The Mixed A_μ-σ Propagator

The anomaly term is $\quad \sim \dfrac{N}{4\pi} F_{03} \cdot \ln \dfrac{\det\sigma}{\det\bar\sigma} = -2i \dfrac{N}{4\pi} F_{03} \cdot \operatorname{Arg}\det\sigma$

When the phase of $\det\sigma$ is small, $\quad 2i \operatorname{Arg}\det\sigma \approx \dfrac{\det\sigma - \det\bar\sigma}{|\det\sigma|}$.

Together with the kinetic terms, this gives,

$$\frac{4\pi}{N}\mathcal{L} \supset \frac{1}{|\det\sigma|^2}\left\{ \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} |\partial_\mu \det\sigma|^2 - F_{03} \cdot \left[\frac{\det\sigma - \det\bar\sigma}{2i}\right] \cdot |\det\sigma| \right\}$$

Neglecting loops, we put $|\det\sigma|$ to its vacuum value Λ

$$\frac{4\pi}{N}\mathcal{L}_{\text{quadratic}} = \frac{1}{4}\frac{F_{\mu\nu}^2}{\Lambda^2} + \frac{1}{2}\frac{|\partial_\mu \det\sigma|^2}{\Lambda^2} - \Lambda^{-1} F_{03} \cdot 2i \operatorname{Im}\det\sigma$$

Redefining $\quad A_\mu \longrightarrow \Lambda \cdot A_\mu, \quad \det\sigma \longrightarrow \Lambda \cdot \det\sigma, \quad$ we find

$$\frac{4\pi}{N}\mathcal{L} = \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2}|\partial_\mu \det\sigma|^2 - 2i\Lambda \cdot F_{03} \cdot \det\pi,$$

where $\quad \det\pi \equiv \operatorname{Im}\det\sigma$

Now omit the real part,

$$\frac{4\pi}{\tilde{\nu}} \mathcal{L} = \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2}\left(\partial_\mu \sqrt{2}\,\pi\right)^2 - 2i\Lambda \cdot f_{03} \cdot \sqrt{2}\,\pi \,,$$

We also don't really need $\frac{4\pi}{\tilde{\nu}}$ for these calculations,

$$\mathcal{L}_{\text{Quadratic}} = \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2}\left(\partial_\mu \sqrt{2}\,\pi\right)^2 - 2i\Lambda \cdot \epsilon_{\mu\nu}\,\partial_\mu A_\nu \cdot \sqrt{2}\,\pi + \frac{1}{2\alpha}\left(\partial_\mu \right.$$

where $\epsilon_{\mu\nu}\,\partial_\mu A_\nu = \partial_0 A_3 - \partial_3 A_0 = f_{03}$

We've also added the gauge-fixing term.

Overall,

$$\mathcal{L}_{\text{quadratic}} = \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 + \frac{1}{2}(\partial_\mu \sqrt{2}\,\pi)^2 - 2i\lambda \epsilon_{\mu\nu}\partial_\mu A_\nu \cdot \sqrt{2}\,\pi =$$

$$= -\frac{1}{2}\begin{pmatrix} A_\mu & \sqrt{2}\,\pi \end{pmatrix} \begin{pmatrix} \Box\left(P_\perp^{\mu\nu} + \frac{1}{\alpha} P_\parallel^{\mu\nu}\right) & -2i\lambda\,\epsilon_{\rho\sigma}\partial_\rho \delta_\sigma^\mu \\ 2i\lambda\,\epsilon_{\rho\sigma}\partial_\rho\delta_\sigma^\nu & \Box \end{pmatrix} \begin{pmatrix} A_\nu \\ \sqrt{2}\,\pi \end{pmatrix}$$

Pay attention to the overall factor of $-1/2$

Although we work in the Euclidean space, we

denote $\partial_\mu \partial^\mu \equiv \Box$

As customary, here $P_\parallel^{\mu\nu} = \frac{1}{\Box}\partial_\mu \partial_\nu$

$$P_\perp^{\mu\nu} = \delta^{\mu\nu} - P_\parallel^{\mu\nu}$$

So the kinetic matrix is

$$\begin{array}{cc} & \longleftarrow \nu \longrightarrow \\ \mu & \begin{pmatrix} \square\left(\theta_{\perp}^{\mu\nu} + \frac{1}{\alpha}\theta_{\parallel}^{\mu\nu}\right) & -2i\lambda\, \epsilon_{\rho\sigma}\, \partial_{\rho}\, \delta_{\sigma}^{\mu} \\ & \\ 2i\lambda\, \epsilon_{\rho\sigma}\, \partial_{\rho}\, \delta_{\sigma}^{\nu} & \square \end{pmatrix} \end{array}$$

The propagator matrix must have the shape

$$\begin{array}{cc} & \longleftarrow \xi \longrightarrow \\ \nu & \begin{pmatrix} & \\ & \end{pmatrix} \end{array}$$

Here's our guess for the first column of the propagator,

$$\begin{pmatrix} x\, \beta_{\perp}^{\nu\xi} \;+\; y\, \beta_{\parallel}^{\nu\xi} \\ \\ z \cdot \epsilon_{x\lambda} \cdot \partial_x \partial_\lambda^{\xi} \end{pmatrix}$$

<u>First row times first column:</u>

$$x\Box\, \mathcal{P}_\perp^{\mu\xi} + \frac{y\Box}{\alpha}\, \mathcal{P}_\parallel^{\mu\xi} -$$

$$-2i\Lambda\, \epsilon_{\rho\sigma}\, \partial_\rho\, \delta_\sigma^{\mu} \cdot z\, \epsilon_{\varkappa\lambda}\, \partial_\varkappa\, \delta_\lambda^{\xi} =$$

$$\boxed{\epsilon_{\rho\sigma}\,\epsilon_{\varkappa\lambda} = \delta_\rho^{[\varkappa}\,\delta_\sigma^{\lambda]}}$$

$$= x\Box\, \mathcal{P}_\perp^{\mu\xi} + \frac{y\Box}{\alpha} \cdot \mathcal{P}_\parallel^{\mu\xi} -$$

$$-2i\Lambda\, \delta_{[\rho}^{\varkappa}\, \delta_{\sigma]}^{\lambda} \cdot z \cdot \partial_\rho\, \delta_\sigma^{\mu} \cdot \partial_\varkappa\, \delta_\lambda^{\xi} =$$

$$= x\Box\, \mathcal{P}_\perp^{\mu\xi} + \frac{y\Box}{\alpha}\, \mathcal{P}_\parallel^{\mu\xi} -$$

$$-2i\Lambda z\, \underbrace{\partial_{[\rho}\, \delta_{\sigma]}^{\mu} \cdot \partial_\rho\, \delta_\sigma^{\xi}}_{\Box\delta^{\mu\xi} - \partial^\mu\partial^\xi = \Box\,\mathcal{P}_\perp^{\mu\xi}} =$$

$$= \left(x\Box - 2i\Lambda z\Box\right) \mathcal{P}_\perp^{\mu\xi} + \frac{y\Box}{\alpha}\, \mathcal{P}_\parallel^{\mu\xi} \equiv \delta^{\mu\xi}$$

We have

$$\left.\begin{aligned} x\,\text{Ш} - 2i\lambda z\,\text{Ш} &= 1 \\ y/\alpha\,\text{Ш} &= 1 \end{aligned}\right\}$$

$$\underline{x = 1/\text{Ш} + 2i\lambda z}$$

$$\underline{y = \alpha/\text{Ш}}$$

Second row times first column:

$$2i\lambda\, \epsilon_{\rho\sigma}\, \partial_\rho \delta_\sigma^{\nu} \cdot \left(x\, \delta_{1}^{\nu\xi} + y\, \delta_{11}^{\nu\xi} \right) +$$

$$+ \ \square \cdot z\, \epsilon_{\varkappa\lambda}\, \partial_\varkappa \delta_\lambda^{\xi} \qquad =$$

Now $\partial_{[\rho} \delta_{\sigma]}^{\nu} \cdot \partial_\nu \partial_\xi = 0$ ← annihilates the parallel parts

so, out of bracket $(x \ldots + y \ldots)$ only

$x \cdot \delta^{\nu\xi}$ will contribute

$$= 2i\lambda x \cdot \epsilon_{\rho\sigma}\, \partial_\rho \delta_\sigma^{\xi} + \square z \cdot \epsilon_{\varkappa\lambda} \cdot \partial_\varkappa \delta_\lambda^{\xi} = 0.$$

$$\Rightarrow \quad 2i\lambda x + \square z = 0$$

From p.7, $x = 1/\Omega + 2i\Lambda z$

$$\Rightarrow \quad x \cdot \Omega = 1 + 2i\Lambda z \Omega \qquad (9.1)$$

coupled now with $2i\Lambda x + \Omega z = 0$ (what we just found)

$$\Rightarrow \quad \Omega z = -2i\Lambda x$$

Plug it into (9.1)

$$x\Omega = 1 + 2i\Lambda(-2i\Lambda x)$$

or

$$x\Omega = 1 + 4\Lambda^2 \cdot x$$

$$\Rightarrow \quad x = \frac{1}{\Omega - 4\Lambda^2}$$

$$\Rightarrow \quad z = -\frac{2i\Lambda x}{\Omega} = -\frac{2i\Lambda}{\Omega(\Omega - 4\Lambda^2)}$$

$y = \alpha/\Omega$ (see p.7)

Hypothesis about the second column:

$$\begin{pmatrix} -z \cdot \varepsilon_{\varkappa\lambda}\, d_\varkappa\, \delta_\lambda^{\nu} \\ \\ \\ u/n \end{pmatrix}$$

z should turn out to be the same as on p. 9

First row times second column:

$$\Box\left(\delta_\perp^{\mu\nu} + \frac{1}{\alpha}\,\delta_\parallel^{\mu\nu}\right)\cdot(-z)\,\epsilon_{\varkappa\lambda}\,\partial_\varkappa\,\delta_\lambda^{\nu} - 2i\Lambda\,\epsilon_{\rho\sigma}\,\partial_\rho\,\delta_\sigma^{\mu}\cdot u/\Box$$

$=$ again, only $\delta^{\mu\nu}$ survives $=$

$$= -z\,\Box\cdot\epsilon_{\varkappa\lambda}\,\partial_\varkappa\,\delta_\lambda^{\nu} - 2i\Lambda\,\epsilon_{\rho\sigma}\,\partial_\rho\,\delta_\sigma^{\mu}\,u/\Box = 0$$

$$\Rightarrow \quad -z\,\Box - 2i\Lambda\,u/\Box = 0$$

$$\underline{\underline{z = -\frac{2i\Lambda u}{\Box^2}}} \qquad (11.1)$$

Second row times second column:

$$2i\lambda\, \epsilon_{\rho\sigma}\, \partial_\rho \delta_\sigma^{\nu} \cdot (-z)\, \epsilon_{\varkappa\lambda}\, \partial_\varkappa \delta_\lambda^{\nu} \quad + \quad u \quad =$$

$$= \quad -2i\Lambda z \cdot \delta_\rho^{[\varkappa} \delta_\sigma^{\lambda]} \quad \partial_\rho \delta_\sigma^{\nu} \cdot \partial_\varkappa \delta_\lambda^{\nu} \quad + \quad u \quad =$$

$$= \quad -2i\Lambda z \cdot \underbrace{\delta_\rho^{\varkappa} \delta_\sigma^{\lambda}}_{\text{apply them}} \cdot \partial_{[\rho} \delta_{\sigma]}^{\nu} \cdot \partial_\varkappa \delta_\lambda^{\nu} \quad + \quad u \quad =$$

$$= \quad -2i\Lambda z \cdot \partial_{[\rho} \delta_{\sigma]}^{\nu} \cdot \partial_\rho \delta_\sigma^{\nu} \quad + \quad u \quad =$$

$$= \quad -2i\Lambda z \cdot \left[\delta^{\nu\nu} \Box \; - \; \partial_\rho \partial_\sigma \delta^{\rho\sigma} \right] \quad + \quad u \quad =$$

$$= \quad -2i\Lambda z \cdot \Box \; + \; u \quad = \quad 1$$

$$\underline{u \; - \; 2i\Lambda z \cdot \Box \; = \; 1} \qquad (12.1)$$

Equations (11.1) and (12.1):

$$z = -\frac{2i\lambda u}{\Omega^2}$$

$$u - 2i\lambda z \cdot \Omega = 1$$

$$\downarrow$$

$$u = 1 + 2i\lambda z \cdot \Omega$$

$$\Rightarrow \quad z = -\frac{2i\lambda}{\Omega^2}\left(1 + 2i\lambda z \cdot \Omega\right) = -\frac{2i\lambda}{\Omega^2} + (+4\lambda^2)\frac{z}{\Omega}$$

$$z\left(1 - 4\lambda^2/\Omega\right) = -\frac{2i\lambda}{\Omega^2}$$

or, $$z = -\frac{2i\lambda}{\Omega(\Omega - 4\lambda^2)}$$

Again, as was anticipated.

Then

$$u = -\frac{\Omega^2}{2i\lambda} z = -\frac{\Omega^2}{2i\lambda} \cdot (-1) \frac{2i\lambda}{\Omega(\Omega - 4\lambda^2)} =$$

$$= \frac{\Omega}{\Omega - 4\lambda^2}$$

Together,

$$z = -\frac{2i\lambda}{\Omega(\Omega - 4\lambda^2)}$$

$$u = \frac{\Omega}{\Omega - 4\lambda^2} \Rightarrow \text{element}(2,2) \text{ is}$$

$$u/\Omega = \frac{1}{\Omega - 4\lambda^2}$$

The first column is (p. 9) The second column is (p. 54)

$$\begin{pmatrix} \dfrac{B_1^{\nu\xi}}{\Omega - 4\lambda^2} + \dfrac{\alpha}{\Omega} B_{11}^{\nu\xi} \\[2em] -\dfrac{2i\lambda}{\Omega - 4\lambda^2}\, \epsilon_{\varkappa\lambda} \dfrac{\delta_\varkappa \delta_\lambda^{\xi}}{\Omega} \end{pmatrix} \qquad \begin{pmatrix} \dfrac{2i\lambda}{\Omega - 4\lambda^2}\, \epsilon_{\varkappa\lambda} \dfrac{\delta_\varkappa \delta_\lambda^{\nu}}{\Omega} \\[2em] \dfrac{1}{\Omega - 4\lambda^2} \end{pmatrix}$$

Altogether, the propagator matrix is

$$
\begin{array}{cc}
 & \xi \\
\nu & \begin{pmatrix}
\dfrac{P_\perp^{\nu\xi}}{\square - 4\lambda^2} + \dfrac{\alpha P_\parallel^{\nu\xi}}{\square} & \dfrac{i\lambda}{\square - 4\lambda^2} \cdot \epsilon_{\kappa\lambda} \dfrac{\partial^{\kappa} \delta^{\lambda\nu}}{\square} \\
 & \\
-\dfrac{i\lambda}{\square - 4\lambda^2} \epsilon_{\kappa\lambda} \dfrac{\partial^{\kappa} \delta^{\lambda\xi}}{\square} & \dfrac{1}{\square - 4\lambda^2}
\end{pmatrix}
\end{array}
$$