Nuclear Physics B234 (1984) 106–124

CLOSURE OF THE GAUGE ALGEBRA, GENERALIZED LIE EQUATIONS AND FEYNMAN RULES

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Received 15 July 1982
(Final version received 1 August 1983)

A method is given by which an open gauge algebra can always be closed and even made abelian. As a preliminary the generalized Lie equations for the open group are obtained. The Feynman rules for gauge theories with open algebras are derived by reducing the gauge theory to a non-gauge one.

1. Introduction and summary

It is known since the appearance of supergravity that the transformations of invariance in gauge theories do not generally form a Lie group. Moreover, the gauge algebra is generally open, and this is the source of difficulties. The central problem in extended supergravity is to find auxiliary fields closing the algebra off the mass shell. However there is no regular way to find auxiliary fields, and it is not clear in which cases they can be found. In this situation it may be useful to carry out the general analysis of gauge transformations and clarify the status of open algebras in gauge theories. This is the purpose of the present work.

In the present paper we consider the general theory of gauge fields in the spirit of De Witt [1]. We shall use the condensed notation for the field:

$$g^i, \qquad i = 1,\dots,n.$$

In fact this means that we work with a finite-dimensional model, but the results are equally valid for euclidean field theory. For simplicity we shall also suppose that all g^i are bosons. The generalization to the Bose-Fermi case presents no difficulty.

Let $\mathcal{S}(g)$ be the action of the field g. It is supposed that $\mathcal{S}(g)$ has at least one stationary point g_0 and is regular (infinitely differentiable) in its neighbourhood.

Generally

$$\operatorname{rank} \left. \frac{\partial^2 S(g)}{\partial g^i \partial g^k} \right|_{g=g_0} = n - m, \qquad 0 \leqslant m < n, \tag{1}$$

and we suppose* that m linearly independent Noether identities hold in a neighbourhood of the stationary point:

$$\frac{\partial S(g)}{\partial g^i} R^i_\alpha(g) = 0, \qquad \alpha = 1, \dots, m, \tag{2}$$

$$\operatorname{rank} R^i_\alpha|_{g=g_0} = m, \tag{3}$$

where R^i_α are some regular functions.

Our consideration will be purely local. The conclusions will generally be true only in some (finite) neighbourhood of the stationary point. This is sufficient if in quantum theory we confine ourselves to the quasiclassical (loop) expansion.

Under the above conditions one can prove [2, 3], that the Lie bracket of vectors R^i_α is generally of the following form:

$$\frac{\partial R^i_\alpha}{\partial g^k} R^k_\beta - \frac{\partial R^i_\beta}{\partial g^k} R^k_\alpha = T^\gamma_{\beta\alpha} R^i_\gamma + E^{ik}_{\beta\alpha} \frac{\partial S(g)}{\partial g^k}, \tag{4}$$

where $T^\gamma_{\beta\alpha}$ and $E^{ik}_{\beta\alpha}$ are some regular functions antisymmetric in α, β and i, k. These are the commutation relations of the open gauge algebra. Only if $E = 0$ and $T =$ const, has one the Lie algebra. The case $E = 0$, $T \neq$ const was analyzed in ref. [4] at both algebraic and group levels. Eq. (4) is only the lowest-order relation of the open algebra. All algebraic consequences of (4) were derived in refs. [2, 3], and in [3] the generating equation was found, containing all structure relations of the open algebra.

The vectors R^i_α may be interpreted as generators of infinitesimal gauge transformations

$$\delta g^i = R^i_\alpha \delta\theta^\alpha, \qquad \delta S(g) = 0,$$

with parameters θ^α, leaving the action invariant. However the question arises whether finite gauge transformations exist, because if the algebra is open, the Lie equations are non-integrable.

* There can be two reasons for the degeneracy of the matrix (1). One is the presence of Noether identities, and another one is the non-linearity of $S(g)$ at the stationary point. An example of the latter possibility is $S(g) = g^4$. However a field theory of the latter type would admit neither a particle interpretation nor a quasiclassical expansion. By postulating (3) we exclude such theories.

Besides that one may notice that generators R^i_α are defined only by Noether identities (2), (3), and this definition admits a considerable arbitrariness. Any new generators of the form

$$\mathcal{R}^i_\alpha = \Lambda^\beta_\alpha\left(R^i_\beta + K^{im}_\beta \frac{\partial \mathcal{S}(g)}{\partial g^m} \right), \tag{5}$$

make up an equivalent basis if $\Lambda^\beta_\alpha(g)$ is regular and invertible, and $K^{im}_\beta(g)$ is regular and antisymmetric in i, m. Then, perhaps the non-closure of the algebra is the effect of the unsuccessful choice of the basis of generators. In other words, is the open algebra an independent mathematical object?

The present paper contains the answers to the above questions. They are the following. The finite gauge transformations always exist. However they satisfy generally not the Lie equations, but some generalized equations given below. The generalized Lie equations form an infinite sequence which reminds one of the infinite sequence of structure relations of the gauge algebra. Using the formalism of antifields and antibrackets we construct the generating function for the open group just as it was done in ref. [3] for the algebra.

Further, the open algebra is not an independent mathematical object. The function K in eq. (5) can always be chosen in such a way that the new generators will form a closed algebra. Moreover, after the algebra is closed, the matrix Λ in eq. (5) can always be chosen in such a way, that the new generators will form the abelian algebra. Of course, such Λ will be field dependent, so there is no contradiction with the theory of Lie groups. Furthermore, such a reparametrization of the field g can always be found, so that the action will not depend on m field variables, and the gauge theory will be reduced to a non-gauge one. We find explicitly the exact solution for Λ and the approximate solution for K.

Thus any gauge theory is abelian. There exist only open or closed, abelian or non-abelian, bases of generators. However additional requirements of locality and relativistic covariance in field theory, condition the existence of distinguished parametrizations and distinguished bases of generators which are generally non-abelian and open. For this reason the working technique remains that of ref. [3]. We show how the quantization rules of ref. [3] can be derived by reducing the gauge theory to a non-gauge one.

2. Generalized Lie equations

Let us consider the following initial-value problem for the ordinary differential equation:

$$\frac{\mathrm{d}g^i}{\mathrm{d}x} = R^i_\alpha(g)\theta^\alpha, \qquad g^i|_{x=0} = \varphi^i, \tag{6}$$

where $\theta^\alpha \neq 0$ are some parameters, and φ^i are some initial data. Eq. (6) is evidently integrable at least locally. Let us put $x = 1$ (or any other fixed value) in the solution.

Then (6) defines a function

$$g^i = g^i(\varphi, x\theta)|_{x=1}, \tag{7}$$

at least for sufficiently small θ, One can show, using the method* of ref. [4], that the function (7) considered as a function of θ satisfies also the following equation in partial derivatives:

$$-\frac{\partial g^m}{\partial \theta^\beta} + \Lambda^\alpha_\beta \left(R^m_\alpha(g) + K^{mn}_\alpha \frac{\partial S(g)}{\partial g^n} \right) = 0, \tag{8}$$

$$g^m|_{\theta=0} = \varphi^m. \tag{9}$$

The coefficients of this equation:

$$\Lambda = \Lambda(\varphi, \theta, x)|_{x=1}, \qquad K = K(\varphi, \theta, x)|_{x=1}, \tag{10}$$

are the solutions of two other problems for ordinary differential equations:

$$\frac{\mathrm{d}\Lambda^\alpha_\beta}{\mathrm{d}x} = \delta^\alpha_\beta + T^\alpha_{\mu\nu}(g)\theta^\nu \Lambda^\mu_\beta, \qquad \Lambda^\alpha_\beta|_{x=0} = 0, \tag{11}$$

$$\frac{\mathrm{d}K^{mn}_\beta}{\mathrm{d}x} = -\Lambda^{-1\gamma}_\beta K^{mn}_\gamma + \left[T^\gamma_{\alpha\beta}(g) K^{mn}_\gamma + \frac{\partial R^m_\alpha}{\partial g^i} K^{in}_\beta - \frac{\partial R^n_\alpha}{\partial g^i} K^{im}_\beta + E^{mn}_{\beta\alpha}(g) \right] \theta^a,$$

$$K^{mn}_\beta|_{x=0} = 0. \tag{12}$$

Here $T(g)$ and $E(g)$ are functions entering the commutation relations (4) of the open algebra, and g is the solution of (6).

The same method can be used to show that the coefficients (10) considered as functions of θ satisfy the following equations in partial derivatives:

$$\frac{\partial \Lambda^\alpha_\beta}{\partial \theta^\delta} - \frac{\partial \Lambda^\alpha_\delta}{\partial \theta^\beta} + T^\alpha_{\mu\nu}(g)\Lambda^\mu_\delta \Lambda^\nu_\beta = -\Lambda^\mu_\delta \Lambda^\nu_\beta H^{\alpha i}_{\mu\nu} \frac{\partial S(g)}{\partial g^i}, \tag{13}$$

$$\frac{\partial K^{mn}_\beta}{\partial \theta^\gamma} \Lambda^{-1\gamma}_\delta - \frac{\partial K^{mn}_\delta}{\partial \theta^\gamma} \Lambda^{-1\gamma}_\beta - E^{mn}_{\beta\delta}(g) + K^{mn}_\gamma \left[T^\gamma_{\beta\delta}(g) + H^{\gamma k}_{\beta\delta} \frac{\partial S(g)}{\partial g^k} \right]$$

$$+ \frac{\partial^2 S(g)}{\partial g^i \partial g^k} \left(K^{mi}_\beta K^{kn}_\delta - K^{mi}_\delta K^{kn}_\beta \right) + \left\{ \frac{\partial R^m_\beta}{\partial g^i} K^{in}_\delta - \frac{\partial R^m_\delta}{\partial g^i} K^{in}_\beta + R^m_\alpha H^{\alpha n}_{\beta\delta} \right\}$$

$$-\{ m \leftrightarrow n \} = \tfrac{3}{2} \tilde{L}^{mnk}_{\beta\delta} \frac{\partial S(g)}{\partial g^k}, \tag{14}$$

* One applies $\partial/\partial\theta$ to eq. (6) and shows that the left-hand side of (8) satisfies the homogeneous differential equation in x with zero initial data if the coefficients Λ and K satisfy eqs. (11), (12).

where the new coefficients: $H^{\alpha i}_{\beta\delta}$ and $\tilde{L}^{mnk}_{\beta\delta}$ are the solutions of certain initial-value problems for ordinary differential equations★. Again one can show that $H^{\alpha i}_{\beta\delta}$ and $\tilde{L}^{mnk}_{\beta\delta}$ as functions of θ satisfy some equations in partial derivatives in which next-order new coefficients arise, and so on.

An infinite sequence of equations in θ arises, of which eqs. (8), (13) and (14) are the lowest-order ones. This sequence defines the open group. Each equation of this sequence is formally the integrability condition for the previous equation, but the integrability of the whole sequence is guaranteed by the above construction.

Eq. (8) is the generalized Lie equation. Eq. (13) is the generalized Maurer-Cartan equation. Eq. (14) and all higher-order equations are the new ones. The solution of generalized Lie equations is the infinite sequence of structure functions on the group space:

$$g^m, \Lambda^\alpha_\beta, K^{mn}_\beta, H^{\alpha i}_{\beta\delta}, \tilde{L}^{mnk}_{\beta\delta}, \text{etc.} \tag{15}$$

The first of them defines the finite gauge transformations for any gauge theory. The properties

$$\operatorname{rank}\frac{\partial g^i}{\partial \varphi^k} = n, \qquad \operatorname{rank}\frac{\partial g^i}{\partial \theta^\alpha} = m, \tag{16}$$

of the solution are also guaranteed. In particular, the last equality of (16) follows from (3) and (8) if the initial data (9) belong to a sufficiently small neighbourhood of the stationary point.

Note, that it is in fact sufficient to solve consecutively the ordinary differential equations (6), (11), (12), etc. to find the structure functions (15). However equations in partial derivatives are covariant with respect to reparametrizations of the group space, while the ordinary differential equations correspond to the particular (canonical) choice of the group parametrization:

$$\theta^\beta \Lambda^\alpha_\beta = x\theta^\alpha, \qquad \theta^\beta K^{mn}_\beta = 0, \qquad \text{etc.} \tag{17}$$

3. Abelization of gauge theory

First of all we note that in the framework of the present local consideration there is no Gribov problem [5]. Admissible gauge conditions always exist locally. Functions $X^\alpha(g)$ are admissible gauge conditions if they are regular and

$$\det Q^\alpha_\beta|_{g=g_0} \neq 0, \qquad Q^\alpha_\beta \equiv \frac{\partial X^\alpha}{\partial g^i} R^i_\beta. \tag{18}$$

★ Both $H^{\alpha i}_{\beta\delta}$ and $\tilde{L}^{mnk}_{\beta\delta}$ are antisymmetric in β, δ, and $\tilde{L}^{mnk}_{\beta\delta}$ possesses the cyclic antisymmetry in m, n, k

The existence of such $X^\alpha(g)$ in a neighbourhood of the stationary point is guaranteed by eq. (3).

Let us choose any admissible set of gauge conditions and impose it upon the initial data (9) to the Lie equation (8):

$$X^\mu(\varphi) = 0. \tag{19}$$

Since, owing to (18),

$$\operatorname{rank} \frac{\partial X^\mu}{\partial \varphi^i} = m, \tag{20}$$

eq. (19) defines some $(n-m)$-dimensional surface:

$$\varphi^i = F^i(\xi^A), \qquad X^\mu(F(\xi)) \equiv 0, \qquad \operatorname{rank} \frac{\partial F^i}{\partial \xi^A} = n - m, \tag{21}$$

where ξ^A, $A = 1, \dots, (n-m)$ are the parameters on this surface. Then the solution (7) of the Lie equation takes the form

$$g^i = g^i(F(\xi), \theta), \tag{22}$$

and ξ^A play the role of independent initial data.

Let us regard eq. (22) as the equation defining a reparametrization of the field g:

$$g^i \to \bar{g}^i = \{\xi^A, \theta^\mu\}. \tag{23}$$

One can prove using eqs. (18)–(22) that this reparametrization is regular and invertible. Therefore eq. (22) can be solved with respect to ξ and θ:

$$\xi^A = \xi^A(g), \qquad \theta^\mu = \theta^\mu(g). \tag{24}$$

The properties of these functions follow from the relations for the jacobian matrix:

$$\text{(a)} \quad \frac{\partial g^i}{\partial \theta^\nu} \frac{\partial \theta^\mu}{\partial g^i} = \delta^\mu_\nu, \qquad \text{(b)} \quad \frac{\partial g^i}{\partial \theta^\mu} \frac{\partial \xi_A}{\partial g^i} = 0. \tag{25}$$

Eq. (25a) shows that functions $\theta^\mu(g)$ behave like abelian fields: they experience constant shifts under gauge transformations. Eq. (25b) shows that functions $\xi^A(g)$ are invariants of the gauge group. One can easily prove that these invariants are functionally independent, and that any invariant is a function of ξ^A. There are exactly $(n-m)$ invariants in any gauge theory.*

* The method of constructing gauge invariants with the aid of the Lie equation was suggested in ref. [6].

Since the action of a theory is gauge invariant, it can only be a function of $(n-m)$ invariants ξ^A:

$$\mathbb{S}(g)=\mathbb{S}(F(\xi))\equiv\bar{\mathbb{S}}(\xi). \tag{26}$$

This means that in the parametrization $\bar{g}$ the action does not depend on m field variables θ^μ and becomes a non-gauge action.

In the parametrization $\bar{g}$ the Noether identities take the form:

$$\bar{\mathfrak{R}}^i{}_\alpha\frac{\partial\bar{\mathbb{S}}}{\partial\bar{g}^i}=\frac{\partial}{\partial\theta^\alpha}\bar{\mathbb{S}}(\xi)=0, \qquad \bar{\mathfrak{R}}^i{}_\alpha=\delta^i_\alpha, \tag{27}$$

and the corresponding generators $\bar{\mathfrak{R}}$ form the abelian algebra. Since the Lie bracket of vector fields is a vector, the abelian generators in the initial parametrization g can be obtained with the aid of the vector transformation rule:

$$\mathfrak{R}^i{}_\alpha(g)=\frac{\partial g^i}{\partial\bar{g}^k}\bar{\mathfrak{R}}^k{}_\alpha=\frac{\partial}{\partial\theta^\alpha}g^i(F(\xi),\theta). \tag{28}$$

The right-hand side here should be expressed through g according to (23). This finishes the construction of the abelian basis of generators for any gauge theory.

The comparison of (28) with (8) leads to the representation (5) for the abelian generator:

$$\text{(a)}\quad \mathfrak{R}^i{}_\alpha=\Lambda^\beta_\alpha(g)r^i_\beta, \qquad \text{(b)}\quad r^i_\beta=R^i_\beta+K^{im}_\beta(g)\frac{\partial\mathbb{S}(g)}{\partial g^m}. \tag{29}$$

One finds that the functions $\Lambda(g)$ and $K(g)$, entering this representation, are the lowest-order structure functions (10) of the open group, converted into the functions of g by the imposition of conditions (19):

$$\Lambda(g)\equiv\Lambda(\varphi,\theta)|_{X(\varphi)=0}=\Lambda(F(\xi(g)),\theta(g)),$$

$$K(g)\equiv K(\varphi,\theta)|_{X(\varphi)=0}=K(F(\xi(g)),\theta(g)). \tag{30}$$

In the same way the imposition of conditions (19) converts all the rest structure functions (15) on the group space into the functions of g. The structure functions as functions of g satisfy differential equations which can be obtained from the group equations (13), (14), etc. by the replacement:

$$\frac{\partial}{\partial\theta^\beta}=\Lambda^\alpha_\beta\left(R^i_\alpha+K^{im}_\alpha\frac{\partial\mathbb{S}}{\partial g^m}\right)\frac{\partial}{\partial g^i}. \tag{31}$$

Eq. (14) for $K(g)$ can be used to verify directly that the generators r^i_α defined in eq. (29b) form the closed algebra:

$$\frac{\partial r^i_\alpha}{\partial g^k} r^k_\beta - \frac{\partial r^i_\beta}{\partial g^k} r^k_\alpha = \tau^\gamma_{\beta\alpha}(g) r^i_\gamma , \tag{32}$$

with

$$\tau^\gamma_{\beta\alpha} = T^\gamma_{\beta\alpha}(g) + H^{\gamma m}_{\beta\alpha}(g) \frac{\partial \mathfrak{S}(g)}{\partial g^m} . \tag{33}$$

In terms of r and τ eq. (13) becomes the usual Maurer-Cartan equation. As an equation for $\Lambda(g)$ it guarantees the commutativity of $\mathfrak{R}'^i_\alpha$.

Given the generators r^i_α of the closed algebra, the Maurer-Cartan equation for $\Lambda(g)$ admits the exact solution:

$$\Lambda^{-1\alpha}_\beta(g) = \frac{\partial X^\alpha(g)}{\partial g^m} r^m_\beta(g) . \tag{34}$$

Here $X^\alpha(g)$ may be any set of admissible gauge conditions; Λ^α_β is the ghost propagator in an arbitrary gauge.

The solution for $K(g)$ can be found approximately according to the following scheme. One first solves the ordinary differential equations (6), (11) and (12). Next one converts the obtained functions on the group space into the functions of g. This does not require the explicit introduction of variables ξ and can be done as follows. Having the functions (7) and (10), one solves eq. (7) with respect to φ:

$$\varphi^i = \varphi^i(g, x\theta) , \tag{35}$$

and inserts the solution into eq. (19):

$$X^\mu(\varphi(g, x\theta)) = 0 . \tag{36}$$

Solving of (36) with respect to $x\theta$ gives the function $\theta(g)$. The insertion of $\theta(g)$ into (35) gives $\varphi(g)$. The insertion of $\theta(g)$ and $\varphi(g)$ into (10) gives $\Lambda(g)$ and $K(g)$. All these operations can be done approximately by the expansion in powers of x.

The above approximate procedure will be considerably improved if we replace the basis R^i_α used in the initial equation (6) by the modified basis:

$$R'^i_\alpha = R^i_\beta Q^{-1\beta}_\alpha , \tag{37}$$

with Q defined in (18). Then T and E entering eqs. (11) and (12) will be replaced by

$$T'^\gamma_{\alpha\beta} = 0 , \qquad E'^{ik}_{\alpha\beta} = E^{mn}_{\mu\nu} P^i_m P^k_n Q^{-1\mu}_\alpha Q^{-1\nu}_\beta , \tag{38}$$

where

$$P^i_m = \delta^i_m - R^i_\gamma Q^{-1\gamma}_\sigma \frac{\partial X^\sigma}{\partial g^m}\,.$$

Let Λ' and K' be the solutions of eqs. (11) and (12) in the modified basis. Then the generators r^i_α which form the closed algebra can be found as

$$r^i_\alpha = R^i_\alpha + Q^\beta_\alpha K'^{im}_\beta \frac{\partial \mathbb{S}}{\partial g^m}\,, \tag{39}$$

where $K' = K'(g)$. The advantage of the modified scheme is that, firstly, we obtain the exact solution of eq. (36):

$$x\theta^\mu = X^\mu(g)\,. \tag{40}$$

Secondly, the solution for Λ' is trivial:

$$\Lambda'^\alpha_\beta = x\delta^\alpha_\beta\,, \tag{41}$$

and thirdly, K' possesses the property:

$$\frac{\partial X^\mu(g)}{\partial g^i} K'^{im}_\alpha(g) = 0\,. \tag{42}$$

As a result we obtain the explicit solution for the closed algebra (32) as an expansion in powers of an arbitrary gauge $X^\mu(g)$:

$$r^i_\mu = R^i_\mu + \tfrac{1}{2} X^\alpha Q^{-1\beta}_\alpha P^i_m E^{mn}_{\mu\beta} \frac{\partial \mathbb{S}}{\partial g^n} + \mathrm{O}(X^2)\,, \tag{43}$$

$$\tau^\gamma_{\mu\nu} = T^\gamma_{\mu\nu} + Q^{-1\gamma}_\beta \frac{\partial X^\beta}{\partial g^p} E^{pn}_{\mu\nu} \frac{\partial \mathbb{S}}{\partial g^n} + \mathrm{O}(X)\,. \tag{44}$$

Finally, using (39) and (42) in (34) we obtain the exact solution for Λ:

$$\Lambda^\alpha_\beta = Q^{-1\alpha}_\beta\,. \tag{45}$$

Thus any gauge theory can be abelized and reduced to a non-gauge one. An arbitrary set of gauge conditions enters the procedure of abelization. The transition to another set is equivalent to the transition to another basis of invariants ξ i.e. to a reparametrization of the non-gauge action $\bar{\mathbb{S}}(\xi)$.

4. Canonical generator of the open group

In ref. [3] a phase space of fields Φ^A and antifields Φ^*_A was introduced*,

$$\varepsilon(\Phi^*_A) = \varepsilon(\Phi^A) + 1,$$

with the following operation called "antibrackets":

$$(X, Y) = \frac{\partial_r X}{\partial \Phi^A} \frac{\partial_\ell Y}{\partial \Phi^*_A} - \frac{\partial_r X}{\partial \Phi^*_A} \frac{\partial_\ell Y}{\partial \Phi^A}.$$

It was shown that all relations of an arbitrary open algebra are contained in the master equation:

$$(S, S) = 0. \tag{46}$$

For the description of the gauge algebra the content of Φ should be

$$g^i, C^\alpha \subset \Phi^A, \qquad g^*_i, C^*_\alpha \subset \Phi^*_A, \tag{47}$$

where g^i is the initial gauge field and C^α is an auxiliary fermionic field (ghost); g^*_i and C^*_α are corresponding antifields. It is convenient to introduce a notion of the ghost number and ascribe the following values of this number to auxiliary fields:

$$\mathrm{gh}(C^\alpha) = 1, \qquad \mathrm{gh}(g^*_i) = -1, \qquad \mathrm{gh}(C^*_\alpha) = -2. \tag{48}$$

The solution of (46) with the condition $\varepsilon(S) = 0$, $\mathrm{gh}(S) = 0$, and boundary conditions:

$$\text{(a)} \quad S|_{C=0} = \mathcal{S}(g), \qquad \text{(b)} \quad \left. \frac{\partial_\ell}{\partial g^*_i} \frac{\partial_r S}{\partial C^\alpha} \right|_{C=0} = R^i_\alpha(g), \tag{49}$$

is the generating function for structure coefficients of the gauge algebra:

$$S(\Phi, \Phi^*) = \mathcal{S}(g) + g^*_i R^i_\alpha C^\alpha + \tfrac{1}{2} C^*_\gamma T^\gamma_{\alpha\beta} C^\alpha C^\beta$$
$$+ \tfrac{1}{4} g^*_n g^*_m E^{mn}_{\alpha\beta} C^\alpha C^\beta + \mathrm{O}(C^3). \tag{50}$$

The relations which the master equation imposes upon the coefficients of the expansion (50) are the relations (2), (4) etc. of the open algebra [3].

Let us consider an arbitrary finite canonical transformation in the space of fields and antifields:

$$\Phi' = \Phi - \frac{\partial_\ell F}{\partial \Phi'^*}, \qquad \Phi^* = \Phi'^* - \frac{\partial_r F}{\partial \Phi}, \tag{51}$$

* $\varepsilon(X)$ denotes the Grassman parity of X; ∂_r and ∂_ℓ are right and left derivatives.

where

$$F = F(\Phi, \Phi'^*), \qquad \varepsilon(F) = 1, \tag{52}$$

is some fermionic generator. Any canonical transformation preserves the form of the master equation and the Grassman parity of the solution. If we require that

$$\mathrm{gh}(F) = -1, \qquad F|_{C=0} = 0, \tag{53}$$

then the ghost number of S and the boundary condition (49a) will also be preserved. However the boundary condition (49b) will change, and this change will correspond to a general change (5) of the basis of generators. Therefore under the conditions (53) a canonical transformation of S is equivalent to the transition to another basis of the gauge algebra.

We know already that there always exists a basis in which the algebra is closed. In the closed basis (32) the solution of the master equation is linear in antifields:

$$S_0(\Phi, \Phi^*) = \mathbb{S}(g) + g_i^* r_\alpha^i C^\alpha + \tfrac{1}{2} C_\gamma^* \tau_{\alpha\beta}^\gamma C^\alpha C^\beta . \tag{54}$$

We may now look for a canonical transformation from the closed basis (54) to an arbitrary open basis (50):

$$S(\Phi, \Phi^*) = S_0(\Phi', \Phi'^*) . \tag{55}$$

Eq. (55) may be considered as an equation for the generator of such a canonical transformation. The solution for the generator can be expanded in powers of C:

$$F(\Phi, \Phi'^*) = \tfrac{1}{2} g_n'^* g_m'^* K_\beta^{mn}(g) C^\beta + \tfrac{1}{2} g_m'^* C_\alpha'^* H_{\beta\gamma}^{\alpha m}(g) C^\beta C^\gamma$$

$$+ \tfrac{1}{8} g_m'^* g_n'^* g_r'^* L_{\beta\gamma}^{rnm}(g) C^\beta C^\gamma + \mathrm{O}(C^3) . \tag{56}$$

It turns out that the coefficients of this expansion are just the structure functions (15) of the open group in the variables $g^\star$. The relations which eq. (55) imposes upon the coefficients of the expansion (56) are the generalized Lie equations with the replacement (31). The first of these relations is (29b). The second one is (33). The

$\star$ Only $L_{\beta\delta}^{mnr}$ is shifted with respect to $\tilde{L}_{\beta\delta}^{mnr}$ by the quantity:

$$\tilde{L}_{\beta\delta}^{mnr} = L_{\beta\delta}^{mnr} + \left[\left(K_\gamma^{mn} H_{\beta\delta}^{\gamma r} + \frac{\partial K_\delta^{mn}}{\partial g^t} K_\beta^{rt} - \frac{\partial K_\beta^{mn}}{\partial g^t} K_\delta^{rt} \right) + \text{cycl. perm. } (m, n, r) \right] ,$$

but the cyclic antisymmetry is preserved.

third one is eq. (14) for $K(g)$, and so on. The fermionic generator (56) is the generating function of the open group.

We shall need below some general properties of the jacobian of a canonical transformation in the space of fields and antifields:

$$J = \mathrm{Ber}\frac{\partial(\Phi, \Phi^*)}{\partial(\Phi', \Phi'^*)}. \tag{57}$$

As explained in ref. [3] this jacobian is non-trivial.

Let us consider the operator*

$$\Delta = \frac{\partial_r}{\partial \Phi^A}\frac{\partial_r}{\partial \Phi^*_A}(-1)^{\varepsilon_A+1}, \qquad \varepsilon_A \equiv \varepsilon(\Phi^A). \tag{58}$$

It possesses the properties:

$$\Delta^2 \equiv 0, \qquad \varepsilon(\Delta) = 1,$$
$$\Delta(X, Y) = (X, \Delta Y) - (-1)^{\varepsilon_Y}(\Delta X, Y). \tag{59}$$

The operator Δ is non-covariant under canonical transformations and transforms as follows:

$$\Delta X = \Delta' X + \tfrac{1}{2}(X, \ln J). \tag{60}$$

Using the properties of Δ one can prove the following property of J:

$$\Delta' \ln J = -\tfrac{1}{4}(\ln J, \ln J), \tag{61}$$

or

$$\Delta' \sqrt{J} = 0.$$

Another property of a canonical transformation is:

$$\mathrm{Ber}\frac{\partial \Phi}{\partial \Phi'} = \mathrm{Ber}\frac{\partial \Phi^*}{\partial \Phi'^*} = J^{1/2}. \tag{62}$$

One more result we shall need is the following. Let ζ_A be functions on the phase space, such that

$$(\zeta_A, \zeta_B) = 0, \qquad \mathrm{Ber}\frac{\partial \zeta}{\partial \Phi^*} \neq 0. \tag{63}$$

* This operator coincides with Δ of ref. [3] when applied to bosons.

Then the equation $\zeta_A = 0$ is equivalent to

$$\Phi^*_A = \frac{\partial \Psi(\Phi)}{\partial \Phi^A},$$

where Ψ is some fermion. If one subjects such ζ_A to a canonical transformation, then

$$\operatorname{Ber}\frac{\partial \zeta}{\partial \Phi'^*} = J^{1/2}\operatorname{Ber}\frac{\partial \zeta}{\partial \Phi^*}. \tag{64}$$

The latter equality generalizes (62).

5. Derivation of Feynman rules

Since any gauge theory can be reduced to a non-gauge one, the Feynman rules for gauge theories with open algebras [2, 3] should be derivable from Feynman rules for non-gauge theories. In terms of the non-gauge action (26) the functional integral of a theory is of the form:

$$Z = \int \mathrm{d}\xi\, M(\xi) \exp\left\{\frac{i}{\hbar}\mathbb{S}(\xi)\right\}, \tag{65}$$

where $M(\xi)$ is an unknown measure which can be found only by the canonical quantization*. It is only known that this measure is gauge invariant: $M(\xi) = M^{\mathrm{inv}}(g)$.

Let $\Psi^\alpha(g)$ be any admissible set of gauge conditions. Then in the parametrization ξ, θ the equations $\Psi^\alpha = 0$ are solvable with respect to θ^α. Let $\theta^\alpha = f^\alpha(\xi)$ be the solution. Then (65) can be transformed as follows:

$$\begin{aligned} Z &= \int \mathrm{d}\xi\, \mathrm{d}\theta\, \delta(\theta - f(\xi)) M(\xi) \exp\left\{\frac{i}{\hbar}\mathbb{S}(\xi)\right\} \\ &= \int \mathrm{d}g\, \delta(\Psi(g)) \exp\left\{\frac{i}{\hbar}\mathbb{S}(g)\right\} M^{\mathrm{inv}}(g) \\ &\quad \times \det\frac{\partial \Psi}{\partial \theta} \det\frac{\partial(\xi, \theta)}{\partial g}. \end{aligned} \tag{66}$$

The Lie equation and equations of the replacement (23) give:

$$\det\frac{\partial \Psi^\alpha}{\partial \theta^\mu} = \det\frac{\partial \Psi^\alpha}{\partial g^i}\mathfrak{R}^i_\mu = \left(\det\frac{\partial \Psi^\alpha}{\partial g^i} r^i_\mu\right)(\det \Lambda), \tag{67}$$

$$\mathfrak{R}^i_\mu \frac{\partial}{\partial g^i} \ln\det\frac{\partial g}{\partial(\xi, \theta)} = \frac{\partial}{\partial g^i}\mathfrak{R}^i_\mu. \tag{68}$$

* The number of independent variables ξ is $(n - m)$, while the number of independent degrees of freedom in a local gauge theory is $(n - 2m)$, (counting functions of time in both cases). Therefore the measure $M(\xi)$ compensating this difference dynamically should be non-local. In simple theories this non-local measure is constant and can be omitted.

Using the Maurer-Cartan equation in (68) one finds:

$$\ln\det\frac{\partial(\xi,\theta)}{\partial g} = -\ln\det\Lambda - \mathcal{M}(g), \tag{69}$$

where

$$r^i_\mu\frac{\partial\mathcal{M}(g)}{\partial g^i} = \frac{\partial r^i_\mu}{\partial g^i} + \tau^\alpha_{\mu\alpha}. \tag{70}$$

As a result the functional integral (66) takes the form:

$$Z=\int \mathrm{d}g\,\mathrm{d}\pi\,\mathrm{d}\bar{C}\,\mathrm{d}C\exp\left\{\frac{i}{\hbar}\left[\mathcal{S}(g)+\pi_\alpha\Psi^\alpha(g)+\bar{C}_\alpha\frac{\partial\Psi^\alpha}{\partial g^i}r^i_\mu C^\mu + i\hbar\mathcal{M}(g)\right]\right\}, \tag{71}$$

where auxiliary integration variables: bosons π_α and fermions $\bar{C}_\alpha$, C^μ are introduced, and the unknown $M^{\mathrm{inv}}(g)$ is absorbed by the arbitrariness in the definition of $\mathcal{M}(g)$ by eq. (70).

The expression (71) reproduces the Faddeev-Popov quantization rules for closed algebras, corrected by the presence of the measure. However the generators r^i_μ of the closed basis and the measure $\mathcal{M}$, entering the expression (71), are generally non-local. Further transformations are needed to restore in (71) the initially given open basis which we suppose to be local. For this purpose we enlarge the phase space (47) by including in Φ all integration variables of (71):

$$g^i, C^\alpha, \bar{C}_\beta, \pi_\mu \subset \Phi, \qquad g^*_i, C^*_\alpha, \bar{C}^{*\beta}, \pi^{*\mu}\subset\Phi^*. \tag{72}$$

In the enlarged phase space we define the gauge fermion

$$\Psi(\Phi) = \bar{C}_\alpha\Psi^\alpha, \tag{73}$$

and the action

$$W_0(\Phi,\Phi^*) = S_0(\Phi,\Phi^*) + \bar{C}^{*\alpha}\pi_\alpha + i\hbar\mathcal{M}(g), \tag{74}$$

where $S_0(\Phi, \Phi^*)$ is given by (54). Then the functional integral (71) can be identically rewritten as

$$Z=\int \mathrm{d}\Phi\,\mathrm{d}\Phi^*\,\delta\left(\Phi^* - \frac{\partial\Psi}{\partial\Phi}\right)\exp\left\{\frac{i}{\hbar}W_0(\Phi,\Phi^*)\right\}, \tag{75}$$

and $W_0(\Phi, \Phi^*)$ satisfies the equation

$$\tfrac{1}{2}(W_0, W_0) = i\hbar \Delta W_0, \tag{76}$$

in consequence of the master equation for S_0 and eq. (70) for $\mathfrak{M}(g)$.

It remains for us to make the canonical transformation of integration variables in (75), converting the closed basis S_0 into the initial open basis S. As shown above, such a canonical transformation exists, and its generator is that of the open group. Since functions

$$\zeta_A = \Phi^*_A - \frac{\partial \Psi}{\partial \Phi^A}, \tag{77}$$

possess the properties (63), and these properties are canonically invariant, we find that under a canonical transformation

$$\delta\left(\Phi^* - \frac{\partial \Psi}{\partial \Phi}\right) = J^{-1/2} \delta\left(\Phi'^* - \frac{\partial \Psi'}{\partial \Phi'}\right), \tag{78}$$

where Ψ' is some new gauge fermion, and J is the jacobian (57). As a result the transformed integral (75) takes the form:

$$Z = \int \mathrm{d}\Phi' \, \mathrm{d}\Phi'^* \, \delta\left(\Phi'^* - \frac{\partial \Psi'}{\partial \Phi'}\right) \exp\left\{\frac{i}{\hbar} W(\Phi', \Phi'^*)\right\}, \tag{79}$$

where

$$W(\Phi', \Phi'^*) = W_0(\Phi, \Phi^*) + \frac{\hbar}{i} \ln J^{1/2}. \tag{80}$$

Using the properties (59)–(62) of the operator Δ and the jacobian J, one can show that the above W satisfies the same equation (76) as W_0 does, but in primed variables:

$$\tfrac{1}{2}(W, W)' = i\hbar \Delta' W. \tag{81}$$

Expressions (79) and (81) reproduce the quantization rules of ref. [3]. These rules generalize the rules of ref. [2] and give the form of multi-ghost couplings for the case of any open gauge algebra. The vertices of these couplings are just the structure coefficients of the open algebra. In particular, the four-ghost terms in the action are of the form:

$$E^{ik}_{\alpha\beta}(g) C^\beta C^\alpha \frac{\partial \bar{C}_\mu \Psi^\mu(g)}{\partial g^k} \frac{\partial \bar{C}_\nu \Psi^\nu(g)}{\partial g^i}. \tag{82}$$

For $N=1$ supergravity these terms were first obtained in refs. [7–9]. In the case when auxiliary fields can be introduced to close the algebra one can use the Feynman-DeWitt or Faddeev-Popov quantization rules. In this case only the two-ghost terms are present in the action, but integrating over auxiliary fields one recovers the four-ghost couplings (82) [10–12]. Couplings of more than four ghosts cannot arise in the usual formalism of auxiliary fields. This shows that auxiliary fields cannot be introduced (at least in the usual way) when the *rank of the algebra* is higher than 2 [10], i.e. when $E^{ik}_{\alpha\beta}$ is not the highest-order surviving coefficient of the algebra.

6. Conclusion

The present work gives the method for the construction of group invariants in gauge theories. We have shown that the Noether identities: (2), (3) always guarantee the invariance of the action under an m-parameter group of gauge transformations. The problem is, however, that the "generators" R^i_α, which appear in the Noether identities, may not generate *any* transformations of invariance, and the equations $\delta g^i = R^i_\alpha \delta\theta^\alpha$ may be purely formal (non-integrable). This is the case when R^i_α form an open algebra. In this case the invariants of gauge transformations ($\xi(g)$) are solutions of the equations

$$\frac{\partial\xi(g)}{\partial g^i} r^i_\alpha \equiv \frac{\partial\xi(g)}{\partial g^i}\left(R^i_\alpha + K^{im}_\alpha(g)\frac{\partial \mathbb{S}}{\partial g^m}\right) = 0\,, \tag{83}$$

and not

$$\frac{\partial\xi(g)}{\partial g^i} R^i_\alpha = 0\,. \tag{84}$$

This fact is essential for obtaining the invariants when auxiliary fields and tensor calculus are absent. Eqs. (83) and (84) coincide only for one invariant: the classical action $\mathbb{S}(g)$.

Recently the gauge-independent and parametrization-independent effective action for the mean field was constructed in gauge theories [13, 14]. The methods of the present work were essentially used in this construction.

The main results of the present work were reported at the 2nd Moscow Seminar "Quantum gravity" in October 1981 [15]. Some similar arguments were presented there by Voronov and Tyutin [16, 17].

7. Examples

To illustrate the construction of the closed basis r^i_α we shall consider $N=1$ supergravity. The supersymmetry transformations in the local basis R^i_α are of the following form in this theory [18,19]:

$$\delta\psi_\rho = \mathfrak{D}_\rho \varepsilon, \qquad \delta e^a_\mu = -\tfrac{1}{2} i \bar{\psi}_\mu \gamma^a \varepsilon, \tag{85}$$

where $\mathfrak{D}_\rho$ is the spinor-field covariant derivative with torsion. Eqs. (85) give the explicit form of the condensed equations $\delta g^i = R^i_\alpha \delta\theta^\alpha$ with $\delta\theta^\alpha = \varepsilon(x)$. The transformations (85) generate the open algebra [20]. The structure coefficient $E^{ik}_{\alpha\beta}$ of this algebra can be found in [7]. Below we use the results and conventions of ref. [7].

Eqs. (85) *are not in fact the infinitesimal form of any actual transformations.* The actual transformations of invariance are described by the equations

$$\delta' g^i = r^i_\alpha \delta\theta^\alpha, \tag{86}$$

where r^i_α are the generators in the closed basis. The exact equation for r^i_α and the method for its approximate solution are given above. In the lowest-order approximation one can use eq. (43) generalized to the Bose-Fermi case. Choosing the gauge conditions X, entering eq. (43), as

$$X = \gamma^\lambda \psi_\lambda, \tag{87}$$

we obtain

$$\delta'\psi_\rho = \mathfrak{D}_\rho \varepsilon + \tfrac{1}{32} i \left(\delta^\nu_\rho I - \mathfrak{D}_\rho Q^{-1} \gamma^\nu \right) B_\nu(\varepsilon) + \mathrm{O}(X^2), \tag{88a}$$

$$\delta' e^a_\mu = -\tfrac{1}{2} i \bar{\psi}_\mu \gamma^a \varepsilon - \tfrac{1}{64} \bar{\psi}_\mu \gamma^a Q^{-1} \gamma^\nu B_\nu(\varepsilon) + \mathrm{O}(X^2), \tag{88b}$$

$$\begin{aligned} B_\nu(\varepsilon) = {} & \left(\bar{\varepsilon}\gamma^\alpha Q^{-1} X\right)\left(g_{\nu\mu}\gamma_\alpha + \gamma_\nu\gamma_\alpha\gamma_\mu - \gamma_\mu\gamma_\alpha\gamma_\nu\right) G^\mu \\ & - 4\left(\bar{\varepsilon}\gamma_5 \sigma_{\nu\mu} Q^{-1} X\right)\gamma_5 G^\mu + 4\left(\bar{\varepsilon}\sigma_{\nu\mu} Q^{-1} X\right) G^\mu \\ & + 2 g_{\nu\mu}\left(\bar{\varepsilon}\sigma_{\alpha\beta} Q^{-1} X\right)\sigma^{\alpha\beta} G^\mu. \end{aligned} \tag{89}$$

Here

$$G^\mu = \varepsilon^{\mu\alpha\beta\delta} \gamma_5 \gamma_\alpha \mathfrak{D}_\beta \psi_\delta, \tag{90}$$

and Q is the spinor-field operator defined as

$$Q\varepsilon \equiv \delta(\varepsilon) X = \gamma^\mu \mathfrak{D}_\mu \varepsilon + \tfrac{1}{2} i \gamma^\nu \psi_\mu \left(\bar{\psi}_\nu \gamma^\mu \varepsilon\right). \tag{91}$$

One can verify by direct computation that the commutator $[\delta'(\varepsilon_1), \delta'(\varepsilon_2)]$ of the transformations (88) is closed with accuracy $\mathrm{o}(X)$. Note, however, that the transformations (88) are non-local.

To illustrate the construction of the abelian generators $\mathcal{R}^i_{\ \alpha}$ we may consider the Yang-Mills theory. The local transformations of invariance are of the following form in this theory:

$$\delta A^a_\mu = \nabla^{ab}_\mu \varepsilon^b, \qquad \nabla^{ab}_\mu = \partial_\mu \delta^{ab} + \lambda f^{acb} A^c_\mu, \qquad \lambda = \text{const}, \tag{92}$$

where ε^b are the parameters. The transformations (92) generate the closed non-abelian algebra with structure constants f^{abc}.

If $R^i_{\ \alpha}$ are generators of a closed algebra, then the abelian basis of the generators can always be found as

$$\mathcal{R}^i_{\ \alpha} = R^i_\beta Q^{-1\beta}_\alpha, \tag{93}$$

where $Q^{-1\beta}_\alpha$ is the ghost propagator in any gauge. The transformations

$$\delta' g^i = \mathcal{R}^i_{\ \alpha} \delta\theta^\alpha \tag{94}$$

with the generators (93) are exactly commutative.

Choosing gauge conditions for the Yang-Mills field as either

$$X^a_1 = \partial_\mu A^a_\mu, \tag{95}$$

or

$$X^a_2 = n_\mu A^a_\mu, \qquad n_\mu = \text{const}, \tag{96}$$

we obtain two different abelian bases in the Yang-Mills theory:

$$\delta'_1 A^a_\mu = \nabla^{ac}_\mu Q^{-1cb}_1 \varepsilon^b, \tag{97}$$

$$\delta'_2 A^a_\mu = \nabla^{ac}_\mu Q^{-1cb}_2 \varepsilon^b. \tag{98}$$

Here Q^{ab}_1 and Q^{ab}_2 are operators defined as

$$Q^{ab}_1 = \partial_\mu \nabla^{ab}_\mu, \qquad Q^{ab}_2 = n_\mu \nabla^{ab}_\mu.$$

The commutativity of the transformations (97), as well as (98), can be verified by direct computation. However we notice, that the transformations (97) are non-local, while the transformations (98) are local in the directions orthogonal to n_μ but are not relativistic covariant. A theory, in which one can find a local, covariant and abelian basis of generators, is what is usually called an "abelian theory".

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