Detailed calculation of the complete two-loop Higgs-Yukawa beta function in an arbitrary α gauge

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(Received 16 May 1983)

We present a complete calculation of the two-loop contributions to the Higgs-Yukawa beta function, giving the evaluations of individual Feynman diagrams. We calculate in an arbitrary α gauge, and in a range of subtraction schemes that includes minimal subtraction (MS) and modified minimal subtraction ($\overline{\text{MS}}$). We compute the beta function explicitly for the Weinberg-Salam theory, but our results should be readily adaptable to the computation of the beta function in other theories.

INTRODUCTION

The renormalization-group evolution of Yukawa couplings is important in many contexts and in some cases one needs to know this evolution quite precisely. For instance, we have recently carried out a complete two-loop calculation of the bottom-quark mass within the framework of the SU(5) model.[1] By computing to this accuracy we were able to set limits on the number of generations, and we found that three and four, but not more than four, generations gave predictions consistent with the observed value of M_b. This conclusion disagreed with the results of an earlier partial two-loop calculation.[2]

Limits on the number of generations have also been derived in this way for supersymmetric unified theories.[3]

Another example is the prediction of the masses of heavy fermions ($\lesssim 300$ GeV). We have calculated these quantities to two loops by making use of properties of the Yukawa renormalization-group equations—the existence of so-called infrared pseudofixed points.[4] Two-loop precision is desirable here because the fermions are heavy, which means that their Yukawa couplings can be large.

More recently in supergravity theories the Yukawa evolution has been found to be important in determining the low-energy spontaneous-symmetry-breaking scale.[5]

The evolution of a Yukawa coupling is of course described by the corresponding beta function. For our SU(5) prediction of M_b we calculated the complete two-loop beta function for this coupling in the Weinberg-Salam theory. This calculation included all strong, electroweak, Yukawa, and scalar self-coupling terms, and was performed in an arbitrary α gauge, and in a range of subtraction schemes comprising minimal subtraction (MS) and modified minimal subtraction ($\overline{\text{MS}}$). We have briefly reported the results elsewhere.[6]

In this paper we present the details of our calculation of the Higgs-Yukawa beta function, β_{HY}, giving the evaluation of individual Feynman diagrams. Though we compute the beta function within the context of the Weinberg-Salam theory, our results should be readily adaptable to the computation of β_{HY} in other theories. Our explicit calculation of the gauge dependence should be particularly useful in any such computation as a check. For the Weinberg-Salam theory the requirement that the gauge dependence eventually cancel gave us a total of 66 additive checks on the correctness of our results.

We also calculate the two-loop anomalous dimensions for the Higgs bosons and the fermions. The Higgs anomalous dimension, though gauge variant, will be useful in calculating β_λ for the scalar self-coupling. The anomalous dimensions have been in part calculated previously; we agree with the earlier results.[7]

The organization of the paper is as follows. In Sec. I, we describe the overall framework of our calculation, and give the Lagrangian we employ. In Sec. II, we briefly explain how the beta function is extracted from a computation of the infinite parts of the diagrams. In Sec. III, we describe our subtraction schemes and the way in which the scheme dependence cancels diagram by diagram.

Finally, in Sec. IV, we present the contributions of the individual diagrams to the beta function, and our results for the Higgs and fermion anomalous dimensions.

I. FRAMEWORK

We are interested in computing the Higgs-Yukawa beta function for the standard Weinberg-Salam theory. To calculate we work with a slightly generalized version of this theory, so as to gain additional checks on our results as well as broader applicability.

(1) We take the color gauge group to be SU(N_c), $N_c \geq 3$, with fermions in either the fundamental or singlet representation (for quarks and leptons, respectively).

(2) Left-handed fermions and also the Higgs bosons are taken to be in N_2-dimensional representations of $\text{SU(2)}_{\text{weak}}$, with $N_2 \geq 2$. However, we include just the usual *pairs* of SU(2)-singlet right-handed fermions (e.g., top and bottom). (Thus N_2-2 of the left-handed fermions in a multiplet have no right-handed partner and remain massless.)

(3) The left-handed fermions are assigned arbitrary hypercharge y. Their right-handed partners then have hypercharge $y+1$ $(y-1)$ for an upper (lower) flavor. The Higgs bosons have as usual hypercharge $+1$.

(4) We assume that there are N_H Higgs bosons, only one of which couples to the fermions.

(5) N_g denotes the number of generations.

Also important are the following features of our calculation. It is carried out in an arbitrary α gauge, where the gauge propagator is $-(i/k^2)[g_{\mu\nu}-(1-\alpha)k_\mu k_\nu/k^2]$. We use dimensional regularization, discussing a range of MS-type schemes. The second-order term in β is both gauge and scheme independent for these schemes, but the explicit cancellation of the dependence in individual sectors gives us stringent checks on the validity of our results. There are in all 66 additive checks due to gauge invariance, and the sum of each diagram separately with its counterterms is scheme independent (for the range of schemes we consider).

Lastly, because we are concerned with the high-energy evolution of coupling constants, we work with the high-energy symmetric form of the theory. There is no unphysical Higgs particle and the W bosons are massless. Also we take the Higgs particle to be massless. As usual we ignore the hierarchy problem, assuming that quadratic divergences can simply be dropped.

For completeness we give below our working Lagrangian. We write down terms for a single generation only. The gauge coupling constants, g_1, g_2 and g_3, refer to the U(1), weak and colored groups, respectively. λ_2^a and λ_3^a are matrices, elements of the Lie algebra, for the weak and colored groups. They are normalized so that in the fundamental representation $\mathrm{Tr}(\lambda^2)=\frac{1}{2}$.

The Higgs-Yukawa coupling constants are labeled g_t and g_b for an upper and lower flavor. The matrix $i\sigma_2$ is a generalization of the usual Pauli matrix, and allows the complex-conjugate Higgs field $\phi^\dagger$ to give mass to the t quark.

The Lagrangian is

$$\begin{aligned}\mathscr{L}=&\bar{\psi}_L i\gamma_\mu\left[\partial^\mu+ig_1\frac{y}{2}B^\mu+ig_2W^\mu\cdot\lambda_2+ig_3G^\mu\cdot\lambda_3\right]\psi_L\\&+\bar{T}_r i\gamma_\mu\left[\partial^\mu+ig_1\frac{(y+1)}{2}B^\mu+ig_3G^\mu\cdot\lambda_3\right]T_r+\bar{B}_r i\gamma_\mu\left[\partial^\mu+ig_1\frac{(y-1)}{2}B^\mu+ig_3G^\mu\cdot\lambda_3\right]B_r\\&+|(\partial_\mu+ig_1\tfrac{1}{2}B_\mu+ig_2W_\mu\cdot\lambda_2)\phi|^2-\frac{\lambda}{6}|\phi^\dagger\phi|^2-g_B(\bar{\psi}_L\phi B_r+\mathrm{H.c.})-g_T[\bar{\psi}_L(i\sigma_2)\phi^\dagger T_r+\mathrm{H.c.}]\\&-\tfrac{1}{4}(F^1_{\mu\nu}F_1^{\mu\nu}+F^2_{\mu\nu}F_2^{\mu\nu}+F^3_{\mu\nu}F_3^{\mu\nu})-\frac{1}{2\alpha}[(\partial_\mu B^\mu)^2+(\partial_\mu W^\mu)^2+(\partial_\mu G^\mu)^2]\\&+(\partial^\mu\eta_2^{a+})(\delta_{ab}\partial_\mu-g_2C_2^{abc}W_\mu^b)\eta_2^c+(\partial^\mu\eta_3^{a\dagger})(\delta_{ab}\partial_\mu-g_3C_3^{abc}G_\mu^b)\eta_3^c\ .\end{aligned}\tag{1}$$

With this Lagrangian, then, for a three-point diagram with an incoming B_r and an outgoing B_L, the neutral Higgs particle is incoming.

Note that we choose the same gauge for all gauge fields.

II. EXTRACTING THE BETA FUNCTION

The quantities actually computed are the $1/\epsilon$ poles in the two-point (Higgs-particle and fermion self-energies) and the three-point (Yukawa) Green's functions. (Here $\epsilon=4-d$, where d is the number of dimensions.) In an MS-type scheme these define the renormalization factors Z_{ψ_L}, Z_{ψ_r}, Z_ϕ, and Z_{coupling} where

$$\begin{aligned}&\psi^{L,r}_{\text{un}}=Z_{\psi_{L,r}}{}^{1/2}\psi^{L,r}_{\text{ren}}\ ,\\&\phi_{\text{un}}=Z_\phi{}^{1/2}\phi_{\text{ren}}\ ,\\&g_{\text{un}}\bar{\psi}^L_{\text{un}}\phi_{\text{un}}\psi^r_{\text{un}}=Z_{\text{coupling}}g_{\text{ren}}\bar{\psi}^L_{\text{ren}}\phi_{\text{ren}}\psi^r_{\text{ren}}\ .\end{aligned}\tag{2}$$

For an MS-type scheme β can be obtained very simply from these quantities.

Suppose we are interested in β for g_T. The relevant Z is Z_{g_T}, defined by

$$g_T^{\text{un}}=Z_{g_T}g_T^{\text{ren}}\ .\tag{3}$$

If Z_{g_T} is expanded as

$$Z_{g_T}=1+Z^s_{g_T}\frac{1}{\epsilon}+\cdots\ ,\tag{4}$$

Then β is given by[8]

$$\beta_T=\tfrac{1}{2}g_T\left[g_T\frac{d}{dg_T}+g_B\frac{d}{dg_B}+\sum_{i=1}^{3}g_i\frac{d}{dg_i}+2\lambda\frac{d}{d\lambda}\right]Z^s_{g_T}\ .\tag{5}$$

(The expression in parentheses is intended to represent a sum over *all* coupling constants, including those for fermions of other generations.)

Now since

$$Z_{g_T}=Z_{\text{coupling}}/(Z_{\psi_L}Z_{\psi_r}Z_\phi)^{1/2}$$

expanding the Z's in $1/\epsilon$ gives

$$Z^s_{g_T}=Z^s_{\text{coupling}}-\tfrac{1}{2}(Z^s_{\psi_L}+Z^s_{\psi_r}+Z^s_\phi)\tag{6}$$

for the leading terms. The superscript s indicates the coefficient of the simple pole. Also, the operator in parentheses in Eq. (5) acts very simply on Z_g^s, so that

$$\beta_{g_T}^{(1)}=g_T[Z^{1,s}_{\text{coupling}}-\tfrac{1}{2}(Z^{1,s}_{\psi_L}+Z^{1,s}_{\psi_r}+Z^{1,s}_{\phi})]\ ,$$
$$\beta_{g_T}^{(2)}=2g_T[Z^{2,s}_{\text{coupling}}-\tfrac{1}{2}(Z^{2,s}_{\psi_L}+Z^{2,s}_{\psi_r}+Z^{2,s}_{\phi})]\ , \tag{7}$$

where Z^1 and $\beta^{(1)}$, Z^2 and $\beta^{(2)}$, are the one- and two-loop terms of Z and β. Thus β has been related to a simple combination of the $1/\epsilon$ poles of the two- and three-point Green's functions.

By a similar argument, one can show that the two-loop anomalous dimensions are

$$\gamma_{\psi_{L,r}}=-Z^s_{\psi_{L,r}},\ \ \gamma_\phi=-Z^s_\phi\ . \tag{8}$$

Referring to Eq. (7), one sees that the anomalous dimensions are exactly the appropriate self-energy contributions to the beta function.

III. SCHEME AND GAUGE DEPENDENCE

The beta function through second order is gauge and scheme independent for the range of schemes we employ. Furthermore, our Lagrangian contains several arbitrary parameters, and our results must be gauge invariant regardless of what values they have. Therefore individual terms of β, e.g., the coefficients of N_c, N_2, y, y^2, y^3, etc., will be gauge invariant by themselves. The 80 odd diagrams can be grouped into overlapping subclasses, where each subclass contributes to a different term and is separately gauge invariant for the correct combination of its diagrams. As mentioned earlier, we have then a total of 66 additive checks on our calculation.

The scheme independence of the second-order term of β gives additional checks. It turns out that the sum of each diagram with its counterterms is individually scheme independent for our schemes.

The schemes that we utilize[9] can be described as minimal subtraction with a modified definition of d-dimensional integration. The counterterms are chosen as usual to exactly cancel the poles in $1/\epsilon$, but the standard formulas for integrals are multiplied by some smooth function of d that goes to 1 at $d=4$. For example, we define

$$\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2+m^2)^\alpha}=\frac{1}{N(d)}(\text{standard definition})$$
$$=\frac{1}{N(d)}\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(\alpha-d/2)}{\Gamma(\alpha)(m^2)^{\alpha-d/2}}\ , \tag{9}$$

where $N(d)$ is such a function. In this approach MS corresponds to chosing $N(d)=1$, and $\overline{\text{MS}}$ to $N(d)=(4\pi)^{4-d}/\Gamma(d-3)$.

The great advantage of these schemes is that the simple expression for β derived earlier in Eq. (7) remains valid.

Scheme independence then follows from the fact that when one computes in one of these schemes, say in MS, the coefficient of $1/\epsilon^2$ for each diagram is $-\frac{1}{2}$ times the corresponding counterterm coefficient. It is easy to see how this relationship arises computationally. Crudely, it is due to the fact that when calculating a counterterm one is really computing two independent integrals: one to determine the counterterm insertion, and the second to evaluate a diagram with this insertion. On the other hand, for a two-loop diagram the two integrations are linked by their dependence on a common momentum.

The relationship of $1/\epsilon^2$ coefficients leads to the cancellation of scheme dependence as follows. The two-loop diagram in our scheme has one $1/N(d)$ factor for each loop integration—two in all. But for the counterterm one of these integrals is truncated. The pole part only of the counterterm insertion is used, and this is proportional to $1/N(4)$ or 1. Thus the insertion is independent of the behavior of N away from $d=4$, and the whole counterterm is proportional to $1/N(d)$ only. Then if a diagram is evaluated to be

$$\frac{1}{N^2(\epsilon)}\left[\frac{A}{\epsilon^2}+\frac{B}{\epsilon}+O(1)\right]=\frac{A}{\epsilon^2}+\frac{B-2N_1A}{\epsilon}+O(1) \tag{10}$$

with $N=1+N_1\epsilon+O(\epsilon^2)$ and the sum of its counterdiagrams is

$$\frac{1}{N(\epsilon)}\left[\frac{-2A}{\epsilon^2}+\frac{C}{\epsilon}+O(1)\right]=\frac{-2A}{\epsilon^2}+\frac{C+2N_1A}{\epsilon}+O(1)\ , \tag{11}$$

the total is

$$\frac{-A}{\epsilon^2}+\frac{B+C}{\epsilon}+O(1)\ . \tag{12}$$

The simple pole, which alone contributes to β, is independent of N.

One would not expect cancellation of scheme dependence by this mechanism in higher loops.

IV. CONTRIBUTIONS OF THE DIAGRAMS TO β_B

In this section we give the contributions of individual diagrams to β_B, the beta function for g_B (rather than the actual value of the diagrams). Here g_B represents the coupling of the Higgs particle to any down-type fermion. β for g_T can be obtained by letting $g_B \leftrightarrow g_T$ and $y\to -y$ in each generation.

Note that a listed contribution represents the *sum* of the illustrated diagram with its associated counterterm diagrams, where the latter are obtained by replacing a divergent subdiagram with minus its $1/\epsilon$ pole part.

We use the following notation.

(1) g_H represents any Higgs-Yukawa coupling; g_i is any gauge coupling.

(2) $N_3=N_c$ for a quark, $N_3=1$ for a lepton. $D_{3(2)}$ is the dimension of the adjoint representation for $SU(N_c)$ [SU(2)]. Thus $D_3={N_c}^2-1$ and $D_2=3$.

(3) We denote Casimir constants as follows. $R_{3(2)}$ is the Casimir constant for the fundamental representation of $SU(N_c)$ [SU(2)]. Thus, e.g., $\lambda_3^a\lambda_3^a=R_3\mathbb{1}$ for λ_3 in the fundamental representation of $SU(N_c)$. $G_{3(2)}$ is the Casimir constant for the adjoint representation of $SU(N_c)$ [SU(2)].

For the standard theory, with $N_c=3$,

$$R_3=\tfrac{4}{3},\quad R_2=\tfrac{3}{4},\quad G_3=3,\quad G_2=2\ .$$

For a lepton, we set $R_3=0$.

(4) For diagrams with fermion loops, fermions of any generation can contribute. We use barred coupling constants, e.g., $\bar g_B$, to indicate that a sum over *all* fermions with the corresponding T_3 should be performed. Similarly, $\bar y$ or $\bar N_3$ signifies that in this sum the appropriate values of y or N_2 should be substituted. For example,

$$\begin{aligned}\bar g_B{}^2\bar N_3\bar y\equiv{}& g_B{}^2N_c\tfrac{1}{3}+g_\tau{}^2(1)(-1)\\&+G_S{}^2N_c\tfrac{1}{3}+g_\mu{}^2(1)(-1)\cdots\ .\end{aligned}\tag{13}$$

(5) The gauge propagator is $-(i/k^2)[g_{\mu\nu}-(1-\alpha)k_\mu k_\nu/k^2]$. To obtain β for g_T, take $g_B\leftrightarrow g_T$, $\bar g_B\leftrightarrow\bar g_T, y\to -y, \bar y\to-\bar y$.

In part IV A, we give the Higgs-particle self-energy contributions and their total, the Higgs-particle anomalous dimension. In IV B, we give the fermion self-energy contributions and anomalous dimensions, separating right-handed from left-handed, and in IV C, the three-point results.

A. Higgs-particle self-energy contributions

The relevant diagrams appear in Figs. 1, 2, 3, and 4(a) and 4(b). All expressions should be multiplied by $g_B/(16\pi^2)^2$ to get the true contribution to β_B. The $g_H{}^5$ contributions to β_B derive from the diagrams in Fig. 1:

$$\begin{aligned}&1(\text{a}):\ [\tfrac{1}{2}(\bar g_B{}^4+\bar g_T{}^4)(N_2+1)+\bar g_B{}^2\bar g_T{}^2]\bar N_3\{-\tfrac{3}{2}\}\ ,\\&1(\text{b}):\ (-\bar g_B{}^2\bar g_T{}^2)\bar N_3\{-2\}\ .\end{aligned}\tag{14}$$

The $g_H{}^3g_i{}^2$ terms come from the diagrams in Fig. 2:

$$\begin{aligned}2(\text{a}):\ &\{-[g_3{}^2R_3(\bar g_B{}^2+\bar g_T{}^2)]\bar N_3\\&-\tfrac{1}{2}[g_2{}^2R_2(\bar g_B{}^2+\bar g_T{}^2)]\bar N_3\\&-\tfrac{1}{4}g_1{}^2(\bar g_B{}^2[\bar y^2-\bar y+\tfrac{1}{2}]\\&+\bar g_T{}^2[\bar y^2+\bar y+\tfrac{1}{2}])\bar N_3\}\{\alpha\}\ ,\\2(\text{b}):\ &\{-\tfrac{1}{2}[g_3{}^2R_3(\bar g_B{}^2+\bar g_T{}^2)]\bar N_3-\tfrac{1}{8}[g_1{}^2(\bar g_B{}^2[\bar y^2-\bar y]\\&+\bar g_T{}^2[\bar y^2+\bar y])]\bar N_3\}\{-10-2\alpha\}\ ,\\2(\text{c}):\ &\{-\tfrac{1}{2}[g_2{}^2R_2(\bar g_B{}^2+\bar g_T{}^2)]\bar N_3\\&-\tfrac{1}{8}[g_1{}^2(\bar g_B{}^2+\bar g_T{}^2)]\bar N_3\}\{-7+5\alpha\}\ ,\\2(\text{d}):\ &\{-[g_2{}^2R_2(\bar g_B{}^2+\bar g_T{}^2)]\bar N_3\\&-\tfrac{1}{4}[g_1{}^2(\bar g_B{}^2+\bar g_T{}^2)]\bar N_3\}\{1-3\alpha\}\ .\end{aligned}\tag{15}$$

FIG. 1. $g_H{}^4$ contributions to the Higgs-particle anomalous dimension.

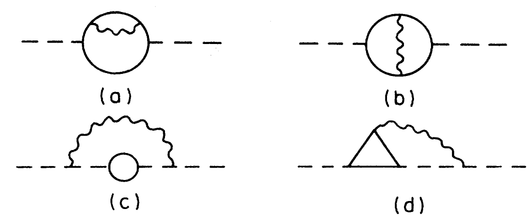


FIG. 2. $g_H{}^2g_i{}^2$ contributions to the Higgs-particle anomalous dimension.

Figure 3 contributes to terms in β_B proportional to $g_Bg_i{}^4$,

$$\begin{aligned}3(\text{a}):\ &-2[g_2{}^4R_2G_2]\{\tfrac{53}{48}+\tfrac{11}{8}\alpha+\tfrac{1}{2}\alpha^2\}\ ,\\3(\text{b}):\ &-2[g_2{}^4R_2G_2]\{\tfrac{11}{48}-\tfrac{1}{8}\alpha^2\}\ ,\\3(\text{c}):\ &-2[g_2{}^4R_2G_2]\{\tfrac{63}{16}-\tfrac{15}{8}\alpha-\alpha^2\}\ ,\\3(\text{d}):\ &\{-[g_2{}^4R_2\bar R_2N_2/D_2]N_H-\tfrac{1}{16}[g_1{}^4N_2N_H]\}\{-\tfrac{11}{6}\}\ ,\\3(\text{e}):\ &\{\tfrac{1}{2}[g_2{}^4R_2\bar R_2N_2/D_2]\bar N_3\\&+\tfrac{1}{32}[g_1{}^4(\bar y^2(N_2+2)+2)]\bar N_3\}\{\tfrac{10}{3}\}\ ,\\3(\text{f}):\ &\{-[g_2{}^4R_2{}^2]-\tfrac{1}{2}[g_1{}^2g_2{}^2R_2]\\&-\tfrac{1}{16}[g_1{}^4]\}\{\tfrac{9}{2}-3\alpha+\tfrac{1}{2}\alpha^2\}\ ,\\3(\text{g}):\ &\{4[g_2{}^4(R_2{}^2-\tfrac{1}{4}R_2G_2)]\\&+2[g_1{}^2g_2{}^2R_2]+\tfrac{1}{4}[g_1{}^4]\}\{\tfrac{9}{8}-\tfrac{1}{2}\alpha-\alpha^2\}\ ,\\3(\text{h}):\ &0\ ,\\3(\text{i}):\ &\{-2[g_2{}^4(R_2{}^2-\tfrac{1}{4}R_2G_2)]\\&-[g_1{}^2g_2{}^2R_2]-\tfrac{1}{8}[g_1{}^4]\}\{-\tfrac{3}{2}-\tfrac{1}{2}\alpha^2\}\ ,\\3(\text{j}):\ &\{-[g_2{}^4(R_2{}^2-\tfrac{1}{2}R_2G_2)]\\&-\tfrac{1}{2}[g_1{}^2g_2{}^2R_2]-\tfrac{1}{16}[g_1{}^4]\}\{\tfrac{3}{2}+\alpha-\tfrac{7}{2}\alpha^2\}\ .\end{aligned}\tag{16}$$

The following are contributions involving the four-point coupling:

$$\begin{aligned}&4(\text{a}):\ -\frac{\lambda^2}{9}(N_2+1)\{-\tfrac{1}{2}\}\ ,\\&4(\text{b}):\ 0\ .\end{aligned}\tag{17}$$

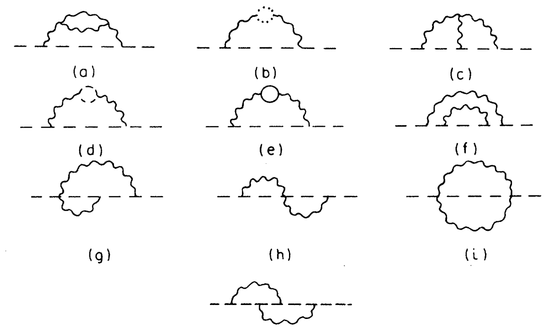
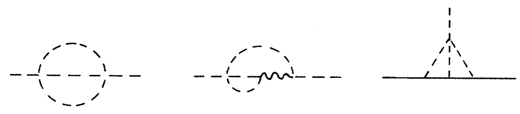
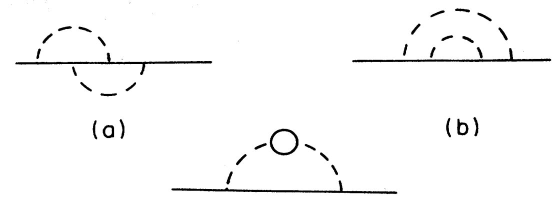


FIG. 3. $g_i{}^4$ contributions to the Higgs-particle anomalous dimension.



(a) (b) (c)

FIG. 4. Contributions to β involving the scalar self-coupling.



(a) (b) (c)

FIG. 5. $g_H{}^4$ contributions to the fermion anomalous dimension.

The total contribution to β_{g_B} from the two-loop Higgs-particle self-energy diagrams is $g_B/(16\pi^2)^2$ times

$$\begin{aligned}&-\tfrac{3}{4}(\bar g_B{}^4+\bar g_T{}^4)(N_2+1)\bar N_3+\tfrac12(\bar g_B{}^2\bar g_T{}^2)\bar N_3+5g_3{}^2R_3(\bar g_B{}^2+\bar g_T{}^2)\bar N_3\\&\quad+\tfrac52 g_2{}^2R_2(\bar g_B{}^2+\bar g_T{}^2)\bar N_3+\tfrac58 g_1{}^2[\bar g_B{}^2(2\bar y^2-2\bar y+1)\bar N_3+\bar g_T{}^2(2\bar y^2+2\bar y+1)\bar N_3]\\&\qquad+g_2{}^4R_2G_2\{-\tfrac{35}{3}+2\alpha+\tfrac14\alpha^2\}+\tfrac{11}{6}g_2{}^4\frac{R_2{}^2N_2}{D_2}N_H+\tfrac{10}{6}g_2{}^4\frac{R_2{}^2N_2\bar N_3}{D_2}\\&\qquad\quad+g_2{}^4R_2{}^2\tfrac32+g_1{}^2g_2{}^2R_2\tfrac34+\tfrac{10}{96}g_1{}^4[\bar y^2(N_2+2)+2]\bar N_3+\tfrac{11}{96}g_1{}^4N_2N_H+g_1{}^4\tfrac{3}{32}+\lambda^2\frac{(N_2+1)}{18}\;.\end{aligned}\tag{18}$$

Equation (18) also gives the two-loop Higgs-particle anomalous dimension.

B. Fermion self-energy contributions

The fermion self-energy diagrams appear in Figs. 5—7. We give the results separately for the right- and left-handed self-energies where these are different. A single listing represents, as usual, the sum of equal right-handed and left-handed contributions. A factor $g_B/(16\pi^2)^2$ has been suppressed throughout.

The $g_H{}^5$ contributions to β_B derive from Fig. 5.

$$\begin{aligned}&5(\text{a}):\ 0\,,\\&5(\text{b}):\ -[2g_B{}^4+g_B{}^2g_T{}^2+g_T{}^4]N_2\{\tfrac18\}\,,\\&5(\text{c}):\ \tfrac12(\bar g_B{}^2+\bar g_T{}^2)[g_B{}^2(N_2+1)+g_T{}^2]\bar N_3\{-\tfrac32\}\,.\end{aligned}\tag{19}$$

The $g_H{}^3g_i{}^2$ contributions derive from Fig. 6.

	Left	Right
6(a):	$\{[g_3{}^2R_3(g_B{}^2+g_T{}^2)]+\tfrac14[g_1{}^2(g_B{}^2(y-1)^2+g_T{}^2(y+1)^2)]\}\{\tfrac14\alpha\}$,	$\{[g_3{}^2R_3+g_2{}^2R_2+g_1{}^2y^2/4]g_B{}^2N_2\}\{\tfrac14\alpha\}$,
6(b):	$\{[g_3{}^2R_3+g_2{}^2R_2+g_1{}^2y^2/4](g_B{}^2+g_T{}^2)\}\{-\tfrac34\alpha\}$,	$\{[g_3{}^2R_3+g_1{}^2(y-1)^2/4]g_B{}^2N_2\}\{-\tfrac34\alpha\}$,
6(c):	$[g_2{}^2R_2+\tfrac14g_1{}^2](g_B{}^2+g_T{}^2)\{\tfrac54+\tfrac14\alpha\}$,	$[g_2{}^2R_2+\tfrac14g_1{}^2]g_B{}^2N_2\{\tfrac54+\tfrac14\alpha\}$,
6(d):	$\tfrac14g_1{}^2[g_T{}^2(y+1)-g_B{}^2(y-1)]\{3-\tfrac12\alpha\}$	$[g_2{}^2R_2+\tfrac14g_1{}^2y]g_B{}^2N_2\{3-\tfrac12\alpha\}$
6(e):	$\{-2[g_2{}^2R_2(g_B{}^2+g_T{}^2)]+\tfrac12[g_1{}^2(g_T{}^2-g_B{}^2)y]\}\times\{-\tfrac34-\tfrac14\alpha\}$,	$\tfrac12[g_1{}^2g_B{}^2N_2(y-1)]\{-\tfrac34-\tfrac14\alpha\}$,
6(f):	$\{2[g_3{}^2R_3(g_B{}^2+g_T{}^2)]+\tfrac12[g_1{}^2(g_B{}^2(y^2-y)+g_T{}^2(y^2+y))]\}\{-1+\tfrac14\alpha\}$.	$\{2[g_3{}^2R_3g_B{}^2N_2]+\tfrac12[g_1{}^2g_B{}^2N_2(y^2-y)]\}\{-1+\tfrac14\alpha\}$. (20)

The $g_Hg_i{}^4$ terms arise from Fig. 7:

	Left	Right
7(a):	$-\left[g_2{}^4\dfrac{R_2{}^2}{D_2}+\dfrac{1}{16}g_1{}^4y^2\right]N_2N_H\{\tfrac12\}$,	$-\tfrac1{16}[g_1{}^4N_2(y-1)^2]N_H\{\tfrac12\}$,
7(b):	$\{\tfrac12[g_3{}^4\bar R_3R_3(N_2+2)/D_3]\bar N_3+\tfrac12[g_2{}^4R_2{}^2N_2/D_2]\bar N_3+\tfrac1{32}[g_1{}^4(\bar y^2(N_2+2)+2)y^2]\bar N_3\}\{-2\}$,	$\{\tfrac12[g_3{}^4R_3\bar R_3(N_2+2)/D_3]\bar N_3+\tfrac1{32}[g_1{}^4(\bar y^2(N_2+2)+2)(y-1)^2]\bar N_3\}\{-2\}$,
7(c):	$\{[g_3{}^4R_3G_3]+[g_2{}^4R_2G_2]\}\{\tfrac18+\tfrac14\alpha^2\}$,	$[g_3{}^4R_3G_3]\{\tfrac18+\tfrac14\alpha^2\}$,
7(d):	$\{\tfrac12[g_3{}^4R_3G_3]+\tfrac12[g_2{}^4R_2G_2]\}\{\tfrac{25}4-\tfrac32\alpha-\tfrac12\alpha^2\}$,	$\tfrac12[g_3{}^4R_3G_3]\{\tfrac{25}4-\tfrac32\alpha-\tfrac12\alpha^2\}$,
7(e):	$\tfrac12[g_3{}^4R_3G_3+g_2{}^4R_2G_2]\{\tfrac92+\tfrac{11}2\alpha+\alpha^2\}$,	$\tfrac12[g_3{}^4R_3G_3]\{\tfrac92+\tfrac{11}2\alpha+\alpha^2\}$, (21)

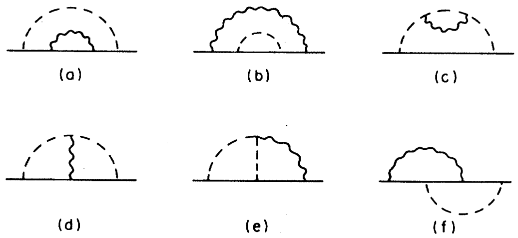


FIG. 6. $g_H{}^2 g_i{}^2$ contributions to the fermion anomalous dimension.

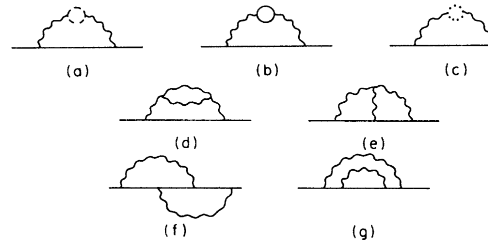


FIG. 7. $g_i{}^4$ contributions to the fermion anomalous dimension.

$$\begin{aligned}
7(\text{f}):\quad &\{-g_3{}^2(R_3{}^2-\tfrac{1}{2}G_3R_3)-2[g_2{}^2g_3{}^2R_2R_3] && \{-g_3{}^2(R_3{}^2-\tfrac{1}{2}G_3R_3)\\
&-\tfrac{1}{2}[g_1{}^2g_3{}^2R_3y^2]-[g_2{}^4(R_2{}^2-\tfrac{1}{2}G_2R_2)] && -\tfrac{1}{2}[g_1{}^2g_3{}^2R_3(y-1)^2]-\tfrac{1}{16}[g_1{}^4(y-1)^4]\}\{\tfrac{3}{2}-\tfrac{1}{2}\alpha^2\},\\
&-\tfrac{1}{2}[g_1{}^2g_2{}^2R_2y^2]-\tfrac{1}{16}[g_1{}^4y^4]\}\{\tfrac{3}{2}-\tfrac{1}{2}\alpha^2\},\\
7(\text{g}):\quad &\{-[g_3{}^4R_3{}^2]-2[g_2{}^2g_3{}^2R_2R_3]-\tfrac{1}{2}[g_1{}^2g_3{}^2R_3y^2] && \{-[g_3{}^4R_3{}^2]-\tfrac{1}{2}[g_1{}^2g_3{}^2R_3(y-1)^2]\\
&-[g_2{}^4R_2{}^2]-\tfrac{1}{2}[g_1{}^2g_2{}^2R_2y^2]-\tfrac{1}{16}[g_1{}^4y^4]\}\{\tfrac{1}{2}\alpha^2\}. && -\tfrac{1}{16}[g_1{}^4(y-1)^4]\}\{\tfrac{1}{2}\alpha^2\}.
\end{aligned}$$

The total left-handed fermionic self-energy contribution is $g_B/(16\pi^2)^2$ times

$$\begin{aligned}
&-\tfrac{1}{2}N_H[g_2{}^4R_2\bar{R}_2/D_2]N_2-\tfrac{1}{32}N_H[g_1{}^4y^2]N_2-[g_3{}^4R_3\bar{R}_3(N_2+2)/D_3]\bar{N}_3\\
&\quad-[g_2{}^4R_2\bar{R}_2N_2/D_2]\bar{N}_3-\tfrac{1}{16}\{g_1{}^4[\bar{y}^2(N_2+2)+2]y^2\}\bar{N}_3\\
&\quad+g_3{}^4[-\tfrac{3}{2}R_3{}^2+(\tfrac{25}{4}+2\alpha+\tfrac{1}{4}\alpha^2)R_3G_3]-3[g_2{}^2g_3{}^2R_2R_3]-\tfrac{3}{4}[g_1{}^2g_3{}^2R_3y^2]\\
&\quad+g_2{}^4[-\tfrac{3}{2}R_2{}^2+(\tfrac{25}{4}+2\alpha+\tfrac{1}{4}\alpha^2)R_2G_2]-\tfrac{3}{4}[g_1{}^2g_2{}^2R_2y^2]-\tfrac{3}{32}[g_1{}^4y^4]\\
&\quad-2g_3{}^2R_3(g_B{}^2+g_T{}^2)+\tfrac{11}{4}g_2{}^2R_2(g_B{}^2+g_T{}^2)+g_1{}^2[g_B{}^2(-\tfrac{1}{2}y^2+\tfrac{1}{8}y+\tfrac{17}{16})+g_T{}^2(-\tfrac{1}{2}y^2-\tfrac{1}{8}y+\tfrac{17}{16})]\\
&\quad-(\tfrac{1}{8}g_B{}^4+\tfrac{1}{16}g_B{}^2g_T{}^2+\tfrac{1}{16}g_T{}^4)N_2-\tfrac{3}{8}(\bar{g}_B{}^2+\bar{g}_T{}^2)[g_B{}^2(N_2+1)+g_T{}^2]\bar{N}_3\ . \qquad (22)
\end{aligned}$$

The right-handed contribution is $g_B/(16\pi^2)^2$ times

$$\begin{aligned}
&-(\tfrac{1}{8}g_B{}^4+\tfrac{1}{16}g_B{}^2g_T{}^2+\tfrac{1}{16}g_T{}^4)N_2-\tfrac{3}{8}(\bar{g}_B{}^2+\bar{g}_T{}^2)(g_B{}^2(N_2+1)+g_T{}^2)\bar{N}_3\\
&\quad-2[g_3{}^2R_3g_B{}^2N_2]+\tfrac{17}{4}[g_2{}^2R_2g_B{}^2N_2]+[g_1{}^2g_B{}^2N_2](-\tfrac{1}{2}y^2+\tfrac{7}{8}y+\tfrac{11}{16})\\
&\quad-\tfrac{1}{32}[g_1{}^4N_2(y-1)^2]N_H-[g_3{}^4R_3\bar{R}_3(N_2+2)/D_3]N_3-\tfrac{1}{16}\{g_1{}^4[\bar{y}^2(N_2+2)+2](y-1)^2\}\bar{N}_3\\
&\quad+g_3{}^4R_3G_3[\tfrac{1}{4}\alpha^2+2\alpha+\tfrac{25}{4}]-\tfrac{3}{2}[g_3{}^4R_3{}^2]-\tfrac{3}{4}[g_1{}^2g_3{}^2R_3(y-1)^4]-\tfrac{3}{32}[g_1{}^4(y-1)^4]\ . \qquad (23)
\end{aligned}$$

The expressions in (22) and (23) represent also the two-loop fermion anomalous dimensions.

C. Three-point contributions

These arise from the diagrams of Figs. 8, 9, 10, and 4(c). A factor $g_B/(16\pi^2)^2$ has been suppressed throughout. The $g_H{}^5$ contributions of Fig. 8 are

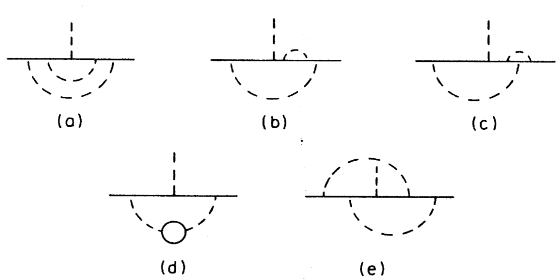


FIG. 8. $g_H{}^5$ contributions to the Higgs-Yukawa β.

$$\begin{aligned}
&8(\text{a}):\ -2[g_B{}^2g_T{}^2]\{1\}\ ,\\
&8(\text{b}):\ 2[g_T{}^4(N_2+1)+g_T{}^2g_B{}^2]\{\tfrac{1}{2}\}\ ,\\
&8(\text{c}):\ 0\ , \qquad (24)\\
&8(\text{d}):\ -(\bar{g}_B{}^2+\bar{g}_T{}^2)g_T{}^2\bar{N}_3\{-2\}\ ,\\
&8(\text{e}):\ -2[g_B{}^4]\{-1\}\ .
\end{aligned}$$

The $g_H{}^3g_i{}^2$ contributions of Fig. 9 are

$$\begin{aligned}
9(\text{a}):\ &0\ ,\\
9(\text{b}):\ &\{2[g_3{}^2R_3(g_B{}^2(N_2+1)+g_T{}^2)]\\
&+\tfrac{1}{2}[g_1{}^2(g_B{}^2[N_2y^2+(y-1)^2]\\
&\qquad+g_T{}^2[y^2-1])]\}\{\tfrac{1}{2}-\tfrac{1}{2}\alpha\}\ ,\\
9(\text{c}):\ &\{-4[g_3{}^2R_3g_T{}^2]+2[g_2{}^2R_2g_T{}^2]\\
&-\tfrac{1}{2}[g_1{}^2g_T{}^2(2y^2-1)]\}\{3-\alpha\}\ ,
\end{aligned}$$

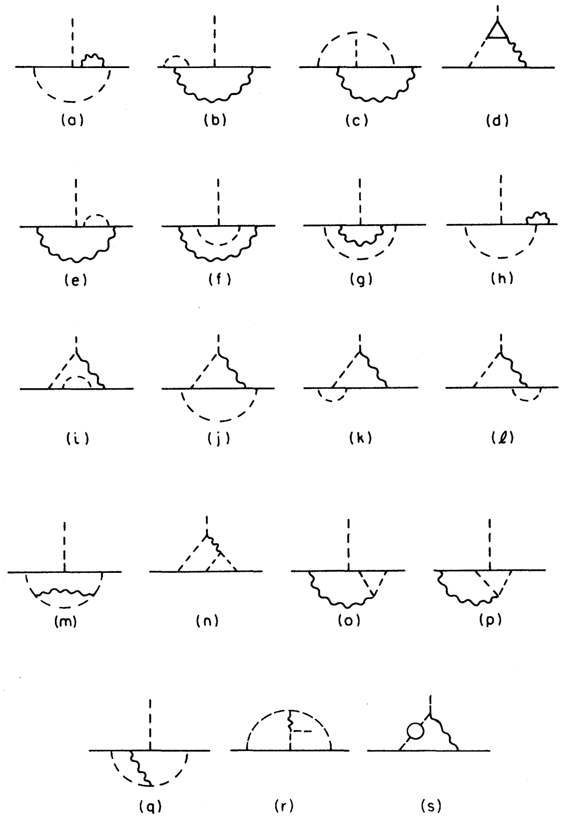


FIG. 9. $g_H{}^3 g_i{}^2$ contributions to the Higgs-Yukawa β.

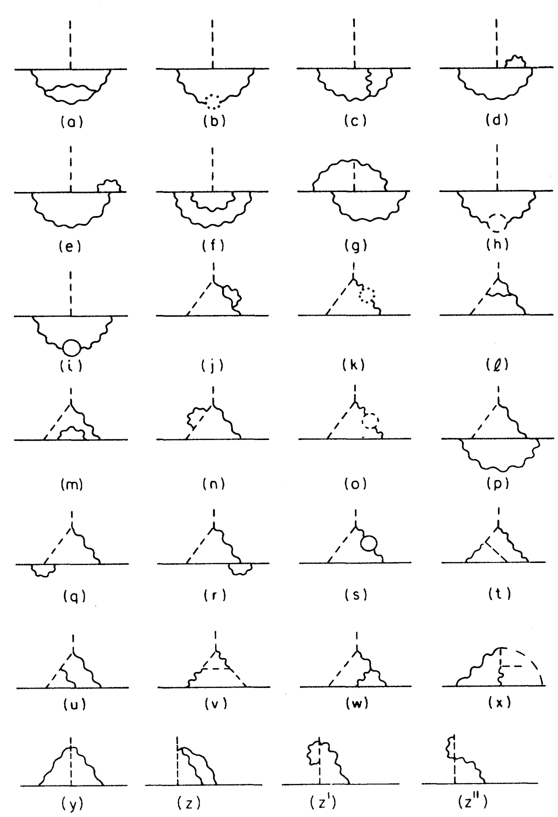


FIG. 10. $g_i{}^4$ contributions to the Higgs-Yukawa β.

$$
\begin{aligned}
&9(\mathrm{d}):\ \{-[g_2{}^2R_2(\bar g_B{}^2+\bar g_T{}^2)]\bar N_3 \\
&\qquad -\tfrac{1}{4}[g_1{}^2(\bar g_B{}^2+\bar g_T{}^2)]\bar N_3\}\{2\alpha\}\ ,\\
&9(\mathrm{e}):\ \{2[g_3{}^2R_3+\tfrac{1}{4}g_1{}^2(y^2-y)]\\
&\qquad \times(g_B{}^2(N_2+1)+g_T{}^2)\}\{\tfrac{5}{2}+\tfrac{1}{2}\alpha\}\ ,\\
&9(\mathrm{f}):\ \{-2[g_3{}^2R_3g_T{}^2]-\tfrac{1}{2}[g_1{}^2g_T{}^2(y^2-y)]\}\{5+\alpha\}\ ,\\
&9(\mathrm{g}):\ \{-2[g_3{}^2R_3g_T{}^2]-\tfrac{1}{2}[g_1{}^2g_T{}^2(y^2+y)]\}\{1+\alpha\}\ ,\\
&9(\mathrm{h}):\ \{-4[g_3{}^2R_3g_T{}^2]-[g_1{}^2g_T{}^2y^2]\}\{-2\}\ ,\\
&9(\mathrm{i}):\ \{2[g_2{}^2R_2(g_B{}^2+g_T{}^2)]\\
&\qquad -\tfrac{1}{2}[g_1{}^2(g_B{}^2N_2(y-1)-(g_B{}^2+g_T{}^2)y)]\}\{\tfrac{1}{2}\alpha\}\ ,\\
&9(\mathrm{j}):\ 0\ , \qquad\qquad (25)\\
&9(\mathrm{k}):\ 0\ ,\\
&9(\mathrm{l}):\ \{-\tfrac{1}{2}g_1{}^2[-g_B{}^2N_2y+g_B{}^2(y-1)\\
&\qquad +g_T{}^2(y+1)]\}\{\tfrac{1}{2}\alpha\}\ ,\\
&9(\mathrm{m}):\ \{-2[g_2{}^2R_2g_T{}^2]-\tfrac{1}{2}g_1{}^2g_T{}^2\}\{1+\alpha\}\ ,\\
&9(\mathrm{n}):\ \{2[g_2{}^2R_2(g_B{}^2+g_T{}^2)]\\
&\qquad -\tfrac{1}{2}[g_1{}^2(-g_B{}^2(N_2+1)+g_T{}^2)]\}\{-\tfrac{1}{2}\alpha\}\ ,\\
&9(\mathrm{o}):\ \{\tfrac{1}{2}[g_1{}^2(g_B{}^2N_2y+(g_T{}^2-g_B{}^2)(y-1))]\}\{\tfrac{5}{2}+\tfrac{1}{2}\alpha\}\ ,\\
&9(\mathrm{p}):\ \{+2[g_2{}^2R_2g_T{}^2]-\tfrac{1}{2}[g_1{}^2g_T{}^2(2y-1)]\}\{\alpha\}\ ,\\
&9(\mathrm{q}):\ \{-2[g_2{}^2R_2g_T{}^2]-\tfrac{1}{2}[g_1{}^2g_T{}^2(2y+1)]\}\{3-\alpha\}\ ,\\
&9(\mathrm{r}):\ 0\ ,\\
&9(\mathrm{s}):\ \{-[g_2{}^2R_2(\bar g_B{}^2+\bar g_T{}^2)]\bar N_3\\
&\qquad -\tfrac{1}{4}[g_1{}^2(\bar g_B{}^2+\bar g_T{}^2)]\bar N_3\}\{-2\alpha\}\ .
\end{aligned}
$$

The diagrams of Fig. 10 contribute terms of the form $g_H g_i{}^4$:

$$10(\mathrm{a}):\ [g_3{}^4R_3G_3]\{-\tfrac{32}{3}-4\alpha-\alpha^2\}\ ,$$

$$10(\mathrm{b}):\ 2[g_3{}^4R_3G_3]\{-\tfrac{7}{12}-\tfrac{1}{4}\alpha^2\}\ ,$$

$$10(\mathrm{c}):\ 2[g_3{}^4R_3G_3]\{-12+\alpha+\alpha^2\}\ ,$$

$$\begin{aligned}
&10(\mathrm{d}){:}\ \ \{-4[g_3{}^4R_3{}^2]-2[g_2{}^2g_3{}^2R_2R_3]-\tfrac{1}{2}[g_1{}^2g_3{}^2R_3(4y^2-4y+1)]-\tfrac{1}{2}[g_1{}^2g_2{}^2R_2(y^2-y)]\\
&\qquad -\tfrac{1}{8}[g_1{}^4(y^3(y-1)+(y-1)^3y)]\}\{-2\alpha\}\ ,\\
&10(\mathrm{e}){:}\ \ \{-4[g_3{}^4(R_3{}^2-\tfrac{1}{2}R_3G_3)]-2[g_2{}^2g_3{}^2R_2R_3]-\tfrac{1}{2}[g_1{}^2g_3{}^2R_3(4y^2-4y+1)]\\
&\qquad -\tfrac{1}{2}[g_1{}^2g_2{}^2R_2(y^2-y)]-\tfrac{1}{8}[g_1{}^4(y^3(y-1)+(y-1)^3y)]\}\{-6+2\alpha\}\ ,\\
&10(\mathrm{f}){:}\ \ \{-2[g_3{}^4R_3{}^2]-[g_1{}^2g_3{}^2R_3(y^2-y)]-\tfrac{1}{8}[g_1{}^4y^2(y-1)^2]\}\{9+6\alpha+\alpha^2\}\ ,\\
&10(\mathrm{g}){:}\ \ \{-2[g_3{}^4(R_3{}^2-\tfrac{1}{2}R_3G_3)]-[g_1{}^2g_3{}^2R_3(y^2-y)]-\tfrac{1}{8}[g_1{}^4y^2(y-1)^2]\}\{3-6\alpha-\alpha^2\}\ ,\\
&10(\mathrm{h}){:}\ \ -\tfrac{1}{8}[g_1{}^4N_2N_H(y^2-y)]\{-\tfrac{7}{3}\}\ ,\\
&10(\mathrm{i}){:}\ \ \left\{g_3{}^4\bar{R}_3R_3\frac{(N_2+2)}{D_3}\bar{N}_3+\tfrac{1}{16}[g_1{}^4(\bar{y}^2(N_2+2)+2)(y^2-y)]\bar{N}_3\right\}\{\tfrac{16}{3}\}\ ,\\
&10(\mathrm{j}){:}\ \ [g_2{}^4R_2G_2]\{\tfrac{1}{2}\alpha^2\}\ ,\\
&10(\mathrm{k}){:}\ \ [2g_2{}^4R_2G_2]\{-\tfrac{1}{4}\alpha^2\}\ ,\\
&10(\mathrm{l}){:}\ \ [g_2{}^4R_2G_2]\{-\tfrac{1}{2}\alpha^2\}\ ,\\
&10(\mathrm{m}){:}\ \ 0\ ,\\
&10(\mathrm{n}){:}\ \ \{-2[g_2{}^4R_2{}^2]-[g_1{}^2g_2{}^2R_2]-\tfrac{1}{8}[g_1{}^4]\}\{\alpha+\alpha^2\}\ ,\\
&10(\mathrm{o}){:}\ \ 0\ ,\\
&10(\mathrm{p}){:}\ \ \{-2[g_2{}^2g_3{}^2R_2R_3]-\tfrac{1}{2}[g_1{}^2g_3{}^2R_3]-\tfrac{1}{2}[g_1{}^2g_2{}^2R_2(y^2-y)]-\tfrac{1}{8}[g_1{}^4(y^2-y)]\}\{2\alpha\}\ ,\\
&10(\mathrm{q}){:}\ \ \{-2[g_2{}^2g_3{}^2R_2R_3]-\tfrac{1}{2}[g_1{}^2g_3{}^2R_3]-\tfrac{1}{2}[g_1{}^2g_2{}^2R_2(y^2-y)]-\tfrac{1}{8}[g_1{}^4(y^2-y)]\}\{-2\alpha\}\ ,\\
&10(\mathrm{r}){:}\ \ 0\ ,\\
&10(\mathrm{s}){:}\ \ 0\ ,\\
&10(\mathrm{t}){:}\ \ \{-[g_1{}^2g_2{}^2R_2(y-1)]-\tfrac{1}{4}[g_1{}^4(y^2-y)]\}\{\alpha^2\}\ ,\\
&10(\mathrm{u}){:}\ \ \{-2[g_2{}^4R_2{}^2]-[g_1{}^2g_2{}^2R_2y]-\tfrac{1}{8}[g_1{}^4(2y^2-2y+1)]\}\{3\alpha-\alpha^2\}\ ,\\
&10(\mathrm{v}){:}\ \ \{-2g_2{}^4(R_2{}^2-\tfrac{1}{2}R_2G_2)-[g_1{}^2g_2{}^2R_2]-\tfrac{1}{8}[g_1{}^4]\}\{\alpha^2\}\ ,\\
&10(\mathrm{w}){:}\ \ [g_2{}^4R_2G_2]\{-\tfrac{1}{2}\alpha-\tfrac{1}{2}\alpha^2\}\ ,\\
&10(\mathrm{x}){:}\ \ \{-2[g_2{}^4(R_2{}^2-\tfrac{1}{2}R_2G_2)]-[g_1{}^2g_2{}^2R_2y]-\tfrac{1}{8}[g_1{}^4(y^2+(y-1)^2)]\}\{-3\alpha\}\ ,\\
&10(\mathrm{y}){:}\ \ \{[g_1{}^2g_2{}^2R_2(y-1)]+\tfrac{1}{4}[g_1{}^4(y^2-y)]\}\{3+\alpha^2\}\ ,\\
&10(\mathrm{z}){:}\ \ \{4[g_2{}^4(R_2{}^2-\tfrac{1}{4}R_2G_2)]+2[g_1{}^2g_2{}^2R_2y]+\tfrac{1}{4}[g_1{}^4(2y^2-2y+1)]\}\{-\tfrac{3}{2}-\tfrac{1}{2}\alpha^2\}\ ,\\
&10(\mathrm{z}'){:}\ \ \{4[g_2{}^4(R_2{}^2-\tfrac{1}{4}R_2G_2)]+2[g_1{}^2g_2{}^2R_2]+\tfrac{1}{4}[g_1{}^4]\}\{\tfrac{1}{2}\alpha+\alpha^2\}\ ,\\
&10(\mathrm{z}''){:}\ \ 0\ .
\end{aligned}\tag{26}$$

Diagram (4c) contributes

$$4(\mathrm{c}){:}\ \ -\tfrac{2}{3}\lambda g_B{}^2(N_2+1)\{1\}\ .\tag{27}$$

Finally, the complete two-loop beta function for g_B is $g_B/(16\pi^2)^2$ times

$$
\begin{aligned}
&-\bar{N}_3\{\tfrac{3}{4}(N_2+1)(\bar{g}_B{}^4+\bar{g}_T{}^4)-\tfrac{1}{2}\bar{g}_B{}^2\bar{g}_T{}^2+\tfrac{3}{4}(N_2+1)(\bar{g}_B{}^2+\bar{g}_T{}^2)g_B{}^2-\tfrac{5}{4}(\bar{g}_B{}^2+\bar{g}_T{}^2)g_T{}^2\}\\
&\quad -g_B{}^4\left[\frac{N_2}{4}-2\right]-g_B{}^2g_T{}^2\left[\frac{N_2}{8}+1\right]+g_T{}^4(\tfrac{7}{8}N_2+1)\\
&\quad +5g_3{}^2R_3(\bar{g}_B{}^2+\bar{g}_T{}^2)N_3-16g_3{}^2R_3g_T{}^2+4g_3{}^2R_3(g_B{}^2(N_2+1)+g_T{}^2)\\
&\quad +\tfrac{17}{4}g_2{}^2R_2g_B{}^2N_2+\tfrac{5}{2}g_2{}^2R_2(\bar{g}_B{}^2+\bar{g}_T{}^2)\bar{N}_3+\tfrac{11}{4}g_2{}^2R_2(g_B{}^2+g_T{}^2)\\
&\quad -2g_2{}^2R_2g_T{}^2+\tfrac{5}{4}g_1{}^2[\bar{g}_B{}^2(\bar{y}^2-\bar{y}+\tfrac{1}{2})+\bar{g}_T{}^2(\bar{y}^2+\bar{y}+\tfrac{1}{2})]\bar{N}_3\\
&\quad -\tfrac{1}{2}g_1{}^2[g_B{}^2N_2(-2y^2-\tfrac{7}{4}y-\tfrac{11}{8})+g_B{}^2(-2y^2+\tfrac{23}{4}y-\tfrac{41}{8})+g_T{}^2(-2y^2+\tfrac{1}{4}y+\tfrac{7}{8})]\\
&\quad -\tfrac{1}{2}g_1{}^2g_T{}^2(8y^2+2y+1)-\tfrac{3}{8}g_1{}^4(\tfrac{1}{2}y^4-y^3-\tfrac{1}{2}y^2+y+1)-\tfrac{1}{8}g_1{}^4N_2N_H(-\tfrac{2}{3}+\tfrac{11}{6}y-\tfrac{11}{6}y^2)\\
&\quad +\tfrac{1}{16}g_1{}^4\bar{N}_3[\bar{y}^2(N_2+2)+2](\tfrac{2}{3}-\tfrac{10}{3}y+\tfrac{10}{3}y^2)-\tfrac{1}{2}g_1{}^2g_2{}^2R_2(\tfrac{9}{2}+6y-\tfrac{9}{2}y^2)-\tfrac{1}{2}g_1{}^2g_3{}^2R_3(3y^2-3y-\tfrac{9}{2})\\
&\quad +\tfrac{4}{9}g_2{}^4R_2\bar{R}_2\bar{N}_2N_H+\tfrac{2}{9}g_2{}^4\bar{R}_2R_2\bar{N}_2\bar{N}_3-2g_2{}^4[3R_2{}^2+\tfrac{47}{24}G_2R_2]+9g_2{}^2g_3{}^2R_2R_3+g_3{}^4(-3R_3{}^2-\tfrac{97}{3}G_3R_3)\\
&\quad +\tfrac{10}{3}g_3{}^4\bar{R}_3R_3\frac{\bar{N}_3}{D_3}(N_2+2)-\tfrac{2}{3}\lambda g_B{}^2(N_2+1)+\frac{\lambda^2}{18}(N_2+1)\;. \qquad (28)
\end{aligned}
$$

When the sum over the loop fermions is performed, β_B is

$$
\begin{aligned}
&\frac{g_B}{(16\pi^2)^2}\left\{\left[-\tfrac{3}{2}N_c(N_2+1)-\frac{N_2}{4}+2\right]g_B{}^4+\left[N_c[1-\tfrac{3}{4}N_2]-\frac{N_2}{8}-1\right]g_B{}^2g_T{}^2\right.\\
&\quad -\tfrac{3}{4}(N_2+1)g_B{}^2g_\tau{}^2-\tfrac{3}{4}N_c(N_2+1)g_B{}^2(g_C{}^2+g_S{}^2)+g_B{}^2g_3{}^2R_3(5N_c+4N_2+4)+g_B{}^2g_2{}^2R_2(N_2\tfrac{17}{4}+N_c\tfrac{5}{2}+\tfrac{11}{4})\\
&\quad +g_B{}^2g_1{}^2(N_c\tfrac{25}{72}+N_2\tfrac{157}{144}+\tfrac{247}{144})+g_T{}^4[N_c(\tfrac{1}{2}-\tfrac{3}{4}N_2)+\tfrac{7}{8}N_2+1]\\
&\quad +\tfrac{5}{4}g_T{}^2g_\tau{}^2+\tfrac{5}{4}g_T{}^2(g_S{}^2+g_C{}^2)N_c+g_T{}^2g_3{}^2R_3(N_c5-12)\\
&\quad +g_T{}^2g_2{}^2R_2(N_c\tfrac{5}{2}+\tfrac{3}{4})+g_T{}^2g_1{}^2(N_c\tfrac{85}{72}-\tfrac{79}{48})-\tfrac{3}{4}g_\tau{}^4(N_2+1)+\tfrac{5}{2}g_\tau{}^2g_2{}^2R_2+\tfrac{25}{8}g_\tau{}^2g_1{}^2+\tfrac{1}{2}g_S{}^2g_C{}^2N_c\\
&\quad +5(g_S{}^2+g_C{}^2)g_3{}^2R_3+\tfrac{5}{2}(g_S{}^2+g_C{}^2)g_2{}^2R_2N_c+\tfrac{25}{72}g_S{}^2g_1{}^2N_c+\tfrac{85}{72}g_C{}^2g_1{}^2N_c\\
&\quad -\tfrac{3}{4}(g_S{}^4+g_C{}^4)N_c(N_2+1)+g_3{}^4\left[R_3{}^2\frac{[N_C(N_2+2)N_G]}{2D_3}\frac{20}{3}-3R_3{}^2+R_3G_3(-\tfrac{97}{3})\right]\\
&\quad +9g_3{}^2g_2{}^2R_3R_2+\tfrac{31}{12}g_3{}^2g_1{}^2R_3+g_2{}^4\{R_2{}^2[N_2N_H\tfrac{4}{9}+(N_c+1)N_2N_G\tfrac{2}{9}-6]-G_2R_2\tfrac{47}{12}\}\\
&\quad \left.-3g_2{}^2g_1{}^2R_2-\tfrac{2}{3}\lambda g_B{}^2(N_2+1)+\frac{\lambda^2}{18}(N_2+1)+\frac{g_1{}^4}{216}\left[7N_HN_2-N_G\left[\frac{N_c}{9}(N_2+20)+N_2+4\right]-101\right]\right\}. \qquad (29)
\end{aligned}
$$

N_g is the number of generations, and N_H is the number of Higgs particles. The g_S , g_C, and g_τ terms are intended to represent terms involving upper and lower couplings of other generations and the leptonic coupling. Similarly, β_T is $g_T/(16\pi^2)^2$ times

$$
\begin{aligned}
&g_T{}^4\left[-\tfrac{3}{2}N_c(N_2+1)-\frac{N_2}{4}+2\right]+g_T{}^2g_B{}^2\left[N_c(1-\tfrac{3}{4}N_2)-\frac{N_2}{8}-1\right]\\
&\quad -\tfrac{3}{4}g_T{}^2g_\tau{}^2(N_2+1)-\tfrac{3}{4}g_T{}^2(g_C{}^2+g_S{}^2)(N_c)(N_2+1)+g_T{}^2g_3{}^2R_3(5N_c+4N_2+4)\\
&\quad +g_T{}^2g_2{}^2R_2(N_2\tfrac{17}{4}+N_c\tfrac{5}{2}+\tfrac{11}{4})+g_T{}^2g_1{}^2(N_c\tfrac{85}{72}+N_2\tfrac{73}{144}+\tfrac{523}{144})\\
&\quad +g_B{}^4[N_c(\tfrac{1}{2}-\tfrac{3}{4}N_2)+\tfrac{7}{8}N_2+1]+\tfrac{5}{4}g_B{}^2g_\tau{}^2+\tfrac{5}{4}g_B{}^2(g_C{}^2+g_S{}^2)N_c\\
&\quad +g_B{}^2g_3{}^2R_3(5N_c-12)+g_B{}^2g_2{}^2R_2(\tfrac{5}{2}N_c+\tfrac{3}{4})+g_B{}^2g_1{}^2(N_c\tfrac{25}{72}-\tfrac{43}{48})\\
&\quad -\tfrac{3}{4}g_\tau{}^4(N_2+1)+\tfrac{5}{2}g_\tau{}^2g_2{}^2R_2+\tfrac{25}{8}g_\tau{}^2g_1{}^2-\tfrac{3}{4}(g_C{}^4+g_S{}^4)N_c(N_2+1)\\
&\quad +\tfrac{1}{2}g_C{}^2g_S{}^2N_c+5(g_C{}^2+g_S{}^2)g_3{}^2R_3N_c+\tfrac{5}{2}(g_C{}^2+g_S{}^2)g_2{}^2R_2N_c\\
&\quad +\tfrac{85}{72}g_C{}^2g_1{}^2N_c+\tfrac{25}{72}g_S{}^2g_1{}^2N_c+g_3{}^4\left[R_3{}^2\left[\frac{(N_GN_c)(N_2+2)}{2D_3}\frac{20}{3}-3\right]+R_3G_3(-\tfrac{97}{3})\right]
\end{aligned}
$$

$$+9g_3{}^2g_2{}^2R_3R_2+\tfrac{19}{12}g_3{}^2g_1{}^2R_3+g_2{}^4\{R_2{}^2[N_2N_H\tfrac{4}{9}+N_2N_G(N_c+1)\tfrac{2}{9}-6]+G_2R_2(-\tfrac{47}{12})\}$$
$$-g_2{}^2g_1{}^2R_2+\tfrac{29}{216}g_1{}^4N_G\left[\frac{N_c}{9}(N_2+20)+N_2+4\right]-\tfrac{2\lambda}{3}g_T{}^2(N_2+1)+\frac{\lambda^2}{18}(N_2+1)\ . \qquad (30)$$

Finally, β_τ for g_τ is $g_\tau/(16\pi^2)^2$ times

$$g_\tau{}^4\left[-\tfrac{3}{2}(N_2+1)-\frac{N_2}{4}+2\right]-\tfrac{3}{4}g_\tau{}^2(g_B{}^2+g_T{}^2)N_c(N_2+1)-\tfrac{3}{4}g_\tau{}^2g_\mu{}^2(N_2+1)$$
$$+g_\tau{}^2g_2{}^2R_2(N_2\tfrac{17}{4}+\tfrac{21}{4})+g_\tau{}^2g_1{}^2(\tfrac{13}{16}N_2+\tfrac{153}{16})-\tfrac{3}{4}(g_B{}^4+g_T{}^4)N_c(N_2+1)+\tfrac{1}{2}g_B{}^2g_T{}^2N_c+5(g_T{}^2+g_B{}^2)g_3{}^2R_3N_c$$
$$+\tfrac{5}{2}(g_T{}^2+g_B{}^2)g_2{}^2R_2N_c+\tfrac{25}{72}g_B{}^2g_1{}^2N_c+\tfrac{85}{72}g_T{}^2g_1{}^2N_c$$
$$-\tfrac{3}{4}g_\mu{}^4+\tfrac{5}{2}g_\mu{}^2g_2{}^2R_2+\tfrac{25}{8}g_\mu{}^2g_1{}^2+g_2{}^4\{R_2{}^2[N_2N_H\tfrac{4}{9}+(N_c+1)N_2N_G\tfrac{2}{9}-6]+R_2G_2(-\tfrac{47}{12})\}$$
$$+3g_2{}^2g_1{}^2R_2+g_1{}^4\left[N_G\left[N_c\frac{(N_2+20)}{9}+4+N_2\right]\tfrac{11}{24}+N_2N_H\tfrac{13}{24}-\tfrac{3}{8}\right]-\frac{2\lambda}{3}g_\tau{}^2(N_2+1)+\frac{\lambda^2}{18}(N_2+1)\ . \qquad (31)$$

g_μ represents the coupling for a lepton μ of a different generation than τ.

The specialization of the above results to the standard theory ($N_3=2$, $N_2=2$, etc.) has already been given in Ref. 6.

ACKNOWLEDGMENTS

We would like to thank Chris Hill for his encouragement and advice, and Pat Oleck for her patience and significant contributions. We also wish to thank Marie Machacek and Mike Vaughn, both for their hospitality, and for many discussions of matters computational.

*Operated by Universities Research Association, Inc. under contract with the United States Department of Energy.

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