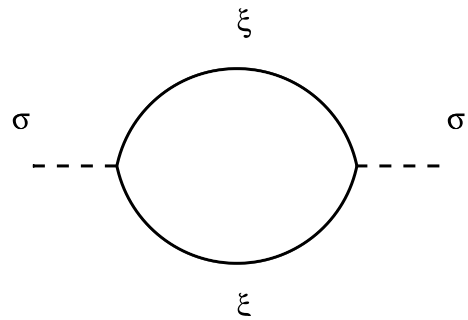

Figure 1: Kinetic term for the for σ field.

1 Notes with regards to SBY

1.1 Different effective Lagrangians

In this section we will comment on the relation between the effective Lagrangian derived in Sect. **??** from the large-N expansion and the Veneziano–Yankielowicz effective Lagrangian based on anomalies and supersymmetry. For simplicity we will set $m_i = 0$ in this section. Generalization to $m_i \neq 0$ is straightforward. We assume the heterotic deformation to be small, $u \ll 1$.

The $1/N$ expansion allows one to derive an honest-to-god effective Lagrangian for the field σ, valid both in its kinetic and potential parts. The leading order in $1/N$ in the potential part is determined by the diagram depcited in Fig. 1 which gives

$$\mathcal{L}_{\rm kin} = \frac{N}{4\pi}\,\frac{1}{2|\sigma|^2}\,|\,\partial_\mu \sigma\,|^2\,. \tag{1}$$

The virtual ξ momenta saturating the loop integral are of the order of the ξ mass $\sqrt{2}|\sigma|$. Up to a numerical coefficient this result is obvious since the field σ has mass-dimension 1.

The potential part following from calculations in Sect. **??** is

$$\mathcal{L}_{\rm pot} = \frac{N}{4\pi}\left\{\Lambda^2 + 2|\sigma|^2\left[\ln\frac{2|\sigma|^2}{\Lambda^2} - 1 + u\right]\right\}. \tag{2}$$

All corrections to (1) and (2) are suppressed by powers of $1/N$. For what follows it is convenient to introduce a dimensionless variable

$$\mathcal{S} = \frac{\sqrt{2}\sigma}{\Lambda}\,. \tag{3}$$

Then the large-N effective Lagrangian of the σ field takes the form NEPRAVILNO?

$$\mathcal{L}_{\rm eff} = \frac{N}{4\pi}\left\{\frac{1}{2\,|\mathcal{S}|^2}\;|\partial_\mu \mathcal{S}\,|^2 + \Lambda^2\left[1+|\mathcal{S}|^2\left(\ln|\mathcal{S}|^2-1+u\right)\right]\right\}. \tag{4}$$

On the other hand, the Veneziano–Yankielowicz method [?] produces an effective Lagrangian in the Pickwick sense. It realizes, in a superpotential, the anomalous Ward identities of the underlying theory and other symmetries, such as supersymmetry, and gives no information on the kinetic part. In the CP$(N-1)$ models the Veneziano–Yankielowicz superpotential $\mathcal{W}_{\rm VY} = \Sigma\ln\Sigma$ (for twisted superfields) was obtained in [1, ?, ?]. In terms of the scalar potential for the σ field the Veneziano–Yankielowicz construction has the form

$$V_{VY} = \frac{e_\sigma^2}{2}\left|\frac{N}{2\pi}\ln\frac{\sqrt{2}\,\sigma}{\Lambda}\right|^2 + \frac{N}{4\pi}u\,2|\sigma|^2\,. \tag{5}$$

The kinetic term (that's where e_σ^2 comes from) was not determined; however, we can take it in the form obtained in the large-N expansion, see (1), since it is scale invariant and, hence, does not violate Ward identities.

Combining

$$e_\sigma^2 = \frac{4\pi}{N}\,2|\sigma|^2 \tag{6}$$

(see [?]) with (5) we arrive at

$$\begin{aligned}
\mathcal{L}_{VY} &= \frac{N}{4\pi}\,\frac{1}{2|\sigma|^2}\,|\,\partial_\mu\sigma\,|^2 + \frac{N}{4\pi}\left\{2\cdot 2|\sigma|^2\left|\ln\frac{\sqrt{2}\,\sigma}{\Lambda}\right|^2 + 2|\sigma|^2\,u\right\} \\
&= \frac{N}{4\pi}\left\{\frac{1}{2\,|\mathcal{S}|^2}\;|\partial_\mu\mathcal{S}\,|^2 + \Lambda^2\left[2|\mathcal{S}|^2\;|\ln\mathcal{S}|^2 + |\mathcal{S}|^2\,u\right]\right\}.
\end{aligned} \tag{7}$$

It is obvious that the potential in (4) is drastically different from that in (7). For instance, (4) contains a single log, while (7) has the square of this logarithm. We will comment on the difference and the reasons for its appearence [?] later. Now, let us have a closer look at the minima of (4) and (7). The variable $\mathcal{S}$ is complex, and there iare N solutions which differ by the phase,

$$\mathcal{S}_* = |\mathcal{S}_*|\exp\left(\frac{2\pi k}{N}\right)\,,\qquad k=0,1,...,N-1\,, \tag{8}$$

N equivalent vacua. This feature is obvious, and we will omit the phase setting $k = 0$. Thus, we focus on a real solution. The minimum of (4) lies at

$$\mathcal{S}_* = e^{-u/2} \tag{9}$$

while the corresponding value of $V_{\rm eff}$ is

$$V_{\rm eff}(\mathcal{S}_*) = \frac{N}{4\pi}\,\Lambda^2\left(1 - e^{-u}\right). \tag{10}$$

At the same time, the minimum of (7) lies at

$$\mathcal{S}_* = \exp\left(-\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{u}{2}}\right) = e^{-u/2}\left(1 - \frac{u^2}{4} + ...\right) \tag{11}$$

implying that

$$\begin{aligned} V_{\rm VY}(\mathcal{S}_*) &= \frac{N}{4\pi}\,\Lambda^2\,\left(1 - \sqrt{1-2u}\right)\exp\left(-1 + \sqrt{1-2u}\right) \\ &= \frac{N}{4\pi}\,\Lambda^2\left(1 - e^{-u}\right)\left(1 - \frac{u^2}{6} + ...\right). \end{aligned} \tag{12}$$

The σ masses are

$$m_\sigma^2 = \begin{cases} 4\Lambda^2\,e^{-u}\,(1-u)\,, \\ 4\Lambda^2\,e^{-u}\,(1-u)\,(1-u^2+...)\,, \end{cases} \tag{13}$$

for (4) and (7) respectively. The positions of the minima, the σ masses as well as the vacuum energy densities in these two cases differ by $O(u^2)$ in relative units. They coincide in the leading and next-to-leading orders in u, however.

There are two questions to be discussed: (i) why the effective Lagrangians (4) and (7), being essentially different, predict identical vacuum parameters in the leading and next-to-leading order in u; and (ii) why the parameters extracted from the $1/N$ and Veneziano–Yankielowicz Lagrangians diverge from each other at $O(u^2)$ and higher orders.

The answer to the first question can be found in [1]. While the $1/N$ Lagrangian is defined unambiguously, the Veneziano–Yankielowicz method determines only the superpotential part of the action. The kinetic part remains ambiguous. We got used to the fact that variations of the kinetic part affect only terms with derivatives, which are totally irrelevant for the potential part. This is not the case in supersymmtry. The correct statement is that variations of the kinetic part term, in addition to derivative terms, contains terms with $F\bar{F}$, which vanish in the vacuum ($F = 0$) but alter the form of the potential outside the vacuum points (minima of the potential). The only requirement to the kinetic term is that it should obey all Ward identities (including anomalous) of the underlying microscopic theory. For instance, in the case at hand, the simplest choice $\ln\bar{\Sigma}\,\ln\Sigma$ does the job. However,

$$\ln\bar{\Sigma}\,\ln\Sigma\left[1+\frac{(\bar{D}^2\ln\bar{\Sigma})\,(D^2\ln\Sigma)}{\bar{\Sigma}\Sigma}\right]$$

does the job as well. In this latter case there is an additional factor

$$\left[1+\bar{F}F/(\bar{\sigma}^2\sigma^2)+...\right]$$

which reduces to 1 in the points where $F = 0$ and changes the expression for F (and, hence, the scalar potential) outside minima (i.e. at $F \neq 0$).

The answer to the second question is even more evident. The Veneziano–Yankielowicz Lagrangian (7) reflects the Ward identites of the unperturbed CP($N-1$) model. That's the reason why the predictions following from this Lagrangian fail at the level $O(u^2)$, but are valid at the level $O(u)$. We remind the reader that it was shown in [**?**] that the vacuum energy density at the level $O(u)$ is determined by the bifermion condensate in the conventional (unperturbed) CP($N-1$) model.

One last remark is in order here. The kinetic term (1) is not canonic and singular at $\sigma = 0$, implying that this point should be analyzed separately. One can readily cast (1) in the canonic form by a change of variables. Upon this transformation $\sigma \to \tilde{\sigma} = 2\ln\sqrt{2}\sigma/\Lambda$ (assuming for simplicity σ to be real and positivde), the transformed potential (2) develops an extremum at $\sigma = 0$ (i.e. $\tilde{\sigma} \to -\infty$). This extremum is maximum rather than minimum. Indeed, at $u = 0$

$$\tilde{\mathcal{L}}_{\rm pot} = \frac{N\Lambda^2}{4\pi^2}\,(\tilde{\sigma}-1)\,e^{\tilde{\sigma}} + {\rm const.} \qquad (14)$$

It is curious to note that (14) exactly coincides with the (two-dimensional) dilaton effective Lagrangian derived in [2] on the basis of the most general (anomalous) scale Ward identities.

References

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[2] A. Migdal and M. Shifman, Phys. Lett. B **114**, 445 (1982).