Dear Nick,

We have analysed our knowledge of the problem with CP^{N-1} and now we think that we agree with most of the points that you make in your paper. Our general understanding and current problems are listed below, but at this point our tasks may essentially overlap with yours, finding the CMS. We find it reasonable to suggest to combine our efforts. We will treat of course any your decision with respect. Even if you choose not to, there may still be results which we may need to communicate back and forth.

Here is our general understanding of the appearance of the strong coupling states in CP^{N-1} with $\mathcal{Z}_N$ masses. There are N Argyres-Douglas points sitting evenly on the circle. Take the usual tower of dyonic kinks

$$\mathcal{D}^{(n)} \;=\; \mathcal{W}_1 \;-\; \mathcal{W}_0 \;+\; i\,n\,(m_1 - m_0)\,. \tag{1}$$

Pick one AD point, say the one which we usually call AD^0:

$$m_0^{\mathrm{AD}^0} \;=\; e^{i\pi/N}\,. \tag{2}$$

Let us fix the conventions so that the pure "monopole" state ($\mathcal{D}^{(n)}$ with $n = 0$) becomes massless at this point. The branch cuts that weave the complex plane of m^0 in agreement with the logarithms and N-th power roots in (1) work such that at the next clock-wise AD point $e^{-i\pi/N}$ (that is, $\mathrm{AD}^{(1)}$), it is the dyon $\mathcal{D}^{(1)}$ that becomes massless, then $\mathcal{D}^{(2)}$ at $\mathrm{AD}^{(2)}$, etc. Going the opposite direction, counter-clockwise, the dyon $\mathcal{D}^{(-1)}$ becomes massless at $\mathrm{AD}^{(N-1)}$, the $\mathcal{D}^{(-2)}$ at the next one, etc. That is, to reinstate, it is ordinary dyonic kinks which become massless at the Argyres-Douglas points which are N of. They are not any kind of bound states.

However, once becoming massless and passing through into the strong coupling region (into the primary CMS), these "dyons" become the mirror-symmetry kinks. There are N of them. They become the bound states. To see that one takes all N of these kinks simultaneously and pushes them outside through the one location, say $\mathrm{AD}^{(0)}$ (or, in fact, through any other place on the primary CMS curve). Now, out of the N states, there was one that was massless at $\mathrm{AD}^{(0)}$, and we called it the monopole. That state went in through $\mathrm{AD}^{(0)}$ and went out the same way, without surpassing any monodromy lines, so it obviously is still massless at $\mathrm{AD}^{(0)}$, and hence is the same monopole. All the other ones, when exiting through $\mathrm{AD}^{(0)}$ do not become massless. If you can imagine that, we dragged different dyonic states (1) from $\mathrm{AD}^{(0)}$ to the same $\mathrm{AD}^{(0)}$, each along a different closed path, so that each became massless at its designated AD point — and the logarithms have combined into such a monodromy as to turn those dyons into the bound states

$$(\mathcal{M}\cdot\mathcal{Q}_k)\,. \tag{3}$$

If, instead, we had exited the CMS through a different AD point, say $\mathrm{AD}^{(1)}$, then we would end up with the bound states involving the other quarks

$$(\mathcal{M}\cdot\mathcal{Q}_{k1})\,, \qquad \text{with} \quad \mathcal{Q}_{k1} \;=\; i\,(m_k\,-\,m_1)\,. \tag{4}$$

An example of the branch-sliced complex plane of m_0 is shown in the picture that we attach. There, all (hyperbolic-looking) branch cuts are related to their logarithms by subscripts $\sigma_* - m_k$. There are also two branch cuts of the cubic root (for σ_k), which connect AD points, and are deformed in the picture so as to pass near the origin.

Although a detailed analysis of the above monodromy was performed for CP^2 only, I am sure that it works the same way for other N. Again, the outcome is, the states that are massless at the AD points are ordinary dyons, but the states that sit at the strong coupling near the origin are the "bound states".

Our recent analysis has concentrated on the weak coupling expansion of the kink masses. To be able to see the bound states quasi-classically (that is, not relying on the BPS relation tying the mass to the central charge), one needs to calculate the one-loop correction to the classical mass, account for the anomaly, and for the eigen-value of the corresponding fermionic non-zero mode (which was found in your 1999 paper too). And, compare all this to the large-m expansion of the exact superpotential, fixing all the constants. This part is almost complete now, but we still list it below.

In terms of the non-zero modes, again, we re-derived the fermionic eigen-mode equation; we also derived the bosonic eigen-mode equation and showed that the two non-zero modes coincide and therefore have the same normalizability condition as your Eq. (35).

These are the questions which we think are left in this problem.

- *Concurrent decay modes for odd N*, here in CP^2 for simplicity:

$$(\mathcal{D}^{(\nu)}\cdot\mathcal{Q}_2) \quad\longrightarrow\quad \mathcal{D}^{(\nu)} \quad+\quad \mathcal{Q}_2 \tag{5}$$

  and

$$(\mathcal{D}^{(\nu)}\cdot\mathcal{Q}_2) \quad\longrightarrow\quad \mathcal{D}^{(\nu+1)} \quad+\quad \mathcal{Q}_{21}\,, \tag{6}$$

  where

$$Q_{21} \quad=\quad i\,(m_2 \,-\, m_1) \tag{7}$$

  is the "third" quark. Which one occurs first?

  We are worried by the fact that the decay curves for the above processes look disconnected. They should ideally be closed curves, otherwise one can escape the decay by sneaking through the hole. Furthermore, the curves should not pass through the origin (even though, contradictory enough, zero is the solution of the CMS conditions for these curves), since the mirror theory dictates that there cannot be decays in a small vicinity of the origin.

  Another point is that, to our opinion, the absence of KS relations for the process considered in our April's paper

$$(\mathcal{D}^{(\nu)}\cdot\mathcal{Q}_2) \quad\longrightarrow\quad (\mathcal{M}\cdot\mathcal{Q}_2) \quad+\quad \nu\cdot\mathcal{Q}_1 \tag{8}$$

  should be reflected in the fact that the above two decays (5) and (6) happen *before* the (8) one.

- *Decay curves at even N*,

$$(\mathcal{M}\cdot\mathcal{Q}_k) \quad\longrightarrow\quad \mathcal{M} \quad+\quad \mathcal{Q}_k\,, \qquad\qquad k \;=\; 2\,,\,...\; N-1\,. \tag{9}$$

  as well as for odd N with k other than $(N+1)/2$. Unlike the above processes, this process happens on the way from the strong coupling to the weak coupling, since the bound states are not observed semiclassically.

- *Other bound states?* Essentially, in the strongest coupling region, there are only N states. But there might be intermediate regions where bound states with more than one quark exist. There may be many such states. They could even open additional decay channels for the above single-quark bound states.

- *Semiclassical expansion of the soliton mass at the order* $O(r^0)$, and matching with the one-loop result. We have done this expansion for CP^2 carefully, for all complex masses m^0, and were able to match it with the semi-classical result up to the order of r (*i.e.* the logarithm). There is still a constant term which needs to be carefully interpreted

We are looking forward to hearing your opinion,

Sincerely,

Alexei Yung, Misha Shifman and Pasha Bolokhov