FTPI-MINN-09/45, UMN-TH-2828/09
December 21/DRAFT

Large-N Solution of the Heterotic CP($N-1$) Model with Twisted Masses

Pavel A. Bolokhov[a,b], **Mikhail Shifman**[c] **and Alexei Yung**[c,d]

[a] *Theoretical Physics Department, St.Petersburg State University, Ulyanovskaya 1, Peterhof, St.Petersburg, 198504, Russia*
[b] *Department of Physics and Astronomy, University of Victoria, Victoria, BC, V8P 1A1 Canada*
[c] *William I. Fine Theoretical Physics Institute, University of Minnesota, Minneapolis, MN 55455, USA*
[d] *Petersburg Nuclear Physics Institute, Gatchina, St. Petersburg 188300, Russia*

Abstract

We address a number of unanswered questions in the $\mathcal{N}=(0,2)$-deformed CP($N-1$) model with twisted masses. In particular, we complete the program of solving CP($N-1$) model with twisted masses in the large-N limit. In hep-th/0512153 nonsupersymmetric version of the model with the Z_N symmetric twisted masses was analyzed in the framework of Witten's method. In arXiv:0803.0698 this analysis was extended: the large-N solution of the heterotic $\mathcal{N}=(0,2)$ CP($N-1$) model with no twisted masses was found. Here we solve this model with the twisted masses switched on. Dynamical scenarios at large and small m are studied (m is the twisted mass scale). We found three distinct phases and two phase transitions on the m plane. Two phases with the spontaneously broken Z_N-symmetry are separated by a phase with unbroken Z_N. This latter phase is characterized by a unique vacuum and confinement of all U(1) charged fields ("quarks"). In the broken phases (one of them is at strong coupling) there are N degenerate vacua and no confinement, similarly to the situation in the $\mathcal{N}=(2,2)$ model. Supersymmetry is spontaneously broken everywhere except a circle $|m|=\Lambda$ in the Z_N-unbroken phase.

Related issues are considered. In particular, we discuss the mirror representation for the heterotic model in a certain limiting case.

Contents

1 Introduction

Two-dimensional CP($N-1$) models with twisted masses emerged as effective low-energy theories on the worldsheet of non-Abelian strings in a class of four-dimensional $\mathcal{N}=2$ gauge theories with unequal (s)quark masses [1, 2, 3, 4] (for reviews see [5]). Deforming these models in various ways (i.e. breaking supersymmetry down to $\mathcal{N}=1$ and to nothing) one arrives at nonsupersymmetric or heterotic CP($N-1$) models.[1] These two-dimensional models are very interesting on their own, since they exhibit nontrivial dynamics with or without phase transitions as one varies the twisted mass scale. In this paper we will present the large-N solution of the $\mathcal{N}=(0,2)$ CP($N-1$) model with twisted masses. As a warm up exercise we analyze this model in the limit of vanishing heterotic deformation, i.e. the $\mathcal{N}=(2,2)$ CP($N-1$) model with twisted masses (at $N\to\infty$). Our solution of the undeformed model exhibits two regimes – the strong coupling regime and the Higgs regime – with the crossover between them. We determine and discuss the Argyres–Douglas point.

Then we proceed to the large-N solution of the heterotic deformation. Both solutions (with and without deformation) that we present here are based on the method developed by Witten [6, 7] (see also [8]) and extended in [9] to include the heterotic deformation. For certain purposes we find it convenient to invoke the mirror representation [10, 11].

An $\mathcal{N}=(0,2)$ CP($N-1$) $\times C$ model on the string world sheet in the bulk theory deformed by $\mu\mathcal{A}^2$ was suggested by Edalati and Tong [12]. It was derived from the bulk theory in [13] (see also [14, 15]). Finally, the heterotic $\mathcal{N}=(0,2)$ CP($N-1$) model with twisted masses was formulated in [16]. Its derivation from the microscopic bulk theory is under way [17].

We report a number of exciting and quite unexpected results in the heterotically deformed CP($N-1$) model with twisted masses. The model has two adjustable parameters: one describing the strength of the heterotic deformation, and the other, m/Λ, sets the scale of the twisted masses. Dynamics of the model drastically changes as we vary the value of m, the parameter defining the twisted masses

$$m_k = m\exp\left(i\frac{2\pi\,k}{N}\right), \qquad k=0,1,2,...,N-1\,.$$

[1]Strictly speaking, the full derivation of the heterotic CP($N-1$) model with twisted masses, valid for arbitrary values of the deformation parameters, from the microscopic bulk theory, is still absent.

We discover three distinct phases on the m plane. In the first and the third phases, occurring at small and large values of $|m|$, the Z_N-symmetry of the model is spontaneously broken. Correspondingly, there are N degenerate vacua and no confinement. In appearance, this is akin to what happens in the undeformed $\mathcal{N} = (2, 2)$ model. However, the nature of these two phases is quite different. The first one occurs at strong coupling (small $|m|$) while the third one at weak coupling (large $|m|$). In essence, this is the Higgs phase. Surprisingly, at intermediate values of $|m|$ we find the Coulomb/confining phase, with unbroken Z_N-symmetry and unique vacuum. It is thoroughly investigated and the reasons for the photon to remain massless revealed. Moreover, we find that at $|m| = \Lambda$ (in the the Coulomb/confining phase) the vacuum of the model is supersymmetric, while for all other values of m supersymmetry is spontaneously broken. At small and large values of the heterotic deformation parameter our solution is analytic. At intermediate values of this parameter it is semi-analytic: at certain stages we have to resort to numerical calculations.

There are two phase transitions between the three distinct phases. We thoroughly analyze these phase transitions and argue that they are of the second kind.

In addition to the large-N solution we address a number of related issues. For instance, in the limit of small heterotic deformation parameter we build the mirror representation for the $\mathcal{N} = (0, 2)$ CP(1) model.

The organization of the paper is as follows. In Sect. 2 we review general aspects of CP($N-1$) models. Section 3 introduces our basic heterotic CP($N-1$) model with twisted masses in two alternative formulations: gauged and geometric. Section 4 presents the large-N solution of the CP($N-1$) model with twisted masses and no heterotic deformation. In Sect. 5 we add a small deformation and analyze its impact on the large-N solution. In Sect. 6 we find the analytic large-N solution of the heterotic CP($N-1$) model with twisted masses in the limit of large deformation. Three phases and two phase transitions are identified. Intermediate values of the heterotic deformation parameter are studied semi-analytically and numerically. In Sect. 7 we focus on the second phase, namely, the Coulomb/confining regime. We explain here why, unlike two other phases, the photon remains massless. Section 8 is devoted to related issues, such as the mirror representation for the heterotic model, the Veneziano–Yankielowicz vs. large-N effective Lagrangians, and the evolution process of the BPS spectrum as a function of m. Section 9 briefly summarizes our findings.

2 Generalities

{genera}

$\mathcal{N} = (2,2)$ supersymmetric CP$(N-1)$ sigma model was originally constructed [18] in terms of $\mathcal{N} = 1$ superfields. Somewhat later it was realized [19] that $\mathcal{N} = 1$ supersymmetry is automatically elevated up to $\mathcal{N} = 2$ provided the target manifold of the sigma model in question is Kählerian (for reviews see [20, 21]). The Witten index [22] of the CP$(N-1)$ model is N, implying unbroken supersymmetry and N degenerate vacua at zero energy density. The CP$(N-1)$ manifold is compact; therefore, superpotential is impossible. One can introduce mass terms, however, through the twisted masses [23]. The model is asymptotically free [24], a dynamical scale Λ is generated through dimensional transmutation. If the scale of the twisted masses is much larger than Λ, the theory is at weak coupling. Otherwise it is at strong coupling. A priori, there are N distinct twisted mass parameters. However, in the absence of the heterotic deformation one of them is unobservable (see below). In this case the model is characterized by the coupling constant g^2, the vacuum angle θ and the twisted mass parameters m_1, m_2, ..., m_N with the constraint

$$m_1 + m_2 + ... + m_N = 0\,. \tag{2.1}$$ {one}

By introducing a heterotic deformation, generally speaking, we eliminate the above constraint. The twisted masses are arbitrary complex parameters. Of special interest in some instances (for example, in studying possible phase transitions) is the Z_N symmetric choice

$$m_k = m \exp\left(\frac{2\pi\, i\, k}{N}\right)\,, \qquad k = 0, 1, 2, ..., N-1\,. \tag{2.2}$$ {two}

The set (2.2) will be referred to as the Z_N-symmetric masses. Then the constraint (2.1) is automatically satisfied. Without loss of generality m can be assumed to be real and positive.

Where necessary, we mark the bare coupling constant by the subscript 0 and introduce the inverse parameter β as follows:

$$\beta = \frac{1}{g_0^2}\,. \tag{2.3}$$

At large N, in the 't Hooft limit, the parameter β scales as N.

There are two equivalent languages commonly used in the description of the CP$(N-1)$ model: the geometric language ascending to [19] (see also [21]), and the so-called gauged formulation ascending to [6, 7]. Both have their convenient and less convenient sides. We will discuss both formulations although construction of the $1/N$ expansion is more convenient within the framework of the gauged formulation. At $|m|/\Lambda \to 0$ the elementary fields of the gauged formulation (they form an N-plet) are in one-to-one correspondence with the kinks in the geometric formulation. The multiplicity of kinks – the fact they they enter in N-plets – can be readily established [25] using the mirror representation [10]. We will discuss this in more detail later.

3 The model

{mmod}

3.1 Gauged formulation, no heterotic deformation

In this section we will briefly review the gauged formulation [6, 7] of the $\mathcal{N} = (2, 2)$ CP$(N-1)$ model with twisted masses [23], i.e. we set the heterotic deformation coupling $\gamma = 0$. This formulation is built on an N-plet of complex scalar fields n^i where $i = 1, 2, ..., N$. We impose the constraint

$$\bar{n}_i\, n^i = 2\beta\,. \tag{3.1}$$ {m31}

This leaves us with $2N - 1$ real bosonic degrees of freedom. To eliminate one extra degree of freedom we impose a local U(1) invariance $n^i(x) \to e^{i\alpha(x)} n^i(x)$. To this end we introduce a gauge field A_μ which converts the partial derivative into the covariant one,

$$\partial_\mu \to \nabla_\mu \equiv \partial_\mu - i\, A_\mu\,. \tag{3.2}$$ {m32}

The field A_μ is auxiliary; it enters in the Lagrangian without derivatives. The kinetic term of the n fields is

$$\mathcal{L} = \left|\nabla_\mu n^i\right|^2\,. \tag{3.3}$$ {m33}

The superpartner to the field n^i is an N-plet of complex two-component spinor fields ξ^i,

$$\xi^i = \begin{cases} \xi^i_R \\ \xi^i_L \end{cases}\,, \tag{3.4}$$ {m34}

subject to the constraint

$$\bar{n}_i\, \xi^i = 0\,, \qquad \bar{\xi}_i\, n^i = 0\,. \tag{3.5}$$ {npxi}

Needless to say, the auxiliary field A_μ has a complex scalar superpartner σ and a two-component complex spinor superpartner λ; both enter without derivatives. The full $\mathcal{N}=(2,2)$ -symmetric Lagrangian is [2]

$$
\begin{aligned}
\mathcal{L} \;&=\; \frac{1}{e_0^2}\left(\frac{1}{4}F_{\mu\nu}^2+|\partial_\mu\sigma|^2+\frac{1}{2}D^2+\bar\lambda\, i\bar\sigma^\mu\partial_\mu\,\lambda\right)+i\,D\left(\bar n_i n^i-2\beta\right)\\
&+\; \left|\nabla_\mu n^i\right|^2+\bar\xi_i\, i\bar\sigma^\mu\nabla_\mu\,\xi^i+2\sum_i\left|\sigma-\frac{m_i}{\sqrt2}\right|^2|n^i|^2\\
&+\; i\sqrt2\sum_i\left(\sigma-\frac{m_i}{\sqrt2}\right)\bar\xi_{Ri}\,\xi_L^i+i\sqrt2\,\bar n_i\left(\lambda_R\xi_L^i-\lambda_L\xi_R^i\right)\\
&+\; i\sqrt2\sum_i\left(\bar\sigma-\frac{\bar m_i}{\sqrt2}\right)\bar\xi_{Li}\,\xi_R^i+i\sqrt2\,n^i\left(\bar\lambda_L\bar\xi_{Ri}-\bar\lambda_R\bar\xi_{Li}\right), \qquad (3.6)
\end{aligned}
$$

where m_i are twisted mass parameters, and the limit $e_0^2\to\infty$ is implied. Moreover, Znak poslednih chlenov v dvuh poslednih strochkah ne sovpapadet s Pashey!

$$
\bar\sigma^\mu=\{1,\,i\sigma_3\}\,, \qquad (3.7)
$$

{wtpi3}

see Appendix A.

It is clearly seen that the auxiliary field σ enters in (3.6) only through the combination

$$
\sigma-\frac{m_i}{\sqrt2}\,. \qquad (3.8)
$$

{combi}

By an appropriate shift of σ one can always redefine the twisted mass parameters in such a way that the constraint (2.1) is satisfied. The U(1) gauge symmetry is built in. This symmetry eliminates one bosonic degree of freedom, leaving us with $2N-2$ dynamical bosonic degrees of freedom inherent to CP$(N-1)$ model.

3.2 Geometric formulation, $\tilde\gamma=0$

Here we will briefly review the $\mathcal{N}=(2,2)$ supersymmetric CP$(N-1)$ models in the geometric formulation. The target space is the $N-1$-dimensional Kähler manifold parametrized by the fields ϕ^i, $\phi^{\dagger\,\bar j}$, $i,\bar j=1,\ldots,N-1$, which are the lowest components of the chiral and antichiral superfields

$$
\Phi^i(x^\mu+i\bar\theta\gamma^\mu\theta),\qquad \bar\Phi^{\bar j}(x^\mu-i\bar\theta\gamma^\mu\theta)\,, \qquad (3.9)
$$

{wtpi4}

[2]This is, obviously, the Euclidean version.

where [3]

$$x^\mu = \{t, z\}, \qquad \bar\theta = \theta^\dagger \gamma^0, \qquad \bar\psi = \psi^\dagger\gamma^0$$
$$\gamma^0 = \gamma^t = \sigma_2\,, \qquad \gamma^1 = \gamma^z = i\sigma_1\,, \qquad \gamma_5 \equiv \gamma^0\gamma^1 = \sigma_3\,. \tag{3.10}$$

With no twisted mass the Lagrangian is [19] (see also [26])

$$\mathcal{L}_{m=0} = \int d^4\theta K(\Phi,\bar\Phi) = G_{i\bar j}\left[\partial^\mu \bar\phi^{\bar j}\,\partial_\mu\phi^i + i\bar\psi^{\bar j}\gamma^\mu \mathcal{D}_\mu\psi^i\right] - \frac{1}{2}\,R_{i\bar j k\bar l}\,(\bar\psi^{\bar j}\psi^i)(\bar\psi^{\bar l}\psi^k). \tag{3.11}$$ {eq:kinetic}

where

$$G_{i\bar j} = \frac{\partial^2 K(\phi,\,\bar\phi)}{\partial\phi^i\partial\bar\phi^{\bar j}} \tag{3.12}$$ {wtpi6}

is the Kähler metric, and $R_{i\bar j k\bar l}$ is the Riemann tensor [27],

$$R_{i\bar j k\bar m} = -\frac{g_0^2}{2}\left(G_{i\bar j}G_{k\bar m} + G_{i\bar m}G_{k\bar j}\right)\,. \tag{3.13}$$ {640}

Moreover,

$$\mathcal{D}_\mu\psi^i = \partial_\mu\psi^i + \Gamma^i_{kl}\partial_\mu\phi^k\psi^l$$

is the covariant derivative. The Ricci tensor $R_{i\bar j}$ is proportional to the metric [27],

$$R_{i\bar j} = \frac{g_0^2}{2}\,N\,G_{i\bar j}\,. \tag{3.14}$$ {eq:RG}

For the massless CP$(N-1)$ model a particular choice of the Kähler potential

$$K_{m=0} = \frac{2}{g_0^2}\ln\left(1 + \sum_{i,\bar j=1}^{N-1}\bar\Phi^{\bar j}\delta_{\bar j i}\Phi^i\right) \tag{3.15}$$ {eq:kahler}

corresponds to the round Fubini–Study metric.

Let us briefly remind how one can introduce the twisted mass parameters [23, 28]. The theory (3.11) can be interpreted as an $\mathcal{N}=1$ theory of $N-1$ chiral superfields in four dimensions. The theory possesses $N-1$ distinct U(1) isometries parametrized by t^a, $a = 1,\ldots,N-1$. The Killing vectors of the isometries can be expressed via derivatives of the Killing potentials $D^a(\phi,\bar\phi)$,

$$\frac{d\phi^i}{d\,t_a} = -iG^{i\bar j}\,\frac{\partial D^a}{\partial\bar\phi^{\bar j}}\,, \qquad \frac{d\bar\phi^{\bar j}}{d\,t_a} = iG^{i\bar j}\,\frac{\partial D^a}{\partial\phi^i}\,. \tag{3.16}$$ {eq:KillD}

[3] In the Euclidean space $\bar\psi$ becomes an independent variable.

This defines the U(1) Killing potentials, up to additive constants.

The $N-1$ isometries are evident from the expression (3.15) for the Kähler potential,

$$\delta\phi^i = -i\delta t_a (T^a)^i_k (\phi)^k\,, \qquad \delta\bar\phi^{\bar j} = i\delta t_a (T^a)^{\bar j}_{\bar l}\bar\phi^{\bar l}\,, \qquad a=1,\dots,N-1\,, \tag{3.17}$$ {eq:iso}

(together with the similar variation of fermionic fields), where the generators T^a have a simple diagonal form,

$$(T^a)^i_k = \delta^i_a\delta^a_k\,, \qquad a=1,\dots,N-1\,. \tag{3.18}$$

The explicit form of the Killing potentials D^a in CP$(N-1)$ with the Fubini–Study metric is

$$D^a = \frac{2}{g_0^2}\,\frac{\bar\phi\, T^a\phi}{1+\bar\phi\,\phi}\,, \qquad a=1,\dots,N-1\,. \tag{3.19}$$ {eq:KillF}

Here we use the matrix notation implying that ϕ is a column ϕ^i and $\bar\phi$ is a row $\bar\phi^{\bar j}$.

The isometries allow us to introduce an interaction with $N-1$ *external* U(1) gauge superfields V_a by modifying, in a gauge invariant way, the Kähler potential (3.15),

$$K_{m=0}(\Phi,\bar\Phi) \to K_m(\Phi,\bar\Phi,V)\,. \tag{3.20}$$ {eq:mkahler}

For CP$(N-1)$ this modification takes the form

$$K_m = \frac{2}{g_0^2}\ln\left(1+\bar\Phi\,\mathrm{e}^{V_aT^a}\Phi\right)\,. \tag{3.21}$$ {eq:mkahlerp}

In every gauge multiplet V_a let us retain only the A^a_x and A^a_y components of the gauge potentials taking them to be just constants,

$$V_a = -m_a\bar\theta(1+\gamma_5)\theta - \bar m_a\bar\theta(1-\gamma_5)\theta\,, \tag{3.22}$$ {wtpi1}

where we introduced complex masses m_a as linear combinations of constant U(1) gauge potentials,

$$\begin{aligned} m_a &= A^a_y + iA^a_x\,, \qquad \bar m_a = m_a^* = A^a_y - iA^a_x\,, \\ a &= 1,2,...,N-1\,. \end{aligned} \tag{3.23}$$

The introduction of the twisted masses does not break $\mathcal{N}=2$ supersymmetry. To see this one can note that the mass parameters can be viewed as the lowest components of the twisted chiral superfields $D_2\bar D_1 V_a$.

Now we can go back to two dimensions implying that there is no dependence on x and y in the chiral fields. It gives us the Lagrangian with the twisted masses included [23, 28]:

$$\begin{aligned}\mathcal{L}_m &= \int d^4\theta\, K_m(\Phi, \bar{\Phi}, V) = G_{i\bar{j}}\, g_{MN} \left[\mathcal{D}^M \bar{\phi}^{\bar{j}}\, \mathcal{D}^N \phi^i + i\, \bar{\psi}^{\bar{j}} \gamma^M\, D^N \psi^i\right] \\ &- \frac{1}{2} R_{i\bar{j}k\bar{l}}\, (\bar{\psi}^{\bar{j}} \psi^i)(\bar{\psi}^{\bar{l}} \psi^k)\,, \end{aligned} \tag{3.24}$$

where $G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K_m|_{\theta=\bar{\theta}=0}$ is the Kähler metric and summation over M includes, besides $M = \mu = 0, 1$, also $M = +, -$. The metric g_{MN} and extra gamma-matrices are

$$g_{MN} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix}, \qquad \gamma^+ = -i(1 + \gamma_5)\,, \quad \gamma^- = i(1 - \gamma_5)\,. \tag{3.25}$$ {eq:metric}

The gamma-matrices satisfy the following algebra:

$$\bar{\Gamma}^M \Gamma^N + \bar{\Gamma}^N \Gamma^M = 2 g^{MN}\,, \tag{3.26}$$

where the set $\bar{\Gamma}^M$ differs from Γ^M by interchanging of the $+, -$ components, $\bar{\Gamma}^{\pm} = \Gamma^{\mp}$. The gauge covariant derivatives $\mathcal{D}^M$ are defined as

$$\begin{aligned} &\mathcal{D}^\mu \phi = \partial^\mu \phi\,, \qquad \mathcal{D}^+ \phi = -\bar{m}_a T^a \phi\,, \qquad \mathcal{D}^- \phi = m_a T^a \phi\,, \\ &\mathcal{D}^\mu \bar{\phi} = \partial^\mu \bar{\phi}\,, \qquad \mathcal{D}^+ \bar{\phi} = \bar{\phi}\, T^a \bar{m}_a\,, \qquad \mathcal{D}^- \bar{\phi} = -\bar{\phi}\, T^a m_a\,, \end{aligned} \tag{3.27}$$

and similarly for $\mathcal{D}^M \psi$, while the general covariant derivatives $D^M \psi$ are

$$D^M \psi^i = \mathcal{D}^M \psi^i + \Gamma^i_{kl}\, \mathcal{D}^M \phi^k\, \psi^l\,. \tag{3.28}$$

3.3 Gauged formulation, switching on the heterotic deformation

{gfsothd}

The general formulation of $\mathcal{N} = (0, 2)$ gauge theories in two dimensions was addressed by Witten in [7], see also [29]. In order to deform the CP$(N-1)$ model breaking

$\mathcal{N} = (2,2)$ down to $\mathcal{N} = (0,2)$ we must introduce a right-handed spinor field ζ_R whose target space is C (with a bosonic superpartner $\mathcal{F}$), which is coupled to other fields as follows [12, 13]:

$$\begin{aligned} \Delta\mathcal{L} &= \bar{\zeta}_R\, i\partial_L\, \zeta_R + \bar{\mathcal{F}}\,\mathcal{F} \\ &+ 2i\,\omega\,\bar{\lambda}_L\,\zeta_R + 2i\,\bar{\omega}\,\bar{\zeta}_R\,\lambda_L - 2i\,\omega\,\mathcal{F}\,\sigma - 2i\,\bar{\omega}\,\bar{\mathcal{F}}\,\bar{\sigma}\,, \end{aligned} \tag{3.29}$$

where we define

$$\omega = \sqrt{2\beta}\,\delta\,. \tag{3.30}$$ {deffp}

This term must be added to the $\mathcal{N} = (2,2)$ Lagrangian (3.6). It is quite obvious that the dependence on (3.8) is gone. The deformation term (3.29) has a separate dependence on σ, not reducible to the combination (3.8). Therefore, for a generic choice, all N twisted mass parameters m_1, m_2, ..., m_N become observable, Eq. (2.1) is no longer valid.

Eliminating $\mathcal{F}$, $\bar{\mathcal{F}}$ and $\bar{\lambda}$, λ we get

$$\Delta\mathcal{L} = 4\,|\omega|^2\,|\sigma|^2\,, \tag{3.31}$$ {deffpp}

while the constraints (3.5) are replaced by

$$\begin{aligned} \bar{n}_i\,\xi^i_L &= 0\,, \qquad \bar{\xi}_{Li}\,n^i = 0\,, \\ \bar{n}_i\,\xi^i_R &= \sqrt{2}\,\omega\zeta_R\,, \qquad \bar{\xi}_{Ri}\,n^i = \sqrt{2}\,\bar{\omega}\bar{\zeta}_R\,. \end{aligned} \tag{3.32}$$

We still have to discuss how the parameter ω is related to other deformation parameters (which are equivalent to ω), and their N dependence. We want to single out appropriate powers of N so that the large-N limit will be smooth. The parameter δ in Eq. (3.29) is N-independent. Therefore, ω scales as $\sqrt{N}$.

One can restore the original form of the constraints (3.5) by shifting the ξ_R fields, namely,

$$\xi'_R = \xi_R - \sqrt{2}\,\bar{\delta}\,n\,\bar{\zeta}_R\,, \qquad \bar{\xi}'_R = \bar{\xi}_R - \sqrt{2}\,\delta\,\bar{n}\,\zeta_R\,. \tag{3.33}$$ {wtpi7}

This obviously changes the normalization of the kinetic term for ζ_R, which we can bring back to its canonic form by a rescaling ζ_R,

$$\zeta_R \to (1 - |\tilde{\gamma}|^2)\,\zeta_R\,, \tag{3.34}$$ {wtpi8}

where the relation between $\tilde{\gamma}$ and δ is given in Eq. (3.43). As a result of these transformations, the following Lagrangian emerges:

$$
\begin{aligned}
\mathcal{L} \;&=\; \overline{\zeta}_R\, i\partial_L\, \zeta_R + \omega\, i\partial_L \overline{n}\xi_R\, \zeta_R + \bar{\omega}\, \overline{\xi}_R i\partial_L n\, \overline{\zeta}_R + |\tilde{\gamma}|^2\, \overline{\xi}_L \xi_L\, \overline{\zeta}_R \zeta_R \\
&+\; i\omega\, m^l\, \bar{n}_l\, \xi_L^l\, \zeta_R - i\omega\, \bar{m}^l\, \bar{\xi}_{Ll} n^l\, \overline{\zeta}_R \\
&+\; 2\beta \Bigg\{ |\partial_k n|^2 + (\overline{n}\partial_k n)^2 + \overline{\xi}_R i\partial_L \xi_R + \overline{\xi}_L i\partial_R \xi_L - (\overline{n} i\partial_R n)\overline{\xi}_L \xi_L - (\overline{n} i\partial_L n)\overline{\xi}_R \xi_R \\
&+\; (1-|\tilde{\gamma}|^2)\, \overline{\xi}_L \xi_R\, \overline{\xi}_R \xi_L - \overline{\xi}_R \xi_R\, \overline{\xi}_L \xi_L \\
&+\; \sum_l |m^l|^2 \left|n^l\right|^2 - i m^l\, \bar{\xi}_{Rl} \xi_L^l \;-\; i\bar{m}^l\, \bar{\xi}_{Ll} \xi_R^l \\
&-\; (1-|\tilde{\gamma}|^2) \left(\left| \sum m^l |n^l|^2 \right|^2 - i m^l\, |n^l|^2 (\overline{\xi}_R \xi_L) - i\bar{m}^l\, |n^l|^2 (\overline{\xi}_L \xi_R) \right) \Bigg\} .
\end{aligned} \tag{3.35}
$$

The sums over l above run from $l = 1$ to N. If the masses are chosen Z_N-symmetrically, see (2.2), this Lagrangian is explicitly Z_N-symmetric, see Appendix C.

If all $m_l = 0$, the model (3.35) reduces to the $\mathcal{N} = (0, 2)$ CP$(N-1)$ model derived in [13], see (3.6). Later on we will examine other special choices for the the mass terms. Here we will only note that with all $m_l \neq 0$ the masses of the boson and fermion excitations following from (3.35) split. Say, in the l_0-th vacuum

$$
\begin{aligned}
M^{(l)}_{\rm ferm} \;&=\; m^l - m^{l_0} + |\tilde{\gamma}|^2\, m^{l_0} \,, \\
\left| M^{(l)}_{\rm bos} \right| \;&=\; \sqrt{\left| M^{(l)}_{\rm ferm} \right|^2 - |\tilde{\gamma}|^4\, |m^{l_0}|^2} \,, \\
l \;&=\; 1, 2, ..., N; \quad l \neq l_0 \,.
\end{aligned} \tag{3.36}
$$

The model (3.35) still contains redundant fields. In particular, there are N bosonic fields n^l and N fermionic ξ^l, whereas the number of physical degrees of freedom is $2 \times (N-1)$. One can readily eliminate the redundant fields, say, n^N and ξ^N, by exploiting the constraints (3.5). Then explicit Z_N-symmetry will be lost, of course. It will survive as an implicit symmetry.

3.4 Geometric formulation, $\tilde{\gamma} \neq 0$

{gftgnz}

The parameter of the heterotic deformation in the geometric formulation will be denoted by $\tilde{\gamma}_{(M)}$ (the tilde appears here for historical reasons; perhaps, in the future it will be reasonable to omit it; subscript $_{(M)}$ will stress that this section works with Minkowski notations).

To obtain the Lagrangian of the heterotically deformed model we act as follows [16]: we start from (3.11), add the right-handed spinor field ζ_R, with the same kinetic term as in Sect. 3.3, and add the bifermion terms

$$\frac{g_0}{\sqrt{2}}\left[\tilde{\gamma}_{(M)}\,\zeta_R G_{i\bar{j}}\left(i\partial_L\bar{\phi}^{\bar{j}}\right)\psi_R^i + \bar{\tilde{\gamma}}_{(M)}\,\bar{\zeta}_R G_{i\bar{j}}\left(i\partial_L\phi^i\right)\bar{\psi}_R^{\bar{j}}\right]. \tag{3.37}$$

Next, we change the four-fermion terms exactly in the same way this was done in [13], namely

$$\begin{aligned}
&- \frac{1}{2}R_{i\bar{j}k\bar{l}}\left[\left(\bar{\psi}^{\bar{j}}\psi^i\right)\left(\bar{\psi}^{\bar{l}}\psi^k\right)\left(\bar{\psi}^{\bar{j}}\psi^i\right)\left(\bar{\psi}^{\bar{l}}\psi^k\right)\right] \quad \longrightarrow \\
&- \frac{g_0^2}{2}\left(G_{i\bar{j}}\psi_R^{\dagger\bar{j}}\,\psi_R^i\right)\left(G_{k\bar{m}}\psi_L^{\dagger\bar{m}}\,\psi_L^k\right) + \frac{g_0^2}{2}\left(1-|\tilde{\gamma}_{(M)}|^2\right)\left(G_{i\bar{j}}\psi_R^{\dagger\bar{j}}\,\psi_L^i\right)\left(G_{k\bar{m}}\psi_L^{\dagger\bar{m}}\,\psi_R^k\right)\,, \\
&- \frac{g_0^2}{2}\,|\tilde{\gamma}_{(M)}|^2\,\left(\zeta_R^\dagger\,\zeta_R\right)\left(G_{i\bar{j}}\,\psi_L^{\dagger\bar{j}}\psi_L^i\right),
\end{aligned} \tag{3.38}$$

where the first line represents the last term in Eq. (3.11), and we used the identity (3.13). If one of the twisted masses from the set $\{m_1,\, m_2,\, ...,\, m_N\}$ vanishes (say, $m^N = 0$), then this is the end of the story. The masses m_a in Eqs. (3.22) and (3.23) are $\{m_1,\, m_2,\, ...,\, m_{N-1}\}$.

However, with more general twisted mass sets, for instance, for the Z_N-symmetric masses (2.2), one arrives at a more contrived situation since one should take into account an extra contribution. Occurrence of this contribution can be seen [16] in a relatively concise manner using the superfield formalism of [13],

$$\Delta\mathcal{L} \sim M\int \mathcal{B}\; d\bar{\theta}_L\, d\theta_R + \text{H.c.}\,, \tag{3.39}$$

{tftpi1}

where $\mathcal{B}$ is a (dimensionless) $\mathcal{N} = (0,2)$ superfield [4]

$$\mathcal{B} = \left\{\zeta_R\left(x^\mu + i\bar{\theta}\gamma^\mu\theta\right) + \sqrt{2}\theta_R\mathcal{F}\right\}\bar{\theta}_L\,. \tag{3.40}$$

{tftpi2}

[4]This means that $\mathcal{B}$ is the superfield only with respect to the right-handed transformations.

The parameter M appearing in (3.39) has dimension of mass; in fact, it is proportional to m^N.

As a result, the heterotically deformed CP$(N-1)$ Lagrangian with all N twisted mass parameters included can be written in the following general form:

$$\mathcal{L} = \mathcal{L}_\zeta + \mathcal{L}_{m=0} + \mathcal{L}_m\,, \tag{3.41}$$ {tftpi3}

where the notation is self-explanatory. The expression for $\mathcal{L}_m$ is quite cumbersome. We will not reproduce it here, referring the interested reader to [16]. For convenience, we present here the first two terms,

$$\begin{aligned}
\mathcal{L}_\zeta + \mathcal{L}_{m=0} \qquad &= \zeta_R^\dagger\, i\partial_L\, \zeta_R + \left[\tilde{\gamma}_{(M)}\,\frac{g_0}{\sqrt{2}}\,\zeta_R\, G_{i\bar{j}}\left(i\,\partial_L\phi^{\dagger\,\bar{j}}\right)\psi_R^i + \text{H.c.}\right] \\
&-\frac{g_0^2}{2}\,|\tilde{\gamma}_{(M)}|^2\,\left(\zeta_R^\dagger\,\zeta_R\right)\left(G_{i\bar{j}}\,\psi_L^{\dagger\,\bar{j}}\psi_L^i\right) \\
&+G_{i\bar{j}}\left[\partial_\mu\phi^{\dagger\,\bar{j}}\,\partial_\mu\phi^i + i\bar{\psi}^{\bar{j}}\gamma^\mu D_\mu\psi^i\right] \\
&-\frac{g_0^2}{2}\left(G_{i\bar{j}}\psi_R^{\dagger\,\bar{j}}\,\psi_R^i\right)\left(G_{k\bar{m}}\psi_L^{\dagger\,\bar{m}}\,\psi_L^k\right) \\
&+\frac{g_0^2}{2}\left(1-|\tilde{\gamma}_{(M)}|^2\right)\left(G_{i\bar{j}}\psi_R^{\dagger\,\bar{j}}\,\psi_L^i\right)\left(G_{k\bar{m}}\psi_L^{\dagger\,\bar{m}}\,\psi_R^k\right)\,,
\end{aligned} \tag{3.42}$$

where we used (3.13). The above Lagrangian is $\mathcal{N}=(0,2)$-supersymmetric at the classical level. Supersymmetry is spontaneously broken by nonperturbative effects [12, 9]. Inclusion of $\mathcal{L}_m$ spontanesously breaks supersymmetry at the classical level (see Eq. (2.11) in [16]).

The relation between $\tilde{\gamma}$ and δ is as follows [16]:

$$i\,\tilde{\gamma}_{(M)} = \tilde{\gamma}_{(E)} = \sqrt{2}\frac{\delta}{\sqrt{1+2|\delta|^2}}, \tag{3.43}$$ {tftpi4}

implying that $\tilde{\gamma}$ does *not* scale with N in the 't Hooft limit. See Appendix B for details on relation between Euclidean and Minkowski notations.

4 Large-N solution of the CP($N-1$) model with twisted masses

{lnscptm}

In this section we present the large-N solution of the $\mathcal{N}=(2,2)$ supersymmetric CP($N-1$) model with twisted masses (3.6). We consider a special case of mass deformation (2.2) preserving the Z_N symmetry of the model. The $\mathcal{N}=(2,2)$ model with the vanishing twisted masses, as well as nonsupersymmetric CP($N-1$) model, were solved by Witten in the large-N limit [6]. The same method was used in [30] to study nonsupersymmetric CP($N-1$) model with twisted mass. In this section we will generalize this analysis to solve the $\mathcal{N}=(2,2)$ theory with twisted masses included.

First let us very briefly review the physics of nonsupersymmetric CP($N-1$) model revealed by the large-N solution [30]. In the limit of vanishing masses, the CP($N-1$) model is known to be a strongly coupled asymptotically free field theory [24]. A dynamical scale Λ is generated as a result of dimensional transmutation. At large N it can be solved by virtue of the $1/N$ expansion [6]. The solution exhibits a "composite massless photon" coupled to N quanta n^i, each with charge $1/\sqrt{N}$ with respect to this photon. In two dimensions the corresponding Coulomb potential is long-range. It causes linear confinement, so that only the $\bar{n}\,n$ pairs show up in the spectrum [31, 6]. This is the reason why we will refer to this phase as "Coulomb/confining." In the Coulomb/confining phase the vacuum is unique and the Z_N symmetry is unbroken.

On the other hand, if the mass deformation parameter m is $\gg \Lambda$, the model is at weak coupling, the field n develops a vacuum expectation value (VEV), there are N physically equivalent vacua, in each of which the Z_N symmetry is spontaneously broken. We refer to this regime as the Higgs phase.

In Ref. [32] it was argued that (nonsupersymmetric) twisted mass deformed CP($N-1$) model undergoes a phase transition when the value of the mass parameter is $\sim \Lambda$, to the Higgs phase with the broken Z_N symmetry. In [30] this result was confirmed by the explicit large-N solution. (Previously the issue of two phases and phase transitions in related models was addressed by Ferrari [33, 34].)

In the $\mathcal{N}=(2,2)$ supersymmetric CP($N-1$) model, generally speaking, we do not expect a phase transition in the twisted mass to occur. In this section we confirm this expectation demonstrating that the Z_N symmetry is broken at all values of the twisted mass. (See, however, the end of Sect. 4.2.) Still, the theory has two distinct regimes, the Higgs regime at large m and the strong-coupling one at small m.[5]

Since the action (3.6) is quadratic in the fields n^i and ξ^i we can integrate over these fields and then minimize the resulting effective action with respect to the fields from the gauge multiplet. The large-N limit ensures the corrections to the saddle point approximation to be small. In fact, this procedure boils down to calculating a small set of one-loop graphs with the n^i and ξ^i fields propagating in loops.

In the Higgs regime the field n^{i_0} develops a VEV. One can always choose $i_0 = 0$ and denote $n^{i_0} \equiv n$. The field n, along with σ, are our order parameters that distinguish between the strong coupling and Higgs regimes. These parameters show a rather dramatic crossover behavior when we move from one regime to another.

Therefore, we do not want to integrate over n *a priori*. Instead, we will stick to the following strategy: we integrate over $N-1$ fields n^i with $i \neq 0$. The resulting effective action is to be considered as a functional of $n^0 \equiv n$, D and σ. To find the vacuum configuration, we will then minimize the effective action with respect to n, D and σ.

The fields n^i and ξ^i $(i = 1, ... N-1)$ enter the Lagrangian quadratically,

$$\begin{aligned}
\Delta\mathcal{L} \;&=\; \overline{n}_i \left(-\partial_k^2 \;+\; \left|\sqrt{2}\sigma - m^i\right|^2 + i\,D \right) n^i + ... \\
&+\;\; \left(\overline{\xi}_{Ri}\;\overline{\xi}_{Li}\right) \begin{pmatrix} i\,\partial_L & i\left(\sqrt{2}\sigma - m^i\right) \\ i\left(\sqrt{2}\overline{\sigma} - \overline{m}^i\right) & i\,\partial_R \end{pmatrix} \begin{pmatrix} \xi^i_R \\ \xi^i_L \end{pmatrix} + ..., \qquad (4.1)
\end{aligned}$$

where the ellipses denote terms which contain neither n nor ξ fields. Hence, integration over n^i and ξ^i in (3.6) yields the following ratio of the determinants:

$$\frac{\prod_{i=1}^{N-1} \det\,\left(-\partial_k^2 + \left|\sqrt{2}\sigma - m_i\right|^2\right)}{\prod_{i=1}^{N-1} \det\,\left(-\partial_k^2 + iD + \left|\sqrt{2}\sigma - m_i\right|^2\right)}, \qquad (4.2) \quad \{\texttt{det}\}$$

where we dropped the gauge field A_k. The determinant in the denominator comes from the boson loops while that in the numerator from the fermion loops. Note,

[5]At finite N there is no phase transition between these regimes. Instead, one has a crossover. This is explained after Eq. (4.23).

that the n^i mass squared is given by $iD + |\sqrt{2}\sigma - m_i|^2$ while that of fermions ξ^i is $|\sqrt{2}\sigma - m_i|^2$. If supersymmetry is unbroken (i.e. $D = 0$) these masses are equal, and the ratio of the determinants reduces to unity, as it should be, of course.

Calculation of the determinants in Eq. (4.2) is straightforward. We easily get the following contribution to the effective Lagrangian:

$$\begin{aligned} \Delta\mathcal{L} &= \sum_{i=1}^{N-1} \frac{1}{4\pi} \left\{ \left(iD + \left|\sqrt{2}\sigma - m_i\right|^2 \right) \left(\ln \frac{M_{\rm uv}^2}{iD + \left|\sqrt{2}\sigma - m_i\right|^2} + 1 \right) \right. \\ &\left. - \left|\sqrt{2}\sigma - m_i\right|^2 \left(\ln \frac{M_{\rm uv}^2}{\left|\sqrt{2}\sigma - m_i\right|^2} + 1 \right) \right\}, \end{aligned} \tag{4.3}$$

where quadratically divergent contributions from bosons and fermions do not depend on D and σ and cancel each other. Here $M_{\rm uv}$ is an ultraviolet (UV) cutoff. The bare coupling constant $2\beta_0$ in (3.6) can be parametrized as

$$2\beta_0 = \frac{N}{4\pi} \ln \frac{M_{\rm uv}^2}{\Lambda^2}. \tag{4.4}$$

Substituting this expression in (3.6) and adding the one-loop correction (4.3) we see that the term proportional to $iD \ln M_{\rm uv}^2$ is canceled, and the effective action is expressed in terms of the renormalized coupling constant,

$$2\beta_{\rm ren} = \frac{1}{4\pi} \sum_{i=1}^{N-1} \ln \frac{iD + \left|\sqrt{2}\sigma - m_i\right|^2}{\Lambda^2}. \tag{4.5}$$ {coupling}

Assembling all contributions together and dropping the gaugino fields λ we get the effective potential as a function of n, D and σ fields in the form

$$\begin{aligned} V_{\rm eff} &= \int d^2x \left\{ \left(iD + \left|\sqrt{2}\sigma - m_0\right|^2 \right) |n|^2 \right. \\ &- \frac{1}{4\pi} \sum_{i=1}^{N-1} \left(iD + \left|\sqrt{2}\sigma - m_i\right|^2 \right) \ln \frac{iD + \left|\sqrt{2}\sigma - m_i\right|^2}{\Lambda^2} \\ &\left. + \frac{1}{4\pi} \sum_{i=1}^{N-1} \left|\sqrt{2}\sigma - m_i\right|^2 \ln \frac{\left|\sqrt{2}\sigma - m_i\right|^2}{\Lambda^2} + \frac{1}{4\pi} iD\,(N-1) \right\}. \end{aligned} \tag{4.6}$$

Now, to find the vacua, we must minimize the effective potential (4.6) with respect to n, D and σ. In this way we arrive at the set of the vacuum equations,

$$|n|^2 \;=\; 2\,\beta_{\rm ren}\,, \tag{4.7}$$

$$\Big(iD + |\sqrt{2}\sigma - m_0|^2\Big)\;n \;=\; 0\,, \tag{4.8}$$

$$(\sqrt{2}\sigma - m_0)|n|^2 - \frac{1}{4\pi}\sum_{i=1}^{N-1}(\sqrt{2}\sigma - m^i)\,\ln\frac{iD + \left|\sqrt{2}\sigma\,-\,m^i\right|^2}{\left|\sqrt{2}\sigma - m^i\right|^2} \;=\; 0\,, \tag{4.9}$$

where $2\beta_{\rm ren}$ is determined by Eq. (4.5).

From Eq. (4.8) it is obvious that there are two options: either

$$iD \;+\; \left|\sqrt{2}\sigma - m_0\right|^2 \;=\; 0 \tag{4.10}$$ {higgsph22}

or

$$n \;=\; 0\,. \tag{4.11}$$ {strongph22}

These two distinct solutions correspond to the Higgs and the strong-coupling regimes of the theory, respectively. Equations (4.7)–(4.9) represent our *master set* which determines the vacua of the theory.

4.1 The Higgs regime

{hireg}

Consider first the Higgs regime. For large m we have the solution

$$D \;=\; 0, \qquad \sqrt{2}\sigma \;=\; m_0, \qquad |n|^2 \;=\; 2\beta_{\rm ren}\,. \tag{4.12}$$ {higgsvac}

The first condition here, $D = 0$, means that $\mathcal{N} = (2,2)$ supersymmetry is not broken and the vacuum energy is zero. Integrating over n's and ξ's we fixed $n^0 \equiv n$. Clearly, alternatively we could have fixed any other n^{i_0}. Then, instead of (4.12), we would get

$$D \;=\; 0, \qquad \sqrt{2}\sigma \;=\; m_{i_0}, \qquad |n^{i_0}|^2 \;=\; 2\beta_{\rm ren}\,, \tag{4.13}$$ {higgsvacN}

demonstrating the presence of N degenerate vacua. The discrete chiral Z_{2N} symmetry (C.6) is broken by these VEV's down to Z_2. Substituting the above expressions for D and σ in (4.5) we get the renormalized coupling

$$2\beta_{\rm ren} \;=\; \frac{1}{4\pi}\sum_{i=1}^{N-1} \ln\,\frac{|m_0-m_i|^2}{\Lambda^2} \;=\; \frac{N}{2\pi}\,\ln\,\frac{m}{\Lambda}\,, \tag{4.14}$$ {22higgscoupli

where we calculated the sum over i in the large-N limit for the special choice of masses (2.2).

In each vacuum there are $2(N-1)$ elementary excitations[6] with the physical masses

$$M_i \;=\; |m_i - m_{i_0}|\,, \qquad i \neq i_0\,. \tag{4.15}$$ {elmass}

In addition to the elementary excitations, there are kinks (domain "walls" which are particles in two dimensions) interpolating between these vacua. Their masses scale as

$$M_i^{\rm kink} \;\sim\; \beta_{\rm ren}\, M_i\,. \tag{4.16}$$ {kinkmass}

The kinks are much heavier than elementary excitations at weak coupling. Note that they have nothing to do with Witten's n solitons [6] identified as the n^i fields at strong coupling, see Sect. 8.3.

Since $|n^{i_0}|^2 \,=\, 2\beta_{\rm ren}$ is positively defined we see that the crossover point is at $m=\Lambda$. Below this point, the VEV of the n field vanishes, and we are in the strong coupling regime.

4.2 The strong coupling regime

{tscreg}

For small m the solutions of Eqs. (4.7)–(4.9) can be readily found,

$$D \;=\; 0, \qquad n \;=0\,, \qquad 2\beta_{\rm ren} \;=\; \frac{1}{4\pi}\sum_{i=1}^{N-1}\ln\,\frac{\left|\sqrt{2}\sigma - m_i\right|^2}{\Lambda^2} = 0\,. \tag{4.17}$$ {strongvac}

Much in the same way as in the Higgs regime, the condition $D=0$ means that $\mathcal{N}=(2,2)$ supersymmetry remains unbroken.

Note that at large N, the summation in (4.17) can be extended to include the $i=0$ term,

$$2\beta_{\rm ren} \;=\; \frac{1}{4\pi}\sum_{i=0}^{N-1}\ln\,\frac{\left|\sqrt{2}\sigma - m_i\right|^2}{\Lambda^2} = 0\,. \tag{4.18}$$ {strongvacp}

[6]Here we count real degrees of freedom. The action (3.6) contains N complex fields n^i. The phase of n^{i_0} can be eliminated from the very beginning. The condition $|n^i|^2 = 2\beta$ eliminates one extra field.

Then Eq. (4.18), as well as the subsequent (4.19), are valid both for small and large m.

The last equation can be identically rewritten as

$$\prod_{i=0}^{N-1}\left|\sqrt{2}\sigma - m_i\right| = \Lambda^N . \tag{4.19}$$ {Witcond}

Although we derived this equation in the large-N approximation it is, in fact, exact, since precisely this equation follows from the Veneziano–Yankielovicz effective Lagrangian which was obtained for $\mathcal{N}=(2,2)$ CP$(N-1)$ model with the inclusion of twisted masses, in [35, 36, 7, 37, 28]. The Veneziano–Yankielovicz Lagrangian implies (4.19) at finite N.

For the Z_N-symmetric masses Eq. (4.19) can be solved. Say, for even N one can rewrite this equation in the form

$$\left|\left(\sqrt{2}\sigma\right)^N - m^N\right| = \Lambda^N , \tag{4.20}$$ {tumvn}

due to the fact that with the masses given in (2.2)

$$\begin{aligned}
\sum m_i &= 0 ,\\
\sum_{i,j;\, i\neq j} m_i m_j &= 0 ,\\
&\dots\\
\sum_{i_1,i_2,\dots,i_{N-1}} m_{i_1} m_{i_2} \dots m_{i_{N-1}} &= 0 , \qquad (i_1 \neq i_2 \neq \dots \neq i_{N-1}) .
\end{aligned} \tag{4.21}$$

Equation (4.20) has N solutions

$$\sqrt{2}\sigma = \left(\Lambda^N + m^N\right)^{1/N} \exp\left(\frac{2\pi\, i\, k}{N}\right), \quad k = 0, \dots, N-1, \tag{4.22}$$ {22sigma}

where we assumed for simplicity that $m \equiv m_0$ is real and positive. (This is by no means necessary; we will relax this assumption at the end of this section.) This solution shows the presence of N degenerate vacua. Since σ is nonzero in all these vacua the discrete chiral Z_{2N} symmetry is broken down to Z_2 in the strong-coupling regime, much in the same way as in the Higgs regime. This should be contrasted

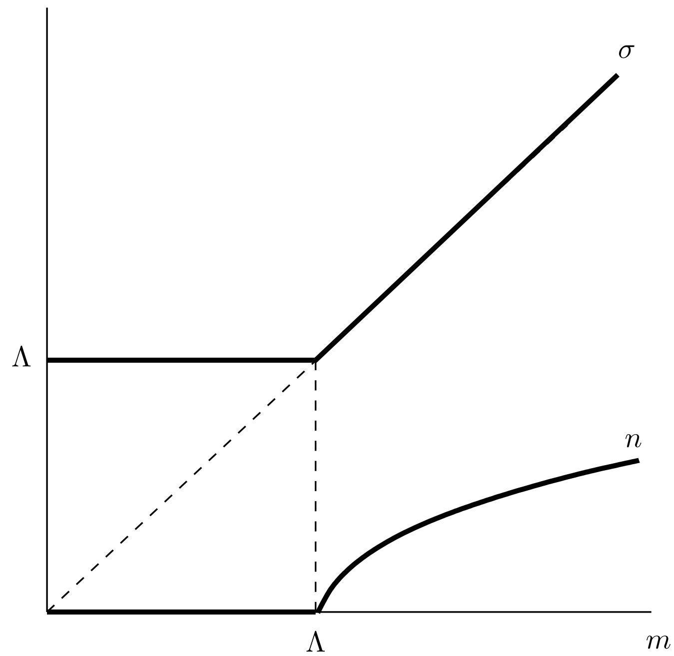

Figure 1: Plots of n and σ VEVs (thick lines) vs. m in the $\mathcal{N} = (2,2)$ CP$(N-1)$ model with twisted masses as in (2.2). {fig22nsigma}

with the large-N solution of the nonsupersymmetric massive CP$(N-1)$ model [30]. In the latter case, $\sigma = 0$ in the strong coupling phase, therefore, the theory has a single vacuum state in which the Z_{2N} symmetry is restored. This is a signal of a phase transition separating the Higgs and Coulomb/confining phases in the nonsupersymmetric massive CP$(N-1)$ model [30].

In fact, in the large-N approximation the formula (4.22) can be rewritten as

$$\sqrt{2}\sigma \;=\; \exp\left(\frac{2\pi\, i\, k}{N}\right) \times \left\{ \begin{array}{ll} \Lambda, & m < \Lambda \\ m, & m > \Lambda \end{array} \right. , \qquad k = 0, ..., N-1 \tag{4.23}$$ {22sigmaapp}

with the exponential accuracy $O\left(e^{-N}\right)$. Note that at large m this formula reproduces our result (4.13) obtained in the Higgs regime. In the limit $m \to 0$ it gives Witten's result [6].

The VEVs n and σ as functions of m are plotted in Fig. 1. These plots suggest that we have discontinuities in derivatives over m for both order parameters. Taken at its face value, this would signal a phase transition, of course. We note, however, that the exact formula (4.22) shows a smooth behavior in σ. Therefore, we interpret the discontinuity in (4.23) as an artifact of the large-N approximation. The crossover transition between the two regimes becomes exceedingly more pronounced as we increase N and turns into the second-order phase transition in the limit $N \to \infty$. We stress again that the Z_{2N} symmetry is broken down to Z_2 in the both regimes.

There is one interesting special point in Eq. (4.20). Relaxing the requirement of reality of the parameter m_0 we can choose the product [7]

$$\prod_{i=0}^{N-1} (-m_i) \;=\; \Lambda^N \,. \qquad (4.24)$$ {ADpoint}

At this particular point Eq. (4.20) reduces to $\sigma^N = 0$ with the solution

$$\sigma \;=\; 0\,. \qquad (4.25)$$ {ADsigma}

All N vacua coalesce! This is a two-dimensional "reflection" of the four-dimensional Argyres–Douglas (AD) point [38, 39].

4.3 Generic twisted masses and the Argyres–Douglas points

{genadp}

In this subsection we briefly describe the AD points in the nondeformed $\mathcal{N} = (2,2)$ CP$(N-1)$ model. At these points one or more kinks interpolating between different vacua of the model become *massless*. These points determine a nontrivial conformal regime in the theory. The complexified version of the vacuum equation (4.19) appropriate for a generic choice of the twisted mass parameters is

$$\prod_{i=0}^{N-1} \left(\sqrt{2}\sigma - m_i\right) \;=\; \Lambda^N \,. \qquad (4.26)$$ {Witcondc}

Let us have a closer look at this equation given a set of arbitrary masses. Our task is to find the values of mass parameters such that two roots of this equation coalesce, $\sigma_1 = \sigma_2$. Near the common value of σ Eq. (4.26) can be simplified, namely,

$$\left(\sqrt{2}\sigma - m_{12}\right)^2 - \frac{\Delta m_{12}^2}{4} \;=\; \Lambda_{\rm eff}^2 \;\equiv\; \frac{\Lambda^N}{\prod_{i\neq 1,2}(m_{12} - m_i)}, \qquad (4.27)$$ {sigmaeqAD}

[7]The complex version of Eq. (4.20) is in agreement with the exact superpotential for the $\mathcal{N} = (2,2)$ CP$(N-1)$ model [35, 36, 7, 37, 28].

where

$$m_{12} \;=\; \frac{1}{2}(m_1+m_2), \qquad \Delta m_{12} \;=\; m_1-m_2 \tag{4.28}$$ {m12}

Equation (4.27) gives

$$\sqrt{2}\,\sigma_{1,2} \;=\; m_{12} \;\pm\; \sqrt{\frac{\Delta m_{12}^2}{4} \;+\; \Lambda_{\rm eff}^2}. \tag{4.29}$$

Two vacua coalesce if the squere root vanishes,

$$-\Delta m_{12}^2 \prod_{i\neq 1,2}(m_{12}-m_i) \;=\; 4\Lambda^N\,. \tag{4.30}$$ {AD}

At this AD point one of N kinks interpolating between the vacua at σ_1 and σ_2 becomes massless.

Similarly, one can consider more complicated AD points in which more than two vacua coalesce. At these AD points more than one kink becomes massless. The point (4.24) corresponds to a very special regime in which all N vacua coalesce (for the Z_N-symmetric choice of masses on the circle (2.2)). At this point in the mass parameter space one of N kinks interpolating between each two "neighboring" vacua becomes massless. This AD point was studied previously in [47]. We remind that the $\mathcal{N}=(2,2)$ supersymmetric CP$(N-1)$ model is an effective theory on the world sheet of the non-Abelian string in $\mathcal{N}=2$ SQCD (with the U(N) gauge group and $N_f=N$ flavors [1, 2, 3, 4]). Therefore, the massless kinks at the AD points in two dimensions correspond to massless confined monopoles at the AD points in four-dimensional bulk theory.

We pause here to make a remark unrelated to the Argyres–Douglas points. Assume one has a (nearly) generic set of twisted masses subject to a single constraint

$$\prod_{i=0}^{N-1}(-m_i) = \Lambda^N\,. \tag{4.31}$$ {weass}

Then Eq. (4.26) has a solution $\sigma=0$, with other $N-1$ solutions $\sigma\neq 0$. Now, if we introduce the heterotic deformation $\sim\sigma^2$, the vacuum $\sigma=0$ remains supersymmetric, while in all other vacua supersymmetry is broken.

5 CP$(N-1)$ model at small heterotic deformations

{hecpnsm}

Now, we switch on the heterotic deformation which breaks $\mathcal{N} = (2,2)$ supersymmetry down to $\mathcal{N} = (0,2)$. In this section we will assume this deformation to be small limiting ourselves to the lowest nontrivial order in the heterotic deformation. All preparatory work was carried out in Sect. 4. Therefore, here we can focus on the impact of the heterotic deformation *per se*.

To determine the effective action allowing us to explore the vacuum structure of the heterotic model, just as in Sect. 4, we integrate over all but one given n^l field (and its superpartner ξ^l). One can alway choose this fixed (unintegrated) field to be $n^0 \equiv n$. Assuming σ and D to be constant background fields, and evaluating the determinants one arrives at the following effective potential:

$$
\begin{aligned}
V_{\rm eff} \;\; = \;\; & \int d^2x \bigg\{ \left(iD \; + \; \left|\sqrt{2}\sigma - m_0\right|^2 \right) |n|^2 \\
& - \; \frac{1}{4\pi} \sum_{i=1}^{N-1} \left(iD \; + \; \left|\sqrt{2}\sigma - m^i\right|^2 \right) \ln \frac{iD \, + \, \left|\sqrt{2}\sigma - m^i\right|^2}{\Lambda^2} \\
& + \; \frac{1}{4\pi} \sum_{i=1}^{N-1} \left|\sqrt{2}\sigma - m^i\right|^2 \ln \frac{\left|\sqrt{2}\sigma - m^i\right|^2}{\Lambda^2} \; + \; \frac{1}{4\pi}\, iD\,(N-1) \; + \; \frac{N}{2\pi} \cdot u \left|\sigma\right|^2 \bigg\} ,
\end{aligned}
\tag{5.1}
$$

where we have introduced a deformation parameter

$$
u \;\; \equiv \;\; \frac{8\pi}{N} \, |\omega|^2 \, . \tag{5.2}
$$

{u}

Note that although $|\omega|^2$ grows as $O(N)$ for large N, the parameter u does not scale with N and so is more appropriate for the role of an expansion parameter.

The above expression for $V_{\rm eff}$ replicates Eq. (4.6) except for the last term representing the heterotic deformation. Now, to find the vacua, we must minimize the effective potential (5.1) with respect to n, D and σ. The set of the vacuum equations is

$$|n|^2 - \frac{1}{4\pi}\sum_{i=1}^{N-1} \ln \frac{iD + \left|\sqrt{2}\sigma - m^i\right|^2}{\Lambda^2} = 0\,, \tag{5.3}$$

$$\left(iD + \left|\sqrt{2}\sigma - m_0\right|^2\right) n = 0\,, \tag{5.4}$$

$$\left(\sqrt{2}\sigma - m_0\right)|n|^2 - \frac{1}{4\pi}\sum_{i=1}^{N-1} \left(\sqrt{2}\sigma - m^i\right) \ln \frac{iD + \left|\sqrt{2}\sigma \,-\, m^i\right|^2}{\left|\sqrt{2}\sigma \,-\, m^i\right|^2} + \frac{N}{4\pi}\cdot u\,\sqrt{2}\sigma = 0\,. \tag{5.5}$$

It is identical to the master set of Sect. 4 with the exception of the last term in Eq. (5.5). Equation (5.4) is the same; hence we have the same two options: either

$$iD + \left|\sqrt{2}\sigma - m_0\right|^2 = 0 \tag{5.6}$$ {higgsph}

or

$$n = 0\,. \tag{5.7}$$ {strongph}

Since the deformation parameter is assumed to be small, we will solve these equations perturbatively, expanding in powers of u,

$$\begin{aligned} n &= n^{(0)} + u\cdot n^{(1)} + \dots\,, \\ iD &= iD^{(0)} + u\cdot iD^{(1)} + \dots\,, \\ \sigma &= \sigma^{(0)} + u\cdot \sigma^{(1)} + \dots\,. \end{aligned} \tag{5.8}$$

Here $n^{(0)}$, $D^{(0)}$ and $\sigma^{(0)}$ constitute the solution of the $\mathcal{N} = (2,2)$ CP$(N-1)$ sigma model, in particular $D^{(0)} = 0$ in both cases (5.6) and (5.7) corresponding to the Higgs and the strong-coupling regimes of the theory, respectively. We remind that the mass parameters are chosen according to (2.2).

5.1 The Higgs regime

{subshr}

The large-N supersymmetric solution of the $\mathcal{N} = (2,2)$ CP$(N-1)$ sigma model in the Higgs phase is given in Eqs. (4.13) and (4.14). Expanding Eqs. (5.3) – (5.5) to the first order in u, we calculate

$$
\begin{aligned}
iD^{(0)} &= 0\,, \qquad iD^{(1)} = 0\,, \qquad iD^{(2)} = -\,|\sqrt{2}\sigma^{(1)}|^2\,,\\
\sqrt{2}\sigma^{(0)} &= m, \qquad \sqrt{2}\sigma^{(1)} = -\,\frac{N}{4\pi}\,\frac{m}{|n^{(0)}|^2}\,,\\
|n^{(0)}|^2 &= 2\beta_{\rm ren}\,, \qquad n^{(1)} = -\,\frac{2\,m}{\overline{n}^{(0)}\,|n^{(0)}|^2}\,\frac{N}{32\pi^2}\sum_{i=1}^{N-1}\frac{1}{m-m^i}\,.
\end{aligned}
\tag{5.9}
$$

With masses from (2.2) we then obtain

$$
\sum_{i=1}^{N-1}\frac{1}{m-m^i} \;=\; \frac{N-1}{2m} \;=\; \frac{N}{2m} \;+\; O(1)\,. \tag{5.10}
$$

{higgseqpp}

Using this, we simplify the solution (5.9),

$$
\begin{aligned}
\sqrt{2}\sigma &= m\left(1 \;-\; \frac{u/2}{\ln m/\Lambda}\right) \;+\; \ldots\,,\\
iD &= -\,m^2\left(\frac{u/2}{\ln m/\Lambda}\right)^2 \;+\; \ldots\,,\\
n &= \sqrt{2\beta^{(0)}_{\rm ren}}\left(1 \;-\; \frac{u/8}{(\ln m/\Lambda)^2}\right) \;+\; \ldots\,.
\end{aligned}
\tag{5.11}
$$

This is in the Higgs phase, where

$$
2\beta^{(0)}_{\rm ren} \;=\; \frac{N}{2\pi}\ln\left(\frac{m}{\Lambda}\right)\,.
$$

Eqs. (5.11) agree with numerical calculations of the solution of the vacuum equations in the Higgs phase.

5.2 Strong coupling

{subsestrco}

Our starting point is the zeroth order in u solution (Sect. 4),

$$
n^{(0)} \;=\; 0\,, \quad iD^{(0)} \;=\; 0\,, \quad \sqrt{2}\sigma^{(0)} \;=\; \widetilde{\Lambda}\cdot e^{i\frac{2\pi l}{N}}\,, \tag{5.12}
$$

{zosol}

where

$$
\widetilde{\Lambda} \;=\; \sqrt[N]{\Lambda^N + m^N} \;=\; \Lambda\left(1 \;+\; O\left(e^{-N}\right)\right) \;\; \text{at} \;\; N \;\to\; \infty\,. \tag{5.13}
$$

{tilla}

At strong coupling n vanishes exactly, not only in the zeroth order in u. Omitting the details, the first order solution to the vacuum equations (5.3), (5.5) is given by (in conjunction with $n = 0$)

$$
\begin{aligned}
D^{(0)} &= 0\,, \qquad iD^{(1)} = \frac{\sqrt{2}\sigma^{(0)}}{\frac{1}{N}\sum_{i=1}^{N-1}\frac{1}{\sqrt{2}\overline{\sigma}^{(0)} - \overline{m}^i}}\,, \qquad (5.14)\\
\sqrt{2}\sigma^{(1)}\,\frac{1}{N}\sum_{i=1}^{N-1}\frac{1}{\sqrt{2}\sigma^{(0)} - m^i} \;+\; \text{h.c.} &= -\,\sqrt{2}\sigma^{(0)}\,\frac{\sum_{i=1}^{N-1}\frac{1}{|\sqrt{2}\sigma^{(0)} - m^i|^2}}{\sum_{i=1}^{N-1}\frac{1}{\sqrt{2}\overline{\sigma}^{(0)} - \overline{m}^i}}\,.
\end{aligned}
$$

We use the following relations to simplify the above formulas when masses are set as in (2.2):

$$
\begin{aligned}
\frac{1}{N}\sum_{k=0}^{N-1}\frac{1}{1 - \alpha\, e^{\frac{2\pi i k}{N}}} &= \frac{1}{1 - \alpha^N} \;\simeq\; 1\,,\\
\frac{1}{N}\sum_{k=0}^{N-1}\frac{1}{(1+\alpha^2) - 2\,\alpha\cos\frac{2\pi k}{N}} &= \frac{1}{1 - \alpha^2}\,\frac{1 + \alpha^N}{1 - \alpha^N} \;\simeq\; \frac{1}{1 - \alpha^2}\,. \qquad (5.15)
\end{aligned}
$$

{vspfor}

This enables us to present the results for $m \ll \Lambda$ in the following quite simple form:

$$
\begin{aligned}
n &= 0\,, \qquad iD = u\,\Lambda^2 + \ldots\,,\\
\sqrt{2}\sigma &= \Lambda\, e^{\frac{2\pi i l}{N}}\left(1 - \frac{u}{2}\,\frac{\Lambda^2}{\Lambda^2 - m^2}\right) + \ldots\,. \qquad (5.16)
\end{aligned}
$$

We complement these formulas for the strong coupling phase by finding the approximate solutions now as expansions in m^2 parameter, assuming m to be small. We obtain

$$
\begin{aligned}
\sqrt{2}\sigma &= e^{\frac{2\pi i l}{N}}\Lambda\left(e^{-u/2} - \frac{m^2}{\Lambda^2}\,\mathrm{sh}\,u/2\right) + \ldots\,,\\
iD &= \Lambda^2\left(1 - e^{-u}\right) + O\left(\frac{m^4}{\Lambda^4}\right)\,, \qquad (5.17)
\end{aligned}
$$

{smstr}

where u does not need to be (too) small anymore.

Just a brief look at the Higgs phase solution (5.11) and the strong coupling phase solutions (5.16) and (5.17) reveals, that these expansions blow up when one approaches $m \approx \Lambda$! While the exact solutions are expected to be finite for all m, our approximations cannot be trusted at $m = \Lambda$. This is the first sign that something is going on at these values of masses. As we will later see from the large-u solutions, as well as from the numerical solution of the vacuum equations, the theory experiences a double phase transition as m goes from the area below Λ towards $m \gg \Lambda$.

6 Heterotic CP$(N-1)$ model at large deformations

{hetdefld}

Now it is time to study equations (5.3) – (5.5) in the opposite limit of large values of the deformation parameter $u \gg 1$. We will see that our theory has three distinct phases separated by two phase transitions:

(i) Strong coupling phase with the broken Z_N symmetry at small m;

(ii) Coulomb/confining Z_N-symmetric phase at intermediate m (the coupling constant is strong in this phase as in the case (i));

(iii) Higgs phase at large m where the Z_N symmetry is again broken.

As previously, we assume that mass parameters are chosen in accordance with (2.2).

6.1 Strong coupling phase with broken Z_N

{scpwbz}

This phase occurs at very small masses, namely,

$$m \le \Lambda\, e^{-u/2}\,, \qquad u \gg 1\,. \tag{6.1}$$ {scphmass}

In this phase we have

$$|n| = 0, \qquad 2\beta_{\rm ren} = \frac{1}{4\pi}\sum_{i=1}^{N-1} \ln\,\frac{iD + |\sqrt{2}\sigma - m_i|^2}{\Lambda^2} = 0\,. \tag{6.2}$$ {scphn}

As we will see momentarily, σ is exponentially small in this phase. Masses are also small. Then the second equation in (6.2) gives

$$iD \;\approx\; \Lambda^2\,. \tag{6.3}$$ {scphD}

With this value of iD we can rewrite Eq. (5.5) in the form

$$\sum_{i=1}^{N-1}\left(\sqrt{2}\sigma - m_i\right)\ln\,\frac{\Lambda^2}{\left|\sqrt{2}\sigma - m_i\right|^2} = N\left(\sqrt{2}\sigma\right)u\,. \tag{6.4}$$ {scpheq3}

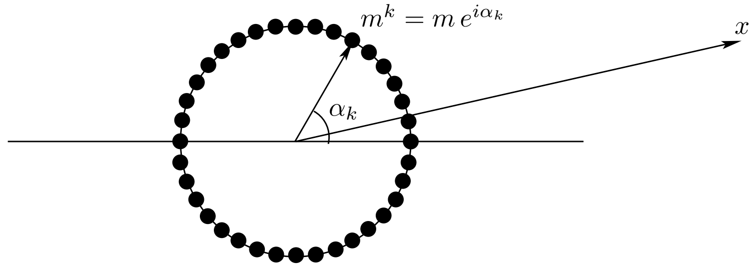

Figure 2: Electrostatic analog problem. The circle of radius m (see Eq. (2.2)) is homogeneously populated by "electric charges," namely, $\alpha_k = 2\pi\, k/N$ where $k = 0, 1, 2, ..., N-1$. We then must calculate the electrostatic potential at the point x. {circ}

The following trick is very convenient for solving this equation.

Let us consider an auxiliary problem from static electrodynamics in two dimensions. Assume we have N equal "electric charges" evenly distributed over the circle depicted in Fig. 2. In the limit of large N one can consider this distribution to be continuous (and homogenius). The task is to find the electrostatic potential at the point x on the plane.

It is not difficult to calculate the potential of a charged circle of radius m centered at the origin in two-dimensional electrostatics. Representing x by a complex number we get (in the large-N limit)

$$\frac{1}{N}\sum_{i=0}^{N-1} \ln\,|x - m_i|^2 = \begin{cases} \ln\,|x|^2, & |x| > m \\ \ln\, m^2, & |x| < m \end{cases}. \tag{6.5}$$ {chargedcircle

Now, to obtain the left-hand side of (6.4) we must integrate (6.5) over x and then substitute $x = \sqrt{2}\sigma$. In this way we arrive at

$$\frac{1}{N}\sum_{i=0}^{N-1}\left(\sqrt{2}\sigma - m_i\right)\ln\,\frac{\Lambda^2}{|\sqrt{2}\sigma - m_i|^2} = \begin{cases} \sqrt{2}\sigma\,\ln\,\frac{\Lambda^2}{|\sqrt{2}\,\sigma|^2} - \frac{m^2}{\sqrt{2}\,\bar{\sigma}}\,, & |\sqrt{2}\sigma| > m \\ \sqrt{2}\sigma\,\left(\ln\,\frac{\Lambda^2}{m^2} - 1\right), & |\sqrt{2}\sigma| < m \end{cases}. \tag{6.6}$$ {usefulformula

Outside the circle the potential is the same as that of the unit charge at the origin. Inside the circle the potential is constant.

Substituting Eq. (6.6) in (6.4) at $m < |\sqrt{2}\sigma|$ (i.e. outside the circle) we get

$$\sqrt{2}\langle\sigma\rangle = e^{\frac{2\pi i}{N}k}\,\Lambda\, e^{-u/2}, \qquad k = 0, ..., (N-1). \tag{6.7}$$ {scphsigma}

The vacuum value of σ is exponentially small at large u. The bound $m < |\sqrt{2}\sigma|$ translates into the condition (6.1) for m.

We see that we have N degenerate vacua in this phase. The chiral Z_{2N} symmetry is broken down to Z_2, the order parameter is $\langle\sigma\rangle$. Moreover, the absolute value of σ in these vacua does not depend on m. In fact, this solution coincides with the one obtained in [9] for $m = 0$. This phase is quite similar to the strong coupling phase of the $\mathcal{N} = (2,2)$ CP$(N-1)$ model, see (4.23). The difference is that the absolute value of σ depends now on u and becomes exponentially small in the limit $u \gg 1$.

The vacuum energy is positive (see Eq. (6.3)) – supersymmetry is broken. We will present a plot of the vacuum energy as a function of m below, in Sect. 6.2.

6.2 Coulomb/confining phase

{subscoulco}

Now we increase m above the bound (6.1). From (6.6) we see that the exponentially small solution to Eq. (6.4) no longer exist. The only solution is

$$\langle\sigma\rangle = 0\,. \tag{6.8}$$ {confsigma}

In addition, Eq. (6.2) implies

$$|n| = 0, \qquad iD = \Lambda^2 - m^2\,. \tag{6.9}$$ {confnD}

This solution describes a single Z_N symmetric vacuum. All other vacua are lifted and become quasivacua (metastable at large N). This phase is quite similar to the Coulomb/confining phase of nonsupersymmetric CP$(N-1)$ model without twisted masses [6]. The presence of small splittings between quasivacua produces a linear rising confining potential between kinks that interpolate between, say, the true vacuum and the lowest quasivacuum [32], see also the review [40]. As was already mentioned, this linear potential was interpreted, long time ago [31, 6], as the Coulomb interaction, see the next section for a more detailed discussion.

As soon as we have a phase with the broken Z_N symmetry at small m, and the Z_N-symmetric phase at intermediate m the theory experiences a phase transition that separates these phases. As a rule, one does not have phase transitions in supersymmetric theories. However, in the model at hand supersymmetry is badly broken (in fact, it is broken already at the classical level [16]); therefore, the emergence of a phase transition is not too surprising.

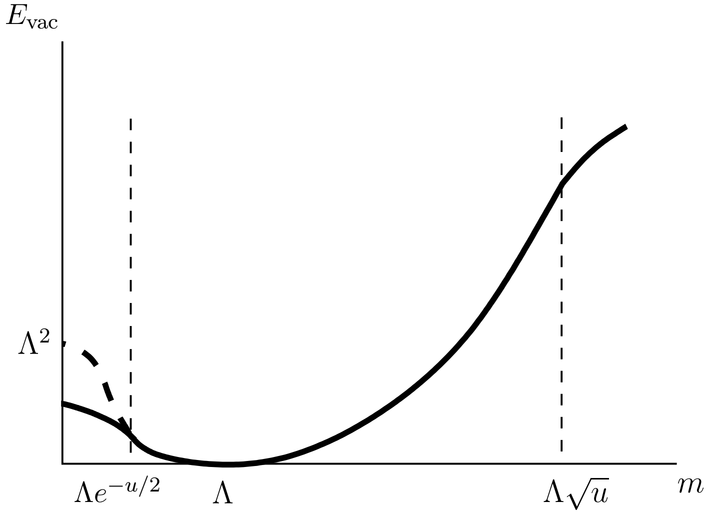

Figure 3: Vacuum energy density vs. m. {figvacE}

We can calculate the vacuum energy explicitly to see the degree of supersymmetry breaking. Substituting (6.8) and (6.9) in the effective potential (5.1) we get

$$E_{\rm vac}^{\rm Coulomb} \;=\; \frac{N}{4\pi}\left[\Lambda^2 \;-\; m^2 \;+\; m^2\ln\frac{m^2}{\Lambda^2}\right]. \tag{6.10}$$ {confEvac}

The behavior of the vacuum energy density $E_{\rm vac}$ vs. m is shown in Fig. 3.

$E_{\rm vac}$ is positive at generic values of m, as it should be in the case of the spontaneous breaking of supersymmetry. Observe, however, that the vacuum energy density *vanishes* at $m = \Lambda$. This is a signal of $\mathcal{N} = (0,2)$ supersymmetry restoration. To check that this is indeed the case – supersymmetry is dynamically restored at $m = \Lambda$ – we can compare the masses of the bosons n^i and their fermion superpartners ξ^i. From (3.6) we see that the difference of their masses reduces to iD. Now, Eq. (6.9) shows that iD vanishes exactly at $m = \Lambda$.

This is a remarkable phenomenon: while $\mathcal{N} = (0,2)$ supersymmetry is broken at the classical level at $m = \Lambda$, it gets restored at the quantum level at this particular point in the parameter space. This observation is implicit in [41] where a Veneziano–Yankielowicz-type (VY-type) superpotential [42] for $\mathcal{N} = (2,2)$ CP$(N-1)$ model (see [35, 36, 7]) was extrapolated to the $\mathcal{N} = (0,2)$ case.

We pause here to make an explanatory remark regarding Fig. 3 and Eq. (6.10). The plot of $E_{\rm vac}^{\rm Coulomb}$ is presented in this figure assuming the parameter m to be real (we follow this assumption in the bulk of the paper). In fact, m can be viewed as a complex parameter, the phase of m being interpreted as a θ angle. A straightforward examination shows that for the complex values of m Eq. (6.10) must be replaced by

$$E_{\rm vac}^{\rm Coulomb} \;=\; \frac{N}{4\pi}\left[\Lambda^2 \;-|m|^2 \;+|m|^2\ln\frac{|m|^2}{\Lambda^2}\right].$$

This means that the vacuum is supersymmetric (i.e. $E_{\rm vac}^{\rm Coulomb} = 0$) on the curve $|m|^2 = \Lambda^2$.

Now we turn our attention to what happens with σ at the strong/Coulomb phase transition point. More detail on that will be given in Section 6.4 with the help of numerical calculations, however, at large u the behaviour of σ can be analysed just by inspecting Eq. (6.4). To solve this equation we can evaluate the sum in it using Eq. (6.6). In place of the massive quantities σ, Λ and m it is convenient to introduce dimensionless variables

$$\mathcal{S} \;=\; \frac{\sqrt{2}\sigma}{\Lambda}\,, \qquad\qquad \mu \;=\; m/\Lambda\,. \tag{6.11}$$ {csdef}

Then Eq. (6.4) turns into

$$\mu^2 \;=\; -\,\mathcal{S}^2\left(u \;+\; \ln S^2\right). \tag{6.12}$$ {mu-s}

Instead of solving for $\mathcal{S}$ one can use (6.12) as a solution for m with respect to σ. Figure 4 illustrates the dependence (6.12). We now treat this graph as the dependence of σ on m. In particular, for zero masses, $\sqrt{2}\sigma = \Lambda e^{-u/2}$, as it should be. Following from right to left, as the mass grows, σ decreases, and at some point the curve bends downward. This happens at $\mu^2 = \mathcal{S}^2 = e^{-(1+u)}$, as can be seen from Eq. (6.12). At this point the derivative $\partial\sigma/\partial m$ becomes infinite, which is indicative of a phase transition. The behaviour of σ near this point is circle-like,

$$\sqrt{2}\sigma \;\simeq\; \Lambda\, e^{-\frac{1+u}{2}} \;+\; \frac{1}{2}\Lambda\sqrt{e^{-(u+1)} \;-\; m^2/\Lambda^2}\,. \tag{6.13}$$ {scirc}

Equations (6.12) and (6.13) together with Fig. 4 are approximate, but qualitatively they very closely demonstrate what happens to σ in reality. Namely, σ monotonically decreases with increase of mass, until it experiences a vertical bend, at which point it drops down to zero, the fact that cannot be seen from Eq. (6.12). Rather, this is seen in the Coulomb phase, and reproduced with numerical solution in Section 6.4. Despite the fact that apparently σ experiences a drop, the energy density does not (see Fig. 6), and therefore the phase transition is of the second order.

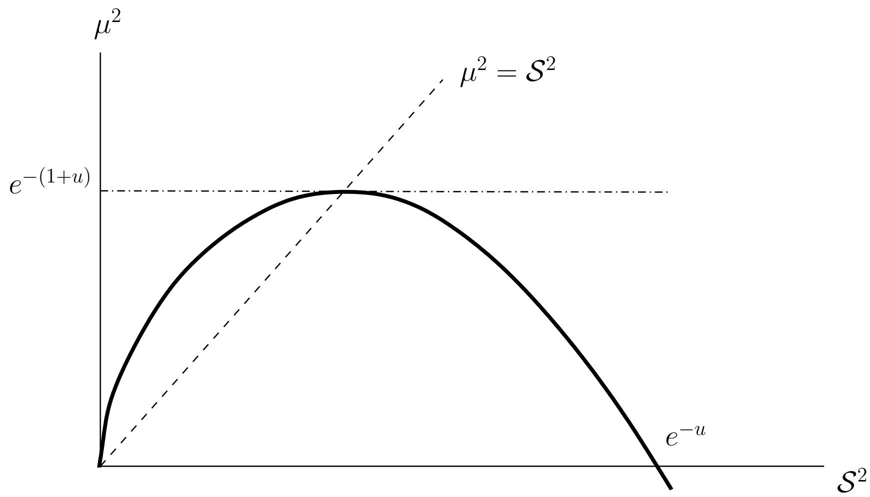

Figure 4: Dependence of m^2 versus σ^2. The rescaled variables as in text are $\mu = m/\Lambda$ and $\mathcal{S} = \sqrt{2}\sigma/\Lambda$. {fig:mus}

6.3 Higgs phase

{subshiggph}

The Higgs phase occurs in the model under consideration at large m. Below we will show that the model is in the Higgs phase at

$$m \;>\; \sqrt{u}\Lambda\,, \qquad \text{if } u \;\gg\; 1\,. \tag{6.14}$$ {Hphmass}

In this phase $|n|$ develops a VEV. From Eq. (5.4) we see that

$$iD \;=\; -\left|\sqrt{2}\sigma - m_0\right|^2. \tag{6.15}$$ {HphD}

To begin with, let us examine Eqs. (5.3) – (5.5) far to the right from the boundary (6.14), i.e. at $m \gg \sqrt{u}\Lambda$. In this regime we can drop the second logarithmic term in (5.5). This will be confirmed shortly. The first term is much larger because it is proportional to $\beta_{\rm ren}$ which is large in the quasiclassical region (see Eqs. (4.7) and (4.12)). Then Eq. (5.5) reduces to

$$(\sqrt{2}\sigma \;-\; m_0)\,2\beta_{\rm ren} \;+\; \frac{N}{4\pi}\,u\,\sqrt{2}\sigma \;\;=\;\; 0\,, \tag{6.16}$$

implying, at large u

$$\sqrt{2}\sigma \;\;=\;\; \left(\frac{8\pi}{N}\beta_{\rm ren}\right)\,\frac{m_0}{u}, \tag{6.17}$$ {Hphsigma}

where we take into account that $|\sigma| \ll m$, the fact justified *a posteriori*. Equation (6.17) applies to the $k=0$ vacuum. It is obvious that the solution for other $N-1$ vacua can be obtained from (6.17) by replacing $m_0 \to m_{i_0}$ where $i_0 = 1, ..., (N-1)$.

Thus, we have N degenerate vacua again. In each of them $|\sigma|$ is small ($\sim m/u$) but nonvanishing. The Z_{2N} chiral symmetry is again broken down to Z_2. Clearly, the Higgs phase is separated form the Coulomb/confining phase (where Z_{2N} is unbroken) by a phase transition.

To get the vacuum expectation value of n^0 we must analyze the logarithms in Eq. (5.3) and (5.5) with a better accuracy: σ in the numerators cannot be neglected. We must keep the terms linear in σ. Since the solution for σ is real, see (6.17), we can rewrite the logarithm in (5.3) as follows:

$$
\begin{aligned}
\ln\left(iD + \left|\sqrt{2}\sigma - m^i\right|^2\right) &= \ln\left(2\sqrt{2}\sigma\,\mathrm{Re}(m_0 - m_i)\right) \\
&= \ln\left[4\sqrt{2}\sigma\, m\, \sin^2\left(\frac{\alpha_k}{2}\right)\right], \qquad \alpha_k = \frac{2\pi\, k}{N}, \quad k = 1, ..., N-1\,. \qquad (6.18)
\end{aligned}
$$

where α_k is the phase of m_k, see Fig. 2. On the other hand, Eq. (4.14) can be presented in the form

$$
\frac{1}{4\pi}\sum_{k=1}^{N-1} \ln\left[4\, m^2\, \sin^2\left(\frac{\alpha_k}{2}\right)\right] = \frac{N}{4\pi}\, \ln m^2\,. \qquad (6.19) \quad \{\texttt{hujtwo}\}
$$

Thus, we conclude that

$$
\begin{aligned}
|n|^2 &= 2\beta_{\rm ren} = \frac{N}{4\pi}\, \ln\frac{\sqrt{2}\sigma\, m}{\Lambda^2} \\
&\sim \frac{N}{4\pi}\, \ln\frac{m^2}{u\,\Lambda^2} \qquad (6.20)
\end{aligned}
$$

in each of the N vacua in the Higgs phase. Here the last (rather rough) estimate follows from (6.17).

Our next task is to get an equation for $\beta_{\rm ren}$ (*en route*, we will relax the constraint $m \gg \sqrt{u}\Lambda$). To this end we must examine Eq. (5.5), including the logarithm into consideration. We will expand the numerator neglecting $O(\sigma^2)$ terms, while in the denominator we can set $\sigma = 0$ right away. Then the summation in (5.5) can be readily performed using the formula

$$
\frac{1}{N}\sum_{k=0}^{N-1} (m_0 - m_k) \ln\left[4\, m^2\, \sin^2\left(\frac{\alpha_k}{2}\right)\right] = m\left(\ln m^2 + 1\right), \qquad (6.21) \quad \{\texttt{hujfour}\}
$$

which follows, in turn, from Eq. (6.6). As a result, we arrive at

$$\sqrt{2}\sigma\, u = m\left(\frac{8\pi}{N}\,\beta_{\rm ren} + 1\right). \tag{6.22}$$ {hujfive}

The only approximation here is $u \gg 1$, plus, of course, Eq. (6.14). Combining Eqs. (6.22) and (6.20) we obtain the following relation for $\beta_{\rm ren}$:

po vidu NSVZ beta funktsiya!

$$\frac{8\pi}{N}\beta_{\rm ren} - \ln\left(\frac{8\pi}{N}\,\beta_{\rm ren} + 1\right) = \ln\frac{m^2}{u\Lambda^2}\,. \tag{6.23}$$ {Hphbetaeq}

Strictly speaking, Eq. (6.23) has two solutions at large m, deep inside the Higgs domain. The smaller solution corresponds to negative $\beta_{\rm ren}$. Since $|n|^2 = 2\beta_{\rm ren}$ is positively defined we keep only the larger one. At $m \gg \sqrt{u}\Lambda$ $\beta_{\rm ren}$ is large and is given (with the logarithmic accuracy) by the last estimate in Eq. (6.20). As we reduce m, at $m = \sqrt{u}\Lambda$, two solutions of (6.23) coalesce. At smaller m they become complex. Thus $m = \sqrt{u}\Lambda$ is indeed the phase transition point to the Coulomb/confining phase. At this point $\beta_{\rm ren} = 0$, which coincides with its value in the Coulomb/confining phase, see (6.9). Thus, $\beta_{\rm ren}$ is continuous at the point of the phase transition, while its derivative over m is discontinuous.

Calculating the vacuum energy in this phase we get

$$E_{\rm vac}^{\rm Higgs} \;=\; \frac{N}{4\pi}\left(m^2\ln\frac{m^2}{\Lambda^2} \;+\; \Lambda^2 \;-\; m^2 \;+\; O\left(\frac{m^2}{u^2}\ln\frac{m^2}{\Lambda^2}\right)\right). \tag{6.24}$$ {HiggsEvac}

The vacuum energy density in all phases is displayed in Fig. 3. We can see that the expression for energy (6.24) at large u concides with Eq. (6.10), which signifies that the phase transition is of the second order. In Section 6.4.2 we will analyse the transition in more detail.

6.4 Numerical evaluation

{subsnum}

6.4.1 Strong/Coulomb phase transition

To see what happens at the strong-to-Coulomb phase transition we solve the vacuum equations (5.3) – (5.5) numerically, with $n = 0$. Figure 5 shows the dependence of σ on m in the strong coupling phase. The main graph is compared to the small-m solution (5.17) and the large-u dependence (6.12) (still, solved numerically). One convinces that the large-u solution given by Eqs. (6.4) and (6.12) describes the true

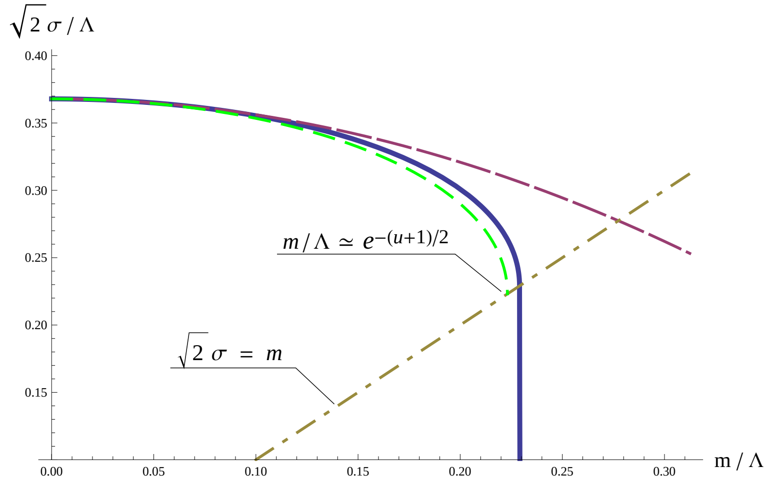

Figure 5: The dependence of σ on m. The solid curve shows the numerical solution of the equations (5.3) and (5.5) in the strong coupling phase ($|n| = 0$) for $u = 2$. The upper dashed curve shows the small-m limit (5.17). The lower dashed curve refers to the large-u solution obtained as a numerical evaluation of Eq. (6.12). The dash-dot $\sqrt{2}\sigma = m$ line is given for reference. To the right of the drop, σ is identically zero. {fig:numsigm}

solution really well. The curve for $\sqrt{2}\sigma$ monotonically decreases from the value $\Lambda e^{-u/2}$, until it meets the line $\sqrt{2}\sigma = m$, at which point σ turns down 90^o and drops vertically to zero. This is the strong-Coulomb phase transition point, which is approximately located at $m = \sqrt{2}\sigma \simeq e^{-(1+u)/2}$. To the right of the phase transition point σ is identically zero.

Figure 6 shows the energy density in the strong coupling and Coulomb/confining phase. The curve clearly displays that the energy does not experience any jumps at the phase transition point $m \simeq e^{-(1+u)/2}$. That is, the phase transition is at most of the second order. The curve, however, does apparently experience a break of incline at that point. To the right of the phase transition, the numerical curve exactly overlays the Coulomb phase energy density Eq. (6.10).

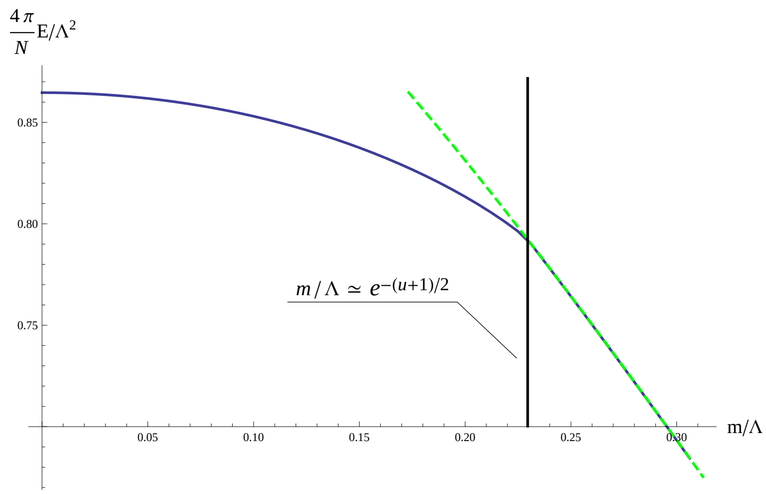

Figure 6: Energy density versus m in the strong coupling and Coulomb phase for $u = 2$. The dashed curve displays the energy density of the Coulomb/confining phase, Eq. (6.10). {fig:numEm}

6.4.2 Higgs phase

{subsnumhiggs}

We revoke the dimensionless variables $\mathcal{S}$ and μ introduced in Eq. (6.11). The system of vacuum equations (5.3)-(5.5), written in terms of these variables, becomes

$$\frac{1}{N}\sum_{k=1}^{N-1}\left\{ (\mu^k - \mu)\,\ln\left(|\mathcal{S}-\mu^k|^2 \,-\, |\mathcal{S}-\mu|^2\right) \;+\; (\mathcal{S}-\mu^k)\,\ln|\mathcal{S}-\mu^k|^2 \right\} \;+$$
$$+\;u\cdot\mathcal{S} \;=\; 0\,. \tag{6.25}$$

Here $\mu_k \;=\; \mu\, e^{i2\pi k/N}$. Evaluating the sum, one arrives at an algebraic equation

$$\left(1 \,+\, u \,+\, \ln\mu^2\right)\mathcal{S} \;=\; \mu\,\Big(1 \,+\, \ln(\mu\mathcal{S})\Big)\,. \tag{6.26}$$ {higgs_vaceq}

We can solve Eq. (6.26) numerically. The result is summarized in Fig. 7 which shows the dependence of $\mathcal{S}$ and $|n|^2$ on μ. The latter dependence can be inferred from Eq. (6.20),

$$\frac{4\pi}{N}\,|n|^2 \;=\; \ln\frac{\sqrt{2}\sigma\, m}{\Lambda^2} \;=\; \ln\mu\mathcal{S}\,. \tag{6.27}$$ {higgs_numn}

Vanishing of $|n|^2$ at certain $\mu_* = m_*/\Lambda$ delineates the Higgs and the Coulomb/confining

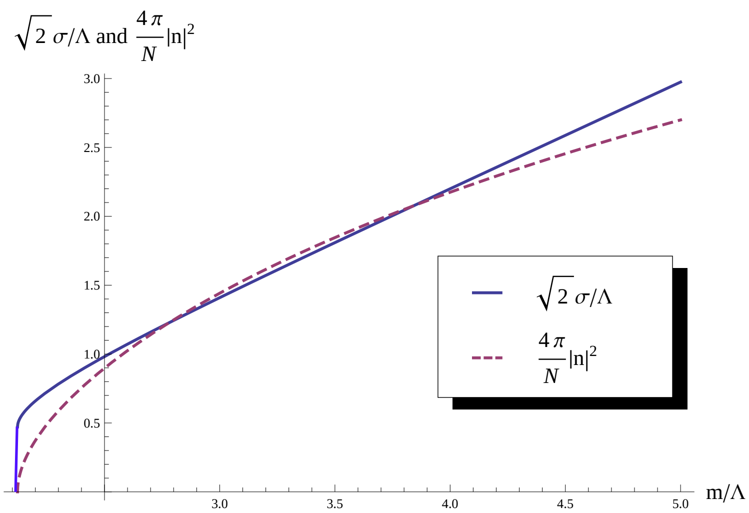

Figure 7: Dependence of σ and coupling constant $|n|^2$ on m in the Higgs phase for $u = 2$. The former dependence is obtained from Eq. (6.26), while the latter is then reconstructed from Eq. (6.27). {fig:numsignm}

phases. At that point, σ experiences a vertical slope. To the left of the phase transition point, $|n|^2$ becomes negative — the analysis of Eq. (6.26) is not valid. Strictly speaking, Higgs phase vacuum equations become invalid as soon as $|n|^2$ reaches zero, one needs to deal with the Coulomb phase.

Demanding the derivative $\partial \mathcal{S}/\partial \mu$ in Eq. (6.26) to be infinite, one arrives at an equation governing the phase boundary:

$$\mu_*^2 \;-\; \ln \mu_*^2 \;=\; 1 \;+\; u\,, \tag{6.28}$$ {higgs_phbnd}

together with a useful relation

$$\mu_* \;=\; \frac{1}{\mathcal{S}_*}\,. \tag{6.29}$$ {higgs_cs}

This first equation has two solutions, see Fig. 8. The larger solution determines the Coulomb/confining-Higgs phase boundary. Its asymptotics at large u, $\mu^2 \gg \ln \mu^2$, is given by (*cf.* (6.14))

$$\mu_{*H} \;\simeq\; \sqrt{1+u}\,. \tag{6.30}$$

We can now use Eq. (6.26) to derive a vacuum equation for the coupling constant $2\beta_{\rm ren} \;=\; |n|^2$. From Eq. (6.27) one can see that the coupling constant vanishes at

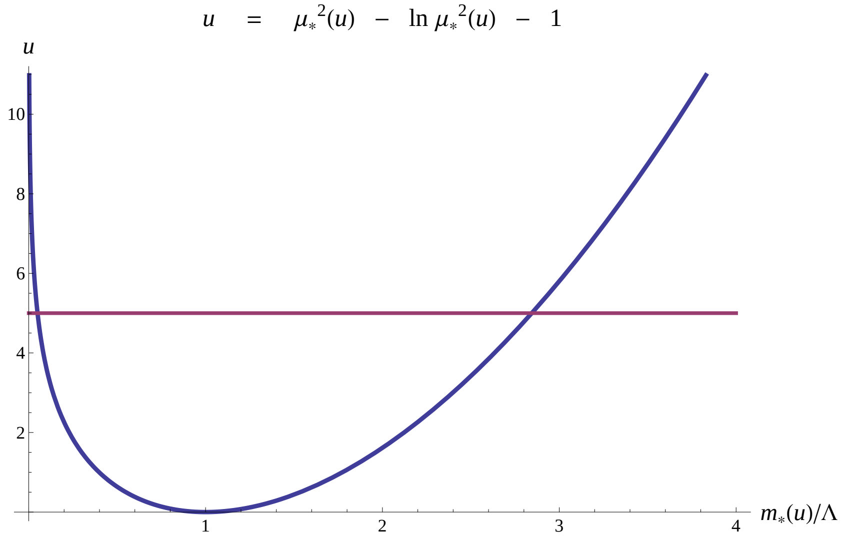

Figure 8: Graph displaying the dependence of the Coulomb/confining-Higgs phase transition location μ_{*H} (see Eq. (6.28)) on deformation parameter u. For each non-zero value of u there are two solutions, the greater of which represents μ_{*H}. {fig:higgsbord

the point (6.29), confirming once again that a phase transition takes place. Plugging Eq. (6.27) into Eq. (6.26), one arrives at

$$\frac{8\pi}{N}\beta_{\rm ren} \;-\; \ln\left(\frac{8\pi}{N}\beta_{\rm ren} \,+\, 1\right) \;=\; \ln\,\frac{\mu^2}{1 \,+\, u \,+\, \ln\mu^2}\,. \qquad (6.31)$$ {higgs_beta}

At large u, one recovers Eq. (6.23), and therefore the same analysis of the solutions sketched after Eq. (6.23) applies to Eq. (6.31) — there are two solutions, only one of which is physical. This is the solution shown in Fig. 7.

We now turn to the question of the energy density in the Higgs phase. In terms of the dimensionless variables, expression (5.1) can be brought into the form

$$\begin{aligned}\frac{4\pi}{N}\,\frac{E_{\rm vac}^{\rm Higgs}}{\Lambda^2} \;&=\; \frac{1}{N}\sum\left|\mathcal{S} \,-\, \mu^k\right|^2\,\ln\left|\mathcal{S} \,-\, \mu^k\right|^2 \\ &-\; \frac{1}{N}\sum\mathcal{S}\left\{\,(\mu \,-\, \mu^k) \,+\, (\mu \,-\, \overline{\mu}^k)\,\right\}\,\ln\,\Big(2\mu\mathcal{S}\,(1 \,-\, \cos\alpha_k)\Big) \qquad (6.32) \\ &-\; (\mathcal{S} \,-\, \mu)^2 \,+\, u\,\mathcal{S}^2\,.\end{aligned}$$

Evaluating the sums for large N, and using vacuum equation (6.26), one comes to

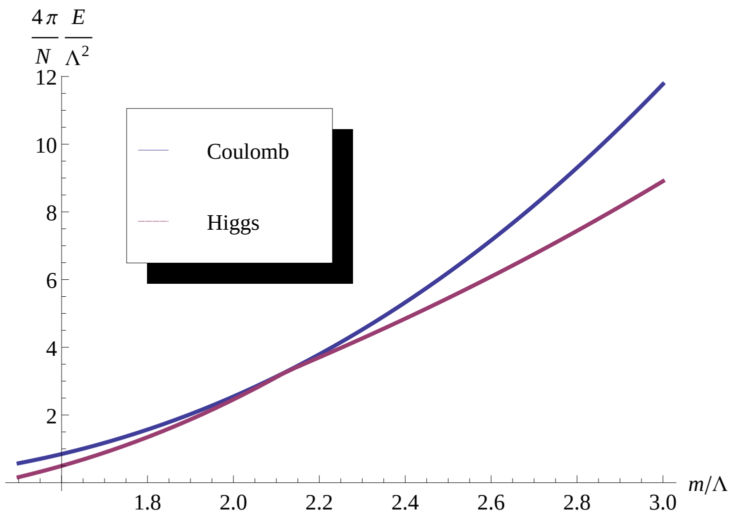

Figure 9: Energy densities for Coulomb/confining and Higgs phases for $u = 2$. Conjunction occurs exactly at $\mu = \mu_*$. {fig:higgsener

the expression

$$\frac{4\pi}{N}\,\frac{E_{\rm vac}^{\rm Higgs}}{\Lambda^2} \;=\; (\mu^2 \,-\, \mathcal{S}^2)\,\ln\mu^2 \;-\; (\mu \,-\, \mathcal{S})^2 \;-\; u\,\mathcal{S}^2\,. \tag{6.33}$$ {higgs_energy}

It is now straightforward to see that this energy density matches the Coulomb phase energy density at the phase transition point μ_*. One eliminates $\mathcal{S}$ from Eq. (6.33) using relation (6.29), and resolves the logarithm $\ln\mu_*^2$ via (6.28). The result is

$$\frac{4\pi}{N}\,\frac{E_{\rm vac}^{\rm Coulomb}(\mu_*)}{\Lambda^2} \;=\; \frac{4\pi}{N}\,\frac{E_{\rm vac}^{\rm Higgs}(\mu_*)}{\Lambda^2} \;=\; \mu_*^4 \;-\; (2\,+\,u)\mu_*^2 \;+\; 1\,. \tag{6.34}$$

Figure 9 illustrates this.

7 More on the Coulomb/confining phase

{moccp}

As was shown above (Sects. 6.1 and 6.3), both in the strong coupling and Higgs phases the Z_{2N} symmetry is spontaneously broken down to Z_2 while in the Coulomb/confining phase this symmetry remains unbroken (see Sect. 6.2). In the former two phases we have N degenerate vacua, while in the later phase the theory has

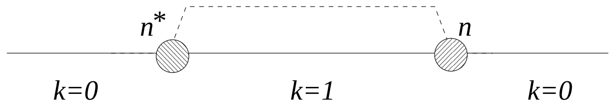

Figure 10: Linear confinement of the kink-antikink pair. The solid straight line represents the ground state. The dashed line shows the vacuum energy density of the lowest quasivacuum. {fig:conf}

a single vacuum. Just like in non-supersymmetric CP$(N-1)$ model [32] the vacua split, and $N-1$ would-be vacua become quasivacua, see [40] for a review. The vacuum splitting can be understood as a manifestation of the Coulomb/confining linear potential between the kinks [31, 6] that interpolate between the true vacum, and say, the lowest quasivacuum. The force is attractive in the kink-antikink pairs leading to formation of weakly coupled bound states (weak coupling is the manifestation of the $1/N$ suppression of the confining potential, see below). The charged kinks (i.e. the n quanta) are eliminated from the spectrum. This is the reason why the n fields were called "quarks" by Witten [6]. The spectrum of the theory consists of $\bar{n}n$-"mesons." The picture of confinement of n's is shown in Fig. 10.

This Coulomb/confinig linear potential can appear only provided that thephoton remains massless, which is certainly true in the pure bosonic CP$(N-1)$ model. As was pointed out by Witten [6], in the supersymmetric $\mathcal{N}=(2,2)$ CP$(N-1)$ model the photon acquires a nonvanishing mass due to a chiral coupling to fermions. In particular, with vanishing twisted masses the photon is massive in both $\mathcal{N}=(2,2)$ [6] and $\mathcal{N}=(0,2)$ [9]. Below we will calculate the photon mass in the model (3.6) and show that it does vanish in the Coulomb/confinig phase considered in Sect. 6.2. This is in accord with the unbroken Z_N symmetry detected in this phase. It guarantees self-consistency of the picture.

To this end we start from the one-loop effective action which is a function of fields from the gauge supermultiplet (A_k, σ and λ). After integration over n^i and ξ^i in the strong coupling or Coulomb/confining phases (at $n=0$) the bosonic part of this effective action takes the form [9]

$$S_{\text{eff}} = \int d^2x \left\{ \frac{1}{4e_\gamma^2} F_{\mu\nu}^2 + \frac{1}{e_\sigma^2} |\partial_\mu \sigma|^2 + V(\sigma) + \sqrt{2}(\bar{b}\delta\sigma - b\delta\bar{\sigma})\, F^* \right\}, \tag{7.1}$$ {effaction}

where F^* is the dual gauge field strength,

$$F^* = \frac{1}{2}\varepsilon_{\mu\nu} F_{\mu\nu}\,, \tag{7.2}$$

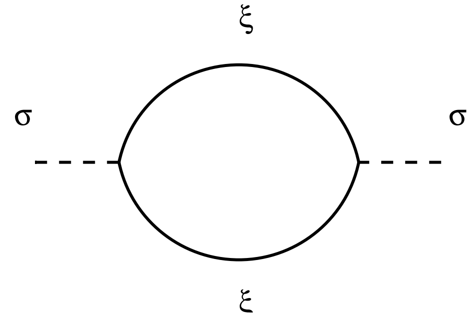

Figure 11: The wave function renormalization for σ. {fig:esigma}

while $V(\sigma)$ can be obtained from (5.1) by eliminating D by virtue of its equation of motion (5.3). This was done in the closed form for $m = 0$ in [9], see also Sec. 8.2. Here e_γ^2 and e_σ^2 and b are the coupling constants which determine the wave function renormalization for the photon, σ and sigma-photon mixing respectively ($\delta\sigma$ is the quantum fluctuation of the field σ around its VEV). These couplings are given by one-loop graphs which we will consider below. In the $m = 0$ case these graphs were calculated in [9].

The wave-function renormalizations of the fields from the gauge supermultiplet are, in principle, momentum-dependent. We calculate them below in the low-energy limit assuming the external momenta to be small. The wave-function renormalization for σ is given by the graph in Fig. 11. A straightforward calculation yields

$$\frac{1}{e_\sigma^2} = \frac{1}{4\pi} \sum_{i=0}^{N-1} \frac{1}{|\sqrt{2}\sigma + m_i|^2} \,. \tag{7.3}$$ {esigma}

The above graph is given by the integral over the momenta of the ξ fermions propagating in the loop. The integral is saturated at momenta of the order of the ξ mass $|\sqrt{2}\sigma + m_i|$.

The wave function renormalization for the gauge field was calculated by Witten in [6] for zero masses. The generalization to the case of nonzero masses takes the form

$$\frac{1}{e_\gamma^2} = \frac{1}{4\pi} \sum_{i=0}^{N-1} \left[\frac{1}{3}\, \frac{1}{iD + |\sqrt{2}\sigma + m_i|^2} + \frac{2}{3}\, \frac{1}{|\sqrt{2}\sigma + m_i|^2} \right] . \tag{7.4}$$ {egamma}

The right-hand side in Eq. (7.4) is given by two graphs in Fig. 12, with bosons n^i and fermions ξ^i in the loops. The first term in (7.4) comes from bosons while the

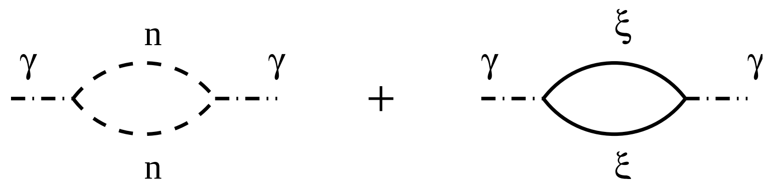

Figure 12: The wave function renormalization for the gauge field. {fig:photon}

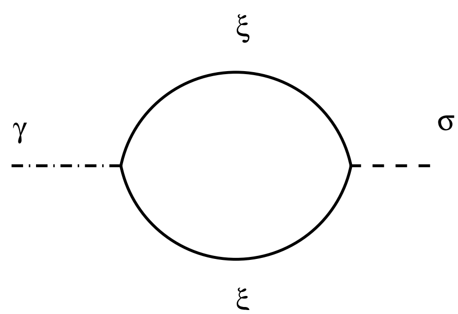

Figure 13: Photon–σ mixing. {fig:photsig}

second one is due to fermions.

We see that although both A_μ and σ were introduced in (3.6) as auxiliary fields (in the limit $e_0^2 \to \infty$) after renormalization both couplings, e_γ^2 and e_σ^2, become finite. This makes these fields physical [6].

The $(\delta\sigma)\, F^*$ mixing was calculated by Witten in [6] in the massless $\mathcal{N} = (2,2)$ theory. This mixing is due to the chiral fermion couplings which can make the photon massive in two dimensions. In the effective action this term is represented by the mixing of the gauge field with the fluctuation of σ. It is given by the graph in Fig. 13. Direct calculation gives for the coupling b

$$b = \frac{1}{2\pi} \sum_{i=0}^{N-1} \frac{\sqrt{2}\sigma + m_i}{|\sqrt{2}\sigma + m_i|^2}\,. \tag{7.5}$$ {gamma}

With the given value of b the photon mass is

$$m_{\rm ph} = e_\sigma e_\gamma |b|\,. \tag{7.6}$$ {phmass}

We see that in the strong coupling phase, with $\sigma \neq 0$, the photon mass does not vanish. However, in the Coulomb/confining phase in which the VEV of σ vanishes,

$b = 0$, and the photon is massless, as expected,

$$m_{\rm ph} = 0\,. \tag{7.7}$$

The above circumstance can be readily understood from symmetry considerations too. The field $\delta\sigma$ transforms nontrivially under Z_{2N} symmetry; therefore, the nonvanishing value of b in (7.1) breaks this symmetry. However, Z_{2N} symmetry is restored in the Coulomb/confining phase. Hence, b should be zero in this phase, and it is.

If the photon mass vanishes, generating a linear potential, it is not difficult to find the splittings between the quasivacua; they are determined by the value of e_γ^2. From (7.4) we estimate

$$E_{k+1} - E_k \sim e_\gamma^2 = \frac{4\pi}{N}\left\{\begin{array}{ll} 3\Lambda^2, & m \ll \Lambda \\ \frac{3}{2}m^2, & m \gg \Lambda \end{array}\right. . \tag{7.8}$$ {split}

We see that the splitting is a $1/N$ effect. This was expected.

8 Related issues

{relais}

In this section we address a few questions which are not necessarily confined to the large-N limit. Rather, we focus on some general features of our results, with the intention to provide some useful clarifications/illustrations.

8.1 Remarks on the mirror representation for the heterotic CP(1) in the limit of small deformation[8]

razobratsya s normirovkoy γ

It's $\gamma = \frac{\tilde{\gamma}}{\sqrt{2}g_0}$

In this section we will set all twisted masses to zero. The geometric representatition of the heterotic $\mathcal{N} = (0,2)$ CP(1) model is as follows [13]:

$$L_{\rm heterotic} = \zeta_R^\dagger\, i\partial_L\, \zeta_R + \left[\gamma_{(M)}\,\zeta_R\, R\left(i\,\partial_L\phi^\dagger\right)\psi_R + {\rm H.c.}\right] \tag{8.1}$$

$$-g_0^2|\gamma_{(M)}|^2\left(\zeta_R^\dagger\,\zeta_R\right)\left(R\,\psi_L^\dagger\psi_L\right) + G\left\{\partial_\mu\phi^\dagger\,\partial^\mu\phi + \frac{i}{2}\big(\psi_L^\dagger\,\overleftrightarrow{\partial_R}\,\psi_L + \psi_R^\dagger\,\overleftrightarrow{\partial_L}\,\psi_R\big)\right.$$

$$\left. -\frac{i}{\chi}\left[\psi_L^\dagger\psi_L\big(\phi^\dagger\,\overleftrightarrow{\partial_R}\,\phi\big) + \psi_R^\dagger\,\psi_R\big(\phi^\dagger\,\overleftrightarrow{\partial_L}\,\phi\big)\right] - \frac{2(1-g_0^2|\gamma_{(M)}|^2)}{\chi^2}\,\psi_L^\dagger\,\psi_L\,\psi_R^\dagger\,\psi_R\right\}, \tag{8.2}$$

[8]Subscript $_{(M)}$ of $\gamma_{(M)}$ will indicate that we are working in Minkowski space in this section.

where the field ζ_R appearing in the first line is the spinor field on C, a necessary ingredient of the $\mathcal{N}=(0,2)$ deformation [12]. Here G is the metric, R is the Ricci tensor and $\chi \equiv 1+\phi\,\phi^\dagger$,

$$G = \frac{2}{g_0^2\,\chi^2}\,, \qquad R = \frac{2}{\chi^2}\,, \tag{8.3}$$ {fsmetrone}

cf. Eq. (3.14).

We assume the deformation parameter γ to be small (it is dimensionless) and work to the leading order in γ, neglecting $O(\gamma^2)$ effects in the superpotential. The kinetic terms of the CP(1) fields ϕ and ψ contain $\frac{1}{g^2}$ in the normalization while γ in the first line is defined in conjunction with the Ricci tensor, so that there is no $\frac{1}{g^2}$ in front of this term. This convention is important for what follows.

Now, let us remember that the undeformed $\mathcal{N}=(2,2)$ CP(1) model has a mirror representation [10, 11], a Wess–Zumino model with the superpotential

$$\mathcal{W}_{\rm mirror} = \Lambda\left(Y + \frac{1}{Y}\right)\,, \tag{8.4}$$ {AABone}

where Λ is the dynamical scale of the CP(1) model. The question is: "what is the mirror representation of the deformed model (8.2), to the leading order in γ?"

Surprisingly, this question has a very simple answer. To find the answer let us observe that the term of the first order in γ in (8.2) is nothing but the superconformal anomaly in the unperturbed $\mathcal{N}=(2,2)$ model (it is sufficient to consider this anomaly in the unperturbed model since we are after the leading term in γ in the mirror representation). More exactly, in the $\mathcal{N}=(2,2)$ CP(1) model [43, 44]

$$\gamma_\mu J^\mu_\alpha = -\frac{\sqrt{2}}{2\pi}\,R\,\left(\partial_\nu\phi^\dagger\right)\left(\gamma^\nu\psi\right)_\alpha\,, \tag{8.5}$$ {six}

where J^μ_α is the supercurrent. In what follows, for simplicity, numerical factors like 2 or π will be omitted. Equation (8.5) implies that the $O(\gamma)$ deformation term in (8.2) can be written as

$$\Delta\mathcal{L} = \gamma_{(M)}\zeta_R\left(\gamma_\mu\,J^\mu\right)_L \tag{8.6}$$ {seven}

Since (8.5) has a geometric meaning we can readily rewrite this term in the mirror representationin terms of $\mathcal{W}_{\rm mirror}$. Indeed, in the generalized $\mathcal{N}=(0,2)$ Wess–Zumino model the term proportional to $\gamma_{(M)}\zeta_R$ is [45]

$$\Delta\mathcal{L} = \zeta_R\psi_L\,\mathcal{H}' \tag{8.7}$$ {arione}

where $\mathcal{H}$ is the h-superpotential. Moreover,

$$(\gamma_\mu J^\mu)_L = \mathcal{W}' \, (\psi_Y)_L + O(\gamma) \,. \tag{8.8}$$ {aritwo}

Substituting Eq. (8.8) in (8.6) and comparing with (8.7) we conclude that

$$\mathcal{H} = \gamma_{(M)} \, \mathcal{W}_{\rm mirror} \,. \tag{8.9}$$ {arithree}

In principle, one could have added a constant on the right-hand side, but this would ruin the Z_2 symmetry inherent to the $\mathcal{N} = (0,2)$ CP(1) Lagrangian. The constant must be set at zero. The scalar potential of the $\mathcal{N} = (0,2)$ mirror Wess–Zumino model is [45]

$$V = |\mathcal{W}'|^2 + |\mathcal{H}|^2 = |\mathcal{W}'_{\rm mirror}|^2 + |\gamma_{(M)}|^2 \, |\mathcal{W}_{\rm mirror}|^2 \,. \tag{8.10}$$ {arifour}

where $\mathcal{W}_{\rm mirror}$ is given in (8.4). The second equality here is valid in the small-deformation limit.

At $\gamma \neq 0$ it is obvious that $V > 0$ and supersymmtry is broken. The Z_2 symmetry apparent in (8.10) is spontaneously broken too: we have two degenerate vacua.

8.2 Different effective Lagrangians

{defefl}

In this section we will comment on the relation between the effective Lagrangian derived in Sect. 5 from the large-N expansion and the Veneziano–Yankielowicz effective Lagrangian based on anomalies and supersymmetry. For simplicity we will set $m_i = 0$ in this section. Generalization to $m_i \neq 0$ is straightforward. We assume the heterotic deformation to be small, $u \ll 1$.

The $1/N$ expansion allows one to derive an honest-to-god effective Lagrangian for the field σ, valid both in its kinetic and potential parts. The leading order in $1/N$ in the potential part is determined by the diagram depicted in Fig. 11 which gives at zero twisted masses

$$\mathcal{L}_{\rm kin} = \frac{N}{4\pi} \, \frac{1}{2|\sigma|^2} \, |\, \partial_\mu \sigma \,|^2 \,, \tag{8.11}$$ {fdop1}

see (7.3). The virtual ξ momenta saturating the loop integral are of the order of the ξ mass $\sqrt{2}|\sigma|$. Up to a numerical coefficient this result is obvious since the field σ has mass-dimension 1.

The potential part following from calculations in Sect. 5 is

$$\mathcal{L}_{\rm pot} = \frac{N}{4\pi}\left\{\Lambda^2 + 2|\sigma|^2\left[\ln\frac{2|\sigma|^2}{\Lambda^2} - 1 + u\right]\right\}. \qquad (8.12)$$ {fdop2}

All corrections to (8.11) and (8.12) are suppressed by powers of $1/N$. For what follows it is convenient to introduce a dimensionless variable

$$\mathcal{S} = \frac{\sqrt{2}\sigma}{\Lambda}. \qquad (8.13)$$

Then the large-N effective Lagrangian of the σ field takes the form NEPRAVILNO?

$$\mathcal{L}_{\rm eff} = \frac{N}{4\pi}\left\{\frac{1}{2\,|\mathcal{S}|^2}\,|\partial_\mu\mathcal{S}\,|^2 + \Lambda^2\left[1 + |\mathcal{S}|^2\left(\ln|\mathcal{S}|^2 - 1 + u\right)\right]\right\}. \qquad (8.14)$$ {fdop3}

On the other hand, the Veneziano–Yankielowicz method [42] produces an effective Lagrangian in the Pickwick sense. It realizes, in a superpotential, the anomalous Ward identities of the underlying theory and other symmetries, such as supersymmetry, and gives no information on the kinetic part. In the CP$(N-1)$ models the Veneziano–Yankielowicz superpotential $\mathcal{W}_{\rm VY} = \Sigma\ln\Sigma$ (for twisted superfields) was obtained in [35, 36, 7]. In terms of the scalar potential for the σ field the Veneziano–Yankielowicz construction has the form

$$V_{VY} = \frac{e_\sigma^2}{2}\left|\frac{N}{2\pi}\ln\frac{\sqrt{2}\,\sigma}{\Lambda}\right|^2 + \frac{N}{4\pi}u\,2|\sigma|^2. \qquad (8.15)$$ {fdop4}

The kinetic term (that's where e_σ^2 comes from) was not determined; however, we can take it in the form obtained in the large-N expansion, see (8.11), since it is scale invariant and, hence, does not violate Ward identities.

Combining

$$e_\sigma^2 = \frac{4\pi}{N}\,2|\sigma|^2 \qquad (8.16)$$ {fdop5}

(see [9]) with (8.15) we arrive at

$$\begin{aligned}\mathcal{L}_{VY} &= \frac{N}{4\pi}\,\frac{1}{2|\sigma|^2}\,|\,\partial_\mu\sigma\,|^2 + \frac{N}{4\pi}\left\{2\cdot 2|\sigma|^2\left|\ln\frac{\sqrt{2}\,\sigma}{\Lambda}\right|^2 + 2|\sigma|^2\,u\right\} \\ &= \frac{N}{4\pi}\left\{\frac{1}{2\,|\mathcal{S}|^2}\,|\partial_\mu\mathcal{S}\,|^2 + \Lambda^2\left[2|\mathcal{S}|^2\,\left|\ln\mathcal{S}\right|^2 + |\mathcal{S}|^2\,u\right]\right\}. \qquad (8.17)\end{aligned}$$

It is obvious that the potential in (8.14) is drastically different from that in (8.17). For instance, (8.14) contains a single log, while (8.17) has the square of this logarithm. We will comment on the difference and the reasons for its appearence [?] later. Now, let us have a closer look at the minima of (8.14) and (8.17). The variable $\mathcal{S}$ is complex, and there iare N solutions which differ by the phase,

$$\mathcal{S}_* = |\mathcal{S}_*| \exp\left(\frac{2\pi k}{N}\right), \qquad k = 0, 1, ..., N-1\,, \tag{8.18}$$

N equivalent vacua. This feature is obvious, and we will omit the phase setting $k=0$. Thus, we focus on a real solution. The minimum of (8.14) lies at

$$\mathcal{S}_* = e^{-u/2} \tag{8.19}$$ {wdop1}

while the corresponding value of $V_{\rm eff}$ is

$$V_{\rm eff}(\mathcal{S}_*) = \frac{N}{4\pi}\,\Lambda^2\left(1 - e^{-u}\right). \tag{8.20}$$ {wdop2}

At the same time, the minimum of (8.17) lies at

$$\mathcal{S}_* = \exp\left(-\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{u}{2}}\right) = e^{-u/2}\left(1 - \frac{u^2}{4} + ...\right) \tag{8.21}$$ {wdop3}

implying that

$$\begin{aligned} V_{\rm VY}(\mathcal{S}_*) &= \frac{N}{4\pi}\,\Lambda^2\,\left(1 - \sqrt{1-2u}\right)\exp\left(-1 + \sqrt{1-2u}\right) \\ &= \frac{N}{4\pi}\,\Lambda^2\left(1 - e^{-u}\right)\left(1 - \frac{u^2}{6} + ...\right). \end{aligned} \tag{8.22}$$

The σ masses are

$$m_\sigma^2 = \begin{cases} 4\Lambda^2\, e^{-u}\,(1-u)\,, \\ 4\Lambda^2\, e^{-u}\,(1-u)\,(1 - u^2 + ...)\,, \end{cases} \tag{8.23}$$ {wdop4}

for (8.14) and (8.17) respectively. The positions of the minima, the σ masses as well as the vacuum energy densities in these two cases differ by $O(u^2)$ in relative units. They coincide in the leading and next-to-leading orders in u, however.

There are two questions to be discussed: (i) why the effective Lagrangians (8.14) and (8.17), being essentially different, predict identical vacuum parameters in the leading and next-to-leading order in u; and (ii) why the parameters extracted from the $1/N$ and Veneziano–Yankielowicz Lagrangians diverge from each other at $O(u^2)$ and higher orders.

The answer to the first question can be found in [35]. While the $1/N$ Lagrangian is defined unambiguously, the Veneziano–Yankielowicz method determines only the superpotential part of the action. The kinetic part remains ambiguous. We got used to the fact that variations of the kinetic part affect only terms with derivatives, which are totally irrelevant for the potential part. This is not the case in supersymmtry. The correct statement is that variations of the kinetic part term, in addition to derivative terms, contains terms with $F\bar{F}$, which vanish in the vacuum ($F = 0$) but alter the form of the potential outside the vacuum points (minima of the potential). The only requirement to the kinetic term is that it should obey all Ward identities (including anomalous) of the underlying microscopic theory. For instance, in the case at hand, the simplest choice $\ln\bar{\Sigma}\,\ln\Sigma$ does the job. However,

$$\ln\bar{\Sigma}\,\ln\Sigma\left[1+\frac{(\bar{D}^2\ln\bar{\Sigma})\,(D^2\ln\Sigma)}{\bar{\Sigma}\Sigma}\right]$$

does the job as well. In this latter case there is an additional factor

$$\left[1+\bar{F}F/(\bar{\sigma}^2\sigma^2)+...\right]$$

which reduces to 1 in the points where $F = 0$ and changes the expression for F (and, hence, the scalar potential) outside minima (i.e. at $F \neq 0$).

The answer to the second question is even more evident. The Veneziano–Yankielowicz Lagrangian (8.17) reflects the Ward identites of the unperturbed CP($N-1$) model. That's the reason why the predictions following from this Lagrangian fail at the level $O(u^2)$, but are valid at the level $O(u)$. We remind the reader that it was shown in [13] that the vacuum energy density at the level $O(u)$ is determined by the bifermion condensate in the conventional (unperturbed) CP($N-1$) model.

One last remark is in order here. The kinetic term (8.11) is not canonic and singular at $\sigma = 0$, implying that this point should be analyzed separately. One can readily cast (8.11) in the canonic form by a change of variables. Upon this transformation $\sigma \to \tilde{\sigma} = 2\ln\sqrt{2}\sigma/\Lambda$ (assuming for simplicity σ to be real and positivde), the transformed potential (8.12) develops an extremum at $\sigma = 0$ (i.e. $\tilde{\sigma} \to -\infty$). This extremum is maximum rather than minimum. Indeed, at $u = 0$

$$\tilde{\mathcal{L}}_{\rm pot} = \frac{N\Lambda^2}{4\pi^2}\,(\tilde{\sigma}-1)\,e^{\tilde{\sigma}} + \text{const.} \qquad (8.24)$$ {mdop7}

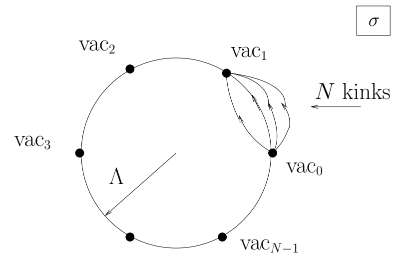

Figure 14: The kinks are represented by the n fields at $|m_i| < \Lambda$. {nkin}

It is curious to note that (8.24) exactly coincides with the (two-dimensional) dilaton effective Lagrangian derived in [48] on the basis of the most general (anomalous) scale Ward identities.

8.3 When the n fields can be considered as solitons {wtnfcb}

Long ago Witten showed [6] that the n fields in fact describe kinks interpolating between two neighboring vacua picked up from the set of N degenerate supersymmetric vacua of the $\mathcal{N} = (2,2)$ sigma model. The above statement refers to the model with no twisted masses. (See Fig. 14). Here we will discuss the physical status of these states, and the BPS spectrum at large, as the twisted mass parameter evolves towards large values, $|m|/\Lambda \gg 1$. N will be assumed to be large so that we can use the large-N solutions.

In the undeformed $\mathcal{N} = (2,2)$ theory two distinct regimes are known to exist. At large m the theory is in the Higgs regime, while at small m it is in the strong coupling regime. There is no phase transition between the two regimes, since the global Z_N symmetry of the model is spontanepously broken in both. An apparent dsicontinuity of, say, the derivative of the vacuum expectation of σ at $m = \Lambda$ (see Fig. 1) is an artifact of the $N \to \infty$ limit. However, the BPS spectrum experiences a drastic change in passing from small to large m. In particular, at $|m| < \Lambda$, the masses M_i of the n^i fields in the vacuum $|0\rangle$ are

$$M_i^2 = 2\left|\Lambda - \frac{m_i}{\sqrt{2}}\right|^2, \qquad i = 0, 1, ..., N-1\,. \tag{8.25}$$ {nkma}

This follows from Eq. (3.6). The mass degeneracy of the kink N-plet is gone since the twisted mass terms (2.2) break the SU(N) symmetry of the model leaving intact only $U(1)^{N-1}$ and Z_N-symmetries.

On the other hand, at large m, in the Higgs regime at weak coupling, the n fields no longer describe solitons; rather they represent elementary excitations. In each of the N vacua there are $2(N-1)$ real elementary excitations. Say, consider the vacuum in which n^0 develops a VEV. The phase of n^0 is eaten by the Higgs mechanism providing the mass to the photon field. The modulus of n^0 is excluded from dynamics by the constraint $|n^0|^2 = 2\beta$. The elementary excitations are described by n^1, n^2, ..., n^{N-1}. The same equation (3.6) implies that the masses of these elementary excitations are[9] (at large N)

$$M_k^2 = |m_0 - m_k|^2 = \left(m\,\frac{2\pi k}{N}\right)^2, \qquad k = 1, 2, ..., N-1\,. \tag{8.26}$$ {nkmap}

since the vacuum expectation of σ in the vacuum under consideration is $\sigma \approx m_0 \equiv m$. For the values of k that do not scale with N, each such mass in (8.26) scales as $1/N$ at large N. This is the consequence of our Z_N-symmetric choice of the twisted mass parameters (2.2).

What about the kink masses in the Higgs regime? (For brevity we will refer to them as the weak-coupling regime (WCR) kinks, although this is not quite precise. Indeed, if m is in the Higgs domain but $\sim$ a few$\,\times\,\Lambda$, the coupling constant is of order 1.) The masses of the WCR kinks can be found from the Veneziano–Yankielowicz effective superpotential [37], which is exact in the model under consideration,

$$\mathcal{W}_{\rm VY} = \frac{1}{4\pi}\sum_{i=0}^{N-1}\left(\sqrt{2}\,\sigma - m_i\right)\ln\frac{\sqrt{2}\,\sigma - m_i}{\Lambda} - \frac{N}{4\pi}\sqrt{2}\,\sigma\,. \tag{8.27}$$ {widu}

For definiteness N is assumed to be even. One should be careful with the logarithmic function. It is obvious that (8.27) is defined up to $im_p/2$ times any integer number. For the kinks interpolating between the vacua $|0\rangle$ and $|p\rangle$ one has [37]

$$M_{\rm kink} = 2\,|\Delta\mathcal{W}| = 2\left|\mathcal{W}_{\rm VY}(\sigma_0) - \mathcal{W}_{\rm VY}(\sigma_p) + \frac{i}{2}\,(m_0 - m_p)\,q\right|, \tag{8.28}$$ {widup}

where the logarithmic function in (8.27) is defined with the cut along the negative semi-axis, and q is an arbitrary integer. Since the vacuum values of sigma satisfy

$$(\sqrt{2}\sigma)^N = m^N + \Lambda^N$$

[9] In fact, equation $M_k = |m_0 - m_k|$ is exact.

and $|m/\Lambda| > 1$, we have $\sqrt{2}\sigma_p = m_p$, with the exponential accuracy. If $\sigma_0 = m_0 = m$ and $|p| < N/4$, in calculating $\mathcal{W}_{\rm VY}(\sigma_0,\, \sigma_p)$ we do not touch the cut of the logarithm, and then we can use the following expression [10]

$$
\begin{aligned}
\mathcal{W}_{\rm VY}(\sigma) &= \frac{1}{4\pi}\,\sqrt{2}\sigma\,\ln\left(\frac{\sqrt{2}\sigma}{e\Lambda}\right)^N + \frac{1}{4\pi\,N}\,\sqrt{2}\sigma\,F_N(y)\,, \\
F_N(y) &\equiv \sum_{k=1}^{\infty}\,\frac{N}{k\,(kN-1)}\,y^k \\
y &= \left(\frac{m}{\sqrt{2}\,\sigma}\right)^N\,.
\end{aligned}
\tag{8.29}
$$

Note that F_N is finite at $N\to\infty$ and $|y|\le 1$. As a result, at $|p| < N/4$ we arrive at

$$
M_{\rm kink} = \frac{N\,m}{\pi}\,\sin\frac{\pi\,|p|}{N}\left|\ln\frac{m}{e\Lambda} + 2\pi i\frac{q}{N}\,.\right|\,. \tag{8.30}
$$

{chet4}

The lightest mass is obtained by setting $q = 0$,

$$
M_{\rm kink*} = \frac{N\,m}{\pi}\,\sin\frac{\pi\,|p|}{N}\left|\ln\frac{m}{e\Lambda}\right| \tag{8.31}
$$

{chet4w}

At $p\sim N^0$ and $|m/\Lambda|\gg 1$ the kink mass does not scale with N. In addition, it is enhanced by the large logarithm $\ln m/e\Lambda \sim (Ng^2)^{-1}$ compared to the masses of the elementary excitations. Each kink has a tower of excitations on top of it, corresponding to $q\neq 0$. The existence of excitations is due to the fact that in additition to the topological charge, the kinks can have U(1) charges [28]. Excitations decay on the curve of marginal stability (CMS) $|m| = e\Lambda$, see [50].

Now consider $|p|$ larger than $N/4$. Equation (8.29) is no longer valid. Instead, in the limit $N=\infty$ we have $(\frac{1}{4}\le x\le\frac{3}{4})$

$$
\mathcal{W}_{\rm VY}(\sigma_p = m_p) \to \frac{m_p}{4\pi}\,\ln\left(\frac{m}{e^{f_1(x)+if_2(x)}\,\Lambda}\right)^N \tag{8.32}
$$

{chet2}

[10]One might be tempted to use Eq. (6.6) here, but this does not work since in this expression we have a different argument of the logarithm.

where

$$f_1 = -\cos 4\pi\, x\,, \qquad f_2 = \begin{cases} 4\pi\left(x - \frac{1}{4}\right) + \sin 4\pi\, x\,, & \frac{1}{4} \le x \le \frac{1}{2}\,, \\ 4\pi\left(x - \frac{3}{4}\right) + \sin 4\pi\, x\,, & \frac{1}{2} \le x \le \frac{3}{4} \end{cases} \tag{8.33}$$

$$x = \frac{p}{N}\,. \tag{8.34}$$

For kinks interpolating between the vacua $|0\rangle$ and $|p\rangle$ with $\frac{N}{4} \le p \le \frac{3N}{4}$ we thus get

$$\begin{aligned} M_{\rm kink} &= \frac{N\, m}{\pi}\,(\sin \pi\, x) \\ &\times \left| \ln \frac{m}{\Lambda e^{(1+f_1)/2}} - f_2\,\cot \pi x - \frac{i}{2}\left[f_2 + (1-f_1)\cot \pi x\right] + 2\pi i \frac{q}{N} \right| \end{aligned} \tag{8.35}$$

In particular, the "most composite" kink corresponding to $p = N/2$ and $x = 1/2$. For such kinks

$$M_{\rm kink} = \frac{N\, m}{\pi}\left| \ln \frac{m}{\Lambda} + 2\pi i \frac{\tilde{q}}{N} \right|\,. \tag{8.36}$$ {chet3}

The tower of excitations collapses at $|m| = \Lambda$. This is another CMS (to which the AD point belongs). Generally speaking, there are $N/2$ curves of marginal stability corresponding to distinct values of p in the interval $(N/4, 3N/4)$ and such values of m for which the real part in the second line in (8.35) vanishes. They are presented by co-axial circles in the m^N plane with radii r^N where $r \in [1, e]$. There is a cut in the m^N plane along the negative semi-axis [50]. At $|m| < \Lambda$ all excitations decay, and the kinks are represented by the n fields.

So far we ignored the heterotic deformation. Since the latter generally speaking breaks supersymmetry (spontaneously), the analysis presented above is no longer valid. However, for small heterotic deformation, the overall picture is likely to stay intact as long as we are in the Z_N broken phase. We still have N degenerate vacua (albeit they are at nonvanishing energy), and solitons of various kinds which interpolate between them. The BPS saturation-based consideration with regards to the soliton masses and CMS is no longer applicable, and these masses and curves will be distorted compared to the $\mathcal{N} = (2, 2)$ case. These distortions should be small, however.

Probably, it makes sense to

1) check equations;

2) $\tilde{q}$ seems to contain 1/4

3) For each p take such value of q which minimizes the kink mass. Then plot $M_{\mathrm{kink}*}$ as a function of x for some value of m/Λ and N. Say, $m/\Lambda = 4$ or 5.

9 Conclusions

{conclu}

In this paper we presented the large-N solution of the two-dimensional heterotic $\mathcal{N} = (0, 2)$ CP($N - 1$) model. Our studies were motivated by the fact that this model emerges on the world sheet of non-Abelian strings supported in a class of four-dimensional $\mathcal{N} = 1$ Yang–Mills theories. The non-trivial dynamics which we observed – with three distinct phases, confinement and no confinement, and two phase transitions – must somehow reflect dynamics of appropriate four-dimensional theoris. If so, we open a window to a multitude of unexplored dynamical scenarios in $\mathcal{N} = 1$ theories. But this is a topic for a separate investigation.

Acknowledgments

We are grateful to A. Gorsky, Andrey Losev, Victor Mikhailov, and A. Vainshtein for very useful discussions. The work of PAB was supported in part by the NSF Grant No. PHY-0554660. PAB is grateful for kind hospitality to FTPI, University of Minnesota, where a part of this work was done. The work of MS was supported in part by DOE grant DE-FG02-94ER408. The work of AY was supported by FTPI, University of Minnesota, by RFBR Grant No. 09-02-00457a and by Russian State Grant for Scientific Schools RSGSS-11242003.2.

A Notations in Euclidean Space

{app:eucl}

Since CP$(N-1)$ sigma model can be obtained as a dimensional reduction from four-dimensional theory, we present first our four-dimensional notations. The indices of four-dimensional spinors are raised and lowered by the SU(2) metric tensor,

$$\psi_\alpha = \epsilon_{\alpha\beta}\,\psi^\beta, \qquad \overline{\psi}_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\,\overline{\psi}^{\dot\beta}, \qquad \psi^\alpha = \epsilon^{\alpha\beta}\,\psi_\beta, \qquad \overline{\psi}^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\,\overline{\psi}_{\dot\beta}\,, \tag{A.1}$$

where

$$\epsilon_{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{and} \qquad \epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{A.2}$$

The contractions of the spinor indices are short-handed as

$$\lambda\psi = \lambda_\alpha\,\psi^\alpha\,, \qquad \overline{\lambda\psi} = \overline{\lambda}^{\dot\alpha}\,\overline{\psi}_{\dot\alpha}\,. \tag{A.3}$$

The sigma matrices for the euclidean space we take as

$$\sigma_\mu^{\alpha\dot\alpha} = \Big(\mathbf{1}, \quad -i\,\tau^k\Big)^{\alpha\dot\alpha}, \qquad \overline{\sigma}_{\dot\alpha\alpha\,\mu} = \Big(\mathbf{1}, \quad i\,\tau^k\Big)_{\dot\alpha\alpha}, \tag{A.4}$$

where τ^k are the Pauli matrices.

Reduction to two dimensions can be conveniently done by picking out x^0 and x^3 as the world sheet (or "longitudinal") coordinates, and integrating over the orthogonal coordinates. The two-dimensional derivatives are the defined to be

$$\partial_R = \partial_0 + i\partial_3\,, \qquad \partial_L = \partial_0 - i\partial_3\,. \tag{A.5}$$

One then identifies the lower-index spinors as the two-dimensional left- and right-handed chiral spinors

$$\xi_R = \xi_1\,, \qquad \xi_L = \xi_2\,, \qquad \overline{\xi}_R = \overline{\xi}_{\dot 1}\,, \qquad \overline{\xi}_L = \overline{\xi}_{\dot 2}\,. \tag{A.6}$$

For two-dimensional variables, the CP$(N-1)$ indices are written as upper ones

$$n^l\,, \quad \xi^l\,,$$

and lower ones for the conjugate moduli

$$\overline{n}_l\,, \quad \overline{\xi}_l\,,$$

where $l = 1, ..., N$. In the geometric formulation of CP$(N-1)$, global indices are written upstairs in both cases, only for the conjugate variables the indices with bars are used

$$\phi^i\,,\ \psi^i\,, \qquad \overline{\phi^{\bar\imath}}\,,\ \overline{\psi^{\bar\imath}}\,, \qquad\qquad i,\ \bar\imath = 1, ..., N-1\,,$$

and the metric $g_{i\bar\jmath}$ is used to contract them.

B Minkowski versus Euclidean formulation

{app:mink}

In the bulk of the paper we use both, Minkowski and Euclidean conventions. It is useful to summarize the transition rules. If the Minkowski coordinates are

$$x^\mu_M = \{t,\, z\}\,, \tag{B.1}$$ {appeone}

the passage to the Euclidean space requires

$$t \to -i\tau\,, \tag{B.2}$$ {appe2}

and the Euclidean coordinates are

$$x^\mu_M = \{\tau,\, z\}\,. \tag{B.3}$$ {appe3}

The derivatives are defined as follows:

$$\begin{aligned} \partial^M_L &= \partial_t + \partial_z\,, \qquad \partial^M_R = \partial_t - \partial_z\,, \\ \partial^E_L &= \partial_\tau - i\partial_z\,, \qquad \partial^E_R = \partial_\tau + i\partial_z\,. \end{aligned} \tag{B.4}$$

The Dirac spinor is

$$\Psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \tag{B.5}$$ {appe5}

In passing to the Eucildean space $\Psi^M = \Psi^E$; however, $\bar\Psi$ is transformed,

$$\bar\Psi^M \to i\bar\Psi^E\,. \tag{B.6}$$ {appe6}

Moreover, Ψ^E and $\bar\Psi^E$ are *not* related by the complex conjugation operation. They become independent variables. The fermion gamma matrices are defined as

$$\bar\sigma^\mu_M = \{1,\, -\sigma_3\}\,, \qquad \bar\sigma^\mu_E = \{1,\, i\sigma_3\}\,. \tag{B.7}$$ {appe7}

Finally,

$$\mathcal{L}_E = -\mathcal{L}_M(t = -i\tau, \ldots). \tag{B.8}$$

With this notation, formally, the fermion kinetic terms in $\mathcal{L}_E$ and $\mathcal{L}_M$ coincide. We use the following equivalent definitions of the heterotic deformation terms

$$\frac{g_0}{\sqrt{2}}\tilde\gamma_{(M)}\,\zeta_R G_{i\bar j}\left(i\partial_L\bar\phi^{\bar j}\right)\psi^i_R\,, \qquad \frac{1}{g_0^2}\,\widetilde\gamma_{(E)}\,\chi^a_R\,(i\partial_L S^a)\zeta_R\,, \qquad \frac{2}{g_0^2}\,\widetilde\gamma_{(E)}\,(i\partial_L\overline{n})\xi_R\zeta_R \tag{B.9}$$ {appabiferm}

in Minkowski and Euclidean spaces correspondingly. The following transition rule applies,

$$\widetilde{\gamma}_{(M)} = -i\,\widetilde{\gamma}_{(E)}\,. \tag{B.10}$$ {appe8}

Everywhere where there is no menace of confusion we omit the super/subscripts M, E. The first two terms in Eq. (B.9) originally were introduced in Ref. [13], with a constant

$$\gamma \;=\; \widetilde{\gamma}/(\sqrt{2}g_0)\,. \tag{B.11}$$

C Global symmetries of the CP($N-1$) model with Z_N-symmetric twisted masses [11]

{app:symm}

In the absence of the twisted masses the model is SU(N) symmetric. The twisted masses (2.2) explicitly break this symmetry of the Lagrangian (3.6) down to U$(1)^{N-1}$,

$$\begin{aligned} n^\ell \;&\to\; e^{i\alpha_\ell} n^\ell\,, \quad \xi^\ell_R \to e^{i\alpha_\ell}\xi^\ell_R \quad \xi^\ell_L \to e^{i\alpha_\ell}\xi^\ell_L\,, \quad \ell = 1,2,...,N\,, \\ \sigma \;&\to\; \sigma\,, \quad \lambda_{R,L} \to \lambda_{R,L}\,. \end{aligned} \tag{C.1}$$

where α_ℓ are N constant phases different for different ℓ.

Next, there is a global vectorial U(1) symmetry which rotates all fermions ξ^ℓ in one and the same way, leaving the boson fields intact,

$$\begin{aligned} \xi^\ell_R \;&\to\; e^{i\beta}\xi^\ell_R\,, \quad \xi^\ell_L \to e^{i\beta}\xi^\ell_L\,, \quad \ell = 1,2,...,N\,, \\ \lambda_R \;&\to\; e^{-i\beta}\lambda_R\,, \quad \lambda_L \to e^{-i\beta}\lambda_L\,, \\ n^\ell \;&\to\; n^\ell\,, \quad \sigma \to \sigma\,. \end{aligned} \tag{C.2}$$

Finally, there is a discrete Z_{2N} symmetry which is of most importance for our purposes. Indeed, let us start from the axial U$(1)_R$ transformation which would be a symmetry of the classical action at $m = 0$ (it is anomalous, though, under quantum corrections),

$$\begin{aligned} \xi^\ell_R \;&\to\; e^{i\gamma}\xi^\ell_R\,, \quad \xi^\ell_L \to e^{-i\gamma}\xi^\ell_L\,, \quad \ell = 1,2,...,N\,, \\ \lambda_R \;&\to\; e^{i\gamma}\lambda_R\,, \quad \lambda_L \to e^{-i\gamma}\lambda_L\,, \quad \sigma \to e^{2i\gamma}\sigma\,, \\ n^\ell \;&\to\; n^\ell\,. \end{aligned} \tag{C.3}$$

[11] See also the Appendix in Ref. [46].

With m switched on and the chiral anomaly included, this transformation is no longer the symmetry of the model. However, a discrete Z_{2N} subgroup survives both the inclusion of anomaly and $m \neq 0$. This subgroup corresponds to

$$\gamma_k = \frac{2\pi i k}{2N}\,, \quad k = 1, 2, ..., N\,. \tag{C.4}$$ {appe12}

with the simultaneous shift

$$\ell \to \ell - k\,. \tag{C.5}$$ {appe13}

In other words,

$$\begin{aligned}
\xi_R^\ell &\to e^{i\gamma_k}\xi_R^{\ell-k}\,, \quad \xi_L^\ell \to e^{-i\gamma_k}\xi_L^{\ell-k}\,,\\
\lambda_R &\to e^{i\gamma_k}\lambda_R\,, \quad \lambda_L \to e^{-i\gamma_k}\lambda_L\,, \quad \sigma \to e^{2i\gamma_k}\sigma\,,\\
n^\ell &\to n^{\ell-k}\,.
\end{aligned} \tag{C.6}$$

This Z_{2N} symmetry relies on the particular choice of masses given in (2.2).

When we switch on the heterotic deformation, the Z_N transformations (C.6) must be supplemented by

$$\zeta_R \to e^{-i\gamma_k}\,\zeta_R\,. \tag{C.7}$$ {bee35p}

The symmetry of the Lagrangian (3.35) remains intact.

The order parameters for the Z_N symmetry are as follows: (i) the set of the vacuum expectation values $\{\langle n^0\rangle,\ \langle n^1\rangle,\ ...\ \langle n^{N-1}\rangle\}$ and (i) the bifermion condensate $\langle\bar\xi_{R,\ell}\,\xi_L^\ell\rangle$. Say, a nonvanishing value of $\langle n^0\rangle$ or $\langle\bar\xi_{R,\ell}\,\xi_L^\ell\rangle$ implies that the Z_{2N} symmetry of the action is broken down to Z_2. The first order parameter is more convenient for detection at large m while the second at small m.

It is instructive to illustrate the above conclusions in the geometrical formulation of the sigma model. namely, in (for simplicity we will consider CP(1); generalization to CP$(N-1)$ is straightforward). In components the Lagrangian of the model is

$$\begin{aligned}
&\mathcal{L}_{CP(1)} = G\Big\{\partial_\mu\bar\phi\,\partial^\mu\phi - |m|^2\bar\phi\,\phi + \frac{i}{2}\big(\psi_L^\dagger\overleftrightarrow{\partial}_R\psi_L + \psi_R^\dagger\overleftrightarrow{\partial}_L\psi_R\big)\\
&-i\,\frac{1-\bar\phi\,\phi}{\chi}\,\big(m\,\psi_L^\dagger\psi_R + \bar m\psi_R^\dagger\psi_L\big)\\
&-\frac{i}{\chi}\,\big[\psi_L^\dagger\psi_L\big(\bar\phi\overleftrightarrow{\partial}_R\phi\big) + \psi_R^\dagger\psi_R\big(\bar\phi\overleftrightarrow{\partial}_L\phi\big)\big]\\
&-\frac{2}{\chi^2}\,\psi_L^\dagger\,\psi_L\,\psi_R^\dagger\,\psi_R\Big\}\,,
\end{aligned} \tag{C.8}$$

where

$$\chi = 1 + \bar{\phi}\,\phi\,, \quad G = \frac{2}{g_0^2\,\chi^2}\,. \tag{C.9}$$

The Z_2 transformation corresponding to (C.6) is

$$\phi \to -\frac{1}{\bar{\phi}}\,, \qquad \psi_R^\dagger \psi_L \to -\psi_R^\dagger \psi_L\,. \tag{C.10}$$ {bee40}

The order parameter which can detect breaking/nonbreaking of the above symmetry is

$$\frac{m}{g_0^2}\left(1 - \frac{g_0^2}{2\pi}\right)\frac{\bar{\phi}\,\phi - 1}{\bar{\phi}\,\phi + 1} - iR\psi_R^\dagger \psi_L\,. \tag{C.11}$$

Under the transformation (C.10) this order parameter changes sign. In fact, this is the central charge of the $\mathcal{N} = 2$ sigma model, including the anomaly [43, 44].

References

[1] A. Hanany and D. Tong, JHEP **0307**, 037 (2003) [hep-th/0306150].

[2] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, Nucl. Phys. B **673**, 187 (2003) [hep-th/0307287].

[3] M. Shifman and A. Yung, Phys. Rev. D **70**, 045004 (2004) [hep-th/0403149].

[4] A. Hanany and D. Tong, JHEP **0404**, 066 (2004) [hep-th/0403158].

[5] D. Tong, Annals Phys. **324**, 30 (2009) [arXiv:0809.5060 [hep-th]]; M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, J. Phys. A **39**, R315 (2006) [arXiv:hep-th/0602170]; K. Konishi, Lect. Notes Phys. **737**, 471 (2008) [arXiv:hep-th/0702102]; M. Shifman and A. Yung, *Supersymmetric Solitons,* (Cambridge University Press, 2009).

[6] E. Witten, Nucl. Phys. B **149**, 285 (1979).

[7] E. Witten, Nucl. Phys. B **403**, 159 (1993) [hep-th/9301042].

[8] A. D'Adda, P. Di Vecchia and M. Lüscher, Nucl. Phys. B **152**, 125 (1979).

[9] M. Shifman and A. Yung, Phys. Rev. D **77**, 125017 (2008) [arXiv:0803.0698 [hep-th]].

[10] K. Hori and C. Vafa, *Mirror symmetry,* arXiv:hep-th/0002222.

[11] E. Frenkel and A. Losev, Commun. Math. Phys. **269**, 39 (2007) [arXiv:hep-th/0505131].

[12] M. Edalati and D. Tong, JHEP **0705**, 005 (2007) [arXiv:hep-th/0703045].

[13] M. Shifman and A. Yung, Phys. Rev. D **77**, 125016 (2008) [arXiv:0803.0158 [hep-th]].

[14] P. A. Bolokhov, M. Shifman and A. Yung, Phys. Rev. D **79**, 085015 (2009) (Erratum: Phys. Rev. D 80, 049902 (2009)) [arXiv:0901.4603 [hep-th]].

[15] P. A. Bolokhov, M. Shifman and A. Yung, Phys. Rev. D **79**, 106001 (2009) (Erratum: Phys. Rev. D 80, 049903 (2009)) [arXiv:0903.1089 [hep-th]].

[16] P. A. Bolokhov, M. Shifman and A. Yung, *Heterotic* $\mathcal{N}=(0,2)$ $CP(N-1)$ *Model with Twisted Masses,* arXiv:0907.2715 [hep-th].

[17] M. Shifman and A. Yung, work in progress.

[18] E. Witten, Phys. Rev. D **16**, 2991 (1977); P. Di Vecchia and S. Ferrara, Nucl. Phys. B **130**, 93 (1977).

[19] B. Zumino, Phys. Lett. B **87**, 203 (1979).

[20] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Phys. Rept. **116**, 103 (1984).

[21] A. M. Perelomov, Phys. Rept. **174**, 229 (1989).

[22] E. Witten, Nucl. Phys. B **202**, 253 (1982).

[23] L. Alvarez-Gaumé and D. Z. Freedman, Commun. Math. Phys. **91**, 87 (1983); S. J. Gates, Nucl. Phys. B **238**, 349 (1984); S. J. Gates, C. M. Hull and M. Roček, Nucl. Phys. B **248**, 157 (1984).

[24] A. M. Polyakov, Phys. Lett. B **59**, 79 (1975).

[25] A. Ritz, M. Shifman and A. Vainshtein, Phys. Rev. D **66**, 065015 (2002) [arXiv:hep-th/0205083].

[26] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, Second Edition, Princeton University Press, 1992.

[27] S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, Academic Press, New York, 1978.

[28] N. Dorey, JHEP **9811**, 005 (1998) [hep-th/9806056].

[29] E. Witten, *Two-dimensional models with (0,2) supersymmetry: Perturbative aspects,* arXiv:hep-th/0504078.

[30] A. Gorsky, M. Shifman and A. Yung, Phys. Rev. D **73**, 065011 (2006) [hep-th/0512153].

[31] S. R. Coleman, Annals Phys. **101**, 239 (1976).

[32] A. Gorsky, M. Shifman and A. Yung, Phys. Rev. D **71**, 045010 (2005) [hep-th/0412082].

[33] F. Ferrari, JHEP **0205** 044 (2002) [hep-th/0202002].

[34] F. Ferrari, Phys. Lett. **B496** 212 (2000) [hep-th/0003142]; JHEP **0106**, 057 (2001) [hep-th/0102041].

[35] A. D'Adda, A. C. Davis, P. DiVeccia and P. Salamonson, Nucl. Phys. **B222** 45 (1983).

[36] S. Cecotti and C. Vafa, Comm. Math. Phys. **157** 569 (1993) [hep-th/9211097].

[37] A. Hanany and K. Hori, Nucl. Phys. B **513**, 119 (1998) [arXiv:hep-th/9707192].

[38] P. C. Argyres and M. R. Douglas, Nucl. Phys. **B448**, 93 (1995) [arXiv:hep-th/9505062].

[39] P. C. Argyres, M. R. Plesser, N. Seiberg, and E. Witten, Nucl. Phys. **B461**, 71 (1996) [arXiv:hep-th/9511154].

[40] M. Shifman and A. Yung, Rev. Mod. Phys. **79** 1139 (2007) [arXiv:hep-th/0703267].

[41] D. Tong, JHEP **0709**, 022 (2007) [arXiv:hep-th/0703235].

[42] G. Veneziano and S. Yankielowicz, Phys. Lett. B **113**, 231 (1982).

[43] A. Losev and M. Shifman, Phys. Rev. D **68**, 045006 (2003) [arXiv:hep-th/0304003].

[44] M. Shifman, A. Vainshtein and R. Zwicky, J. Phys. A **39**, 13005 (2006) [arXiv:hep-th/0602004].

[45] M. Shifman and A. Yung, $\mathcal{N} = (0, 2)$ *Deformation of the* $\mathcal{N} = (2, 2)$ *Wess–Zumino model in two dimensions*, arXiv:0912.3836 [hep-th].

[46] M. Shifman and A. Yung, Phys. Rev. D **79**, 105006 (2009) [arXiv:0901.4144 [hep-th]].

[47] D. Tong, JHEP **0612**, 051 (2006) [arXiv:hep-th/0610214].

[48] A. Migdal and M. Shifman, Phys. Lett. B **114**, 445 (1982).

[49] A. Kovner and M. A. Shifman, Phys. Rev. D **56**, 2396 (1997) [arXiv:hep-th/9702174].

[50] S. Olmez and M. Shifman, J. Phys. A **40**, 11151 (2007) [arXiv:hep-th/0703149].