A note on AD point vs. CMS curve in supersymmetric CP$(N-1)$ model

The effective superpotential in the mirror representation reads [1, 2]

$$\mathcal{W}(\sigma) \;=\; \frac{N}{4\pi}\left(\sigma \;-\; \frac{1}{N}\sum m_j\,\ln(\sigma\;+\;m_j)\right)\,. \tag{1}$$ {Wgen}

Its vacua are located at

$$\sigma_{(p)} \;=\; \sqrt[N]{1\;+\;(-m_0)^N}\;e^{2\pi i p/N}\,. \tag{2}$$ {sigmap}

Both the vacua $\sigma_{(p)}$ and the masses m_j satisfy the Z_N symmetry. The Argyres-Douglas point is, therefore when all σ's vanish:

$$\sigma_0 \;=\; \sigma_{(p)} \;=\; \ldots \;=\; 0 \qquad\qquad \forall\, p. \tag{3}$$

The CMS condition is the equality of the phases of the fundamental and topological masses,

$$Z \;=\; -\,i\left((m_k\;-\;m_l)\;+\;(m_D^{(k)}\;-\;m_D^{(l)})\right)\,, \tag{4}$$

where the latter are

$$m_D^{(k)} \;=\; 2i\,\mathcal{W}(\sigma_{(k)})\,, \tag{5}$$

and, therefore

$$Z \;=\; -\,i\left((m_k\;-\;m_l)\;+\;2i\,\left\{\mathcal{W}(\sigma_{(k)})\;-\;\mathcal{W}(\sigma_{(l)})\right\}\right)\,. \tag{6}$$ {Z}

It seems that, as $\sigma\to 0$, the topological part of (6) goes to zero and the equality of phases — *i.e.* the CMS condition — is satisfied.

However, equation (1) for $\mathcal{W}(\sigma)$ does not mean that literally

$$\mathcal{W}(\sigma_{(p)}) \;=\; \frac{N}{4\pi}\left(\sigma_{(p)} \;-\; \frac{1}{N}\sum m_j\,\ln(\sigma_{(p)}\;+\;m_j)\right)\,,$$

as in general, each logarithm has a cut and admits an addition of $2\pi i$ times an integer. The latter integer constants are fixed by imposing the Z_N symmetry on the vacuum values of the superpotential,

$$\mathcal{W}(\sigma_{(p)}) \;=\; e^{2\pi i p/N}\,\mathcal{W}(\sigma_0)\,, \tag{7}$$ {WZN}

i.e. the vacuum values also sit on the circle. The vacuum values therefore could only cancel in (6) if this circle shrinks to zero, which does not happen.

The value $\mathcal{W}(\sigma_{(p)})$ can be related to the value in the zeroth vacuum $\mathcal{W}(\sigma_0)$ as in [1], by substituting

$$\sigma_{(p)} \quad = \quad e^{2\pi i p/N}\, \sigma_0 \,.$$

However, analysis of branch cuts of the logarithms shows that in this case

$$\mathcal{W}(\sigma_{(p)}) \quad \neq \quad e^{2\pi i p/N}\, \mathcal{W}(\sigma_0)$$

if $\mathcal{W}(\sigma)$ is just taken literally from Eq. (1). The above expression is understood to be taken in the limit $|\sigma_0| \to 0$, *i.e.* near the AD point.

The limit of the AD point is useful, since then one can correctly neglect the phases of the σ's in the logarithms ($\arg(\sigma + m_j) \;\approx\; \arg m_j$), and analyse which of the logarithms may exceed their branch cuts. Imposition of Z_N symmetry then demands to cancel the corresponding constants that pop out. The result is,

$$\begin{aligned}
\text{true}\ \mathcal{W}(\sigma_{(p)}) \quad &= \quad \frac{N}{4\pi}\left(\sigma_{(p)} \;-\; \frac{1}{N}\sum m_j\, \ln(\sigma_{(p)} \;+\; m_j)\right) \quad + \\
&+ \quad \frac{i}{2}\begin{cases} \displaystyle\sum_{j>n-|p|}^{n} (-m_j)\,, & p<0 \\ \\ \displaystyle\sum_{j=-n}^{j<-n+p} m_j\,, & p>0 \end{cases} \qquad , \qquad\qquad (8) \quad \texttt{\{Wtrue\}} \\
&[\lim\ \sigma_0 \;\to\; 0]\,, \\
&[\text{for CP}(N-1) \text{ with } \textit{odd } N].
\end{aligned}$$

The above equation is meant to apply in the limit where masses m_j approach the AD point $\sigma_0 = 0$ (therefore, one can simply throw away the σ's on the RHS of (8)). This equation is derived for the case of odd N. On one hand, Eq. (8) restores the Z_N symmetry, on the other hand, it allows to efficiently compare the vacuum values of the superpotential (at least in the AD point).

The conventions for the masses and vacua in Eq. (8) are as follows: masses are numbered as $-(N-1)/2$, ..., 0, ..., $+(N-1)/2$, with the same numbering for the vacua $\sigma_{(p)}$. Therefore, m_0 and σ_0 are real and positive. The cuts of each logarithm in (1) are assumed to be directed in the real negative direction, with the phase of the argument varying from $-i\pi$ to $+i\pi$.

Equation (8) means that for the p-th vacuum, p masses have to be added (or subtracted, depending on the sign of p) to the initial expression (1). Again, it was derived for the case of odd N, when both m_0 and σ_0 can be made real. [But the confidence is that a very similar result holds for *even* N as well — p masses have to be added. The following discussion was not peformed for even N, however].

Using Eq. (8) it is easy to show that for theory with odd N the phase of $m_D^{(k)} - m_D^{(l)}$ in the limit $\sigma_0 \to 0$ always differs from the phase of $m^k - m^l$ by a factor of i, and therefore they can never be made equal. Therefore, the AD point does not lie on the CMS.

On the other hand, for CP(1) theory ($N = 2$), this does not happen. The masses m_j are imaginary at the AD point (see Eq. (2)),

$$m_{(1)} \quad \to \quad i\,, \qquad\qquad m_{(-1)} \quad \to \quad -\,i\,,$$

while the superpotential is real (the sum of imaginary masses times their logarithms),

$$\mathcal{W}(\sigma_{(1)}) \quad = \quad -\,\mathcal{W}(\sigma_{(-1)}) \quad \in \quad \mathcal{R}\,.$$

The extra factor of $2i$ in Eq. (6) therefore balances the phases, and the AD point is on the CMS. What happens in general CP$(N-1)$ with even $N = 2m$ remains to be seen.

References

[1] S. Olmez and M. Shifman, J. Phys. A **40**, 11151 (2007) [arXiv:hep-th/0703149].

[2] N. Dorey, JHEP **9811**, 005 (1998) [arXiv:hep-th/9806056].