Matrix Form of the 6D Hypertensor Action

Let's group the relations b/w $[\varphi^A, \phi^A]$ and ϕ^{jB} into matrix form:

$$\begin{Bmatrix} \varphi^A \\ \phi^A \end{Bmatrix} = \begin{Bmatrix} g^A{}_{jB} \\ f_{AjB} \end{Bmatrix} \begin{Bmatrix} \phi^{jB} \end{Bmatrix} \qquad \begin{Bmatrix} \bar\varphi_A \\ \bar\phi^A_A \end{Bmatrix} = \begin{Bmatrix} \bar g_A{}^{jB} \\ \bar f^{AjB} \end{Bmatrix} \begin{Bmatrix} \bar\phi_{jB} \end{Bmatrix}$$

$$\phi^i = \Xi^i{}_{jB}\, \phi^{jB} \qquad \bar\phi_i = \bar\Xi_i{}^{jB}\, \bar\phi_{jB} >$$

Then

$$\mathcal{L}_F = G_C{}^D \Big[-i \big\{ g^C{}_{iA}\, \bar g_D{}^{jB} + f_{DiA} \bar f^{CjB} \big\} \phi^{iA} \not\partial \bar\phi_{jB} -$$

$$- \frac{1}{2} \big\{ f_{DiA}\, g^C{}_{jB} - g^C{}_{iA} f_{DjB} \big\} \phi^{iA} \partial \phi^{jB}$$

$$- \frac{1}{2} \big\{ \bar f^{CjB}\, \bar g_D{}^{iA} - \bar g_D{}^{jB} \bar f^{CiA} \big\} \bar\phi_{jB} \partial \bar\phi_{iA} \Big]$$

$$\mathcal{L}_H = -i\, \phi^{T i} \begin{Bmatrix} G & \\ & G^T \end{Bmatrix}_i{}^k \not\partial \bar\Phi_k + \frac{1}{2} \Phi^{T i} \begin{Bmatrix} & G \\ -G^T & \end{Bmatrix}_{ik} \partial \Phi^k +$$

$$\Phi^i = \begin{Bmatrix} \varphi^A \\ \phi_A \end{Bmatrix},$$

$$+ \frac{1}{2} \bar\Phi_i \begin{Bmatrix} 0 & G^T \\ -G & 0 \end{Bmatrix} \partial \bar\Phi_k ;$$

this comes from

$$\mathcal{L}_F = G_A{}^B \Big\{ \big[-i\varphi^A \not\partial \bar\varphi_B - i \phi_B \not\partial \phi^A \big] - \frac{1}{2} \big[\phi_B \partial \varphi^A - \varphi^A \partial \phi_B \big] -$$

$$- \frac{1}{2} \big[\bar\phi^A \partial \bar\varphi_B - \bar\varphi_B \partial \bar\phi^A \big] \Big\}$$

We now transfer to hypermultiplet Ψ:

$$\phi^i = \Xi^i{}_{jB}\,\Psi^{jB}; \quad \text{or} \quad \phi = \Xi\Psi$$

$$\mathcal{L}_H = -i\,\bar{\Psi}^T \Xi^T \begin{pmatrix} G & \\ & G^T \end{pmatrix} \partial\!\!\!/\, \bar{\Xi}\bar{\Psi} -$$

$$-\frac{1}{2}\Psi^T \Xi^T \begin{pmatrix} 0 & -G \\ G^T & 0 \end{pmatrix} \partial\, \Xi \Psi - \qquad \begin{pmatrix} G & \\ & G^T \end{pmatrix} \equiv g$$

$$-\frac{1}{2}\bar{\Psi}^T \bar{\Xi}^T \begin{pmatrix} 0 & -G^T \\ G & 0 \end{pmatrix} \partial\, \bar{\Xi}\bar{\Psi}, \qquad \begin{pmatrix} & -G \\ G^T & \end{pmatrix} \equiv h$$

$$\begin{pmatrix} & -G^T \\ G & \end{pmatrix} \equiv h^*$$

From here

$$K_{iA}{}^{jB} = \Xi^T g \bar{\Xi}; \quad \text{- hermitean}$$

$$H_{iA,jB} - H_{jB,iA} = \Xi^T \begin{pmatrix} 0 & -G \\ G^T & 0 \end{pmatrix} \Xi = \left[\Xi^T h\, \Xi\right]_{jBiA}$$

$$\bar{H}^{iAjB} - \bar{H}^{jBiA} = \bar{\Xi}^T \begin{pmatrix} 0 & -G^T \\ G & 0 \end{pmatrix} \bar{\Xi} = \left[\bar{\Xi}^T h^* \bar{\Xi}\right]^{jBiA}$$

Evidently, can't expect that h or h^* are (anti) holomorphic.

Let's now subst the ansatz $G = EE^{\dagger}$. $\Rightarrow G^{T} = E^{*}E^{T}$,

$$g = \begin{bmatrix} G & \\ & G^{T} \end{bmatrix} = \begin{bmatrix} EE^{\dagger} & \\ & E^{*}E^{T} \end{bmatrix} = \begin{bmatrix} E & \\ & E^{*} \end{bmatrix} \begin{bmatrix} \mathbb{1} & \\ & \mathbb{1} \end{bmatrix} \begin{bmatrix} E^{\dagger} & \\ & E^{T} \end{bmatrix};$$

$$h = \begin{bmatrix} 0 & -G \\ G^{T} & 0 \end{bmatrix} \overset{?}{=} \begin{bmatrix} E & \\ & E^{*} \end{bmatrix} \begin{bmatrix} & -\mathbb{1} \\ \mathbb{1} & \end{bmatrix} \begin{bmatrix} E^{T} & \\ & E^{\dagger} \end{bmatrix} =$$

$E^{\dagger}$ [illegible] $E^{*}E^{T}$ [illegible]

$$h^{*} \overset{?}{=} \begin{bmatrix} E^{*} & \\ & \end{bmatrix} \begin{bmatrix} & +\mathbb{1} \\ -\mathbb{1} & \end{bmatrix} \begin{bmatrix} E^{\dagger} & \\ & E^{T} \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & E^{*} \\ -E & 0 \end{bmatrix} \begin{bmatrix} E^{\dagger} & \\ & E \end{bmatrix} = \begin{bmatrix} & \\ -EE^{\dagger} & 0 \end{bmatrix} = \begin{bmatrix} & \\ -G & 0 \end{bmatrix}$$ [illegible]

Now denote,
$$D \equiv \begin{bmatrix} E & \\ & E^{*} \end{bmatrix}; \quad D^{T} = \begin{bmatrix} E^{T} & \\ & E^{\dagger} \end{bmatrix};$$
$$\mathcal{E} \equiv \begin{bmatrix} & -\mathbb{1} \\ \mathbb{1} & \end{bmatrix};$$

$$\Rightarrow \quad g = D \cdot D^{\dagger}, \quad h = D\mathcal{E}D^{T}, \quad h^{*} = D^{*}\mathcal{E}D^{\dagger},$$

$$\Rightarrow \quad \mathcal{K} = \bar{\Xi}^T D \cdot D^\dagger \bar{\bar{\Xi}}; \qquad \mathcal{H}_{[AjB]} = \bar{\Xi}^T D E D^T \bar{\Xi};$$

[kähler metric]

$$\mathcal{F}^{[AjB]} = \bar{\Xi}^\dagger D^* E D^\dagger \bar{\bar{\Xi}};$$

we then introduce $\mathcal{J} = \bar{\Xi}^T D$,

$$\Leftrightarrow \qquad \mathcal{K} = \mathcal{J} \cdot \mathcal{J}^\dagger; \quad \mathcal{H}_{[\,]} = \mathcal{J} \cdot E \mathcal{J}^T; \quad \mathcal{F}^{[\,]} = \mathcal{J}^* E \mathcal{J}^\dagger;$$

It's left to show that $\mathcal{J}$ is holomorphic:

$$\mathcal{J} = \bar{\Xi}^T D = \begin{pmatrix} g_j^T & f_j^T \end{pmatrix} \begin{pmatrix} E & \\ & E^* \end{pmatrix} = \begin{pmatrix} \underbrace{g_j^T E}_{\text{holomorphic}} & f_j^T E^* \end{pmatrix};$$

$$f_j^T E^* = f_{A_j B} E^{*B}{}_C;$$

It c.b. shown that $f_j^T E^*$ is holomorphic:

we recall that there's a connection $-f_{AkC} = \rho_{AB}\, g^B{}_{kC}$,

i.e. $-f_k = \hat{\rho} \cdot g_k$, $\Rightarrow f_j = -\hat{\rho}\, g_j = -E^{\dagger -1} \epsilon E^\dagger g_j$,

and so $f_j^T E^* = -g_j^T E \epsilon^T E^{*-1} \cdot E^* = +g_j^T E \epsilon^T$

$= \textbf{holomorphic} \qquad \Rightarrow \mathcal{J}$ is holomorphic