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APPLICATION OF INVARIANT RENORMALIZATION TO THE NON-LINEAR CHIRAL INVARIANT PION LAGRANGIAN IN THE ONE-LOOP APPROXIMATION

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Abstract: The functional for the one-loop diagrams for the non-linear chiral invariant pion Lagrangian is calculated invariantly using methods strongly reminiscent of gauge-invariant renormalization techniques in QED. It is shown that the restrictions chiral invariance puts on the renormalization counterterms do not suffice to extract finite results from the one-loop diagrams. The problem of the Adler condition in the presence of a symmetry breaking mass term is considered.

1. INTRODUCTION

Non-linear Lagrangians like the chiral invariant pion Lagrangian are known to be non-renormalizable. Simple power-counting arguments show that the divergences of the graphs become worse and worse as the number of internal loops increases, necessitating an infinite number of counterterms. Therefore, non-linear Lagrangians have until recently only been used in the tree approximation where all the current algebra results can be easily reproduced [1]. However, strictly speaking, those results cannot be true above threshold, because the amplitudes calculated from tree diagrams are real and, therefore, unitarity is a priori violated above threshold. Recently, there have been attempts to unitarize the amplitudes calculated from non-linear Lagrangians, i.e., to use these Lagrangians as a basis for a real quantum field theory despite the fact that the theories are non-renormalizable in the usual sense. The main motivation for such an approach is that chiral invariance is expected to put strong constraints on the counterterms, so that one may hope to extract finite results from the one-loop diagrams, for instance. Thus, one looks at a certain restricted class of diagrams beyond the tree approximation and tries to find the restrictions chiral invariance puts on the divergent terms. This can be done by using an invariant renormalization method, so that at each stage of the calculation the

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invariance is manifest and the counterterms themselves will be chiral invariant. Then further cancellations of counterterms cannot be expected to result from chiral invariance only.

In this paper we calculate the invariant counterterms for the one-loop diagrams. Using an expression for the one-loop graphs, which was derived in ref. [2], we are able to reduce the problem of evaluating the counterterms to a similar problem in QED. The equations turn out to be both invariant with respect to a pion field redefinition, and also covariant with respect to a kind of gauge group. The invariant renormalization with respect to this gauge group can be done either in momentum space [3] or in coordinate space [4].

As our basic Lagrangian is chiral invariant, the mass of the pion is necessarily zero. Hence, the well-known infrared difficulties of massless QED [5, 6] will show up in the renormalization procedure. However, in reality the pion has a non-vanishing mass. We therefore separate the infra-red divergences from the ultra-violet ones and remark that the former disappear once a mass term is included. Furthermore, it can be verified that the ultra-violet divergences produced by such a mass term do not invalidate our final conclusion.

In sect. 2 the necessary equations for calculating one-loop diagrams are set up and the similarity to QED is emphasized. In sect. 3 we calculate the invariant counterterms for the non-linear pion Lagrangian. There will only be two new terms which have to be added to the original Lagrangian. Furthermore, we discuss the question whether the renormalized Lagrangian in the presence of a symmetry breaking mass term allows a partially conserved axial-vector current, in other words, whether the amplitudes calculated with the renormalized Lagrangian in the tree-graph approximation fulfil the Adler condition in the soft pion limit. It is argued that this will not be the case in general. In sect. 4 we recalculate the one-loop functional with the modified Lagrangian and show that it produces an infinite number of additional counterterms. Thus, chiral invariance is not enough to get finite results from the one-loop diagrams.

2. THE ONE-LOOP FUNCTIONAL

In this section we develop an invariant representation for the self-energy functional Σ_π containing all one-loop diagrams for the non-linear chiral invariant pion Lagrangian. This Lagrangian can be written [7] as

$$L = \tfrac{1}{2} g_{ij}(\pi)\, \partial_\mu \pi^i \partial^\mu \pi^j\,, \qquad i, j = 1, 2, 3 \quad ; \tag{2.1}$$

$g_{ij}(\pi)$ is the metric in a curved isospace with constant curvature f_π^{-2} and we have introduced $\partial_\mu \equiv \partial/\partial x^\mu$; similarly, derivatives with respect to the pion field π^i will be denoted ∂_i or $_{,i}$, covariant derivatives in isospace $_{;i}$.

From the results of ref. [2] we immediately conclude that the functional describing all connected one-loop diagrams can be written as

$$\Sigma_\pi = -\tfrac{1}{2} i \log \mathrm{Det}(S_{ab} G_0) , \tag{2.2}$$

with

$$S_{ab} = e_a^k S_{;kl} e_b^l , \qquad S = \int L \, \mathrm{d}x ; \tag{2.3}$$

$S_{ab}(\pi)$ is a functional of the classical pion field. $e_a^k(\pi)$ $(a = 1, 2, 3)$ are dreibein fields associated with the metric g_{ij}, i.e.,

$$e_a^k e_a^l = g^{kl} , \qquad e_a^i e_{ib} = \delta_{ab} , \tag{2.4}$$

and are defined up to a rotation in the Euclidean R_3 (indices $a, b, \ldots$). G_0 is a Green's function for the d'Alembert equation and we will specify later which Green's function has to be chosen. The symbol Det stands for the determinant in the functional sense and we have used the convenient notation whereby the space-time coordinates are included in the indices k, l and a, b, e.g., ref. [4].

With S a chiral scalar and therefore

$$S_{;k} = S_{,k} \equiv \frac{\delta S}{\delta \pi^k(x)} ,$$

we have

$$S_{;k} = -g_{ik} \Box \pi^i - \Gamma_{k,ij} \partial_\mu \pi^i \partial^\mu \pi^j , \tag{2.5}$$

where $\Gamma_{k,ij} = \frac{1}{2}(g_{ik,j} + g_{jk,i} - g_{ij,k})$ are the Christoffel symbols for the metric g_{ij}. The expression

$$S_{;kl} = S_{,kl} - \Gamma_{kl}^m S_{,m} \tag{2.6}$$

leads to

$$\begin{aligned} S_{;kl} = &-g_{kl} \Box \, \delta(x-x') - 2 \partial_\mu \pi^i \Gamma_{k,il} \partial^\mu \delta(x-x') \\ &- \{ \partial_l \Gamma_{k,ij} \partial_\mu \pi^i \partial^\mu \pi^j + g_{ik,l} \Box \pi^i + \Gamma_{kl}^m S_{,m} \} \, \delta(x-x') . \end{aligned} \tag{2.7}$$

In the following we will occasionally omit the δ function in expressions like (2.7) and treat the equations as operator equations in x space.

Furthermore,

$$S_{ab} = e_a^k S_{;kl} e_b^l = -\delta_{ab} \Box - 2 \rho_{\mu,ab} \partial^\mu - B_{ab} , \tag{2.8}$$

where

$$\rho_{\mu,ab} = e_a^k \Gamma_{k,il} \partial_\mu \pi^i e_b^l + g_{kl} e_a^k \partial_\mu e_b^l \,, \tag{2.9}$$

$$\begin{aligned} B_{ab} &= e_a^k \{ \partial_l \Gamma_{k,ij} \partial_\mu \pi^i \partial^\mu \pi^j + g_{ik,l} \Box \pi^i + \Gamma_{kl}^m S_{,m} \} e_b^l \\ &\quad + g_{kl} e_a^k \Box e_b^l + 2 e_a^k \Gamma_{k,il} \partial_\mu \pi^i \partial^\mu e_b^l \,. \end{aligned} \tag{2.10}$$

As we have been working in a covariant manner, it is obvious that (2.8) can be written in such a way as to exhibit the covariance explicitly. Indeed

$$S_{ab} = - \{ (\partial_\mu + \rho_\mu)_{ac} (\partial^\mu + \rho^\mu)_{cb} + C_{ab} \} \,, \tag{2.11}$$

where $(\partial_\mu)_{ac} \equiv \delta_{ac} \partial_\mu$ and

$$\begin{aligned} C_{ab} &= e_a^k e_b^l \, \partial_\mu \pi^i \partial^\mu \pi^j \{ \partial_l \Gamma_{k,ij} - \partial_i \Gamma_{k,lj} \\ &\quad - g^{mn} \Gamma_{m,il} \Gamma_{k,nj} + g^{mn} \Gamma_{n,ij} \Gamma_{k,lm} - g_{nk,l} g^{nm} \Gamma_{m,ij} + g_{kn,j} g^{nm} \Gamma_{m,il} \} \\ &= e_a^k e_b^l \partial_\mu \pi^i \partial^\mu \pi^j R_{kjli} \,, \end{aligned} \tag{2.12}$$

and R_{kjli} is the Riemannian curvature tensor for the metric g_{ij}.

The following remarks concerning (2.11) should be kept in mind. C_{ab} and $\rho_{\mu,ab}$ are invariant under a pion field redefinition, which is obvious for C_{ab} from (2.12) and can immediately be checked for $\rho_{\mu,ab}$. On the other hand, C_{ab} is also a tensor in Euclidean three-space, i.e., transforms covariantly under rotations of the dreibein fields. Finally, $\rho_{\mu,ab}$ transforms under the same rotations in such a way that $(\partial_\mu + \rho_\mu)_{ab}$ transforms covariantly, i.e., $\rho_{\mu,ab}$ plays the role of a gauge field with respect to the gauge transformations which rotate the dreibein fields [7].

For later use let us define

$$\rho_{\mu\nu,ab} = (\partial_\mu \rho_\nu - \partial_\nu \rho_\mu + \rho_\mu \rho_\nu - \rho_\nu \rho_\mu)_{ab} \,, \tag{2.13}$$

which can be written

$$\rho_{\mu\nu,ab} = R_{klij} e_a^k e_b^l \partial_\mu \pi^i \partial_\nu \pi^j \,. \tag{2.14}$$

With the help of (2.11) the one-loop self-energy functional Σ_π can now be calculated in a chiral invariant manner and the divergent parts necessitating renormalization counterterms may then be isolated. As a matter of fact, we have essentially reduced the problem to a problem of QED where the evaluation of the self-energy functional leads to a similar formula. To extract the divergent terms invariantly also with respect to the dreibein rotations we can use the gauge invariant methods of Schwinger [3] and DeWitt [4, 5], working in momentum and co-ordinate space, respectively.

3. INVARIANT RENORMALIZATION OF THE BASIC LAGRANGIAN

The Green's function G_{ab} is defined by

$$S_{ab}G_{bc} = -\delta_{ac} , \tag{3.1}$$

or

$$[(\partial_\mu + \rho_\mu)^2_{ab} + C_{ab}]\, G_{bc} = \delta_{ac} .$$

To calculate the determinant of G_{ab} in a gauge-independent way we employ the proper-time formalism in coordinate space [4,5]. Since the pion is massless in a chiral invariant theory, we will encounter infra-red divergences in addition to the ultra-violet divergences. However, in reality the pion is massive. Including a mass term in (2.1) produces an additional term $m^2 D_{ab}$ in (3.1). D_{ab} will give rise to further ultra-violet divergences which are, of course, not chiral invariant, but those terms vanish in the limit $m \to 0$ in agreement with the basic condition that the renormalization procedure must not produce any non-invariant terms.

However, in addition to the ultra-violet divergences a mass term also produces terms in the one-loop self-energy functional which are not invariant and which do not vanish in the limit of zero pion mass (anomalous terms). Since we are interested in the possible cancellations of ultra-violet divergences here, we will not consider those anomalous terms in the following.

We make use of the following representation for the Green's function [3–5]

$$G_{ab} = -i \int_0^\infty \mathrm{d}s\, \langle x, s | x', 0\rangle_{ab}\, \mathrm{e}^{-im^2 s} . \tag{3.2}$$

As we are interested in the Feynman Green's function for the time being, a $-i\epsilon$ attached to m^2 is understood in (3.2).

The transition amplitude $\langle x, s | x', 0\rangle_{ab}$ is seen to satisfy the Schrödinger equation

$$i\, \frac{\partial}{\partial s} \langle x,s|x',0\rangle_{ab} = -[(\partial_\mu + \rho_\mu)^2_{ac} + \widetilde{C}_{ac}]\, \langle x,s|x',0\rangle_{cb} \tag{3.3}$$

and the boundary condition

$$\langle x,0|x',0\rangle_{ab} = \delta_{ab}\delta(x-x') , \tag{3.4}$$

where $\widetilde{C}_{ab} = C_{ab} + m^2(D_{ab} - \delta_{ab})$. Eq. (3.3) can be solved by the ansatz

$$\langle x, s | x', 0\rangle = -i(4\pi)^{-2} s^{-2}\, \mathrm{e}^{i\sigma/2s} \sum_{n=0}^{\infty} a_n (is)^n \,, \tag{3.5}$$

where $\sigma = \frac{1}{2}(x-x')^2$ and the a_n are matrices with indices a, b and x, x'.

To satisfy the boundary condition (3.4) one requires

$$\lim_{x' \to x} a_{0,ab}(x, x') = \delta_{ab} \,. \tag{3.6}$$

inserting (3.5) into (3.3) gives

$$\partial^\mu \sigma\, a_{0;\mu} = 0 \,, \tag{3.7}$$

and the recursion relations

$$\partial^\mu \sigma\, a_{n+1;\mu} + (n+1) a_{n+1} = a_{n;\mu}{}^{\mu} + \widetilde{C} a_n \,, \quad n = 0, 1, 2, \dots \,, \tag{3.8}$$

where $a_{n;\mu} \equiv (\partial_\mu + \rho_\mu) a_n$.

Let us note in passing that (3.6) and (3.7) are just the defining equations for the so-called gauge-dependent function [3, 4].

Repeated differentiation of (3.7) and going to the limit $x' \to x$ leads to

$$\lim_{x' \to x} a_{0;\mu} = 0 \,, \tag{3.9a}$$

$$\lim_{x' \to x} a_{0;\mu\nu} = -\tfrac{1}{2} \rho_{\mu\nu} \,, \tag{3.9b}$$

$$\lim_{x' \to x} a_{0;\mu}{}^{\mu}{}_{\nu}{}^{\nu} = \tfrac{1}{2} \rho_{\mu\nu} \rho^{\mu\nu} \,. \tag{3.9c}$$

In the following only $a_1(x,x)$ and $a_2(x,x)$ will be needed. From (3.8)

$$a_1(x,x) = \lim_{x' \to x} a_{0;\mu}{}^{\mu} + \widetilde{C} = \widetilde{C} \,, \tag{3.10}$$

$$2a_2(x,x) = a_{1;\mu}{}^{\mu}(x,x) + \widetilde{C} a_1(x,x) \,; \tag{3.11}$$

whereas (3.8) differentiated twice gives

$$\begin{aligned} 3a_{1,\mu}{}^{\mu}(x,x) &= \lim_{x' \to x} a_{0;\mu}{}^{\mu}{}_{\nu}{}^{\nu} + \widetilde{C}_{;\nu}{}^{\nu} \\ &= \tfrac{1}{2} \rho_{\mu\nu} \rho^{\mu\nu} + \widetilde{C}_{;\nu}{}^{\nu} . \end{aligned} \tag{3.12}$$

Thus

$$a_2(x,x) = \tfrac{1}{6}\left(\tfrac{1}{2} \rho_{\mu\nu} \rho^{\mu\nu} + \widetilde{C}_{;\mu}{}^{\mu} \right) + \tfrac{1}{2} \widetilde{C}^2 \,. \tag{3.13}$$

We are now in a position to evaluate the renormalization counterterms by calculating $\frac{1}{2} i \log \operatorname{Det} G$ and extracting the ultra-violet divergent terms. If we would use the Feynman Green's function for this purpose, we would get in addition to real counterterms divergent terms which are imaginary. This is related to the presence of so-called acausal loops and has been discussed extensively by DeWitt and others [4, 5]. Such terms also appear in the momentum space method, when one performs a rotation through 90° in the energy plane to convert from Minkowski space to Euclidean space; if the integral converges, this procedure is legitimate, but when the integral diverges the contribution from the arc at infinity corresponds exactly to one of the imaginary terms in co-ordinate space. Discarding those imaginary terms, as one usually does in momentum space calculations, corresponds precisely to eliminating the acausal loops in coordinate space. DeWitt [4] has given the recipe for calculating the corrected one-loop self-energy functional $\widetilde{\Sigma}_\pi$, which is

$$\widetilde{\Sigma}_\pi = \tfrac{1}{2}i \log \mathrm{Det}\, GG_0^{-1} - \tfrac{1}{2}i \log \mathrm{Det}\, \bar{G}\bar{G}_0^{-1} = T_\pi - T_\pi^0 \,, \tag{3.14}$$

where G is the Feynman Green's function and $\bar{G}$ is one half the sum of the retarded and the advanced Green's functions. Thus

$$\begin{aligned} T_\pi &\equiv \tfrac{1}{2}i(\mathrm{Tr}\log G - \mathrm{Tr}\log \bar{G}) \\ &= \tfrac{1}{2}\,\mathrm{Im}\int_0^\infty \mathrm{d}s\, s^{-1}\, \mathrm{Tr}\langle x, s | x', 0\rangle\, \mathrm{e}^{-im^2 s} \\ &= \tfrac{1}{2}\,\mathrm{Im}\int_0^\infty \mathrm{d}s\, s^{-1}\, \mathrm{e}^{-im^2 s} \int \mathrm{d}x\ \mathrm{tr}\,\langle x, s | x, 0\rangle \,. \end{aligned} \tag{3.15}$$

Using (3.5) and going to the limit of zero mass, one finds

$$\begin{aligned} T_\pi = (32\pi^2)^{-1} \{ &\int_0^\infty \mathrm{d}s\, s^{-3} + \int_0^\infty \mathrm{d}s\, s^{-2} \int \mathrm{d}x\ \mathrm{tr}\, a_1(x,x) \\ &+ \int_0^\infty \mathrm{d}s\, s^{-1} \int \mathrm{d}x\ \mathrm{tr}\, a_2(x,x) + \ldots \} \,, \end{aligned} \tag{3.16}$$

where we have omitted all terms that are not ultra-violet divergent; as mentioned above, those terms are infra-red divergent and/or anomalous when $m \to 0$.

The first term in (3.16) is cancelled by T_π^0 and we finally get

$$\widetilde{\Sigma}_\pi = c_1 \int \mathrm{d}x\ \mathrm{tr}\, a_1(x,x) + c_2 \int \mathrm{d}x\ \mathrm{tr}\, a_2(x,x) + \ldots \tag{3.17}$$

As s has dimensions $(\mathrm{mass})^{-2}$, c_1 is quadratically divergent, c_2 is logarithmically divergent in the usual momentum space terminology.

Using (3.10), (3.13) and the fact that isospace has constant curvature f_π^{-2} which implies [8]

$$R_{ijkl} = f_\pi^{-2}(g_{ik}g_{jl} - g_{il}g_{jk}) \tag{3.18}$$

we calculate (with $m = 0$)

$$\mathrm{tr}\, a_1(x,x) = C_{aa}(x) = \partial_\mu \pi^i \partial^\mu \pi^j R^k{}_{jki} = 2f_\pi^{-2} g_{ij} \partial_\mu \pi^i \partial^\mu \pi^j \,, \tag{3.19}$$

$$\begin{aligned}\mathrm{tr}\, a_2(x,x) &= \tfrac{1}{12}\, \rho_{\mu\nu,ab} \rho^{\mu\nu}{}_{ba} + \tfrac{1}{6}\, C_{;\mu}{}^{\mu}{}_{aa} + \tfrac{1}{2}\, C_{ab} C_{ba} \\ &= \partial_\mu \pi^i \partial^\mu \pi^r \partial_\nu \pi^j \partial^\nu \pi^m \left(\tfrac{1}{12} R_{klij} R^{lk}{}_{rm} + \tfrac{1}{2} R_{kilr} R^{l}{}_{j}{}^{k}{}_{m}\right) + \tfrac{1}{6}\, C_{;\mu}{}^{\mu}{}_{aa} \\ &= f_\pi^{-4} \{ \tfrac{2}{3} (g_{rj} \partial_\mu \pi^r \partial_\nu \pi^j)(g_{im} \partial^\mu \pi^i \partial^\nu \pi^m) \\ &\quad + \tfrac{1}{3} (g_{ir} \partial_\mu \pi^i \partial^\mu \pi^r)(g_{jm} \partial_\nu \pi^j \partial^\nu \pi^m)\} + \tfrac{1}{6}\, C_{;\nu}{}^{\nu}{}_{aa} \,.\end{aligned} \tag{3.20}$$

A straightforward calculation shows that

$$C_{;\mu}{}^{\mu}{}_{aa} = \Box (R_{ij} \partial_\mu \pi^i \partial^\mu \pi^j) \,; \tag{3.21}$$

thus $C_{;\mu}{}^{\mu}{}_{aa}$ is a total divergence and does not contribute to the integral over $\mathrm{tr}\, a_2(x,x)$. Therefore

$$\begin{aligned}\widetilde{\Sigma}_\pi &= 2\, c_1 f_\pi^{-2} \int \mathrm{d}x\, g_{ij} \partial_\mu \pi^i \partial^\mu \pi^j \\ &\quad + \tfrac{1}{3} c_2 f_\pi^{-4} \{ 2 \int \mathrm{d}x\, (g_{ij} \partial_\mu \pi^i \partial_\nu \pi^j)\, (g_{kl} \partial^\mu \pi^k \partial^\nu \pi^l) \\ &\quad + \int \mathrm{d}x\, (g_{ij} \partial_\mu \pi^i \partial^\mu \pi^j)^2 \} + \ldots\end{aligned} \tag{3.22}$$

Note that the divergent terms are manifestly chiral invariant and gauge invariant with respect to rotations of the dreibein fields. This establishes once more the invariant nature of the procedure we have used.

The quadratically divergent term has the form of the original action and may be combined with it to give a pion field strength renormalization and a renormalization of the pion decay constant f_π.

The logarithmically divergent terms are new ones. The next step consists, therefore, in adding counterterms of this type to the original action and checking whether the new theory is already renormalized, i.e., yields finite results in the one-loop approximation. We will show in sect. 4 that this is not the case; rather, an infinite number of divergent terms will appear.

In some recent papers the validity of the Adler condition in higher-order calculations, based on the chiral invariant pion Lagrangian (2.1), has been discussed. By using naive Feynman rules Charap [9] came to the conclusion that in all but the tree-graph diagrams the Adler condition is violated even in the symmetric case. Soon afterwards it was found [10] that the naive Feynman rules have to be modified and that the amplitudes calculated with the correct Feynman rules do indeed vanish at the Adler point.

We, therefore, consider it an interesting question to ask whether our renormalization procedure is compatible with the Adler condition in the presence of a symmetry breaking mass term. It is easily checked that the term $\frac{1}{2}\widetilde{C}^2$ in a_2 (3.13) will produce non-invariant expressions so that contrary, e.g., to the linear σ model, which is renormalizable [11], non-invariant counterterms are necessary. These in turn produce terms in the PCAC relation which contain derivatives of the pion field and can therefore not be eliminated by a pion field redefinition. Thus, the axial-vector current derived from the invariant part of the renormalized action in the usual way cannot be used to establish the Adler condition. However, Adler's theorem would still hold if one could find another axial-vector current that fulfils a PCAC relation with no derivatives on the right-hand side. The most straightforward way would be to add terms to the axial-vector current which might cancel the unwanted terms with pion field derivatives in the PCAC relation. This turns out not to be possible, mainly because of the presence of the terms $(g_{ij}\partial_\mu\pi^i\partial_\nu\pi^j)(g_{kl}\partial^\mu\pi^k\partial^\nu\pi^l)$ and $(g_{ij}\partial_\mu\pi^i\partial^\mu\pi^j)^2$ in (3.22), which have to be included in the renormalized Lagrangian and therefore appear in the field equations. In sect. 4 we will demonstrate that it is just the presence of those terms which makes the theory non-renormalizable.

The physically relevant consequences of the violation of the Adler condition for on-shell amplitudes would, however, only be additional terms of the order m^2, which are formally of the same order as the usual on-shell corrections to the soft-pion limit. Thus, the extra terms vanish in the limit $m \to 0$, so that the Adler condition is fulfilled for $m = 0$. However, because of the infra-red divergence of the one-loop self-energy functional this limit is only a formal one, especially in view of the fact that there are non-invariant (anomalous) pieces in the infra-red divergent part of $\widetilde{\Sigma}_\pi$.

4. NON-RENORMALIZABILITY OF THE MODIFIED LAGRANGIAN

The generalized action S' which we consider in this section has the form

$$\begin{aligned} S' = {} & \tfrac{1}{2}\int \mathrm{d}x\, g_{ij}\partial_\mu\pi^i\partial^\mu\pi^j \\ & + Z_1 f_\pi^{-4}\{2\int \mathrm{d}x\,(g_{ij}\partial_\mu\pi^i\partial_\nu\pi^j)(g_{kl}\partial^\mu\pi^k\partial^\nu\pi^l) \\ & + \int \mathrm{d}x\,(g_{ij}\partial_\mu\pi^i\partial^\mu\pi^j)^2\}\,. \end{aligned} \tag{4.1}$$

To evaluate the one-loop self-energy functional we calculate as before

$$S'_{ab} = e_a^k S'_{;kl} e_b^l\,, \tag{4.2}$$

which can be written as

$$S'_{ab} = -\{G^{\mu\nu}(\partial_\mu + \rho'_\mu)(\partial_\nu + \rho'_\nu)\}_{ab} + \ldots \tag{4.3}$$

where ρ'_μ is the analogue to ρ_μ in sect. 2 and there are no second derivatives in the terms which are omitted in (4.3). As we will only consider the highest divergences of S' in the one-loop approximation, the precise form of ρ'_μ and the left out terms in (4.3) do not concern us here. For the highest divergences, which will turn out to be of the quartic type, only G_{ab} is important which we write as

$$G^{\mu\nu}{}_{ab} = \eta^{\mu\nu}\delta_{ab} + h^{\mu\nu}{}_{ab} , \tag{4.4}$$

where

$$h^{\mu\nu}{}_{ab} = 4Z_1 f_\pi^{-4} \{\eta^{\mu\nu}[\delta_{ab}g_{ij}\partial_\lambda\pi^i\partial^\lambda\pi^j + 2e_{ai}e_{bj}\partial_\lambda\pi^i\partial^\lambda\pi^j]$$
$$+ 2\delta_{ab}g_{ij}\partial^\mu\pi^i\partial^\nu\pi^j + 4e_{ai}e_{bj}\partial^\mu\pi^i\partial^\nu\pi^j\} . \tag{4.5}$$

Now

$$\Sigma'_\pi = -\tfrac{1}{2} i \log \operatorname{Det}(S'_{ab}G_0)$$
$$= -\tfrac{1}{2} i \operatorname{Tr}\log[\delta_{ab}\delta(x-x') - h^{\mu\nu}{}_{ab}\partial_\mu\partial_\nu G_0 + \ldots]$$
$$= \tfrac{1}{2} i \operatorname{Tr}\sum_{n=1}^{\infty} \frac{1}{n}(QG_0)^n + \ldots , \tag{4.6}$$

where $Q_{ab} = h^{\mu\nu}{}_{ab}\partial_\mu\partial_\nu$ and again omitting terms with less than two derivatives. Written out in detail, the nth term in (4.6) has the form

$$\operatorname{Tr}(QG_0)^n = \operatorname{tr}\int \mathrm{d}x_1 \ldots \mathrm{d}x_n\, h^{\mu_1\mu_2}(x_1)h^{\mu_3\mu_4}(x_2)\ldots h^{\mu_{2n-1}\mu_{2n}}(x_n)$$
$$\times\, \partial^1_{\mu_1}\partial^1_{\mu_2}G_0(x_1,x_2)\,\partial^2_{\mu_3}\partial^2_{\mu_4}G_0(x_2,x_3)\ldots\partial^n_{\mu_{2n-1}}\partial^n_{\mu_{2n}}G_0(x_n,x_1) \tag{4.7}$$

and $\partial^k_{\mu_i} \equiv (\partial/\partial x_k{}^{\mu_i})$.

The most divergent contribution of (4.7) may now be isolated as follows. We expand all $h^{\mu_{2k-1}\mu_{2k}}(x_k)$ for $k \geqslant 2$ around x_1 and retain only the first terms of these expansions, i.e., $h^{\mu_{2k-1}\mu_{2k}}(x_1)$. Then, the integrations over $x_2, \ldots, x_n$ can be done leading to

$$\frac{1}{(2\pi)^4}\int \mathrm{d}p\, \frac{p_{\mu_1}p_{\mu_2}\ldots p_{\mu_{2n}}}{(p^2 - i\epsilon)^n} = iZ_2\Pi_{\mu_1\ldots\mu_{2n}} , \tag{4.8}$$

where $\Pi_{\mu_1 \dots \mu_{2n}}$ is the correctly normalized totally symmetric tensor constructed from products of $\eta_{\mu_i \mu_k}$

$$\Pi_{\mu_1 \dots \mu_{2n}} = \frac{1}{2^n (n+1)!} (\eta_{\mu_1 \mu_2} \eta_{\mu_3 \mu_4} \dots \eta_{\mu_{2n-1} \mu_{2n}} + \text{perm.}) , \qquad (4.9)$$

and we have used

$$\frac{1}{(2\pi)^4} \int \mathrm{d}p = - \lim_{\Delta \to \infty} \int \mathrm{d}p \, \frac{\Lambda^6}{(p^2 - \Lambda^2)^3} = \lim_{\Lambda \to \infty} \frac{i \Lambda^4}{32 \pi^2} \equiv i Z_2 . \qquad (4.10)$$

In exchanging the limit $\Lambda \to \infty$ and the integration over momentum space in (4.10) we automatically get rid of the contributions from the arc at infinity in the energy plane which result from a Wick rotation applied to (4.8). This is in accordance with the procedure in sect. 3 by which one eliminates the acausal loops.

Thus,

$$\widetilde{\Sigma}'_\pi = - Z_2 \int \mathrm{d}x \sum_{n=1}^{\infty} \operatorname{tr} P_n + \dots , \qquad (4.11)$$

with

$$P_n = \frac{1}{2n} \Pi_{\mu_1 \mu_2 \dots \mu_{2n}} h^{\mu_1 \mu_2} h^{\mu_3 \mu_4} \dots h^{\mu_{2n-1} \mu_{2n}} . \qquad (4.12)$$

Using the definition (4.5) of $h^{\mu\nu}_{ab}$ we get

$$\operatorname{tr} P_1 = 15 Z_1 f_\pi^{-4} g_{ij} \partial_\mu \pi^i \partial^\mu \pi^j , \qquad (4.13a)$$

$$\operatorname{tr} P_2 = \tfrac{10}{3} Z_1^2 f_\pi^{-8} \{ 19 (g_{ij} \partial_\mu \pi^i \partial^\mu \pi^j)^2$$

$$+ 14 (g_{ij} \partial_\mu \pi^i \partial_\nu \pi^j)(g_{kl} \partial^\mu \pi^k \partial^\nu \pi^l) \} , \qquad (4.13b)$$

and so on.

Since all the terms that appear in $\operatorname{tr} P_n$ have positive sign there are no cancellations possible and we may conclude that in the most divergent part of $\widetilde{\Sigma}'_\pi$, which is quartically divergent because of (4.10), arbitrary powers of $g_{ij} \partial_\mu \pi^i \partial^\mu \pi^j$ and $g_{ij} \partial_\mu \pi^i \partial_\nu \pi^j$ occur. Therefore, an infinite number of counterterms would be necessary to renormalize (4.1).

Let us finally emphasize once more the crucial point of our approach. Our method of renormalization ensures that the counterterms are explicitly chiral invariant. Therefore, no further cancellations of divergent quantities are possible which would be due to the invariance of the Lagrangians (2.1) and (4.1). In conclusion, we may state that the usual renormalization techniques do not provide a means of extracting finite results from the one-loop diagrams for the general chiral invariant pion Lagrangian (2.1).

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