Nuclear Physics B106 (1976) 475–492

MAGNETIC MONOPOLES IN SU(3) GAUGE THEORIES

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Received 19 December 1975

't Hooft constructed a finite mass unit monopole by (a) embedding the electromagnetic U(1) gauge theory in an SU(2) theory *via* a Higgs triplet mechanism, (b) seeking solutions invariant under simultaneous rotations in ordinary and SU(2) space. We consider an SU(3) gauge theory broken down to U(2) by an octet Higgs vector and find two finite-mass stable solutions corresponding to the two ways of embedding SU(2) in SU(3). One solution is the pure unit monopole but the other, which promises to be lighter, has $\frac{1}{2}$ unit of magnetic charge (thus violating Dirac's condition, naïvely applied) as well as being a source of isomagnetic flux (associated with the unbroken SU(2)).

1. Introduction

There are a variety of reasons for the present interest in monopoles:

(i) The hope that they may provide end points for the magnetic vortex line model of dual strings [1].

(ii) More generally they may be relevant to the confinement problem.

(iii) U(1) monopoles arise naturally when the U(1) gauge group is embedded in a SU(2) gauge symmetry broken down to U(1) by an isovector Higgs field [2]. The monopole charge is not apparent in the original Lagrangian and is seen to be topological in nature [3]. As well as being stable, when smoothed out, the monopole is

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heavy and strongly interacting. Maybe the hadrons, together with their quantum numbers, arise thus as solitons of the unified gauge theories of weak and electromagnetic interactions.

Nevertheless, there appear to be fundamental difficulties blocking the realisation of these ideas and it seems to us that more information about possible generalisation of monopoles should be sought by looking at explicit examples. We shall consider the most natural gauge group beyond SU(2), namely SU(3), since it already has much potentially interesting new structure, which, as we shall see, does lead to something new. By analogy with SU(2), we take the Higgs field to be an octet. The form of its self-interaction (assumed renormalizable) forces its ground states to be all related to the 8-axis by a SU(3) rotation [4, 5]. Then the unbroken gauge symmetry corresponding to its little group consists of a U(1) (in the direction of the Higgs field), which we identify as the electromagnetic gauge group, times an SU(2). Thus we have a natural framework for finding SU(2) monopoles (as well as U(1) ones) if they exist. Actually, general arguments [6] say that pure SU(2) monopoles cannot exist (because SU(2) is simply connected), and indeed, we shall see in some detail why this is so here. In particular, we shall find that there are topologically distinct ways in which the Higgs field can behave at spatial infinity and that these distinct ways are characterized by the monopole charge associated with the U(1) gauge group mentioned. This is expected; what is unexpected is that in terms of Dirac units we find solutions with one unit and one half unit. By Dirac unit we mean $1/2q$ where q is the smallest U(1) electric charge occurring in the theory, namely the quark charge. (If we had taken the gauge particle charge $Q = 3q$ as the smallest unit of electric charge, we would have solutions with 3 or $\frac{3}{2}$ Dirac units which equally well violate Dirac's argument that all monopole charges must be integral multiples of his unit.)

Our solution with $g = \frac{1}{2}$ $(1/2q)$ which violates Dirac's result does not actually violate his argument, since it has an SU(2) component, unlike the $g = 1$ $(1/2q)$ solution mentioned, which is purely U(1). As explained in the text, the two components, when added vectorially, do satisfy Dirac's condition, suitably interpreted. We also find other solutions; for instance, a mixed solution with $g = 1$ $(1/2q)$, which probably decays into the pure monopole with the same monopole charge.

't Hooft's SU(2) monopole solution had a very special symmetry. It was invariant under simultaneous rotations in real space and isotopic space, i.e. rotations generated by $\boldsymbol{J} + \boldsymbol{T}$, where $\boldsymbol{J}$ is the angular momentum and $\boldsymbol{T}$ the isospin generator. We look for all solutions with the same symmetry (since symmetric solutions should have lower energies, and this seems to be the maximal non-trivial symmetry we can achieve). There are essentially two different ways of identifying $\boldsymbol{T}$ among the SU(3) generators, $(\frac{1}{2}\lambda_1, \frac{1}{2}\lambda_2, \frac{1}{2}\lambda_3)$ or $(\lambda_2, \lambda_5, \lambda_7)$. These generate SU(2) and SO(3) groups, respectively, which are subgroups of SU(3).

Another important aspect of our procedure is that we look for "point solutions" first. These are singular at the origin and are characterized by the fact that the Higgs field is unexcited (so that its covariant derivative vanishes) and that the gauge field tensor exhibits an inverse square law radial field. Having found these solutions (which are tabulated) we use them as guides to the asymptotic behaviour at large radii to be satisfied by smoothed out finite energy solutions.

In sect. 2 we set up the general gauge formalism, explaining our general approach, with reference to the SU(2) case. In sect. 3 we find all possible Higgs fields satisfying the above invariance (there are six possibilities), and we discuss their topological properties. In sect. 4 we find all possible gauge potentials with the aforementioned invariance, and use the remaining gauge invariance to eliminate many of the unknown functions and suggest the judicious choice of variables. The general Lagrangian in the presence of the above symmetry is found and used to establish the point solutions.

These are discussed in sect. 5 together with the possibility of smoothing them out.

2. General approach

Consider the gauge theory for a compact connected Lie group G, furnished with a Higgs field ϕ_i and a Dirac field ψ. When the ϕ is in its ground state, the symmetry H which survives the Higgs mechanism unbroken is the little group of ϕ. We shall assume that ϕ_i belongs to the self-adjoint representation of G, just as the gauge potential W_i^μ. If Q_i denotes the generators of G, then the generators of H consist of the Q_i commuting with $\hat{\phi}_i Q_i$ ($\hat{\phi}_i = \phi_i/\sqrt{\phi^2}$). Then H automatically has a U(1) factor generated by $\hat{\phi}_i Q_i$ itself, which we shall identify as the electromagnetic gauge group since this is the only invariant possibility.

The Lagrangian is

$$\mathcal{L} = -\tfrac{1}{4}(G_{\mu\nu}^2)^2 + \tfrac{1}{2}(\mathcal{D}_\mu\phi)^2 + i\bar{\psi}\gamma\cdot\mathcal{D}\,\psi - V(\phi)\,, \tag{2.1}$$

and the equations of motion,

$$\mathcal{D}^\mu G_{\mu\nu}^i = ef^{ijk}\,(\mathcal{D}_\nu\phi)_j\,\phi_k + e\bar{\psi}\gamma_\nu\,D(Q_i)\,\psi\,, \tag{2.2}$$

$$\mathcal{D}^\mu\,\mathcal{D}_\mu\phi_i = \partial V/\partial\phi_i\,, \tag{2.3}$$

$$\gamma^\mu\,\mathcal{D}_\mu\,\psi = 0\,, \tag{2.4}$$

where $\mathcal{D}^\mu$ denotes the appropriate covariant derivatives,

$$(\mathcal{D}^\mu)_{\alpha\beta} = \partial^\mu\delta_{\alpha\beta} - ie\,W_i^\mu\,D_{\alpha\beta}(Q^i)\,, \tag{2.5}$$

and $D(Q^i)$ denotes the generator Q^i in the appropriate representation.

We now ask how we can see the gauge theory for H embedded within the above theory, and our answer is to put ϕ in its ground state in a covariant way by imposing

$$\mathcal{D}^\mu\phi = 0\,, \qquad \partial V/\partial\phi = 0\,. \tag{2.6}$$

The defining property of the gauge field tensor $G^i_{\mu\nu}$ is

$$([\mathcal{D}_\mu, \mathcal{D}_\nu]\ \phi)_i = ief_{ijk}\, G^j_{\mu\nu}\phi^k\,,$$

and (2.6) therefore implies that $G_{\mu\nu}$ is in the little group of ϕ, i.e. the only non-vanishing components of $G^i_{\mu\nu}$ are indeed those corresponding to the generators of H.

The electromagnetic U(1) gauge invariant derivative of a field with charge q is

$$\partial_\mu - iq\,A_\mu\,,$$

where A_μ is the electromagnetic potential. If $A_\mu = \hat{\phi}^i W^i_\mu$ we see, comparing with (2.5), that the electric charges of the particles are the eigenvalues of $eD(Q\hat{\phi})$. We shall use this result repeatedly.

Notice that if (2.6) is satisfied, the equations of motion (2.2)–(2.4) reduce to

$$\mathcal{D}^\mu G^i_{\mu\nu} = 0\,,$$

if in addition ψ vanishes.

't Hooft [2] considered G = O(3) so that H = U(1), and so (2.6) then implies

$$G^i_{\mu\nu} = \hat{\phi}^i\, F_{\mu\nu}\,, \tag{2.7}$$

where by (2.2),

$$\partial^\mu F_{\mu\nu} = e\hat{\phi}_i \bar{\psi}\, \tfrac{1}{2}\tau_i\gamma_\nu\psi\,, \tag{2.8}$$

$$\partial^\mu F^*_{\mu\nu} = 0\,. \tag{2.9}$$

These are the Maxwell equations. There is a difference, however, because the gauge potential is, as a consequence of (2.6),

$$\boldsymbol{W}_\mu = (1/e)\,\partial_\mu\,\hat{\boldsymbol{\phi}}\wedge\hat{\boldsymbol{\phi}} + A_\mu\hat{\boldsymbol{\phi}}, \tag{2.10}$$

where A_μ is arbitrary. As choice of gauge, the gauge $W_0 = 0$ is particularly convenient, since it corresponds, using the constraint (2.6), to a time-independent scalar field ϕ and the electromagnetic gauge choice $A_0 = 0$. Then we find

$$\boldsymbol{G}_{\mu\nu} = \hat{\boldsymbol{\phi}}(\partial_\mu A_\nu - \partial_\nu A_\mu - (1/e)\,\hat{\boldsymbol{\phi}}\cdot\partial_\mu\hat{\boldsymbol{\phi}}\wedge\partial_\nu\hat{\boldsymbol{\phi}})\,, \tag{2.11}$$

which is as predicted eq. (2.7). A_μ is like the ordinary electromagnetic four-potential if a gauge is chosen in which $\hat{\phi}$ is a constant vector. There are situations in which this is not possible; for example, if

$$\hat{\boldsymbol{\phi}} = \boldsymbol{n}(M)\,,$$

where

$$n(M) = (\sin\theta\cos M\phi, \sin\theta\sin M\phi, \cos\theta), \qquad M = 0, \pm 1, \pm 2\dots . \tag{2.12}$$

It is impossible to rotate $\hat{\phi}$ to a constant vector continuously in θ and ϕ if $M \neq 0$. If $A_\mu = 0$, we find that (2.8) is satisfied with $\psi = 0$, since

$$F_{\mu\nu} = (M/e)\, \epsilon_{\mu\nu\sigma}\, \hat{r}_\sigma / r^2 , \tag{2.13}$$

i.e. we have a point magnetic monopole of strength M/e described without any Dirac strings. The only singularity of any field quantity is at the origin.

$M = 1$ is the simplest case; then $\boldsymbol{n}(1) = \hat{r}$, and we shall say that we have one Dirac unit of monopole charge, since the isospinor field ψ has electric charge $e_0 = \frac{1}{2}e$, and $g_0 = 1/e$ is then the smallest monopole charge allowed by Dirac's condition

$$e_0 g_0 = \tfrac{1}{2} . \tag{2.14}$$

In order to construct a monopole solution which is regular everywhere and has finite mass, the condition (2.6) must be relaxed so that it is valid only asymptotically for large radii. For $M = 1$ the field (2.12) has a special symmetry: it is invariant under simultaneous space and isotopic rotations. Roughly,

$$[\boldsymbol{J} + \boldsymbol{T}, \phi] = 0 . \tag{2.15}$$

Equally,

$$\partial\phi/\partial t = 0 , \qquad W_0 = 0 . \tag{2.16}$$

These conditions, particularly (2.15), are both very significant and mysterious at the same time. Eq. (2.15) couples together space-rotation (J_i) and isotopic rotation (T_i) generators. After imposing (2.16), the field equations (2.2) and (2.4) have an $O_J(3) \times O_T(3)$ symmetry, and $\boldsymbol{J} + \boldsymbol{T}$ generates an O(3) subgroup of this. Thus eq. (2.15) can be maintained, independently of (2.6) and applied to W^i_μ also, and will lead to a significant simplification of eqs. (2.2)–(2.4).

't Hooft argued that the resultant equations had a solution with finite mass and monopole charge $g = 1/e$. This procedure does not work if $|M| > 1$, and we have failed to establish that non-singular finite mass solutions exist then.

Our aim in this paper is to generalize the procedure outlined here to the SU(3) case. The fundamental things we keep are (2.6), when looking for point solutions or at the asymptotic behaviour of regular solutions, together with eqs. (2.15) and (2.16).

The work of Wu and Yang [8], discussing solutions of pure SU(2) Yang-Mills theory, has been extended to SU(3) by Wu and Wu [8], and has been further considered by Marciano and Pagels and by Chakrabarti. The authors of ref. [5] also suggest the ansatz (3.9a) below for an SU(3) Higgs field.

3. The octet Higgs field

There are two fundamental difference between SU(3) and SU(2), which will underly the differences in the corresponding monopole spectra. One is that different octet vectors (of unit length) cannot, in general, be SU(3) rotated into each other, whereas triplet vectors of unit length can always be SU(2) rotated into each other. The other is that there are two distinct ways of embedding SU(2) in SU(3) (as we shall see later).

It will be much simpler to work with 3×3 Hermitian traceless matrices Φ rather than octet vectors ϕ_i,

$$\Phi = \sum_{i=1}^{8} \tfrac{1}{2}\lambda_i \phi_i\,, \qquad \phi_i = \mathrm{Tr}\,(\lambda_i \Phi)\,. \tag{3.1}$$

If $\hat{\phi}_i$ is normalized, $\hat{\phi}_i = \phi_i/\sqrt{\phi^2}$,

$$\hat{\phi}^2 = 1\,, \qquad \mathrm{Tr}\,(\hat{\Phi}^2) = \tfrac{1}{2}\,. \tag{3.2}$$

If Φ is SU(3) rotated to Φ',

$$\Phi' = U\,\Phi\,U^{+}, \qquad UU^{+} = 1, \qquad \det U = 1\,, \tag{3.3}$$

and its eigenvalues are invariant. Thus Φ's with different eigenvalues lie on distinct SU(3) orbits. It is the Higgs field self-interaction $V(\phi)$ which determines these eigenvalues when ϕ is in its ground state. In octet language the most general renormalizable possibility is

$$V(\phi) = p\,(1 - \phi^2)^2 + q\,(\phi_i \pm \sqrt{3}\,\mathrm{d}_{ijk}\phi_j\phi_k)^2, \qquad p,q > 0 \tag{3.4}$$

(given that, for convenience, ϕ is normalized in its ground state). We shall henceforth choose the plus sign. Obviously $V(\phi) \geqslant 0$ and vanishes only if $\phi_i^2 = 1$ and $\phi_i = -\sqrt{3}d_{ijk}\phi_j\phi_k$, i.e. if ϕ_i lies on the SU(3) orbit of δ_{i8}. Thus in its ground state Φ always has the same eigenvalues as $\frac{1}{2}\lambda_8$, namely

$$\tfrac{1}{2}\sqrt{\tfrac{1}{3}}, \quad \tfrac{1}{2}\sqrt{\tfrac{1}{3}}, \quad -\sqrt{\tfrac{1}{3}}, \tag{3.5}$$

and we shall say it is "λ_8-like" [4,5].

This has two important consequences. First, all electric charges are quantized, since they are all integral multiplies of the smallest eigenvalue of the quark charge operator $e\Phi$, which means multiples of $e_0 = \frac{1}{2}\sqrt{\frac{1}{3}}e$, the smallest quark charge. Secondly, the unbroken symmetry group, which is the little group of Φ, has the structure $U(1) \times SU(2)$ (or more correctly U(2)), whereas if no two eigenvalues of Φ are equal it is $U(1) \times U(1)$. It is interesting to have the SU(2) symmetry, because we can then hope to find pure SU(2) monopole (if they exist).

If Φ is not in its ground state, it is no longer clear what is the quark charge operator, but a likely candidate (analogous to the SU(2) choice) is $e\hat{\Phi}$. In general, the eigenvalues, and hence the electric charges, can vary continuously and will not be quantized. This is unsatisfactory, and it appears that we can only recognize conventional electromagnetic theory embedded in the SU(3) gauge theory when the Higgs field is in its ground state. In particular, unlike the SU(2) case, no electromagnetic tensor can be defined in general.

We will look for Higgs fields satisfying (2.15), simply because this is the simplest possibility to treat and corresponds to what 't Hooft did for SU(2). Thus we must ask how the SU(2) (or SO(3) generators T_i can be embedded in SU(3). There are essentially two ways, corresponding to the two possible decompositions of a quark triplet into SU(2) irreducible representations,

$$(3) = (2) + 1 \,, \tag{3.6a}$$

$$(3) = (3) \,. \tag{3.6b}$$

The corresponding decompositions for octets are

$$(8) = (3) + (2) + (2) + (1) \,, \tag{3.7a}$$

$$(8) = (5) + (3) \,, \tag{3.7b}$$

explicitly,

$$\Phi(r) = \alpha(r)\psi_1 + \beta(r)\psi_2 \,, \tag{3.8a}$$

$$\psi_1 = \sum_1^3 \tfrac{1}{2}\lambda_i r_i \,, \qquad \psi_2 = \tfrac{1}{2}\lambda_8 \,, \tag{3.9a}$$

or

$$\Phi(r) = A(r)\phi_1 + B(r)\phi_2 \,, \tag{3.8b}$$

$$(\phi_1)_{\alpha\beta} = \hat{r}_\alpha \hat{r}_\beta - \tfrac{1}{3}\delta_{\alpha\beta}\hat{r} \,, \qquad (\phi_2)_{\alpha\beta} = i\epsilon_{\alpha\beta\gamma}\hat{r}_\gamma \,. \tag{3.9b}$$

These satisfy

$$[J + T, \Phi] = 0 \,, \tag{3.10}$$

where $(J)_{\sigma\rho} = \delta_{\sigma\rho}(-ir \wedge \Delta)$ since J is purely orbital when acting on a Lorentz scalar ϕ, and

$$(T_1, T_2, T_3) = (\tfrac{1}{2}\lambda_1, \tfrac{1}{2}\lambda_2, \tfrac{1}{2}\lambda_3) \tag{3.11a}$$

$$= (\lambda_1, -\lambda_5, \lambda_2) \,, \tag{3.11b}$$

respectively, corresponding to the cases (a) and (b) in eqs. (3.7)–(3.9). We may think of these as the "U spin" and "nuclear physics" [9] embeddings of SU(2) in SU(3), respectively. These are the most general solutions of (3.10).

If we further demand that Φ satisfy (2.6), and so is λ_8-like, we find

$$\Phi = \tfrac{1}{2}(\pm\sqrt{3}\,\psi_1 - \psi_2) \,, \tag{3.12}$$

or $$\Phi = \psi_2 \tag{3.13}$$

$$\text{or} \quad \Phi = \tfrac{1}{4}\sqrt{3}\,(\phi_1 \pm \phi_2) \tag{3.14}$$

$$\text{or} \quad \Phi = -\tfrac{1}{2}\sqrt{3}\,\phi_1\ . \tag{3.15}$$

Before proceeding further, we should ask if these possibilities are topologically distinct in the following sense. Let us regard eqs. (3.12)–(3.15) as mappings from the unit sphere in space ($\hat{r}$) to the set of possible scalar fields Φ of λ_8 type; then the mappings are topologically distinct if they cannot be continuously distorted into one another. More precisely, the orbit of λ_8 in SU(3) is the manifold SU(3)/U(2), where U(2) is the little group of λ_8, and we must decide to which second homotopy class each of the above mappings corresponds. That the second homotopy group is *non-trivial* follows from the relation [6] $\pi_2[\mathrm{SU(3)/U(2)}] = \pi_1[\mathrm{U(2)}] = \pi_1[\mathrm{U(1)}] = Z$, since SU(3) is simply connected. We have been unable to decide this question directly, but observe that each of the six possibilities above can be transformed by an SU(3) rotation, continuously, as a function of the Euler angles θ,ϕ, into one or another of

$$\Phi(M) = -\tfrac{1}{4}\lambda_8 + \tfrac{1}{4}\sqrt{3}\ \boldsymbol{\lambda}\cdot\boldsymbol{n}(M)\ , \qquad M = 0, \pm 1, \pm 2 \tag{3.16}$$

((3.15) and (3.13) can be continuously transformed into one another in this sense, see below), and we believe that the integers M label the homotopy classes, referred to above, with (3.16) defining typical members of each class. $\boldsymbol{n}(M)$ is the vector defined in eq. (2.12).

The standard form (3.16) has the disadvantage that it commutes with $\boldsymbol{J} + \boldsymbol{T}$ only if $M = \pm 1$. Later on, we shall construct point solutions (several of which may correspond to a single ϕ), and it will turn out that the electromagnetic monopole charge of the point solution is always $-\sqrt{3}M/2e$ i.e. $-\frac{1}{2}M$ Dirac units $\sqrt{3}/e$). Besides explicit verification, we have no general proof of this statement, but it is probably the clue to constructing monopole solutions for $|M| > 2$.

The required matrices U such that

$$U\,\Phi\,U^+ = \Phi(M)\ , \qquad UU^+ = 1\ , \qquad \det U = 1\ , \tag{3.17}$$

are I and

$$V = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

respectively, for (3.12), and $VU(\theta,\phi)$ and $U(\theta,\phi)$ respectively rotate $\frac{1}{4}\sqrt{3}\,(\phi_1 \pm \phi_2)$, (3.14) into $\phi(\mp 2)$ where

$$U(\theta,\phi) = \begin{pmatrix} X \begin{pmatrix} \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\theta}} \end{pmatrix} \\ \hat{r} \end{pmatrix} \tag{3.18}$$

and $\hat{r}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$ are the unit vectors associated with the spherical polar coordinate system and X is the (2×2) matrix

$$X = \sqrt{\tfrac{1}{2}} \begin{pmatrix} e^{i(\phi - \frac{1}{2}\theta)} & ie^{i(\phi+\frac{1}{2}\theta)} \\ ie^{-i(\phi+\frac{1}{2}\theta)} & e^{-i(\phi-\frac{1}{2}\theta)} \end{pmatrix} .$$

The matrix $U(\theta, \phi)$ given by (3.18) is indeed everywhere continuous in θ and ϕ on the unit sphere and has thc property that

$$U(\theta, \phi)\,\hat{r} = \hat{z} ,$$

something which an SO(3) matrix cannot do. When applied to (3.15) *via* (3.17), $U(\theta, \phi)$ yields (3.13), which can further be rotated to $\Phi(0)$ by $U = \exp(\frac{1}{2}i\theta\lambda_2)$. That the "gauge equivalence" is a homotopic equivalence follows from the mathematical fact that the second homotopy group of SU(3) is trivial, i.e. $\pi_2[\mathrm{SU}(3)] = 1$. Then there must exist a family of continuous gauge transformations $U(t; \theta, \phi)$, $0 \leqslant t \leqslant 1$, such that $U(0; \theta, \phi) = 1$, $U(1; \theta, \phi) = U(\theta, \phi)$ above, and the homotopic equivalence is established.

4. Solutions to the gauge field equations

We mentioned the assumption that the Higgs octet field satisfies

$$[\boldsymbol{J} + \boldsymbol{T}, \Phi] = 0 . \tag{4.1}$$

We shall now assume, in addition, that the octet gauge field potential satisfies the analogous equation

$$[\boldsymbol{J} + \boldsymbol{T}, W] = 0 . \tag{4.2}$$

It is understood that the spin angular momentum part of $\boldsymbol{J}$ acts on the Lorentz index of the W's and that the representation of $\boldsymbol{T}$ is either (3.11a) or (3.11b), corresponding to whether the Higgs field considered is (3.8a) or (3.8b).

The field equations (2.2)–(2.4) are invariant under $\mathrm{O}_J(3) \times \mathrm{O}_T(3)$, i.e. the group generated by $\boldsymbol{J}$ and $\boldsymbol{T}$ separately. Thus we can expect a considerable simplification of these equations if we seek solutions invariant under the subgroup $\mathrm{O}_{J+T}(3)$ generated by $\boldsymbol{J} + \boldsymbol{T}$. Further, these solutions, having a high degree of symmetry, would be expected to have the lowest energies.

Our first object is to show that if (4.1) and (4.2) are satisfied, W_μ is actually very limited. The first step is to show that the most general possibility for W_μ in the two cases (3.11a) and (3.11b) are, respectively,

$$eW_\mu = \gamma(r)\,\partial_\mu\psi_1 + i(1 - \delta(r))[\psi_1, \partial_\mu\psi_1] + \zeta(r)\frac{\hat{r}_\mu}{r}\,\psi_1 + \eta(r)\frac{\hat{r}_\mu}{r}\,\psi_2 , \tag{4.3a}$$

$$\begin{aligned} eW_\mu = {} & C_1(r)\,\partial_\mu\phi_1 + i(1 - D_1(r))[\phi_1, \partial_\mu\phi_1] + C_2(r)\,\partial_\mu\phi_2 \\ & - iD_2(r)[\phi_2, \partial_\mu\phi_1] + E_1(r)\frac{\hat{r}_\mu}{r}\,\phi_1 + E_2(r)\frac{\hat{r}_\mu}{r}\,\phi_2 . \end{aligned} \tag{4.3b}$$

Thus we have expanded W_μ in terms of basic functions satisfying (4.2) with coefficients spherically symmetric functions $\gamma(r)$, $\delta(r)$ etc., which will be determined by the equations of motion. The chosen basis promises to be useful since it is related to the Φ basis, (3.8a) and (3.8b), and because of the algebraic properties discussed in the appendix.

To count the number of basis elements, we examine how W_μ transforms under $\boldsymbol{S}+\boldsymbol{T}$, where $\boldsymbol{S}$ is the spin part of $\boldsymbol{J}$, and $\boldsymbol{T}$ is given by (3.11a) or (3.11b). In case (a) W_μ must transform (according to (3.7a)) as a

$$(3) \times ((3) + (1)) = (5) + (3) + (1) + (3)\,,$$

and so have four terms, as indeed (4.3a) has. In case (b) (3.7b) gives

$$(3) \times ((5) + (3)) = (7) + (5) + (3) + (5) + (3) + (1)\,.$$

Eq. (5.2b) should have six terms as indeed it does. That the basic functions in (4.3) are linearly dependent, follows from the orthogonality under the scalar product $\mathrm{Tr}(X^\mu Y_\mu)$.

We shall now show that many of the invariant functions $\gamma(r)$, $\delta(r)$ etc. are redundant, owing to gauge invariance. The invariance (4.1) is partly a choice of gauge, since we can easily SU(3) gauge transform Φ so that it is not true. On the other hand, it does not determine the gauge uniquely since there remains a U(1) $\times$ U(1) subgroup of gauge transformation leaving Φ (as given by (3.8)) invariant. This can be used to eliminate some of the coefficients in the W_μ expansion (4.3) as we shall now see. Under any SU(3) gauge transformation U, the gauge potential transforms as

$$W'_\mu = UW_\mu U^+ + (i/e)\, U\partial_\mu U^+\,. \tag{4.4}$$

We parametrize the U(1) $\times$ U(1) little group of Φ (3.8a) as

$$U(r) = \mathrm{e}^{i\theta(r)\hat{\boldsymbol{r}}\cdot\frac{1}{2}\boldsymbol{\lambda}}\,\mathrm{e}^{i\phi(r)\frac{1}{2}\lambda_8}\,. \tag{4.5a}$$

This yields

$$\zeta' = \zeta - r(\partial\theta/\partial r)\,, \qquad \eta' = \eta - r(\partial\phi/\partial r)\,, \tag{4.6a}$$

$$\gamma' + i\delta' = \mathrm{e}^{i\theta}\,(\gamma + i\delta)\,.$$

Many conclusions can be drawn from this. Gauge-invariant expressions, such as the energy density, can only depend on invariant combinations such as $\gamma^2+\delta^2$ and not all on η. If the expression involves only first derivatives, $\partial\zeta/\partial r$ cannot occur. Hence ζ will be eliminated trivially by its equation of motion. γ can be gauge-transformed away. This will leave only one effective degree of freedom, $\delta(r)$.

Note that W_μ has a gauge-invariant part, $i[\psi_1, \partial_\mu\psi_1]/e$. Choosing $\gamma = 0$, by our choice of gauge we find, using the results of the appendix,

$$eG_{\mu\nu} = \frac{(\delta^2-1)\,\epsilon_{\mu\nu\sigma}\hat{r}_\sigma\psi_1}{r^2} - \frac{i}{r}\frac{\mathrm{d}\delta}{\mathrm{d}r}[\psi_1, \hat{r}_\mu\partial_\nu\psi_1 - \hat{r}_\nu\partial_\mu\psi_1] \tag{4.7a}$$

$$-\frac{\zeta\delta}{r}(\hat{r}_\mu\partial_\nu\psi_1 - \hat{r}_\nu\partial_\mu\psi_1)\,,$$

$$\mathcal{D}_\mu\Phi = \alpha\delta\ \partial_\mu\psi_1 + \frac{\mathrm{d}\alpha}{\mathrm{d}r}\hat{r}_\mu\psi_1 + \frac{\mathrm{d}\beta}{\mathrm{d}r}\hat{r}_\mu\psi_2\,, \tag{4.8a}$$

$$H = -\int \mathrm{d}^3x\,\mathcal{L} = \int \mathrm{d}^3x\left(\frac{1}{e^2r^2}\left(\frac{\mathrm{d}\delta}{\mathrm{d}r}\right)^2 + \frac{(\delta^2-1)^2}{2e^2r^4} + \frac{\alpha^2\delta^2}{r^2}\right.$$
$$\left. + \frac{1}{2}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}r}\right)^2 + \frac{1}{2}\left(\frac{\mathrm{d}\beta}{\mathrm{d}r}\right)^2 + p(1-\alpha^2-\beta^2)^2 + q[\alpha^2(1+2\beta)^2 + (\beta+\alpha^2-\beta^2)^2]\right) \tag{4.9a}$$

In the last equation, ζ has been eliminated by its own, trivial, equation of motion, $\zeta = 0$. The equations of motion for α, β, and derived from H, are indeed those found by direct substitution of (3.8a) and (4.3a) into the field equations. This is because their little group was a subgroup of the invariance group of the equations.

If ϕ is given by (3.8b) we can parametrize its little group by

$$U = \mathrm{e}^{i\theta_1(r)\phi_1}\,\mathrm{e}^{i\theta_2(r)\phi_2}\,. \tag{4.5b}$$

It is shown in the appendix that in a way very similar to the preceding, it is possible to choose a gauge in which $C_1 = C_2 = 0$. The derivatives of E_1 and E_2 cannot occur in the energy density. In the chosen gauge, the E_1 and E_2 equations of motion will simply state $E_1 = E_2 = 0$ and hence these can be ignored with impunity. Retaining just D_1 and D_2, we find

$$eG_{\mu\nu} = -(6\phi_1D_1D_2 + \phi_2(D_1^2 + D_2^2 - 1))\frac{\epsilon_{\mu\nu\sigma}\hat{r}_\sigma}{r^2}$$
$$-i\left[\frac{\mathrm{d}D_1}{\mathrm{d}r}\phi_1 + \frac{\mathrm{d}D_2}{\mathrm{d}r}\phi_2, r_\mu\partial_\nu\phi_1 - r_\nu\partial_\mu\phi_1\right]\,, \tag{4.7b}$$

$$\mathcal{D}_\mu\Phi = \partial_\mu\phi_1\,(AD_1 + BD_2) + \partial_\mu\phi_2\,(AD_2 + BD_1) + \hat{r}_\mu\phi_1\frac{\mathrm{d}A}{\mathrm{d}r} + \hat{r}_\mu\phi_2\frac{\mathrm{d}B}{\mathrm{d}r}\,, \tag{4.8b}$$

$$\frac{\mathrm{d}^3H}{\mathrm{d}^3x} = -\mathcal{L} = \frac{4}{e^2r^2}\left[\left(\frac{\mathrm{d}D_1}{\mathrm{d}r}\right)^2 + \left(\frac{\mathrm{d}D_2}{\mathrm{d}r}\right)^2 + \frac{(D_1^2+D_2^2-1)^2 + 12D_1^2D_2^2}{2r^2}\right]$$
$$+\frac{2}{3}\left(\frac{\mathrm{d}A}{\mathrm{d}r}\right)^2 + 2\left(\frac{\mathrm{d}B}{\mathrm{d}r}\right)^2 + \frac{4}{r^2}((AD_1+BD_2)^2 + (AD_2+BD_1)^2)$$
$$+p(4(B^2+\tfrac{1}{3}A^2)-1)^2 + 4q(\tfrac{1}{3}(A+2\sqrt{\tfrac{1}{3}}(A^2-3B^2))^2 + B^2(1-4\sqrt{\tfrac{1}{3}}A)^2)\,, \tag{4.9b}$$

Note that our expressions for H((4.9a,b)) are automatically positive, being sums of squares. Despite the more complicated situations they are simpler than 't Hooft's SU(2) expression, because the group theory has lead us to a better choice of variables (and in particular the isolation of the "1" in δ and D_1).

According to our procedure, explained earlier, we shall look for "point solutions" singular at the origin, and satisfying (2.6). This means that Φ is one or the other of (3.12)–(3.15) and that each coefficient in (4.8a) vanishes. There are few possibilities.

In case (a) we see from eq. (4.8a) that α and β are constant. Furthermore, either α or δ vanishes. If α vanishes, Φ is (3.12), and the δ equation of motion reads

$$\delta'' = \delta(\delta^2 - 1)/r^2 ,$$

with constant solutions $\delta = 0$ or 1. If α does not vanish, Φ is (3.12) and δ vanishes. Using (4.7a) we find the three distinct "point" solutions presented in the first three rows of columns I to II of table 1. Before discussing the meaning of these solutions we shall find the point solutions corresponding to the "nuclear physics" embedding of SU(2) in SU(3).

Now, eq. (2.6) implies that Φ is given by (3.14) or (3.15) and that each coefficient in (4.8b) vanishes. (3.15) then implies $D_1 = D_2 = 0$ (solution V), whereas (3.14) implies $D_1 = \mp D_2$. The D_1 equation of motion then implies

$$D_1'' = D_1(8D_1^2 - 1)/r^2 ,$$

with constant solutions $D_1 = 0$ (solution VI) or $D_1 = \pm\sqrt{\tfrac{1}{8}}$ (solution IV).

Table 1 presents these solutions for ϕ and W_μ in the first two rows, and the resultant $G_{\mu\nu}$ in the third row. Notice that this $G_{\mu\nu}$ always turns out to be a linear combination of ψ_1 and ψ_2 (or ϕ_1 and ϕ_2) times $\epsilon_{\mu\nu\sigma}\hat{r}_\sigma/r^2$, which indicates an inverse square law radial magnetic field. The linear combination of ψ's (or ϕ's) tells us the magnitude and direction of this flux in SU(3) space. To interpret it, we re-express it as a linear combination of the relevant Higgs field Φ, defining the electromagnetic direction and χ the normalized linear combination of ψ_1 and ψ_2 (or ϕ_1 and ϕ_2) orthogonal to ϕ, i.e.

$$\mathrm{Tr}\,(\Phi^2) = \mathrm{Tr}\,(\chi^2) = \tfrac{1}{2}, \qquad \mathrm{Tr}\,(\Phi\chi) = 0 .$$

By the argument of sect. 2, $G_{\mu\nu}$ must lie in the little group of Φ, an U(1) $\times$ SU(2) subgroup with Φ defining the U(1) direction. Hence χ denotes a direction in the SU(2) space. Therefore, the numerical coefficients of Φ and χ specify the electromagnetic and isomagnetic monopole charges, respectively. These are given in rows 4 and 5, expressed as multiples of what we call "Dirac units", namely the smallest possible value g_0, these magnetic charges may have according to a naive application of Dirac's quantization condition

$$e_0 g_0 = \tfrac{1}{2} ,$$

where e_0 is the smallest charge. Now the charges are the eigenvalues of the charge operator, and their eigenvalues vary with direction in SU(3) space, as we saw in sect. 3. For a direction n^i ($(n^i)^2 = 1$) the charge operator is $\tfrac{1}{2}en_i\lambda_i$, as we saw in

Table 1
Point solutions

T_1, T_2, T_3	$\lambda_1, \lambda_2, \lambda_3$ embedding			$\lambda_7, -\lambda_2, \lambda_5$ embedding		
Solution	I	II	III	IV	V	VI
Φ	ψ_2 (3.13)	ψ_2 (3.13)	$\frac{1}{2}(-\psi_2 \mp \sqrt{3}\psi_1)$ (3.12)	$\frac{1}{4}\sqrt{3}(\phi_1 \pm \phi_2)$ (3.14)	$-\frac{1}{2}\sqrt{3}\,\phi_1$ (3.15)	$\frac{1}{4}\sqrt{3}(\phi_1 \pm \phi_2)$ (3.14)
W_μ	$\delta = 1$	$\delta = 0$	$\delta = 0$	$D_1 = \mp D_2 = \sqrt{\frac{1}{8}}$	$D_1 = D_2 = 0$	$D_1 = D_2 = 0$
G where $G_{\mu\nu} = G \dfrac{\epsilon_{\mu\nu\sigma}\hat{r}_\sigma}{r^2}$	0	$-\frac{1}{e}\psi_1 = -\frac{1}{e}\chi$	$-\frac{1}{e}\psi_1 = \frac{\pm 1}{2e}(\sqrt{3}\phi + \chi)$	$\pm\frac{\sqrt{3}}{e}\phi$	$\phi_2 = \frac{2}{e}\chi$	$\frac{\phi_2}{e} = \pm\left(\frac{\sqrt{3}}{e}\phi - \frac{1}{e}\chi\right)$
(Electro) magnetic charge in Dirac units of $\sqrt{3}/e$	0	0	$\pm\frac{1}{2}$	± 1	0	± 1
Isomagnetic charge in Dirac units of $1/e$	0	-1	$\pm\frac{1}{2}$	0	2	1
Total magnetic charge in Dirac units appropriate to SU(3) direction of $G_{\mu\nu}$	0	1	1	1	2	2
Energy density $dH/d^3x = \frac{1}{2}\mathrm{Tr}(G_{\mu\nu}G_{\mu\nu})$ in units of $3/2e^2r^4$	0	$\frac{1}{3}$	$\frac{1}{3}$	1	$\frac{4}{3}$	$\frac{4}{3}$
Interpretation and conjectured stability	vacuum	pure isopole unstable into I	mixed "semipole" stable	pure monopole stable	pure isopole unstable into I	mixed isopole unstable into IV

sect. 2. If n^i is δ_{i8} (or related to it by an SU(3) rotation), then $e_0 = \frac{1}{2}\sqrt{\frac{1}{3}}e$. So $g_0 = \sqrt{3}/e$. If $n^i = \delta_{i3}$ (or is any SU(2) direction), then $e_0 = \frac{1}{2}e$, so $g_0 = 1/e$ as stated in table 1.

In making these statements we have used the charges in the quark (triplet) representation, since these are the smallest, but none of our conclusions would be altered if we referred to the charges of the gauge particles (which lie in the octet representation). Then the λ_8-like e_0 would have been multiplied by 3.

In row 6 we have looked at the total strength ($2\mathrm{Tr}G^2$) expressed as multiples of units appropriate to the direction of G(λ_3-like for solutions II, III, V, and VI and λ_8-like for IV).

Row 7 specifies the energy density with solution IV specifying the unit.

5. Discussion of the point solutions and their smoothing out

We shall now discuss the significance of the 6 point solutions in our table and conclude that columns III and IV are the most significant objects. Column IV is the object we should always have expected; a pure electromagnetic monopole with no isopole component. It is the only such object, and like 't Hooft's SU(2) solution, it has the lowest unit of magnetic charge allowed by Dirac's quantization condition, i.e.

$$e_0 g_0 = \pm\tfrac{1}{2}, \qquad g_0 = \pm\sqrt{3}/e ,$$

since the quark with lowest charge e_0 has $e_0 = \frac{1}{2}\sqrt{\frac{1}{3}}e$.

Column III is the surprise: it has lower energy density than IV, in fact a fraction of one third, and its magnetic charge is $\frac{1}{2}g_0$, thereby violating Dirac's condition. Associated with the magnetic flux is an "isomagnetic flux", a circumstance not envisaged by Dirac in his argument. Notice that the strength of the total magnetic charge listed in row 5 is always 0, 1 or 2, when expressed in appropriate units, so that in this sense Dirac's condition is satisfied. We conjecture that if Dirac's argument were applied to the situation where both pure magnetic and isomagnetic flux were possible, then this would be the appropriate condition. We can illustrate this by the following example. Consider a quark moving in the field of a fixed "semipole" (solution III). If it is isosinglet, its charge is $-2e_0 = -\sqrt{\frac{1}{3}}e$ and no isoelectric flux emerges so the only contribution to the field angular momentum is pure electromagnetic. Then Dirac's argument is valid and is satisfied,

$$(-2e_0)\left(\tfrac{1}{2}g_0\right) = \tfrac{1}{2} .$$

If the quark is isodoublet, its charge is e_0, and it is a source of isoelectric flux which now interacts with the isomagnetic flux of the semipole to add to the purely electromagnetic contribution to the field angular momentum. Thus Dirac's argument is invalidated just when the result was invalid.

Our solutions support the idea that the purely magnetic charge is determined solely by the homotopy class of the Higgs field at large distances [6]. In fact, in each case, the magnetic charge is $-\frac{1}{2}M$ in Dirac units, where M is derived from the appropriate Higgs field by gauge-transforming it into the form (3.16);

$$\Phi(M) = \tfrac{1}{4}(-\lambda_8 + \sqrt{3}\ \boldsymbol{\lambda}\cdot \boldsymbol{n}(M))\ , \qquad M = 0, \pm 1, \pm 2$$

as discussed in sect. 3. In fact, the following point solution corroborates this for all $M = 0, \pm 1, \pm 2, \pm 3 \ldots$,

$$\Phi = \Phi(M)\ , \qquad W_\mu = \tfrac{4}{3}(i/e)\ [\phi(M), \partial_\mu \phi(M)]\ ,$$

$$G_{\mu\nu} = -\frac{M}{er^2}\,\epsilon_{\mu\nu\sigma}\,\hat{r}_\sigma\,\tfrac{1}{2}\boldsymbol{n}(M)\cdot\boldsymbol{\lambda} = \frac{\epsilon_{\mu\nu\sigma}\hat{r}_\sigma}{r^2}\left(-\tfrac{1}{2}\sqrt{3}\frac{M}{e}\Phi - \frac{M}{\partial e}\chi\right).$$

Solutions I, III and VI are the special cases $M = 0, \pm 1, \pm 2$.

Our point solutions are, of course, singular at the origin and have infinite energy. We shall now discuss the possibility of smoothing out this singularity to obtain solutions with finite energy, with the point solutions serving as guides to the asymptotic behaviour at large radii. The situation is more complicated than the SU(2) case [2], since now, corresponding to each set of equations of motion (arising from varying the energy integral (4.9a) or (4.9b)) there are several possible boundary conditions. In the space of the functions $A(r)\ B(r)\ldots$ there are presumably local minima (yielding solutions to the equations of motion) corresponding to each possible boundary condition. These solutions should have finite positive energy, since we can choose test functions satisfying the boundary conditions, which give finite energy and hence an upper bound on the energy. The energy integrals, being positive definite, obviously have some lower bound (except for the vacuum solution I). Quantum mechanically these solutions may not be stable, since one could "tunnel" into another if the energy needed to interpolate in the function space from one to another was finite. The energy integrals suggest that interpolation between asymptotically distinct Higgs field requires an infinite energy jump, but that an interpolation between (IV, VI) or (I,II) is possible because only the W_μ components change and the relevant terms have a $1/r^4$ factor. An interpolation between V and I may also be possible, since the Higgs fields are homotopic, and so can be interpolated keeping the Higgs self-interaction zero asymptotically (as we saw earlier). This interpolation must violate the symmetry (4.1). This leaves us with 3 classes of solution (I,II,V) III and (IV, VI) which cannot possibly decay into each other since they have distinct electromagnetic monopole charges, 0, $\pm\frac{1}{2}$ and ± 1 in Dirac units, and this monopole charge is strictly conserved. By 't Hooft's argument [2] there must be a smoothed-out finite-energy solution for each of these classes. For the (I,II,V) classes it is obviously the vacuum, for the III "semi-pole" class something which looks like III asymptotically. For the (IV,VI) class the solution probably looks like a pure monopole (IV) at large radii, since it has lower energy density than VI in the Coulomb tail, but which it is may depend on the details of the

Higgs self-interaction. Likewise the "semipole" should be lighter than the monopole, but further investigation is needed.

Many other questions may be asked concerning the nature of the force between two of our solutions, the construction of solutions of higher monopole charges, and the extension to higher groups and different sorts of Higgs fields.

We should like to thank many friends in CERN and Copenhagen for discussions.

Appendix

Algebraic properties of the basis functions in the gauge potential decompositions

The verification of the equations in sect. 4 is facilitated if one is aware of the remarkable algebraic structure possessed by the basis functions in the decompositions (4.3) of the gauge potential W_μ.

In the embedding (a) of SU(2) in SU(3) ($T_1, T_2, T_3 = \frac{1}{2}\lambda_1, \frac{1}{2}\lambda_2, \frac{1}{2}\lambda_3$), the W_μ decomposition can be written

$$eW_\mu = \gamma(r)\,\partial_\mu\psi_1 + (\delta - 1)\alpha_\mu + \zeta\frac{r_\mu\psi_1}{r} + \eta\frac{r_\mu\psi_2}{r}\,,$$

where we find

$$[\psi_1, \partial_\mu\psi_1] = i\alpha_\mu\,, \qquad [\psi_1, \alpha_\mu] = -i\partial_\mu\psi_1\,, \tag{A.1a}$$

while ψ_2 commutes with ψ_1, $\partial_\mu\psi_1$ and α_μ. So the ψ_1 and ψ_2 parts of W_μ transform separately under the two factors in U(4.5a). The transformation of the ψ_2 part is trivial, giving the η equation in (4.6a), whereas the remaining equations follow from the fact that, by (A.1a), $(\partial_\mu\psi_1, \alpha_\mu)$ is a doublet under ψ_1, and the fact that

$$i\mathrm{e}^{-i\theta\phi}\partial_\mu \mathrm{e}^{i\theta\phi} = \phi\partial_\mu\theta + \sin\theta\;\partial_\mu\phi + i(1 - \cos\theta)[\phi, \partial_\mu\phi]\,, \tag{A.2}$$

providing there exists a quantity c such that

$$4(\phi + c)^2 = 1\,, \qquad \partial_\mu c = 0\,, \qquad [\phi, c] = 0\,. \tag{A.3}$$

Hence

$$\phi = \tfrac{1}{2}\boldsymbol{\lambda}\cdot\hat{\boldsymbol{r}}\,, \qquad c = \tfrac{1}{3}(1 - \sqrt{3}\,\lambda_8)\,.$$

In the evaluation of $G_{\mu\nu}$ the following is helpful:

$$[\partial_\mu\psi_1, \partial_\nu\psi_1] = [\alpha_\mu, \alpha_\nu] = ih_{\mu\nu}\,, \tag{A.4a}$$

$$[\partial_\mu\psi_1, \alpha_\nu] + [\alpha_\mu, \partial_\nu\psi_1] = 0\,,$$

$$h_{\mu\nu} = \psi_1 \ \epsilon_{\mu\nu\sigma} r_\sigma / r^2 \ .$$

The first equation, involving $\partial_\mu \psi_1$, must be proved explicitly; the others follow by repeated commutation with ψ_1 and use of (A.1a).

It is convenient to write the (b) – "nuclear physics" embedding (T_1, T_2, T_3 = λ_7, $-\lambda_5$, λ_2) – decomposition of W_μ as

$$eW_\mu = \sum_{\epsilon=\pm 1} (C_\epsilon \partial_\mu \phi_\epsilon + (D_\epsilon - 1) A_\mu^\epsilon + E_\epsilon r_\mu \ \phi_\epsilon / r) \ ,$$

where

$$\tfrac{1}{2} [\phi_1 + \epsilon \phi_2] = \phi_\epsilon , \qquad \epsilon = \pm 1 \ ,$$

and

$$X_\epsilon = X_1 + \epsilon X_2 \ , \qquad \text{for} \qquad X = C, D \text{ or } E \ ,$$

in terms of the notation of (4.3b). Then the following algebraic properties emerge,

$$[\phi^\epsilon, \partial_\mu \phi^\epsilon] = i A_\mu^\epsilon \ , \qquad [\phi^\epsilon, A_\mu^\epsilon] = -i \partial_\mu \phi^\epsilon \ , \tag{A.1b}$$

while $\phi^{-\epsilon}$ commutes with ϕ^ϵ, $\partial_\mu \phi^\epsilon$, A_μ^ϵ. If we rewrite the gauge transformation (4.5b) as $U = \exp(i\theta_+ \phi_+) \exp(i\theta_- \phi_-)$, we see that the + and – parts of W_μ transforms separately. In fact, applying (A.2) and (A.3), we find (taking $c = \frac{1}{6}$ to satisfy (A.3))

$$C_\epsilon' + iD_\epsilon' = \mathrm{e}^{i\theta_\epsilon} (C_\epsilon + iD_\epsilon) \ , \qquad E_\epsilon' = E_\epsilon - r \frac{\mathrm{d}\theta_\epsilon}{\mathrm{d}r} \ ,$$

which shows that we can choose a gauge in which $C_\epsilon = 0$. In this gauge the E equation of motion will read $E_\epsilon = 0$, so only D_ϵ is left.

In evaluating $G_{\mu\nu}$ the following is helpful:

$$[\partial_\mu \phi^\epsilon, \partial_\nu \phi^\epsilon] = [A_\mu^\epsilon, A_\nu^\epsilon] = i H_{\mu\nu}^\epsilon \ ,$$

where

$$H_{\mu\nu}^\epsilon = -\tfrac{1}{2} (\phi_2 + 3\epsilon \phi_1) \ \epsilon_{\mu\nu\sigma} \frac{\hat{r}_\sigma}{r^2} \ .$$

Otherwise

$$[X^\mu, Y^\nu] + [Y^\mu, X^\nu] = 0 \ ,$$

if $X_\mu Y_\nu$ are unequal and chosen from $\partial_\mu \phi^\epsilon, A_\mu^\epsilon$. The first step is to prove explicitly the result for expressions involving only $\partial_\mu \phi^\epsilon$ and not A_μ. Then the remaining results follow by repeated commutation with ϕ^ϵ and use of (A.1b).

The relations involving $\partial_\mu \phi_1$, $\partial_\mu \phi_2$ alone must be proved explicitly. Then all the other relations (A.2b) follow by repeated commutation with ϕ_1 and ϕ_2 and use of (A.1b). Note also that

$$\partial_\mu A^\epsilon_\nu - \partial_\nu A^\epsilon_\mu = 2H^\epsilon_{\mu\nu} .$$

Once these results are known it is easy to verify the equations of the main text.

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