie algebra

$$
\begin{aligned}
&[L_n, L_m] = (n-m) L_{n+m} + \tfrac{1}{2} D\, n(n^2-1)\, \delta_{n,-m} \\
&\{G_r^a, G_s^b\} = \{\bar{G}_r^a, \bar{G}_s^b\} = 0 \\
&\{G_r^a, \bar{G}_s^b\} = 2\delta^{ab} L_{r+s} + 2(r-s)\, \sigma_i^{ab} T_{r+s}^i + \tfrac{1}{2} D(4r^2-1)\, \delta_{r,-s} \\
&[L_n, T_m^i] = -\, m T_{n+m}^i \\
&[L_n, G_r^a] = (\tfrac{1}{2} n - r)\, G_{n+r}^a \\
&[T_n^i, G_r^a] = \tfrac{1}{2} \sigma^{iab} G_{n+r}^b \\
&[T_n^i, T_m^j] = \mathrm{i} \epsilon^{ijk} T^k{}_{n+m} + \delta^{ij} \tfrac{1}{2} D\, n \delta_{n,-m}
\end{aligned}
\tag{6.40}
$$

where σ_{ab}^i are the Pauli matrices and G^a is a complex spinor.

We would like to end this section by remarking that the SU (2) string, in spite of the fact that it admits a Fock space realization, does not have a sensible quantum theory because it is not ghost-free for any value of the space–time dimension. This is also the case for the non canonical models based on higher O(N) symmetries.

As a consequence, the only consistent string model, based on a supersymmetry, is the U (1) string, besides the Neveu–Schwarz–Ramond string.

In the next subsection, we will describe the superfield formulation of these two supersymmetric string models.

6.4. Supersymmetric strings: the Neveu–Schwarz–Ramond and the Colour-U(1) models

In this subsection, we will consider in some details the Lagrangian formulation of supersymmetric strings. Let us start with the Neveu–Schwarz–Ramond [91, 102] string. This model, in the gauge where the Lagrangian is linearized, is actually invariant under the graded Lie algebra in which the anticommuting quantity is a Majorana spinor θ_α.

The conformal transformations of this algebra in the parameter space are given by

$$
\begin{aligned}
&\delta \xi_\mu = u_\mu \\
&\delta \theta = \tfrac{1}{4} \gamma_\mu \not{\partial} u^\mu \theta
\end{aligned}
\tag{6.41}
$$

where $u_\mu(\tau,\sigma)$ is constrained by the equation

$$\partial_\mu u_\nu + \partial_\nu u_\mu - \tfrac{1}{2} g_{\mu\nu}\partial_\rho u^\rho = 0 \tag{6.42}$$

which implies

$$\tfrac{1}{2}(u^0(\tau,\sigma) + u^1(\tau,\sigma)) = u(\tau + \sigma)$$
$$\tfrac{1}{2}(u^0(\tau,\sigma) - u^1(\tau,\sigma)) = v(\tau - \sigma) \tag{6.43}$$

and u,v are arbitrary functions.

The supergauge transformations are

$$\delta\xi_\mu = -\,\mathrm{i}\alpha\gamma_\mu\theta, \qquad \delta\theta = \alpha \tag{6.44}$$

where $\alpha(\tau,\sigma)$ is an anticommuting Majorana spinor constrained by the equation

$$\gamma^\mu\gamma^\nu\partial_\mu\alpha = 0 \tag{6.45}$$

which implies

$$\alpha_1(\tau,\sigma) = \alpha_1(\tau - \sigma), \qquad \alpha_2(\tau,\sigma) = \alpha_2(\tau + \sigma). \tag{6.46}$$

The N.S.R. Lagrangian is described in terms of a real superfield [43]

$$X(\xi_\mu,\theta_\alpha) = \phi(\xi) + \mathrm{i}\theta\psi(\xi) + \tfrac{1}{2}\,\mathrm{i}\theta\theta F(\xi) \tag{6.47}$$

which, under supergauges, undergoes the following transformation

$$X'(\xi,\theta) = X(\xi - \mathrm{i}\alpha\gamma\theta,\ \theta + \alpha) \tag{6.48}$$

that is, in terms of component fields

$$\delta\phi = \mathrm{i}\alpha\psi$$
$$\delta\psi = \partial\!\!\!/\phi\alpha + F\alpha$$
$$\delta F = \mathrm{i}\alpha\partial\!\!\!/\psi. \tag{6.49}$$

Neglecting for a moment the problem arising from the finite length of the string, one can immediately write a linearized action invariant under (6.48). In terms of the covariant derivative

$$\mathrm{D}_\alpha = \frac{\partial}{\partial\theta^\alpha} + \mathrm{i}(\gamma^\mu\theta)_\alpha\partial_\mu \tag{6.50}$$

the action reads

$$A = \frac{1}{8\pi\alpha'}\int\mathrm{d}\tau\int_0^\pi\mathrm{d}\sigma\int\mathrm{d}^2\theta\ \mathrm{D}^\alpha X\mathrm{D}_\alpha X = \frac{1}{2\pi\alpha'}\int\mathrm{d}\tau\int_0^\pi\mathrm{d}\sigma\,[-\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}\,\mathrm{i}\psi\partial\!\!\!/\psi + \tfrac{1}{2}F^2]\,. \tag{6.51}$$

From (6.51), one gets the equations of motion

$$\mathrm{D}^\alpha\mathrm{D}_\alpha X = 0 \tag{6.52}$$

or in terms of component fields

$$\Box\phi = \partial\!\!\!/\psi = F = 0. \tag{6.53}$$

At the boundaries of the string a bit more care is needed. In fact (6.51) is not invariant under the whole graded Lie algebra given by (6.41) and (6.44) but only under the transformations that leave the ends of the string $\sigma = 0$ and $\sigma = \pi$ fixed. This implies

$$u(\tau) = v(\tau); \qquad u(\tau + 2\pi) = u(\tau), \tag{6.54}$$

$$\alpha_1(\tau) = -\alpha_2(\tau); \qquad \alpha_1(\tau + 2\pi) = \mp\alpha_1(\tau). \tag{6.55}$$

The restrictions on the parameters given by (6.54), (6.55) imply the following boundary conditions for the fields

$$\left.\frac{\partial\phi(\tau,\sigma)}{\partial\sigma}\right|_{\sigma=0,\pi} = 0, \qquad \begin{aligned} &\psi_1(\tau) = \psi_2(\tau) \\ &\psi_1(\tau + 2\pi) = \mp\psi_1(\tau). \end{aligned} \tag{6.56}$$

The antiperiodicity (periodicity) of the field ψ shows that actually the upper (lower) sign in (6.55) gives the Neveu–Schwarz (Ramond) sector of the N.S.R. model.

Now we will discuss the gauge conditions of the theory. As pointed out by Zumino [140], the linearized Lagrangian (6.51) can be derived from a Lagrangian invariant under general coordinate transformations in superspace $(\tau, \sigma, \theta_\alpha)$. In this framework, the constraints are a direct consequence of the invariance properties of the Lagrangian.

In the gauge where the Lagrangian is linearized these constraints give the vanishing of the supercurrent that generates the local invariance group of the action. In the N.S.R. string, the supercurrent is given by the following superfield bilinear

$$V_{\mu\alpha}(\xi,\theta) = \gamma^\rho\gamma_\mu \mathrm{D}_\alpha X \partial_\rho X. \tag{6.57}$$

Its vanishing implies the usual gauge and supergauge conditions at the classical level

$$\begin{aligned} &\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}g_{\mu\nu}\partial^\rho\phi\partial_\rho\phi + \tfrac{1}{4}\mathrm{i}\bar{\psi}(\gamma_\mu\partial_\nu + \gamma_\nu\partial_\mu)\psi = 0 \\ &\gamma^\mu\gamma_\rho\partial_\mu\phi\psi = 0. \end{aligned} \tag{6.58}$$

In quantum theory the constraints (6.58) give rise to a ghost free model for D (=space–time dimension) = 10.

We consider now the Lagrangian formulation of the U(1) string. This system, from the physical point of view, can be regarded as a modified N.S.R. model in which, also, a charge is distributed along the string.

However, the implementation of a new quantum number drastically changes the gauge algebra and the physics of the string.

Using the more suitable U(1) complex notations, the local graded Lie group of the U(1) string is given, in the parameter space, by the following transformations

$$\delta\xi = u(\xi), \qquad \delta\theta = \frac{\dot{u}(\xi)}{2}\theta \qquad \text{(conformal)} \tag{6.59}$$

$$\left.\begin{aligned} &\delta\xi = \tfrac{1}{2}\mathrm{i}(\alpha(\xi)\bar{\theta} + \bar{\alpha}(\xi)\theta) \\ &\delta\theta = \alpha(\xi) + \tfrac{1}{2}\mathrm{i}\dot{\alpha}(\xi)\bar{\theta}\theta \\ &\delta\bar{\theta} = \bar{\alpha}(\xi) - \tfrac{1}{2}\mathrm{i}\dot{\bar{\alpha}}(\xi)\bar{\theta}\theta \end{aligned}\right\} \qquad \text{(supergauge)} \tag{6.60}$$

$$\left.\begin{aligned} &\delta\xi = 0 \\ &\delta\theta = -\mathrm{i}T(\xi)\theta \\ &\delta\bar{\theta} = \mathrm{i}T(\xi)\bar{\theta} \end{aligned}\right\} \qquad \text{(local U(1) rotations)} \tag{6.61}$$

where $\theta = \theta^1 + \mathrm{i}\theta^2$, $\alpha = \alpha^1 + \mathrm{i}\alpha^2$.

Using the shifted variable $\xi^- = \xi - \frac{1}{2}\,\mathrm{i}\theta\bar\theta$, (6.59), (6.60), (6.61) actually simplify into

$$\begin{array}{llll} \delta\xi^- = u(\xi^-), & \delta\theta = \frac{1}{2}\dot u(\xi^-)\theta, & \delta\bar\theta = \frac{1}{2}\dot u(\xi^-)\bar\theta & \text{(conformal)} \\ \delta\xi^- = \mathrm{i}\bar\alpha(\xi^-)\theta, & \delta\theta = \alpha(\xi^-), & \delta\bar\theta = \bar\alpha(\xi^-) + \mathrm{i}\dot{\bar\alpha}(\xi^-)\theta\bar\theta & \text{(supergauge)} \\ \delta\xi^- = 0, & \delta\theta = -\,\mathrm{i}T(\xi^-)\theta, & \delta\bar\theta = \mathrm{i}T(\xi^-)\bar\theta & \text{(U(1) rotations)} \end{array} \tag{6.62}$$

Eqs. (6.62) show that the U(1) graded Lie algebra can be realized in the smaller parametric space (ξ^-,θ) where θ is a complex anticommuting variable.

According to this remark, we can construct the U(1) string Lagrangian, defining, on the parametric space $(\xi_\mu,\theta^i_\alpha)$, a scalar complex superfield $S(\xi_\mu,\theta_\alpha,\bar\theta_\alpha)$ with the restriction that $S(\xi - \frac{1}{2}\mathrm{i}\bar\theta\gamma\theta,\ \theta,\bar\theta)$ is independent of $\bar\theta$. This statement can be formulated in terms of a covariant equation:

$$\bar{\mathrm{D}}_\alpha S = 0 \tag{6.63}$$

where $\bar{\mathrm{D}}_\alpha = \partial/\partial\bar\theta^\alpha + \frac{1}{2}\mathrm{i}(\not\partial\theta)_\alpha$ is a covariant derivative which, together with its complex conjugate $\mathrm{D}_\alpha = \partial/\partial\theta^\alpha + \frac{1}{2}\mathrm{i}(\not\partial\bar\theta)_\alpha$ anticommutes with the supergauge charges.

Eq. (6.63) implies the following form for S

$$S(\xi^-,\theta) = A(\xi^-) + \mathrm{i}\theta^\alpha\bar\psi_\alpha(\xi^-) + \tfrac{1}{2}\mathrm{i}\theta^\alpha\theta_\alpha F(\xi^-) \tag{6.64}$$

A, ψ, F being complex fields.

Under a supergauge transformation, these fields undergo the following change

$$\delta A = \mathrm{i}\alpha\bar\psi, \qquad \delta\bar\psi = \alpha F + \not\partial A\bar\alpha, \qquad \delta F = \mathrm{i}\bar\alpha\not\partial\psi. \tag{6.65}$$

The string action is given by

$$\begin{aligned} A &= \frac{1}{16\pi\alpha'}\int \mathrm{d}\tau \int_0^\pi \mathrm{d}\sigma \int \mathrm{d}^2\theta\ \mathrm{d}^2\bar\theta\ \bar S(\xi^+,\bar\theta)\,S(\xi^-,\theta) \\ &= \frac{1}{2\pi\alpha'}\int \mathrm{d}\tau \int_0^\pi \mathrm{d}\sigma\left[-\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}(\partial_\mu C)^2 - \tfrac{1}{2}\mathrm{i}\psi^i\not\partial\psi^i + \tfrac{1}{2}(E^2 + D^2)\right] \end{aligned} \tag{6.66}$$

where

$$A = \phi + \mathrm{i}C, \quad \psi = \psi^1 + \mathrm{i}\psi^2, \quad F = E - \mathrm{i}D.$$

The superfield equations are

$$DDS = 0 \tag{6.67}$$

which imply

$$\Box\phi = \Box C = \not\partial\psi^i = E = D = 0. \tag{6.68}$$

Now, we have to discuss the boundary conditions of the model.

There are two types of boundary conditions

$$\begin{array}{ll} \text{at } \sigma = 0 & \alpha_1 + \alpha_2 = 0 \\ \text{at } \sigma = \pi & \alpha_1 - \mathrm{e}^{\mathrm{i}\phi}\alpha_2 = 0 \text{ or} \end{array} \tag{6.69}$$

$$\text{at } \sigma = 0 \qquad \alpha_1 + \alpha_2 = 0$$

$$\text{at } \sigma = \pi \qquad \alpha_1 - e^{i\phi}\bar{\alpha}_2 = 0 \tag{6.70}$$

where ϕ is an arbitrary phase, completely irrelevant due to the U(1) local invariance.

From (6.69) one gets the following conditions

$$\left.\frac{\partial A(\tau,\sigma)}{\partial\sigma}\right|_{0,\pi} = 0, \tag{6.71}$$

i.e. $A(\tau) = B(\tau)$ and $A(\tau + 2\pi) = A(\tau)$, being

$$A(\tau,\sigma) = \tfrac{1}{2}\{A(\tau+\sigma) + B(\tau-\sigma)\}$$

$$\psi_1(\tau) = \psi_2(\tau), \qquad \psi_1(\tau + 2\pi) = e^{i\phi}\,\psi_1(\tau). \tag{6.72}$$

In spite of the fact that ψ depends on an arbitrary phase, which means that the Fourier expansion of ψ can contain any type of frequencies $n + \epsilon$ (ϵ is any real number depending on the choice of ϕ), the physical properties of the models (for example the spectrum) are ϕ-independent; in fact, only one model is originated from the boundary conditions (6.69) which resembles the Neveu–Schwarz model. This model is ghost and tachyon free for the critical dimension $D = 2$ and contains only one physical state of zero mass, the ground state.

Another model is originated from the boundary conditions (6.70). In this case, one gets a boson and a fermion with integral frequencies and a boson and a fermion with half-integral frequencies. The physical properties of this model have not been investigated yet.

As a final step, we have to derive the non-linear constraints which follow from general reparametrization invariance in the superspace $(\tau,\sigma,\theta^i_\alpha)$. We learnt that these constraints, in the linearized gauge, are just equivalent to the vanishing of the supercurrent multiplet. In the U(1) string, the latter is given by the following vector superfield

$$V_\mu(\xi_\mu,\theta^i_\alpha) = i\bar{D}\bar{S}\gamma_\mu DS. \tag{6.73}$$

In terms of component fields, one gets

$$\bar{\psi}\gamma_\mu\psi = 0, \qquad \not{\partial}A\gamma_\mu\psi = 0,$$

$$\partial_\mu\bar{A}\,\partial_\nu A - \tfrac{1}{2}ig_{\mu\nu}\partial^\rho\bar{A}\,\partial_\rho A + \tfrac{1}{4}i\bar{\psi}(\gamma_\mu\partial_\nu + \gamma_\nu\partial_\mu)\psi = 0. \tag{6.74}$$

The second equation in (6.74) leads to two fermionic constraints while the first and the third ones to two bosonic constraints. These constraints are just the vanishing of the Noether currents respectively associated with U(1), supergauge and conformal transformations. The last two constraints are just generalizations of the gauge and supergauge conditions of the Neveu–Schwarz model, while the first constraint is of a new type and is connected with the charge distribution introduced on the string.

At the quantum level, the vanishing of the local current implies that physical states (excitations of the string) must be U(1) singlets.

The fact that the U(1) internal symmetry introduced as part of a graded Lie algebra must be a colour symmetry is then closely related to the fact that it belongs to the gauge group of the string. Colour confinement is simply achieved in the present framework as a gauge constraint on physical states.

Conventions and notations

We use a space–time metric of the form:

$$g_{11} = g_{22} = g_{33} = -g_{00} = 1, \quad g_{\mu\nu} = 0, \quad \mu \neq \nu.$$

The γ matrices are in a Majorana representation. They are real and satisfy the properties $\gamma_i^2 = -\gamma_0^2 = 1$. Moreover γ_5 and the Ricci tensor $\epsilon^{\mu\nu\rho\sigma}$ are defined as follows:

$$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3, \qquad \epsilon^{0123} = 1.$$

If ψ is a self-conjugate (Majorana) spinor (this means that its four components are real in the Majorana representation of the γ-matrices), its adjoint is defined as follows: $\bar{\psi}_\alpha = \psi_\beta\gamma^0_{\beta\alpha} = (\psi^{\mathrm{T}}\gamma^0)_\alpha$. Sometimes we use the equivalent notation $\bar{\psi}_\alpha = \psi^\alpha$ so that $\psi^\alpha\chi_\alpha = \psi^{\mathrm{T}}\gamma^0\chi$. We use Van der Waerden notations for the Weyl two-component formalism [48].

For the metric matrix $\epsilon^{\alpha\beta}$ we have $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$, $\epsilon^{12} = 1$ (same for dotted indices), $\sigma_{\mu\alpha\dot{\beta}} = (1, \sigma_i)$. For a two component spinor we have $(\psi_\alpha)^* = \bar{\psi}_{\dot{\alpha}}$, $(\chi_{\dot{\alpha}})^* = \bar{\chi}_\alpha$ (same for upper indices). Moreover $\chi_\alpha = \epsilon_{\alpha\beta}\chi^\beta$. For any 4×4 matrix Γ define its adjoint $\tilde{\Gamma} = -\gamma_0\Gamma^{\mathrm{T}}\gamma_0$; then for any couple of (anticommuting) Majorana spinors one has

$$\bar{\alpha}_1\Gamma\alpha_2 = \bar{\alpha}_2\tilde{\Gamma}\alpha_1.$$

Now $\tilde{\gamma}_A = \gamma_A$ for $1, \gamma_5, \gamma_5\gamma_\mu$; $\tilde{\gamma}_A = -\gamma_A$ for $\gamma_\mu, \gamma_\mu\gamma_\nu$ $(\mu < \nu)$.

For any γ_A define γ^A such that $\gamma_A\gamma^A = 1$. Then the following (Fierz) rearrangement formula holds

$$(\bar{\alpha}_1\alpha_3)\alpha_2 = -\tfrac{1}{4}\sum_A (\bar{\alpha}_1\gamma_A\alpha_2)\gamma^A\alpha_3,$$

$\alpha_1, \alpha_2, \alpha_3$ being any triplet of (anticommuting) Majorana spinors.

References

[1] E.S. Abers and B.W. Lee, Phys. Reports 9C (1973) 1.
[2] M. Ademollo et al., Phys. Letters 62B (1976) 105;
M. Ademollo et al., Nucl. Phys. B111 (1976) 77; B114 (1976) 297.
[3] V.P. Akulov, D.V. Volkov and V.A. Soroka, JETP Letters 22 (1975) 187.
[4] R. Arnowitt, P. Nath and B. Zumino, Phys. Letters 56B (1975) 81.
[5] R. Arnowitt and P. Nath, Proc. Conf. on Riddle of Gravitation, Syracuse 1975, to be published Gen. Rel. Grav. 1976.
[6] R. Arnowitt and P. Nath, Phys. Rev. Letters 36 (1976) 1526.
[7] W.A. Bardeen, unpublished.
[8] C. Becchi, A. Rouet and R. Stora, Comm. Math. Phys. 42 (1975) 127.
[9] F.A. Berezin, the Method of Second Quantization (Acad. Press, New York, 1966).
[10] F.A. Berezin and G.I. Katz, Matemat. Sbornik (USSR) 82 (1970) 343.
[11] S. Browne, Phys. Letters 59B (1975) 253.
[12] S. Browne, L. O'Raifeartaigh and T. Sherry, Nucl. Phys. B99 (1975) 150.
[13] D.M. Capper, J. Phys. G2 (1976) 141.
[14] D.M. Capper and G. Leibbrandt, Nucl. Phys. B85 (1975) 492.
[15] D.M. Capper and G. Leibbrandt, Nucl. Phys. B85 (1975) 503.
[16] D.M. Capper and M. Ramón Medrano, J. Phys. G2 (1976) 269.
[17] D.M. Capper, A. Salam and J. Strathdee, Nucl. Phys. B107 (1976) 137.
[18] L.N. Chang, K.I. Macrae and F. Mansouri, Phys. Rev. D13 (1976) 235.

[19] S. Coleman and J. Mandula, Phys. Rev. 159 (1967) 1251.
[20] L. Corwin, Y. Neéman and S. Sternberg, Rev. Mod. Phys. 47 (1975) 573.
[21] T.L. Cutright and G.I. Ghandour, Phys. Letters 59B (1975) 387; preprint CALT 68-531 (Aug. 1976).
[22] R. Delbourgo, Nuovo Cimento 25A (1975) 646.
[23] R. Delbourgo, J. Phys. G1 (1975) 800.
[24] R. Delbourgo, V.B. Prasad, J. Phys. G1 (1975) 377.
[25] R. Delbourgo and M. Ramón Medrano, Nucl. Phys. B110 (1976) 473.
[26] R. Delbourgo, A. Salam and J. Strathdee, Phys. Letters 51B (1975) 475.
[27] S. Deser and B. Zumino, Phys. Letters 62B (1976) 335.
[28] B. de Wit, Phys. Rev. D11 (1975) 898.
[29] B. de Wit, Phys. Rev. D12 (1975) 1628.
[30] B. de Wit and D.Z. Freedman, Phys. Rev. Letters 35 (1975) 827.
[31] B. de Wit and D.Z. Freedman, Phys. Rev. D12 (1975) 2286.
[32] P.H. Dondi and M. Sohnius, Nucl. Phys. B81 (1974) 317.
[33] F. Englert and R. Brout, Phys. Rev. Letters 13 (1964) 321.
[34] P. Fayet, Nucl. Phys. B78 (1974) 14.
[35] P. Fayet, Nucl. Phys. B90 (1975) 104.
[36] P. Fayet, Phys. Letters 58B (1975) 67.
[37] P. Fayet, Nuovo Cimento 31A (1976) 626.
[38] P. Fayet, Nucl. Phys. B113 (1976) 135.
[39] P. Fayet, Phys. Rev. D14 (1976) 3557.
[40] P. Fayet, in preparation.
[41] P. Fayet and J. Iliopoulos, Phys. Letters 51B (1974) 461.
[42] S. Ferrara, Nucl. Phys. B77 (1974) 73.
[43] S. Ferrara, Lett. Nuov. Cim. 13 (1975) 629.
[44] S. Ferrara, Rivista del Nuovo Cimento 6 (1975) 105.
[45] S. Ferrara, J. Iliopoulos and B. Zumino, Nucl. Phys. B77 (1974) 413.
[46] S. Ferrara and O. Piguet, Nucl. Phys. B93 (1975) 261.
[47] S. Ferrara and E. Remiddi, Phys. Letters 53B (1974) 347.
[48] S. Ferrara, J. Wess and B. Zumino, Phys. Letters 51B (1974) 239.
[49] S. Ferrara and B. Zumino, Nucl. Phys. B79 (1974) 413.
[50] S. Ferrara and B. Zumino, Nucl. Phys. B87 (1975) 207.
[51] R.J. Firth and J.D. Jenkins, Nucl. Phys. B85 (1975) 525.
[52] E.S. Fradkin and O.K. Kalashnikov, Nucl. Phys. B98 (1975) 259.
[53] P.H. Frampton, Dual Resonance Models (Benjamin, 1974).
[54] D.Z. Freedman, P. van Nieuwenhuizen and S. Ferrara, Phys. Rev. D13 (1976) 3214.
[55] D.Z. Freedman and P. van Nieuwenhuizen, Phys. Rev. D14 (1976) 912.
[56] P.G.O. Freund and I. Kaplansky, J. Math. Phys. 17 (1976) 228.
[57] K. Fujikawa and W. Lang, Nucl. Phys. B88 (1975) 61.
[58] K. Fujikawa and W. Lang, Nucl. Phys. B88 (1975) 77.
[59] H. Georgi and S.L. Glashow, Phys. Rev. Letters 28 (1972) 1494.
[60] J.L. Gervais and B. Sakita, Nucl. Phys. B34 (1971) 632.
[61] F. Gliozzi and J. Scherk, private communication.
[62] P. Goddard, Nucl. Phys. B88 (1975) 429.
[63] J. Goldstone, Nuovo Cimento 19 (1961) 154.
[64] J. Goldstone, A. Salam and S. Weinberg, Phys. Rev. 127 (1962) 965.
[65] Yu.A. Gol'fand and E.P. Likhtman, JETP Letters 13 (1971) 323.
[66] D. Gross, Proc. 17th Int. Conf. on High-Energy Physics, London 1974, ed. J.R.Smith (Rutherford Lab., Chilton, Didcot, Oxon, U.K.).
[67] D. Grosser, Nucl. Phys. B92 (1975) 120; B94 (1975) 513.
[68] R. Haag, J.T. Łopuszański and M. Sohnius, Nucl. Phys. B88 (1975) 257.
[69] P.W. Higgs, Phys. Rev. Letters 13 (1964) 508.
[70] J. Honerkamp, F. Krause, M. Scheunert and M. Schlindwein, Phys. Letters 53B (1974) 60.
[71] J. Honerkamp, F. Krause, M. Scheunert and M. Schlindwein, Nucl. Phys. B95 (1975) 397.
[72] M. Huq, Phys. Rev. D14 (1976) 3548.
[73] J. Iliopoulos, Proc. Intern. Storage Ring Meeting Flaine, France (Feb. 1976) p. 35.
[74] J. Iliopoulos and B. Zumino, Nucl. Phys. B76 (1974) 310.
[75] M. Jacob, ed., Dual Theory (North-Holland, 1974).

[76] D.R.T. Jones, Nucl. Phys. B87 (1975) 127.
[77] V.G. Kac, Functional Analysis and its Applications 9 (1975) 91 (Russian).
[78] T.W.B. Kibble, Phys. Rev. 155 (1967) 1554.
[79] W. Lang and J. Wess, Nucl. Phys. B81 (1974) 249.
[80] E.P. Likhtman, JETP Letters 21 (1975) 109.
[81] E.P. Likhtman, JETP Letters 22 (1975) 57.
[82] G. Mack, Nucl. Phys. B5 (1968) 499.
[83] G. Mack and A. Salam, Ann. of Phys. 53 (1969) 174.
[84] G.B. Mainland and K. Tanaka, Phys. Rev. D12 (1975) 2394.
[85] G.B. Mainland and K. Tanaka, Phys. Rev. D13 (1976) 2318.
[86] W. Nahm, V. Rittenberg and M. Scheunert, Phys. Letters 61B (1976) 383; J. Math. Phys. 17 (1976) 868; 17 (1976) 1626; 17 (1976) 1640.
[87] S. Nandi, Nucl. Phys. B97 (1975) 178.
[88] P. Nath, Proc. Conf. on Gauge Theories and Modern Field Theory, Northeastern University, 1975, eds. R. Arnowitt and P. Nath (MIT Press) p. 281.
[89] P. Nath and R. Arnowitt, Phys. Letters 56B (1975) 177.
[90] P. Nath and R. Arnowitt, Proc. French Phys. Soc. Dijon (1975), J. de Phys. 37, C2-85 (1976).
[91] A. Neveu and J.H. Schwarz, Nucl. Phys. B31 (1971) 86.
[92] V.I. Ogievetsky and E. Sokatchev, JETP Letters 23 (1976) 58.
[93] L. O'Raifeartaigh, Phys. Letters 56B (1975) 41.
[94] L. O'Raifeartaigh, Nucl. Phys. B89 (1975) 418.
[95] L. O'Raifeartaigh, Comm. DIAS, Series A, N° 22 (1975).
[96] L. O'Raifeartaigh, Nucl. Phys. B96 (1975) 331.
[97] L. O'Raifeartaigh and G. Parravicini, Nucl. Phys. B111 (1976) 516.
[98] A. Pais and V. Rittenberg, J. Math. Phys. 16 (1975) 2062; 17 (1976) 598.
[99] O. Piguet and A. Rouet, Nucl. Phys. B99 (1975) 458.
[100] O. Piguet and A. Rouet, Nucl. Phys. B108 (1976) 265.
[101] O. Piguet and M. Schweda, Nucl. Phys. B92 (1975) 334.
[102] P. Ramond, Phys. Rev. D3 (1971) 2415.
[103] W. Rühl and B.C. Yunn, Fortschritte der Physik 23 (1975) 431.
[104] W. Rühl and B.C. Yunn, Fortschritte der Physik 23 (1975) 451.
[105] A. Salam, Proc. 8th Nobel Symposium (Wiley, New York, 1968).
[106] A. Salam and J. Strathdee, Nucl. Phys. B76 (1974) 477.
[107] A. Salam and J. Strathdee, Nucl. Phys. B80 (1974) 499.
[108] A. Salam and J. Strathdee, Nucl. Phys. B84 (1975) 127.
[109] A. Salam and J. Strathdee, Nucl. Phys. B86 (1975) 142.
[110] A. Salam and J. Strathdee, Nucl. Phys. B87 (1975) 85.
[111] A. Salam and J. Strathdee, Nucl. Phys. B97 (1975) 293.
[112] A. Salam and J. Strathdee, Phys. Letters 49B (1974) 465.
[113] A. Salam and J. Strathdee, Phys. Letters 51B (1974) 353.
[114] A. Salam and J. Strathdee, Phys. Rev. D11 (1975) 1521.
[115] A. Salam and J. Strathdee, Trieste preprint IC/75/38.
[116] J. Scherk, Rev. Mod. Phys. 47 (1975) 123.
[117] A.A. Slavnov, Teor. Mat. Fiz. (USSR) 23 (1975) 3.
[118] A.A. Slavnov, Nucl. Phys. B97 (1975) 155.
[119] A.A. Slavnov, Dubna preprint E2-9389 (Dec. 1975).
[120] E. Sokatchev, Nucl. Phys. B99 (1975) 96.
[121] M. Suzuki, Nucl. Phys. B83 (1974) 269.
[122] H.S. Tsao, Phys. Letters 53B (1974) 381.
[123] P. Van Nieuwenhuizen, ITP-SB-75-46, Invited talk at the Marcel Grossmann meeting on the recent progress of the fundamentals of general relativity, Trieste Univ. (July 1975).
[124] D.V. Volkov and V.P. Akulov, Phys. Lett. 46B (1973) 109; Teor. Mat. Fiz. (USSR) 18 (1974) 39.
[125] D.V. Volkov and V.A. Soroka, JETP Letters 18 (1973) 312; Teor. Mat. Fiz. (USSR) 20 (1974) 291.
[126] S. Weinberg, Phys. Rev. Letters 19 (1967) 1264.
[127] S. Weinberg, Phys. Rev. Letters 29 (1972) 1698.
[128] S. Weinberg, Phys. Letters 62B (1976) 111.
[129] J. Wess, Lecture Notes in Physics 37 (Springer Verlag, Berlin, 1974) p. 352.
[130] J. Wess and B. Zumino, Nucl. Phys. B70 (1974) 39.

[131] J. Wess and B. Zumino, Phys. Letters 49B (1974) 52.
[132] J. Wess and B. Zumino, Nucl. Phys. B78 (1974) 1.
[133] P. West, Nucl. Phys. B91 (1975) 289.
[134] P. West, Nucl. Phys. B106 (1976) 219.
[135] G. Woo, Phys. Rev. D12 (1975) 975.
[136] G. Woo, Lett. Nuov. Cim. 13 (1975) 546.
[137] B. Zumino, in: Renormalization and Invariance in QFT, ed. E.R. Caianiello (Plenum Press, 1973) p. 367.
[138] B. Zumino, Proc. 17th Int. Conf. on High-Energy Physics, London 1974, ed. J.R. Smith (Rutherford Lab, Chilton, Didcot, Oxon, U.K.).
[139] B. Zumino, Nucl. Phys. B89 (1975) 535.
[140] B. Zumino, Proc. Conf. on Gauge Theories and Modern Field Theory, Northeastern University, 1975, eds. R. Arnowitt and P. Nath (MIT Press) p. 255.

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I. Global supersymmetry: models for unified interactions

Low-energy theorems have been considered when there exist other boson–fermion mass degeneracies than the photon–neutrino one, and explicit examples have been carried out in detail [150]. With notations of subsection 5.2, relations are obtained between the decay amplitudes for $z \to \bar{\nu} e_- w_+$ and $E_0^* \to e_- w_+$ (the neutral particles z and E_0^* being degenerate in mass); $z \to \bar{\nu} e_- w_+ \gamma$ and $z \to \bar{\nu} e_- w_+, E_0^* \to e_- w_+ \gamma$ (radiative process with mass degeneracy). The decay amplitude for $E_+ \to \nu e_+ e_- w_+$ vanishes with the four-momentum of the neutrino, since there is no contribution from a mass-degenerate boson–fermion pair.

Progresses have been made recently towards realistic applications of supersymmetry to particle physics [147]. We have seen already that the electron and muon neutrinos are unlikely to be Goldstone particles; one way out is to have the Goldstone spinor completely decoupled, as discussed in subsection 5.3.

A very important problem is that supersymmetric theories involve a lot of scalar particles, whereas ordinary charged-current weak interactions appear as mediated by vector particles only. A solution has been obtained, which makes use of model [35] presented in subsection 5.2. The fermions of this model are no more interpreted as leptons of the electron sector, but rather as belonging to a new class of leptons; they are now denoted by ℓ_-, L_-, ℓ_0, L_0, ν_γ. These new leptons have their own quantum number (R-number) defined by means of R-invariance, and carried by the spinorial generator in the supersymmetry algebra. The massless two-component fermion ν_γ, associated with the photon under supersymmetry, will be called "photonic neutrino". It may be that the anomalous e–μ events [152] are due to the production and decay of a pair of leptons such as ℓ_-, according to $e_+ e_- \to \ell_+ \ell_- \to e_\pm \mu_\mp$ + unobserved neutrinos (see fig. 6).

New chiral superfields are introduced; they describe known leptons (ν_e, e, ν_μ, μ) and quarks ($p, n, p', \lambda, \ldots$) together with scalar particles, which are heavy owing to supersymmetry breaking [147].

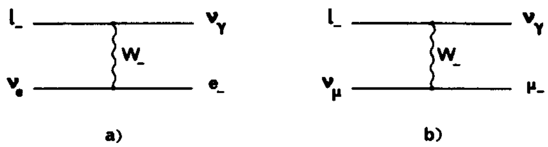

Fig. 6. Some diagrams contributing to the decay of the new lepton ℓ_-.

The mechanism used in that paper (limit of a spontaneously broken supersymmetric theory) did not allow for the super-Yukawa coupling inducing zeroth-order mass terms for electron, muon and quarks. This can be cured easily if one uses a different symmetry breaking mechanism: arbitrary mass terms for scalar particles are added, which break supersymmetry explicitly, but may also be viewed as the result of a spontaneous breaking in a non-physical sector [119, 153]; a similar method [154] has been applied to the model of ref. [147]; compatible with the above super-Yukawa coupling, it allows for massive electron, muon and quarks.

Electron, muon and baryon numbers are conserved; they are carried both by usual leptons and quarks, and by the heavy scalar particles associated with them. Scalar bosons can be exchanged between fermions when leptons of the new class are involved; but μ-decay and β-decays are due to a W_- exchange (fig. 7), scalar exchanges being negligible. The usual phenomenology of weak and electromagnetic interactions, including neutral current effects, is recovered.

On the other hand processes such as $\nu_\gamma e_- \to \nu_\gamma e_-$ or $\nu_\gamma p \to \nu_\gamma p$ are due to exchanges of scalar bosons carrying both R-number and lepton-or-baryon-number (see fig. 8).

If the energy is not large enough to create heavy leptons, neutrinos of the new type can only be produced in pairs, owing to R-number conservation ($e_+e_- \to \nu_\gamma\bar{\nu}_\gamma$, etc.). How can we detect these neutrinos, once produced? At low energy they can only scatter with matter, as shown in fig. 8; this is an effective neutral current effect, hard to observe.

Strong interactions are due to gluon vector superfields (for example a SU(3) colour octet) interacting with quark superfields. Besides ordinary spin 1-gluons and spin $\frac{1}{2}$-quarks, they involve (massless) spin $\frac{1}{2}$-gluons and (heavy) spin 0-quarks. Spin $\frac{1}{2}$ gluons couple spin $\frac{1}{2}$-quarks to heavy spin-0 quarks.

In conclusion it seems likely that supersymmetry is a relevant framework for a gauge theory of weak, electromagnetic and strong interactions. However, the supersymmetry breaking mechanism which has been used is not fully satisfactory. A true spontaneous breaking, with the appearance of a physical Goldstone neutrino, should be preferable. Such a program can be realized [148]; then, to the crucial question "why Goldstone neutrinos have not been observed?", we can give the same answer as given above for the photonic neutrino ν_γ: it belongs to a new class of leptons, with its own conserved quantum number, associated with R-invariance.

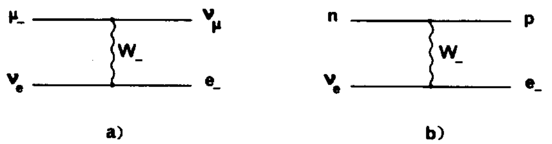

Fig. 7. Diagrams contributing: a) to μ-decay, b) to β-decays.

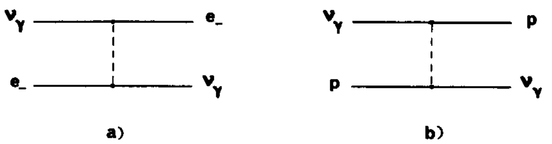

Fig. 8. Some diagrams contributing to $\nu_\gamma e_-$ and $\nu_\gamma p$ scatterings. A spin-0 quark (carrying both R-number and baryon-number) is exchanged in b.

II. Local supersymmetry: supergravity theories and their interaction with matter

In these last months great progress has been achieved in the development of local supersymmetry (supergravity).

The pure gauge Lagrangian for supergravity [54, 27] has been coupled, using different techniques, to the previously known multiplets of global supersymmetry. The local coupling to the spin $(1,\frac{1}{2})$ vector multiplet has been first derived [169, 168], both for the abelian (Maxwell) and non-abelian (Yang–Mills) case, using the second-order formulation of general relativity. The same coupling has been derived in the first-order formulation (with torsion) by means of a self-consistency approach [176]. The local supersymmetric Lagrangian for the massless spin $(\frac{1}{2},0^{\pm})$ scalar multiplet has been constructed in refs. [168, 167]. An explicit mass term for the scalar multiplet has been shown to give problems [167] in the sense that a nonpolynomial Lagrangian seems to be required by local invariance.

The algebraic structure of these theories has been explored by studying the commutator algebra [176, 167]. These commutators give (modulo equations of motion) general coordinate transformations, field-dependent Lorentz and supersymmetry transformations, i.e. they reproduce known symmetries of these systems. The unsatisfactory point is that the algebra is closed only after the field equations have been used. This is certainly a consequence of the fact that auxiliary fields, which are needed in order to have a linear representation of the algebra, have not yet been introduced. Work along the direction of a complete understanding of the structure of these auxiliary fields in the context of local supersymmetry is in progress [157].

The superspace approach should be a convenient tool for the understanding of the auxiliary field structure, because they appear there in a natural way [177, 190]. As far as the superspace approach is concerned, it has been shown that the pure supergravity Lagrangian can be reobtained from a Riemannian geometry in a singular limit [183, 184]. The supergravity equations of motion can be obtained more directly if a non-Riemannian geometry in superspace is introduced [190].

A model with a local chiral symmetry has been constructed [173]. This model is obtained by adding the Fayet–Iliopoulos term [41] to the supergravity Lagrangian coupled to the abelian vector multiplet [169]. Parity conservation and local supersymmetry require the spin 1 field to be an axial vector, which turns out to be the gauge field of the U(1) chiral group. Local invariance requires also a cosmological term with cosmological constant $\lambda \sim e^2\kappa^{-2}$, e being the chiral gauge coupling constant.

In the models discussed so far, in which supergravity is coupled to matter multiplets, all local internal symmetries are introduced in an almost trivial way, because they commute with the spinorial generators of the supersymmetry algebra (with the exception of the U(1) chiral group previously discussed). The spin 1 fields of the (matter) vector multiplets are precisely the gauge fields of these internal symmetries.

A promising result along the direction of building unified theories has been certainly the recent development of extended supergravity theories. The spin $\frac{3}{2}$ gauge field now carries an internal symmetry index. These theories arise from the gauging of an extended supersymmetry algebra [111, 38] in which internal and space–time symmetries are non-trivially mixed. These algebras were considered in detail, at the global level, in sections 2 and 4.7.

Extended supergravity theories can be regarded as ordinary couplings of (real) supergravity with several matter multiplets. The main ingredient is the presence of the massless spin $(\frac{3}{2},1)$ spinor multiplet [185] which is needed to increase the number of spin $\frac{3}{2}$ gauge fields. The spinor multiplet, together with lower spin multiplets, joins the gauge spin $(2,\frac{3}{2})$ multiplet of ordinary supergravity to build the larger gauge multiplet of extended supersymmetry (this is exactly analogous to the mechanism described in subsection 4.7, where the spin $(\frac{1}{2},0^{\pm})$ multiplet joins the gauge spin $(1,\frac{1}{2})$ multiplet of the ordinary supersymmetric (Yang–Mills) gauge theory, to build the gauge hypermultiplet). The precise number of ordinary multiplets contained in an extended multiplet depends on the number of spinorial charges and it is a group property.

Supergravity theories with extended local supersymmetry can exist with an O(N) (real) internal symmetry up to $N = 8$ [178]. The existence of $N_{\mathrm{MAX}} = 8$ is related to the fact that λ_{MAX} (maximum helicity) = 2; otherwise no (interacting) gauge theory could be constructed [181].

Extended supergravity theories have been explicitly constructed for $N = 2$ [172] and $N = 3$ [170, 174]. Partial results for $N = 4$ also exist [163]. The internal symmetry groups are not gauged if κ (the gravitational coupling) is the only non-vanishing coupling constant, and therefore they appear as global symmetries. The spin $\frac{3}{2}$ and spin 1 fields interact via a magnetic-moment type coupling with strength κ.

Surprisingly enough the gauging of the symmetry group under which the spin $\frac{3}{2}$ fields transform automatically introduces an apparent mass term for the Rarita–Schwinger field $m \sim e\kappa^{-1}$ together with a cosmological constant $\lambda \sim e^2\kappa^{-2} \sim m^2$, e being the gauge coupling constant [175]. The mass term for the spin $\frac{3}{2}$ field as well as the accompanying cosmological term can be introduced also in pure supergravity without extended supersymmetry [186].

However, there is an exception to the previous mechanism, i.e. the case in which O(2) is the internal symmetry group [171]. Then by introduction of a massive scalar multiplet (hypermultiplet) [38] with mass m, the spin 1 field of the gravitational multiplet can be coupled minimally to the scalar multiplet with gauge coupling $e = m\kappa$. No mass for the spin $\frac{3}{2}$ field, nor a cosmological term arise. The reason is that the gauged O(2) group is not the group under which the Rarita–Schwinger transforms but rather the central charge of the extended supersymmetry algebra [68]. It may be that this exceptional situation which occurs just for the abelian extended supersymmetry is related to the same exceptional situation which arises in the superspace approach of local supersymmetry [184, 155].

The main result in the theory of local supersymmetry is possibly the renormalizability property of quantum supergravity [162, 187, 182]. It has been shown that ordinary supergravity and extended supergravity theories, in the absence of matter multiplets, are one-loop renormalizable [162, 189] and arguments have been given in favour of full renormalizability at higher loops [180, 164]. These results provide, in particular, the first example of finite quantum corrections to the S-matrix of particles interacting through gravitation. The pure O(2)-supergravity theory [172], which contains the Maxwell–Einstein system as a subtheory, constitutes the first example of a finite S-matrix for photon–photon scattering due to gravitation.

Ordinary supergravity in interaction with (supersymmetric) matter is one-loop non-renormalizable [188, 189]. This result would pose serious problems for building unified theories, although there are not convincing arguments that the same negative result holds in any extended supergravity theory. The difficulties in constructing realistic models unifying gravitation with other particle interactions lie in the fact that the gauge-group of ordinary interactions is bigger than O(8) and therefore cannot be gauged by the vector bosons belonging to the same irreducible multiplet as the graviton.

Locally supersymmetric theories have also been considered in one and two space–time dimensions. The former case corresponds to the description of spinning particles [156, 166, 159, 161], the latter to strings with fermionic degrees of freedom [91, 102, 2]. These Lagrangians are not only invariant under general coordinate transformations but also under local supersymmetry transformations [165, 158, 160]. Finally the connection among the spinor dual model at the critical dimension and the several existing supergravity theories has been clarified [179].

References

[141] L. Brink, J. Scherk and J.H. Schwarz, preprint CALT 68-574 (Dec. 1976).
[142] T.E. Clark, O. Piguet and K. Sibold, Nucl. Phys. B119 (1977) 292.
[143] T.E. Clark, O. Piguet and K. Sibold, preprint, Renormalization Theory in Superspace (1977).
[144] M. Daniel, Phys. Letters 65B (1976) 246;
M. Daniel and C.N. Ktorides, Nucl. Phys. B115 (1976) 313.
[145] P.H. Dondi, J. Phys. A8 (1975) 1298.
[146] P.H. Dondi and H. Nicolai, preprint, Lattice Supersymmetry (1976).
[147] P. Fayet, Phys. Letters 64B (1976) 159; Proc. 18th Int. Conf. on High-Energy Physics, Tbilisi 1976, ed. Dubna (USSR) vol. 2, T 8.
[148] P. Fayet, to be submitted for publication.
[149] W. Lang, Nucl. Phys. B114 (1976) 123.
[150] G.B. Mainland, E. Tagasuki and K. Tanaka, Phys. Rev. D14 (1976) 1065.
[151] L. Mezincescu and V.I. Ogievetsky, Dubna preprint E2-8277 (Sept. 1974); Usp. Fiz. Nauk 117 (1975) 637.
[152] M.L. Perl et al., Phys. Rev. Letters 35 (1975) 1489.
[153] A.A. Slavnov, Proc. 18th Int. Conf. on High-Energy Physics, Tbilisi 1976, ed. Dubna (USSR) vol. 2, T 12; preprint MPI-PAE/P Th 43/76 (Dec. 1976).
[154] M.F. Sohnius, preprint MPI-PAE/P Th 3/77 (Jan. 1977).
[155] R. Arnowitt and P. Nath, Phys. Rev. D15 (1977) 1033;
M.H. Friedman and Y. Srivastava, Phys. Rev. D15 (1977) 1026.
[156] F.A. Berezin and M.S. Marinov, preprint ITEP 43 (1976).
[157] P. Breitenlohner, preprints MPI-PAE 29/76, 38/76;
C. Teitelboim, Princeton preprint 1977.
[158] L. Brink, P. Di Vecchia and P. Howe, Phys. Letters 65B (1976) 471.
[159] L. Brink, P. Di Vecchia and P. Howe, Nordita preprint NBI-HE-76-8.
[160] L. Brink and J.H. Schwarz, preprint CALT 68-568 (1976).
[161] R. Casalbuoni, Phys. Letters 62B (1976) 49; Nuovo Cimento 33A (1976) 115, 33A (1976) 389.
[162] A. Das and D.Z. Freedman, Nucl. Phys. B114 (1976) 271.
[163] A. Das, Stony Brook preprint ITP-SB-77-4 (1977).
[164] S. Deser, J.H. Kay and K.S. Stelle, Phys. Rev. Letters 38 (1977) 527.
[165] S. Deser and B. Zumino, Phys. Letters 65B (1976) 369.
[166] S. Deser, B. Zumino, L. Brink, P. Di Vecchia and P. Howe, Phys. Letters 64B (1976) 435.
[167] S. Ferrara, D.Z. Freedman, P. van Nieuwenhuizen, P. Breitenlohner, F. Gliozzi and J. Scherk, Phys. Rev. D15 (1977) 1013.
[168] S. Ferrara, F. Gliozzi, J. Scherk and P. van Nieuwenhuizen, Nucl. Phys. B117 (1976) 333.
[169] S. Ferrara, J. Scherk and P. van Nieuwenhuizen, Phys. Rev. Letters 37 (1976) 1035.
[170] S. Ferrara, J. Scherk and B. Zumino, Phys. Letters B66 (1977) 35.
[171] S. Ferrara, J. Scherk and B. Zumino, preprint PTENS 76/24, Nucl. Phys. B, to be published.
[172] S. Ferrara and P. van Nieuwenhuizen, Phys. Rev. Letters 37 (1976) 1669.
[173] D.Z. Freedman, Phys. Rev. D15 (1977) 1173.
[174] D.Z. Freedman, Phys. Rev. Letters 38 (1976) 105.
[175] D.Z. Freedman and A. Das, Stony Brook preprint ITP-SB-76-64 (1976).
[176] D.Z. Freedman and J.H. Schwarz, Phys. Rev. D15 (1977) 1007.
[177] M. Gell-Mann and Y. Ne'eman, unpublished.
[178] M. Gell-Mann and Y. Ne'eman, unpublished.
[179] F. Gliozzi, J. Scherk and D. Olive, Phys. Letters 65B (1976) 282.
[180] M.T. Grisaru, Phys. Letters 66B (1977) 75.
[181] M.T. Grisaru, H.N. Pendleton and P. van Nieuwenhuizen, Phys. Rev. D15 (1977) 996.
[182] M.T. Grisaru, P. van Nieuwenhuizen and J.A.M. Vermaseren, Phys. Rev. Letters 37 (1976) 1662.
[183] P. Nath and R. Arnowitt, Phys. Letters 65B (1976) 73.
[184] Y. Ne'eman, Tel-Aviv preprint (1976).
[185] V.I. Ogievtsky and E. Sokatchev, Dubna preprints E 2-9984, E 2-9985 (1976).
[186] P.K. Townsend, Stony Brook preprint ITP-SB-77-2 (1977).
[187] P.K. Townsend and P. van Nieuwenhuizen, Stony Brook preprint ITP-SB-76-60 (1976).
[188] P. van Nieuwenhuizen and J.A.M. Vermaseren, Phys. Letters 65B (1976) 263.
[189] P. van Nieuwenhuizen and J.A.M. Vermaseren, Stony Brook preprint ITP-SB-77-8 (1977).
[190] J. Wess and B. Zumino, Phys. Letters 66B (1977) 361.