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PHYSICAL REVIEW D VOLUME 9, NUMBER 6 15 MARCH 1974

Functional evaluation of the effective potential*

R. Jackiw

Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

(Received 2 November 1973)

By use of the path-integral formulation of quantum mechanics, a series expansion for the effective potential is derived. Each order of the series corresponds to an infinite set of conventional Feynman diagrams, with a fixed number of loops. As an application of the formalism, three calculations are performed. For a set of n self-interacting scalar fields, the effective potential is computed to the two-loop approximation. Also, all loops are summed in the leading-logarithmic approximation when n gets large. Finally, the effective potential for scalar, massless electrodynamics is derived in an arbitrary gauge. It is found that the potential is gauge-dependent, and a specific gauge is exhibited in which all one-loop effects disappear.

I. INTRODUCTION

The effective potential for a field theory (that is the generating functional for zero-momentum single-particle irreducible Green's functions[1]), introduced by Euler, Heisenberg, and Schwinger, is useful in studies of spontaneous symmetry breaking, as was first pointed out by Jona-Lasinio,[2] and more recently by several authors.[3,4] Calculation of this object has proceeded by summing infinite series of Feynman graphs at zero momentum.[3,4] Obviously this is an onerous task, especially when several interactions are present which complicate the combinatorial factors that multiply each graph. Moreover, the calculation has been only performed in the one-loop approximation, since higher-loop contributions appear extremely difficult to evaluate.

However, it is important to be able to study the higher-order multiloop graphs, if not explicitly, at least in general terms. Two circumstances can be envisioned where multiloop graphs are needed. The one-loop approximation is very simple; indeed it will be seen that it is not typical of the higher-order terms. Thus it may be that relevant effects do not set in until the two-loop level. More importantly, bound states which, as has been recently suggested, can provide a mechanism for spontaneous mass generation[5] can never be observed in a finite order of the loop expansion. Necessarily they require at least an infinite subset of all orders.

In this paper, I shall use the Feynman path-integral method to obtain a simple formula for the effective potential. The formula has the advantage of summing all the relevant Feynman graphs to a given order of the loop expansion. Furthermore, in a natural way it generates all orders of the loop expansion, representing each order by a *finite* number of graphs. Before stating the result, some notation must be introduced.

Consider a theory described by a Lagrangian $\mathcal{L}$ depending on a set of fields $\phi_a(x)$ and construct the classical action,

$$I(\phi)=\int d^4x\,\mathfrak{L}\{\phi_a(x)\}\ . \qquad (1.1)$$

[We work exclusively with Bose fields. For gauge fields $I(\phi)$ contains a gauge term.] Next define another Lagrangian by the following procedure:

$$I(\hat{\phi}+\phi)-I(\hat{\phi})-\int d^4x\,\phi_a(x)\,\frac{\delta I(\phi)}{\delta\phi_a(x)}\bigg|_{\phi=\hat{\phi}} \equiv \int d^4x\,\hat{\mathfrak{L}}\{\hat{\phi}_a;\phi_a(x)\}\ . \qquad (1.2)$$

In the above, the shifting field $\hat{\phi}_a$ is a constant, x-independent object. Assuming that the original Lagrangian $\mathfrak{L}$ was composed of terms quadratic, cubic, and quartic in $\phi_a(x)$, the new Lagrangian $\hat{\mathfrak{L}}$ will similarly have quadratic and higher terms in $\phi_a(x)$. The quadratic terms define a new propagator for the fields, while the higher powers comprise an interaction Lagrangian $\hat{\mathfrak{L}}_I$, with "coupling constants" which depend on $\hat{\phi}_a$. Thus (1.2) may be written as

$$\int d^4x\,\hat{\mathfrak{L}}\{\hat{\phi}_a;\phi_a(x)\} = \int d^4x\,d^4y\,\tfrac{1}{2}\phi_a(x)i\mathfrak{D}^{-1}{}_{ab}\{\hat{\phi};x,y\}\phi_b(y) + \int d^4x\,\hat{\mathfrak{L}}_I\{\hat{\phi}_a;\phi_a(x)\}\ . \qquad (1.3)$$

The propagator, denoted by $\mathfrak{D}_{ab}\{\hat{\varphi};x,y\}$, also satisfies

$$i\mathfrak{D}^{-1}{}_{ab}\{\hat{\phi};x,y\}=\frac{\delta^2 I(\phi)}{\delta\phi_a(x)\delta\phi_b(y)}\bigg|_{\phi=\hat{\phi}}\ . \qquad (1.4)$$

Since $\hat{\phi}_a$ is a constant, the propagator is a function of relative coordinates only, and may be Fourier transformed:

$$i\mathfrak{D}^{-1}{}_{ab}\{\hat{\phi};k\}=\int d^4x\,e^{ikx}i\mathfrak{D}^{-1}{}_{ab}\{\hat{\phi};x,0\}\ . \qquad (1.5)$$

[In my convention the free-field spin-zero propagator is $i/(k^2-\mu^2+i\epsilon)$.]

The formula for the effective potential $V(\hat{\phi})$ is

$$V(\hat{\phi})=V_0(\hat{\phi})-\tfrac{1}{2}i\hbar\int\frac{d^4k}{(2\pi)^4}\ln\det i\mathfrak{D}^{-1}{}_{ab}\{\hat{\phi};k\} + i\hbar\left\langle\exp\left(\frac{i}{\hbar}\int d^4x\,\hat{\mathfrak{L}}_I\{\hat{\phi}_a,\phi_a(x)\}\right)\right\rangle\ . \qquad (1.6)$$

The first term is the classical tree approximation. The second term is the contribution of all graphs with one closed loop, where the determinant operates on the indices (a,b) which can refer to internal or spin degrees of freedom. The last term summarizes the following operation. Compute the vacuum expectation value of

$$T\exp\left(\frac{i}{\hbar}\int d^4x\,\hat{\mathfrak{L}}_I\{\hat{\phi}_a,\phi_a(x)\}\right)\ ,$$

using conventional Feynman rules, with $\mathfrak{D}_{ab}\{\hat{\phi};k\}$ as the propagator; keep only connected single-particle irreducible graphs; delete an over-all factor of space-time volume $\int d^4x$. This gives

$$\left\langle\exp\left(\frac{i}{\hbar}\int d^4x\,\hat{\mathfrak{L}}_I\{\hat{\phi}_a,\phi_a(x)\}\right)\right\rangle .$$

I have retained the factor $\hbar$ in the definition of the S matrix (though not in the parameters of the Lagrangian). The reason is that an expansion in $\hbar$ is equivalent to the loop expansion.[6] The tree contribution, $V_0(\hat{\phi})$, is independent of $\hbar$; the one-loop determinant is proportional to $\hbar$. The remaining term is of order $\hbar^2$. This is seen by rescaling the field: $\phi_a(x)\to(\hbar)^{1/2}\phi_a(x)$. Since $\hat{\mathfrak{L}}_I$ may contain a term cubic in $\phi_a(x)$, $(i/\hbar)\hat{\mathfrak{L}}_I$ is at least of order $\hbar^{1/2}$. Fractional powers of $\hbar$ cannot occur in perturbation theory; thus $(i/\hbar)\hat{\mathfrak{L}}_I$ begins with order $\hbar$, and the additional factor $\hbar$ in (1.6) makes the last term $O(\hbar^2)$.

After rescaling by $(\hbar)^{1/2}$, one can expand the exponential and evaluate

$$\left\langle\exp\left(\frac{i}{\hbar}\int d^4x\,\hat{\mathfrak{L}}_I\{\hat{\phi}_a,(\hbar)^{1/2}\phi_a(x)\}\right)\right\rangle$$

in powers of $\hbar$. Ordinary diagrammatic analysis of the vacuum amplitude can be applied, and the graphical series given in Fig. 1 is found for $V(\hat{\phi})$. The dot is the free term; the single unadorned circle is the logarithm of the determinant. The remaining terms are conventional: Lines represent the propagator $\mathfrak{D}_{ab}\{\hat{\phi};k\}$, and two kinds of vertices occur, cubic and quartic. These are vertices of the shifted Lagrangian $\hat{\mathfrak{L}}_I$; the cubic vertex in general depends on $\hat{\phi}_a$. (We deal with a

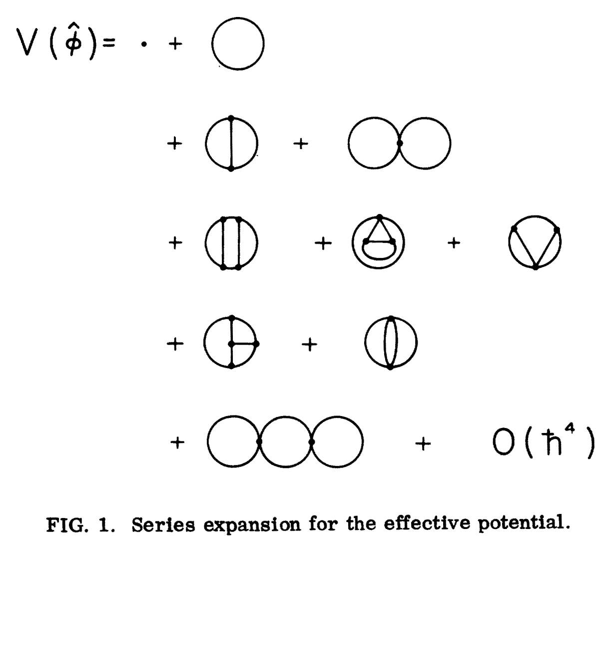

FIG. 1. Series expansion for the effective potential.

renormalizable theory where no higher field polynomials are present in the Lagrangian.)

Thus the problem of computing the effective potential to a definite order in $\hbar$ is reduced to finite analytic operations. The two-loop contribution involves two graphs; even the three-loop terms comprise a still manageable six graphs. We have not given the combinatorial factors that multiply each graph; these depend on the model and can be deduced with Wick's theorem. Note that the one-loop term is strikingly different from all the others; it involves a logarithm.[7]

The remainder of this paper is organized as follows. In Sec. II, the formula (1.6) is derived by functional methods. As an explicit exercise, in Sec. III we evaluate the effective potential $V(\hat{\phi})$ for a set of self-interacting spin-zero fields. Previous results[3,4] of the one-loop calculation are regained, and the two-loop contribution is given. Also a formula for $V(\hat{\phi})$ is obtained which is exact in the leading logarithmic approximation as the number of fields gets large. In Sec. IV, we repeat the Coleman-Weinberg[3] calculation for massless electrodynamics. These authors computed the one-loop potential in the Landau gauge, since in that gauge the infinite set of Feynman graphs simplifies considerably. As our formalism dispenses with infinite summations, we are in a position to compute in arbitrary gauges. We find that $V(\hat{\phi})$ is *gauge-dependent on the one-loop level*. This raises questions, discussed in Sec. IV, about the physical significance of the mathematical properties of $V(\hat{\phi})$ in a gauge theory. Section V is devoted to concluding remarks.

II. PATH-INTEGRAL DERIVATION OF THE EFFECTIVE POTENTIAL

We derive the formula (1.6) by an application of Feynman's path integral. Many of the steps will be obvious to those practiced in this technique. Nevertheless, they are included here for completeness. The argument will proceed in two stages. First, we shall give a very explicit demonstration, valid to order $\hbar^2$. Then a more general argument will establish the result to all orders.

A. Preliminary definitions

For a field theory described by a Lagrangian $\mathcal{L}\{\phi_a(x)\}$, the connected generating functional $W(J)$ is defined by

$$Z(J)=\exp\left[\frac{i}{\hbar}W(J)\right]=\langle 0^+|0^-\rangle_J\ . \tag{2.1}$$

$Z(J)$ is the vacuum persistence amplitude in the presence of a linear interaction with an external source $J_a(x)$, i.e., a term $J_a(x)\phi^a(x)$ is added to $\mathcal{L}\{\phi_a(x)\}$. The effective action $\Gamma(\overline{\phi})$ is obtained from $W(J)$ by a Legendre transform:

$$\begin{gathered}\Gamma(\overline{\phi})=W(J)-\int d^4x\,\frac{\delta W(J)}{\delta J_a(x)}J_a(x)\ ,\\ \frac{\delta W(J)}{\delta J_a(x)}\equiv\overline{\phi}_a(x)\ .\end{gathered} \tag{2.2}$$

$\Gamma(\overline{\phi})$ generates single-particle irreducible connected graphs. Note in particular that the connected, single-particle irreducible vacuum graphs are given by $\Gamma(0)$, which is $W(J)$ evaluated at that value of $J_a(x)$ where $\delta W(J)/\delta J_a(x)$, or equivalently $\delta Z(J)/\delta J_a(x)$, vanishes. We record this expression here, since we shall need it later:

$$W(J)|_{\delta Z(J)/\delta J_a(x)=0}=\text{irreducible, connected vacuum graphs.} \tag{2.3}$$

The effective potential $V(\hat{\phi})$ is defined from $\Gamma(\overline{\phi})$ by setting $\overline{\phi}_a(x)$ to be a constant field $\hat{\phi}_a$. An over-all factor of space-time volume must be removed, and we have

$$\Gamma(\hat{\phi})=-V(\hat{\phi})\int d^4x\ . \tag{2.4}$$

The path-integral representation for $Z(J)$ is

$$Z(J)=\int d\phi\,\exp\left(\frac{i}{\hbar}\{I(\phi)+\phi J\}\right)\ . \tag{2.5}$$

For notational simplicity, we shall suppress all indices, and space-time variables will be treated as suppressed indices as well. Thus ϕJ stands for $\int d^4x\,\phi_a(x)J_a(x)$. Similarly, all differentiation is functional. Also we ignore a J-independent normalization factor, which is present in (2.5).

B. First stage of the proof to order $\hbar^2$

It is well known that an expansion of $W(J)$ in powers of $\hbar$ is the loop expansion.[8] Moreover, in the functional integral this corresponds to a stationary-phase evaluation,[9] which is achieved by translating ϕ in (2.5) by ϕ^0, where ϕ^0 is chosen to satisfy the classical equations of motion:

$$\left.\frac{\partial I(\phi)}{\partial\phi}\right|_{\phi=\phi^0}=-J\ . \tag{2.6}$$

This defines ϕ^0 as a functional of J. Also we have

$$I(\phi+\phi^0)=I(\phi^0)-J\phi+\tfrac{1}{2}\phi i\mathfrak{D}^{-1}\phi+I\{\phi^0;\phi\}\ , \tag{2.7}$$

where

$$i\mathfrak{D}^{-1}=\left.\frac{\partial^2 I(\phi)}{\partial\phi^2}\right|_{\phi=\phi^0} \tag{2.8}$$

[compare (1.4)]. Hence

$$Z(J)=\exp\left(\frac{i}{\hbar}[I(\phi^0)+\phi^0 J]\right)\times\int d\phi\,\exp\left(\frac{i}{\hbar}[\tfrac{1}{2}\phi i\mathfrak{D}^{-1}\phi+I\{\phi^0;\phi\}]\right)=\exp\left(\frac{i}{\hbar}[I(\phi^0)+\phi^0 J]\right)\mathrm{Det}^{-1/2}(i\mathfrak{D}^{-1})Z_2(J)\,, \tag{2.9a}$$

$$Z_2(J)=\frac{\int d\phi\,\exp((i/\hbar)[\frac{1}{2}\phi i\mathfrak{D}^{-1}\phi+I\{\phi^0;\phi\}])}{\int d\phi\,\exp[(i/\hbar)(\frac{1}{2}\phi i\mathfrak{D}^{-1}\phi)]}\,, \tag{2.9b}$$

$$W(J)=I(\phi^0)+\phi^0 J+\tfrac{1}{2}i\hbar\ln\mathrm{Det}(i\mathfrak{D}^{-1})+W_2(J)\,, \tag{2.10a}$$

$$W_2(J)=-i\hbar\ln Z_2(J)\,. \tag{2.10b}$$

In the above, we have used the fundamental path integral

$$\int d\phi\,\exp\left(\frac{i}{\hbar}\frac{1}{2}\phi M\phi\right)=(\mathrm{Det}M)^{-1/2}\,.$$

The determinant is taken in the functional sense. $W_2(J)$ is of order $\hbar^2$ (hence the subscript). The argument for this is the same as in the Introduction. Observe also that $W_2(J)$ is the connected vacuum amplitude for a theory described by the action $\frac{1}{2}\phi i\mathfrak{D}^{-1}\phi+I\{\phi^0;\phi\}$. It is *not* single-particle irreducible. The denominator in (2.9b) ensures proper normalization and indicates that the propagator in this theory is $\mathfrak{D}$. This is a translationally noninvariant propagator, since it depends on ϕ^0; see (2.8). Furthermore, the "coupling constants" of this theory also involve ϕ^0; they represent position-dependent, nonlocal "interactions."

If we were computing $W(J)$, (2.10a) would provide a loop expansion in powers of $\hbar$. However, we are interested in $\Gamma(\overline{\phi})$, and a Legendre transform must be performed. We need to evaluate $\partial W(J)/\partial J$. If $\hbar$ were zero, it would simply follow from (2.6) and (2.10) that

$$\frac{\partial W(J)}{\partial J}=\frac{\partial I(\phi^0)}{\partial\phi^0}\frac{\partial\phi^0}{\partial J}+\phi^0+J\frac{\partial\phi^0}{\partial J}=\phi^0\,.$$

Hence we shall set

$$\frac{\partial W(J)}{\partial J}\equiv\overline{\phi}\,,\qquad \phi^0=\overline{\phi}+\phi^1\,, \tag{2.11}$$

where ϕ^1 is a functional of $\overline{\phi}$, to be determined later. It is of order $\hbar$. To complete the evaluation of $\Gamma(\overline{\phi})$ from $W(J)$ given by (2.10a), J must be eliminated in favor of $\overline{\phi}$. This is to be done from (2.11) which defines, via the dependence of ϕ^0 on J [see (2.6)], a relationship between J and $\overline{\phi}$.

Fortunately, we need not solve this very implicit equation. Observe that (2.10a) shows that $W(J)$ depends on J implicitly through ϕ^0 and explicitly only in the second term. Both $i\mathfrak{D}^{-1}$ and $W_2(J)$ are in fact functionals only of ϕ^0, as is seen from (2.8), (2.9b), and (2.10b). Moreover, (2.6) can be taken to define J as a functional of ϕ^0, rather than vice versa, a fact which we make explicit by writing $J(\phi^0)$. Thus eliminating J in favor of $\overline{\phi}$ in (2.10a) is equivalent to eliminating ϕ^0 in favor of $\overline{\phi}$, a task easily achieved with the help of (2.11). Consequently, we arrive at the result

$$\Gamma(\overline{\phi})=W(J)-\overline{\phi}J(\phi^0)=I(\overline{\phi}+\phi^1)+\phi^1 J(\overline{\phi}+\phi^1)+\Gamma_1(\overline{\phi}+\phi^1)\,. \tag{2.12}$$

Here $\Gamma_1(\phi^0)$ stands for the last two terms in (2.10a), and the subscript reminds us that Γ_1 is first order in $\hbar$. Next we use (2.6) again to find to order $\hbar^2$

$$\begin{aligned}\Gamma(\overline{\phi})&=I(\overline{\phi})+\phi^1\frac{\partial I(\overline{\phi})}{\partial\overline{\phi}}+\tfrac{1}{2}\phi^1\frac{\partial^2 I(\overline{\phi})}{\partial\overline{\phi}^2}\phi^1+\phi^1 J(\overline{\phi})\\&\quad+\phi^1\frac{\partial J(\overline{\phi})}{\partial\overline{\phi}}\phi^1+\Gamma_1(\overline{\phi})+\phi^1\frac{\partial\Gamma_1(\overline{\phi})}{\partial\overline{\phi}}+O(\hbar^3)\\&=I(\overline{\phi})+\Gamma_1(\overline{\phi})+\tfrac{1}{2}\phi^1 i\mathfrak{D}^{-1}\phi^1\\&\quad+\phi^1\frac{\partial J(\overline{\phi})}{\partial\overline{\phi}}\phi^1+\phi^1\frac{\partial\Gamma_1(\overline{\phi})}{\partial\overline{\phi}}+O(\hbar^3)\,.\end{aligned} \tag{2.13}$$

Equation (2.13) shows that to order $\hbar$, $\Gamma(\overline{\phi})$ is simply given by

$$\Gamma(\overline{\phi})=I(\overline{\phi})+\tfrac{1}{2}i\hbar\ln\mathrm{Det}(i\mathfrak{D}^{-1})+O(\hbar^2)\,. \tag{2.14}$$

This result, which is known,[10] summarizes all one-loop calculations. The terms of order $\hbar^2$ are of two distinct types. First, there are the last three terms in (2.13) which arise from the fact that ϕ^0 does not coincide with $\overline{\phi}$. Second, there is the order-$\hbar^2$ contribution to $\Gamma_1(\overline{\phi})$ itself. From (2.10a) this is seen to be W_2, the sum of all connected vacuum graphs in the theory governed by the action,

$$\tfrac{1}{2}\phi i\mathfrak{D}^{-1}\phi+I\{\overline{\phi};\phi\}=I(\phi+\overline{\phi})-I(\overline{\phi})-\phi\frac{\partial I(\overline{\phi})}{\partial\overline{\phi}}\,.$$

(Since W_2 is itself of order $\hbar^2$, we may freely set ϕ^0 equal to $\overline{\phi}$ in W_2.) To complete the argument to order $\hbar^2$, it will now be shown that the single-particle *reducible* graphs in W_2 cancel against the last three terms in (2.13) involving ϕ^1.

First, we seek a formula for ϕ^1. It is sufficient to determine ϕ^1 to lowest order in $\hbar$. From (2.10a) and (2.11) it follows that

$$\frac{\partial W(J)}{\partial J}=\phi^0+\frac{\partial\Gamma_1(\overline{\phi})}{\partial\overline{\phi}}\frac{\partial\overline{\phi}}{\partial J}=\overline{\phi}=\phi^0-\phi^1\,,\qquad \phi^1=-\frac{\partial\Gamma_1(\phi^0)}{\partial\phi^0}\frac{\partial\phi^0}{\partial J}+O(\hbar^2)\,. \tag{2.15}$$

From (2.6) we have, upon differentiation with respect to ϕ^0,

$$\frac{\partial^2 I(\phi^0)}{\partial \phi^{0 2}} = -\frac{\partial J(\phi^0)}{\partial \phi^0} = i\mathfrak{D}^{-1} \tag{2.16a}$$

or, equivalently,

$$i\mathfrak{D} = \frac{\partial \phi^0}{\partial J} \; . \tag{2.16b}$$

Thus

$$\phi^1 = -i\, \frac{\partial \Gamma_1(\phi^0)}{\partial \phi^0}\, \mathfrak{D} + O(\hbar^2)$$
$$= -i\mathfrak{D}\, \frac{\partial \Gamma_1(\phi^0)}{\partial \phi^0} + O(\hbar^2) \; . \tag{2.17}$$

Order of factors does not matter since $\mathfrak{D}$ is symmetric. Therefore, the last three terms in (2.13) are given to lowest order by

$$\tfrac{1}{2}\phi^1 i\mathfrak{D}^{-1}\phi^1 + \phi^1\, \frac{\partial J(\phi^0)}{\partial \phi^0}\, \phi^1 + \phi^1\, \frac{\partial \Gamma_1(\phi^0)}{\partial \phi^0}$$
$$= \tfrac{1}{2}\left(-i\, \frac{\partial \Gamma_1(\phi^0)}{\partial \phi^0}\, \mathfrak{D}\right)(i\mathfrak{D}^{-1})\left(-i\mathfrak{D}\, \frac{\partial \Gamma_1(\phi^0)}{\partial \phi^0}\right)$$
$$+ \left(-i\, \frac{\partial \Gamma_1(\phi^0)}{\partial \phi^0}\, \mathfrak{D}\right)(-i\mathfrak{D}^{-1})\left(-i\mathfrak{D}\, \frac{\partial \Gamma_1(\phi^0)}{\partial \phi^0}\right)$$
$$+ \left(-i\, \frac{\partial \Gamma_1(\phi^0)}{\partial \phi^0}\, \mathfrak{D}\right)\frac{\partial \Gamma_1(\phi^0)}{\partial \phi^0} + O(\hbar^3)$$
$$= -\frac{i}{2}\, \frac{\partial \Gamma_1(\phi^0)}{\partial \phi^0}\, \mathfrak{D}\, \frac{\partial \Gamma_1(\phi^0)}{\partial \phi^0} \; . \tag{2.18}$$

FIG. 2. Single-particle reducible vacuum graph contributing to $W_2(J)$ but not to $\Gamma_1(\bar{\phi})$.

It is this quantity which will cancel against the single-particle reducible term in W_2. Note that (2.10a) implies

$$\frac{\partial \Gamma_1(\phi^0)}{\partial \phi^0} = \tfrac{1}{2} i\hbar\, \mathrm{Tr}\mathfrak{D}\, \frac{\partial \mathfrak{D}^{-1}}{\partial \phi^0} + O(\hbar^2) \; . \tag{2.19}$$

The trace operation, indicated by Tr, is functional.

We now turn to a computation of W_2 to order $\hbar^2$. Continuing the expansion (2.7), we see that

$$I\{\phi^0;\phi\} = \frac{1}{3!}\,\phi^3\, \frac{\partial^3 I(\phi^0)}{\partial \phi^{0 3}} + \frac{1}{4!}\,\phi^4\, \frac{\partial^4 I(\phi^0)}{\partial \phi^{0 4}}$$
$$= \frac{1}{3!}\,\phi^3\, \frac{\partial i\mathfrak{D}^{-1}}{\partial \phi^0} + \frac{1}{4!}\,\phi^4\, \frac{\partial^2 i\mathfrak{D}^{-1}}{\partial \phi^{0 2}} \; . \tag{2.20}$$

(We shall assume that no higher-order terms are present; it is easy to check that they do not matter in any case.) Thus we are led by (2.9b) and (2.10b) to an evaluation of

$$\left\langle 0 \left| T \exp\left[\frac{i}{\hbar}\left(\frac{1}{3!}\,\phi^3\, \frac{\partial i\mathfrak{D}^{-1}}{\partial \phi^0} + \frac{1}{4!}\,\phi^4\, \frac{\partial^2 i\mathfrak{D}^{-1}}{\partial \phi^{0 2}}\right)\right] \right| 0 \right\rangle_{\text{connected}} \; . \tag{2.21}$$

Rescaling the field $\phi \to (\hbar)^{1/2}\phi$, expanding the exponential, and keeping nonvanishing, connected terms through order $\hbar$, we are left with

$$i\hbar\left\langle 0 \left| T\, \frac{1}{4!}\,\phi^4 \right| 0 \right\rangle \frac{\partial^2 i\mathfrak{D}^{-1}}{\partial \phi^{0 2}} - \tfrac{1}{2}\hbar \left\langle 0 \left| T\left(\frac{1}{3!}\,\phi^3\, \frac{\partial i\mathfrak{D}^{-1}}{\partial \phi^0}\right)\left(\frac{1}{3!}\,\phi^3\, \frac{\partial i\mathfrak{D}^{-1}}{\partial \phi^0}\right) \right| 0 \right\rangle_{\text{connected}} \; . \tag{2.22a}$$

The first term has no single-particle reducible graphs. The only connected structure is the single-particle irreducible contribution, pictured as the third diagram in Fig. 1. The second term contains the irreducible graph pictured as the second diagram of Fig. 1. Also there is a single-particle reducible diagram given in Fig. 2. To evaluate it, we reintroduce all the suppressed indices in (2.22a):

$$-\tfrac{1}{2}\hbar\, \frac{1}{(3!)^2}\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 d^4x_5 d^4x_6\, \frac{\partial i\mathfrak{D}^{-1}{}_{ab}\{\phi^0; x_1, x_2\}}{\partial \phi^0_c(x_3)}\, \frac{\partial i\mathfrak{D}^{-1}{}_{de}\{\phi^0; x_4, x_5\}}{\partial \phi^0_f(x_6)}$$
$$\times \langle 0 | T\phi_a(x_1)\phi_b(x_2)\phi_c(x_3)\phi_d(x_4)\phi_e(x_5)\phi_f(x_6) | 0 \rangle \; . \tag{2.22b}$$

The "vertices" are symmetric in all their arguments. Consequently, Wick's theorem gives, for the reducible part,

$$\tfrac{1}{2}\hbar\, \frac{1}{(3!)^2}\, 9\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 d^4x_5 d^4x_6\, \frac{\partial \mathfrak{D}^{-1}{}_{ab}\{\phi^0; x_1, x_2\}}{\partial \phi^0_c(x_3)}\, \frac{\partial \mathfrak{D}^{-1}{}_{de}\{\phi^0; x_4, x_5\}}{\partial \phi^0_f(x_6)}$$
$$\times\, \mathfrak{D}_{ba}\{\phi^0; x_2, x_1\}\mathfrak{D}_{cf}\{\phi^0; x_3, x_6\}\mathfrak{D}_{ed}\{\phi^0; x_5, x_4\} = \tfrac{1}{8}\hbar\left(\mathrm{Tr}\mathfrak{D}\, \frac{\partial \mathfrak{D}^{-1}}{\partial \phi^0}\right)\mathfrak{D}\left(\mathrm{Tr}\mathfrak{D}\, \frac{\partial \mathfrak{D}^{-1}}{\partial \phi^0}\right) \; . \tag{2.22c}$$

W_2 is $-i\hbar$ times the above; using (2.19) finally gives

$$\frac{i}{2}\frac{\partial\Gamma_1(\phi^0)}{\partial\phi^0}\mathfrak{D}\frac{\partial\Gamma_1(\phi^0)}{\partial\phi^0}\,. \tag{2.22d}$$

As promised, this cancels (2.18).

The following assertion has now been proved, to order $\hbar^2$. The effective action $\Gamma(\overline{\phi})$ is given by the free term, by the functional determinant, and by the set of connected, single-particle irreducible vacuum graphs in a theory described by the action

$$I(\phi+\overline{\phi})-I(\overline{\phi})-\phi\,\frac{\partial I(\overline{\phi})}{\partial\overline{\phi}}\quad.$$

For arbitrary, space-time-dependent fields $\overline{\phi}$, no practical computation can be performed. However, as we spell out explicitly below, for constant fields, the calculations can be simply carried out.

C. Second stage of the proof to all orders in $\hbar$

The above tediously explicit argument clearly cannot be extended profitably to higher orders. I shall now give a much simpler, albeit nonexplicit, proof which makes no perturbative approximation. We again translate the field ϕ in the functional integral (2.5) by the quantity ϕ^0; however, we no longer impose the condition that ϕ^0 satisfy the classical equations of motion. Rather it is some functional of J, whose properties will be determined presently. It is trivially true that

$$Z(J)=\exp\left(\frac{i}{\hbar}[I(\phi^0)+\phi^0 J]\right)Z_1(J)\,,$$
$$Z_1(J)=\int d\phi\,\exp\left(\frac{i}{\hbar}[I(\phi+\phi^0)-I(\phi^0)+\phi J]\right), \tag{2.23}$$

$$W(J)=I(\phi^0)+\phi^0 J+W_1(J)\,,$$
$$W_1(J)=-i\hbar\ln Z_1(J)\,. \tag{2.24}$$

We compute $\partial W(J)/\partial J$ in order to effect the Legendre transform which defines $\Gamma(\overline{\phi})$:

$$\overline{\phi}\equiv\frac{\partial W(J)}{\partial J}=\phi^0+\left[\frac{\partial I(\phi^0)}{\partial\phi^0}+\frac{\partial W_1}{\partial\phi^0}+J(\phi^0)\right]\frac{\partial\phi^0}{\partial J}\quad. \tag{2.25}$$

As before, we view J as a functional of ϕ^0, hence W_1 depends on J through its dependence on ϕ^0.[11] We now demand that ϕ^0 be that functional of J (or vice versa, that J be that functional of ϕ^0) which makes it true that

$$\frac{\partial I(\phi^0)}{\partial\phi^0}+\frac{\partial W_1}{\partial\phi^0}+J(\phi^0)=0\,. \tag{2.26}$$

In that case $\overline{\phi}=\phi^0$ and $\Gamma(\overline{\phi})$ is simply

$$\Gamma(\overline{\phi})=I(\overline{\phi})+W_1\,. \tag{2.27}$$

It would appear that to evaluate W_1 as a functional of $\overline{\phi}$ would require the solution of the terribly implicit set of equations (2.24) and (2.26). Progress can be made by substituting (2.26) into (2.24):

$$W_1=-i\hbar\ln\int d\phi\,\exp\left\{\frac{i}{\hbar}\left[I(\phi+\phi^0)-I(\phi^0)-\phi\,\frac{\partial I(\phi^0)}{\partial\phi^0}-\phi\,\frac{\partial W_1}{\partial\phi^0}\right]\right\}\,. \tag{2.28}$$

All reference to J has disappeared and W_1 is found to satisfy the above functional, integro-differential equation.

Let us next introduce the objects

$$Z(\phi^0;K)=\int d\phi\,\exp\left\{\frac{i}{\hbar}\left[I(\phi+\phi^0)-I(\phi^0)-\phi\,\frac{\partial I(\phi^0)}{\partial\phi^0}+\phi K\right]\right\}, \tag{2.29a}$$

$$W(\phi^0;K)=-i\hbar\ln Z(\phi^0;K)\,. \tag{2.29b}$$

Clearly, $W_1=W(\phi^0;K)|_{K=-\partial W_1/\partial\phi^0}$. Recall also that apart from an over-all normalization factor, the connected single-particle irreducible vacuum graphs for the theory governed by the action

$$I(\phi+\phi^0)-I(\phi^0)-\phi\,\frac{\partial I(\phi^0)}{\partial\phi^0}$$

are given by $W(\phi^0;K)$, evaluated at that value of K which makes $\partial Z(\phi^0;K)/\partial K$ vanish [see (2.3)].

We now show that $\partial Z(\phi^0;K)/\partial K$ at $K=-\partial W_1/\partial\phi^0$ does indeed vanish. Since this is the value of K at which we seek $W(\phi^0;K)$, this establishes

$$W_1=W(\phi^0;K)|_{K=-\partial W_1/\partial\phi^0}$$

to be the sum of all connected single-particle irreducible vacuum graphs in the theory governed by the action

$$I(\phi+\phi^0)-I(\phi^0)-\phi\,\frac{\partial I(\phi^0)}{\partial\phi^0}$$

(apart from the afore-mentioned normalization factor).

The derivative of $Z(\phi^0;K)$ is

$$\frac{\partial Z(\phi^0;K)}{\partial K}=\frac{i}{\hbar}\int d\phi\,\phi\,\exp\left\{\frac{i}{\hbar}\left[I(\phi+\phi^0)-I(\phi^0)-\phi\,\frac{\partial I(\phi^0)}{\partial\phi^0}+\phi K\right]\right\}. \tag{2.30a}$$

Hence the principal result will follow if we can show that

$$0=\int d\phi\,\phi\exp\left\{\frac{i}{\hbar}\left[I(\phi+\phi^0)-I(\phi^0)-\phi\,\frac{\partial I(\phi^0)}{\partial\phi^0}-\phi\,\frac{\partial W_1}{\partial\phi^0}\right]\right\}. \tag{2.30b}$$

To establish the validity of (2.30b), differentiate (2.28) with respect to ϕ^0:

$$\frac{\partial W_1}{\partial\phi^0}=\frac{1}{Z_1}\int d\phi\left\{\frac{\partial}{\partial\phi^0}I(\phi+\phi^0)-\frac{\partial I(\phi^0)}{\partial\phi^0}-\phi\,\frac{\partial^2}{\partial\phi^{02}}[I(\phi^0)+W_1]\right\}\exp\left\{\frac{i}{\hbar}\left[I(\phi+\phi^0)-I(\phi^0)-\phi\,\frac{\partial I(\phi^0)}{\partial\phi^0}-\phi\,\frac{\partial W_1}{\partial\phi^0}\right]\right\}. \tag{2.31a}$$

The first term in the first set of curly brackets may also be written in the following way:

$$\begin{aligned}\int d\phi\,\frac{\partial}{\partial\phi^0}I(\phi+\phi^0)\exp\left\{\frac{i}{\hbar}\left[I(\phi+\phi^0)-I(\phi^0)-\phi\,\frac{\partial I(\phi^0)}{\partial\phi^0}-\phi\,\frac{\partial W_1}{\partial\phi^0}\right]\right\}\\
=-i\hbar\int d\phi\,\frac{\partial}{\partial\phi}\exp\left\{\frac{i}{\hbar}\left[I(\phi+\phi^0)-I(\phi^0)-\phi\,\frac{\partial I(\phi^0)}{\partial\phi^0}-\phi\,\frac{\partial W_1}{\partial\phi^0}\right]\right\}\\
+\int d\phi\left\{\frac{\partial I(\phi^0)}{\partial\phi^0}+\frac{\partial W_1}{\partial\phi^0}\right\}\exp\left\{\frac{i}{\hbar}\left[I(\phi+\phi^0)-I(\phi^0)-\phi\,\frac{\partial I(\phi^0)}{\partial\phi^0}-\phi\,\frac{\partial W_1}{\partial\phi^0}\right]\right\}.\end{aligned} \tag{2.31b}$$

The first integral vanishes due to a functional integration by parts. Reinserting the remainder into (2.31a), we find

$$\frac{\partial W_1}{\partial\phi^0}=\frac{1}{Z_1}\int d\phi\left\{\frac{\partial W_1}{\partial\phi^0}-\phi\,\frac{\partial^2}{\partial\phi^{02}}[I(\phi^0)+W_1]\right\}\exp\left\{\frac{i}{\hbar}\left[I(\phi+\phi^0)-I(\phi^0)-\phi\,\frac{\partial I(\phi^0)}{\partial\phi^0}-\phi\,\frac{\partial W_1}{\partial\phi^0}\right]\right\}$$

or

$$\begin{aligned}0=&\frac{\partial^2}{\partial\phi^{02}}[I(\phi^0)+W_1]\\
&\times\int d\phi\,\phi\exp\left\{\frac{i}{\hbar}\left[I(\phi+\phi^0)-I(\phi^0)\right.\right.\\
&\left.\left.-\phi\,\frac{\partial I(\phi^0)}{\partial\phi^0}-\phi\,\frac{\partial W_1}{\partial\phi^0}\right]\right\}.\end{aligned} \tag{2.31c}$$

The first factor clearly is not zero (it is the inverse propagator), hence the second factor vanishes as is wanted.

Finally, we come to the matter of the normalization. Perturbation theory conventionally assumes that a purely kinetic term (a quadratic action) does not yield a vacuum-to-vacuum transition. This is not true of $Z(\phi^0;K)$, as defined by (2.29a). To make it so, $Z(\phi^0;K)$ should be divided by

$$\int d\phi\exp\left(\frac{i}{\hbar}\,\phi i\mathfrak{D}^{-1}\phi\right);$$

hence W_1, when evaluated in conventional perturbation theory, acquires the additional term $\frac{1}{2}i\hbar\ln\mathrm{Det}\,i\mathfrak{D}^{-1}$. The argument is now complete and the assertion at the end of Sec. II B is established to all orders.

D. Final step of the argument

To complete the evaluation of the effective potential, we set $\overline{\phi}$ in $\Gamma(\phi)$ to a constant field $\hat{\phi}$. Now it is possible to convert the functional expressions to ordinary integrals. Clearly, the classical contribution is

$$I(\hat{\phi})=-V_0(\hat{\phi})\int d^4x\,. \tag{2.32}$$

The propagator $\mathfrak{D}\{\hat{\phi};x,y\}$ becomes translation-invariant, and diagonal in the momentum representation. Thus the functional determinant is easily evaluated:

$$\begin{aligned}\ln\mathrm{Det}\,i\mathfrak{D}^{-1}&=\mathrm{Tr}\,\mathrm{Ln}\,i\mathfrak{D}^{-1}\\
&=\mathrm{Tr}(2\pi)^4\delta^4(k-l)\ln i\mathfrak{D}^{-1}\{\hat{\phi};k\}\,.\end{aligned} \tag{2.33a}$$

The logarithm on the right-hand side in the first line is functional; however, since $i\mathfrak{D}^{-1}$ is proportional to the unit operator in the momentum representation, $(2\pi)^4\delta(k-l)$, the logarithm in the second equality is an ordinary function of the matrix $i\mathfrak{D}^{-1}\{\hat{\phi},k\}$. Evaluating the remaining functional trace leaves

$$\begin{aligned}\mathrm{Tr}(2\pi)^4\delta^4(k-l)\ln i\mathfrak{D}^{-1}\{\hat{\phi};k\}\\
&=\int\frac{d^4k}{(2\pi)^4}(2\pi)^4\delta^4(k-l)\Big|_{l=k}\mathrm{Tr}\ln i\mathfrak{D}^{-1}\{\hat{\phi};k\}\\
&=\int\frac{d^4k}{(2\pi)^4}\ln\det i\mathfrak{D}^{-1}\{\hat{\phi};k\}\int d^4x\,.\end{aligned} \tag{2.33b}$$

Finally the vacuum graphs are now easy to evaluate, since the vertices are no longer x-dependent, though they do still depend on $\hat{\phi}$. Translation invariance ensures that an over-all factor $\int d^4x$ will be present in all these graphs; this factor is removed in the definition of $V(\hat\phi)$. Thus Eq. (1.6) for $V(\hat\phi)$ is proved.

Although a complete evaluation of $\Gamma(\overline{\phi})$ for arbitrary $\overline{\phi}(x)$ is not in general possible, even in the one-loop approximation, due to the intractability of the functional determinant, it is possible, with additional effort, to give an expansion of $\Gamma(\overline{\phi})$ in terms of derivatives of $\overline{\phi}$ (Ref. 3):

$$\Gamma(\overline{\phi}) = -\int d^4x\, V(\overline{\phi}(x)) + \int d^4x\, V_a^\mu(\overline{\phi}(x))\partial_\mu \overline{\phi}_a(x) + \cdots \,. \tag{2.34a}$$

By setting $\overline{\phi}_a(x) = \hat\phi_a + x_\mu \hat\phi_a^\mu$, $\hat\phi_a$ and $\hat\phi_a^\mu$ constant,

$$\Gamma(\overline{\phi}) = -V(\hat\phi)\int d^4x - V'(\hat\phi)\hat\phi^\mu \int d^4x\, x_\mu + V^\mu(\hat\phi)\hat\phi_\mu \int d^4x + \cdots \,. \tag{2.34b}$$

When this form of $\overline{\phi}(x)$ is substituted in our expression for $\Gamma(\overline{\phi})$, (2.14) for the one-loop term or (1.6) for the complete loop series, an expansion in $\hat\phi^\mu$ can be evaluated, and equated term by term with (2.34b). This gives a determination of $V^\mu(\hat\phi)$, and the procedure can be continued with an attending increase of computational tedium.

III. SELF-INTERACTING SPIN-ZERO FIELDS

A. Preliminaries

As an example of the general procedure, I shall calculate the effective potential for a theory of n spinless fields ϕ_a, with an O(n)-invariant interaction. For simplicity, the fields are taken to be massless. The computation is performed exactly in the two-loop approximation. Also the contribution of leading logarithms to all loops is summed in the limit $n \to \infty$.

The Lagrangian is

$$\mathcal{L}\{\phi_a(x)\} = \tfrac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a - \frac{\lambda}{4!}\phi^4 - \frac{\delta\mu^2}{2}\phi^2 - \frac{\delta\lambda}{4!}\phi^4 + \tfrac{1}{2}z\partial_\mu\phi_a\partial^\mu\phi_a \,,$$

$$\phi^2 = \phi_a\phi_a \,, \quad \phi^4 = (\phi^2)^2 \,. \tag{3.1a}$$

The counterterms $\delta\mu^2$, $\delta\lambda$, and z are given by power series in $\hbar$, beginning with order $\hbar$:

$$\begin{aligned} \delta\mu^2 &= \hbar\delta\mu_1^2 + \hbar^2\delta\mu_2^2 + \cdots \,, \\ \delta\lambda &= \hbar\delta\lambda_1 + \hbar^2\delta\lambda_2 + \cdots \,, \\ z &= \hbar z_1 + \hbar^2 z_2 + \cdots \,. \end{aligned} \tag{3.1b}$$

It is known that $z_1 = 0$,[3] and to the order we are working z_2 plays no role; hence we ignore the entire wave-function renormalization counterterm. The shifted Lagrangian is

$$\begin{aligned} \mathcal{L}\{\hat\phi_a;\phi_a(x)\} = {}& \tfrac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a \\ & - \tfrac{1}{2}\phi_a[(\delta\mu^2 + \tfrac{1}{6}(\lambda+\delta\lambda)\hat\phi^2)\delta_{ab} + \tfrac{1}{3}\hat\phi_a\hat\phi_b]\phi_b \\ & - \frac{\lambda+\delta\lambda}{6}\hat\phi_a\phi_a\phi^2 - \frac{\lambda+\delta\lambda}{4!}\phi^4 \,. \end{aligned} \tag{3.2}$$

This determines the propagator; in momentum space it is

$$i\mathfrak{D}^{-1}{}_{ab}\{\hat\phi;k\} = [k^2 - (\delta\mu^2 + \tfrac{1}{6}(\lambda+\delta\lambda)\hat\phi^2)]\delta_{ab} - \tfrac{1}{3}(\lambda+\delta\lambda)\phi_a\phi_b \,,$$

$$\mathfrak{D}_{ab}\{\hat\phi;k\} = \frac{i}{k^2 - \delta\mu^2 - \frac{1}{2}(\lambda+\delta\lambda)\hat\phi^2 + i\epsilon}\,\frac{\hat\phi_a\hat\phi_b}{\hat\phi^2} + \frac{i}{k^2 - \delta\mu^2 - \frac{1}{6}(\lambda+\delta\lambda)\hat\phi^2 + i\epsilon}\left[\delta_{ab} - \frac{\hat\phi_a\hat\phi_b}{\hat\phi^2}\right] \,. \tag{3.3}$$

The field shift gives rise to a nondiagonal "mass term" $(\lambda+\delta\lambda)(\hat\phi^2\frac{1}{6}\delta_{ab} + \frac{1}{3}\hat\phi_a\hat\phi_b)$. Also a cubic interaction $\frac{1}{6}(\lambda+\delta\lambda)\hat\phi_a\phi_a\phi^2$ is induced; the "coupling constant" depends on $\hat\phi_a$.

B. The effective potential in the two-loop approximation

The zeroth-order effective potential is just the classical expression

$$V_0(\hat\phi) = \frac{\lambda+\delta\lambda}{4!}\hat\phi^4 + \frac{\delta\mu^2}{2}\hat\phi^2 \,. \tag{3.4}$$

In first order, according to (1.6), we have

$$\begin{aligned} V_1(\hat\phi) &= -\frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\ln\det i\mathfrak{D}^{-1}{}_{ab}\{\hat\phi;k\} \\ &= -\frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\ln[k^2 - m_1^2 + i\epsilon] \times [k^2 - m_2^2 + i\epsilon]^{n-1} \,, \end{aligned} \tag{3.5a}$$

$$m_1^2 = \delta\mu^2 + \tfrac{1}{2}(\lambda+\delta\lambda)\hat\phi^2 \,,$$

$$m_2^2 = \delta\mu^2 + \tfrac{1}{6}(\lambda+\delta\lambda)\hat\phi^2 \,.$$

The integration contour is rotated, the integral is evaluated with a cutoff at $k^2 = \Lambda^2$, and the final result is

$$V_1(\hat\phi)=\frac{\hbar}{64\pi^2}\left[\Lambda^4\ln\left(1+\frac{m_1^2}{\Lambda^2}\right)-m_1^4\ln\left(\frac{\Lambda^2}{m_1^2}+1\right)+\Lambda^2 m_1^2\right]$$
$$+\frac{\hbar(n-1)}{64\pi^2}\left[\Lambda^4\ln\left(1+\frac{m_2^2}{\Lambda^2}\right)-m_2^4\ln\left(\frac{\Lambda^2}{m_2^2}+1\right)+\Lambda^2 m_2^2\right]\quad. \tag{3.5b}$$

A constant of order Λ^4, independent of all parameters, has been dropped. In the present formalism no infrared divergences are encountered since a mass term in induced by the shift of the field.

Renormalization is carried out as follows. We substitute (3.1b) into (3.5b) and expand to order $\hbar^2$. It is demanded that as $\Lambda^2\to\infty$, the effective potential be finite and that the mass remain zero to order $\hbar$. This forces the following conditions, in the limit $\Lambda^2\to\infty$:

$$\delta\mu_1^2=-\frac{\Lambda^2\lambda}{2^5\pi^2}[1+\tfrac{1}{3}(n-1)]\ ,$$
$$\delta\lambda_1=\frac{3\lambda^2}{2^5\pi^2}\left[\ln\frac{2\Lambda^2e^{1/2}}{\lambda}+\tfrac{1}{9}(n-1)\ln\frac{6\Lambda^2e^{1/2}}{\lambda}\right]+\delta\bar\lambda_1\ . \tag{3.6}$$

Here $\delta\bar\lambda_1$ is an unspecified but finite quantity. The effective potential now becomes

$$V(\hat\phi)=\frac{\lambda+\delta\bar\lambda}{4!}\hat\phi^4+\tfrac{1}{2}\delta\bar\mu^2\hat\phi^2+\frac{\hbar\lambda^2}{2^8\pi^2}[1+\tfrac{1}{9}(n-1)]\hat\phi^4\ln\hat\phi^2$$
$$+\hbar^2 a\hat\phi^2\ln\hat\phi^2+\hbar^2 b\hat\phi^4\ln\hat\phi^2+O(\hbar^2)\ ,$$
$$\delta\bar\mu^2=\hbar^2\delta\mu_2^2\ , \tag{3.7a}$$
$$\delta\bar\lambda=\hbar\delta\bar\lambda_1+\hbar^2\delta\lambda_2\ .$$

We have adjusted $\delta\mu_2^2$ and $\delta\lambda_2$ by adding various infinite constants so that (3.7a) is true as written. This can be done, since at the present stage, they are still undetermined. The important point is that a and b are fixed in terms of other parameters:

$$a=-\frac{\lambda^2\Lambda^2}{2^{11}\pi^4}[1+\tfrac{1}{3}(n-1)]^2\ ,$$
$$b=\frac{\lambda\,\delta\bar\lambda_1}{2^7\pi^2}[1+\tfrac{1}{9}(n-1)]$$
$$+\frac{3\lambda^3}{2^{12}\pi^4}[1+\tfrac{1}{9}(n-1)]$$
$$\times\left[\ln\frac{2\Lambda^2e^{1/2}}{\lambda}+\tfrac{1}{9}(n-1)\ln\frac{6\Lambda^2e^{1/2}}{\lambda}\right]\cdot \tag{3.7b}$$

For the renormalization program to be successful, the two-loop contribution must cancel the infinite parts of a and b.

Now for the two-loop effects: According to (1.6) and (3.2) we seek the order-$\hbar^2$ connected, single-particle irreducible graphs of

$$i\hbar\left\langle 0\left|T\exp\left\{-\frac{i}{\hbar}\lambda\int d^4x\left[\tfrac{1}{6}\hat\phi_a\phi_a(x)\phi^2(x)+\frac{1}{4!}\phi^4(x)\right]\right\}\right|0\right\rangle\ . \tag{3.8}$$

We have deleted the counterterms, since they play no role in this approximation. Upon rescaling $\phi\to(\hbar)^{1/2}\phi$, expanding the exponential to the relevant order, and applying Wick's theorem, we are left with the two integrals

$$I_1=\tfrac{1}{24}\hbar^2\lambda\int\frac{d^4k\,d^4l}{(2\pi)^8}[\mathfrak{D}_{aa}\{\hat\phi;k\}\mathfrak{D}_{bb}\{\hat\phi;l\}+2\mathfrak{D}_{ab}\{\hat\phi;k\}\mathfrak{D}_{ba}\{\hat\phi;l\}]\ , \tag{3.9a}$$

$$I_2=-\tfrac{1}{36}i\hbar^2\lambda^2\hat\phi_a\hat\phi_b\int\frac{d^4k\,d^4l}{(2\pi)^8}[\mathfrak{D}_{ab}\{\hat\phi;k+l\}\mathfrak{D}_{cd}\{\hat\phi;k\}\mathfrak{D}_{dc}\{\hat\phi;l\}+2\mathfrak{D}_{ac}\{\hat\phi;k\}\mathfrak{D}_{cd}\{\hat\phi;k+l\}\mathfrak{D}_{db}\{\hat\phi;l\}]\ . \tag{3.9b}$$

The first corresponds to the double bubble of Fig. 1, while the second is the "radiatively" corrected single bubble. After a rotation to Euclidean space, we find

$$I_1=\tfrac{1}{24}\hbar^2\lambda\left[\int\frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2+m_1^2}+\frac{n-1}{k^2+m_2^2}\right)\right]^2+\tfrac{1}{12}\hbar^2\lambda\left[\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2+m_1^2}\right]^2+\tfrac{1}{12}\hbar^2\lambda(n-1)\left[\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2+m_2^2}\right]^2\ , \tag{3.10a}$$

$$I_2=-\tfrac{1}{36}\hbar^2\lambda^2\hat\phi^2\int\frac{d^4k\,d^4l}{(2\pi)^8}\left[\frac{3}{(k^2+m_1^2)(l^2+m_1^2)([k+l]^2+m_1^2)}+\frac{n-1}{(k^2+m_2^2)(l^2+m_2^2)([k+l]^2+m_1^2)}\right]\ . \tag{3.10b}$$

Note that I_2 is negative, and that I_1 always dominates over I_2 for sufficiently large n.

A straightforward calculation gives, in the limit $\Lambda^2\to\infty$,

$$I_1 = \frac{\hbar^2\lambda^2\Lambda^2}{2^{11}\pi^4}[1+\tfrac{1}{3}(n-1)]^2\hat\phi^2\ln\hat\phi^2 + \frac{\hbar^2\lambda^3}{2^{12}\pi^4}\left[\{1+\tfrac{1}{9}(n-1)\}\ln\frac{\lambda}{2\Lambda^2} + \tfrac{1}{27}(n-1)\{5+(n-1)\}\ln\frac{\lambda}{6\Lambda^2}\right]\hat\varphi^4\ln\hat\varphi^2$$
$$+\frac{\hbar^2\lambda^3}{2^{13}\pi^4}[1+\tfrac{8}{27}(n-1)+\tfrac{1}{27}(n-1)^2]\,\hat\phi^4\ln^2\hat\phi^2+\cdots\,, \tag{3.11a}$$

$$I_2 = \frac{\hbar^2\lambda^3}{2^{11}\pi^4}\left[\ln\frac{\lambda}{2\Lambda^2 e} + \tfrac{2}{27}(n-1)\ln\frac{\lambda}{6\Lambda^2 e} + \tfrac{1}{9}(n-1)\ln\frac{\lambda}{2\Lambda^2 e}\right]\hat\phi^4\ln\hat\phi^2 + \frac{\hbar^2\lambda^3}{2^{12}\pi^4}[1+\tfrac{5}{27}(n-1)]\hat\phi^4\ln^2\hat\phi^2+\cdots\,. \tag{3.11b}$$

The dots represent a quadratic polynomial in $\hat\phi^2$ with infinite coefficients, which I have not calculated since it is unnecessary; it can be absorbed in the counterterms.

Upon combining I_1+I_2 with $V(\hat\phi)$ given by (3.7), one sees that the infinite terms occurring in $a\hat\phi^2\ln\hat\phi^2$ and $b\hat\phi^4\ln\hat\phi^2$ cancel, as they should, and the (unrenormalized) potential, through order $\hbar^2$, is

$$V(\hat\phi) = \frac{1}{4!}[\lambda+\delta\bar\lambda-\alpha\hbar^2]\hat\phi^4+\tfrac{1}{2}[\delta\bar\mu^2-\beta\hbar^2]\hat\phi^2+\frac{\hbar\lambda^2}{2^8\pi^2}[1+\tfrac{1}{9}(n-1)]\hat\phi^4\ln\hat\phi^2+\frac{3\hbar^2\lambda^3}{2^{13}\pi^4}[1+\tfrac{1}{9}(n-1)]^2\hat\phi^4\ln^2\hat\phi^2$$
$$+\frac{\hbar^2\lambda\,\delta\bar\lambda_1}{2^7\pi^2}[1+\tfrac{1}{9}(n-1)]\hat\phi^4\ln\hat\phi^2+\frac{3\hbar^2\lambda^3}{2^{13}\pi^4}\{[1+\tfrac{1}{9}(n-1)]^2-\tfrac{4}{3}-\tfrac{20}{81}(n-1)\}\hat\phi^4\ln\hat\phi^2+O(\hbar^3)\,. \tag{3.12}$$

The contributions involving α and β arise from the infinite quadratic polynomial present in I_1+I_2. Renormalization is effected by demanding that, through order $\hbar^2$, the mass remain zero. This forces

$$\delta\bar\mu^2=\hbar^2\delta\mu_2{}^2=\hbar^2\beta\,. \tag{3.13a}$$

Also the potential must be finite; hence

$$\hbar^2\delta\lambda_2=\hbar^2\alpha+\hbar^2\delta\bar\lambda_2\,, \tag{3.13b}$$

where $\delta\bar\lambda_2$ is unspecified but finite. Consequently, the renormalized potential becomes

$$V(\hat\phi)=\frac{\bar\lambda}{4!}\hat\phi^4(1+\bar\lambda a_n\ln\hat\phi^2+\bar\lambda^2a_n{}^2\ln^2\hat\phi^2+\bar\lambda^2b_n\ln\hat\phi^2)$$
$$+O(\hbar^3)\,. \tag{3.14}$$

Here

$$a_n=\frac{3\hbar}{32\pi^2}[1+\tfrac{1}{9}(n-1)]\,, \tag{3.15a}$$

$$b_n=a_n{}^2-\frac{3\hbar^2}{2^8\pi^4}[1+\tfrac{5}{27}(n-1)]\,, \tag{3.15b}$$

and

$$\bar\lambda=\lambda+\delta\bar\lambda=\lambda+\hbar\delta\bar\lambda_1+\hbar^2\delta\bar\lambda_2+\cdots\,. \tag{3.15c}$$

Note that for large n, $b_n/a_n{}^2\to1$. The remaining counterterm, $\delta\bar\lambda$, is unspecified but finite. Its value is a matter of convenience; if one follows Coleman and Weinberg[3] and requires

$$\left.\frac{\partial^4V(\hat\phi)}{\partial\hat\phi^4}\right|_{\hat\phi^2=M^2}=\lambda\,, \tag{3.16}$$

then the effective potential can be parameterized in the following fashion:

$$V(\hat\phi)=\frac{\lambda}{4!}\hat\phi^4\left[1+\lambda a_n\left(\ln\frac{\hat\phi^2}{M^2}-\tfrac{25}{6}\right)+\lambda^2a_n{}^2\left(\ln\frac{\hat\phi^2}{M^2}-\tfrac{25}{6}\right)^2\right.$$
$$\left.+\lambda^2b_n\left(\ln\frac{\hat\phi^2}{M^2}-\tfrac{25}{6}\right)+\lambda^2a_n{}^2\tfrac{205}{36}\right]$$
$$+O(\hbar^3)\,. \tag{3.17}$$

It is easy to verify the above is parameterization-invariant, and that the order-$\hbar$ term agrees with previous calculations.[12]

C. The limit of many Bose fields

Exact calculation beyond two loops is clearly prohibitively difficult. Nevertheless, the availability of an exact graphical expansion, Eq. (1.6) and Fig. 1, permits a sensible and consistent approximation scheme, which sums manageable subsets of graphs. Consider for example the limit of large n, a limit familiar from statistical mechanics. I have already remarked that in the two-loop calculation the double bubble, given by (3.10a) and (3.11a), dominates over the radiatively corrected single bubble, (3.10b) or (3.11b): The former is proportional to n^2, and the latter to n. It is easy to see that this situation persists in every order of the loop expansion: On the p-loop level, the p-fold iteration of the bubble is proportional to n^p, while other graphs are at most of order n^{p-1}.

We are thus led to the approximation of keeping in each order of the loop expansion the term dominant in n. This is a sensible approximation if n is large; it corresponds to considering only graphs of the form depicted in Fig. 3. Each graph gives a contribution to $V(\hat\phi)$ proportional to a power of $\ln\hat\phi^2$. If in each order of the expansion one makes

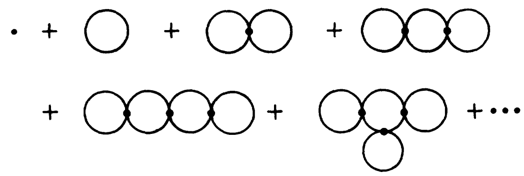

FIG. 3. Graphs that dominate $V(\hat{\phi})$ in large-n limit.

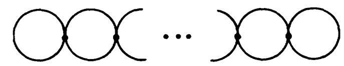

FIG. 4. Linearly iterated bubbles which give leading logarithms that dominate $V(\hat{\phi})$ in large-n limit.

the further approximation of keeping only the leading power of the logarithm, the set of relevant graphs simplifies further. Consider, for example, the last two graphs of Fig. 3, which are both fourth-order in the loop expansion. The first leads to $\ln^4\hat{\phi}^2$, while the second gives only $\ln^3\hat{\phi}^2$. This is because the first diagram has four divergent integrals, while the second has only three divergent integrals since the three-vertex loop represents a convergent integral. Thus in the leading logarithmic approximation we need sum only linearly iterated bubbles, i.e., graphs of the general form given in Fig. 4. According to (1.6) and (3.2), we seek the linearly iterated bubbles contributing to

$$i\hbar\left\langle 0\left|T\exp\left[-\frac{i\bar{\lambda}}{4!\hbar}\int d^4x\,\phi^4(x)\right]\right|0\right\rangle = i\hbar\sum_{p=1}^{\infty}\left(-\frac{i\bar{\lambda}\hbar}{4!}\right)^p\int d^4x_1\cdots d^4x_p \times\langle 0|T\phi^4(x_1)\cdots\phi^4(x_p)|0\rangle,$$

$$\bar{\lambda}=\lambda+\delta\lambda\ . \quad (3.18)$$

On the right-hand side the field has been rescaled as usual. It is unnecessary to use the complete propagator (3.3) in the evaluation of (3.18). Since factors of n can only arise from $\delta_{aa}=n$, it is sufficient to take for the propagator

$$D_{ab}(k)=\frac{i\delta_{ab}}{k^2-m^2+i\epsilon}\,,\quad m^2=\tfrac{1}{6}\lambda\hat{\phi}^2+\delta\mu^2\ . \quad (3.19)$$

To evaluate the graph of Fig. 4, observe that its structure is the following. At either end there is a tadpole given by

$$\int\frac{d^4k}{(2\pi)^4}D_{aa}(k)=\frac{n}{16\pi^2}\left(\Lambda^2-m^2\ln\frac{\Lambda^2+m^2}{m^2}\right). \quad (3.20)$$

Between the tadpoles there are $p-2$ two-line bubbles. Each bubble is

$$\int\frac{d^4k}{(2\pi)^4}D_{ab}(k)D_{ba}(k) = -\frac{in}{16\pi^2}\left(\ln\frac{\Lambda^2+m^2}{m^2}-\frac{\Lambda^2}{\Lambda^2+m^2}\right)\ . \quad (3.21)$$

Finally, we must calculate the combinatorial factor. This is done as follows. The two tadpoles will arise from the self-contraction by two of the p factors of ϕ^4 in (3.18). Choose the two. This can happen in $\frac{1}{2}p(p-1)$ ways. Each of these may be rewritten by Wick's theorem,

$$\phi^4=2:\phi^2:\int\frac{d^4k}{(2\pi)^4}D_{aa}(k)+\cdots\,, \quad (3.22)$$

where the omitted terms do not involve the desired tadpole. Thus we find that for present purposes

$$\int d^4x_1\cdots d^4x_p\langle 0|T\phi^4(x_1)\cdots\phi^4(x_p)|0\rangle = 2p(p-1)\left[\int\frac{d^4k}{(2\pi)^4}D_{aa}(k)\right]^2\int d^4x_1\cdots d^4x_p\langle 0|T:\phi^2(x_1)::\phi^2(x_2):\phi^4(x_3)\cdots\phi^4(x_p)|0\rangle\ . \quad (3.23\text{a})$$

In the subsequent evaluation each of the two $:\phi^2:$ must contract as a unit into one of the $p-2$ ϕ^4 factors. Only in this way do we arrive at the propagator combination $D_{ab}(k)D_{ba}(k)$ which leads to the linear multiple bubble. Consequently (3.23a) may be replaced by

$$2p(p-1)\left[\int\frac{d^4k}{(2\pi)^4}D_{aa}(k)\right]^2\int d^4x_1\cdots d^4x_p 2^{p-2}(p-2)!\langle 0|T:\phi^2(x_1)::\phi^2(x_2):|0\rangle\cdots\langle 0|T:\phi^2(x_{p-1})::\phi^2(x_p):|0\rangle$$

$$=2^{p-1}p!\left[\int\frac{d^4k}{(2\pi)^4}D_{aa}(k)\right]^2\left[2\int\frac{d^4l}{(2\pi)^4}D_{bc}(l)D_{cb}(l)\right]^{p-1}\int d^4x\ . \quad (3.23\text{b})$$

[The case $p=1$ is special, but the final result (3.23b) is correct in that instance also.]

Thus the contribution of (3.18) to the effective potential is found to be

$$\frac{36}{4!}\bar{\lambda}a_n^2\left(\Lambda^2-m^2\ln\frac{\Lambda^2+m^2}{m^2}\right)^2 \times\left[1-\bar{\lambda}a_n\left(\ln\frac{m^2}{m^2+\Lambda^2}+\frac{\Lambda^2}{m^2+\Lambda^2}\right)\right]^{-1},$$

$$a_n=\frac{\hbar n}{96\pi^2}. \quad (3.24)$$

This is to be combined with the zeroth-order (3.4) and the one-loop (3.5b) terms in $V(\hat{\phi})$. The appearance of the formula may be simplified somewhat by rescaling the cutoff and the mass counterterm by

$$\delta\mu^2 \to \tfrac{1}{6}\bar{\lambda}\delta\mu^2,$$
$$\Lambda^2 \to \tfrac{1}{6}\lambda\Lambda^2.$$

The final expression is

$$4!\,V(\hat{\phi})=\bar{\lambda}\hat{\phi}^4+2\bar{\lambda}\delta\mu^2\hat{\phi}^2+\frac{\bar{\lambda}^2a_n\mu^2\Lambda^4}{\Lambda^2+\mu^2} +\bar{\lambda}^2a_n\Lambda^4\ln\left(1+\frac{\mu^2}{\Lambda^2}\right)+\bar{\lambda}^2a_n\mu^4f(\mu^2) +\frac{\bar{\lambda}^3a_n^2g^2(\mu^2)}{1-\bar{\lambda}a_nf(\mu^2)}+\text{constant}, \quad (3.25)$$

$$f(\mu^2)=-\ln\left(\frac{\Lambda^2}{\mu^2}+1\right)+\frac{\Lambda^2}{\Lambda^2+\mu^2}, \quad (3.26a)$$

$$g(\mu^2)=\Lambda^2-\mu^2\ln\left(\frac{\Lambda^2}{\mu^2}+1\right), \quad (3.26b)$$

$$\mu^2=\hat{\phi}^2+\delta\mu^2. \quad (3.26c)$$

The constant is adjusted so that $V(0)=0$. In the above, $\delta\mu^2$ and $\bar{\lambda}$ are determined iteratively from the renormalization conditions. Also wave-function renormalization must be performed. In the leading logarithmic approximation everything simplifies enormously. We find[13]

$$V(\hat{\phi})=\frac{1}{4!}\bar{\lambda}\hat{\phi}^4\frac{1}{1-\bar{\lambda}a_n\ln\hat{\phi}^2}. \quad (3.27)$$

IV. SCALAR, MASSLESS QUANTUM ELECTRODYNAMICS

A. Preliminaries

As a second application of the formalism, I shall calculate $V(\hat{\phi})$ to order $\hbar$ in scalar, massless quantum electrodynamics. This is the theory considered by Coleman and Weinberg.[3] However, in contrast with these authors, I compute in an arbitrary gauge, to expose the gauge dependence of the various formulas.

It is clear that $\Gamma(\overline{\phi})$ is gauge-dependent: $\Gamma(\overline{\phi})$ is the generating function for connected, single-particle irreducible Green's functions of charged fields and these are gauge-dependent. $V(\hat{\phi})$ generates the Green's functions at zero momentum. This is a special value; indeed for massless particles it is a point on the mass shell. Thus it might be hoped that $V(\hat{\phi})$ is gauge-independent. Nevertheless, we show that $V(\hat{\phi})$ is gauge-dependent on the one-loop level. The reason for this gauge dependence may be understood when it is recalled that the field shift, which is performed for the evaluation of $V(\hat{\phi})$, induces a "mass" term to the Bose fields. Once the Bose fields are massive, the zero momentum point no longer corresponds to the mass shell, and the Green's functions are gauge-dependent.

Yet a further consideration illuminates the gauge dependence. The equation of motion for the gauge field in the presence of charge-bearing sources [these are required in a computation of $\Gamma(\overline{\phi})$] involves a nonconserved current. Consequently, gauge degrees of freedom are necessarily excited, and the gauge cannot be fixed arbitrarily. Hence, the effective potential, which summarizes the dynamics of the theory, reflects the gauge dependence.

The classical action, written in terms of real fields, apart from counterterms, is

$$I(\phi,A^\mu)=\int d^4x\,d^4y\left[\tfrac{1}{2}A^\mu(x)i\Delta^{-1}{}_{\mu\nu}(x-y)A^\nu(y)+\tfrac{1}{2}\phi_a(x)iD^{-1}{}_{ab}(x-y)\phi_b(y)\right] +\int d^4x\left[-e\epsilon_{ab}\partial_\mu\phi_a(x)\phi_b(x)A^\mu(x)+\tfrac{1}{2}e^2\phi^2(x)A^2(x)-(\lambda/4!)\phi^4(x)\right]. \quad (4.1)$$

Here ϵ_{ab} is the two-dimensional antisymmetric tensor, $\phi^2=\phi_1{}^2+\phi_2{}^2$, $\phi^4=(\phi^2)^2$. The first integral in (4.1) contains the kinetic terms. $D_{ab}(x-y)$ is the free boson propagator and $\Delta_{\mu\nu}(x-y)$ is the free photon propagator. Specification of the latter requires a choice of gauge. For the present, it is unnecessary to exhibit explicitly this choice. For simplicity I have assumed that the gauge is translation-invariant and does not require gauge-compensating ghost fields. Hence in momentum space we have

$$D_{ab}(k)=\delta_{ab}\frac{i}{k^2-i\epsilon}, \quad (4.2a)$$

$$\Delta_{\mu\nu}(k)=-\frac{i}{k^2+i\epsilon}\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2+i\epsilon}\right)+\text{gauge terms}. \quad (4.2b)$$

To initiate the computation, the fields are to be shifted by a constant field. Since we seek only $V(\hat{\phi})$, it is unnecessary to shift the photon field:

$$\int d^4x\,\hat{\mathcal{L}}\{\hat\phi_a;\phi_a(x),A^\mu(x)\}=I(\phi+\hat\phi,A^\mu)-I(\hat\phi,0)-\int d^4x\phi_a(x)\frac{\delta I(\phi,0)}{\delta\phi_a(x)}\bigg|_{\phi=\hat\phi}-\int d^4x A^\nu(x)\frac{\delta I(\hat\phi,A^\mu)}{\delta A^\nu(x)}\bigg|_{A^\mu=0}$$

$$=\int d^4x\,d^4y[\tfrac{1}{2}A^\mu(x)i\bar\Delta^{-1}{}_{\mu\nu}\{\hat\phi;x,y\}A^\nu(y)+\tfrac{1}{2}\phi_a(x)i\mathfrak{D}^{-1}{}_{ab}\{\hat\phi;x,y\}\phi_b(y)$$

$$+A^\mu(x)M_{\mu a}\{\hat\phi;x,y\}\phi_a(y)]+\cdots\,, \tag{4.3}$$

$$i\bar\Delta^{-1}{}_{\mu\nu}\{\hat\phi;x,y\}=i\Delta^{-1}{}_{\mu\nu}(x-y)+e^2\hat\phi^2g_{\mu\nu}\delta^4(x-y)\,, \tag{4.4a}$$

$$i\mathfrak{D}^{-1}{}_{ab}\{\hat\phi;x,y\}=iD^{-1}{}_{ab}(x-y)-\lambda\,(\tfrac{1}{6}\hat\phi^2\delta_{ab}+\tfrac{1}{3}\hat\phi_a\hat\phi_b)\delta^4(x-y)\,, \tag{4.4b}$$

$$M^{\mu}_{a}\{\hat\phi;x,y\}=-e\epsilon_{ab}\hat\phi_b\partial^\mu\delta^4(x-y)\,. \tag{4.4c}$$

The dots in (4.3) indicate that cubic and quartic interaction terms have been dropped. They are not required: We are computing only to order $\hbar$. It is seen that the effect of the shift is to provide a "mass" for the photon $e^2\hat\phi^2$ and a nondiagonal "mass" for the boson $\frac{1}{6}\lambda\hat\phi^2\delta_{ab}+\frac{1}{3}\lambda\hat\phi_a\hat\phi_b$. Also a boson-photon transition is induced by M^μ_a.

According to the general theory, we have

$$\Gamma(\hat\phi)=I(\hat\phi,0)-i\hbar\ln Z_1(\hat\phi)+O(\hbar^2)\,,$$

$$Z_1(\hat\phi)=\int d\phi_a dA^\mu\exp\frac{i}{\hbar}\int d^4x\,\hat{\mathcal{L}}\{\hat\phi_a;\phi_a(x),A^\mu(x)\}\,. \tag{4.5}$$

Since $\hat{\mathcal{L}}\{\hat\phi_a;\phi_a,A^\mu\}$ is quadratic in ϕ_a and A^μ [see (4.3)] the functional integral is elementary. The answer, obtained by first integrating over ϕ_a and then over A^μ, is

$$Z_1(\hat\phi)=\mathrm{Det}^{-1/2}(i\mathfrak{D}^{-1})\,\mathrm{Det}^{-1/2}(i\bar\Delta^{-1}+iN)\,. \tag{4.6}$$

Here the determinants are still functional. The matrix N is defined by

$$N^{\mu\nu}\{\hat\phi;x,y\}=\int d^4z\,d^4w M^\mu_a\{\hat\phi;x,z\}\mathfrak{D}_{ab}\{\hat\phi;z,w\}\times M^\nu_b\{\hat\phi;y,w\}\,. \tag{4.7}$$

B. The effective potential

For constant field $\hat\phi$, there is translation invariance of the theory. Consequently, in the momentum representation the functional determinants are diagonal, and we find for the effective potential

$$V(\hat\phi)=V_0(\hat\phi)-\frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\ln\det(i\mathfrak{D}^{-1}{}_{ab}\{\hat\phi;k\})-\frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\ln\det(i\bar\Delta^{-1}{}_{\mu\nu}\{\hat\phi;k\}+iN_{\mu\nu}\{\hat\phi;k\})\,. \tag{4.8}$$

The first integral accounts for the Bose loops and has been already encountered in the previous discussion, Sec. III:

$$i\mathfrak{D}^{-1}{}_{ab}\{\hat\phi;k\}=k^2\delta_{ab}-\tfrac{1}{6}\lambda\hat\phi^2\delta_{ab}-\tfrac{1}{3}\lambda\hat\phi_a\hat\phi_b\,, \tag{4.9a}$$

$$\mathfrak{D}_{ab}\{\hat\phi;k\}=\frac{i}{k^2-\frac{1}{2}\lambda\hat\phi^2+i\epsilon}\frac{\hat\phi_a\hat\phi_b}{\hat\phi^2}+\frac{i}{k^2-\frac{1}{6}\lambda\hat\phi^2+i\epsilon}\left(\delta_{ab}-\frac{\hat\phi_a\hat\phi_b}{\hat\phi^2}\right)\,, \tag{4.9b}$$

$$\det(i\mathfrak{D}^{-1}{}_{ab}\{\hat\phi;k\})=(k^2-\tfrac{1}{2}\lambda\hat\phi^2)(k^2-\tfrac{1}{6}\lambda\hat\phi^2)\,, \tag{4.9c}$$

$$-\frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\ln\det(i\mathfrak{D}^{-1}{}_{ab}\{\hat\phi;k\})=\frac{1}{4!}\hat\phi^4\left\{\frac{5}{48\pi^2}\bar\lambda^2\hbar\ln\hat\phi^2\right\}. \tag{4.10}$$

In Eq. (4.10), $\bar\lambda=\lambda+\delta\lambda$ and the mass has been renormalized to zero.

The second integral in (4.8) describes the photon loops and the photon-boson transitions:

$$i\bar\Delta^{-1}{}_{\mu\nu}\{\hat\phi;k\}=(-k^2+e^2\hat\phi^2)g_{\mu\nu}+k_\mu k_\nu\;+\text{gauge terms}\,, \tag{4.11a}$$

$$N^{\mu\nu}\{\hat\phi;k\}=\int d^4x\,e^{ikx}N^{\mu\nu}\{\hat\phi;x,0\}=M^\mu_a\{\hat\phi;k\}\mathfrak{D}_{ab}\{\hat\phi;k\}M^\nu_b\{\hat\phi;-k\}\,, \tag{4.11b}$$

$$M^\mu_a\{\hat\phi;k\}=\int d^4x\,e^{ikx}M^\mu_a\{\hat\phi;x,0\}=iek^\mu\epsilon_{ab}\hat\phi_b\,, \tag{4.11c}$$

$$N^{\mu\nu}\{\hat\phi;k\}=\frac{ie^2k^\mu k^\nu\hat\phi^2}{k^2-\frac{1}{6}\lambda\hat\phi^2+i\epsilon}\,, \tag{4.11d}$$

$$i\bar\Delta^{-1}{}_{\mu\nu}\{\hat\phi;k\}+iN_{\mu\nu}\{\hat\phi;k\}=(-k^2+e^2\hat\phi^2)\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2+i\epsilon}\right)-k_\mu k_\nu\left[\frac{e^2\frac{1}{6}\lambda\hat\phi^4}{(k^2+i\epsilon)(k^2-\frac{1}{6}\lambda\hat\phi^2+i\epsilon)}\right]+\text{gauge terms}\,. \tag{4.11e}$$

The Bose-photon mixing term $N_{\mu\nu}\{\hat\phi;k\}$ is an effective gauge. To complete the evaluation, a choice of gauge in (4.11) must be made. It is straightforward to show that the most general choice of photon gauge which is translation-invariant and does not require gauge-compensating ghosts leads to an inverse photon propagator of the form

$$i\Delta^{-1}{}_{\mu\nu}(k) = -k^2 g_{\mu\nu} + k_\mu k_\nu + d_\mu(k) d_\nu(-k)\ , \qquad (4.12)$$

where $d_\mu(k)$ is an arbitrary vector which satisfies $k^\mu d_\mu(k) \neq 0$. Therefore it follows that

$$\begin{aligned} &\ln \det(i\bar{\Delta}^{-1}{}_{\mu\nu}\{\hat\phi; k\} + iN_{\mu\nu}\{\hat\phi; k\})\\ &\quad = 2\ln[k^2 - e^2\hat\phi^2] - \ln[k^2 - \tfrac{1}{6}\lambda\hat\phi^2] + \ln\left[(k^2 - e^2\hat\phi^2)\left(\tfrac{1}{6}e^2\lambda\hat\phi^4 - \frac{(d\cdot k)^2}{k^2 + i\epsilon}[k^2 - \tfrac{1}{6}\lambda\hat\phi^2]\right) - \tfrac{1}{6}e^2\lambda\hat\phi^4\left(d^2 - \frac{(d\cdot k)^2}{k^2+i\epsilon}\right)\right]\ ,\\ &\qquad d^2 = d^\mu(k)d_\mu(-k),\quad (k\cdot d)^2 = [k_\mu d^\mu(k)][k_\nu d^\nu(-k)]\ . \end{aligned} \qquad (4.13)$$

As promised, there is a gauge dependence, i.e., a dependence on $d^\mu(k)$. The gauge-dependent contribution is proportional to $e^2\frac{1}{6}\lambda\hat\phi^4$, which is the mass induced to the photon $(e^2\hat\phi^2)$ times one of the masses induced to the boson $(\frac{1}{6}\lambda\hat\phi^2)$. To see this, observe that if the $e^2\frac{1}{6}\lambda\hat\phi^4$ term is ignored, the logarithms in (4.13) become

$$3\ln[k^2 - e^2\hat\phi^2] + \ln\left[-\frac{(d\cdot k)^2}{k^2 + i\epsilon}\right]\ .$$

Now the gauge term may be dropped, since it is independent of $\hat\phi$.

For an explicit evaluation of $V(\hat\phi)$, we must specify the k dependence of $d^\mu(k)$. An especially simple choice is the class of Lorentz gauges $d^\mu(k) = (1/\sqrt{\alpha}\,)k^\mu$. In this case (4.13) becomes (apart from unimportant constants)

$$3\ln[k^2 - e^2\hat\phi^2] - \ln[k^2 - \tfrac{1}{6}\lambda\hat\phi^2] + \ln[k^4 - \tfrac{1}{6}\lambda k^2\hat\phi^2 + \alpha\tfrac{1}{6}e^2\lambda\hat\phi^4]$$

and, apart from a polynomial in $\hat\phi^2$,

$$-\tfrac{1}{2}i\hbar\int\frac{d^4k}{(2\pi)^4}\ln\det[i\bar\Delta^{-1}{}_{\mu\nu}\{\hat\phi; k\} + iN_{\mu\nu}\{\hat\phi; k\}] = \frac{\hat\phi^4}{4!}\frac{\hbar}{8\pi^2}[9e^4 - \alpha e^2\lambda]\ln\hat\phi^2\ . \qquad (4.14)$$

The effective potential thus is

$$V(\hat\phi) = \frac{\hat\phi^4}{4!}\left[\bar\lambda + \frac{\hbar}{8\pi^2}(\tfrac{5}{6}\bar\lambda^2 + 9e^4 - \alpha e^2\bar\lambda)\ln\hat\phi^2\right]\ . \qquad (4.15)$$

This agrees with the Coleman-Weinberg calculation for $\alpha = 0$, in the Landau gauge.[3] [I have, of course, dropped $\hat\phi^2$ and $\hat\phi^4$ terms, as they are renormalized. However, for negative α and imaginary $d^\mu(k)$, the $\hat\phi^4$ term has a complex coefficient.]

The gauge dependence of the effective potential may also be seen in another way. Consider the four-boson scattering amplitude, at zero momentum. The one-loop, order-λe^2 contribution is summarized by Fig. 5. The integral representation for this quantity is proportional to

$$e^2\lambda\int\frac{d^4k}{(2\pi)^4}\frac{k_\mu k_\nu\,\Delta^{\mu\nu}(k)}{k^4} = \frac{\alpha e^2\lambda}{16\pi^2}\int_0^\infty\frac{dk^2}{k^2}\ , \qquad (4.16)$$

which is gauge-dependent. The reason for the gauge dependence is clear: To obtain the physical, gauge-independent scattering amplitude, one must add wave-function renormalization graphs of the form depicted in Fig. 6. At zero momentum they survive; however, they are not single-particle irreducible, hence by definition they are excluded from $V(\hat\phi)$. (It is not difficult to show that the single-particle irreducible e^4 graphs contributing to the scattering amplitude are separately gauge-invariant at zero momentum.)

C. Discussion

The observation that $V(\hat\phi)$ is gauge-dependent for a gauge theory raises a question concerning the physical significance of any mathematical properties of $V(\hat\phi)$. I have already remarked that $V(\hat\phi)$ becomes complex for covariant Lorentz gauges with $\alpha < 0$. It is also true that the whole concept of an effective potential can be destroyed, since one can also work with nontranslation-invariant gauges. In that circumstance $\Gamma(\hat\phi)$ for constant fields $\hat\phi$ is given by $\Gamma(\hat\phi) = \int d^4x\, V(\hat\phi; x)$, and a local effective potential cannot be defined.

None of these peculiarities of $V(\hat\phi)$ are especially disturbing if one considers the symmetric solution to the theory at $\hat\phi = 0$. However, the search for a minimum in $V(\hat\phi)$ away from $\hat\phi = 0$ is not a gauge-invariant procedure. Indeed, in the above example, all one-loop effects can be made to disappear by the choice

$$\alpha = \frac{5}{6}\frac{\bar\lambda}{e^2} + 9\frac{e^2}{\bar\lambda}\ . \qquad (4.17)$$

At the present time I do not know whether the

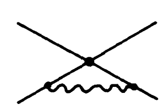

FIG. 5. Gauge-dependent contribution to $V(\hat\phi)$.

gauge dependence of $V(\hat{\phi})$ is a serious defect. If one adopts the view that the loop expansion must be converted to an ordinary perturbative expansion in the coupling constant, then to order e^4 there is no gauge dependence in (4.15). $\alpha e^2 \bar{\lambda}$ is of order e^6, since $\bar{\lambda}$ is assumed to be of order e^4. In this context a choice of gauge like (4.17) is inadmissible since it clearly mixes up orders of perturbation theory. (However, it is not always possible to reexpand in the coupling constant.[4]) Thus it is not clear whether the *physical* consequence of the Coleman-Weinberg[3] calculation is questioned. Their physically interesting result is a formula for the ratio of the spontaneously generated mass of the Higgs particle to the vector-meson mass. If spontaneous symmetry breaking by radiative corrections is a physical effect, it presumably is gauge-independent, and the mass ratio can be computed in any gauge. However, there is no proof that radiative spontaneous symmetry breaking in gauge theories is a gauge-invariant phenomenon.[14] Clearly, a calculation of $V(\hat{\phi})$ to order e^6 will be illuminating. This requires a two-loop calculation which is now in progress.

V. SUMMARY AND CONCLUSIONS

The main purpose of this paper is to develop techniques for studying the effective potential beyond lowest order, so that bound-state phenomenon can be examined. The expansion in Sec. II for $V(\hat{\phi})$ will serve that purpose. Especially interesting are zero-mass bound states in view of their role in spontaneous mass generation.[5] When there is a zero-mass bound state, single-particle irreducible Green's functions have a pole at zero momentum. Yet hopefully, the effective potential is well defined. I expect that the singularity in momentum space becomes replaced by a singularity in $\hat{\phi}$ space. This was observed by Coleman and Weinberg[3] in connection with infrared divergences. Also, our computation of $V(\hat{\phi})$ for n Bose fields, in the limit of large n, shows how many loop effects sum up to produce a singularity in $V(\hat{\phi})$; see (3.25) and (3.27).

Although the development was confined to Bose fields, Fermi fields can be treated analogously. One difference, however, is that the functional determinant which summarizes the one-loop graphs enters with a different power: Rather than $\mathrm{Det}^{-1/2}$, we have, for fermions, Det. The reason is that

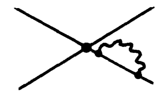

FIG. 6. External wave-function renormalization graph which removes gauge dependence of Fig. 5.

the basic functional integral for Fermi fields is

$$\int d\psi\, d\bar{\psi} \exp \frac{i}{\hbar} \bar{\psi} M \psi = \mathrm{Det} M .$$

Also, ghost loops in gauge theories are handled quite naturally by the present formalism. Of course, one is not interested in generating Green's functions with external ghost lines; this simply means that the ghost field need not be shifted.

In the course of applying the formalism to various examples, it was demonstrated that for gauge theories $V(\hat{\phi})$ is gauge-dependent. This raises the following question about spontaneous symmetry breaking by radiative corrections: Suppose $V(\hat{\phi})$ is found to possess a minimum at a nonzero value of $\hat{\phi}$. Is this minimum present in all gauges, or is it an artifact of the choice of gauge? Furthermore, are physical amplitudes, evaluated at nonvanishing $\hat{\phi}$, gauge-invariant?[14] The Coleman-Weinberg[3] example of massless, scalar electrodynamics is inconclusive, since their calculation is approximate: Only one-loop graphs are considered, and the answer is reliable only to order e^4, while the gauge dependence appears in e^6. To be sure there is a gauge in which the one-loop minimum disappears; see (4.17). However, that gauge introduces inverse powers of the coupling constant, which emphasize higher orders. These have not, as yet, been computed.

ACKNOWLEDGMENT

This research was performed at the Department of Applied Mathematics and Theoretical Physics of Cambridge University, and I am grateful for the kind hospitality which was extended by the members of that laboratory. Conversations with I. Drummond, K. Johnson, and H. Schnitzer were especially helpful, and are happily acknowledged. H. Schnitzer has also researched some of the topics discussed here, and communicated his results to me. I also benefited from a correspondence with B. DeWitt concerning his investigations of the effective potential. Some of my calculations were checked by L. Dolan and E. Tomboulis.

*Work supported in part through funds provided by the U. S. Atomic Energy Commission under Contract No. AT(11-1)-3069.

[1]The reader who is unfamiliar with the effective potential can turn to Eqs. (2.1) to (2.4), where it is defined.

[2]G. Jona-Lasinio, Nuovo Cimento 34, 1790 (1964).

[3]S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973).

[4]I. Drummond, DAMPT report (unpublished).

[5]R. Jackiw and K. Johnson, Phys. Rev. D 8, 2386 (1973); J. M. Cornwall and R. E. Norton, *ibid.* 8, 3338 (1973).

[6]Y. Nambu, Phys. Lett. 26B, 626 (1968).

[7]Similar results for scalar-field theories have been obtained by B. DeWitt. Although the general case is also known to him, it has not been published. Discussion of his research can be found in B. DeWitt, in *Magic Without Magic: John Archibald Wheeler, a Collection of Essays in Honor of His 60th Birthday*, edited by John R. Klauder (Freeman, San Francisco, 1972).

[8]There exists an alternate definition of the effective potential due to J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1962). For nongauge theories this is equivalent to the definition given in the text. For gauge theories, differences arise and our definition seems to be more flexible. In this context see S. Weinberg, Phys. Rev. D 7, 1888 (1973) as well as L. Dolan and R. Jackiw, Phys. Rev. D (to be published).

[9]R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

[10]See for example Ref. 7 or B. Lee and J. Zinn-Justin, Phys. Rev. D 5, 3121 (1972).

[11]The argument of W_1 is not indicated explicitly, since we now wish to view W_1 as a functional of ϕ^0 rather than J, but do not want to introduce new notation.

[12]For $n=1$ and 2, the one-loop contribution is given by Coleman and Weinberg, Ref. 3, and by Drummond, Ref. 4; for arbitrary n, the $O(\hbar)$ term was also found by E. Weinberg, Ph.D. thesis, Harvard, 1973 (unpublished).

[13]The leading logarithm approximation can also be obtained by renormalization-group methods; see Coleman and Weinberg, Ref. 3, and Weinberg, Ref. 12.

[14]That the *conventional* Higgs mechanism leads to a gauge-invariant S matrix has been shown by several authors, most recently by T. Appelquist, J. Carazzone, T. Goldman, and H. R. Quinn, Phys. Rev. D 8, 1747 (1973). References to earlier work by 't Hooft, Lee, Veltman, and others can be found in this paper.