Physical effects of the Immirzi parameter

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The Immirzi parameter is a constant appearing in the version of the general relativity action utilized as a starting point for the loop quantization of gravity. The parameter is commonly believed not to show up in the equations of motion, because it appears in front of a term in the action that vanishes on shell. We show that in the presence of fermions, instead, the Immirzi term of the action does not vanish on shell, and the Immirzi parameter appears in the equations of motion. It is the coupling constant of a parity violating four-fermion interaction. Therefore the nontriviality of the Immirzi parameter leads to effects that are observables in principle, even independently from nonperturbative quantum gravity.

Connection variables play an important role in the nonperturbative quantization of general relativity. Ashtekar realized that the gravitational field can be effectively described in terms of a selfdual $SL(2,\mathbb{C})$ Yang-Mills-like connection and its conjugate electric field, satisfying appropriate reality conditions [1], and loop quantum gravity [2] started as a canonical quantization of general relativity using these self-dual variables. In order to overcome difficulties related to the implementation of the reality conditions in the quantum theory and to the noncompatness of $SL(2,\mathbb{C})$, the attention has later shifted to a real $SU(2)$ connection, known as the Barbero connection [3]. An action functional that leads directly to both the Ashtekar or Barbero formalism is the following, called the Holst action [4]

$$S[e,A] = \frac{1}{16\pi G}\left(\int d^4x\ ee^a_I e^b_J\ F^{IJ}_{ab} - \frac{1}{\gamma}\int d^4x\ ee^a_I e^b_J\ {}^*F^{IJ}_{ab}\right). \tag{1}$$

Here $I, J \ldots = 0,1,2,3$ are internal Lorentz indices and $a, b \ldots = 0,1,2,3$ are space-time indices. The field e^I_a is the tetrad field, e is its determinant and e^a_I its inverse; A^{IJ}_a is a Lorentz connection. F is the curvature of A and ${}^*F$ is its dual, defined by ${}^*F^{IJ} = \frac{1}{2}\epsilon^{IJ}{}_{KL}F^{KL}$. The coupling constant γ is the Immirzi parameter [5]. The choice $\gamma = i$ leads to the self-dual Ashtekar canonical formalism, while a real γ leads to the $SU(2)$ Barbero connection.

The first term in (1) is the familiar tetrad-Palatini action of general relativity. The second term does not affect the equations of motion, for the following reason. The equations of motion for the connection turn out to be Cartan's first structure equation

$$D_{[a}e^I_{b]} = 0, \tag{2}$$

where D_a is the covariant derivative defined by A. The solution of this equation is that A is the torsion-free spin-connection $\omega[e]$ of the tetrad field e. If we thus replace A by $\omega[e]$ in (1), the first term becomes the tetrad expression of the Einstein-Hilbert action, while the second term is identically zero, due to the Bianchi identities $R_{[abc]d} = 0$. Stationarity with respect to the variation of the tetrad yield then the Einstein equations. Therefore, as it is often stressed, the Immirzi parameter γ does not appear to have any effect on the equations of motion.

The parameter plays an important role in loop quantum gravity, where the spectrum of quantum geometry operators is modulated by its value. For instance, the area of a surface and the volume of a space region are quantized in units of $\gamma\ell_p^2$ and $\gamma^{3/2}\ell_p^3$ respectively. Furthermore, the nonperturbative calculation of the entropy of a black hole appears to yield a result compatible with Hawking's semi-classical formula only for a specific value of γ. See [6] for recent evaluations and references. The role of the Immirzi parameter is often compared with the role of the Θ angle in QCD—which also appears as a constant in front of a term in the action with no effect on the equations of motion: a parameter that governs only nonperturbative quantum effects. See for instance [7]. In this letter we point out that this is in fact not the case in general.

The catch is that the second term in (1) vanishes only when equation (2) is satisfied, but equation (2) is modified in the presence of fermions (or, more in general, matter field that couple to the connection). In the presence of a fermion field, (1) becomes

$$S[e,A,\psi] = S[e,A] + \int d^4x\ e\ \overline{\psi}\,\gamma^I e^a_I D_a\psi, \tag{3}$$

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where γ^I are the Dirac matrices; and (2) becomes

$$D_{[a}e^I_{b]} = fermion\ current. \tag{4}$$

The fermion current acts as a source for a torsion component in the connection.

In the following we compute the equations of motion of (3), namely the equations of motion of general relativity coupled with a fermionic field in the presence of a nonvanishing Immirzi parameter. We solve explicitly the equations for the connection. By inserting the solution into the action we obtain an effective action that contains a parity-violating four-fermion interaction. (The fact that the second term in (1) violates parity is discussed for instance in [8]). The coupling constant of this term depends explicitly on the Immirzi parameter.

Let us start by introducing the tensor

$$p^{IJ}{}_{KL} = \frac{1}{2}\epsilon^{IJ}{}_{KL} - \frac{1}{\gamma}(\delta^I_K\delta^J_L - \delta^I_L\delta^J_K) \tag{5}$$

and its inverse

$$p^{-1}{}_{KL}{}^{IJ} = \frac{\gamma}{\gamma^2+1}\left(-\frac{1}{2}\epsilon^{IJ}{}_{KL} + \gamma(\delta^I_K\delta^J_L - \delta^I_L\delta^J_K)\right). \tag{6}$$

Using this, the action (3) can be written in the form

$$S[e, A, \psi] = \frac{1}{16\pi G}\int d^4x\ e e^a_I e^b_J\ p^{IJ}{}_{KL}F^{KL}_{ab} + \int d^4x\ e\ \overline{\psi}\,\gamma^I e^a_I D_a\psi, \tag{7}$$

and the equation of motion for the connection reads

$$p^{IJ}{}_{KL}D_b(e\, e^b_I e^a_J) = 8\pi G e T_{KL}{}^a, \tag{8}$$

where the fermion current $T_{KL}{}^a$ is defined by

$$T_{KL}{}^a = \frac{\delta S[e, A, \psi]}{\delta A^{KL}_a}. \tag{9}$$

Recalling that $D_a\psi = \partial_a\psi - A^{KL}_a\gamma_K\gamma_L\psi$, and using the identity $\gamma^A\gamma^{[B}\gamma^{C]} = -i\epsilon^{ABCD}\ \gamma_5\gamma_D + 2\eta^{A[B}\gamma^{C]}$, we can write this current as

$$T_{KL}{}^a = 2e^a{}_{[K}\ j_{{}_v L]} - ie^a_I\epsilon^I{}_{KLJ}\ j^J_{{}_a} \tag{10}$$

where

$$j^K_{{}_v} \ = \ \overline{\psi}\,\gamma^K\,\psi, \tag{11}$$

$$j^K_{{}_a} \ = \ \overline{\psi}\,\gamma_5\gamma^K\,\psi \tag{12}$$

are a vector and an axial fermion currents.

Using the inverse tensor (6) equation (8) gives

$$D_b(e\, e^b_I e^a_J) = 8\pi G e\ p^{-1}{}_{IJ}{}^{KL}T_{KL}{}^a \equiv 8\pi G e\ J_{IJ}{}^a. \tag{13}$$

Equation (13) can be solved for the connection. For this, we write the connection in the form $A^{IJ}_a = \omega[e]^{IJ}_a + C_a{}^{IJ}$, where ω is the torsion free spin connection determined by e, namely the solution of (2), and C is the torsion. Using this, (13) gives

$$C_{a[K}{}^a e_{L]}{}^b + C_{[KL]}{}^a = 8\pi G\ J_{IK}{}^a, \tag{14}$$

Notice that we transform internal and spacetime indices into each other, using the tetrad field, and preserving the horizontal order of the indices. This equation can be solved by contracting the indices, and then summing terms with cyclical permutation of the indices: it is easy to verify that the solution is

$$C_a{}^{IJ} = 8\pi G\frac{\gamma}{\gamma^2+1}\left(-2e^{[I}_a\ (\gamma j^{J]}_{{}_v} - ij^{J]}_{{}_a}) + ie^K_a\,\epsilon_K{}^{KIJ}{}_L(\gamma j^L_{{}_a} - ij^L_{{}_v})\right). \tag{15}$$

We can now obtain an equivalent action by replacing A with $\omega + C$ in (3). It is easy to see that the terms linear in the fermion current are total derivatives, leaving

$$S[e,\psi] = S[e] + S_f[e,\psi] + S_{int}[e,\psi], \tag{16}$$

where the first two terms are the standard second order tetrad action of general relativity with fermions,

$$S[e] + S_f[e,\psi] = \frac{1}{16\pi G}\int d^4x\ e e^a_I e^b_J\ F^{IJ}_{ab}[\omega[e]] \quad + \quad \int d^4x\ e\ \overline{\psi}\,\gamma^I e^a_I D_a[\omega[e]]\psi \tag{17}$$

and the interaction term can be obtained by a tedious but straightforward calculation as

$$S_{int}[e,\psi] = -72\pi G\frac{\gamma}{\gamma^2+1}\int d^4x\ e\ \left(\gamma\ j_{\scriptscriptstyle v}\!\cdot\! j_{\scriptscriptstyle v}\ - 2i\ j_{\scriptscriptstyle v}\!\cdot\! j_{\scriptscriptstyle a}\ + \gamma\ j_{\scriptscriptstyle a}\!\cdot\! j_{\scriptscriptstyle a}\right). \tag{18}$$

where $j\cdot j \equiv j_I j^I$. This term describes a four-fermion interaction mediated by a non-propagating torsion.

In the limit $\gamma \to \infty$, the four-fermion term becomes the standard parity-invariant torsion interaction term of Einstein-Cartan theory. The characteristic effect of a nonvanishing Immirzi parameter term in Holst action is therefore the parity violating term $j_{\scriptscriptstyle v}\!\cdot\! j_{\scriptscriptstyle a}$, which reads explicitly as

$$S_{pv}[e,\psi] = \frac{144\, i\,\pi\,\gamma\, G}{\gamma^2+1}\ \int d^4x\ e\ (\overline{\psi}\gamma_A\psi)\ (\overline{\psi}\gamma_5\gamma^A\psi). \tag{19}$$

In conclusion, general relativity admits the natural formulation given by the action (3), widely used as a starting point for the nonperturbative quantization of the theory. This formulation includes a four-fermion interaction mediated by the torsion, which includes the parity violating component (19) whose coupling constant is determined by the Immirzi parameter. The four-fermion interaction is weak, because it is suppressed by one power of the Newton constant. It is therefore compatible with all present observations, as far as we can see. The interaction is present also on a flat spacetime.

The value of the Immirzi parameter is therefore observable in principle also independently from its effect on the nonperturbative quantum theory. The analogy with the Θ angle of QCD is, in this regard, misleading (see also [9]). The Immirzi parameter is precisely the ratio between the strengths of the parity preserving and the parity violating four fermion interactions.

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