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Large-N Solution of the Heterotic CP$(N-1)$ Model with Twisted Masses

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Abstract

We address a number of unanswered questions on the $\mathcal{N} = (0, 2)$ deformation of the CP$(N-1)$ model with twisted masses. In particular, we complete the program of solving CP$(N-1)$ model with twisted masses in the large-N limit. In hep-th/0512153 nonsupersymmetric version of the model with the Z_N symmetric twisted masses was analyzed in the framework of Witten's method. In arXiv:0803.0698 this analysis was extended: we presented the large-N solution of the heterotic $\mathcal{N} = (0, 2)$ CP$(N-1)$ model with no twisted masses. Here we solve the heterotic $\mathcal{N} = (0, 2)$ CP$(N-1)$ model with nonvanishing twisted masses. Dynamical scenarios at large and small m are studied (m is the twisted mass scale). In a certain limiting case we discuss the mirror representation for the heterotic model.

1 Introduction

Two-dimensional CP($N-1$) models with twisted masses emerged as effective low-energy theories on the worldsheet of non-Abelian strings in a class of four-dimensional $\mathcal{N}=2$ gauge theories with unequal (s)quark masses [1, 2, 3, 4] (for reviews see [5]). Deforming these models in various ways (i.e. breaking supersymmetry down to $\mathcal{N}=1$ and to nothing) one arrives at nonsupersymmetric or heterotic CP($N-1$) models.[1] These two-dimensional models are very interesting on their own, since they exhibit nontrivial dynamics with or without phase transitions as one varies the twisted mass scale. In this paper we will present the large-N solution of the $\mathcal{N}=(0,2)$ CP($N-1$) model with twisted masses. As a warm up exercise we analyze this model in the limit of vanishing heterotic deformation, i.e. the $\mathcal{N}=(2,2)$ CP($N-1$) model with twisted masses (at $N\to\infty$). Both solutions that we report here are based on the method developed by Witten [6, 7] (see also [8]) and extended in [9] to include the heterotic deformation. For certain purposes we find it convenient to invoke the mirror representation [10, 11]. An $\mathcal{N}=(0,2)$ CP($N-1$) $\times C$ model on the string world sheet in the bulk theory deformed by $\mu\mathcal{A}^2$ was suggested by Edalati and Tong [12]. It was derived from the bulk theory in [13] (see also [14, 15]). Finally, the heterotic $\mathcal{N}=(0,2)$ CP($N-1$) model with twisted masses was formulated in [16].

2 Generalities

$\mathcal{N}=(2,2)$ supersymmtric CP($N-1$) sigma model was originally constructed [17] in terms of $\mathcal{N}=1$ superfields. Somewhat later it was realized [18] that $\mathcal{N}=1$ supersymmetry is automatically elevated up to $\mathcal{N}=2$ provided the target manifold of the sigma model in question is Kählerian (for reviews see [19, 20]). The Witten index [21] of the CP($N-1$) model is N, implying unbroken supersymmetry and N degenerate vacua at zero energy density. The CP($N-1$) manifold is compact; therefore, superpotential is impossible. One can introduce mass terms, however, through the twisted masses [22]. The model is asymptotically free [23], a dynamical scale Λ is generated through dimensional transmutation. If the scale of the twisted masses is much larger than Λ, the theory is at weak coupling. Otherwise it is at strong

[1] Strictly speaking, the full derivation of the heterotic CP($N-1$) model with twisted masses, valid for arbitrary values of the deformation parameters, from the microscopic bulk theory, is still absent. However, at small values of the deformation parameters, such a derivation is quite straightforward.

coupling. A priori, there are N distinct twisted mass parameters. However, in the absence of the heterotic deformation one of them is unobservable (see below). In this case the model is characterized by the coupling constant g^2, the vacuum angle θ and the twisted mass parameters m_1, m_2, ..., m_N with the constraint

$$m_1 + m_2 + ... + m_N = 0\,. \tag{2.1}$$ {one}

Introducing a heterotic deformation, generally speaking, we eliminate the above constraint. The twisted masses are arbitrary complex parameters. Of special interest in some instances (for example, in studying possible phase transitions) is the Z_N symmetric choice

$$m_k = m \exp\left(\frac{2\pi\, i\, k}{N}\right)\,, \qquad k = 0, 1, 2, ..., N-1\,. \tag{2.2}$$ {two}

The set (2.2) will be referred to as the Z_N-symmetric masses. Then the constraint (2.1) is automatically satisfied. Without loss of generality m can be assumed to be real and positive.

Where necessary, we mark the bare coupling constant by the subscript 0 and introduce the inverse parameter β as follows:

$$\beta = \frac{1}{g_0^2}\,. \tag{2.3}$$

At large N, in the 't Hooft limit, the parameter β scales as N.

There are two equivalent languages commonly used in description of the CP($N-1$) model: the geometric language ascending to [18] (see also [20]), and the so-called gauged formulation ascending to [6, 7]. Both have their convenient and less convenient sides. We will discuss both formulations although construction of the $1/N$ expansion is more convenient within the framework of the gauged formulation. At $|m|/\Lambda \to 0$ the elementary fields of the gauged formulation (they form an N-plet) are in one-to-one correspondence with the kinks in the geometric formulation. The multiplicity of kinks – the fact they they enter in N-plets – can be readily established [24] using the mirror representation [10]. We will discuss this in more detail later.

3 The model

3.1 Gauged formulation, no heterotic deformation

In this section we will briefly review the gauged formulation [6, 7] of the $\mathcal{N} = (2,2)$ CP$(N-1)$ model with twisted masses [22], i.e. we set the heterotic deformation coupling $\gamma = 0$. This formulation is built on an N-plet of complex scalar fields n^i where $i = 1, 2, ..., N$. We impose the constraint

$$\bar{n}_i\, n^i = 2\beta\,. \tag{3.1}$$

This leaves us with $2N-1$ real bosonic degrees of freedom. To eliminate one extra degree of freedom we impose a local U(1) invariance $n^i(x) \to e^{i\alpha(x)} n^i(x)$. To this end we introduce a gauge field A_μ which converts the partial derivative into the covariant one,

$$\partial_\mu \to \nabla_\mu \equiv \partial_\mu - i\, A_\mu\,. \tag{3.2}$$

The field A_μ is auxiliary; it enters in the Lagrangian without derivatives. The kinetic term of the n fields is

$$\mathcal{L} = \left|\nabla_\mu n^i\right|^2\,. \tag{3.3}$$

The superpartner to the field n^i is an N-plet of complex two-component spinor fields ξ^i,

$$\xi^i = \begin{cases} \xi^i_R \\ \xi^i_L \end{cases}\,, \tag{3.4}$$

subject to the constraint

$$\bar{n}_i\, \xi^i = 0\,, \qquad \bar{\xi}_i\, n^i = 0\,. \tag{3.5}$$ {npxi}

Needless to say, the auxiliary field A_μ has a complex scalar superpartner σ and a two-component complex spinor superpartner λ; both enter without derivatives. The full $\mathcal{N}=(2,2)$ -symmetric Lagrangian is[2]

$$
\begin{aligned}
\mathcal{L} \;=\;& \frac{1}{e_0^2}\left(\frac{1}{4}F_{\mu\nu}^2 + |\partial_\mu\sigma|^2 + \frac{1}{2}D^2 + \bar\lambda\, i\bar\sigma^\mu\partial_\mu\,\lambda\right) + i\,D\left(\bar n_i n^i - 2\beta\right) \\
+\;& \left|\nabla_\mu n^i\right|^2 + \bar\xi_i\, i\bar\sigma^\mu\nabla_\mu\,\xi^i + 2\sum_i\left|\sigma - \frac{m_i}{\sqrt2}\right|^2 |n^i|^2 \\
+\;& i\sqrt2\sum_i\left(\sigma - \frac{m_i}{\sqrt2}\right)\bar\xi_{Ri}\,\xi^i_L + i\sqrt2\,\bar n_i\left(\lambda_R\xi^i_L - \lambda_L\xi^i_R\right) \\
+\;& i\sqrt2\sum_i\left(\bar\sigma - \frac{\bar m_i}{\sqrt2}\right)\bar\xi_{Li}\,\xi^i_R + i\sqrt2\, n^i\left(\lambda_L\bar\xi_{Ri} - \lambda_R\bar\xi_{Li}\right),
\end{aligned}
\tag{3.6}
$$

where m_i are twisted mass parameters, and the limit $e_0^2\to\infty$ is implied. Moreover, Znak poslednih chlenov v dvuh poslednih strochkah ne sovpapadet s Pashey!

$$
\bar\sigma^\mu = \{1,\, i\sigma_3\}\,, \tag{3.7}
$$ {wtpi3}

see Appendix A.

It is clearly seen that the auxiliary field σ enters in (3.6) only through the combination

$$
\sigma - \frac{m_i}{\sqrt2}\,. \tag{3.8}
$$ {combi}

By an appropriate shift of σ one can always redefine the twisted mass parameters in such a way that the constraint (2.1) is satisfied. The U(1) gauge symmetry is built in. This symmetry eliminates one bosonic degree of freedom, leaving us with $2N-2$ dynamical bosonic degrees of freedom inherent to CP$(N-1)$ model.

3.2 Geometric formulation, $\tilde\gamma = 0$

Here we will briefly review the $\mathcal{N}=(2,2)$ supersymmetric CP$(N-1)$ models in the geometric formulation. The target space is the $N-1$-dimensional Kähler manifold parametrized by the fields ϕ^i, $\phi^{\dagger\,\bar j}$, $i,\bar j = 1,\ldots,N-1$, which are the lowest components of the chiral and antichiral superfields

$$
\Phi^i(x^\mu + i\bar\theta\gamma^\mu\theta), \qquad \bar\Phi^{\bar j}(x^\mu - i\bar\theta\gamma^\mu\theta)\,, \tag{3.9}
$$ {wtpi4}

[2] This is, obviously, the Euclidean version.

where [3]

$$x^\mu = \{t, z\}, \qquad \bar\theta = \theta^\dagger \gamma^0, \qquad \bar\psi = \psi^\dagger\gamma^0$$

$$\gamma^0 = \gamma^t = \sigma_2\,, \qquad \gamma^1 = \gamma^z = i\sigma_1\,, \qquad \gamma_5 \equiv \gamma^0\gamma^1 = \sigma_3\,. \tag{3.10}$$

With no twisted mass the Lagrangian is [18] (see also [26])

$$\mathcal{L}_{m=0} = \int d^4\theta K(\Phi,\bar\Phi) = G_{i\bar j}\left[\partial^\mu \bar\phi^{\bar j}\,\partial_\mu\phi^i + i\bar\psi^{\bar j}\gamma^\mu\mathcal{D}_\mu\psi^i\right] - \frac{1}{2}\,R_{i\bar j k\bar l}\,(\bar\psi^{\bar j}\psi^i)(\bar\psi^{\bar l}\psi^k). \tag{3.11}$$ {eq:kinetic}

$$G_{i\bar j} = \frac{\partial^2 K(\phi,\,\bar\phi)}{\partial\phi^i\partial\bar\phi^{\bar j}} \tag{3.12}$$ {wtpi6}

is the Kähler metric, $R_{i\bar j k\bar l}$ is the Riemann tensor [27],

$$R_{i\bar j k\bar m} = -\frac{g_0^2}{2}\left(G_{i\bar j}G_{k\bar m} + G_{i\bar m}G_{k\bar j}\right)\,. \tag{3.13}$$ {640}

Moreover,

$$\mathcal{D}_\mu\psi^i = \partial_\mu\psi^i + \Gamma^i_{kl}\partial_\mu\phi^k\psi^l$$

is the covariant derivative. The Ricci tensor $R_{i\bar j}$ is proportional to the metric [27],

$$R_{i\bar j} = \frac{g_0^2}{2}\,N\,G_{i\bar j}\,. \tag{3.14}$$ {eq:RG}

For the massless CP($N-1$) model a particular choice of the Kähler potential

$$K_{m=0} = \frac{2}{g_0^2}\ln\left(1 + \sum_{i,\bar j=1}^{N-1}\bar\Phi^{\bar j}\delta_{\bar j i}\Phi^i\right) \tag{3.15}$$ {eq:kahler}

corresponds to the round Fubini–Study metric.

Let us briefly remind how one can introduce the twisted mass parameters [22, 25]. The theory (3.11) can be interpreted as an $\mathcal{N}=1$ theory of d chiral superfields in four dimensions. The theory possesses $N-1$ distinct U(1) isometries parametrized by t^a, $a = 1,\ldots,N-1$. The Killing vectors of the isometries can be expressed via derivatives of the Killing potentials $D^a(\phi,\bar\phi)$,

$$\frac{d\phi^i}{d\,t_a} = -iG^{i\bar j}\,\frac{\partial D^a}{\partial\bar\phi^{\bar j}}\,, \qquad \frac{d\bar\phi^{\bar j}}{d\,t_a} = iG^{i\bar j}\,\frac{\partial D^a}{\partial\phi^i}\,. \tag{3.16}$$ {eq:KillD}

[3] In the Euclidean space $\bar\psi$ becomes an independent variable.

This defines U(1) Killing potentials, up to additive constants.

The $N-1$ isometries are evident from the expression (3.15) for the Kähler potential,

$$\delta\phi^i = -i\delta t_a (T^a)^i_k (\phi)^k\,, \qquad \delta\bar\phi^{\,\bar j} = i\delta t_a (T^a)^{\bar j}_{\bar l}\bar\phi^{\,\bar l}\,, \qquad a=1,\dots,N-1\,, \tag{3.17}$$ {eq:iso}

(together with the similar variation of fermionic fields), where the generators T^a have a simple diagonal form,

$$(T^a)^i_k = \delta^i_a\delta^a_k\,, \qquad a=1,\dots,N-1\,. \tag{3.18}$$

The explicit form of the Killing potentials D^a in CP($N-1$) with the Fubini–Study metric is

$$D^a = \frac{2}{g_0^2}\,\frac{\bar\phi\, T^a\phi}{1+\bar\phi\,\phi}\,, \qquad a=1,\dots,N-1\,. \tag{3.19}$$ {eq:KillF}

Here we use the matrix notation implying that ϕ is a column ϕ^i and $\bar\phi$ is a row $\bar\phi^{\bar j}$.

The isometries allow us to introduce an interaction with $N-1$ *external* U(1) gauge superfields V_a by modifying, in a gauge invariant way, the Kähler potential (3.15),

$$K_{m=0}(\Phi,\bar\Phi) \to K_m(\Phi,\bar\Phi,V)\,. \tag{3.20}$$ {eq:mkahler}

For CP($N-1$) this modification takes the form

$$K_m = \frac{2}{g_0^2}\ln\left(1+\bar\Phi\,\mathrm{e}^{V_a T^a}\Phi\right)\,. \tag{3.21}$$ {eq:mkahlerp}

In every gauge multiplet V_a let us retain only the A^a_x and A^a_y components of the gauge potentials taking them to be just constants,

$$V_a = -m_a\bar\theta(1+\gamma_5)\theta - \bar m_a\bar\theta(1-\gamma_5)\theta\,, \tag{3.22}$$ {wtpi1}

where we introduced complex masses m_a as linear combinations of constant U(1) gauge potentials,

$$\begin{aligned} m_a &= A^a_y + iA^a_x\,, \qquad \bar m_a = m_a^* = A^a_y - iA^a_x\,,\\ a &= 1,2,...,N-1\,. \end{aligned} \tag{3.23}$$

In spite of the explicit θ dependence the introduction of masses does not break $\mathcal{N}=2$ supersymmetry. One way to see this is to notice that the mass parameters can be viewed as the lowest components of the twisted chiral superfields $D_2\bar D_1 V_a$.

Now we can go back to two dimensions implying that there is no dependence on x and y in the chiral fields. It gives us the Lagrangian with the twisted masses included [22, 25]:

$$\mathcal{L}_m = \int d^4\theta \, K_m(\Phi, \bar{\Phi}, V) = G_{i\bar{j}} \, g_{MN} \left[\mathcal{D}^M \bar{\phi}^{\bar{j}} \, \mathcal{D}^N \phi^i + i \, \bar{\psi}^{\bar{j}} \gamma^M \, D^N \psi^i \right] - \frac{1}{2} R_{i\bar{j}k\bar{l}} \, (\bar{\psi}^{\bar{j}} \psi^i)(\bar{\psi}^{\bar{l}} \psi^k) \,, \tag{3.24}$$

where $G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K_m|_{\theta=\bar{\theta}=0}$ is the Kähler metric and summation over M includes, besides $M = \mu = 0, 1$, also $M = +, -$. The metric g_{MN} and extra gamma-matrices are

$$g_{MN} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix} , \qquad \gamma^+ = -i(1+\gamma_5) \,, \quad \gamma^- = i(1-\gamma_5) \,. \tag{3.25}$$

{eq:metric}

The gamma-matrices satisfy the following algebra:

$$\bar{\Gamma}^M \Gamma^N + \bar{\Gamma}^N \Gamma^M = 2 g^{MN} \,, \tag{3.26}$$

where the set $\bar{\Gamma}^M$ differs from Γ^M by interchanging of the $+, -$ components, $\bar{\Gamma}^{\pm} = \Gamma^{\mp}$. The gauge covariant derivatives $\mathcal{D}^M$ are defined as

$$\begin{aligned} &\mathcal{D}^\mu \phi = \partial^\mu \phi \,, \qquad \mathcal{D}^+ \phi = -\bar{m}_a T^a \phi \,, \qquad \mathcal{D}^- \phi = m_a T^a \phi \,, \\ &\mathcal{D}^\mu \bar{\phi} = \partial^\mu \bar{\phi} \,, \qquad \mathcal{D}^+ \bar{\phi} = \bar{\phi} \, T^a \bar{m}_a \,, \qquad \mathcal{D}^- \bar{\phi} = -\bar{\phi} \, T^a m_a \,, \end{aligned} \tag{3.27}$$

and similarly for $\mathcal{D}^M \psi$, while the general covariant derivatives $D^M \psi$ are

$$D^M \psi^i = \mathcal{D}^M \psi^i + \Gamma^i_{kl} \, \mathcal{D}^M \phi^k \, \psi^l \,. \tag{3.28}$$

3.3 Gauged formulation, switching on the heterotic deformation

{gfsothd}

The general formulation of $\mathcal{N} = (0, 2)$ gauge theories in two dimensions was addressed by Witten in [7], see also [36]. In order to deform the CP$(N-1)$ model breaking

$\mathcal{N} = (2,2)$ down to $\mathcal{N} = (0,2)$ we must introduce a right-handed spinor field ζ_R whose target space is C (with a bosonic superpartner $\mathcal{F}$), which is coupled to other fields as follows [12, 13]:

$$\begin{aligned} \Delta\mathcal{L} &= \bar{\zeta}_R\, i\partial_L\, \zeta_R + \bar{\mathcal{F}}\,\mathcal{F} \\ &- 2i\,\omega\,\bar{\lambda}_L\,\zeta_R - 2i\,\bar{\omega}\,\bar{\zeta}_R\,\lambda_L + 2i\,\omega\,\mathcal{F}\,\sigma + 2i\,\bar{\omega}\,\bar{\mathcal{F}}\,\bar{\sigma}\,, \end{aligned} \tag{3.29}$$

where we define

$$\omega = \sqrt{2\beta}\,\delta\,. \tag{3.30}$$ {deffp}

This term must be added to the $\mathcal{N} = (2,2)$ Lagrangian (3.6). It is quite obvious that the dependence on (3.8) is gone. The deformation term (3.29) has a separate dependence on σ, not reducible to the combination (3.8). Therefore, for a generic choice, all N twisted mass parameters m_1, m_2, ..., m_N become observable, Eq. (2.1) is no longer valid.

Eliminating $\mathcal{F}$, $\bar{\mathcal{F}}$ and $\bar{\lambda}$, λ we get

$$\Delta\mathcal{L} = 4\,|\omega|^2\,|\sigma|^2\,, \tag{3.31}$$ {deffpp}

while the constraints (3.5) are replaced by

$$\begin{aligned} \bar{n}_i\,\xi^i_L &= 0\,, \qquad \bar{\xi}_{Ri}\,n^i = 0\,, \\ \bar{n}_i\,\xi^i_R &= -\sqrt{2}\,\omega\zeta_R\,, \qquad \bar{\xi}_{Ri}\,n^i = -\sqrt{2}\,\bar{\omega}\bar{\zeta}_R\,. \end{aligned} \tag{3.32}$$

We still have to discuss how the parameter ω is related to other deformation parameters (which are equivalent to ω), and their N dependence. We want to single out appropriate powers of N so that the large-N limit will be smooth. The parameter δ in Eq. (3.29) is N-independent. Therefore, ω scales as $\sqrt{N}$.

One can restore the original form of the constraints (3.5) by shifting the ξ_R fields, namely,

$$\xi'_R = \xi_R + \sqrt{2}\,\bar{\delta}\,n\,\bar{\zeta}_R\,, \qquad \bar{\xi}'_R = \bar{\xi}_R + \sqrt{2}\,\delta\,\bar{n}\,\zeta_R\,. \tag{3.33}$$ {wtpi7}

This obviously changes the normalization of the kinetic term for ζ_R, which we can bring back to its canonic form by a rescaling ζ_R,

$$\zeta_R \to (1 - |\gamma|^2)\,\zeta_R\,. \tag{3.34}$$ {wtpi8}

As a result of these transformations, the following Lagrangian emerges:

$$
\begin{aligned}
\mathcal{L} &= \overline{\zeta}_R\, i\partial_L\, \zeta_R - \omega\, i\partial_L \overline{n}\xi_R\, \zeta_R - \bar{\omega}\, \overline{\xi}_R i\partial_L n\, \overline{\zeta}_R + |\tilde{\gamma}|^2\, \overline{\xi}_L\xi_L\, \overline{\zeta}_R\zeta_R \\
&- i\omega\, m^l\, \bar{n}_l\, \xi_L^l\, \zeta_R + i\omega\, \bar{m}^l\, \bar{\xi}_{Ll} n^l\, \overline{\zeta}_R \\
&+ 2\beta\Big\{ |\partial_k n|^2 + (\overline{n}\partial_k n)^2 + \overline{\xi}_R i\partial_L \xi_R + \overline{\xi}_L i\partial_R \xi_L - (\overline{n} i\partial_R n)\overline{\xi}_L\xi_L - (\overline{n} i\partial_L n)\overline{\xi}_R\xi_R \\
&+ (1-|\tilde{\gamma}|^2)\, \overline{\xi}_L\xi_R\, \overline{\xi}_R\xi_L - \overline{\xi}_R\xi_R\, \overline{\xi}_L\xi_L \\
&+ \sum_l |m^l|^2\, \left|n^l\right|^2 - im^l\, \bar{\xi}_{Rl}\xi_L^l \ - \ i\bar{m}^l\, \bar{\xi}_{Ll}\xi_R^l \\
&- (1-|\tilde{\gamma}|^2)\left(\left|\sum m^l |n^l|^2\right|^2 - im^l\, |n^l|^2 (\overline{\xi}_R\xi_L) - i\bar{m}^l\, |n^l|^2 (\overline{\xi}_L\xi_R) \right)\Big\}\,.
\end{aligned}
\tag{3.35}
$$

The sums over l above run from $l = 1$ to N. If the masses are chosen Z_N-symmetrically, see (2.2), this Lagrangian is explicitly Z_N-symmetric, see Appendix B.

If all $m_l = 0$, the model (3.35) reduces to the $\mathcal{N} = (0, 2)$ CP$(N-1)$ model derived in [13], see (3.6). Later on we will examine other special choices for the the mass terms. Here we will only note that with all $m_l \neq 0$ the masses of the boson and fermion excitations following from (3.35) split. Say, in the l_0-th vacuum

$$
\begin{aligned}
M_{\rm ferm}^{(l)} &= m^l - m^{l_0} + |\tilde{\gamma}|^2\, m^{l_0}\,, \\
\left|M_{\rm bos}^{(l)}\right| &= \sqrt{\left|M_{\rm ferm}^{(l)}\right|^2 - |\tilde{\gamma}|^4\, |m^{l_0}|^2}\,, \\
l &= 1, 2, ..., N; \quad l \neq l_0\,.
\end{aligned}
\tag{3.36}
$$

The model (3.35) still contains redundant fields. In particular, there are N bosonic fields n^l and N fermionic ξ^l, whereas the number of physical degrees of freedom is $2 \times (N-1)$. One can readily eliminate the redundant fields, say, n^N and ξ^N, by exploiting the constraints (3.5). Then explicit Z_N-symmetry will be lost, of course. It will survive as an implicit symmetry.

3.4 Geometric formulation, $\tilde{\gamma} \neq 0$

{gftgnz}

The parameter of the heterotic deformation in the geometric formulation will be denoted by $\tilde{\gamma}$ (the tilde appears here for historical reasons; perhaps, in the future it will be reasonable to omit it).

To obtain the Lagrangian of the heterotically deformed model we act as follows [16]: we start from (3.11), add the right-handed spinor field ζ_R, with the same kinetic term as in Sect. 3.3, and add the bifermion terms

$$\frac{\tilde{\gamma}\, g_0}{\sqrt{2}} \left[\zeta_R G_{i\bar{j}} \left(i\partial_L \bar{\phi}^{\bar{j}} \right) \psi_R^i + \bar{\zeta}_R G_{i\bar{j}} \left(i\partial_L \phi^i \right) \bar{\psi}_R^{\bar{j}} \right]. \tag{3.37}$$

Next, we change the four-fermion terms exactly in the same way this was done in [13], namely

$$\begin{aligned} &- \frac{1}{2} R_{i\bar{j}k\bar{l}} \left[\left(\bar{\psi}^{\bar{j}} \psi^i \right) \left(\bar{\psi}^{\bar{l}} \psi^k \right) \left(\bar{\psi}^{\bar{j}} \psi^i \right) \left(\bar{\psi}^{\bar{l}} \psi^k \right) \right] \\ &\to -\frac{g_0^2}{2} \left(G_{i\bar{j}} \psi_R^{\dagger\bar{j}} \psi_R^i \right) \left(G_{k\bar{m}} \psi_L^{\dagger\bar{m}} \psi_L^k \right) + \frac{g_0^2}{2} \left(1 - |\tilde{\gamma}|^2 \right) \left(G_{i\bar{j}} \psi_R^{\dagger\bar{j}} \psi_L^i \right) \left(G_{k\bar{m}} \psi_L^{\dagger\bar{m}} \psi_R^k \right) , \\ &- \frac{g_0^2}{2} |\tilde{\gamma}|^2 \left(\zeta_R^\dagger \zeta_R \right) \left(G_{i\bar{j}}\, \psi_L^{\dagger\bar{j}} \psi_L^i \right), \end{aligned} \tag{3.38}$$

where the first line represents the last term in Eq. (3.11), and we used the identity (3.13). If one of the twisted masses from the set $\{m_1, m_2, ..., m_N\}$ vanishes (say, $m^N = 0$), then this is the end of the story. The masses m_a in Eqs. (3.22) and (3.23) are $\{m_1, m_2, ..., m_{N-1}\}$.

However, with more general twisted mass sets, for instance, for the Z_N-symmetric masses (2.2), one arrives at a more contrived situation since one should take into account an extra contribution. Occurrence of this contribution can be seen [16] in a relatively concise manner using the superfield formalism of [13],

$$\Delta\mathcal{L} \sim M \int \mathcal{B}\, d\bar{\theta}_L\, d\theta_R + \text{H.c.}\,, \tag{3.39}$$ {tftpi1}

where $\mathcal{B}$ is a (dimensionless) $\mathcal{N} = (0, 2)$ superfield [4]

$$\mathcal{B} = \left\{ \zeta_R \left(x^\mu + i\bar{\theta}\gamma^\mu\theta \right) + \sqrt{2}\theta_R \mathcal{F} \right\} \bar{\theta}_L\,. \tag{3.40}$$ {tftpi2}

[4]This means that $\mathcal{B}$ is the superfield only with respect to the right-handed transformations.

The parameter M appearing in (3.39) has dimension of mass; in fact, it is proportional to m^N.

As a result, the heterotically deformed CP$(N-1)$ Lagrangian with all N twisted mass parameters included can be written in the following general form:

$$\mathcal{L} = \mathcal{L}_\zeta + \mathcal{L}_{m=0} + \mathcal{L}_m\,, \tag{3.41}$$ {tftpi3}

where the notation is self-explanatory. The expression for $\mathcal{L}_m$ is quite combersome. We will not reproduce it here, referring the interested reader to [16]. For convenience, we present here the first two terms,

$$\begin{aligned}
\mathcal{L}_\zeta + \mathcal{L}_{m=0} \qquad &= \zeta_R^\dagger\, i\partial_L\, \zeta_R + \left[\tilde{\gamma}\,\frac{g_0}{\sqrt{2}}\,\zeta_R\, G_{i\bar{j}}\,\left(i\,\partial_L\phi^{\dagger\,\bar{j}}\right)\psi_R^i + \text{H.c.}\right] \\
&-\frac{g_0^2}{2}\,|\tilde{\gamma}|^2\,\left(\zeta_R^\dagger\,\zeta_R\right)\left(G_{i\bar{j}}\,\psi_L^{\dagger\,\bar{j}}\psi_L^i\right) \\
&+G_{i\bar{j}}\left[\partial_\mu\phi^{\dagger\,\bar{j}}\,\partial_\mu\phi^i + i\bar{\psi}^{\bar{j}}\gamma^\mu D_\mu\psi^i\right] \\
&-\frac{g_0^2}{2}\left(G_{i\bar{j}}\psi_R^{\dagger\,\bar{j}}\,\psi_R^i\right)\left(G_{k\bar{m}}\psi_L^{\dagger\,\bar{m}}\,\psi_L^k\right) \\
&+\frac{g_0^2}{2}\left(1-|\tilde{\gamma}|^2\right)\left(G_{i\bar{j}}\psi_R^{\dagger\,\bar{j}}\,\psi_L^i\right)\left(G_{k\bar{m}}\psi_L^{\dagger\,\bar{m}}\,\psi_R^k\right)\,,
\end{aligned} \tag{3.42}$$

where we used (3.13). The above Lagrangian is $\mathcal{N}=(0,2)$-supersymmetric at the classical level. Supersymmetry is spontaneously broken by nonperturbative effects [12, 9]. Inclusion of $\mathcal{L}_m$ spontanesously breaks supersymmetry at the classical level [16], see Eq. (**??**).

The relation between $\tilde{\gamma}$ and δ is as follows [16]:

$$-i\,\tilde{\gamma}_M = \tilde{\gamma}_E = \sqrt{2}\frac{\delta_E}{\sqrt{1+2|\delta|^2}}, \tag{3.43}$$ {tftpi4}

implying that $\tilde{\gamma}$ does *not* scale with N in the 't Hooft limit.

4 Large-N solution of the CP$(N-1)$ model with twisted masses

In this section we present a large N solution of $\mathcal{N}=(2,2)$ supersymmetric $CP(N-1)$ model with twisted masses (3.6). We consider a special case of mass deformation (2.2) preserving a Z_N symmetry of the model. The $\mathcal{N}=(2,2)$ model with zero twisted masses as well as nonsupersymmetric $CP(N-1)$ model were solved by Witten in the large-N limit [6]. The same method was used in [34] to study nonsupersymmetric $CP(N-1)$ model with twisted mass. In this section we will generalize Witten's analysis to solve the $\mathcal{N}=(2,2)$ theory with twisted masses.

First let us briefly review the physics of non-supersymmetric CP$(N-1)$ model reveled by large N solution in [34]. In the limit of vanishing mass deformation, the $CP(N-1)$ model is known to be a strongly coupled asymptotically free field theory [28]. A dynamical scale Λ is generated as a result of dimensional transmutation. However, at large N it can be solved by virtue of $1/N$ expansion [6]. The solution found by Witten exhibits a composite photon, coupled to N quanta n, each with charge $1/\sqrt{N}$ with respect to this photon. In two dimensions the corresponding potential is long-range. It causes linear confinement, so that only n^*n pairs show up in the spectrum [41, 6]. This is the reason why we refer to this phase as "Coulomb/confining." In the Coulomb/confining phase the vacuum is unique and the Z_N symmetry is unbroken.

On the other hand, if the mass deformation parameter is $m \gg \Lambda$, the model is at weak coupling, the field n develops a vacuum expectation value (VEV), there are N physically equivalent vacua, in each of which the Z_N symmetry is spontaneously broken. We refer to this regime as the Higgs phase.

In Ref. [29] it was argued that non-supersymmetric twisted mass deformed $CP(N-1)$ model undergoes a phase transition when the value of the mass parameter is $\sim \Lambda$ to the Higgs phase with broken Z_N symmetry. In [34] this result was confirmed by large N solution. The issue of two phases and phase transitions in related models was also addressed by Ferrari [30, 31].

In $\mathcal{N}=(2,2)$ supersymmetric CP$(N-1)$ model we do not expect a phase transition with respect to twisted mass to occur. Below in this section we confirm this expectation showing that Z_N symmetry is broken at all values of twisted mass. Still the theory has two different regimes, the Higgs regime at large m and the strong-coupling one at small m.

Since the action (3.6) is quadratic in the fields n^i and ξ^i we can integrate over these fields and then minimize the resulting effective action with respect to the fields from the gauge multiplet. The large-N limit ensures the corrections to the saddle point approximation to be small. In fact, this procedure boils down to calculating a small set of one-loop graphs with the n^i and ξ^i fields propagating in loops.

In the Higgs regime the field n^{i_0} develop a VEV. One can always choose $i_0 = 0$ and denote $n^{i_0} \equiv n$. The field n, along with σ, are our order parameters that distinguish between the strong coupling and the Higgs regimes. These parameters show rather dramatic crossover behavior when we move from one regime to another.

Therefore, we do not want to integrate over n *a priory*. Instead, we will stick to the following strategy: we integrate over $N-1$ fields n^i with $i \neq 0$. The resulting effective action is to be considered as a functional of n, D and σ. To find the vacuum configuration, we will minimize the effective action with respect to n, D and σ.

Integration over n^i and ξ^i in (3.6) yields the following determinants:

$$\frac{\prod_{i=1}^{N-1} \det\left(-\partial_k^2 + |\sqrt{2}\sigma - m_i|^2\right)}{\prod_{i=1}^{N-1} \det\left(-\partial_k^2 + iD + |\sqrt{2}\sigma - m_i|^2\right)}, \tag{4.1}$$ {det}

where we dropped the gauge field A_k. The determinant in the denominator here comes from the boson loops while the one in the numerator from fermion loops. Note, that the n^i mass is given by $iD + |\sqrt{2}\sigma - m_i|^2$ while that of fermions ξ^i is $2|\sqrt{2}\sigma - m_i|^2$. If supersymmetry is unbroken (i.e. $D = 0$) these masses are equal, and the product of the determinants reduces to unity, as it should be.

Calculation of the determinants in Eq. (4.1) is straightforward. We easily get the following contribution to the effective action:

$$\begin{aligned} &\sum_{i=1}^{N-1} \frac{1}{4\pi} \left\{ \left(iD + |\sqrt{2}\sigma - m_i|^2\right) \left[\ln \frac{M_{\text{uv}}^2}{iD + |\sqrt{2}\sigma - m_i|^2} + 1\right] \right. \\ - \quad & \left. |\sqrt{2}\sigma - m_i|^2 \left[\ln \frac{M_{\text{uv}}^2}{|\sqrt{2}\sigma - m_i|^2} + 1\right] \right\}, \end{aligned} \tag{4.2}$$

where quadratically divergent contributions from bosons and fermions do not depend on D and σ and cancel each other. Here M_{uv} is an ultraviolet (UV) cutoff. The bare coupling constant $2\beta_0$ in (3.6) can be parametrized as

$$2\beta_0 = \frac{N}{4\pi} \ln \frac{M_{\text{uv}}^2}{\Lambda^2}. \tag{4.3}$$

Substituting this expression in (3.6) and adding the one-loop correction (4.2) we see that the term proportional to $iD \ln\, M_{\rm uv}^2$ is canceled out, and the effective action is expressed in terms of the renormalized coupling constant,

$$2\beta_{\rm ren} = \frac{1}{4\pi} \sum_{i=1}^{N-1} \ln \frac{iD + |\sqrt{2}\sigma - m_i|^2}{\Lambda^2}\,. \qquad (4.4)$$

{coupling}

Assembling all contributions together and dropping gauginos λ we get the effective potential as a function of n, D and σ fields in the form

$$\begin{aligned} V_{\rm eff} \;&=\; \int d^2x \left\{ \left(iD \;+\; 2\left|\sigma - \frac{m_0}{\sqrt{2}}\right|^2 \right) |n|^2 \right. \\ &-\; \frac{1}{4\pi} \sum_{i=1}^{N-1} \left(iD \;+\; 2\left|\sigma - \frac{m_i}{\sqrt{2}}\right|^2 \right) \ln \frac{iD \;+\; 2|\sigma - \frac{m_i}{\sqrt{2}}|^2}{\Lambda^2} \\ &+\; \frac{1}{4\pi} \sum_{i=1}^{N-1} 2\left|\sigma - \frac{m_i}{\sqrt{2}}\right|^2 \ln \frac{2\,|\sigma - \frac{m_i}{\sqrt{2}}|^2}{\Lambda^2} + \frac{1}{4\pi}\, iD\,(N-1) \end{aligned} \qquad (4.5)$$

Now, to find the vacua, we must minimize the effective potential (4.5) with respect to n, D and σ. In this way we arrive at the set of the vacuum equations,

$$|n|^2 = 2\beta_{\rm ren}\,, \qquad (4.6)$$

$$\left(iD \;\;+\;\; 2\left|\sigma - \frac{m_0}{\sqrt{2}}\right|^2 \right) n = 0\,, \qquad (4.7)$$

$$\left(\sigma - \frac{m_0}{\sqrt{2}}\right)|n|^2 - \frac{1}{4\pi} \sum_{i}^{N-1} \left(\sigma - \frac{m^i}{\sqrt{2}}\right) \ln \frac{iD + 2\left|\sigma \;-\; \frac{m^i}{\sqrt{2}}\right|^2}{2\left|\sigma - \frac{m^i}{\sqrt{2}}\right|^2} \;=\; 0\,, \qquad (4.8)$$

where $2\beta_{\rm ren}$ is determined by Eq. (4.4).

From Eq. (4.7) it is obvious that there are two options, either

$$iD + 2\left|\sigma - \frac{m_0}{\sqrt{2}}\right|^2 = 0 \qquad (4.9)$$

{higgsph22}

or

$$n = 0\,. \tag{4.10}$$
{strongph22}

These two distinct solutions correspond to the Higgs and the strong-coupling regimes of the theory.

Equations (4.6)-(4.8) represent our *master set* which determines the vacua of the theory.

Higgs regime

Consider first the Higgs regime. For large m we have

$$D = 0, \qquad \sqrt{2}\sigma = m_0, \qquad |n|^2 = 2\beta_{\rm ren}. \tag{4.11}$$
{higgsvac}

The first condition here, $D = 0$ means that the $\mathcal{N} = (2,2)$ supersymmetry is not broken and the vacuum energy is zero. Integrating over n's and ξ's we fixed $n^0 \equiv n$. Clearly we can fix any other n^{i_0}. Then (4.11) takes the form

$$D = 0, \qquad \sqrt{2}\sigma = m_{i_0}, \qquad |n^{i_0}|^2 = 2\beta_{\rm ren} \tag{4.12}$$
{higgsvacN}

showing presence of N degenerate vacua. The discrete chiral Z_{2N} symmetry (B.6) is broken by these VEV's down to Z_2. Substituting expressions for D and σ in (4.4) we get for the renormalized coupling

$$2\beta_{\rm ren} = \frac{1}{4\pi}\sum_{i=1}^{N-1} \ln\,\frac{|m_{i_0} - m_i|^2}{\Lambda^2} = \frac{N}{2\pi}\,\ln\,\frac{m}{\Lambda}\,, \tag{4.13}$$
{22higgscoupli

where we have calculated the sum over i in the large N limit for the special choice of masses in (2.2).

There are $2(N-1)$ elementary excitations[5] with physical masses

$$M_i = |m_i - m_{i_0}|\,, \qquad i \neq i_0\,. \tag{4.14}$$
{elmass}

Besides, there are kinks (domain "walls" which are particles in two dimensions) interpolating between these vacua. Their masses scale as

$$M_i^{\rm kink} \sim \beta_{\rm ren}\, M_i\,. \tag{4.15}$$
{kinkmass}

[5]Here we count real degrees of freedom. The action (3.6) contains N complex fields n^i. The phase of n^{i_0} can be eliminated from the very beginning. The condition $|n^i|^2 = 2\beta$ eliminates one more field.

The kinks are much heavier than elementary excitations at weak coupling. Note that they have nothing to do with Witten's n solitons [6] identified as solitons at strong coupling.

Since $|n^{i_0}|^2 = 2\beta_{\rm ren}$ is positively defined we see that the crossover point is at $m = \Lambda$. Below this point VEV of n is zero and we are in the strong coupling regime.

Strong coupling regime

For small m solutions of equations (4.6)-(4.8) can be readily found,

$$D = 0, \qquad n = 0, \qquad 2\beta_{\rm ren} = \frac{1}{4\pi}\sum_{i=1}^{N-1} \ln\frac{|\sqrt{2}\sigma - m_i|^2}{\Lambda^2} = 0. \tag{4.16}$$ {strongvac}

Much in the same way as in the Higgs regime, the condition $D = 0$ means that the $\mathcal{N} = (2,2)$ supersymmetry is not broken.

The last equation can be rewritten as

$$\prod_{i=0}^{N-1} |\sqrt{2}\sigma - m_i| \;=\; \Lambda^N \tag{4.17}$$ {Witcond}

Note, that although we derived this equation in the large N approximation it is, in fact, exact. I follows from the exact superpotential for $\mathcal{N} = (2,2)$ CP$(N-1)$ model [32, 33, 7, 35, 25].

For the special choice of masses in (2.2) equation (4.17) can be solved. Assuming for simplicity that $N = 2^p$ we rewrite this equation in the form

$$|(\sqrt{2}\sigma)^N - m^N| = \Lambda^N, \tag{4.18}$$

which has N solutions

$$\sqrt{2}\sigma = \left(\Lambda^N + m^N\right)^{1/N} \exp\left(\frac{2\pi\, i\, k}{N}\right), \quad k = 0, ..., N-1, \tag{4.19}$$ {22sigma}

where we assumed for simplicity that $m \equiv m_0$ is real and positive. This solution shows the presence of N degenerate vacua. Since σ is nonzero in all these vacua the discrete chiral Z_{2N} symmetry is broken down to Z_2 in the strong coupling regime much in the same way as in the Higgs regime. This should be contrasted with large N solution of the non-supersymmetric massive CP$(N-1)$ model [34]. In the

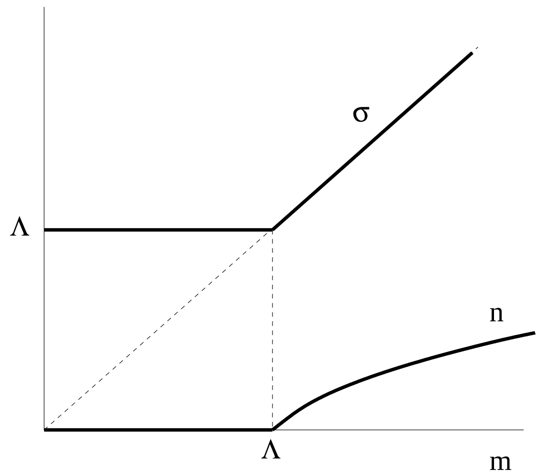

Figure 1: Plots of n and σ VEV's (thick lines) vs. m. {fig22nsigma}

latter model $\sigma = 0$ in the strong coupling phase, so the theory has only one single vacuum state and Z_{2N} symmetry is restored. This is a signal of a phase transition separating Higgs and Coulomb/confining phase in the non-supersymmetric massive CP$(N-1)$ model [34].

In fact, in the large N approximation result in (4.19) can be rewritten as

$$\sqrt{2}\sigma = \exp\left(\frac{2\pi\, i\, k}{N}\right)\left[\begin{array}{cc} \Lambda & m < \Lambda \\ m & m > \Lambda \end{array}\right., \qquad k = 0, ..., N-1 \tag{4.20}$$ {22sigmaapp}

with the exponential accuracy with respect to N. Note that at large m this formula reproduces our result (4.12) obtained in the Higgs regime. In the limit $m \to 0$ it gives Witten's result obtained in [6]

The behavior of VEV's n and σ as functions of m are plotted in Fig. 1. These plots suggest that we have discontinuities in derivatives with respect to m for both order parameters. This would signal phase transition, of course. We note, however, that exact formula (4.19) shows a smooth behavior of σ. Therefore we interpret the discontinuity in (4.20) as an artifact of large N approximation. The crossover transition between two regimes become more and more pronounced as we increase N and turn into a phase transition in the limit $N \to \infty$. We stress again that Z_{2N} symmetry is broken down to Z_2 in both regimes.

There is one interesting special regime in Eq. (4.17). If [6]

$$\prod_{i=0}^{N-1} (-m_i) \,=\, \Lambda^N \qquad (4.21)$$ {ADpoint}

all N vacua coalesce at

$$\sigma = 0. \qquad (4.22)$$ {ADsigma}

At this point in the mass parameter space one of N kinks interpolating between two "neighboring" vacua becomes massless. This point determines a non-trivial conformal regime in the theory. This is a two dimensional image of the four dimensional Argyres-Douglas point [38, 39].

5 Heterotic CP$(N-1)$ model at small deformations

To build construct an effective action allowing us to explore the vacuum structure of the model under consideration, we integrate over all but one given n^l filed (and its superpartner ξ^l). One can alway choose this fixed (unintegrated) field to be the last one, n^N. To the leading order in $1/N$ we can (and will) ignore the gauge field A_μ. The fields n^i and ξ^i $(i = 1, \dots N-1)$ enter the Lagrangian quadratically,

$$\begin{aligned} \mathcal{L} \;&=\; \overline{n}_i \left(-\partial_k^2 \;+\; 2\left|\sigma - \frac{m^i}{\sqrt{2}}\right|^2 + i\,D \right) n^i + \dots \\ &+\; \left(\overline{\xi}_{Ri}\;\overline{\xi}_{Li} \right) \begin{pmatrix} i\,\partial_L & i\sqrt{2}\left(\sigma - \frac{m^i}{\sqrt{2}}\right) \\ i\sqrt{2}\left(\overline{\sigma} - \frac{\overline{m}^i}{\sqrt{2}}\right) & i\,\partial_R \end{pmatrix} \begin{pmatrix} \xi_R^i \\ \xi_L^i \end{pmatrix} + \dots, \end{aligned} \qquad (5.1)$$

where the ellipses denote terms which contain neither n nor ξ fields. Integrating over the n, ξ fields produces the determinant

$$\prod_{i=1}^{N-1} \frac{\det\left(-\partial_k^2 + 2\left|\sigma - \frac{m^i}{\sqrt{2}}\right|^2 \right)}{\det\left(-\partial_k^2 + 2\left|\sigma - \frac{m^i}{\sqrt{2}}\right|^2 + i\,D \right)}\,. \qquad (5.2)$$ {determ}

[6]Strictly speaking the phase does not follow from (4.17). It is written down below using the complex version of Eq. (4.17), which follows from the exact superpotential for $\mathcal{N} = (2,2)$ CP$(N-1)$ model [32, 33, 7, 35, 25].

Assuming σ and D to be constant background fields, introducing the notation

$$n^N \equiv n\,, \qquad m^N \equiv m_0\,, \tag{5.3}$$ {nnotat}

and evaluating this determinant, one arrives at the following effective potential:

$$\begin{aligned}
V_{\rm eff} \;\; = \;\; & \int d^2x \left\{ \left(iD \;+\; 2\left|\sigma - \frac{m_0}{\sqrt{2}}\right|^2\right) |n|^2 \right. \\
- \;\; & \frac{1}{4\pi}\sum_{i=1}^{N-1}\left(iD \;+\; 2\left|\sigma - \frac{m^i}{\sqrt{2}}\right|^2\right) \ln\frac{iD \;+\; 2|\sigma - \frac{m^i}{\sqrt{2}}|^2}{\Lambda^2} \\
+ \;\; & \frac{1}{4\pi}\sum_{i=1}^{N-1} 2\left|\sigma - \frac{m^i}{\sqrt{2}}\right|^2 \ln\frac{2\,|\sigma - \frac{m^i}{\sqrt{2}}|^2}{\Lambda^2} + \frac{1}{4\pi}\, iD\,(N-1) + 4\,|\omega|^2\,\left|\sigma\right|^2
\end{aligned} \tag{5.4}$$

Now, to find the vacua, we must minimize the effective potential (5.4) with respect to n, D and σ. In this way we one arrive at the set of the vacuum equations,

$$|n|^2 - \frac{1}{4\pi}\sum_i^{N-1} \ln\frac{iD + 2\left|\sigma - \frac{m^i}{\sqrt{2}}\right|^2}{\Lambda^2} = 0\,, \tag{5.5}$$

$$\left(iD \;\;+\;\; 2\left|\sigma - \frac{m_0}{\sqrt{2}}\right|^2\right) n = 0\,, \tag{5.6}$$

$$\left(\sigma - \frac{m_0}{\sqrt{2}}\right)|n|^2 - \frac{1}{4\pi}\sum_i^{N-1}\left(\sigma - \frac{m^i}{\sqrt{2}}\right) \ln\frac{iD + 2\left|\sigma \;-\; \frac{m^i}{\sqrt{2}}\right|^2}{2\left|\sigma - \frac{m^i}{\sqrt{2}}\right|^2} + 2\,|\omega|^2\sigma \;\;=\;\; 0\,. \tag{5.7}$$

From Eq. (5.6) it is obvious that there are two options, either

$$iD + 2\,\left|\sigma - \frac{m_0}{\sqrt{2}}\right|^2 = 0 \tag{5.8}$$ {higgsph}

or

$$n = 0\,. \tag{5.9}$$ {strongph}

These two distinct solutions correspond to the Higgs and the strong-coupling regimes of the theory.

As a first step, we solve these equations perturbatively, assuming $|\omega|^2$ to be a small parameter,

$$\begin{aligned} n &= n^{(0)} + |\omega|^2 \, n^{(1)} + \dots \,, \\ iD &= iD^{(0)} + |\omega|^2 \, iD^{(1)} + \dots \,, \\ \sigma &= \sigma^{(0)} + |\omega|^2 \, \sigma^{(1)} + \dots \,. \end{aligned} \tag{5.10}$$

Here $n^{(0)}$, $D^{(0)}$ and $\sigma^{(0)}$ constitute the solution of the $\mathcal{N} = (2,2)$ CP$(N-1)$ sigma model, in particular $D^{(0)} = 0$ in both phases.

To obtain simple expressions for the solution we will assume the masses to be sitting on the circle,

$$m^k \;=\; m \cdot e^{i2\pi k/N} \,, \qquad\qquad k \;=\; 0, \dots N-1 \,.$$

5.1 The Higgs regime

The large-N supersymmetric solution of the $\mathcal{N} = (2,2)$ CP$(N-1)$ sigma model in the Higgs phase is

$$\begin{aligned} n^{(0)} &= \sqrt{r_{\rm ren}^{(0)}} \,, \qquad\qquad \textit{the phase of } n^{(0)} \textit{ is not determined} \\ iD^{(0)} &= 0 \,, \\ \sigma^{(0)} &= \frac{m_0}{\sqrt{2}} \,, \end{aligned}$$

where $r_{\rm ren}^{(0)}$ is the renormalized coupling of the unperturbed theory,

$$r_{\rm ren}^{(0)} \;=\; \frac{N}{2\pi} \log m/\Lambda \,.$$

Expanding Eqs. (5.5) – (5.7) to the first order in $|\omega|^2$, we obtain

$$\begin{aligned}
\overline{n}^{(0)}\, n^{(1)} \;+\; \text{h.c.} \;&=\; \frac{1}{4\pi}\sum_i^{N-1} \frac{iD^{(1)} \;+\; 2\left(\overline{\sigma}^{(0)}\,\sigma^{(1)} \;-\; \overline{\sigma}^{(1)}\,\dfrac{m^i}{\sqrt{2}} \;+\; \text{h.c.}\right)}{2\left|\sigma^{(0)} - \dfrac{m^i}{\sqrt{2}}\right|^2}\,, \\
iD^{(1)} \;&=\; 0\,, \qquad\qquad (5.11) \quad \{\texttt{higgseq}\} \\
\sigma^{(1)}\,|n^{(0)}|^2 \;-\; \frac{1}{4\pi}\,iD^{(1)}\sum_i^{N-1}\frac{1}{2\left(\overline{\sigma}^{(0)} - \dfrac{\overline{m}^i}{\sqrt{2}}\right)} \;+\; 2\,\sigma^{(0)} \;&=\; 0\,.
\end{aligned}$$

We can see that iD vanishes to the first order in $|\omega|^2$. Thus, for iD the first order of expansion is not sufficient, since we know that supersymmetry is broken and hence $iD \neq 0$. This variable, however, is easy to recover in the Higgs phase given the corresponding expansion of σ via Eq. (5.8).

The solution of Eqs. (5.11) can be written as

$$\begin{aligned}
iD^{(0)} \;&=\; 0\,, & iD^{(1)} \;&=\; 0\,, & iD^{(2)} \;&=\; -\,2\,|\sigma^{(1)}|^2\,, \\
\sigma^{(0)} \;&=\; \frac{m_0}{\sqrt{2}}\,, & \sigma^{(1)} \;&=\; -\,\frac{2\sigma^{(0)}}{|n^{(0)}|^2}\,, \\
|n^{(0)}|^2 \;&=\; r^{(0)}_{\text{ren}}\,, & n^{(1)} \;&=\; -\,\frac{2\,m^0}{\overline{n}^{(0)}\,|n^{(0)}|^2}\,\frac{1}{4\pi}\sum_i^{N-1}\frac{1}{m^0 - m^i}\,.
\end{aligned}$$

To simplify these expressions, as noted before, we put the masses on the circle, which gives then

$$\sum_{i=1}^{N-1}\frac{1}{m^0 - m^i} \;=\; \frac{N-1}{2m} \;=\; \frac{N}{2m} \;+\; O(1)\,.$$

Finally, we obtain

$$\begin{aligned}
\sigma \;&=\; \frac{m^0}{\sqrt{2}}\left(1 \;-\; \frac{2\,|\omega|^2}{|n^{(0)}|^2}\right) \;+\; \ldots\,, \\
iD \;&=\; -\,4\,\frac{m_0^2}{(r^{(0)}_{\text{ren}})^2}\,|\omega|^4 \;+\; \ldots\,, \\
n \;&=\; \sqrt{r^{(0)}_{\text{ren}}} \;-\; \frac{N}{4\pi}\,\frac{1}{r^{(0)}_{\text{ren}}\,\overline{n}^{(0)}}\,|\omega|^2 \;+\; \ldots
\end{aligned}$$

for the Higgs phase, where

$$r^{(0)}_{\rm ren} \;=\; \frac{N}{2\pi}\log m/\Lambda\,.$$

5.2 Strong coupling

The zeroth order in $|\omega|^2$ solution is

$$\begin{aligned} n^{(0)} &= 0\,, \\ iD^{(0)} &= 0\,, \\ \sigma^{(0)} &= \widetilde{\Lambda}\cdot e^{i\frac{2\pi l}{N}}\,, \qquad\qquad \widetilde{\Lambda} \;=\; \sqrt[N]{\Lambda^N \,+\, m^N}\,, \end{aligned}$$

for some fixed $l = 0, \ldots N-1$. Furthermore, in this phase n is known exactly,

$$n \;=\; 0\,.$$

The rest two equations in (5.5)-(5.7) give at the first order in $|\omega|^2$

$$\sum_i^{N-1} \frac{iD^{(1)} \,+\, 2\left(\overline{\sigma}^{(0)}\,\sigma^{(1)} \,-\, \sigma^{(1)}\,\frac{\overline{m}^i}{\sqrt{2}} \,+\, \text{h.c.}\right)}{2\left|\sigma^{(0)} - \frac{m^i}{\sqrt{2}}\right|^2} \;=\; 0\,,$$

$$\frac{1}{4\pi}\sum_i^{N-1}\frac{iD^{(1)}}{2\left(\overline{\sigma}^{(0)} - \frac{\overline{m}^i}{\sqrt{2}}\right)} \;=\; 2\,\sigma^{(0)}\,.$$

The solution to these equations are given by

$$n \;=\; 0\,, \qquad\qquad iD^{(0)} \;=\; 0\,,$$

$$iD^{(1)} \;=\; 8\pi\cdot\frac{2\sigma^{(0)}}{\displaystyle\sum_i^{N-1}\frac{1}{\overline{\sigma}^{(0)} - \overline{m}^i/\sqrt{2}}}\,, \qquad\qquad i \;=\; 1, \ldots N-1\,, \tag{5.12}$$ {strongeq}

$$\sigma^{(1)}\cdot\sum_i^{N-1}\frac{1}{\sigma^{(0)} - \frac{m^i}{\sqrt{2}}} \;+\; \text{h.c.} \;=\; -\,8\pi\cdot\sigma^{(0)}\cdot\frac{\displaystyle\sum_i^{N-1}\frac{1}{\left|\sigma^{(0)} - \frac{m^i}{\sqrt{2}}\right|^2}}{\displaystyle\sum_i^{N-1}\frac{1}{\overline{\sigma}^{(0)} - \frac{\overline{m}^i}{\sqrt{2}}}}\,.$$

We use the following relations to simplify the above solution in the case when the masses are distributed on a circle,

$$\sum_{k=0}^{N-1} \frac{1}{1 \,-\, \alpha\, e^{\frac{2\pi i k}{N}}} \;=\; \frac{N}{1 \,-\, \alpha^N}\,,$$
$$\sum_{k=0}^{N-1} \frac{1}{(1+\alpha^2) \,-\, 2\,\alpha \cos\frac{2\pi k}{N}} \;=\; \frac{1}{1\,-\,\alpha^2}\left(\frac{2\,N}{1\,-\,\alpha^N} \,-\, N\right)\,.$$

This gives,

$$\sum_{k=1}^{N-1} \frac{1}{\sigma^{(0)} - \frac{m^k}{\sqrt{2}}} \;=\; -\,\frac{1}{\sigma^{(0)} - \frac{m}{\sqrt{2}}} \,+\, N\,\frac{\widetilde{\Lambda}^N}{\Lambda^N}\,\frac{1}{\sigma^{(0)}}\,,$$
$$\sum_{k=1}^{N-1} \frac{1}{\left|\sigma^{(0)} - \frac{m^k}{\sqrt{2}}\right|^2} \;=\; -\,\frac{1}{\left|\sigma^{(0)} - \frac{m}{\sqrt{2}}\right|^2} \,+\, \frac{2\,N}{\widetilde{\Lambda}^2 - m^2}\cdot\frac{\widetilde{\Lambda}^N + m^N}{\Lambda^N}\,.$$

In fact we will only need the leading-N contribution from them above relations. In particular, if we substitute these relations into Eq. (5.12) directly, we will not have iD real. The reason *perhaps* is that the $\sigma^{(0)}$ solution is only valid up to $O(1/N)$ contributions. Therefore, we ignore the O(1) contributions versus O(N).

$$iD^{(1)} \;=\; \frac{16\pi}{N}\,\frac{\Lambda^N}{\widetilde{\Lambda}^N}\,|\sigma^{(0)}|^2\,,$$
$$|\sigma^{(1)}| \;=\; -\,\frac{8\pi}{N}\,\frac{\Lambda^N}{\widetilde{\Lambda}^N}\,\frac{\widetilde{\Lambda}^3}{\widetilde{\Lambda}^2 - m^2}\left[1 \,+\, \frac{m^N}{\widetilde{\Lambda}^N}\right]\cdot\left(\cos\!\left(\frac{2\pi l}{N} - \varphi\right)\right)^{-1}\,,$$

where φ is the arbitrary phase of $\sigma^{(1)}$. We can choose this phase to be the same as that of $\sigma^{(0)}$,

$$\varphi \;\equiv\; \frac{2\pi l}{N}$$

for the cosine to disappear. If we take the limit of small masses

$$\frac{m^N}{\Lambda^N} \;\ll\; 1\,,$$

and also note that

$$\widetilde{\Lambda} \;=\; \Lambda$$

with exponential accuracy in $1/N$, we can further simplify the result, and arrive at

$$\begin{aligned} n &= 0 \\ iD &= u\,\Lambda^2 + \ldots, \qquad\qquad \text{for } \frac{m^N}{\Lambda^N} \ll 1, \\ \sqrt{2}\sigma &= \Lambda \cdot e^{\frac{2\pi i l}{N}} \left[1 - \frac{u}{2} \frac{\Lambda^2}{\Lambda^2 - m^2} \right]. \end{aligned}$$

where

$$u \equiv \frac{8\pi}{N} |\omega|^2 \tag{5.13}$$ {u}

Note that although $|\omega|^2$ grows as $O(N)$ for large N, the coefficients of $|\omega|^2$-corrections are suppressed by the corresponding power of $1/N$ so that the corrections are neutral in N.

6 Heterotic CP$(N-1)$ model at large deformations

Now we study equations (5.5)-(5.7) in the limit of large values of deformation parameter $u \gg 1$. We will see that our theory has three different phases separated by two phase transitions:

- Strong coupling phase with broken Z_N symmetry at small m
- Coulomb/confining Z_N-symmetric phase at intermediate m (coupling is also strong in this phase)
- Higgs phase at large m, where Z_N symmetry is again broken

We assume that masses are distributed on a circle, see (2.2).

Strong coupling phase with broken Z_N

This phase occurs at very small masses,

$$m \leq \Lambda\, e^{-u/2} \tag{6.1}$$ {scphmass}

In this phase we have

$$|n| = 0, \qquad 2\beta_{\rm ren} = \frac{1}{4\pi}\sum_{i=1}^{N-1} \ln \frac{iD + |\sqrt{2}\sigma - m_i|^2}{\Lambda^2} = 0. \tag{6.2}$$ {scphn}

As we will see below σ is exponentially small in this phase. Masses are also small. Then the second equation here gives

$$iD \approx \Lambda^2. \tag{6.3}$$ {scphD}

With this value of iD equation (5.7) can be written as

$$\sum_{i=1}^{N-1} \left(\sqrt{2}\sigma - m_i\right) \ln \frac{\Lambda^2}{|\sqrt{2}\sigma - m_i|^2} = N\left(\sqrt{2}\sigma\right) u \tag{6.4}$$ {scpheq3}

To solve this equation we use the following trick. The "potential of a charged circle" of radius m in two dimensions with the center at the origin calculated at the point in the two dimensional plane, represented by a complex variable x reads

$$\frac{1}{N}\sum_{i=0}^{N-1} \ln |x - m_i|^2 = \left[\begin{array}{ll} \ln |x|^2, & |x| > m \\ \ln m^2, & |x| < m \end{array} \right. \tag{6.5}$$ {chargedcircle

in the large N-limit. Integrating this formula with respect to x we find

$$\frac{1}{N}\sum_{i=0}^{N-1}\left(\sqrt{2}\sigma - m_i\right)\ln\frac{\Lambda^2}{|\sqrt{2}\sigma - m_i|^2} = \left[\begin{array}{ll} \sqrt{2}\sigma\,\ln\frac{\Lambda^2}{|\sqrt{2}\sigma|^2}, & |\sqrt{2}\sigma| > m \\ \sqrt{2}\sigma\left(\ln\frac{\Lambda^2}{m^2} - 1\right), & |\sqrt{2}\sigma| < m \end{array}\right. \tag{6.6}$$ {usefulformula

Substituting this equation in (6.4) at small m, $m < |\sqrt{2}\sigma|$ we get

$$\sqrt{2}\langle\sigma\rangle = e^{\frac{2\pi i}{N}k}\,\Lambda\, e^{-u/2}, \qquad k = 0, ..., (N-1). \tag{6.7}$$ {scphsigma}

The vacuum value of σ is exponentially small at large u. The bound $m < |\sqrt{2}\sigma|$ translates into the condition (6.1) for m.

We see that we have N degenerate vacua in this phase. The chiral Z_{2N} symmetry is broken down to Z_2, the order parameter is $\langle\sigma\rangle$. Moreover, the absolute value of σ in these vacua does not depend on m. In fact this solution coincides with the one obtained in [9] for $m = 0$. This phase is quite similar to the strong coupling phase of $\mathcal{N} = (2,2)$ CP$(N-1)$ model, see (4.20). The difference is that the absolute value of σ depends now on u and becomes exponentially small in the limit $u \gg 1$.

Coulomb/confining phase

Now we increase m beyond the bound (6.1). From (6.6) we see that the exponentially small solution to Eq. (6.4) no longer exist. The only solution is

$$\langle\sigma\rangle = 0. \tag{6.8}$$ {confsigma}

We also have

$$|n| = 0, \qquad iD = \Lambda^2 - m^2 \tag{6.9}$$ {confnD}

as follows from (6.2).

This solution shows that we have only one Z_N symmetric vacuum now. All other vacua are lifted and become quasivacua (metastable at large N). This phase is quite similar to the Coulomb/confining phase of non-supersymmetric CP$(N-1)$ model found by Witten [6] at zero masses. Presence of small splitting between quasivacua produces a linear rising confining potential between kinks which interpolates between, say, true vacuum and the lowest quasivacuum [29] see also review [40]. As we already mentioned this linear potential was also interpreted as a Coulomb interaction long time ago [41, 6], see next section for a more detailed discussion.

As soon as we have a phase with broken Z_N symmetry at small m and Z_N symmetric phase at intermediate m we have a phase transition separating these phases. Usually one does not have phase transitions in supersymmetric theories. However, in the model at hand the supersymmetry is badly broken (in fact it is broken already at the classical level) therefore the emergence of a phase transition is not too much surprising.

We can calculate the vacuum energy to see the supersymmetry breaking explicitly. Substituting (6.8) and (6.9) into the effective potential (5.4) we get

$$E_{vac} = \frac{N}{4\pi}[\Lambda^2 - m^2 + m^2 \ln \frac{m^2}{\Lambda^2}]. \qquad (6.10)$$ {confEvac}

We plot the vacuum energy vs. m in Fig. **??**.

Now observe that although positive at generic values of m the vacuum energy vanishes at $m = \Lambda$. This is a signal of $\mathcal{N} = (0,2)$ supersymmetry restoration. To check this conclusion we can compare masses of boson n^i and its fermion superpartner ξ^i. From (3.6) we see that difference of their masses equals to iD. Now (6.9) shows that iD vanishes precisely at $m = \Lambda$.

Thus we see an amazing phenomenon: while $\mathcal{N} = (0,2)$ supersymmetry was broken at the classical level it gets restored at the quantum level at the particular point in the parameter space. Condition $|m| = \Lambda$ shows that we are precisely at the $\mathcal{N} = (0,2)$ descendant of the $\mathcal{N} = (2,2)$ Argyres-Douglas point (4.21). $\mathcal{N} = (0,2)$ supersymmetry gets restored at the Argyres-Douglas point. This observation was first made in [37] using a Veneziano-Yankielowicz-type superpotential for $\mathcal{N} = (2,2)$ CP$(N-1)$ model [32, 33, 7] extrapolated to the $\mathcal{N} = (0,2)$ supersymmetric case.

Higgs phase

The Higgs phase occurs in our model at large m. As we show below the model goes into the Higgs phase at

$$m > \sqrt{u}\Lambda \qquad (6.11)$$ {Hphmass}

and we keep u large. In this phase $|n|$ develop VEV. From Eq. (5.6) we see that

$$iD = -|\sqrt{2}\sigma - m_0|. \qquad (6.12)$$ {HphD}

First we study equations (5.5)-(5.7) at $m \gg \sqrt{u}\Lambda$. In this regime we can drop the second logarithmic term in (5.7). The first term is much larger because it is proportional to $\beta_{\rm ren}$ which is large in the quasiclassical region. Then Eq. (5.7) reduces to

$$(\sqrt{2}\sigma - m_0)\, 2\beta_{\rm ren} + \frac{N}{4\pi}\, u\, \sqrt{2}\sigma = 0, \tag{6.13}$$

which at large u gives

$$\sqrt{2}\sigma = \left(\frac{8\pi}{N}\beta_{\rm ren}\right) \frac{m_{i_0}}{u}, \tag{6.14}$$ {Hphsigma}

where we take into account that we can have nonzero VEV for any n^{i_0}, $i_0 = 0, ..., (N-1)$. Thus we have N degenerative vacua again. In each of them σ is small ($\sim m/u$) but nonzero. The Z_{2N} chiral symmetry is again broken down to Z_2. Clearly the Higgs phase is separated form the Coulomb/confining phase (where Z_{2N} is unbroken) by a phase transition.

In each of N vacua n^{i_0} develop VEV,

$$|n^{i_0}|^2 = 2\beta_{\rm ren} = \frac{1}{4\pi}\sum_{i\neq i_0} \ln\frac{|\sqrt{2}\sigma - m_i|^2 - |\sqrt{2}\sigma - m_{i_0}|^2}{\Lambda^2}. \tag{6.15}$$ {Hphn}

Since σ is small we can expand the r.h.s. in powers of σ. The leading contribution vanishes, while calculating the contribution proportional to σ (given by (6.14)) we get the following equation for $\beta_{\rm ren}$

$$\frac{8\pi}{N}\beta_{\rm ren} - \ln\left(\frac{8\pi}{N}\beta_{\rm ren}\right) = \ln\frac{m^2}{u\Lambda^2} + \text{const}, \tag{6.16}$$ {Hphbetaeq}

where we can neglect constant term in the first order approximation at $m \gg \sqrt{u}\Lambda$.

This equation has two solutions at large m. Keeping only the larger one (we expect β to be large) we get approximately

$$2\beta_{\rm ren} \approx \frac{N}{4\pi}\ln\frac{m^2}{u\Lambda^2}. \tag{6.17}$$ {Hphbeta}

As we reduce m, at $m \sim \sqrt{u}\Lambda$ two solutions of (6.16) coalesce and at smaller m become complex. Since $|n^{i_0}|^2 = 2\beta_{\rm ren}$ is positively defined this signals the point of a phase transition to the Coulomb/confining phase.

To determine the point of the phase transition with better accuracy we have to take into account the logarithmic term in Eq. (5.7)...

Calculating the vacuum energy in this phase we get

$$E_{\rm vac} = ... \tag{6.18}$$

7 More on the Coulomb/confining phase

8 Remarks on mirror for heterotic CP(1)

The geometric representatition of the heterotic CP(1) model is the following deformation of the $\mathcal{N}=(2,2)$ model [13]:

$$L_{\text{heterotic}} = \zeta_R^\dagger\, i\partial_L\,\zeta_R + \left[\gamma\,\zeta_R\,R\left(i\,\partial_L\phi^\dagger\right)\psi_R + \text{H.c.}\right] - g_0^2|\gamma|^2\left(\zeta_R^\dagger\,\zeta_R\right)\left(R\,\psi_L^\dagger\psi_L\right)$$

$$+G\left\{\partial_\mu\phi^\dagger\,\partial^\mu\phi + \frac{i}{2}\big(\psi_L^\dagger\,\overleftrightarrow{\partial_R}\,\psi_L + \psi_R^\dagger\,\overleftrightarrow{\partial_L}\,\psi_R\big)\right.$$

$$\left.-\frac{i}{\chi}\left[\psi_L^\dagger\psi_L\big(\phi^\dagger\,\overleftrightarrow{\partial_R}\,\phi\big) + \psi_R^\dagger\,\psi_R\big(\phi^\dagger\,\overleftrightarrow{\partial_L}\,\phi\big)\right] - \frac{2(1-g_0^2|\gamma|^2)}{\chi^2}\,\psi_L^\dagger\,\psi_L\,\psi_R^\dagger\,\psi_R\right\}, \qquad (8.19)$$

where the field ζ_R appearing in the first line is the spinor field on C which appears, with necessity in the $\mathcal{N}=(0,2)$ model [12]. Here G is the metric while R is the Ricci tensor. To begin with, we let us assume the deformation parameter γ to be small (it is dimensionless) and work to the leading order in γ.

Note that

$$K\Big|_{\theta=\bar\theta=0} = \frac{2}{g_0^2}\,\ln\chi\,,$$

$$G = G_{1\bar 1} = \partial_\phi\partial_{\phi^\dagger}\,K\Big|_{\theta=\bar\theta=0} = \frac{2}{g_0^2\,\chi^2}\,,$$

$$\Gamma = \Gamma^1_{11} = -2\,\frac{\phi^\dagger}{\chi}\,,\quad \bar\Gamma = \Gamma^{\bar 1}_{\bar 1\bar 1} = -2\,\frac{\phi}{\chi}\,,$$

$$R \equiv R_{1\bar 1} = -G^{-1}R_{1\bar 1 1\bar 1} = \frac{2}{\chi^2}\,, \qquad (8.20)$$

where the following notation is implied:

$$\chi \equiv 1 + \phi\,\phi^\dagger\,. \qquad (8.21)$$

Thus, the kinetic terms of the CP(1) fields ϕ and ψ contain $\frac{1}{g^2}$ in the normalization while γ in the first line is defined in conjunction with the Ricci tensor, so that there is no $\frac{1}{g^2}$ in front of this term. This convention is important for what follows.

Now, let us remember that the undeformed CP(1) model has a mirror representation [10, 11]

$$\mathcal{L}_{\text{mirror}} = \int d^4\theta\,Y^\dagger\,Y + \int d^2\theta\,\mathcal{W}_{\text{mirror}}(Y) + \text{H.c.} \qquad (8.22) \quad \{\texttt{five}\}$$

where

$$\mathcal{W}_{\rm mirror} = \Lambda \left(e^{Y} + e^{-Y}\right) \,, \tag{8.23}$$

and Λ is the dynamical scale of the CP(1) model.

The question is: "what is the mirror representation of the deformed model (8.19), to the leading order in γ?"

Surprisingly, this question has a very simple answer. To find the answer let us observe that the term of the first order in γ in (8.19) is nothing but the superconformal anomaly in the unperturbed CP(1) model (it is sufficient to consider this anomaly in the unperturbed model since we are after the leading term in γ in the mirror representation). More exactly, in the CP(1) model

$$\gamma_\mu J^\mu_\alpha = -\frac{\sqrt{2}}{2\pi}\, R\, \left(\partial_\nu \phi^\dagger\right) \left(\gamma^\nu \psi\right)_\alpha \,, \tag{8.24}$$ {six}

where J^μ_α is the supercurrent. In what follows, for simplicity, numerical factors like 2 or π will be omitted This means that the deformation term in (8.19) can be written as

$$\Delta\mathcal{L} = \gamma \zeta_R \left(\gamma_\mu\, J^\mu\right)_L \tag{8.25}$$ {seven}

Since (8.24) has a geometric meaning we can readily rewrite this term in the mirror representation (8.22). Indeed, in the generalized Wess-Zumino model

$$\left(\gamma_\mu\, J^\mu\right)_L = \frac{\partial \mathcal{W}_{\rm mirror}}{\partial Y}\, \psi_L \,. \tag{8.26}$$ {eight}

This equation is not quite exact, but it is good enough for the time being.

Now, combining (8.25) and (8.26) we get the following deformation of $\mathcal{L}_{\rm mirror}$ (in the leading order in γ):

$$\Delta\mathcal{L}_{\rm mirror} = \gamma\, \zeta_R\, \psi_L\, \frac{\partial \mathcal{W}_{\rm mirror}}{\partial Y} + \text{H.c.} \tag{8.27}$$

In superfields this is equivalent to adding to (8.22) the following terms:

$$\Delta\mathcal{L}_{\rm mirror} \int d^4\theta \mathcal{B}\, \mathcal{B}^\dagger + \gamma \int d\theta_L^\dagger\, \theta_L\, d\theta_L d\theta_R\; \mathcal{B}\, \mathcal{W}_{\rm mirror} + \text{H.c.} \tag{8.28}$$

This adds up to the scalar potential the term

$$\Delta V = |\gamma|^2 \left|\mathcal{W}_{\rm mirror}\right|^2 \,. \tag{8.29}$$

Note that a constant shift in $\mathcal{W}_{\rm mirror}$ becomes an observable physical effect.

9 Different effective Lagrangians

10 Conclusions

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Appendix A:
Minkowski vesrus Euclidean formulation

In the bulk of the paper we use both, Minkowski and Euclidean conventions. It is useful to summarize the transition rules. If the Minkowski coordinates are

$$x^\mu_M = \{t,\, z\}\,, \tag{A.1}$$

the passage to the Euclidean space requires

$$t \to -i\tau\,, \tag{A.2}$$

and the Euclidean coordinates are

$$x^\mu_M = \{\tau,\, z\}\,. \tag{A.3}$$

The derivatives are defined as follows:

$$\begin{aligned} \partial^M_L &= \partial_t + \partial_z\,, \qquad \partial^M_R = \partial_t - \partial_z\,, \\ \partial^E_L &= \partial_\tau - i\partial_z\,, \qquad \partial^M_R = \partial_\tau + i\partial_z\,. \end{aligned} \tag{A.4}$$

The Dirac spinor is

$$\Psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \tag{A.5}$$

In passing to the Eucildean space $\Psi^M = \Psi^E$; however, $\bar\Psi$ is transformed,

$$\bar\Psi^M \to i\bar\Psi^E\,. \tag{A.6}$$

Moreover, Ψ^E and $\bar\Psi^E$ are *not* related by the complex conjugation operation. They become independent variables. The fermion gamma matrices are defined as

$$\bar\sigma^\mu_M = \{1,\, -\sigma_3\}\,, \qquad \bar\sigma^\mu_E = \{1,\, i\sigma_3\}\,. \tag{A.7}$$

Finally,

$$\mathcal{L}_E = -\mathcal{L}_M(t = -i\tau, ...). \tag{A.8}$$

With this notation, formally, the fermion kinetic terms in $\mathcal{L}_E$ and $\mathcal{L}_M$ coincide. If we want the heterotic deformation term to have the same form in $\mathcal{L}_E$ and $\mathcal{L}_M$ we must (and do) transform the heterotic deformation parameter as follows:

$$\gamma_M = i\,\gamma_E\,. \tag{A.9}$$

Everywhere where there is no menace of confusion we omit the super/subscripts M, E. It is obvious from the context where the Euclidean or Minkowski formulation is implied.

Appendix B:
Global symmetries of the CP($N-1$) model with Z_N-symmetric twisted masses

Now let us discuss global symmetries of this model. In the absence of the twisted masses the model was SU(N) symmetric. The twisted masses (**??**) explicitly break this symmetry down to U$(1)^{N-1}$,

$$
\begin{aligned}
n^\ell &\to e^{i\alpha_\ell} n^\ell\,, \quad \xi_R^\ell \to e^{i\alpha_\ell}\xi_R^\ell \quad \xi_L^\ell \to e^{i\alpha_\ell}\xi_L^\ell\,, \quad \ell = 1,2,...,N\,,\\
\sigma &\to \sigma\,, \quad \lambda_{R,L} \to \lambda_{R,L}\,.
\end{aligned} \tag{B.1}
$$

where α_ℓ are N constant phases different for different ℓ.

Next, there is a global vectorial U(1) symmetry which rotates all fermions ξ^ℓ in one and the same way, leaving the boson fields intact,

$$
\begin{aligned}
\xi_R^\ell &\to e^{i\beta}\xi_R^\ell\,, \quad \xi_L^\ell \to e^{i\beta}\xi_L^\ell\,, \quad \ell = 1,2,...,N\,,\\
\lambda_R &\to e^{-i\beta}\lambda_R\,, \quad \lambda_L \to e^{-i\beta}\lambda_L\,,\\
n^\ell &\to n^\ell\,, \quad \sigma \to \sigma\,.
\end{aligned} \tag{B.2}
$$

Finally, there is a discrete Z_{2N} symmetry which is of most importance for our purposes. Indeed, let us start from the axial U$(1)_R$ transformation which would be a symmetry of the classical action at $m=0$ (it is anomalous, though, under quantum corrections),

$$
\begin{aligned}
\xi_R^\ell &\to e^{i\gamma}\xi_R^\ell\,, \quad \xi_L^\ell \to e^{-i\gamma}\xi_L^\ell\,, \quad \ell = 1,2,...,N\,,\\
\lambda_R &\to e^{i\gamma}\lambda_R\,, \quad \lambda_L \to e^{-i\gamma}\lambda_L\,, \quad \sigma \to e^{2i\gamma}\sigma\,,\\
n^\ell &\to n^\ell\,.
\end{aligned} \tag{B.3}
$$

With m switched on and the chiral anomaly included, this transformation is no longer the symmetry of the model. However, a discrete Z_{2N} subgroup survives both the inclusion of anomaly and $m \neq 0$. This subgroup corresponds to

$$
\gamma_k = \frac{2\pi i k}{2N}\,, \quad k = 1,2,...,N\,. \tag{B.4}
$$

with the simultaneous shift

$$\ell \to \ell - k\,. \tag{B.5}$$

In other words,

$$\begin{aligned} \xi_R^\ell &\to e^{i\gamma_k}\xi_R^{\ell-k}\,, \quad \xi_L^\ell \to e^{-i\gamma_k}\xi_L^{\ell-k}\,, \\ \lambda_R &\to e^{i\gamma_k}\lambda_R\,, \quad \lambda_L \to e^{-i\gamma_k}\lambda_L\,, \quad \sigma \to e^{2i\gamma_k}\sigma\,, \\ n^\ell &\to n^{\ell-k}\,. \end{aligned} \tag{B.6}$$

This Z_{2N} symmetry relies on the particular choice of masses given in (**??**).

The order parameters for the Z_N symmetry are as follows: (i) the set of the vacuum expectation values $\{\langle n^1\rangle,\ \langle n^2\rangle,\ ...\langle n^N\rangle\}$ and (i) the bifermion condensate $\langle \xi^\dagger_{R,\,\ell}\xi_L^\ell\rangle$. Say, a nonvanishing value of $\langle n^1\rangle$ or $\langle \xi^\dagger_{R,\,\ell}\xi_L^\ell\rangle$ implies that the Z_{2N} symmetry of the action is broken down to Z_2. The first order parameter is more convenient for detection at large m while the second at small m.

It is instructive to illustrate the above conclusions in a different formulation of the sigma model, namely, in the geometrical formulation (for simplicity we will consider CP(1); generalization to CP$(N-1)$ is straightforward). In components the Lagrangian of the model is

$$\begin{aligned} &\mathcal{L}_{CP(1)} = G\Big\{\partial_\mu\bar\phi\,\partial^\mu\phi - |m|^2\bar\phi\,\phi + \frac{i}{2}\big(\psi_L^\dagger\overset{\leftrightarrow}{\partial}_R\psi_L + \psi_R^\dagger\overset{\leftrightarrow}{\partial}_L\psi_R\big) \\ &-i\,\frac{1-\bar\phi\,\phi}{\chi}\,\big(m\,\psi_L^\dagger\psi_R + \bar m\psi_R^\dagger\psi_L\big) \\ &-\frac{i}{\chi}\left[\psi_L^\dagger\psi_L\big(\bar\phi\overset{\leftrightarrow}{\partial}_R\phi\big) + \psi_R^\dagger\psi_R\big(\bar\phi\overset{\leftrightarrow}{\partial}_L\phi\big)\right] \\ &-\frac{2}{\chi^2}\,\psi_L^\dagger\,\psi_L\,\psi_R^\dagger\,\psi_R\Big\}\,, \end{aligned} \tag{B.7}$$

where

$$\chi = 1 + \bar\phi\,\phi\,, \quad G = \frac{2}{g_0^2\,\chi^2}\,. \tag{B.8}$$

The Z_2 transformation corresponding to (B.6) is

$$\phi \to -\frac{1}{\bar\phi}\,, \qquad \psi_R^\dagger\psi_L \to -\psi_R^\dagger\psi_L\,. \tag{B.9}$$ {bee40}

The order parameter which can detect breaking/nonbreaking of the above symmetry is

$$\frac{m}{g_0^2}\left(1-\frac{g_0^2}{2\pi}\right)\frac{\bar{\phi}\,\phi-1}{\bar{\phi}\,\phi+1}-iR\psi_R^{\dagger}\psi_L\,. \tag{B.10}$$

Under the transformation (B.9) this order parameter changes sign. In fact, this is the central charge of the $\mathcal{N}=2$ sigma model, including the anomaly [**?**].

Now, what changes if instead of the $\mathcal{N}=2$ model we will consider nonsupersymmetric CP($N-1$) model with twisted masses? Then the part of the Lagrangian (3.6) containing fermions must be dropped. The same must be done in the Z_2 order parameter. As was shown in [29, 34], now at $m>\Lambda$ the Z_2 symmetry is broken, while at $m<\Lambda$ unbroken. A phase transition takes place.

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