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Low-Energy Effective Action of the Supersymmetric CP($N-1$) Model in the Large-N Limit

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Abstract

We perform a supersymmetric completion ...

1 Introduction

2 Supersymmetrization

Our potential is,

$$V_{\text{eff}} \;=\; -\frac{N}{4\pi}\Big\{\big(\big|\sqrt{2}\sigma\big|^2+iD\big)\log\big(\big|\sqrt{2}\sigma\big|^2+iD\big)-iD-\big|\sqrt{2}\sigma\big|^2\log\big|\sqrt{2}\sigma\big|^2\Big\}. \quad (2.1)$$

It can be cast in integral forms which will prove useful below,

$$\begin{aligned}\frac{4\pi}{N}V_{\text{eff}} \;&=\; -\int_0^{iD} dt\,\ln\big(\big|\sqrt{2}\sigma\big|^2\,+\,t\,\big) \;=\\ &=\; -\big|\sqrt{2}\sigma\big|^2\left(\ln\big|\sqrt{2}\sigma\big|^2\cdot x\;+\;\int_0^x \ln\left(1+x\right)dx\right). \end{aligned}\quad (2.2)$$

It can also be represented as a series,

$$\frac{4\pi}{N}V_{\text{eff}} \;=\; -\,iD\,\ln\big|\sqrt{2}\sigma\big|^2\;+\;\sum_{k\geq 1}\frac{(-1)^k}{k\,(k+1)}\,iD\left(\frac{iD}{\big|\sqrt{2}\sigma\big|^2}\right)^k. \quad (2.3)$$

The first term comes from Witten's superpotential. The first term of the series (with $k\,=\,1$) is taken into account by the kinetic term. Thus we can write our effective action as

$$\begin{aligned}\frac{4\pi}{N}\mathcal{L} \;&=\; -\int d^4\theta\,\frac{1}{2}\Big|\ln\sqrt{2}\Sigma\Big|^2\;-\;i\int d^2\tilde\theta\,\Big(\sqrt{2}\Sigma\,\ln\sqrt{2}\Sigma\;-\;\sqrt{2}\Sigma\Big)\;+\\ &+\;\frac{i}{2}\int d^2\tilde\theta\,S\sum_{k\geq 1}\frac{(-1)^k}{k\,(k+1)\,(k+2)}\left(\frac{S}{\sqrt{2}\Sigma}\right)^k\;+\;\text{h.c.},\end{aligned}\quad (2.4)$$

where

$$S \;=\; \frac{i}{2}\,\overline{D}_R D_L\ln\sqrt{2}\overline{\Sigma}\,,\qquad\qquad \overline{S}\;=\;\frac{i}{2}\,\overline{D}_L D_R\ln\sqrt{2}\Sigma\,. \quad (2.5)$$

The series in the second line can be written as a D-term. We get

$$\begin{aligned}\frac{4\pi}{N}\mathcal{L} \;&=\; -\int d^4\theta\,\frac{1}{2}\Big|\ln\sqrt{2}\Sigma\Big|^2\;-\;i\int d^2\tilde\theta\,\Big(\sqrt{2}\Sigma\,\ln\sqrt{2}\Sigma\;-\;\sqrt{2}\Sigma\Big)\;+\\ &+\;\frac{1}{2}\int d^4\theta\,\ln\sqrt{2}\overline{\Sigma}\sum_{k\geq 1}\frac{(-1)^k}{k\,(k+1)\,(k+2)}\left(\frac{S}{\sqrt{2}\Sigma}\right)^k\;+\;\text{h.c.}\end{aligned}\quad (2.6)$$

Here we can see explicitly that the only F-term is the Witten's superpotential, while all other terms are D-terms.

The last term can also be re-written as an analytic expression,

$$\frac{4\pi}{N}\,\mathcal{L} \;=\; -\int d^4\theta\,\frac{1}{2}\left|\ln\sqrt{2}\Sigma\right|^2 \;-\; i\int d^2\tilde{\theta}\,\left(\sqrt{2}\Sigma\,\ln\sqrt{2}\Sigma \;-\; \sqrt{2}\Sigma\right) \;- \\ -\;\frac{i}{4}\int d^2\tilde{\theta}\,\left(\frac{\left(S+\sqrt{2}\Sigma\right)^2}{S}\,\ln\left(1+\frac{S}{\sqrt{2}\Sigma}\right) \;-\; \sqrt{2}\Sigma\right) \;+\; \text{h.c.} \tag{2.7}$$

We wrote this expression containing the logarithm as a (twisted) superpotential, but it can be easily put into a D-term form should a need arise.

Before proceeding with finding the component version of the above expressions, let us first write out the component expansion of the "known" part of the action — that is, of the Witten's superpotential and the "superkinetic" term,

$$\begin{aligned}
\frac{4\pi}{N}\,\mathcal{L}_{(0)} \;&=\; -\int d^4\theta\,\left(\frac{1}{2}\left|\ln\sqrt{2}\Sigma\right|^2 \;+\right. \\
&\quad \left. +\; i\,d^2\tilde{\theta}\,\left(\sqrt{2}\Sigma\,\ln\sqrt{2}\Sigma \;-\; \sqrt{2}\Sigma\right) \;+\; i\,d^2\bar{\tilde{\theta}}\,\left(\sqrt{2}\overline{\Sigma}\,\ln\sqrt{2}\overline{\Sigma} \;-\; \sqrt{2}\overline{\Sigma}\right)\right) \;= \\
&=\; -\,iD\,\log|\sqrt{2}\sigma|^2 \;-\; \frac{1}{2}\,\frac{\left(iD\right)^2}{\left|\sqrt{2}\sigma\right|^2} \;-\; F_{03}\,\log\frac{\sqrt{2}\sigma}{\sqrt{2}\overline{\sigma}} \;+ \\
&+\; \frac{\left|\partial_\mu\sigma\right|^2 \;+\; \frac{1}{2}\,F_{03}^2 \;+\; \frac{1}{2}\left(\overline{\lambda}_R\,i\overleftrightarrow{\mathcal{D}}_L\lambda_R \;+\; \overline{\lambda}_L\,i\overleftrightarrow{\mathcal{D}}_R\lambda_L\right)}{\left|\sqrt{2}\sigma\right|^2} \;- \\
&-\; 2\,\frac{i\sqrt{2}\sigma\overline{\lambda}_R\lambda_L \;+\; i\sqrt{2}\,\overline{\sigma\lambda}_L\lambda_R}{\left|\sqrt{2}\sigma\right|^2} \;+\; 2\,\frac{\overline{\lambda}_R\lambda_L\overline{\lambda}_L\lambda_R}{\left|\sqrt{2}\sigma\right|^4} \;+ \\
&+\; \frac{(iD+F_{03})\,i\sqrt{2}\sigma\overline{\lambda}_R\lambda_L \;+\; (iD-F_{03})\,i\sqrt{2}\,\overline{\sigma\lambda}_L\lambda_R}{\left|\sqrt{2}\sigma\right|^4}\,.
\end{aligned} \tag{2.8}$$

The long derivatives here are *(not sure why they have different signs, shouldn't)*

$$\begin{aligned}
\mathcal{D}_L \;&=\; \partial_L \;-\; \partial_L\ln\sqrt{2}\sigma \\
\mathcal{D}_R \;&=\; \partial_R \;+\; \partial_R\ln\sqrt{2}\sigma\,,
\end{aligned} \tag{2.9}$$

with conjugate expressions for the left-ward derivatives $\overleftarrow{\mathcal{D}}_L$ and $\overleftarrow{\mathcal{D}}_R$. These derivatives include the Christoffel symbol $1/\sigma$ which arises from the fact that we have a Kähler potential $\big|\log\Sigma\big|^2$. In coordinates Σ, $\overline{\Sigma}$ the latter potential generates a metric $1/|\sigma|^2$ which is seen in the denominators in the expressions above. However, the Riemann tensor is zero for this metric, reflecting the observation that the target space is in fact flat — if one adopts $\log\Sigma$ and $\log\overline{\Sigma}$ as the coordinates, the metric disappears completely. Still, we prefer to work with the coordinates Σ and $\overline{\Sigma}$.

The first two terms in the component expansion in (2.8) form the leading part of the effective potential (2.3). The third term is the anomaly*, may need more discussion*. The terms on the fourth line in (2.8) are the kinetic terms, while the terms on the rest of the lines are Yukawa-like and quartic couplings.

We now calculate the component expansion of the series term in (2.4) in the two–derivative approximation. That is, we only retain terms with no more than two spacetime derivatives for bosons, and terms with no more than one spacetime derivative for fermions. We do so by performing the superspace integration on each of the terms in the series and then assembling the terms into logarithms. Rather than listing the resulting expression for the series by itself, we present the component expression for the whole Lagragian (2.4), which appears more compact:

$$
\begin{aligned}
\mathcal{L}_{\text{two deriv}} \;\;=\;\; & \frac{\left|\partial_\mu \sigma\right|^2}{\left|\sqrt{2}\sigma\right|^2} \;-\; F_{03}\,\log\frac{\sqrt{2}\sigma}{\sqrt{2}\overline{\sigma}} \;+\; \frac{4\pi}{N}\,V_{\text{eff}} \;- \\
- \;\; & \frac{F_{03}^2}{\left|\sqrt{2}\sigma\right|^2}\left(2\,\frac{\ln(1+x)\,-\,x}{x^2} \;+\; \frac{1}{2}\,\frac{1}{1\,+\,x}\right) \;- \\
- \;\; & \frac{\overline{\lambda}_R\, i\overleftrightarrow{\mathcal{D}}_L\lambda_R \;+\; \overline{\lambda}_L\, i\overleftrightarrow{\mathcal{D}}_R\lambda_L}{\left|\sqrt{2}\sigma\right|^2}\,\frac{\ln(1+x)\,-\,x}{x^2} \;- \\
- \;\; & 2\,\frac{i\sqrt{2}\sigma\overline{\lambda}_R\lambda_L \;+\; i\sqrt{2}\,\overline{\sigma}\overline{\lambda}_L\lambda_R}{\left|\sqrt{2}\sigma\right|^2}\,\frac{\ln(1+x)}{x} \;+ \\
+ \;\; & 4\,\frac{\overline{\lambda}_R\lambda_L\overline{\lambda}_L\lambda_R}{\left|\sqrt{2}\sigma\right|^4}\left(\frac{\ln(1+x)\,-\,x}{x^2} \;+\; \frac{1}{1\,+\,x}\right) \;+ \\
+ \;\; & \frac{1}{4}\,\Box\,\log|\sqrt{2}\sigma|^2\cdot\frac{(1-x^2)\ln(1+x)\,-\,x}{x^2} \;- \\
- \;\; & 2\,\frac{F_{03}\big(i\sqrt{2}\sigma\overline{\lambda}_R\lambda_L \;-\; i\sqrt{2}\,\overline{\sigma}\overline{\lambda}_L\lambda_R\big)}{\left|\sqrt{2}\sigma\right|^4}\,\frac{\ln(1+x)\,-\,x}{x^2}\,.
\end{aligned}
\tag{2.10}
$$

Here we have introduced a shorthand for the series expansion quantity

$$
x \;\;=\;\; \frac{iD}{\left|\sqrt{2}\sigma\right|^2}\,. \tag{2.11}
$$

We notice that the effect of adding the series to Eq. (2.8) is the multiplication of all terms by certain functions of x. In fact, the whole expression (2.8) is the leading order approximation (or subleading for some of the terms) in x of the two-derivative Langrangian (2.10).

Now we exclude the auxiliary field D to obtain the effective action as a function of σ. Due to the very complicated dependence of the action (2.10) on D, we can only do so with the accuracy of keeping up to two space-time derivatives. In order to make the procedure of resolution of D easier, we switch to using the variable x. The Lagrangian (2.10) can be split into three pieces:

$$
\mathcal{L} \;\;=\;\; \mathcal{L}_{(0)}(x) \;\;+\;\; \mathcal{L}_{(1)}(x) \;\;+\;\; \mathcal{L}_{(2)}(x)\,, \tag{2.12}
$$

each explicitly containing, correspondingly, none, one and two space-time derivatives. In particular,

$$\frac{4\pi}{N}\,\mathcal{L}_{(0)}(x) \;=\; \frac{\left|\partial_\mu \sigma\right|^2}{\left|\sqrt{2}\sigma\right|^2} \;-\; F_{03}\,\log\frac{\sqrt{2}\sigma}{\sqrt{2}\overline{\sigma}} \;+\; \frac{4\pi}{N}\,V_{\text{eff}}(x)\,. \tag{2.13}$$

Although this expression actually does contain space-time derivatives in the kinetic term for σ and in the anomaly term, the latter terms are independent of x and are unaltered during the course of resolution of x. The first-order part $\mathcal{L}_{(1)}(x)$ only includes the term on the fourth line in (2.10), *i.e.* the Yukawa-like coupling. Each part in the expansion (2.12), besides the explicit derivatives, will also contain derivatives implictly, via x. More specifically, we split the solution (to be found) of the equations of motion as,

$$x \;=\; x_0 \;+\; x_1 \;+\; \dots\,. \tag{2.14}$$

Expanding the Lagrangian perturbatively in the number of space-time derivatives, and using the equation of motion for x, one can show that to the second order in space-time derivatives the Lagrangian is

$$\mathcal{L}_{\text{two deriv}}(\sigma) \;=\; \mathcal{L}_{(0)}(x_0) \;+\; \mathcal{L}_{(1)}(x_0) \;+\; \mathcal{L}_{(2)}(x_0) \;-\; \frac{1}{2}\,\frac{\partial^2 \mathcal{L}_{(0)}}{\partial x^2}\bigg|_{x_0} \cdot x_1^2\,. \tag{2.15}$$

Here x_0 is the minimum of the potential $V_{\text{eff}}(x)$,

$$x_0 \;=\; \frac{1 \,-\, \left|\sqrt{2}\sigma\right|^2}{\left|\sqrt{2}\sigma\right|^2}\,, \tag{2.16}$$

while x_1 is

$$x_1 \;=\; -\,\frac{\partial \mathcal{L}_{(1)}}{\partial x}(x_0)\,\left(\frac{\partial^2 \mathcal{L}_{(0)}}{\partial x^2}(x_0)\right)^{-1}\,. \tag{2.17}$$

Higher terms in x are not needed for our purposes. It is only due to the presence of the Yukawa-like coupling in Eq. (2.10) — which effectively is a one-derivative term — that the expression (2.15) includes the last term depending on x_1^2. That term, however, is important as it modifies the coefficient of the quartic fermionic coupling in Eq. (2.10).

It is now straightforward to calculate expression (2.15). For the sake of presenting the results, it is helpful to introduce a variable r,

$$r \;=\; \left|\sqrt{2}\sigma\right|^2 \tag{2.18}$$

and a function $v_{\rm eff}(r)$,

$$v_{\rm eff}(r) \;=\; \frac{4\pi}{N}\, V_{\rm eff}(\sigma) \;=\; r\,\ln r \;+\; 1 \;-\; r\,. \tag{2.19}$$

The result for Eq. (2.15) can now be given as,

$$\begin{aligned}
&\frac{4\pi}{N}\,\mathcal{L}_{\text{two deriv}}(\sigma, A_\mu, \lambda) \;= \\
&= \;\frac{\left|\partial_\mu\sigma\right|^2}{r} \;-\; F_{03}\,\log\frac{\sqrt{2}\sigma}{\sqrt{2}\overline{\sigma}} \;+\; v_{\rm eff}(r) \;- \\
&+\; 2\,F_{03}^2\left(\frac{v_{\rm eff}(r)}{(1-r)^2} \;-\; \frac{r}{4}\right) \;+\; \left(\overline{\lambda}_R\, i\overleftrightarrow{\mathcal{D}_L}\lambda_R \;+\; \overline{\lambda}_L\, i\overleftrightarrow{\mathcal{D}_R}\lambda_L\right)\frac{v_{\rm eff}(r)}{(1-r)^2} \;+ \\
&+\; 2\left(i\sqrt{2}\sigma\overline{\lambda}_R\lambda_L \;+\; i\sqrt{2}\,\overline{\sigma\lambda}_L\lambda_R\right)\frac{1-r}{r}\,\ln r \;- \\
&-\; 4\,\overline{\lambda}_R\lambda_L\overline{\lambda}_L\lambda_R\left(\frac{v_{\rm eff}(r)}{r\,(1-r)^2} \;-\; \frac{1}{r} \;+\; \frac{r\left(1\;-\;r\;-\;\ln r\right)^2}{(1-r)^4}\right) \;- \\
&-\; \frac{1}{4}\,\Box\,\ln r\left(\frac{r\,v_{\rm eff}(r)}{(1-r)^2} \;-\; \ln r\right) \;+ \\
&+\; 2\,F_{03}\left(i\sqrt{2}\sigma\overline{\lambda}_R\lambda_L \;-\; i\sqrt{2}\,\overline{\sigma\lambda}_L\lambda_R\right)\frac{v_{\rm eff}(r)}{r\,(1-r)^2}\,.
\end{aligned} \tag{2.20}$$

Interestingly, most of the "coefficients" of the terms in the Lagrangian (2.20) contain the potential $v_{\rm eff}(r)$ in this or that form.

- Confirm the form of the long derivatives

3 Conclusion

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