1 1-loop Effective Action

We take the Lagrangian of the heterotic CP$(N-1)\times$C sigma model with twisted masses as

$$
\begin{aligned}
\mathcal{L}_{1+1} \;=\;& -\frac{1}{8e^2}\,F_{RL}^2 \;+\; \frac{1}{e^2}\,|\partial_k\sigma|^2 \;+\; \frac{1}{2e^2}\,D^2 \;+\; \frac{1}{e^2}\,\overline{\lambda}_R\, i\partial_L\,\lambda_R \;+\; \frac{1}{e^2}\overline{\lambda}_L\, i\partial_R\,\lambda_L \\
&+\; |\nabla n|^2 \;+\; 2\Big|\sigma - \frac{m^l}{\sqrt{2}}\Big|^2\,\big|n^l\big|^2 \;+\; iD\,\Big(\big|n^l\big|^2 - r_0\Big) \\
&+\; \overline{\xi}_R\, i\nabla_L\xi_R \;+\; \overline{\xi}_L\, i\nabla_R\xi_L \;+\; i\sqrt{2}\,\Big(\sigma - \frac{m^l}{\sqrt{2}}\Big)\overline{\xi}_{Rl}\xi^l_L \;+\; i\sqrt{2}\,\Big(\overline{\sigma} - \frac{\overline{m}^l}{\sqrt{2}}\Big)\overline{\xi}_{Ll}\xi^l_R \\
&+\; i\sqrt{2}\,\overline{\xi_{[R}\,\lambda_{L]}}\,n \;-\; i\sqrt{2}\,\overline{n}\,\lambda_{[R}\,\xi_{L]} \qquad (1.1)\quad \{\texttt{sigma\_full}\} \\
&+\; \overline{\zeta}_R\, i\partial_L\,\zeta_R \;+\; \overline{\mathcal{F}}\,\mathcal{F} \\
&-\; 2i\,\omega\cdot\overline{\lambda}_L\,\zeta_R \;-\; 2i\,\overline{\omega}\cdot\overline{\zeta}_R\,\lambda_L \;+\; 2i\,\omega\cdot\mathcal{F}\,\sigma \;+\; 2i\,\overline{\omega}\cdot\overline{\mathcal{F}}\,\overline{\sigma}\,, \\
& l \;=\; 1,\ldots N\,.
\end{aligned}
$$

We have defined here

$$
\omega \;=\; \sqrt{r_0}\,\delta\,, \qquad\qquad r_0 \;=\; 2\beta\,.
$$

To build the low-energy effective action, we integrate over all but one n^l and ξ^l. We also ignore the gauge field A_μ. Variables n^i and ξ^i, $i = 1,\ldots N-1$, enter the Lagrangian quadratically,

$$
\begin{aligned}
\mathcal{L} \;\supset\;& \overline{n}_i\,\bigg(-\partial_k^2 \;+\; 2\,\Big|\sigma - \frac{m^i}{\sqrt{2}}\Big|^2 \;+\; i\,D\bigg)\,n^i \\
&+\; \begin{pmatrix}\overline{\xi}_{Ri} & \overline{\xi}_{Li}\end{pmatrix}
\begin{pmatrix} i\,\partial_L & i\,\sqrt{2}\,\Big(\sigma - \dfrac{m^i}{\sqrt{2}}\Big) \\ i\,\sqrt{2}\,\Big(\overline{\sigma} - \dfrac{\overline{m}^i}{\sqrt{2}}\Big) & i\,\partial_R \end{pmatrix}
\begin{pmatrix}\xi^i_R \\ \xi^i_L\end{pmatrix}.
\end{aligned}
$$

Integrating over these variables produces the determinant

$$
\prod_{i=1}^{N-1}\frac{\det\bigg(-\partial_k^2 \;+\; 2\,\Big|\sigma - \dfrac{m^i}{\sqrt{2}}\Big|^2\bigg)}{\det\bigg(-\partial_k^2 \;+\; 2\,\Big|\sigma - \dfrac{m^i}{\sqrt{2}}\Big|^2 \;+\; i\,D\bigg)}\,.
$$

Evaluating this determinant at one loop, and denoting $n^N \equiv n$, and its mass $m_0 \equiv$

m^N, one arrives at the effective potential

$$\begin{aligned} V_{\rm eff} \;=\; \int d^2x \Bigg(& \left(iD \,+\, 2\left|\sigma - \frac{m_0}{\sqrt{2}}\right|^2 \right) |n|^2 \\ & - \frac{1}{4\pi} \sum_{i=1}^{N-1} \left(iD \,+\, 2\left|\sigma - \frac{m^i}{\sqrt{2}}\right|^2 \right) \ln \frac{iD + 2|\sigma - \frac{m^i}{\sqrt{2}}|^2}{\Lambda^2} \\ & + \frac{1}{4\pi} \sum_{i=1}^{N-1} 2\left|\sigma - \frac{m^i}{\sqrt{2}}\right|^2 \ln \frac{2\,|\sigma - \frac{m^i}{\sqrt{2}}|^2}{\Lambda^2} \,+\, \frac{1}{4\pi}\, iD\,(N-1) \\ & + 4\,|\omega|^2\,|\sigma|^2 \Bigg) . \end{aligned} \tag{1.2}$$ {Veff}

Minimizing the potential (1.2) with respect to n, D and σ, one arrives at the vacuum equations

$$|n|^2 \;-\; \frac{1}{4\pi} \sum_{i}^{N-1} \log \frac{iD \,+\, 2\left|\sigma - \frac{m^i}{\sqrt{2}}\right|^2}{\Lambda^2} \;=\; 0 \tag{1.3}$$ {eff1}

$$\left(iD \;+\; 2\left|\sigma - \frac{m_0}{\sqrt{2}}\right|^2 \right) n \;=\; 0 \tag{1.4}$$ {eff2}

$$\left(\sigma - \frac{m_0}{\sqrt{2}}\right)|n|^2 \;-\; \frac{1}{4\pi} \sum_{i}^{N-1} \left(\sigma - \frac{m^i}{\sqrt{2}}\right) \log \frac{iD \,+\, 2\left|\sigma \,-\, \frac{m^i}{\sqrt{2}}\right|^2}{2\left|\sigma - \frac{m^i}{\sqrt{2}}\right|^2} \;+\; 2\,|\omega|^2\sigma \;=\; 0\,. \tag{1.5}$$ {eff3}

Immediately we observe that Eq. (1.4) implies two distinct cases,

- $$iD \,+\, 2\left|\sigma - \frac{m_0}{\sqrt{2}}\right|^2 \;=\; 0 \tag{1.6}$$ {higgsph}

  and

- $$n \;=\; 0\,. \tag{1.7}$$ {strongph}

These cases correspond to the Higgs and the strong-coupling phases of the theory. We solve these equations perturbatively, assuming $|\omega|^2$ a small parameter,

$$\begin{aligned} n \;&=\; n^{(0)} \;+\; |\omega|^2\, n^{(1)} \;+\; \dots\,, \\ iD \;&=\; iD^{(0)} \;+\; |\omega|^2\, iD^{(1)} \;+\; \dots\,, \\ \sigma \;&=\; \sigma^{(0)} \;+\; |\omega|^2\, \sigma^{(1)} \;+\; \dots\,. \end{aligned}$$

Here $n^{(0)}$, $D^{(0)}$ and $\sigma^{(0)}$ constitute the solution of the $\mathcal{N}=(2,2)$ CP$(N-1)$ sigma model, in particular $D^{(0)}=0$ in both phases.

To obtain simple expressions for the solution we will assume the masses to be sitting on the circle,

$$m^k \;=\; m\cdot e^{i2\pi k/N}\,, \qquad\qquad k \;=\; 0,\ldots N-1\,.$$

2 Higgs Phase

The large-N supersymmetric solution of the $\mathcal{N}=(2,2)$ CP$(N-1)$ sigma model in the Higgs phase is

$$\begin{aligned}
n^{(0)} &= \sqrt{r^{(0)}_{\rm ren}}\,, \qquad\qquad \textit{the phase of } n^{(0)} \textit{ is not determined}\\
iD^{(0)} &= 0\,,\\
\sigma^{(0)} &= \frac{m_0}{\sqrt{2}}\,,
\end{aligned}$$

where $r^{(0)}_{\rm ren}$ is the renormalized coupling of the unperturbed theory,

$$r^{(0)}_{\rm ren} \;=\; \frac{N}{2\pi}\log m/\Lambda\,.$$

Expanding equations (1.3)-(1.5) to the first order in $|\omega|^2$, we obtain

$$\begin{aligned}
\overline{n}^{(0)}\,n^{(1)} \;+\; \text{h.c.} \;&=\; \frac{1}{4\pi}\sum_i^{N-1}\frac{iD^{(1)} \;+\; 2\left(\overline{\sigma}^{(0)}\,\sigma^{(1)} \;-\; \overline{\sigma}^{(1)}\,\frac{m^i}{\sqrt{2}} \;+\; \text{h.c.}\right)}{2\left|\sigma^{(0)}-\frac{m^i}{\sqrt{2}}\right|^2}\,,\\
iD^{(1)} \;&=\; 0\,, \qquad\qquad (2.8)\ \{\texttt{higgseq}\}\\
\sigma^{(1)}\,|n^{(0)}|^2 \;-\; \frac{1}{4\pi}\,iD^{(1)}\sum_i^{N-1}&\frac{1}{2\left(\overline{\sigma}^{(0)}-\frac{\overline{m}^i}{\sqrt{2}}\right)} \;+\; 2\,\sigma^{(0)} \;=\; 0\,.
\end{aligned}$$

We can see that iD vanishes to the first order in $|\omega|^2$. Thus, for iD the first order of expansion is not sufficient, since we know that supersymmetry is broken and hence $iD\;\neq\;0$. This variable, however, is easy to recover in the Higgs phase given the corresponding expansion of σ via Eq. (1.6).

The solution to equations (2.8) can be written as

$$iD^{(0)} = 0\,, \qquad iD^{(1)} = 0\,, \qquad iD^{(2)} = -\,2\,|\sigma^{(1)}|^2\,,$$

$$\sigma^{(0)} = \frac{m_0}{\sqrt{2}}\,, \qquad \sigma^{(1)} = -\,\frac{2\sigma^{(0)}}{|n^{(0)}|^2}\,,$$

$$|n^{(0)}|^2 = r^{(0)}_{\rm ren}\,, \qquad n^{(1)} = -\,\frac{2\,m^0}{\overline{n}^{(0)}\,|n^{(0)}|^2}\,\frac{1}{4\pi}\sum_i^{N-1}\frac{1}{m^0-m^i}\,.$$

To simplify these expressions, as noted before, we put the masses on the circle, which gives then

$$\sum_{i=1}^{N-1}\frac{1}{m^0-m^i} = \frac{N-1}{2m} = \frac{N}{2m} + O(1)\,.$$

Finally, we obtain

$$\sigma = \frac{m^0}{\sqrt{2}}\left(1 - \frac{2\,|\omega|^2}{|n^{(0)}|^2}\right) + \ldots\,,$$

$$iD = -\,4\,\frac{m_0^2}{(r^{(0)}_{\rm ren})^2}\,|\omega|^4 + \ldots\,,$$

$$n = \sqrt{r^{(0)}_{\rm ren}} - \frac{N}{4\pi}\,\frac{1}{r^{(0)}_{\rm ren}\,\overline{n}^{(0)}}\,|\omega|^2 + \ldots$$

for the Higgs phase, where

$$r^{(0)}_{\rm ren} = \frac{N}{2\pi}\log m/\Lambda\,.$$

3 Strong Coupled Phase

The zeroth order in $|\omega|^2$ solution is

$$n^{(0)} = 0\,,$$

$$iD^{(0)} = 0\,,$$

$$\sigma^{(0)} = \frac{\widetilde{\Lambda}}{\sqrt{2}}\cdot e^{i\frac{2\pi l}{N}}\,, \qquad \widetilde{\Lambda} = \sqrt[N]{\Lambda^N + m^N}\,,$$

for some fixed $l = 0, \ldots N-1$. Furthermore, in this phase n is known exactly,

$$n = 0\,.$$

The rest two equations in (1.3)-(1.5) give at the first order in $|\omega|^2$

$$\sum_i^{N-1} \frac{iD^{(1)} \;+\; 2\left(\overline{\sigma}^{(0)}\,\sigma^{(1)} \;-\; \sigma^{(1)}\,\frac{\overline{m}^i}{\sqrt{2}} \;+\; \text{h.c.}\right)}{2\left|\sigma^{(0)} - \frac{m^i}{\sqrt{2}}\right|^2} \;=\; 0\,,$$

$$\frac{1}{4\pi}\sum_i^{N-1} \frac{iD^{(1)}}{2\left(\overline{\sigma}^{(0)} - \frac{\overline{m}^i}{\sqrt{2}}\right)} \;=\; 2\,\sigma^{(0)}\,.$$

The solution to these equations are given by

$$n \;=\; 0\,, \qquad\qquad iD^{(0)} \;=\; 0\,,$$

$$iD^{(1)} \;=\; 8\pi\cdot\frac{2\sigma^{(0)}}{\sum_i^{N-1}\frac{1}{\overline{\sigma}^{(0)} - \overline{m}^i/\sqrt{2}}}\,, \qquad\qquad i \;=\; 1,\ldots N-1\,, \tag{3.9}$$ {strongeq}

$$\sigma^{(1)}\cdot\sum_i^{N-1}\frac{1}{\sigma^{(0)} - \frac{m^i}{\sqrt{2}}} \;+\; \text{h.c.} \;=\; -\;8\pi\cdot\sigma^{(0)}\cdot\frac{\sum_i^{N-1}\frac{1}{\left|\sigma^{(0)} - \frac{m^i}{\sqrt{2}}\right|^2}}{\sum_i^{N-1}\frac{1}{\overline{\sigma}^{(0)} - \frac{\overline{m}^i}{\sqrt{2}}}}\,.$$

We use the following relations to simplify the above solution in the case when the masses are distributed on a circle,

$$\sum_{k=0}^{N-1}\frac{1}{1\;-\;\alpha\,e^{\frac{2\pi i k}{N}}} \;=\; \frac{N}{1\;-\;\alpha^N}\,,$$

$$\sum_{k=0}^{N-1}\frac{1}{(1+\alpha^2)\;-\;2\,\alpha\cos\frac{2\pi k}{N}} \;=\; \frac{1}{1\;-\;\alpha^2}\left(\frac{2\,N}{1\;-\;\alpha^N}\;-\;N\right)\,.$$

This gives,

$$\sum_{k=1}^{N-1}\frac{1}{\sigma^{(0)} - \frac{m^k}{\sqrt{2}}} \;=\; -\;\frac{1}{\sigma^{(0)} - \frac{m}{\sqrt{2}}} \;+\; N\,\frac{\widetilde{\Lambda}^N}{\Lambda^N}\,\frac{1}{\sigma^{(0)}}\,,$$

$$\sum_{k=1}^{N-1}\frac{1}{\left|\sigma^{(0)} - \frac{m^k}{\sqrt{2}}\right|^2} \;=\; -\;\frac{1}{\left|\sigma^{(0)} - \frac{m}{\sqrt{2}}\right|^2} \;+\; \frac{2\,N}{\widetilde{\Lambda}^2 - m^2}\cdot\frac{\widetilde{\Lambda}^N + m^N}{\Lambda^N}\,.$$

In fact we will only need the leading-N contribution from them above relations. In particular, if we substitute these relations into Eq. (3.9) directly, we will not have

iD real. The reason *perhaps* is that the $\sigma^{(0)}$ solution is only valid up to $O(1/N)$ contributions. Therefore, we ignore the O(1) contributions versus O(N).

$$
\begin{aligned}
iD^{(1)} &= \frac{16\pi}{N}\,\frac{\Lambda^N}{\widetilde{\Lambda}^N}\,|\sigma^{(0)}|^2\,, \\
|\sigma^{(1)}| &= -\,\sqrt{2}\,\frac{2\pi}{N}\,\frac{\Lambda^N}{\widetilde{\Lambda}^N}\,\frac{\widetilde{\Lambda}^3}{\widetilde{\Lambda}^2 - m^2}\left[1 \,+\, \frac{m^N}{\widetilde{\Lambda}^N}\right]\cdot\left(\cos\Big(\frac{2\pi l}{N} - \varphi\Big)\right)^{-1},
\end{aligned}
$$

where φ is the arbitrary phase of $\sigma^{(1)}$. We can choose this phase to be the same as that of $\sigma^{(0)}$,

$$
\varphi \;\equiv\; \frac{2\pi l}{N}
$$

for the cosine to disappear. If we take the limit of small masses

$$
\frac{m^N}{\Lambda^N} \;\ll\; 1\,,
$$

and also note that

$$
\widetilde{\Lambda} \;=\; \Lambda
$$

with exponential accuracy in $1/N$, we can further simplify the result, and arrive at

$$
\begin{aligned}
n &= 0 \\
iD &= |\omega|^2\cdot\frac{8\pi}{N}\cdot\Lambda^2 \,+\, \ldots\,, \qquad\qquad \text{for } \frac{m^N}{\Lambda^N} \;\ll\; 1\,, \\
\sigma &= \frac{\Lambda}{\sqrt{2}}\cdot e^{\frac{2\pi i l}{N}} \,-\, |\omega|^2\,\sqrt{2}\,\frac{2\pi}{N}\,\frac{\Lambda^3}{\Lambda^2 - m^2}\,e^{\frac{2\pi i l}{N}}\,.
\end{aligned}
$$

Note that although $|\omega|^2$ grows as $O(N)$ for large N, the coefficients of $|\omega|^2$-corrections are suppressed by the corresponding power of $1/N$ so that the corrections are neutral in N.

References

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