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Propagator for a Complex Scalar Field in the Euclidean Space

$$\mathcal{L} = |\partial_\mu \phi|^2 + m^2 |\phi|^2$$

$$\mathcal{L} + \phi f + \bar{\phi}\bar{f} = \bar{\phi}\underbrace{\left(-\Box + m^2\right)}_{= M}\phi + \phi f + \bar{\phi}\bar{f} =$$

$$= \bar{\phi} M \phi + \phi f + \bar{\phi}\bar{f} \Rightarrow$$

redefine $\phi \rightarrow \phi + M^{-1}\bar{f}$, $\bar{\phi} \rightarrow \bar{\phi} + fM^{-1}$

$$\Rightarrow (\bar{\phi} + fM^{-1}) M (M^{-1}\bar{f} + \phi) + f\phi - fM^{-1}\bar{f} +$$

$$+ \bar{\phi}\bar{f} - fM^{-1}\bar{f} =$$

$$= + \cancel{\bar{\phi}\bar{f}} + \bar{\phi}M\phi + \cancel{f\phi} + \cancel{fM^{-1}\bar{f}} +$$

$$+ \cancel{f\phi} - \cancel{fM^{-1}\bar{f}} + \cancel{\bar{\phi}\bar{f}} - \cancel{fM^{-1}\bar{f}} =$$

$$= \bar{\phi} M \phi - f M^{-1} \bar{f}$$

$$\int d\phi\, d\bar{\phi}\; e^{-\mathcal{L} + \bar{\phi} f + \bar{f} \phi} =$$

$$= \int d\phi\, d\bar{\phi}\; e^{-\bar{\phi} M \phi + f M^{-1} \bar{f}} =$$

$$= e^{f M^{-1} \bar{f}} \cdot \int d\phi\, d\bar{\phi}\; e^{-\bar{\phi} M \phi}$$

the factor will get normalized away

$$Z_0(f, \bar{f}) = e^{f M^{-1} \bar{f}}$$

$$D(x, y) = \frac{\int d\phi\, d\bar{\phi}\; \bar{\phi}(x) \phi(y)\, e^{-\mathcal{L}}}{\int d\phi\, d\bar{\phi}\; e^{-\mathcal{L}}} =$$

$$= \frac{\delta}{\delta \bar{f}(x)} \frac{\delta}{\delta f(y)} Z(f, \bar{f}) \Big|_{f \bar{f} = 0} = M^{-1}(y, x)$$

What is $M^{-1}(y,x)$?

$$\int M^{-1}(y,z)\, M(z,x)\, dz \equiv \delta(y-x)$$

For $M = (-\Box_x + m^2)\,\delta(x-z)$

$$\int dz\; M^{-1}(y,z)\,\delta(x-z)(-\Box_x + m^2)\, dz$$

Let's apply it to a function $\phi(x)$:

$$f(y) \equiv \int M^{-1}(y,z)\,\delta(x-z)\,(-\Box_x + m^2)\,\phi(x)\, dx\, dz =$$

$$= \int M^{-1}(y,z)\,\delta(x-z)(-\Box_x + m^2)\,\frac{dp}{(2\pi)^d}\,\phi(p)\,e^{ipx}\, dx\, dz =$$

$$= \int M^{-1}(y,z)\,\frac{dq}{(2\pi)^d}\,e^{iq(x-z)}\,(-\Box_x + m^2)\,\frac{dp}{(2\pi)^d}\,\phi(p)\,e^{ipx}\, dx\, dz =$$

$$= \int M^{-1}(y,z)\,\frac{dq}{(2\pi)^d}\,e^{iq(x-z)}\,(p^2+m^2)\,\frac{dp}{(2\pi)^d}\,dx\,dz\,\phi(p)\,e^{ipx} =$$

$$\text{---}\; e^{ir(y-z)}\,\frac{M^{-1}(r)\,dr}{(2\pi)^d}$$

$$\int dx\; e^{ipx}\; e^{iqx} = (2\pi)^d\,\delta(p+q)$$

$$= \int M^{-1}(y,z)\,\frac{dq}{(2\pi)^d}\,e^{-iqz}\cdot \cancel{(2\pi)^d}\,\delta(p+q)\,(p^2+m^2)\,\frac{dp}{\cancel{(2\pi)^d}}\,dz\,\phi(p) =$$

$$= \int M^{-1}(y,z) \frac{dq}{(2\pi)^d} e^{-iqz} (q^2+m^2)\, dz\, \phi(q) =$$

$$= \int e^{ir(y-z)} \frac{M^{-1}(r)}{(2\pi)^d} dr \cdot \frac{dq}{(2\pi)^d} e^{-iqz} (q^2+m^2)\, \phi(q)\, dz =$$

$$\int e^{-irz-iqz}\, dz = (2\pi)^d \delta(r+q)$$

$$= \int \cancel{(2\pi)^d} \delta(r+q)\, e^{iry}\, M^{-1}(r) \frac{dr}{\cancel{(2\pi)^d}} \frac{dq}{(2\pi)^d} (q^2+m^2)\, \phi(-q) =$$

$$= \int e^{iry} M^{-1}(r) \frac{dr}{(2\pi)^d} (r^2+m^2)\, \phi(r) =$$

now we put $M^{-1}(r) = \dfrac{1}{r^2+m^2}$

$$= \int e^{iry} \frac{dr}{(2\pi)^d} \phi(r) = \phi(y)$$

Then we get that

$$M^{-1}(y,z) = \int \frac{dp}{(2\pi)^d} \frac{e^{ip(y-z)}}{p^2+m^2}$$

Once again,

$$\boxed{M^{-1}(y,z) = \int \frac{dp}{(2\pi)^d} \frac{e^{ip(y-z)}}{p^2+m^2}}$$

The propagator,

$$D(x,y) = \frac{\int d\phi\, d\bar{\phi}\, \bar{\phi}(x)\phi(y)\, e^{-\mathcal{L}}}{\int d\phi\, d\bar{\phi}\, e^{-\mathcal{L}}} =$$

$$= M^{-1}(y,x) = \int \frac{dp}{(2\pi)^d} \frac{e^{ip(y-x)}}{p^2+m^2}$$