# The Boundary of the Strong Phase

$$\begin{cases} \frac{1}{N}\sum \ln\{i\Delta + |S-\mu^k|^2\} = 0 & (1.1) \\ \frac{1}{N}\sum (S-\mu^k)\ln \frac{i\Delta + |S-\mu^k|^2}{|S-\mu^k|^2} = u \cdot S & (1.2) \end{cases}$$

Assumptions about the solution (based on numerical evaluation):

- solution $\{ i\Delta(\mu), S(\mu) \}$ in a range of $\mu$ is reversible (e.g. $i\Delta = i\Delta(S)$, etc.)
- at the phase transition point $\mu^*$ $\frac{\partial S}{\partial \mu} \to \infty$, and probably so does $\frac{\partial i\Delta}{\partial \mu}$
- the functions $i\Delta(\mu)$, $S(\mu)$ themselves are finite at the phase transition $\mu^*$

Looking for the phase transition:

- rewrite (1.1), (1.2) as

Eq(1.1) ⇒ $i\Delta = F(S, \mu)$ in order to exclude $i\Delta$

Eq.(1.2) ⇒ $i\Delta = G(S, \mu)$

where $F$, $G$ are "known" but unknown (uncalculable) functions.

- on the solution,

$$i\Delta(\mu) = F(S(\mu), \mu)$$

$$i\Delta(\mu) = G(S(\mu), \mu)$$

or $F(S(\mu), \mu) = G(S(\mu), \mu)$ ⇐ valid in a range of $\mu$

then $$\frac{dF}{d\mu} = \frac{\partial F}{\partial S}\cdot\frac{\partial S}{\partial \mu} + \frac{\partial F}{\partial \mu}$$

$$\shortparallel$$

$$\frac{dG}{d\mu} = \frac{\partial G}{\partial S}\cdot\frac{\partial S}{\partial \mu} + \frac{\partial G}{\partial \mu}$$

- again,

$$\frac{\partial F}{\partial S} \cdot \frac{\partial S}{\partial \mu} + \frac{\partial F}{\partial \mu} = \frac{\partial G}{\partial S} \cdot \frac{\partial S}{\partial \mu} + \frac{\partial G}{\partial \mu}$$

now, at phase transition $\mu^*$, $\left.\frac{\partial S}{\partial \mu}\right|_{\mu^*} \to \infty$

as $\mu \to \mu^*$, $\frac{\partial S}{\partial \mu} \to \infty$, which means

$$\left.\frac{\partial F}{\partial S}\right|_{\mu^*} = \left.\frac{\partial G}{\partial S}\right|_{\mu^*}, \text{ and } \left.\frac{\partial F}{\partial \mu}\right|_{\mu^*} = \left.\frac{\partial G}{\partial \mu}\right|_{\mu^*}$$

Note that F and G are still the same "is", but expressed from different equations.

We now turn to analyzing the equations.

Consider Eq. (1.1):

$$\frac{1}{N}\sum\left\{\ln i\Delta + |S-\mu^k|^2\right\} = 0$$

Diff. w.r.t. $\mu$:

$$\frac{1}{N}\sum\frac{\frac{\partial S}{\partial\mu}\left\{\partial\frac{i\Delta}{\partial S} + 2S - (\mu^k+\bar{\mu}^k)\right\} + \frac{\partial i\Delta}{\partial\mu} + 2\mu - S(\lambda^k+\bar{\lambda}^k)}{i\Delta + |S-\mu^k|^2} = 0 \qquad (4.3)$$

Here $\lambda^k = \frac{\mu^k}{\mu}$ and we used

$$\frac{\partial|S-\mu^k|^2}{\partial\mu} = \frac{\partial\left(S^2 - S(\mu^k+\bar{\mu}^k) + \mu^2\right)}{\partial\mu} =$$

$$= -2S\cdot\frac{\partial S}{\partial\mu} - (\mu^k+\bar{\mu}^k)\cdot\frac{\partial S}{\partial\mu} - S(\lambda^k+\bar{\lambda}^k) + 2\mu =$$

$$= \left\{2S - (\mu^k+\bar{\mu}^k)\right\}\frac{\partial S}{\partial\mu} - S(\lambda^k+\bar{\lambda}^k) + 2\mu$$

Remind, here by $\frac{\partial i\Delta}{\partial S}$ and $\partial\frac{i\Delta}{\partial\mu}$ we understand $\frac{\partial F}{\partial S}$ and $\frac{\partial F}{\partial\mu}$, as by this equation (i.e. (1.1)) $i\Delta$ is constrained to be $i\Delta = F(S,\mu)$, and Eq. (4.3) is just a "rewriting" of $\frac{\partial i\Delta}{\partial\mu} = \frac{\partial F}{\partial S}\frac{\partial S}{\partial\mu} + \frac{\partial F}{\partial\mu}$

At phase transition point $\mu^*$, $\frac{\partial S}{\partial \mu} = \infty$, so Eq. (4.3)

then yields two equations:

$$\frac{1}{N} \sum \frac{\frac{\partial i\Delta}{\partial \mu} + 2\mu - S(\lambda^k + \bar{\lambda}^k)}{i\Delta + |S - \mu^k|^2} = 0$$

and

$$\frac{1}{N} \sum \frac{\frac{\partial i\Delta}{\partial S} + 2S - (\mu^k + \bar{\mu}^k)}{i\Delta + |S - \mu^k|^2} = 0 \qquad (5.1)$$

Introducing

$$A \equiv \frac{1}{N} \sum \frac{1}{i\Delta + |S - \mu^k|^2}$$

$$B \equiv \frac{1}{N} \sum \frac{\mu^k}{i\Delta + |S - \mu^k|^2} \qquad \text{note } A, B \in \mathbb{R}$$

(we will need later)

$$C \equiv \frac{1}{N} \sum \frac{(\mu^k)^2}{i\Delta + |S - \mu^k|^2}$$

All $A$, $B$ and $C$ are functions of $(i\Delta, S, \mu)$, and on the solution — just functions of $\mu$.

We then re-arrange Eq. (5.1):

$$\frac{\partial\, i\Lambda}{\partial \mu} = 2\,{}^{B}\!/_{A} \cdot {}^{S}\!/_{\mu} - 2\mu \qquad (6.1)$$

and

$$\frac{\partial \cdot i\Lambda}{\partial S} = 2\,{}^{B}\!/_{A} - 2S \qquad (6.3)$$

Again remind, here we understand

$$\frac{\partial\, i\Lambda}{\partial \mu} \equiv \frac{\partial F}{\partial \mu}$$

and $\frac{\partial\, i\Lambda}{\partial S} = \frac{\partial F}{\partial S}$

But at phase transition point we will equate them to

$\frac{\partial G}{\partial \mu}$ and $\frac{\partial G}{\partial S}$ respectively.

We do a similar analysis to Eq.(1.2) now, applying both "equations of motions" where possible to simplify the expressions.

Rewrite Eq. (1.2):

$$\frac{1}{N}\sum(S-\mu^k)\ln\left[i\Delta+|S-\mu^k|^2\right]-\frac{1}{N}\sum(S-\mu^k)\ln|S-\mu^k|^2=u\cdot S$$

we know

$$\frac{1}{N}\sum(S-\mu^k)\ln|S-\mu^k|^2=S\ln|S|^2+\mu^2/S \quad \text{for } |S|>\mu$$

we assume $\mathbf{S}$ real here

$$\Rightarrow \frac{1}{N}\sum(S-\mu^k)\ln\left[i\Delta+|S-\mu^k|^2\right]-\mathbf{S}\ln S^2-\mu^2/\mathbf{S}=u\cdot\mathbf{S}$$

$\mathbf{S}$ contribution, obviously, drops out due to Eq(1.1)

We then have

$$\frac{1}{N}\sum \mu^k \cdot \ln\left[i\Delta + |S-\mu^k|^2\right] + S\ln S^2 + \mu^2/S + u\cdot S = 0 \quad (8.1)$$

Now we diff. w.r.t. $\mu$:

$$\frac{1}{N}\sum \lambda^k \cdot \ln(i\Delta + |S-\mu^k|^2) + \frac{1}{N}\sum \mu^k \cdot \frac{\partial}{\partial\mu}\ln\left[i\Delta + |S-\mu^k|^2\right] +$$

$$+ \frac{\partial S}{\partial \mu}\left[\ln S^2 + 2 - \mu^2/S^2 + u\right] + 2\mu/S = 0 \quad (8.3)$$

The first term, $\frac{1}{N}\sum \lambda^k \cdot \ln(i\Delta + |S-\mu^k|^2)$ we can again express from (8.1)

$$\frac{1}{N}\sum \lambda^k \cdot \ln(i\Delta + |S-\mu^k|^2) = \frac{1}{\mu}\frac{1}{N}\sum \mu^k \cdot \ln(i\Delta + |S-\mu^k|^2) =$$

$$= -\,S/\mu \ln S^2 - \mu/S - u\cdot S/\mu$$

Then,

$$\frac{1}{N}\sum \mu^k \cdot \frac{\partial}{\partial\mu} \ln\left[i\Delta + |\delta-\mu^k|^2\right] - \qquad (9.1)$$

$$- \delta/\mu \ln \delta^2 + \mu/\delta - u\cdot \delta/\mu + \frac{\partial\delta}{\partial\mu}\left[\ln\delta^2 + 2 - \mu^2/\delta^2 + u\right] = 0$$

Now,

$$\frac{1}{N}\sum \mu^k \cdot \frac{\partial}{\partial\mu} \ln\left[i\Delta + |\delta-\mu^k|^2\right] =$$

$$= \frac{1}{N}\sum \mu^k \frac{\frac{\partial\delta}{\partial\mu}\left[\frac{\partial i\Delta}{\partial\delta} + 2\delta - (\mu^k+\bar{\mu}^k)\right] + \frac{\partial i\Delta}{\partial\mu} - \delta(\lambda^k+\bar{\lambda}^k) + 2\mu}{i\Delta + |\delta-\mu^k|^2} \qquad (9.3)$$

Here $\frac{\partial i\Delta}{\partial\mu} = \frac{\partial G}{\partial\mu}$ and $\frac{\partial i\Delta}{\partial\delta} = \frac{\partial G}{\partial\delta}$

We plug this whole thing into Eq. (9.1)

But first we use simplifying notations $A$, $B$ and $C$ to reduce the size of Expr. (9.3):

$$\frac{1}{N} \sum \mu^{k} \cdot \frac{\partial}{\partial \mu} \ln \left[ i\Lambda + |S - \mu \psi|^2 \right] =$$

$$= \frac{\partial S}{\partial \mu} \left[ B \cdot \left( \frac{\partial i\Lambda}{\partial S} + 2S \right) - A \cdot \mu^2 - C \right] +$$

$$+ B \left( \frac{\partial i\Lambda}{\partial \mu} + 2\mu \right) - \mu S \cdot A - S/\mu \cdot C$$

Now, still, before plugging this into Eq. (9.1) we equate $\frac{\partial i\Lambda}{\partial S} \equiv \frac{\partial G}{\partial S}$ here to $\frac{\partial F}{\partial S}$, and $\frac{\partial i\Lambda}{\partial \mu} \equiv \frac{\partial G}{\partial \mu}$ to $\frac{\partial F}{\partial \mu}$, from Eqs. (6.1) and (8.3) respectively, at $\mu = \mu^*$,

$$B \left( \partial \frac{i\Lambda}{\partial S} + 2S \right) = 2 B^2 / A$$

$$B \left( \frac{\partial i\Lambda}{\partial \mu} + 2\mu \right) = 2 B^2 / A \cdot S/\mu$$

This gives,

$$\frac{1}{N}\sum \mu^4 \cdot \frac{\partial}{\partial\mu} \ln\left[i\Delta + |S-\mu^4|^2\right] =$$

$$= \frac{\partial S}{\partial\mu}\left[2B^2/A - A\cdot\mu^2 - C\right] +$$

$$+ 2B^2/A \cdot S/\mu - \mu S \cdot A - S/\mu \cdot C$$

Finally, now, use this in Eq. (9.1),

$$2B^2/A \cdot S/\mu - A\cdot\mu S - S/\mu \cdot C - S/\mu \ln S^2 + \mu/S - u\cdot S/\mu +$$

$$+\left[\ln S^2 + 2 - \mu^2/S^2 + u + 2B^2/A - A\cdot\mu^2 - C\right]\frac{\partial S}{\partial\mu} = 0$$

at $\mu = \mu^*$

Invoking the assumption $\left.\frac{\delta S}{\delta \mu}\right|_{\mu^*} \to \infty$, we arrive at two equations:

$$\ln S^2 - \mu^2/S^2 + u + \left[ C + A\cdot\mu^2 - 2\, B^2/A \right] = 0$$

and

$$\ln S^2 + 2 - \mu^2/S^2 + u - \left[ C + A\cdot\mu^2 - 2\, B^2/A \right] = 0$$

This yields two conditions:

$$\Rightarrow \quad \ln S^2 - \mu^2/S^2 + 1 + u = 0$$

and

at $\mu = \mu^*$

$$\Rightarrow \quad C + A\cdot\mu^2 - 2\, B^2/A = 1$$

If we accept that $S = \mu$ at phase transition, then

$$\ln \mu_*^2 + u = 0, \quad \text{or} \quad \mu_* = e^{-u/2}$$