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Kinetic Term for σ in the Effective Action

Proper Time Method

$$Z = \int dn\, d\bar{n}\, e^{-\bar{n} M n - |n|^2 \phi} = \int dn\, d\bar{n}\, e^{-\bar{n}(-\Box + m^2 + \phi) n}$$

where $m^2 = iD$, $\phi = |\nabla\sigma|^2$

$$e^{-\Gamma} = Z \propto \frac{1}{\det^N(-\Box + m^2 + \phi)}$$

$$\Gamma = \ln \det^N(-\Box + m^2 + \phi) = N \operatorname{Tr} \ln\left[-\Box + m^2 + \phi\right] =$$

$$= N \operatorname{Tr} \int dm^2 \frac{1}{-\Box + m^2 + \phi} = N \operatorname{Tr} \int dm^2 \int_0^\infty e^{-(-\Box + m^2 + \phi)s}\, ds =$$

$$= -N \operatorname{Tr} \int_0^\infty \frac{ds}{s} e^{-(-\Box + m^2 + \phi)s} =$$

$$= -N \int \langle x| \frac{ds}{s} e^{-(m^2+\phi-\Box)s} |x\rangle \, dx =$$

$$1 = \int \frac{dp}{(2\pi)^d} |p\rangle\langle p|$$

$$= -N \int \langle x|p\rangle \frac{ds}{s} \frac{dp}{(2\pi)^d} dx \, e^{-(m^2+\phi-\Box)s} \langle p|x\rangle =$$

$$\langle p|x\rangle = e^{ipx}$$

$$= -N \int \frac{dx\,dp}{(2\pi)^d} \langle x|p\rangle \frac{ds}{s} e^{-(m^2+\phi-\Box)s} \langle p|x\rangle =$$

$$= -N \int \frac{dx\,dp}{(2\pi)^d} \frac{ds}{s} e^{-ipx} e^{-(m^2+\phi-\Box)s} e^{ipx} =$$

$$= -N \int \frac{dx\,dp}{(2\pi)^d} \frac{ds}{s} e^{-m^2 s} e^{-ipx} e^{(\Box-\phi)s} e^{ipx}$$

$$e^{(\Omega - \phi)s} =$$

- $[\Omega s, -\phi s] = -s^2\left(\Omega(\phi) + 2\partial_\mu\phi\,\partial_\mu\right)$

need only two derivatives

- $[\Omega s\,[\Omega s, -\phi s]] =$

$= -s^3\,[\Omega, \Omega(\phi) + 2\partial_\mu\phi\,\partial_\mu] =$

$\simeq -s^3\,[\Omega, 2\partial_\mu\phi]\,\partial_\mu \simeq -4s^3\,\partial_\mu\partial_\nu(\phi)\cdot\partial_\mu\partial_\nu$

- $[-\phi s\,[\Omega s, -\phi s]] = s^3\,[\phi, \Omega(\phi) + 2\partial_\mu\phi\,\partial_\mu] =$

$= s^3\, 2\partial_\mu\phi\cdot[\phi\ \partial_\mu] = -2s^3\,\partial_\mu\phi\cdot\partial_\mu\phi$

- all triple commutators will have too many derivatives or will vanish

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$$e^{X+Y} = \left(e^{X/N} e^{Y/N}\right)^N_{N\to\infty}$$

Zassenhaus formula

$$e^{X+Y} = e^X e^Y e^{-\frac{1}{2}[X,Y]} e^{\frac{1}{3}[Y[X,Y]] + \frac{1}{6}[X[X,Y]]} \cdots\cdots$$

In our case,

$$= e^{(\Box-\phi)s} = e^{-\phi s} e^{\Box s} e^{-\frac{1}{2}[-\phi s, \Box s]} \cdot e^{\frac{1}{3}[\Box s[-\phi s, \Box s]] + \frac{1}{6}[-\phi s[-\phi s, \Box s]]} =$$

$$= e^{-\phi s} e^{\Box s} e^{\frac{1}{2}[\Box s, -\phi s]} e^{-\frac{1}{3}[\Box s[\Box s, -\phi s]] - \frac{1}{6}[-\phi s[\Box s, -\phi s]]} =$$

$$= e^{-\phi s} e^{\Box s} e^{-\frac{s^2}{2}(\Box\phi + 2\partial_\mu\phi\partial_\mu)} \times$$

$$\times e^{-\frac{1}{3}(-4)s^3 \partial_\mu\partial_\nu(\phi)\partial_\mu\partial_\nu} e^{-\frac{1}{6}(-2)s^3 \partial_\mu\phi\partial_\mu\phi} =$$

$$= e^{-\phi s} e^{\Box s} e^{-\frac{s^2}{2}(\Box\phi + 2\partial_\mu\phi\partial_\mu)} \times$$

$$\times e^{\frac{4}{3}s^3 \partial_\mu\partial_\nu(\phi)\partial_\mu\partial_\nu} \cdot e^{\frac{1}{3}s^3 \partial_\mu\phi\partial_\mu\phi} =$$

$$= e^{-\phi s + \frac{1}{3} s^3 \partial_\mu \phi \partial_\mu \phi} \times$$

$$\times e^{\Box s} e^{-\frac{s^2}{2}(\Box \phi + 2 \partial_\mu \phi \partial_\mu)} \times e^{\frac{4}{3} s^3 \partial_\mu \partial_\nu (\phi) \partial_\mu \partial_\nu} =$$

$$= e^{-\phi s + \frac{1}{3} s^3 \partial_\mu \phi \partial_\mu \phi} \times e^{-\frac{s^2}{2} \Box \phi}$$

$$\times e^{\Box s} \underbrace{e^{-s^2 \partial_\mu \phi \cdot \partial_\mu}}_{\text{need only up to 2nd power}} e^{\frac{4}{3} s^3 \partial_\mu \partial_\nu \phi \cdot \partial_\mu \partial_\nu} =$$

$$= e^{-\phi s - \frac{s^2}{2} \Box \phi + \frac{1}{3} s^3 \partial_\mu \phi \partial_\mu \phi} \times$$

$$\times e^{\Box s} \left[1 - s^2 \partial_\mu \phi \partial_\mu + \frac{s^4}{2} \partial_\mu \phi \partial_\nu \phi \partial_\mu \partial_\nu \right] \times$$

$$\times \underbrace{e^{\frac{4}{3} s^3 \partial_\mu \partial_\nu \phi \partial_\mu \partial_\nu}}_{\text{only 1st power}} =$$

$$\approx e^{-\phi s}\left(1 - \frac{s^2}{2}\Box\phi + \frac{1}{3}s^3(\partial_\mu\phi)^2\right)\cdot$$

$$\cdot\, e^{\Box s}\left\{1 - s^2\partial_\mu\phi\,\partial_\mu + \frac{s^4}{2}\partial_\mu\phi\,\partial_\nu\phi\,\partial_\mu\partial_\nu\right\}\cdot$$

$$\cdot\left\{1 + \frac{4}{3}s^3\,\partial_\mu\partial_\nu\phi\,\partial_\mu\partial_\nu\right\} =$$

$$\approx e^{-\phi s}\left\{1 - \frac{s^2}{2}\Box\phi + \frac{1}{3}s^3(\partial_\mu\phi)^2\right\}\cdot$$

$$\cdot\, e^{\Box s}\left\{1 - s^2\partial_\mu\phi\,\partial_\mu + \frac{s^4}{2}\partial_\mu\phi\,\partial_\nu\phi\,\partial_\mu\partial_\nu + \right.$$

$$\left. + \frac{4}{3}s^3\,\partial_\mu\partial_\nu\phi\,\partial_\mu\partial_\nu + \dots\right\} =$$

$$+ \; e^{\Box s}(-s^2)\,\partial_\mu\phi\,\partial_\mu =$$

$$= \left[1 + \Box s + \tfrac{1}{2}(\Box s)^2 + \dots\right](-s^2)\,\partial_\mu\phi\,\partial_\mu =$$

$$= (-s^2)\left[1 + \Box s + \tfrac{1}{2}(s\Box)^2 + \dots\right]\partial_\mu\phi\,\partial_\mu =$$

$$= -s^2 \sum_{j=0}^{\infty} \frac{s^j}{j!}\,\Box^j \cdot \partial_\mu\phi\;\partial_\mu =$$

$$= -s^2 \sum_{j\geq 0} \frac{s^j}{j!}\left\{\partial_\mu\phi\,\Box^j\,\partial_\mu + 2j\,\partial_\mu\partial_\nu\phi\,\partial_\mu\partial_\nu\,\Box^{j-1}\right\} =$$

$$= -s^2 \sum_{j\geq 0}\left[\partial_\mu\phi\,\partial_\mu \cdot \frac{(s\Box)^j}{j!} + 2\,\frac{s^j}{(j-1)!}\,\partial_\mu\partial_\nu\phi \cdot \overbrace{\Box^{j-1}}^{\partial_\mu\partial_\nu}\right] =$$

$$= -s^2\left[\partial_\mu\phi\,\partial_\mu \cdot e^{s\Box} + 2\sum_{j\geq 1}\partial_\mu\partial_\nu\phi\,\partial_\mu\partial_\nu\, s \cdot \frac{s^{j-1}\,\Box^{j-1}}{(j-1)!}\right] =$$

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$$= -s^2 \left[\partial_\mu \phi \, \partial_\mu \, e^{s\Box} + 2s \, \partial_\mu \partial_\nu \phi \, \partial_\mu \partial_\nu \, e^{s\Box} \right]$$

$$e^{\Box s} (-s^2) \partial_\mu \phi \, \partial_\mu =$$

$$= -s^2 \cdot \partial_\mu \phi \, \partial_\mu \, e^{s\Box} \cancel{+} \, 2s^3 \, \partial_\mu \partial_\nu \phi \, \partial_\mu \partial_\nu \, e^{s\Box} =$$

$$= -s^2 \partial_\mu \phi \, \partial_\mu \, e^{s\Box} - 2s^3 \, \partial_\mu \partial_\nu \phi \, \partial_\mu \partial_\nu \, e^{s\Box}$$

$$\simeq e^{-\phi s}\left\{1 - \frac{s^2}{2}\Box\phi + \frac{1}{3}s^3(\partial_\mu\phi)^2\right\}\cdot$$

$$\cdot\left\{1 - s^2\partial_\mu\phi\,\partial_\mu - 2s^3\partial_\mu\partial_\nu\phi\,\partial_\mu\partial_\nu + \frac{s^4}{2}\partial_\mu\phi\,\partial_\nu\phi\,\partial_\mu\partial_\nu + \right.$$

$$\left. + \frac{4}{3}s^3\partial_\mu\partial_\nu\phi\,\partial_\mu\partial_\nu + \dots\right\} e^{s\Box} \simeq$$

$$\simeq e^{-\phi s}\left\{1 - \frac{s^2}{2}\Box\phi + \frac{1}{3}s^3(\partial_\mu\phi)^2\right\}\cdot$$

$$\cdot\left\{1 - s^2\partial_\mu\phi\,\partial_\mu - \frac{2}{3}s^3\partial_\mu\partial_\nu\phi\,\partial_\mu\partial_\nu + \frac{s^4}{2}\partial_\mu\phi\,\partial_\nu\phi\,\partial_\mu\partial_\nu\right\}\cdot$$

$$\cdot e^{s\Box} \simeq$$

$$\simeq e^{-\phi s}\left\{1 - s^2\partial_\mu\phi\,\partial_\mu - \frac{s^2}{2}\Box\phi + \right.$$

$$\left. + \frac{s^3}{3}(\partial_\mu\phi)^2 - \frac{2}{3}s^3\partial_\mu\partial_\nu\phi\,\partial_\mu\partial_\nu + \frac{s^4}{2}\partial_\mu\phi\,\partial_\nu\phi\,\partial_\mu\partial_\nu\right\} e^{s\Box} \simeq$$

$$\simeq e^{(\Box-\phi)s}$$

$$\int \frac{dp}{(2\pi)^d}\, e^{-ipx}\, e^{(\Box - \phi)s}\, e^{ipx} =$$

$$= \int \frac{dp}{(2\pi)^d} e^{-ipx} \cdot$$

$$\cdot\, e^{-\phi s}\Big[1 - s^2 \partial_\mu \phi\, \partial_\mu - \frac{s^2}{2}\Box\phi +$$

$$+ \frac{s^3}{3}(\partial_\mu \phi)^2 - 2/_3\, s^3 \partial_\mu \partial_\nu \phi \cdot \partial_\mu \partial_\nu + \frac{s^4}{2}\, \partial_\mu \phi\, \partial_\nu \phi\, \partial_\mu \partial_\nu \Big] e^{s\Box} \cdot$$

$$\cdot\, e^{ipx} = \text{averaging} =$$

$$= \int \frac{dp}{(2\pi)^d} e^{-\phi s}\Big\{1 \cancel{+} \frac{s^2}{2}\boxed{p^2}\phi +$$

$$+ \frac{s^3}{3}(\partial_\mu \phi)^2 + \frac{1}{3} s^3 \Box\phi \cdot p^2 - \frac{s^4}{4}(\partial_\mu \phi)^2 p^2\Big] e^{-sp^2} =$$

$$= e^{-\phi s} \int \frac{dp}{(2\pi)^d}\Big[e^{-sp^2}\cdot\Big[1 - \frac{s^2}{2}\Box\phi + \frac{s^3}{3}(\partial_\mu \phi)^2\Big] +$$

$$+ p^2 e^{-sp^2}\Big[\frac{s^3}{3}\Box\phi - \frac{s^4}{4}(\partial_\mu \phi)^2\Big]\Big] =$$

$$= \frac{e^{-\phi s}}{4\pi}\left\{\int dp^2\, e^{-sp^2} * \left\{1 - \frac{s^2}{2}\Box\phi + \frac{s^3}{3}(\partial_\mu\phi)^2\right\} + \right.$$

$$\left. + \int dp^2\, p^2 e^{-sp^2}\left\{\frac{s^3}{3}\Box\phi - \frac{s^4}{4}(\partial_\mu\phi)^2\right\}\right\} =$$

$$\int dp^2\, e^{-sp^2} = -\frac{1}{s}e^{-sp^2}\Big|_0^\infty = 1/s$$

$$\int dp^2\, p^2 e^{-sp^2} = -\frac{\partial}{\partial s}\int dp^2 e^{-sp^2} = 1/s^2$$

$$= \frac{e^{-\phi s}}{4\pi}\left\{1/s - \frac{s}{2}\Box\phi + \frac{s^2}{3}(\partial_\mu\phi)^2 + \frac{s}{3}\Box\phi - \frac{s^2}{4}(\partial_\mu\phi)^2\right\} =$$

$$= \frac{e^{-\phi s}}{4\pi}\left\{1/s - \frac{s}{6}\Box\phi + \frac{s^2}{12}(\partial_\mu\phi)^2\right\} =$$

$$= \int \frac{dp}{(2\pi)^d} e^{-ipx} e^{(\Box-\phi)s} e^{ipx}$$

$$\Gamma = -N \int dx \frac{ds}{s} e^{-m^2 s} \cdot \frac{e^{-s\phi}}{4\pi} \left[\frac{1}{s} - \frac{s}{6} \square\phi + \frac{s^2}{12} (\partial_\mu \phi)^2 \right] =$$

$$= -\frac{N}{4\pi} \int dx \, e^{-(m^2+\phi)s} \left[\frac{1}{s^2} - \frac{1}{6} \square\phi + \frac{s}{12} (\partial_\mu \phi)^2 \right] ds =$$

$$\int \frac{e^{-as}}{s^2} ds = -\frac{e^{-as}}{s} \Big|_0^\infty \cancel{+} \int \frac{a e^{-as}}{s} ds =$$

$$= -\frac{e^{-as}}{s} \cancel{+} a \int \int da \, e^{-as} \, ds =$$

$$= -\frac{e^{-as}}{s} \cancel{+} a \int da \, \frac{1}{a} e^{-as} =$$

$$= \frac{e^{-as_0}}{s_0} + a \ln a$$

$$\int_0^\infty e^{-as} ds = -\frac{1}{a} e^{-as} \Big|_0^\infty = 1/a$$

$$\int_0^\infty s \, e^{-as} ds = -\partial/\partial a \int e^{-as} ds = 1/a^2$$

$$= -\frac{N}{4\pi}\int dx \left\{ \frac{e^{-s_0(\phi+m^2)}}{s_0} + (\phi+m^2)\ln(\phi+m^2) - \right.$$

$$\left. - \frac{1}{6}\Box\phi \cdot \frac{1}{\phi+m^2} + \frac{1}{12}(\partial_\mu\phi)^2\frac{1}{(\phi+m^2)^2} \right\} \supset$$

$$\supset -\frac{N}{4\pi}\int dx \left\{ \frac{1}{6}\frac{(-\Box\phi)}{\phi+m^2} + \frac{1}{12}\frac{(\partial_\mu\phi)^2}{(\phi+m^2)^2} \right\} =$$

$$\Rightarrow \quad = \int dx \left(+\frac{N}{48\pi}\right)\frac{(\partial_\mu\phi)^2}{(\phi+m^2)^2} \subset \Gamma$$