Exercise IV.13

"BUSSTEPP Lectures On Supersymmetry"
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Let $W(\Phi)$ be given by

$$W(\Phi) = \mu\Phi^2 + \nu\Phi^3.$$

Determine μ and ν in such a way that the action obtained by adding to the action (59) the superpotential term

$$\int d^2\theta\, W(\Phi) + \int d^2\bar\theta\, \overline{W(\Phi)}$$

and eliminating the auxiliary field via its equation of motion we recover the Wess-Zumino model, under the identification $\phi = \frac{1}{2}[S + iP]$ and $\psi^\alpha = [\chi^\alpha, \bar\chi_{\dot\alpha}]$

Background

Massive interacting Wess-Zumino model:

$$(**)\quad \mathcal{L} = -\frac{1}{2}(\partial S)^2 - \frac{1}{2}(\partial P)^2 - \frac{1}{2}\bar\psi\partial\!\!\!/\psi - \frac{1}{2}mS^2 - \frac{1}{2}mP^2 - \frac{1}{2}m\bar\psi\psi - \lambda\left[\bar\psi[S - P\gamma^5]\psi + \frac{1}{2}\lambda[S^2+P^2]^2 + mS[S^2+P^2]\right];$$

$$\int d^4x\, d^2\theta\, d^2\bar\theta\, \{2\bar\Phi\Phi\} \longmapsto -2\partial_\mu\phi\partial^\mu\bar\phi + 2F\bar F + \frac{i}{2}[\chi\partial\!\!\!/\bar\chi + \bar\chi\bar\partial\!\!\!/\chi] + \underbrace{\frac{1}{2}\partial_\mu[\phi\partial^\mu\bar\phi + \bar\phi\partial^\mu\phi]}_{\text{not relevant}};$$

$(***)$

So, starting from $\Phi = \phi(y) + \theta\chi(y) + \theta^2 F(y)$ we calculate $\int d^2\theta\, \Phi^2$ and $\int d^2\theta\, \Phi^3$. Note, $\int d^2\theta\, \Phi^2(x) =$

$= \int d^2\theta\, e^{-iU}[\phi + \theta\chi + \theta^2 F]^{2 \text{ or } 3}$ since U contains both θ and $\bar\theta$, while $\int d^2\theta$ picks up the component of θ^2, putting all other terms zero $= \int d^2\theta\, [\phi + \theta\chi + \theta^2 F]^{2 \text{ or } 3}$.

$$\Phi^2 = [\phi(y) + \theta\chi + \theta^2 F][\phi + \theta\chi + \theta^2 F] =$$

$$= \phi^2 + 2\phi\cdot\theta\chi + 2\theta^2 F\phi - \tfrac{1}{2}\theta^2\chi^2;$$

$$\int d^2\theta\, \Phi^2 = 2F\phi - \tfrac{1}{2}\chi^2;$$

$$\Phi^3\Big|_{\theta^2} = [\phi + \theta\chi + \theta^2 F][\phi^2 + 2\phi\cdot\theta\chi + \theta^2\{2F\phi - \tfrac{1}{2}\chi^2\}]\Big|_{\theta^2} =$$

$$= \theta^2[2F\phi^2 - \tfrac{1}{2}\phi\chi^2 + F\phi^2 - \phi\cdot\chi^2] = \theta^2\cdot 3[F\phi^2 - \tfrac{1}{2}\phi\chi^2].$$

So we get the following extra terms added to our Lagrangian:

$$\mu[2F\phi - \tfrac{1}{2}\chi^2] + 3v[F\phi^2 - \tfrac{1}{2}\phi\chi^2] +$$
$$+ \mu[2\overline{F\phi} - \tfrac{1}{2}\bar\chi^2] + 3v[\overline{F\phi^2} - \tfrac{1}{2}\bar\phi\bar\chi^2]$$

Now we can instantly check the mass term for Majorana ψ, because χ is not restricted anyhow by the auxiliary field F:

$$-\tfrac{1}{2}m\bar\psi\psi = -\tfrac{1}{2}m\,(\chi_\alpha, -\bar\chi^{\dot\alpha})\begin{bmatrix}\chi^\alpha \\ \bar\chi_{\dot\alpha}\end{bmatrix} = -\tfrac{1}{2}m\chi_\alpha\chi^\alpha + \tfrac{1}{2}m\bar\chi^{\dot\alpha}\bar\chi_{\dot\alpha} =$$

$$= \tfrac{1}{2}m\chi\chi + \tfrac{1}{2}m\bar\chi\bar\chi$$

while we get $-\tfrac{1}{2}\mu\chi^2 - \tfrac{1}{2}\mu\bar\chi^2$; there is a problem here, because we'll later see that μ should actually be equal to m.

Also, we can check for interaction of χ with ϕ (that is, (3)

ψ with S and P). Again, we'll leave it for later to show that

ν must be equal $\frac{4}{3}\lambda$.

$$-\frac{3}{2}\nu\phi\chi^2 = -2\lambda\cdot\frac{1}{2}(S+iP)\chi^2 = -\lambda(S+iP)\chi^2 = $$

$$= \text{back} = -2\lambda\phi\chi^2.$$

Let's check the answer now.

$$-\lambda\bar{\Psi}(S-P\gamma^5)\psi = -\lambda(\chi_\alpha \quad \bar{\chi}^{\dot{\alpha}})\begin{pmatrix} S+iP & \\ & S-iP\end{pmatrix}\begin{pmatrix}\chi^\alpha \\ \bar{\chi}_{\dot{\alpha}}\end{pmatrix} =$$

$$\gamma^5 = \begin{pmatrix} -i & \\ & i\end{pmatrix}$$

$$= -\lambda(\chi_\alpha, \bar{\chi}^{\dot{\alpha}})\begin{pmatrix} 2\phi & \\ & 2\bar{\phi}\end{pmatrix}\begin{pmatrix}\chi^\alpha \\ \bar{\chi}_{\dot{\alpha}}\end{pmatrix} = -2\lambda(\phi\chi_\alpha, -\bar{\phi}\bar{\chi}^{\dot{\alpha}})\begin{pmatrix}\chi^\alpha \\ \bar{\chi}_{\dot{\alpha}}\end{pmatrix} =$$

$$= 2\lambda\phi\chi^2 + 2\lambda\bar{\phi}\bar{\chi}^2;$$

Which differs by a sign with what we've got above.

Now we write all auxiliary terms, to resolve for F and $\overline{F}$:

$$\mu \cdot 2F\phi + 3\nu F\phi^2 + \mu \cdot 2\overline{F\phi} + 3\nu \overline{F\phi^2} + 2\overline{F}F$$

From this we obtain

$$F = -\mu\overline{\phi} - \frac{3}{2}\nu\overline{\phi}^2$$
$$\overline{F} = -\mu\phi - \frac{3}{2}\nu\phi^2$$

Substituting it back, we notice that the whole expression must be equal $-2\overline{F}F$, that is

$$(*) \quad -2\overline{F}F = -2[\mu^2\phi\overline{\phi} + \frac{3}{2}\mu\nu\phi\overline{\phi}^2 + \frac{3}{2}\mu\nu\overline{\phi}\phi^2 + \frac{9}{4}\nu^2\phi^2\overline{\phi}^2] =$$
$$= -2\mu^2\phi\overline{\phi} - 3\mu\nu\phi\overline{\phi}[\phi + \overline{\phi}] - \frac{9}{2}\nu^2\phi^2\overline{\phi}^2;$$

Now we check $(*)$ term by term against $(**)$:

$$-2\mu^2\phi\overline{\phi} = -2\mu^2\frac{1}{4}[S+iP][S-iP] = -\frac{1}{2}\mu^2S^2 - \frac{1}{2}\mu^2P^2$$

which shows that, indeed, $\boldsymbol{\mu = m}$. Thus all massive terms have been accounted for. It's only left to check two terms:

$$-\frac{1}{2}\lambda^2[S^2+P^2]^2 \quad \text{and} \quad -\lambda m S[S^2+P^2]$$

Let's consider

$$-\frac{9}{2}\nu^2\phi^2\bar{\phi}^2 = -\frac{9}{2}\nu^2(\phi\bar{\phi})^2 = -\frac{9}{2}\nu^2\frac{1}{16}[S^2+P^2]^2 \equiv -\frac{1}{2}\lambda^2[S^2+P^2]^2,$$

or, $$\nu^2\cdot\frac{9}{16} = \lambda^2;$$

$$\nu = \frac{4}{3}\lambda$$

And, finally,

$$-3\mu\nu\phi\bar{\phi}[\phi+\bar{\phi}] = -3\mu\nu\frac{1}{4}[S^2+P^2]\frac{[S+iP+S-iP]}{2} =$$

$$= -3m\cdot\frac{4}{3}\lambda\cdot\frac{1}{4}\cdot S[S^2+P^2] = -\lambda m S[S^2+P^2].$$

Thus, also adding results from the Exercise IV.10, we can accomodate for all terms in (**) except for

$$-\frac{1}{2}m\bar{\psi}\psi - \lambda\bar{\psi}[S-P\gamma^5]\psi.$$

Note that the kinetic term $-\frac{1}{2}\bar{\psi}\not{\partial}\psi$ is perfectly fitted:

$$-\frac{1}{2}[\chi_\alpha, -\bar{\chi}^{\dot{\alpha}}]\partial_\mu\begin{bmatrix} & -i\sigma^{\mu\alpha\dot{\alpha}} \\ i\bar{\sigma}^\mu_{\dot{\alpha}\alpha} & \end{bmatrix}\begin{bmatrix}\chi^\alpha \\ \bar{\chi}_{\dot{\alpha}}\end{bmatrix} =$$

$$= -\frac{1}{2}[-i\bar{\chi}^{\dot{\alpha}}\bar{\sigma}^\mu_{\dot{\alpha}\alpha}, -i\chi_\alpha\sigma^{\mu\alpha\dot{\alpha}}]\partial_\mu\begin{bmatrix}\chi^\alpha \\ \bar{\chi}_{\dot{\alpha}}\end{bmatrix} =$$

$$= \frac{1}{2}i(\bar{\chi}\bar{\not{\partial}}\chi + \chi\not{\partial}\bar{\chi}),$$ which is in perfect agreement with (***).