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Low-Energy Effective Action of the Supersymmetric CP($N-1$) Model in the Large-N Limit

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Abstract

Motivated by an old would-be paradox we found a solution of supersymmetric CP($N-1$) models in superfields. Our main target is the Kähler potential, since the superpotential term was exactly known since early 1990s. To this end we used the large-N expansion to perform a supersymmetric calculation in the leading order in $1/N$. The models considered are $\mathcal{N}=(2,2)$ basic CP($N-1$) model and its nonminimal $\mathcal{N}=(0,2)$ deformation various aspects of which are being actively studied since 2007. We also extend the above models by adding twisted masses.

1 Introduction

{sec1}

Supersymmetric and non-supersymmetric CP($N-1$) models in two dimensions are exactly solvable in the large-N limit [1, 2, 3]. Moreover, the superpotential part can be found exactly (for any N) [4, 5, 6] in terms of the so-called twisted superfield Σ (to be defined below) in the form

$$\mathcal{L}_{\rm sp} \;=\; {\rm const}\cdot\left(\int d\theta_R d\bar{\theta}_L\left(\sqrt{2}\Sigma\,\log\sqrt{2}\Sigma\;-\;\sqrt{2}\Sigma\right)\;+\;{\rm h.c.}\right). \qquad (1.1)$$ {1}

For brevity below we will sometimes refer to above expression as the Witten superpotential. Equation (1.1) encodes all information about the anomalies of the CP($N-1$) model. In this sense it is akin to the Veneziano–Yankielowicz superpotential [7] in $\mathcal{N}=1$ super-Yang-Mills theory in four dimensions. However, the Veneziano-Yankielowicz result admittedly presents an effective superpotential suitable only for determination of the vacuum structure while (1.1) presents the exact superpotential part of the solution of the CP($N-1$) model.

The scalar superpotential for the CP($N-1$) models in the large-N limit was found in [3, 8] (from a nonsupersymmetric calculation) to have the form

$$V \;=\; \frac{N}{4\pi}\left\{\Lambda^2\;+\;\left|\sqrt{2}\sigma\right|^2\left(\log\frac{\left|\sqrt{2}\sigma\right|^2}{\Lambda^2}\;-\;1\right)\right\}, \qquad (1.2)$$ {2}

where σ is the lowest component of Σ. In [8] it was checked that the critical points of (1.1) coincide with the minima of the scalar potential (1.2). Both expressions lead to zero-energy ground states, *i.e.* to supersymmetric vacua.

Rather often an apparent paradox is pointed out in comparing (1.1) and (1.2). Usually it is assumed that the Kähler term in the superfield formulation has the simplest possible form invariant under scale transformations (and those related to the scale transformations by supersymmetry), namely,

$$\mathcal{L}_{\rm K} \;=\; {\rm const}\cdot\int d^4\theta\,\log\sqrt{2}\Sigma\;\log\sqrt{2}\bar{\Sigma}\,. \qquad (1.3)$$ {3}

Then, eliminating the D term (the last component of Σ) by using equations of motion for D, we would obviously arrive at

$$V \;=\; {\rm const}\cdot\left(\left|\sqrt{2}\sigma\right|^2\,\log\left|\sqrt{2}\sigma\right|^2\right)^2. \qquad (1.4)$$

It is clear that this expression, being proportional to the square of a logarithm, cannot coincide with (1.2).

A way out from this paradox was pointed out in [8]. There it was suggested that the minimal form of the Kähler term (1.3) is not complete. The Kähler term is in fact more complicated (although still compatible with scale invariance) — there is an extra contribution to $\mathcal{L}_{\rm K}$ having a special feature of vanishing at $D = 0$, *i.e.* in supersymmetric vacua. However, the result for the full $\mathcal{L}_{\rm K}$ was not derived. Here we close this gap carrying out a supersymmetric calculation of both $\mathcal{L}_{\rm K}$ and $\mathcal{L}_{\rm sp}$ to the leading order in $1/N$. While the latter coincides with (1.1), as was expected, the expression for $\mathcal{L}_{\rm K}$ in the large-N solution brings a surprise.[1] The Kähler potential obtained depends not only on Σ and $\overline{\Sigma}$, as is usually the case, but also on superderivatives of these superfields. This is the reason why scale invariance apparent in (1.3) can be maintained in additional terms $\Delta\mathcal{L}_{\rm K}$.

Combining our expression for the full Kähler potential with (1.1) we recover the scalar potential (1.2).

The outline of the paper is as follows. Section 2 is introductory. Here we introduce our basic notation, in particular, twisted superfields[2], formulate the problem in more detail, and essentially "guess" the supersymmetric large-N solution in terms of twisted superfields. In Sec. 3 we expand the superfield action in components in the two-derivative approximation. At the end of that section we derive the scalar potential (1.2) from the superpotential (1.1) and the Kähler potential we found in our supersymmetric calculation based on the large-N expansion. Section 4 treats the nonminimal heterotic deformation of CP$(N-1)$ models worked out in [9, 10], see also [3, 8, 11], to the leading order in $1/N$. In Sec. 5 we repeat the procedure with non vanishing twisted masses introduced in a standard way. Finally, Section 6 summarizes our results and conclusions.

[1]Note that unlike $\mathcal{L}_{\rm sp}$ the Kähler potential cannot be exactly established on general grounds and requires an actual and rather cumbersome calculation which, fortunately, can be performed to the leading order in $1/N$.

[2]Further notation can be found in Appendix.

2 Supersymmetrization of the Effective Potential

{ssuper}

We start from the supersymmetric two-dimensional CP^{N-1} model,

$$\begin{aligned}
\mathcal{L}_{(2,2)} \;=\;& \frac{1}{4e^2}\,F_{kl}^2 \;+\; \frac{1}{e^2}\,|\partial_k\sigma|^2 \;+\; \frac{1}{2e^2}\,D^2 \;+\; \frac{1}{e^2}\,\overline{\lambda}_R\,i\partial_L\,\lambda_R \;+\; \frac{1}{e^2}\overline{\lambda}_L\,i\partial_R\,\lambda_L \;+ \\
+\;& |\nabla n|^2 \;+\; |\sqrt{2}\sigma|^2\,\left|n^l\right|^2 \;+\; iD\left(\left|n^l\right|^2 - 2\beta\right) \;+ \\
+\;& \overline{\xi}_R\,i\nabla_L\xi_R \;+\; \overline{\xi}_L\,i\nabla_R\xi_L \;+\; i\sqrt{2}\sigma\,\overline{\xi}_{Rl}\xi_L^l \;+\; i\sqrt{2}\overline{\sigma}\,\overline{\xi}_{Ll}\xi_R^l \;+ \\
+\;& i\sqrt{2}\,\overline{\xi_{[R}\,\lambda_{L]}}\,n \;-\; i\sqrt{2}\,\overline{n}\,\lambda_{[R}\,\xi_{L]}\,, \qquad l \;=\; 1,\ldots N\,.
\end{aligned} \tag{2.5}$$

{sigma22}

The notations are explained in [12], although they are of no great importance for our discussion. In [3], the large-N effective potential of the theory was found by integrating over n^l fields and their superpartners ξ:

$$V_{\rm eff} \;=\; -\frac{N}{4\pi}\left\{\left(\left|\sqrt{2}\sigma\right|^2 + iD\right)\log\left(\left|\sqrt{2}\sigma\right|^2 + iD\right) - iD - \left|\sqrt{2}\sigma\right|^2\log\left|\sqrt{2}\sigma\right|^2\right\}. \tag{2.6}$$

{Veff}

We quickly note that this expression can be cast in integral forms which will appear useful further on in our paper,

$$\begin{aligned}
\frac{4\pi}{N}\,V_{\rm eff} \;&=\; -\int_0^{iD} dt\,\ln\left(\left|\sqrt{2}\sigma\right|^2 + t\right) \;= \\
&=\; -\left|\sqrt{2}\sigma\right|^2\left(\ln\left|\sqrt{2}\sigma\right|^2\cdot x \;+\; \int_0^x \ln\left(1+x\right)dx\right).
\end{aligned} \tag{2.7}$$

{Vint}

Here x is a variable that is slightly more convenient to use than D,

$$x \;=\; \frac{iD}{\left|\sqrt{2}\sigma\right|^2}\,. \tag{2.8}$$

The above expression is now easy to write also as a series in x,

$$\frac{4\pi}{N}\,V_{\rm eff} \;=\; x\left|\sqrt{2}\sigma\right|^2\left(-\ln\left|\sqrt{2}\sigma\right|^2 \;+\; \sum_{k\geq 1}\frac{(-1)^k}{k\,(k+1)}\,x^k\right). \tag{2.9}$$

{Vser}

In this form, the potential precisely agrees with the effective action found in [2], in the limit where space-time derivatives of fields are discarded.

Our goal is to find a supersymmetric expression written explicitly in superfields, the *constant* bosonic part of which would coincide with (2.9).

Although both σ and D are part of the $\mathcal{N} = (2,2)$ supermultiplet V, it is more convenient to work in terms of the *twisted* chiral superfield Σ instead of V,

$$\Sigma \;=\; \frac{i}{\sqrt{2}}\, D_L\, \overline{D}_R\, V\,, \qquad\qquad \overline{\Sigma} \;=\; \frac{i}{\sqrt{2}}\, D_R\, \overline{D}_L\, V\,. \tag{2.10}$$

{defSigma}

A general definition of a twisted chiral superfield is that

$$D_L\,\Sigma \;=\; \overline{D}_R\,\Sigma \;=\; 0\,. \tag{2.11}$$

Field σ naturally comes as the lowest component of superfield Σ defined as in (2.10),

$$\Sigma \;=\; \sigma \;-\; \sqrt{2}\,\theta_R\overline{\lambda}_L \;+\; \sqrt{2}\,\overline{\theta}_L\lambda_R \;+\; \sqrt{2}\,\theta_R\overline{\theta}_L\,\Big(D \;-\; i\,F_{03}\Big)\,. \tag{2.12}$$

So, essentially, all occurences of field σ in (2.9) can be replaced with Σ.

In order to "supersymmetrize" field D we need to get to higher components of Σ. This is achieved by introducing a twisted chiral superfield S,

$$S \;=\; \frac{i}{2}\,\overline{D}_R D_L \ln\sqrt{2}\Sigma\,, \qquad\qquad \overline{S} \;=\; \frac{i}{2}\,\overline{D}_L D_R \ln\sqrt{2}\Sigma\,. \tag{2.13}$$

Now it appears that the ratio $x \;=\; iD/|\sigma|^2$ is best supersymmetrized [2] by the ratio of S and Σ,

$$x \;=\; \frac{iD}{\big|\sqrt{2}\sigma\big|^2} \quad\longrightarrow\quad \frac{S}{\sqrt{2}\Sigma}\,. \tag{2.14}$$

Using twisted superfields one can construct *twisted superpotentials*

$$\int d^2\tilde{\theta}\,\widetilde{\mathcal{W}}(\Sigma) \;=\; \frac{1}{2}\,\overline{D}_L\,D_R\,\widetilde{\mathcal{W}}(\Sigma)\Big|\,, \tag{2.15}$$

and regular $d^4\theta$ integrals — of course, provided there is something non-twisted chiral in the integrand.

Armed with these definitions, we can now translate series (2.9) into the language of superfields. The first term in Eq. (2.9) comes from the so-called Witten's superpotential,

$$i\int d^2\tilde{\theta}\,\Big(\sqrt{2}\Sigma\,\ln\sqrt{2}\Sigma \;-\; \sqrt{2}\Sigma\Big)\,. \tag{2.16}$$

The series in (2.9) can be shown to sit in the highest component of the following expression,

$$\frac{i}{2}\, S \sum_{k\geq 1} \frac{(-1)^k}{k\,(k+1)\,(k+2)} \left(\frac{S}{\sqrt{2}\Sigma}\right)^k . \tag{2.17}$$

This, actually, accounts for all terms in the series in Eq. (2.9) but the first one. That one happens to be part of a separate, "superkinetic" term $|\ln\Sigma|^2$ — the choice for such a name will be clear later.

Altogether, the effective action in an explicitly supersymmetric form can be written as,

$$\begin{aligned}\frac{4\pi}{N}\,\mathcal{L} \;&=\; -\int d^4\theta\,\frac{1}{2}\left|\ln\sqrt{2}\Sigma\right|^2 \;-\; i\int d^2\tilde{\theta}\,\left(\sqrt{2}\Sigma\,\ln\sqrt{2}\Sigma \;-\; \sqrt{2}\Sigma\right) \;+\\ &+\; \frac{i}{2}\int d^2\tilde{\theta}\, S \sum_{k\geq 1} \frac{(-1)^k}{k\,(k+1)\,(k+2)} \left(\frac{S}{\sqrt{2}\Sigma}\right)^k \;+\; \text{h.c.}.\end{aligned} \tag{2.18}$$

{sseries}

The series in the second line can be written as a D-term. We get

$$\begin{aligned}\frac{4\pi}{N}\,\mathcal{L} \;&=\; -\int d^4\theta\,\frac{1}{2}\left|\ln\sqrt{2}\Sigma\right|^2 \;-\; i\int d^2\tilde{\theta}\,\left(\sqrt{2}\Sigma\,\ln\sqrt{2}\Sigma \;-\; \sqrt{2}\Sigma\right) \;+\\ &+\; \frac{1}{2}\int d^4\theta\,\ln\sqrt{2}\bar{\Sigma} \sum_{k\geq 1} \frac{(-1)^k}{k\,(k+1)\,(k+2)} \left(\frac{S}{\sqrt{2}\Sigma}\right)^k \;+\; \text{h.c.}\end{aligned} \tag{2.19}$$

Here we can see explicitly that the only F-term is the Witten's superpotential, while all other terms are D-terms. Finally, the above series obviously can be re-summed, and not quite surprisingly will form a logarithm, as we started from a logarithm in Eq. (2.6),

$$\begin{aligned}\frac{4\pi}{N}\,\mathcal{L} \;&=\; -\int d^4\theta\,\frac{1}{2}\left|\ln\sqrt{2}\Sigma\right|^2 \;-\; i\int d^2\tilde{\theta}\,\left(\sqrt{2}\Sigma\,\ln\sqrt{2}\Sigma \;-\; \sqrt{2}\Sigma\right) \;-\\ &-\; \frac{1}{4}\int d^4\theta\,\ln\sqrt{2}\bar{\Sigma}\left[\left(1+\frac{\sqrt{2}\Sigma}{S}\right)^2 \ln\left(1+\frac{S}{\sqrt{2}\Sigma}\right) \;-\; \frac{\sqrt{2}\Sigma}{S}\right] \;+\; \text{h.c.}\end{aligned} \tag{2.20}$$

{Lsuper}

The above expression consists of three parts — the "superkinetic" term, Witten's superpotential and an additional logarithmic D-term. Both the superkinetic and the logarithmic D-terms form the Kähler potential of the target space. For a minute, let us pretend that the complicated logarithmic D-term did not exist, and let us find the component expansion of the superkinetic term and Witten's potential,

$$
\begin{aligned}
\frac{4\pi}{N}\,\mathcal{L}_{(0)} \;&=\; -\int d^4\theta\,\Bigg(\frac{1}{2}\Big|\ln\sqrt{2}\Sigma\Big|^2 \;+\\
&+\; i\,d^2\tilde\theta\,\Big(\sqrt{2}\Sigma\,\ln\sqrt{2}\Sigma \;-\; \sqrt{2}\Sigma\Big) \;+\; i\,d^2\bar{\tilde\theta}\,\Big(\sqrt{2}\overline{\Sigma}\,\ln\sqrt{2}\overline{\Sigma} \;-\; \sqrt{2}\overline{\Sigma}\Big)\Bigg) \;=\\
&=\; -\,iD\,\log|\sqrt{2}\sigma|^2 \;-\; \frac{1}{2}\,\frac{\big(iD\big)^2}{\big|\sqrt{2}\sigma\big|^2} \;-\; F_{03}\,\log\frac{\sqrt{2}\sigma}{\sqrt{2}\overline{\sigma}} \;+\\
&+\; \frac{\big|\partial_\mu\sigma\big|^2 \;+\; \frac{1}{2}\,F_{03}^2 \;+\; \frac{1}{2}\Big(\,\overline{\lambda}_R\, i\overleftrightarrow{\mathcal{D}}_L\lambda_R \;+\; \overline{\lambda}_L\, i\overleftrightarrow{\mathcal{D}}_R\lambda_L\Big)}{\big|\sqrt{2}\sigma\big|^2} \;-\\
&-\; 2\,\frac{i\sqrt{2}\sigma\overline{\lambda}_R\lambda_L \;+\; i\sqrt{2}\,\overline{\sigma}\overline{\lambda}_L\lambda_R}{\big|\sqrt{2}\sigma\big|^2} \;+\; 2\,\frac{\overline{\lambda}_R\lambda_L\overline{\lambda}_L\lambda_R}{\big|\sqrt{2}\sigma\big|^4} \;+\\
&+\; \frac{(iD+F_{03})\,i\sqrt{2}\sigma\overline{\lambda}_R\lambda_L \;+\; (iD-F_{03})\,i\sqrt{2}\,\overline{\sigma}\overline{\lambda}_L\lambda_R}{\big|\sqrt{2}\sigma\big|^4}\,.
\end{aligned}
\tag{2.21}
$$

{L0}

The long derivatives here are

$$
\begin{aligned}
\mathcal{D}_L \;&=\; \partial_L \;-\; \partial_L\ln\sqrt{2}\sigma\,,\\
\mathcal{D}_R \;&=\; \partial_R \;-\; \partial_R\ln\sqrt{2}\overline{\sigma}\,,
\end{aligned}
\tag{2.22}
$$

with conjugate expressions for the left-ward derivatives $\overleftarrow{\mathcal{D}}_L$ and $\overleftarrow{\mathcal{D}}_R$. These derivatives include the Christoffel symbol $1/\sigma$ which arises from the fact that we have a Kähler potential $\big|\log\Sigma\big|^2$. This potential is singular at zero, as the n^l fields of the gauge formulation of the sigma model (that were integrated out in [3] to find (2.6)) would become massless at $\sigma = 0$. In coordinates Σ, $\overline{\Sigma}$ the Kähler potential generates a metric $1/|\sigma|^2$ which is seen in the denominators in the expressions above. However, the Riemann tensor is zero for this metric, reflecting the observation that the target space is in fact flat — if one adopts $\log\Sigma$ and $\log\overline{\Sigma}$ as the coordinates, the metric disappears completely. This also explains why the terms in Eq. (2.21) that come from the superkinetic term are scale invariant — the logarithmic field $\log\Sigma$ depends on scale-invariant ratios $\overline{\lambda}_L/\sigma$, λ_R/σ *etc.* Still, we prefer to work with the coordinates Σ and $\overline{\Sigma}$.

The first two terms in the component expansion in (2.21) form the leading part of the effective potential (2.9). The third term is the anomaly*, may need more discussion*. The terms on the fourth line in (2.21) are the kinetic terms, while the terms on the rest of the lines are Yukawa-like and quartic couplings. Although we have only written the "trivial" part of the action, it already contains quite a bit of low-energy information about the theory. We will now see that the complicated logarithmic D-term gives a correction to Eq. (2.21).

3 Effective Action up to Two Derivatives

{saction}

Full component expansion of the effective action (2.20) is given in Appendix B. However, since at low energies we are only interested in the lowest level of momenta, we can limit ourselves to only the lowest powers of space-time derivatives — second for bosons and first for fermions.

To obtain the low energy action in the two-derivative approximation, it is more convenient to work with the series representation (2.18). We perform the superspace integration on each of the terms in the series, drop everything beyond one or two derivatives as needed, and then assemble the resulting terms back into logarithms.

We find,

$$
\begin{aligned}
\frac{4\pi}{N}\,\mathcal{L}_{\text{two deriv}} \;=\;& \frac{\left|\partial_\mu\sigma\right|^2}{\left|\sqrt{2}\sigma\right|^2} \;-\; F_{03}\,\log\frac{\sqrt{2}\sigma}{\sqrt{2}\overline{\sigma}} \;+\; \frac{4\pi}{N}\,V_{\text{eff}} \;-\\
&-\; \frac{F_{03}^2}{\left|\sqrt{2}\sigma\right|^2}\left(2\,\frac{\ln(1+x)\,-\,x}{x^2} \;+\; \frac{1}{2}\,\frac{1}{1\,+\,x}\right) \;-\\
&-\; \frac{\overline{\lambda}_R\, i\overleftrightarrow{\mathcal{D}}_L\lambda_R \;+\; \overline{\lambda}_L\, i\overleftrightarrow{\mathcal{D}}_R\lambda_L}{\left|\sqrt{2}\sigma\right|^2}\,\frac{\ln(1+x)\,-\,x}{x^2} \;-\\
&-\; 2\,\frac{i\sqrt{2}\sigma\overline{\lambda}_R\lambda_L \;+\; i\sqrt{2}\,\overline{\sigma\lambda}_L\lambda_R}{\left|\sqrt{2}\sigma\right|^2}\,\frac{\ln(1+x)}{x} \;+\\
&+\; 4\,\frac{\overline{\lambda}_R\lambda_L\overline{\lambda}_L\lambda_R}{\left|\sqrt{2}\sigma\right|^4}\left(\frac{\ln(1+x)\,-\,x}{x^2} \;+\; \frac{1}{1\,+\,x}\right) \;+\\
&+\; \frac{1}{4}\,\Box\,\log|\sqrt{2}\sigma|^2\cdot\frac{(1-x^2)\ln(1+x)\,-\,x}{x^2} \;-\\
&-\; 2\,\frac{F_{03}\big(i\sqrt{2}\sigma\overline{\lambda}_R\lambda_L \;-\; i\sqrt{2}\,\overline{\sigma\lambda}_L\lambda_R\big)}{\left|\sqrt{2}\sigma\right|^4}\,\frac{\ln(1+x)\,-\,x}{x^2}\,.
\end{aligned}
\qquad (3.23)\quad \{\texttt{L2d}\}
$$

Comparing Eqs (3.23) and (2.21) we notice that the effect of adding the supersymmetric series to Eq. (2.21) is the multiplication of all terms by certain functions of x. In fact, the whole expression (2.21) is the leading order approximation (or subleading for some of the terms) in x of the two-derivative Langrangian (3.23).

Now we exclude the auxiliary field D to obtain the effective action as a function of σ only. Due to the very complicated dependence of the action (3.23) on D, we can only do so with the accuracy of keeping up to two space-time derivatives. In order to make the procedure of resolution of D easier, we switch to using the variable x. The Lagrangian (3.23) can be split into three pieces:

$$
\mathcal{L} \;=\; \mathcal{L}_{(0)}(x) \;+\; \mathcal{L}_{(1)}(x) \;+\; \mathcal{L}_{(2)}(x)\,, \qquad (3.24)\quad \{\texttt{Lpert}\}
$$

each explicitly containing, correspondingly, none, one and two space-time derivatives. In particular,

$$
\frac{4\pi}{N}\,\mathcal{L}_{(0)}(x) \;=\; \frac{\left|\partial_\mu\sigma\right|^2}{\left|\sqrt{2}\sigma\right|^2} \;-\; F_{03}\,\log\frac{\sqrt{2}\sigma}{\sqrt{2}\overline{\sigma}} \;+\; \frac{4\pi}{N}\,V_{\text{eff}}(x)\,. \qquad (3.25)
$$

Although this expression actually does contain space-time derivatives in the kinetic term for σ and in the anomaly term, the latter terms are independent of x and are unaltered during the course of resolution of x. The first-order part $\mathcal{L}_{(1)}(x)$ only includes the term on the fourth line in (3.23), *i.e.* the Yukawa-like coupling. Each part in the expansion (3.24), besides the explicit derivatives, will also contain derivatives implictly, via x. More specifically, we split the solution (to be found) of the equations of motion as,

$$x \;=\; x_0 \;+\; x_1 \;+\; \ldots . \tag{3.26}$$

Expanding the Lagrangian perturbatively in the number of space-time derivatives, and using the equation of motion for x, one can show that to the second order in space-time derivatives the Lagrangian is

$$\mathcal{L}_{\text{two deriv}}(\sigma) \;=\; \mathcal{L}_{(0)}(x_0) \;+\; \mathcal{L}_{(1)}(x_0) \;+\; \mathcal{L}_{(2)}(x_0) \;-\; \frac{1}{2}\frac{\partial^2 \mathcal{L}_{(0)}}{\partial x^2}\bigg|_{x_0} \cdot x_1^2 . \tag{3.27}$$ {Lx}

Here x_0 is the minimum of the potential $V_{\text{eff}}(x)$,

$$x_0 \;=\; \frac{1 - \left|\sqrt{2}\sigma\right|^2}{\left|\sqrt{2}\sigma\right|^2} , \tag{3.28}$$

while x_1 is

$$x_1 \;=\; -\,\frac{\partial \mathcal{L}_{(1)}}{\partial x}(x_0)\left(\frac{\partial^2 \mathcal{L}_{(0)}}{\partial x^2}(x_0)\right)^{-1} . \tag{3.29}$$

Higher terms in x are not needed for our purposes. It is only due to the presence of the Yukawa-like coupling in Eq. (3.23) — which effectively is a one-derivative term — that the expression (3.27) includes the last term depending on x_1^2. That term, however, is important as it modifies the coefficient of the quartic fermionic coupling in Eq. (3.23).

It is now straightforward to calculate expression (3.27). For the sake of presenting the results, it is helpful to introduce a variable r,

$$r \;=\; \left|\sqrt{2}\sigma\right|^2 \tag{3.30}$$

and a function $v_{\text{eff}}(r)$,

$$v_{\text{eff}}(r) \;=\; \frac{4\pi}{N} V_{\text{eff}}(\sigma) \;=\; r\ln r \;+\; 1 \;-\; r\,. \tag{3.31}$$

The result for Eq. (3.27) can now be given as,

$$\frac{4\pi}{N}\,\mathcal{L}_{\text{two deriv}}(\sigma, A_\mu, \lambda) \;=\;$$

$$= \;\frac{\left|\partial_\mu\sigma\right|^2}{r} \;-\; F_{03}\,\log\frac{\sqrt{2}\sigma}{\sqrt{2}\overline{\sigma}} \;+\; v_{\text{eff}}(r) \;-$$

$$+\;\; 2\,F_{03}^2\left(\frac{v_{\text{eff}}(r)}{(1-r)^2} \;-\; \frac{r}{4}\right) \;+\; \left(\overline{\lambda}_R\, i\overleftrightarrow{\mathcal{D}_L}\lambda_R \;+\; \overline{\lambda}_L\, i\overleftrightarrow{\mathcal{D}_R}\lambda_L\right)\frac{v_{\text{eff}}(r)}{(1-r)^2} \;+$$

$$+\;\; 2\left(i\sqrt{2}\sigma\overline{\lambda}_R\lambda_L \;+\; i\sqrt{2}\,\overline{\sigma\lambda}_L\lambda_R\right)\frac{1-r}{r}\,\ln r \;- \tag{3.32}$$

$$-\;\; 4\,\overline{\lambda}_R\lambda_L\overline{\lambda}_L\lambda_R\left(\frac{v_{\text{eff}}(r)}{r\,(1-r)^2} \;-\; \frac{1}{r} \;+\; \frac{r\left(1\,-\,r\,-\,\ln r\right)^2}{(1-r)^4}\right) \;-$$

$$-\;\; \frac{1}{4}\,\Box\,\ln r\left(\frac{r\,v_{\text{eff}}(r)}{(1-r)^2} \;-\; \ln r\right) \;+$$

$$+\;\; 2\,F_{03}\left(i\sqrt{2}\sigma\overline{\lambda}_R\lambda_L \;-\; i\sqrt{2}\,\overline{\sigma\lambda}_L\lambda_R\right)\frac{v_{\text{eff}}(r)}{r\,(1-r)^2}\,.$$

{Lsigma}

Interestingly, most of the "coefficients" of the terms in the Lagrangian (3.32) contain the potential $v_{\text{eff}}(r)$ in one or another form.

4 Heterotic Deformation

{shet}

The heterotic deformation is introduced using $\mathcal{N} = (0, 2)$ superfields and essentially follows the guidelines of [9]. Before describing this deformation in detail we need to introduce the necessary superfield machinery consistent with our notations.

4.1 $\mathcal{N} = (0, 2)$ superfields

We define $\mathcal{N} = (0, 2)$ superspace via reduction of $\mathcal{N} = (2, 2)$ superspace by putting

$$\theta_L \;=\; \overline{\theta}_L \;=\; 0\,. \tag{4.33}$$

Each chiral and twisted-chiral $\mathcal{N} = (2, 2)$ superfield this way splits into two $\mathcal{N} = (0, 2)$ superfields. While chiral superfields are usually described as being dependent on a “holomorphic” variable y^μ,

$$y^\mu \;=\; x^\mu \;+\; i\,\overline{\theta\sigma}_\mu\theta\,, \tag{4.34}$$

and twisted-chiral as dependent on a twisted variable $\widetilde{y}^\mu$,

$$\widetilde{y}^\mu \;=\; x^\mu \;+\; i\,\overline{\theta\widetilde{\sigma}}_\mu\theta\,, \tag{4.35}$$

(see Appendix A for details and notations), the distinction between the two variables vanishes upon reduction to $\mathcal{N}=(0,2)$ superspace. As a result, $\mathcal{N}=(0,2)$ superspace defines only one kind of holomorphic superfields — chiral $\mathcal{N}=(0,2)$ superfields, which depend upon the reduced variables υ^μ

$$\begin{aligned}\upsilon^0 \;&=\; x^0 \;+\; i\,\overline{\theta}_R\theta_R\\ \upsilon^3 \;&=\; x^3 \;+\; i\,\overline{\theta}_R\theta_R\,.\end{aligned} \tag{4.36}$$

To make a distinction with $\mathcal{N}=(2,2)$ superfields, we denote the $\mathcal{N}=(0,2)$ superfields by symbols with a hat on top — $\hat{\sigma}$, $\hat{\xi}$, *etc.* Of immediate interest to us is the *positional* superfield $Z(y)$ which splits into two $\mathcal{N}=(0,2)$ superfields — $\hat{z}(\upsilon)$ and $\hat{\zeta}(\upsilon)$,

$$\begin{aligned}\hat{z}(\upsilon) \;&=\; z \;-\; \sqrt{2}\,\theta_R\,\zeta_L\,, \qquad & \hat{\zeta}(\upsilon) \;&=\; \zeta_R \;+\; \sqrt{2}\,\theta_R\,\mathcal{F}\,,\\ \overline{\hat{z}}(\overline{\upsilon}) \;&=\; \overline{z} \;+\; \sqrt{2}\,\overline{\theta_L\zeta}_L\,, \qquad & \overline{\hat{\zeta}}(\overline{\upsilon}) \;&=\; \overline{\psi}_R \;+\; \sqrt{2}\,\overline{\theta_R\mathcal{F}}\,,\end{aligned} \tag{4.37}$$

and the twisted chiral superfield $\Sigma(\widetilde{y})$, which splits into $\hat{\sigma}(\upsilon)$ and $\hat{\lambda}(\upsilon)$,

$$\begin{aligned}\hat{\sigma}(\upsilon) \;&=\; \sigma \;-\; \sqrt{2}\,\theta_R\,\overline{\lambda}_L \qquad & \hat{\lambda}(\upsilon) \;&=\; \lambda_R \;+\; i\,\theta_R\,(iD \,+\, F_{03})\\ \overline{\hat{\sigma}}(\overline{\upsilon}) \;&=\; \overline{\sigma} \;+\; \sqrt{2}\,\overline{\theta}_R\,\lambda_L \qquad & \overline{\hat{\lambda}}(\overline{\upsilon}) \;&=\; \overline{\lambda}_R \;+\; i\,\overline{\theta}_R\,(iD \,-\, F_{03})\,.\end{aligned} \tag{4.38}$$

4.2 Construction of the deformation

The heterotic deformation is introduced using the fermionic translational degree of freedom ζ_R, which, as we just described, sits in the supermultiplet $\hat{\zeta}$. The first thing this new degree of freedom needs is the kinetic term

$$\int d^2\theta_R\;\hat{\zeta}_R\,\overline{\hat{\zeta}}_R \;=\; \overline{\zeta}_R\,i\partial_L\zeta_R \;+\; \overline{\mathcal{F}}\,\mathcal{F}\,. \tag{4.39}$$

In the $\mathcal{N} = (0,2)$ space, a superpotential, as a holomorphic function $J(\hat{\sigma})$ is constructed using an arbitrary fermionic multiplet. It is obvious that for our purposes $\hat{\zeta}$ is the right fermionic superfield, giving rise to terms like

$$\int d\theta_R \, \hat{\zeta} \, J(\hat{\sigma}) \, . \tag{4.40}$$

As for the superpotential function $J(\hat{\sigma})$ itself, it comes from the four-dimensional deformation superpotential — if applicable — taken as a function of $\hat{\sigma}$,

$$J(\hat{\sigma}) \quad = \quad \frac{\partial \, \mathcal{W}_{\text{4-d}}(\hat{\sigma})}{\partial \hat{\sigma}} \, . \tag{4.41}$$

Remind, that originally $\mathcal{W}_{\text{4-d}}$ is a function of $\mathcal{A}$. We stick to the quadratic deformation, generically without making references to the four-dimensional bulk theory, in which case

$$J(\sqrt{2}\hat{\sigma}) \quad = \quad \delta \cdot \sqrt{2}\hat{\sigma} \, , \tag{4.42}$$

where δ is the parameter of deformation.

Altogether, the part of the Lagrangian involving the supertranslational sector is

$$\mathcal{L}_{\text{het}} \quad = \quad \int d^2\theta_R \, \hat{\zeta}_R \, \hat{\overline{\zeta}}_R \quad - \quad i \int d\theta_R \, \hat{\zeta} \cdot J(\sqrt{2}\hat{\sigma}) \quad - \quad i \int d\overline{\theta}_R \, \hat{\overline{\zeta}} \cdot \overline{J}(\sqrt{2}\hat{\overline{\sigma}}) \, . \tag{4.43}$$

In components, this is

$$\begin{aligned} \mathcal{L}_{\text{het}} \quad = \quad & \overline{\zeta}_R \, i\partial_L \zeta_R \quad + \quad \overline{\mathcal{F}} \, \mathcal{F} \quad - \quad i \sqrt{2}\delta \, \mathcal{F} \cdot \sqrt{2}\sigma \quad - \quad i \sqrt{2}\overline{\delta} \, \overline{\mathcal{F}} \cdot \sqrt{2}\overline{\sigma} \quad + \\ & + \quad i \sqrt{2}\delta \cdot \sqrt{2}\overline{\lambda}_L \zeta_R \quad + \quad i \sqrt{2}\overline{\delta} \cdot \sqrt{2}\overline{\zeta}_R \lambda_L \, . \end{aligned} \tag{4.44}$$

With the auxiliary field $\mathcal{F}$ excluded, this produces a quadratic potential for σ,

$$\mathcal{L}_{\text{het}} \quad = \quad \overline{\zeta}_R \, i\partial_L \zeta_R \; + \; |\sqrt{2}\delta|^2 \left|\sqrt{2}\sigma\right|^2 \; + \; i \sqrt{2}\delta \cdot \sqrt{2}\overline{\lambda}_L \zeta_R \; + \; i \sqrt{2}\overline{\delta} \cdot \sqrt{2}\overline{\zeta}_R \lambda_L \, . \tag{4.45}$$

Since the introduction of this deformation did not involve the variables n^l or ξ^l of the original CP^{N-1} theory, the above terms are just added on top of (3.32) (paying due respect to the factor $N/4\pi$ in the latter equation) to produce the solution of the deformed theory.

5 Twisted Masses

{stwist}

Generalization to the theory with twisted masses is quite trivial. In essense, every instance of σ in the Lagrangian is replaced with the difference $\sigma - m_k$. Since all masses are generically different, the overall factor $N/4\pi$ is replaced by $1/4\pi$ and a sum over k.

It is straightforward to write a superfield generalization of this. In Eq. (2.20), every occurence of superfield Σ should be replaced with $\Sigma - m_k$, and, in addition, we have to introduce N fields S_k. It is of absolutely no effort to also include the heterotic deformation (for which, however, replacement $\sigma \to \sigma - m_k$ is not done). The overall supersymmetric form of the effective Lagrangian with twisted masses and heterotic deformation is,

$$
\begin{aligned}
4\pi\,\mathcal{L} \;\; = \;\; & - \sum_k \Bigg(\int d^4\theta \, \frac{1}{2} \Big| \ln\big(\sqrt{2}\Sigma \, - \, m_k\big) \Big|^2 \;+ \\
& + \; i \int d^2\tilde{\theta} \, \Big(\big(\sqrt{2}\Sigma \, - \, m_k\big) \, \ln\big(\sqrt{2}\Sigma \, - \, m_k\big) \; - \; \big(\sqrt{2}\Sigma \, - \, m_k\big) \Big) \; + \\
& + \; \frac{1}{4} \int d^4\theta \, \ln\big(\sqrt{2}\overline{\Sigma} \, - \, \overline{m}_k\big) \times \\
& \qquad \times \Bigg(\Big(1 + \frac{\sqrt{2}\Sigma \, - \, m_k}{S_k} \Big)^2 \ln\Bigg(1 + \frac{S_k}{\sqrt{2}\Sigma \, - \, m_k} \Bigg) \; - \; \frac{\sqrt{2}\Sigma \, - \, m_k}{S_k} \Bigg) \Bigg) \; + \\
& + \; 4\pi \int d^2\theta_R \; \hat{\zeta}_R \, \bar{\hat{\zeta}}_R \;\; - \;\; 4\pi i \int d\theta_R \; \hat{\zeta} \cdot J(\sqrt{2}\hat{\sigma}) \;\; + \;\; \text{h.c.}
\end{aligned}
\tag{5.46}
$$

{hetmass}

Hermitean conjugate is understood here to be added only to those terms that require it (in particular, the first term in Eq. (5.46) does not need a conjugate). We have introduced the obvious notation,

$$
S_k \;\; = \;\; \frac{i}{2} \, \overline{D}_R D_L \, \ln\big(\sqrt{2}\overline{\Sigma} - \overline{m}_k\big) \, , \qquad \qquad \overline{S}_k \;\; = \;\; \frac{i}{2} \, \overline{D}_L D_R \, \ln\big(\sqrt{2}\Sigma - m_k\big) \, . \tag{5.47}
$$

6 Conclusions

{sfinal}

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A Notations

{app:notations

A.1 Twisted superfields

Twisted superfields exist in two dimensions and are defined by a "twisted" chirality condition

$$D_L\,\Sigma \;=\; \overline{D}_R\,\Sigma \;=\; 0\,. \tag{A.1}$$ {gentwist}

Analogously, for twisted anti-chiral superfields,

$$D_R\,\overline{\Sigma} \;=\; \overline{D}_L\,\overline{\Sigma} \;=\; 0\,. \tag{A.2}$$ {genantitwist}

The reason these conditions look so similar to those for chiral superfields is that there is no ideomatic difference between chiral and twisted-chiral superfields. To say more, they are interchanged by the action of mirror symmetry — the transposition of supercharges turns condition (A.1) into the one for a chiral superfield. This way, as in the case with the chiral superfields, the constraints (A.1) and (A.2) are solved by letting the superfields be arbitrary functions of "chiral" variables $\widetilde{y}^\mu$,

$$\begin{aligned}
\Sigma &= \Sigma(\widetilde{y}^\mu) & \overline{\Sigma} &= \overline{\Sigma}(\overline{\widetilde{y}}^\mu)\\
\widetilde{y}^0 &= x^0 + i\left(\overline{\theta}_R\theta_R - \overline{\theta}_L\theta_L\right) & \overline{\widetilde{y}}^0 &= x^0 - i\left(\overline{\theta}_R\theta_R - \overline{\theta}_L\theta_L\right)\\
\widetilde{y}^3 &= x^3 + \left(\overline{\theta}_R\theta_R + \overline{\theta}_L\theta_L\right) & \overline{\widetilde{y}}^3 &= x^3 - \left(\overline{\theta}_R\theta_R + \overline{\theta}_L\theta_L\right)\,,
\end{aligned}$$

after which they will have the usual "chiral" component expansion

$$\begin{aligned}
\Sigma(\widetilde{y}) &= \sigma(\widetilde{y}) - \sqrt{2}\,\theta_R\overline{\lambda}_L + \sqrt{2}\,\overline{\theta}_L\lambda_R + \sqrt{2}\,\theta_R\overline{\theta}_L\,\widetilde{F}\\
\overline{\Sigma}(\overline{\widetilde{y}}) &= \overline{\sigma}(\overline{\widetilde{y}}) - \sqrt{2}\,\theta_L\overline{\lambda}_R + \sqrt{2}\,\overline{\theta}_R\lambda_L + \sqrt{2}\,\theta_L\overline{\theta}_R\,\overline{\widetilde{F}}\,.
\end{aligned}$$

Here we understand that each function on the right hand side depends on $\widetilde{y}^\mu$ and $\overline{\widetilde{y}}^\mu$, correspondingly.

Exactly the same way as with chiral superfields, one constructs *twisted superpotentials* $\widetilde{\mathcal{W}}(\Sigma)$, just as functions that depend on Σ holomorphically. One then performs the twisted $d^2\tilde{\theta}$ integration as,

$$\int d^2\tilde{\theta}\,\widetilde{\mathcal{W}}(\Sigma) \;=\; \frac{1}{2}\,\overline{D}_L\,D_R\,\widetilde{\mathcal{W}}(\Sigma)\Big|\,, \qquad \int d^2\bar{\tilde{\theta}}\,\overline{\widetilde{\mathcal{W}}}(\overline{\Sigma}) \;=\; \frac{1}{2}\,\overline{D}_R\,D_L\,\overline{\widetilde{\mathcal{W}}}(\overline{\Sigma})\Big|\,. \tag{A.3}$$

And, of course, one can perform the full superspace integration of twisted superfields, provided that this holomorphicity is broken (*e.g.* by putting both chiral and anti-chiral factors),

$$\int d^4\theta \, \overline{\Sigma} \, \Sigma \, , \tag{A.4}$$

or the result will obviously be a total derivative.

One famous example of a twisted superfield is the *fieldstrength* of a $\mathcal{N} = (2,2)$ gauge supermultiplet V,

$$\Sigma \;=\; \frac{i}{\sqrt{2}} D_L \, \overline{D}_R \, V \, , \qquad\qquad \overline{\Sigma} \;=\; \frac{i}{\sqrt{2}} D_R \, \overline{D}_L \, V \, . \tag{A.5}$$

In components it takes the form

$$\begin{aligned} \Sigma(\widetilde{y}) \;&=\; \sigma(\widetilde{y}) \;-\; \sqrt{2}\,\theta_R \overline{\lambda}_L \;+\; \sqrt{2}\,\overline{\theta}_L \lambda_R \;+\; \sqrt{2}\,\theta_R \overline{\theta}_L \Big(D \;-\; i\,F_{03} \Big) \\ \overline{\Sigma}(\overline{\widetilde{y}}) \;&=\; \overline{\sigma}(\overline{\widetilde{y}}) \;-\; \sqrt{2}\,\theta_L \overline{\lambda}_R \;+\; \sqrt{2}\,\overline{\theta}_R \lambda_L \;+\; \sqrt{2}\,\theta_L \overline{\theta}_R \Big(D \;+\; i\,F_{03} \Big) \, . \end{aligned}$$

A.2 $\mathcal{N} = (0, 2)$ superfields

We define $\mathcal{N} = (0,2)$ superspace via reduction of $\mathcal{N} = (2,2)$ superspace by putting

$$\theta_L \;=\; \overline{\theta}_L \;=\; 0 \, . \tag{A.6}$$

Each chiral and twisted-chiral $\mathcal{N} = (2,2)$ superfield this way splits into two $\mathcal{N} = (0,2)$ superfields. They still, however, retain their property of holomorphicity. While chiral superfields are usually described as being dependent on a "holomorphic" variable y^μ,

$$y^\mu \;=\; x^\mu \;+\; i\,\overline{\theta\sigma}_\mu \theta \, , \tag{A.7}$$

and twisted-chiral as dependent on a twisted variable $\widetilde{y}^\mu$,

$$\widetilde{y}^\mu \;=\; x^\mu \;+\; i\,\overline{\theta\widetilde{\sigma}}_\mu \theta \, , \tag{A.8}$$

the distinction between the two variables vanishes upon reduction to $\mathcal{N} = (0,2)$ superspace. As a result, $\mathcal{N} = (0,2)$ superspace defines only one kind of holomorphic superfields — chiral $\mathcal{N}=(0,2)$ superfields, which depend upon the reduced variables υ^μ

$$\begin{aligned}\upsilon^0 &= x^0 + i\,\overline{\theta}_R\theta_R\\ \upsilon^3 &= x^3 + i\,\overline{\theta}_R\theta_R\,.\end{aligned}\tag{A.9}$$

To make a distinction with $\mathcal{N}=(2,2)$ superfields, we denote the $\mathcal{N}=(0,2)$ superfields by symbols with a hat on top — $\hat{\sigma}$, $\hat{\xi}$, *etc.*

Chiral superfields $\Phi(y)$ split into $\mathcal{N}=(0,2)$ superfields $\hat{\phi}(\upsilon)$ and $\hat{\xi}(\upsilon)$,

$$\begin{aligned}\Phi(y) &\longrightarrow \hat{\phi}(\upsilon) + \sqrt{2}\,\theta_L\,\hat{\xi}(\upsilon)\,,\\ \overline{\Phi}(\overline{y}) &\longrightarrow \overline{\hat{\phi}}(\overline{\upsilon}) - \sqrt{2}\,\overline{\theta}_L\,\overline{\hat{\xi}}(\overline{\upsilon})\,,\end{aligned}\tag{A.10}$$ {Phisplit}

while twisted-chiral superfields $\Sigma(\widetilde{y})$ split into $\hat{\sigma}(\upsilon)$ and $\hat{\lambda}(\upsilon)$,

$$\begin{aligned}\Sigma(\widetilde{y}) &\longrightarrow \hat{\sigma}(\upsilon) + \sqrt{2}\,\overline{\theta}_L\,\hat{\lambda}(\upsilon)\,,\\ \overline{\Sigma}(\overline{\widetilde{y}}) &\longrightarrow \overline{\hat{\sigma}}(\overline{\upsilon}) - \sqrt{2}\,\theta_L\,\overline{\hat{\lambda}}(\upsilon)\,.\end{aligned}\tag{A.11}$$ {Sigmasplit}

We alert that relations (A.10) and (A.11) have only symbolical meaning demonstrating the effect of splitting, while there is no equality: the right-hand sides have incomplete dependence on θ_L, $\overline{\theta}_L$.

The individual $\mathcal{N}=(0,2)$ superfields have a quite straightforward component expansion,

$$\begin{aligned}\hat{\phi}(\upsilon) &= \phi - \sqrt{2}\,\theta_R\,\psi_L\,, &\qquad \hat{\xi}(\upsilon) &= \psi_R + \sqrt{2}\,\theta_R\,F\,,\\ \overline{\hat{\phi}}(\overline{\upsilon}) &= \overline{\phi} + \sqrt{2}\,\overline{\theta_L\psi}_L\,, &\qquad \overline{\hat{\xi}}(\overline{\upsilon}) &= \overline{\psi}_R + \sqrt{2}\,\overline{\theta_R F}\,,\end{aligned}\tag{A.12}$$

and similarly do the ones that arise from splitting of the the twisted chiral superfield $\Sigma(\widetilde{y})$,

$$\begin{aligned}\hat{\sigma}(\upsilon) &= \sigma - \sqrt{2}\,\theta_R\,\overline{\lambda}_L\,, &\qquad \hat{\lambda}(\upsilon) &= \lambda_R - \theta_R\,\widetilde{F}\,,\\ \overline{\hat{\sigma}}(\overline{\upsilon}) &= \overline{\sigma} + \sqrt{2}\,\overline{\theta}_R\,\lambda_L\,, &\qquad \overline{\hat{\lambda}}(\overline{\upsilon}) &= \overline{\lambda}_R - \overline{\theta}_R\,\overline{\widetilde{F}}\,.\end{aligned}\tag{A.13}$$

It is interesting to note that the simple structure of splitting shown in (A.10) and (A.11) makes fermionic superfields in $\mathcal{N}=(0,2)$ superspace much more ubiqutious than in $\mathcal{N} = (2,2)$ superspace. And the first example to this is the $\mathcal{N} = (0,2)$ superpotential. It has to be fermionic because the integration over *half* of the $\mathcal{N} = (0,2)$ superspace $d\theta_R$ is such. A superpotential can be constructed using an arbitrary holomorphic function — say $J(\hat{\sigma})$, and by multiplying it by an arbitrary (but still chiral) fermionic multiplet — say $\hat{\rho}$,

$$\int d\theta_R \, \hat{\rho} \, J(\hat{\sigma}) \,. \tag{A.14}$$

"Full" superspace integrals can conventionally be built using both chiral and antichiral fields,

$$\int d^2\theta_R \, \hat{\xi} \, \hat{\overline{\xi}} \quad = \quad \overline{\psi}_R \, i\partial_L \psi_R \quad + \quad \overline{F} \, F \,. \tag{A.15}$$

B Component Expansion of the Effective Action

{app:expansion

Here we give the complete component expansion of expression (2.20). Typically, one would be interested in a specific limit of this expression, such as the bosonic part of it, or the constant bosonic part, or an approximation in the certain number of space-time derivatives (some approximations, however, are easier to derive from the series representation (2.18)).

We make a remark that, according to [2], the whole fermionic part of the below effective Lagrangian can be obtained from its bosonic part simply via a replacement

$$\sqrt{2}\sigma \quad \longrightarrow \quad \sqrt{2}\sigma \quad + \quad 2\,\frac{i\,\overline{\lambda}_L \lambda_R}{iD + F_{03}} \,. \tag{B.16}$$

It is useful to know the lowest component of the superfield S,

$$s \quad = \quad S\Big| \quad = \quad \frac{\sqrt{2}\overline{\sigma}\,(iD \, - \, F_{03}) \; - \; 2\,i\,\overline{\lambda}_R \lambda_L}{\left(\sqrt{2}\,\overline{\sigma}\right)^2} \,,$$

$$\overline{s} \quad = \quad \overline{S}\Big| \quad = \quad \frac{\sqrt{2}\sigma\,(iD \, + \, F_{03}) \; - \; 2\,i\,\overline{\lambda}_L \lambda_R}{\left(\sqrt{2}\,\sigma\right)^2} \,.$$

In the expression below, however, we extensively make use of the lowest component of the ratio $S/(\sqrt{2}\Sigma)$, which we denote as p,

$$p \;=\; \frac{S}{\sqrt{2}\Sigma}\bigg| \;=\; \frac{1}{\left|\sqrt{2}\sigma\right|^2}\left(iD \,-\, F_{03} \;-\; \frac{2\,i\sqrt{2}\sigma\overline{\lambda}_R\lambda_L}{\left|\sqrt{2}\sigma\right|^2}\right),$$

$$\overline{p} \;=\; \frac{\overline{S}}{\sqrt{2}\overline{\Sigma}}\bigg| \;=\; \frac{1}{\left|\sqrt{2}\sigma\right|^2}\left(iD \,+\, F_{03} \;-\; \frac{2\,i\sqrt{2}\overline{\sigma}\overline{\lambda}_L\lambda_R}{\left|\sqrt{2}\sigma\right|^2}\right).$$

We have,

$$\frac{4\pi}{N}\mathcal{L} \;=\; iD \;-\; F_{03}\,\log\frac{\sqrt{2}\sigma}{\sqrt{2}\overline{\sigma}} \;-\; i\,\frac{\sqrt{2}\sigma\overline{\lambda}_R\lambda_L \;+\; \sqrt{2}\overline{\sigma}\overline{\lambda}_L\lambda_R}{\left|\sqrt{2}\sigma\right|^2} \;-$$

$$-\;\frac{1}{4}\ln\sqrt{2}\sigma\,\Box\,\ln\sqrt{2}\sigma \;-\; \frac{1}{4}\ln\sqrt{2}\overline{\sigma}\,\Box\,\ln\sqrt{2}\overline{\sigma} \;-$$

$$-\;\frac{1}{2}\left(iD \,+\, F_{03} \,+\, \frac{1}{2}\Box\ln\sqrt{2}\overline{\sigma}\right)\ln\left(\left|\sqrt{2}\sigma\right|^2 + iD - F_{03} - 2i\,\frac{\overline{\lambda}_R\lambda_L}{\sqrt{2}\overline{\sigma}}\right) \;-$$

$$-\;\frac{1}{2}\left(iD \,-\, F_{03} \,+\, \frac{1}{2}\Box\ln\sqrt{2}\sigma\right)\ln\left(\left|\sqrt{2}\sigma\right|^2 + iD + F_{03} - 2i\,\frac{\overline{\lambda}_L\lambda_R}{\sqrt{2}\sigma}\right)$$

$$+\;\frac{-\,\overline{\lambda}_R\,i\overleftarrow{\mathcal{D}}_L\lambda_R \,+\, \overline{\lambda}_L\,i\mathcal{D}_R\lambda_L \,-\, \frac{1}{2}\,i\,\sqrt{2}\overline{\sigma}\overline{\lambda}_L\lambda_R \,+\, \frac{1}{2}\,\frac{1}{\sqrt{2}\overline{\sigma}}\,i\,\overline{\lambda}_R\overleftarrow{\mathcal{D}}_L\,\mathcal{D}_R\lambda_L}{\left|\sqrt{2}\sigma\right|^2 + iD - F_{03} - 2i\,\frac{\overline{\lambda}_R\lambda_L}{\sqrt{2}\overline{\sigma}}}$$

$$+\;\frac{\overline{\lambda}_R\,i\mathcal{D}_L\lambda_R \,-\, \overline{\lambda}_L\,i\overleftarrow{\mathcal{D}}_R\lambda_L \,-\, \frac{1}{2}\,i\,\sqrt{2}\sigma\overline{\lambda}_R\lambda_L \,+\, \frac{1}{2}\,\frac{1}{\sqrt{2}\sigma}\,i\,\overline{\lambda}_L\overleftarrow{\mathcal{D}}_R\,\mathcal{D}_L\lambda_R}{\left|\sqrt{2}\sigma\right|^2 + iD + F_{03} - 2i\,\frac{\overline{\lambda}_L\lambda_R}{\sqrt{2}\sigma}}$$

$$-\;\frac{1}{4}\,\frac{\left|\sqrt{2}\sigma\right|^2\left(iD + F_{03} + \Box\ln\sqrt{2}\overline{\sigma}\right)}{\left|\sqrt{2}\sigma\right|^2 + iD - F_{03} - 2i\,\frac{\overline{\lambda}_R\lambda_L}{\sqrt{2}\overline{\sigma}}} \;-\; \frac{1}{4}\,\frac{\left|\sqrt{2}\sigma\right|^2\left(iD - F_{03} + \Box\ln\sqrt{2}\sigma\right)}{\left|\sqrt{2}\sigma\right|^2 + iD + F_{03} - 2i\,\frac{\overline{\lambda}_L\lambda_R}{\sqrt{2}\sigma}}$$

$$-\;\frac{\left|\sqrt{2}\sigma\right|^2}{2}\left(1 + \frac{1}{p}\right)\frac{-\,\overline{\lambda}_R\,i\overleftarrow{\mathcal{D}}_L\lambda_R \,+\, \overline{\lambda}_L\,i\mathcal{D}_R\lambda_L \,+\, i\,\sqrt{2}\overline{\sigma}\overline{\lambda}_L\lambda_R \,+\, \frac{1}{\sqrt{2}\overline{\sigma}}\,i\,\overline{\lambda}_R\overleftarrow{\mathcal{D}}_L\,\mathcal{D}_R\lambda_L}{\left(\left|\sqrt{2}\sigma\right|^2 + iD - F_{03} - 2i\,\frac{\overline{\lambda}_R\lambda_L}{\sqrt{2}\overline{\sigma}}\right)^2}$$

$$
\begin{aligned}
&- \frac{\left|\sqrt{2}\sigma\right|^2}{2}\left(1+\frac{1}{\overline{p}}\right)\frac{\overline{\lambda}_R\, i\mathcal{D}_L\lambda_R - \overline{\lambda}_L\, i\overleftarrow{\mathcal{D}}_R\lambda_L + i\sqrt{2}\sigma\overline{\lambda}_R\lambda_L + \frac{1}{\sqrt{2}\sigma}\, i\,\overline{\lambda}_L\overleftarrow{\mathcal{D}}_R\,\mathcal{D}_L\lambda_R}{\left(\left|\sqrt{2}\sigma\right|^2 + iD + F_{03} - 2i\,\frac{\overline{\lambda}_L\lambda_R}{\sqrt{2}\sigma}\right)^2} \\
&+ \frac{1}{2\,p^3}\Bigg(-p^2\left(iD + F_{03}\right) + \frac{1}{2}\,p\,\Box\ln\sqrt{2}\overline{\sigma} - 2\,p^2\, i\,\frac{\overline{\lambda}_L\lambda_R}{\sqrt{2}\sigma} + \\
&\qquad + 2\,i\,\frac{1}{\sqrt{2}\sigma}\,\frac{\overline{\lambda}_R\overleftarrow{\mathcal{D}}_L\,\mathcal{D}_R\lambda_L}{\left(\sqrt{2}\overline{\sigma}\right)^2} - 2\,p\,\frac{\overline{\lambda}_L\, i\mathcal{D}_R\lambda_L - \overline{\lambda}_R\, i\overleftarrow{\mathcal{D}}_L\lambda_R}{\left|\sqrt{2}\sigma\right|^2}\Bigg)\cdot\ln\left(1+p\right) \\
&+ \frac{1}{2\,\overline{p}^3}\Bigg(-\overline{p}^2\left(iD - F_{03}\right) + \frac{1}{2}\,\overline{p}\,\Box\ln\sqrt{2}\sigma - 2\,\overline{p}^2\, i\,\frac{\overline{\lambda}_R\lambda_L}{\sqrt{2}\overline{\sigma}} + \\
&\qquad + 2\,i\,\frac{1}{\sqrt{2}\overline{\sigma}}\,\frac{\overline{\lambda}_L\overleftarrow{\mathcal{D}}_R\,\mathcal{D}_L\lambda_L}{\left(\sqrt{2}\sigma\right)^2} - 2\,\overline{p}\,\frac{\overline{\lambda}_R\, i\mathcal{D}_L\lambda_R - \overline{\lambda}_L\, i\overleftarrow{\mathcal{D}}_R\lambda_L}{\left|\sqrt{2}\sigma\right|^2}\Bigg)\cdot\ln\left(1+\overline{p}\right) \\
&+ \frac{1}{2\,p^2}\, i\left(\frac{-\sqrt{2}\overline{\sigma}\overline{\lambda}_L - \overline{\lambda}_R\overleftarrow{\mathcal{D}}_L}{\left|\sqrt{2}\sigma\right|^2 + iD - F_{03} - 2i\,\frac{\overline{\lambda}_R\lambda_L}{\sqrt{2}\overline{\sigma}}} + \frac{\overline{\lambda}_L}{\sqrt{2}\sigma}\right)\left(2\,p\,\lambda_R + \frac{\mathcal{D}_R\lambda_L}{\sqrt{2}\overline{\sigma}}\right) \\
&+ \frac{1}{2\,\overline{p}^2}\, i\left(\frac{-\sqrt{2}\sigma\overline{\lambda}_R - \overline{\lambda}_L\overleftarrow{\mathcal{D}}_R}{\left|\sqrt{2}\sigma\right|^2 + iD + F_{03} - 2i\,\frac{\overline{\lambda}_L\lambda_R}{\sqrt{2}\sigma}} + \frac{\overline{\lambda}_R}{\sqrt{2}\overline{\sigma}}\right)\left(2\,\overline{p}\,\lambda_L + \frac{\mathcal{D}_L\lambda_R}{\sqrt{2}\sigma}\right) \\
&+ \frac{1}{2\,p^2}\, i\left(-2\,p\,\overline{\lambda}_L + \frac{\overline{\lambda}_R\overleftarrow{\mathcal{D}}_L}{\sqrt{2}\overline{\sigma}}\right)\left(\frac{\sqrt{2}\overline{\sigma}\lambda_R - \mathcal{D}_R\lambda_L}{\left|\sqrt{2}\sigma\right|^2 + iD - F_{03} - 2i\,\frac{\overline{\lambda}_R\lambda_L}{\sqrt{2}\overline{\sigma}}} - \frac{\lambda_R}{\sqrt{2}\sigma}\right) \\
&+ \frac{1}{2\,\overline{p}^2}\, i\left(-2\,\overline{p}\,\overline{\lambda}_R + \frac{\overline{\lambda}_L\overleftarrow{\mathcal{D}}_R}{\sqrt{2}\sigma}\right)\left(\frac{\sqrt{2}\sigma\lambda_L - \mathcal{D}_L\lambda_R}{\left|\sqrt{2}\sigma\right|^2 + iD + F_{03} - 2i\,\frac{\overline{\lambda}_L\lambda_R}{\sqrt{2}\sigma}} - \frac{\lambda_L}{\sqrt{2}\overline{\sigma}}\right) \\
&- \frac{1}{4\,p}\left|\sqrt{2}\sigma\right|^2\frac{iD + F_{03} + \Box\ln\sqrt{2}\overline{\sigma}}{\left|\sqrt{2}\sigma\right|^2 + iD - F_{03} - 2i\,\frac{\overline{\lambda}_R\lambda_L}{\sqrt{2}\overline{\sigma}}} + \frac{1}{4}\left|\sqrt{2}\sigma\right|^2\frac{\overline{p}}{p}
\end{aligned} \tag{B.17}
$$

$$- \quad \frac{1}{4\,\overline{p}} \left|\sqrt{2}\sigma\right|^2 \frac{iD \,-\, F_{03} \,+\, \Box \ln \sqrt{2}\sigma}{\left|\sqrt{2}\sigma\right|^2 \,+\, iD \,+\, F_{03} \,-\, 2i\,\frac{\overline{\lambda}_L \lambda_R}{\sqrt{2}\sigma}} \quad + \quad \frac{1}{4} \left|\sqrt{2}\sigma\right|^2 \frac{p}{\overline{p}}\,.$$

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