Consistency Conditions for Kaluza-Klein Anomalies

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General consistency requirements on anomalies in $D = 2n$ dimensions are derived with topological techniques. It is shown that for a non-Abelian gauge theory the group-theoretic piece of the anomaly is an overall multiplicative factor for any even $D \geq 4$.

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In constructing any theory of particle interactions, one wishes to remove all ambiguities as well as any unnecessary freedom from the Lagrangian. It is well known that it is often just too naive to write down some simple Lagrangians since the theory may lead to inconsistencies. In certain theories, constraints necessary to remove ambiguities arise subtly, though naturally, from requirements of consistency. The most familiar and most relevant examples for elementary-particle physics are gauge theories.

The only known constraint on Yang-Mills theories with fermions in four space-time dimensions is the absence of triangle anomalies. This constraint has proven fruitful in that it appears at least partially to determine the observed spectrum of fermions—quarks and leptons. That is, given the low-energy effective gauge group [SU(3) ⊗ SU(2) ⊗ U(1)] and given the existence of nontrivial fermions one is led to a minimal spectrum coincident with that observed in each generation (e.g., u, d, ν_e, e).

There has been recent interest in extending this study of anomalies to include more space-time dimensions.[1] The motivation for studying higher-dimensional theories arises principally from the desire to include a consistent quantum gravity for which one strong candidate is the ten-dimensional string. (Another candidate theory which incorporates fermions is extended supergravity in four dimensions.) Although in the low-energy, zero-slope limit the string theory approximates a nonrenormalizable theory,[2] the very-high-mass states provide a cutoff which probably renders the theory finite even for $D=10$. Such a cutoff cannot remove anomalies which receive contributions only from zero-mass states.[3]

Here we shall not restrict our analysis to the $D=10$ superstring theory,[4] but will address a more general question of what governs the anomaly in dimension $D=2n$. It will be shown that global differential-geometric considerations can completely determine both the group-theoretic and field-theoretic forms of the anomaly up to a trivial multiplicative constant. Higher-order loop corrections cannot do more than renormalize this constant (if that).

We are interested in the highest Chern class in $D=2n$ dimensions since this is what determines the anomaly. This is because for the surface term on the sphere S^{2n-1}, the only surviving topological invariant is this $2n$th Chern class.[5] For $D=4$, the relevant four-form is[5]

$$d^*J = F \wedge F, \tag{1}$$

$$= d(A\wedge dA + \tfrac{2}{3} A\wedge A\wedge A), \tag{2}$$

where the asterisk denotes the Hodge star and F is the curvature two-form in Yang-Mills theory given by

$$F = dA + A\wedge A. \tag{3}$$

The reason for using an index-free differential-geometric notation should become perfectly clear below when we derive the D-form for $D>4$ dimensions.

Introducing gauge group and Lorentz indices, we have the familiar form for the topological current density,

$$J_\mu{}^a = C\epsilon_{\mu\alpha\beta\gamma} \mathrm{Tr}\,\Lambda^a (A_\alpha \partial_\beta A_\gamma + \tfrac{2}{3} A_\alpha A_\beta A_\gamma) \tag{4}$$

$$= C\,S\,\mathrm{Tr}(\Lambda^a\Lambda^b\Lambda^c)\epsilon_{\mu\alpha\beta\gamma}[A_\alpha{}^b \partial_\beta A_\gamma{}^c + \tfrac{2}{3} i f^{cde} A_\alpha{}^b A_\beta{}^d A_\gamma{}^e]. \tag{5}$$

Here the Λ^a are the generators of the gauge group written in an appropriate basis to be discussed below. The notation $S\mathrm{Tr}$ denotes the totally symmetrized trace

$$S\mathrm{Tr}(\Lambda^{a_1}\Lambda^{a_2}\dots\Lambda^{a_p}) = \frac{1}{p!}\sum_{\substack{\text{perms}\\ \{a_i\}}} \mathrm{Tr}(\Lambda^{a_1}\Lambda^{a_2}\cdots\Lambda^{a_p}), \tag{6}$$

where $i=1,2,\dots,p$. Equation (5) is an important consistency requirement for the axial anomaly in $D=4$ and exhibits the tensorial structure dictated by Lorentz invariance and gauge invariance under a non-Abelian group.[6] In particular, it reveals that the symmetric trace over the group generators factors out and hence provides the well-known condition[7] for absence of triangle anomalies.

In the light of recent interest in Kaluza-Klein theory, we have generalized the above relations to higher even dimensions (odd dimensions have no anomalies though they do have interesting topological subtleties; see, e.g., Deser, Jackiw, and Templeton[8]).

We have obtained the general tensorial structure explicitly for $D=6$, 8, and 10, and, in particular, have derived the factorization property of the group-theoretic piece of the anomaly in these higher dimensions. Consequently, a compact statement of the generalized no-anomaly condition can be presented.

In $D=6$ one has

$$d{*}J = F\wedge F\wedge F, \tag{7}$$

$$= d(A\wedge dA\wedge dA + A\wedge A\wedge A\wedge dA + \tfrac{3}{5}A\wedge A\wedge A\wedge A\wedge A), \tag{8}$$

and converting to normal physics notation gives

$$\partial_\mu J_\mu{}^a = S\mathrm{Tr}(\Lambda^a\Lambda^b\Lambda^c\Lambda^d)\epsilon_{\mu\alpha\beta\gamma\delta\epsilon}\,\partial_\mu[A_\alpha{}^b\partial_\beta A_\gamma{}^c\partial_\delta A_\epsilon{}^d + iA_\alpha{}^bA_\beta{}^kA_\gamma{}^l\partial_\delta A_\epsilon{}^d f^{ckl} - \tfrac{3}{5}A_\alpha{}^bA_\beta{}^kA_\gamma{}^lA_\delta{}^mA_\epsilon{}^n f^{ckl}f^{dmn}]. \tag{9}$$

The case $D=10$ is most relevant for the case of string theory. Here the ten-form corresponding to the fifth Chern class is simply

$$d{*}J = F\wedge F\wedge F\wedge F\wedge F \tag{10}$$

$$\begin{aligned} = d(&A\wedge dA\wedge dA\wedge dA\wedge dA + \tfrac{5}{3}A\wedge A\wedge A\wedge dA\wedge dA\wedge dA + 2A\wedge A\wedge A\wedge A\wedge A\wedge dA\wedge dA \\ &+ \tfrac{10}{7}A\wedge A\wedge A\wedge A\wedge A\wedge A\wedge A\wedge dA + \tfrac{5}{9}A\wedge A\wedge A\wedge A\wedge A\wedge A\wedge A\wedge A\wedge A), \end{aligned} \tag{11}$$

and hence in indicial notation

$$\begin{aligned} \partial_\mu J_\mu{}^a = {}& S\mathrm{Tr}(\Lambda^a\Lambda^b\Lambda^c\Lambda^d\Lambda^e\Lambda^f)\epsilon_{\alpha\beta\gamma\delta\epsilon\kappa\lambda\mu\nu\rho} \\ &\times\partial_\alpha[A_\beta{}^b\partial_\gamma A_\delta{}^c\partial_\epsilon A_\kappa{}^d\partial_\lambda A_\mu{}^e\partial_\nu A_\rho{}^f + \tfrac{5}{3}iA_\beta{}^bA_\gamma{}^iA_\delta{}^j\partial_\epsilon A_\kappa{}^d\partial_\lambda A_\mu{}^e\partial_\nu A_\rho{}^f f^{cij} \\ &\quad - 2A_\beta{}^bA_\gamma{}^iA_\delta{}^jA_\epsilon{}^kA_\kappa{}^l\partial_\lambda A_\mu{}^e\partial_\nu A_\rho{}^f f^{cij}f^{dkl} \\ &\quad - \tfrac{10}{7}iA_\beta{}^bA_\gamma{}^iA_\delta{}^jA_\epsilon{}^kA_\kappa{}^lA_\lambda{}^mA_\mu{}^n\partial_\nu A_\rho{}^f f^{cij}f^{dkl}f^{emn} \\ &\quad + \tfrac{5}{9}A_\beta{}^bA_\gamma{}^iA_\delta{}^jA_\epsilon{}^kA_\kappa{}^lA_\lambda{}^mA_\mu{}^nA_\nu{}^pA_\rho{}^q f^{cij}f^{dkl}f^{emn}f^{fpq}]. \end{aligned} \tag{12}$$

We see that, in general, the $D=10$ anomaly in Eq. (12) is vanishing provided

$$S\mathrm{Tr}(\Lambda^a\Lambda^b\Lambda^c\Lambda^d\Lambda^e\Lambda^f) = 0 \tag{13}$$

for generators in the basis appropriate for the fermion representation.

The derivation of Eq. (11) and hence Eq. (12) in index-free notation is straightforward as the reader may verify. We have obtained the analog of Eq. (12) in $D=8$ (pentagon anomaly) and $D \geqslant 12$ but will not write them explicitly here.[9]

We have derived tractable forms for the symmetrized traces which generalize the well-known $D=4$ formula, and will summarize the results below. We consider fermions in only the fundamental antisymmetric representations $[k]$ of SU(N) since such representations play a special role in model building. Recall that for $D=4$ one obtains

$$S\mathrm{Tr}(\Lambda^a\Lambda^b\Lambda^c) = A_3 S\,\mathrm{Tr}(\lambda^a\lambda^b\lambda^c), \tag{14}$$

where λ^a are generators in the defining representation and

$$A_3 = \binom{N-3}{k-1} - \binom{N-3}{k-2}. \tag{15}$$

For the cases $D=6$ and 10 we find results as follows (details of our derivations will be provided in a longer paper[9] under preparation):

$$S\,\mathrm{Tr}(\Lambda^a\Lambda^b\Lambda^c\Lambda^d) = A_4\, S\,\mathrm{Tr}(\lambda^a\lambda^b\lambda^c\lambda^d) + A_4{}'\, S[\mathrm{Tr}(\lambda^a\lambda^b)\,\mathrm{Tr}(\lambda^c\lambda^d)], \tag{16}$$

$$S\,\mathrm{Tr}(\Lambda^a\Lambda^b\Lambda^c\Lambda^d\Lambda^e\Lambda^f) = A_6\, S\,\mathrm{Tr}(\lambda^a\lambda^b\lambda^c\lambda^d\lambda^e\lambda^f) + A_6{}'\, S[\mathrm{Tr}(\lambda^a\lambda^b)\,\mathrm{Tr}(\lambda^c\lambda^d\lambda^e\lambda^f)] + A_6{}''S[\mathrm{Tr}(\lambda^a\lambda^b)\,\mathrm{Tr}(\lambda^c\lambda^d)\,\mathrm{Tr}(\lambda^e\lambda^f)] + A_6{}'''\, S[\mathrm{Tr}(\lambda^a\lambda^b\lambda^c)\mathrm{Tr}(\lambda^d\lambda^e\lambda^f)], \tag{17}$$

in which

$$A_p(N,k) = \sum_{l=0}^{k} \binom{N}{l} (-1)^{k+l+1}(k-l)^{p-1}, \tag{18}$$

$$A_4{}'(N,k) = 3\binom{N-4}{k-2}, \tag{19}$$

$$A_6{}'(N,k) = 15 A_4(N-2, k-1), \tag{20}$$

$$A_6{}''(N,k) = 15\binom{N-6}{k-3}, \tag{21}$$

$$A_6{}'''(N,k) = \tfrac{2}{3} A_6{}'(N,k) + \tfrac{4}{3} A_6{}''(N,k). \tag{22}$$

Note that $A_p(N,k)$ in Eq. (18) is closely related to the Eulerian numbers of combinatorial analysis.[9] The above formulas are useful for model building and have been used elsewhere.[1,9] Explicit evaluation of one-loop anomalies in higher dimensions is provided in the preceding Letter.[10]

In summary, we have successfully written the consistency conditions satisfied by anomalies for any even D. The tensorial structure is dictated by the geometry and, most important, the no-anomaly condition generalizes from $D=4$ to $D=2n$ where the requirement becomes simply the vanishing of the totally symmetrized trace of $n+1$ group generators written in the fermion representation.

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[2]Note that although a fundamental gauge theory in more than four dimensions is *a priori* inconsistent because of nonrenormalizability, it can provide a sensible effective approximation to a higher-dimensional string theory if the latter is completely free of ultraviolet infinities.

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