Harmonic superspace

Martin Roček

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In dimension 4, it is well known that we have two possible measures: $\int d^2\theta$ behaves like two spinorial derivatives and gives terms such as $\psi\psi$ or ϕF (mass and potential terms), and $\int d^4\theta$ behaves like four spinorial derivatives and gives terms like ψ^4, $\psi\partial\psi$, or $(\partial\phi)^2$ (kinetic terms).

What happens if we have more than four thetas? We need to introduce subspaces.

The simplest example of this is the chiral measure $\int d^2\theta$ we just described in four dimensions. This makes sense in $\mathfrak{p}^{*|4}$ because part of the algebra of spinor derivatives, $\{\bar{D}, \bar{D}\} = 0$, serves as an integrability condition for the notion of chiral superfields defined by $\bar{D}\phi = 0$.

In the case of more than four thetas, we focus on four dimensions and lower. (For higher dimensions, see the work of N. Berkovits; there we need either spacetime twistors or pure spinors). We consider extended supersymmetry, which means we take

$$\theta \in \Gamma(\Pi S \otimes \mathbb{C}^N)$$

(where "N" measures the amount of SUSY). For arbitrary N, there is a lot of recent work on this, particularly by P. Howe and others. I'll focus on $D = 4$, $N = 2$, which is very similar to $D = 2$,$N = 4$. So we have

$$\theta^{\alpha a} \in \Gamma(\Pi S_+ \otimes \mathbb{C}^2),$$
$$\bar{\theta}^{\dot{\alpha}}{}_a \in \Gamma(\Pi S_- \otimes \mathbb{C}^2),$$

The spinor derivatives obey

$$\{D_{\alpha a}, D_{\beta b}\} = 0\ , \quad \{\bar{D}_{\dot{\alpha}}{}^a, D_{\beta b}\} = i\delta^a_b\partial_{\beta\dot{\alpha}}$$

We introduce $u^a \in \mathbb{C}^2$, and form

$$\nabla_\alpha = u^a D_{\alpha a}, \quad \bar{\nabla}_{\dot{\alpha}} = u^a \bar{D}_{\dot{\alpha}}{}^b \epsilon_{ab}.$$

This is a (graded) abelian subalgebra analogous to the subalgebra generated by chiral derivatives, and we can consider harmonic superfields that are annihilated by $\nabla, \bar{\nabla}$. Note that the overall scale of u^a doesn't matter, so the complex parameters u^a can be thought of as homogeneous coordinates on $\mathbb{P}^1$. The traditional harmonic superspace people (Sokatchev et. al., Buchbinder, Ketov, Howe, etc.) study representations of supersymmetry and construct actions by regarding this as the 2-sphere and doing harmonic analysis on this sphere. The projective superspace people (Gates-Hull-MR, Lindström-MR, Gonzalez-Rey, etc.) regard this as the Riemann sphere and focus on complex analysis. This has a very direct relation to twistors.

We can choose inhomogenous coordinates in a patch: $\zeta = u^2/u^1 \in \mathbb{P}^1$.

Examples: (1) $N = 2$ Super Yang-Mills theory. First, let's review $N = 1$ Yang–Mills theory in $D = 4$. A chiral superfield $\phi \in V$ naturally gets an action of the complexified gauge group by $g \in G_{\mathbb{C}}$:

$$\begin{aligned}\phi &\to g\phi \\ \bar{\phi} &\to \bar{\phi}\bar{g}\ ,\end{aligned}$$

where g is itself chiral; this is natural because the product of chiral superfields is chiral ($\bar{D}\phi = 0 \Rightarrow \bar{D}g\phi = 0$ if $\bar{D}g = 0$; the group is complexified because chiral superfields cannot be real). Since $\bar{g}$ is *anti*chiral, we introduce what used to be call a prepotential h before Seiberg and Witten introduced a different use of the word; h is hermitian, and transforms under the *non*compact part of the complexified gauge group: $h \to \bar{g}^{-1}hg^{-1}$. Then the gauge covariant derivatives (superconnections) can be written as

$$\bar{D}_{\dot{\alpha}} \quad \text{and} \quad h^{-1}D_\alpha h$$

Note: physicists usually write $h = e^V$ and write the group element as $g = e^{i\Lambda}$.

Now we are ready to go to $N = 2$. We write the projective superspace derivatives as

$$\begin{aligned}\nabla_\alpha &= D_{\alpha 1} + \zeta D_{\alpha 2} \\ \bar{\nabla}_{\dot{\alpha}} &= \bar{D}_{\dot{\alpha}}{}^2 - \zeta \bar{D}_{\dot{\alpha}}{}^1.\end{aligned}$$

Notice that $\bar{\nabla}$ is the conjugate of ∇ under a involution that is the composition of complex conjugation with the antipodal map on $\mathbb{P}^1$: $\bar{\zeta} \to -\frac{1}{\zeta}$, up to a projective factor.

We define polar (arctic and antarctic) multiplets Υ and $\bar{\Upsilon}$: We introduce Υ with

$$\nabla_\alpha \Upsilon = \bar{\nabla}_{\dot{\alpha}} \Upsilon = 0$$

and $\Upsilon(\zeta = 0)$ is regular; similarly, $\bar{\Upsilon}$ is regular at $1/\zeta = 0$. The natural group action is again complex with an *arctic* group element $g \in G_{\mathbb{C}}$: $\Upsilon \to g\Upsilon$, $\bar{\Upsilon} \to \bar{\Upsilon}\bar{g}$.

As in the $N = 1$ case, we introduce a hermitian prepotential h with $\nabla_\alpha h = \bar{\nabla}_{\dot{\alpha}} h = 0$; h is *tropical*: it is regular away from $\zeta = 0$ and $\frac{1}{\zeta} = 0$. Again it transforms as then

$$h \to \bar{g}^{-1}hg^{-1}.$$

In the abelian case, we can always factor

$$h = h_- h_+$$

where h_+ is regular at $\zeta = 0$ and h_- is regular at $\frac{1}{\zeta} = 0$. *However*, $\nabla_\alpha h_\pm \neq 0$. In the nonabelian case, this is a from of the Riemann-Hilbert problem, but we can imagine starting with the factored form and defining h in terms of $h_\pm$.

Since $\nabla_\alpha h = 0$ we have

$$h_-^{-1} \nabla_\alpha h_- = -(\nabla_\alpha h_+) h_+^{-1} \ ;$$

comparing powers of ζ on both sides of the equation, we find

$$h_-^{-1} \nabla_\alpha h_- = \mathcal{D}_{1\alpha} + \zeta \mathcal{D}_{2\alpha} \equiv \mathcal{D}_\alpha(\zeta) \ .$$

The usual chiral field strength of $N = 2$ super Yang-Mills theory can be expressed in terms of these covariant derivatives as

$$\{\mathcal{D}_\alpha(\zeta_1), \mathcal{D}_\beta(\zeta_2)\} = \epsilon_{\alpha\beta}(\zeta_1 - \zeta_2) W \ ;$$

(W, though it was introduced much earlier, should be familiar from Seiberg–Witten theory). Similarly, the vector covariant derivative is independent of ζ:

$$\{\mathcal{D}_\alpha(\zeta_1), \bar{\mathcal{D}}_{\dot\beta}(\zeta_2)\} = i(\zeta_1 - \zeta_2) \mathcal{D}_{\alpha\dot\beta} \ .$$

Hypermultiplets have many different off-shell representations; a particularly useful set is the "$O(2n)$ multiplets":

$$\eta_{(2n)} = \sum_0^{2n} \eta_{(2n)i} \zeta^i.$$

We impose the reality condition $\bar\eta_{(2n)} = (-1)^n \zeta^{2n} \eta_{(2n)}(-\frac{1}{\zeta})$ and of course we have $\nabla_\alpha \eta = 0$, $\bar\nabla_{\dot\alpha} \eta = 0$.

We need to choose a measure; any measure that is linearly independent of $\nabla(\zeta)$ is fine; a convenient one is the $N = 1$ measure (because it allows us to immediately recognize the $N = 1$ content of the theory), to write the $N = 2$ supersymmeytric Lagrangian:

$$D_1^2 \bar{D}_1^2 \int_C \frac{d\zeta}{2\pi i \zeta} f(\eta, \zeta).$$

Examples: A_k ALE spaces. We work with $O(2)$ multiplet, and choose the function f to be:

$$f = \frac{1}{\zeta} \sum_{i=0}^{k} (\eta - b_i) \ln(\eta - b_i).$$

with $b_i = c_i + \zeta r_i - \zeta^2 \bar{c}_i$ (where c_i are complex constants and r_i are real constants).

The $O(2)$ multiplet satisfies

$$\begin{aligned}
&D_{\alpha 1} \eta_0 = 0 \quad \eta_1 = \bar\eta_1 \\
&D_{\alpha 2} \eta_0 = -D_{\alpha 1} \eta_1 \qquad \Longrightarrow \ D_1^2 \eta_1 = 0 \\
&D_{\alpha 2} \eta_1 = -D_{\alpha 1} \eta_2 \equiv D_{\alpha 1} \bar\eta_0 \\
&D_{\alpha 2} \bar\eta_0 = 0.
\end{aligned}$$

To do ALF instead of ALE, modify f to:

$$f = \frac{1}{\zeta} \sum_{i=0}^{k} (\eta - b_i) \ln(\eta - b_i) + \frac{1}{2\zeta^2} \eta^2.$$

Other examples: The $O(4)$ multiplet gives rise to D_k type ALE spaces and the Atiyah–Hitchin metric on the moduli space of two centered SU(2) monopoles.

There is nice generalization by Conor Houghton to the general multi-monopole space, involving the $O(2n)$ multiplet (see references at the end of this talk).

Let us go back to the polar multiplets. Formally, we can write the same kind of Lagrangian:

$$D_1^2 \bar{D}_1^2 \int_C \frac{d\zeta}{2\pi i \zeta} f(\Upsilon, \bar{\Upsilon}, \zeta).$$

However, in the general case, I don't know how to do the relevant contour integral. Notice that for constant ζ, $f(\Upsilon, \bar{\Upsilon})$ resembles a Kähler potential; indeed, I conjecture that the corresponding hyperkähler manifold is locally the cotangent bundle of the Kähler manifold with Kähler potential f. This conjecture seems to hold in the cases where the hyperkähler manifold is constructed by a hyperkähler quotient of a vector space, which is this language looks simply like a Kähler quotient of in terms of the Υ variables. More details can be found in the first reference below.l

I am very happy to thank Dave Morrison for providing the TeX source of a preliminary version of these notes.

Suggested references for projective superspace:

A recent summary can be found in: Appendix B of hep-th/0101161.

The Conor Houghton article is: hep-th/9910212.

The Atiyah-Hitchin metric is constructed in: hep-th/9512075.

N=2 super Yang-Mills in projective superspace is constructed in: Commun.Math.Phys.128:191,1990.

The A-series ALE spaces are discussed in our original paper with Hitchin: Commun.Math.Phys.108:535,1

Different projective multiplets were introduced in: Commun.Math.Phys.115:21,1988.