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SUPERSYMMETRIC YANG-MILLS THEORIES *

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Yang-Mills theories with simple supersymmetry are constructed in 2, 4, 6, and 10 dimensions, and it is argued that these are essentially the only cases possible. The method of dimensional reduction is then applied to obtain various Yang-Mills theories with extended supersymmetry in two and four dimensions. It is found that all possible four-dimensional Yang-Mills theories with extended supersymmetry are obtained in this way.

1. Introduction

Three types of supersymmetric Yang-Mills theories in four dimensions are known. In the first one that was found [1] the infinitesimal parameter of the supersymmetry transformation is a Majorana spinor ("simple" supersymmetry). In the second one [2] it is a Dirac spinor ("complex" supersymmetry). In the third case it consists of four Majorana (or Weyl) spinors [3]. This last model was obtained recently by applying the method of dimensional reduction to a supersymmetric Yang-Mills theory in ten-dimensional space-time.

The goal of this paper is to classify all the possible supersymmetric Yang-Mills theories in both two and four dimensions. The interest in four dimensions is obvious, of course, as one of these schemes may be part of a correct theory. The two-dimensional cases are also emphasized because of the possibility of coupling such Yang-Mills multiplets to a corresponding two-dimensional supergravity theory [4] in order to get a modified string theory. Our technique consists of two stages. In the first stage Yang-Mills theories with simple supersymmetry are constructed for all space-time dimensions in which it is possible. Then in the second stage each of the higher-

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dimension theories is squeezed into two or four dimensions by the "method of dimensional reduction", which is briefly reviewed.

In four dimensions it turns out that the only possible schemes are the three listed in the first paragraph. However, it is emphasized that each case can be formulated in two alternative (but equivalent) ways. In the first one the Fermi fields are Majorana spinors and the internal symmetry is either O(1), O(2) or O(4). In the second one the Fermi fields are Weyl spinors (i.e., they have definite handedness like neutrinos) and the internal symmetry is either U(1), U(2) or SU(4). In two dimensions there are four possible types of supersymmetric Yang-Mills theories. They are characterized by Fermi transformation parameters that are (i) a Majorana spinor, (ii) a Dirac spinor, (iii) an SU(2) doublet of Dirac spinors, and (iv) an SU(4) quartet of Dirac spinors, respectively.

In sect. 2 the possible space-time dimensions for Yang-Mills theories with simple supersymmetry are investigated. The Lagrangian for each allowed case is given explicitly. In sect. 3 the four-dimensional theory is squeezed into a two-dimensional theory. Sects. 4 and 5 involve similar manipulations to get from 6 to 4 or 2 and from 10 to 4 or 2 dimensions respectively. Sect. 6 gives some general discussion of the results.

2. Construction of supersymmetric Yang-Mills theories in 2, 4, 6 and 10 space-time dimensions

We start by considering a Yang-Mills theory with spin-$\frac{1}{2}$ particles in the regular representation of the gauge group, which can be chosen, for example, to be any semisimple Lie group. The action is

$$S = \int \mathrm{d}^D x \; \{-\tfrac{1}{4}F_{\mu\nu}^{\;a} F^{\mu\nu a} + i\bar{\lambda}^a \gamma \cdot D\lambda^a\} \,. \tag{2.1}$$

We use the metric $g_{\mu\nu} = \mathrm{diag}(+1, -1, ..., -1)$ and

$$F_{\mu\nu}^{\;a} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f_{abc} A_\mu^b A_\nu^c \,, \tag{2.2}$$

$$D_\mu \lambda^a = \partial_\mu \lambda^a - g f_{abc} A_\mu^c \lambda^b \,. \tag{2.3}$$

We first examine for which dimensions the action (2.1) can be supersymmetric without the addition of other fields. Theories which transform as linear representations of supersymmetry must have the same number of bosonic and fermionic degrees of freedom. For D dimensions the massless vector particle has $D-2$ degrees of freedom while a Dirac spinor has $2^{D/2}$. These numbers do not match for any dimension. We can, however, reduce the number of Fermi degrees of freedom by requiring the spinors to satisfy supplementary conditions such as the Majorana condition

$$\lambda = C\bar{\lambda}^T \,, \tag{2.4}$$

where C is the charge conjugation operator, or the Weyl "left- or right-handedness" condition

$$\lambda = \pm\gamma_5\lambda\ . \tag{2.5}$$

It has been proven [3] that the Majorana condition is only possible when the dimension of space-time is 2 or 4 modulo 8, while the Weyl condition can be imposed for any even number of dimensions. The two conditions can be simultaneously imposed if the number of dimensions is 2 modulo 8. Each condition reduces the number of degrees of freedom for the Fermi field by a factor 2. We then find that for $D = 4$ with either the Majorana or the Weyl condition and for $D = 6$ with the Weyl condition, we have the same number of Fermi and Bose degrees of freedom. For $D = 10$ we have also the same number provided we impose both the Weyl and the Majorana conditions for the spinors. Hence for $D = 4$, 6 and 10 the action (2.1) might be supersymmetric without the addition of further fields. In order to check the invariance, consider the transformations

$$\begin{aligned}
\delta A_\mu^a &= i\bar{\alpha}\gamma_\mu\lambda^a - i\bar{\lambda}^a\gamma_\mu\alpha\ ,\\
\delta\lambda^a &= \sigma_{\mu\nu}F^{\mu\nu a}\alpha\ ,\\
\delta\bar{\lambda}^a &= -\bar{\alpha}\sigma_{\mu\nu}F^{\mu\nu a}\ ,
\end{aligned} \tag{2.6}$$

By a straightforward calculation we get (up to total derivatives) after cancelling $F\partial\lambda$ terms,

$$\delta L = -igf_{abc}\bar{\lambda}^a\gamma_\mu\lambda^b\delta A^{\mu c} = gf_{abc}[\bar{\lambda}^a\gamma_\mu\lambda^b\bar{\alpha}\gamma^\mu\lambda^c - \bar{\lambda}^a\gamma_\mu\lambda^b\bar{\lambda}^c\gamma^\mu\alpha]\ . \tag{2.7}$$

We note that in order to have invariance both terms must equal zero. (They are equal in the Majorana case.) To check when this is true we Fierz rearrange the terms. We introduce a complete set of matrices $1, \gamma^\mu, \sigma^{\mu\nu}, \sigma^{\mu\nu\rho}, ..., \sigma^{\mu_1\mu_2\cdots\mu_D} = \epsilon^{\mu_1\mu_2\cdots\mu_D}\gamma_{D+1}$, where $\sigma^{\mu_1\cdots\mu_n}$ is the completely antisymmetrized product of $\gamma^{\mu_1}\ ...\ \gamma^{\mu_n}$, normalized so that $\sigma^{\mu_1\cdots\mu_n}\sigma_{\mu_1\cdots\mu_n} = \binom{D}{n}$. We define

$$T_n = f_{abc}\bar{\lambda}^a\sigma^{\mu_1\cdots\mu_n}\lambda^b\bar{\alpha}\sigma_{\mu_1\cdots\mu_n}\lambda^c\ ,$$

and apply a Fierz transformation to get the following relations

$$T_n = -\sum T_m F_{mn}\ , \tag{2.8}$$

where we used the antisymmetry of f_{abc}. The Fierz matrix F_{mn} is given by [5]

$$F_{mn} = (-1)^{mn+1}2^{-D/2}\oint\frac{\mathrm{d}z}{2\pi i}\ z^{-n-1}(1+z)^{D-m}(1-z)^m\ . \tag{2.9}$$

The action (2.1) will be supersymmetric if

$$T_1 = -\sum T_m F_{m1} \tag{2.10}$$

can be shown to equal zero. Some of the T_m's are zero if we choose the Majorana or the Weyl conditions on the spinor λ. We further note from (2.9) that

$$F_{D/2\ 1} = 0 \,. \tag{2.11}$$

In table 1 we list the various terms in (2.10) and the conditions under which they vanish for $D = 10$. The other dimensions can be obtained by cutting the table after T_D.

For $D = 4$ and the Majorana condition we find directly that $T_1 = -\frac{1}{2}T_1 = 0$ and the action (2.1) is supersymmetric, as has been known for some time [1]. In fact for this dimension the Majorana and the Weyl conditions are equivalent so we could use Weyl spinors instead. For $D = 6$ and the Weyl condition eq. (2.10) reads

$$T_1 = -\tfrac{1}{2}(T_1 - T_5) \,, \tag{2.12}$$

but $T_5 = -T_1$ because of the Weyl condition so $T_1 = 0$ and the action (2.1) is supersymmetric. For $D = 10$ and both the Majorana and the Weyl conditions eq. (2.10) reads

$$T_1 = -\tfrac{1}{4}(T_1 - T_9) \,. \tag{2.13}$$

The Weyl condition this time leads to $T_1 = -T_9$ so $T_1 = 0$ and the action (2.1) is again supersymmetric.

There are still additional possibilities for finding supersymmetric Yang-Mills theories by including more fields in the action (2.1). For example, when $D = 2$ the vector

Table 1

	Vanish due to Majorana condition	Vanish due to Weyl condition
T_0	×	×
T_1		
T_2		×
T_3	×	
T_4	×	×
T_5		
T_6		×
T_7	×	
T_8	×	×
T_9		
T_{10}		×

field carries no physical degrees of freedom while a Majorana spinor carries one. Hence the theory might be supersymmetric with the addition of one scalar field. The remaining term in δL (2.7) can, according to our table, be Fierz rearranged to a term of the type $gf_{abc}\bar{\lambda}^a\gamma_5\lambda^b\bar{\alpha}\gamma_5\lambda^c$. Hence we should try a pseudoscalar field and indeed the correct action, which has been known for some time [6], is

$$S = \int d^2 x \ \{-\tfrac{1}{4}F^{\mu\nu a}F_{\mu\nu a} + \tfrac{1}{2}i\bar{\lambda}^a\gamma\cdot D\lambda^a + \tfrac{1}{2}D_\mu P^a D^\mu P^a + \tfrac{1}{2}gP^a\bar{\lambda}^b\gamma_5\lambda^c f_{abc}\}\ , \tag{2.14}$$

with the supersymmetry transformation formulas

$$\begin{aligned} \delta A_\mu^a &= i\bar{\alpha}\gamma_\mu\lambda^a\ , \\ \delta\lambda^a &= \sigma_{\mu\nu}F^{\mu\nu a}\alpha - i\gamma_5\gamma\cdot DP^a\alpha\ , \\ \delta P^a &= \bar{\alpha}\gamma_5\lambda^a\ . \end{aligned} \tag{2.15}$$

For $D = 8$ we could ask if a corresponding addition of fields is possible. For this dimension there are 6 bosonic and 8 fermionic degrees of freedom when the Weyl condition is used. Hence we need two more bosonic ones. It is not easy to guess how this can be arranged to work. However, we have a foolproof deductive procedure that gives the correct answer. Namely, by applying the method of dimensional reduction to the 10-dimensional supersymmetric Yang-Mills model we obtain a model in 8 dimensions. The result of this analysis is summarized in sect. 5 for completeness.

We could further ask whether there exist supersymmetric Yang-Mills theories for odd numbers of dimensions. For example, consider $D = 3$. In this case it is appropriate to use the ordinary Dirac matrices of $D = 4$, but now only γ^0, γ^1 and γ^2 are related to space-time. To still be able to use the whole Dirac algebra one could introduce spinors for which $\gamma^3\lambda = \lambda$. Alternatively, we could choose the condition to be $\gamma^5\lambda = \lambda$. Then, however, the three-dimensional theory looks the same as the one we would get from the four-dimensional one expressed in terms of Weyl spinors when we perform a reduction to three dimensions. In this sense we do not get a significantly different theory. The same is true for $D = 5$ and 7.

For $D > 10$ the number of Fermi degrees of freedom is so large that a huge number of additional Bose fields is required for the theories to have a chance of being supersymmetric. We believe that there are actually no supersymmetric Yang-Mills theories for $D > 10$.

3. Dimensional reduction from 4 to 2 space-time dimensions

We will now show in detail the reduction of the simple supersymmetric Yang-Mills theory in four dimensions to two dimensions. The idea [7] is to interpret two of the space dimensions, assumed to be compact, as internal degrees of freedom. The simplest way to do this, which involves no space-time curvature, is to introduce two radii R^i such that $0 \leqslant x_2 \leqslant R_2$ and $0 \leqslant x_3 \leqslant R_3$, and demand the fields to be periodic in x_2 and x_3 between 0 and R_2, R_3. Then Fourier resolving the fields with respect to x_2 and x_3 gives an infinite number of two-dimensional fields. By letting R_2 and R_3 approach zero and rescaling the fields we keep only the zeroth coefficient in the Fourier expansion and are left with a finite number of fields. We note that this technique does not destroy the supersymmetry of the theory. The transformations need only to be reexpressed in terms of the new fields. More sophisticated techniques [7] of dimensional reduction are also available in curved space-time and may be useful in supergravity models with more than four dimensions.

Consider the four-dimensional theory

$$S = \int \mathrm{d}^4 x \left[-\tfrac{1}{4} F_{\mu\nu}^{\ a} F^{\mu\nu a} + \tfrac{1}{2} i \bar{\lambda}^a \Gamma \cdot D \lambda^a\right] , \tag{3.1}$$

with the supersymmetry transformations

$$\delta A_\mu^a = i\bar{\alpha}\Gamma_\mu \lambda^a , \qquad \delta\lambda^a = \Sigma_{\mu\nu} F^{\mu\nu a} , \tag{3.2}$$

where λ is a Majorana spinor. In carrying out dimensional reductions we always use capital Γ's for the Dirac matrices in the higher dimension and lower case γ's for the lower dimension. A representation for Γ^μ which directly exposes the two-dimensional Dirac matrices is

$$\begin{aligned} &\Gamma^\mu = \gamma^\mu \otimes I_2 , \qquad \mu = 0, 1 , \\ &\Gamma^{2,3} = \gamma_3 \otimes i\sigma_{1,2} , \end{aligned} \tag{3.3}$$

where I_N represents the $N \times N$ unit matrix, and we use the representation

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma_3 = \gamma^0\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{3.4}$$

We write the spinor λ as two two-component spinors

$$\lambda = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$$

and impose the Majorana constraint (2.4) on it with

$$C_4 = -i\Gamma^2\Gamma^0 = -\gamma_1 \otimes \sigma_1 = -C_2 \otimes \sigma_1 . \tag{3.5}$$

This gives the structure

$$\lambda = \begin{pmatrix} \chi \\ -\tilde{\chi} \end{pmatrix} \qquad \text{where } \tilde{\chi} = C_2 \bar{\chi}^T . \tag{3.6}$$

Note that χ is not a Majorana spinor, but a full two-dimensional Dirac spinor instead. We further decompose the vector fields as

$$A^a_\mu = A_\mu{}^a \qquad \text{for } \mu = 0, 1 \,, \tag{3.7a}$$

$$\Phi^a = A^a_3 + iA^a_2 \,. \tag{3.7b}$$

The expressions (3.3)–(3.7) can now be inserted into (3.1) and (3.2) and the dimensional reduction performed to obtain the two-dimensional theory

$$\begin{aligned} S = \int \mathrm{d}^2 x [& -\tfrac{1}{4} F_{\mu\nu}{}^a F^{\mu\nu a} + \tfrac{1}{2} D_\mu \Phi^a D^\mu \Phi^{a*}] \\ & + i\bar{\chi}^a \gamma \cdot D\chi^a + \tfrac{1}{2} i g f_{abc} (\bar{\chi}^a \gamma_5 \tilde{\chi}^b \Phi^c - \bar{\tilde{\chi}}^a \gamma_5 \chi^b \Phi^{c*}) + \tfrac{1}{8} g^2 (f_{abc} \Phi^b \Phi^{*c})^2 \,, \end{aligned} \tag{3.8}$$

with the supersymmetry transformations

$$\begin{aligned} \delta A^a_\mu &= i[\bar{\alpha}\gamma_\mu \chi^a - \bar{\chi}^a \gamma_\mu \alpha] \,, \\ \delta \Phi^a &= -2i\bar{\tilde{\alpha}}\gamma_5 \chi^a \,, \\ \delta \Phi^{*a} &= -2i\bar{\alpha}\gamma_5 \tilde{\chi}^a \,, \\ \delta \chi^a &= (\sigma_{\mu\nu} F^{\mu\nu a} - \tfrac{1}{2} g f_{abc} \Phi^b \Phi^{*c})\alpha - \gamma \cdot D\Phi^a \gamma_5 \tilde{\alpha} \,, \\ \delta \tilde{\chi}^a &= (\sigma_{\mu\nu} F^{\mu\nu a} + \tfrac{1}{2} g f_{abc} \Phi^b \Phi^{*c})\tilde{\alpha} - \gamma \cdot D\Phi^{*a} \gamma_5 \alpha \,. \end{aligned} \tag{3.9}$$

In this way we have derived a two-dimensional Yang-Mills theory with a complex supersymmetry. Note that the complex scalar field Φ has fermion number equal to two. The supersymmetric "O(1)" theory [6] is a special case of this theory when χ is restricted to be a Majorana spinor and Φ to be real.

4. Dimensional reduction of the 6-dimensional theory

The six-dimensional theory can be reduced to either 4 or 2 dimensions. We do each case separately since for each one we want to choose a representation of the Dirac algebra which exposes the Dirac matrix of the lower dimension. As in the preceding section we use Γ^μ for the dimension we start with and γ^μ for the dimension we reduce to. Consider the reduction from 6 to 4. The six-dimensional action is

$$\int \mathrm{d}^6 \chi \{ -\tfrac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + i\bar{\lambda}^a \Gamma \cdot D\lambda^a \} \,, \tag{4.1}$$

with λ satisfying the Weyl condition (2.5). The supersymmetry transformations are

$$\delta A_\mu^a = i(\bar{\alpha}\Gamma_\mu\lambda^a - \bar{\lambda}^a\Gamma_\mu\alpha)\,, \qquad \delta\lambda^a = \Sigma_{\mu\nu}F^{\mu\nu a}\alpha\,, \qquad \delta\bar{\lambda}^a = -\bar{\alpha}\Sigma_{\mu\nu}F^{\mu\nu a}\,. \tag{4.2}$$

We choose the representation

$$\begin{aligned}\Gamma^\mu &= \gamma^\mu \otimes I_2\,, \qquad \mu = 0, 1, 2, 3\,,\\ \Gamma^{4,5} &= \gamma_5 \otimes i\sigma_{1,2}\,,\\ \Gamma^7 &= \gamma_5 \otimes \sigma_3\,.\end{aligned} \tag{4.3}$$

The Weyl condition forces λ to have the structure

$$\lambda = \begin{pmatrix} L\chi \\ R\chi \end{pmatrix},$$

$$L = \tfrac{1}{2}(1+\gamma_5), \qquad R = \tfrac{1}{2}(1-\gamma_5)\,, \tag{4.4}$$

where χ is a four-dimensional Dirac spinor.

We further define

$$\begin{aligned}A_\mu{}^a &= A_\mu^a, \qquad \mu = 0, 1, 2, 3\,,\\ A_4^a &= S^a\,,\\ A_5^a &= P^a\,.\end{aligned} \tag{4.5}$$

Inserting (4.3)–(4.5) into (4.1) and (4.2) and squeezing to four dimensions we reach the theory

$$\begin{aligned}S = \int d^4x\,\{&-\tfrac{1}{4}F_{\mu\nu}^{\ a}F^{\mu\nu a} + \tfrac{1}{2}D_\mu S^a D^\mu S^a + \tfrac{1}{2}D_\mu P^a D^\mu P^a\\ &+ i\bar{\chi}^a\gamma\cdot D\chi^a + igf_{abc}\bar{\chi}^a\chi^b S^c + gf_{abc}\bar{\chi}^a\gamma_5\chi^b P^c - \tfrac{1}{2}g^2(f_{abc}S^bP^c)^2\}\,,\end{aligned} \tag{4.6}$$

with the supersymmetry transformations

$$\begin{aligned}\delta A_\mu^a &= i(\bar{\alpha}\gamma_\mu\chi^a - \bar{\chi}^a\gamma^\mu\alpha)\,,\\ \delta P^a &= \bar{\chi}^a\gamma_5\alpha - \bar{\alpha}\gamma_5\chi^a\,,\\ \delta S^a &= i(\bar{\chi}^a\alpha - \bar{\alpha}\chi^a)\,,\\ \delta\chi^a &= (\sigma_{\mu\nu}F^{\mu\nu a} + igf_{abc}\gamma_5 P^b S^c + i\gamma\cdot DP^a\gamma_5 - \gamma\cdot DS^a)\alpha\,.\end{aligned} \tag{4.7}$$

This is exactly the complex supersymmetric four-dimensional theory previously discussed by Ferrara and Zumino and Fayet [2].

Instead of formulating the theory with one Dirac spinor, one can formulate it in terms of a pair of Weyl spinors. Fayet has observed that in this case the theory exhibits an SU(2) invariance. In addition it has a U(1) chiral invariance obtained by combining chiral transformations of χ^a and complex phase transformations of $S^a + iP^a$.

In order to reduce (4.1) and (4.2) to two dimensions we choose the representation

$$\begin{aligned}
&\Gamma^\mu = \gamma^\mu \otimes I_4\,, \qquad \mu = 0, 1\,,\\
&\Gamma^i = \gamma_3 \otimes \rho_{i-2}, \quad i = 2, 3, 4, 5\,,\\
&\Gamma^7 = \gamma_3 \otimes \rho_5\,,
\end{aligned} \tag{4.8}$$

where $\rho_0, \rho_1, \rho_2, \rho_3$ and ρ_5 are 4×4 matrices chosen to equal four-dimensional γ-matrices such that $\rho_{1,2,3} = \gamma^{1,2,3}, \rho_0 = i\gamma^0, \rho_5 = \gamma_5$.

This time the Weyl condition leads to

$$\lambda = \sqrt{\tfrac{1}{2}}\begin{bmatrix} \chi_1 \\ \chi_2 \\ \gamma_5\chi_1 \\ \gamma_5\chi_2 \end{bmatrix}, \tag{4.9}$$

with $\chi_{1,2}$ two-dimensional Dirac spinors. Furthermore we decompose the vector field as

$$\begin{aligned}
&A^a_\mu = A^a_\mu\,, \qquad \mu = 0, 1\,,\\
&A^a_2 = P^a\,,\\
&A^a_i = \phi^a_{i-2}, \quad i = 3, 4, 5\,,
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
S = \int \mathrm{d}^2x\, \{ &-\tfrac{1}{4}F_{\mu\nu}^{\ a}F^{\mu\nu a} + \tfrac{1}{2}D_\mu P^a D^\mu P^a + \tfrac{1}{2}D_\mu\phi^a_i D^\mu\phi^a_i + i\bar{\chi}^a\gamma\cdot D\chi^a \\
&+ gf_{abc}\bar{\chi}^a\gamma_5\chi^b P^c - igf_{abc}\bar{\chi}^a\tau^i\chi^b\phi^c_i - g^2(f_{abc}P^bS^c)^2 - \tfrac{1}{2}g^2(f_{abc}\phi^b_i\phi^c_i)^2\}\,,
\end{aligned} \tag{4.11}$$

with the supersymmetry transformations

$$\begin{aligned}
&\delta A^a_\mu = i(\bar{\alpha}\gamma_\mu\chi^a - \bar{\chi}^a\gamma_\mu\alpha)\,,\\
&\delta P^a = \bar{\chi}^a\gamma_3\alpha - \bar{\alpha}\gamma_3\chi^a\,,
\end{aligned}$$

$$\delta\phi_i^a = i(\bar{\alpha}\tau^i\chi^a - \bar{\chi}^a\tau^i\alpha) ,$$

$$\delta\chi^a = \{\sigma_{\mu\nu}F^{\mu\nu a} - i\gamma_3\gamma\cdot DP^a + \gamma\cdot D\phi_i^a\tau^i$$
$$- igf_{abc}P^b\phi_i^c\tau^i\gamma_3 + \frac{i}{2}g\epsilon_{ijk}f_{abc}\phi_i^b\phi_i^c\tau^k\}\,\alpha . \tag{4.12}$$

We have now reached a theory in which supersymmetry is unified with an internal SU(2) symmetry. However, the SU(2) symmetry transforms a pair of Dirac spinors. If we reexpressed the theory in terms of four Majorana spinors, instead, an O(4) invariance is exhibited provided that γ_5 transformations are included in the O(4). Actually it is possible to check that (4.12) is O(4) invariant, but obviously the O(4) transformations cannot commute with parity since the three scalars and the pseudoscalar are combined in a quartet. Which invariance is more appropriate to emphasize depends on ones prejudices about the definition of parity, which in turn should be determined by the physical context (if any) in which the model is to be used.

5. Dimensional reduction of the 10-dimensional theory

In this section we follow the same procedure as in the preceding section. We start with the action

$$S = \int d^{10}x\{-\tfrac{1}{4}F_{\mu\nu}^{\ a}F^{\mu\nu a} + \tfrac{1}{2}i\bar{\lambda}^a\Gamma\cdot D\lambda^a\} , \tag{5.1}$$

where λ satisfies both the Majorana (2.4) and the Weyl (2.5) conditions. The supersymmetry transformations are, as before,

$$\delta A_\mu^a = i\bar{\alpha}\Gamma_\mu\lambda^a , \qquad \delta\lambda^a = \Sigma_{\mu\nu}F^{\mu\nu a}\alpha . \tag{5.2}$$

To squeeze this theory into four dimensions we choose the representation

$$\Gamma^\mu = \gamma^\mu\otimes I_8 , \qquad \mu = 0,1,2,3 ,$$

$$\Gamma^{ij} = \gamma_5\otimes\begin{pmatrix}0 & \rho^{ij}\\ \rho_{ij} & 0\end{pmatrix} , \qquad i,j = 1,2,3,4 , \tag{5.3}$$

where we use a notation Γ^{ij} which is antisymmetric in i and j, and i and j run from 1 to 4 instead of $\Gamma^4, \Gamma^5, ..., \Gamma^9$. ρ^{ij} is a 4 X 4 matrix with matrix elements

$$(\rho^{ij})_{kl} = \delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il} , \tag{5.4a}$$

$$(\rho_{ij})_{kl} = \tfrac{1}{2}\epsilon_{ijmn}(\rho^{mn})_{kl} = \epsilon_{ijkl} . \tag{5.4b}$$

In this representation

$$\Gamma_{11} = \Gamma_0\Gamma_1 \dots \Gamma_9 = \gamma_5 \otimes \begin{pmatrix} I_4 & 0 \\ 0 & I_4 \end{pmatrix} , \tag{5.5}$$

and the charge conjugation operator is

$$C_{10} = C \otimes \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix} , \tag{5.6}$$

where C is the charge conjugation operator for four dimensions. Imposing both the Weyl and the Majorana conditions on λ results in the structure

$$\lambda = \begin{bmatrix} L\chi^1 \\ \vdots \\ L\chi^4 \\ R\widetilde{\chi}_1 \\ \vdots \\ R\widetilde{\chi}_4 \end{bmatrix}, \qquad \widetilde{\chi}_i = C\overline{\chi}^{iT} , \tag{5.7}$$

where L and R were defined in (4.4). Anticipating an SU(4) structure we write

$$\lambda = \begin{pmatrix} L\chi^i \\ R\widetilde{\chi}_i \end{pmatrix}, \tag{5.8}$$

with χ^i transforming as a 4 and χ_i as a $\overline{4}$. Similarly we decompose the vector field as

$$A^a_\mu = A^a_\mu , \qquad \mu = 0, 1, 2, 3 , \tag{5.9a}$$

$$\phi^a_{i4} = (A^a_{i+3} + iA^a_{i+6})/\sqrt{2}, \qquad i = 1, 2, 3 , \tag{5.9b}$$

$$\phi^{jka} = \tfrac{1}{2}\epsilon^{jklm}\phi^a_{lm} = (\phi^a_{jk})^* . \tag{5.9c}$$

This means that ϕ_{ij} transforms as a 6 of SU(4).

We can now introduce (5.3)–(5.9) into (5.1) and (5.2) and reduce the dimension to four to obtain the theory

$$S = \int \mathrm{d}^4x \{ -\tfrac{1}{4}F_{\mu\nu a}F^{\mu\nu a} + \tfrac{1}{2}D_\mu\phi^a_{ij}D^\mu\phi^a_{ij} + i\overline{\chi}^a\gamma\cdot DL\chi^a$$
$$-\tfrac{1}{2}igf_{abc}(\overline{\widetilde{\chi}}^{ai}L\chi^{jb}\phi^c_{ij} - \overline{\chi}^a_i R\widetilde{\chi}^b_j\phi^{ijc}) - \tfrac{1}{4}g^2(f_{abc}\phi^b_{ij}\phi^c_{kl})(f_{ade}\phi^{ijd}\phi^{kle})\} , \tag{5.10}$$

which is invariant under the supersymmetry transformations.

$$
\begin{aligned}
\delta A_\mu^a &= i(\bar{\alpha}_i \gamma_\mu L \chi^{ia} - \bar{\chi}_i^a \gamma_\mu L \alpha^i) , \\
\delta \phi_{ij} &= i(\bar{\alpha}_j R \widetilde{\chi}_i^a - \bar{\alpha}_i R \widetilde{\chi}_j^a + \epsilon_{ijkl} \bar{\alpha}^k L \chi^{la}) , \\
\delta L \chi^{ia} &= \sigma_{\mu\nu} F^{\mu\nu a} L \alpha^i - \gamma \cdot D\phi^{ija} R \widetilde{\alpha}_j + \tfrac{1}{2} g f_{abc} \phi_b^{ik} \phi_{kj}^c L \alpha^j , \\
\delta R \widetilde{\chi}_i^a &= \sigma_{\mu\nu} F^{\mu\nu a} R \widetilde{\alpha}_i + \gamma \cdot D\phi_{ij}^a L \alpha^i + \tfrac{1}{2} g f_{abc} \phi_{ik}^b \phi_c^{kj} R \widetilde{\alpha}_j .
\end{aligned}
\tag{5.11}
$$

We have now obtained a Yang-Mills theory combining an SU(4) symmetry with supersymmetry for which the infinitesimal parameters of the transformations transform as a left-handed 4 and a right-handed $\bar{4}$ of SU(4). A problem with this model is that no fermion number commuting with SU(4) can be defined because of the condition (5.9c). This model is really only SU(4) invariant and not U(4) invariant. This is precisely the case where the classification theorem of Haag et al. [10] permits the internal symmetry group to be either U(4) or SU(4), whereas in general it has to be U(N) for $N \neq 4$.

A different formulation of the same theory was originally obtained [3] by choosing a representation in which the four-dimensional spinors are Majorana instead of Weyl spinors. Then the resulting symmetry is SO(4) instead of SU(4). The two cases are actually the same and the apparent difference just resides in the interpretation of the spinor. Of course, the physics looks different since the SU(4) description appears to break parity while the SO(4) one appears to reduce the internal symmetry group instead. If it were not for the problem of fermion number discussed above, one might be tempted to identify the SU(4) of this model with physical flavor.

To reduce the ten-dimensional theory down to two dimensions we use a representation which is patterned after the representations of (5.3) as much as possible.

$$
\begin{aligned}
\Gamma^\mu &= \gamma^\mu \otimes I_2 \otimes I_8 , && \mu = 0, 1 , \\
\Gamma^{2,3} &= \gamma_3 \otimes i\sigma_{1,2} \otimes I_8 , \\
\Gamma^{ij} &= \gamma_3 \otimes \sigma_3 \otimes \begin{pmatrix} 0 & \rho^{ij} \\ \rho_{ij} & 0 \end{pmatrix}, && i, j = 1, 2, 3, 4 ,
\end{aligned}
\tag{5.12}
$$

with ρ^{ij} and ρ_{ij} as in (5.4).

This time we find

$$
\Gamma_{11} = \gamma_3 \otimes \sigma_3 \otimes \begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix} , \tag{5.13}
$$

$$
C_{10} = \gamma^1 \otimes \sigma_1 \otimes \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix} . \tag{5.14}
$$

The Weyl and Majorana conditions this time lead to the following structure for the spinor

$$\lambda = \begin{bmatrix} L\chi^i \\ R\chi^i \\ R\widetilde{\chi}_i \\ L\widetilde{\chi}_i \end{bmatrix}, \qquad i = 1, 2, 3, 4 \ , \tag{5.15}$$

where L and R are $\frac{1}{2}(1 \pm \gamma_3)$. χ^i and $\widetilde{\chi}_i$ are Dirac spinors in two dimensions transforming as a 4 and a $\overline{4}$ under SU(4).

We decompose the vector field as

$$\begin{aligned} A^a_\mu &= A^a_\mu, \qquad \mu = 0, 1 \ , \\ A^a_2 &= S^a, \\ A^a_3 &= P^a, \end{aligned} \tag{5.16}$$

$$\begin{aligned} \phi^a_{i4} &= (A^a_{i+3} + iA^a_{i+6})/\sqrt{2}, \qquad i = 1, 2, 3 \ , \\ \phi^{jka} &= \tfrac{1}{2}\epsilon^{jklm}\phi^a_{lm} = (\phi^a_{jk})^* \ . \end{aligned} \tag{5.17}$$

With the formulae (5.12) – (5.17) and the usual reduction technique we end up with the following theory

$$\begin{aligned} S = \int \mathrm{d}^2x [& -\tfrac{1}{4}F_{\mu\nu}^{\ a}F^{\mu\nu a} + \tfrac{1}{2}D_\mu S^a D^\mu S^a + \tfrac{1}{2}D_\mu P^a D^\mu P^a \\ & + \tfrac{1}{2}D_\mu \phi^a_{ij} D^\mu \phi^{ija} + i\overline{\chi}^a_i \gamma \cdot D\chi^{ia} + gf_{abc}(\overline{\chi}^a_i \gamma_5 \chi^{ib} P^c + i\overline{\chi}^a_i \chi^{ib} S^c) \\ & - \tfrac{1}{2}igf_{abc}(\overline{\widetilde{\chi}}^{ia} \chi^{jb} \phi^{\ c}_{ij} - \overline{\chi}^a_i \widetilde{\chi}^b_j \phi^{ij}_c) - \tfrac{1}{4}g^2(f_{abc}\phi^{\ b}_{ij}\phi^{\ c}_{kl})(f_{ade}\phi^{ijd}\phi^{kle}) \\ & - \tfrac{1}{2}g^2(f_{abc}S^bP^c)^2 - \tfrac{1}{2}g^2(f_{abc}S^b\phi^{\ c}_{ij})(f_{ade}S^d\phi^{ije}) - \tfrac{1}{2}g^2(f_{abc}P^b\phi^{\ c}_{ij})(f_{ade}P^d\phi^{ije})] \ , \end{aligned} \tag{5.18}$$

which is invariant under

$$\begin{aligned} \delta A^a_\mu &= i(\overline{\alpha}_i \gamma_\mu \chi^{ia} - \overline{\chi}^a_i \gamma_\mu \alpha^i) \ , \\ \delta P^a &= \overline{\chi}^a_i \gamma_5 \alpha^i - \overline{\alpha}_i \gamma_5 \chi^{ia} \ , \\ \delta S^a &= i(\overline{\chi}^a_i \alpha^i - \overline{\alpha}_i \chi^{ia}), \\ \delta \phi^a_{ij} &= i(\overline{\alpha}_j \widetilde{\chi}^a_i - \overline{\alpha}_i \widetilde{\chi}^a_j + \epsilon_{ijkl}\overline{\widetilde{\alpha}}^k \chi^{la}) \ , \end{aligned}$$

$$\begin{aligned}\delta\chi^{ia} &= \sigma_{\mu\nu}F^{\mu\nu a} + i\gamma\cdot DP^a\gamma_5\alpha^i - \gamma\cdot DS^a\alpha^i \\ &\quad - \gamma\cdot D\phi^{ija}\widetilde{\alpha}_j + gf_{abc}(iP^bS^c\gamma_5\alpha^i - iP^b\phi^{ijc}\gamma_5\widetilde{\alpha}_j - S^b\phi_c^{ij}\widetilde{\alpha}_j + \tfrac{1}{2}\phi_b^{ik}\phi_{kj}^{c}\alpha^j)\,.\end{aligned} \tag{5.19}$$

Also this time we get an SU(4) symmetry combined with the supersymmetry, such that the infinitesimal parameters of the transformation are Dirac spinors transforming as 4 and $\overline{4}$ of SU(4).

Since we are trying to be complete in our enumeration of supersymmetric Yang-Mills theories, a few words about $D = 8$ are in order. A $D = 8$ theory can be obtained by squeezing the ten-dimensional theory into eight dimensions. In doing this the 32-component spinor λ which starts out as both Majorana and Weyl in ten dimensions ends up as a 16-component Weyl spinor χ in eight dimensions. Since the procedure is the same as in all the other reductions that have been described, we omit the details. One finds that A_8 and A_9 should be combined into a complex scalar field Φ carrying two units of fermion number. Altogether one obtains

$$\begin{aligned}S = \int d^8x\,\{&-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\overline{\chi}\gamma\cdot D\chi + \tfrac{1}{2}D_\mu\Phi D^\mu\Phi^* \\ &+ \tfrac{1}{2}g^2(f_{abc}\Phi^b\Phi^{*c})^2 + \tfrac{1}{2}gf_{abc}(\overline{\chi}^a\widetilde{\chi}^b\Phi^c - \overline{\widetilde{\chi}}^a\chi^b\Phi^{*c})\}\,,\end{aligned} \tag{5.20}$$

where χ is left-handed ($\chi = \gamma_9\chi$) and $\widetilde{\chi} = C\overline{\chi}^T$ is right-handed. The supersymmetry transformation formulas of this theory are

$$\begin{aligned}\delta A_\mu &= i(\overline{\alpha}\gamma_\mu\chi - \overline{\chi}\gamma_\mu\alpha)\,, \\ \delta\Phi &= 2\overline{\widetilde{\alpha}}\chi\,, \\ \delta\chi &= \sigma_{\mu\nu}F^{\mu\nu}\alpha + i\gamma\cdot D\Phi\widetilde{\alpha} - \tfrac{1}{2}gf_{abc}\Phi^b\Phi^{*c}\alpha\,.\end{aligned} \tag{5.21}$$

This theory is not very interesting from the point of view of learning new possibilities in four dimensions since it obviously reduces to the same theory there as the ten-dimensional one.

6. Discussion

There are three different types of supersymmetric Yang-Mills theories in four dimensions, which correspond to theories with simple supersymmetry in 4, 6, and 10 dimensions. Since these theories contain 1, 2, and 4 Majorana spinor fields, respectively, they correspond to O(1), O(2), and O(4) supersymmetry. We can understand that these are the only possibilities by listing the helicity content of the various possible irreducible representations and adding the *CPT* conjugate for those representations

Table 2

J_Z	O(1)	O(2)	O(3)	O(4)
1	1	1	1	1
$\frac{1}{2}$	1	2	3 + 1	4
0		1 + 1	3 + 3	6
$-\frac{1}{2}$	1	2	1 + 3	4
−1	1	1	1	1

that are not self conjugate (see table 2) ★. We see, in particular, that the O(3) and O(4) cases have the same content and are in fact equivalent. This is why no additional O(3) model turned up in our analysis. Also, O(N) groups with $N > 4$ are excluded since we are assuming (essentially as a matter of definition) that a Yang-Mills multiplet contains no spins greater than one.

Supergravity can presumably be formulated for N values all the way up to 8 (only $N = 1, 2, 3$ have been done explicitly so far [8]). However, if one insists on adding interaction with Yang-Mills "matter" multiplets, then one is restricted to $N \leqslant 4$. The alternative hope, of course, is that it is possible to incorporate all matter in the irreducible gravity multiplet. If that is possible, then none of the models discussed here is relevant.

For two dimensions we found four types of Yang-Mills theories based on O(1), O(2), SU(2), and SU(4) supersymmetry. For the O(1) and O(2) cases there exists a locally supersymmetric string theory to which the Yang-Mills theory can couple [4]. The same may be true in the SU(2) case, although there is some uncertainty whether it accommodates a locally supersymmetric string Lagrangian. In any case the classification theorem of dual model gauge algebras [9] implies that there is no string theory with a local SU(4) supersymmetry invariance. It is interesting that whereas in four dimensions there are more possibilities for supergravity theories than for supersymmetric Yang-Mills theories, the reverse is true in two dimensions.

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