Vacuum Energy in the Higgs Phase

$$\frac{4\pi}{N}\frac{E_{vac}}{\Lambda^2} = \frac{1}{N}\sum |S-\mu^k|^2 \log |S-\mu^k|^2 -$$

$$-\frac{1}{N}\sum S'\left[(\mu-\mu^k)+(\bar\mu-\bar\mu^k)\right]\log\left\{4\mu S \sin^2\frac{\alpha_k}{2}\right\} -$$

$$-(S-\mu)^2 + u S'^2$$

we know $\frac{1}{N}\sum(\mu-\mu^k)\ln\left(4\mu^2\sin^2\frac{\alpha_k}{2}\right) = \mu\left(\ln\mu^2+1\right)$ (1.1)

$\Rightarrow$ two terms to consider

- $\frac{1}{N}\sum |S-\mu^k|^2 \log|S-\mu^k|^2 = \frac{1}{N}\sum\left[S^2+\mu^2 - S(\mu^k+\bar\mu^k)\right]\log|S-\mu^k|^2 =$

$$= \frac{1}{N}\sum\left[\mu^2 - S^2 + S^2-\mu^k S' + S^2 - \bar\mu^k S'\right]\log|S-\mu^k|^2 =$$

$$= \frac{1}{N}\sum\left[(\mu^2-S'^2) + S'(S-\mu^k) + S'(S-\bar\mu^k)\right]\log|S-\mu^k|^2 =$$

= we can now use the summation formulas, $|S|<|\mu|$ =

$$\boxed{\frac{1}{N}\sum(S-\mu^k)\ln|S-\mu^k|^2 = S\{1+\ln\mu^2\},\ |S|<|\mu| \quad (1.5)}$$

$$= (\mu^2-S'^2)\log\mu^2 +$$

$$+ 2S'^2\left[1+\ln\mu^2\right] = \mu^2\ln\mu^2 + S'^2\ln\mu^2 + 2S'^2 =$$

$$= (\mu^2+S'^2)\ln\mu^2 + 2S'^2$$

- $-\frac{1}{N}\sum \delta \left[(\mu-\mu^k)+(\mu-\bar{\mu}^k)\right] \operatorname{Log}\left\{4\mu\delta \cdot \sin^2\frac{\alpha_k}{2}\right\} =$

$$
\left|\begin{array}{l}
\frac{1}{N}\sum(\mu-\mu^k)\operatorname{Log}\left(4\mu\delta\cdot\sin^2\frac{\alpha_k}{2}\right) = \\
= \frac{1}{N}\sum(\mu-\mu^k)\left\{\operatorname{Log}\left(4\mu^2\cdot\sin^2\frac{\alpha_k}{2}\right)+\operatorname{Log}\delta/\mu\right\} = \\
= \mu(\ln\mu^2+1)+\mu\operatorname{Log}\delta/\mu = \\
= \mu(1+\ln(\mu\delta))
\end{array}\right.
$$

$$= -2\mu\delta(1+\ln(\mu\delta));$$

Altogether,

$$\frac{4\pi}{N}\frac{E_{vac}}{\Lambda^2} = (\mu^2+\delta^2)\ln\mu^2+2\delta^2-2\mu\delta(1+\ln(\mu\delta)) =$$

$$= (\delta-\mu)^2+u\delta^2$$

next we use the vacuum equations $(1+u+\ln\mu^2)\delta = \mu(1+\ln(\mu\delta))$

$$\frac{4G}{N}\frac{E_{vac}}{\Lambda^2} = (\mu^2 + S^2)\ln\mu^2 + 2S^2 - 2S^2(1+u+\ln\mu^2) + (S-\mu)^2 + u S^2 =$$

$$= \mu^2\ln\mu^2 + S^2\ln\mu^2 + \cancel{2S^2} - \cancel{2S^2} - u\cdot 2S^2 - 2S^2\ln\mu^2$$

$$\underbrace{- S^2 - \mu^2 + 2\mu S}_{\text{keep}} + u S^2 =$$

$$= \mu^2\ln\mu^2 - S^2\ln\mu^2 - uS^2 - (\mu - S)^2 =$$

$$= (\mu^2 - S^2)\ln\mu^2 - (\mu - S)^2 - uS^2$$

$$\boxed{\frac{4G}{N}\frac{E_{vac}}{\Lambda^2} = (\mu^2 - S^2)\ln\mu^2 - (\mu - S)^2 - uS^2}$$

(4)

At the Coulomb-Higgs phase transition,

$$\mu_* = 1/s_* \quad \text{and} \quad \mu_*^2 - \ln \mu_*^2 = 1 + u$$

$$\Rightarrow \ln \mu_*^2 = \mu_*^2 - 1 - u$$

$$\mathcal{E}_{\text{Higgs}}\Big|^{\text{phase transition}} = \frac{4\pi}{N} \frac{E_{vac}}{\Lambda^2} =$$

$$= \left(\mu^2 - \frac{1}{\mu^2}\right)\left\{\mu^2 - 1 - u\right\} - (\mu - 1/\mu)^2 - u\frac{1}{\mu^2} =$$

$$= \mu^4 - \mu^2 - u\mu^2 - 1 + \cancel{\frac{1}{\mu^2}} + \cancel{u\frac{1}{\mu^2}} - \mu^2 - \cancel{\frac{1}{\mu^2}} + 2 - \cancel{u\frac{1}{\mu^2}} =$$

$$= \mu_*^4 - \underline{2\mu_*^2} - u\mu_*^2 + 1$$

$$\mathcal{E}_{\text{Coulomb}}\Big|_{\text{phase transition}} = 1 - \mu^2 + \mu^2 \ln \mu^2 =$$

$$= 1 - \mu^2 + \mu^2\left[\mu^2 - 1 - u\right] =$$

$$= 1 - \mu^2 + \mu^4 - \mu^2 - u\mu^2 = \mu_*^4 - \underline{2\mu_*^2} - u\mu_*^2 + 1$$

exact same answer