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2D – 4D Correspondence: Towers of Kinks Versus Towers of Monopoles in $\mathcal{N} = 2$ Theories

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Abstract

1 Introduction

Recently we revisited the problem of the BPS kink spectrum in the supersymmetric CP($N-1$) l model with $\mathcal{Z}_N$-symmetric twisted masses [6] in connection with the studies of the curves of marginal stability. We derived the BPS spectrum by combining three requirements: (i) at small values of the mass terms, i.e. at strong coupling, the solution implied by the mirror representation [1, 2], in the first order in the twisted masses [3]; (ii) consistency in the Argyres–Douglas points, and (iii) quasiclassical limit which had been analyzed previously in [8]. Our analysis is based on the superpotential of the Veneziano–Yankielowicz type that is exact in the BPS sector. This potential is presented by a multibranch (and, hence, multivalued) function. Therefore, a disambiguation is necessary. The combination of the above three requirements led us to an unambiguous prediction. A a surprising finding is: in the $\mathcal{N}=2$ CP($N-1$) model with $\mathcal{Z}_N$-symmetric twisted masses there are $N-1$ towers of BPS saturated kinks. The previous studies in the literature mention a single tower. Only this single tower is seen in the quasiclassical analysis in [8].

Since the CP($N-1$) model with $\mathcal{Z}_N$ -symmetric twisted masses appears as a low-energy theory on the world sheet of non-Abelian strings [9] supported in certain four-dimensional $\mathcal{N}=2$ gauge theories with $N=N_f$, the prediction for the BPS spectrum in two dimensions can be elevated to four dimensions. Thus, our formula simultaneously describes confined monopoles in the Higgs phase of the four-dimensional gauge theory, as explained e.g. in the review paper [10]. Thus, we predict that at large values of the mass differences of the (s)quark fields (which translate into the twisted masses in 2D) these monopoles appear in the spectrum in the form of the same $N-1$ towers.

In this paper we will discuss the origin and the physical meaning of the phenomenon of $N-1$ towers for kinks in 2D/monopoles in 4D. To avoid bulky notation and excessive technicalities we will mostly focus on the simplest nontrivial example, that of CP(2). Generalization to CP($N-1$) is conceptually straightforward. We will briefly discuss it at the end.

For arbitrary N the Z_N symmetric twisted mass parameters are defined as

$$m_k = m_0 \cdot e^{2\pi i k/N}\,, \qquad k=0,1,...,N-1\,; \tag{1.1}$$

the set of the mass parameters depends on a single complex parameter m_0. In what follows we will assume m_0 to be real. For $N=3$ we have three mass parameters, and two masses in the geometric formulation (they can be viewed as mass terms of the elementary fermion excitations), namely

$$m_0, \;\; m_1 = m_0 e^{2\pi i/3}, \;\; m_2 = m_0 e^{-2\pi i/3};$$
$$M_1 = m_1 - m_0\,, \qquad M_2 = m_2 - m_0\,. \tag{1.2}$$

The master formula to be used below takes the form

$$\begin{aligned} M_{\rm kink} &= U_0(m_0) + i\,\vec N\cdot\vec m\,, \\ U_0(m_0) &= -\frac{1}{2\pi}\left(e^{2\pi i/3} - 1\right) \\ &\times \left\{3\,\sqrt[3]{m_0^3\,+\,\Lambda^3} + \sum_{j=0}^{2} m_j\,\ln\,\frac{\sqrt[3]{m_0^3\,+\,\Lambda^3}\,-\,m_j}{\Lambda}\right\}, \end{aligned} \tag{1.3}$$

where

$$\vec m = \{m_0,\;...,\;m_{N-1}\}\,, \tag{1.4}$$

and $\vec N$ is an integer-valued vector determined in [6]. In fact, there are two such vectors (hence, two towers mentioned above),

$$\begin{aligned} \vec N_1 &= \{-n\,+\,1,\;\; n,\;\; 0\}\,, \\ \vec N_2 &= \{-n,\;\; n,\;\; 1\}\,, \end{aligned} \tag{1.5}$$

where n is an integer parameter.

With our parameter values $U_0(m_0)$ is an explicit single-valued function. The multivaluedness resides in (1.5). Indeed,

$$\begin{aligned} \vec m\,\vec N_1 &= nM_1 + m_0 = n'\,M_1 + m_1\,, \\ \vec m\,\vec N_2 &= n\,M_1 + m_2\,, \\ n' &= n - 1\,. \end{aligned} \tag{1.6}$$

2 Formulation

The classical expression for the central charge has two contributions [4]: the Noether and the topological terms,

$$\mathcal{Z} \;=\; i\,M_a\,q^a \;+\; \int dz\,\partial_z\,O\,, \qquad\qquad a \;=\; 1, \ldots N-1\,. \tag{2.1}$$

where M^a are the twisted masses (in the geometric formulation),

$$M_a \;=\; m^a \;-\; m^0\,, \tag{2.2}$$

m^a (a=1,2, ..., N) are the masses in the gauge formulation, and the operator O consists of two parts, canonical and anomalous,

$$O \;=\; O_{\text{canon}} + O_{\text{anom}}, \tag{2.3}$$

$$O_{\text{canon}} \;=\; \sum_{a=1}^{N-1} M_a D^a, \tag{2.4}$$

$$O_{\text{anom}} \;=\; -\frac{N\,g_0^2}{4\pi}\left(\sum_{a=1}^{N-1} M_a D^a + g_{i\bar{j}}\,\bar{\psi}^{\bar{j}}\,\frac{1-\gamma_5}{2}\,\psi^i\right). \tag{2.5}$$

Moreover, the Noether charges q^a can be obtained from $N-1$ U(1) currents J^a_μ defined as [1]

$$\begin{aligned} J^a_{RL} \;&=\; g_{i\bar{j}}\,\overline{\phi}^{\bar{j}}\,(T^a)^{i\bar{j}}\,i\,\overleftrightarrow{\partial}_{RL}\,\phi^i \\ &+\; \frac{1}{2}\,g_{i\bar{j}}\,\overline{\psi}^{\bar{m}}_{LR}\,\left((T^a)^{\bar{p}}_{\bar{m}}\,\delta^{\bar{j}}_{\bar{p}} \;+\; \overline{\phi}^{\bar{r}}\,(T^a)^{\bar{k}}_{\bar{r}}\,\Gamma^{\bar{j}}_{\bar{k}\bar{m}}\right)\,\psi^i_{LR} \\ &+\; \frac{1}{2}\,g_{i\bar{j}}\,\overline{\psi}^{\bar{j}}_{LR}\,\left(\delta^i{}_p\,(T^a)^p{}_m \;+\; \Gamma^{\,i}_{mk}\,(T^a)^k{}_r\,\phi^r\right)\,\psi^m_{LR} \end{aligned} \tag{2.6}$$

in the geometric representation, and

$$\begin{aligned} J^a_{RL} \;&=\; i\,\overline{n}_a\,\overleftrightarrow{\partial}_{RL} n^a \;-\; |n^a|^2\cdot i\,(\overline{n}\,\overleftrightarrow{\partial}\,n) \\ &+\; \overline{\xi}^a_{LR}\,\xi^a_{LR} \;-\; |n^a|^2\cdot(\overline{\xi}_{LR}\,\xi_{LR}) \end{aligned} \tag{2.7}$$

[1] There is a typo in the definition of these currents in [4].

in the gauged formulation. Here

$$(T^a)^i_k = \delta^i_a \delta^a_k \,, \quad \text{(no summation over } a!) \tag{2.8}$$

and a similar expression for the overbarred indices. Finally, D^a are the Killing potentials,

$$D^a \quad = \quad r_0 \, \frac{\overline{\phi}\; T^a \, \phi}{1 \; + \; |\phi|^2} \quad = \quad r_0 \, \frac{\overline{\phi}^a \, \phi^a}{1 \; + \; |\phi|^2} \,. \tag{2.9}$$

The generators T^a always pick up the a-th component. In this expression,

$$r_0 = \frac{2}{g_0^2} \tag{2.10}$$

is a popular alternative notation for the sigma model coupling.

Note that Eq. (2.9) contains the bare coupling. It is clear that the one-loop correction must (and will) convert the bare coupling into the renormalized coupling. The anomalous part O_{anom} is obtained at one loop. Therefore, in the one-loop approximation for the central charge it is sufficient to treat O_{anom} in the lowest order. Moreover, the bifermion term in O_{anom} plays a role only in the two-loop approximation. As a result, to calculate the central charge at one loop it is sufficient to analyze the one-loop correction to O_{canon}. The latter is determined by the tadpole graphs in Fig. XXX. As usual, the simplest way to perform the calculation is the background field method. The part of the central charge under consideration is determined by the value of the fields at the spatial infinities. In the CP(2) model to be considered below there are three vacua and three possible ways of interpolation between them. All kinks are equivalent. We will choose a particular kink corresponding to the following boundary conditions:

$$\phi^1(z=-\infty) = 0 \,, \qquad \phi^1(z=+\infty) = \infty \,, \qquad \phi^2 = 0 \,. \tag{2.11}$$

We split the field ϕ into two parts,

$$\phi = \phi_{\text{b}} + \phi_{\text{qu}} \,, \tag{2.12}$$

and expand D^a in ϕ_{qu} keeping terms quadratic in ϕ_{qu}.

A digression is in order regarding the one-loop central charge in the gauged formulation. D^a receives a one-loop contribution, due to r, as is most easily seen in the gauge representation of this operator,

$$D^a \;=\; r \cdot \overline{n}_a \, n^a \,, \qquad\qquad |n_l|^2 \;=\; 1\,. \tag{2.13}$$

The renormalized operator contains the running coupling

$$r \;=\; r_0 \;-\; \frac{N}{2\pi} \ln \frac{M_{\rm uv}}{|M^a|}\,. \tag{2.14}$$

In general, it is some typical mass scale that appears in the denominator of the logarithm, but, *e.g.* in CP(2) with $\mathcal{Z}_3$ twisted mass terms all masses and all mass differences have equal magnitude. The UV cut-off $M_{\rm UV}$ and bare constant r_0 can be turned into the strong coupling scale Λ:

$$r \;=\; \frac{N}{2\pi} \ln \frac{|M^a|}{\Lambda}\,. \tag{2.15}$$

We will be looking for the semi-classical expression for the central charge in the presence of the soliton interpolating between vacua (0) and (1)

$$\phi^1(z) \;=\; e^{|M^1|z}\,, \qquad\qquad \phi^2(z) \;=\; \phi^3(z) \;=\; \ldots \;=\; 0\,. \tag{2.16}$$

That this is the right kink can be seen in the gauged formulation,

$$\begin{aligned}
n^0 \;&=\; \frac{1}{\sqrt{1 \;+\; e^{2|M^1|z}}}\,,\\
n^1 \;&=\; \frac{e^{|M^1|z}}{\sqrt{1 \;+\; e^{2|M^1|z}}}\,,\\
n^2 \;&=\; 0\,,\\
&\;\vdots\\
n^k \;&=\; 0\,,\\
&\;\vdots\\
n^{N-1} \;&=\; 0\,.
\end{aligned} \tag{2.17}$$

In this background, D^a taken at the edges of the worldsheet yields just the coupling constant:

$$D^a\Big|_{-\infty}^{+\infty} \;=\; r\,. \tag{2.18}$$

Therefore, the topological contribution to the central charge of the kink is

$$\mathcal{Z} \;\supset\; \frac{N}{2\pi}\,M^1\,\ln\frac{|M^a|}{\Lambda}\,. \tag{2.19}$$

As for the Noether contribution, the quantization of the "angle" coordinate of the kink gives

$$i\,n\,M^1\,, \tag{2.20}$$

with $q^1 \;=\; n$ an integer number. As for the other q^k, the kink does not have fermionic zero-modes of ψ^k with $k = 2,\, 3,\, ...\, N-1$. However, we will argue that there is a *non*-zero mode relevant to the problem of multiple towers that we consider (in fact, the existence of this nonzero mode was noted by Dorey *et al.* [5]). This mode describes a bound state of the kink and a fermion ψ^k.

3 Semiclassical calculation of the central charge in CP(2)

If the twisted masses M_a satisfy the condition

$$|M_a| \gg \Lambda\,, \tag{3.21}$$

then we find ourselves at weak coupling where the one-loop calculation of the central charges will be sufficient for our purposes. This calculation can be carried out in a straightforward manner for all CP$(N-1)$ models, but for the sake if simplicity we will limit ourselves to CP(2). Generalization to larger N is quite obvious.

In CP(2) there are two twisted mass parameters, M_1 and M_2, as shown in Fig. XXX. Accordingly, there are two U(1) charges, see Eq. (2.6). The Noether charges are not renormalized; therefore we will focus on the topological part represented by the Killing potentials, which are renormalized.

3.1 Topological contribution

One-loop calculations are most easily performed using the background field method. For what follows it is important that $\phi_{\rm b}^2 \equiv 0$ for the kink under consideration. If so, all off-diagonal elements of the metric $g_{i\bar{j}}$ vanish, while the diagonal elements take the form

$$g^{\rm b}_{1\bar{1}} \equiv g_{1\bar{1}}\Big|_{\phi_{\rm b}} = \frac{2}{g_0^2}\,\frac{1}{\chi^2}\,,$$
$$g^{\rm b}_{2\bar{2}} \equiv g_{2\bar{2}}\Big|_{\phi_{\rm b}} = \frac{2}{g_0^2}\,\frac{1}{\chi}\,, \tag{3.22}$$

where

$$\chi = 1 + \left|\phi_{\rm b}^1\right|^2 \tag{3.23}$$

At the boundaries $\phi_{\rm b}^{1,2}$ take their (vacuum) coordinate-independent values; therefore, the Lagrangian for the quantum fields can be written as

$$\mathcal{L} = g^{\rm b}_{1\bar{1}}\left|\partial^\mu \phi^1_{\rm qu}\right|^2 + g^{\rm b}_{2\bar{2}}\left|\partial^\mu \phi^2_{\rm qu}\right|^2 + ... \tag{3.24}$$

where the ellipses stand for the terms irrelevant for our calculation.

The Killing potentials can be expanded in the same way. Under the condition $\phi_{\rm b}^2 \equiv 0$ we arrive at

$$D^1 \;=\; D^1\Big|_{\phi_{\rm b}} + \frac{2}{g_0^2}\,\frac{1-|\phi_{\rm b}^1|^2}{\chi^3}\,\left|\phi^1_{\rm qu}\right|^2 - \frac{2}{g_0^2}\,\frac{|\phi_{\rm b}^1|^2}{\chi^2}\,\left|\phi^2_{\rm qu}\right|^2 + \dots\,,$$
$$D^2 \;=\; 0\,. \tag{3.25}$$

Equation (3.24) implies that the Green's functions of the quantum fields are

$$\left\langle \phi^1_{\rm qu}, \phi^1_{\rm qu}\right\rangle = \frac{g_0^2\,\chi^2}{2}\,\frac{i}{k^2-|M|^2}\,, \qquad \left\langle \phi^2_{\rm qu}, \phi^2_{\rm qu}\right\rangle = \frac{g_0^2\,\chi}{2}\,\frac{i}{k^2-|M|^2}\,. \tag{3.26}$$

where

$$|M| \equiv |M_1| \equiv |M_2|\;. \tag{3.27}$$

Now, combining (3.25) and (3.26) to evaluate the tadpoles graphs of Fig. XXX with

$\phi^1_{\rm qu}$ and $\phi^2_{\rm qu}$ running inside we arrive at

$$\begin{aligned} D^1_{\rm one-loop} &= \frac{1}{4\pi}\,\ln\frac{|M_{\rm uv}|^2}{|M|^2} \\ &\times \left(\frac{1-|\phi^1_{\rm b}|^2}{\chi} - \frac{|\phi^1_{\rm b}|^2}{\chi}\right)^{\phi^1_{\rm b}=0}_{\phi^1_{\rm b}=\infty} \\ &= \frac{1}{4\pi}\,\ln\frac{|M_{\rm uv}|^2}{|M|^2}\,(2+1)\,. \end{aligned} \tag{3.28}$$

where $M_{\rm uv}$ is an ultraviolet cut-off (e.g. the Pauli-Villars regulator mass). The first and second terms in the parentheses come from the $\phi^1_{\rm qu}$ and $\phi^2_{\rm qu}$ loops, respectively. In the general case of the CP($N-1$) model one must replace $2+1$ by $2+1\times(N-2)=N$.

This information allows us to obtain the contribution of the Killing potential to the central charge at one loop, namely,

$$\Delta_{\rm K}\mathcal{Z} = -2M_1\left[\frac{1}{g_0^2} - \frac{3}{4\pi}\left(\ln\left|\frac{M_{\rm uv}}{M}\right| + 1\right)\right] \tag{3.29}$$

Note that the renormalized coupling in the case at hand is [7]

$$\frac{1}{g^2} = \frac{1}{g_0^2} - \frac{3}{4\pi}\ln\left|\frac{M_{\rm uv}}{M}\right| \equiv \frac{3}{4\pi}\ln\left|\frac{M}{\Lambda}\right|\,. \tag{3.30}$$

For the generic CP($N-1$) model the coefficient 3 in front of the logarithm in (3.30) is repalced by N. Equation (3.30) serves as a (standard) definition of the dynamical scale parameter Λ in perturbation theory.

3.2 Contribution of the Noether charges

This is not the end of the story, however. We must add to $\Delta_{\rm K}\mathcal{Z}$ a part of the central charge associated with the Noether terms in (2.1), which accounts for the quantziation of the fermion zero modes as well as effects of the θ term.

3.3 Weak-coupling expansion

We can start from the known superpotential of the Veneziano–Yankielowicz type, and the spectrum that it generates. It gives the exact solution of the CP($N-1$)

model with twisted masses in the BPS sector. In our case of $\mathcal{Z}_N$ symmetric twisted masses this superpotential is given in [6].

Here we narrow down to the case of CP(2), with masses that are $\mathcal{Z}_3$ symmetric. The central charge determining the BPS spectrum is given by the difference of the values of the superpotential in two vacua. The general formula adjusted for CP(2) is [6] as follows:

$$\mathcal{Z}\Big|_{-\infty}^{+\infty} \quad = \quad U_0(m_0) \;+\; i\,n\,M_1 \;+\; i\left\{\begin{array}{l} m_0 \\ m_2 \end{array}\right.$$

$$\xrightarrow{|m_0|\to\infty} \quad -\frac{3}{2\pi}\,M^1\left\{\ln\frac{|M_1|}{\Lambda}-1\right\}+\frac{1}{4\sqrt{3}}\,M_1\;+i\,n\,M_1+i\left\{\begin{array}{l} m_0 \\ m_2 \end{array}\right. +\ldots \tag{3.31}$$

where the ellipsis represents suppressed terms dying off as inverse powers of the large mass parameter. As was mentioned, we assume in Eq. (3.31) the parameter m^0 to be real and positive. (This assumption is inessential and can be easily lifted but we will not do it in this paper.)

It is not difficult to rearrange the last double-valued term presenting it as a linear combination of m_0+m_2 and M_2. Then Equation (3.31) takes the form

$$\mathcal{Z} = -\frac{3}{2\pi}\,M_1\left(\ln\frac{|M_1|}{\Lambda}-1\right)+i\,n\,M_1-\frac{i}{4}\,M_1\mp\frac{i}{2}\,M_2\,. \tag{3.32}$$

Theoretically it is possible to redefine Λ by switching on the θ term, which can be introduced as a phase of Λ, namely $\Lambda\to\Lambda e^{-i\theta/3}$. The Veneziano–Yankielowicz superpotential is defined with a "non-perturbative" $\Lambda_{\rm np}$, which may be related to the perturbative $\Lambda_{\rm pt}$ as

$$\Lambda_{\rm np}^3 = -i\,\Lambda_{\rm pt}^3\,. \tag{3.33}$$

The phase has to be settled later, OK?

4 Quasiclassical Kink Solution in CP(2)

In this section we will briefly discuss the kink solution in CP(2). In fact, the bosonic part (and its quantization), as well as the fermion zero mode in ψ^1 are the same as in CP(1) [10]. The crucial difference is the occurrence of a localized (but nonzero!) mode in ψ^2.

4.1 What to expect for the kink mass

Let us return to Eq. (3.32). For the time being we will ignore the second and the third terms is this formula. The second term can be nullified by setting $n = 0$. The third term, as discussed above, can be absorbed into the θ angle. Then $\mathcal{Z}$ takes the form

$$\mathcal{Z} = -\left\{ \frac{3}{2\pi} M_1 \left(\ln \frac{|M_1|}{\Lambda} - 1 \right) \pm \frac{i}{2} M_2 \right\} \tag{4.1}$$

and $M_{\rm kink} = |\mathcal{Z}|$. To simplify further discussion now it is convenient to change phase conventions. We will assume M_1 to be real and negative, $-M_1 \equiv |M_1|$. This can be always done. This will change the θ angle, but we will defer consideration of the θ-induced effects until later.

If M_1 is real and negative, then $\mathrm{Re}M_2$ can be absorbed into θ, while $\mathrm{Im}M_2$ shifts the absolute value of $|\mathcal{Z}|$ and, hence, the kink mass. For the time being we can ignore the unit term in the parentheses in (4.1) associated with the anomaly in the central charge, and focus on the effect due to $\mathrm{Im}M_2$,

$$\begin{aligned} \mathrm{Im}M_2 &= -\frac{\sqrt{3}}{2}\,|M_1| = M_2 - \frac{M_1}{2}\,, \\ M_{\rm kink} &= \frac{2|M_1|}{g^2} \pm \mathrm{Im}M_2 = \frac{2|M_1|}{g^2} \mp \frac{\sqrt{3}}{2}\,|M_1|\,. \end{aligned} \tag{4.2}$$

4.2 Kink solution

In the classical kink solution the field ϕ^2 is not involved. The BPS equation for ϕ^1 is the same as in CP(1), namely,

$$\partial_z \phi = |M| \phi\,. \tag{4.3}$$

The solution of this equation can be written as

$$\phi(z) = e^{|M|(z-z_0) - i\alpha}\,. \tag{4.4}$$

Here z_0 is the kink center while α is an arbitrary phase related to $\mathrm{U}(1)_1$. In fact, these two parameters enter only in the combination $|m|z_0 + i\alpha$. The kink center is complexified.

The effect of the modulus α explains the occurrence of nM_1 from the Noether part.

4.2.1 Quantization of the bosonic moduli

To carry out conventional quasiclassical quantization we, as usual, assume the moduli z_0 and α in Eq. (4.4) to be (weakly) time-dependent, substitute (4.4) in the bosonic part of the Lagrangian, integrate over z and arrive at

$$\mathcal{L}_{\rm QM} = -M_{\rm kink} + \frac{M_{\rm kink}}{2}\dot{z}_0^2 + \left\{\frac{1}{g^2|M|}\,\dot{\alpha}^2 - \frac{\theta}{2\pi}\,\dot{\alpha}\right\}. \tag{4.5}$$

The first term is the classical kink mass, the second describes free motion of the kink along the z axis. The term in the braces is most interesting. The variable α is compact. Its very existence is related to the exact U(1) symmetry of the model. The energy spectrum corresponding to α dynamics is quantized. It is not difficult to see that

$$E_{[\alpha]} = \frac{g^2|M|}{4}\; q^2_{\rm U(1)}\,, \tag{4.6}$$

where $q_{\rm U(1)}$ is the U(1) charge of the soliton,

$$q_{\rm U(1)} = k + \frac{\theta}{2\pi}\,,\qquad k = \text{ an integer}\,. \tag{4.7}$$

The kink U(1) charge is no longer integer in the presence of the θ term, it is shifted by $\theta/(2\pi)$.

4.3 Fermions in quasiclassical consideration

First we will focus on the zero modes of ψ^1 in the kink background (4.4). The coefficients in front of the fermion zero modes will become (time-dependent) fermion moduli, for which we are going to build corresponding quantum mechanics. There are two such moduli, $\bar{\eta}$ and η.

The equations for the fermion zero modes are

$$\begin{aligned}
&\partial_z\psi^1_L \quad -\frac{2}{\chi}\left(\bar{\phi}^1\partial_z\phi^1\right)\psi^1_L - i\,\frac{1-\bar{\phi}^1\phi^1}{\chi}\,|M|e^{i\beta}\psi^1_R = 0\,,\\
&\partial_z\psi^1_R \quad -\frac{2}{\chi}\left(\bar{\phi}^1\partial_z\phi^1\right)\psi^1_R + i\,\frac{1-\bar{\phi}^1\phi^1}{\chi}\,|M|e^{-i\beta}\psi^1_L = 0
\end{aligned} \tag{4.8}$$

(plus similar equations for $\bar{\psi}$; since our operator is Hermitean we do not need to consider them separately.)

It is not difficult to find solution to these equations, either directly, or using supersymmetry. Indeed, if we know the bosonic solution (4.4), its fermionic superpartner — and the fermion zero modes are such superpartners — is obtained from the bosonic one by those two supertransformations which act on $\bar{\phi}$, ϕ nontrivially. In this way we conclude that the functional form of the fermion zero mode must coincide with the functional form of the boson solution (4.4). Concretely,

$$\begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = \eta \left(\frac{g^2|M|}{2} \right)^{1/2} \begin{pmatrix} -ie^{-i\beta} \\ 1 \end{pmatrix} e^{|M|(z-z_0)} \tag{4.9}$$

and

$$\begin{pmatrix} \bar{\psi}^1_R \\ \bar{\psi}^1_L \end{pmatrix} = \bar{\eta} \left(\frac{g^2|M|}{2} \right)^{1/2} \begin{pmatrix} ie^{i\beta} \\ 1 \end{pmatrix} e^{|M|(z-z_0)}\,, \tag{4.10}$$

where the numerical factor is introduced to ensure proper normalization of the quantum-mechanical Lagrangian. Another solution which asymptotically, at large z, behaves as $e^{3|M|(z-z_0)}$ must be discarded as non-normalizable.

Now, to perform quasiclassical quantization we follow the standard route: the moduli are assumed to be time-dependent, and we derive quantum mechanics of moduli starting from the original Lagrangian. Substituting the kink solution and the fermion zero modes for Ψ one gets

$$\mathcal{L}'_{\rm QM} = i\,\bar{\eta}\dot{\eta}\,. \tag{4.11}$$

In the Hamiltonian approach the only remnants of the fermion moduli are the anticommutation relations

$$\{\bar{\eta}\eta\} = 1\,, \quad \{\bar{\eta}\bar{\eta}\} = 0\,, \quad \{\eta\eta\} = 0\,, \tag{4.12}$$

which tell us that the wave function is two-component (i.e. the kink supermultiplet is two-dimensional). One can implement Eq. (4.12) by choosing e.g. $\bar{\eta} = \sigma^+$, $\eta = \sigma^-$.

4.4 Combining bosonic and fermionic moduli

Quantum dynamics of the kink at hand is summarized by the Hamiltonian

$$H_{\rm QM} = \frac{M_{\rm kink}}{2}\,\dot{\bar{\zeta}}\,\dot{\zeta} \tag{4.13}$$

acting in the space of two-component wave functions. The variable ζ here is a complexified kink center,

$$\zeta = z_0 + \frac{i}{|M|}\,\alpha\,. \tag{4.14}$$

For simplicity, we set the vacuum angle $\theta = 0$ for the time being (it will be reinstated later).

The original field theory we deal with has four conserved supercharges. Two of them, $\mathcal{Q}$ and $\bar{\mathcal{Q}}$, act trivially in the critical kink sector. In moduli quantum mechanics they take the form

$$\mathcal{Q} = \sqrt{M_0}\,\dot{\zeta}\eta\,, \qquad \bar{\mathcal{Q}} = \sqrt{M_0}\,\dot{\bar{\zeta}}\bar{\eta}\,; \tag{4.15}$$

they do indeed vanish provided that the kink is at rest. Superalgebra describing kink quantum mechanics is $\{\bar{\mathcal{Q}}\,\mathcal{Q}\} = 2H_{\rm QM}$. This is nothing but Witten's $\mathcal{N} = 1$ supersymmetric quantum mechanics (two supercharges). The realization we deal with is peculiar and distinct from that of Witten. Indeed, the standard Witten quantum mechanics includes one (real) bosonic degree of freedom and two fermionic, while we have two bosonic degrees of freedom, x_0 and α. Nevertheless, superalgebra remains the same due to the fact that the bosonic coordinate is complexified.

Finally, to conclude this section, let us calculate the U(1) charge of the kink states. Starting from the expression for the $\mathrm{U}(1)_1$ current we substitute the fermion zero modes and get [2]

$$\Delta q_{\rm U(1)} = \frac{1}{2}[\bar{\eta}\eta] \tag{4.16}$$

(this is to be added to the bosonic part, Eq. (4.7)). Given that $\bar{\eta} = \sigma^+$ and $\eta = \sigma^-$ we arrive at $\Delta q_{\rm U(1)} = \frac{1}{2}\sigma_3$. This means that the $\mathrm{U}(1)_1$ charges of two kink states in the supermultiplet split from the value given in Eq. (4.7): one has the $\mathrm{U}(1)_1$ charge

$$k + \frac{1}{2} + \frac{\theta}{2\pi}\,,$$

and another

$$k - \frac{1}{2} + \frac{\theta}{2\pi}\,.$$

[2] To set the scale properly, so that the U(1) charge of the vacuum state vanishes, one must antisymmetrize the fermion current, $\bar{\psi}\gamma^\mu\psi \;\to\; (1/2)\left(\bar{\psi}\gamma^\mu\psi - \bar{\psi}^c\gamma^\mu\psi^c\right)$ where the superscript c denotes C conjugation.

4.5 Bound States of ψ^2 with the eigenvalue $\sqrt{3}|M_1|/2$

To find the non-zero mode, we write out the linearized Dirac equations in the background of the ϕ^1 kink. For convenience, we rescale the variable z into a dimensionless variable s:

$$s \;=\; 2\,|M^1|\,z\,. \tag{4.17}$$

Then the kink takes the form

$$\phi^1(s) \;=\; e^s\,, \qquad \text{and} \quad \phi^k(s) \;=\; 0 \qquad \text{for} \;\; k \;>\; 1\,, \tag{4.18}$$

or

$$\begin{aligned} n^0 \;&=\; \frac{1}{\sqrt{1\,+\,e^s}}\,, \\ n^1 \;&=\; \frac{e^{s/2}}{\sqrt{1\,+\,e^s}}\,, \\ n^2 \;&=\; 0\,, \\ &\;\vdots \\ n^k \;&=\; 0\,, \\ &\;\vdots \\ n^{N-1} \;&=\; 0\,. \end{aligned} \tag{4.19}$$

The masses will also turn dimensionless by the same factor,

$$\mu^l \;=\; \frac{m^l}{2|M^1|}\,, \qquad \text{and} \qquad \mu^a_G \;=\; \frac{M^a}{2|M^1|}\,, \tag{4.20}$$

written both for geometric and gauge formulations.

The linearized Dirac equations for the fermion ψ^k with $k \;>\; 1$ then look like

$$\begin{aligned} \Big\{\partial_s \;-\; |\mu^1_G|\,f(s)\Big\}\,\psi^k_R \;+\; i\,\Big(\mu^1_G\,f(s) \;-\; \mu^k_G\Big)\cdot\psi^k_L \;&=\; i\,\lambda\,\psi^k_L \\ \Big\{\partial_s \;-\; |\mu^1_G|\,f(s)\Big\}\,\psi^k_L \;-\; i\,\Big(\overline{\mu}^1_G\,f(s) \;-\; \overline{\mu}^k_G\Big)\cdot\psi^k_R \;&=\; -\,i\,\overline{\lambda}\,\psi^k_R\,. \end{aligned} \tag{4.21}$$

Here $f(s)$ is a real function

$$f(s) \quad = \quad \frac{e^s}{1 \ + \ e^s} \,. \tag{4.22}$$

Eigenvalue λ is zero for zero-modes, or gives the energy for non-zero modes. If one starts from the gauged formulation, one arrives at a simpler system, which can be obtained from the above one by redefinition of the functions. That is, the conversion between the geometric and gauge formulations is precisely such as to remove the inhomogeneous term from the figure brackets,

$$\begin{aligned} \partial_s \, \xi^k_R \quad + \quad i \, \left(\mu^1_G \, f(s) \ - \ \mu^k_G \right) \cdot \xi^k_L \quad &= \quad\ \ i \, \lambda \, \xi^k_L \\ \partial_s \, \xi^k_L \quad - \quad i \, \left(\overline{\mu}^1_G \, f(s) \ - \ \overline{\mu}^k_G \right) \cdot \xi^k_R \quad &= \quad - \, i \, \overline{\lambda} \, \xi^k_R \,. \end{aligned} \tag{4.23}$$

This system does not allow normalizable zero modes. However, there is a normalizable non-zero mode with the energy given by the absolute value of

$$\lambda \quad = \quad - \, \mu^k_G \ + \ \frac{1}{2} \mu^1_G \,. \tag{4.24}$$

The mode is

$$\begin{aligned} \xi^k_R \quad &= \quad \left(\frac{e^{\alpha s}}{1 \ + \ e^s} \right)^{|\mu^1_G|} \\ \xi^k_L \quad &= \quad - \, i \, \frac{\overline{\mu}^1_G}{|\mu^1_G|} \cdot \xi^k_R \,. \end{aligned} \tag{4.25}$$

It is BPS-saturated. That it is BPS can be seen from the expansion of the central charge

$$| \, r \cdot M^1 \ + \ i \, M^k \, | \quad = \quad r \cdot |M^1| \quad - \quad |M^k| \cdot \sin \mathrm{Arg} \, \frac{M^k}{M^1} \quad + \quad \ldots \,, \tag{4.26}$$

in the large coupling constant r. This is the central charge of the bound state of a fermion and the kink as discovered by Dorey *et al.* [5], written semi-classically.

5 Matching the Central Charges

The following central charges need to meet the correspondence.

- The 4-dimensional central charge (at the root of the baryonic Higgs branch),

$$\mathcal{Z} \;=\; i\,\vec{n}_m \cdot \vec{a}_D \;+\; i\,\vec{n}_e \cdot \vec{a} \;+\; i\,m^a \cdot S^a \;+\; i\,m^k\,\vec{w}^k \tag{5.1}$$

- For magnetic charge one, and electric charge $\vec{\alpha}_1$ this gives, up to normalization,

$$\mathcal{Z} \;=\; i\,a_D(m_0) \;+\; i\,(m^1 \,-\, m^0)\,n \;+\; i\,(m^k \,-\, m^0)\,. \tag{5.2}$$

- Our 2-d expression for the central charge gives

$$\mathcal{Z} \;=\; U_0(m_0) \;+\; i\,(m^1 \,-\, m^0)\,n \;+\; i\,m^k\,. \tag{5.3}$$

  It could be that the four-dimensional Λ differs from the two-dimensional one, although then one of them would have to depend on the masses.

- The above two-dimensional charge, when expanded, gives

$$\mathcal{Z} \;=\; \frac{3}{2\pi}\,(m^1 - m^0)\left\{\ln\frac{|m^1 \,-\, m^0|}{\Lambda} - 3\right\} \;+\; i\,m^k \;-\; \frac{1}{4\sqrt{3}}\,(m^1 - m^0) \;+\; ... \tag{5.4}$$

- The perturbative result gives

$$\mathcal{Z} \;=\; \frac{3}{2\pi}\,(m^1 \,-\, m^0)\left\{\ln\frac{|m^1 \,-\, m^0|}{\Lambda} \,-\, 3\right\} \;+\; i\,(m^k \,-\, m^0) \tag{5.5}$$

- The original classical expression is

$$\mathcal{Z} \;=\; i\,(m^k \,-\, m^0)\,q^k \;+\; (m^k \,-\, m^0)\cdot D^k\Big|_{-\infty}^{+\infty} \tag{5.6}$$

All these expressions must agree with each other

6 Conclusion

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