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INSTANTON EFFECTS IN SUPERSYMMETRIC THEORIES

V.A. NOVIKOV, M.A. SHIFMAN, A.I. VAINSHTEIN and V.I. ZAKHAROV

Institute of Theoretical and Experimental Physics, Moscow 117259, USSR

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We calculate some simplest n-point functions in supersymmetric Yang-Mills theories contributed by instantons. The result is not vanishing and we discuss some implications of this. In particular, the dynamics of the supersymmetric theories must exhibit some unusual features.

1. Introduction

In this paper we consider some simple correlation functions contributed by instantons in supersymmetric theories. Namely, for the $N=1$ supersymmetric Yang-Mills (SYM) theory we find

$$f(x_1, x_2) = \langle 0 | T\{ W^2(x_1), W^2(x_2) \} | 0' \rangle , \tag{1}$$

while in the $N=2$ case we concentrate on

$$f(x_1, x_2, x_3, x_4) = \langle 0 | T\{ W^2(x_1), W^2(x_2), SS(x_3), SS(x_4) \} | 0' \rangle . \tag{2}$$

Here $W^2 = W^\alpha W_\alpha = \varepsilon_{\alpha\beta} W^\alpha W^\beta$; $\varepsilon_{12} = -1$ and W^α is the spinor superfield built of the gluon field strength tensor and the gluino field (see eq. (8)); S is the matter scalar superfield and $|0\rangle, |0'\rangle$ are the perturbation theory vacuum states with a unit difference in topological charge.

The choice of (1) and (2) is not without motivation. The point is that the change in the topological charge is to be accompanied by a certain change in chirality. The functions (1) and (2) are the simplest ones to satisfy the chirality selection rule.

There are some specific problems with instantons in supersymmetric theories. Superficially, they defy supersymmetry by producing an effective multifermion interaction without its bosonic counterpart [1–3]. Although the problem has been eventually resolved [4] the solution is not so trivial and calls for integration over the instanton size to recover supersymmetry. Thus, the very notion of effective lagrangian induced by instantons of small size turns out to be incompatible with supersymmetry (provided that the underlying classical theory is conformal invariant, as is the case with gluodynamics). Under these circumstances, it seemed to us interesting to trace a non-vanishing instanton contribution to some n-point functions.

Our results are as follows. Rather unexpectedly, the two-point function (1) turns out to be a constant

$$f(x_1, x_2) = \text{const}. \tag{3}$$

Moreover, if $x_2 - x_1$ is small, the integral over the instanton size is saturated by instantons of small size, $\rho \sim |x_2 - x_1|$, and by current theoretical standards the calculation is reliable.

Furthermore, supersymmetry guarantees that if $f(x_1, x_2)$ is a constant at small $|x_2 - x_1|$ it remains the same for all $|x_2 - x_1|$. This would bring us in conflict with the general principles of field theory requiring from the correlation functions that they fall off at large distances.

At the moment we have in mind two alternative ways out of this difficulty. First, one can make a guess that fluctuations of large scale (of order of confinement radius) produce contributions to (1) which exactly cancels the instantons. To this end one must have

$$\langle 0|W^4(0)|0'\rangle_{\substack{\text{large scale}\\ \text{fluctuations}}} = C\Lambda^6, \tag{4}$$

where Λ is the standard scale parameter of QCD-type theories and the constant C is well defined. Then one can arrange for $f(x_1, x_2) = 0$.

The other possibility is that there are no instantons in the physical vacuum so that the average over the physical vacuum differs drastically from the average over the $|0\rangle, |0'\rangle$ states. Note that instantons of arbitrarily small size must be forbidden in the vacuum in this case.

As for the four-point function (2) the result is the following: $f(x_1, x_2, x_3, x_4)$ does not vanish and is a non-trivial function of the coordinates. This demonstrates that instantons in supersymmetric theories in a way are no worse than in QCD. The result is not without a surprise, however, since, as mentioned above instantons are usually thought to produce a multifermion interaction alone. We demonstrate that the four-point function (2) has both fermionic and bosonic non-vanishing components and fully respects supersymmetry.

In order to make our assertions more transparent it is worth reformulating some of them in the component language. In the $N = 1$ SYM theory our analysis reduces to computation of one-instanton contributions to a purely fermionic two-point function

$$\langle \varphi^a_\alpha(x)\varphi^{a\alpha}(x), \varphi^b_\beta(0)\varphi^{b\beta}(0)\rangle .$$

Here φ denotes the gluino field, a and α are the color and Lorentz indices, respectively ($a = 1,2,3$; $\alpha = 1,2$). The one-instanton contribution is non-vanishing. At the same time, it is obviously absent in the corresponding bosonic two-point functions. Still, this result is perfectly consistent with supersymmetry.

The paper is organized in the following way. In sect. 2 we introduce superfield notation for the instanton field. In sect. 3 we evaluate the two-point function (1) and discuss some implications. In sect 4 we consider the $N = 2$ example, eq. (2).

2. Instanton superfield

In this section we derive some simple expressions for the instanton field in superfield notation. They are useful in evaluating n-point functions.

Let us make first some general remarks on two-point functions generated by chiral superfields of the same chirality. The chiral superfield is a function of θ and x_L (for a review see refs. [5, 6]):

$$\phi_c(x_L, \theta),$$

where θ is the standard fermionic coordinate and

$$x_{L\mu} = x_\mu - i\theta\sigma_\mu\bar{\theta}, \qquad \sigma_\mu = (1, \sigma_i).$$

Under the supersymmetry transformations

$$\theta \to \theta + \varepsilon, \qquad \bar{\theta} \to \bar{\theta} + \bar{\varepsilon}, \qquad x_\mu \to x_\mu + i\varepsilon\sigma_\mu\bar{\theta} - i\theta\sigma_\mu\bar{\varepsilon},$$

we see that x_L does not vary under the ε infinitesimal transformations.

Now, because of the Poincaré invariance any two-point function depends only on the difference $x_1 - x_2$. The generalization of this statement to the case of supersymmetry is that the argument of any two-point function is $(x_1 - x_2)_{\text{inv}}$ where

$$(x_1 - x_2)_\mu^{\text{inv}} = (x_1 - x_2)_\mu - \tfrac{1}{2}i(\theta_1 + \theta_2)\sigma_\mu(\bar{\theta}_1 - \bar{\theta}_2) + \tfrac{1}{2}i(\theta_1 - \theta_2)\sigma_\mu(\bar{\theta}_1 + \bar{\theta}_2).$$

One can readily check explicitly that this combination possesses the following nice property: $(x_1 - x_2)^{\text{inv}}$ is organized in such a way that it does not vary if both fermionic coordinates are shifted, $\theta_{1,2} \to \theta_{1,2} + \varepsilon$ or $\bar{\theta}_{1,2} \to \bar{\theta}_{1,2} + \bar{\varepsilon}$.

However, if we deal only with, say, left-handed chiral fields they must depend on $(x_{1,2})_L$, and $(x_1 - x_2)^{\text{inv}}$ must reduce to $(x_{1L} - x_{2L})$. Rewriting $(x_1 - x_2)^{\text{inv}}$ as

$$(x_1 - x_2)_\mu^{\text{inv}} = (x_{1L} - x_{2L})_\mu + i(\theta_1 - \theta_2)\sigma_\mu(\bar{\theta}_1 + \bar{\theta}_2),$$

we see that $(x_1 - x_2)^{\text{inv}}$ reduces to $(x_{1L} - x_{2L})$ provided that $\delta^2(\theta_1 - \theta_2)$ is inserted. Thus, the general form of two-point functions in this case is

$$\Phi[(x_{1L}, \theta_1), (x_{2L}, \theta_2)] = f(x_{1L} - x_{2L})(\theta_1 - \theta_2)^2 + \text{const.}, \tag{5}$$

where we have also reserved for a constant which satisfies all the constraints by itself.

To introduce instantons consider the supersymmetric Yang-Mills theory. The simplest case is the $N=1$ theory with the lagrangian given by

$$\mathfrak{L} = -\tfrac{1}{4}\left(G^a_{\mu\nu}\right)^2 - i\varphi^a\sigma_\mu D^{ab}_\mu\bar{\varphi}^b + (\cdots), \tag{6}$$

where $G^a_{\mu\nu}$ is the gluon field strength tensor, D^{ab}_μ is the covariant derivative, a is the colour index ($a = 1,2,3$ in case of SU(2) as the gauge group), and φ^a are Weyl spinors in adjoint representation of the colour group. Finally, the dots denote the gauge fixing term and ghosts. These terms violate supersymmetry explicitly in the Wess-Zumino supergauge we are using. Moreover, in the classical case supersymmetry transformations are also assumed to be accompanied by some gauge transformations. To avoid complications associated with this explicit breaking of supersymmetry we shall consider gauge-invariant n-point functions only.

The instanton solution for $G^a_{\mu\nu}$ is given by

$$\left[G^a_{\mu\nu}\right]^{\mathrm{cl}} = -\frac{4}{g}\eta_{a\mu\nu}\frac{\rho^2}{\left[(x-x_0)^2+\rho^2\right]^2}, \tag{7}$$

where g is the coupling constant, $\eta_{a\mu\nu}$ are the 't Hooft symbols, x_0 and ρ are the position and size of the instanton, respectively.

The field strength tensor $G^a_{\mu\nu}$ and the spinors φ^a are embedded into the superfield W^a_α (see, e.g. ref. [5]):

$$W^a_\alpha(x_{\mathrm{L}},\theta) = 4i\varphi^a_\alpha(x_{\mathrm{L}}) + 2\eta_{b\mu\nu}G^a_{\mu\nu}(\sigma^b\theta)_\alpha - 4\theta^2\left(\sigma_\mu D^{ab}_\mu\bar{\varphi}^b\right)_\alpha, \tag{8}$$

where σ^b are the Pauli matrices.

Furthermore, to ensure gauge invariance (and supersymmetry, see above) consider $W^2 = \varepsilon_{\alpha\beta}W^\alpha W^\beta$. For W^2 associated with the instanton it is natural to write, following eq. (5)

$$\left[W^2(x_{\mathrm{L}},\theta)\right]_{\text{instanton}} = \frac{3\cdot 2^{10}}{g^2}\frac{\rho^4}{\left[(x_{\mathrm{L}}-x_0)^2+\rho^2\right]^4}(\theta-\theta_0)^2. \tag{9}$$

Indeed, the θ^2 component here is proportional to G^2 and is directly calculable in terms of $(G^a_{\mu\nu})^{\mathrm{cl}}$ (see eq. (7)). The rest is reconstructed on symmetry grounds.

The meaning of the parameter θ_0 (fermionic coordinate of the instanton centre) is readily identified. It is nothing else but the expansion parameter for the fermionic zero modes*. Indeed, completing θ^2 to $(\theta-\theta_0)^2$ we introduce non-vanishing $(\theta)^1$

* The normalization of the zero mode generated in this way is not standard, however. To compensate for this we shall insert an extra factor g for each zero mode into the instanton measure, see eq. (13). Let us also notice that in ref. [4] the parameter analogous to θ_0 was denoted by α.

and $(\theta)^0$ components of W^2 and the fermionic field in these components is just the fermionic zero mode. These components could be generated, of course, by a direct application of the supersymmetry transformations as well.

Still, eq. (9) is incomplete. It misses the effect of the superconformal fermionic zero modes (for nomenclature and discussion see refs. [7,4]). As far as supersymmetry is concerned these zero modes are something foreign in the sense that they do not have symmetry meaning; the answer can be supersymmetric if these modes are omitted altogether. Nevertheless, we must keep them to reproduce standard instanton calculus. To this end we introduce an external fermionic parameter $\bar{\beta}$.

In this way we come to the final result for the $W^2_{\text{instanton}}$:

$$\left[W^2(x_{\mathrm{L}},\theta)\right]_{\text{instanton}} = \frac{3\cdot 2^{10}}{g^2}\frac{\rho^4}{\left[(x_{\mathrm{L}}-x_0)^2+\rho^2\right]^4}\left[(\theta-\theta_0)-(x_{\mathrm{L}}-x_0)_\mu\sigma_\mu\bar{\beta}\right]^2. \tag{10}$$

Eq. (10) reproduces now all the zero modes. Note, however, that the symmetry properties become a bit more involved. Indeed, expression (10) does not fit the general form (5) and therefore does not respect supersymmetry automatically.

Performing supersymmetry transformations we find that they induce the following changes of $x_0, \theta_0, \rho, \bar{\beta}$:

$$x_0' - x_0 = 2i\theta_0\sigma\bar{\varepsilon}, \qquad \rho'-\rho = \rho\left(-2i\bar{\varepsilon}\bar{\beta}+3\bar{\varepsilon}^2\bar{\beta}^2\right),$$
$$\theta_0'-\theta_0 = 0, \qquad \bar{\beta}'-\bar{\beta} = -4i\bar{\beta}\left(\bar{\varepsilon}\bar{\beta}\right). \tag{11}$$

From these we conclude once more that (x_0, θ_0) are the ordinary (left-handed) coordinates. The pair $(\rho, \bar{\beta})$, on the other hand, is a kind of a superfield parameter. In particular, it is clear that it would be inconsistent with supersymmetry to integrate over $\bar{\beta}$ without integrating over ρ. Note that this observation is not new but a mere repetition of the recent analysis [4] in the superfield language.

3. The $N=1$ case

It is now straightforward to evaluate the two-point function (1) in a manifestly supersymmetric way.

On general grounds we have (see eq. (9))

$$\langle 0|T\{W^2(x_{1\mathrm{L}},\theta_1), W^2(x_{2\mathrm{L}},\theta_2)\}|0'\rangle = f(x_{2\mathrm{L}}-x_{1\mathrm{L}})(\theta_1-\theta_2)^2+\text{const.} \tag{12}$$

Moreover, it is clear that only the constant term has a chance to survive. Indeed, the two-point function considered vanishes in perturbation theory because of conservation of chirality. Furthermore, for instantons to be effective we need four fermionic sources to eliminate zeros of the instanton determinant due to the fermionic zero modes. These sources are provided by the $(\varphi_\alpha^a)^2$ components of $W^2(x_{1L},\theta_1)$, $W^2(x_{2L},\theta_2)$ and by counting θ we find that this component is not present in the first term.

All this is reproduced by an explicit calculation:

$$\langle 0|T\{W^2(x_{1L},\theta_1),W^2(x_{2L},\theta_2)\}|0'\rangle$$

$$= Cg^4\left(\sqrt{\frac{2\pi}{\alpha_s}}\right)^8 \Lambda^6 \exp\left(-\frac{2\pi}{\alpha_s(\Lambda)}\right)\int d^4x_0\, d^2\theta_0\, d^2\bar\beta\, d\rho^2$$

$$\times \frac{3^2 2^{20}}{g^4}\rho^8 \frac{[\theta_1-\theta_0-(x_{1L}-x_0,\sigma)\bar\beta]^2[\theta_2-\theta_0-(x_{2L}-x_0,\sigma)\bar\beta]^2}{[(x_{1L}-x_0)^2+\rho^2]^4[(x_{2L}-x_0)^2+\rho^2]^4}, \tag{13}$$

where we have substituted eq. (10) for W^2 and accounted for the instanton measure. The latter is responsible, in particular, for the emergence of the dimensional parameter Λ^6.

It is convenient to integrate over θ_0 and $\bar\beta$ first:

$$\int d^2\theta_0\, d^2\bar\beta\, [\theta_1-\theta_0-(x_{1L}-x_0,\sigma)\bar\beta]^2[\theta_2-\theta_0-(x_{2L}-x_0,\sigma)\bar\beta]^2 = 4(x_{2L}-x_{1L})^2. \tag{14}$$

To complete the integration we need to integrate over x_0 and ρ. Note, however, that the integrand is a sign-definite function so that it is clear that the result of the integration is not vanishing.

More explicitly,

$$\int d\rho^2\, d^4x_0 \frac{\rho^8(x_{2L}-x_{1L})^2}{[(x_{1L}-x_0)^2+\rho^2]^4[(x_{2L}-x_0)^2+\rho^2]^4}$$

$$= \tfrac{10}{3}\pi^2\int d\alpha\, \alpha^3(1-\alpha)^3\int d\rho^2 \frac{\rho^8(x_{2L}-x_{1L})^2}{[(x_{2L}-x_{1L})^2\alpha(1-\alpha)+\rho^2]^6} = \tfrac{1}{45}\pi^2, \tag{15}$$

$$\langle 0|T\{W^2(x_{1L},\theta_1),W^2(x_{2L},\theta_2)\}|0'\rangle_{\text{instanton}} = c\left(\frac{2\pi}{\alpha_s}\right)^4 \exp\left(-\frac{2\pi}{\alpha_s(\Lambda)}\right)\Lambda^6, \tag{16}$$

which is our final result.

Let us discuss now the implications of (16). We see that the two-point function calculated in the instanton approximation exhibits pathological behaviour: it does not fall off at large $x_2 - x_1$, $|x_2 - x_1| \to \infty$. Clearly, this is an unphysical result since no exchange of physical particles could reproduce such behaviour*. Note that the integration over the instanton size is saturated at $\rho \sim |x_2 - x_1|$. Therefore, in the instanton approximation itself, the lack of the fall-off at large $x_2 - x_1$ is due to existence of instantons of *arbitrary large scale*.

Of course, it is not for the first time that instantons result in some unreasonable answer. There are some special circumstances, however, which call for a more careful consideration.

The point is that the instanton calculation is perfectly consistent by itself in the case considered. It respects supersymmetry, is both ultraviolet and infrared stable and stands true to any order in perturbation theory (we discuss the last point in detail in ref. [8]). So, it is our prejudice against arbitrary large scale fluctuations of the colour field that makes the result of the instanton calculation unacceptable. This prejudice has all the chances to be justified.

The problem becomes even more acute if we start from arbitrarily small $x_2 - x_1$. Because of the asymptotic freedom the contribution of short distances is well under control and the instanton contribution is reliably calculable. On the other hand, $f(x) = \text{const}$ generated by the instantons must be removed somehow already at arbitrarily small $(x_1 - x_2)$. Otherwise, the supersymmetry would extend this constant to large $(x_1 - x_2)$ as well, see eq. (12).

How can one resolve the paradox? Thus far we were discussing direct instantons of small size, $\rho \sim |x_1 - x_2|^{-1} \to 0$. The quantity figuring in eq. (16), in the language of the operator expansion, is actually the direct instanton contribution to the unit operator. This is not the end of the story, however. Assume that there are some other non-perturbative fluctuations in the vacuum with a fixed large size. Then they can induce

$$\langle 0| W^2(0) W^2(0)|0'\rangle_{\text{l.s.f.}} \neq 0\,,$$

where the subscript l.s.f. marks large scale fluctuations. If so,

$$\langle 0|T\{W^2(x_{1\text{L}}, \theta_1) W^2(x_{2\text{L}}, \theta_2)\}|0'\rangle = \text{const.} + \langle 0| W^2(0) W^2(0)|0'\rangle_{\text{l.s.f.}}\,, \qquad |x_1 - x_2| \ll \Lambda\,, \quad (17)$$

where the constant in the right-hand side is due to the small-size instantons in the unit operator, and, apart from the unit operator, we have accounted also for the operator W^4. The coefficient in front of W^4 is obviously equal to unity in the limit $|x_1 - x_2| \to 0$.

* Usually, if some correlator tends to a constant for large separation of the points, this corresponds to disconnected graphs, or a vacuum insertion. Instantons are not disconnected, however.

From eqs. (16) and (17) we conclude that if

$$\langle 0|W^2(0)W^2(0)|0'\rangle_{\mathrm{l.s.f.}} = -\mathrm{const.}\left(\frac{2\pi}{\alpha_s}\right)^4 \Lambda^6 \exp\left(-\frac{2\pi}{\alpha_s(\Lambda)}\right), \tag{18}$$

then there is no apparent inconsistency in the theory.

It is worth noting that what we call $\langle 0|W^4|0'\rangle_{\mathrm{l.s.f.}}$ is not to be identified with the total vacuum expectation value of the operator W^4. The latter is contributed to in principle by small-size instantons as well. If we define $\langle \mathrm{vac}|W^4|\mathrm{vac}\rangle$ as

$$\lim_{x_1 \to x_2} \langle 0|T\{W^2(x_1,\theta_1)W^2(x_2,\theta_2)\}|0'\rangle,$$

then the condition imposed on the theory results in complete cancellation of large- and small-scale contributions to $\langle 0|W^4|0'\rangle$. Under such a definition

$$\langle 0|W^2(0)W^2(0)|0'\rangle = 0. \tag{19}$$

It is amusing that the cancellation must take place between fluctuations of some characteristic scale and instantons of *zero* size. Indeed, in the limit $(x_2 - x_1) \to 0$ the direct instantons saturating eq. (16) should have $\rho \to 0$. Such a phenomenon has no parallel in ordinary QCD.

There is one more condition which must be checked within the picture considered now. The point is that the result for $f(x_1, x_2)$ discussed above corresponds to a constant in x space or $\delta(q)$ in momentum space. At the same time for large momentum transfers the only behavior consistent with supersymmetry is trivial: the correlation function (1) should vanish identically (see the discussion following eq. (12)). For $|q^2| \gg \Lambda^2$ one can try the standard operator expansion which expresses the correlation function (1) in terms of various local operators times coefficients containing all q dependence. Since we have assumed that $\langle 0|W^4|0'\rangle_{\mathrm{l.s.f.}} \neq 0$ we must check that the operator W^4 does not appear in the expansion, or, in other words, the graph of fig. 1 and others of that type yield a vanishing expansion coefficient. We have convinced ourselves that this condition is satisfied in lowest order in α_s, and the coefficient in front of the operator W^4, or φ^4, is zero. Apparently, supersymmetry protects itself, and the operator W^4 appears only in relations like (17) at $q = 0$.

It is worth emphasizing that so far we have assumed instantons of small size to be allowed in the physical vacuum. A radical solution of all problems discussed above would emerge if there were no mixture of states with different topological charge in the physical vacuum, and all one-instanton contributions did not arise at all. Such a situation is realized in supersymmetric quantum mechanics, for instance. Although one can find non-vanishing instanton contributions in the presence of external sources (Salomonson and Van Holten; for a more detailed discussion and references see ref. [3]), instantons do not affect the evolution of the system: only

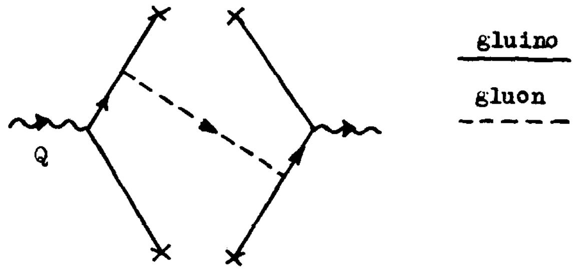

Fig. 1. The simplest graph determining the operator expansion coefficients in the two-point function $\langle 0|T\{W^2(x_1,\theta_1)W^2(x_2,\theta_2)\}|0\rangle$ for large-momentum transfer. The momentum transfer Q flows along the lines marked by arrows.

instanton–anti-instanton pairs and double instantons are important. In supersymmetric quantum mechanics, however, one readily identifies a conservation law responsible for this phenomenon. There is no apparent parallel to this in SYM theories.

Finally, we would like to notice that the constancy of the correlation function (16) allows one to establish a stronger result referring to a gauge non-invariant four-point function

$$F^{a_1a_2a_3a_4}_{\alpha_1\alpha_2\alpha_3\alpha_4} = \langle 0|T\{\varphi^{a_1}_{\alpha_1}(x_1)\varphi^{a_2}_{\alpha_2}(x_2)\varphi^{a_3}_{\alpha_3}(x_3)\varphi^{a_4}_{\alpha_4}(x_4)\}|0'\rangle .$$

We shall show that in the one-instanton approximation $F^{a_1a_2a_3a_4}_{\alpha_1\alpha_2\alpha_3\alpha_4}$ also reduces to a constant which can be eliminated within the framework of the same scenario as discussed above. The cancellation is achieved with the help of the same vacuum value

$$\langle 0|\varphi^4|0'\rangle_{\mathrm{l.s.f.}} .$$

Let us outline the proof. The general structure of $F^{a_1a_2a_3a_4}_{\alpha_1\alpha_2\alpha_3\alpha_4}$ is obviously the following

$$F \sim \delta^{a_1a_2}\delta^{a_3a_4}\int \mathrm{d}\rho^2\,\mathrm{d}^4x_0\,\rho^8 \prod_{i=1}^{4} \frac{1}{\left[(x_i-x_0)^2+\rho^2\right]^2}$$

$$\times (x_1-x_0)_{\alpha_1\dot\beta}(x_2-x_0)^{\dot\beta}{}_{\alpha_2}\varepsilon_{\alpha_3\alpha_4} + \text{permutations}.$$

Introducing the Feynman parameterization one can perform x_0 and ρ integration. Simple dimensional arguments imply

$$F \sim \int \prod_{i=1}^{4} \mathrm{d}\xi_i\,\delta\left(1-\sum_{i=1}^{4}\xi_i\right)\left[C_1 \ln\Delta^2 + C_2 + C_3\Delta^{-2}\cdot(\text{bilinear in } x_i \text{ structures})\right].$$

Here C_1, C_2, C_3 are some constants and

$$\Delta^2=\sum_{i=1}^{4}\xi_i(1-\xi_i)x_i^2-\sum_{i,j=1}^{4}\xi_i\xi_j(x_ix_j).$$

On the other hand, if $x_1=x_2$, $x_3=x_4$, or $x_1=x_3$, $x_2=x_4$ or $x_1=x_4$, $x_2=x_3$ the correlation function F should reduce to a constant (see eq. (16)). This requirement fixes C_1 and C_3, namely $C_1=C_3=0$. Thus,

$$F^{a_1a_2a_3a_4}_{\alpha_1\alpha_2\alpha_3\alpha_4}=\text{const}.$$

4. The $N=2$ case

Consider now the theory with one matter superfield

$$S=s+\psi\theta+\cdots,$$

where ψ and s are spinor and scalar fields respectively. Now instantons have eight zero modes and we turn to the four-point function (2) as a possible candidate for non-trivial instanton contribution.

In superfield notation the extra piece in the lagrangian is

$$\delta\mathcal{L}=\int d^4\theta\,\bar{S}\,e^{-gV^a\tau^a}S,$$

where V^a is the real vector superfield incorporating the gluon field.

To evaluate the four-point function (2) we need an analogue of eq. (13) for the SS case. It is rather clear that we can use for this purpose the following expression

$$S^2(x_{3L},\theta_3)=\frac{c}{\left[(x_{3L}-x_0)^2+\rho^2\right]^4}\left[\theta_3-\theta_0-(x_{3L}-x_0,\sigma)\bar{\beta}\right]^2$$
$$\times\left[\alpha_1+(x_{3L}-x_0,\sigma)\bar{\beta}_1\right]^2. \tag{20}$$

Here $\alpha_1, \bar{\beta}_1$ are some new parameters. Eq. (20) correctly reproduces the effect of the fermion zero modes, and the parameters $\alpha_1, \bar{\beta}_1$ are introduced just for this purpose. The other terms correspond to the zero modes of the scalar field. These could be also obtained by a direct application of supertransformations to the fermion zero modes (for further discussion see below). Moreover, we have already learned that the fermion coordinate of the instanton centre includes the $(x_L-x_0,\sigma)\bar{\beta}$ term and we have accounted for this in eq. (20).

Using eqs. (13), (20) ensures that supersymmetry is observed as far as the $(Q\varepsilon)$ transformations are concerned (Q is the generator of the supersymmetry transformations). As for the $\bar{Q}\bar{\varepsilon}$ transformations they result in some reshuffle of the parameters $(\bar{\beta}, \rho; \alpha_1, \bar{\beta}_1)$ and the symmetry restores only upon integration over these parameters. We will not go into detailed discussion of this point. It is worth noting that we do not make explicit the $N=2$ supersymmetry and treat the gluino and matter spinor field on a different footing.

After these preliminary remarks we can readily write down the four-point function (2):

$$\Phi[(x_{1L}\theta_1),(x_{2L}\theta_2),(x_{3L}\theta_3),(x_{4L}\theta_L)]$$

$$= c\Lambda^4\left(\frac{2\pi}{\alpha_s}\right)^4 \exp\left(-\frac{2\pi}{\alpha_s(\Lambda)}\right)$$

$$\times \int d^4x_0\, d^2\theta_0 \frac{d\rho}{\rho} d^2\bar{\beta}\, d^2\alpha_1\, d^2\bar{\beta}_1 \left[\alpha_1 + (x_{3L}-x_0,\sigma)\bar{\beta}_1\right]^2$$

$$+\left[\alpha_1 + (x_{4L}-x_0,\sigma)\bar{\beta}_1\right]^2 \times \prod_{i=1}^{4} \frac{\rho^4\left[\theta_i - \theta_0 - (x_{iL}-x_0,\sigma)\bar{\beta}\right]^2}{\left[(x_{iL}-x_0)^2+\rho^2\right]^4}. \quad (21)$$

Let us concentrate first on the $\theta_3^2\theta_4^2$ component. It corresponds to the case when purely fermionic components are kept in all the superfields. Performing integration over $\theta_0, \bar{\beta}, \alpha_1, \bar{\beta}_1$ we come to

$$\int d^2\theta_3\, d^2\theta_4\, \theta_1^2\, d^2\theta_1\, \theta_2^2\, d^2\theta_2\, \Phi[(x_{1L}\theta_1),(x_{2L}\theta_2),(x_{3L}\theta_3),(x_{4L},\theta_L)]$$

$$= C\int d^4x_0\, d\rho\, \rho^{15}(x_{2L}-x_{1L})^2(x_{4L}-x_{3L})^2 \prod_{i=1}^{4}\left[(x_{iL}-x_0)^2+\rho^2\right]^{-4}. \quad (22)$$

Explicit calculation becomes more cumbersome but we do not actually need the numerical answer. It suffices to note that the integrand has a definite sign and the result of integration cannot be zero for this reason:

$$\int d^2\theta_3\, d^2\theta_4\, \theta_1^2\, d^2\theta_1\, \theta_2^2\, d^2\theta_2\, \Phi[(x_1\theta_1),(x_2\theta_2),(x_3\theta_3),(x_4\theta_4)] \neq 0. \quad (23)$$

Moreover, it is not a constant as is seen from the dimensional argument.

The $\theta_3^2\theta_4^2$ component is not the only one which is not vanishing. In a similar way we can prove that

$$\int \theta_3^2\, d^2\theta_3\, \theta_4^2\, d^2\theta_4\, d^2\theta_1\, d^2\theta_2\, \Phi[(x_1\theta_1),(x_2\theta_2),(x_3\theta_3),(x_4\theta_4)]$$

$$= \int d^2\theta_3\, d^2\theta_4\, \theta_1^2\, d^2\theta_1\, \theta_2^2\, d^2\theta_2\, \Phi[(x_1\theta_1),(x_2\theta_2),(x_3\theta_3),(x_4\theta_4)]. \quad (24)$$

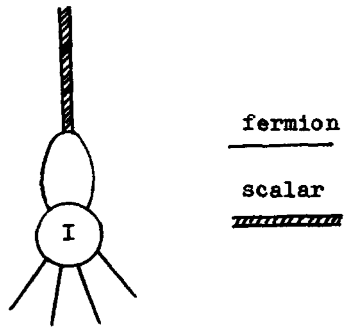

Fig. 2. Annihilation of two fermion zero modes by an external scalar source. In the superfield formalism this diagram corresponds to substitution of the scalar field zero mode which eliminates two zeros of the instanton determinant.

This $\theta_1^2\theta_2^2$ component corresponds to a purely bosonic field correlator, with no sign of the fermionic zero modes left.

This symmetry between the fermionic and bosonic components is of course inherent to expression (21) which is manifestly supersymmetric (at least with respect to the ε, not $\bar{\varepsilon}$, transformations). However, one can wonder how this symmetry comes about from the standard instanton calculation despite the eight fermionic zero modes.

Still, the standard calculation does agree with (24). The point is that with a change from the $(\varphi_\alpha^a)^2$ component of W^2 to the $G^a\varphi^\alpha$ or to the $(G_{\mu\nu}^a)^2$ component we gain an extra $1/g$ or $1/g^2$ factor since $(G_{\mu\nu}^a)^{\mathrm{cl}}$ is proportional to $1/g$. Therefore, we must keep the next order correction in g as well. At this point we should consider the graphs of the type represented in fig. 2 which convert two fermionic zero modes into a scalar field and describe annihilation of two fermionic zero modes by a scalar external source. The graphs are generated by interaction of the gluino with the matter scalar and spinor fields. One readily finds that just this kind of interaction is implicit in eq. (21).

Moreover, it is quite straightforward to check that the standard calculus and eq. (21) give the same result. The point is that the integration over the extra interaction coordinate implicit in fig. 2 produces exactly the same result as a direct substitution of the scalar field "zero mode" into SS, as is done in eq. (21). By the scalar field zero mode we understand the result of application of the supertransformation to the fermionic zero mode.

Indeed, the scalar field s satisfies the equation

$$D^2 s = ig\varphi \times \psi\,, \tag{25}$$

where ψ and φ are the spinor fields of matter and gluino, respectively. Moreover, using the Green function for the scalar field in the presence of the instanton we can solve (25):

$$s(x) = ig\int G(x, y)\varphi(y) \times \bar{\psi}(y)\,\mathrm{d}^4y, \tag{26}$$

where $G(x, y)$ is the Green function and the fermionic fields are considered as given, which is true in this order in the coupling constant. Expression (26) for the scalar field is in one-to-one correspondence with the graph of fig. 2. We can integrate over d^4y either explicitly or guess the result using the supersymmetry, using the fact that the scalar and spinor fields are in the same supermultiplet. Eq. (21) uses the latter way.

To summarize, the superfield and ordinary formalism result of course in the same final answer but the use of the superfields allows us to shorten the derivation.

5. Conclusions

Thus we have shown that instantons in supersymmetric theories generate some non-trivial n-point functions. In case of the $N = 1$ theory the main lesson is that the dynamics must be quite unusual. The options are: (a) a certain correlator, corresponding to connected graphs, does not fall off at large distances; (b) there is conspiracy between the large-scale fluctuations and instantons of zero size to cancel each other's contributions; or (c) there is a new selection rule which forbids the presence of instantons in the physical vacuum.

In the case of the $N = 2$ calculation the lesson is that the vacua with different topological charges are connected via instantons in the presence of external sources.

Moreover, there is no actual problem with the apparent boson-fermion asymmetry due to the fermionic zero modes and the supersymmetry. In more detail, there are two different sets of Ward identities depending on the chirality of the supercharge generating the transformations. In the case of the $(Q\varepsilon)$ transformations one must account for the scalar field zero modes which, formally, substitute for two fermionic zero modes. We have traced the same mechanism within the standard instanton calculus as well. In the case of the $(\bar{Q}\bar{\varepsilon})$ transformations, collective coordinates, like the instanton size, change. Supersymmetry recovers upon integration over all the collective coordinates.

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