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Phases of $N = 2$ theories in two dimensions

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By looking at phase transitions which occur as parameters are varied in supersymmetric gauge theories, a natural relation is found between sigma models based on Calabi–Yau hypersurfaces in weighted projective spaces and Landau–Ginzburg models. The construction permits one to recover the known correspondence between these types of models and to greatly extend it to include new classes of manifolds and also to include models with (0,2) world-sheet supersymmetry. The construction also predicts the possibility of certain physical processes involving a change in the topology of space-time.

1. Introduction

The present paper is based on systematically exploiting one simple idea which is familiar in $N = 1$ supersymmetric theories in four dimensions and which we will therefore first state in that context. We consider renormalizable gauge theories constructed from vector (gauge) multiplets and charged chiral multiplets. If the gauge group includes a U(1) factor, one of the possible supersymmetric interactions is the Fayet–Iliopoulos D term. In terms of a vector superfield V, this term can be written as

$$-r \int \mathrm{d}^4x \, \mathrm{d}^4\theta \, V. \tag{1.1}$$

The potential energy for scalar components s_i, $i = 1, \ldots, k$, of chiral multiplets S_i of charge n_i is then

$$U(s_i) = \frac{1}{2e^2} D^2 + \sum_i \left| \frac{\partial W}{\partial s_i} \right|^2 , \tag{1.2}$$

where e is the gauge coupling, W is a holomorphic function known as the superpotential, and

$$D = -e^2 \left(\sum_i n_i |s_i|^2 - r \right). \tag{1.3}$$

Actually, D can be interpreted as the hamiltonian function for the U(1) action on a copy of $\mathbb{C}^k$ (on which the s_i are coordinates and which we endow with the

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Kähler form $\omega = -i\sum_j ds_j \wedge d\bar{s}_j$). The parameter r corresponds to the familiar possibility of adding a constant to the hamiltonian.

As r is varied, such a system will typically undergo phase transitions, in many cases leaving supersymmetry unbroken but otherwise changing the pattern of massless fields. This is a familiar and important story in phenomenology of renormalizable supersymmetric theories in four dimensions. Our intention in the present paper is to study the same models, but dimensionally reduced to $D = 2$. The two-dimensional version of the story is particularly rich, because milder anomaly cancellation conditions give more freedom in constructing models and because of the usual special properties of massless fields in two dimensions.

As a special case which originally motivated this investigation, we will recover the familiar correspondence between Calabi–Yau and Landau–Ginzburg models [1–3]. The usual arguments for this correspondence are heuristic arguments of universality [1–3] including the simple fact that the Calabi–Yau and Landau–Ginzburg models are $N = 2$ models parametrized by the same data, a heuristic path integral argument [3], and equivalence of the cc chiral rings, which is discussed in the papers just cited and in refs. [4–6].

From our construction we will recapture essentially all the concrete evidence for the Calabi–Yau/Landau–Ginzburg correspondence along with new features. The argument will be carried out without assuming or proving conformal invariance of either Calabi–Yau or Landau–Ginzburg models; the picture is certainly richer when supplemented by (well-known) independent arguments for conformal invariance.

We will also find many generalizations of the usual C–Y/L–G correspondence, involving more general classes of Calabi–Yau manifolds (hypersurfaces in products of projective spaces and in more general toric and other varieties, and intersections of such hypersurfaces). One novelty is that in general one must consider gauged Landau–Ginzburg models as well as ordinary ones. Moreover, we will get an extension of the C–Y/L–G correspondence to models with (0,2) world-sheet supersymmetry. Hitherto only the case of (2,2) supersymmetry has been considered. (0,2) world-sheet supersymmetry is of phenomenological interest as it can naturally lead, for instance, to models with space-time grand unified gauge group SU(5) or SO(10) rather than E_6.

Variation of the coefficients of the Fayet–Iliopoulos D terms can also induce phase transitions involving changes in the topology of a Calabi–Yau manifold. This will be investigated from a local point of view only in subsect. 5.5. For reasons explained in sect. 4, the topology changes in question preserve the birational equivalence class of the Calabi–Yau manifold. Precisely this situation has been studied globally by Aspinwall, Greene, and Morrison [7] who by using mirror symmetry (and finding pairs of topologically different but birationally equivalent manifolds with the same mirror) obtained very precise and detailed evidence for the possibility of physical processes with change of topology. The present problem turned out to have an unexpectedly close relation to their investigation, and I benefited from discussions with them, especially in subsect. 5.5. The results of ref. [7] and of subsect. 5.5 are the first concrete evidence for the (long-suspected) occurrence in string theory of processes with change of the space-time topology.

The present paper is based on familiar representations of the relevant sigma models [8], except that we will consider linear sigma models (rather than the nonlinear ones to which they reduce at low energies); this gives a useful added freedom of maneuver.

While this work was in progress, I received a paper by Thaddeus [9] who uses a mathematical idea closely related to that of the present paper to prove the Verlinde formula for the dimension of the space of conformal blocks of the SU(2) WZW model.

2. Field theory background

In this section, I will present the field theory background to our subsequent analysis. Most of the detailed formulas of this section are not needed for reading the paper, and are presented for reference and to be more self-contained.

$N = 2$ supersymmetry in two dimensions can be obtained by dimensional reduction from $N = 1$ supersymmetry in four-dimensional space-time. A number of good books exist [10–14]. Our conventions for four-dimensional superfields will be those of ref. [10].

In superspace with coordinates $x^m, \theta^\alpha, \overline{\theta}^{\dot\alpha}$, supersymmetry is realized geometrically by the operators

$$\begin{aligned} Q_\alpha &= \frac{\partial}{\partial\theta^\alpha} - i\sigma^m_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}\frac{\partial}{\partial x^m}, \\ \overline{Q}_{\dot\alpha} &= -\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} + i\sigma^m_{\alpha\dot\alpha}\theta^\alpha\frac{\partial}{\partial x^m}. \end{aligned} \tag{2.1}$$

α and $\dot\alpha$ are the two chiralities of spinor indices; also one writes $\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta$, $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$, where ϵ is the antisymmetric tensor with $\epsilon^{12} = -\epsilon_{12} = 1$; and similarly for dotted indices. The tensors $\sigma^m_{\alpha\dot{\alpha}}$ are in the representation used in ref. [10]

$$\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$
$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.2}$$

The metric is $\eta^{mn} = \mathrm{diag}(-1, 1, 1, 1)$. The supersymmetry generators of eq. (2.1) commute with the operators

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^m_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^m},$$
$$\overline{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} - i\sigma^m_{\alpha\dot{\alpha}}\theta^\alpha\frac{\partial}{\partial x^m} \tag{2.3}$$

which are used in writing lagrangians.

The simplest type of superfield is a chiral superfield Φ which obeys $\overline{D}_{\dot{\alpha}}\Phi = 0$ and can be expanded

$$\Phi(x,\theta) = \phi(y) + \sqrt{2}\theta^\alpha\psi_\alpha(y) + \theta^\alpha\theta_\alpha F(y), \tag{2.4}$$

where $y^m = x^m + i\theta^\alpha\sigma^m_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}$. The complex conjugate of Φ is an antichiral multiplet $\overline{\Phi}$, obeying $D_\alpha\overline{\Phi} = 0$, and with an expansion

$$\overline{\Phi} = \overline{\phi}(\overline{y}) + \sqrt{2}\overline{\theta}_{\dot{\alpha}}\overline{\psi}^{\dot{\alpha}}(\overline{y}) + \overline{\theta}_{\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}F(\overline{y}), \tag{2.5}$$

where $\overline{y}^m = x^m - i\theta^\alpha\sigma^m_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}$. In general, the $\theta = 0$ component of a chiral field labeled by a capital letter $\Phi, S, P, \ldots$ will be denoted by the corresponding lower case letter $\phi, s, p, \ldots$.

To formulate gauge theory, one introduces a gauge field in superspace, replacing the differential operators D_α, $\overline{D}_{\dot{\alpha}}$, and $\partial_m = \partial/\partial x^m$ by gauge covariant derivatives $\mathcal{D}_\alpha$, $\overline{\mathcal{D}}_{\dot{\alpha}}$, and $\mathcal{D}_m$. One imposes however a severe restriction on the superspace gauge fields. To permit charged chiral superfields to exist, one needs the integrability of the equation $\overline{\mathcal{D}}_{\dot{\alpha}}\Phi = 0$. So one requires

$$0 = \{\overline{\mathcal{D}}_{\dot{\alpha}}, \overline{\mathcal{D}}_{\dot{\beta}}\} = \{\mathcal{D}_\alpha, \mathcal{D}_\beta\}. \tag{2.6}$$

These conditions ensure that, with a suitable partial gauge fixing,

$$\mathcal{D}_\alpha = \mathrm{e}^{-V}\mathcal{D}_\alpha\mathrm{e}^V,$$
$$\overline{\mathcal{D}}_{\dot{\alpha}} = \mathrm{e}^V\overline{\mathcal{D}}_{\dot{\alpha}}\mathrm{e}^{-V}, \tag{2.7}$$

where V is a real Lie algebra valued function on superspace; V is known as a vector superfield. One also imposes the further constraint

$$\{\mathcal{D}_\alpha, \overline{\mathcal{D}}_{\dot\alpha}\} = -2i\sigma^m_{\alpha\dot\alpha}\mathcal{D}_m, \tag{2.8}$$

so that the superspace gauge field is entirely determined in terms of V.

There is still a residual gauge invariance, which for gauge group U(1) – in which case V is simply a single real function on superspace – takes the form

$$V \to V + i(\Lambda - \overline{\Lambda}), \tag{2.9}$$

Λ being a chiral superfield. A chiral superfield Φ of charge Q transforms under this residual gauge invariance as

$$\Phi \to \exp(-iQ\Lambda)\cdot\Phi. \tag{2.10}$$

One can partially fix this residual gauge invariance (by going to what is called Wess–Zumino gauge) to put V in the form

$$V = -\theta^\alpha\sigma^m_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}v_m + i\theta^\alpha\theta_\alpha\overline{\theta}_{\dot\alpha}\overline{\lambda}^{\dot\alpha} - i\overline{\theta}_{\dot\alpha}\overline{\theta}^{\dot\alpha}\theta^\alpha\lambda_\alpha + \tfrac{1}{2}\theta^\alpha\theta_\alpha\overline{\theta}_{\dot\alpha}\overline{\theta}^{\dot\alpha}D. \tag{2.11}$$

In Wess–Zumino gauge, one still retains the ordinary gauge invariance. This corresponds to $\Lambda = -a(x)$ (a real function of x only) with the usual transformation laws $\Phi \to \exp(iQa)\Phi$, $v \to v - \mathrm{d}a$.

In four dimensions, the basic gauge invariant field strength of the abelian gauge field is $[\mathcal{D}_\alpha, \mathcal{D}_m]$. In two dimensions there is a more basic invariant that we will introduce later.

In Wess–Zumino gauge, though the physical content of the theory is relatively transparent, the supersymmetry transformation laws are relatively complicated because supersymmetry transformations (generated by Q_α and $\overline{Q}_{\dot\alpha}$) must be accompanied by gauge transformations to preserve the Wess–Zumino gauge. The transformation laws become still more complicated when dimensionally reduced to two dimensions*. I will now write down the reduced transformation laws, in Wess–Zumino gauge, for reference, but I hasten to reassure the reader that this paper can be read without detailed familiarity with the following formulas.

In making the reduction, I will take the fields to be independent of x^1 and x^2. Thus, the components v_1 and v_2 of the gauge field in the x^1 and x^2 directions, along with all the other fields, are functions of x^0, x^3 only; I write $\sigma = (v_1 - iv_2)/\sqrt{2}$, $\overline{\sigma} = (v_1 + iv_2)/\sqrt{2}$. For the remaining space-time coordinates, I will adopt a two-dimensional notation, writing $x^0 = y^0$, $x^3 = y^1$. After dimensional reduction, it is convenient to label the fermion components as

* All conventions will be those that follow by dimensional reduction from four-dimensional conventions of ref. [10]. I adopted this approach because the models we will study arise so naturally by reduction from four dimensions. Also, there do not seem to be any standard conventions for component expansions of vector superfields in two dimensions. Chiral superfields in two dimensions are often described by conventions that differ from the present ones by factors of $\sqrt{2}$; these factors can be absorbed in rescaling the fields in a fairly obvious way.

$(\psi^1, \psi^2) = (\psi^-, \psi^+)$ and $(\psi_1, \psi_2) = (\psi_-, \psi_+)$ (so $\psi^- = \psi_+$, $\psi^+ = -\psi_-$), and similarly for dotted components. The dimensionally reduced transformation laws of the vector multiplet under a supersymmetry transformation with parameters $\epsilon_\pm, \bar{\epsilon}_\pm$ are (by reducing the formulas on p. 50 of ref. [10])

$$
\begin{aligned}
\delta v_m &= i\bar{\epsilon}\sigma_m\lambda + i\epsilon\sigma_m\bar{\lambda}, \\
\delta\sigma &= -i\sqrt{2}\bar{\epsilon}_+\lambda_- - i\sqrt{2}\epsilon_-\bar{\lambda}_+, \\
\delta\bar{\sigma} &= -i\sqrt{2}\epsilon_+\bar{\lambda}_- - i\sqrt{2}\bar{\epsilon}_-\lambda_+, \\
\delta D &= -\bar{\epsilon}_+(\partial_0 - \partial_1)\lambda_+ - \bar{\epsilon}_-(\partial_0 + \partial_1)\lambda_- + \epsilon_+(\partial_0 - \partial_1)\bar{\lambda}_+ + \epsilon_-(\partial_0 + \partial_1)\bar{\lambda}_-, \\
\delta\lambda_+ &= i\epsilon_+ D + \sqrt{2}(\partial_0 + \partial_1)\bar{\sigma}\epsilon_- - v_{01}\epsilon_+, \\
\delta\lambda_- &= i\epsilon_- D + \sqrt{2}(\partial_0 - \partial_1)\sigma\epsilon_+ + v_{01}\epsilon_-, \\
\delta\bar{\lambda}_+ &= -i\bar{\epsilon}_+ D + \sqrt{2}(\partial_0 + \partial_1)\sigma\bar{\epsilon}_- - v_{01}\bar{\epsilon}_+, \\
\delta\bar{\lambda}_- &= -i\bar{\epsilon}_- D + \sqrt{2}(\partial_0 - \partial_1)\bar{\sigma}\bar{\epsilon}_+ + v_{01}\bar{\epsilon}_-,
\end{aligned}
\tag{2.12}
$$

with $v_{01} = \partial_0 v_1 - \partial_1 v_0$.

In the presence of gauge fields, a chiral superfield Φ in a given representation of the gauge group is a superfield obeying $\bar{\mathcal{D}}_{\dot\alpha}\Phi = 0$, where the covariant derivative in $\bar{\mathcal{D}}_{\dot\alpha}$ is taken in the appropriate representation. If we write

$$\Phi = \mathrm{e}^V \Phi_0, \tag{2.13}$$

then Φ_0 obeys $\bar{D}_{\dot\alpha}\Phi_0 = 0$ and has a theta expansion of the form given in (2.4). The transformation laws for these component fields come out to be

$$
\begin{aligned}
\delta\phi &= \sqrt{2}\,(\epsilon_+\psi_- - \epsilon_-\psi_+)\,, \\
\delta\psi_+ &= i\sqrt{2}(D_0 + D_1)\phi\bar{\epsilon}_- + \sqrt{2}\epsilon_+ F - 2Q\phi\bar{\sigma}\bar{\epsilon}_+, \\
\delta\psi_- &= -i\sqrt{2}(D_0 - D_1)\phi\bar{\epsilon}_+ + \sqrt{2}\epsilon_- F + 2Q\phi\sigma\bar{\epsilon}_-, \\
\delta F &= -i\sqrt{2}\bar{\epsilon}_+(D_0 - D_1)\psi_+ - i\sqrt{2}\bar{\epsilon}_-(D_0 + D_1)\psi_- \\
&\quad + 2Q(\bar{\epsilon}_+\bar{\sigma}\psi_- + \bar{\epsilon}_-\sigma\psi_+) + 2iQ\phi(\bar{\epsilon}_-\bar{\lambda}_+ - \bar{\epsilon}_+\bar{\lambda}_-)\,.
\end{aligned}
\tag{2.14}
$$

The transformation laws of the antichiral multiplet $\bar{\Phi}$ are the complex conjugate of these.

Twisted chiral superfields; Gauge field strength. One of the novelties that appears in two dimensions, relative to four, is that in addition to chiral superfields, obeying $\bar{D}_+\Phi = \bar{D}_-\Phi = 0$, it is possible to have twisted chiral superfields, obeying $\bar{D}_+\Sigma = D_-\Sigma = 0$ [16–19]. Sigma models containing both chiral and twisted chiral superfields are quite lovely. Since mirror symmetry turns chiral multiplets into twisted chiral multiplets, it is likely that consideration of appropriate models containing multiplets of both types is helpful for understanding mirror symmetry.

More important for our present purposes, however, is the fact (exploited in ref. [20] in explaining $N = 2$ duality as an abelian mirror symmetry) that in two dimensions the basic gauge invariant field strength of the superspace gauge field is a twisted chiral superfield. This quantity is

$$\Sigma = \frac{1}{2\sqrt{2}}\{\overline{\mathcal{D}}_+, \mathcal{D}_-\}, \tag{2.15}$$

which according to the Bianchi identities is annihilated by $\overline{\mathcal{D}}_+$ and $\mathcal{D}_-$. In the abelian case, one has

$$\begin{aligned}\Sigma &= \frac{1}{\sqrt{2}}\overline{D}_+ D_- V \\ &= \sigma - i\sqrt{2}\theta^+\overline{\lambda}_+ - i\sqrt{2}\overline{\theta}^-\lambda_- + \sqrt{2}\theta^+\overline{\theta}^-(D - iv_{01}) \\ &\quad - i\overline{\theta}^-\theta^-(\partial_0 - \partial_1)\sigma - i\theta^+\overline{\theta}^+(\partial_0 + \partial_1)\sigma + \sqrt{2}\overline{\theta}^-\theta^+\theta^-(\partial_0 - \partial_1)\overline{\lambda}_+ \\ &\quad + \sqrt{2}\theta^+\overline{\theta}^-\overline{\theta}^+(\partial_0 + \partial_1)\lambda_- - \theta^+\overline{\theta}^-\theta^-\overline{\theta}^+(\partial_0{}^2 - \partial_1{}^2)\sigma.\end{aligned} \tag{2.16}$$

(The complicated terms involving derivatives are determined from the first three terms by the twisted chiral condition $\overline{\mathcal{D}}_+\Sigma = \mathcal{D}_-\Sigma = 0$; they could be eliminated by the twisted version of $x^m \to y^m + i\theta^\alpha\sigma^m_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}$.)

Lagrangians. It is straightforward to write lagrangians in superspace. We will consider only super-renormalizable theories that will be particularly simple representatives of their universality classes.

These lagrangians are of the form

$$L = L_{\text{kin}} + L_W + L_{\text{gauge}} + L_{D,\theta} \tag{2.17}$$

where the four terms are respectively the kinetic energy of chiral superfields, the superpotential interaction, the kinetic energy of gauge fields, and the Fayet–Iliopoulos term and theta angle, all constructed as follows.

For simplicity, we write the formulas for the case of an abelian gauge group $U(1)^s$, described as above by vector superfields $V_a, a = 1, \ldots, s$. This will be the case of primary interest. We assume that there are k chiral superfields Φ_i of charges $Q_{i,a}$. The Φ_i can be interpreted as coordinates on a copy of $Z = \mathbb{C}^k$; and their kinetic energy is determined by a Kähler metric on Z. For superrenormalizability, this metric should be flat. The lagrangian corresponding to such a metric is, with a suitable choice of coordinates,

$$L_{\text{kin}} = \int d^2y\, d^4\theta \sum_i \overline{\Phi}_i\Phi_i = \int d^2y d^4\theta \sum_i \overline{\Phi}_{0,i} \exp\left[2\sum_a Q_{i,a}V_a\right]\Phi_{0,i}. \tag{2.18}$$

(The $\Phi_{0,i}$ are defined as in (2.13) by $\Phi_i = \exp(\sum_a Q_{i,a}V_a)\Phi_{0,i}$.) In components this becomes

$$L_{\text{kin}} = \sum_i \int d^2y \Big(-D_\rho\overline{\phi}_i D^\rho \phi_i + i\overline{\psi}_{-,i}(D_0 + D_1)\psi_{-,i} + i\overline{\psi}_{+,i}(D_0 - D_1)\psi_{+,i}$$
$$+ |F_i|^2 - 2\sum_a \overline{\sigma}_a \sigma_a Q_{i,a}{}^2 \overline{\phi}_i \phi_i - \sqrt{2}\sum_a Q_{i,a}\left(\overline{\sigma}_a \overline{\psi}_{+i}\psi_{-i} + \sigma_a \overline{\psi}_{-i}\psi_{+i}\right)$$
$$+ \sum_a D_a Q_{i,a}\overline{\phi}_i\phi_i - \sum_a i\sqrt{2}Q_{i,a}\overline{\phi}_i(\psi_{-,i}\lambda_{+,a} - \psi_{+,i}\lambda_{-,a})$$
$$- \sum_a i\sqrt{2}Q_{i,a}\phi_i(\overline{\lambda}_{-,a}\overline{\psi}_{+,i} - \overline{\lambda}_{+,a}\overline{\psi}_{-,i}) \Big), \tag{2.19}$$

where the world-sheet metric is $ds^2 = -(dy^0)^2 + (dy^1)^2$.

The other part of the lagrangian involving the chiral superfields is constructed from a gauge invariant holomorphic function W on Z known as the superpotential. In our models, W will always be a polynomial. The corresponding part of the lagrangian is

$$L_W = -\int d^2y d\theta^+ d\theta^- \; W(\Phi_i)|_{\overline{\theta}^+ = \overline{\theta}^- = 0} - \text{h.c.} \tag{2.20}$$

This is integrated over only half of the odd coordinates, an operation that is supersymmetric only because the Φ_i are chiral superfields and W is holomorphic. In components,

$$L_W = -\int d^2y \left(F_i \frac{\partial W}{\partial \phi_i} + \frac{\partial^2 W}{\partial\phi_i \partial\phi_j}\psi_{-,i}\psi_{+,j}\right) - \text{h.c.} \tag{2.21}$$

The gauge kinetic energy is constructed from the twisted chiral superfields Σ_a defined as above. Introducing gauge coupling constants e_a, $a = 1, \dots, s$, the gauge kinetic energy is

$$L_{\text{gauge}} = -\sum_a \frac{1}{4e_a{}^2}\int d^2y d^4\theta \, \overline{\Sigma}_a \Sigma_a. \tag{2.22}$$

In components this is

$$L_{\text{gauge}} = \sum_a \frac{1}{e_a{}^2}\int d^2y \Big(\tfrac{1}{2}v_{01,a}{}^2 + \tfrac{1}{2}D_a{}^2 + i\overline{\lambda}_{+,a}(\partial_0 - \partial_1)\lambda_{+,a}$$
$$+ i\overline{\lambda}_{-,a}(\partial_0 + \partial_1)\lambda_{-,a} - |\partial_\rho \sigma_a|^2 \Big). \tag{2.23}$$

Certain additional terms involving the gauge multiplet will play an essential role. For simplicity I write the following formulas for the case of a single gauge multiplet V, that is a gauge group that is just U(1). The generalization is obtained simply by summing over the various components, as in (2.23).

By inspection of the supersymmetry transformation laws, one can see that the

Fayet–Iliopoulos term

$$-r\int d^2y\ D \tag{2.24}$$

is supersymmetric. This is usually written in superspace as $-\int d^2y\ d^4\theta\ V$, but that is an unsatisfactory representation as V is not gauge invariant. I will give a better version momentarily. Another important coupling is the theta angle,

$$\frac{\theta}{2\pi}\int dv = \frac{\theta}{2\pi}\int d^2y\ v_{01}. \tag{2.25}$$

θ should be regarded as an angular variable (the physics is periodic in θ with period 2π, but perhaps not smooth or even continuous in θ), since the quantity $\int dv/2\pi$ measures the first Chern class of the abelian gauge field and is always integral (with appropriate boundary conditions).

The above interactions can be neatly written in superspace using the twisted chiral superfield Σ. Indeed,

$$\begin{aligned}\int d^2y\, d\theta^+ d\overline{\theta}^- \,\Sigma|_{\theta^-=\overline{\theta}^+=0} &= \sqrt{2}\int d^2y\,(D - iv_{01})\,,\\ \int d^2y\, d\theta^- d\overline{\theta}^+ \overline{\Sigma}|_{\theta^+=\overline{\theta}^-=0} &= \sqrt{2}\int d^2y\,(D + iv_{01})\,.\end{aligned} \tag{2.26}$$

From (2.26) it follows that we can write

$$\begin{aligned}L_{D,\theta} &= \int d^2y\left(-rD + \frac{\theta}{2\pi}v_{01}\right)\\ &= \frac{it}{2\sqrt{2}}\int d^2y d\theta^+ d\overline{\theta}^-\ \Sigma|_{\theta^-=\overline{\theta}^+=0} - \frac{i\overline{t}}{2\sqrt{2}}\int d^2y d\theta^- d\overline{\theta}^+\ \overline{\Sigma}\Big|_{\theta^+=\overline{\theta}^-=0}\end{aligned} \tag{2.27}$$

with

$$t = ir + \frac{\theta}{2\pi}. \tag{2.28}$$

Twisted chiral superpotential. Eq. (2.27) has a generalization that may be unfamiliar as it has no close analog in four dimensions. (See however refs. [21,20] for closely related matters.) We can introduce a holomorphic "twisted superpotential" $\widetilde{W}(\Sigma)$ and write

$$\Delta L = \int d^2y d\theta^+ d\overline{\theta}^-\ \widetilde{W}(\Sigma)\Big|_{\theta^-=\overline{\theta}^+=0} + \text{h.c.} \tag{2.29}$$

In components, this is

$$\Delta L = \int d^2y\left(\sqrt{2}\widetilde{W}'(\sigma)(D - iv_{01}) + 2\widetilde{W}''(\sigma)\overline{\lambda}_+\lambda_-\right) + \text{h.c.} \tag{2.30}$$

This reduces to $L_{D,\theta}$ precisely if $\widetilde{W}$ is the linear function $\widetilde{W}(x) = itx/2\sqrt{2}$.

In this paper, we will assume that the microscopic $\widetilde{W}$ function is linear, partly to ensure the R invariance that is discussed presently. Even so, we will find that a more elaborate twisted superpotential may be generated by quantum corrections. The following comments will be useful background for that discussion.

Consider the twisted superpotential $\widetilde{W}(x) = -x \ln x \cdot p/\sqrt{2}$ (p being a constant). We compute

$$\Delta L = -p \int d^2y \left((\ln\sigma + 1)(D - iv_{01}) + \sqrt{2}\frac{1}{\sigma}\bar{\lambda}_+\lambda_- \right) - \text{h.c.} \tag{2.31}$$

Because $\ln\sigma$ is only well-defined modulo $2\pi i\mathbb{Z}$, (2.31) is not well-defined as a real-valued functional. However, in quantum mechanics it is good enough if ΔL is well-defined modulo $2\pi i$. Since $\int d^2y\ v_{01}$ takes values in $2\pi\mathbb{Z}$, ΔL is well-defined modulo $2\pi i$ if and only if

$$4\pi p \in \mathbb{Z}. \tag{2.32}$$

Because (2.31) has a singularity at $\sigma = 0$, its possible generation by quantum corrections would occur, if at all, only in an approximation whose validity would be limited to large σ. This is precisely what we will find. For future use, let us note that if σ is large and slowly varying, then the addition of (2.31) to the lagrangian would have the effect of shifting r to an effective value

$$r_{\text{eff}} = r + 2p \ln|\sigma|. \tag{2.33}$$

Symmetries. The models that we have constructed have $N = 2$ supersymmetry in two dimensions, that is two left-moving supersymmetries (acting on $\theta^-, \bar{\theta}^-$) and two right-moving ones (acting on $\theta^+, \bar{\theta}^+$). It is natural to ask if the models also have left- and right-moving R symmetries. A right-moving R-symmetry is a U(1) symmetry under which $\theta^+ \to e^{i\alpha}\theta^+$, $\bar{\theta}^+ \to e^{-i\alpha}\bar{\theta}^+$, while $\theta^-, \bar{\theta}^-$ are invariant. A left-moving R-symmetry obeys the analogous condition with + and − exchanged.

Left- and right-moving R symmetries are important because they are part of the $N = 2$ superconformal algebra. If present, they give important information about possible conformally invariant limits of the models under discussion.

The existence of such left- and right-moving symmetries in the models under discussion is fairly natural from the four-dimensional point of view. Many four-dimensional supersymmetric theories would have a single R-symmetry, with charges +1 for θ^+ and θ^-, and −1 for $\bar{\theta}^{\pm}$. In addition, in the process of dimensional reduction to two dimensions, we potentially get a second U(1) symmetry – corresponding to rotations of the extra dimensions. Under this symmetry, θ^+ and θ^- transform oppositely. By taking suitable combinations of these operations, one may hope to find the left- or right-moving R-symmetry, under which θ^+ or θ^- should be invariant.

It is easy to find these symmetries explicitly in the above two-dimensional formulas. First we suppress the superpotential interaction L_W, and we also assume that any twisted superpotential interaction, if present, is a linear term of the specific form $L_{D,\theta}$. Under these assumptions, the left- and right-moving R symmetries are easy to find explicitly. For instance, there is a right-moving R-symmetry ($\epsilon^+ = -\epsilon_-$ has charge 1, $\epsilon^- = \epsilon_+$ is invariant) under which the non-zero charge assignments are as follows: $(\psi_{+i}, F_i, \sigma_a, \lambda_{-a})$ have charges $(-1, -1, 1, 1)$; their complex conjugates have opposite charge; and other fields are invariant. A left-moving R-symmetry ($\epsilon^- = \epsilon_+$ has charge 1, $\epsilon^+ = -\epsilon_-$ is invariant) likewise can be constructed under which $(\psi_{-i}, F_i, \sigma_a, \lambda_{+a})$ have charges $(-1, -1, -1, 1)$ and other fields are neutral. Let us call the charges generating the right- and left-moving symmetries J_{R} and J_{L}.

One of the most important properties is that J_{R} and J_{L} may be anomalous. The reason for this is that J_{R}, for instance, couples to right-moving charged fermions $\psi_{+i}, \overline{\psi}_{+i}$, but the left-moving fermions that it couples to are neutral. The condition that there should be no gauge anomalies that would spoil conservation of J_{R} is that

$$\sum_i Q_{i,a} = 0, \qquad \text{for } a = 1, \dots, s. \tag{2.34}$$

The same condition is needed to ensure that J_{L} is actually a valid symmetry. (In any event, regardless of the $Q_{i,a}$, the sum $J_{\mathrm{L}} + J_{\mathrm{R}}$ is conserved.) We will see later that various other good things happen when (2.34) is imposed.

Before going on, it is important to note that the models under discussion, if we ignore the superpotential, have additional (non-anomalous) symmetries that commute with both left- and right-moving supersymmetries and therefore could be added to the J_{L} and J_{R} introduced above. These are global transformations of the chiral superfields of the form

$$\Phi_i \to \exp(i\alpha k_i)\Phi_i, \tag{2.35}$$

with arbitrary k_i; they commute with supersymmetry since a common transformation is assumed for each component of Φ_i.

Are these R symmetries still valid in the presence of a superpotential? The couplings coming from $\int \mathrm{d}^2\theta\, W$ have charge 1 under J_{R} as we have defined it hitherto. To save the situation, we must add to J_{R} a transformation of the form (2.35) under which $W \to \mathrm{e}^{-i\alpha} W$. The superpotential W is said to be quasi-homogeneous if k_i exist so that W transforms as indicated. Thus, by adding additional terms to the original J_{R} (and analogous additional terms for J_{L}), one can find left- and right-moving R symmetries precisely in case (2.34) holds and W is quasi-homogeneous.

The twisted superpotential violates R-invariance unless it takes the linear form that leads to what we have called $L_{D,\theta}$. Therefore, in the R-invariant case, the quantum corrections cannot generate a more complicated twisted chiral superpotential.

3. The C–Y/L–G correspondence

We now turn to our problem of shedding some light on the correspondence between Calabi–Yau and Landau–Ginzburg models. In doing so, we will consider only the simplest situation, with a gauge group that is simply $\mathrm{G} = \mathrm{U}(1)$, and thus a single vector superfield V. The generalizations will be relegated to sect. 4.

Happily, of all the formulas of sect. 2, we primarily need only a few of the simpler ones. The fields D and F_i enter in the lagrangian (2.17) as "auxiliary fields", without kinetic energy. One can solve for them by their equations of motion to get

$$D = -e^2 \left(\sum_i Q_i |\phi_i|^2 - r \right),$$
$$F_i = \frac{\partial W}{\partial \phi_i}. \tag{3.1}$$

The potential energy for the dynamical scalar fields ϕ_i, σ can then be written

$$U(\phi_i, \sigma) = \frac{1}{2e^2} D^2 + \sum_i |F_i|^2 + 2\bar{\sigma}\sigma \sum_i {Q_i}^2 |\phi_i|^2. \tag{3.2}$$

3.1. THE MODEL

Now, let me explain the situation on which we will focus. We take the chiral superfields Φ_i to be n fields S_i of charge 1, and one field P of charge $-n$. This obviously ensures condition (2.34) for anomaly cancellation and R-invariance. Lower-case letters s_i and p will denote the bosonic fields in the supermultiplets S_i and P. We take the superpotential to be the gauge invariant function

$$W = P \cdot G(S_1, \dots, S_n), \tag{3.3}$$

with G a homogeneous polynomial of degree n. Notice that this W is quasi-homogeneous; one can pick the k_i in (2.35) to be -1 for P and 0 for S_i or (by adding a multiple of the gauge generator) 0 for P and $-1/n$ for S_i. The superpotentials we consider later in this paper will always be quasi-homogeneous in a similar obvious way; this will not be spelled out in detail in subsequent examples.

We want to choose G to be "transverse" in the sense that the equations

$$0 = \frac{\partial G}{\partial S_1} = \dots = \frac{\partial G}{\partial S_n} \tag{3.4}$$

have no common root except at $S_i = 0$. This property ensures that if we think of the S_i as homogeneous variables, then the hypersurface X in $\mathbb{CP}^{n-1}$ defined by the equation $G = 0$ is smooth. A generic homogeneous polynomial has this property. An analogous transversality condition enters in each of our later models.

For the model just introduced, the bosonic potential (3.2) is

$$U = |G(s_i)|^2 + |p|^2 \sum_i \left|\frac{\partial G}{\partial s_i}\right|^2 + \frac{1}{2e^2} D^2 + 2|\sigma|^2 \left(\sum_i |s_i|^2 + n^2 |p|^2 \right) \tag{3.5}$$

with

$$D = -e^2 \left(\sum_i \bar{s}_i s_i - n\bar{p}p - r \right). \tag{3.6}$$

Now, let us discuss the low-energy physics for various values of r. First, we take $r \gg 0$. In this case, obtaining $D = 0$ requires that the s_i cannot all vanish. That being so, vanishing of the term $|p|^2 \sum_i |\partial_i G|^2$ in the potential requires (in view of transversality of G) that $p = 0$. Vanishing of D therefore gives us precisely

$$\sum_i \bar{s}_i s_i = r. \tag{3.7}$$

Dividing the space of solutions of (3.7) by the gauge group U(1), we get precisely a copy of complex projective space $\mathbb{CP}^{n-1}$, with Kähler class proportional to r. Finally, we must set $G = 0$ to ensure the vanishing of the first term in the formula (3.5) for the potential, and $\sigma = 0$ to ensure the vanishing of the last term.

So the space of classical vacua is isomorphic to the hypersurface $\mathrm{X} \subset \mathbb{CP}^{n-1}$ defined by $G = 0$. It is a well-known fact that a smooth hypersurface of degree k in $\mathbb{CP}^{n-1}$ is a Calabi–Yau manifold if and only if $k = n$. We have not accidentally stumbled upon the Calabi–Yau condition. We picked P to be of charge $-n$, and hence G of degree n, to ensure the condition (2.34) for anomaly-free R-invariance. Anomaly free R-invariance of the underlying model ensures such invariance of the effective low-energy sigma model; but for sigma models, anomaly free R-invariance is equivalent to the Calabi–Yau condition.

All modes other than oscillations tangent to X have masses at tree level. (The gauge field v gets a mass from the Higgs mechanism; masses for p, σ, and modes of s_i not tangent to X are visible in (3.4).) The low-energy theory therefore is a sigma model with target space X and Kähler class proportional to r.

Now we consider the case of $r \ll 0$. In this case, the vanishing of D requires $p \neq 0$. The vanishing of $|p|^2 \sum_i |\partial_i G|^2$ then requires (given transversality of G) that all $s_i = 0$. This being so, the modulus of p must in fact be $|p| = \sqrt{-r/n}$. By a gauge transformation, one can fix the argument of p to vanish. So the theory has a unique classical vacuum, up to gauge transformation. In expanding around this vacuum, the s_i are massless (for $n \geqslant 3$). These massless fields are governed by an effective superpotential that can be determined by integrating out the massive field p; integrating out p simply means in this case setting p to its expectation value. So the effective superpotential of the low energy theory is $\widetilde{W} = \sqrt{-r} \cdot W(s_i)$. The factor of $\sqrt{-r}$ is inessential, as it can be absorbed in rescaling the s_i. This effective superpotential has (for $n \geqslant 3$) a degenerate critical point at the origin, where it vanishes up to nth order. A theory with a unique classical vacuum state governed by a superpotential with a degenerate critical point is usually called a Landau–Ginzburg theory.

In this case, we actually get a Landau–Ginzburg orbifold, that is an orbifold of a Landau–Ginzburg theory, for the following reason. The vacuum expectation value of p does not completely destroy the gauge invariance; rather, it breaks U(1) down to a $\mathbb{Z}_n$ subgroup E that acts by $s_i \to \zeta s_i$, where ζ is an nth root of 1. This residual gauge invariance means that what we get is actually a $\mathbb{Z}_n$ orbifold of a Landau–Ginzburg theory. This follows from a general and elementary though perhaps not universally recognized fact; orbifolds (as the term is usually used in quantum field theory) are equivalent to theories with a finite gauge group.

In fact, in calculating the path integral on a Riemann surface Σ, instead of expanding about the absolute minimum of the action at $p = \sqrt{-r}$, $s_i = v = 0$, and gauge transformations thereof, we can expand around configurations that are gauge equivalent to $p = \sqrt{-r}$, $s_i = v = 0$ only locally. In such configurations, the gauge fields v may have monodromies. As the monodromies must leave p invariant, they take values in the group E of nth roots of unity. The construction of the low-energy approximation to the path integral involves a sum over the possible E-valued monodromies. But this sum over monodromies is precisely the usual operation of summing over E-twists in all channels by which one constructs the quantum field theory of the E-orbifold. Thus, in general, orbifolds can be regarded as a special case of gauge theories in which the gauge group is a finite group.

Of course, both for $r \gg 0$ and for $r \ll 0$, the integration over the massive fields to get an effective theory for the massless fields cannot just be done classically; we must consider quantum corrections to the sigma model or Landau–Ginzburg orbifold obtained above. We will look at this more closely in subsect. 3.2; suffice it to say here that in the R-invariant case, for sufficiently big $|r|$, integrating out the massive fields just corrects the parameters in the effective low-energy theory without a qualitative change in the nature of the system.

The moral of the story. The above construction suggests that, rather than Landau–Ginzburg being "equivalent" to Calabi–Yau, they are two different phases of the same system. In fact, from this point of view Landau–Ginzburg looks like the analytic continuation of Calabi–Yau to negative Kähler class. We will re-examine the moral of our story after some further analysis.

3.2. THE "SINGULARITY"

To extract any precise conclusions about the relation between the Landau–Ginzburg and Calabi–Yau models, we will have to deal with the apparent singularity at $r = 0$ that separates them.

The problem is not that there might be a phase transition at $r = 0$. In our applications, we are interested either in quantizing the theory on a circle (compact, finite volume), or on performing path integrals on a compact Riemann surface. Either way, we will not see the usual singularities associated with phase transitions; those depend crucially on having infinite world-sheet volume. The only singularities we might see would be due to failure of effective compactness of the target space.

For instance, if the σ field were absent (we will see in sect. 6 that it can be suppressed preserving $(0,2)$ supersymmetry), there could be no singularity at $r = 0$ in finite world-sheet volume. For without the σ field, the only possible zero of the classical bosonic potential at $r = 0$ would be at $s_i = p = 0$. Moreover, the locus in which the classical potential is less than any given number T would likewise be compact. These conditions mean that in quantization on a circle, quantum states of any given energy decay exponentially at large values of the fields, so that the spectrum is discrete. Such a discrete spectrum varies continuously – though perhaps in a difficult-to-calculate fashion – with the parameters.

The σ field changes the picture even for $r \neq 0$. Though the space of classical vacua is compact for any r, and was determined above, the space on which the classical potential is less than some value T is compact only for T small enough. In fact, for $s_i = p = 0$, the classical potential has the constant value $D^2/2e^2 = e^2r^2/2$, independent of σ. This means that only the low-lying part of the spectrum – at energies such that the region at infinity is inaccessible – is discrete. If the theory is quantized on a circle of radius R, then at energies above some critical T, the theory has a continuous spectrum, coming from wave functions supported near $s_i = p = 0, |\sigma| \gg 0$. Semi-classically, the critical value of T appears to be

$$T_{\rm cr} = \frac{e^2r^2}{2} \cdot 2\pi R. \tag{3.8}$$

Before going on, let us assess the reliability of this estimate. The non-zero value of T came from the expectation value of the D field in the region s_i, p near zero, $|\sigma| \gg 0$. In that region, the chiral superfields, including s_i, p and their superpartners, have masses proportional to $|\sigma|$; the gauge multiplet (σ, v, and λ) is massless. If one simply sets s_i and p to zero, the gauge multiplet is described by a free field theory and (except for a subtlety about the θ angle that we come to later) there are no quantum corrections at all to the classical value (3.8). Naively, integrating out fields with masses proportional to $|\sigma|$ will give only corrections proportional to negative powers of $|\sigma|$ and should not change that conclusion. This will be so if the ultraviolet behavior is good enough.

To see what really happens, consider a general model with chiral superfields Φ_i of charge Q_i. The one-loop correction to the vacuum expectation value of

$-D/e^2 = \sum_i Q_i|\phi_i|^2 - r$ will be

$$\delta\left\langle -\frac{D}{e^2}\right\rangle = \sum_i Q_i \int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{k^2+2|\sigma|^2+\ldots}. \tag{3.9}$$

(The ... are σ-independent contributions to the masses.) This vanishes for large σ if

$$\sum_i Q_i = 0. \tag{3.10}$$

If (3.10) is not obeyed, then (3.9) diverges. To renormalize the theory, we subtract the value of δD at, say, $|\sigma| = \mu$. The subtraction is interpreted as a redefinition of the parameter r. After the subtraction, one has for the large σ behavior

$$\begin{aligned}\delta\left\langle -\frac{D}{e^2}\right\rangle &= \sum_i Q_i \int \frac{\mathrm{d}^2k}{(2\pi)^2}\left(\frac{1}{k^2+2|\sigma|^2} - \frac{1}{k^2+2\mu^2}\right)\\ &= \sum_i Q_i \cdot \left(\frac{\ln(\mu/|\sigma|)}{2\pi}\right).\end{aligned} \tag{3.11}$$

This formula diverges for $|\sigma| \to \infty$. (The formula also diverges for σ near 0, but this effect is misleading; for σ near 0, the point $s_i = p = 0$ about which we expanded is unstable and the approximations have been inappropriate.) The divergence means that quantum corrections are large in this region of field space if $\sum_i Q_i \neq 0$, and crucial properties cannot be read off from the classical lagrangian. The correction to D can be interpreted in terms of a σ-dependent effective value of r,

$$r_{\text{eff}} \sim r + \frac{1}{2\pi}\sum_i Q_i \cdot \ln(|\sigma|/\mu). \tag{3.12}$$

One may wonder what superspace interaction can have this σ-dependent effective value of r as one of its consequences. Looking back to equation (2.33), the answer is clear. We have generated a twisted chiral superpotential, of precisely the form considered in (2.31), with

$$4\pi p = \sum_i Q_i \tag{3.13}$$

– in agreement with our general observation that $4\pi p$ must be an integer. (In the case of the $\mathbb{CP}^{n-1}$ model, to which we will specialize later, this effect was first computed from a different viewpoint by D'Adda et al. [21]. The general formula has also been conjectured [22].) I leave it to the curious reader to verify, by further computations in the large-σ region, the σ-dependent effective value of θ that is also predicted by this twisted chiral superpotential.

Before discussing the consequences of this computation, let us assess its relevance. Is the specific quantum correction that we just evaluated the only one that matters? In fact, for large σ, this theory is a theory of massless free fields weakly coupled by superrenormalizable interactions to fields of mass proportional to σ. Perturbation theory will be uniformly valid (and in fact increasingly good) for large σ, unless there are ultraviolet divergences. (Such divergences can be translated upon renormalization into effects that grow for large σ, as we have just seen.) Because of the superrenormalizability of these models, and the non-renormalization theorems of the superpotential, the only divergent quantum correction is the one-loop renormalization of r that we have just encountered. (A roughly analogous fact in four-dimensional renormalizable supersymmetric theories is that the only quadratic divergence is the one-loop renormalization of r, again proportional to $\sum_i Q_i$.) That is why this is the important quantum correction.

$\sum_i Q_i = 0$. Let us now discuss the consequences of the above computation. First we assume that $\sum_i Q_i = 0$. In that case, for large σ we can simply throw away the chiral superfields, with their masses of order $|\sigma|$. We still have to look at the effective theory of the massless gauge multiplet; this multiplet has the effective lagrangian $L_{\text{gauge}} + L_{D,\theta}$. This effective theory is a free theory, and supersymmetry ensures that the usual quantum corrections to the ground-state energy from zero-point fluctuations in σ and λ cancel. But for the gauge field v, an important subtlety arises. The lagrangian for v is

$$L = \int_\Sigma d^2y \left(\frac{1}{2e^2} v_{01}^2 + \frac{\theta}{2\pi} v_{01} \right). \tag{3.14}$$

The significance of the θ term in abelian gauge theory in two dimensions was explained long ago [23]. It induces a constant electric field in the vacuum, equal to the electric field due to a charge of strength $\theta/2\pi$. This constant electric field contributes to the energy of the vacuum. This contribution has the value that one might guess by simply minimizing the lagrangian with respect to v_{01}, namely $(e^2/2) \cdot (\theta/2\pi)^2$. More exactly, this is so as long as $|\theta| \leqslant \pi$. Otherwise, by a process involving pair creation, the contribution to the energy can be reduced to the minimum value of $(e^2/2) \cdot ((\theta/2\pi) - n)^2$ for $n \in \mathbb{Z}$. The contribution of the theta term to the vacuum energy density is thus in general

$$\frac{e^2}{2} \left(\frac{\widetilde{\theta}}{2\pi} \right)^2 \tag{3.15}$$

where $|\widetilde{\theta}| \leqslant \pi$ and $\widetilde{\theta} - \theta \in 2\pi\mathbb{Z}$. In particular the vacuum energy density is a continuous and periodic but not smooth function of θ with period 2π.

We thus can determine the *exact*, quantum-corrected analog of (3.8): the minimum energy density of a quantum state at large σ is

$$U = \frac{e^2}{2}\left(r^2 + \left(\frac{\tilde{\theta}}{2\pi}\right)^2\right). \tag{3.16}$$

The minimum energy of a quantum state obtained in quantizing the gauge multiplet on a circle of circumference $2\pi R$ is

$$T'_{\text{cr}} = U \cdot 2\pi R. \tag{3.17}$$

The quantum energy spectrum is discrete and continuously varying for energies below T'_{cr}, for at such energies the target space is effectively compact, the region of large σ being inaccessible. As long as either r or θ is non-zero, T'_{cr} is non-zero, and there is a range of energies with such a discrete spectrum.

Another statement along a similar lines is important in some of our applications. Instead of considering the energy H, one could consider $H + P$, P being the momentum. Since the vacuum has $P = 0$ and particle excitations contribute a non-negative amount to $H + P$, the conclusion is the same: the spectrum is discrete for $H + P < T'_{\text{cr}}$.

In all of our applications, when we try to actually learn something from the relation between Landau–Ginzburg and Calabi–Yau models, we will use some sort of generalized topological reasoning. In the simplest case, we consider the index of one of the supercharges (equal to $\text{Tr}(-1)^F$ or the elliptic genus, as explained below). Given a discrete and continuously varying spectrum for H (or $H + P$) below some positive constant, these indices can be computed by counting the states of $H = 0$ (or $H + P = 0$), and in particular are well-defined and invariant under deformation. Therefore, these quantities are independent of r and θ except at $r = \theta = 0$, and we can interpolate safely from $r > 0$ to $r < 0$ by taking $\theta \neq 0$.

We will also consider more delicate quantities involving couplings of states of $H = 0$ (or $H + P = 0$). Some of these quantities, related to Yukawa couplings, for instance, are predicted to be independent of r, θ, while others are predicted to vary holomorphically in t. These predictions depend on arguments (see ref. [24], for instance) that rest ultimately on integration by parts in field space. These arguments might fail if the integrand of the path integral were to behave badly for $\sigma \to \infty$. However, as long as $T'_{\text{cr}} \neq 0$, wave functionals of states of $H = 0$ (or $H + P = 0$) vanish exponentially for $\sigma \to \infty$. Path-integral representations of couplings of such states are then highly convergent, and integration by parts in field space is justified.

In sum, then, the only real singularity, for the topological quantities in finite volume, is at $r = \theta = 0$. By taking $\theta \neq 0$, we can continue happily from Calabi–Yau to Landau–Ginzburg.

What happens if $\sum_i Q_i \neq 0$. Now we want to analyze what happens if

$\sum_i Q_i \neq 0$.

In our classical lagrangian, the twisted superpotential for the gauge multiplet Σ was simply a linear function $\widetilde{W}(\Sigma) = it\Sigma/2\sqrt{2}$. One might suspect that just as an ordinary W leads to a bosonic potential $\sum_i |\partial_i W|^2$, a twisted superpotential would lead to a potential energy

$$\left|\frac{\partial \widetilde{W}(\sigma)}{\partial \sigma}\right|^2 \tag{3.18}$$

This is almost true, except for a factor that comes from the normalization we used for the gauge multiplet. Indeed, the potential energy in (3.16) is $U = 4e^2|\widetilde{W}_Q{}'|^2$ with $\widetilde{W}_Q(x) = i\widehat{t}x/2\sqrt{2}$; here $\widehat{t} = t + n$, with $n \in \mathbb{Z}$ chosen to minimize U. Notice that the mechanism for appearance of this (almost) $|\widetilde{W}'|^2$ term in the energy is rather subtle; it arises partly from integrating out the D auxiliary field and partly from the effects of the θ angle.

The formula (3.18) is still valid, for the same reasons (except for the same modifications), for computing the potential energy induced by the more general twisted superpotential

$$\widetilde{W} = \frac{1}{\sqrt{2}}\left(\frac{it\Sigma}{2} - \frac{\sum_i Q_i}{2\pi}\Sigma \ln(\Sigma/\mu)\right) \tag{3.19}$$

that incorporates the quantum correction. After all, for large σ we simply have a theory with a weak σ dependence in the effective values of r and θ, and we can compute the dependence of the energy on r and θ just as we did when these were strictly constant. So the effective potential for σ is just

$$U(\sigma) = \frac{e^2}{2}\left|i\widehat{t} - \sum_i \frac{Q_i}{2\pi}\left(\ln(\sigma/\mu) + 1\right)\right|^2 . \tag{3.20}$$

Again $\widehat{t} = t + n$, where $n \in \mathbb{Z}$ is to be chosen to minimize $U(\sigma)$. $U(\sigma)$ is therefore a continuous but not smooth function of σ and t.

Now we can see that our analysis of the correspondence between sigma models and Landau–Ginzburg models must be modified when $\sum_i Q_i \neq 0$ because the semi-classical determination of the vacuum structure does not apply. For $\sum_i Q_i > 0$, there are new vacuum states, not predicted in the semi-classical reasoning, for $r \ll 0$; and for $\sum_i Q_i < 0$, there are such new states for $r \gg 0$. These new ground states are determined by

$$i\widehat{t} - \frac{\sum_i Q_i}{2\pi}\left(\ln(\sigma/\mu) + 1\right) = 0 \tag{3.21}$$

and there are precisely $|\sum_i Q_i|$ of them. The new solutions are explicitly

$$\sigma = \frac{\mu}{e}\exp\left(\frac{2\pi i\widehat{t}}{\sum_i Q_i}\right); \tag{3.22}$$

there are $|\sum_i Q_i|$ of them because of the freedom $\hat{t} \to \hat{t} + n$. Of course, the only relevant solutions of (3.21) are those that arise for large σ, where the approximations are valid; this is why we need r large and of the appropriate sign.

Eq. (3.20) grows at infinity like $|\ln \sigma|^2$. This ensures, in finite world-sheet volume, that the model has no singularity at any r or θ, as long as $\sum_i Q_i \neq 0$*. Thus, a generalization of the sigma model/L–G correspondence, taking account the extra states at infinity, holds even when $\sum_i Q_i \neq 0$.

For illustration, let us see how the new states enter in the $\mathbb{C}\mathrm{P}^{n-1}$ model. We consider a model with gauge group U(1) and n chiral superfields S_i, all of charge 1. Gauge invariance requires the superpotential to be $W = 0$. The classical potential is simply

$$U(s_i, \sigma) = \frac{e^2}{2}\left(\sum_i \bar{s}_i s_i - r\right)^2 + 2|\sigma|^2 \sum_i |s_i|^2. \tag{3.23}$$

For $r \gg 0$, a classical vacuum must have $\sigma = 0$, $\sum_i |s_i|^2 = r$. The space of such vacua, up to gauge transformation, is a copy of $\mathbb{C}\mathrm{P}^{n-1}$ with Kähler class proportional to r, and the model at low energies is a sigma model with this target space. For $r \ll 0$, it appears on the other hand that supersymmetry is spontaneously broken. This poses a problem. For $r \gg 0$, the supersymmetric index $\mathrm{Tr}(-1)^F$ of the theory** is equal to n (the Euler characteristic of $\mathbb{C}\mathrm{P}^{n-1}$), while spontaneous supersymmetry breaking would mean that the index would be 0 for $r \ll 0$. Given the $|\ln \sigma|^2$ behavior of the potential at infinity, vacuum states cannot flow to or from infinity, and the index should be constant. To solve the problem, we need only note that $\sum_i Q_i = n$. So the expected n zero-energy states are found by setting $s_i = 0$ and σ to a solution of (3.21).

Another continuation of the $\mathbb{C}\mathrm{P}^{n-1}$ sigma model to negative r was discussed recently [25].

The moral, revisited. Our arguments have been sensitive only to singularities crude enough as to be visible even in finite world-sheet volume. What is the phase structure in infinite world-sheet volume? Are Calabi–Yau and Landau–Ginzburg separated by a true phase transition, at or near $r = 0$?

There is no reason that the answer to this question has to be "universal," that is, independent of the path one follows in interpolating from Calabi–Yau to Landau–Ginzburg in a multiparameter space of not necessarily conformally

* This is in accord with the fact that for the A model or the half-twisted model, introduced in subsect. 3.3, the instanton sums are always finite when $\sum_i Q_i \neq 0$ because the anomalous R-symmetry ensures that any given amplitude receives contributions only from finitely many values of the instanton numbers. When $\sum_i Q_i = 0$, the instanton sums are infinite sums that can have singularities; we will see an example in subsect. 5.5.

** Its definition is well known and is given below.

invariant theories. Along a suitable path, there may well be a sharply defined phase transition, while along another path there might not be one.

The situation would then be similar to the relation between the gas and "liquid" phases of a fluid. We would have one system which in one limit is well described as Calabi–Yau; in another limit it is well described by Landau–Ginzburg. But these would be two different limits of the same system, with no order parameter or inescapable singularity separating them; and so one might say (as is customary) that Landau–Ginzburg and Calabi–Yau are "equivalent".

It is believed that the theories we are studying all flow in the infrared to conformal field theories. If so, a particularly interesting type of interpolation from Landau–Ginzburg to Calabi–Yau would be via conformally invariant theories. Is there a continuous family of conformal field theories interpolating from Landau–Ginzburg to Calabi–Yau? This is the only way that Landau–Ginzburg and Calabi–Yau could truly be "equivalent" as conformal field theories. In the case of topology change (which we will see to be analogous in subsect. 5.5), mirror symmetry gives [7] a direct indication that the answer is "Yes". In the present context, we can attempt to argue (not rigorously) as follows. First of all, any singularity or discontinuity of a family of conformal field theories that holds on the real line must already hold on the circle, since conformally the real line is just the circle with a point deleted. Now, in the above, for linear sigma models we identified all of the singularities that occur in finite volume: the only such singularity is at $r = \theta = 0$. The key point to avoid a singularity was the non-vanishing of the large-σ minimum energy density U, given in (3.16). This was an exact formula, not depending on small e. To approach the infrared limit, we should scale $e \to$ infinity, and it would appear that U only grows in this limit. The potential trouble is that the definition of U involved a large-σ limit which might not commute with the large-e limit.

In any event, semi-classically, just like liquid and gas, Calabi–Yau and Landau–Ginzburg look like different "phases," and I will use that convenient, informal language.

3.3. APPLICATIONS

Whether r is positive or negative, the linear sigma models investigated here are super-renormalizable rather than conformally invariant. Even classically, to reduce the $r \gg 0$ theory to a Calabi–Yau nonlinear sigma model, or to reduce the $r \ll 0$ theory to a Landau–Ginzburg model, we have to take the limit in which e (the gauge coupling) goes to infinity.

But our arguments depended on having $e < \infty$. Moreover, $e \ll 1$ is really the region in which the model is under effective control, the semi-classical arguments being reliable.

If we do take $e \to \infty$ with $r \gg 0$, we will get a Calabi–Yau non-linear sigma model with target a hypersurface X. The Kähler metric g on X (along with the corresponding Kähler form ω) is induced from the embedding in $\mathbb{C}\mathrm{P}^{n-1}$ (with its standard metric). It is known that this metric on X does *not* give a conformally invariant model. It is believed that there is (for large enough r) a unique Kähler metric g' on X, with a Kähler form ω' cohomologous to ω, that gives a conformally invariant sigma model. The two models with metric g and g' differ by a superspace interaction of the form

$$\int \mathrm{d}^2x\mathrm{d}^4\theta \; T, \tag{3.24}$$

where the function T obeys $\omega - \omega' = -i\partial\overline{\partial}T$.

Consequently, even if we set $e = \infty$, the linear sigma model considered here is not expected to coincide with the conformally invariant nonlinear sigma model. They differ by the term (3.24), and moreover the function T is unknown. So how can we learn anything relevant to the conformally invariant model?

One answer is that the T coupling, like the effects of finite e, are believed to be "irrelevant" in the technical sense of the renormalization group; that is, these effects are expected to disappear in the infrared. This is of theoretical importance – for instance it enters in heuristic discussions of the Landau–Ginzburg/Calabi–Yau correspondence – but difficult to use as a basis for calculation.

A more practical answer depends on the fact that $N = 2$ theories in two dimensions are richly endowed with quantities that are invariant under adding to the lagrangian a term of the form $\int \mathrm{d}^4\theta \, (\ldots)$ (with ... a *gauge invariant* operator). Such quantities include $\mathrm{Tr}(-1)^F$, the elliptic genus, and the observables of two twisted topological field theories (the A model and the B model), and of a so far little studied hybrid, the half-twisted model. Let us call these the quasi-topological quantities. Any such quantities are invariant under the addition of a term such as (3.24) to the lagrangian, so they can be computed using the "wrong" Kähler metric on X.

Moreover, once one is reconciled to studying primarily the quasi-topological quantities, it is easy to see that our problem with the e dependence can be resolved in essentially the same way. The e-dependent term in the lagrangian is

$$-\frac{1}{4e^2}\int \mathrm{d}^2y\mathrm{d}^4\theta \; \overline{\Sigma}\Sigma, \tag{3.25}$$

so the quasi-topological quantities are independent of e. The dependence on the Fayet–Iliopoulos and theta terms can be similarly analyzed, with the result that some quasi-topological terms are invariant under these couplings and others vary holomorphically in t.

Indices. The simplest quasi-topological quantities are simply the indices of various operators (in canonical quantization on a circle). Let H and P be, as above, the hamiltonian and momentum of the theory, and let $H_\pm = (H \pm P)/2$. It is possible to find a linear combination Q of the supercharges $Q_\pm$ and $\overline{Q}_\pm$ such that Q is self-adjoint and $Q^2 = H$. Let $(-1)^F$ be the operator that counts the number of fermions modulo two, and so anti-commutes with Q. The object $\mathrm{Tr}(-1)^F \mathrm{e}^{-\beta H}$ is independent of β by standard arguments (involving a pairing of states at non-zero H that comes from multiplication by Q). This quantity is the index of Q (or more properly of the piece of Q that maps states even under $(-1)^F$ to odd states). It is usually written simply

$$\mathrm{Tr}(-1)^F. \tag{3.26}$$

This index is independent of r and θ (as long as we keep away from the singularity at $r = \theta = 0$) because of the discrete spectrum for H below a critical value; so it has the same value for Landau–Ginzburg as for Calabi–Yau.

A similar but much more refined invariant can be constructed using the left- and right-moving R-symmetry of the theory. Let $W_\mathrm{L} = \oint J_\mathrm{L}$ and $W_\mathrm{R} = \oint J_\mathrm{R}$ be the R charges; and let

$$\begin{aligned}(-1)^{F_\mathrm{R}} &= \exp(i\pi W_\mathrm{R}),\\ (-1)^{F_\mathrm{L}} &= \exp(i\pi W_\mathrm{L}).\end{aligned} \tag{3.27}$$

Then $(-1)^F = (-1)^{F_\mathrm{L}}(-1)^{F_\mathrm{R}}$. Let $Q_\mathrm{R} = (Q_+ + \overline{Q}_+)/2$, so Q_R anticommutes with $(-1)^{F_\mathrm{R}}$ and commutes with $(-1)^{F_\mathrm{L}}$, and ${Q_\mathrm{R}}^2 = H_+$. By a standard argument (a pairing of states at $H_+ \neq 0$ that comes from multiplication by Q_R), the quantity

$$\mathrm{Tr}(-1)^{F_\mathrm{R}} q^{H_-} \mathrm{e}^{i\theta W_\mathrm{L}} \exp(-\beta H_+) \tag{3.28}$$

is independent of β; we denote this as

$$F(q,\theta) = \mathrm{Tr}(-1)^{F_\mathrm{R}} q^{H_-} \mathrm{e}^{i\theta W_\mathrm{L}}. \tag{3.29}$$

This quantity, which can be interpreted as the index of Q_R, is essentially the elliptic genus of the model [26–28]; it has interesting modular properties that can be established from its path-integral representation. Given the discrete spectrum for H_+ below a critical value, $F(q,\theta)$ is invariant under variation of supersymmetric parameters, so it is a topological invariant and moreover has the same value for Calabi–Yau and Landau–Ginzburg. (In the Landau–Ginzburg case, this function has not been previously studied; it has quite interesting properties explored in ref. [54].)

In case $\sum_i Q_i$ is even but not zero, $F(q,\theta)$ is still an invariant provided we restrict θ to integer multiples of $2\pi\left(\sum_i Q_i\right)^{-1}$ (corresponding to anomaly-free discrete R symmetries) and provided we take due account of the vacuum states at large σ. The even-ness of $\sum_i Q_i$ is needed to ensure that $(-1)^{F_\mathrm{R}}$ is not anomalous; at low energies, it results in the target space of the $r \gg 0$ sigma model being a spin manifold, a natural requirement in defining the elliptic genus.

Continuation to euclidean world-sheet; twisting. So far in this paper, the two-dimensional world-sheet has had a lorentzian signature. For our remaining applications, a euclidean world-sheet is more natural. We make the analytic continuation to a euclidean signature world-sheet by setting $y^0 = -iy^2$; the line element $\mathrm{d}s^2 = -(\mathrm{d}y^0)^2 + (\mathrm{d}y^1)^2$ becomes $\mathrm{d}s^2 = (\mathrm{d}y^1)^2 + (\mathrm{d}y^2)^2$.

These linear sigma models not only are not topologically invariant; they are not even conformally invariant, except (putatively) in the infrared limit. However, as for any $N = 2$ supersymmetric models in two dimensions, topological field theories can be constructed by a simple twisting procedure [29,24,30]. This involves substituting the stress tensor by

$$T_{\alpha\beta} \to T_{\alpha\beta} + \tfrac{1}{4}\left(\epsilon_\alpha{}^\gamma \partial_\gamma (J_{\mathrm{R}\beta} \pm J_{\mathrm{L}\beta}) + \alpha \leftrightarrow \beta\right). \tag{3.30}$$

One obtains two models, depending on the choice of sign. For the + and − sign respectively one gets the so-called B model and A model. The B model is anomalous unless $\sum_i Q_i = 0$, so our discussion of it is limited to that case. There is no such restriction for the A model (but the A model has very special properties when $\sum_i Q_i = 0$, for only then the instanton sums are infinite).

The point of the twisting procedure is that certain supercharges come to have Lorentz spin zero if Lorentz transformations are defined using the modified stress tensor. These are $\overline{Q}_+$ and $\overline{Q}_-$ for the B model, and Q_- and $\overline{Q}_+$ for the A model. As these quantities are Lorentz invariant, they remain symmetries when the models are formulated on an arbitrary Riemann surface Σ of genus g. Thus either the A or the B model, formulated on any surface Σ, has a $(0|2)$-dimensional supergroup F of symmetries.

Let $Q = \overline{Q}_+ + \overline{Q}_-$ or $Q = Q_- + \overline{Q}_+$ for the B or A model; then in either case $Q^2 = 0$. It is natural to try to think of Q as a sort of BRST operator, considering physical states to be the cohomology classes of Q and the observables to be correlation functions of Q invariant vertex operators. One can show, as in classical Hodge theory, that BRST cohomology classes have F-invariant representatives; this leads to some useful simplifications.

In either of the twisted models, one has $T_{\alpha\beta} = \{Q, \Lambda_{\alpha\beta}\}$ for some $\Lambda_{\alpha\beta}$; this condition ensures that the correlation functions of Q-invariant operators are independent of the metric. So the twisted models become topological field theories; we call their observables the topological correlation functions. As explained in detail in ref. [24], the topological observables of the A and B models include as special cases the *ac* and *cc* chiral rings of the untwisted model, which determine the low-energy Yukawa couplings of the string compactification defined by the untwisted model.

Any perturbation of the lagrangian of the form $\int \mathrm{d}^2y\, \mathrm{d}^4\theta \ldots$ can be written as $\{Q, \ldots\}$, so the topological observables are invariant under such perturbations. In particular, the topological observables are independent of the gauge coupling e. For other couplings the basic rule is as follows. In the B model, any term in the lagrangian that can be written as $\int d\overline{\theta}^{+} \ldots$ or $\int d\overline{\theta}^{-} \ldots$ is $\{Q, \ldots\}$, while in the A model this is true for any term that can be written as $\int d\theta^{-} \ldots$ or $\int d\overline{\theta}^{+} \ldots$.

Looking at the definition of the superpotential couplings,

$$L_W = \int d^2y d\theta^+ d\theta^- \; W(\Phi_i)|_{\overline{\theta}^+ = \overline{\theta}^- = 0} + \int d^2y d\overline{\theta}^- d\overline{\theta}^+ \; \overline{W}(\overline{\Phi})\Big|_{\theta^+ = \theta^- = 0}, \tag{3.31}$$

we see that the observables of the A model are independent of the superpotential while the observables of the B model vary holomorphically with W.

Similarly, recalling the definition of the Fayet–Iliopoulos and theta couplings,

$$L_{D,\theta} = \frac{it}{2\sqrt{2}} \int d^2y d\theta^+ d\overline{\theta}^- \; \Sigma|_{\theta^- = \overline{\theta}^+ = 0} - \frac{i\overline{t}}{2\sqrt{2}} \int d^2y d\theta^- d\overline{\theta}^+ \; \overline{\Sigma}\Big|_{\theta^+ = \overline{\theta}^- = 0}, \tag{3.32}$$

we see that the topological observables of the B model are independent of r and θ while those of the A model vary holomorphically in t.

The A model has a greatly enriched variant in which one ignores Q_- and considers $\overline{Q}_+$ as a BRST operator; thus physical states or vertex operators are cohomology classes of $\overline{Q}_+$. (The space of physical states of the half-twisted model is infinite dimensional; the elliptic genus (3.29) is essentially the index of the $\overline{Q}_+$ operator.) This variant of the A model is called the half-twisted model. The half-twisted model is not a topological field theory; if this model is formulated on a Riemann surface Σ, its correlation functions are conformally invariant (even if the untwisted model was not) and vary holomorphically with the complex structure of Σ. The correlation functions of the half-twisted model are invariant under change in e and vary holomorphically with both W and t, by essentially the above arguments.

Since the interpolation between Calabi–Yau and Landau–Ginzburg is obtained by varying t, we can draw some conclusions: the topological observables of the B model are the same for Calabi–Yau and Landau–Ginzburg; for the A and half-twisted models, Landau–Ginzburg is an analytic continuation of Calabi–Yau.

3.4. THE TWISTED MODELS IN DETAIL

We will now examine more closely the B and A models. (The half-twisted model can be considered together with the A model in what follows.) The basic tool is a sort of fixed-point theorem discussed in ref. [24], sect. 5. We recall that either the A or B model has a $(0|2)$-dimensional supergroup F of symmetries, with fermionic generators that we will call Q_σ. F-invariant representatives can be picked for all of the vertex operators, so F can be regarded as a symmetry of the path integral.

The evaluation of the path integral for topological correlation functions can be localized on the space Z of fixed points of F, that is the space of points in field space for which $\{Q_\sigma, A\} = 0$ for every field A and each σ. If the Q_σ have a simple zero along Z, then the path integral over modes normal to Z gives a simple factor of ± 1 due to cancellations between bosons and fermions. Otherwise, the path integral reduces to an integral over a finite-dimensional space of directions in field space in which the Q_σ vanish to higher than first order.

The B model. In the B model, by considering $\{\overline{Q}_\pm, \overline{\lambda}_\pm\} = 0$, one finds $D = v_{12} = 0$*. Let Φ_α be the chiral superfields with components $\phi_\alpha, \psi_\alpha, F_\alpha$ and charges Q_α. By setting $\{\overline{Q}_\pm, \overline{\psi}_{\alpha\pm}\} = 0$, one gets $F = 0$. From $\{\overline{Q}_\pm, \psi_{\alpha\pm}\} = 0$, one learns that $D\phi_\alpha / Dy^i = 0$, and that $Q_\alpha \phi_\alpha \sigma = 0$ for all α. For either $r \gg 0$ or $r \ll 0$, the equations $D = F_\alpha = 0$ require that the quantities $Q_\alpha \phi_\alpha$ are not all zero, so the last equation implies that $\sigma = 0$.

In sum then, for the B model, an F fixed point is simply a constant map from the world-sheet Σ to the space of classical vacua. For $r \gg 0$, the space of classical vacua is the target space X of the low-energy sigma model. In this case, the generators Q_σ have simple zeros along their space Z $\cong$ X of zeroes, so the path integral reduces to an integral over X. This integral can be analyzed as in refs. [24], sect. 4; correlation functions can be evaluated in terms of periods of differential forms on X. For $r \ll 0$, the space of classical vacua is a point, so the Q_σ have only one zero, but this zero is degenerate; the path integral reduces to a finite-dimensional integral analyzed by Vafa [31].

The A model. For the A model, the equations $\{Q_\sigma, \lambda_\pm\} = 0$ and $\{Q_\sigma, \overline{\lambda}_\pm\} = 0$ give

$$\begin{aligned} D + v_{12} &= 0\,, \\ \mathrm{d}\sigma &= 0\,. \end{aligned} \tag{3.33}$$

The equations $\{Q_\sigma, \psi_{\alpha\pm}\} = 0$ and $\{Q_\sigma, \overline{\psi}_{\alpha\pm}\} = 0$ similarly give

$$\begin{aligned} \left(\frac{D}{Dy^1} + i\frac{D}{Dy^2}\right)\phi_\alpha &= 0\,, \\ F_\alpha &= 0\,, \\ Q_\alpha \phi_\alpha \sigma &= 0\,. \end{aligned} \tag{3.34}$$

The last equation in (3.34) and the last equation in (3.33) imply, together, that $\sigma = 0$, since for the models of interest the other equations do not permit $Q_\alpha \phi_\alpha$ to vanish identically for each α. Altogether, the surviving equations are

$$\begin{aligned} \left(\frac{D}{Dy^1} + i\frac{D}{Dy^2}\right)\phi_\alpha &= 0\,, \\ F_\alpha &= 0\,, \\ D + v_{12} &= 0\,. \end{aligned} \tag{3.35}$$

* Recall the Wick rotation $y^0 \to -iy^2$, so $v_{01} \to -iv_{12}$.

From the identities

$$\int d^2y D_i\overline{\phi}_\alpha D^i\phi_\alpha = \int d^2y (D_1 - iD_2)\overline{\phi}_\alpha (D_1 + iD_2)\phi_\alpha - \int d^2y\, Q_\alpha \overline{\phi}_\alpha \phi_\alpha v_{12}, \tag{3.36}$$

$$\frac{1}{2e^2}\int d^2y({v_{12}}^2 + D^2) = \frac{1}{2e^2}\int d^2y(v_{12} + D)^2 - \frac{1}{e^2}\int d^2y\ Dv_{12}, \tag{3.37}$$

and $D = -e^2(\sum_\alpha Q_\alpha\overline{\phi}_\alpha\phi_\alpha - r)$, we find that if $D + v_{12} = (D_1 + iD_2)\phi_\alpha = 0$, then

$$\int d^2y \left(\sum_\alpha D_i\overline{\phi}_\alpha D^i\phi_\alpha + \frac{1}{2e^2}\left({v_{12}}^2 + D^2\right)\right) = -r\int d^2y\ v_{12}. \tag{3.38}$$

The left-hand side coincides with the bosonic part of the lagrangian of the theory, modulo $\sigma = F_\alpha = 0$, and apart from the theta term. So the action of a solution of (3.35) is

$$L = -2\pi itN, \tag{3.39}$$

where N is the instanton number

$$N = -\frac{1}{2\pi}\int d^2y\ v_{12}, \tag{3.40}$$

and $t = ir + \theta/2\pi$; the substitution of r by $-it$ takes account of the theta term in the action.

Since the left-hand side of (3.38) is positive definite, it follows that solutions of (3.35) necessarily have $N \geqslant 0$ if $r > 0$, and $N \leqslant 0$ if $r < 0$. The "instanton expansion" for the evaluation of any topological observable is therefore an expansion in instantons, of the general form

$$\sum_{k\geqslant 0} a_k \exp(2\pi ikt), \tag{3.41}$$

for $r \to \infty$, or an expansion in anti-instantons, of the general form

$$\sum_{k\geqslant 0} b_k \exp(-2\pi ikt), \tag{3.42}$$

for $r \to -\infty$.

A standard vanishing theorem (a line bundle of negative degree cannot have a non-zero holomorphic section) says that if $(D_1 + iD_2)\phi_\alpha = 0$, then

$$\phi_\alpha = 0 \quad \text{unless} \quad \text{sign}(Q_\alpha) = \text{sign}(N). \tag{3.43}$$

This follows from (3.36) if one uses the invariance of $(D_1 + iD_2)\phi = 0$ under complex gauge transformations (discussed below) to set v_{12} = constant.

Let us now look more closely at the structure of the instantons and anti-instantons, for the case relevant to the simplest form of the C–Y/L–G correspondence: the case in which there are n chiral superfields S_i of charge 1, and one chiral superfield P of charge $-n$, and the superpotential is $W = PG(S_i)$.

$r \ll 0$. The case of $r \ll 0$ is easier, so we consider it first. In view of (3.43), $s_i = 0$, but $p \neq 0$. Vanishing of s_i ensures that $F_\alpha = 0$. The remaining equations are

$$(D_1 + iD_2)p = 0,$$
$$v_{12} = e^2(-n|p|^2 - r). \tag{3.44}$$

These are the equations of the Nielsen–Olesen abelian vortex line, and the qualitative properties of the solutions are well known.

The p field is massive, with a mass proportional to $e\sqrt{r}$, and Compton wavelength $\lambda \sim 1/e\sqrt{r}$. Because the p field has charge $-n$, the solutions of (3.44) can have instanton number $-k/n$ for arbitrary integer $k \geqslant 0$. In an anti-instanton field of instanton number $-k/n$, the p field vanishes at k (generically distinct) points $x_1 \ldots x_k \in \Sigma$, and the energy density of the anti-instanton field is concentrated within a distance of order λ from those points. To a low energy observer, probing distances much longer than λ, the field looks like it is concentrated at k points; it looks like a superposition of k point anti-instantons each of instanton number $-1/n$.

Around one of the x_a, where the p field has a simple zero, it changes in phase by 2π. The s_i, of charge 1, therefore change in phase by $-2\pi/n$. Therefore, from the point of view of the low-energy effective theory of the massless fields s_i, it looks like "twist fields," about which the s_i change in phase by $-2\pi/n$, have been inserted at the k points x_a. Of course, in a global situation, on a compact Riemann surface Σ, the total change in phase of the s_i must be a multiple of 2π, so k must be divisible by n. This was to be expected; it is the standard quantization of the instanton number in the presence of fields of charge 1.

We see therefore the interpretation of the parameters r, θ in the Landau–Ginzburg theory of $r \ll 0$. There is a gas of twist fields, with a chemical potential such that an amplitude with k twist fields receives a factor of

$$\exp(-2\pi i t k/n). \tag{3.45}$$

For vacuum amplitudes, or amplitudes with insertions of ordinary (untwisted) fields, the number k of twist fields in the gas must be divisible by n. It is however known in the theory of the Landau–Ginzburg model that the A model has BRST invariant observables in the twisted sectors. A correlation function with, say, s twist fields receives contributions only from values of k such that $k+s$ is divisible by n. For given s, let k_0 be the smallest non-negative integer with $k_0 + s$ divisible by n. A correlation function $\langle T_1 \ldots T_s\rangle$ of s twist fields is then proportional for $r \to -\infty$ to

$$\exp(-2\pi i t k_0/n) \tag{3.46}$$

and in particular vanishes at $r = -\infty$ unless $k_0 = 0$. This means that at $r = -\infty$ there is a selection rule not present otherwise: $\langle T_1 \ldots T_s\rangle = 0$ unless s is divisible by n. This selection rule corresponds to a quantum $\mathbb{Z}_n$ symmetry that holds only at $r = -\infty$.

To recapitulate, let us emphasize that for $r \ll 0$, the low-energy theory is a Landau–Ginzburg theory, which does not have (smooth) instantons. The construction of smooth instanton solutions depends on the massive fields; the instanton solutions are smooth at a fundamental level, but look like point singularities in the low-energy theory.

The case $r \gg 0$. If conversely $r \gg 0$, we are dealing with instantons rather than anti-instantons; (3.43) forces $p = 0$. With the superpotential being $W = PG(S_i)$, the equations to be obeyed are

$$\begin{aligned}(D_1 + iD_2)s_i &= 0\,,\\ G(s_i) &= 0\,,\\ v_{12} &= e^2\left(\sum_i |s_i|^2 - r\right).\end{aligned} \tag{3.47}$$

We will presently analyze the import of these equations, but first, without any mathematical formalism, let me state the result that will arise. The low-energy theory for $r \gg 0$ is a sigma model with target space a Calabi–Yau hypersurface $\mathrm{X} \subset \mathbb{C}\mathrm{P}^{n-1}$. We should certainly expect to find the instantons of the sigma model, that is the holomorphic maps of Σ to X. However (just as for $r \ll 0$), we will also find additional instantons that are perfectly smooth objects at the fundamental level, with the massive fields present, but look like singular objects in the effective low-energy theory. These additional instantons have structure varying on a scale proportional to the Compton wavelengths of the massive fields (and other structure varying more smoothly).

The presence of these additional "singular" instantons means that the instanton expansion of the linear sigma model studied in this paper cannot be expected to coincide with the standard instanton expansion of the conformally invariant nonlinear sigma model to which it reduces at low energies. However, conventional renormalization theory tells us how they will be related. We should think about the effects of integrating out the point-like instanton structures (along with all other effects of the massive fields) to get an effective theory of the massless fields. This theory must be a sigma model of some kind, with some effective values of r and θ and some effective defining equation $G = 0$. (All other parameters are believed to be "irrelevant" in the sense of the renormalization group.) Since the A model observables do not depend on the particular polynomial G, the effect of integrating out the point instantons should be simply to change the effective values of r and θ. Thus, the instanton expansion of the linear sigma model should differ from the conventional instanton expansion of the nonlinear sigma model by a redefinition of the variable t.

Now I come to the mathematical analysis of (3.47). Obviously, these equations are invariant under gauge transformations

$$s_i \to e^{i\lambda} s_i, \quad v \to v - d\lambda, \tag{3.48}$$

with real λ. Actually, it is easy to see that the first two equations in (3.47) are invariant under (3.48) even if λ is *complex*. It is a very beautiful fact [32] that the last equation in (3.47) can be interpreted as a condition that fixes the complex gauge invariance of the first two equations. In other words, the space of solutions of the first two equations, up to a complex gauge transformation, is the same as the space of solutions of the set of three equations, up to a real gauge transformation. (This can be regarded as an infinite-dimensional analog of the relation between complex and symplectic quotients that will be summarized in sect. 4.)

So we can study simply the first two equations in (3.47), up to complex gauge transformations. To make the analysis particularly simple, we will assume also that the world-sheet Σ is of genus zero (the main case in standard applications of the A model). This leads to simplification because any complex line bundle on Σ is $\mathcal{O}(k)$ for some k. If u, v are homogeneous coordinates for $\Sigma \cong \mathbb{CP}^1$, then a holomorphic section of $\mathcal{O}(k)$ is just a holomorphic function of u, v that is homogeneous of degree k. I will later use the same name $\mathcal{O}(k)$ to denote the analogous line bundle over $\mathbb{CP}^{n-1}$.

Up to a complex gauge transformation, the first equation in (3.47) simply says that the s_i are holomorphic sections of $\mathcal{O}(k)$, k being the instanton number. Thus $s_i = s_i(u, v)$ are polynomials in u and v, homogeneous of degree k, explicitly

$$s_i(u,v) = s_{i,k}u^k + s_{i,k-1}u^{k-1}v + \ldots + s_{i,0}v^k. \tag{3.49}$$

The overall scaling

$$s_i(u,v) \to t s_i(u,v), \quad t \in \mathbb{C}^* \tag{3.50}$$

corresponds to the complex gauge transformations with constant gauge parameter; these are isomorphisms of $\mathcal{O}(k)$. The second equation in (3.47) says that

$$G(s_1(u,v), \ldots, s_n(u,v)) = 0. \tag{3.51}$$

The moduli space of solutions of equations (3.47) is the space of polynomials $s_1(u,v), \ldots, s_n(u,v)$ of degree k obeying (3.51), modulo (3.50).

Now let us consider the moduli space of the sigma model, that is the moduli space of degree-k holomorphic maps $\Phi : \Sigma \to X$, X being the hypersurface $G = 0$ in $\mathbb{CP}^{n-1}$. This sigma model moduli space is easy to describe. Φ having degree k means by definition that $\Phi^*(\mathcal{O}(1)) = \mathcal{O}(k)$. The homogeneous coordinates s_i of $\mathbb{CP}^{n-1}$ are holomorphic sections of $\mathcal{O}(1)$, so $\Phi^*(s_i)$ are holomorphic sections of $\mathcal{O}(k)$. This means that any degree-k holomorphic map $\Phi : \mathbb{CP}^1 \to \mathbb{CP}^{n-1}$ is of the form

$$(u,v) \to (s_1(u,v), \ldots, s_n(u,v)) \tag{3.52}$$

with degree-k polynomials $s_1(u,v),\dots s_n(u,v)$ which must obey (3.51). Since in $\mathbb{CP}^{n-1}$ the s_i are not permitted to vanish simultaneously, these polynomials must have no common zeroes. Conversely, if the s_i obey (3.51) and have no common zeroes, the formula (3.52) describes a degree-k holomorphic map $\mathbb{CP}^1 \to \mathbb{CP}^{n-1}$.

The relation between the two moduli spaces is now clear. Every sigma model instanton comes from a solution of (3.47), but (3.47) has extra solutions, in which the $s_i(u,v)$ do have common zeroes for some u, v, not both zero. These extra solutions are smooth objects in the underlying gauge theory, but near the common zeroes of the s_i they have a scale of variation of order $1/e\sqrt{r}$; this is the "point-like" behavior promised in our introductory remarks. These instantons show "point-like" behavior only near the common zeroes of the s_i; elsewhere they look like sigma model instantons.

One might think that the condition that the s_i have common zeroes would be obeyed only very exceptionally, so that the point-like instantons would perhaps be irrelevant. As a preliminary to seeing that this is not so, let us work out the dimension of the instanton moduli space. Equation (3.51) asserts the vanishing of a polynomial in u, v homogeneous of degree kn, which therefore has an expansion $G(u,v) = a_{kn}u^{kn} + a_{kn-1}u^{kn-1}v + \dots + a_0 v^{kn}$. The $kn+1$ a's can be written out explicitly as polynomials of degree n in the $(k+1)n$ coefficients $s_{i,r}$. Allowing also for the scaling relation (3.50), the dimension of the instanton moduli space is expected to be $(k+1)n - (kn+1) - 1 = n-2$ if everything is generic. It is known that this is the actual dimension of the sigma model moduli space, for generic G. (In the usual case of target space a quintic three-fold in $\mathbb{CP}^4$, $n-2=3$, and the dimension is reduced to zero after dividing by the action of $\mathrm{SL}(2,\mathbb{C})$ on Σ.)

If now $s_i(u,v)$ are a collection of degree-k polynomials, obeying (3.51) and having no common zeros, then $\tilde{s}_i(u,v) = (\alpha u + \beta v)s_i(u,v)$ are a collection of degree-$(k+1)$ polynomials, obeying (3.51), and with a common zero where $\alpha u + \beta v = 0$. After allowing for an irrelevant scaling of α, β, the $\tilde{s}_i$ depend on one more parameter than the s_i; thus, while the sigma model instantons depend on $n-2$ parameters, the point-like instantons with one common zero depend on $n-1$ parameters. (And the point-like instantons with w common zeros depend on $n-2+w$ parameters.) The point-like instantons are in no way exceptional, and they must be considered in integrating out the massive fields to get an effective action for the massless fields. Conventional renormalization lore suggests, as noted earlier, that their effect is to induce a reparametrization of the variable t.

4. Some geometrical background

The construction of sect. 3 has many generalizations, some of which we will explore in sect. 5. First, however, I want to explain some geometrical background that sheds some light on the subject. These facts are not strictly necessary for reading sect. 5, but clarify some of the issues, especially when we come to considering changes of topology in subsect. 5.5. In effect we will be explaining, in an elementary and *ad hoc* way, the relation (see ref. [33] or [34, p. 158]) between symplectic and holomorphic quotients.

Consider the manifold $\mathrm{Y} = \mathbb{C}^{n+1}$ with coordinates $s_1, \dots, s_n$ and p, and with the $\mathbb{C}^*$ action

$$\begin{aligned} s_i &\to \lambda s_i , \\ p &\to \lambda^{-n} p \end{aligned} \tag{4.1}$$

for $\lambda \in \mathbb{C}^*$. We want to form a quotient of Y by $\mathbb{C}^*$. The relevance of this will gradually become clear. Since $\mathbb{C}^*$ acts freely on Y minus the origin O, one might think that one could straightforwardly form a reasonable quotient $(\mathrm{Y} - \mathrm{O})/\mathbb{C}^*$. For a free action of a compact group, a reasonable quotient always exists, but the story is quite different for non-compact groups.

Let P be a point in Y with $p = 0$, and let P' be a point in Y with $s_1 = \dots = s_n = 0$. Under the $\mathbb{C}^*$ action, P can be brought arbitrarily close to the origin (by taking $\lambda \to 0$), and the same can be done to P' (by taking $\lambda \to \infty$). Even if we delete the origin in forming the quotient, any neighborhood of P has a $\mathbb{C}^*$ orbit that intersects any neighborhood of P'. This means that the set theoretic quotient $(\mathrm{Y}-\mathrm{O})/\mathbb{C}^*$, with its natural induced topology, is not a Hausdorff space.

Let Y_1 be the subset of Y with $p = 0$ and the s_i not all 0; let Y_2 be the subset $s_i = 0$, $p \neq 0$. The $\mathbb{C}^*$ orbits that come arbitrarily close to the origin are (apart from O itself) the orbits in Y_1 or Y_2. If $\widetilde{\mathrm{Y}} = \mathrm{Y} - (\mathrm{Y}_1 \cup \mathrm{Y}_2 \cup \mathrm{O})$, then since $\widetilde{\mathrm{Y}}$ only contains "good" $\mathbb{C}^*$ orbits (that are closed and bounded away from the origin), and $\mathbb{C}^*$ acts freely on $\widetilde{\mathrm{Y}}$, the quotient space $\widetilde{\mathrm{Y}}/\mathbb{C}^*$ is a manifold, just as if $\mathbb{C}^*$ were compact. Deleting Y_1, Y_2, and O is too crude, however; by suitably including some of the ill-behaved orbits one can obtain natural partial compactifications of $\widetilde{\mathrm{Y}}/\mathbb{C}^*$.

In the simple case that we have considered, it is not hard to guess how to do this. And there is a systematic theory (geometric invariant theory [34]; for an introduction see Newstead's book [35]). Instead of studying this situation purely mathematically, let us return to the physics problem of subsect. 3.1. In that problem, Y was endowed with the Kähler metric $\mathrm{d}\tau^2 = \sum_i |\mathrm{d}s_i|^2 + |\mathrm{d}p|^2$. This Kähler metric is not $\mathbb{C}^*$ invariant, so $\mathbb{C}^*$ is not a symmetry group of the $N = 2$ supersymmetric field theory studied in sect. 3. The metric is, however, invariant under the maximal compact subgroup $\mathrm{U}(1) \subset \mathbb{C}^*$ (whose action is given by (4.1) with $|\lambda| = 1$). This U(1) was used in sect. 3 as the gauge group. The Kähler form associated with the Kähler metric $\mathrm{d}\tau^2$ is $\omega = -i \sum_i \mathrm{d}\bar{s}_i \wedge \mathrm{d}s_i - i\mathrm{d}\bar{p} \wedge \mathrm{d}p$.

The U(1) action on Y preserves this Kähler form. Since Y is simply connected, there inevitably is a hamiltonian function that generates by Poisson brackets the U(1) action on Y; it is simply

$$\widetilde{D} = -\frac{D}{e^2} = \sum_i |s_i|^2 - n|p|^2 - r, \tag{4.2}$$

with r being an arbitrary constant. The D function that played such an important role in sect. 3 is thus simply the generator, in this sense, of the gauge group.

In sect. 3, it was natural to set $D = 0$, so as to minimize the energy, and then divide by the gauge group. The combined operation of setting $D = 0$ and then dividing by U(1) gives what is called the symplectic quotient of Y by U(1). The symplectic quotient of Y by U(1), often denoted Y//U(1), depends on r, of course. Even the topology of Y//U(1) changes when r passes through certain distinguished values at which Y//U(1) is singular – in our example this occurs only at $r = 0$. And when the topology is not changing, the symplectic structure of Y//U(1) still depends on r. Y//U(1) has a natural symplectic structure obtained by restricting ω to $D = 0$ and then projecting to the quotient Y//U(1) = $\{D = 0\}$/U(1). This natural symplectic structure determines an element of $H^2(Y//U(1), \mathbb{R})$ which depends linearly on r; this is related to the Duistermaat–Heckman integration formula [36]. See also ref. [55].

However, if we consider not Y but $\widetilde{Y}$, certain simplifications arise: $\widetilde{Y}$//U(1) is in a certain sense naturally independent of r. To see this, we consider the D function restricted to a particular $\mathbb{C}^*$ orbit. It is

$$\widetilde{D}(\lambda) = |\lambda|^2 \sum_i |s_i|^2 - n|\lambda|^{-2n}|p|^2 - r. \tag{4.3}$$

Because we are in $\widetilde{Y}$, the coefficients of $|\lambda|^2$ and $|\lambda|^{-2n}$ are both non-zero. $\widetilde{D}(\lambda)$ is a monotonic function of $|\lambda|$ which goes to $+\infty$ for $\lambda \to \infty$ and to $-\infty$ for $\lambda \to 0$. It follows that $\widetilde{D}(\lambda) = 0$ for a unique value of $|\lambda|$.

The given $\mathbb{C}^*$ orbit contributes, of course, precisely one point to the quotient $\widetilde{Y}/\mathbb{C}^*$. It also contributes precisely one point to $\widetilde{Y}$//U(1), since $|\lambda|$ is uniquely determined by requiring $D = 0$, and the argument of λ is absorbed in the U(1) action. Therefore, $\widetilde{Y}$//U(1) coincides naturally with $\widetilde{Y}/\mathbb{C}^*$.

From the symplectic point of view, there is no mystery about how to include the bad points – the origin O and the points in Y_1 and Y_2. The result, however, depends on r:

(i) For $r > 0$, setting $D = 0$ is possible in Y_1 but not for the other bad points. The symplectic quotient Q = Y_1//U(1) is a copy of $\mathbb{CP}^{n-1}$, with Kähler form proportional to r. So Y//U(1) is the union of $\widetilde{Y}/\mathbb{C}^*$ with this $\mathbb{CP}^{n-1}$.

(ii) For $r = 0$, the origin is the only bad point with $D = 0$. Its symplectic quotient is a single point, and Y//U(1) is the union of $\widetilde{Y}/\mathbb{C}^*$ with this point.

(iii) For $r < 0$, setting $D = 0$ is possible in Y_2 but not for the other bad points. The symplectic quotient $Y_2//U(1)$ is a single point, and $Y//U(1)$ is the union of $\tilde{Y}/\mathbb{C}^*$ with this point.

Since $Q = Y_1//U(1)$ is the same as $Y_1/\mathbb{C}^*$ (both being $\mathbb{CP}^{n-1}$), and similarly $Y_2//U(1) = Y_2/\mathbb{C}^*$ (both being a point), we can restate these results holomorphically. Let us do this in detail for $r > 0$ and for $r < 0$:

(i) For $r > 0$, $Y//U(1)$ is the same as $Z = Y'/\mathbb{C}^*$, where $Y' = \tilde{Y} \cup Y_1$ is the union of the points at which the s_i are not all zero. Z is fibered over $\mathbb{CP}^{n-1}$ (by forgetting p). The fiber is a copy of the complex plane, parametrized by p. Z is therefore a complex line bundle over $\mathbb{CP}^{n-1}$; in fact, in view of the transformation law in (4.1), Z is the total space of the line bundle $\mathcal{O}(-n)$. Z is actually a non-compact Calabi–Yau manifold. To see this, begin with the equation $\sum_i Q_i = 0$ that played such an important role in sect. 3 (the Q_i being here the exponents in (4.1)). This equation ensures that the $(n+1)$-form $\Theta = ds_1 \wedge \ldots \wedge ds_n \wedge dp$ is $\mathbb{C}^*$ invariant. Contracting Θ with the vector field generating the $\mathbb{C}^*$ action, we get an everywhere non-zero holomorphic n-form Θ' whose restriction to Y' is the pullback of a holomorphic n-form on $Z = Y'/\mathbb{C}^*$.

(iii) For $r < 0$, $Y//U(1)$ is the same as $Y''/\mathbb{C}^*$ where $Y'' = \tilde{Y} \cup Y_2$ is the region with $p \neq 0$. We can therefore use the $\mathbb{C}^*$ action on (s_i, p) to set $p = 1$. This leaves a residual invariance under the subgroup of $U(1)$ defined by $\lambda^n = 1$. Dividing by this group, $Y//U(1)$ for $r < 0$ is the same as $Z' = \mathbb{C}^n/\mathbb{Z}_n$. This is again a non-compact Calabi–Yau manifold, by virtue of $\sum_i Q_i = 0$.

There are two main conclusions:

(a) $Y//U(1)$ carries a natural complex structure for any r. It is evident "physically" that this must be so, since the low-energy limit of the models studied in sect. 3 is (if we set the superpotential to zero) an $N = 2$ sigma model with this target space. (The analog of this reasoning for $N = 4$ leads to the celebrated hyper-Kähler quotient [18].)

(b) The various complex manifolds $Y//U(1)$ for r positive, negative, or zero are all equivalent to each other on dense open sets, since on a dense open set they all coincide with $Y'/\mathbb{C}^*$. The technical term for this is that these complex manifolds are birationally equivalent. In fact, we can be more specific. Z is obtained by blowing up the origin in Z' – replacing the origin in $Z' = \mathbb{C}^n/\mathbb{Z}_n$ (which originated as $Y_2//U(1)$) with a copy of $\mathbb{CP}^{n-1} \cong Y_1//U(1)$.

Though I have illustrated the ideas in a very special case, these conclusions are general [33,34]. Given an action of a compact group G on, for instance, $Y = \mathbb{C}^n$ or on a projective variety (endowed with a choice of Kähler metric) the various possible symplectic quotients $Y//G$ (obtained from different choices of D functions) carry natural complex structures and are all birationally equivalent to one another. Indeed, they can all be identified on a dense open set with $\tilde{Y}/G_{\mathbb{C}}$, where $\tilde{Y}$ is a suitable dense open set in Y, and $G_{\mathbb{C}}$ is the complexification of G.

The technique of sect. 3 is really a technique for interpolating between sigma models with birationally equivalent target spaces, obtained by varying the D

functions. This technique can be applied in the absence of any superpotential, in which case it leads directly to relations between some sigma models with birationally equivalent targets. We will pick up this theme in subsect. 5.5 in connection with transitions in the topology of space-time.

On the other hand, if one begins with a $\mathbb{C}^*$-invariant holomorphic function W on Y – serving as a superpotential – then W will descend to a holomorphic function $\widehat{W}$ – serving as a superpotential – on any of the symplectic quotients Y//G. Thus, any relation between sigma models with birationally equivalent targets obtained by the method of sect. 3 extends to the case in which these spaces are endowed with "common" superpotentials, that is superpotentials that come from a common underlying gauge invariant function on Y. In general, birational equivalence is the condition under which it makes sense to speak in this way of the "same" holomorphic function on different complex manifolds.

For instance, in the case treated in sect. 3, the underlying superpotential was $W = pG(s_i)$, with G being homogeneous of degree n. For $r > 0$, W restricts and descends to a holomorphic function $\widehat{W}$ on the symplectic quotient Z. The effective theory after integrating out the massive gauge multiplet is a theory with target space Z and superpotential $\widehat{W}$; by essentially the computations of sect. 3, it reduces at low energies to the sigma model of the Calabi–Yau hypersurface $G = 0$ in $\mathbb{C}\mathrm{P}^{n-1} \subset \mathrm{Z}$.

On the other hand, for $r < 0$, W restricts and descends to the holomorphic function $G(s_i)$ on $\mathrm{Z}' = \mathbb{C}^n/\mathbb{Z}_n$. This is the superpotential of the familiar Landau–Ginzburg orbifold.

In this way, the birational equivalence between Z and Z′ lies behind the C–Y/L–G correspondence. The basic relation is the one between quantum field theory on Z and on Z′; the C–Y/L–G correspondence arises upon examining this relation in the presence of a particular common superpotential.

5. Generalizations

Our task in this section is to describe some generalizations of the Calabi–Yau/Landau–Ginzburg correspondence. We will first consider fairly immediate generalizations, involving (i) hypersurfaces in a weighted projective space; (ii) hypersurfaces in products of projective spaces and in more general toric varieties; (iii) hypersurfaces in grassmannians; (iv) intersections of hypersurfaces in any of those spaces. In each case, the discussion will be rather brief because most of the discussion in sect. 3 carries over with obvious modifications. Finally, in subsect. 5.5, we will discuss the occurrence in this framework of transitions between manifolds of different topology. The framework of sect. 4 is useful background, especially for subsect. 5.5.

5.1. HYPERSURFACES IN WEIGHTED PROJECTIVE SPACE

The most obvious slight generalization of what we have already done in sect. 3 is the following. Consider a U(1) gauge theory with n chiral superfields S_i, $i = 1,\dots,n$, of charge q_i, and one more chiral superfield P of charge $-\sum_i q_i$. By rescaling the U(1) generator, one can assume that the q_i are relatively prime.

The above choice of charges ensures that $\sum_i Q_i = 0$, a condition which as we saw in sect. 3 leads to anomaly-free left- and right-moving R-symmetry and hence to Calabi–Yau manifolds at low energies. In our subsequent examples, we will always ensure in a similar way that $\sum_i Q_i = 0$, for each U(1) charge, without pointing this out explicitly in each case.

We take the superpotential to be

$$W = PG(S_i), \tag{5.1}$$

where G is a polynomial of charge $q = \sum_i q_i$, transverse in the sense that the equations

$$\frac{\partial G}{\partial S_1} = \dots = \frac{\partial G}{\partial S_n} = 0 \tag{5.2}$$

are obeyed only at $S_1 = \dots = S_n = 0$. (Existence of such a G is a severe restriction on the q_i, analyzed in ref. [37,56], but if any G of charge q obeys this condition, the generic one does.) The part of the lagrangian containing the chiral superfields is thus

$$\int d^2y d^4\theta \left(\sum_i \overline{S}_i e^{2V} S_i + \overline{P} e^{-2qV} P\right) - \int d^2y d^2\theta PG(S_i) - \int d^2y d^2\overline{\theta}\ \overline{P}\ \overline{G}(\overline{S}_i). \tag{5.3}$$

The potential energy is much as before

$$U(A_i,\sigma) = \frac{1}{2e^2}D^2 + \sum_\alpha |F_\alpha|^2 + 2\overline{\sigma}\sigma\left(\sum_i q_i{}^2|s_i|^2 + q^2|p|^2\right), \tag{5.4}$$

with

$$D = -e^2\left(\sum_i q_i|s_i|^2 - q|p|^2 - r\right) \tag{5.5}$$

and

$$\sum_\alpha |F_\alpha|^2 = |G|^2 + |p|^2 \sum_i \left|\frac{\partial G}{\partial s_i}\right|^2 . \tag{5.6}$$

Since we will have to write similar formulas many times, we will adopt a few conventions. The symbol $\sum_\alpha$ will always involve a sum over all chiral superfields Φ_α, with auxiliary fields F_α. The terms in the potential involving σ always have the effect of giving σ a mass and forcing its vacuum expectation value to vanish (away from phase boundaries; the behavior near phase boundaries is always as explained in subsect. 3.2). Knowing this, σ can be dropped from the discussion, and the potential can be written only at $\sigma = 0$.

The analysis proceeds as in subsect. 3.1. One step in finding the ground-state structure is to set $D = 0$ and divide by the gauge group U(1) (forming the symplectic quotient of $\mathbb{C}^{n+1}$ by U(1), in the language of sect. 4). If $r \gg 0$, the locus $D = 0$ divided by U(1) is a copy of the weighted projective space $\mathrm{W}\mathbb{C}\mathrm{P}^{n-1}_{q_1,\ldots,q_n}$, with Kähler class proportional to r. The low-energy theory is a sigma model whose target space is the hypersurface $G = 0$ in that weighted projective space. This hypersurface is a Calabi–Yau manifold because of the underlying R-invariance. The assumption that the q_i are relatively prime means that (away from singularities of the weighted projective space) the gauge symmetry is completely broken.

On the other hand, for $r \ll 0$, the field p gets a vacuum expectation value, breaking the gauge group down to $\mathbb{Z}_q$. There is a unique vacuum, with $s_i = 0$ and $p = \sqrt{-r/q}$, up to a gauge transformation. The low-energy theory is a $\mathbb{Z}_q$ orbifold with superpotential $G(s_1,\ldots,s_n)$. We have recovered the standard correspondence between Calabi–Yau hypersurfaces in weighted projective space and Landau–Ginzburg orbifolds.

5.2. HYPERSURFACES IN TORIC VARIETIES

Now I want to consider a case in which we will discover something new, instead of just recovering known results. To avoid cluttering up the notation, I begin by considering a very special case of a hypersurface in a product of projective spaces, say $\mathbb{C}\mathrm{P}^{n-1} \times \mathbb{C}\mathrm{P}^{m-1}$.

First of all, to describe $\mathbb{C}\mathrm{P}^{n-1} \times \mathbb{C}\mathrm{P}^{m-1}$, we consider a $\mathrm{U}(1) \times \mathrm{U}(1)$ gauge theory with two vector superfields, say V_1 and V_2, with gauge couplings e_1 and e_2. There are therefore two independent Fayet–Iliopoulos terms, in components

$$L_D = -\int \mathrm{d}^2 y\, (r_1 D_1 + r_2 D_2)\,. \tag{5.7}$$

We introduce superfields S_i, $i = 1,\ldots,n$, of charges $(1,0)$, T_j, $j = 1,\ldots,m$, of charges $(0,1)$, and one more superfield P of charges $(-n,-m)$. The kinetic energy of these fields is thus

$$\int \mathrm{d}^2 y \mathrm{d}^4\theta \left(\sum_i \overline{S}_i \mathrm{e}^{2V_1} S_i + \sum_j \overline{T}_j \mathrm{e}^{2V_2} T_j + \overline{P} \mathrm{e}^{-2nV_1 - 2mV_2} P \right). \tag{5.8}$$

For the superpotential, we take

$$W = PG(S_i, T_j), \tag{5.9}$$

where G is a polynomial homogeneous of degree n in the S_i and of degree m in the T_j, and transverse in the sense that the equations

$$0 = \frac{\partial G}{\partial S_i} = \frac{\partial G}{\partial T_j} \tag{5.10}$$

have no solutions unless either the S_i or the T_j are all zero. The potential energy of this theory is (at $\sigma = 0$)

$$\begin{aligned} U(s_i, t_j) = & \frac{e_1{}^2}{2} \left(\sum_i |s_i|^2 - n|p|^2 - r_1 \right)^2 + \frac{e_2{}^2}{2} \left(\sum_j |t_j|^2 - m|p|^2 - r_2 \right)^2 \\ & + \sum_\alpha |F_\alpha|^2, \end{aligned} \tag{5.11}$$

with

$$\sum_\alpha |F_\alpha|^2 = |G|^2 + |p|^2 \left(\sum_i \left| \frac{\partial G}{\partial s_i} \right|^2 + \sum_j \left| \frac{\partial G}{\partial t_j} \right|^2 \right). \tag{5.12}$$

Now, let us analyze the vacuum structure. Transversality of G means that (5.12) vanishes precisely if either (i) $p = 0$ and $G = 0$; or (ii) $s_i = 0, i = 1, \ldots, n$, with suitable values of other fields; or (iii) $t_j = 0, j = 1, \ldots, m$, with suitable values of other fields. These three choices correspond to three phases of the theory, which we will call phase I, phase II, and phase III.

Which phase is realized depends on the values of the r_i. Upon setting (5.11) to zero, one finds that phase I is realized for $r_1, r_2 \geqslant 0$; phase II for $r_1 \leqslant 0$, $nr_2 - mr_1 \geqslant 0$; and phase III for $r_2 \leqslant 0$, $nr_2 - mr_1 \leqslant 0$. The phase boundaries are thus the half-lines $r_1 = 0, r_2 \geqslant 0$; $r_2 = 0, r_1 \geqslant 0$; and $r_1, r_2 \leqslant 0$, $nr_2 - mr_1 = 0$ in the $r_1 - r_2$ plane. These phase boundaries divide the plane into regions of phase I, II, or III.

Phase I is the familiar Calabi–Yau phase. In this phase, the expectation values of s_i and t_j break the gauge group completely. In the vacuum, $p = 0$, $\sum_i |s_i|^2 = r_1$, $\sum_j |t_j|^2 = r_2$. Upon dividing by the gauge group $\mathrm{U}(1) \times \mathrm{U}(1)$, the s_i and t_j determine a point in $\mathbb{CP}^{n-1} \times \mathbb{CP}^{m-1}$ with Kähler classes proportional to r_1, r_2. Since the vanishing of (5.12) requires also $G = 0$ (and all modes not tangent to the solution space of $G = 0$ are massive), the low-energy theory is a sigma model of the hypersurface $G = 0$ in $\mathbb{CP}^{n-1} \times \mathbb{CP}^{m-1}$. Because G is of bidegree (n, m), this is actually a Calabi–Yau hypersurface.

Now let us consider phase II. In this region, p and t_j have vacuum expectation values, spontaneously breaking the gauge group to $\mathbb{Z}_n$. The expectation values obey

$$|p| = \sqrt{-r_1/n}, \qquad \sum_j |t_j|^2 = r_2 - \frac{mr_1}{n}. \tag{5.13}$$

By a gauge transformation, one can assume that $p > 0$. This leaves the residual gauge invariance

$$t_j \to \exp(i\theta)\cdot t_j, \qquad s_i \to \exp(-i(m/n)\theta)\cdot s_i. \tag{5.14}$$

The value of t_j, after imposing the second equation in (5.13) and the gauge invariance in (5.14), determines a point in $\mathbb{C}\mathrm{P}^{m-1}$. Because s_i transforms as in (5.14) under this gauge transformation, the s_i should be considered to define not a point in $\mathbb{C}^n$ but a point in the fiber of an n-dimensional complex vector bundle Y over $\mathbb{C}\mathrm{P}^{m-1}$. (If m/n is not an integer, this is not an ordinary vector bundle but a V-bundle, the vector bundle analog of an orbifold.) Setting p to its vacuum expectation value, there is an effective superpotential given by the holomorphic function $W_{\mathrm{eff}} = \sqrt{-r_1/n}\cdot G(s_i, t_j)$ on Y. Phase II is thus a peculiar hybrid of a $\mathbb{C}\mathrm{P}^{m-1}$ sigma model in the t directions and a Landau–Ginzburg orbifold in the s directions. This is our first encounter with such a hybrid, but we will see that hybrids of various kinds are ubiquitous.

Phase III can, of course, be treated in the same way and is a similar hybrid of a $\mathbb{C}\mathrm{P}^{n-1}$ sigma model and a Landau–Ginzburg orbifold.

One may wonder if it is possible to describe a Calabi–Yau hypersurface in a product of projective spaces by a more ordinary Landau–Ginzburg model. The closest we can come to this is to go to the phase boundary between phase II and phase III, setting $r_1, r_2 < 0$ and $mr_1 - nr_2 = 0$. Then there is up to gauge transformation a unique classical vacuum with $s_i = t_j = 0$, $p = \sqrt{-r_1/n} = \sqrt{-r_2/m}$. The vacuum expectation value of p breaks the gauge group to $\mathrm{H} = \mathrm{U}(1)\times \mathbb{Z}_d$, where d is the greatest common divisor of n and m. The low-energy theory is a *gauged* Landau–Ginzburg model, with gauge group H.

As an example, take $n = m = 3$, with $G(S_i, T_j)$ homogeneous of degree $(3,3)$. On the boundary between phase II and phase III, the unbroken gauge group is $\mathrm{H} = \mathrm{U}(1)\times\mathbb{Z}_3$; U(1) acts as $S_i \to \exp(i\theta)\cdot S_i$, $T_j \to \exp(-i\theta)\cdot T_j$, and $\mathbb{Z}_3$ multiplies the S_i by a cube root of unity while leaving T_j invariant. The low-energy theory is described by the effective action

$$\begin{aligned} L_{\mathrm{eff}} = L_{\mathrm{gauge}} &+ \int \mathrm{d}^2y\,\mathrm{d}^4\theta\left(\overline{S}_i \mathrm{e}^{2V} S_i + \overline{T}_j \mathrm{e}^{-2V} T_j - rV\right) \\ &- \int \mathrm{d}^2 y\left(\int \mathrm{d}^2\theta \langle p\rangle G + \text{h.c.}\right), \end{aligned} \tag{5.15}$$

where the parameter r measures the distance from the II–III phase boundary. Precisely for $r = 0$, the model has an isolated classical vacuum at the origin and unbroken gauge invariance. The model is then a gauged Landau–Ginzburg model at low energies.

Gauged Landau–Ginzburg models can be analyzed like ordinary ones. For ordinary Landau–Ginzburg models, the contribution of the untwisted sectors to the (c, c) chiral ring (the chiral ring of the B model) is the ring of polynomials $A(S_i, T_j)$ in the chiral fields modulo the usual relations generated by $\partial G/\partial S_i$,

$\partial G/\partial T_j$. For a Landau–Ginzburg orbifold, one requires that A be invariant under the appropriate discrete symmetry. For a gauged Landau–Ginzburg model, A must be gauge invariant. (As we learned in subsect. 3.1, orbifolds are simply a special case of gauge theories with finite gauge group.) In the present case, gauge invariance can be imposed as follows: a U(1) invariant homogenous polynomial in S_i and T_j must be of equal degree in S_i and T_j, while $\mathbb{Z}_3$ invariance says that this degree must be divisible by three. The chiral ring thus consists of polynomials of bidegree $(0,0)$, $(3,3)$, $(6,6)$, and $(9,9)$. It seems unlikely that this model can be described by an ordinary (ungauged) Landau–Ginzburg model. Without the requirement of continuous gauge invariance, the (c,c) chiral ring would be infinite dimensional.

More general toric varieties. A generalization is to consider a model with N chiral superfields $S_1, \dots, S_N$, and one more chiral superfield P, and with gauge group $\mathrm{H} = \mathrm{U}(1)^d$. In the previous example, $N = n + m$ and $d = 2$. One can think of the S_i as linear functions on $\mathrm{V} = \mathbb{C}^N$. We suppose that the jth copy of U(1) acts by $S_i \to \exp(i\theta q_{ij}) \cdot S_i$ with integers q_{ij}, and $P \to \exp(-i\theta q_j)P$, with $q_j = \sum_i q_{ij}$. We pick the superpotential to be $W = PG(S_1, \dots, S_N)$ where G has charge q_j under the jth copy of U(1). The potential energy is

$$U(s_1, \dots, s_n, p) = \sum_j \frac{1}{2e_j{}^2} D_j{}^2 + \sum_\alpha |F_\alpha|^2 \tag{5.16}$$

with

$$D_j = -e_j{}^2 \left(\sum_i q_{ij}|s_i|^2 - q_j|p|^2 - r_j \right) \tag{5.17}$$

and

$$\sum_\alpha |F_\alpha|^2 = |G|^2 + |p|^2 \sum_i \left| \frac{\partial G}{\partial s_i} \right|^2 . \tag{5.18}$$

If one temporarily restricts to $p = 0$, then setting the D_j to 0 and dividing by H, one obtains a so-called symplectic quotient of $\mathrm{V} = \mathbb{C}^N$ by H, as discussed in sect. 4. This quotient, call it Z, is of real dimension $2(N - d)$, and obviously admits an action of a group $\mathrm{F} = \mathrm{U}(1)^{N-d}$ (the phase rotations of the s_i that have not been included in H). Actually (once the r_j are picked), Z is naturally a Kähler manifold – this is obvious in the formalism of $N = 2$ supersymmetry; mathematically, the complex structure on Z can be constructed using the relation between symplectic and holomorphic quotients, as sketched in sect. 4. Using the complex structure on Z, the F action on Z can be analytically continued to an action of $\mathrm{F}_{\mathbb{C}} = (\mathbb{C}^*)^{N-d}$*.

* The $\mathrm{F}_{\mathbb{C}}$ action on $\mathrm{Z} = \mathrm{V}//\mathrm{H}$ can be constructed concretely as follows. The F action on V is generated by transformations $s_i \to \lambda^{e_i} s_i$ with $\lambda \in \mathrm{U}(1)$ and certain exponents e_i. By permitting $\lambda \in \mathbb{C}^*$, one extends the F action on V to an $\mathrm{F}_{\mathbb{C}}$ action; as this commutes with H, it descends to an $\mathrm{F}_{\mathbb{C}}$ action on $\mathrm{V}/\mathrm{H}_{\mathbb{C}}$, which as we know from sect. 4 coincides with $\mathrm{V}//\mathrm{H}$ on a dense open set.

Z and $F_{\mathbb{C}}$ have the same complex dimension $N-d$ and $F_{\mathbb{C}}$ acts freely on a dense open set in Z; Z is therefore a so-called toric variety. Under some restriction on the q_{ij}, Z is compact. This is true if for some j the q_{ij} are all positive. (It is true more generally if for some positive numbers m_j, one has $\sum_j q_{ij} m_j > 0$ for all i; this however is not a very essential generalization as one can reduce to the previous case by an automorphism of the gauge group $U(1)^d$.) Conversely, every compact toric variety can be obtained in this way [38, 39].

The equation $G = 0$ defines a hypersurface X in Z. As a hypersurface in a toric variety, X is called a toric hypersurface. If we pick G to be transverse in the sense that the equations

$$G = \frac{\partial G}{\partial S_i} = 0 \tag{5.19}$$

are satisfied only for all $S_i = 0$, then this hypersurface is smooth away from singularities of Z; we henceforth consider only this case.

With Z and X as above and suitable r_j (for instance $r_j > 0$ precisely for some value of j for which $q_j > 0$), the condition $U = 0$ forces the s_i to not all vanish. Then setting U to zero (and using the transversality of G), p and G must vanish so the locus of classical ground states up to gauge transformation is precisely the hypersurface $X \subset Z$. The low-energy theory is a sigma model with this target space.

On the other hand, by varying the r_j, the system will undergo many phase transitions. Some of these phase transitions will involve changes in the topology of X, preserving the interpretation of the model as a Calabi–Yau sigma model (at low energies), while changing the topology of space-time. The local behavior involved in such topology change will be discussed in subsect. 5.5. Other phases of the system will be hybrids of sigma models and Landau–Ginzburg models; one type of hybrid appeared in the example treated at the beginning of this subsection. Finally, if we set $r_j = -rq_j$ (with some fixed r), then there is a unique classical vacuum up to gauge transformation; it has $|p|^2 = r$, $s_i = 0$. In this case, we get a description as a gauged Landau–Ginzburg model, generalizing the case treated above. The gauge group is $U(1)^{d-1} \times \mathbb{Z}_n$, for some n. It is worth noting that Batyrev [40] gave a kind of Landau–Ginzburg description of the cohomology of a general toric hypersurface; the facts we have just explained presumably give a field theoretic setting for his construction.

A general comment. A smooth toric variety Z of dimension $N - d$, with $H^2(Z, \mathbb{R})$ being d-dimensional, can always be obtained as the symplectic quotient of $\mathbb{C}^N$ by $U(1)^d$. The symplectic quotient $\mathbb{C}^N//U(1)^d$ depends on d parameters $r_1, \ldots, r_d$, the constant terms that can be added to the D functions. In keeping with the Duistermaat–Heckman theorem, starting from a fixed symplectic structure on $\mathbb{C}^N$, the natural induced symplectic structure on $Z = \mathbb{C}^N//U(1)^d$ varies linearly with the r_i. In other words, in some basis e_i of $H^2(Z, \mathbb{R})$, the Kähler class of $Z = \mathbb{C}^N//U(1)^d$ is $[\omega] = \sum_i r_i e_i$.

Now consider the Calabi–Yau hypersurface $X \subset Z$. $H^2(X,\mathbb{R})$ has some generators that can be found by restricting the cohomology of Z to X. The restriction of $H^2(Z,\mathbb{R})$ may however give only part of $H^2(X,\mathbb{R})$ (as in the example studied in ref. [7]). In this case, the most general Kähler metric on X is not conveniently obtained by embedding in Z, and the r_i are only a subset of natural linear coordinates parametrizing the Kähler class of X. When this occurs, the phase diagram we have constructed is not the full phase diagram of sigma models with target space X, but its restriction to the subspace spanned by the r_i.

5.3. HYPERSURFACES IN GRASSMANNIANS

Here we will consider a simple application of similar reasoning applied to a model with a non-abelian gauge group. In fact, we will find a correspondence between sigma models with target space a Calabi–Yau hypersurface in a grassmannian and gauged Landau–Ginzburg models, this time with a non-abelian gauge group. (Calabi–Yau hypersurfaces in grassmannians are discussed in the book by Hubsch [41].) In contrast to our previous experience, in this example the occurrence of a gauged Landau–Ginzburg model will be stable, not limited to special values of the parameters, and therefore perhaps more interesting. Further generalizations, using more complicated gauge groups or gauge groups in other representations, should be obvious.

To begin with, we need a convenient realization of a sigma model with target space a grassmannian, say the grassmannian of k planes in complex n space. To achieve this, we start with kn chiral superfields $S^i{}_\lambda$, $i = 1,\dots,k$, $\lambda = 1,\dots,n$. We think of the $S^i{}_\lambda$ as matrix elements of a $k \times n$ matrix S; $\overline{S}$ will denote its adjoint. The group $G = U(k)$ acts on the S's by

$$S^i{}_\lambda \to M^i{}_{i'} S^{i'}{}_\lambda, \tag{5.20}$$

for $M^i{}_{i'} \in U(k)$. To write a model with gauge group $U(k)$, we introduce a $k \times k$ hermitian matrix V of vector superfields. We take the lagrangian to be the gauge and matter kinetic energy plus a Fayet–Iliopoulos term for the central factor $U(1) \subset U(k)$. The potential energy is (at $\sigma = 0$)

$$U(s^i{}_\lambda) = \frac{1}{2e^2}\mathrm{Tr} D^2 \tag{5.21}$$

where

$$-\frac{D}{e^2} = s\overline{s} - r \tag{5.22}$$

(that is, $-D/e^2$ is a $k \times k$ hermitian matrix with matrix elements $-(D/e^2)^i{}_{i'} = \sum_\lambda s^i{}_\lambda \overline{s}^\lambda{}_{i'} - r\delta^i{}_{i'}$). Vanishing of D can be given the following interpretation. Consider the $s^i{}_\lambda$ for fixed i as defining a vector $w^{(i)}$ in an n-dimensional vector space $W \cong \mathbb{C}^n$. (If you wish, pick a fixed basis $e^1 \dots e^n$ of W and set $w^{(i)} =$

$\sum_\lambda s^i{}_\lambda e^\lambda$.) Then the equation $D = 0$ asserts * that the $w^{(i)}/\sqrt{r}$ are orthonormal. These vectors therefore span a k-dimensional subspace F of W. Upon dividing by the gauge group $U(k)$, the gauge invariant information contained in the expectation values of the $s^i{}_\lambda$ is merely the choice of F. The space of all F's is by definition the grassmannian $G(k,n)$ of k-planes in complex n space. At low energies, the model is thus simply a sigma model with this target space. In mathematical terms, using the language of sect. 4, the above construction amounts to obtaining $G(k,n)$ as the symplectic quotient of $\mathbb{C}^{kn}$ by $U(k)$**.

To obtain a Calabi–Yau hypersurface in this grassmannian, we introduce one more complex superfield P, transforming under (5.20) as $P \to (\det M)^{-n} P$. One then takes the superpotential to be $W = PG(S^i{}_\lambda)$, where G is a polynomial that transforms as $G \to (\det M)^n G$. The formula for D is modified to

$$-\frac{D}{e^2} = s\bar{s} - kn|p|^2 - r\,. \tag{5.23}$$

The additional terms that appear in the scalar potential are

$$\sum_\alpha |F_\alpha|^2 = |G|^2 + |p|^2 \sum_{i,\lambda} \left| \frac{\partial G}{\partial s^i{}_\lambda} \right|^2 . \tag{5.24}$$

For $r \gg 0$, the model can be analyzed in a now familiar fashion. Assuming that G is chosen so that the hypersurface $X \subset G(k,n)$ given by $G = 0$ is smooth, vanishing of the potential implies that $p = G = 0$; the low-energy theory is a sigma model with target space X. For $r \ll 0$, under the same assumptions, the model has a unique classical vacuum (up to gauge transformation) with $\langle p \rangle = \sqrt{-r/kn}$, $\langle s^i{}_\lambda \rangle = 0$. The expectation value of p breaks the gauge group from $U(k)$ to the subgroup H of $U(k)$ consisting of matrices whose determinant is an nth root of 1. (H is an extension of $SU(k)$ by $\mathbb{Z}_n$.) The model at low energies is a gauged Landau–Ginzburg model with massless chiral superfields $S^i{}_\lambda$, effective superpotential $W_{\text{eff}} = \langle p \rangle G(S^i{}_\lambda)$, and gauge group H.

An example. Here is a convenient and amusing example (see ref. [41], p. 101). One of the simplest examples of a Calabi–Yau manifold X is an intersection $G_1 = G_2 = 0$ in $\mathbb{CP}^5$, with G_1 and G_2 being respectively polynomials homogeneous of degree two and of degree four in homogeneous coordinates $T_1, \ldots, T_6$ of $\mathbb{CP}^5$. It does not seem that this model has an ordinary Landau–Ginzburg realization. However, it has a gauged Landau–Ginzburg phase, since (for a smooth quadric G_1) the locus $G_1 = 0$ in $\mathbb{CP}^5$ is equivalent to a copy of

* For $r > 0$. For $r < 0$, the renormalization effect studied in subsect. 3.2 is important in understanding the model; the vacuum states all lie at large σ.

** Alternatively, the grassmannian can be realized as a holomorphic quotient as follows. By allowing $M \in GL(k,\mathbb{C})$, (5.20) defines a $GL(k,\mathbb{C})$ action on $\mathbb{C}^{kn}$; the grassmannian is the quotient by $GL(k,\mathbb{C})$ of a dense open set in $\mathbb{C}^{kn}$ consisting of good orbits that contain zeros of the moment map. These are precisely the orbits on which the vectors $w^{(i)} = s^i{}_\lambda e^\lambda$ are linearly independent.

G(2,4). Hence, X can be realized as a hypersurface in G(2,4), and studied as above.

This may be seen explicitly as follows. Instead of thinking of the T_i as a six-component "column vector," arrange them as components of a 4×4 antisymmetric tensor $T_{\lambda\beta} = -T_{\beta\lambda}$, $\lambda, \beta = 1, \ldots, 4$. There is no invariant information in the choice of the quadratic polynomial G_1, assuming it is non-degenerate, since all non-degenerate quadrics are equivalent up to a linear change of coordinates. A convenient choice of G_1 is

$$G_1 = \epsilon^{\alpha\beta\gamma\delta} T_{\alpha\beta} T_{\gamma\delta} \tag{5.25}$$

with $\epsilon^{\alpha\beta\gamma\delta}$ the Levi-Civita tensor. The equation $G_1 = 0$ can be solved in terms of eight complex variables $S^i{}_\lambda$, $i = 1,2$, $\lambda = 1, \ldots, 4$, by

$$T_{\lambda\delta} = \epsilon_{ij} S^i{}_\lambda S^j{}_\delta \tag{5.26}$$

with ϵ_{ij} again the Levi-Civita tensor. Two $S^i{}_\lambda$'s give T's that are proportional (and so define the same point in $\mathbb{C}P^5$) precisely if they differ by $S^i{}_\lambda \to M^i{}_{i'} S^{i'}{}_\lambda$, M being in GL(2, $\mathbb{C}$). Since this equivalence relation on the S's is the one that leads to G(2,4), this completes the explanation of the isomorphism of the quadric $G_1 = 0$ in $\mathbb{C}P^5$ with G(2,4).

We want to study the Calabi–Yau manifold X given by the equations $G_1 = G_2 = 0$ in $\mathbb{C}P^5$, with $G_1(T_\lambda)$ as above and G_2 a quartic polynomial in the T's. Under the substitution (5.26), $G_2(T)$ becomes an eighth order, SL(2, $\mathbb{C}$) invariant polynomial $\widetilde{G}(S)$. X is equivalent to the hypersurface $\widetilde{G}(S) = 0$ in G(2,4). So according to our general discussion of hypersurfaces in grassmannians, the sigma model with target space X can be studied by studying the U(2) gauge theory with chiral superfields $S^i{}_\lambda$ and superpotential $W = P\widetilde{G}(S)$. Its phases can be found as at the beginning of this section and are the Calabi–Yau phase and a gauged Landau–Ginzburg phase.

5.4. INTERSECTIONS OF HYPERSURFACES

Now we will briefly discuss the application of these ideas to intersections of hypersurfaces.

To avoid cluttering the notation, we will consider only the simplest case of an intersection of hypersurfaces in a ordinary projective space, say $\mathbb{C}P^{n-1}$ with projective coordinates $s_1, \ldots, s_n$. We consider hypersurfaces H_a, $a = 1, \ldots, k$ defined by the vanishing of homogeneous polynomials G_a of degree q_a. The intersection X of the H_a is a Calabi–Yau manifold if $\sum_a q_a = n$. The condition that X is smooth is that the H_a intersect transversely in the sense that for any complex numbers p_a, not all zero, the equations

$$G_a = \sum_a p_a \frac{\partial G_a}{\partial s_i} = 0 \tag{5.27}$$

have a common solution only for $s_1 = \ldots = s_n = 0$.

To describe the hypersurface X along the general lines of the present paper, we consider a U(1) gauge theory with chiral superfields S_i of charge 1 and P_a of charge $-q_a$. We take the superpotential to be $W = \sum_a P_a G_a(S_1, \ldots, S_n)$. The ordinary potential is then

$$U = \frac{1}{2e^2} D^2 + \sum_\alpha |F_\alpha|^2 \tag{5.28}$$

with

$$D = -e^2 \left(\sum_i |s_i|^2 - \sum_a q_a |p_a|^2 - r \right) \tag{5.29}$$

and

$$\sum_\alpha |F_\alpha|^2 = \sum_a |G_a|^2 + \sum_i \left| \sum_a p_a \frac{\partial G_a}{\partial s_i} \right|^2 . \tag{5.30}$$

There are two phases. For $r > 0$, one uses the transversality condition (5.27) to show (as we have done earlier in similar problems) that the space of classical ground states is the variety X. For $r < 0$, the p's have an expectation value, and the s's do not. The space of ground states is then a weighted projective space $\mathrm{W} = \mathrm{W}\mathbb{C}\mathrm{P}^{k-1}_{a_1,\ldots,a_n}$, and the low-energy theory is a hybrid Landau–Ginzburg/sigma model on a vector bundle over W.

Intersections of hypersurfaces in more general toric varieties (or grassmannians, etc.) can be treated by obvious extensions of these remarks. In general, one will get many phases, of all the types we have seen (Calabi–Yau, Landau–Ginzburg, gauged Landau–Ginzburg, and various kinds of hybrid). One may wonder if there are any cases in which one gets an ordinary Landau–Ginzburg phase as opposed to a sigma model/L–G hybrid. This occurs only in special cases, some of which are of phenomenological interest. I will next describe such a case.

Landau–Ginzburg models for intersections of hypersurfaces. In $\mathbb{C}\mathrm{P}^{n-1} \times \mathbb{C}\mathrm{P}^{m-1}$, with homogeneous coordinates $s_1, \ldots, s_n$ and $t_1, \ldots, t_m$ for the two factors, we will consider the variety X defined by $G_1 = G_2 = 0$, the G's being bi-homogeneous polynomials in s_i and in t_j. If $G_a, a = 1, 2$ is bi-homogeneous in s_i, t_j of degree (q_a, q_a'), then X is a Calabi–Yau manifold if $q_1 + q_2 = n, q_1' + q_2' = m$. The condition that X is smooth is that for any complex numbers p_a not both zero, the equations

$$0 = G_1 = G_2 = \sum_a p_a \frac{\partial G_a}{\partial s_i} = \sum_a p_a \frac{\partial G_a}{\partial t_j} \tag{5.31}$$

have no common solution unless the s_i or the t_j are all zero. (Values of (s_i, t_j) with all s_i or all t_j zero do not determine a point in $\mathbb{C}\mathrm{P}^{n-1} \times \mathbb{C}\mathrm{P}^{m-1}$.) This transversality condition, however, is not sufficient to lead in our usual construction to a Landau–Ginzburg model; it will lead to a hybrid Landau–Ginzburg/sigma model similar to some we have seen above.

To construct a model with the sigma model of X as one of its phases, we consider a $U(1) \times U(1)$ gauge theory, with chiral fields S_i, $i = 1, \ldots, n$ of charge $(1,0)$, T_j, $j = 1, \ldots, m$ of charge $(0,1)$, and two more fields P_a, $a = 1,2$, of charge $(-q_a, -q_a')$. We pick the superpotential to be $W = \sum_a P_a G_a(S_i, T_j)$. The relevant parts of the classical potential are

$$\sum_a \frac{1}{2e_a{}^2} D_a{}^2 = \frac{e_1{}^2}{2} \left(\sum_i |s_i|^2 - \sum_a q_a |p_a|^2 - r_1 \right)^2 + \frac{e_2{}^2}{2} \left(\sum_j |t_j|^2 - \sum_a q_a' |p_a|^2 - r_2 \right)^2 \tag{5.32}$$

and

$$\sum_\alpha |F_\alpha|^2 = |G_1|^2 + |G_2|^2 + \sum_i \left| \sum_a p_a \frac{\partial G_a}{\partial s_i} \right|^2 + \sum_j \left| \sum_a p_a \frac{\partial G_a}{\partial t_j} \right|^2 . \tag{5.33}$$

Setting all this to zero, one finds for $r_1, r_2 \gg 0$ a phase which describes at low energies a sigma model with target space X. In general there is a perhaps interesting phase diagram with several phases, but none of these describes a Landau–Ginzburg model. For instance, for $r_1, r_2 \ll 0$, setting D_a to zero requires p_1, p_2 to not both vanish. The vanishing of the F_α is then equivalent to (5.31), and the transversality condition asserts that this requires that the s_i are all zero or the t_j are all zero – but not necessarily both. If the s_i are all zero, the t_j are not uniquely determined, and vice versa. One therefore has not an isolated classical vacuum, leading to a Landau–Ginzburg model, but a family of vacua with additional massless particles, corresponding to a hybrid model. This particular hybrid model actually has a reducible target space (one branch with $s_i = 0$ and one with $t_j = 0$) and a singularity (other than the usual orbifold singularities) at the intersection $s_i = t_j = 0$.

A Landau–Ginzburg model arises only if n, m and the q_a, q_a' are such that, for suitable values of r_1 and r_2, vanishing of the classical potential forces the s_i *and* t_j all to vanish. This occurs only in very special cases*. Some of these special cases are of phenomenological interest. For instance, take $n = 4$, $m = 3$, $(q_1, q_1') = (3,0)$, $(q_2, q_2') = (1,3)$. We take

$$G_1 = \sum_{i=1}^{4} S_i{}^3, \qquad G_2 = \sum_{j=1}^{3} S_i T_i{}^3. \tag{5.34}$$

* Apparently the relevant cases are precisely $n = m$, $(q_1, q_1') = (n-1, 1)$, $(q_2, q_2') = (1, n-1)$; or alternatively $n \geqslant m$, $(q_1, q_1') = (n-1, 0)$, $(q_2, q_2') = (1, m)$.

We have

$$\sum_a \frac{1}{2e_a^2} D_a^2 = \frac{e_1^2}{2}\left(\sum_i |s_i|^2 - 3|p_1|^2 - |p_2|^2 - r_1\right)^2 + \frac{e_2^2}{2}\left(\sum_j |t_j|^2 - 3|p_2|^2 - r_2\right)^2 \tag{5.35}$$

We consider the region $r_1 \ll r_2 \ll 0$. Vanishing of (5.35) requires that $p_2 \neq 0$ and that either (a) $p_1 \neq 0$, or (b) $p_1 = 0$, and the t_j are not all zero. Case (b) is incompatible with (5.31), which immediately implies that the t_j are all zero if $p_1 = 0$, $p_2 \neq 0$. In case (a), (5.31) is easily seen to imply that $s_i = t_j = 0$ for all i and j. In this case, $|p_1|$ and $|p_2|$ are uniquely determined by (5.35). So for this range of r_1 and r_2, the model has a unique classical vacuum up to gauge transformation and reduces at low energies to a Landau–Ginzburg orbifold. The model is actually a $\mathbb{Z}_9$ orbifold, as the vacuum expectation values of p_1 and p_2 break the gauge group to $\mathbb{Z}_9$. The effective superpotential for the massless superfields S_i, T_j is (setting the $\langle p_a \rangle$ to 1 by rescaling the variables)

$$W_{\text{eff}} = \sum_a G_a = \sum_{j=1}^{3}\left(S_j{}^3 + S_j T_j{}^3\right) + S_4{}^3. \tag{5.36}$$

This model is of interest for two reasons. First of all, a suitable quotient of the variety X by a finite group gives one of the first known and simplest three generation models, constructed originally by Schimmrigk [42]. Second, the Landau–Ginzburg description shows that the model is exactly soluble at a particular point in its moduli space. Indeed, $S^3 + ST^3$ is the superpotential of the minimal model with E_7 modular invariant, while S^3 is that of the minimal model with A_1 modular invariant, so the model is a tensor product of three E_7 minimal models and one A_1 model. It is in fact one of Gepner's original models [43].

5.5. CHANGE OF TOPOLOGY

It has been a long-standing and fascinating question whether in string theory physical processes can occur in which the topology of space-time changes. (Examples have been known, involving duality or mirror symmetry, of non-classical equivalences between different topologies, but that is a somewhat different question.) Recently, Aspinwall, Greene, and Morrison [7] undertook to use mirror symmetry to show that the answer is affirmative, at least in one special situation. Though their starting point is quite different, their analysis ultimately involves the same phase diagrams that we have been examining in this paper.

These phase diagrams can be very complicated in general and – as shown in detail in ref. [7] in a particular example – can contain a variety of Calabi–Yau phases, with different smooth target spaces. Upon varying the Fayet–Iliopoulos terms, transitions occur between the different target space-times. By arguments of subsect. 3.2, the transitions are continuous if the θ angles are generic. In string theory, the Fayet–Iliopoulos parameters (like all operators in the world-sheet lagrangian) are interpreted as expectation values of some fields in space-time; they are dynamical variables, free to change in time. This gives a mechanism for physical topology change.

The transitions that arise this way are transitions between topologically distinct but birationally equivalent Calabi–Yau manifolds. The reasons for this were sketched in sect. 4; in brief, the various symplectic quotients of a given target space that one can construct by varying the Fayet–Iliopoulos parameters can all be identified generically (on a dense open set) with a fixed complex quotient of the same target.

While a global situation has been analyzed in ref. [7], my discussion here will be more modest in scope, focusing on the local mechanism of topology change. In other words, we will analyze how, by varying the constant term in the D function, we obtain a change of topology in a local region of space-time. The reader should think of the space-time we will consider as being embedded as part of a compact Calabi–Yau manifold, whose topology will change in the process we will consider.

The example that we will consider is in complex dimension three and is not as special as it may appear. It is the generic "flop" (or birational transformation not changing c_1) considered in Mori theory (of birational classification of complex manifolds; for a review see ref. [44]) in complex dimension three. It is possible that by introducing enough parameters, any birational transformation between three-dimensional Calabi–Yau manifolds can be constructed as a succession of transformations each locally isomorphic to the one we will analyze.

We consider a U(1) gauge theory with two chiral superfields A_i, $i = 1, 2$ of charge 1 and two chiral superfields B_j, $j = 1, 2$, of charge -1. We can think of them as coordinates on $\mathrm{V} \cong \mathbb{C}^4$. We take the superpotential to be zero. The ordinary potential energy for the bosonic components is (at $\sigma = 0$)

$$U(a_i, b_j) = \frac{e^2}{2}\left(\sum_i |a_i|^2 - \sum_j |b_j|^2 - r\right)^2 . \tag{5.37}$$

The U(1) action on V is

$$\begin{aligned} a_i &\to \lambda a_i \\ b_j &\to \lambda^{-1} b_j \end{aligned} \tag{5.38}$$

with $\lambda \in \mathrm{U}(1)$; it can be extended to a $\mathbb{C}^*$ action by permitting $\lambda \in \mathbb{C}^*$.

The low-energy effective space-time is obtained as usual by setting $U = 0$ and dividing by U(1); it is in other words $\mathrm{V}//\mathrm{U}(1)$, the symplectic quotient of V by U(1) in the language of sect. 4. There are really three possible symplectic quotients, corresponding to $r > 0$, $r = 0$, and $r < 0$. We will call them Z_+, Z_0, and Z_-.

Z_+, Z_0, and Z_- are all birationally isomorphic in their natural complex structures according to the arguments of sect. 4. In fact, let $\widetilde{V}$ be the region of V in which the a_i are not both zero, and the b_j are not both zero. Then by precisely the arguments of sect. 4, the symplectic quotient $\widetilde{V}//U(1)$ can be naturally identified – for any r – with the holomorphic quotient $\widetilde{V}/\mathbb{C}^*$. Z_+, Z_0, and Z_- are partial compactifications of $\widetilde{V}/\mathbb{C}^*$ obtained by including contributions of some of the bad $\mathbb{C}^*$ orbits on which $a_1 = a_2 = 0$ or $b_1 = b_1 = 0$. In fact, these bad orbits are of three types: the origin O with $a_i = b_j = 0$; the set V_1 of orbits with $b_j = 0$ but the a_i not both zero; and the set V_2 of orbits with $a_i = 0$ but the b_j not both zero.

By including the appropriate bad orbits, Z_+, Z_0, and Z_- can be described as follows:

(i) For $r > 0$, the $\mathbb{C}^*$ orbits that do not contain zeros of D are the orbits in O and V_2. Other $\mathbb{C}^*$ orbits (whether in $\widetilde{V}$ or V_1) each contribute precisely one point to the symplectic quotient Z_+. Hence $Z_+ = (\widetilde{V} \cup V_1)/\mathbb{C}^*$. Here $\widetilde{V} \cup V_1$ is precisely the region in V in which the a_i are not both zero. The values of the a_i, up to scaling by $\mathbb{C}^*$, determine a point in a copy of $\mathbb{CP}^1$ which we will call $\mathbb{CP}^1_a$. Z_+ is fibered over $\mathbb{CP}^1_a$ by forgetting the values of the b_j; since the values of the b_j are arbitrary, the fiber is a copy of $\mathbb{C}^2$. The zero section of $Z_+ \to \mathbb{CP}^1_a$, that is the locus $b_1 = b_2 = 0$, is a genus zero holomorphic curve, in fact an embedding $\mathbb{CP}^1_a \subset Z_+$. We henceforth identify $\mathbb{CP}^1_a$ with its image under this embedding. For $b_j = 0$, vanishing of D gives $\sum_i |a_i|^2 = r$, so the Kähler form of $\mathbb{CP}^1_a$ is proportional to r.

(ii) For $r = 0$, the only bad $\mathbb{C}^*$ orbit that contributes is O; it contributes a single point to Z_0, and that point is a singularity. We will examine the singularity more closely later.

(iii) The structure for $r < 0$ is similar to that for $r > 0$ with the roles of a_i and b_j reversed. In particular, $Z_- = (\widetilde{V} \cup V_2)/\mathbb{C}^*$. Here $\widetilde{V} \cup V_2$ is the region of V in which the b_j are not both zero. The values of the b_j determine a point in a copy of $\mathbb{CP}^1$ that we will call $\mathbb{CP}^1_b$. Z_- is fibered over $\mathbb{CP}^1_b$ with fiber a copy of $\mathbb{C}^2$ parametrized by the valued of the a_i. The zero section $a_1 = a_2 = 0$ of $Z_- \to \mathbb{CP}^1_b$ gives a genus zero curve in Z_- which is an embedding $\mathbb{CP}^1_b \subset Z_-$. This curve – which we identify with $\mathbb{CP}^1_b$ – has Kähler form proportional to $-r$.

Z_+, Z_0, and Z_- are all non-compact Calabi–Yau manifolds. Indeed, the holomorphic four-form $\Theta = da_1 \wedge da_2 \wedge db_1 \wedge db_2$ on V is $\mathbb{C}^*$ invariant (because $\sum_i Q_i = 0$); contracting it with the vector field generating the $\mathbb{C}^*$ action gives an everywhere non-zero holomorphic three form whose restriction to the appropriate region of V is the pullback of a holomorphic volume form on Z_+, Z_0, or Z_-.

In passing from $r > 0$ to $r < 0$, a change in topology occurs. $\mathbb{CP}^1_a$ shrinks to zero size as $r \to 0$ from above, and is replaced by $\mathbb{CP}^1_b$ for $r < 0$. Classically the interpolation from $r > 0$ to $r < 0$ passes through a singularity at $r = 0$, but according to the argument of subsect. 3.2, there is no such singularity in the sigma model as long as the θ angle is generic. We have achieved the promised smooth change in topology in sigma models of (non-compact) Calabi–Yau manifolds.

It is true that (by exchanging the a's and b's), Z_+ and Z_- are actually isomorphic. However, in a global situation, with our discussion applied to a portion of some compact Calabi–Yau manifold, the replacement of Z_+ by Z_- or of $\mathbb{CP}^1_a$ by $\mathbb{CP}^1_b$ entails a change in the topology of space-time.

To give a new perspective on the model, let us now consider more closely the nature of the singularity at the origin of the exceptional symplectic quotient Z_0.

Let x, y, z, and t be coordinates on a copy of $\mathbb{C}^4$ that we will call W. The formulas

$$\begin{aligned} x &= a_1 b_1 , \\ y &= a_2 b_2 , \\ z &= a_1 b_2 , \\ t &= a_2 b_1 . \end{aligned} \tag{5.39}$$

give a $\mathbb{C}^*$ invariant map $V \to W$. By restricting these formulas to $\widetilde{V} \cup O$ and dividing by $\mathbb{C}^*$, we get a map from $Z_0 \to W$. (By starting with $\widetilde{V} \cup V_1$ or $\widetilde{V} \cup V_2$ we would get maps $Z_\pm \to W$.) It is evident that the image of Z_0 in W lies in the affine quadric Q defined by

$$xy - zt = 0. \tag{5.40}$$

In fact, the map defined in (5.39) is an isomorphism between Z_0 and Q.

This assertion can be justified as follows:

(a) To prove that the map is surjective, we must show that if x, y, z, and t obey (5.40), then (5.39) is satisfied for some values of a_i, b_j. If $x = y = z = t = 0$, we take $a_i = b_j = 0$. If, say, $x \neq 0$, we pick $a_1 = 1$ and then iteratively solve (5.39) for a_2, b_1, b_2.

(b) To prove that the map is injective, we must show that if x, y, z, and t obey (5.40), then (5.39) determines the a_i, b_j uniquely up to the action of $\mathbb{C}^*$. If $x = y = z = t = 0$, then (5.39) requires that either the a_i or the b_j are both zero. But in that case, for a_i and b_j to define a point in Z_0, they must all be zero, so $x = y = z = t = 0$ is the image only of the point $O \in Z_0$. Otherwise, if say $x \neq 0$, then (5.39) requires $a_1 \neq 0$. We can use the $\mathbb{C}^*$ action to set $a_1 = 1$, and then (5.39) uniquely determines a_2, b_1, b_2. This completes the proof of isomorphism of Z_0 and Q.

The singularity of Q at $x = y = z = t = 0$ is the simplest type of isolated singularity of a three-dimensional Calabi–Yau manifold (see ref. [41], p. 121). This singularity can be resolved by blowing up the origin in Q, but that would ruin the Calabi–Yau condition. There are two minimal ways to resolve the singularity of Q while preserving the Calabi–Yau condition. The two choices are to replace the singular point by a copy of $\mathbb{CP}^1$ which should be either $\mathbb{CP}^1_a$ or $\mathbb{CP}^1_b$ as constructed above. Thus the two "small resolutions" of Q as a Calabi–Yau manifold are precisely Z_+ and Z_-. The topology-changing transition that we found above was a transition between these two small resolutions.

The instanton sum. To get more insight, we want to study the behavior of physical observables in the transition from Z_+ to Z_-, or more exactly from X_+ to X_- where $X_\pm$ are compact Calabi–Yau manifolds that coincide outside of a region (or perhaps finitely many similar regions) where they look like $Z_\pm$. The simplest physical observables to analyze are low-energy Yukawa couplings. Yukawa couplings of massless multiplets derived from $H^{2,1}(X_\pm)$ are independent of r and so should be invariant under the transition from X_+ to X_-. Of more interest are the Yukawa couplings of massless multiplets associated with $H^{1,1}(X_\pm)$. These are determined by complicated instanton sums; we want to see what happens to those sums in interpolating from $r > 0$ to $r < 0$.

Writing down the full instanton sums on $X_\pm$ would require very detailed information about the rational curves. Comparing the instantons of $X_\pm$ is much easier. All that happens in the transition from X_+ to X_- is that one genus zero curve, called $\mathbb{CP}^1_a$ above, is lost, and replaced by another genus zero curve, above called $\mathbb{CP}^1_b$. Our problem is simply to evaluate the contributions for $r > 0$ from $\mathbb{CP}^1_a$ and its multiple covers, and compare to the contributions for $r < 0$ from $\mathbb{CP}^1_b$ and its multiple covers.

First of all, if $H^{2,0}(X_\pm) = 0$ (as for almost all Calabi–Yau manifolds), then $H^{1,1}(X_\pm)$ can be identified with the group of divisors on $X_\pm$, up to linear equivalence. Moreover, as X_+ and X_- differ only in complex codimension two, divisors on X_+ can be naturally identified with divisors on X_-. This gives a natural isomorphism between the groups $H^{1,1}(X_\pm)$, which we will therefore call simply $H^{1,1}(X)$.

It may appear that we need some detailed information about $H^{1,1}(X)$, but happily that is not so. If E_1, E_2, and E_3 are three divisors, the contribution of a rational curve C to the corresponding Yukawa coupling is proportional to

$$(C, E_1)(C, E_2)(C, E_3) \tag{5.41}$$

where (C, E) is the intersection number of the curve C and the divisor E. Therefore $\mathbb{CP}^1_a$ and $\mathbb{CP}^1_b$ and their covers contribute only to Yukawa couplings of states associated with divisors that they meet. This means that any divisor which, up to linear equivalence, can be chosen not to intersect $\mathbb{CP}^1_a$ and $\mathbb{CP}^1_b$ is not sensitive to the interpolation from X_+ to X_-. We therefore need not understand the divisor classes of $X_\pm$; it is enough to understand the divisor classes of $Z_\pm$ or equivalently of Z_0.

The divisor class group of Z_0 is calculated in Hartshorne's book [45], example II.6.6.1 and exercise II.6.5. The result is that any divisor on Z_0 (or $Z_\pm$) is a multiple of the divisor E given by $a_1 = 0$. (Not being $\mathbb{C}^*$ invariant, a_1 is a section of a line bundle rather than a function, so the fact that E is the divisor of a_1 does not make it trivial.) For example, the divisor E′ given by $b_1 = 0$ obeys

$$\mathrm{E} + \mathrm{E}' = 0 \tag{5.42}$$

since the product $a_1 b_1 = x$ is a function on Z_0, and $\mathrm{E} + \mathrm{E}'$ is the divisor of this function.

Therefore, the only Yukawa coupling that we need to evaluate is the three point function of the multiplet associated with E. To evaluate the contributions of $\mathbb{C}\mathrm{P}^1_a$ and $\mathbb{C}\mathrm{P}^1_b$ to this three point coupling, we need to know their intersection number with E. Indeed,

$$(\mathbb{C}\mathrm{P}^1_a, \mathrm{E}) = 1, \tag{5.43}$$

since $\mathbb{C}\mathrm{P}^1_a$ and E meet transversely in one point (represented on V by the $\mathbb{C}^*$ orbit $a_1 = b_1 = b_2 = 0$, $a_2 \neq 0$). Likewise $(\mathbb{C}\mathrm{P}^1_b, \mathrm{E}') = 1$, and in view of (5.42), it follows that

$$(\mathbb{C}\mathrm{P}^1_b, \mathrm{E}) = -1. \tag{5.44}$$

(E and $\mathbb{C}\mathrm{P}^1_b$ do not meet transversely, so this number cannot be determined by just counting intersection points.)

Apart from these intersection numbers, to evaluate the instanton sums we need to know the instanton action, which for an instanton C contributes a factor $\exp(2\pi i \tau)$, where $-2\pi i \tau = \int_C \omega$; here ω is a two form representing a complexified Kähler form on X (its real part is the ordinary Kähler form, and its imaginary part incorporates the θ angle). We therefore need $-2\pi i t_a = \int_{\mathbb{C}\mathrm{P}^1_a} \omega$, $-2\pi i t_b = \int_{\mathbb{C}\mathrm{P}^1_b} \omega$. Because every divisor is locally equivalent to a multiple of E, we can assume that ω is a multiple of the Poincaré dual of E (plus terms that do not contribute to t_a or t_b). In view of (5.43) and (5.44) we get therefore the important result

$$t_b = -t_a. \tag{5.45}$$

We will now compute on $\mathrm{X}_\pm$ the Yukawa coupling of three fields associated with the divisor E. We will do the calculation as a function of t_a or equivalently t_b. In the calculation on X_+, the physical region is $\mathrm{Im}\, t_a > 0$, since if the curve $\mathbb{C}\mathrm{P}^1_a$ is to exist, as it does on X_+, it must have positive area. The physical region for the calculation on X_- is likewise $\mathrm{Im}\, t_b > 0$ or equivalently $\mathrm{Im}\, t_a < 0$. What we want to do is to compute the instanton sum on X_+ for $\mathrm{Im}\, t_a > 0$ and compare its analytic continuation to the instanton sum on X_- for $\mathrm{Im}\, t_a < 0$.

The actual computation is not difficult. On X_+, we have to take account of $\mathbb{C}\mathrm{P}^1_a$ and its multiple covers. Using the basic formula for the instanton contribution [46] and the formula for the contributions of multiple covers conjectured by Candelas et al. [47] and justified by Aspinwall and Morrison [48], the contribution of $\mathbb{C}P^1_a$ and its multiple covers to the Yukawa coupling is

$$\lambda_+ = (\mathbb{C}P^1_a, E)^3 \sum_{n=1}^{\infty} e^{2\pi i n t_a} = \frac{e^{2\pi i t_a}}{1 - e^{2\pi i t_a}}. \tag{5.46}$$

The nth term is the contribution of the nth cover of $\mathbb{C}P^1_a$; the series converges for $\operatorname{Im} t_a > 0$.

On X_-, we have to likewise take account of $\mathbb{C}P^1_b$ and its covers. The formula analogous to (5.46) (convergent now for $\operatorname{Im} t_a < 0$) is

$$\lambda_- = (\mathbb{C}P^1_b, E)^3 \sum_{n=1}^{\infty} e^{2\pi i n t_b} = -\frac{e^{-2\pi i t_a}}{1 - e^{-2\pi i t_a}}. \tag{5.47}$$

One might naively think that the hypothesis of smooth continuation from X_+ to X_- should mean that $\lambda_+ = \lambda_-$, but in fact according to the above formulas

$$\lambda_+ - \lambda_- = -1. \tag{5.48}$$

The discrepancy has a natural interpretation, however. In addition to the instanton sums, the Yukawa couplings receive a constant "classical" contribution from the intersection pairings of X_+ and X_-. The birationally equivalent manifolds X_+ and X_- have different cohomology rings and different intersection pairings. It must be the case that the difference in intersection pairings between X_+ and X_- is $+1$, cancelling (5.48).

This is easy to verify. Instead of comparing the triple intersection number (E, E, E), we can just as well look at (E, E', E'), since restricted to $V_\pm$, $E' = -E$. The difference in the classical intersection pairing between X_+ and X_- can be measured by counting intersections in V_+ and V_-. We have to be careful, though, to use the same representatives for E, E', E' in V_+ as in V_-; if the divisors are shifted by linear equivalence, the number of intersections outside of $V_\pm$ might change. So we represent E by $a_1 = 0$ and the two copies of E′ by $b_1 = 0$ and $b_2 = 0$. In V_+, these divisors meet transversely at the one point $a_1 = b_1 = b_2 = 0$, so the intersection number is $+1$. In V_-, b_1 and b_2 never both vanish, so the intersection number is 0. So we get the expected excess intersection number in X_+ relative to X_- of $+1$.

So the Yukawa couplings on X_- are analytic continuations of those of X_+, as expected. Moreover, the fact that (5.46) and (5.47) have a pole at $t_a = 0$ as their only singularity is in accord with the argument of subsect. 3.2 according to which r and θ (essentially the imaginary and real parts of t_a) must both vanish to get a singularity.

6. Extension to (0, 2) models

Models with $(0,2)$ supersymmetry (that is, two right-moving and no left-moving supersymmetries, as opposed to the $(2,2)$ models studied above) are important phenomenologically. In the context of compactification of string theory to four dimensions, they lead naturally to SU(5) or SO(10) rather than E_6 as effective the grand unified gauge group. These models have the reputation of being much harder to study than (2,2) models, because the analogs of the Gepner soluble models, the Landau–Ginzburg correspondence, etc., have not been known. In this section we will construct the (0,2) version of the Landau–Ginzburg correspondence. This is a straightforward matter of working out the structure of the appropriate (0,2) superfields and then repeating the procedure of sect. 3.

6.1. (0,2) SUPERFIELDS

We will work in (0,2) superspace, with bosonic coordinates y^α, $\alpha = 1,2$, and fermionic coordinates $\theta^+, \overline{\theta}^+$ *. The supersymmetry generators are

$$
\begin{aligned}
Q_+ &= \frac{\partial}{\partial\theta^+} + i\overline{\theta}^+\left(\frac{\partial}{\partial y^0} + \frac{\partial}{\partial y^1}\right), \\
\overline{Q}_+ &= -\frac{\partial}{\partial\overline{\theta}^+} - i\theta^+\left(\frac{\partial}{\partial y^0} + \frac{\partial}{\partial y^1}\right).
\end{aligned}
\tag{6.1}
$$

These commute with

$$
\begin{aligned}
D_+ &= \frac{\partial}{\partial\theta^+} - i\overline{\theta}^+\left(\frac{\partial}{\partial y^0} + \frac{\partial}{\partial y^1}\right), \\
\overline{D}_+ &= -\frac{\partial}{\partial\overline{\theta}^+} + i\theta^+\left(\frac{\partial}{\partial y^0} + \frac{\partial}{\partial y^1}\right),
\end{aligned}
\tag{6.2}
$$

which are used in constructing lagrangians.

I make no claim to describing here all possible (0,2) models, only those that can be conveniently described by certain types of (0,2) superfields.

The gauge multiplet. We want to introduce gauge fields in superspace. The gauge covariant derivatives will be called $\mathcal{D}_+, \overline{\mathcal{D}}_+$, and $\mathcal{D}_\alpha = D/Dy^\alpha$. We assume the constraints on the superspace gauge fields

$$
\begin{aligned}
\mathcal{D}_+{}^2 &= \overline{\mathcal{D}}_+{}^2 = 0 \\
\mathcal{D}_+\overline{\mathcal{D}}_+ + \overline{\mathcal{D}}_+\mathcal{D}_+ &= 2i(\mathcal{D}_0 + \mathcal{D}_1).
\end{aligned}
\tag{6.3}
$$

(These equations, for instance, permit the existence of the (0,2) chiral superfields that we introduce later.) The first two equations mean that

$$
\begin{aligned}
\mathcal{D}_+ &= e^{-\Psi}D_+e^{\Psi} \\
\overline{\mathcal{D}}_+ &= e^{\overline{\Psi}}\overline{D}_+e^{-\overline{\Psi}},
\end{aligned}
\tag{6.4}
$$

* There have been previous discussions relevant to what follows [49–51], but a few points made below are new and others are reviewed for completeness.

with Ψ a Lie algebra valued function. By a gauge transformation one can assume that Ψ is real. One can also gauge away terms in Ψ that are independent of θ^+ or of $\overline{\theta}^+$. In particular, one can go to an analog of Wess–Zumino gauge in which $\Psi = \theta^+\overline{\theta}^+(v_0 + v_1)$, with $v_0 + v_1$ a function of the y^α only. With this partial gauge fixing,

$$\begin{aligned} \mathcal{D}_0 + \mathcal{D}_1 &= \partial_0 + \partial_1 + i(v_0 + v_1), \\ \mathcal{D}_+ &= \frac{\partial}{\partial\theta^+} - i\overline{\theta}^+(\mathcal{D}_0 + \mathcal{D}_1), \\ \overline{\mathcal{D}}_+ &= -\frac{\partial}{\partial\overline{\theta}^+} + i\theta^+(\mathcal{D}_0 + \mathcal{D}_1). \end{aligned} \tag{6.5}$$

The rest of the superspace gauge field is

$$\mathcal{D}_0 - \mathcal{D}_1 = \partial_0 - \partial_1 + iV, \tag{6.6}$$

with some function V that can be expanded

$$V = v_0 - v_1 - 2i\theta^+\overline{\lambda}_- - 2i\overline{\theta}^+\lambda_- + 2\theta^+\overline{\theta}^+D. \tag{6.7}$$

The transformation laws of $v_\alpha, \lambda_-, \overline{\lambda}_-$, and D can be deduced from these formulas (using the supersymmetry generators (6.2) accompanied by gauge transformations to preserve the form of (6.5)) and are precisely the (0,2) truncation of the transformation laws (2.12) of the (2,2) gauge multiplet – except that some fields are missing. The missing fields form a separate (0,2) multiplet that will be identified later.

The basic gauge invariant field strength is $\Upsilon = [\overline{\mathcal{D}}_+, \mathcal{D}_0 - \mathcal{D}_1]$. The natural superspace action for the (0,2) gauge multiplet is

$$L_{\text{gauge}} = \frac{1}{8e^2}\int \mathrm{d}^2y\mathrm{d}\theta^+\mathrm{d}\overline{\theta}^+\ \mathrm{Tr}\,\overline{\Upsilon}\Upsilon \tag{6.8}$$

This can be evaluated in components to give (in the U(1) case, for simplicity)

$$\frac{1}{e^2}\int \mathrm{d}^2y\left(\frac{1}{2}v_{01}{}^2 + i\overline{\lambda}_-(\partial_0 + \partial_1)\lambda_- + \frac{1}{2}D^2\right). \tag{6.9}$$

The chiral multiplet. There are two types of matter multiplets to consider. One is a bose field Φ, in some representation of the gauge group, obeying

$$\overline{\mathcal{D}}_+\Phi = 0. \tag{6.10}$$

We will call this the (0,2) chiral multiplet. The chiral multiplet has a θ expansion

$$\Phi = \phi + \sqrt{2}\theta^+\psi_+ - i\theta^+\overline{\theta}^+(D_0 + D_1)\phi. \tag{6.11}$$

(Here D_α is now of course the "ordinary" gauge-covariant derivative at $\theta^+ = \overline{\theta}^+ = 0$.) If Φ is a field of charge Q, interacting with the abelian gauge multiplet described in components above, the action is

$$L_{\mathrm{ch}} = -\frac{i}{2}\int \mathrm{d}^2y\,\mathrm{d}^2\theta\;\overline{\Phi}(\mathcal{D}_0 - \mathcal{D}_1)\Phi$$
$$= \int \mathrm{d}^2y \Big(-|D_\alpha\phi|^2 + \overline{\psi}_+ i(D_0 - D_1)\psi_+$$
$$-iQ\sqrt{2}\overline{\phi}\lambda_-\psi_+ + iQ\sqrt{2}\overline{\psi}_+\overline{\lambda}_-\phi + QD\overline{\phi}\phi\Big). \tag{6.12}$$

The Fermi multiplet. The other type of matter multiplet is an anticommuting, negative chirality spinor field Λ_-, in some representation of the gauge group, obeying

$$\overline{\mathcal{D}}_+\Lambda_- = \sqrt{2}E, \tag{6.13}$$

where E is some superfield obeying

$$\overline{\mathcal{D}}_+E = 0. \tag{6.14}$$

We will call Λ_- a Fermi multiplet. The θ expansion of the Fermi multiplet is

$$\Lambda_- = \lambda_- - \sqrt{2}\theta^+G - i\theta^+\overline{\theta}^+(D_0 + D_1)\lambda_- - \sqrt{2}\overline{\theta}^+E. \tag{6.15}$$

In turn E has a theta expansion. In the important case that $E = E(\Phi_i)$ is a holomorphic function of some chiral superfields Φ_i (with expansions as in (6.11)), one has

$$E(\Phi_i) = E(\phi_i) + \sqrt{2}\theta^+\frac{\partial E}{\partial\phi_i}\psi_{+,i} - i\theta^+\overline{\theta}^+(D_0 + D_1)E(\phi_i). \tag{6.16}$$

The natural action of the Fermi multiplet is

$$L_{\mathrm{F}} = -\frac{1}{2}\int \mathrm{d}^2y\mathrm{d}^2\theta\;\overline{\Lambda}_-\Lambda_-. \tag{6.17}$$

If E is as in (6.16), then the component expansion is

$$L_{\mathrm{F}} = \int \mathrm{d}^2y\Big(i\overline{\lambda}_-(D_0 + D_1)\lambda_- + |G|^2 - |E(\phi_i)|^2$$
$$-\Big(\overline{\lambda}_-\frac{\partial E}{\partial\phi_i}\psi_{+,i} + \frac{\partial\overline{E}}{\partial\overline{\phi}_i}\overline{\psi}_{+,i}\lambda_-\Big)\Big). \tag{6.18}$$

Other terms in the action. Other terms in the action will be of the form

$$\int \mathrm{d}^2y\mathrm{d}\theta^+\;(\ldots)|_{\overline{\theta}^+=0} + \mathrm{h.c.}, \tag{6.19}$$

where $\ldots$ is some anticommuting superfield annihilated by $\overline{\mathcal{D}}_+$. These terms, which cannot be written as integrals over all of superspace, are of particular importance.

For instance, the gauge field strength Υ obeys $\overline{\mathcal{D}}_+\Upsilon = 0$. So in the abelian case we can write

$$\begin{aligned} L_{D,\theta} &= \frac{t}{4}\int \mathrm{d}^2y\mathrm{d}\theta^+ \, \Upsilon|_{\overline{\theta}^+=0} + \text{h.c.} \\ &= \frac{it}{2}\int \mathrm{d}^2y(D - iv_{01}) - \frac{i\bar{t}}{2}\int \mathrm{d}^2y(D + iv_{01}). \end{aligned} \tag{6.20}$$

For the other main example, let $\Lambda_{-,a}$ be some Fermi superfields with

$$\overline{\mathcal{D}}_+\Lambda_{-,a} = \sqrt{2}E_a, \tag{6.21}$$

E_a being some chiral superfields. And let J^a be some chiral superfields with

$$E_aJ^a = 0. \tag{6.22}$$

Then

$$\overline{\mathcal{D}}_+(\Lambda_{-,a}J^a) = 0. \tag{6.23}$$

So we can introduce another term in the action; it is the $(0,2)$ analog of the superpotential:

$$L_J = -\frac{1}{\sqrt{2}}\int \mathrm{d}^2y\mathrm{d}\theta^+ \, \Lambda_{-,a}J^a|_{\overline{\theta}^+=0} - \text{h.c.} \tag{6.24}$$

If we suppose that E_a and J^a are holomorphic functions of chiral superfields Φ_i, we get

$$L_J = -\int \mathrm{d}^2y\left(G_aJ^a(\phi_i) + \lambda_{-,a}\psi_{+,i}\frac{\partial J^a}{\partial\phi_i}\right) - \text{h.c.} \tag{6.25}$$

Combining all this, we take the lagrangian to be

$$L = L_{\text{gauge}} + L_{\text{ch}} + L_{\text{F}} + L_{D,\theta} + L_J. \tag{6.26}$$

After eliminating the auxiliary fields D and G_a, the bosonic potential turns out to be

$$\mathrm{U}(\phi_i) = \frac{e^2}{2}\left(\sum_i Q_i|\phi_i|^2 - r\right)^2 + \sum_a |E_a|^2 + \sum_a |J^a|^2. \tag{6.27}$$

The $(0,2)$ analog of the C–Y/L–G correspondence will be found – in a by now familiar fashion – by studying the vacuum structure as a function of r.

Reduction of $(2,2)$ *multiplets.* Now let us discuss the reduction of $(2,2)$ multiplets to $(0,2)$ multiplets.

First, consider the $(2,2)$ gauge multiplet, described in Wess–Zumino gauge by a scalar function $V(y,\theta^\pm,\overline{\theta}^\pm)$ with the expansion of eq. (2.11). Part of this multiplet makes up the $(0,2)$ gauge multiplet described above. The remaining fields are σ, λ_+, and $\overline{\lambda}_+$, and make up a $(0,2)$ chiral multiplet. In fact, this multiplet is simply $\Sigma' = \Sigma|_{\theta^-=\overline{\theta}^-=0}$, where $\Sigma = \sigma + \ldots$ is the gauge invariant field strength of the $(2,2)$ gauge multiplet, introduced in equation (2.15)*.

* To put the θ expansion of Σ' in the standard form (6.11) for a $(0,2)$ chiral multiplet, one must absorb a factor of i in $\overline{\lambda}_+$.

Now consider a $(2,2)$ chiral multiplet Φ. Φ decomposes under $(0,2)$ supersymmetry into two multiplets. First, there is a $(0,2)$ chiral multiplet

$$\Phi' = \Phi|_{\theta^- = \bar{\theta}^- = 0}. \tag{6.28}$$

Second, let

$$\Lambda_- = \frac{1}{\sqrt{2}} \mathcal{D}_- \Phi|_{\theta^- = \bar{\theta}^- = 0}. \tag{6.29}$$

Then

$$\overline{\mathcal{D}}_+ \Lambda_- = \frac{1}{\sqrt{2}} \{\overline{\mathcal{D}}_+, \mathcal{D}_-\}\Phi|_{\theta^- = \bar{\theta}^- = 0}. \tag{6.30}$$

If, for instance, the gauge group is $\mathrm{U}(1)$, and Φ is of charge Q, this amounts to

$$\overline{\mathcal{D}}_+ \Lambda_- = 2iQ\Sigma'\Phi'. \tag{6.31}$$

So Λ_- is a Fermi multiplet with

$$E = iQ\sqrt{2}\Sigma'\Phi'. \tag{6.32}$$

Finally, we want the $(0,2)$ reduction of the superpotential of a $(2,2)$ model. Consider a $(2,2)$ model with chiral superfields Φ_i and a superpotential $W(\Phi_i)$. Under $(0,2)$ supersymmetry, the Φ_i split, as we have just seen, into $(0,2)$ chiral superfields Φ_i', and Fermi superfields $\Lambda_{-,i}$. The latter obey $\overline{\mathcal{D}}_+ \Lambda_{-,i} = \sqrt{2}E_i$, where from (6.32),

$$E_i = iQ_i\sqrt{2}\Sigma'\Phi_i'. \tag{6.33}$$

While in $(2,2)$ supersymmetry the scalar and Yukawa couplings of chiral superfields are determined by a single function W, in $(0,2)$ supersymmetry we must specify a collection of functions J^i, one for each $\Lambda_{-,i}$, obeying $J^i E_i = 0$, with E_i given above. The J^i that arise in reduction of a $(2,2)$ model are simply

$$J^i = \frac{\partial W}{\partial \Phi_i'}. \tag{6.34}$$

This can be found by comparing equation (2.21) for couplings derived from a superpotential in $(2,2)$ models to equation (6.25) for the couplings determined by the J's. With J^i as in (6.34) and E_i as in (6.33) the equation $E_i J^i = 0$ is a consequence of the gauge invariance of W.

6.2. SOME MODELS

Now, let us apply this machinery to some sigma models with $(0,2)$ supersymmetry. For simplicity, we will consider only the case in which the target space is a hypersurface X in $\mathbb{CP}^{n-1}$.

We know from sect. 3 which superfields we need to construct a super-renormalizable $(2,2)$ model flowing at low energies to the sigma model of X. For our present purposes, we must decompose those superfields into $(0,2)$ multiplets, with a view to eventually constructing $(0,2)$ deformations of the models of sect. 3 – and their Landau–Ginzburg description.

So we consider a $U(1)$ gauge theory in $(0,2)$ superspace, with the following $(0,2)$ chiral multiplets: n fields S_i of charge 1*, one field P of charge $-n$, and one neutral field Σ (from the reduction of the $(2,2)$ gauge multiplet). In addition, we want n Fermi multiplets $\Lambda_{-,i}$, of charge 1, with

$$E_i = i\sqrt{2}\Sigma S_i, \tag{6.35}$$

and one more Fermi multiplet $\Lambda_{-,0}$, of charge $-n$, with

$$E_0 = -in\sqrt{2}\Sigma P. \tag{6.36}$$

To complete the specification of the model, we need to pick additional functions J^i, $i = 1,\dots,n$, and J^0, with

$$E_i J^i + E_0 J^0 = 0. \tag{6.37}$$

In the $(2,2)$ case, we had the superpotential $W = PG(S_i)$, with G a homogeneous nth order polynomial obeying the usual transversality condition. In $(0,2)$ language, this choice according to (6.34) leads to

$$\begin{aligned} J^i &= P\frac{\partial G}{\partial S_i}, \\ J^0 &= G. \end{aligned} \tag{6.38}$$

To obtain $(0,2)$ models, we will change these choices while preserving (6.37) as well as gauge invariance. We will also assume that J^0 remains independent of P while J^i remains linear in P. (Other terms allowed by gauge invariance would be of higher order in the chiral superfields and apparently "irrelevant" in the renormalization group sense. The same is true of possible modifications of the E's.) So we may as well preserve $J^0 = G$ as the definition of G. It is however possible to change the J^i. The general possibility is

$$\begin{aligned} J^i &= P\frac{\partial G}{\partial S_i} + PG^i, \\ J^0 &= G, \end{aligned} \tag{6.39}$$

where the G^i are polynomials of degree $n-1$ in the S_k chosen so that

$$S_i G^i = 0. \tag{6.40}$$

So we obtain a family of $(0,2)$ models parametrized, in addition to the usual data, by the G^i. Pulling together the relevant formulas, the potential energy of

* All supermultiplets here will be $(0,2)$ supermultiplets, and I omit the primes from the notation.

the theory is

$$U(s_i,p) = \frac{e^2}{2}\left(\sum_i |s_i|^2 - n|p|^2 - r\right)^2 + |G|^2 + |p|^2 \sum_i \left|\frac{\partial G}{\partial s_i} + G^i\right|^2$$
$$+ 2|\sigma|^2\left(\sum_i |s_i|^2 + n^2|p|^2\right). \tag{6.41}$$

Here the bosonic components of S_i, P, and Σ have been called s_i, p, and σ. We can therefore analyze the r dependence just as in the $(2,2)$ case. For $r \gg 0$ we get a phase in which the low-energy theory is a $(0,2)$ sigma model with target space X. For $r \ll 0$, we get a $(0,2)$ Landau–Ginzburg phase. We have generalized the C–Y/L–G correspondence to $(0,2)$ models. Let us look at the two phases more closely.

The Landau–Ginzburg phase. For $r \ll 0$, p gets an expectation value, just as in the $(2,2)$ case, breaking the gauge group to $\mathbb{Z}_n$. The model is therefore a $\mathbb{Z}_n$ orbifold.

The massless multiplets are the chiral multiplets S_i and $\Lambda_{-,i}$; other multiplets are massive. The effective values of E_i and J^i in the low-energy theory are obtained by setting σ and p to their vacuum expectation values, namely 0 and $\sqrt{-r/n}$. So one has in the effective $(0,2)$ Landau–Ginzburg model

$$E_i = 0$$
$$J^i = \langle p\rangle\left(\frac{\partial G}{\partial s_i} + G^i\right). \tag{6.42}$$

Of course these obey $E_i J^i = 0$.

The Calabi–Yau phase. For $r \gg 0$ the massless bose modes are the modes tangent to X. The right-moving massless fermions are just the $(0,2)$ partners of the massless bosons.

But what about left-moving massless fermions? Here the situation is more interesting. The Fermi component of $\Lambda_{-,0}$ is massive, but certain modes of $\lambda_{-,i}$ (the Fermi components of the superfields $\Lambda_{-,i}$) are massless. By inspection of the above formulas for the lagrangian, the massless modes can be seen to obey

$$\sum_i \bar{s}_i \lambda_{-,i} = 0 \tag{6.43}$$

and also

$$\sum_i \left(\frac{\partial G}{\partial s_i} + G^i\right)\lambda_{-,i} = 0. \tag{6.44}$$

Let us put these formulas in the general format of $(0,2)$ sigma models. In general, massless left-moving fermions are sections of some holomorphic vector bundle F over the target space X. This means that it must be possible to describe the massless modes by equations that vary holomorphically in the s_i. Now, (6.44) is holomorphic in s_i, but (6.43) is not. To achieve more understanding, we should replace (6.43) by an equation that has the same consequences but varies holomorphically in s. This is easily done. Instead of the constraint (6.43), introduce a gauge invariance, saying that two modes of λ_i are considered equivalent if they are related by a transformation

$$\lambda_i \to \lambda_i + s_i\lambda, \tag{6.45}$$

for some λ. Then (6.43) can be regarded as a gauge fixing condition; each orbit of the invariance (6.45) has a unique representative obeying (6.43).

To put (6.45) and (6.44) in their theoretical context, introduce the usual line bundles $\mathcal{O}(k)$ over $\mathbb{CP}^{n-1}$, restricted to X, and consider a sequence of maps of vector bundles over X,

$$0 \to \mathcal{O} \xrightarrow{\alpha} \oplus_{i=1}^{n} \mathcal{O}(1) \xrightarrow{\beta} \mathcal{O}(n) \to 0, \tag{6.46}$$

with α being the map $\lambda \to s_i\lambda$, and β the map

$$\lambda_i \to \lambda_i \left(\frac{\partial G}{\partial s_i} + G^i \right). \tag{6.47}$$

(6.46) is a complex of vector bundles (that is $\beta\alpha = 0$), but it is not an exact sequence. On the contrary, $\ker\beta/\mathrm{Im}\,\alpha$ is a rank $n-2$ holomorphic vector bundle F over X.

For $G^i = 0$, F reduces to the tangent bundle TX of X. In general, F is a deformation of TX, and conversely, all deformations of TX as a holomorphic vector bundle over X can be described as in (6.46). This family of deformations of TX has been discussed previously [52, volume II, pp. 461-3]. The choice of F (together with the rest of the data) determines a $(0,2)$ model with target X, so the G^i parametrize a family of such models; we have found the Landau–Ginzburg description of that family.

One may wonder what concrete results follow from this $(0,2)$ analog of the usual C–Y/L–G correspondence. One answer, along the lines of subsect. 3.3, is that $\mathrm{Tr}(-1)^F$ and the elliptic genus are invariant under the transition from Calabi–Yau to Landau–Ginzburg models, while for the half-twisted $(0,2)$ model (which determines low-energy Yukawa couplings) Landau–Ginzburg is an analytic continuation of Calabi–Yau. (The A and B topological field theories do not have $(0,2)$ analogs.) Another answer, along the lines of subsect. 5.5, is that topology-changing processes involving $(0,2)$ models can be analyzed just as for $(2,2)$ models.

To complete the story, perhaps I should point out that in $(0,2)$ sigma models with left-moving massless fermions taking values in some vector bundle F over the target space X, the objects E_a and J^a are geometrically to be understood as holomorphic sections of F and of F* (the dual of F), respectively. If non-zero, these sections give masses to some of the modes. In the particular example we have been studying with X a hypersurface in $\mathbb{C}\mathrm{P}^{n-1}$ and F as above, the effective E_a and J^a are 0. They could hardly be otherwise, as in these examples F and F* have no non-zero global holomorphic sections. But in the general study of $(0,2)$ sigma models, when F is such that F or F* have global sections, the consideration of E_a and J^a is likely to be of great importance.

Conformal invariance? In sect. 3, we constructed the C–Y/L–G correspondence for $(2,2)$ models without having to know whether conformal invariance is valid on either side. In the present section, we have done the same for $(0,2)$ models.

For the $(0,2)$ models, the situation is believed to be quite different from that for $(2,2)$ models. The $(2,2)$ models studied in sect. 3 are all believed to have conformally invariant fixed points to which they flow in the infrared. Order by order in perturbation theory, the same is true for the $(0,2)$ models analyzed above. But in many cases, these approximate $(0,2)$ conformal fixed points are destabilized by nonperturbative corrections [46,53]. For instance, for the particular case we have been looking at closely, it is believed that this occurs for generic choices of the polynomials G, G^i. Little is understood, from the world-sheet point of view, about the nature of the breakdown of conformal invariance.

We have, for illustrative purposes, studied in detail only $(0,2)$ models that can be constructed as perturbations of $(2,2)$ models. It would be extremely interesting to study other $(0,2)$ models using the techniques in this paper.

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