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Heterotic $\mathcal{N} = (0, 2)$ $CP(N-1)$ model with twisted masses

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We present a two-dimensional heterotic $\mathcal{N} = (0, 2)$ $CP(N-1)$ model with twisted masses. It is supposed to describe internal dynamics of non-Abelian strings in massive $\mathcal{N} = 2$ SQCD with $\mathcal{N} = 1$-preserving deformations. We present gauge and geometric formulations of the world-sheet theory and check its $\mathcal{N} = (0, 2)$ supersymmetry. It turns out that the set of twisted masses in the heterotic model has N complex mass parameters, rather than $N - 1$. In the general case, when all mass parameters are nonvanishing, $\mathcal{N} = (0, 2)$ supersymmetry is *spontaneously* broken already at the classical level. If at least one of the above mass parameters vanishes, then $\mathcal{N} = (0, 2)$ is unbroken at the classical level. The spontaneous breaking of supersymmetry in this case occurs through nonperturbative effects.

I. INTRODUCTION

The recent discovery of non-Abelian strings [1–4] supported by certain four-dimensional supersymmetric gauge theories opened an avenue to understanding of a number of dynamical issues which could not have been addressed previously. Here we will focus on one particular aspect: models describing low-energy dynamics on the world sheet of various non-Abelian strings.

The starting point was [2–4] $\mathcal{N} = 2$ super-QCD (SQCD) with the gauge group U(N), N_f massive quark hypermultiplets and the Fayet–Iliopoulos (FI) ξ term for the U(1) factor (for a review see [5]). If $\xi \gg \Lambda^2$ this bulk theory can be treated quasiclassically. Furthermore, for $N_f = N$ critical flux tube solutions exist (BPS-saturated both at the classical and quantum levels) which, in addition to the conventional (super)translational moduli are characterized by orientational and superorientational moduli. Low-energy dynamics of these moduli fields is described by a two-dimensional $\mathcal{N} = (2, 2)$ sigma model with the $CP(N-1)$ target space. If the mass terms of the quark supermultiplets are different, the $\mathcal{N} = (2, 2)$ world-sheet model acquires twisted masses [6]. Still, $\mathcal{N} = (2, 2)$ supersymmetry (SUSY) on the world sheet is preserved, which guarantees complete decoupling of the (super)translational and (super)orientational sectors of the world-sheet model. The (super)translational sector is presented by a free $\mathcal{N} = (2, 2)$ field theory.

Moving towards $\mathcal{N} = 1$ bulk theories one discovers [7–9] a novel class of deformations of the world-sheet $\mathcal{N} = (2, 2)$ supersymmetric $CP(N-1)$ model, currently known as *heterotic* $CP(N-1)$ model. Assume we deform basic $\mathcal{N} = 2$ SQCD[1] by the superpotential mass term for the adjoint supermultiplet,

$$\mathcal{W}_{3+1} = \frac{\mu}{2}[\mathcal{A}^2 + (\mathcal{A}^a)^2], \qquad (1.1)$$

where μ is a common mass parameter for the chiral superfields in $\mathcal{N} = 2$ gauge supermultiplets, U(1) and SU(N), respectively. The subscript $3 + 1$ tells us that the deformation superpotential (1.1) refers to the four-dimensional bulk theory. Then the $\mathcal{N} = 2$ supersymmetry in the bulk is lost, as it is explicitly broken down to $\mathcal{N} = 1$. Gone with $\mathcal{N} = 2$ in the bulk is $\mathcal{N} = (2, 2)$ SUSY on the world sheet. Decoupling of the (super)translational and (super)orientational sectors disappears. Instead, the two fermionic moduli from the supertranslational sector (right-handed spinors) get connected with the superorientational fermionic moduli. The corresponding coupling constants are presented by one complex number [7] related [8,9] to the deformation parameter μ in Eq. (1.1). The heterotic coupling of the supertranslational moduli fields reduces the world-sheet supersymmetry from $\mathcal{N} = (2, 2)$ to $\mathcal{N} = (0, 2)$. This is the origin of the alternative name, $\mathcal{N} = (0, 2)$ $CP(N-1) \times$ C model. The heterotic $CP(N-1)$ model has rich dynamics (it was solved [10] at large N) leading to the spontaneous breaking of $\mathcal{N} = (0, 2)$ SUSY due to nonperturbative effects.[2]

This heterotic model follows from the bulk theory described above provided all mass terms of the quark hypermultiplets are set equal. If they are unequal, it is reasonable to expect that, just as in the $\mathcal{N} = 2$ case, the inequality of masses in the bulk will manifest itself on the world sheet as twisted masses. Leaving aside (for future studies) derivation of the heterotic $CP(N-1)$ model with twisted masses

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[1]The gauge group is assumed to be U(N) and we take N quark hypermultiplets with *equal* mass terms.

[2]The latter statement refers to the particular form of the bulk deformation quoted in Eq. (1.1).

on the world sheet from the $\mathcal{N} = 1$-deformed microscopic theory in the bulk, we will focus in this work on the heterotic $CP(N-1)$ model with twisted masses *per se*. To the best of our knowledge nobody has ever discussed this model. We address two questions, (i) whether or not $\mathcal{N} = (0, 2)$ supersymmetry at the classical level survives the introduction of the twisted masses into heterotic $CP(N-1)$, and (ii) construction of the corresponding Lagrangian both in the gauged and geometrical formulations. The large-N solution of the model will be the next step [11].

Our findings can be summarized as follows. If the $\mathcal{N} = (2, 2)$ model with twisted masses contains $N - 1$ free mass parameters, its heterotic deformation allows one to introduce two rather than one extra complex parameters. One of them is obvious: a complex coupling (δ or $\tilde{\gamma}$, see below[3]) regulating the strength of the heterotic deformation. The second complex parameter is less obvious, being an extra mass parameter. It turns out that the set of twisted masses in the heterotic version has N complex mass parameters rather than $N - 1$. In the general case, when all mass parameters are nonvanishing, $\mathcal{N} = (0, 2)$ supersymmetry is *spontaneously* broken already at the classical level. If at least one of the above mass parameters vanishes the $\mathcal{N} = (0, 2)$ is unbroken at the classical level. The spontaneous breaking of SUSY occurs in this case through nonperturbative effects [11].

The paper is organized as follows. In Sec, II we briefly review the twisted-mass deformed $\mathcal{N} = (2, 2)\ CP(N-1)$ model in the gauge formulation. Then we show how one can introduce, additionally, the heterotic deformation of the type discussed above. Although $\mathcal{N} = (2, 2)$ SUSY is destroyed by the combination of the two deformations, $\mathcal{N} = (0, 2)$ is demonstrated to survive. We explain why, in addition to the heterotic deformation parameter, an extra mass parameter appears (extra compared to the $\mathcal{N} = (2, 2)$ model). In Sec. III we follow the same avenue to obtain the Lagrangian of the twisted-mass deformed heterotic $\mathcal{N} = (2, 2)$ model in the geometric formulation. Section IV summarizes our results and outlines the program of future research in the given direction.

II. GAUGE FORMULATION

In this section we first review the two-dimensional $\mathcal{N} = (2, 2)\ CP(N-1)$ sigma model with twisted masses in the gauge formulation and then present its $\mathcal{N} = (0, 2)$ deformation.

A. $\mathcal{N} = (2, 2)\ CP(N-1)$ model

The two-dimensional supersymmetric $\mathcal{N} = (2, 2)$ $CP(N-1)$ model is known to describe internal dynamics of non-Abelian strings in $\mathcal{N} = 2$ super-QCD with the $U(N)$ gauge group and N flavors of quarks [1–4], see also reviews [5,12–14]. In the gauge formulation, this model with twisted masses $m^l\ (l = 1, \ldots N)$ is given by the strong-coupling limit ($e^2 \to \infty$) of the following U(1) gauge theory [15]:

$$\mathcal{L}_{(2,2)} = \frac{1}{4e^2}F_{kl}^2 + \frac{1}{e^2}|\partial_k \sigma|^2 + \frac{1}{2e^2}D^2 + \frac{1}{e^2}\bar{\lambda}_R i\partial_L \lambda_R + \frac{1}{e^2}\bar{\lambda}_L i\partial_R \lambda_L + 2\beta\left(|\nabla n|^2 + 2\left|\sigma - \frac{m^l}{\sqrt{2}}\right|^2 |n^l|^2 + iD(|n^l|^2 - 1)\right.$$

$$\left. + \bar{\xi}_R i\nabla_L \xi_R + \bar{\xi}_L i\nabla_R \xi_L + i\sqrt{2}\left(\sigma - \frac{m^l}{\sqrt{2}}\right)\bar{\xi}_{Rl}\xi_L^l + i\sqrt{2}\left(\bar{\sigma} - \frac{\bar{m}^l}{\sqrt{2}}\right)\bar{\xi}_{Ll}\xi_R^l + i\sqrt{2}\overline{\xi_{[R}\lambda_{L]}}n - i\sqrt{2}\bar{n}\lambda_{[R}\xi_{L]}\right)$$

$$l = 1, \ldots N, \tag{2.1}$$

where

$$\nabla_k = \partial_k - iA_k, \qquad \nabla_{R,L} = \nabla_0 \pm i\nabla_3,$$
$$\lambda_{[R}\xi_{L]} = \lambda_R \xi_L - \lambda_L \xi_R, \tag{2.2}$$

while x_k ($k = 0, 3$) denotes the two coordinates on the string world sheet. We assume that the string is stretched in the x_3 direction. In the above Lagrangian n^l are N complex scalar fields of the $CP(N-1)$ model and $\xi_{R,L}^l$ are their fermionic superpartners. All fields of the gauge supermultiplet, namely, the gauge field A_k, complex scalar σ, fermions $\lambda_{R,L}$ and auxiliary field D are not dynamical in the limit $e^2 \to \infty$. They can be eliminated via algebraic equa-

tions of motion. In particular, integration over D and λ gives the standard $CP(N-1)$ model constraints

$$|n^l|^2 = 1, \qquad \bar{n}_l \xi_{R,L}^l = 0. \tag{2.3}$$

Parameters m^l in Eq. (2.1) are the twisted masses.

A comment is in order here on our summation conventions for the $CP(N-1)$ indices l, *etc.*, since they become nontrivial once the twisted masses are introduced. The sum in l is always implied if the index is written more than once. In the places where this can cause ambiguity we put the summation sign explicitly. Furthermore, we specify the range of variation of $CP(N-1)$ indices in the end of equations. Finally, we omit the summation sign in those terms where the sum is obvious, such as the kinetic terms $\bar{\xi}i\nabla\xi$.

Another comment refers to the twisted masses. From Eq. (2.1) it is obvious that by virtue of the shift of the σ

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[3]In this paper we denote the parameter of deformation of the heterotic world sheet by $\tilde{\gamma}$ which is related to the analogous parameter γ originally introduced in [8], as $\tilde{\gamma} = \sqrt{2/\beta}\gamma$.

field one can always impose an additional condition

$$\sum_{l=1}^{N} m^l = 0. \tag{2.4}$$

Therefore, in fact, there are $N - 1$ independent mass parameters in the $\mathcal{N} = (2, 2)$ version of the model.

We will use two normalizations of the physical fields n, ξ, ζ in this paper. To prove supersymmetry of (2.1) it is easier to include the factor of 2β into the big bracket by redefining the corresponding fields, so that $|n|^2 = 2\beta$.

However, in order to determine the correspondence between the above model and the geometric formulation of $CP(N - 1)$ model with twisted masses it is easier to leave it outside, so that $|n^l|^2 = 1$.

The model (2.1), apart from the two-dimensional Fayet–Iliopoulos (FI) term $-iD$, is nothing but the dimensionally reduced $\mathcal{N} = 1$ four-dimensional SQED. From this fact one obtains the following transformation laws of the component fields under $\mathcal{N} = (2, 2)$ supersymmetry, with transformation parameters $\epsilon_{R,L}$ and $\bar{\epsilon}_{R,L}$:

$$\delta A_{R,L} = 2i(\epsilon_{R,L}\bar{\lambda}_{R,L} - \bar{\epsilon}_{R,L}\lambda_{R,L}), \qquad \delta\sigma = -\sqrt{2}(\epsilon_R\bar{\lambda}_L - \bar{\epsilon}_L\lambda_R), \qquad \delta\bar{\sigma} = +\sqrt{2}(\bar{\epsilon}_R\lambda_L - \epsilon_L\bar{\lambda}_R),$$

$$\delta\lambda_R = -\epsilon_R \cdot D - \frac{1}{2}\epsilon_R \cdot F_{RL} - i\sqrt{2}\partial_R\sigma \cdot \epsilon_L \qquad \delta\lambda_L = -\epsilon_L \cdot D + \frac{1}{2}\epsilon_L \cdot F_{RL} - i\sqrt{2}\partial_L\sigma \cdot \epsilon_R$$

$$\delta\bar{\lambda}_R = -\bar{\epsilon}_R \cdot D + \frac{1}{2}\bar{\epsilon}_R \cdot F_{RL} + i\sqrt{2}\partial_R\bar{\sigma} \cdot \bar{\epsilon}_L \qquad \delta\bar{\lambda}_L = -\bar{\epsilon}_L \cdot D - \frac{1}{2}\bar{\epsilon}_L \cdot F_{RL} + i\sqrt{2}\partial_L\sigma \cdot \bar{\epsilon}_R$$

$$\delta D = i\epsilon_R\partial_L\bar{\lambda}_R + i\bar{\epsilon}_R\partial_L\lambda_R + i\epsilon_L\partial_R\bar{\lambda}_L + i\bar{\epsilon}_L\partial_R\lambda_L$$

$$\delta n = -\sqrt{2}\epsilon_{[R}\xi_{L]} \qquad \delta\bar{n} = +\sqrt{2}\overline{\epsilon_{[R}\xi_{L]}} \qquad \delta\xi_R^l = -i\sqrt{2}\bar{\epsilon}_L\nabla_R n^l + \sqrt{2}\epsilon_R F^l - 2i\bar{\epsilon}_R\left(\sigma - \frac{m^l}{\sqrt{2}}\right)n^l \tag{2.5}$$

$$\delta\xi_L^l = +i\sqrt{2}\bar{\epsilon}_R\nabla_L n^l + \sqrt{2}\epsilon_L F^l + 2i\bar{\epsilon}_L\left(\bar{\sigma} - \frac{\bar{m}^l}{\sqrt{2}}\right)n^l \qquad \delta\bar{\xi}_{lR} = +i\sqrt{2}\epsilon_L\nabla_R\bar{n}_l + \sqrt{2}\bar{\epsilon}_R\bar{F}_l - 2i\epsilon_R\left(\bar{\sigma} - \frac{\bar{m}^l}{\sqrt{2}}\right)\bar{n}_l$$

$$\delta\bar{\xi}_{lL} = -i\sqrt{2}\epsilon_R\nabla_L\bar{n}_l + \sqrt{2}\bar{\epsilon}_L\bar{F}_l + 2i\epsilon_L\left(\sigma - \frac{m^l}{\sqrt{2}}\right)\bar{n}_l$$

$$\delta F^l = -i\sqrt{2}(\bar{\epsilon}_L\nabla_R\xi_L^l + \bar{\epsilon}_R\nabla_L\xi_R^l) - 2i\overline{\epsilon_{[R}\lambda_{L]}}n^l - 2i\left(\bar{\epsilon}_R\left(\sigma - \frac{m^l}{\sqrt{2}}\right)\xi_L^l + \bar{\epsilon}_L\left(\bar{\sigma} - \frac{\bar{m}^l}{\sqrt{2}}\right)\xi_R^l\right)$$

$$\delta\bar{F}_l = -i\sqrt{2}(\epsilon_R\nabla_L\bar{\xi}_{lR} + \epsilon_L\nabla_R\bar{\xi}_{lL}) + 2i\bar{n}_l\epsilon_{[R}\lambda_{L]} + 2i\left(\epsilon_L\left(\sigma - \frac{m^l}{\sqrt{2}}\right)\bar{\xi}_{lR} + \epsilon_R\left(\bar{\sigma} - \frac{\bar{m}^l}{\sqrt{2}}\right)\bar{\xi}_{lL}\right),$$

where F^l are F components of the (n^l, ξ^l) supermultiplet, while $F_{RL} = -2iF_{03}$ is a convenient notation for the gauge field strength.

Obviously, with vanishing twisted masses, the theory (2.1) is invariant under the massless version of the supertransformations (2.5). The masses themselves can be considered just as constant "background" gauge fields of the "original" $U(1)^{N-1}$ four-dimensional SQED, directed in the (x_1, x_2)-plane [16,17]. After dimensional reduction they become constant "σ"s. Hence, supersymmetry should be preserved by twisted-mass deformations,[4] and it indeed is [6].

B. Heterotic $\mathcal{N} = (0, 2)$ $CP(N - 1) \times C$ model

As was mentioned, in $\mathcal{N} = 2$ supersymmetric bulk theory, the translational sector of the world-sheet model on the non-Abelian string decouples from the orientational one. The translational sector is associated with the position of the string x_{0i} ($i = 1, 2$) in the orthogonal plane (x_1, x_2); it also includes its fermionic superpartners ζ_L and ζ_R. The orientational sector is described by the $\mathcal{N} = (2, 2)$ $CP(N - 1)$ model (2.1). Once $\mathcal{N} = 2$ breaking deformation (1.1) is added in the bulk theory, this decoupling no longer takes place [7]. The translational sector becomes mixed with the orientational one. In fact, the fields x_{0i} and ζ_L are still free and do decouple. At the same time, the right-handed translational modulus ζ_R becomes coupled to the orientational sector.

Our next step is to combine the heterotic $\mathcal{N} = (0, 2)$ deformations of massless $CP(N - 1) \times C$ model studied in [7–10] with the twisted-mass deformed $CP(N - 1)$ model (2.1).

Let us parenthetically note that, as was shown in [7,8], the BPS nature of the non-Abelian string solution is preserved only if the critical points of the bulk deformation superpotential coincide with the quark masses. This is related to the fact that, if the above condition is fulfilled, only the quark scalar fields whose masses are related by $\mathcal{N} = 1$ supersymmetry to masses of the gauge bosons are

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[4]The vector supermultiplet $(A_k, \sigma, \lambda, D)$, with only σ non-vanishing and constant, is obviously invariant under $\mathcal{N} = (2, 2)$ supersymmetry.

excited on the string solution. Other quark fields, with different masses, identically vanish. If the above condition is not satisfied, other quark fields must be nonzero on the solution. This immediately spoils the BPS saturation of the string solution.

Clearly, the only critical point of the superpotential (1.1) is at zero. Therefore, for a generic choice of nonvanishing quark mass terms, the above condition is not met. Thus, we expect that the "BPS-ness" of the non-Abelian string solutions is lost. Below we will see how this four-dimensional perspective is translated into the language of the two-dimensional world-sheet theory. We will see that it

manifests itself in the spontaneous breaking of $\mathcal{N} = (0, 2)$ supersymmetry in the $CP(N - 1) \times \mathbb{C}$ model on the string world sheet. This happens already at the classical level. Note that in the massless case studied in [8,9] the above condition is met, and $\mathcal{N} = (0, 2)$ supersymmetry is preserved in the world-sheet model at the classical level. Still, it turns out to be spontaneously broken by quantum (non-perturbative) effects [8,10,18].

In the gauge formulation, the two-dimensional $\mathcal{N} = (0, 2)$ $CP(N - 1) \times \mathbb{C}$ model with twisted masses is given by the strong-coupling ($e^2 \to \infty$) limit of the following U(1) theory

$$
\begin{aligned}
\mathcal{L}_{(0,2)} = & -\frac{1}{8e^2} F_{RL}^2 + \frac{1}{e^2} |\partial_k \sigma|^2 + \frac{1}{2e^2} D^2 + \frac{1}{e^2} \bar{\lambda}_R i \partial_L \lambda_R + \frac{1}{e^2} \bar{\lambda}_L i \partial_R \lambda_L \\
& + 2\beta \Big(|\nabla n|^2 + 2 \left| \sigma - \frac{m^l}{\sqrt{2}} \right|^2 |n^l|^2 + iD(|n^l|^2 - 1) + \bar{\xi}_R i \nabla_L \xi_R + \bar{\xi}_L i \nabla_R \xi_L \\
& + i\sqrt{2} \Big(\sigma - \frac{m^l}{\sqrt{2}} \Big) \bar{\xi}_{Rl} \xi_L^l + i\sqrt{2} \Big(\bar{\sigma} - \frac{\bar{m}^l}{\sqrt{2}} \Big) \bar{\xi}_{Ll} \xi_R^l + i\sqrt{2} \overline{\xi_{[R} \lambda_{L]}} n - i\sqrt{2} \bar{n} \lambda_{[R} \xi_{L]} + \bar{\zeta}_R i \partial_L \zeta_R + \bar{\mathcal{F}} \mathcal{F} \\
& + 2i \frac{\partial^2 \hat{\mathcal{W}}}{\partial \sigma^2} \bar{\lambda}_L \zeta_R + 2i \frac{\partial^2 \bar{\hat{\mathcal{W}}}}{\partial \bar{\sigma}^2} \bar{\zeta}_R \lambda_L - 2i \frac{\partial \hat{\mathcal{W}}}{\partial \sigma} \mathcal{F} - 2i \bar{\mathcal{F}} \frac{\partial \bar{\hat{\mathcal{W}}}}{\partial \bar{\sigma}} \Big), \qquad l = 1, \ldots N,
\end{aligned}
\tag{2.6}
$$

where $\hat{\mathcal{W}}(\Sigma)$ is an $\mathcal{N} = (2, 2)$ breaking superpotential function. The hat over $\mathcal{W}$ will remind us that this superpotential refers to a two-dimensional theory, rather than a four-dimensional one. In particular, in this paper we consider $\hat{\mathcal{W}}(\Sigma)$ to be quadratic,

$$
\hat{\mathcal{W}}(\Sigma) = \frac{1}{2} \delta \Sigma^2.
$$

The deformation parameter δ was introduced in [8].[5]

We prove $\mathcal{N} = (0, 2)$ supersymmetry of the Lagrangian (2.6) in two steps (for now we absorb the factor of 2β into the definition of n^l, ξ^l, ζ_R and $\mathcal{F}$). The mass deformation and the heterotic deformation are independent of each other, as we will shortly prove. Therefore, we can consider first the theory deformed only by the twisted masses, and then add $\mathcal{N} = (0, 2)$ terms.

As the first step, we discard the superpotential $\hat{\mathcal{W}}$. Then the theory splits into two decoupled sectors—orientational and translational. The orientational sector (2.1) was already

considered in Sec. II A. The translational sector is free

$$
\bar{\zeta}_R i \partial_L \zeta_R + \bar{\mathcal{F}} \mathcal{F},
$$

and invariant under the right-handed supersymmetry [8],

$$
\begin{aligned}
\delta \zeta_R = \sqrt{2} \epsilon_R \mathcal{F}, \qquad & \delta \bar{\zeta}_R = \sqrt{2} \bar{\epsilon}_R \bar{\mathcal{F}}, \\
\delta \mathcal{F} = -i\sqrt{2} \bar{\epsilon}_R \partial_L \zeta_R, \qquad & \delta \bar{\mathcal{F}} = -i\sqrt{2} \epsilon_R \partial_L \bar{\zeta}_R.
\end{aligned}
\tag{2.7}
$$

Thus, the direct sum of the two sectors preserves $\mathcal{N} = (0, 2)$ supersymmetry.

The final step is to restore the heterotic deformation $\hat{\mathcal{W}}$. The $\mathcal{N} = (2, 2)$ fields that mix with the translational sector by means of $\hat{\mathcal{W}}$ are λ_L and σ. The supertransformations of the latter do not involve the masses m^l, e.g. $\delta \sigma = -\sqrt{2}(\epsilon_R \bar{\lambda}_L - \bar{\epsilon}_L \lambda_R)$, see Eq. (2.5). As a result, the heterotic deformation and the twisted-mass deformation are indeed independent of each other.

Finally, now we can assert that the $\hat{\mathcal{W}}$ terms are invariant under the overall right-handed supersymmetry (2.5) and (2.7), for arbitrary superpotential functions $\hat{\mathcal{W}}(\sigma)$. With the heterotic deformation switched on, $\hat{\mathcal{W}}(\sigma) \neq 0$, the shift property is lost in (2.6): the shift of σ is no longer a symmetry of the theory. Hence, there are N independent twisted-mass parameters; physically measurable quantities depend on all of them. In particular, in each Higgs vacuum at weak coupling, $N - 1$ parameters define masses of excitations, while one "extra" parameter determines the vacuum energy.

---

[5]The relation δ of the world-sheet theory to the parameter μ in the four-dimensional superpotential (1.1) was studied in [8,9]. To be more exact, this relation was derived in [8,9] for massless version of the theory. Since the mass deformation seems to be independent of $\mathcal{N} = (0, 2)$ deformation, we expect that the same relation will hold in the massive theory. The proof of this is left for future work. In [8,9] we obtained that at small μ the parameter $\delta = \text{const} \frac{g_2^2 \mu}{m_W}$, while at large μ we found $\delta = \text{const} \frac{\mu}{|\mu|} \sqrt{\ln \frac{g_2^2 \mu}{m_W}}$. Here g_2^2 is the SU(N) gauge coupling, while m_W is the mass of the SU(N) gauge boson of the bulk theory.

By analogy with massive *nonsupersymmetric* $CP(N - 1)$ model studied in [19] in the large-N approximation, we expect that the model (2.6) exhibits two phases, separated by a crossover transition, namely, the week-coupling Higgs phase at large masses and the strong-coupling phase at small masses. Detailed study of dynamics of the heterotic $CP(N - 1) \times C$ model (2.6) is left for future work [11]. Here we just comment on the week-coupling Higgs phase.

If the twisted masses are large, $|m^l| \gg \Lambda$ (where Λ is the dynamical scale of the world-sheet theory), the coupling constant β is frozen at the scale of the order of $|m^l|$. The theory is at weak coupling and can be studied perturbatively. We have N vacua. In each of them, the vacuum expectation value (VEV) of n^l is given by

$$\langle n^l \rangle = \delta^{ll_0}, \qquad l_0 = 1, \ldots, N. \qquad (2.8)$$

As follows from (2.6), in order to find VEV of the σ field in the l_0-th vacuum, we have to minimize the following potential

$$2 \left| \sigma - \frac{m^{l_0}}{\sqrt{2}} \right|^2 + 4 |\delta|^2 |\sigma|^2 \qquad (2.9)$$

with respect to σ. This minimization gives

$$\langle \sigma \rangle = \frac{m^{l_0}}{\sqrt{2}} \frac{1}{1 + 2|\delta|^2}. \qquad (2.10)$$

By substituting this back in the potential (2.6) we get the vacuum energy and masses of all $(N - 1)$ elementary fields n^l and ξ^l ($l \neq l_0$). We have,

$$E_{\text{vac}} = |\tilde{\gamma}|^2 |m^{l_0}|^2, \qquad M_{\text{ferm}}^l = m^l - m^{l_0} + |\tilde{\gamma}|^2 m^{l_0},$$

$$|M_{\text{bos}}^l|^2 = |M_{\text{ferm}}^l|^2 - |\tilde{\gamma}|^4 |m^{l_0}|^2, \qquad l \neq l_0, \qquad (2.11)$$

where we introduced a new parameter $\tilde{\gamma}$ via the relation

$$\frac{1}{1 + 2|\delta|^2} \equiv 1 - |\tilde{\gamma}|^2. \qquad (2.12)$$

Although neither the twisted-mass deformation, nor the heterotic deformation by themselves break supersymmetry completely, when combined, they lead to the spontaneous $\mathcal{N} = (0, 2)$ supersymmetry breaking already at the classical level (unless $m^{l_0} = 0$). Namely, for nonzero masses in each of the Higgs vacua the vacuum energy does not vanish, and the boson masses are different from the fermion masses. As was explained above, this is in accord with our expectations which follow from the bulk theory picture. In particular, in the special case in which all N masses sit on a circle,

$$m^l = m \cdot e^{2\pi l/N}, \qquad l = 1, \ldots, N,$$

the N vacua become degenerate.

Note that supersymmetry restores if one of the masses vanishes. The corresponding vacuum becomes supersymmetric (at the classical level), as is evident from Eq. (2.11),

$$m^{l_0} = 0 \Rightarrow E_{\text{vac}}^{l_0} = 0, \qquad l_0 = 1, \ldots, N.$$

The theory then becomes a heterotic $CP(N - 1) \times C$ model with $N - 1$ twisted-mass parameters.

When $\tilde{\gamma} = 0$, all N vacua become supersymmetric. The heterotic deformation is switched off, and one returns to the twisted-mass deformed $\mathcal{N} = (2, 2)$ $CP(N - 1)$ model. Although formally there are N twisted-mass parameters, it is well-known that the theory has only $N - 1$ physical parameters, more precisely only the mass differences

$$M_{\text{bos}}^i = M_{\text{ferm}}^i = m^i - m^{l_0}$$

enter the spectrum and all other physical quantities. This can be clearly seen from Eq. (2.11) at $\tilde{\gamma} = 0$. This circumstance is in one-to-one correspondence with the σ-shift symmetry of the Lagrangian (2.1). Thus, passing to the heterotic model we acquire not only the heterotic coupling δ or $\tilde{\gamma}$, but, in addition, one extra mass parameter.

To compare the massive heterotic theory in the gauged formulation to the one in the geometric formulation, we will need to eliminate all auxiliary fields from the model. It will be convenient to have the constraint on n^i in the form $|n^l|^2 = 1$. To this end we restore the factor 2β in the model (2.6). Also we understand that the factor 2β naturally arises in the derivation of the sigma model from the string solution in the microscopic bulk theory.

We now eliminate the auxiliary fields from (2.6). As was noted earlier [7], in the $\mathcal{N} = (0, 2)$ theory the right-handed constraint $\bar{n}\xi_R = 0$ is changed,

$$\bar{n}\xi_R \propto \bar{\delta}.$$

One can restore the original form of the constraint by performing a shift of the superorientational variable ξ_R, namely,

$$\xi_R' = \xi_R - \sqrt{2}\bar{\delta}n\bar{\zeta}_R, \qquad \bar{\xi}_R' = \bar{\xi}_R - \sqrt{2}\delta\bar{n}\zeta_R.$$

This obviously changes the normalization of the kinetic term for ζ_R, which we bring back to its canonical form by rescaling ζ_R,

$$\zeta_R \rightarrow \frac{1}{1 + 2|\delta|^2} \zeta_R \equiv (1 - |\tilde{\gamma}|^2)\zeta_R.$$

As a result of all these transformations, the following theory emerges[6]:

---

[6]Note that in this paper $\tilde{\gamma} = \sqrt{2/\beta}\gamma$, where γ was introduced in [8].

$$\frac{\mathcal{L}}{2\beta} = \bar{\zeta}_R i\partial_L \zeta_R + |\partial n|^2 + (\bar{n}\partial_k n)^2 + \bar{\xi}_R i\partial_L \xi_R + \bar{\xi}_L i\partial_R \xi_L - (\bar{n}i\partial_R n)\bar{\xi}_L \xi_L - (\bar{n}i\partial_L n)\bar{\xi}_R \xi_R + \tilde{\gamma}i\partial_L \bar{n}\xi_R \zeta_R + \bar{\tilde{\gamma}}\bar{\xi}_R i\partial_L n\bar{\zeta}_R$$

$$+ |\tilde{\gamma}|^2 \bar{\xi}_L \xi_L \bar{\zeta}_R \zeta_R + (1 - |\tilde{\gamma}|^2)\bar{\xi}_L \xi_R \bar{\xi}_R \xi_L - \bar{\xi}_R \xi_R \bar{\xi}_L \xi_L + \sum_l |m^l|^2 |n^l|^2 - im^l \bar{\xi}_{Rl} \xi_L^l - i\bar{m}^l \bar{\xi}_{Ll} \xi_R^l + i\tilde{\gamma}m^l \bar{n}_l \xi_L^l \zeta_R$$

$$- i\bar{\tilde{\gamma}}\bar{m}^l \bar{\xi}_{Ll} n^l \bar{\zeta}_R - (1 - |\tilde{\gamma}|^2)\left(\left|\sum m^l |n^l|^2\right|^2 - im^l |n^l|^2 (\bar{\xi}_R \xi_L) - i\bar{m}^l |n^l|^2 (\bar{\xi}_L \xi_R)\right), \qquad l = 1, \ldots N. \tag{2.13}$$

A few comments are in order concerning Eq. (2.13). Note that in Eq. (2.6) the massive deformation and the heterotic deformation were independent from each other, and we used this circumstance to prove supersymmetry. Now we see that some terms in Eq. (2.13) depend both on $\tilde{\gamma}$ and m^l. This happens because we have integrated out the auxiliary fields, implying, in turn, that supersymmetry is now realized nonlinearly (see [9] where supertransformations are written for the heterotic $CP(N-1)$ model).

With masses set to zero, the model (2.13) is equivalent to the geometric formulation of the heterotic $\mathcal{N} = (0,2)$

sigma model [8,9]. Our current task is to more closely examine the mass terms in Eq. (2.13). The model still contains redundant fields. In particular, there are N bosonic fields n^l and N fermionic ξ^l, whereas the geometric formulation has $N-1$ corresponding variables, see Sec. III. We can use the constraints

$$\bar{n}_l n^l = 1, \qquad \bar{\xi}_l n^l = 0$$

to get rid of some of them, say n^N and ξ^N. We obtain, for the mass terms,

$$\frac{\mathcal{L}}{2\beta} \supset |\tilde{\gamma}|^2 |m^N|^2 + (|m^i - m^N + |\tilde{\gamma}|^2 m^N|^2 - |\tilde{\gamma}|^4 |m^N|^2)|n^i|^2 - (1 - |\tilde{\gamma}|^2)\left|\sum_i (m^i - m^N)|n^i|^2\right|^2$$

$$- i(m^i - m^N + |\tilde{\gamma}|^2 m^N)\bar{\xi}_{Ri}\xi_L^i - i(\bar{m}^i - \bar{m}^N + |\tilde{\gamma}|^2 \bar{m}^N)\bar{\xi}_{Li}\xi_R^i + i(1 - |\tilde{\gamma}|^2)(m^i - m^N)|n^i|^2(\bar{\xi}_R \xi_L)$$

$$+ i(1 - |\tilde{\gamma}|^2)(\bar{m}^i - \bar{m}^N)|n^i|^2(\bar{\xi}_L \xi_R) + i\tilde{\gamma}(m^i - m^N)\bar{n}_i \xi_L^i \zeta_R - i\bar{\tilde{\gamma}}(\bar{m}^i - \bar{m}^N)\bar{\xi}_{Li} n^i \bar{\zeta}_R$$

$$- i|\tilde{\gamma}|^2 m^N \bar{\xi}_{RN}\xi_L^N - i|\tilde{\gamma}|^2 \bar{m}^N \bar{\xi}_{LN}\xi_R^N, \qquad i = 1, \ldots N-1, \tag{2.14}$$

where we denote

$$(\bar{\xi}\xi) = \bar{\xi}_i \xi^i + \bar{\xi}_N \xi^N, \qquad i = 1, \ldots N-1.$$

Note, that at large values of m^l all N Higgs vacua (2.8) of the theory are still present in the potential (2.14). One of them, with $l_0 = N$, in which $n^i = 0$ for $i = 1, \ldots (N-1)$, is easily seen from (2.14). The vacuum energy and the masses of the fermion and boson elementary excitations match expression (2.11) for $l_0 = N$. Other $(N-1)$ vacua are still present, but not-so-easy to see from (2.14). They are located at $n^{l_0} = 1$ where $l_0 = 1, \ldots (N-1)$ and all n^i with $i \neq l_0$ vanish.

One can easily see these vacua from different equivalent formulations of the theory which emerge if we choose to eliminate the field n^{l_0} rather then n^N. We stress, however, that all N vacua are, in principle, seen from any of these equivalent formulations, and supersymmetry breaking is spontaneous rather than explicit.

III. GEOMETRIC FORMULATION

The $\mathcal{N} = (0,2)$ supersymmetric $CP(N-1)$ model with heterotic and twisted-mass deformations has a geometric description which combines elements of the description of the twisted-mass deformed $\mathcal{N} = (2,2)$ model (see e.g. [20]) and that of the $\mathcal{N} = (0,2)$ massless model [8].

To begin with let us consider the $\mathcal{N} = (2,2)$ $CP(N-1)$ model. In the gauge formulation of this model one has two sets of $N-1$ (anti)chiral superfields Φ^i and $\bar{\Phi}^{\bar{j}}$ ($i, \bar{j} = 1, \ldots, N-1$), the lowest components ϕ^i, $\bar{\phi}^{\bar{j}}$ of which parametrize the target Kähler manifold. The Lagrangian of the $CP(N-1)$ model is given by the following sigma model

$$\mathcal{L} = \int d^4\theta K(\Phi, \bar{\Phi})$$

$$= g_{i\bar{j}}\partial_\mu \phi^i \partial_\mu \bar{\phi}^{\bar{j}} + \frac{1}{2}g_{i\bar{j}}\psi^i i\overleftrightarrow{\bar{\nabla}}\bar{\psi}^{\bar{j}} + \frac{1}{4}R_{i\bar{j}k\bar{l}}\psi^i \psi^j \bar{\psi}^{\bar{k}}\bar{\psi}^{\bar{l}},$$

where $K(\phi, \bar{\phi})$ is the Kähler potential, $g_{i\bar{j}}$ is its Kähler metric

$$g_{i\bar{j}} = \frac{\partial^2 K}{\partial \phi^i \partial \bar{\phi}^{\bar{j}}}, \qquad g^{i\bar{k}} = (g^{-1})^{\bar{k}i},$$

∇_μ is the (target space) covariant derivative,

$$(\nabla_\mu \bar{\psi})^{\bar{j}} = \{\partial_\mu \delta^{\bar{j}}_{\bar{m}} + \Gamma^{\bar{j}}_{\bar{m}\bar{k}}\partial_\mu(\bar{\phi}^{\bar{k}})\}\bar{\psi}^{\bar{m}},$$

$$\Gamma^{\bar{i}}_{\bar{k}\bar{l}} = g^{m\bar{i}}\partial_{\bar{l}}g_{m\bar{k}},$$

$$(\psi\overleftarrow{\bar{\nabla}}_\mu)^i = \psi^m\{\overleftarrow{\partial}_\mu \delta^i_m + \Gamma^i_{mk}\partial_\mu(\phi^k)\},$$

$$\Gamma^i_{kl} = g^{i\bar{m}}\partial_l g_{k\bar{m}}, \tag{3.1}$$

and $R_{ij\bar{k}\bar{l}}$ is the Riemann tensor

$$R_{ij\bar{k}\bar{l}} = \partial_i \partial_{\bar{k}} g_{j\bar{l}} - g^{m\bar{m}} \partial_i g_{j\bar{m}} \partial_{\bar{k}} g_{m\bar{l}}.$$

For the $CP(N-1)$ model one chooses the Kähler potential in the following way:

$$K(\Phi, \bar{\Phi}) = \ln(1 + \bar{\Phi}^{\bar{j}} \delta_{\bar{j}i} \Phi^i),$$

which corresponds to the Fubini–Study metric,

$$g_{i\bar{j}} = \frac{1}{\chi} \left(\delta_{i\bar{j}} - \frac{1}{\chi} \delta_{i\bar{i}} \bar{\phi}^{\bar{i}} \delta_{j\bar{j}} \phi^j \right),$$

where $\chi = 1 + \bar{\phi}^{\bar{j}} \delta_{\bar{j}i} \phi^i$.

In this case,

$$\Gamma^{\bar{i}}_{\bar{k}\bar{l}} = -\frac{\delta^{\bar{i}}_{(\bar{k}} \delta_{\bar{l})i} \phi^i}{\chi}, \qquad \Gamma^i_{kl} = -\frac{\delta^i_{(k} \delta_{l)\bar{i}} \bar{\phi}^{\bar{i}}}{\chi},$$

and the Riemann tensor takes the form

$$R_{ij\bar{k}\bar{l}} = -g_{i(\bar{k}} g_{\bar{l})j}.$$

As was shown in [8], the $\mathcal{N} = (0, 2)$ deformation of the $CP(N-1)$ model can be obtained by introduction of the right-handed supertranslational modulus ζ_R via a "right-handed" supermultiplet $\mathcal{B}$,

$$\mathcal{B} = (\zeta_R + \sqrt{2}\theta_R \mathcal{F})\bar{\theta}_L, \qquad \bar{\mathcal{B}} = \theta_L(\bar{\zeta}_R + \sqrt{2}\bar{\theta}_R \bar{\mathcal{F}}).$$

The latter expressions describe superfields covariant only under the right-handed supersymmetry, while explicitly breaking the left-handed one. In a sense, $\mathcal{B}$ is an analog of the $\mathcal{N} = (0, 2)$ supermultiplet Ξ in the two-dimensional superfield formalism [7]—the above supermultiplet containing only one physical field, which is the supertranslational fermionic variable. $\mathcal{F}$ is the auxiliary component of the $\mathcal{B}$ superfield. The distinction is that $\mathcal{B}$ happens to be a twisted superfield,

$$D_R \mathcal{B} = \bar{D}_L \mathcal{B} = 0.$$

One then constructs the heterotic Lagrangian

$$\mathcal{L}_{(0,2)} = \frac{2}{g_0^2} \int d^4\theta (K(\Phi, \bar{\Phi}) - 2\bar{\mathcal{B}}\mathcal{B} + \sqrt{2}\tilde{\gamma}\mathcal{B}K + \sqrt{2}\bar{\tilde{\gamma}}\,\bar{\mathcal{B}}\,\bar{K}), \tag{3.2}$$

which obviously respects the invariance on the target space $CP(N-1)$. Here

$$\frac{2}{g_0^2} = 2\beta$$

is the coupling constant of the sigma model. The second term in Eq. (3.2) generates the kinetic term for ζ_R, while the last two terms are responsible for the mixing between ζ_R and $\xi_{R,L}$. Explicitly, in components, one has,

$$\frac{\mathcal{L}_{(0,2)}}{2\beta} = \bar{\zeta}_R i\partial_L \zeta_R + g_{i\bar{j}}\partial_\mu \phi^i \partial_\mu \bar{\phi}^{\bar{j}} + \frac{1}{2} g_{i\bar{j}} \psi_R^i i\overleftrightarrow{\nabla}_L \bar{\psi}_R^{\bar{j}}$$

$$+ \frac{1}{2} g_{i\bar{j}} \psi_L^i i\overleftrightarrow{\nabla}_R \bar{\psi}_L^{\bar{j}} + \tilde{\gamma} g_{i\bar{j}}(i\partial_L \bar{\phi}^{\bar{j}}) \psi_R^i \zeta_R$$

$$+ \bar{\tilde{\gamma}} g_{i\bar{j}} \bar{\psi}_R^{\bar{j}}(i\partial_L \phi^i) \bar{\zeta}_R + |\tilde{\gamma}|^2 \bar{\zeta}_R \zeta_R \cdot (g_{i\bar{j}} \bar{\psi}_L^{\bar{j}} \psi_L^i)$$

$$+ (1 - |\tilde{\gamma}|^2)(g_{i\bar{k}} \bar{\psi}_R^{\bar{k}} \psi_L^i)(g_{j\bar{l}} \bar{\psi}_L^{\bar{l}} \psi_R^j)$$

$$- (g_{i\bar{k}} \bar{\psi}_R^{\bar{k}} \psi_R^i)(g_{j\bar{l}} \bar{\psi}_L^{\bar{l}} \psi_L^j). \tag{3.3}$$

This action was originally introduced in [8], although in a slightly different normalization. The two actions match if one normalizes the fermions ζ_R canonically, and takes into account that our deformation parameter $\tilde{\gamma}$ is related to γ of Ref. [8] as

$$\tilde{\gamma} = \sqrt{2}g_0 \cdot \gamma.$$

The geometric form (3.3) can be related to the gauge formulation [Eq. (2.13) with vanishing twisted masses] via the following stereographic projection

$$n^i = \frac{\phi^i}{\sqrt{\chi}}, \qquad \bar{n}_{\bar{i}} = \frac{\bar{\phi}^{\bar{i}}}{\sqrt{\chi}}, \qquad n^N = \frac{1}{\sqrt{\chi}}, \qquad n^N \in \mathcal{R},$$

$$\xi^i = \frac{1}{\sqrt{\chi}} \left(\psi^i - \frac{(\bar{\phi}\psi)}{\chi} \phi^i \right), \qquad \bar{\xi}_{\bar{i}} = \frac{1}{\sqrt{\chi}} \left(\bar{\psi}^{\bar{i}} - \frac{(\bar{\psi}\phi)}{\chi} \bar{\phi}^{\bar{i}} \right),$$

$$\xi^N = -\frac{(\bar{\phi}\psi)}{\chi^{3/2}}, \qquad \bar{\xi}_N = -\frac{(\bar{\psi}\phi)}{\chi^{3/2}}, \tag{3.4}$$

where $i, \bar{i} = 1, \ldots, N-1$ and we shortcut the contractions $(\bar{\psi}\phi) = \delta_{i\bar{j}} \bar{\psi}^{\bar{j}} \phi^i$. Here we chose n^N to be real given an overall phase freedom of the $CP(N-1)$ variables n^l. Also we singled out n^N to be special and equal to $1/\sqrt{\chi}$, which corresponds to picking out one of the N vacua. Indeed, the representation (3.4) is very convenient for analyzing the vacuum lying at $\phi^i = 0$ ($i = 1, 2, \ldots, N-1$). In this representation all other $N-1$ vacua do not disappear, but they lie at infinity in the $\phi^1, \ldots \phi^{N-1}$ parametrization of the target space. Needless to say, the role of n^N in (3.4) can be assumed by any other n^{l_0} ($l_0 = 1, 2, \ldots, N-1$). Then the l_0-th vacuum will be easily accessible, while the N-th one will move to infinity.

A. Twisted masses

The twisted-mass deformation is carried out by formally lifting the theory to four-dimensional space and introducing a set of four-dimensional vector superfields V^i,

$$V^i = A_1^i \theta \sigma_1 \bar{\theta} + A_2^i \theta \sigma_2 \bar{\theta}, \tag{3.5}$$

with only the "transverse" components of the gauge field nonvanishing (remember, we assume that the string is "oriented" in the (x_0, x_3)-plane), and with λ and D equal to zero (λ and D are other components of the vector superfield). The components A_1^i and A_2^i are constant and define the twisted masses through the following relations:

$$m_G^k = -\frac{A_1^k + iA_2^k}{2}, \qquad k = 1, \ldots N - 1. \qquad (3.6)$$

These vector superfields are precisely the same kind of superfields that could give the twisted masses m^l in the model (2.6), see discussion after Eq. (2.5). However, now their number is $N - 1$ instead of N. In particular, as was mentioned, these superfields preserve $\mathcal{N} = (2, 2)$ supersymmetry after dimensional reduction to two dimensions. Until Sec. III B we will not dwell on the obvious fact that the number of the mass parameters in the geometric formulation so far is less (by one) than that of the gauge formulation.

The theory is then gauged with the above vector fields,

$$K(\Phi^i, \bar{\Phi}^{\bar{J}} V^i) = \ln(1 + \bar{\Phi}^{\bar{J}} \delta_{i\bar{j}} e^{V^i} \Phi^i),$$

with the same action as in Eq. (3.2),

$$\mathcal{L}_{(0,2)}^{\text{tw.m.}} = \frac{2}{g_0^2} \int d^4\theta (K(\Phi, \bar{\Phi}, V^i) - 2\bar{\mathcal{B}}\mathcal{B} + \sqrt{2}\tilde{\gamma}\mathcal{B}K + \sqrt{2}\bar{\tilde{\gamma}}\,\bar{\mathcal{B}}\,\bar{K}). \qquad (3.7)$$

The action of the theory (3.7) can be calculated by introducing covariantly-chiral superfields

$$X^i = \Phi^i, \qquad \bar{X}^{\bar{J}} = e^{V^j}\bar{\Phi}^{\bar{J}},$$

in terms of which the Kähler potential takes the original form

$$K = \ln(1 + \bar{X}^i X^i).$$

It turns out that in the above integral one can freely replace D_α and $\bar{D}_\alpha$ with covariant $\nabla_\alpha^{(\bar{j})}$ and $\bar{\nabla}_\alpha^{(i)}$ at any convenient occurrence. This makes calculation of (3.7) straightforward, and the only obvious difference with the massless case comes from the algebra of the covariant derivatives $\nabla_\alpha^{(\bar{j})}$ and $\bar{\nabla}_\alpha^{(i)}$, i.e. from the presence of the constant gauge fields. For calculation of the conjugate term $\sqrt{2}\bar{\tilde{\gamma}}\,\bar{\mathcal{B}}\bar{K}$ one can find convenient to use covariantly-antichiral variables

$$Y^i = e^{V^i}\Phi^i, \qquad Y^{\bar{J}} = \bar{\Phi}^{\bar{J}}.$$

Needless to say that the result is obtained from the massless theory by the "elongation" of the space-time derivatives (however, formally, in four dimensions). Namely, in this way we arrive at

$$
\begin{aligned}
\frac{\mathcal{L}_{(0,2)}^{\text{tw.m.}}}{2\beta} =\; & g_{i\bar{j}}(\nabla_\mu \phi^i)(\nabla_\mu \bar{\phi}^{\bar{j}}) + \frac{1}{2} g_{i\bar{j}} \psi^i i \overleftrightarrow{\nabla}\!\!\!\!\!\nabla \bar{\psi}^{\bar{j}} - \partial_i \partial_{\bar{k}} g_{l\bar{j}} \psi_R^i \psi_L^l \overline{\psi_R^{\bar{k}} \psi_L^{\bar{j}}} + g_{i\bar{j}} F^i \bar{F}^{\bar{j}} - \partial_{\bar{k}} g_{i\bar{j}} F^i \overline{\psi_R^{\bar{k}}\psi_L^{\bar{j}}} + \partial_i g_{l\bar{j}} \psi_R^i \psi_L^l \bar{F}^{\bar{j}} \\
& + \bar{\zeta}_R i\partial_L \zeta_R + \bar{\mathcal{F}}\mathcal{F} + \tilde{\gamma}\mathcal{F}g_{i\bar{j}}\psi_R^i\bar{\psi}_L^{\bar{j}} + \bar{\tilde{\gamma}}\,\bar{\mathcal{F}}\,g_{i\bar{j}}\psi_L^i\bar{\psi}_R^{\bar{j}} - \tilde{\gamma}\mathcal{F}\frac{1}{\chi}\phi^{\bar{j}}(\nabla_1^{\bar{j}} + i\nabla_2^{\bar{j}})\bar{\phi}^{\bar{j}} + \bar{\tilde{\gamma}}\,\bar{\mathcal{F}}\frac{1}{\chi}\bar{\phi}^{\bar{i}}(\nabla_1^i - i\nabla_2^i)\phi^i \\
& - \tilde{\gamma}\zeta_R \cdot g_{i\bar{j}}F^i\bar{\psi}_L^{\bar{j}} + \bar{\tilde{\gamma}}\bar{\zeta}_R \cdot g_{i\bar{j}}\bar{F}^{\bar{j}}\psi_L^i - \tilde{\gamma}\zeta_R\partial_i g_{k\bar{j}}\psi_R^i\psi_L^k \cdot \bar{\psi}_L^{\bar{j}} - \bar{\tilde{\gamma}}\bar{\zeta}_R\partial_{\bar{k}}g_{i\bar{j}}\overline{\psi_R^{\bar{j}}\psi_L^{\bar{k}}} \\
& \cdot \psi_L^i - \tilde{\gamma}\zeta_R g_{i\bar{j}}\psi_R^i i\partial_L\bar{\phi}^{\bar{j}} - \bar{\tilde{\gamma}}\bar{\zeta}_R g_{i\bar{j}}\bar{\psi}_R^{\bar{j}} i\partial_L \phi^i - \tilde{\gamma}\zeta_R g_{i\bar{j}}\psi_L^i(\nabla_1^{\bar{j}} + i\nabla_2^{\bar{j}})\bar{\phi}^{\bar{j}} - \bar{\tilde{\gamma}}\bar{\zeta}_R g_{i\bar{j}}\bar{\psi}_L^{\bar{j}}(\nabla_1^i - i\nabla_2^i)\phi^i. \qquad (3.8)
\end{aligned}
$$

The mass terms here are hidden in the covariant derivatives

$$\nabla_\mu^i = \partial_\mu + \frac{i}{2}A_\mu^i, \qquad \nabla_\mu^{\bar{j}} = \partial_\mu - \frac{i}{2}A_\mu^{\bar{j}}, \qquad (\nabla_\mu \bar{\psi})^{\bar{j}} = \{\nabla_\mu^{\bar{m}}\delta_{\bar{m}}^{\bar{j}} + \Gamma_{\bar{m}\bar{k}}^{\bar{j}}\nabla_\mu^{\bar{k}}(\bar{\phi}^{\bar{k}})\}\bar{\psi}^{\bar{m}},$$

$$(\psi\overleftarrow{\nabla}_\mu)^i = \psi^m\{\overleftarrow{\nabla}_\mu^m \delta_m^i + \Gamma_{mk}^i\nabla_\mu^k(\phi^k)\}, \qquad \Gamma_{\bar{k}\bar{l}}^{\bar{i}} = g^{m\bar{i}}\partial_{\bar{l}}g_{m\bar{k}}, \qquad \Gamma_{kl}^i = g^{i\bar{m}}\partial_l g_{km,} \quad - \quad \mu = 0, \ldots 3.$$

Although the space-time index μ formally runs over all four values, we understand that the derivatives ∂_μ with respect to the transverse coordinates ($\mu = 1, 2$) should be ignored in our two-dimensional theory.

Next, we need to exclude the auxiliary fields F^i and $\mathcal{F}$, since they have no analogs in the gauge formulation of the theory [more precisely, although we did introduce exactly the same $\mathcal{B}$ superfield into the gauge formulation, the highest component $\mathcal{F}$ of that multiplet played a different role in Eq. (2.6) than in Eq. (3.8)]. Also, we substitute the masses, noting that the covariant derivatives in Eq. (3.8) enter in convenient combinations,

$$\nabla_1^{\bar{j}} + i\nabla_2^{\bar{j}} = im_G^{\bar{j}}, \qquad \nabla_1^i - i\nabla_2^i = -i\bar{m}_G^i,$$

$$i, \bar{j} = 1, \ldots N - 1.$$

The F^i-term conditions are the same as in the massless heterotic theory, while the $\mathcal{F}$-term condition gets modified by the masses,

$$\mathcal{F} = -\bar{\tilde{\gamma}}g_{i\bar{j}}\psi_L^i\bar{\psi}_R^{\bar{j}} + i\bar{\tilde{\gamma}}\bar{m}_G^i\frac{\bar{\phi}^i\phi^i}{\chi}. \qquad (3.9)$$

As a result, we obtain,

$$\frac{\mathcal{L}^{\text{tw.m.}}_{(0,2)}}{2\beta} = \bar{\zeta}_R i\partial_L \zeta_R + g_{i\bar{\jmath}}\partial_\mu \phi^i \partial_\mu \bar{\phi}^{\bar{\jmath}} + g_{i\bar{\jmath}} m^i_{G\mu} m^{\bar{\jmath}}_{G\mu} \phi^i \bar{\phi}^{\bar{\jmath}} + \frac{1}{2} g_{i\bar{\jmath}} \psi^i_R i\overleftrightarrow{\nabla}^{(0)}_L \bar{\psi}^{\bar{\jmath}}_R + \frac{1}{2} g_{i\bar{\jmath}} \psi^i_L i\overleftrightarrow{\nabla}^{(0)}_R \bar{\psi}^{\bar{\jmath}}_L$$

$$+ \frac{i}{2} g_{i\bar{\jmath}} \psi^i_L \bar{\bar{m}}_G \bar{\psi}^{\bar{\jmath}}_R + \frac{i}{2} g_{i\bar{\jmath}} \psi^i_R \bar{\bar{\bar{m}}}_G \bar{\psi}^{\bar{\jmath}}_L - (g_{i\bar{\jmath}} \psi^i_R \bar{\psi}^{\bar{\jmath}}_R)(g_{k\bar{l}} \psi^k_L \bar{\psi}^{\bar{l}}_L) + (1-|\tilde{\gamma}|^2)(g_{i\bar{\jmath}} \psi^i_R \bar{\psi}^{\bar{\jmath}}_L)(g_{k\bar{l}} \psi^k_L \bar{\psi}^{\bar{l}}_R)$$

$$+ \tilde{\gamma} g_{i\bar{\jmath}} i\partial_L \bar{\phi}^{\bar{\jmath}} \psi^i_R \zeta_R + \bar{\tilde{\gamma}} g_{i\bar{\jmath}} \bar{\psi}^{\bar{\jmath}}_R i\partial_L \phi^i \bar{\zeta}_R + i\tilde{\gamma} g_{i\bar{\jmath}} m^{\bar{\jmath}}_G \bar{\phi}^{\bar{\jmath}} \psi^i_L \zeta_R - i\bar{\tilde{\gamma}} g_{i\bar{\jmath}} \bar{m}^i_G \bar{\psi}^{\bar{\jmath}}_L \phi^i \bar{\zeta}_R + |\tilde{\gamma}|^2 g_{i\bar{\jmath}} \bar{\psi}^{\bar{\jmath}}_L \psi^i_L \bar{\zeta}_R \zeta_R$$

$$+ i|\tilde{\gamma}|^2 (g_{i\bar{\jmath}} \psi^i_L \bar{\psi}^{\bar{\jmath}}_R) \cdot m^k_G \frac{\bar{\phi}^k \phi^k}{\chi} + i|\tilde{\gamma}|^2 (g_{i\bar{\jmath}} \psi^i_R \bar{\psi}^{\bar{\jmath}}_L) \cdot \bar{m}^{\bar{k}}_G \frac{\bar{\phi}^{\bar{k}} \phi^{\bar{k}}}{\chi} + |\tilde{\gamma}|^2 \cdot \bar{m}^{\bar{\jmath}}_G \frac{\bar{\phi}^{\bar{\jmath}} \phi^{\bar{\jmath}}}{\chi} \cdot m^i_G \frac{\bar{\phi}^i \phi^i}{\chi}. \tag{3.10}$$

Some comments are in order here on the notation used in this expression. First, $\nabla^{(0)}$ denotes the nongauge (but still covariant) derivative (3.1) of the massless theory. The index $\mu = 1, 2$ of the masses m^i_G denotes their real and imaginary parts, respectively, which is consistent with Eq. (3.6). Finally, the matrix $\bar{\bar{m}}_G$ acts on spinors in accordance with the following formula:

$$(m_G \bar{\psi}_{R,L})^{\bar{\jmath}} = (m^{\bar{m}}_G \delta_{\bar{m}}{}^{\bar{\jmath}} + m^{\bar{k}}_G \Gamma^{\bar{\jmath}}_{\bar{m}\bar{k}} \bar{\phi}^{\bar{k}}) \bar{\psi}^{\bar{m}}_{R,L},$$

$$(\psi_{R,L} \bar{m}_G)^i = -\psi^m_{R,L}(m^m_G \delta_m{}^i + m^k_G \Gamma^i_{mk} \phi^k),$$

$$\bar{\bar{m}}_G = \bar{m}_G - \hat{m}_G,$$

with the identical prescription for the conjugate mass matrix $\bar{\bar{\bar{m}}}_G$.

It is instructive to compare the theory (3.10) with the gauge formulation of the twisted-mass deformed heterotic sigma model (2.13). To this end one can use the (inverted) correspondence rules (3.4). For the massless theory, this job was done in [9], where it was shown that one formulation exactly matches onto the other. Therefore, we need to prove the correspondence only for the mass terms. We have

$$\frac{\mathcal{L}^{\text{tw.m.}}_{(0,2)}}{2\beta} \supset g_{i\bar{\jmath}} m^i_{G\mu} m^{\bar{\jmath}}_{G\mu} \phi^i \bar{\phi}^{\bar{\jmath}} + i\frac{1}{2} g_{i\bar{\jmath}} \psi^i_L \bar{m}_G \bar{\psi}^{\bar{\jmath}}_R + i\frac{1}{2} g_{i\bar{\jmath}} \psi^i_R \bar{\bar{m}}_G \bar{\psi}^{\bar{\jmath}}_L + i\tilde{\gamma} g_{i\bar{\jmath}} m^{\bar{\jmath}}_G \bar{\phi}^{\bar{\jmath}} \psi^i_L \zeta_R - i\bar{\tilde{\gamma}} g_{i\bar{\jmath}} \bar{m}^i_G \bar{\psi}^{\bar{\jmath}}_L \phi^i \bar{\zeta}_R$$

$$+ i|\tilde{\gamma}|^2 (g_{i\bar{\jmath}} \psi^i_L \bar{\psi}^{\bar{\jmath}}_R) \cdot m^k_G \frac{\bar{\phi}^k \phi^k}{\chi} + i|\tilde{\gamma}|^2 (g_{i\bar{\jmath}} \psi^i_R \bar{\psi}^{\bar{\jmath}}_L) \cdot \bar{m}^{\bar{k}}_G \frac{\bar{\phi}^{\bar{k}} \phi^{\bar{k}}}{\chi} + |\tilde{\gamma}|^2 \cdot \bar{m}^{\bar{\jmath}}_G \frac{\bar{\phi}^{\bar{\jmath}} \phi^{\bar{\jmath}}}{\chi} \cdot m^i_G \frac{\bar{\phi}^i \phi^i}{\chi}$$

$$= \sum_i |m^i_G|^2 |n^i|^2 - (1-|\tilde{\gamma}|^2) \left| \sum_i m^i_G |n^i|^2 \right|^2 - im^i_G \bar{\xi}_{Ri} \xi^i_L - i\bar{m}^i_G \bar{\xi}_{Li} \xi^i_R + i(1-|\tilde{\gamma}|^2) m^i_G |n^i|^2 (\bar{\xi}_R \xi_L)$$

$$+ i(1-|\tilde{\gamma}|^2) \bar{m}^i_G |n^i|^2 (\bar{\xi}_L \xi_R) + i\tilde{\gamma} m^i_G \bar{n}_i \xi^i_L \zeta_R - i\bar{\tilde{\gamma}} \bar{m}^i_G \bar{\xi}_{Li} n^i \bar{\zeta}_R, \qquad i, \bar{\jmath}, k, \bar{k} = 1, \dots, N-1. \tag{3.11}$$

Here $(\bar{\xi}\xi) = \bar{\xi}_i \xi^i + \bar{\xi}_N \xi^N$, similarly to Eq. (2.14). Comparing Eq. (3.11) with Eq. (2.14), we see that the former does not exactly match the latter. It would match if we set $m^N = 0$ in the gauge formulation. As was discussed in Sec. II, this would be a heterotic $CP(N-1)$ sigma model with $N-1$ twisted-mass parameters, which has a supersymmetric vacuum. Equation (3.10) describes excitations around this vacuum.

That there was a problem with the number of twisted-mass parameters in the geometric formulation was obvious from the beginning, see Eq. (3.6). The number of physical fields is the same in both formulations, but the number of masses is not. Not only that, the theory (3.10) will always be (classically) supersymmetric, whereas (2.13) does break supersymmetry. It is imperative to find a way to introduce one extra mass parameter in the geometric formulation.

B. Spontaneous supersymmetry-breaking geometric formulation

Since $\mathcal{B}$ is a twisted superfield, one can introduce a twisted superpotential of the form

$$\frac{\Delta \mathcal{L}^{\text{tw.m.}}_{(0,2)}}{2\beta} \supset \frac{i}{\sqrt{2}} a \int \mathcal{B} d^2\tilde{\theta} + \text{h.c.}, \qquad d^2\tilde{\theta} = d\bar{\theta}_L d\theta_R$$

(here $i/\sqrt{2}$ is a convenient normalization factor). This creates a linear in $\mathcal{F}$ contribution in the Lagrangian

$$\frac{\Delta \mathcal{L}^{\text{tw.m.}}_{(0,2)}}{2\beta} \supset ia\mathcal{F} + i\bar{a}\,\bar{\mathcal{F}} + \bar{\mathcal{F}}\mathcal{F} + \dots,$$

and, correspondingly, changes the $\mathcal{F}$-term condition (3.9) to the following:

$$\mathcal{F} = -\bar{\tilde{\gamma}} g_{i\bar{\jmath}} \psi^i_L \bar{\psi}^{\bar{\jmath}}_R + i\bar{\tilde{\gamma}} \bar{m}^i_G \frac{\bar{\phi}^i \phi^i}{\chi} - i\bar{a}.$$

Substituting this into Eq. (3.8) produces (a) vacuum energy, and (b) mass shifts both for bosons and fermions. Choosing the appropriate value

$$a = -\tilde{\gamma} m^N,$$

one can now match the masses of elementary excitations to those of the gauge formulation. Overall, the

supersymmetry-breaking theory has the following mass terms:

$$\frac{\mathcal{L}^{\text{tw.m.}}_{(0,2)}}{2\beta} = \int d^4\theta (K(\Phi, \bar\Phi, V^i) - 2\bar{\mathcal{B}}\mathcal{B} - \sqrt{2}\tilde\gamma\mathcal{B}K - \sqrt{2}\bar{\tilde\gamma}\,\bar{\mathcal{B}}\,\bar K) - \frac{i}{\sqrt{2}} \int d^2\tilde\theta \cdot \tilde\gamma m^N \mathcal{B} - \frac{i}{\sqrt{2}} \int d^2\bar{\tilde\theta} \cdot \bar{\tilde\gamma}\bar m^N \bar{\mathcal{B}}$$

$$\supset |\tilde\gamma|^2 |m^N|^2 + g_{i\bar j} m^i_{G\mu} m^{\bar j}_{G\mu} \phi^i \bar\phi^j + |\tilde\gamma|^2 (\bar m^N m^i_G + m^N \bar m^i_G) \frac{\phi^i \bar\phi^i}{\chi} + |\tilde\gamma|^2 \cdot \bar m^{\bar j}_G \frac{\bar\phi^{\bar j}\phi^j}{\chi} \cdot m^i_G \frac{\bar\phi^i \phi^i}{\chi}$$

$$+ i\frac{1}{2} g_{i\bar j} \psi^i_L \vec{\bar m}_G \bar\psi^{\bar j}_R + i\frac{1}{2} g_{i\bar j} \psi^i_R \vec{\bar m}_G \bar\psi^{\bar j}_L + i|\tilde\gamma|^2 m^N g_{i\bar j} \psi^i_L \bar\psi^{\bar j}_R + i|\tilde\gamma|^2 \bar m^N g_{i\bar j} \psi^i_R \bar\psi^{\bar j}_L$$

$$+ i\tilde\gamma g_{i\bar j} m^{\bar j}_G \bar\phi^{\bar j} \psi^i_L \zeta_R - i\bar{\tilde\gamma} g_{i\bar j} \bar m^i_G \bar\psi^{\bar j}_L \phi^i \bar\zeta_R + i|\tilde\gamma|^2 (g_{i\bar j}\psi^i_L \bar\psi^{\bar j}_R) \cdot m^k_G \frac{\bar\phi^k \phi^k}{\chi} + i|\tilde\gamma|^2 (g_{i\bar j}\psi^i_R \bar\psi^{\bar j}_L) \cdot \bar m^{\bar k}_G \frac{\bar\phi^{\bar k} \phi^{\bar k}}{\chi}. \quad (3.12)$$

With all N mass parameters included, the (spontaneous) breaking of SUSY occurs right away, at the classical level.

Under the stereographic projection (3.4) this turns into

$$\frac{\mathcal{L}^{\text{tw.m.}}_{(0,2)}}{2\beta} \supset |\tilde\gamma|^2 |m^N|^2 + (|m^i_G|^2 + |\tilde\gamma|^2 \{\bar m^N m^i_G + m^N \bar m^i_G\})|n^i|^2 - (1 - |\tilde\gamma|^2)\left|\sum_i m^i_G |n^i|^2\right|^2 - i(m^i_G + |\tilde\gamma|^2 m^N)\bar\xi_{Ri}\xi^i_L$$

$$- i(\bar m^i_G + |\tilde\gamma|^2 \bar m^N)\bar\xi_{Li}\xi^i_R + i(1 - |\tilde\gamma|^2)m^i_G |n^i|^2 (\bar\xi_R\xi_L) + i(1 - |\tilde\gamma|^2)\bar m^i_G |n^i|^2 (\bar\xi_L\xi_R) + i\tilde\gamma m^i_G \bar n_i \xi^i_L \zeta_R$$

$$- i\bar{\tilde\gamma}\bar m^i_G \bar\xi_{Li} n^i \bar\zeta_R - i|\tilde\gamma|^2 m^N \bar\xi_{RN}\xi^N_L - i|\tilde\gamma|^2 \bar m^N \bar\xi_{LN}\xi^N_R, \qquad i = 1, \ldots N - 1.$$

We observe that this Lagrangian matches exactly onto the gauge formulation of the heterotic massive sigma model (2.14), provided that we accept

$$m^i_G = m^i - m^N.$$

As an additional check we now show that in the large mass limit, $|m^l| \gg \Lambda$, we can recover all N Higgs vacua (2.8) obtained in the gauge formulation from the geometric formulation (3.12). One of these vacua (with $l_0 = N$) corresponds to $\phi^i = 0$. Other $(N - 1)$ vacua are located at $\phi^{l_0} \to \infty$, $l_0 = 1, \ldots, (N - 1)$ as seen from (3.4). The vacuum energies and boson and fermion masses in these vacua exactly match expressions (2.11) obtained in the gauge formulation.

While it took us some effort to prove $\mathcal{N} = (0, 2)$ supersymmetry of the theory (2.6), the geometric formulation of this theory

$$\frac{\mathcal{L}^{\text{tw.m.}}_{(0,2)}}{2\beta} = \int d^4\theta (K(\Phi, \bar\Phi, V^i) - 2\bar{\mathcal{B}}\mathcal{B}$$
$$- \sqrt{2}\tilde\gamma\mathcal{B}K - \sqrt{2}\bar{\tilde\gamma}\,\bar{\mathcal{B}}\,\bar K)$$
$$- \frac{i}{\sqrt{2}} \int d^2\tilde\theta \cdot \tilde\gamma m^N \mathcal{B} - \frac{i}{\sqrt{2}} \int d^2\bar{\tilde\theta} \cdot \bar{\tilde\gamma}\bar m^N \bar{\mathcal{B}}$$

is manifestly supersymmetric.

IV. CONCLUSIONS

In this paper we considered various two-dimensional supersymmetric sigma models on the $CP(N - 1)$ target space. We constructed an $\mathcal{N} = (0, 2)$ model which combines the twisted-mass deformation with the heterotic deformation following from the bulk theory deformation

(1.1). Rather unexpectedly, in addition to the heterotic coupling parameter, this model contains an extra mass parameter which was unobservable in the absence of the heterotic deformation. If all N mass parameters are non-vanishing, $\mathcal{N} = (0, 2)$ supersymmetry is (spontaneously) broken at the tree level. Setting one mass parameters to zero, we restore supersymmetry at the classical level. At the quantum (nonperturbative) level it is still spontaneously broken.

There are two obvious tasks for the nearest future. First, the $\mathcal{N} = (0, 2)$ model combining the twisted-mass deformation with the heterotic deformation must be fully derived from the microscopic $\mathcal{N} = 1$ Yang–Mills theory in the bulk. Second, it must be solved in the large-N limit. The solution of both problems is within reach.

APPENDIX A: NOTATIONS IN EUCLIDEAN SPACE

Since $CP(N - 1)$ sigma model can be obtained as a dimensional reduction from four-dimensional theory, we present first our four-dimensional notations. The indices of four-dimensional spinors are raised and lowered by the $SU(2)$ metric tensor,

$$\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta, \qquad \bar\psi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta},$$
$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta, \qquad \bar\psi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta}, \tag{A1}$$

where

$$\epsilon_{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and}$$
$$\epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{A2}$$

The contractions of the spinor indices are short-handed as

$$\lambda\psi = \lambda_\alpha\psi^\alpha, \qquad \overline{\lambda\psi} = \bar\lambda^{\dot\alpha}\bar\psi_{\dot\alpha}. \tag{A3}$$

The sigma matrices for the Euclidean space we take as

$$\sigma_\mu^{\alpha\dot\alpha} = \left(\mathbf{1}, \quad -i\tau^k\right)^{\alpha\dot\alpha}, \qquad \bar\sigma_{\dot\alpha\alpha\mu} = \left(\mathbf{1}, \quad i\tau^k\right)_{\dot\alpha\alpha}, \tag{A4}$$

where τ^k are the Pauli matrices.

Reduction to two dimensions can be conveniently done by picking out x^0 and x^3 as the world sheet (or "longitudinal") coordinates, and integrating over the orthogonal coordinates. The two-dimensional derivatives are the defined to be

$$\partial_R = \partial_0 + i\partial_3, \qquad \partial_L = \partial_0 - i\partial_3. \tag{A5}$$

One then identifies the lower-index spinors as the two-dimensional left- and right-handed chiral spinors

$$\xi_R = \xi_1, \qquad \xi_L = \xi_2, \qquad \bar\xi_R = \bar\xi_{\dot1}, \qquad \bar\xi_L = \bar\xi_{\dot2}. \tag{A6}$$

For two-dimensional variables, the $CP(N-1)$ indices are written as upper ones

$$n^l, \qquad \xi^l,$$

and lower ones for the conjugate moduli

$$\bar n_l, \qquad \bar\xi_l,$$

where $l = 1, \ldots, N$. In the geometric formulation of $CP(N-1)$, global indices are written upstairs in both cases, only for the conjugate variables the indices with bars are used

$$\phi^i, \qquad \psi^i, \qquad \bar\phi^{\bar i}, \qquad \bar\psi^{\bar i}, \qquad i, \bar i = 1, \ldots, N-1,$$

and the metric $g_{i\bar j}$ is used to contract them.

APPENDIX B: MINKOWSKI VERSUS EUCLIDEAN FORMULATION

Although this work considers the formulation of heterotic $CP(N-1)$ model in Euclidean space only, a series of our papers work with both Minkowski and Euclidean conventions [8–10]. It is useful to summarize the transition rules. If the Minkowski coordinates are

$$x_M^\mu = \{t, z\}, \tag{B1}$$

the passage to the Euclidean space requires

$$t \to -i\tau, \tag{B2}$$

and the Euclidean coordinates are

$$x_M^\mu = \{\tau, z\}. \tag{B3}$$

The derivatives are defined as follows:

$$\partial_L^M = \partial_t + \partial_z, \qquad \partial_R^M = \partial_t - \partial_z,$$
$$\partial_L^E = \partial_\tau - i\partial_z, \qquad \partial_R^E = \partial_\tau + i\partial_z. \tag{B4}$$

The Dirac spinor is

$$\Psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \tag{B5}$$

In passing to the Euclidean space $\Psi^M = \Psi^E$; however, $\bar\Psi$ is transformed,

$$\bar\Psi^M \to i\bar\Psi^E. \tag{B6}$$

Moreover, Ψ^E and $\bar\Psi^E$ are *not* related by the complex conjugation operation. They become independent variables. The fermion gamma matrices are defined as

$$\bar\sigma_M^\mu = \{1, -\sigma_3\}, \qquad \bar\sigma_E^\mu = \{1, i\sigma_3\}. \tag{B7}$$

Finally,

$$\mathcal{L}_E = -\mathcal{L}_M(t = -i\tau, \ldots). \tag{B8}$$

With this notation, formally, the fermion kinetic terms in $\mathcal{L}_E$ and $\mathcal{L}_M$ coincide. We use the following equivalent definitions of the heterotic deformation terms

$$\frac{g_0}{\sqrt{2}}\tilde\gamma_{(M)}\zeta_R G_{i\bar j}(i\partial_L\bar\phi^{\bar j})\psi_R^i,$$
$$\frac{1}{g_0^2}\tilde\gamma_{(E)}\chi_R^a(i\partial_L S^a)\zeta_R, \tag{B9}$$
$$\frac{2}{g_0^2}\tilde\gamma_{(E)}(i\partial_L\bar n)\xi_R\zeta_R$$

in Minkowski and Euclidean spaces correspondingly. The following transition rule applies,

$$\tilde\gamma_{(M)} = -i\tilde\gamma_{(E)}. \tag{B10}$$

Everywhere where there is no menace of confusion we omit the super/subscripts M, E. The first two terms in Eq. (B9) originally were introduced in Ref. [8], with a constant

$$\gamma = \tilde\gamma/(\sqrt{2}g_0). \tag{B11}$$

In this paper, subscript (E) is always assumed for $\tilde\gamma$.

[1] A. Hanany and D. Tong, J. High Energy Phys. 07 (2003) 037.

[2] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi, and A. Yung, Nucl. Phys. **B673**, 187 (2003).

[3] M. Shifman and A. Yung, Phys. Rev. D **70**, 045004 (2004).

[4] A. Hanany and D. Tong, J. High Energy Phys. 04 (2004) 066.

[5] M. Shifman and A. Yung, Rev. Mod. Phys. **79**, 1139 (2007); M. Shifman and A. Yung, *Supersymmetric Solitons* (Cambridge University Press, Cambridge, England, 2009).

[6] L. Alvarez-Gaumé and D. Z. Freedman, Commun. Math. Phys. **91**, 87 (1983); S. J. Gates, Nucl. Phys. **B238**, 349 (1984); S. J. Gates, C. M. Hull, and M. Roček, Nucl. Phys. **B248**, 157 (1984).

[7] M. Edalati and D. Tong, J. High Energy Phys. 05 (2007) 005.

[8] M. Shifman and A. Yung, Phys. Rev. D **77**, 125016 (2008).

[9] P. A. Bolokhov, M. Shifman, and A. Yung, Phys. Rev. D **79**, 085015 (2009).

[10] M. Shifman and A. Yung, Phys. Rev. D **77**, 125017 (2008).

[11] P. A. Bolokhov, M. Shifman, and A. Yung (work in progress).

[12] D. Tong, arXiv:hep-th/0509216.

[13] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, J. Phys. A **39**, R315 (2006).

[14] D. Tong, Ann. Phys. (N.Y.) **324**, 30 (2009).

[15] E. Witten, Nucl. Phys. **B149**, 285 (1979).

[16] A. Hanany and K. Hori, Nucl. Phys. **B513**, 119 (1998).

[17] N. Dorey, J. High Energy Phys. 11 (1998) 005.

[18] D. Tong, J. High Energy Phys. 09 (2007) 022.

[19] A. Gorsky, M. Shifman, and A. Yung, Phys. Rev. D **73**, 065011 (2006).

[20] M. Shifman, A. Vainshtein, and R. Zwicky, J. Phys. A **39**, 13005 (2006).