Low-Energy Effective Action of CP^{N-1} Model at large N

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Introduction

Bosonic theory

We address the task of solving the two-dimensional CP^{N-1} theory
The non-supersymmetric CP^{N-1} describes a complex vector

$$n^l\,, \qquad\qquad l \;=\; 1\,, \ldots, N$$

subject to the identification

$$\vec{n} \;\sim\; \lambda\,\vec{n}\,, \qquad\qquad \lambda \;\in\; \mathcal{C}$$

In the physical setting $\vec{n}$ arises as *orientational* moduli living on the worldsheet of non-Abelian vortices (with a gauge group SU(N))

The *gauge* formulation for such a theory was introduced by Witten

$$\mathcal{L} \;=\; \frac{1}{4e^2}\,F_{kl}^2 \;+\; \frac{1}{2e^2}\,D^2 \;+\; \left|\nabla n\right|^2 \;+\; iD\big(\left|n^l\right|^2 - 2\beta\,\big)$$

where

$$\nabla_k\, n^l \;=\; (\,\partial_k \;-\; i\,A_k\,)\;n^l$$

In the limit $e \to \infty$ resolution of A_k and D imposes the CP^{N-1} constraint $\vec{n} \;\sim\; \lambda\,\vec{n}$

One of n^l components can be expressed in terms of the other $N-1$, and put to an arbitrary phase — *e.g.* set *real*

$$\mathcal{L} \;=\; \left|\,\partial\, n\,\right|^2 \;+\; \left(\overline{n}\,\partial_k\, n\,\right)^2, \qquad\qquad l \;=\; 1\,,\,\ldots\,,\,N-1$$

$\mathcal{N} = (2,2)$ Supersymmetric Theory

$$
\begin{aligned}
\mathcal{L}_{(2,2)} \;=\; & \frac{1}{4e^2}\,F_{kl}^2 \;+\; \frac{1}{e^2}\left|\partial_k\sigma\right|^2 \;+\; \frac{1}{2e^2}\,D^2 \;+ \\
+\; & \frac{1}{e^2}\,\overline{\lambda}_R\, i\partial_L\,\lambda_R \;+\; \frac{1}{e^2}\overline{\lambda}_L\, i\partial_R\,\lambda_L \;+ \\
+\; & \left|\nabla n\right|^2 \;+\; |\sqrt{2}\sigma|^2\left|n^l\right|^2 \;+\; iD\left(\left|n^l\right|^2 - 2\beta\right) \;+ \\
+\; & \overline{\xi}_R\, i\nabla_L\xi_R \;+\; \overline{\xi}_L\, i\nabla_R\xi_L \;+\; i\sqrt{2}\sigma\,\overline{\xi}_R\xi_L \;+\; i\sqrt{2}\overline{\sigma}\,\overline{\xi}_L\xi_R \;+ \\
+\; & i\sqrt{2}\,\overline{\xi_{[R}\,\lambda_{L]}}\,n \;-\; i\sqrt{2}\,\overline{n}\,\lambda_{[R}\,\xi_{L]}\,, \qquad l \;=\; 1, \ldots N
\end{aligned}
$$

The Exact Superpotential

This theory is known to have an exact *Veneziano-Yankielowicz* type superpotential

$$\int d\theta_R\, d\bar{\theta}_L \left(\sqrt{2}\Sigma \log \sqrt{2}\Sigma \;-\; \sqrt{2}\Sigma \right)$$

also known as *Witten* superpotential, where Σ is a *twisted* superfield

$$\Sigma \;=\; \sigma \;-\; \sqrt{2}\,\theta_R \overline{\lambda}_L \;+\; \sqrt{2}\,\overline{\theta}_L \lambda_R \;+\; \sqrt{2}\,\theta_R \overline{\theta}_L \left(D \;-\; i\,F_{03} \right)$$

and

$$\Sigma \;=\; \frac{i}{\sqrt{2}}\, D_L\, \overline{D}_R\, V$$

We show that at large N one can do better than just superpotential

The Effective Potential

The Effective Scalar Potential

In M.Shifman, A.Yung arXiv:0803.0698 the effective scalar potential was found at large N,

$$V_{\rm eff} \;=\; -\,\frac{N}{4\pi}\Big\{ \big(|\sqrt{2}\sigma|^2 + iD\big)\log\big(|\sqrt{2}\sigma|^2 + iD\big) \;-\; iD \;-\; |\sqrt{2}\sigma|^2\log|\sqrt{2}\sigma|^2 \Big\}$$

Elimination of D leads to

$$V_{\rm eff} \;=\; \frac{N}{4\pi}\Big\{ 1 \;+\; |\sqrt{2}\sigma|^2\Big(\log|\sqrt{2}\sigma|^2 - 1\Big)\Big\}$$

This clearly does not fit into the $\Sigma\log\Sigma$ picture!

Supersymmetric Form

Supersymmetrizing

What is the supersymmetric form of these expressions?

First, scatter them into series in

$$x \;=\; \frac{iD}{\left|\sqrt{2}\sigma\right|^2}$$

as iD is naturally the supersymmetry breaking parameter,

$$\frac{4\pi}{N}\, V_{\text{eff}} \;=\; x\left|\sqrt{2}\sigma\right|^2\left(-\ln\left|\sqrt{2}\sigma\right|^2 \;+\; \sum_{k\geq 1}\frac{(-1)^k}{k\,(k+1)}\,x^k\right)$$

Another representation is

$$-iD\,\ln\left|\sqrt{2}\sigma\right|^2 \;+\; \left|\sqrt{2}\sigma\right|^2\int_0^x \ln\left(1+x\right)dx$$

Now we can promote x to superfields,

$$x \;=\; \frac{iD}{\left|\sqrt{2}\sigma\right|^2} \quad\longrightarrow\quad \frac{S}{\sqrt{2}\Sigma}$$

where

$$S \;=\; \frac{i}{2}\,\overline{D}_R D_L \ln\sqrt{2}\overline{\Sigma}$$

and the lowest part of $S\,/\,\Sigma$ is

$$\left.\frac{S}{\sqrt{2}\Sigma}\right| \;=\; \frac{1}{\left|\sqrt{2}\sigma\right|^2}\left(iD - F_{03} \;-\; \frac{2\,i\sqrt{2}\sigma\overline{\lambda}_R\lambda_L}{\left|\sqrt{2}\sigma\right|^2}\right)$$

So we re-design the series in terms of superfields now (not straightforward)

$$\frac{i}{2}\int d^2\tilde{\theta}\, S \sum_{k\geq 1} \frac{(-1)^k}{k\,(k+1)\,(k+2)} \left(\frac{S}{\sqrt{2}\Sigma}\right)^k$$

This reproduces the original series

$$x\,\left|\sqrt{2}\sigma\right|^2 \sum_{k\geq 1} \frac{(-1)^k}{k\,(k+1)}\, x^k$$

but the first term

$$-\,\frac{1}{2}\left|\sqrt{2}\sigma\right|^2 x^2 \quad = \quad \frac{1}{2}\,\frac{D^2}{\left|\sqrt{2}\sigma\right|^2}$$

Superkinetic term

This D^2 term comes from what we call a *superkinetic term*

$$\frac{1}{2}\,\frac{D^2}{\left|\sqrt{2}\sigma\right|^2} \quad \in \quad -\int d^4\theta\,\frac{1}{2}\left|\,\ln\sqrt{2}\Sigma\,\right|^2$$

Aside of D^2 it also contains the kinetic terms for F_{ik}, λ and σ

The existence of this term has been known as far back as in

A. D'Adda, A. .C. Davis, P. Di Vecchia and P. Salomonson, Nucl. Phys. B 222, 45 (1983)

Putting together this superkinetic term, the Witten's potential and converting our series into a logarithm in a D-term form,

$$
\begin{aligned}
\frac{4\pi}{N}\,\mathcal{L} \;&=\; -\int d^4\theta\,\frac{1}{2}\left|\ln\sqrt{2}\Sigma\right|^2 \;-\; i\int d^2\tilde{\theta}\,\left(\sqrt{2}\Sigma\,\ln\sqrt{2}\Sigma \;-\; \sqrt{2}\Sigma\right) \\
&-\;\frac{1}{4}\int d^4\theta\,\ln\sqrt{2}\overline{\Sigma}\left[\left(1+\frac{\sqrt{2}\Sigma}{S}\right)^2\ln\left(1+\frac{S}{\sqrt{2}\Sigma}\right) \;-\; \frac{\sqrt{2}\Sigma}{S}\right] \;+\; \text{h.c.}
\end{aligned}
$$

Its component expansion is quite huge

But the *constant* bosonic part of it does exactly become

$$
\begin{aligned}
&-\left(\left|\sqrt{2}\sigma\right|^2 + iD\right)\log\left(\left|\sqrt{2}\sigma\right|^2 + iD\right) \;+\; iD \;+\; \left|\sqrt{2}\sigma\right|^2\log\left|\sqrt{2}\sigma\right|^2 \;= \\
&=\; -\,iD\,\ln\left|\sqrt{2}\sigma\right|^2 \;+\; \left|\sqrt{2}\sigma\right|^2\int_0^x \ln\left(1+x\right)dx
\end{aligned}
$$

Component form

Component expansion

We can think of the effective action by limiting to only *two* space-time derivatives – this will yield a much more tractable expression

So we keep *all* powers of the auxiliary field D, while retaining only *two* powers of the derivatives of *physical* fields σ, λ, A_k

In that, we think of fermions as already having half a derivative

$$
\begin{aligned}
\frac{4\pi}{N}\,\mathcal{L}_{\text{two deriv}} \;=\; & \frac{\left|\partial_\mu \sigma\right|^2}{\left|\sqrt{2}\sigma\right|^2} \;-\; F_{03}\,\log\frac{\sqrt{2}\sigma}{\sqrt{2}\overline{\sigma}} \;+\; \frac{4\pi}{N}\,V_{\text{eff}} \;- \\
- \;& \frac{F_{03}^2}{\left|\sqrt{2}\sigma\right|^2}\left(2\,\frac{\ln(1+x)\;-\;x}{x^2} \;+\; \frac{1}{2}\,\frac{1}{1\;+\;x}\right) \;- \\
- \;& \frac{\overline{\lambda}_R\, i\overleftrightarrow{\mathcal{D}}_L \lambda_R \;+\; \overline{\lambda}_L\, i\overleftrightarrow{\mathcal{D}}_R \lambda_L}{\left|\sqrt{2}\sigma\right|^2}\,\frac{\ln(1+x)\;-\;x}{x^2} \;- \\
- \;& 2\,\frac{i\sqrt{2}\sigma\overline{\lambda}_R\lambda_L \;+\; i\sqrt{2}\,\overline{\sigma\lambda}_L\lambda_R}{\left|\sqrt{2}\sigma\right|^2}\,\frac{\ln(1+x)}{x} \;+ \\
+ \;& 4\,\frac{\overline{\lambda}_R\lambda_L\overline{\lambda}_L\lambda_R}{\left|\sqrt{2}\sigma\right|^4}\left(\frac{\ln(1+x)\;-\;x}{x^2} \;+\; \frac{1}{1\;+\;x}\right) \;+ \\
+ \;& \frac{1}{4}\,\Box\,\log\left|\sqrt{2}\sigma\right|^2 \cdot \frac{(1-x^2)\ln(1+x)\;-\;x}{x^2} \;- \\
- \;& 2\,\frac{F_{03}\left(i\sqrt{2}\sigma\overline{\lambda}_R\lambda_L \;-\; i\sqrt{2}\,\overline{\sigma\lambda}_L\lambda_R\right)}{\left|\sqrt{2}\sigma\right|^4}\,\frac{\ln(1+x)\;-\;x}{x^2}\,.
\end{aligned}
$$

Truncation

This action is not supersymmetric — field D which is sitting in x effectively contains one space-time derivative, or two superspace derivatives

$$iD \;\in\; \sqrt{2}\Sigma \;=\; \frac{i}{\sqrt{2}}\, D_L\, \overline{D}_R V$$

So we to "retain" supersymmetry at the level of two space-time derivatives we have to put $D = 0$ wherever there are already two derivatives

The result of this is

$$
\begin{aligned}
&- \;\; iD \,\log|\sqrt{2}\sigma|^2 \;-\; \frac{1}{2}\,\frac{(iD)^2}{|\sqrt{2}\sigma|^2} \;-\; F_{03}\,\log\frac{\sqrt{2}\sigma}{\sqrt{2}\overline{\sigma}} \;+ \\
&+ \;\; \frac{|\partial_\mu\sigma|^2 \;+\; \frac{1}{2}F_{03}^2 \;+\; \frac{1}{2}\left(\,\overline{\lambda}_R\, i\overleftrightarrow{\mathcal{D}}_L\lambda_R \;+\; \overline{\lambda}_L\, i\overleftrightarrow{\mathcal{D}}_R\lambda_L\right)}{|\sqrt{2}\sigma|^2} \;- \\
&- \;\; 2\,\frac{i\sqrt{2}\sigma\overline{\lambda}_R\lambda_L \;+\; i\sqrt{2}\,\overline{\sigma\lambda}_L\lambda_R}{|\sqrt{2}\sigma|^2} \;+\; 2\,\frac{\overline{\lambda}_R\lambda_L\overline{\lambda}_L\lambda_R}{|\sqrt{2}\sigma|^4} \;+ \\
&+ \;\; \frac{(iD+F_{03})\,i\sqrt{2}\sigma\overline{\lambda}_R\lambda_L \;+\; (iD-F_{03})\,i\sqrt{2}\,\overline{\sigma\lambda}_L\lambda_R}{|\sqrt{2}\sigma|^4}\,.
\end{aligned}
$$

This matches the effective one-loop action calculated in M.Shifman, A.Yung arXiv:0803.0698

Integrating D

Another question we can ask is what does the action look like if we leave only the physical fields — A_k, σ and λ?

To do that we must return to the expression with $\log(1+x) - x$, and eliminate x

This is not possible to do exactly — but again, we only need two space-time derivatives

$$\frac{4\pi}{N}\,\mathcal{L}_{\text{two deriv}}(\sigma, A_\mu, \lambda) \;=$$

$$= \;\frac{\left|\partial_\mu\sigma\right|^2}{r} \;-\; F_{03}\,\log\frac{\sqrt{2}\sigma}{\sqrt{2}\overline{\sigma}} \;+\; v_{\text{eff}}(r) \;-$$

$$+\;\; 2\,F_{03}^2\left(\frac{v_{\text{eff}}(r)}{(1-r)^2} \;-\; \frac{r}{4}\right) \;+\; \left(\overline{\lambda}_R\, i\overleftrightarrow{\mathcal{D}_L}\lambda_R \;+\; \overline{\lambda}_L\, i\overleftrightarrow{\mathcal{D}_R}\lambda_L\right)\frac{v_{\text{eff}}(r)}{(1-r)^2} \;+$$

$$+\;\; 2\left(i\sqrt{2}\sigma\overline{\lambda}_R\lambda_L \;+\; i\sqrt{2}\,\overline{\sigma}\overline{\lambda}_L\lambda_R\right)\frac{1-r}{r}\,\ln r \;-$$

$$-\;\; 4\,\overline{\lambda}_R\lambda_L\overline{\lambda}_L\lambda_R\left(\frac{v_{\text{eff}}(r)}{r\,(1-r)^2} \;-\; \frac{1}{r} \;+\; \frac{r\left(1 \;-\; r \;-\; \ln r\right)^2}{(1-r)^4}\right) \;-$$

$$-\;\; \frac{1}{4}\,\Box\,\ln r\left(\frac{r\,v_{\text{eff}}(r)}{(1-r)^2} \;-\; \ln r\right) \;+$$

$$+\;\; 2\,F_{03}\left(i\sqrt{2}\sigma\overline{\lambda}_R\lambda_L \;-\; i\sqrt{2}\,\overline{\sigma}\overline{\lambda}_L\lambda_R\right)\frac{v_{\text{eff}}(r)}{r\,(1-r)^2}\,.$$

where $r \;=\; \left|\sqrt{2}\sigma\right|^2$ and $v_{\text{eff}}(r) \;=\; \frac{4\pi}{N}\,V_{\text{eff}}(\sigma) \;=\; r\ln r \;+\; 1 \;-\; r$

Heterotic and Massive deformations

Heterotic deformation and Twisted masses

Knowing the superfield form of the action it is straightforward to include the twisted masses

$$
\begin{aligned}
4\pi\,\mathcal{L} \;=\; & -\sum_k \Bigg(\int d^4\theta\, \frac{1}{2}\left|\ln\left(\sqrt{2}\Sigma - m_k\right)\right|^2 \;+ \\
& +\; i\int d^2\tilde{\theta}\,\Big(\left(\sqrt{2}\Sigma - m_k\right)\ln\left(\sqrt{2}\Sigma - m_k\right) \;-\; \left(\sqrt{2}\Sigma - m_k\right)\Big) \;+ \\
& +\; \frac{1}{4}\int d^4\theta\, \ln\left(\sqrt{2}\overline{\Sigma} - \overline{m}_k\right) \times \\
& \qquad \times \left[\left(1 + \frac{\sqrt{2}\Sigma - m_k}{S_k}\right)^2 \ln\left(1 + \frac{S_k}{\sqrt{2}\Sigma - m_k}\right) \;-\; \frac{\sqrt{2}\Sigma - m_k}{S_k} \right] \Bigg) \;+ \\
& +\; 4\pi\int d^2\theta_R\, \hat{\zeta}_R\,\hat{\bar{\zeta}}_R \;-\; 4\pi i\int d\theta_R\, \hat{\zeta}\cdot J(\sqrt{2}\hat{\sigma}) \;+\; \text{h.c.}
\end{aligned}
$$

where $\hat{\sigma}(\upsilon) \;=\; \sigma \;-\; \sqrt{2}\,\theta_R\,\overline{\lambda}_L$ and $\hat{z}(\upsilon) \;=\; z \;-\; \sqrt{2}\,\theta_R\,\zeta_L$

Thank you