Veneziano-Yankielowicz Potential

for $\mathcal{N}=(2,2)$ $CP(N-1)$ model

(1)

$$\mathcal{W}_{VY} = \frac{1}{4\pi}\sum_k (\sqrt{2}\sigma - m_k)\ln\frac{\sqrt{2}\sigma - m_k}{\Lambda} - \frac{N}{4\pi}\sqrt{2}\sigma$$

Transform this expression:

$$\mathcal{W}_{VY} = \frac{N\Lambda}{4\pi}\left\{\frac{1}{N}\sum_k (S-\mu_k)\ln(S-\mu_k) - S\right\}$$

The most non-trivial part is to calculate the sum

$$\frac{1}{N}\sum_k (S-\mu_k)\ln(S-\mu_k) =$$

$$\boxed{\begin{aligned} s &\equiv S/\mu = \frac{\sqrt{2}\sigma}{m} \\ \lambda^k &\equiv \frac{\mu_k}{\mu} = \frac{m_k}{m}\end{aligned}}$$

$$= \frac{1}{N}\sum_k \mu(s-\lambda_k)\cdot\left\{\ln\mu + \ln(s-\lambda_k)\right\} =$$

$$= s\cdot\mu\ln\mu + \frac{\mu}{N}\sum(s-\lambda_k)\ln(s-\lambda_k)$$

So we need to evaluate

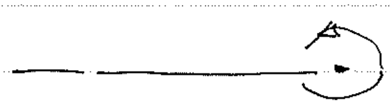
$$\frac{1}{N}\sum_{k}(s-\lambda_k)\ln(s-\lambda_k)$$

We need to separate $\ln(s-\lambda_k)$ into $\ln s + \ln(1-\lambda_k/s)$,

as $\mathrm{Re}\, s > m$ in the Higgs phase.

Also, s is taken in the immediate proximity of the unit circle, as $|\mathrm{Re}\, s| \simeq m$.

Let's direct the cut of the Log along the negative axis:

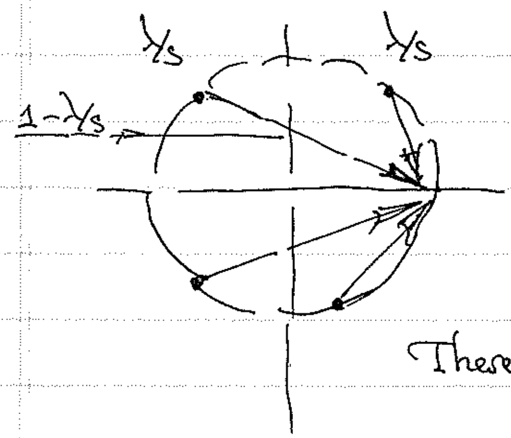


$\varphi = -\pi \ldots +\pi$ for complex numbers

3

For a fixed $s \in S^1$, and varying $\lambda \in S^1$,

λ/s will run over the whole S^1.



For $\operatorname{Arg} \lambda/s = 0..\pi$, $\operatorname{Arg} 1-\lambda/s \in -\pi/2..0$

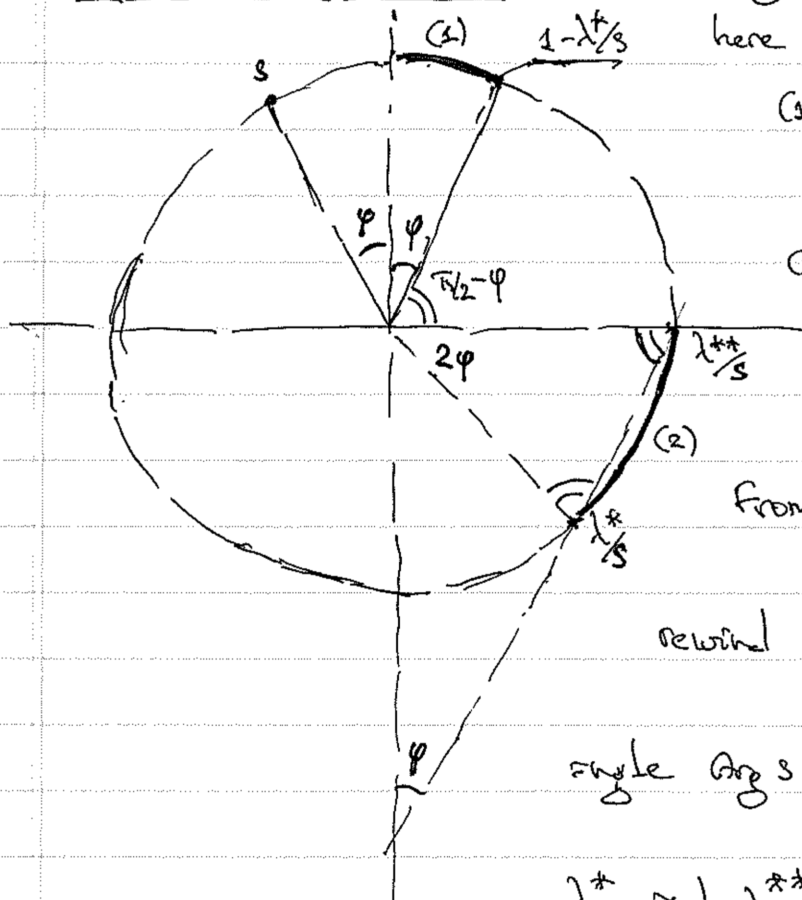
For $\operatorname{Arg} \lambda/s = -\pi..0$, $\operatorname{Arg} 1-\lambda/s \in 0..\pi/2$

Thereby $\operatorname{Arg}(1-\lambda/s) = -\arcsin \dfrac{\operatorname{Im}(\lambda/s)}{|1-\lambda/s|}$

For $|\operatorname{Arg} s| < \pi/2$, $|\operatorname{Arg} s + \operatorname{Arg}(1-\lambda/s)|$ will not exceed π $\Rightarrow$ will not cross the cut of the logarithm.

For $|\operatorname{Arg} s| > \pi/2$, the sum of arguments will exceed π, and consequently, this π will need to be subtracted or added.

Let $\operatorname{Im} s > 0$ first



$\operatorname{Arg}(1-\lambda/s) \in \pi/2-\varphi \,..\, \pi/2$

here $\varphi = \operatorname{Arg} s - \pi/2$

(1) contributions of these $1-\lambda/s$ will exceed π

(2) these λ/s will exceed π in the final logarithm

From points λ^*/s and λ^{**}/s we rewind counterclockwise by the angle $\operatorname{Arg} s = \pi/2+\varphi$ to obtain actual λ^* and λ^{**}.

We have: $\operatorname{Arg} \lambda^* = \pi/2 - \varphi = \pi - \operatorname{Arg} s$

$$\operatorname{Arg} \lambda^{**} = \pi/2 + \varphi = \operatorname{Arg} s$$

That is, for $\operatorname{Arg} \lambda = \pi - \operatorname{Arg} s \,..\, \operatorname{Arg} s$,

$$\ln s + \ln(1-\lambda/s) = \ln(s-\lambda) + 2i\pi$$

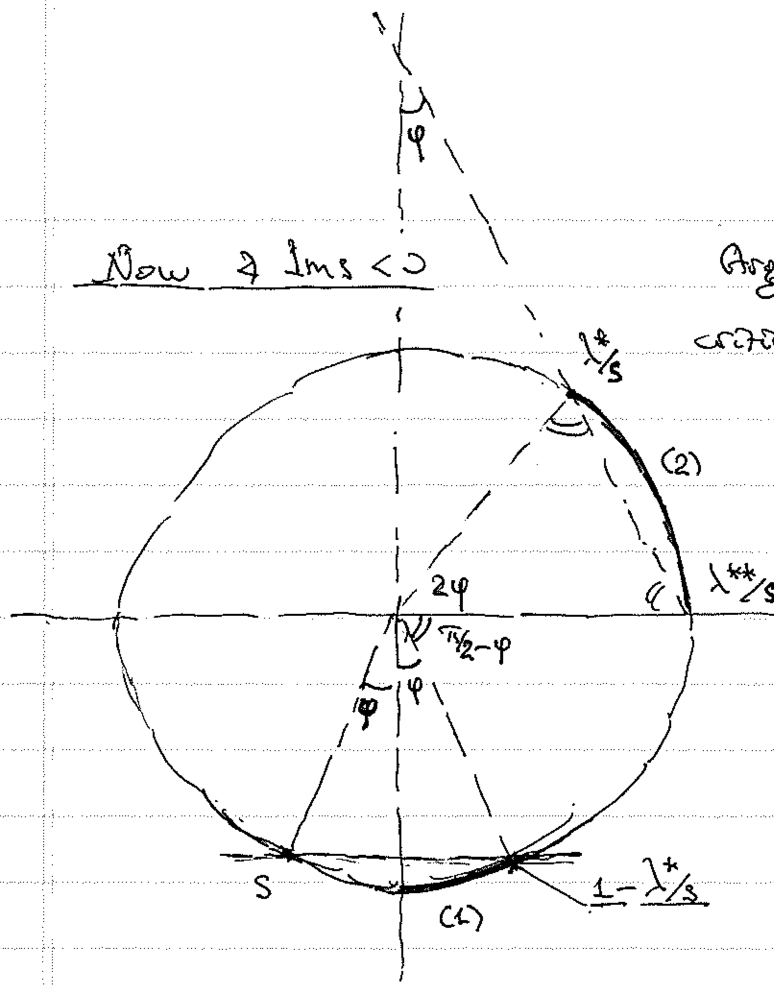
for $\operatorname{Arg} s = +\pi/2 \,..\, +\pi$

(5)

<u>Now $\mathrm{Im}\, s < 0$</u>

$\mathrm{Arg}\, s = -\pi/2 - \varphi$

critical $\mathrm{Arg}(1-\lambda/s) = -\pi/2 \,..\, -(\pi/2-\varphi)$



(1) contributions of these $1-\lambda/s$ will exceed $-\pi$

(2) contributions of these λ/s will exceed $-\pi$ in the final logarithm

From λ^*/s and λ^{**}/s we rewind counterclockwise by $-(\pi/2+\varphi)$,

or <u>clockwise</u> by $\underline{\pi/2+\varphi}$. $\Rightarrow$ we end up with interval

$$\mathrm{Arg}\,\lambda^* = -(\pi/2-\varphi) = -\mathrm{Arg}\, s - \pi$$

$$\mathrm{Arg}\,\lambda^{**} = -(\varphi+\pi/2) = \mathrm{Arg}\, s$$

That is, for $\mathrm{Arg}\,\lambda = \mathrm{Arg}\, s \,..\, -\mathrm{Arg}\, s - \pi$

$$\ln s + \ln(1-\lambda/s) = \ln(s-\lambda) - 2\pi i$$

for $\mathrm{Arg}\, s = -\pi \,..\, -\pi/2$

We have,

$$\ln(s-\lambda) = \ln s + \ln(1-s/\lambda) +$$

$$+ \begin{cases} -2\pi i, & \operatorname{Arg} s = \pi/2 .. \pi,\ \operatorname{Arg}\lambda = \pi - \operatorname{Arg} s .. \operatorname{Arg} s \\ 0, & |\operatorname{Arg} s| < \pi/2 \\ +2\pi i, & \operatorname{Arg} s = -\pi .. -\pi/2,\ \operatorname{Arg}\lambda = \operatorname{Arg} s .. -\operatorname{Arg} s - \pi \end{cases}$$

If $\phi \equiv \operatorname{Arg} s$, then

$$\ln(s-\lambda) = \ln s + \ln(1-s/\lambda) + \begin{cases} -2\pi i, & \phi = \pi/2 .. \pi,\ \operatorname{Arg}\lambda = \pi-\phi .. \phi \\ 0, & |\phi| < \pi/2 \\ +2\pi i, & \phi = -\pi .. -\pi/2,\ \operatorname{Arg}\lambda = \phi .. -\phi-\pi \end{cases}$$

$$\equiv \ln s + \ln(1-s/\lambda) + C$$

where $C = C(s,\lambda) =$ piecewise constant

Now,

$$\frac{1}{N}\sum_k (s-\lambda_k)\ln(s-\lambda_k) \Rightarrow \frac{1}{2\pi i}\oint \frac{d\lambda}{\lambda}(s-\lambda)\ln(s-\lambda) =$$

$$= \frac{1}{2\pi i}\oint \frac{d\lambda}{\lambda}(s-\lambda)\left[\ln s + \ln(1-\lambda/s) + C\right] =$$

$$= \frac{1}{2\pi i}\oint \frac{d\lambda}{\lambda}(s-\lambda)\left(\ln s + C\right) + \frac{1}{2\pi i}\oint \frac{d\lambda}{\lambda}(s-\lambda)\ln(1-\lambda/s)$$

$$\bullet \ \frac{1}{2\pi i} \oint \frac{d\lambda}{\lambda} (s-\lambda)(\ln s + C) =$$

$$= \frac{1}{2\pi i} \oint \frac{d\lambda}{\lambda} (s \ln s + sC) - \frac{1}{2\pi i} \oint d\lambda (\ln s + C) =$$

$$= s \ln s + \frac{1}{2\pi i} \ln\lambda \Big| \cdot sC - \frac{1}{2\pi i} \lambda \cdot \Big| C =$$

$$= s \ln s + \frac{C}{2\pi i} \left[s \ln\lambda - \lambda \right] \Big| =$$

$$= s \ln s + \begin{cases} -\left[s\left(i\phi - i(\pi-\phi)\right) - \left(e^{i\phi} - e^{i(\pi-\phi)}\right)\right], & \phi = \pi/2 .. \pi \\ 0, & |\phi| < \pi/2 \\ +\left[s\left(i(-\phi-\pi) - i\phi\right) - \left(e^{i(-\phi-\pi)} - e^{i\phi}\right)\right], & \phi = -\pi .. -\pi/2 \end{cases}$$

(9)

$$= s\ln s + \begin{cases} -\left[is(2\phi-\pi) - (e^{i\phi}+e^{-i\phi})\right], & \phi = \pi/2 .. \pi \\ 0, & |\phi| < \pi/2 \\ +\left[is(-2\phi-\pi) - (-e^{-i\phi}-e^{i\phi})\right], & \phi = -\pi .. -\pi/2 \end{cases} =$$

$$= s\ln s + \begin{cases} -\left[i(2\phi-\pi)s - 2\cos\phi\right], & \phi = \pi/2 .. \pi \\ 0, & |\phi| < \pi/2 \\ -\left[i(2\phi+\pi)s - 2\cos\phi\right], & \phi = -\pi .. -\pi/2 \end{cases} =$$

$$= s\ln s + \begin{cases} -i(2\phi-\pi)s + 2\cos\phi, & \phi = \pi/2 .. \pi \\ 0, & |\phi| < \pi/2 \\ -i(2\phi+\pi)s + 2\cos\phi, & \phi = -\pi .. -\pi/2 \end{cases} =$$

$$= \frac{1}{2\pi i}\oint \frac{d\lambda}{\lambda}(s-\lambda)(\ln s + C)$$

<u>here $\phi = \operatorname{Arg} s$</u>

$$\bullet \quad \frac{1}{2\pi i} \oint \frac{d\lambda}{\lambda} (s-\lambda) \ln\left(1 - \lambda/s\right) =$$

$$= \frac{1}{2\pi i} \oint \frac{d\lambda}{\lambda} (s-\lambda) \left\{ -\lambda/s - \frac{1}{2}\left(\frac{\lambda}{s}\right)^2 + \dots \right\} =$$

= does not contribute at all $\Rightarrow 0$

Therefore, (see p. 7)

$$\frac{1}{N} \sum_k (s-\lambda_k) \ln(s-\lambda_k) \Rightarrow \frac{1}{2\pi i} \oint \frac{d\lambda}{\lambda} (s-\lambda) \ln(s-\lambda) =$$

$$= s \ln s + \Phi(s),$$

where

$$\Phi(s) = \begin{cases} -i(2\phi - \pi)s + 2\overset{\cos}{\cancel{\sin}}\phi, & \phi = \pi/2 \dots \pi \\ 0, & |\phi| < \pi/2 \\ -i(2\phi + \pi)s + 2\overset{\cos}{\cancel{\sin}}\phi, & \phi = -\pi \dots -\pi/2 \end{cases}$$

$\phi = \operatorname{Arg} s$ $\qquad$ valid for $|s| \approx 1$ only ?

Then we have, see p. 1,

$$\frac{1}{N}\sum (S-\mu_n)\ln(S-\mu_n) =$$

$$= s\cdot\mu\ln\mu + \mu\cdot s\ln s + \mu\,\Phi(s) =$$

$$= S\ln\mu s + \Phi(\mu s) = S\ln S + \Phi(S)$$

a different function

And therefore,

$$\mathcal{W}_{VY}(S) = \frac{N\Lambda}{4\pi}\left[S\ln S - S + \Phi(S) \right]$$

where

$$\Phi(S) = \begin{cases} -i(2\phi-\pi)S + 2\mu\cancel{\sinh}\overset{\cos}{}\phi, & \phi = \pi/2 .. \pi \\ 0, & |\phi| < \pi/2 \\ -i(2\phi+\pi)S + 2\mu\cancel{\sinh}\overset{\cos}{}\phi, & \phi = -\pi .. -\pi/2 \end{cases}$$

(cf. p. 10)

$\phi = \operatorname{Arg} S$; for $|S| = \mu$ only !

Kink masses:

$$M_{kink} = 2\left| \mathcal{W}_{VY}(S_0) - \mathcal{W}_{VY}(S_p) + \frac{i}{2}(m_0 - m_p)q \right|$$

For $|\mathrm{Arg}\, S_p| < \pi/2 \iff |p| < N/4$, $\Phi(S_p) = 0$

And then, for $S_0 = m/\Lambda = \mu$

$$\mathcal{W}_{VY}(S_0) = \frac{N\Lambda}{4\pi}\left[\mu \ln\mu - \mu\right]$$

$$\mathcal{W}_{VY}(S_p) = \frac{N\Lambda}{4\pi}\left[S_p \ln S_p - S_p\right] =$$

$$= \frac{N\Lambda}{4\pi}\mu_p\left[\ln\mu + i\frac{2\pi p}{N} - 1\right] = \qquad \frac{2\pi p}{N} \equiv \phi_p$$

$$= \frac{N\Lambda}{4\pi}\mu e^{i\frac{2\pi p}{N}}\left[(\ln\mu - 1) + i\frac{2\pi p}{N}\right] =$$

again...

$$= \frac{N\Lambda}{4\pi}\mu e^{i\frac{2\pi p}{N}}\left[(\ln\mu - 1) + i\frac{2\pi p}{N}\right] =$$

$$= \frac{N\Lambda}{4\pi}\mu e^{i\phi_p}\left[(\ln\mu - 1) + i\phi_p\right]$$

$$\mathcal{W}_{vq}(S_0) - \mathcal{W}_{vq}(S_p) + \frac{i}{2}(m_0 - m_p)q =$$

$$= \frac{N\Lambda}{4\pi}\Big\{ \mu \ln\mu - \mu -$$

$$- \mu e^{i\phi_p}\left[(\ln\mu - 1) + i\phi_p\right] +$$

$$+ \frac{2\pi i}{N}(\mu - \mu e^{i\phi_p})q \Big\} =$$

$$= \frac{N\Lambda}{4\pi}\Big\{ \mu\ln\mu - \mu -$$

$$- \mu e^{i\phi_p}\ln\mu + \mu e^{i\phi_p} - i\phi_p e^{i\phi_p}\mu +$$

$$+ \mu \cdot i\phi_q - i\phi_q e^{i\phi_p}\cdot\mu \Big\} =$$

$$= \frac{N\Omega\mu}{4\pi}\Big\{ (\ln\mu - 1)(1 - e^{i\phi_p}) +$$

$$+ i\phi_q (1 - e^{i\phi_p}) +$$

$$+ i\phi_p - i\phi_p e^{i\phi_p} - i\phi_p \Big\} =$$

$$= \frac{N\mu}{4\pi}\Big\{ (1 - e^{i\phi_p})\big[(\ln\mu - 1) + i(\phi_p + \phi_q)\big] - i\phi_p \Big\}$$

Looking for $|\ |^2$ of the bracket now:

$$\Big| \ e^{i\phi_p} - 1 = 2i\, e^{i\phi_p/2} \sin\phi_p/2$$

$$|\mathcal{U}_{kink}|^2 \propto \Big| 2i e^{i\phi_p/2} \sin\phi_p/2 \big\{(\ln\mu - 1) + i(\phi_p + \phi_q)\big\} + i\phi_p \Big|^2 =$$

$$= 4\sin^2\phi_p/2 \big[(\ln\mu - 1)^2 + (\phi_p + \phi_q)^2\big] + \phi_p^2 +$$

$$+ 2\phi_p e^{i\phi_p/2} \sin\phi_p/2 \big[(\ln\mu - 1) + i(\phi_p + \phi_q)\big] +$$

$$+ 2\phi_p e^{-i\phi_p/2} \sin\phi_p/2 \big[(\ln\mu - 1) - i(\phi_p + \phi_q)\big] =$$

$$= 4\sin^2\frac{\phi_p}{2}\left[(\ln\mu - 1)^2 + (\phi_p+\phi_q)^2\right] + \phi_p^2 +$$

$$+ 2\phi_p \sin\phi_p/2\,(\ln\mu - 1)\cdot 2\cos\phi_p/2 +$$

$$+ 2\phi_p \sin\phi_p/2\,(\phi_p+\phi_q)(-2\sin\phi_p/2) =$$

$$\Big|\; 2\sin^2\phi_p/2 = 1-\cos\phi_p$$

$$= 2(1-\cos\phi_p)\left[(\ln\mu-1)^2 + (\phi_p+\phi_q)^2\right] + \phi_p^2 +$$

$$+ 2\phi_p(\ln\mu - 1)\cdot\sin\phi_p -$$

$$- 2\phi_p(1-\cos\phi_p)(\phi_p+\phi_q); \qquad =$$

$$= \left| \nabla\omega_{VP}(S_0) - \nabla\omega_{VP}(S_p) + \frac{i}{2}(m_0 - m_p)q \right|^2 \Big/ \left(\frac{\omega m}{4\pi}\right)^2;$$

(16)

$$|\mathcal{Q}_{kink}|^2 = \left(2\frac{N_M}{4\pi}\right)^2 \times$$

$$\times\Big\{ 2(1-\cos\phi_p)\left[(\ln\mu-1)^2 + (\phi_p+\phi_q)^2\right] + \phi_p^2 +$$

$$+ 2\phi_p(\ln\mu - 1)\cdot\sin\phi_p -$$

$$- 2\phi_p(1-\cos\phi_p)(\phi_p+\phi_q) \Big\}$$

for $|\phi_p| < \pi/2$; $\phi_p = \frac{2\pi i p}{N}$; $\phi_q = \frac{2\pi i q}{N}$

$\mu = M/\Lambda$

16a

Easy to find extremum: (w.r.t. q)

$$\frac{\partial}{\partial \phi_q} \propto 2(1-\cos\phi_p) \cdot 2(\phi_p+\phi_q) -$$

$$- 2(1-\cos\phi_p)\phi_p =$$

$$= 2(1-\cos\phi_p)\left[2\phi_p + 2\phi_q - \phi_p\right] \propto$$

$$\propto \phi_p + 2\phi_q \equiv 0$$

$$\Rightarrow \phi_q = -\frac{1}{2}\phi_p \quad \text{or} \quad q = -\frac{1}{2}p$$