On the BPS Spectrum at the Root of the Higgs Branch

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Abstract

We study the BPS spectrum and walls of marginal stability of the $\mathcal{N} = 2$ supersymmetric theory with gauge group $SU(n)$ and $n \leq N_f < 2n$ fundamental flavours at the root of the Higgs branch in four dimensions. The strong coupling spectrum of this theory was conjectured in hep-th/9902134 to coincide with that of the two-dimensional supersymmetric $\mathbb{CP}^{2n-N_f-1}$ sigma model. Using the Kontsevich-Soibelman wall-crossing formula, we start with the conjectured strong-coupling spectrum and extrapolate it to all other regions of the moduli space. In the weakly coupled regime, our results precisely agree with the semiclassical analysis of hep-th/9902134. In addition to the usual dyons, quarks and W-bosons, the resulting weakly-coupled spectrum includes towers of bound states consisting of a dyon and one or more quarks. In the special case of $\mathbb{Z}_n$-symmetric masses, there are bound states with one quark for odd n and no bound states for even n.

1   Introduction

We consider the BPS spectrum of the $\mathcal{N} = 2$ SQCD with gauge group $SU(n)$ and $n \leq N_f < 2n$ fundamental flavours at the root of the Higgs branch in four dimensions. The central charge in this theory was shown in [2] to be the same as in the $\mathcal{N} = (2,2)$ supersymmetric $\mathbb{CP}^{2n-N_f-1}$ sigma model in two dimensions with twisted mass terms (with an appropriate identification of parameters). On this basis, the BPS spectra of the two theories were also conjectured to coincide [1]. Additional support for the conjecture was provided [3], and an explanation for the coincidence of the spectra in terms of vortex strings was given in [8, 9]. In this paper, we present further evidence for this conjecture using the Kontsevich-Soibelman wall-crossing formula for the four-dimensional theory. The finite set of BPS states of the two-dimensional theory in the strong-coupling regime is known [1, 11]. Assuming that the strong-coupling spectrum of the four-dimensional theory is indeed the same, we find the walls of marginal stability and, employing the wall-crossing formula, extrapolate the spectrum to other regions of the moduli space. In the weak-coupling region, we recover the complete semiclassical spectrum derived in [3], thus confirming our starting assumption. We also briefly discuss the relation of our results to recent work on the corresponding two dimensional theory [10].

From our analysis, the following general picture emerges. For a given magnetic charge, there is a ("primary") wall separating the strong-coupling region from the rest of the moduli space. Outside this wall, the spectrum expands and includes an infinite ("primary") tower of dyons as well as quarks and W bosons. In addition, we show that if a particular condition on the complex masses is satisfied, there is one extra ("secondary") tower of bound states consisting of a dyon and one or more quarks. Unlike the primary case, the wall-crossing formula shows that all the states in the extra tower cannot be created at a single wall. Rather, for every bound state in the tower, there is a corresponding ("secondary") wall. We also show that each secondary wall separates the primary wall from the weak-coupling region, so that all walls must be traversed in passing between strong and weak coupling.

A particular configuration of $\mathbb{Z}_n$-symmetric masses, when all n masses form a regular polygon in the complex plane, can be analysed more explicitly: we find that there exists one secondary tower of bound states with one quark for odd n and no bound states for even n.

Let us introduce our conventions in the four-dimensional theory: $N_f = n + \tilde{n}$ is the total number of flavours, $\vec{q}_e$ and $\vec{q}_m$ are the vectors of electric and magnetic charges with n components (counted by I), $\vec{S}$ is the vector of flavour charges with $n + \tilde{n}$ components (counted by i). The central charge is given by

$$Z_{(\vec{q}_e, \vec{q}_m, \vec{M})} = \vec{a}\vec{q}_e + \vec{a}_D \vec{q}_m + \vec{S}\vec{M} = \sum_{I=0}^{n-1} \left(a^I q_{e\,I} + a_{D\,I} q_m^I \right) + \sum_{i=0}^{N_f-1} S_i M_i \tag{1}$$

where $\vec{a}$ is the vacuum expectation value, $\vec{a}_D$ is its magnetic dual, $\vec{M}$ is the vector of flavour masses. We divide $\vec{S}$ and $\vec{M}$ into two pieces: $\vec{s}$ and $\vec{m}$ contain the first n components corresponding to the

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[1]By this we mean that there is a single multiplet of $\mathcal{N} = (2,2)$ SUSY in the two-dimensional theory associated with each massive multiplet of $\mathcal{N} = 2$ SUSY in the four-dimensional theory. With an appropriate identification of parameters the masses of corresponding states agree exactly. For more details see [2, 3]

massless quarks at the root of the Higgs branch, $\vec{\tilde{s}}$ and $\vec{\tilde{m}}$ contain the remaining $\tilde{n}$ components; we distinguish the remaining $\tilde{n}$ flavour components by putting a tilde above their masses, charges, and indices. The root of the Higgs branch is determined by setting $\vec{a} = -\vec{m}$; analogously, we define a magnetic dual mass $\vec{m}_D = -\vec{a}_D(\vec{a} = -\vec{m})$. Therefore, the central charge (1) reduces to,

$$Z_{(\vec{\gamma}_e, \vec{\gamma}_m, \vec{\tilde{s}})} = \vec{m}\vec{\gamma}_e + \vec{m}_D\vec{\gamma}_m + \vec{\tilde{s}}\,\vec{\tilde{m}} = \sum_{I=0}^{n-1} \left(m^I \gamma_{e\,I} + m_{D\,I} \gamma_m^I \right) + \sum_{\tilde{i}=0}^{\tilde{n}-1} s_{\tilde{i}} m_{\tilde{i}} \,, \tag{2}$$

$$\vec{\gamma}_e = -\vec{q}_e + \vec{s}, \quad \vec{\gamma}_m = -\vec{q}_m$$

Now, for each BPS state, the complete set of (electric, magnetic, and flavour) charges is $\gamma = (\vec{\gamma}_e, \vec{\gamma}_m, \vec{\tilde{s}})$; if $\vec{\tilde{s}} = \vec{0}$, we will omit the third entry.

Our approach is mainly based on the Kontsevich–Soibelman wall-crossing formula [4]. For a given charge $\gamma = (\vec{\gamma}_e, \vec{\gamma}_m)$, we define the Kontsevich–Soibelman operator acting on the so-called Darboux coordinates $\mathcal{X}_\beta$ (for any charge β) as

$$\mathcal{K}_\gamma \quad : \quad \mathcal{X}_\beta \to \mathcal{X}_\beta \left(1 - \sigma(\gamma) \mathcal{X}_\gamma \right)^{\langle \beta, \gamma \rangle} \tag{3}$$

where for any pair of charges, $\alpha = (\vec{\alpha}_e, \vec{\alpha}_m)$ and $\beta = (\vec{\beta}_e, \vec{\beta}_m)$, the symplectic product is defined as

$$\langle (\vec{\alpha}_e, \vec{\alpha}_m), (\vec{\beta}_e, \vec{\beta}_m) \rangle = -\vec{\alpha}_e \vec{\beta}_m + \vec{\alpha}_m \vec{\beta}_e \,, \tag{4}$$

and the quadratic refinement is defined as

$$\sigma(\gamma) = (-1)^{\vec{\gamma}_e \vec{\gamma}_m} \,. \tag{5}$$

Let $\Gamma(\vec{M})$ be the set of BPS states, depending on the set of masses $\vec{M}$. We associate the following operator to each point $\vec{M}$ in the moduli space:

$$S = \prod_{\gamma \in \Gamma(\vec{M})} \mathcal{K}_\gamma^{\Omega(\gamma, \vec{M})} = \text{const} \tag{6}$$

where $\Omega(\gamma, \vec{M})$ is the degeneracy of the BPS state with charge γ (giving $+1$ for dyons and -2 for W bosons); all operators (i.e., their BPS rays) are ordered clockwise (equivalently, their central charges as complex vectors are ordered counterclockwise). The statement of the wall-crossing formula is that, although the spectrum and the ordering of operators change across the moduli space, the resulting product S is constant. In principle, knowing the set of charges on one side of the wall of marginal stability, we can compute them on the other side of the wall [5]. In practice, this proceeds via the use of known identities such as (14) and (17) below, which apply for specific values of the symplectic product of the two states whose central charges become aligned at the wall.

2 The wall at strong coupling

Our starting point is the strong-coupling spectrum of the theory. As mentioned above, we start by assuming the spectrum implied by the 2d-4d conjecture of [2, 3]. For the moment, we consider only the BPS states corresponding to kinks interpolating between two neighbouring vacua in the two-dimensional theory. Without loss of generality, we can set the magnetic charge to be equal to $(-1, 1, 0, 0, \dots)$. There are exactly n such states (plus charge conjugates). Electric charges are

determined only up to a fixed shift [3]; for our purposes, it is convenient to choose this shift so that the charges of the states are,

$$
\begin{aligned}
\pm\gamma_1 &= \pm((1,0,0,0,0,\dots),(-1,1,0,0,0,\dots))\,, \\
\pm\gamma_2 &= \pm((0,1,0,0,0,\dots),(-1,1,0,0,0,\dots))\,, \\
\pm\gamma_3 &= \pm((0,0,1,0,0,\dots),(-1,1,0,0,0,\dots))\,, \\
&\dots
\end{aligned}
\tag{7}
$$

The moduli space includes Argyres–Douglas points [15] located on the boundary of the strong-coupling region. One of these states becomes massless at each of these points.

Consider the case of $\mathbb{Z}_n$-symmetric masses. We set

$$
m_I = m_0 \, \exp \frac{2\pi i I}{n}\,,
\tag{8}
$$

where m_0 remains as a free parameter which interpolates between strong and weak coupling. Then, all central charges and walls of marginal stability can be determined as functions of m_0, and the masses automatically obey,

$$
\sum_{I=0}^{n-1} m_I = 0\,.
\tag{9}
$$

A convenient expression for the magnetic dual masses in the $\mathbb{Z}_n$-symmetric case is [10] [2] ,

$$
m_{DI} = e^{2\pi i I/n} \left(n \sqrt[n]{m_0^n + \Lambda^n} + \sum_{k=0}^{n-1} m_0 e^{2\pi i k/n} \log \frac{\sqrt[n]{m_0^n + \Lambda^n} - m_0 e^{2\pi i k/n}}{\Lambda} \right)
\tag{10}
$$

where the branch is fixed by requiring that for $x \in \mathbb{R}_+$, $\sqrt[n]{x} \in \mathbb{R}_+$ and $\log x \in \mathbb{R}$, as in [10]. Then, the Argyres–Douglas points, where the strong-coupling states (7) become massless, are located at [10]

$$
m_0 = \Lambda \, \exp \frac{\pi i (2j+1)}{n}\,, \quad j \in \mathbb{Z}
\tag{11}
$$

(figure 1). As explained in [16], for $\mathbb{Z}_n$-symmetric masses, it is sufficient to consider m_0 belonging to the sector between two neighbouring Argyres–Douglas points, $\Lambda e^{\pi i/n}$ and $\Lambda e^{-\pi i/n}$, where γ_1 and γ_2 from (7) are massless.

Let us find out how the spectrum changes when $\vec{M}$ crosses the primary wall of marginal stability, where the central charges of the first two dyons in (7), γ_1 and γ_2, become aligned:

$$
\frac{Z_{\gamma_1}}{Z_{\gamma_2}} \in \mathbb{R}_+
\tag{12}
$$

(figure 2). Using the wall-crossing formula, we can compute the spectrum on the external side of the wall. The symplectic product is

$$
\langle \gamma_1, \gamma_2 \rangle = -2\,,
\tag{13}
$$

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[2]Although the analysis of [10] is for the corresponding two dimensional theory, the resulting formula for the central charge is identical to that of the four-dimensional theory at the Higgs branch root [2].
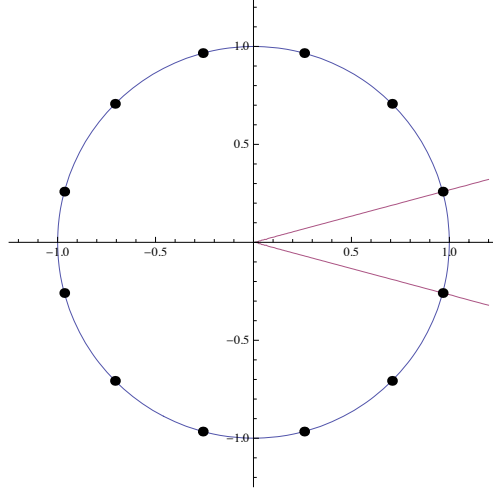
Figure 1: Argyres–Douglas points for $\mathbb{Z}_{12}$-symmetric masses in the m_0 plane.
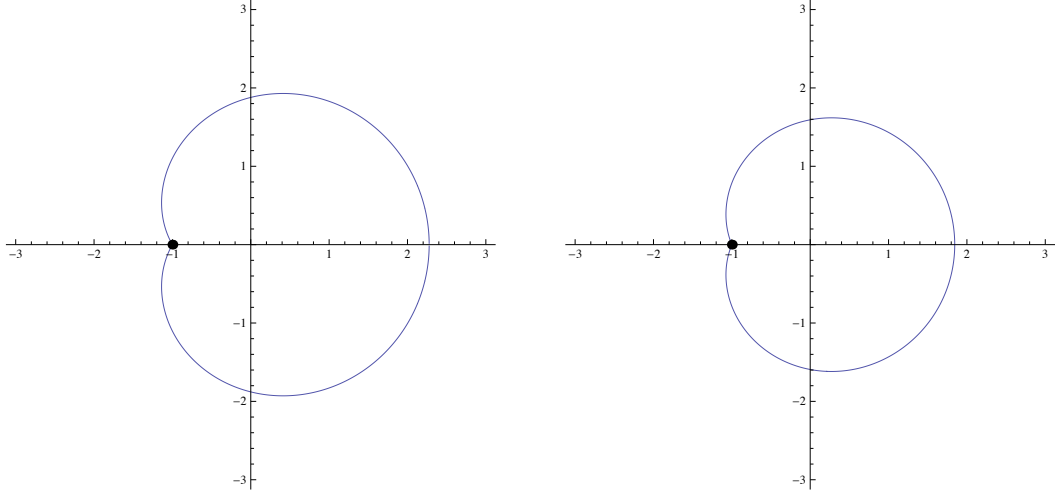

Figure 2: The primary walls of marginal stability for $\mathbb{Z}_2$- and $\mathbb{Z}_3$-symmetric masses [17, 10].

hence, the relevant wall-crossing formula is essentially the same as the formula relating the strong- and weak-coupling spectra of the pure $SU(2)$ theory in four dimensions [3] [4]:

$$\mathcal{K}_{-\gamma_2}\mathcal{K}_{\gamma_1} = \mathcal{K}_{\gamma_1}\mathcal{K}_{2\gamma_1-\gamma_2}\mathcal{K}_{3\gamma_1-2\gamma_2}\mathcal{K}_{4\gamma_1-3\gamma_2}\ldots\mathcal{K}_{\gamma_1-\gamma_2}^{-2}\ldots\mathcal{K}_{3\gamma_1-4\gamma_2}\mathcal{K}_{2\gamma_1-3\gamma_2}\mathcal{K}_{\gamma_1-2\gamma_2}\mathcal{K}_{-\gamma_2}. \tag{14}$$

In our notations, the part of the wall-crossing formula that changes across the wall takes the following form:

$$\begin{aligned}\mathcal{K}_{-((0,1),(-1,1))}\mathcal{K}_{((1,0),(-1,1))} &= \mathcal{K}_{((1,0),(-1,1))}\mathcal{K}_{((2,-1),(-1,1))}\mathcal{K}_{((3,-2),(-1,1))}\mathcal{K}_{((4,-3),(-1,1))}\\ &\ldots\mathcal{K}_{((-1,1),(0,0))}^{-2}\ldots\mathcal{K}_{-((-3,4),(-1,1))}\mathcal{K}_{-((-2,3),(-1,1))}\mathcal{K}_{-((-1,2),(-1,1))}\mathcal{K}_{-((0,1),(-1,1))}\end{aligned} \tag{15}$$

where we display only the first two components electric and magnetic charges, the others being equal to zero. This relation shows that the spectrum outside the wall consists of a tower of dyons and a finite number of quarks and W bosons with charges:

$$\begin{aligned}&\pm\left((-\nu+1,\nu,0,0,\ldots),(-1,1,0,0,\ldots)\right),\\ &\pm\left((-1,1,0,0,\ldots),(0,0,0,0,\ldots)\right).\end{aligned} \tag{16}$$

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[3]This reflects the fact that all the states involved have charges contained in an $SU(2)$ subgroup of the gauge group.

3 Bound states

In fact, the complete BPS spectrum is not limited to the primary tower of states found above: depending on the values of masses, there can also be secondary towers of bound states formed by a dyon and p quarks [3]. Creation (or, conversely, destruction) of these extra states is described by the pentagon formula:

$$\mathcal{K}_{\gamma_1}\mathcal{K}_{\gamma_2} = \mathcal{K}_{\gamma_2}\mathcal{K}_{\gamma_1+\gamma_2}\mathcal{K}_{\gamma_1}\,, \quad \forall \langle \gamma_1, \gamma_2 \rangle = \pm 1 \tag{17}$$

where the new state $\gamma_1 + \gamma_2$ is created from γ_1 and γ_2 where one of the initial states is a quark, and the other one is either a dyon or a bound state consisting of a dyon and $p - 1$ quarks. This process occurs when,

$$\frac{Z_{\gamma_1}}{Z_{\gamma_2}} \in \mathbb{R}_+ \,. \tag{18}$$

We will find the resulting secondary walls and prove that they are located outside the primary wall and have to be crossed as the VEV moves from strong to weak coupling. The restriction on the wedge-product of the two interacting states in (17) allows us to determine which states can combine to form a bound state if the corresponding secondary wall exists.

Starting with the states constructed in the previous section, when $\tilde{n} = 0$, we can see that there can be two possible types of creation processes, both leading to the same set of new states:

$$1: \quad ((-\nu + 1, \nu, 0, 0, 0, \dots), (-1, 1, 0, 0, 0, \dots)) + ((-1, 0, 1, 0, 0, \dots), (0, 0, 0, 0, 0, \dots))$$
$$\leftrightarrow ((-\nu, \nu, 1, 0, 0, \dots), (-1, 1, 0, 0, 0, \dots))\,, \tag{19}$$

$$2: \quad ((-\nu, \nu + 1, 0, 0, 0, \dots), (-1, 1, 0, 0, 0, \dots)) + ((0, -1, 1, 0, 0, \dots), (0, 0, 0, 0, 0, \dots))$$
$$\leftrightarrow ((-\nu, \nu, 1, 0, 0, \dots), (-1, 1, 0, 0, 0, \dots))\,. \tag{20}$$

These are the bound states formed by a dyon and one quark . Explicitly, the walls of marginal stability (18) for these processes are determined by

$$1: \quad \frac{-m_0 + m_2}{(-\nu + 1)m_0 + \nu m_1 - m_{D\,0} + m_{D\,1}} \in \mathbb{R}_+ \,, \tag{21}$$

$$2: \quad \frac{-m_1 + m_2}{-\nu m_0 + (\nu + 1)m_1 - m_{D\,0} + m_{D\,1}} \in \mathbb{R}_+ \,. \tag{22}$$

For general N_f, there are additional bound states involving the remaining $\tilde{n}$ flavours:

$$((-\nu + 1, \nu, 0, 0, 0, \dots), (-1, 1, 0, 0, 0, \dots)) + ((-1, 0, 0, 0, 0, \dots), (0, 0, 0, 0, 0, \dots), (1, 0, 0, \dots))$$
$$\leftrightarrow ((-\nu, \nu, 0, 0, 0, \dots), (-1, 1, 0, 0, 0, \dots), (1, 0, 0, \dots))\,, \tag{23}$$

$$((-\nu, \nu + 1, 0, 0, 0, \dots), (-1, 1, 0, 0, 0, \dots)) + ((0, -1, 0, 0, 0, 0, \dots), (0, 0, 0, 0, 0, \dots), (1, 0, 0, \dots))$$
$$\leftrightarrow ((-\nu, \nu, 0, 0, 0, \dots), (-1, 1, 0, 0, 0, \dots), (1, 0, 0, \dots))\,. \tag{24}$$

They are completely analogous to the ones above: the walls of marginal stability for these processes can be obtained by changing m_2 to $\tilde{m}_0$ in the previous formulae.

As has been discussed above, there can also be bound states formed by a dyon and p quarks:

$$((-\nu + 1 + p, \nu, j_3, j_4, j_5, \dots), (-1, 1, 0, 0, 0, \dots), (\tilde{j}_{\tilde{1}}, \tilde{j}_{\tilde{2}}, \tilde{j}_{\tilde{3}}, \dots)),$$

$$j_i(j_i - 1) = \tilde{j}_{\tilde{i}}(\tilde{j}_{\tilde{i}} - 1) = 0, \quad p + \sum_{i=2}^{n-1} j_i + \sum_{\tilde{i}=0}^{\tilde{n}-1} \tilde{j}_{\tilde{i}} = 0. \tag{25}$$

They exist if, starting with the strong-coupling spectrum and moving into the weak-coupling region, $|p|$ different secondary walls of marginal stability (18) are crossed.

We need to find out if the processes (19) and (20), which we rewrite as

$$1: \quad d_1 + q_1 \leftrightarrow b, \tag{26}$$

$$2: \quad d_2 + q_2 \leftrightarrow b, \tag{27}$$

actually take place when the masses move from strong to weak coupling: to do this, we should check whether the secondary walls (18) are crossed, i.e., if the following conditions are satisfied somewhere outside the primary wall of marginal stability:

$$1: \quad \arg Z_{d_1} = \arg Z_{q_1}, \tag{28}$$

$$2: \quad \arg Z_{d_2} = \arg Z_{q_2}. \tag{29}$$

Note that Z_{q_j} ($j = 1$ or $j = 2$) is independent of the region in the moduli space, and $\arg Z_{d_j}$ changes continuously between the primary wall and the weak-coupling region, therefore, (28) (with $j = 1$) and (29) (with $j = 2$) are satisfied somewhere if in the complex plane, Z_{q_j} lies between the values of Z_{d_j} at the primary wall and in the weak-coupling limit. To check if this is the case, it is convenient to start at the Argyres–Douglas point where d_1 with $\nu = 0$, i.e., γ_1 in (7), becomes massless: near this point, the central charge of a dyon can be approximated as

$$(Z_{d_1})_s \simeq \nu(-m_0 + m_1). \tag{30}$$

Then, we continuously move the masses into the semiclassical region, where

$$(Z_{d_1})_w \simeq i(-m_0 + m_1). \tag{31}$$

From (30) and (31), we have

$$\lim_{g_{\mathrm{eff}} \to 0} \arg \frac{(Z_{d_1})_w}{(Z_{d_1})_s} = \frac{\pi}{2}, \quad \arg \frac{(Z_{d_1})_w}{(Z_{d_1})_s} < \frac{\pi}{2} \tag{32}$$

where the inequality is strict for any g_{eff} as the central charge receives corrections from its electric components at weak coupling. All these statements also hold if d_1 is substituted by d_2. Comparing Z_{q_j} with (30) and (31) and using (32), we conclude that the walls (21, 22) exist if

$$1: \quad \nu > 0: \quad \arg \frac{m_k - m_0}{m_1 - m_0} \in \left(0, \frac{\pi}{2}\right),$$
$$\nu < 0: \quad \arg \frac{m_k - m_0}{m_1 - m_0} \in \left(\frac{\pi}{2}, \pi\right), \tag{33}$$

$$2: \quad \nu > 0: \quad \arg \frac{m_k - m_1}{m_1 - m_0} \in \left(0, \frac{\pi}{2}\right),$$
$$\nu < 0: \quad \arg \frac{m_k - m_1}{m_1 - m_0} \in \left(\frac{\pi}{2}, \pi\right). \tag{34}$$

Since the bound states with $\nu \neq 0$ do not exist at strong coupling, they appear at weak coupling if exactly one of the two walls (28, 29) is crossed. For both $\nu > 0$ and $\nu < 0$, this means that one of the following conditions must be satisfied:

$$0 < \operatorname{Re} \frac{m_k - m_0}{m_1 - m_0} < 1\,,$$
$$0 < \operatorname{Re} \frac{m_k - m_1}{m_1 - m_0} < 1 \tag{35}$$

(again, the inequalities are strict because of (32)). In fact, the conditions are equivalent to each other [4], and the first equation in (35) is precisely the semiclassical constraint derived in [3] from first principles. Finally, we can conclude that the secondary tower of bound states exists if and only if the complex masses obey the equivalent inequalities in (35). Analogously, if (35) holds for p different indices k and $\tilde{k}$, there are towers of bound states with p quarks (25) having $j_k = 1$ and $\tilde{j}_{\tilde{k}} = 1$ for these indices and $j_i = 0$ and $\tilde{j}_{\tilde{i}} = 0$ for all other i and $\tilde{i}$, in accordance with [3].

Applying this result to $\mathbb{Z}_n$-symmetric masses, it is easy to determine which bound states exist in the weak-coupling limit. The constraint (35) reduces to

$$0 < \operatorname{Re} \frac{e^{2\pi ki/n} - 1}{e^{2\pi i/n} - 1} = \operatorname{Re} \frac{e^{2\pi(k-1/2)i/n} - e^{-\pi i/n}}{2i \sin(\pi/n)} < 1 \quad \Longleftrightarrow \quad -1 < \frac{\sin \frac{2\pi k - \pi}{n}}{\sin \frac{\pi}{n}} < 1. \tag{36}$$

Here, $(2\pi k - \pi)/n$ is a multiple of π/n, therefore, the inequality holds only for $(2\pi k - \pi)/n = \pi$, that is, for $k = (n+1)/2$. For $\mathbb{Z}_{2l+1}$-symmetric masses with $l \in \mathbb{N}$, this means that only the bound states formed by one quark with $\gamma_{e\,(l+1)} = 1$ are present (figure 3); for $\mathbb{Z}_{2l}$-symmetric masses, there are no bound states. We can go back to equations (33, 34) to find out which secondary walls of
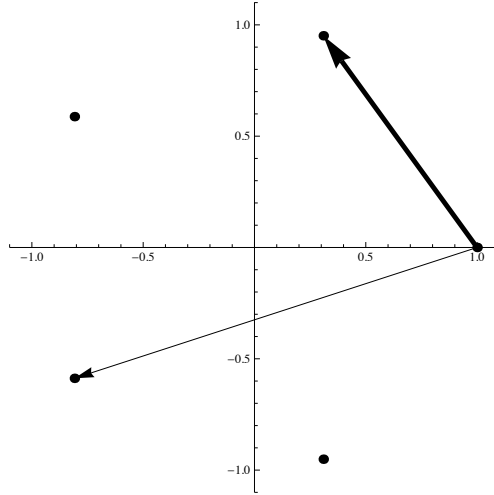

Figure 3: A dyon (thick vector connecting m_0 and m_1, representing its central charge near the massless point at strong coupling for $\nu > 0$) and the quark that can form bound states with it (thin vector connecting m_0 and m_3, equal to its central charge) in the case of $\mathbb{Z}_5$-symmetric masses.

marginal stability exist in the case of $\mathbb{Z}_{2l+1}$-symmetric masses: (19) is realised for $\nu > 0$, (20) is realised for $\nu < 0$, and the corresponding walls are determined by (21) with $\nu > 0$ and (22) with $\nu < 0$ (plotted for $\mathbb{Z}_3$ in figure 4).

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[4] This can be seen by reflecting the complex plane so that m_0 and m_1 exchange their positions.
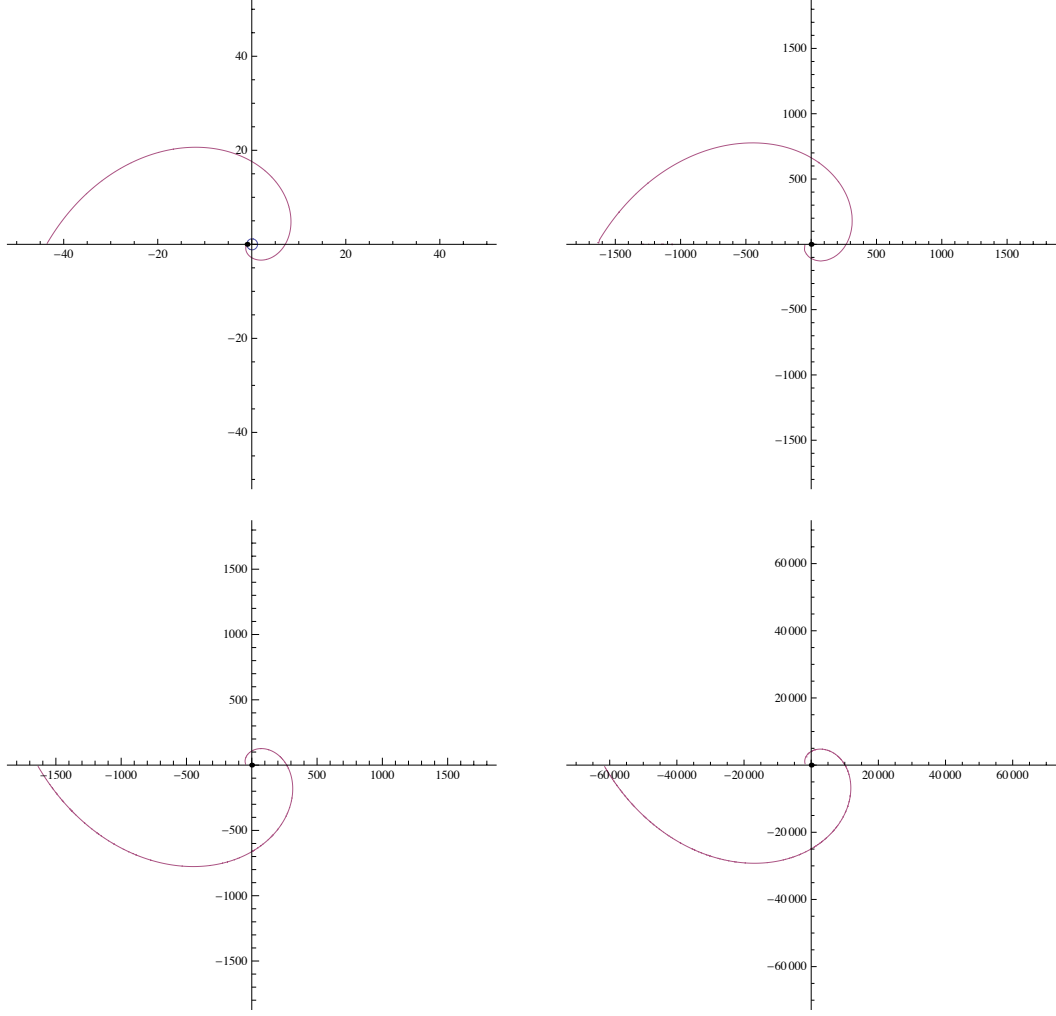
Figure 4: Secondary walls of marginal stability for $\mathbb{Z}_3$-symmetric masses in the $m_0^3/|m_0|^2$ plane (left column: $|\nu| = 1$, right column: $|\nu| = 2$; first raw: bound state with the quark $(-1, 0, 1)$, second raw: bound state with the quark $(0, -1, 1)$). The plots corresponds to the sector in figure 1.

In [10], it was proposed that the weak-coupling spectrum of the corresponding two-dimensional theory must also contain the full set of extra towers of BPS states

$$((-\nu, \nu, 0, 0, \ldots, 0, 1, 0, 0, \ldots), (-1, 1, 0, 0, 0, \ldots)) \tag{37}$$

with electric charge 1 taking all possible positions, corresponding to different towers, and that for each extra tower, there is a single wall of marginal stability where it decays. Although we have nothing to say here about the two-dimensional spectrum [5], our results strongly suggest that not all extra towers appear in the four-dimensional theory [6], and that all these states decay at different walls of marginal stability. For example, the first extra tower corresponding to the proposal of [10] has charges

$$((-\nu, \nu, 1, 0, 0, \ldots), (-1, 1, 0, 0, 0, \ldots)) \tag{38}$$

---

[5] Note, however, that a similar wall-crossing formula is believed to hold for 2d models of this type [18].

[6] The ones that do appear have been interpreted as bound states above.

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for integer ν. The wall-crossing formula certainly forbids the obvious simultaneous decay process for the whole tower of states into the known quarks and dyons of the model:

$$((-\nu, \nu, 1, 0, 0, \dots), (-1, 1, 0, 0, 0, \dots)) \overset{?}{\leftrightarrow} ((0, 0, 1, 0, 0, \dots), (-1, 1, 0, 0, 0, \dots))$$
$$+\nu((-1, 1, 0, 0, 0, \dots), (0, 0, 0, 0, 0, \dots)). \tag{39}$$

To see this, suppose that the spectrum on right-hand side is correct, and we cross the wall in the other direction. The symplectic product of the two charges on the right-hand side is two, which means that the formula (14) should apply. One can then check, however, that this leads to a different decay process into states which have magnetic charges (in one $SU(2)$ subgroup) which are greater than one. Therefore, it is believed that such decay processes are impossible, and that only some extra towers (i.e., only the towers of bound states) appear in this model.

Acknowledgements

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